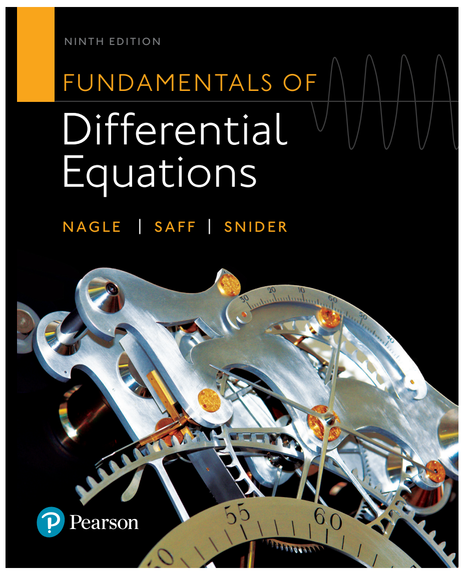
A Solution Manual For

Fundamentals of Differential Equations. By Nagle, Saff and Snider. 9th edition. Boston. Pearson 2018.



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May 15, 2024

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1.1 problem 1

Internal problem ID [4912]
Internal file name [`OUTPUT/4405_Sunday_June_05_2022_01_15_20_PM_58481910/index.tex`]

Book: Fundamentals of Differential Equations. By Nagle, Saff and Snider. 9th edition. Boston. Pearson 2018.
Section: Chapter 2, First order differential equations. Section 2.2, Separable Equations. Exercises. page 46
Problem number: 1.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"first_order_ode_lie_symmetry_calculated"**

Maple gives the following as the ode type

```
[[_homogeneous, `class C`], _dAlembert]
```

$$\boxed{y' - \sin(x + y) = 0}$$

1.1.1 Solving as first order ode lie symmetry calculated ode

Writing the ode as

$$\begin{aligned} y' &= \sin(x + y) \\ y' &= \omega(x, y) \end{aligned}$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x + \omega(\eta_y - \xi_x) - \omega^2\xi_y - \omega_x\xi - \omega_y\eta = 0 \tag{A}$$

The type of this ode is not in the lookup table. To determine ξ, η then (A) is solved using ansatz. Making bivariate polynomials of degree 1 to use as anstaz gives

$$\xi = xa_2 + ya_3 + a_1 \tag{1E}$$

$$\eta = xb_2 + yb_3 + b_1 \tag{2E}$$

Where the unknown coefficients are

$$\{a_1, a_2, a_3, b_1, b_2, b_3\}$$

Substituting equations (1E,2E) and ω into (A) gives

$$\begin{aligned} &b_2 + \sin(x+y)(b_3 - a_2) - \sin(x+y)^2 a_3 \\ &- \cos(x+y)(xa_2 + ya_3 + a_1) - \cos(x+y)(xb_2 + yb_3 + b_1) = 0 \end{aligned} \tag{5E}$$

Putting the above in normal form gives

$$\begin{aligned} &-\cos(x+y)xa_2 - \cos(x+y)xb_2 - \cos(x+y)ya_3 - \cos(x+y)yb_3 - \sin(x+y)^2 a_3 \\ &- \cos(x+y)a_1 - \cos(x+y)b_1 - \sin(x+y)a_2 + \sin(x+y)b_3 + b_2 = 0 \end{aligned}$$

Setting the numerator to zero gives

$$\begin{aligned} &-\cos(x+y)xa_2 - \cos(x+y)xb_2 - \cos(x+y)ya_3 \\ &- \cos(x+y)yb_3 - \sin(x+y)^2 a_3 - \cos(x+y)a_1 \\ &- \cos(x+y)b_1 - \sin(x+y)a_2 + \sin(x+y)b_3 + b_2 = 0 \end{aligned} \tag{6E}$$

Simplifying the above gives

$$\begin{aligned} &b_2 - \frac{a_3}{2} - \cos(x+y)xa_2 - \cos(x+y)xb_2 - \cos(x+y)ya_3 \\ &- \cos(x+y)yb_3 + \frac{a_3\cos(2y+2x)}{2} - \cos(x+y)a_1 \\ &- \cos(x+y)b_1 - \sin(x+y)a_2 + \sin(x+y)b_3 = 0 \end{aligned} \tag{6E}$$

Looking at the above PDE shows the following are all the terms with $\{x, y\}$ in them.

$$\{x, y, \cos(x+y), \cos(2y+2x), \sin(x+y)\}$$

The following substitution is now made to be able to collect on all terms with $\{x, y\}$ in them

$$\{x = v_1, y = v_2, \cos(x+y) = v_3, \cos(2y+2x) = v_4, \sin(x+y) = v_5\}$$

The above PDE (6E) now becomes

$$b_2 - \frac{1}{2}a_3 - v_3v_1a_2 - v_3v_1b_2 - v_3v_2a_3 - v_3v_2b_3 + \frac{1}{2}a_3v_4 - v_3a_1 - v_3b_1 - v_5a_2 + v_5b_3 = 0 \tag{7E}$$

Collecting the above on the terms v_i introduced, and these are

$$\{v_1, v_2, v_3, v_4, v_5\}$$

Equation (7E) now becomes

$$b_2 - \frac{a_3}{2} + (b_3 - a_2) v_5 + (-a_1 - b_1) v_3 + \frac{a_3 v_4}{2} + (-a_2 - b_2) v_1 v_3 + (-a_3 - b_3) v_2 v_3 = 0 \quad \text{(8E)}$$

Setting each coefficients in (8E) to zero gives the following equations to solve

$$\begin{aligned} \frac{a_3}{2} &= 0 \\ -a_1 - b_1 &= 0 \\ -a_2 - b_2 &= 0 \\ -a_3 - b_3 &= 0 \\ b_2 - \frac{a_3}{2} &= 0 \\ b_3 - a_2 &= 0 \end{aligned}$$

Solving the above equations for the unknowns gives

$$\begin{aligned} a_1 &= -b_1 \\ a_2 &= 0 \\ a_3 &= 0 \\ b_1 &= b_1 \\ b_2 &= 0 \\ b_3 &= 0 \end{aligned}$$

Substituting the above solution in the anstaz (1E,2E) (using 1 as arbitrary value for any unknown in the RHS) gives

$$\begin{aligned} \xi &= -1 \\ \eta &= 1 \end{aligned}$$

Shifting is now applied to make $\xi = 0$ in order to simplify the rest of the computation

$$\begin{aligned} \eta &= \eta - \omega(x, y)\,\xi \\ &= 1 - (\sin(x + y))(-1) \\ &= 1 + \sin(x)\cos(y) + \cos(x)\sin(y) \\ \xi &= 0 \end{aligned}$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x, y) \to (R, S)$ where (R, S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y}\right) S(x, y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$R = x$$

S is found from

$$\begin{aligned} S &= \int \frac{1}{\eta} dy \\ &= \int \frac{1}{1 + \sin(x) \cos(y) + \cos(x) \sin(y)} dy \end{aligned}$$

Which results in

$$S = -\frac{2}{\tan\left(\frac{x}{2} + \frac{y}{2}\right) + 1}$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x, y) S_y}{R_x + \omega(x, y) R_y} \tag{2}$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x, y)$ is the right hand side of the original ode given by

$$\omega(x, y) = \sin(x + y)$$

Evaluating all the partial derivatives gives

$$\begin{aligned} R_x &= 1 \\ R_y &= 0 \\ S_x &= \frac{1}{1 + \sin(x + y)} \\ S_y &= \frac{1}{1 + \sin(x + y)} \end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = 1 \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = 1$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$S(R) = R + c_1 \tag{4}$$

To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$-\frac{2}{\tan\left(\frac{x}{2} + \frac{y}{2}\right) + 1} = x + c_1$$

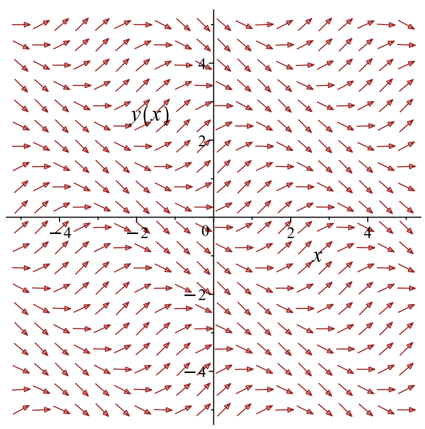
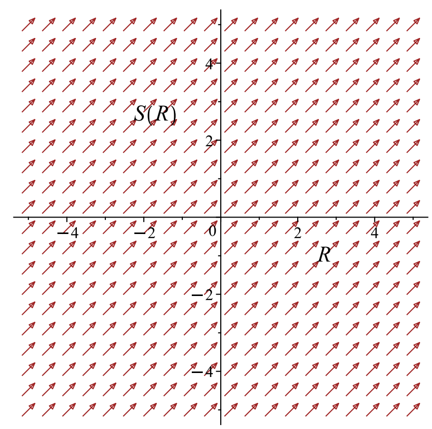
Which simplifies to

$$-\frac{2}{\tan\left(\frac{x}{2} + \frac{y}{2}\right) + 1} = x + c_1$$

Which gives

$$y = -x - 2\arctan\left(\frac{c_1 + x + 2}{x + c_1}\right)$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in x, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dx} = \sin(x + y)$  	$R = x$ $S = -\frac{2}{\tan\left(\frac{x}{2} + \frac{y}{2}\right) + 1}$	$\frac{dS}{dR} = 1$ 

Summary

The solution(s) found are the following

$$y = -x - 2\arctan\left(\frac{c_1 + x + 2}{x + c_1}\right) \tag{1}$$

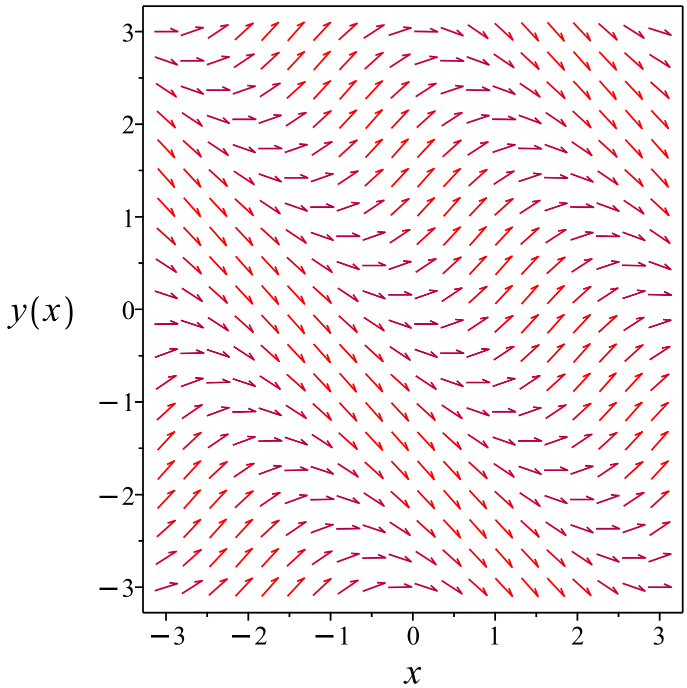


Figure 1: Slope field plot

Verification of solutions

$$y = -x - 2\arctan\left(\frac{c_1 + x + 2}{x + c_1}\right)$$

Verified OK.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
trying homogeneous types:
trying homogeneous C
1st order, trying the canonical coordinates of the invariance group
<- 1st order, canonical coordinates successful
<- homogeneous successful`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 25

```
dsolve(diff(y(x),x)-sin(x+y(x))=0,y(x), singsol=all)
```

$$y(x) = -x - 2\arctan\left(\frac{c_1 - x - 2}{-x + c_1}\right)$$

✓ Solution by Mathematica

Time used: 36.293 (sec). Leaf size: 541

```
DSolve[y'[x]-Sin[x+y[x]]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to -2\arccos\left(\frac{(x+c_1)\sin\left(\frac{x}{2}\right)-(x-2+c_1)\cos\left(\frac{x}{2}\right)}{\sqrt{2}\sqrt{x^2+2(-1+c_1)x+2+c_1{}^2-2c_1}}\right)$$
$$y(x) \to 2\arccos\left(\frac{(x+c_1)\sin\left(\frac{x}{2}\right)-(x-2+c_1)\cos\left(\frac{x}{2}\right)}{\sqrt{2}\sqrt{x^2+2(-1+c_1)x+2+c_1{}^2-2c_1}}\right)$$
$$y(x) \to -2\arccos\left(\frac{(x-2+c_1)\cos\left(\frac{x}{2}\right)-(x+c_1)\sin\left(\frac{x}{2}\right)}{\sqrt{2}\sqrt{x^2+2(-1+c_1)x+2+c_1{}^2-2c_1}}\right)$$
$$y(x) \to 2\arccos\left(\frac{(x-2+c_1)\cos\left(\frac{x}{2}\right)-(x+c_1)\sin\left(\frac{x}{2}\right)}{\sqrt{2}\sqrt{x^2+2(-1+c_1)x+2+c_1{}^2-2c_1}}\right)$$
$$y(x) \to -2\arccos\left(\frac{\cos\left(\frac{x}{2}\right)-\sin\left(\frac{x}{2}\right)}{\sqrt{2}}\right)$$
$$y(x) \to 2\arccos\left(\frac{\cos\left(\frac{x}{2}\right)-\sin\left(\frac{x}{2}\right)}{\sqrt{2}}\right)$$
$$y(x) \to -2\arccos\left(\frac{\sin\left(\frac{x}{2}\right)-\cos\left(\frac{x}{2}\right)}{\sqrt{2}}\right)$$
$$y(x) \to 2\arccos\left(\frac{\sin\left(\frac{x}{2}\right)-\cos\left(\frac{x}{2}\right)}{\sqrt{2}}\right)$$
$$y(x) \to -2\arccos\left(\frac{(x-2)\cos\left(\frac{x}{2}\right)-x\sin\left(\frac{x}{2}\right)}{\sqrt{2}\sqrt{x^2-2x+2}}\right)$$
$$y(x) \to 2\arccos\left(\frac{(x-2)\cos\left(\frac{x}{2}\right)-x\sin\left(\frac{x}{2}\right)}{\sqrt{2}\sqrt{x^2-2x+2}}\right)$$
$$y(x) \to -2\arccos\left(\frac{x\sin\left(\frac{x}{2}\right)-(x-2)\cos\left(\frac{x}{2}\right)}{\sqrt{2}\sqrt{x^2-2x+2}}\right)$$
$$y(x) \to 2\arccos\left(\frac{x\sin\left(\frac{x}{2}\right)-(x-2)\cos\left(\frac{x}{2}\right)}{\sqrt{2}\sqrt{x^2-2x+2}}\right)$$

1.2 problem 2

Internal problem ID [4913]
Internal file name [OUTPUT/4406_Sunday_June_05_2022_01_15_52_PM_16729565/index.tex]

Book: Fundamentals of Differential Equations. By Nagle, Saff and Snider. 9th edition. Boston. Pearson 2018.
Section: Chapter 2, First order differential equations. Section 2.2, Separable Equations. Exercises. page 46
Problem number: 2.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"quadrature"**

Maple gives the following as the ode type

```
[_quadrature]
```

$$\boxed{y' - 4y^2 + 3y = 1}$$

1.2.1 Solving as quadrature ode

Integrating both sides gives

$$\int \frac{1}{4y^2 - 3y + 1} dy = x + c_1$$
$$\frac{2\sqrt{7}\,\arctan\left(\frac{(8y-3)\sqrt{7}}{7}\right)}{7} = x + c_1$$

Solving for y gives these solutions

$$y_1 = \frac{\left(3\sqrt{7} + 7\tan\left(\frac{(x+c_1)\sqrt{7}}{2}\right)\right)\sqrt{7}}{56}$$

Summary
The solution(s) found are the following

$$y = \frac{\left(3\sqrt{7} + 7\tan\left(\frac{(x+c_1)\sqrt{7}}{2}\right)\right)\sqrt{7}}{56} \tag{1}$$

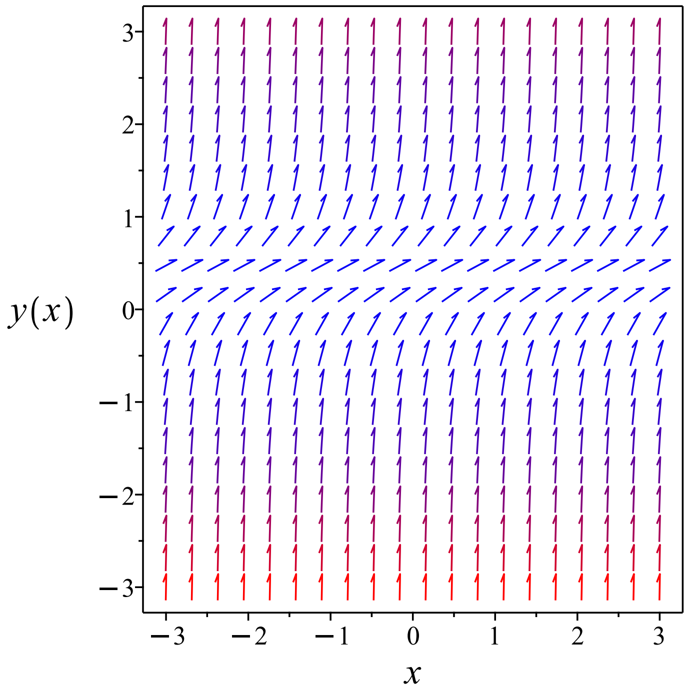


Figure 2: Slope field plot

Verification of solutions

$$y = \frac{\left(3\sqrt{7}+7\tan\left(\frac{(x+c_1)\sqrt{7}}{2}\right)\right)\sqrt{7}}{56}$$

Verified OK.

1.2.2 Maple step by step solution

Let's solve

$y' - 4y^2 + 3y = 1$

- Highest derivative means the order of the ODE is 1

  y'

- Separate variables

  $\frac{y'}{4y^2-3y+1} = 1$

- Integrate both sides with respect to x

  $\int \frac{y'}{4y^2-3y+1}dx = \int 1dx + c_1$

- Evaluate integral

$$\frac{2\sqrt{7}\arctan\left(\frac{(8y-3)\sqrt{7}}{7}\right)}{7} = x + c_1$$

- Solve for y

$$y = \frac{\left(3\sqrt{7}+7\tan\left(\frac{(x+c_1)\sqrt{7}}{2}\right)\right)\sqrt{7}}{56}$$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
<- separable successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 20

```
dsolve(diff(y(x),x)=4*y(x)^2-3*y(x)+1,y(x), singsol=all)
```

$$y(x) = \frac{3}{8} + \frac{\sqrt{7}\tan\left(\frac{(x+c_1)\sqrt{7}}{2}\right)}{8}$$

✓ Solution by Mathematica

Time used: 1.272 (sec). Leaf size: 69

```
DSolve[y'[x]==4*y[x]^2-3*y[x]+1,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \frac{1}{8}\left(3+\sqrt{7}\tan\left(\frac{1}{2}\sqrt{7}(x+c_1)\right)\right)$$
$$y(x) \to \frac{1}{8}\left(3-i\sqrt{7}\right)$$
$$y(x) \to \frac{1}{8}\left(3+i\sqrt{7}\right)$$

1.3 problem 3

Internal problem ID [4914]
Internal file name [`OUTPUT/4407_Sunday_June_05_2022_01_16_02_PM_26610584/index.tex`]

Book: Fundamentals of Differential Equations. By Nagle, Saff and Snider. 9th edition. Boston. Pearson 2018.
Section: Chapter 2, First order differential equations. Section 2.2, Separable Equations. Exercises. page 46
Problem number: 3.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"unknown"**

Maple gives the following as the ode type

`` [`y=_G(x,y')`] ``

Unable to solve or complete the solution.

$$\boxed{s' - t\ln\left(s^{2t}\right) = 8t^2}$$

Unable to determine ODE type.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
trying homogeneous types:
trying Chini
differential order: 1; looking for linear symmetries
trying exact
Looking for potential symmetries
trying inverse_Riccati
trying an equivalence to an Abel ODE
differential order: 1; trying a linearization to 2nd order
--- trying a change of variables {x -> y(x), y(x) -> x}
differential order: 1; trying a linearization to 2nd order
trying 1st order ODE linearizable_by_differentiation
--- Trying Lie symmetry methods, 1st order ---
`, `-> Computing symmetries using: way = 3
`, `-> Computing symmetries using: way = 5
trying symmetry patterns for 1st order ODEs
-> trying a symmetry pattern of the form [F(x)*G(y), 0]
-> trying a symmetry pattern of the form [0, F(x)*G(y)]
-> trying symmetry patterns of the forms [F(x),G(y)] and [G(y),F(x)]
-> trying a symmetry pattern of the form [F(x),G(x)]
-> trying a symmetry pattern of the form [F(y),G(y)]
-> trying a symmetry pattern of the form [F(x)+G(y), 0]
-> trying a symmetry pattern of the form [0, F(x)+G(y)]
-> trying a symmetry pattern of the form [F(x),G(x)*y+H(x)]
-> trying a symmetry pattern of conformal type`
```

✗ Solution by Maple

```
dsolve(diff(s(t),t)=t*ln(s(t)^(2*t))+8*t^2,s(t), singsol=all)
```

No solution found

✓ Solution by Mathematica

Time used: 0.28 (sec). Leaf size: 34

```
DSolve[s'[t]==t*Log[s[t]^(2*t)]+8*t^2,s[t],t,IncludeSingularSolutions -> True]
```

$$s(t) \to \text{InverseFunction}\left[\frac{\text{ExpIntegralEi}(\log(\#1)+4)}{e^4}\&\right]\left[\frac{2t^3}{3}+c_1\right]$$

$$s(t) \to \frac{1}{e^4}$$

1.4 problem 4

Internal problem ID [4915]
Internal file name [`OUTPUT/4408_Sunday_June_05_2022_01_16_12_PM_33003471/index.tex`]

Book: Fundamentals of Differential Equations. By Nagle, Saff and Snider. 9th edition. Boston. Pearson 2018.
Section: Chapter 2, First order differential equations. Section 2.2, Separable Equations. Exercises. page 46
Problem number: 4.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"exact", "separable", "first_order_ode_lie_symmetry_lookup"**

Maple gives the following as the ode type

`[_separable]`

$$\boxed{y' - \frac{y\,\mathrm{e}^{x+y}}{x^2+2} = 0}$$

1.4.1 Solving as separable ode

In canonical form the ODE is

$$\begin{aligned} y' &= F(x,y) \\ &= f(x)g(y) \\ &= \frac{y\,\mathrm{e}^{x}\mathrm{e}^{y}}{x^2+2} \end{aligned}$$

Where $f(x) = \frac{\mathrm{e}^x}{x^2+2}$ and $g(y) = \mathrm{e}^y y$. Integrating both sides gives

$$\frac{1}{\mathrm{e}^y y}\,dy = \frac{\mathrm{e}^x}{x^2+2}\,dx$$

$$\int \frac{1}{\mathrm{e}^{y} y}dy = \int \frac{\mathrm{e}^{x}}{x^{2}+2}dx$$

$$-\operatorname{expIntegral}_{1}\left(y\right) = \frac{i\sqrt{2}\,\mathrm{e}^{i\sqrt{2}}\operatorname{expIntegral}_{1}\left(-x+i\sqrt{2}\right)}{4} - \frac{i\sqrt{2}\,\mathrm{e}^{-i\sqrt{2}}\operatorname{expIntegral}_{1}\left(-x-i\sqrt{2}\right)}{4}+c_{1}$$

Which results in

$$y = \operatorname{RootOf}\left(-i\sqrt{2}\,\mathrm{e}^{i\sqrt{2}}\operatorname{expIntegral}_{1}\left(-x+i\sqrt{2}\right)+i\sqrt{2}\,\mathrm{e}^{-i\sqrt{2}}\operatorname{expIntegral}_{1}\left(-x-i\sqrt{2}\right) - 4\operatorname{expIntegral}_{1}\left(\_Z\right)-4c_{1}\right)$$

Summary
The solution(s) found are the following

$$y = \operatorname{RootOf}\left(-i\sqrt{2}\,\mathrm{e}^{i\sqrt{2}}\operatorname{expIntegral}_{1}\left(-x+i\sqrt{2}\right) + i\sqrt{2}\,\mathrm{e}^{-i\sqrt{2}}\operatorname{expIntegral}_{1}\left(-x-i\sqrt{2}\right)-4\operatorname{expIntegral}_{1}\left(\_Z\right)-4c_{1}\right) \tag{1}$$

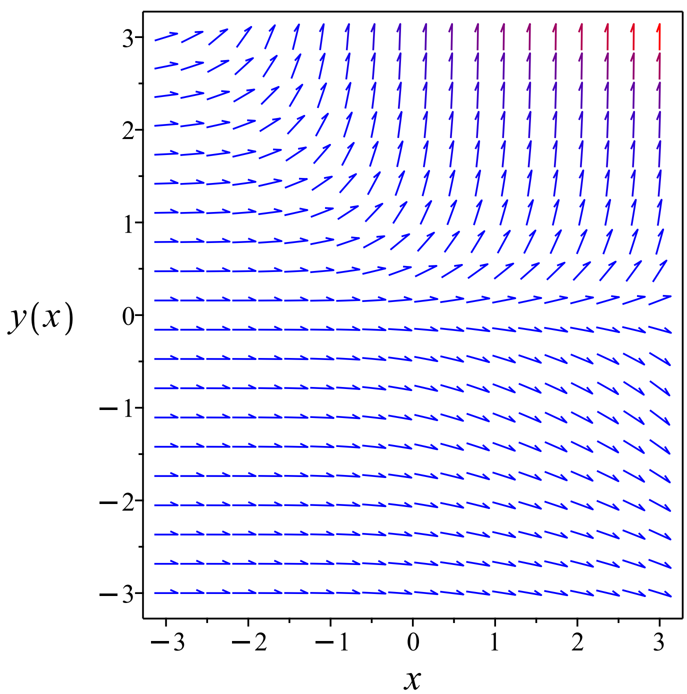


Figure 3: Slope field plot

Verification of solutions

$$y = \operatorname{RootOf}\left(-i\sqrt{2}\, e^{i\sqrt{2}} \operatorname{expIntegral}_1\left(-x+i\sqrt{2}\right)+i\sqrt{2}\, e^{-i\sqrt{2}} \operatorname{expIntegral}_1\left(-x-i\sqrt{2}\right)\right. \\ \left. - 4 \operatorname{expIntegral}_1\left(\_Z\right) - 4c_1\right)$$

Verified OK.

1.4.2 Solving as first order ode lie symmetry lookup ode

Writing the ode as

$$\begin{aligned} y' &= \frac{y\, e^{x+y}}{x^2+2} \\ y' &= \omega(x,y) \end{aligned}$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x + \omega(\eta_y - \xi_x) - \omega^2 \xi_y - \omega_x \xi - \omega_y \eta = 0 \tag{A}$$

The type of this ode is known. It is of type `separable`. Therefore we do not need to solve the PDE (A), and can just use the lookup table shown below to find ξ, η

Table 2: Lie symmetry infinitesimal lookup table for known first order ODE's

ODE class	Form	ξ	η
linear ode	$y' = f(x)y(x) + g(x)$	0	$e^{\int f dx}$
separable ode	$y' = f(x)\, g(y)$	$\frac{1}{f}$	0
quadrature ode	$y' = f(x)$	0	1
quadrature ode	$y' = g(y)$	1	0
homogeneous ODEs of Class A	$y' = f\left(\frac{y}{x}\right)$	x	y
homogeneous ODEs of Class C	$y' = (a + bx + cy)^{\frac{n}{m}}$	1	$-\frac{b}{c}$
homogeneous class D	$y' = \frac{y}{x} + g(x)\, F\left(\frac{y}{x}\right)$	x^2	xy
First order special form ID 1	$y' = g(x)\, e^{h(x)+by} + f(x)$	$\frac{e^{-\int bf(x)dx-h(x)}}{g(x)}$	$\frac{f(x)e^{-\int bf(x)dx-h(x)}}{g(x)}$
polynomial type ode	$y' = \frac{a_1x+b_1y+c_1}{a_2x+b_2y+c_2}$	$\frac{a_1b_2x-a_2b_1x-b_1c_2+b_2c_1}{a_1b_2-a_2b_1}$	$\frac{a_1b_2y-a_2b_1y-a_1c_2-a_2c_1}{a_1b_2-a_2b_1}$
Bernoulli ode	$y' = f(x)\, y + g(x)\, y^n$	0	$e^{-\int (n-1)f(x)dx}y^n$
Reduced Riccati	$y' = f_1(x)\, y + f_2(x)\, y^2$	0	$e^{-\int f_1 dx}$

The above table shows that

$$
\begin{aligned}
\xi(x, y) &= \mathrm{e}^{-x}\left(x^2 + 2\right) \\
\eta(x, y) &= 0
\end{aligned} \tag{A1}
$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x, y) \to (R, S)$ where (R, S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$
\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}
$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y}\right) S(x, y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the

canonical coordinates, where $S(R)$. Since $\eta = 0$ then in this special case

$$R = y$$

S is found from

$$\begin{aligned} S &= \int \frac{1}{\xi} dx \\ &= \int \frac{1}{e^{-x} (x^2 + 2)} dx \end{aligned}$$

Which results in

$$S = \frac{i\sqrt{2}\, e^{i\sqrt{2}} \operatorname{expIntegral}_1\left(-x + i\sqrt{2}\right)}{4} - \frac{i\sqrt{2}\, e^{-i\sqrt{2}} \operatorname{expIntegral}_1\left(-x - i\sqrt{2}\right)}{4}$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x, y) S_y}{R_x + \omega(x, y) R_y} \tag{2}$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x, y)$ is the right hand side of the original ode given by

$$\omega(x, y) = \frac{y\, e^{x+y}}{x^2 + 2}$$

Evaluating all the partial derivatives gives

$$\begin{aligned} R_x &= 0 \\ R_y &= 1 \\ S_x &= \frac{e^x}{x^2 + 2} \\ S_y &= 0 \end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = \frac{e^{-y}}{y} \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = \frac{e^{-R}}{R}$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordinates R, S. Integrating the above gives

$$S(R) = -\operatorname{expIntegral}_1(R) + c_1 \tag{4}$$

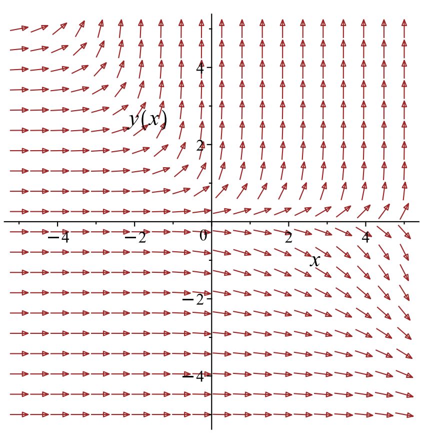
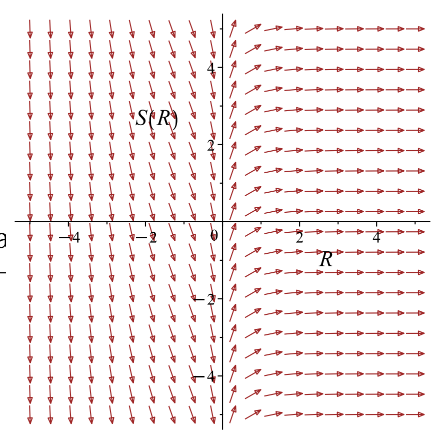
To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$\frac{i\sqrt{2}\left(e^{i\sqrt{2}}\operatorname{expIntegral}_1\left(-x+i\sqrt{2}\right)-e^{-i\sqrt{2}}\operatorname{expIntegral}_1\left(-x-i\sqrt{2}\right)\right)}{4} = -\operatorname{expIntegral}_1(y) + c_1$$

Which simplifies to

$$\frac{i\sqrt{2}\left(e^{i\sqrt{2}}\operatorname{expIntegral}_1\left(-x+i\sqrt{2}\right)-e^{-i\sqrt{2}}\operatorname{expIntegral}_1\left(-x-i\sqrt{2}\right)\right)}{4} = -\operatorname{expIntegral}_1(y) + c_1$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in x, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dx} = \frac{y\,e^{x+y}}{x^2+2}$ 	$R = y$ $S = \frac{i\sqrt{2}\left(e^{i\sqrt{2}}\operatorname{expIntegra}\right.}{}$ ... $\operatorname{al}_1\left(-\right.$	$\frac{dS}{dR} = \frac{e^{-R}}{R}$ 

Summary

The solution(s) found are the following

$$\frac{i\sqrt{2}\left(e^{i\sqrt{2}}\operatorname{expIntegral}_1\left(-x+i\sqrt{2}\right)-e^{-i\sqrt{2}}\operatorname{expIntegral}_1\left(-x-i\sqrt{2}\right)\right)}{4} = -\operatorname{expIntegral}_1(y) + c_1 \tag{1}$$

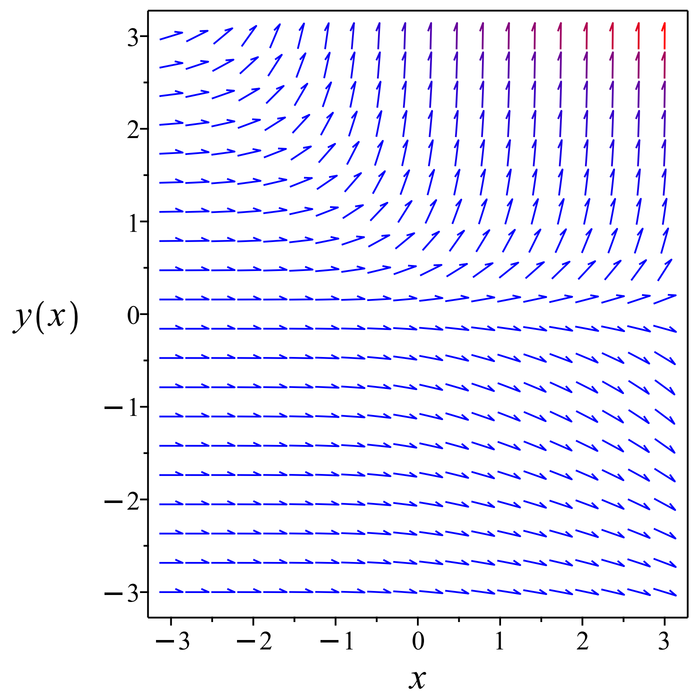


Figure 4: Slope field plot

<u>Verification of solutions</u>

$$\frac{i\sqrt{2}\left(e^{i\sqrt{2}}\,\mathrm{expIntegral}_1\left(-x+i\sqrt{2}\right)-e^{-i\sqrt{2}}\,\mathrm{expIntegral}_1\left(-x-i\sqrt{2}\right)\right)}{4} = -\,\mathrm{expIntegral}_1\left(y\right)+c_1$$

Verified OK.

1.4.3 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x,y)+N(x,y)\frac{dy}{dx}=0 \tag{A}$$

We assume there exists a function $\phi(x,y)=c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y)=0$$

Hence

$$\frac{\partial\phi}{\partial x}+\frac{\partial\phi}{\partial y}\frac{dy}{dx}=0 \tag{B}$$

Comparing (A,B) shows that

$$\begin{aligned}\frac{\partial \phi}{\partial x} &= M \\ \frac{\partial \phi}{\partial y} &= N\end{aligned}$$

But since $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x, y)$ but at least we know now that we can do that since the condition $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x, y)\, \mathrm{d}x + N(x, y)\, \mathrm{d}y = 0 \tag{1A}$$

Therefore

$$\begin{aligned}\left(\frac{\mathrm{e}^{-y}}{y}\right) \mathrm{d}y &= \left(\frac{\mathrm{e}^{x}}{x^2+2}\right) \mathrm{d}x \\ \left(-\frac{\mathrm{e}^{x}}{x^2+2}\right) \mathrm{d}x + \left(\frac{\mathrm{e}^{-y}}{y}\right) \mathrm{d}y &= 0 \qquad (2A)\end{aligned}$$

Comparing (1A) and (2A) shows that

$$\begin{aligned}M(x, y) &= -\frac{\mathrm{e}^{x}}{x^2+2} \\ N(x, y) &= \frac{\mathrm{e}^{-y}}{y}\end{aligned}$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Using result found above gives

$$\begin{aligned}\frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}\left(-\frac{\mathrm{e}^{x}}{x^2+2}\right) \\ &= 0\end{aligned}$$

And

$$
\begin{aligned}
\frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}\left(\frac{\mathrm{e}^{-y}}{y}\right) \\
&= 0
\end{aligned}
$$

Since $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$, then the ODE is exact The following equations are now set up to solve for the function $\phi(x, y)$

$$
\frac{\partial \phi}{\partial x} = M \tag{1}
$$

$$
\frac{\partial \phi}{\partial y} = N \tag{2}
$$

Integrating (1) w.r.t. x gives

$$
\begin{aligned}
\int \frac{\partial \phi}{\partial x} \,\mathrm{d}x &= \int M \,\mathrm{d}x \\
\int \frac{\partial \phi}{\partial x} \,\mathrm{d}x &= \int -\frac{\mathrm{e}^{x}}{x^2+2} \,\mathrm{d}x
\end{aligned}
$$

$$
\phi = \int^{x} -\frac{\mathrm{e}^{\_a}}{\_a^2+2} d\_a + f(y) \tag{3}
$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$
\frac{\partial \phi}{\partial y} = 0 + f'(y) \tag{4}
$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = \frac{\mathrm{e}^{-y}}{y}$. Therefore equation (4) becomes

$$
\frac{\mathrm{e}^{-y}}{y} = 0 + f'(y) \tag{5}
$$

Solving equation (5) for $f'(y)$ gives

$$
f'(y) = \frac{\mathrm{e}^{-y}}{y}
$$

Integrating the above w.r.t y gives

$$\int f'(y)\,\mathrm{d}y = \int \left(\frac{\mathrm{e}^{-y}}{y}\right) \mathrm{d}y$$
$$f(y) = -\operatorname{expIntegral}_1(y) + c_1$$

Where c_1 is constant of integration. Substituting result found above for $f(y)$ into equation (3) gives ϕ

$$\phi = \int^{x} -\frac{\mathrm{e}^{-\_a}}{\_a^2 + 2} d\_a - \operatorname{expIntegral}_1(y) + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = \int^{x} -\frac{\mathrm{e}^{-\_a}}{\_a^2 + 2} d\_a - \operatorname{expIntegral}_1(y)$$

Summary
The solution(s) found are the following

$$\int^{x} -\frac{\mathrm{e}^{-\_a}}{\_a^2 + 2} d\_a - \operatorname{expIntegral}_1(y) = c_1 \tag{1}$$

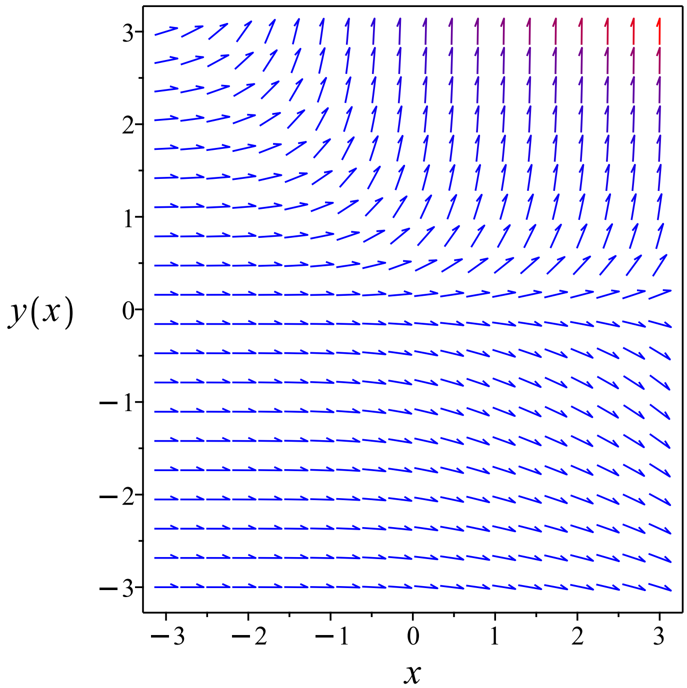


Figure 5: Slope field plot

Verification of solutions

$$\int^{x} -\frac{e^{\_a}}{\_a^2+2} d\_a - \mathrm{expIntegral}_1(y) = c_1$$

Verified OK.

1.4.4 Maple step by step solution

Let's solve

$y' - \frac{y\,e^{x+y}}{x^2+2} = 0$

- Highest derivative means the order of the ODE is 1

  y'

- Separate variables

  $\frac{y'}{e^y y} = \frac{e^x}{x^2+2}$

- Integrate both sides with respect to x

  $\int \frac{y'}{e^y y} dx = \int \frac{e^x}{x^2+2} dx + c_1$

- Evaluate integral

$$-\mathrm{Ei}_1(y) = \frac{\mathrm{I}\sqrt{2}\,\mathrm{e}^{\mathrm{I}\sqrt{2}}\mathrm{Ei}_1\left(-x+\mathrm{I}\sqrt{2}\right)}{4} - \frac{\mathrm{I}\sqrt{2}\,\mathrm{e}^{-\mathrm{I}\sqrt{2}}\mathrm{Ei}_1\left(-x-\mathrm{I}\sqrt{2}\right)}{4} + c_1$$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
<- separable successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 59

```
dsolve(diff(y(x),x)=y(x)*exp(x+y(x))/(x^2+2),y(x), singsol=all)
```

$$\frac{i\sqrt{2}\,\mathrm{e}^{i\sqrt{2}}\,\mathrm{expIntegral}_1\left(-x+i\sqrt{2}\right)}{4} - \frac{i\sqrt{2}\,\mathrm{e}^{-i\sqrt{2}}\,\mathrm{expIntegral}_1\left(-x-i\sqrt{2}\right)}{4} + \mathrm{expIntegral}_1\left(y(x)\right) + c_1 = 0$$

✓ Solution by Mathematica

Time used: 0.932 (sec). Leaf size: 81

```
DSolve[y'[x]==y[x]*Exp[x+y[x]]/(x^2+2),y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \mathrm{InverseFunction}[\mathrm{ExpIntegralEi}(-\#1)\&]\left[c_1 - \frac{ie^{-i\sqrt{2}}\left(e^{2i\sqrt{2}}\,\mathrm{ExpIntegralEi}\left(x-i\sqrt{2}\right) - \mathrm{ExpIntegralEi}\left(x+i\sqrt{2}\right)\right)}{2\sqrt{2}}\right]$$

$$y(x) \to 0$$

1.5 problem 5

Internal problem ID [4916]
Internal file name [`OUTPUT/4409_Sunday_June_05_2022_01_16_21_PM_74557571/index.tex`]

Book: Fundamentals of Differential Equations. By Nagle, Saff and Snider. 9th edition. Boston. Pearson 2018.
Section: Chapter 2, First order differential equations. Section 2.2, Separable Equations. Exercises. page 46
Problem number: 5.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"exact", "separable", "first_order_ode_lie_symmetry_lookup"**

Maple gives the following as the ode type

`[_separable]`

$$\boxed{\left(x y^{2}+3 y^{2}\right) y^{\prime}=2 x}$$

1.5.1 Solving as separable ode

In canonical form the ODE is

$$\begin{aligned} y^{\prime} &= F(x, y) \\ &= f(x) g(y) \\ &= \frac{2 x}{y^{2}(x+3)} \end{aligned}$$

Where $f(x)=\frac{2 x}{x+3}$ and $g(y)=\frac{1}{y^{2}}$. Integrating both sides gives

$$\frac{1}{\frac{1}{y^{2}}} d y=\frac{2 x}{x+3} d x$$

$$\int \frac{1}{\frac{1}{y^2}}dy = \int \frac{2x}{x+3}dx$$
$$\frac{y^3}{3} = 2x - 6\ln(x+3) + c_1$$

Which results in

$$y = (-18\ln(x+3) + 3c_1 + 6x)^{\frac{1}{3}}$$
$$y = -\frac{(-18\ln(x+3) + 3c_1 + 6x)^{\frac{1}{3}}}{2} - \frac{i\sqrt{3}\,(-18\ln(x+3) + 3c_1 + 6x)^{\frac{1}{3}}}{2}$$
$$y = -\frac{(-18\ln(x+3) + 3c_1 + 6x)^{\frac{1}{3}}}{2} + \frac{i\sqrt{3}\,(-18\ln(x+3) + 3c_1 + 6x)^{\frac{1}{3}}}{2}$$

Summary

The solution(s) found are the following

$$y = (-18\ln(x+3) + 3c_1 + 6x)^{\frac{1}{3}} \tag{1}$$
$$y = -\frac{(-18\ln(x+3) + 3c_1 + 6x)^{\frac{1}{3}}}{2} - \frac{i\sqrt{3}\,(-18\ln(x+3) + 3c_1 + 6x)^{\frac{1}{3}}}{2} \tag{2}$$
$$y = -\frac{(-18\ln(x+3) + 3c_1 + 6x)^{\frac{1}{3}}}{2} + \frac{i\sqrt{3}\,(-18\ln(x+3) + 3c_1 + 6x)^{\frac{1}{3}}}{2} \tag{3}$$

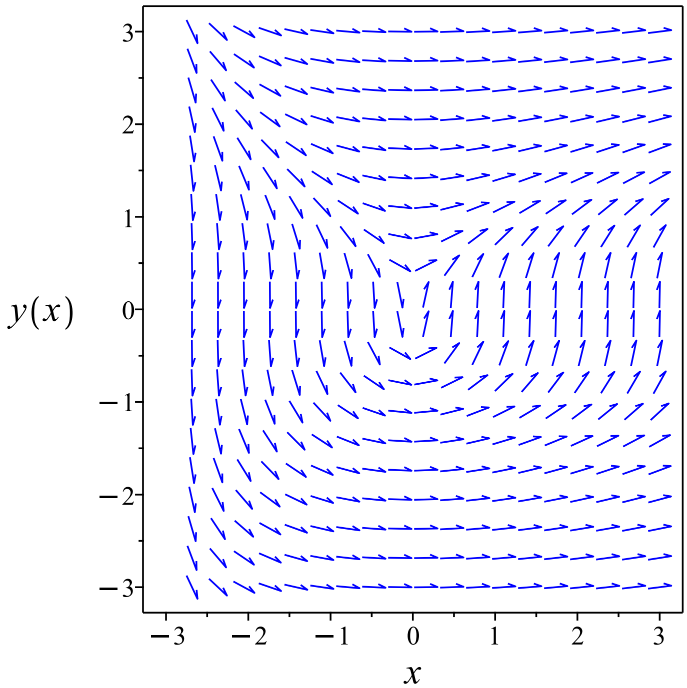


Figure 6: Slope field plot

Verification of solutions

$$y = \left(-18\ln\left(x+3\right)+3c_1+6x\right)^{\frac{1}{3}}$$

Verified OK.

$$y = -\frac{\left(-18\ln\left(x+3\right)+3c_1+6x\right)^{\frac{1}{3}}}{2}-\frac{i\sqrt{3}\,\left(-18\ln\left(x+3\right)+3c_1+6x\right)^{\frac{1}{3}}}{2}$$

Verified OK.

$$y = -\frac{\left(-18\ln\left(x+3\right)+3c_1+6x\right)^{\frac{1}{3}}}{2}+\frac{i\sqrt{3}\,\left(-18\ln\left(x+3\right)+3c_1+6x\right)^{\frac{1}{3}}}{2}$$

Verified OK.

1.5.2 Solving as first order ode lie symmetry lookup ode

Writing the ode as

$$
\begin{aligned}
y' &= \frac{2x}{y^2 (x+3)} \\
y' &= \omega(x,y)
\end{aligned}
$$

The condition of Lie symmetry is the linearized PDE given by

$$
\eta_x + \omega(\eta_y - \xi_x) - \omega^2 \xi_y - \omega_x \xi - \omega_y \eta = 0 \tag{A}
$$

The type of this ode is known. It is of type `separable`. Therefore we do not need to solve the PDE (A), and can just use the lookup table shown below to find ξ, η

Table 5: Lie symmetry infinitesimal lookup table for known first order ODE's

ODE class	Form	ξ	η
linear ode	$y' = f(x)y(x) + g(x)$	0	$e^{\int f dx}$
separable ode	$y' = f(x)\, g(y)$	$\frac{1}{f}$	0
quadrature ode	$y' = f(x)$	0	1
quadrature ode	$y' = g(y)$	1	0
homogeneous ODEs of Class A	$y' = f\left(\frac{y}{x}\right)$	x	y
homogeneous ODEs of Class C	$y' = (a + bx + cy)^{\frac{n}{m}}$	1	$-\frac{b}{c}$
homogeneous class D	$y' = \frac{y}{x} + g(x)\, F\left(\frac{y}{x}\right)$	x^2	xy
First order special form ID 1	$y' = g(x)\, e^{h(x)+by} + f(x)$	$\frac{e^{-\int bf(x)dx - h(x)}}{g(x)}$	$\frac{f(x)e^{-\int bf(x)dx - h(x)}}{g(x)}$
polynomial type ode	$y' = \frac{a_1x + b_1y + c_1}{a_2x + b_2y + c_2}$	$\frac{a_1b_2x - a_2b_1x - b_1c_2 + b_2c_1}{a_1b_2 - a_2b_1}$	$\frac{a_1b_2y - a_2b_1y - a_1c_2 - a_2c_1}{a_1b_2 - a_2b_1}$
Bernoulli ode	$y' = f(x)\, y + g(x)\, y^n$	0	$e^{-\int (n-1)f(x)dx} y^n$
Reduced Riccati	$y' = f_1(x)\, y + f_2(x)\, y^2$	0	$e^{-\int f_1 dx}$

The above table shows that

$$
\begin{aligned}
\xi(x,y) &= \frac{x+3}{2x} \\
\eta(x,y) &= 0 \qquad \text{(A1)}
\end{aligned}
$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x,y) \to (R,S)$ where (R,S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$
\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}
$$

The above comes from the requirements that $\left(\xi\frac{\partial}{\partial x} + \eta\frac{\partial}{\partial y}\right) S(x,y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the canonical coordinates, where $S(R)$. Since $\eta = 0$ then in this special case

$$
R = y
$$

S is found from

$$
\begin{aligned}
S &= \int \frac{1}{\xi} dx \\
&= \int \frac{1}{\frac{x+3}{2x}} dx
\end{aligned}
$$

Which results in

$$
S = 2x - 6\ln(x+3)
$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$
\frac{dS}{dR} = \frac{S_x + \omega(x,y)S_y}{R_x + \omega(x,y)R_y} \tag{2}
$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x,y)$ is the right hand side of the original ode given by

$$
\omega(x,y) = \frac{2x}{y^2(x+3)}
$$

Evaluating all the partial derivatives gives

$$
\begin{aligned}
R_x &= 0 \\
R_y &= 1 \\
S_x &= \frac{2x}{x+3} \\
S_y &= 0
\end{aligned}
$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$
\frac{dS}{dR} = y^2 \tag{2A}
$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$
\frac{dS}{dR} = R^2
$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$
S(R) = \frac{R^3}{3} + c_1 \tag{4}
$$

To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$
2x - 6\ln(x+3) = \frac{y^3}{3} + c_1
$$

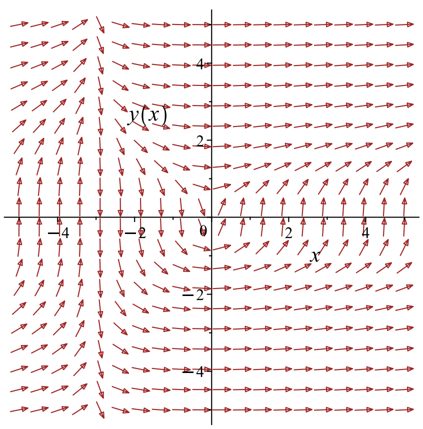
Which simplifies to

$$
2x - 6\ln(x+3) = \frac{y^3}{3} + c_1
$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

<table>
<tr><th>Original ode in x, y coordinates</th><th>Canonical coordinates transformation</th><th>ODE in canonical coordinates (R, S)</th></tr>
<tr><td>

$\frac{dy}{dx} = \frac{2x}{y^2(x+3)}$



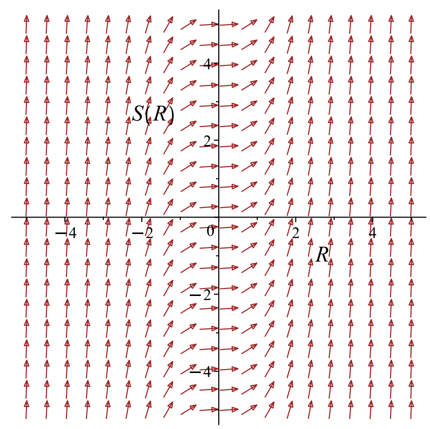
</td><td>

$R = y$

$S = 2x - 6\ln(x+3)$

</td><td>

$\frac{dS}{dR} = R^2$



</td></tr>
</table>

Summary

The solution(s) found are the following

$$2x - 6\ln(x+3) = \frac{y^3}{3} + c_1 \tag{1}$$

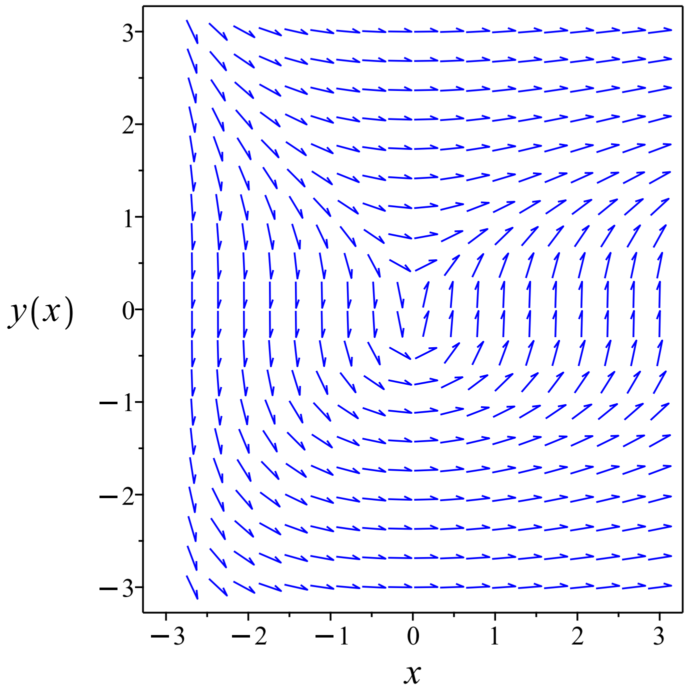


Figure 7: Slope field plot

Verification of solutions

$$2x - 6\ln(x+3) = \frac{y^3}{3} + c_1$$

Verified OK.

1.5.3 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x,y) + N(x,y)\frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x,y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y) = 0$$

Hence

$$\frac{\partial\phi}{\partial x} + \frac{\partial\phi}{\partial y}\frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$\frac{\partial\phi}{\partial x} = M$$
$$\frac{\partial\phi}{\partial y} = N$$

But since $\frac{\partial^2\phi}{\partial x\partial y} = \frac{\partial^2\phi}{\partial y\partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x,y)$ but at least we know now that we can do that since the condition $\frac{\partial^2\phi}{\partial x\partial y} = \frac{\partial^2\phi}{\partial y\partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x,y)\,\mathrm{d}x + N(x,y)\,\mathrm{d}y = 0 \tag{1A}$$

Therefore

$$\left(\frac{y^2}{2}\right)\mathrm{d}y = \left(\frac{x}{x+3}\right)\mathrm{d}x$$
$$\left(-\frac{x}{x+3}\right)\mathrm{d}x + \left(\frac{y^2}{2}\right)\mathrm{d}y = 0 \tag{2A}$$

Comparing (1A) and (2A) shows that

$$M(x,y) = -\frac{x}{x+3}$$
$$N(x,y) = \frac{y^2}{2}$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Using result found above gives

$$\begin{aligned}\frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}\left(-\frac{x}{x+3}\right)\\ &= 0\end{aligned}$$

And

$$
\begin{aligned}
\frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}\left(\frac{y^2}{2}\right) \\
&= 0
\end{aligned}
$$

Since $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$, then the ODE is <u>exact</u> The following equations are now set up to solve for the function $\phi(x, y)$

$$
\frac{\partial \phi}{\partial x} = M \tag{1}
$$

$$
\frac{\partial \phi}{\partial y} = N \tag{2}
$$

Integrating (1) w.r.t. x gives

$$
\begin{aligned}
\int \frac{\partial \phi}{\partial x} \, \mathrm{d}x &= \int M \, \mathrm{d}x \\
\int \frac{\partial \phi}{\partial x} \, \mathrm{d}x &= \int -\frac{x}{x+3} \, \mathrm{d}x
\end{aligned}
$$

$$
\phi = -x + 3\ln(x+3) + f(y) \tag{3}
$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$
\frac{\partial \phi}{\partial y} = 0 + f'(y) \tag{4}
$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = \frac{y^2}{2}$. Therefore equation (4) becomes

$$
\frac{y^2}{2} = 0 + f'(y) \tag{5}
$$

Solving equation (5) for $f'(y)$ gives

$$
f'(y) = \frac{y^2}{2}
$$

Integrating the above w.r.t y gives

$$
\begin{aligned}
\int f'(y) \, \mathrm{d}y &= \int \left(\frac{y^2}{2}\right) \mathrm{d}y \\
f(y) &= \frac{y^3}{6} + c_1
\end{aligned}
$$

Where c_1 is constant of integration. Substituting result found above for $f(y)$ into equation (3) gives ϕ

$$\phi = -x + 3\ln(x+3) + \frac{y^3}{6} + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = -x + 3\ln(x+3) + \frac{y^3}{6}$$

Summary

The solution(s) found are the following

$$-x + 3\ln(x+3) + \frac{y^3}{6} = c_1 \tag{1}$$

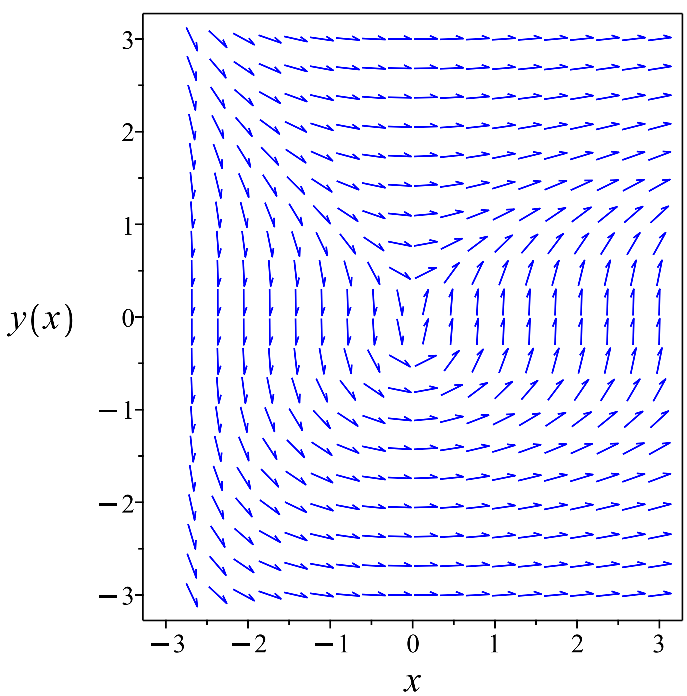


Figure 8: Slope field plot

Verification of solutions

$$-x + 3\ln(x+3) + \frac{y^3}{6} = c_1$$

Verified OK.

1.5.4 Maple step by step solution

Let's solve

$(xy^2+3y^2)\,y'=2x$

- Highest derivative means the order of the ODE is 1

  y'

- Separate variables

  $y'y^2=\frac{2x}{x+3}$

- Integrate both sides with respect to x

  $\int y'y^2dx=\int \frac{2x}{x+3}dx+c_1$

- Evaluate integral

  $\frac{y^3}{3}=2x-6\ln(x+3)+c_1$

- Solve for y

  $y=(-18\ln(x+3)+3c_1+6x)^{\frac{1}{3}}$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
<- Bernoulli successful`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 69

```
dsolve((x*y(x)^2+3*y(x)^2)*diff(y(x),x)-2*x=0,y(x), singsol=all)
```

$$y(x)=(-18\ln(x+3)+c_1+6x)^{\frac{1}{3}}$$

$$y(x)=-\frac{(-18\ln(x+3)+c_1+6x)^{\frac{1}{3}}\,(1+i\sqrt{3})}{2}$$

$$y(x)=\frac{(-18\ln(x+3)+c_1+6x)^{\frac{1}{3}}\,(i\sqrt{3}-1)}{2}$$

✓ Solution by Mathematica

Time used: 0.238 (sec). Leaf size: 85

```
DSolve[(x*y[x]^2+3*y[x]^2)*y'[x]-2*x==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to -\sqrt[3]{-3}\sqrt[3]{2x - 6\log(x+3) + c_1}$$
$$y(x) \to \sqrt[3]{3}\sqrt[3]{2x - 6\log(x+3) + c_1}$$
$$y(x) \to (-1)^{2/3}\sqrt[3]{3}\sqrt[3]{2x - 6\log(x+3) + c_1}$$

1.6 problem 6

Internal problem ID [4917]
Internal file name [`OUTPUT/4410_Sunday_June_05_2022_01_16_30_PM_16331878/index.tex`]

Book: Fundamentals of Differential Equations. By Nagle, Saff and Snider. 9th edition. Boston. Pearson 2018.
Section: Chapter 2, First order differential equations. Section 2.2, Separable Equations. Exercises. page 46
Problem number: 6.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"unknown"**

Maple gives the following as the ode type

```
[_rational, [_Abel, `2nd type`, `class C`]]
```

Unable to solve or complete the solution.

$$\boxed{s^2 + s' - \frac{s+1}{ts} = 0}$$

Unable to determine ODE type.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
trying homogeneous types:
trying Chini
differential order: 1; looking for linear symmetries
trying exact
trying Abel
Looking for potential symmetries
Looking for potential symmetries
Looking for potential symmetries
trying inverse_Riccati
trying an equivalence to an Abel ODE
differential order: 1; trying a linearization to 2nd order
--- trying a change of variables {x -> y(x), y(x) -> x}
differential order: 1; trying a linearization to 2nd order
trying 1st order ODE linearizable_by_differentiation
--- Trying Lie symmetry methods, 1st order ---
`, `-> Computing symmetries using: way = 3
`, `-> Computing symmetries using: way = 4
`, `-> Computing symmetries using: way = 2
trying symmetry patterns for 1st order ODEs
-> trying a symmetry pattern of the form [F(x)*G(y), 0]
-> trying a symmetry pattern of the form [0, F(x)*G(y)]
-> trying symmetry patterns of the forms [F(x),G(y)] and [G(y),F(x)]
-> trying a symmetry pattern of the form [F(x),G(x)]
-> trying a symmetry pattern of the form [F(y),G(y)]
-> trying a symmetry pattern of the form [F(x)+G(y), 0]
-> trying a symmetry pattern of the form [0, F(x)+G(y)]
-> trying a symmetry pattern of the form [F(x),G(x)*y+H(x)]
-> trying a symmetry pattern of conformal type`
```

✗ Solution by Maple

```
dsolve(s(t)^2+diff(s(t),t)=(s(t)+1)/(s(t)*t),s(t), singsol=all)
```

No solution found

✗ Solution by Mathematica

Time used: 0.0 (sec). Leaf size: 0

```
DSolve[s[t]^2+s'[t]==(s[t]+1)/(s[t]*t),s[t],t,IncludeSingularSolutions -> True]
```

Not solved

1.7 problem 7

Internal problem ID [4918]
Internal file name [`OUTPUT/4411_Sunday_June_05_2022_01_16_40_PM_39950676/index.tex`]

Book: Fundamentals of Differential Equations. By Nagle, Saff and Snider. 9th edition. Boston. Pearson 2018.
Section: Chapter 2, First order differential equations. Section 2.2, Separable Equations. Exercises. page 46
Problem number: 7.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"exact", "separable", "first_order_ode_lie_symmetry_lookup"**

Maple gives the following as the ode type

`[_separable]`

$$\boxed{xy' - \frac{1}{y^3} = 0}$$

1.7.1 Solving as separable ode

In canonical form the ODE is

$$\begin{aligned} y' &= F(x,y) \\ &= f(x)g(y) \\ &= \frac{1}{y^3 x} \end{aligned}$$

Where $f(x) = \frac{1}{x}$ and $g(y) = \frac{1}{y^3}$. Integrating both sides gives

$$\frac{1}{\frac{1}{y^3}}\,dy = \frac{1}{x}\,dx$$

$$\int \frac{1}{\frac{1}{y^3}}\,dy = \int \frac{1}{x}\,dx$$

$$\frac{y^4}{4} = \ln(x) + c_1$$

The solution is

$$\frac{y^4}{4} - \ln(x) - c_1 = 0$$

<u>Summary</u>

The solution(s) found are the following

$$\frac{y^4}{4} - \ln(x) - c_1 = 0 \tag{1}$$

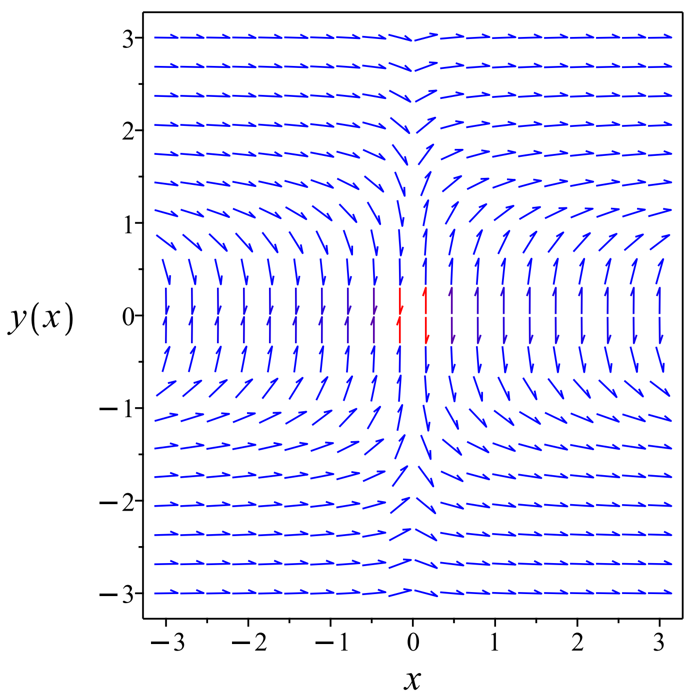


Figure 9: Slope field plot

<u>Verification of solutions</u>

$$\frac{y^4}{4} - \ln(x) - c_1 = 0$$

Verified OK.

1.7.2 Solving as first order ode lie symmetry lookup ode

Writing the ode as

$$
\begin{aligned}
y' &= \frac{1}{y^3 x} \\
y' &= \omega(x, y)
\end{aligned}
$$

The condition of Lie symmetry is the linearized PDE given by

$$
\eta_x + \omega(\eta_y - \xi_x) - \omega^2 \xi_y - \omega_x \xi - \omega_y \eta = 0 \tag{A}
$$

The type of this ode is known. It is of type `separable`. Therefore we do not need to solve the PDE (A), and can just use the lookup table shown below to find ξ, η

Table 8: Lie symmetry infinitesimal lookup table for known first order ODE's

ODE class	Form	ξ	η
linear ode	$y' = f(x)y(x) + g(x)$	0	$e^{\int f dx}$
separable ode	$y' = f(x)\, g(y)$	$\frac{1}{f}$	0
quadrature ode	$y' = f(x)$	0	1
quadrature ode	$y' = g(y)$	1	0
homogeneous ODEs of Class A	$y' = f\left(\frac{y}{x}\right)$	x	y
homogeneous ODEs of Class C	$y' = (a + bx + cy)^{\frac{n}{m}}$	1	$-\frac{b}{c}$
homogeneous class D	$y' = \frac{y}{x} + g(x)\, F\left(\frac{y}{x}\right)$	x^2	xy
First order special form ID 1	$y' = g(x)\, e^{h(x)+by} + f(x)$	$\frac{e^{-\int bf(x)dx - h(x)}}{g(x)}$	$\frac{f(x)e^{-\int bf(x)dx - h(x)}}{g(x)}$
polynomial type ode	$y' = \frac{a_1x+b_1y+c_1}{a_2x+b_2y+c_2}$	$\frac{a_1b_2x - a_2b_1x - b_1c_2 + b_2c_1}{a_1b_2 - a_2b_1}$	$\frac{a_1b_2y - a_2b_1y - a_1c_2 - a_2c_1}{a_1b_2 - a_2b_1}$
Bernoulli ode	$y' = f(x)\, y + g(x)\, y^n$	0	$e^{-\int (n-1)f(x)dx} y^n$
Reduced Riccati	$y' = f_1(x)\, y + f_2(x)\, y^2$	0	$e^{-\int f_1 dx}$

The above table shows that

$$
\begin{aligned}
\xi(x,y) &= x \\
\eta(x,y) &= 0
\end{aligned} \tag{A1}
$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x,y) \to (R,S)$ where (R,S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$
\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}
$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y}\right) S(x,y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the canonical coordinates, where $S(R)$. Since $\eta = 0$ then in this special case

$$
R = y
$$

S is found from

$$
\begin{aligned}
S &= \int \frac{1}{\xi} dx \\
&= \int \frac{1}{x} dx
\end{aligned}
$$

Which results in

$$
S = \ln(x)
$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$
\frac{dS}{dR} = \frac{S_x + \omega(x,y) S_y}{R_x + \omega(x,y) R_y} \tag{2}
$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x,y)$ is the right hand side of the original ode given by

$$
\omega(x,y) = \frac{1}{y^3 x}
$$

Evaluating all the partial derivatives gives

$$\begin{aligned}
R_x &= 0 \\
R_y &= 1 \\
S_x &= \frac{1}{x} \\
S_y &= 0
\end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = y^3 \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = R^3$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$S(R) = \frac{R^4}{4} + c_1 \tag{4}$$

To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$\ln(x) = \frac{y^4}{4} + c_1$$

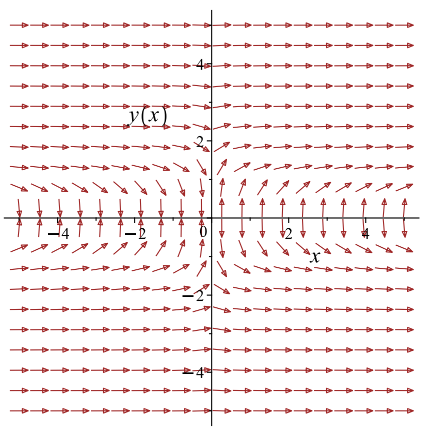
Which simplifies to

$$\ln(x) = \frac{y^4}{4} + c_1$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

<table>
<tr><th>Original ode in x, y coordinates</th><th>Canonical coordinates transformation</th><th>ODE in canonical coordinates (R, S)</th></tr>
<tr><td>

$\frac{dy}{dx} = \frac{1}{y^3 x}$

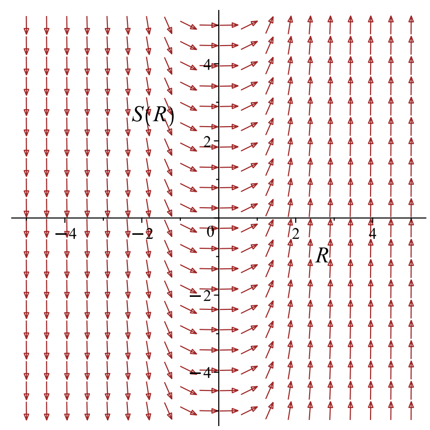


</td><td>

$R = y$
$S = \ln(x)$

</td><td>

$\frac{dS}{dR} = R^3$



</td></tr>
</table>

Summary

The solution(s) found are the following

$$\ln(x) = \frac{y^4}{4} + c_1 \tag{1}$$

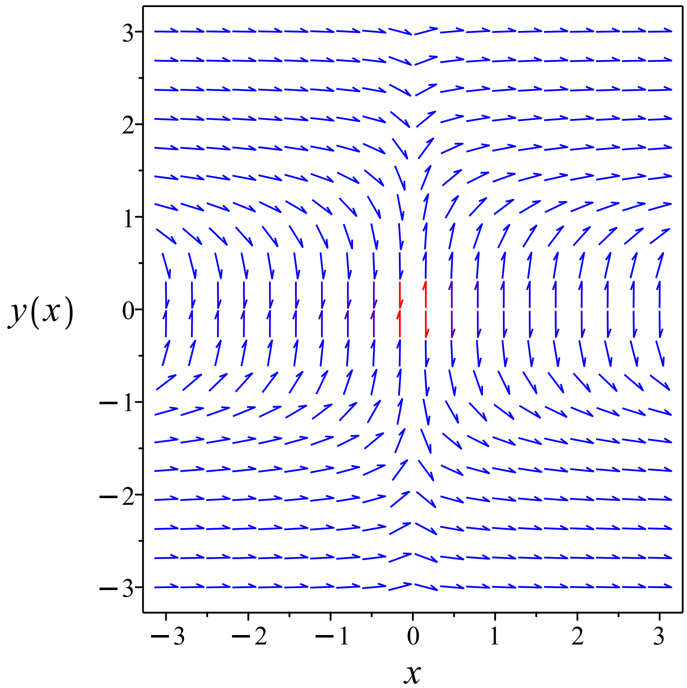


Figure 10: Slope field plot

Verification of solutions

$$\ln(x) = \frac{y^4}{4} + c_1$$

Verified OK.

1.7.3 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x,y) + N(x,y)\frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x,y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y) = 0$$

Hence

$$\frac{\partial \phi}{\partial x} + \frac{\partial \phi}{\partial y}\frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$\frac{\partial \phi}{\partial x} = M$$
$$\frac{\partial \phi}{\partial y} = N$$

But since $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x, y)$ but at least we know now that we can do that since the condition $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x, y)\,\mathrm{d}x + N(x, y)\,\mathrm{d}y = 0 \tag{1A}$$

Therefore

$$\left(y^3\right)\mathrm{d}y = \left(\frac{1}{x}\right)\mathrm{d}x$$
$$\left(-\frac{1}{x}\right)\mathrm{d}x + \left(y^3\right)\mathrm{d}y = 0 \tag{2A}$$

Comparing (1A) and (2A) shows that

$$M(x, y) = -\frac{1}{x}$$
$$N(x, y) = y^3$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Using result found above gives

$$\frac{\partial M}{\partial y} = \frac{\partial}{\partial y}\left(-\frac{1}{x}\right)$$
$$= 0$$

And

$$
\begin{aligned}
\frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}(y^3) \\
&= 0
\end{aligned}
$$

Since $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$, then the ODE is <u>exact</u> The following equations are now set up to solve for the function $\phi(x,y)$

$$
\frac{\partial \phi}{\partial x} = M \tag{1}
$$

$$
\frac{\partial \phi}{\partial y} = N \tag{2}
$$

Integrating (1) w.r.t. x gives

$$
\begin{aligned}
\int \frac{\partial \phi}{\partial x} \,\mathrm{d}x &= \int M \,\mathrm{d}x \\
\int \frac{\partial \phi}{\partial x} \,\mathrm{d}x &= \int -\frac{1}{x} \,\mathrm{d}x
\end{aligned}
$$

$$
\phi = -\ln(x) + f(y) \tag{3}
$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$
\frac{\partial \phi}{\partial y} = 0 + f'(y) \tag{4}
$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = y^3$. Therefore equation (4) becomes

$$
y^3 = 0 + f'(y) \tag{5}
$$

Solving equation (5) for $f'(y)$ gives

$$
f'(y) = y^3
$$

Integrating the above w.r.t y gives

$$
\begin{aligned}
\int f'(y) \,\mathrm{d}y &= \int (y^3) \,\mathrm{d}y \\
f(y) &= \frac{y^4}{4} + c_1
\end{aligned}
$$

Where c_1 is constant of integration. Substituting result found above for $f(y)$ into equation (3) gives ϕ

$$\phi = -\ln(x) + \frac{y^4}{4} + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = -\ln(x) + \frac{y^4}{4}$$

Summary
The solution(s) found are the following

$$\frac{y^4}{4} - \ln(x) = c_1 \tag{1}$$

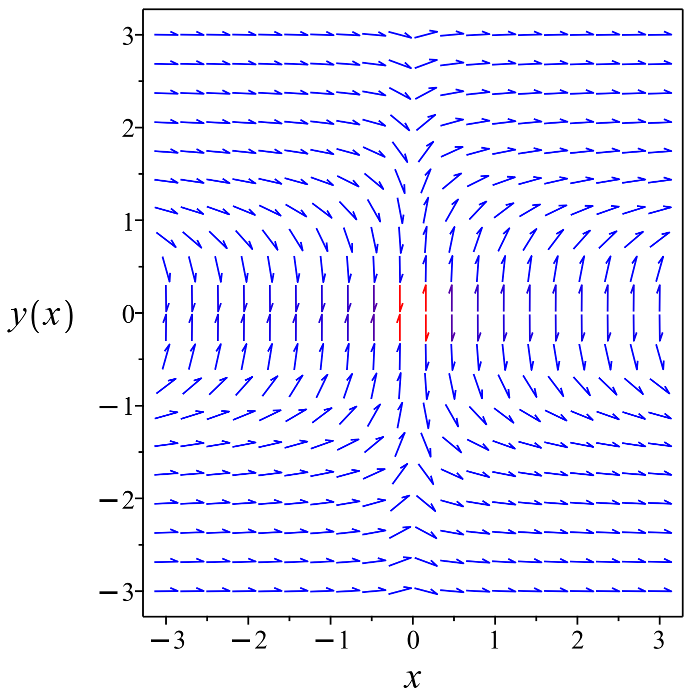


Figure 11: Slope field plot

Verification of solutions

$$\frac{y^4}{4} - \ln(x) = c_1$$

Verified OK.

1.7.4 Maple step by step solution

Let's solve

$xy' - \frac{1}{y^3} = 0$

- Highest derivative means the order of the ODE is 1

  y'

- Separate variables

  $y^3 y' = \frac{1}{x}$

- Integrate both sides with respect to x

  $\int y^3 y' dx = \int \frac{1}{x} dx + c_1$

- Evaluate integral

  $\frac{y^4}{4} = \ln(x) + c_1$

- Solve for y

  $\left\{y = (4\ln(x) + 4c_1)^{\frac{1}{4}}, y = -(4\ln(x) + 4c_1)^{\frac{1}{4}}\right\}$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
<- Bernoulli successful`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 53

```
dsolve(x*diff(y(x),x)=1/y(x)^3,y(x), singsol=all)
```

$$y(x) = (4\ln(x) + c_1)^{\frac{1}{4}}$$
$$y(x) = -(4\ln(x) + c_1)^{\frac{1}{4}}$$
$$y(x) = -i(4\ln(x) + c_1)^{\frac{1}{4}}$$
$$y(x) = i(4\ln(x) + c_1)^{\frac{1}{4}}$$

✓ Solution by Mathematica

Time used: 0.15 (sec). Leaf size: 84

```
DSolve[x*y'[x]==1/y[x]^3,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to -\sqrt{2}\sqrt[4]{\log(x)+c_1}$$
$$y(x) \to -i\sqrt{2}\sqrt[4]{\log(x)+c_1}$$
$$y(x) \to i\sqrt{2}\sqrt[4]{\log(x)+c_1}$$
$$y(x) \to \sqrt{2}\sqrt[4]{\log(x)+c_1}$$

1.8 problem 8

Internal problem ID [4919]
Internal file name [`OUTPUT/4412_Sunday_June_05_2022_01_16_49_PM_73780462/index.tex`]

Book: Fundamentals of Differential Equations. By Nagle, Saff and Snider. 9th edition. Boston. Pearson 2018.
Section: Chapter 2, First order differential equations. Section 2.2, Separable Equations. Exercises. page 46
Problem number: 8.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"exact", "linear", "separable", "homogeneousTypeD2", "first_order_ode_lie_symmetry_lookup"**

Maple gives the following as the ode type

`[_separable]`

$$\boxed{x' - 3x t^2 = 0}$$

1.8.1 Solving as separable ode

In canonical form the ODE is

$$\begin{aligned} x' &= F(t, x) \\ &= f(t)g(x) \\ &= 3x\,t^2 \end{aligned}$$

Where $f(t) = 3t^2$ and $g(x) = x$. Integrating both sides gives

$$\frac{1}{x}\,dx = 3t^2\,dt$$
$$\int \frac{1}{x}\,dx = \int 3t^2\,dt$$
$$\ln(x) = t^3 + c_1$$
$$x = \mathrm{e}^{t^3+c_1}$$
$$= c_1\mathrm{e}^{t^3}$$

Summary

The solution(s) found are the following

$$x = c_1\mathrm{e}^{t^3} \tag{1}$$

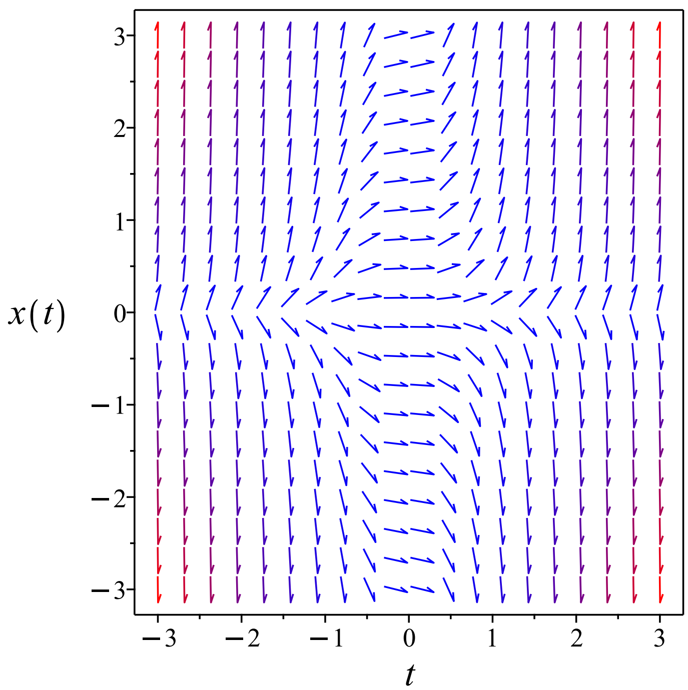


Figure 12: Slope field plot

Verification of solutions

$$x = c_1\mathrm{e}^{t^3}$$

Verified OK.

1.8.2 Solving as linear ode

Entering Linear first order ODE solver. In canonical form a linear first order is

$$x' + p(t)x = q(t)$$

Where here

$$\begin{aligned} p(t) &= -3t^2 \\ q(t) &= 0 \end{aligned}$$

Hence the ode is

$$x' - 3x t^2 = 0$$

The integrating factor μ is

$$\begin{aligned} \mu &= e^{\int -3t^2 dt} \\ &= e^{-t^3} \end{aligned}$$

The ode becomes

$$\begin{aligned} \frac{d}{dt}\mu x &= 0 \\ \frac{d}{dt}\left(e^{-t^3}x\right) &= 0 \end{aligned}$$

Integrating gives

$$e^{-t^3}x = c_1$$

Dividing both sides by the integrating factor $\mu = e^{-t^3}$ results in

$$x = c_1 e^{t^3}$$

Summary
The solution(s) found are the following

$$x = c_1 e^{t^3} \tag{1}$$

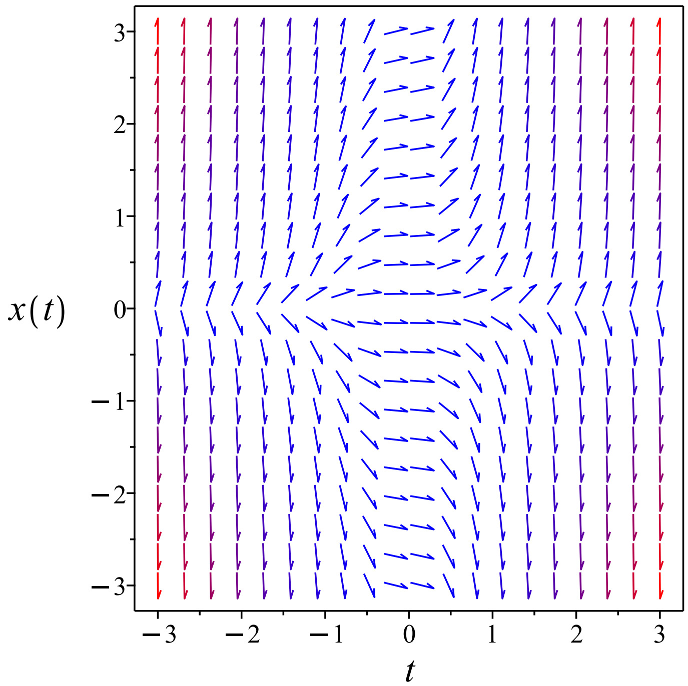


Figure 13: Slope field plot

<u>Verification of solutions</u>

$$x = c_1 e^{t^3}$$

Verified OK.

1.8.3 Solving as homogeneousTypeD2 ode

Using the change of variables $x = u(t)\, t$ on the above ode results in new ode in $u(t)$

$$u'(t)\, t + u(t) - 3u(t)\, t^3 = 0$$

In canonical form the ODE is

$$\begin{aligned} u' &= F(t,u) \\ &= f(t)g(u) \\ &= \frac{u(3t^3-1)}{t} \end{aligned}$$

Where $f(t) = \frac{3t^3-1}{t}$ and $g(u) = u$. Integrating both sides gives

$$
\begin{aligned}
\frac{1}{u}\,du &= \frac{3t^3-1}{t}\,dt \\
\int \frac{1}{u}\,du &= \int \frac{3t^3-1}{t}\,dt \\
\ln(u) &= t^3 - \ln(t) + c_2 \\
u &= \mathrm{e}^{t^3-\ln(t)+c_2} \\
&= c_2\mathrm{e}^{t^3-\ln(t)}
\end{aligned}
$$

Which simplifies to

$$
u(t) = \frac{c_2\mathrm{e}^{t^3}}{t}
$$

Therefore the solution x is

$$
\begin{aligned}
x &= tu \\
&= c_2\mathrm{e}^{t^3}
\end{aligned}
$$

Summary

The solution(s) found are the following

$$
x = c_2\mathrm{e}^{t^3} \tag{1}
$$

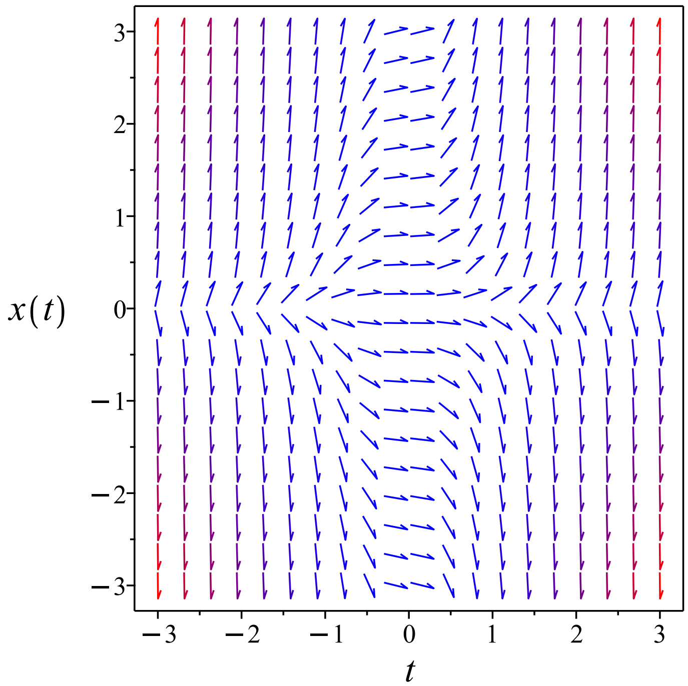


Figure 14: Slope field plot

Verification of solutions

$$x = c_2 e^{t^3}$$

Verified OK.

1.8.4 Solving as first order ode lie symmetry lookup ode

Writing the ode as

$$x' = 3x\,t^2$$
$$x' = \omega(t, x)$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_t + \omega(\eta_x - \xi_t) - \omega^2 \xi_x - \omega_t \xi - \omega_x \eta = 0 \tag{A}$$

The type of this ode is known. It is of type `linear`. Therefore we do not need to solve the PDE (A), and can just use the lookup table shown below to find ξ, η

Table 11: Lie symmetry infinitesimal lookup table for known first order ODE's

ODE class	Form	ξ	η
linear ode	$y' = f(x)y(x) + g(x)$	0	$e^{\int f dx}$
separable ode	$y' = f(x)\, g(y)$	$\frac{1}{f}$	0
quadrature ode	$y' = f(x)$	0	1
quadrature ode	$y' = g(y)$	1	0
homogeneous ODEs of Class A	$y' = f\left(\frac{y}{x}\right)$	x	y
homogeneous ODEs of Class C	$y' = (a + bx + cy)^{\frac{n}{m}}$	1	$-\frac{b}{c}$
homogeneous class D	$y' = \frac{y}{x} + g(x)\, F\left(\frac{y}{x}\right)$	x^2	xy
First order special form ID 1	$y' = g(x)\, e^{h(x)+by} + f(x)$	$\frac{e^{-\int bf(x)dx - h(x)}}{g(x)}$	$\frac{f(x)e^{-\int bf(x)dx - h(x)}}{g(x)}$
polynomial type ode	$y' = \frac{a_1x+b_1y+c_1}{a_2x+b_2y+c_2}$	$\frac{a_1b_2x - a_2b_1x - b_1c_2 + b_2c_1}{a_1b_2 - a_2b_1}$	$\frac{a_1b_2y - a_2b_1y - a_1c_2 - a_2c_1}{a_1b_2 - a_2b_1}$
Bernoulli ode	$y' = f(x)\, y + g(x)\, y^n$	0	$e^{-\int (n-1)f(x)dx} y^n$
Reduced Riccati	$y' = f_1(x)\, y + f_2(x)\, y^2$	0	$e^{-\int f_1 dx}$

The above table shows that

$$
\begin{aligned}
\xi(t, x) &= 0 \\
\eta(t, x) &= e^{t^3}
\end{aligned} \tag{A1}
$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(t, x) \to (R, S)$ where (R, S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$
\frac{dt}{\xi} = \frac{dx}{\eta} = dS \tag{1}
$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial t} + \eta \frac{\partial}{\partial x}\right) S(t, x) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the

canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$R = t$$

S is found from

$$\begin{aligned} S &= \int \frac{1}{\eta} dy \\ &= \int \frac{1}{e^{t^3}} dy \end{aligned}$$

Which results in

$$S = e^{-t^3} x$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_t + \omega(t, x) S_x}{R_t + \omega(t, x) R_x} \tag{2}$$

Where in the above R_t, R_x, S_t, S_x are all partial derivatives and $\omega(t, x)$ is the right hand side of the original ode given by

$$\omega(t, x) = 3x\,t^2$$

Evaluating all the partial derivatives gives

$$\begin{aligned} R_t &= 1 \\ R_x &= 0 \\ S_t &= -3t^2 e^{-t^3} x \\ S_x &= e^{-t^3} \end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = 0 \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for t, x in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = 0$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by

integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$S(R) = c_1 \tag{4}$$

To complete the solution, we just need to transform (4) back to t, x coordinates. This results in

$$e^{-t^3} x = c_1$$

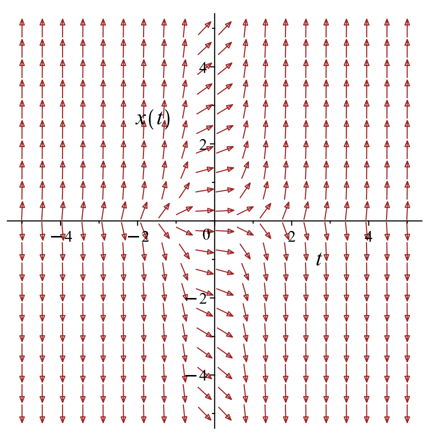
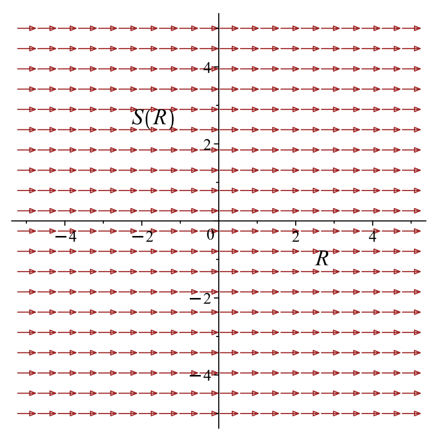
Which simplifies to

$$e^{-t^3} x = c_1$$

Which gives

$$x = c_1 e^{t^3}$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in t, x coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dx}{dt} = 3x\,t^2$ 	$R = t$ $S = e^{-t^3} x$	$\frac{dS}{dR} = 0$ 

Summary
The solution(s) found are the following

$$x = c_1 e^{t^3} \tag{1}$$

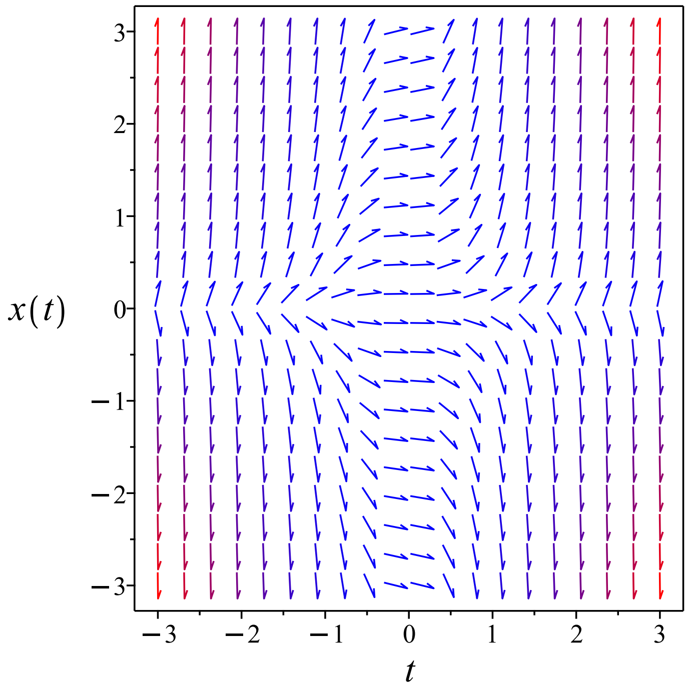


Figure 15: Slope field plot

Verification of solutions

$$x = c_1 e^{t^3}$$

Verified OK.

1.8.5 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x,y) + N(x,y)\frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x,y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y) = 0$$

Hence

$$\frac{\partial\phi}{\partial x} + \frac{\partial\phi}{\partial y}\frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$\begin{aligned}\frac{\partial \phi}{\partial x} &= M \\ \frac{\partial \phi}{\partial y} &= N\end{aligned}$$

But since $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x, y)$ but at least we know now that we can do that since the condition $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(t, x)\,\mathrm{d}t + N(t, x)\,\mathrm{d}x = 0 \tag{1A}$$

Therefore

$$\begin{aligned}\left(\frac{1}{3x}\right)\mathrm{d}x &= \left(t^2\right)\mathrm{d}t \\ \left(-t^2\right)\mathrm{d}t + \left(\frac{1}{3x}\right)\mathrm{d}x &= 0 \qquad \text{(2A)}\end{aligned}$$

Comparing (1A) and (2A) shows that

$$\begin{aligned}M(t, x) &= -t^2 \\ N(t, x) &= \frac{1}{3x}\end{aligned}$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial x} = \frac{\partial N}{\partial t}$$

Using result found above gives

$$\begin{aligned}\frac{\partial M}{\partial x} &= \frac{\partial}{\partial x}\left(-t^2\right) \\ &= 0\end{aligned}$$

And

$$
\begin{aligned}
\frac{\partial N}{\partial t} &= \frac{\partial}{\partial t}\left(\frac{1}{3x}\right) \\
&= 0
\end{aligned}
$$

Since $\frac{\partial M}{\partial x} = \frac{\partial N}{\partial t}$, then the ODE is <u>exact</u> The following equations are now set up to solve for the function $\phi(t, x)$

$$
\frac{\partial \phi}{\partial t} = M \tag{1}
$$

$$
\frac{\partial \phi}{\partial x} = N \tag{2}
$$

Integrating (1) w.r.t. t gives

$$
\begin{aligned}
\int \frac{\partial \phi}{\partial t} \,\mathrm{d}t &= \int M \,\mathrm{d}t \\
\int \frac{\partial \phi}{\partial t} \,\mathrm{d}t &= \int -t^2 \,\mathrm{d}t
\end{aligned}
$$

$$
\phi = -\frac{t^3}{3} + f(x) \tag{3}
$$

Where $f(x)$ is used for the constant of integration since ϕ is a function of both t and x. Taking derivative of equation (3) w.r.t x gives

$$
\frac{\partial \phi}{\partial x} = 0 + f'(x) \tag{4}
$$

But equation (2) says that $\frac{\partial \phi}{\partial x} = \frac{1}{3x}$. Therefore equation (4) becomes

$$
\frac{1}{3x} = 0 + f'(x) \tag{5}
$$

Solving equation (5) for $f'(x)$ gives

$$
f'(x) = \frac{1}{3x}
$$

Integrating the above w.r.t x gives

$$
\begin{aligned}
\int f'(x) \,\mathrm{d}x &= \int \left(\frac{1}{3x}\right) \mathrm{d}x \\
f(x) &= \frac{\ln(x)}{3} + c_1
\end{aligned}
$$

Where c_1 is constant of integration. Substituting result found above for $f(x)$ into equation (3) gives ϕ

$$\phi = -\frac{t^3}{3} + \frac{\ln(x)}{3} + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = -\frac{t^3}{3} + \frac{\ln(x)}{3}$$

The solution becomes

$$x = e^{t^3+3c_1}$$

Summary

The solution(s) found are the following

$$x = e^{t^3+3c_1} \tag{1}$$

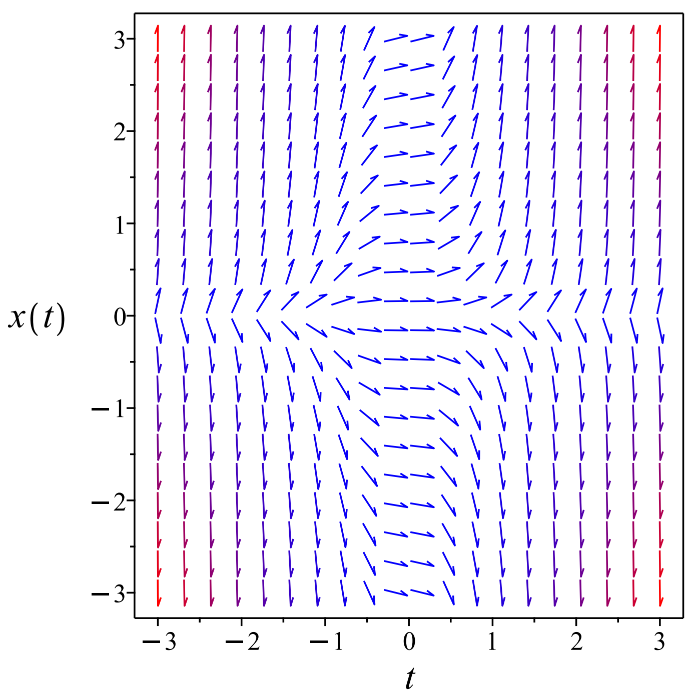


Figure 16: Slope field plot

Verification of solutions

$$x = e^{t^3+3c_1}$$

Verified OK.

1.8.6 Maple step by step solution

Let's solve

$x' - 3xt^2 = 0$

- Highest derivative means the order of the ODE is 1

  x'

- Separate variables

  $\frac{x'}{x} = 3t^2$

- Integrate both sides with respect to t

  $\int \frac{x'}{x}dt = \int 3t^2dt + c_1$

- Evaluate integral

  $\ln(x) = t^3 + c_1$

- Solve for x

  $x = e^{t^3+c_1}$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
<- 1st order linear successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 10

```
dsolve(diff(x(t),t)=3*x(t)*t^2,x(t), singsol=all)
```

$$x(t) = c_1 e^{t^3}$$

✓ Solution by Mathematica

Time used: 0.023 (sec). Leaf size: 18

```
DSolve[x'[t]==3*x[t]*t^2,x[t],t,IncludeSingularSolutions -> True]
```

$$x(t) \to c_1 e^{t^3}$$
$$x(t) \to 0$$

1.9 problem 9

Internal problem ID [4920]
Internal file name [`OUTPUT/4413_Sunday_June_05_2022_01_16_58_PM_89072805/index.tex`]

Book: Fundamentals of Differential Equations. By Nagle, Saff and Snider. 9th edition. Boston. Pearson 2018.
Section: Chapter 2, First order differential equations. Section 2.2, Separable Equations. Exercises. page 46
Problem number: 9.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"exact", "separable", "first_order_ode_lie_symmetry_lookup"**

Maple gives the following as the ode type

`[_separable]`

$$\boxed{x' - \frac{t\,\mathrm{e}^{-t-2x}}{x} = 0}$$

1.9.1 Solving as separable ode

In canonical form the ODE is

$$\begin{aligned} x' &= F(t,x) \\ &= f(t)g(x) \\ &= \frac{t\,\mathrm{e}^{-t}\mathrm{e}^{-2x}}{x} \end{aligned}$$

Where $f(t) = t\,\mathrm{e}^{-t}$ and $g(x) = \frac{\mathrm{e}^{-2x}}{x}$. Integrating both sides gives

$$\frac{1}{\frac{\mathrm{e}^{-2x}}{x}}\,dx = t\,\mathrm{e}^{-t}\,dt$$

$$\int \frac{1}{\frac{e^{-2x}}{x}} dx = \int t\, e^{-t} dt$$

$$\frac{(2x-1)\, e^{2x}}{4} = -(t+1)\, e^{-t} + c_1$$

Which results in

$$x = \frac{\text{LambertW}\left(\left(4c_1 e^{t} - 4t - 4\right) e^{-t-1}\right)}{2} + \frac{1}{2}$$

Summary

The solution(s) found are the following

$$x = \frac{\text{LambertW}\left(\left(4c_1 e^{t} - 4t - 4\right) e^{-t-1}\right)}{2} + \frac{1}{2} \tag{1}$$

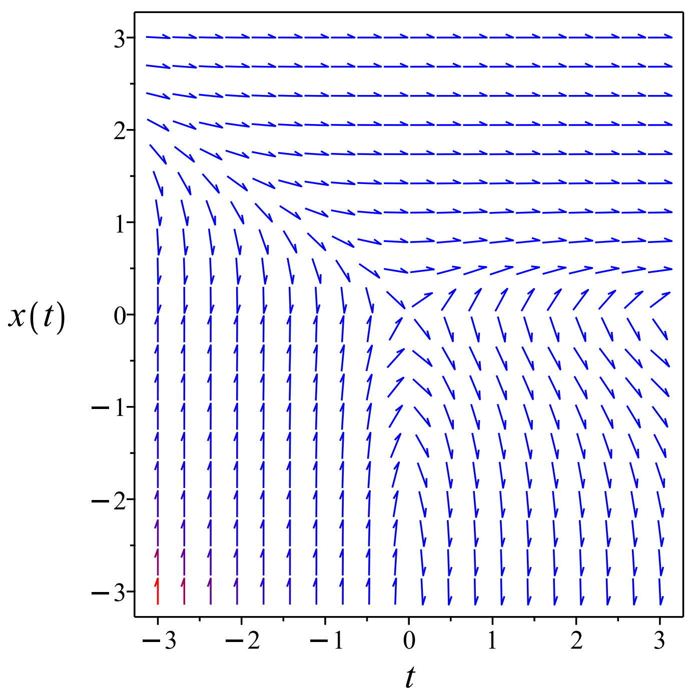


Figure 17: Slope field plot

Verification of solutions

$$x = \frac{\text{LambertW}\left(\left(4c_1 e^{t} - 4t - 4\right) e^{-t-1}\right)}{2} + \frac{1}{2}$$

Verified OK.

1.9.2 Solving as first order ode lie symmetry lookup ode

Writing the ode as

$$
\begin{aligned}
x' &= \frac{t\,\mathrm{e}^{-t-2x}}{x} \\
x' &= \omega(t,x)
\end{aligned}
$$

The condition of Lie symmetry is the linearized PDE given by

$$
\eta_t + \omega(\eta_x - \xi_t) - \omega^2 \xi_x - \omega_t \xi - \omega_x \eta = 0 \tag{A}
$$

The type of this ode is known. It is of type `separable`. Therefore we do not need to solve the PDE (A), and can just use the lookup table shown below to find ξ, η

Table 14: Lie symmetry infinitesimal lookup table for known first order ODE's

ODE class	Form	ξ	η
linear ode	$y' = f(x)y(x) + g(x)$	0	$e^{\int f dx}$
separable ode	$y' = f(x)\,g(y)$	$\frac{1}{f}$	0
quadrature ode	$y' = f(x)$	0	1
quadrature ode	$y' = g(y)$	1	0
homogeneous ODEs of Class A	$y' = f\left(\frac{y}{x}\right)$	x	y
homogeneous ODEs of Class C	$y' = (a + bx + cy)^{\frac{n}{m}}$	1	$-\frac{b}{c}$
homogeneous class D	$y' = \frac{y}{x} + g(x)\,F\left(\frac{y}{x}\right)$	x^2	xy
First order special form ID 1	$y' = g(x)\,e^{h(x)+by} + f(x)$	$\frac{e^{-\int bf(x)dx-h(x)}}{g(x)}$	$\frac{f(x)e^{-\int bf(x)dx-h(x)}}{g(x)}$
polynomial type ode	$y' = \frac{a_1x+b_1y+c_1}{a_2x+b_2y+c_2}$	$\frac{a_1b_2x-a_2b_1x-b_1c_2+b_2c_1}{a_1b_2-a_2b_1}$	$\frac{a_1b_2y-a_2b_1y-a_1c_2-a_2c_1}{a_1b_2-a_2b_1}$
Bernoulli ode	$y' = f(x)\,y + g(x)\,y^n$	0	$e^{-\int(n-1)f(x)dx}y^n$
Reduced Riccati	$y' = f_1(x)\,y + f_2(x)\,y^2$	0	$e^{-\int f_1 dx}$

The above table shows that

$$
\begin{aligned}
\xi(t,x) &= \frac{e^{t}}{t} \\
\eta(t,x) &= 0
\end{aligned}
\tag{A1}
$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(t,x) \to (R,S)$ where (R,S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$
\frac{dt}{\xi} = \frac{dx}{\eta} = dS \tag{1}
$$

The above comes from the requirements that $\left(\xi\frac{\partial}{\partial t} + \eta\frac{\partial}{\partial x}\right) S(t,x) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the canonical coordinates, where $S(R)$. Since $\eta = 0$ then in this special case

$$
R = x
$$

S is found from

$$
\begin{aligned}
S &= \int \frac{1}{\xi} dt \\
&= \int \frac{1}{\frac{e^{t}}{t}} dt
\end{aligned}
$$

Which results in

$$
S = -(t+1)\,e^{-t}
$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$
\frac{dS}{dR} = \frac{S_t + \omega(t,x) S_x}{R_t + \omega(t,x) R_x} \tag{2}
$$

Where in the above R_t, R_x, S_t, S_x are all partial derivatives and $\omega(t,x)$ is the right hand side of the original ode given by

$$
\omega(t,x) = \frac{t\,e^{-t-2x}}{x}
$$

Evaluating all the partial derivatives gives

$$
\begin{aligned}
R_t &= 0 \\
R_x &= 1 \\
S_t &= t\,\mathrm{e}^{-t} \\
S_x &= 0
\end{aligned}
$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$
\frac{dS}{dR} = \mathrm{e}^{2x} x \tag{2A}
$$

We now need to express the RHS as function of R only. This is done by solving for t, x in terms of R, S from the result obtained earlier and simplifying. This gives

$$
\frac{dS}{dR} = \mathrm{e}^{2R} R
$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$
S(R) = \frac{(2R-1)\,\mathrm{e}^{2R}}{4} + c_1 \tag{4}
$$

To complete the solution, we just need to transform (4) back to t, x coordinates. This results in

$$
-(t+1)\,\mathrm{e}^{-t} = \frac{(2x-1)\,\mathrm{e}^{2x}}{4} + c_1
$$

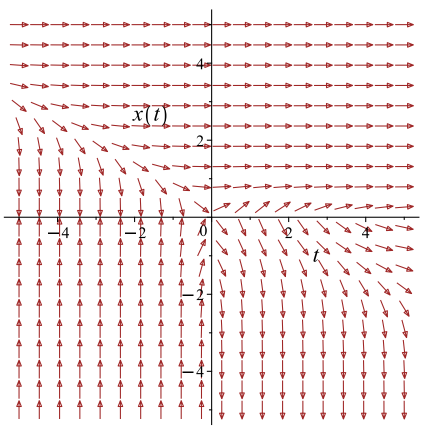
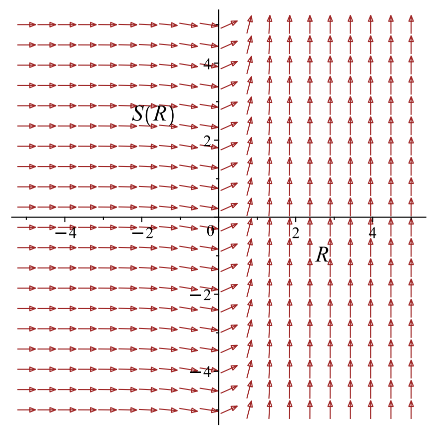
Which simplifies to

$$
-(t+1)\,\mathrm{e}^{-t} = \frac{(2x-1)\,\mathrm{e}^{2x}}{4} + c_1
$$

Which gives

$$
x = \frac{\mathrm{LambertW}\left((-4c_1\mathrm{e}^{t} - 4t - 4)\,\mathrm{e}^{-t-1}\right)}{2} + \frac{1}{2}
$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in t, x coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dx}{dt} = \frac{t\,\mathrm{e}^{-t-2x}}{x}$ 	$R = x$ $S = -(t+1)\,\mathrm{e}^{-t}$	$\frac{dS}{dR} = \mathrm{e}^{2R} R$ 

Summary

The solution(s) found are the following

$$x = \frac{\mathrm{LambertW}\left(\left(-4c_1\mathrm{e}^{t} - 4t - 4\right)\mathrm{e}^{-t-1}\right)}{2} + \frac{1}{2} \tag{1}$$

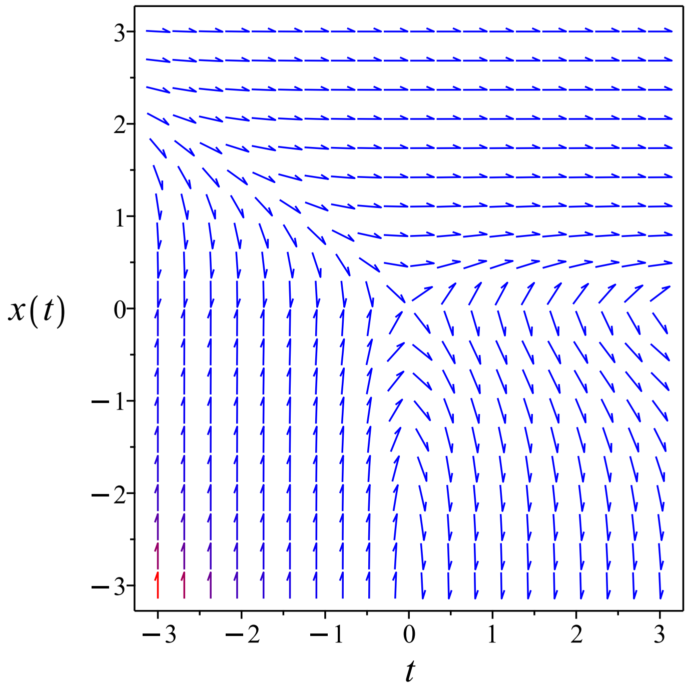


Figure 18: Slope field plot

Verification of solutions

$$x = \frac{\text{LambertW}\left(\left(-4c_1\mathrm{e}^{t} - 4t - 4\right)\mathrm{e}^{-t-1}\right)}{2} + \frac{1}{2}$$

Verified OK.

1.9.3 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x,y) + N(x,y)\frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x,y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y) = 0$$

Hence

$$\frac{\partial \phi}{\partial x} + \frac{\partial \phi}{\partial y}\frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$\begin{aligned}\frac{\partial\phi}{\partial x} &= M\\ \frac{\partial\phi}{\partial y} &= N\end{aligned}$$

But since $\frac{\partial^2\phi}{\partial x\partial y} = \frac{\partial^2\phi}{\partial y\partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x,y)$ but at least we know now that we can do that since the condition $\frac{\partial^2\phi}{\partial x\partial y} = \frac{\partial^2\phi}{\partial y\partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(t,x)\,\mathrm{d}t + N(t,x)\,\mathrm{d}x = 0 \tag{1A}$$

Therefore

$$\begin{aligned}\left(\mathrm{e}^{2x}x\right)\mathrm{d}x &= \left(t\,\mathrm{e}^{-t}\right)\mathrm{d}t\\ \left(-t\,\mathrm{e}^{-t}\right)\mathrm{d}t + \left(\mathrm{e}^{2x}x\right)\mathrm{d}x &= 0 \qquad (2A)\end{aligned}$$

Comparing (1A) and (2A) shows that

$$\begin{aligned}M(t,x) &= -t\,\mathrm{e}^{-t}\\ N(t,x) &= \mathrm{e}^{2x}x\end{aligned}$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial x} = \frac{\partial N}{\partial t}$$

Using result found above gives

$$\begin{aligned}\frac{\partial M}{\partial x} &= \frac{\partial}{\partial x}\left(-t\,\mathrm{e}^{-t}\right)\\ &= 0\end{aligned}$$

And

$$\begin{aligned}\frac{\partial N}{\partial t} &= \frac{\partial}{\partial t}\left(\mathrm{e}^{2x}x\right)\\ &= 0\end{aligned}$$

Since $\frac{\partial M}{\partial x} = \frac{\partial N}{\partial t}$, then the ODE is exact The following equations are now set up to solve for the function $\phi(t, x)$

$$\frac{\partial \phi}{\partial t} = M \tag{1}$$

$$\frac{\partial \phi}{\partial x} = N \tag{2}$$

Integrating (1) w.r.t. t gives

$$\int \frac{\partial \phi}{\partial t} \, \mathrm{d}t = \int M \, \mathrm{d}t$$

$$\int \frac{\partial \phi}{\partial t} \, \mathrm{d}t = \int -t \, \mathrm{e}^{-t} \, \mathrm{d}t$$

$$\phi = (t + 1) \, \mathrm{e}^{-t} + f(x) \tag{3}$$

Where $f(x)$ is used for the constant of integration since ϕ is a function of both t and x. Taking derivative of equation (3) w.r.t x gives

$$\frac{\partial \phi}{\partial x} = 0 + f'(x) \tag{4}$$

But equation (2) says that $\frac{\partial \phi}{\partial x} = \mathrm{e}^{2x} x$. Therefore equation (4) becomes

$$\mathrm{e}^{2x} x = 0 + f'(x) \tag{5}$$

Solving equation (5) for $f'(x)$ gives

$$f'(x) = \mathrm{e}^{2x} x$$

Integrating the above w.r.t x gives

$$\int f'(x) \, \mathrm{d}x = \int (\mathrm{e}^{2x} x) \, \mathrm{d}x$$

$$f(x) = \frac{(2x - 1) \, \mathrm{e}^{2x}}{4} + c_1$$

Where c_1 is constant of integration. Substituting result found above for $f(x)$ into equation (3) gives ϕ

$$\phi = (t + 1) \, \mathrm{e}^{-t} + \frac{(2x - 1) \, \mathrm{e}^{2x}}{4} + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = (t+1)\,\mathrm{e}^{-t} + \frac{(2x-1)\,\mathrm{e}^{2x}}{4}$$

The solution becomes

$$x = \frac{\mathrm{LambertW}\left((4c_1\mathrm{e}^{t} - 4t - 4)\,\mathrm{e}^{-t-1}\right)}{2} + \frac{1}{2}$$

Summary

The solution(s) found are the following

$$x = \frac{\mathrm{LambertW}\left((4c_1\mathrm{e}^{t} - 4t - 4)\,\mathrm{e}^{-t-1}\right)}{2} + \frac{1}{2} \tag{1}$$

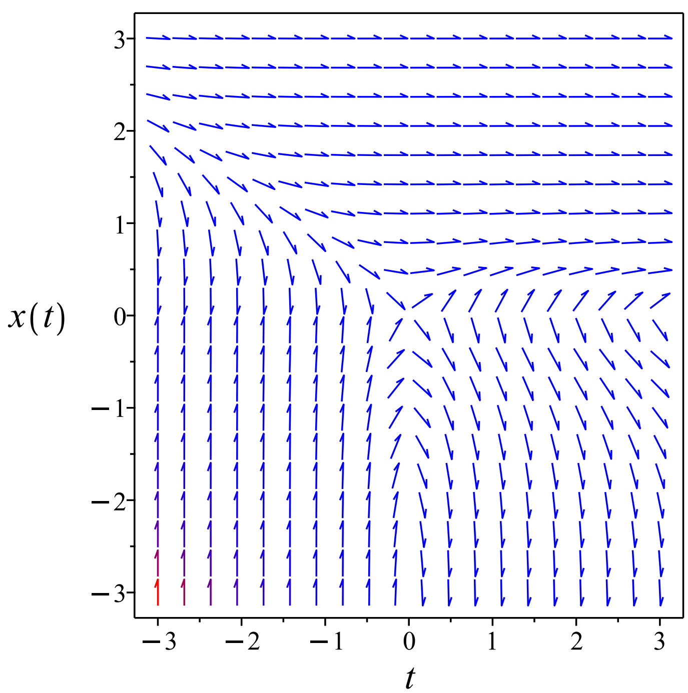


Figure 19: Slope field plot

Verification of solutions

$$x = \frac{\mathrm{LambertW}\left((4c_1\mathrm{e}^{t} - 4t - 4)\,\mathrm{e}^{-t-1}\right)}{2} + \frac{1}{2}$$

Verified OK.

1.9.4 Maple step by step solution

Let's solve

$x' - \frac{t}{x\,e^{t+2x}} = 0$

- Highest derivative means the order of the ODE is 1

  x'

- Separate variables

  $x'(e^{x})^{2}\,x = \frac{t}{e^{t}}$

- Integrate both sides with respect to t

  $\int x'(e^{x})^{2}\,x dt = \int \frac{t}{e^{t}} dt + c_1$

- Evaluate integral

  $\frac{(e^{x})^{2} x}{2} - \frac{(e^{x})^{2}}{4} = -\frac{t+1}{e^{t}} + c_1$

- Solve for x

  $x = \frac{\mathit{LambertW}\left((4c_1 e^{t} - 4t - 4)\,e^{-t-1}\right)}{2} + \frac{1}{2}$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
<- separable successful`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 25

```
dsolve(diff(x(t),t)=t/(x(t)*exp(t+2*x(t))),x(t), singsol=all)
```

$$x(t) = \frac{\mathrm{LambertW}\left(-4(-c_1 e^{t} + t + 1)\,e^{-t-1}\right)}{2} + \frac{1}{2}$$

✓ Solution by Mathematica

Time used: 60.16 (sec). Leaf size: 31

```
DSolve[x'[t]==t/(x[t]*Exp[t+2*x[t]]),x[t],t,IncludeSingularSolutions -> True]
```

$$x(t) \to \frac{1}{2}\left(1 + W\left(-4e^{-t-1}\left(t - c_1 e^t + 1\right)\right)\right)$$

1.10 problem 10

Internal problem ID [4921]
Internal file name [`OUTPUT/4414_Sunday_June_05_2022_01_17_07_PM_38159823/index.tex`]

Book: Fundamentals of Differential Equations. By Nagle, Saff and Snider. 9th edition. Boston. Pearson 2018.
Section: Chapter 2, First order differential equations. Section 2.2, Separable Equations. Exercises. page 46
Problem number: 10.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"exact", "separable", "first_order_ode_lie_symmetry_lookup"**

Maple gives the following as the ode type

`[_separable]`

$$\boxed{y' - \frac{x}{y^2\sqrt{x+1}} = 0}$$

1.10.1 Solving as separable ode

In canonical form the ODE is

$$\begin{aligned} y' &= F(x,y) \\ &= f(x)g(y) \\ &= \frac{x}{y^2\sqrt{x+1}} \end{aligned}$$

Where $f(x) = \frac{x}{\sqrt{x+1}}$ and $g(y) = \frac{1}{y^2}$. Integrating both sides gives

$$\frac{1}{\frac{1}{y^2}}\,dy = \frac{x}{\sqrt{x+1}}\,dx$$

$$
\int \frac{1}{\frac{1}{y^2}} dy = \int \frac{x}{\sqrt{x+1}} dx
$$

$$
\frac{y^3}{3} = \frac{2\sqrt{x+1}\,(-2+x)}{3} + c_1
$$

Which results in

$$
y = \left(2x\sqrt{x+1} - 4\sqrt{x+1} + 3c_1\right)^{\frac{1}{3}}
$$

$$
y = -\frac{\left(2x\sqrt{x+1} - 4\sqrt{x+1} + 3c_1\right)^{\frac{1}{3}}}{2} + \frac{i\sqrt{3}\left(2x\sqrt{x+1} - 4\sqrt{x+1} + 3c_1\right)^{\frac{1}{3}}}{2}
$$

$$
y = -\frac{\left(2x\sqrt{x+1} - 4\sqrt{x+1} + 3c_1\right)^{\frac{1}{3}}}{2} - \frac{i\sqrt{3}\left(2x\sqrt{x+1} - 4\sqrt{x+1} + 3c_1\right)^{\frac{1}{3}}}{2}
$$

Summary

The solution(s) found are the following

$$
y = \left(2x\sqrt{x+1} - 4\sqrt{x+1} + 3c_1\right)^{\frac{1}{3}} \tag{1}
$$

$$
y = -\frac{\left(2x\sqrt{x+1} - 4\sqrt{x+1} + 3c_1\right)^{\frac{1}{3}}}{2} + \frac{i\sqrt{3}\left(2x\sqrt{x+1} - 4\sqrt{x+1} + 3c_1\right)^{\frac{1}{3}}}{2} \tag{2}
$$

$$
y = -\frac{\left(2x\sqrt{x+1} - 4\sqrt{x+1} + 3c_1\right)^{\frac{1}{3}}}{2} - \frac{i\sqrt{3}\left(2x\sqrt{x+1} - 4\sqrt{x+1} + 3c_1\right)^{\frac{1}{3}}}{2} \tag{3}
$$

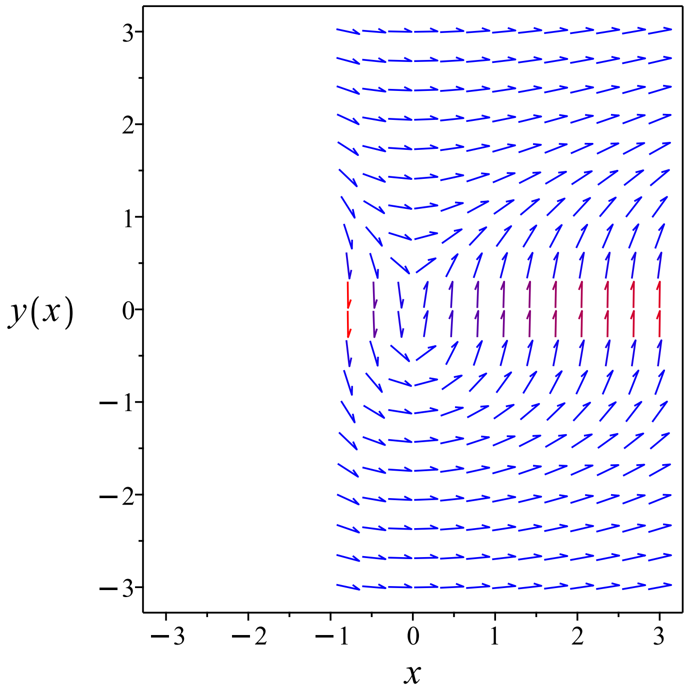


Figure 20: Slope field plot

Verification of solutions

$$y = \left(2x\sqrt{x+1} - 4\sqrt{x+1} + 3c_1\right)^{\frac{1}{3}}$$

Verified OK.

$$y = -\frac{\left(2x\sqrt{x+1} - 4\sqrt{x+1} + 3c_1\right)^{\frac{1}{3}}}{2} + \frac{i\sqrt{3}\left(2x\sqrt{x+1} - 4\sqrt{x+1} + 3c_1\right)^{\frac{1}{3}}}{2}$$

Verified OK.

$$y = -\frac{\left(2x\sqrt{x+1} - 4\sqrt{x+1} + 3c_1\right)^{\frac{1}{3}}}{2} - \frac{i\sqrt{3}\left(2x\sqrt{x+1} - 4\sqrt{x+1} + 3c_1\right)^{\frac{1}{3}}}{2}$$

Verified OK.

1.10.2 Solving as first order ode lie symmetry lookup ode

Writing the ode as

$$
\begin{aligned}
y' &= \frac{x}{y^2\sqrt{x+1}} \\
y' &= \omega(x,y)
\end{aligned}
$$

The condition of Lie symmetry is the linearized PDE given by

$$
\eta_x + \omega(\eta_y - \xi_x) - \omega^2\xi_y - \omega_x\xi - \omega_y\eta = 0 \tag{A}
$$

The type of this ode is known. It is of type `separable`. Therefore we do not need to solve the PDE (A), and can just use the lookup table shown below to find ξ, η

Table 17: Lie symmetry infinitesimal lookup table for known first order ODE's

ODE class	Form	ξ	η
linear ode	$y' = f(x)y(x) + g(x)$	0	$e^{\int f dx}$
separable ode	$y' = f(x)\, g(y)$	$\frac{1}{f}$	0
quadrature ode	$y' = f(x)$	0	1
quadrature ode	$y' = g(y)$	1	0
homogeneous ODEs of Class A	$y' = f\left(\frac{y}{x}\right)$	x	y
homogeneous ODEs of Class C	$y' = (a + bx + cy)^{\frac{n}{m}}$	1	$-\frac{b}{c}$
homogeneous class D	$y' = \frac{y}{x} + g(x)\, F\left(\frac{y}{x}\right)$	x^2	xy
First order special form ID 1	$y' = g(x)\, e^{h(x)+by} + f(x)$	$\frac{e^{-\int bf(x)dx - h(x)}}{g(x)}$	$\frac{f(x)e^{-\int bf(x)dx - h(x)}}{g(x)}$
polynomial type ode	$y' = \frac{a_1x+b_1y+c_1}{a_2x+b_2y+c_2}$	$\frac{a_1b_2x - a_2b_1x - b_1c_2 + b_2c_1}{a_1b_2 - a_2b_1}$	$\frac{a_1b_2y - a_2b_1y - a_1c_2 - a_2c_1}{a_1b_2 - a_2b_1}$
Bernoulli ode	$y' = f(x)\, y + g(x)\, y^n$	0	$e^{-\int(n-1)f(x)dx}y^n$
Reduced Riccati	$y' = f_1(x)\, y + f_2(x)\, y^2$	0	$e^{-\int f_1 dx}$

The above table shows that

$$\begin{aligned}\xi(x,y) &= \frac{\sqrt{x+1}}{x} \\ \eta(x,y) &= 0\end{aligned} \tag{A1}$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x,y) \to (R,S)$ where (R,S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}$$

The above comes from the requirements that $\left(\xi\frac{\partial}{\partial x} + \eta\frac{\partial}{\partial y}\right) S(x,y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the canonical coordinates, where $S(R)$. Since $\eta = 0$ then in this special case

$$R = y$$

S is found from

$$\begin{aligned}S &= \int \frac{1}{\xi} dx \\ &= \int \frac{1}{\frac{\sqrt{x+1}}{x}} dx\end{aligned}$$

Which results in

$$S = \frac{2\sqrt{x+1}\,(-2+x)}{3}$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x,y)S_y}{R_x + \omega(x,y)R_y} \tag{2}$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x,y)$ is the right hand side of the original ode given by

$$\omega(x,y) = \frac{x}{y^2\sqrt{x+1}}$$

Evaluating all the partial derivatives gives

$$
\begin{aligned}
R_x &= 0 \\
R_y &= 1 \\
S_x &= \frac{x}{\sqrt{x+1}} \\
S_y &= 0
\end{aligned}
$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$
\frac{dS}{dR} = y^2 \tag{2A}
$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$
\frac{dS}{dR} = R^2
$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$
S(R) = \frac{R^3}{3} + c_1 \tag{4}
$$

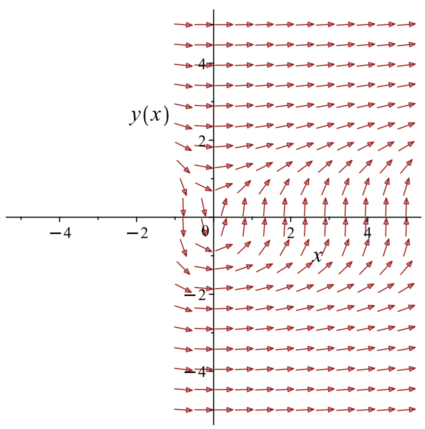
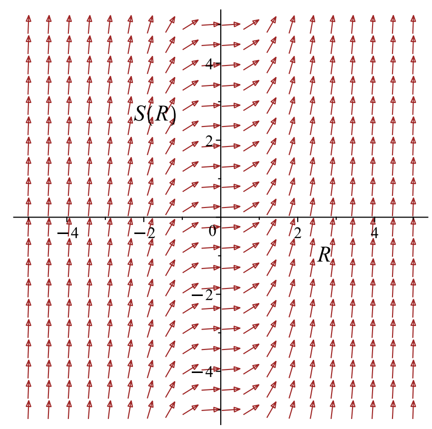
To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$
\frac{2\sqrt{x+1}\,(-2+x)}{3} = \frac{y^3}{3} + c_1
$$

Which simplifies to

$$
\frac{2\sqrt{x+1}\,(-2+x)}{3} = \frac{y^3}{3} + c_1
$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in x, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dx} = \frac{x}{y^2\sqrt{x+1}}$ 	$R = y$ $S = \frac{2\sqrt{x+1}\,(-2+x)}{3}$	$\frac{dS}{dR} = R^2$ 

Summary

The solution(s) found are the following

$$\frac{2\sqrt{x+1}\,(-2+x)}{3} = \frac{y^3}{3} + c_1 \tag{1}$$

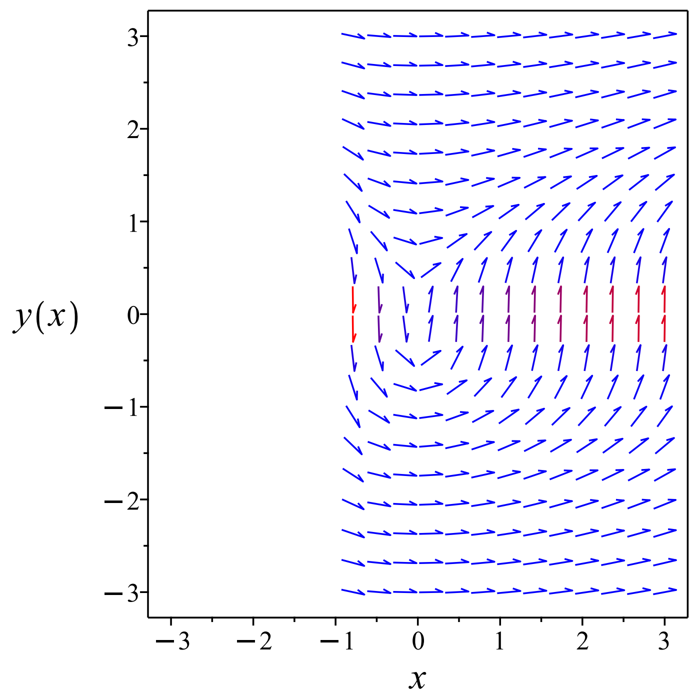


Figure 21: Slope field plot

Verification of solutions

$$\frac{2\sqrt{x+1}\,(-2+x)}{3} = \frac{y^3}{3} + c_1$$

Verified OK.

1.10.3 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x,y) + N(x,y)\frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x,y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y) = 0$$

Hence

$$\frac{\partial\phi}{\partial x} + \frac{\partial\phi}{\partial y}\frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$\frac{\partial \phi}{\partial x} = M$$
$$\frac{\partial \phi}{\partial y} = N$$

But since $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x,y)$ but at least we know now that we can do that since the condition $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x,y)\,\mathrm{d}x + N(x,y)\,\mathrm{d}y = 0 \tag{1A}$$

Therefore

$$\left(y^2\right)\mathrm{d}y = \left(\frac{x}{\sqrt{x+1}}\right)\mathrm{d}x$$
$$\left(-\frac{x}{\sqrt{x+1}}\right)\mathrm{d}x + \left(y^2\right)\mathrm{d}y = 0 \tag{2A}$$

Comparing (1A) and (2A) shows that

$$M(x,y) = -\frac{x}{\sqrt{x+1}}$$
$$N(x,y) = y^2$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Using result found above gives

$$\begin{aligned}\frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}\left(-\frac{x}{\sqrt{x+1}}\right)\\ &= 0\end{aligned}$$

And

$$\frac{\partial N}{\partial x} = \frac{\partial}{\partial x}(y^2)$$
$$= 0$$

Since $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$, then the ODE is exact The following equations are now set up to solve for the function $\phi(x, y)$

$$\frac{\partial \phi}{\partial x} = M \tag{1}$$
$$\frac{\partial \phi}{\partial y} = N \tag{2}$$

Integrating (1) w.r.t. x gives

$$\int \frac{\partial \phi}{\partial x} \, \mathrm{d}x = \int M \, \mathrm{d}x$$
$$\int \frac{\partial \phi}{\partial x} \, \mathrm{d}x = \int -\frac{x}{\sqrt{x+1}} \, \mathrm{d}x$$
$$\phi = -\frac{2\sqrt{x+1}\,(-2+x)}{3} + f(y) \tag{3}$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$\frac{\partial \phi}{\partial y} = 0 + f'(y) \tag{4}$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = y^2$. Therefore equation (4) becomes

$$y^2 = 0 + f'(y) \tag{5}$$

Solving equation (5) for $f'(y)$ gives

$$f'(y) = y^2$$

Integrating the above w.r.t y gives

$$\int f'(y) \, \mathrm{d}y = \int (y^2) \, \mathrm{d}y$$
$$f(y) = \frac{y^3}{3} + c_1$$

Where c_1 is constant of integration. Substituting result found above for $f(y)$ into equation (3) gives ϕ

$$\phi = -\frac{2\sqrt{x+1}\,(-2+x)}{3} + \frac{y^3}{3} + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = -\frac{2\sqrt{x+1}\,(-2+x)}{3} + \frac{y^3}{3}$$

Summary

The solution(s) found are the following

$$-\frac{2\sqrt{x+1}\,(-2+x)}{3} + \frac{y^3}{3} = c_1 \tag{1}$$

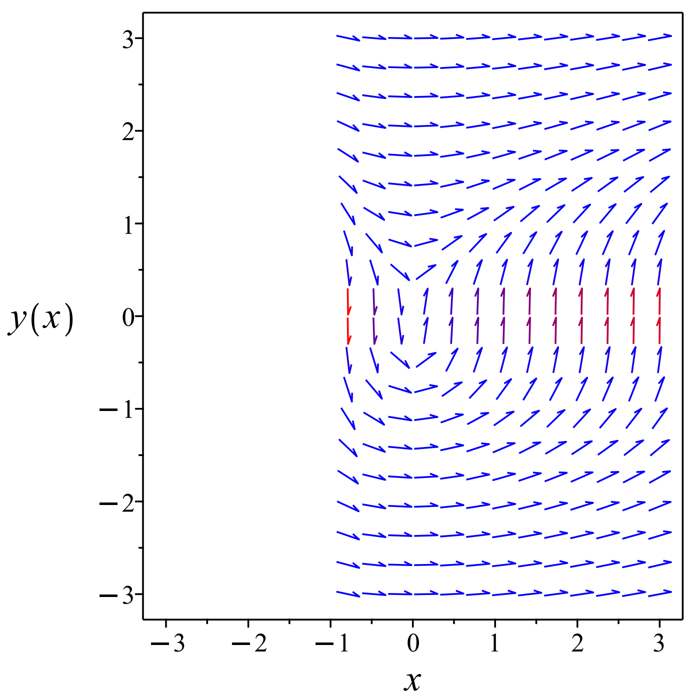


Figure 22: Slope field plot

Verification of solutions

$$-\frac{2\sqrt{x+1}\,(-2+x)}{3} + \frac{y^3}{3} = c_1$$

Verified OK.

1.10.4 Maple step by step solution

Let's solve

$y' - \frac{x}{y^2\sqrt{x+1}} = 0$

- Highest derivative means the order of the ODE is 1

  y'

- Separate variables

  $y'y^2 = \frac{x}{\sqrt{x+1}}$

- Integrate both sides with respect to x

  $\int y'y^2dx = \int \frac{x}{\sqrt{x+1}}dx + c_1$

- Evaluate integral

  $\frac{y^3}{3} = \frac{2\sqrt{x+1}\,(-2+x)}{3} + c_1$

- Solve for y

  $y = \left(2x\sqrt{x+1} - 4\sqrt{x+1} + 3c_1\right)^{\frac{1}{3}}$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
<- Bernoulli successful`
```

✓ Solution by Maple

Time used: 0.015 (sec). Leaf size: 79

```
dsolve(diff(y(x),x)=x/(y(x)^2*sqrt(1+x)),y(x), singsol=all)
```

$$y(x) = \left(2\sqrt{1+x}\,x - 4\sqrt{1+x} + c_1\right)^{\frac{1}{3}}$$
$$y(x) = -\frac{\left((2x-4)\sqrt{1+x} + c_1\right)^{\frac{1}{3}}\left(1+i\sqrt{3}\right)}{2}$$
$$y(x) = \frac{\left((2x-4)\sqrt{1+x} + c_1\right)^{\frac{1}{3}}\left(i\sqrt{3}-1\right)}{2}$$

✓ Solution by Mathematica

Time used: 2.119 (sec). Leaf size: 110

```
DSolve[y'[x]==x/(y[x]^2*Sqrt[1+x]),y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \sqrt[3]{2\sqrt{x+1}x - 4\sqrt{x+1} + 3c_1}$$
$$y(x) \to -\sqrt[3]{-1}\sqrt[3]{2\sqrt{x+1}x - 4\sqrt{x+1} + 3c_1}$$
$$y(x) \to (-1)^{2/3}\sqrt[3]{2\sqrt{x+1}x - 4\sqrt{x+1} + 3c_1}$$

1.11 problem 11

Internal problem ID [4922]
Internal file name [`OUTPUT/4415_Sunday_June_05_2022_01_17_19_PM_99089715/index.tex`]

Book: Fundamentals of Differential Equations. By Nagle, Saff and Snider. 9th edition. Boston. Pearson 2018.
Section: Chapter 2, First order differential equations. Section 2.2, Separable Equations. Exercises. page 46
Problem number: 11.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"exact", "bernoulli", "separable", "first_order_ode_lie_symmetry_lookup"**

Maple gives the following as the ode type

`[_separable]`

$$\boxed{xv' - \frac{1-4v^2}{3v} = 0}$$

1.11.1 Solving as separable ode

In canonical form the ODE is

$$\begin{aligned} v' &= F(x,v) \\ &= f(x)g(v) \\ &= -\frac{4v^2-1}{3vx} \end{aligned}$$

Where $f(x) = -\frac{1}{3x}$ and $g(v) = \frac{4v^2-1}{v}$. Integrating both sides gives

$$\frac{1}{\frac{4v^2-1}{v}}\,dv = -\frac{1}{3x}\,dx$$

$$\int \frac{1}{\frac{4v^2-1}{v}}\,dv = \int -\frac{1}{3x}\,dx$$

$$\frac{\ln(4v^2-1)}{8} = -\frac{\ln(x)}{3} + c_1$$

Raising both side to exponential gives

$$\left(4v^2-1\right)^{\frac{1}{8}} = \mathrm{e}^{-\frac{\ln(x)}{3}+c_1}$$

Which simplifies to

$$\left(4v^2-1\right)^{\frac{1}{8}} = \frac{c_2}{x^{\frac{1}{3}}}$$

Which simplifies to

$$\left(4v^2-1\right)^{\frac{1}{8}} = \frac{c_2 \mathrm{e}^{c_1}}{x^{\frac{1}{3}}}$$

The solution is

$$\left(4v^2-1\right)^{\frac{1}{8}} = \frac{c_2 \mathrm{e}^{c_1}}{x^{\frac{1}{3}}}$$

Summary

The solution(s) found are the following

$$\left(4v^2-1\right)^{\frac{1}{8}} = \frac{c_2 \mathrm{e}^{c_1}}{x^{\frac{1}{3}}} \tag{1}$$

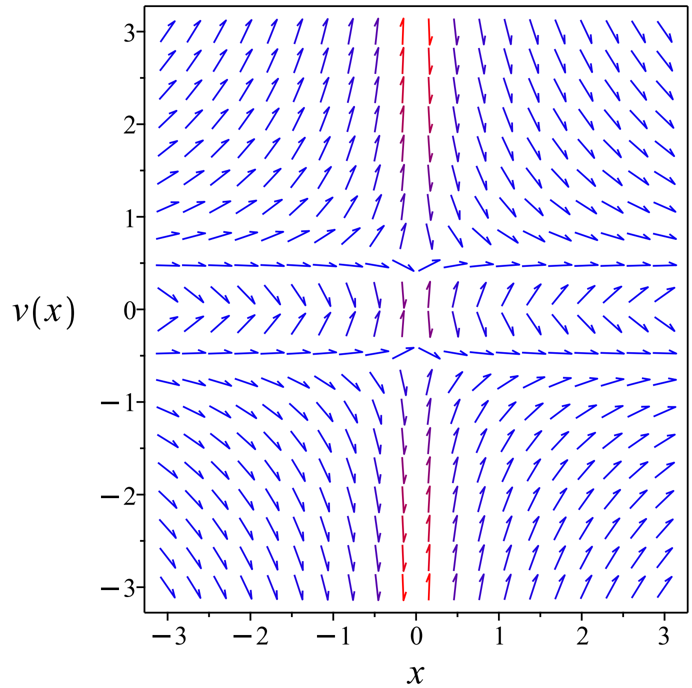


Figure 23: Slope field plot

Verification of solutions

$$(4v^2-1)^{\frac{1}{8}} = \frac{c_2 e^{c_1}}{x^{\frac{1}{3}}}$$

Verified OK.

1.11.2 Solving as first order ode lie symmetry lookup ode

Writing the ode as

$$v' = -\frac{4v^2-1}{3vx}$$
$$v' = \omega(x,v)$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x + \omega(\eta_v - \xi_x) - \omega^2 \xi_v - \omega_x \xi - \omega_v \eta = 0 \tag{A}$$

The type of this ode is known. It is of type `separable`. Therefore we do not need to solve the PDE (A), and can just use the lookup table shown below to find ξ, η

Table 20: Lie symmetry infinitesimal lookup table for known first order ODE's

ODE class	Form	ξ	η
linear ode	$y' = f(x)y(x) + g(x)$	0	$e^{\int f dx}$
separable ode	$y' = f(x)\, g(y)$	$\frac{1}{f}$	0
quadrature ode	$y' = f(x)$	0	1
quadrature ode	$y' = g(y)$	1	0
homogeneous ODEs of Class A	$y' = f\left(\frac{y}{x}\right)$	x	y
homogeneous ODEs of Class C	$y' = (a + bx + cy)^{\frac{n}{m}}$	1	$-\frac{b}{c}$
homogeneous class D	$y' = \frac{y}{x} + g(x)\, F\left(\frac{y}{x}\right)$	x^2	xy
First order special form ID 1	$y' = g(x)\, e^{h(x)+by} + f(x)$	$\frac{e^{-\int bf(x)dx-h(x)}}{g(x)}$	$\frac{f(x)e^{-\int bf(x)dx-h(x)}}{g(x)}$
polynomial type ode	$y' = \frac{a_1x+b_1y+c_1}{a_2x+b_2y+c_2}$	$\frac{a_1b_2x-a_2b_1x-b_1c_2+b_2c_1}{a_1b_2-a_2b_1}$	$\frac{a_1b_2y-a_2b_1y-a_1c_2-a_2c_1}{a_1b_2-a_2b_1}$
Bernoulli ode	$y' = f(x)\, y + g(x)\, y^n$	0	$e^{-\int (n-1)f(x)dx} y^n$
Reduced Riccati	$y' = f_1(x)\, y + f_2(x)\, y^2$	0	$e^{-\int f_1 dx}$

The above table shows that

$$
\begin{aligned}
\xi(x, v) &= -3x \\
\eta(x, v) &= 0
\end{aligned} \tag{A1}
$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x, v) \to (R, S)$ where (R, S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$
\frac{dx}{\xi} = \frac{dv}{\eta} = dS \tag{1}
$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial v}\right) S(x, v) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the

canonical coordinates, where $S(R)$. Since $\eta = 0$ then in this special case

$$R = v$$

S is found from

$$\begin{aligned}S &= \int \frac{1}{\xi} dx \\ &= \int \frac{1}{-3x} dx\end{aligned}$$

Which results in

$$S = -\frac{\ln(x)}{3}$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x,v) S_v}{R_x + \omega(x,v) R_v} \tag{2}$$

Where in the above R_x, R_v, S_x, S_v are all partial derivatives and $\omega(x, v)$ is the right hand side of the original ode given by

$$\omega(x,v) = -\frac{4v^2 - 1}{3vx}$$

Evaluating all the partial derivatives gives

$$\begin{aligned}R_x &= 0 \\ R_v &= 1 \\ S_x &= -\frac{1}{3x} \\ S_v &= 0\end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = \frac{v}{4v^2 - 1} \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for x, v in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = \frac{R}{4R^2 - 1}$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$S(R) = \frac{\ln(4R^2 - 1)}{8} + c_1 \tag{4}$$

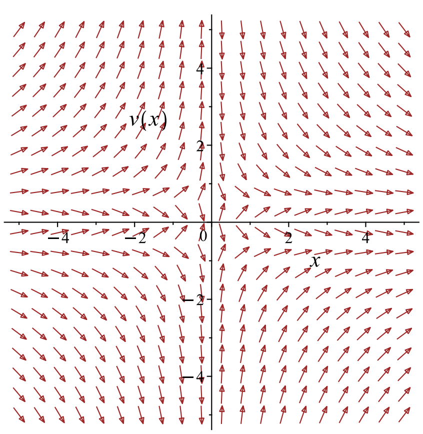
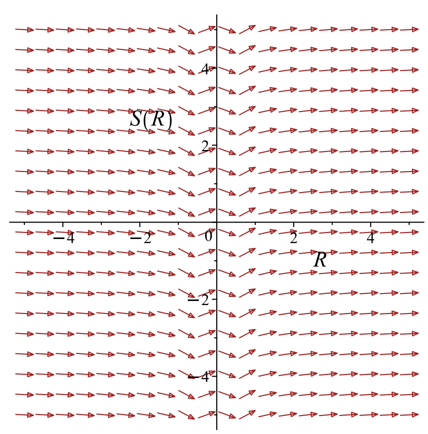
To complete the solution, we just need to transform (4) back to x, v coordinates. This results in

$$-\frac{\ln(x)}{3} = \frac{\ln(4v^2 - 1)}{8} + c_1$$

Which simplifies to

$$-\frac{\ln(x)}{3} = \frac{\ln(4v^2 - 1)}{8} + c_1$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

<table>
<tr><th>Original ode in x, v coordinates</th><th>Canonical coordinates transformation</th><th>ODE in canonical coordinates (R, S)</th></tr>
<tr><td>$\frac{dv}{dx} = -\frac{4v^2-1}{3vx}$

</td><td>$R = v$
$S = -\frac{\ln(x)}{3}$</td><td>$\frac{dS}{dR} = \frac{R}{4R^2-1}$

</td></tr>
</table>

Summary

The solution(s) found are the following

$$-\frac{\ln(x)}{3} = \frac{\ln(4v^2 - 1)}{8} + c_1 \tag{1}$$

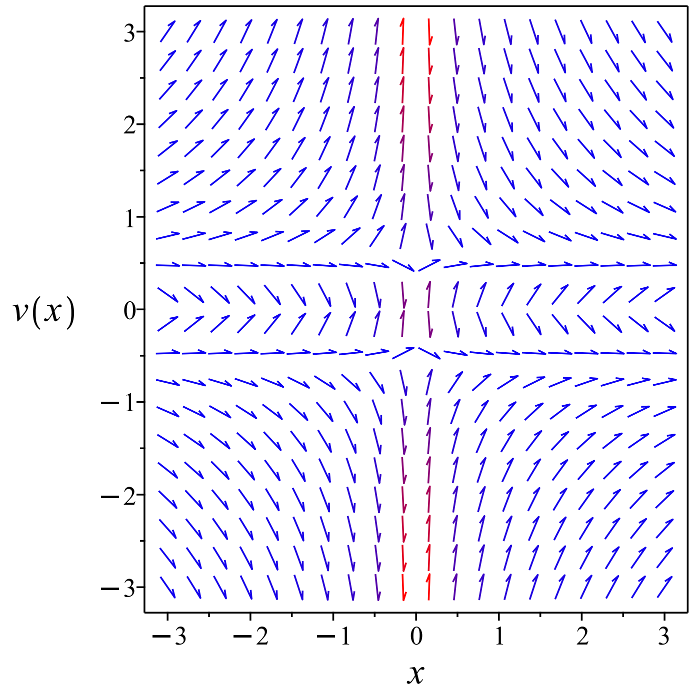


Figure 24: Slope field plot

Verification of solutions

$$-\frac{\ln(x)}{3} = \frac{\ln(4v^2 - 1)}{8} + c_1$$

Verified OK.

1.11.3 Solving as bernoulli ode

In canonical form, the ODE is

$$\begin{aligned} v' &= F(x, v) \\ &= -\frac{4v^2 - 1}{3vx} \end{aligned}$$

This is a Bernoulli ODE.

$$v' = -\frac{4}{3x}v + \frac{1}{3x}\frac{1}{v} \tag{1}$$

The standard Bernoulli ODE has the form

$$v' = f_0(x)v + f_1(x)v^n \tag{2}$$

The first step is to divide the above equation by v^n which gives

$$\frac{v'}{v^n} = f_0(x)v^{1-n} + f_1(x) \tag{3}$$

The next step is use the substitution $w = v^{1-n}$ in equation (3) which generates a new ODE in $w(x)$ which will be linear and can be easily solved using an integrating factor. Backsubstitution then gives the solution $v(x)$ which is what we want.

This method is now applied to the ODE at hand. Comparing the ODE (1) With (2) Shows that

$$
\begin{aligned}
f_0(x) &= -\frac{4}{3x} \\
f_1(x) &= \frac{1}{3x} \\
n &= -1
\end{aligned}
$$

Dividing both sides of ODE (1) by $v^n = \frac{1}{v}$ gives

$$
v'v = -\frac{4v^2}{3x} + \frac{1}{3x} \tag{4}
$$

Let

$$
\begin{aligned}
w &= v^{1-n} \\
&= v^2
\end{aligned} \tag{5}
$$

Taking derivative of equation (5) w.r.t x gives

$$
w' = 2vv' \tag{6}
$$

Substituting equations (5) and (6) into equation (4) gives

$$
\begin{aligned}
\frac{w'(x)}{2} &= -\frac{4w(x)}{3x} + \frac{1}{3x} \\
w' &= -\frac{8w}{3x} + \frac{2}{3x}
\end{aligned} \tag{7}
$$

The above now is a linear ODE in $w(x)$ which is now solved.

Entering Linear first order ODE solver. In canonical form a linear first order is

$$
w'(x) + p(x)w(x) = q(x)
$$

Where here

$$
\begin{aligned}
p(x) &= \frac{8}{3x} \\
q(x) &= \frac{2}{3x}
\end{aligned}
$$

Hence the ode is

$$w'(x) + \frac{8w(x)}{3x} = \frac{2}{3x}$$

The integrating factor μ is

$$\begin{aligned}\mu &= e^{\int \frac{8}{3x} dx} \\ &= x^{\frac{8}{3}}\end{aligned}$$

The ode becomes

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}x}(\mu w) &= (\mu)\left(\frac{2}{3x}\right) \\ \frac{\mathrm{d}}{\mathrm{d}x}\left(x^{\frac{8}{3}} w\right) &= \left(x^{\frac{8}{3}}\right)\left(\frac{2}{3x}\right) \\ \mathrm{d}\left(x^{\frac{8}{3}} w\right) &= \left(\frac{2x^{\frac{5}{3}}}{3}\right) \mathrm{d}x\end{aligned}$$

Integrating gives

$$\begin{aligned}x^{\frac{8}{3}} w &= \int \frac{2x^{\frac{5}{3}}}{3} \mathrm{d}x \\ x^{\frac{8}{3}} w &= \frac{x^{\frac{8}{3}}}{4} + c_1\end{aligned}$$

Dividing both sides by the integrating factor $\mu = x^{\frac{8}{3}}$ results in

$$w(x) = \frac{1}{4} + \frac{c_1}{x^{\frac{8}{3}}}$$

Replacing w in the above by v^2 using equation (5) gives the final solution.

$$v^2 = \frac{1}{4} + \frac{c_1}{x^{\frac{8}{3}}}$$

Solving for v gives

$$\begin{aligned}v(x) &= \frac{\sqrt{x^{\frac{8}{3}}\left(x^{\frac{8}{3}} + 4c_1\right)}}{2x^{\frac{8}{3}}} \\ v(x) &= -\frac{\sqrt{x^{\frac{8}{3}}\left(x^{\frac{8}{3}} + 4c_1\right)}}{2x^{\frac{8}{3}}}\end{aligned}$$

Summary
The solution(s) found are the following

$$v = \frac{\sqrt{x^{\frac{8}{3}} \left(x^{\frac{8}{3}} + 4c_1\right)}}{2x^{\frac{8}{3}}} \tag{1}$$

$$v = -\frac{\sqrt{x^{\frac{8}{3}} \left(x^{\frac{8}{3}} + 4c_1\right)}}{2x^{\frac{8}{3}}} \tag{2}$$

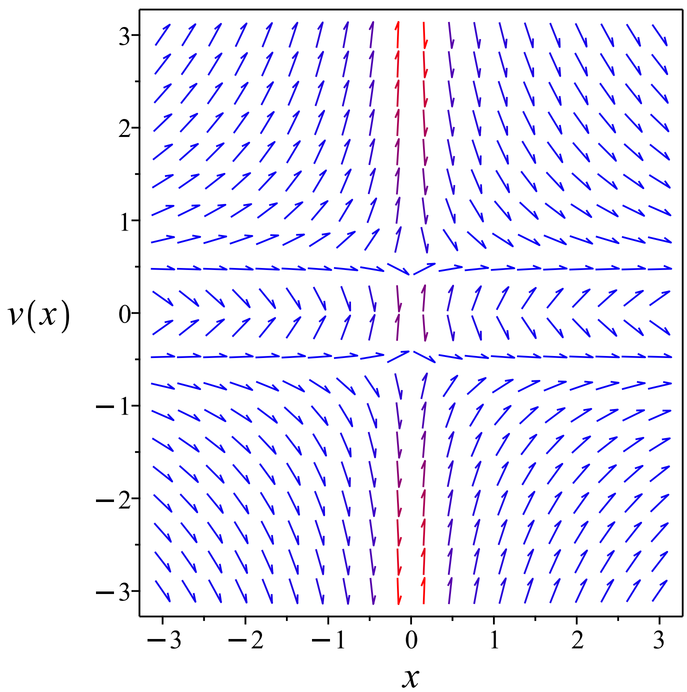


Figure 25: Slope field plot

Verification of solutions

$$v = \frac{\sqrt{x^{\frac{8}{3}} \left(x^{\frac{8}{3}} + 4c_1\right)}}{2x^{\frac{8}{3}}}$$

Verified OK.

$$v = -\frac{\sqrt{x^{\frac{8}{3}} \left(x^{\frac{8}{3}} + 4c_1\right)}}{2x^{\frac{8}{3}}}$$

Verified OK.

1.11.4 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x,y)+N(x,y)\frac{dy}{dx}=0 \tag{A}$$

We assume there exists a function $\phi(x,y)=c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y)=0$$

Hence

$$\frac{\partial\phi}{\partial x}+\frac{\partial\phi}{\partial y}\frac{dy}{dx}=0 \tag{B}$$

Comparing (A,B) shows that

$$\begin{aligned}\frac{\partial\phi}{\partial x}&=M\\ \frac{\partial\phi}{\partial y}&=N\end{aligned}$$

But since $\frac{\partial^2\phi}{\partial x\partial y}=\frac{\partial^2\phi}{\partial y\partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y}=\frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x,y)$ but at least we know now that we can do that since the condition $\frac{\partial^2\phi}{\partial x\partial y}=\frac{\partial^2\phi}{\partial y\partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x,v)\,\mathrm{d}x+N(x,v)\,\mathrm{d}v=0 \tag{1A}$$

Therefore

$$\begin{aligned}\left(-\frac{3v}{4v^2-1}\right)\mathrm{d}v&=\left(\frac{1}{x}\right)\mathrm{d}x\\ \left(-\frac{1}{x}\right)\mathrm{d}x+\left(-\frac{3v}{4v^2-1}\right)\mathrm{d}v&=0 \end{aligned}\tag{2A}$$

Comparing (1A) and (2A) shows that

$$
\begin{aligned}
M(x,v) &= -\frac{1}{x} \\
N(x,v) &= -\frac{3v}{4v^2-1}
\end{aligned}
$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$
\frac{\partial M}{\partial v} = \frac{\partial N}{\partial x}
$$

Using result found above gives

$$
\begin{aligned}
\frac{\partial M}{\partial v} &= \frac{\partial}{\partial v}\left(-\frac{1}{x}\right) \\
&= 0
\end{aligned}
$$

And

$$
\begin{aligned}
\frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}\left(-\frac{3v}{4v^2-1}\right) \\
&= 0
\end{aligned}
$$

Since $\frac{\partial M}{\partial v} = \frac{\partial N}{\partial x}$, then the ODE is exact The following equations are now set up to solve for the function $\phi(x,v)$

$$
\frac{\partial \phi}{\partial x} = M \tag{1}
$$

$$
\frac{\partial \phi}{\partial v} = N \tag{2}
$$

Integrating (1) w.r.t. x gives

$$
\begin{aligned}
\int \frac{\partial \phi}{\partial x} \mathrm{d}x &= \int M \mathrm{d}x \\
\int \frac{\partial \phi}{\partial x} \mathrm{d}x &= \int -\frac{1}{x} \mathrm{d}x
\end{aligned}
$$

$$
\phi = -\ln(x) + f(v) \tag{3}
$$

Where $f(v)$ is used for the constant of integration since ϕ is a function of both x and v. Taking derivative of equation (3) w.r.t v gives

$$
\frac{\partial \phi}{\partial v} = 0 + f'(v) \tag{4}
$$

But equation (2) says that $\frac{\partial \phi}{\partial v} = -\frac{3v}{4v^2-1}$. Therefore equation (4) becomes

$$-\frac{3v}{4v^2-1} = 0 + f'(v) \tag{5}$$

Solving equation (5) for $f'(v)$ gives

$$f'(v) = -\frac{3v}{4v^2-1}$$

Integrating the above w.r.t v gives

$$\int f'(v)\,\mathrm{d}v = \int \left(-\frac{3v}{4v^2-1}\right) \mathrm{d}v$$

$$f(v) = -\frac{3\ln(4v^2-1)}{8} + c_1$$

Where c_1 is constant of integration. Substituting result found above for $f(v)$ into equation (3) gives ϕ

$$\phi = -\ln(x) - \frac{3\ln(4v^2-1)}{8} + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = -\ln(x) - \frac{3\ln(4v^2-1)}{8}$$

Summary

The solution(s) found are the following

$$-\ln(x) - \frac{3\ln(4v^2-1)}{8} = c_1 \tag{1}$$

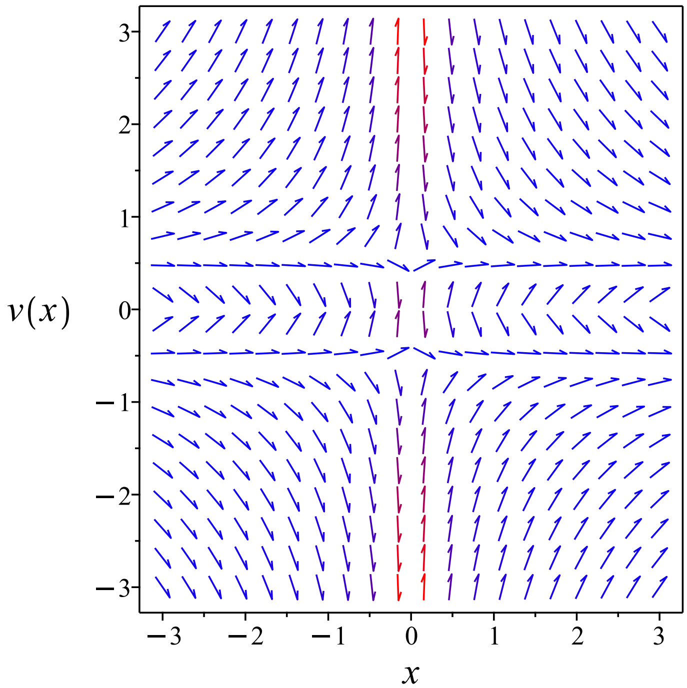


Figure 26: Slope field plot

Verification of solutions

$$-\ln(x) - \frac{3\ln(4v^2-1)}{8} = c_1$$

Verified OK.

1.11.5 Maple step by step solution

Let's solve

$xv' - \frac{1-4v^2}{3v} = 0$

- Highest derivative means the order of the ODE is 1

  v'

- Separate variables

  $\frac{v'v}{1-4v^2} = \frac{1}{3x}$

- Integrate both sides with respect to x

  $\int \frac{v'v}{1-4v^2}dx = \int \frac{1}{3x}dx + c_1$

- Evaluate integral

$$-\frac{\ln(4v^2-1)}{8} = \frac{\ln(x)}{3} + c_1$$

- Solve for v

$$\left\{v = -\frac{\sqrt{x^4 \left(e^{3c_1}\right)^4 + \left(x^2 \left(e^{3c_1}\right)^2\right)^{\frac{2}{3}}}}{2x^2 \left(e^{3c_1}\right)^2}, v = \frac{\sqrt{x^4 \left(e^{3c_1}\right)^4 + \left(x^2 \left(e^{3c_1}\right)^2\right)^{\frac{2}{3}}}}{2x^2 \left(e^{3c_1}\right)^2}\right\}$$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
<- Bernoulli successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 43

```
dsolve(x*diff(v(x),x)=(1-4*v(x)^2)/(3*v(x)),v(x), singsol=all)
```

$$v(x) = -\frac{\sqrt{x^{\frac{8}{3}} \left(x^{\frac{8}{3}} + 4c_1\right)}}{2x^{\frac{8}{3}}}$$

$$v(x) = \frac{\sqrt{x^{\frac{8}{3}} \left(x^{\frac{8}{3}} + 4c_1\right)}}{2x^{\frac{8}{3}}}$$

✓ Solution by Mathematica

Time used: 1.93 (sec). Leaf size: 67

```
DSolve[x*v'[x]==(1-4*v[x]^2)/(3*v[x]),v[x],x,IncludeSingularSolutions -> True]
```

$$v(x) \to -\frac{1}{2}\sqrt{1+\frac{e^{8c_1}}{x^{8/3}}}$$
$$v(x) \to \frac{1}{2}\sqrt{1+\frac{e^{8c_1}}{x^{8/3}}}$$
$$v(x) \to -\frac{1}{2}$$
$$v(x) \to \frac{1}{2}$$

1.12 problem 12

Internal problem ID [4923]
Internal file name [`OUTPUT/4416_Sunday_June_05_2022_01_17_28_PM_33081274/index.tex`]

Book: Fundamentals of Differential Equations. By Nagle, Saff and Snider. 9th edition. Boston. Pearson 2018.
Section: Chapter 2, First order differential equations. Section 2.2, Separable Equations. Exercises. page 46
Problem number: 12.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"exact", "separable", "first_order_ode_lie_symmetry_lookup"**

Maple gives the following as the ode type

`[_separable]`

$$\boxed{y' - \frac{\sec(y)^2}{x^2+1} = 0}$$

1.12.1 Solving as separable ode

In canonical form the ODE is

$$\begin{aligned} y' &= F(x,y) \\ &= f(x)g(y) \\ &= \frac{\sec(y)^2}{x^2+1} \end{aligned}$$

Where $f(x) = \frac{1}{x^2+1}$ and $g(y) = \sec(y)^2$. Integrating both sides gives

$$\frac{1}{\sec(y)^2}\,dy = \frac{1}{x^2+1}\,dx$$

$$\int \frac{1}{\sec(y)^2}\,dy = \int \frac{1}{x^2+1}\,dx$$

$$\frac{\cos(y)\sin(y)}{2} + \frac{y}{2} = \arctan(x) + c_1$$

Which results in

$$y = \text{RootOf}\left(-\sin(\_Z)\cos(\_Z) + 2\arctan(x) + 2c_1 - \_Z\right)$$

Summary

The solution(s) found are the following

$$y = \text{RootOf}\left(-\sin(\_Z)\cos(\_Z) + 2\arctan(x) + 2c_1 - \_Z\right) \tag{1}$$

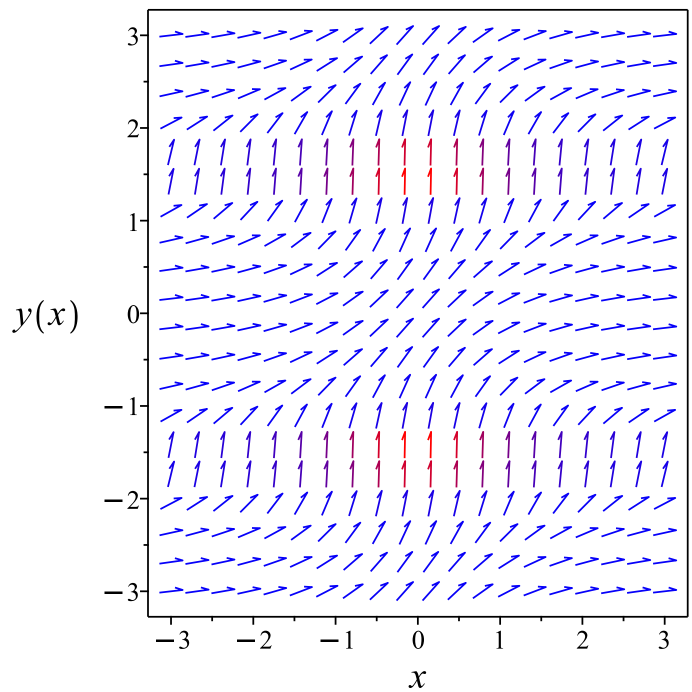


Figure 27: Slope field plot

Verification of solutions

$$y = \text{RootOf}\left(-\sin(\_Z)\cos(\_Z) + 2\arctan(x) + 2c_1 - \_Z\right)$$

Verified OK.

1.12.2 Solving as first order ode lie symmetry lookup ode

Writing the ode as

$$y' = \frac{\sec(y)^2}{x^2+1}$$
$$y' = \omega(x,y)$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x + \omega(\eta_y - \xi_x) - \omega^2\xi_y - \omega_x\xi - \omega_y\eta = 0 \tag{A}$$

The type of this ode is known. It is of type `separable`. Therefore we do not need to solve the PDE (A), and can just use the lookup table shown below to find ξ, η

Table 23: Lie symmetry infinitesimal lookup table for known first order ODE's

ODE class	Form	ξ	η
linear ode	$y' = f(x)y(x) + g(x)$	0	$e^{\int f dx}$
separable ode	$y' = f(x)\, g(y)$	$\frac{1}{f}$	0
quadrature ode	$y' = f(x)$	0	1
quadrature ode	$y' = g(y)$	1	0
homogeneous ODEs of Class A	$y' = f\left(\frac{y}{x}\right)$	x	y
homogeneous ODEs of Class C	$y' = (a + bx + cy)^{\frac{n}{m}}$	1	$-\frac{b}{c}$
homogeneous class D	$y' = \frac{y}{x} + g(x)\, F\left(\frac{y}{x}\right)$	x^2	xy
First order special form ID 1	$y' = g(x)\, e^{h(x)+by} + f(x)$	$\frac{e^{-\int b f(x) dx - h(x)}}{g(x)}$	$\frac{f(x) e^{-\int b f(x) dx - h(x)}}{g(x)}$
polynomial type ode	$y' = \frac{a_1 x + b_1 y + c_1}{a_2 x + b_2 y + c_2}$	$\frac{a_1 b_2 x - a_2 b_1 x - b_1 c_2 + b_2 c_1}{a_1 b_2 - a_2 b_1}$	$\frac{a_1 b_2 y - a_2 b_1 y - a_1 c_2 - a_2 c_1}{a_1 b_2 - a_2 b_1}$
Bernoulli ode	$y' = f(x)\, y + g(x)\, y^n$	0	$e^{-\int (n-1) f(x) dx} y^n$
Reduced Riccati	$y' = f_1(x)\, y + f_2(x)\, y^2$	0	$e^{-\int f_1 dx}$

The above table shows that

$$\begin{aligned}\xi(x,y) &= x^2+1\\ \eta(x,y) &= 0\end{aligned} \tag{A1}$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x,y) \to (R,S)$ where (R,S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}$$

The above comes from the requirements that $\left(\xi\frac{\partial}{\partial x} + \eta\frac{\partial}{\partial y}\right) S(x,y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the canonical coordinates, where $S(R)$. Since $\eta = 0$ then in this special case

$$R = y$$

S is found from

$$\begin{aligned}S &= \int \frac{1}{\xi} dx\\ &= \int \frac{1}{x^2+1} dx\end{aligned}$$

Which results in

$$S = \arctan(x)$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x,y)S_y}{R_x + \omega(x,y)R_y} \tag{2}$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x,y)$ is the right hand side of the original ode given by

$$\omega(x,y) = \frac{\sec(y)^2}{x^2+1}$$

Evaluating all the partial derivatives gives

$$\begin{aligned}
R_x &= 0 \\
R_y &= 1 \\
S_x &= \frac{1}{x^2+1} \\
S_y &= 0
\end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = \cos(y)^2 \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = \cos(R)^2$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$S(R) = \frac{R}{2} + c_1 + \frac{\sin(2R)}{4} \tag{4}$$

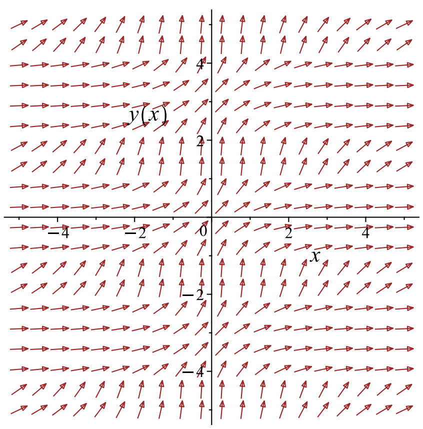
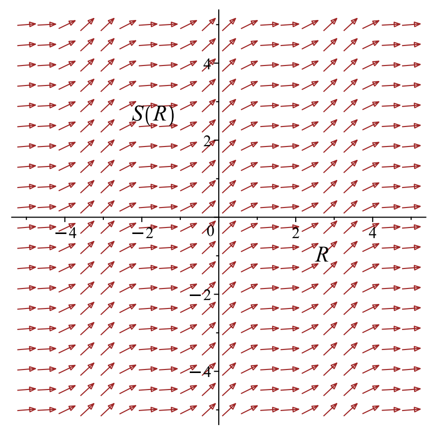
To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$\arctan(x) = \frac{y}{2} + c_1 + \frac{\sin(2y)}{4}$$

Which simplifies to

$$\arctan(x) = \frac{y}{2} + c_1 + \frac{\sin(2y)}{4}$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in x, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dx} = \frac{\sec(y)^2}{x^2+1}$ 	$R = y$ $S = \arctan(x)$	$\frac{dS}{dR} = \cos(R)^2$ 

Summary

The solution(s) found are the following

$$\arctan(x) = \frac{y}{2} + c_1 + \frac{\sin(2y)}{4} \tag{1}$$

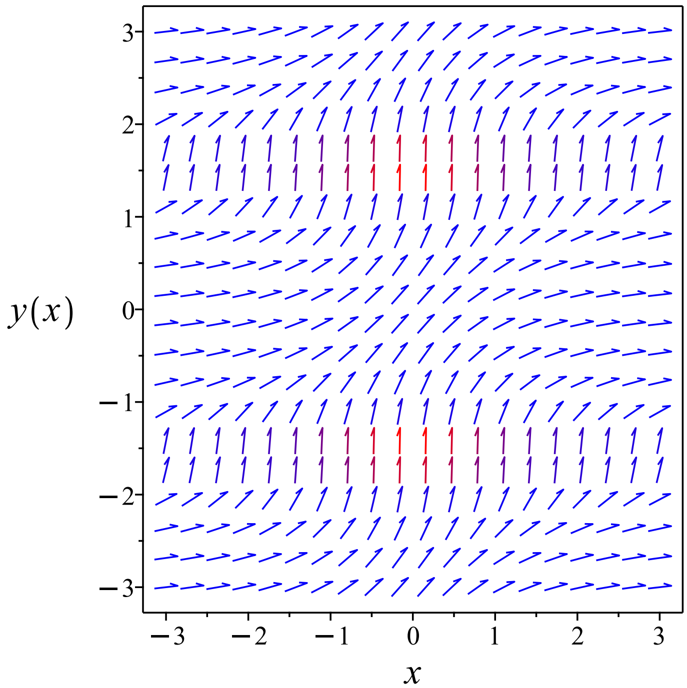


Figure 28: Slope field plot

Verification of solutions

$$\arctan(x) = \frac{y}{2} + c_1 + \frac{\sin(2y)}{4}$$

Verified OK.

1.12.3 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x,y) + N(x,y)\frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x,y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y) = 0$$

Hence

$$\frac{\partial\phi}{\partial x} + \frac{\partial\phi}{\partial y}\frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$
\begin{aligned}
\frac{\partial \phi}{\partial x} &= M \\
\frac{\partial \phi}{\partial y} &= N
\end{aligned}
$$

But since $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ then for the above to be valid, we require that

$$
\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}
$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x, y)$ but at least we know now that we can do that since the condition $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$
M(x, y)\, \mathrm{d}x + N(x, y)\, \mathrm{d}y = 0 \tag{1A}
$$

Therefore

$$
\begin{aligned}
\left(\frac{1}{\sec(y)^2}\right) \mathrm{d}y &= \left(\frac{1}{x^2+1}\right) \mathrm{d}x \\
\left(-\frac{1}{x^2+1}\right) \mathrm{d}x + \left(\frac{1}{\sec(y)^2}\right) \mathrm{d}y &= 0 \qquad (2A)
\end{aligned}
$$

Comparing (1A) and (2A) shows that

$$
\begin{aligned}
M(x, y) &= -\frac{1}{x^2+1} \\
N(x, y) &= \frac{1}{\sec(y)^2}
\end{aligned}
$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$
\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}
$$

Using result found above gives

$$
\begin{aligned}
\frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}\left(-\frac{1}{x^2+1}\right) \\
&= 0
\end{aligned}
$$

And

$$\begin{aligned}
\frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}\left(\frac{1}{\sec\left(y\right)^{2}}\right) \\
&= 0
\end{aligned}$$

Since $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$, then the ODE is exact The following equations are now set up to solve for the function $\phi(x, y)$

$$\frac{\partial \phi}{\partial x} = M \tag{1}$$

$$\frac{\partial \phi}{\partial y} = N \tag{2}$$

Integrating (1) w.r.t. x gives

$$\begin{aligned}
\int \frac{\partial \phi}{\partial x} \, \mathrm{d}x &= \int M \, \mathrm{d}x \\
\int \frac{\partial \phi}{\partial x} \, \mathrm{d}x &= \int -\frac{1}{x^{2}+1} \, \mathrm{d}x
\end{aligned}$$

$$\phi = -\arctan\left(x\right) + f(y) \tag{3}$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$\frac{\partial \phi}{\partial y} = 0 + f'(y) \tag{4}$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = \frac{1}{\sec(y)^{2}}$. Therefore equation (4) becomes

$$\frac{1}{\sec\left(y\right)^{2}} = 0 + f'(y) \tag{5}$$

Solving equation (5) for $f'(y)$ gives

$$f'(y) = \frac{1}{\sec\left(y\right)^{2}}$$

Integrating the above w.r.t y gives

$$\begin{aligned}
\int f'(y) \, \mathrm{d}y &= \int \left(\cos\left(y\right)^{2}\right) \mathrm{d}y \\
f(y) &= \frac{\cos\left(y\right)\sin\left(y\right)}{2} + \frac{y}{2} + c_1
\end{aligned}$$

Where c_1 is constant of integration. Substituting result found above for $f(y)$ into equation (3) gives ϕ

$$\phi = -\arctan(x) + \frac{\cos(y)\sin(y)}{2} + \frac{y}{2} + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = -\arctan(x) + \frac{\cos(y)\sin(y)}{2} + \frac{y}{2}$$

Summary
The solution(s) found are the following

$$-\arctan(x) + \frac{\cos(y)\sin(y)}{2} + \frac{y}{2} = c_1 \tag{1}$$

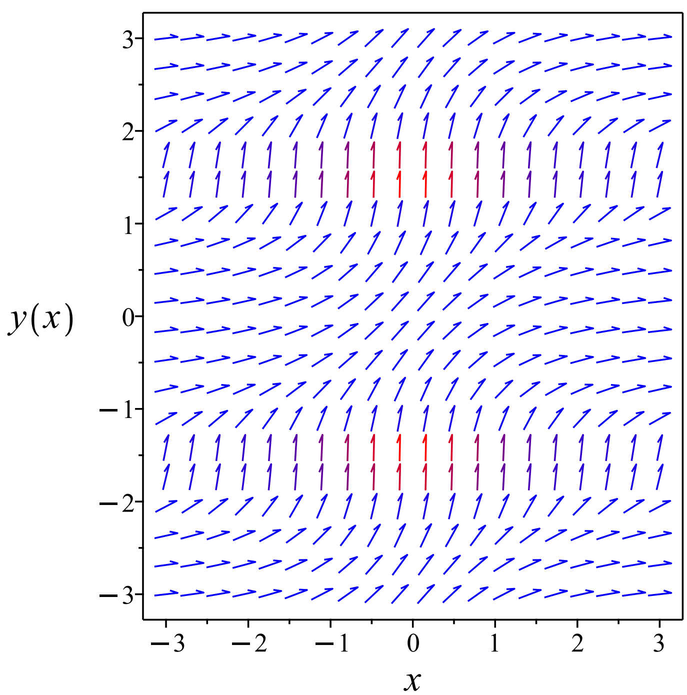


Figure 29: Slope field plot

Verification of solutions

$$-\arctan(x) + \frac{\cos(y)\sin(y)}{2} + \frac{y}{2} = c_1$$

Verified OK.

1.12.4 Maple step by step solution

Let's solve

$y' - \frac{\sec(y)^2}{x^2+1} = 0$

- Highest derivative means the order of the ODE is 1

  y'

- Separate variables

  $\frac{y'}{\sec(y)^2} = \frac{1}{x^2+1}$

- Integrate both sides with respect to x

  $\int \frac{y'}{\sec(y)^2}dx = \int \frac{1}{x^2+1}dx + c_1$

- Evaluate integral

  $\frac{\cos(y)\sin(y)}{2} + \frac{y}{2} = \arctan(x) + c_1$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
<- separable successful`
```

✓ Solution by Maple

Time used: 0.094 (sec). Leaf size: 81

```
dsolve(diff(y(x),x)=sec(y(x))^2/(1+x^2),y(x), singsol=all)
```

$$y(x) = \frac{\arcsin\left(\mathrm{RootOf}\left(\_Z + 2x^2\_Z + \_Zx^4 - x^4\sin\left(-\_Z + 4c_1\right) + 4x^3\cos\left(-\_Z + 4c_1\right) + 6x^2\sin\left(-\_Z + 4\right.\right.\right.}{2}$$

✓ Solution by Mathematica

Time used: 0.517 (sec). Leaf size: 32

```
DSolve[y'[x]==Sec[y[x]]^2/(1+x^2),y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \text{InverseFunction}\left[2\left(\frac{\#1}{2}+\frac{1}{4}\sin(2\#1)\right)\&\right][2\arctan(x)+c_1]$$

1.13 problem 13

Internal problem ID [4924]
Internal file name [`OUTPUT/4417_Sunday_June_05_2022_01_17_37_PM_80153111/index.tex`]

Book: Fundamentals of Differential Equations. By Nagle, Saff and Snider. 9th edition. Boston. Pearson 2018.
Section: Chapter 2, First order differential equations. Section 2.2, Separable Equations. Exercises. page 46
Problem number: 13.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"exact", "separable", "first_order_ode_lie_symmetry_lookup"**

Maple gives the following as the ode type

```
[_separable]
```

$$\boxed{y' - 3x^2(1+y^2)^{\frac{3}{2}} = 0}$$

1.13.1 Solving as separable ode

In canonical form the ODE is

$$\begin{aligned} y' &= F(x,y) \\ &= f(x)g(y) \\ &= 3x^2(y^2+1)^{\frac{3}{2}} \end{aligned}$$

Where $f(x) = 3x^2$ and $g(y) = (y^2+1)^{\frac{3}{2}}$. Integrating both sides gives

$$\frac{1}{(y^2+1)^{\frac{3}{2}}}\,dy = 3x^2\,dx$$

$$\int \frac{1}{(y^2+1)^{\frac{3}{2}}} dy = \int 3x^2 \, dx$$
$$\frac{y}{\sqrt{y^2+1}} = x^3 + c_1$$

The solution is

$$\frac{y}{\sqrt{1+y^2}} - x^3 - c_1 = 0$$

Summary

The solution(s) found are the following

$$\frac{y}{\sqrt{1+y^2}} - x^3 - c_1 = 0 \tag{1}$$

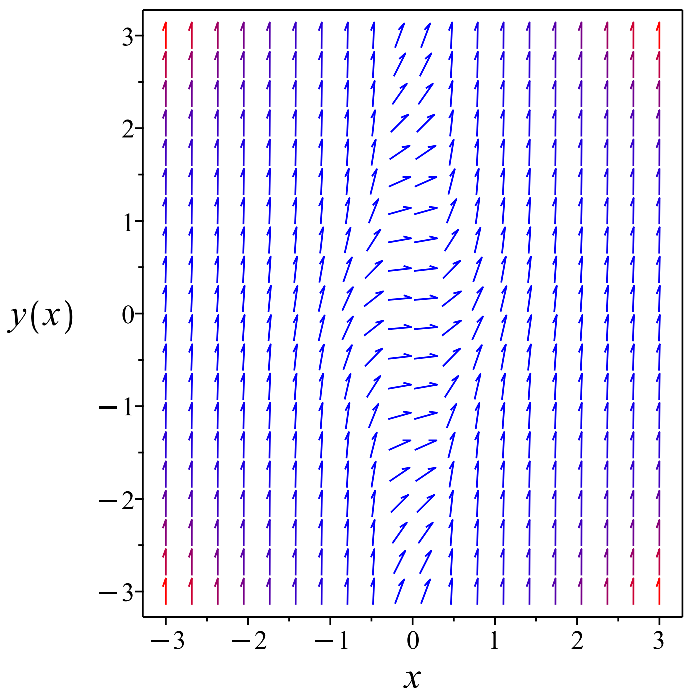


Figure 30: Slope field plot

Verification of solutions

$$\frac{y}{\sqrt{1+y^2}} - x^3 - c_1 = 0$$

Verified OK.

1.13.2 Solving as first order ode lie symmetry lookup ode

Writing the ode as

$$
\begin{aligned}
y' &= 3x^2(y^2+1)^{\frac{3}{2}} \\
y' &= \omega(x,y)
\end{aligned}
$$

The condition of Lie symmetry is the linearized PDE given by

$$
\eta_x + \omega(\eta_y - \xi_x) - \omega^2 \xi_y - \omega_x \xi - \omega_y \eta = 0 \tag{A}
$$

The type of this ode is known. It is of type `separable`. Therefore we do not need to solve the PDE (A), and can just use the lookup table shown below to find ξ, η

Table 26: Lie symmetry infinitesimal lookup table for known first order ODE's

ODE class	Form	ξ	η
linear ode	$y' = f(x)y(x) + g(x)$	0	$e^{\int f dx}$
separable ode	$y' = f(x)\,g(y)$	$\frac{1}{f}$	0
quadrature ode	$y' = f(x)$	0	1
quadrature ode	$y' = g(y)$	1	0
homogeneous ODEs of Class A	$y' = f\left(\frac{y}{x}\right)$	x	y
homogeneous ODEs of Class C	$y' = (a + bx + cy)^{\frac{n}{m}}$	1	$-\frac{b}{c}$
homogeneous class D	$y' = \frac{y}{x} + g(x)\,F\left(\frac{y}{x}\right)$	x^2	xy
First order special form ID 1	$y' = g(x)\,e^{h(x)+by} + f(x)$	$\frac{e^{-\int bf(x)dx - h(x)}}{g(x)}$	$\frac{f(x)e^{-\int bf(x)dx - h(x)}}{g(x)}$
polynomial type ode	$y' = \frac{a_1x+b_1y+c_1}{a_2x+b_2y+c_2}$	$\frac{a_1b_2x - a_2b_1x - b_1c_2 + b_2c_1}{a_1b_2 - a_2b_1}$	$\frac{a_1b_2y - a_2b_1y - a_1c_2 - a_2c_1}{a_1b_2 - a_2b_1}$
Bernoulli ode	$y' = f(x)\,y + g(x)\,y^n$	0	$e^{-\int (n-1)f(x)dx}y^n$
Reduced Riccati	$y' = f_1(x)\,y + f_2(x)\,y^2$	0	$e^{-\int f_1 dx}$

The above table shows that

$$\xi(x, y) = \frac{1}{3x^2}$$
$$\eta(x, y) = 0 \tag{A1}$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x, y) \to (R, S)$ where (R, S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y}\right) S(x, y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the canonical coordinates, where $S(R)$. Since $\eta = 0$ then in this special case

$$R = y$$

S is found from

$$S = \int \frac{1}{\xi} dx$$
$$= \int \frac{1}{\frac{1}{3x^2}} dx$$

Which results in

$$S = x^3$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x, y) S_y}{R_x + \omega(x, y) R_y} \tag{2}$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x, y)$ is the right hand side of the original ode given by

$$\omega(x, y) = 3x^2 \left(y^2 + 1\right)^{\frac{3}{2}}$$

Evaluating all the partial derivatives gives

$$R_x = 0$$
$$R_y = 1$$
$$S_x = 3x^2$$
$$S_y = 0$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = \frac{1}{(y^2+1)^{\frac{3}{2}}} \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = \frac{1}{(R^2+1)^{\frac{3}{2}}}$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$S(R) = \frac{R}{\sqrt{R^2+1}} + c_1 \tag{4}$$

To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$x^3 = \frac{y}{\sqrt{1+y^2}} + c_1$$

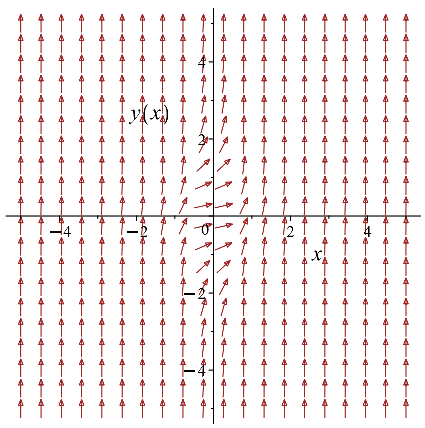
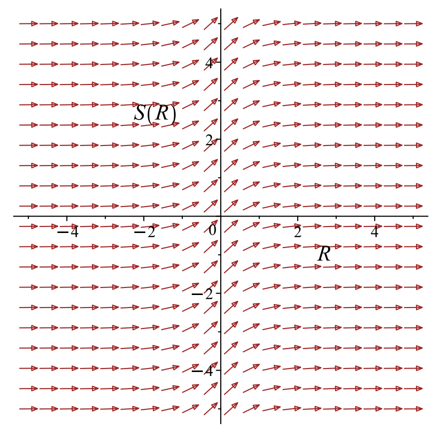
Which simplifies to

$$x^3 = \frac{y}{\sqrt{1+y^2}} + c_1$$

Which gives

$$y = x^3 \sqrt{-\frac{1}{x^6 - 2c_1 x^3 + c_1^2 - 1}} - c_1 \sqrt{-\frac{1}{x^6 - 2c_1 x^3 + c_1^2 - 1}}$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in x, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dx} = 3x^2(y^2+1)^{\frac{3}{2}}$ 	$R = y$ $S = x^3$	$\frac{dS}{dR} = \frac{1}{(R^2+1)^{\frac{3}{2}}}$ 

Summary

The solution(s) found are the following

$$y = x^3\sqrt{-\frac{1}{x^6 - 2c_1x^3 + c_1^2 - 1}} - c_1\sqrt{-\frac{1}{x^6 - 2c_1x^3 + c_1^2 - 1}} \tag{1}$$

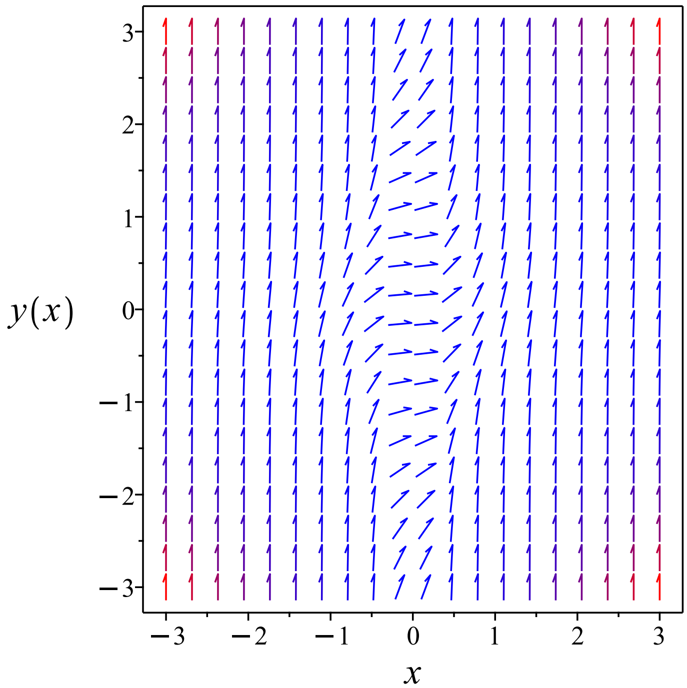


Figure 31: Slope field plot

Verification of solutions

$$y = x^3\sqrt{-\frac{1}{x^6 - 2c_1x^3 + c_1^2 - 1}} - c_1\sqrt{-\frac{1}{x^6 - 2c_1x^3 + c_1^2 - 1}}$$

Verified OK.

1.13.3 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x,y) + N(x,y)\frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x,y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y) = 0$$

Hence

$$\frac{\partial\phi}{\partial x} + \frac{\partial\phi}{\partial y}\frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$
\begin{aligned}
\frac{\partial \phi}{\partial x} &= M \\
\frac{\partial \phi}{\partial y} &= N
\end{aligned}
$$

But since $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ then for the above to be valid, we require that

$$
\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}
$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x, y)$ but at least we know now that we can do that since the condition $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$
M(x, y)\, \mathrm{d}x + N(x, y)\, \mathrm{d}y = 0 \tag{1A}
$$

Therefore

$$
\begin{aligned}
\left(\frac{1}{3\,(y^2+1)^{\frac{3}{2}}}\right) \mathrm{d}y &= (x^2)\, \mathrm{d}x \\
(-x^2)\, \mathrm{d}x + \left(\frac{1}{3\,(y^2+1)^{\frac{3}{2}}}\right) \mathrm{d}y &= 0 \qquad (2A)
\end{aligned}
$$

Comparing (1A) and (2A) shows that

$$
\begin{aligned}
M(x, y) &= -x^2 \\
N(x, y) &= \frac{1}{3\,(y^2+1)^{\frac{3}{2}}}
\end{aligned}
$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$
\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}
$$

Using result found above gives

$$
\begin{aligned}
\frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}(-x^2) \\
&= 0
\end{aligned}
$$

And

$$
\begin{aligned}
\frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}\left(\frac{1}{3\left(y^2+1\right)^{\frac{3}{2}}}\right) \\
&= 0
\end{aligned}
$$

Since $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$, then the ODE is exact The following equations are now set up to solve for the function $\phi(x, y)$

$$
\frac{\partial \phi}{\partial x} = M \tag{1}
$$

$$
\frac{\partial \phi}{\partial y} = N \tag{2}
$$

Integrating (1) w.r.t. x gives

$$
\begin{aligned}
\int \frac{\partial \phi}{\partial x} \, \mathrm{d}x &= \int M \, \mathrm{d}x \\
\int \frac{\partial \phi}{\partial x} \, \mathrm{d}x &= \int -x^2 \, \mathrm{d}x
\end{aligned}
$$

$$
\phi = -\frac{x^3}{3} + f(y) \tag{3}
$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$
\frac{\partial \phi}{\partial y} = 0 + f'(y) \tag{4}
$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = \frac{1}{3(y^2+1)^{\frac{3}{2}}}$. Therefore equation (4) becomes

$$
\frac{1}{3\left(y^2+1\right)^{\frac{3}{2}}} = 0 + f'(y) \tag{5}
$$

Solving equation (5) for $f'(y)$ gives

$$
f'(y) = \frac{1}{3\left(y^2+1\right)^{\frac{3}{2}}}
$$

Integrating the above w.r.t y gives

$$\int f'(y)\,\mathrm{d}y = \int \left(\frac{1}{3\,(y^2+1)^{\frac{3}{2}}}\right) \mathrm{d}y$$

$$f(y) = \frac{y}{3\sqrt{y^2+1}} + c_1$$

Where c_1 is constant of integration. Substituting result found above for $f(y)$ into equation (3) gives ϕ

$$\phi = -\frac{x^3}{3} + \frac{y}{3\sqrt{y^2+1}} + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = -\frac{x^3}{3} + \frac{y}{3\sqrt{y^2+1}}$$

The solution becomes

$$y = x^3\sqrt{-\frac{1}{x^6+6c_1x^3+9c_1^2-1}} + 3c_1\sqrt{-\frac{1}{x^6+6c_1x^3+9c_1^2-1}}$$

Summary

The solution(s) found are the following

$$y = x^3\sqrt{-\frac{1}{x^6+6c_1x^3+9c_1^2-1}} + 3c_1\sqrt{-\frac{1}{x^6+6c_1x^3+9c_1^2-1}} \tag{1}$$

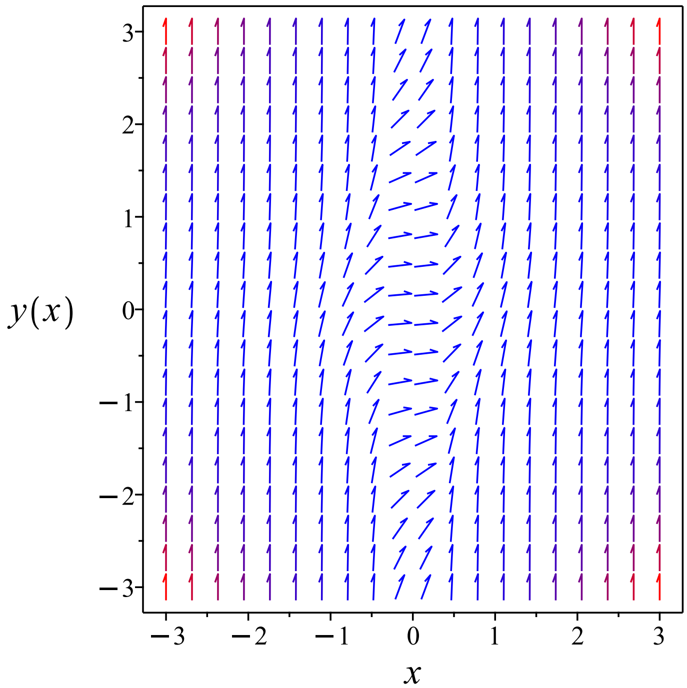


Figure 32: Slope field plot

Verification of solutions

$$y = x^3\sqrt{-\frac{1}{x^6 + 6c_1x^3 + 9c_1^2 - 1}} + 3c_1\sqrt{-\frac{1}{x^6 + 6c_1x^3 + 9c_1^2 - 1}}$$

Verified OK.

1.13.4 Maple step by step solution

Let's solve

$y' - 3x^2(1 + y^2)^{\frac{3}{2}} = 0$

- Highest derivative means the order of the ODE is 1

  y'

- Separate variables

  $\frac{y'}{(1+y^2)^{\frac{3}{2}}} = 3x^2$

- Integrate both sides with respect to x

  $\int \frac{y'}{(1+y^2)^{\frac{3}{2}}}dx = \int 3x^2dx + c_1$

- Evaluate integral

  $\frac{y}{\sqrt{1+y^2}} = x^3 + c_1$

- Solve for y

  $y = x^3\sqrt{-\frac{1}{x^6+2c_1x^3+c_1^2-1}} + c_1\sqrt{-\frac{1}{x^6+2c_1x^3+c_1^2-1}}$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
<- separable successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 20

```
dsolve(diff(y(x),x)=3*x^2*(1+y(x)^2)^(3/2),y(x), singsol=all)
```

$$c_1 + x^3 - \frac{y(x)}{\sqrt{1 + y(x)^2}} = 0$$

✓ Solution by Mathematica

Time used: 0.242 (sec). Leaf size: 83

```
DSolve[y'[x]==3*x^2*(1+y[x]^2)^(3/2),y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to -\frac{i(x^3 + c_1)}{\sqrt{x^6 + 2c_1x^3 - 1 + c_1{}^2}}$$
$$y(x) \to \frac{i(x^3 + c_1)}{\sqrt{x^6 + 2c_1x^3 - 1 + c_1{}^2}}$$
$$y(x) \to -i$$
$$y(x) \to i$$

1.14 problem 14

Internal problem ID [4925]
Internal file name [`OUTPUT/4418_Sunday_June_05_2022_01_17_49_PM_51049474/index.tex`]

Book: Fundamentals of Differential Equations. By Nagle, Saff and Snider. 9th edition. Boston. Pearson 2018.
Section: Chapter 2, First order differential equations. Section 2.2, Separable Equations. Exercises. page 46
Problem number: 14.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"quadrature"**

Maple gives the following as the ode type

```
[_quadrature]
```

$$\boxed{x' - x^3 - x = 0}$$

1.14.1 Solving as quadrature ode

Integrating both sides gives

$$\int \frac{1}{x^3 + x} dx = \int dt$$
$$-\frac{\ln(x^2+1)}{2} + \ln(x) = t + c_1$$

Raising both side to exponential gives

$$e^{-\frac{\ln(x^2+1)}{2} + \ln(x)} = e^{t+c_1}$$

Which simplifies to

$$\frac{x}{\sqrt{x^2+1}} = c_2 e^t$$

Summary

The solution(s) found are the following

$$x = c_2 e^{t} \sqrt{-\frac{1}{c_2^{2} e^{2t} - 1}} \tag{1}$$

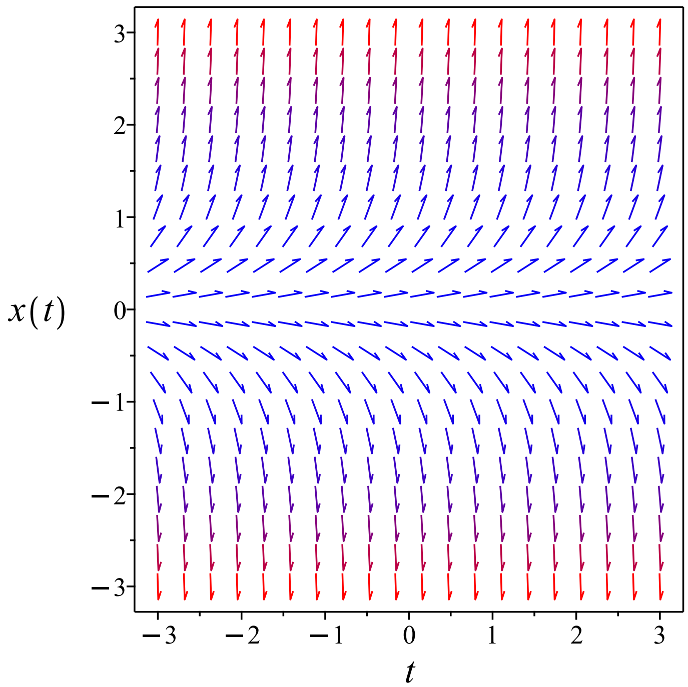


Figure 33: Slope field plot

Verification of solutions

$$x = c_2 e^{t} \sqrt{-\frac{1}{c_2^{2} e^{2t} - 1}}$$

Verified OK.

1.14.2 Maple step by step solution

Let's solve

$x' - x^3 - x = 0$

- Highest derivative means the order of the ODE is 1

  x'

- Separate variables

  $\frac{x'}{x^3+x} = 1$

- Integrate both sides with respect to t

  $\int \frac{x'}{x^3+x}dt = \int 1dt + c_1$

- Evaluate integral

  $-\frac{\ln(x^2+1)}{2} + \ln(x) = t + c_1$

- Solve for x

  $\left\{x = \frac{\sqrt{-(\mathrm{e}^{2t+2c_1}-1)\,\mathrm{e}^{2t+2c_1}}}{\mathrm{e}^{2t+2c_1}-1}, x = -\frac{\sqrt{-(\mathrm{e}^{2t+2c_1}-1)\,\mathrm{e}^{2t+2c_1}}}{\mathrm{e}^{2t+2c_1}-1}\right\}$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
<- Bernoulli successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 29

```
dsolve(diff(x(t),t)-x(t)^3=x(t),x(t), singsol=all)
```

$$x(t) = \frac{1}{\sqrt{e^{-2t}c_1 - 1}}$$
$$x(t) = -\frac{1}{\sqrt{e^{-2t}c_1 - 1}}$$

✓ Solution by Mathematica

Time used: 60.064 (sec). Leaf size: 57

```
DSolve[x'[t]-x[t]^3==x[t],x[t],t,IncludeSingularSolutions -> True]
```

$$x(t) \to -\frac{ie^{t+c_1}}{\sqrt{-1 + e^{2(t+c_1)}}}$$
$$x(t) \to \frac{ie^{t+c_1}}{\sqrt{-1 + e^{2(t+c_1)}}}$$

1.15 problem 15

Internal problem ID [4926]
Internal file name [`OUTPUT/4419_Sunday_June_05_2022_01_18_04_PM_63826328/index.tex`]

Book: Fundamentals of Differential Equations. By Nagle, Saff and Snider. 9th edition. Boston. Pearson 2018.
Section: Chapter 2, First order differential equations. Section 2.2, Separable Equations. Exercises. page 46
Problem number: 15.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"exact", "bernoulli", "separable", "first_order_ode_lie_symmetry_lookup"**

Maple gives the following as the ode type

`[_separable]`

$$\boxed{x y^{2}+y \,{\mathrm e}^{x^{2}} y^{\prime}=-x}$$

1.15.1 Solving as separable ode

In canonical form the ODE is

$$\begin{aligned} y^{\prime} &= F(x,y)\\ &= f(x) g(y)\\ &= -\frac{{\mathrm e}^{-x^{2}} x \left(y^{2}+1\right)}{y} \end{aligned}$$

Where $f(x)=-x \,{\mathrm e}^{-x^{2}}$ and $g(y)=\frac{y^{2}+1}{y}$. Integrating both sides gives

$$\frac{1}{\frac{y^{2}+1}{y}} \,dy = -x \,{\mathrm e}^{-x^{2}} \,dx$$

$$\int \frac{1}{\frac{y^2+1}{y}}\,dy = \int -x\,\mathrm{e}^{-x^2}\,dx$$

$$\frac{\ln\left(y^2+1\right)}{2} = \frac{\mathrm{e}^{-x^2}}{2} + c_1$$

Raising both side to exponential gives

$$\sqrt{y^2+1} = \mathrm{e}^{\frac{\mathrm{e}^{-x^2}}{2}+c_1}$$

Which simplifies to

$$\sqrt{y^2+1} = c_2\mathrm{e}^{\frac{\mathrm{e}^{-x^2}}{2}}$$

The solution is

$$\sqrt{1+y^2} = c_2\mathrm{e}^{\frac{\mathrm{e}^{-x^2}}{2}+c_1}$$

<u>Summary</u>
The solution(s) found are the following

$$\sqrt{1+y^2} = c_2\mathrm{e}^{\frac{\mathrm{e}^{-x^2}}{2}+c_1} \tag{1}$$

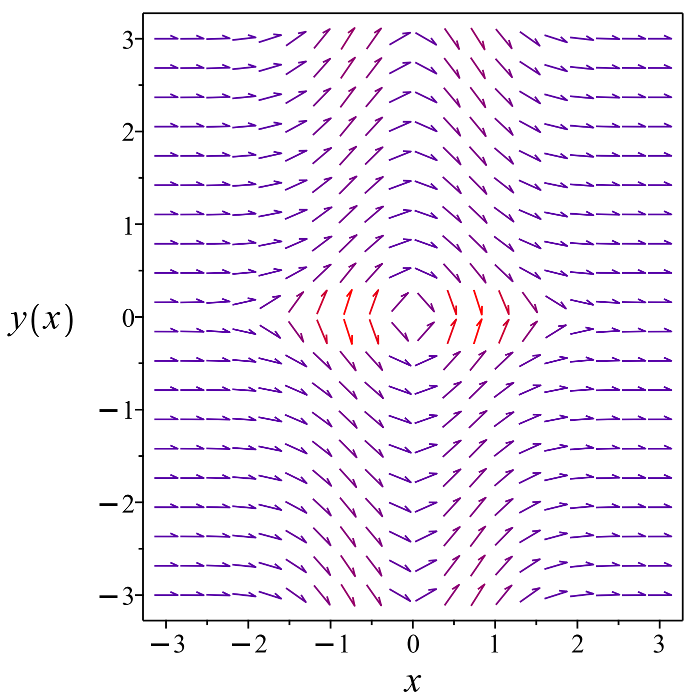


Figure 34: Slope field plot

Verification of solutions

$$\sqrt{1+y^2} = c_2 \mathrm{e}^{\frac{\mathrm{e}^{-x^2}}{2}+c_1}$$

Verified OK.

1.15.2 Solving as first order ode lie symmetry lookup ode

Writing the ode as

$$y' = -\frac{\mathrm{e}^{-x^2} x(y^2+1)}{y}$$
$$y' = \omega(x, y)$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x + \omega(\eta_y - \xi_x) - \omega^2 \xi_y - \omega_x \xi - \omega_y \eta = 0 \tag{A}$$

The type of this ode is known. It is of type `separable`. Therefore we do not need to solve the PDE (A), and can just use the lookup table shown below to find ξ, η

Table 30: Lie symmetry infinitesimal lookup table for known first order ODE's

ODE class	Form	ξ	η
linear ode	$y' = f(x)y(x) + g(x)$	0	$e^{\int f dx}$
separable ode	$y' = f(x)\, g(y)$	$\frac{1}{f}$	0
quadrature ode	$y' = f(x)$	0	1
quadrature ode	$y' = g(y)$	1	0
homogeneous ODEs of Class A	$y' = f\left(\frac{y}{x}\right)$	x	y
homogeneous ODEs of Class C	$y' = (a + bx + cy)^{\frac{n}{m}}$	1	$-\frac{b}{c}$
homogeneous class D	$y' = \frac{y}{x} + g(x)\, F\left(\frac{y}{x}\right)$	x^2	xy
First order special form ID 1	$y' = g(x)\, e^{h(x)+by} + f(x)$	$\frac{e^{-\int bf(x)dx - h(x)}}{g(x)}$	$\frac{f(x)e^{-\int bf(x)dx - h(x)}}{g(x)}$
polynomial type ode	$y' = \frac{a_1x+b_1y+c_1}{a_2x+b_2y+c_2}$	$\frac{a_1b_2x - a_2b_1x - b_1c_2 + b_2c_1}{a_1b_2 - a_2b_1}$	$\frac{a_1b_2y - a_2b_1y - a_1c_2 - a_2c_1}{a_1b_2 - a_2b_1}$
Bernoulli ode	$y' = f(x)\, y + g(x)\, y^n$	0	$e^{-\int (n-1)f(x)dx} y^n$
Reduced Riccati	$y' = f_1(x)\, y + f_2(x)\, y^2$	0	$e^{-\int f_1 dx}$

The above table shows that

$$
\begin{aligned}
\xi(x, y) &= -\frac{e^{x^2}}{x} \\
\eta(x, y) &= 0
\end{aligned}
\tag{A1}
$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x, y) \to (R, S)$ where (R, S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$
\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}
$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y}\right) S(x, y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the

canonical coordinates, where $S(R)$. Since $\eta = 0$ then in this special case

$$R = y$$

S is found from

$$\begin{aligned} S &= \int \frac{1}{\xi} dx \\ &= \int \frac{1}{-\frac{e^{x^2}}{x}} dx \end{aligned}$$

Which results in

$$S = \frac{e^{-x^2}}{2}$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x, y) S_y}{R_x + \omega(x, y) R_y} \tag{2}$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x, y)$ is the right hand side of the original ode given by

$$\omega(x, y) = -\frac{e^{-x^2} x (y^2 + 1)}{y}$$

Evaluating all the partial derivatives gives

$$\begin{aligned} R_x &= 0 \\ R_y &= 1 \\ S_x &= -x\, e^{-x^2} \\ S_y &= 0 \end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = \frac{y}{y^2 + 1} \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = \frac{R}{R^2 + 1}$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordinates R, S. Integrating the above gives

$$S(R) = \frac{\ln(R^2+1)}{2} + c_1 \tag{4}$$

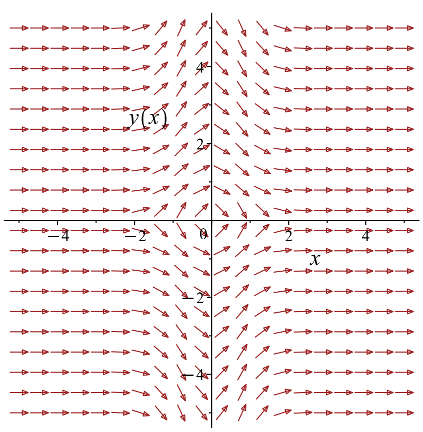
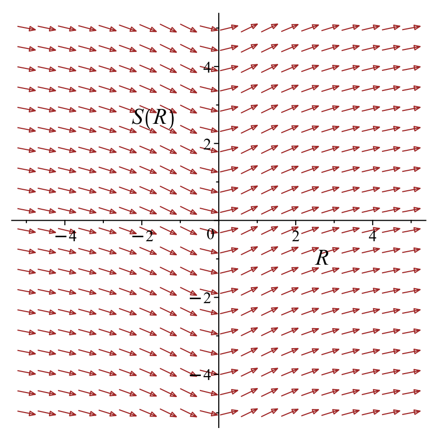
To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$\frac{e^{-x^2}}{2} = \frac{\ln(1+y^2)}{2} + c_1$$

Which simplifies to

$$\frac{e^{-x^2}}{2} = \frac{\ln(1+y^2)}{2} + c_1$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

<table>
<tr><th>Original ode in x, y coordinates</th><th>Canonical coordinates transformation</th><th>ODE in canonical coordinates (R, S)</th></tr>
<tr><td>$\frac{dy}{dx} = -\frac{e^{-x^2} x(y^2+1)}{y}$

</td><td>$R = y$
$S = \frac{e^{-x^2}}{2}$</td><td>$\frac{dS}{dR} = \frac{R}{R^2+1}$

</td></tr>
</table>

Summary

The solution(s) found are the following

$$\frac{e^{-x^2}}{2} = \frac{\ln(1+y^2)}{2} + c_1 \tag{1}$$

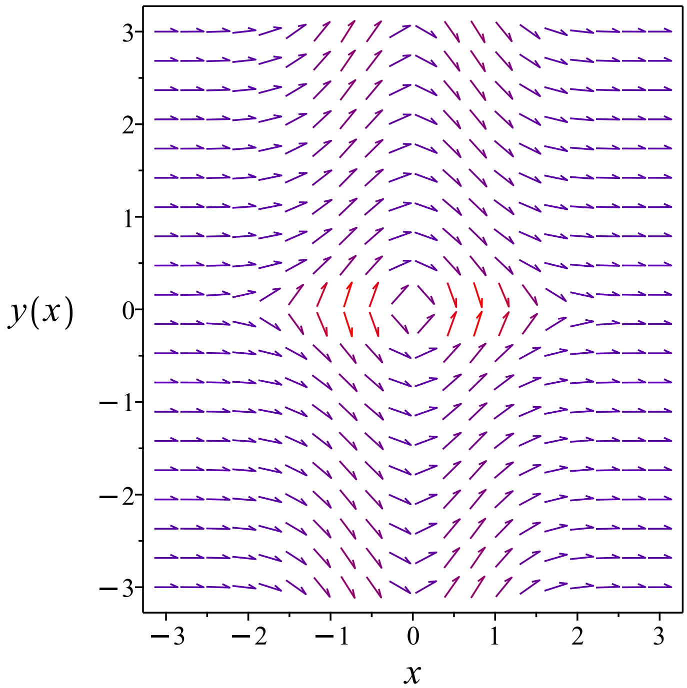


Figure 35: Slope field plot

Verification of solutions

$$\frac{e^{-x^2}}{2} = \frac{\ln(1+y^2)}{2} + c_1$$

Verified OK.

1.15.3 Solving as bernoulli ode

In canonical form, the ODE is

$$\begin{aligned} y' &= F(x,y) \\ &= -\frac{e^{-x^2} x(y^2+1)}{y} \end{aligned}$$

This is a Bernoulli ODE.

$$y' = -x\,e^{-x^2} y - x\,e^{-x^2} \frac{1}{y} \tag{1}$$

The standard Bernoulli ODE has the form

$$y' = f_0(x)y + f_1(x)y^n \tag{2}$$

The first step is to divide the above equation by y^n which gives

$$\frac{y'}{y^n} = f_0(x)y^{1-n} + f_1(x) \tag{3}$$

The next step is use the substitution $w = y^{1-n}$ in equation (3) which generates a new ODE in $w(x)$ which will be linear and can be easily solved using an integrating factor. Backsubstitution then gives the solution $y(x)$ which is what we want.

This method is now applied to the ODE at hand. Comparing the ODE (1) With (2) Shows that

$$\begin{aligned} f_0(x) &= -x\,\mathrm{e}^{-x^2} \\ f_1(x) &= -x\,\mathrm{e}^{-x^2} \\ n &= -1 \end{aligned}$$

Dividing both sides of ODE (1) by $y^n = \frac{1}{y}$ gives

$$y'y = -\mathrm{e}^{-x^2} x\, y^2 - x\,\mathrm{e}^{-x^2} \tag{4}$$

Let

$$\begin{aligned} w &= y^{1-n} \\ &= y^2 \end{aligned} \tag{5}$$

Taking derivative of equation (5) w.r.t x gives

$$w' = 2yy' \tag{6}$$

Substituting equations (5) and (6) into equation (4) gives

$$\begin{aligned} \frac{w'(x)}{2} &= -\mathrm{e}^{-x^2} x w(x) - x\,\mathrm{e}^{-x^2} \\ w' &= -2\,\mathrm{e}^{-x^2} x w - 2x\,\mathrm{e}^{-x^2} \end{aligned} \tag{7}$$

The above now is a linear ODE in $w(x)$ which is now solved.

Entering Linear first order ODE solver. In canonical form a linear first order is

$$w'(x) + p(x)w(x) = q(x)$$

Where here

$$\begin{aligned} p(x) &= 2x\,\mathrm{e}^{-x^2} \\ q(x) &= -2x\,\mathrm{e}^{-x^2} \end{aligned}$$

Hence the ode is

$$w'(x) + 2\,\mathrm{e}^{-x^2} x w(x) = -2x\,\mathrm{e}^{-x^2}$$

The integrating factor μ is

$$
\begin{aligned}
\mu &= \mathrm{e}^{\int 2x\,\mathrm{e}^{-x^2}dx} \\
&= \mathrm{e}^{-\mathrm{e}^{-x^2}}
\end{aligned}
$$

The ode becomes

$$
\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}x}(\mu w) &= (\mu)\left(-2x\,\mathrm{e}^{-x^2}\right) \\
\frac{\mathrm{d}}{\mathrm{d}x}\left(\mathrm{e}^{-\mathrm{e}^{-x^2}} w\right) &= \left(\mathrm{e}^{-\mathrm{e}^{-x^2}}\right)\left(-2x\,\mathrm{e}^{-x^2}\right) \\
\mathrm{d}\left(\mathrm{e}^{-\mathrm{e}^{-x^2}} w\right) &= \left(-2x\,\mathrm{e}^{-x^2-\mathrm{e}^{-x^2}}\right)\,\mathrm{d}x
\end{aligned}
$$

Integrating gives

$$
\begin{aligned}
\mathrm{e}^{-\mathrm{e}^{-x^2}} w &= \int -2x\,\mathrm{e}^{-x^2-\mathrm{e}^{-x^2}}\,\mathrm{d}x \\
\mathrm{e}^{-\mathrm{e}^{-x^2}} w &= -\mathrm{e}^{-\mathrm{e}^{-x^2}} + c_1
\end{aligned}
$$

Dividing both sides by the integrating factor $\mu = \mathrm{e}^{-\mathrm{e}^{-x^2}}$ results in

$$
w(x) = -1 + c_1\mathrm{e}^{\mathrm{e}^{-x^2}}
$$

Replacing w in the above by y^2 using equation (5) gives the final solution.

$$
y^2 = -1 + c_1\mathrm{e}^{\mathrm{e}^{-x^2}}
$$

Solving for y gives

$$
\begin{aligned}
y(x) &= \sqrt{-1 + c_1\mathrm{e}^{\mathrm{e}^{-x^2}}} \\
y(x) &= -\sqrt{-1 + c_1\mathrm{e}^{\mathrm{e}^{-x^2}}}
\end{aligned}
$$

<ins>Summary</ins>

The solution(s) found are the following

$$
y = \sqrt{-1 + c_1\mathrm{e}^{\mathrm{e}^{-x^2}}} \tag{1}
$$

$$
y = -\sqrt{-1 + c_1\mathrm{e}^{\mathrm{e}^{-x^2}}} \tag{2}
$$

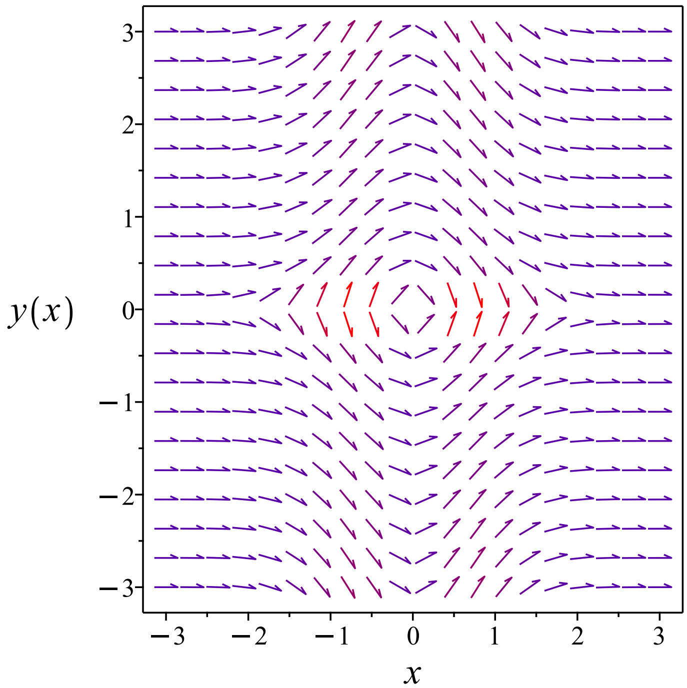


Figure 36: Slope field plot

Verification of solutions

$$y = \sqrt{-1 + c_1 \mathrm{e}^{\mathrm{e}^{-x^2}}}$$

Verified OK.

$$y = -\sqrt{-1 + c_1 \mathrm{e}^{\mathrm{e}^{-x^2}}}$$

Verified OK.

1.15.4 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x, y) + N(x, y)\frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x, y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x, y) = 0$$

Hence

$$\frac{\partial \phi}{\partial x}+\frac{\partial \phi}{\partial y}\frac{dy}{dx}=0 \tag{B}$$

Comparing (A,B) shows that

$$\begin{aligned}\frac{\partial \phi}{\partial x}&=M\\ \frac{\partial \phi}{\partial y}&=N\end{aligned}$$

But since $\frac{\partial^2\phi}{\partial x\partial y}=\frac{\partial^2\phi}{\partial y\partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y}=\frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x,y)$ but at least we know now that we can do that since the condition $\frac{\partial^2\phi}{\partial x\partial y}=\frac{\partial^2\phi}{\partial y\partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x,y)\,\mathrm{d}x+N(x,y)\,\mathrm{d}y=0 \tag{1A}$$

Therefore

$$\begin{aligned}\left(-\frac{y}{y^2+1}\right)\mathrm{d}y&=\left(x\,\mathrm{e}^{-x^2}\right)\mathrm{d}x\\ \left(-x\,\mathrm{e}^{-x^2}\right)\mathrm{d}x+\left(-\frac{y}{y^2+1}\right)\mathrm{d}y&=0 \qquad (2A)\end{aligned}$$

Comparing (1A) and (2A) shows that

$$\begin{aligned}M(x,y)&=-x\,\mathrm{e}^{-x^2}\\ N(x,y)&=-\frac{y}{y^2+1}\end{aligned}$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y}=\frac{\partial N}{\partial x}$$

Using result found above gives

$$\begin{aligned}\frac{\partial M}{\partial y}&=\frac{\partial}{\partial y}\left(-x\,\mathrm{e}^{-x^2}\right)\\ &=0\end{aligned}$$

And

$$
\begin{aligned}
\frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}\left(-\frac{y}{y^2+1}\right) \\
&= 0
\end{aligned}
$$

Since $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$, then the ODE is exact The following equations are now set up to solve for the function $\phi(x,y)$

$$
\frac{\partial \phi}{\partial x} = M \tag{1}
$$

$$
\frac{\partial \phi}{\partial y} = N \tag{2}
$$

Integrating (1) w.r.t. x gives

$$
\begin{aligned}
\int \frac{\partial \phi}{\partial x}\,\mathrm{d}x &= \int M\,\mathrm{d}x \\
\int \frac{\partial \phi}{\partial x}\,\mathrm{d}x &= \int -x\,\mathrm{e}^{-x^2}\,\mathrm{d}x
\end{aligned}
$$

$$
\phi = \frac{\mathrm{e}^{-x^2}}{2} + f(y) \tag{3}
$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$
\frac{\partial \phi}{\partial y} = 0 + f'(y) \tag{4}
$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = -\frac{y}{y^2+1}$. Therefore equation (4) becomes

$$
-\frac{y}{y^2+1} = 0 + f'(y) \tag{5}
$$

Solving equation (5) for $f'(y)$ gives

$$
f'(y) = -\frac{y}{y^2+1}
$$

Integrating the above w.r.t y gives

$$
\begin{aligned}
\int f'(y)\,\mathrm{d}y &= \int \left(-\frac{y}{y^2+1}\right)\mathrm{d}y \\
f(y) &= -\frac{\ln\left(y^2+1\right)}{2} + c_1
\end{aligned}
$$

Where c_1 is constant of integration. Substituting result found above for $f(y)$ into equation (3) gives ϕ

$$\phi = \frac{e^{-x^2}}{2} - \frac{\ln(y^2+1)}{2} + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = \frac{e^{-x^2}}{2} - \frac{\ln(y^2+1)}{2}$$

<u>Summary</u>
The solution(s) found are the following

$$\frac{e^{-x^2}}{2} - \frac{\ln(1+y^2)}{2} = c_1 \tag{1}$$

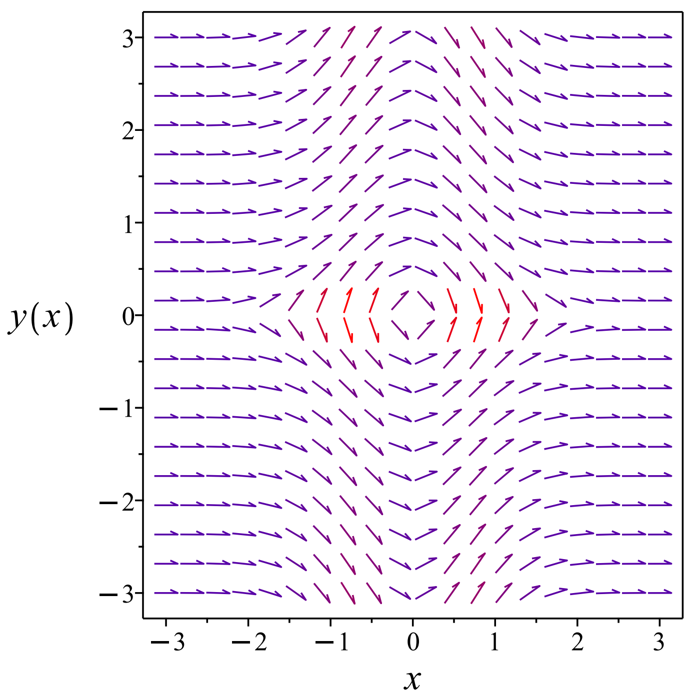


Figure 37: Slope field plot

<u>Verification of solutions</u>

$$\frac{e^{-x^2}}{2} - \frac{\ln(1+y^2)}{2} = c_1$$

Verified OK.

1.15.5 Maple step by step solution

Let's solve

$xy^2+y\,\mathrm{e}^{x^2}y'=-x$

- Highest derivative means the order of the ODE is 1

  y'

- Separate variables

  $\frac{y'y}{1+y^2}=-\frac{x}{\mathrm{e}^{x^2}}$

- Integrate both sides with respect to x

  $\int \frac{y'y}{1+y^2}dx=\int -\frac{x}{\mathrm{e}^{x^2}}dx+c_1$

- Evaluate integral

  $\frac{\ln(1+y^2)}{2}=\frac{1}{2\,\mathrm{e}^{x^2}}+c_1$

- Solve for y

  $\left\{y=\sqrt{-1+\mathrm{e}^{\frac{2c_1\mathrm{e}^{x^2}+1}{\mathrm{e}^{x^2}}}}, y=-\sqrt{-1+\mathrm{e}^{\frac{2c_1\mathrm{e}^{x^2}+1}{\mathrm{e}^{x^2}}}}\right\}$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
<- Bernoulli successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 35

```
dsolve((x+x*y(x)^2)+exp(x^2)*y(x)*diff(y(x),x)=0,y(x), singsol=all)
```

$$y(x)=\sqrt{\mathrm{e}^{\mathrm{e}^{-x^2}}c_1-1}$$
$$y(x)=-\sqrt{\mathrm{e}^{\mathrm{e}^{-x^2}}c_1-1}$$

✓ Solution by Mathematica

Time used: 4.151 (sec). Leaf size: 65

```
DSolve[(x+x*y[x]^2)+Exp[x^2]*y[x]*y'[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to -\sqrt{-1 + e^{e^{-x^2}+2c_1}}$$
$$y(x) \to \sqrt{-1 + e^{e^{-x^2}+2c_1}}$$
$$y(x) \to -i$$
$$y(x) \to i$$

1.16 problem 16

Internal problem ID [4927]
Internal file name [`OUTPUT/4420_Sunday_June_05_2022_01_18_17_PM_58622000/index.tex`]

Book: Fundamentals of Differential Equations. By Nagle, Saff and Snider. 9th edition. Boston. Pearson 2018.
Section: Chapter 2, First order differential equations. Section 2.2, Separable Equations. Exercises. page 46
Problem number: 16.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"exact", "riccati", "separable", "first_order_ode_lie_symmetry_lookup"**

Maple gives the following as the ode type

`[_separable]`

$$\boxed{y\,\mathrm{e}^{\cos(x)}\sin(x) = -\frac{y'}{y}}$$

1.16.1 Solving as separable ode

In canonical form the ODE is

$$\begin{aligned} y' &= F(x,y) \\ &= f(x)g(y) \\ &= -y^2\mathrm{e}^{\cos(x)}\sin(x) \end{aligned}$$

Where $f(x) = -\mathrm{e}^{\cos(x)}\sin(x)$ and $g(y) = y^2$. Integrating both sides gives

$$\frac{1}{y^2}\,dy = -\mathrm{e}^{\cos(x)}\sin(x)\;dx$$

$$\int \frac{1}{y^2}\,dy = \int -\mathrm{e}^{\cos(x)} \sin(x)\,dx$$
$$-\frac{1}{y} = \mathrm{e}^{\cos(x)} + c_1$$

Which results in

$$y = -\frac{1}{\mathrm{e}^{\cos(x)} + c_1}$$

Summary

The solution(s) found are the following

$$y = -\frac{1}{\mathrm{e}^{\cos(x)} + c_1} \tag{1}$$

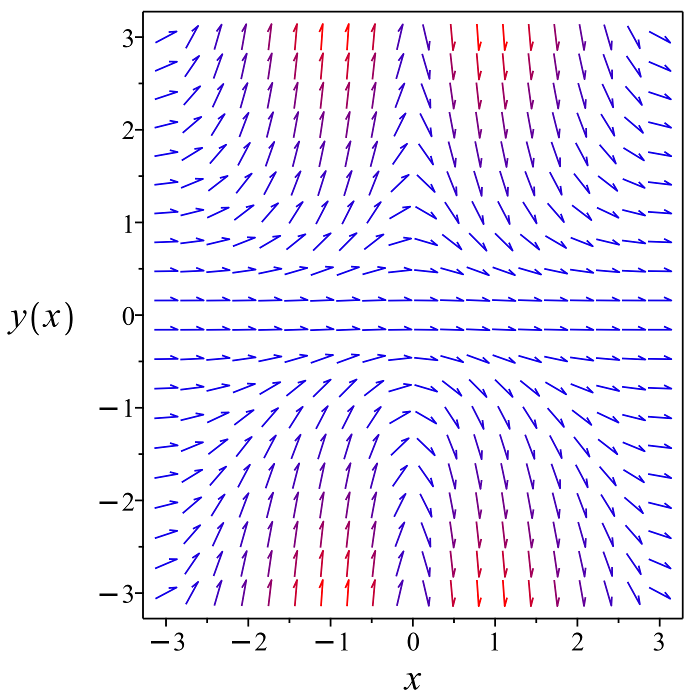


Figure 38: Slope field plot

Verification of solutions

$$y = -\frac{1}{\mathrm{e}^{\cos(x)} + c_1}$$

Verified OK.

1.16.2 Solving as first order ode lie symmetry lookup ode

Writing the ode as

$$\begin{aligned} y' &= -y^2 \mathrm{e}^{\cos(x)} \sin(x) \\ y' &= \omega(x,y) \end{aligned}$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x + \omega(\eta_y - \xi_x) - \omega^2 \xi_y - \omega_x \xi - \omega_y \eta = 0 \tag{A}$$

The type of this ode is known. It is of type `separable`. Therefore we do not need to solve the PDE (A), and can just use the lookup table shown below to find ξ, η

Table 33: Lie symmetry infinitesimal lookup table for known first order ODE's

ODE class	Form	ξ	η
linear ode	$y' = f(x)y(x) + g(x)$	0	$e^{\int f dx}$
separable ode	$y' = f(x)\, g(y)$	$\frac{1}{f}$	0
quadrature ode	$y' = f(x)$	0	1
quadrature ode	$y' = g(y)$	1	0
homogeneous ODEs of Class A	$y' = f\left(\frac{y}{x}\right)$	x	y
homogeneous ODEs of Class C	$y' = (a + bx + cy)^{\frac{n}{m}}$	1	$-\frac{b}{c}$
homogeneous class D	$y' = \frac{y}{x} + g(x)\, F\left(\frac{y}{x}\right)$	x^2	xy
First order special form ID 1	$y' = g(x)\, e^{h(x)+by} + f(x)$	$\frac{e^{-\int bf(x)dx - h(x)}}{g(x)}$	$\frac{f(x)e^{-\int bf(x)dx - h(x)}}{g(x)}$
polynomial type ode	$y' = \frac{a_1x+b_1y+c_1}{a_2x+b_2y+c_2}$	$\frac{a_1b_2x - a_2b_1x - b_1c_2 + b_2c_1}{a_1b_2 - a_2b_1}$	$\frac{a_1b_2y - a_2b_1y - a_1c_2 - a_2c_1}{a_1b_2 - a_2b_1}$
Bernoulli ode	$y' = f(x)\, y + g(x)\, y^n$	0	$e^{-\int (n-1)f(x)dx} y^n$
Reduced Riccati	$y' = f_1(x)\, y + f_2(x)\, y^2$	0	$e^{-\int f_1 dx}$

The above table shows that

$$
\begin{aligned}
\xi(x,y) &= -\frac{\mathrm{e}^{-\cos(x)}}{\sin(x)} \\
\eta(x,y) &= 0
\end{aligned}
\tag{A1}
$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x,y) \to (R,S)$ where (R,S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$
\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}
$$

The above comes from the requirements that $\left(\xi\frac{\partial}{\partial x} + \eta\frac{\partial}{\partial y}\right) S(x,y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the canonical coordinates, where $S(R)$. Since $\eta = 0$ then in this special case

$$
R = y
$$

S is found from

$$
\begin{aligned}
S &= \int \frac{1}{\xi} dx \\
&= \int \frac{1}{-\frac{\mathrm{e}^{-\cos(x)}}{\sin(x)}} dx
\end{aligned}
$$

Which results in

$$
S = \mathrm{e}^{\cos(x)}
$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$
\frac{dS}{dR} = \frac{S_x + \omega(x,y)S_y}{R_x + \omega(x,y)R_y} \tag{2}
$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x,y)$ is the right hand side of the original ode given by

$$
\omega(x,y) = -y^2 \mathrm{e}^{\cos(x)} \sin(x)
$$

Evaluating all the partial derivatives gives

$$\begin{aligned} R_x &= 0 \\ R_y &= 1 \\ S_x &= -\mathrm{e}^{\cos(x)} \sin(x) \\ S_y &= 0 \end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = \frac{1}{y^2} \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = \frac{1}{R^2}$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$S(R) = -\frac{1}{R} + c_1 \tag{4}$$

To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$\mathrm{e}^{\cos(x)} = -\frac{1}{y} + c_1$$

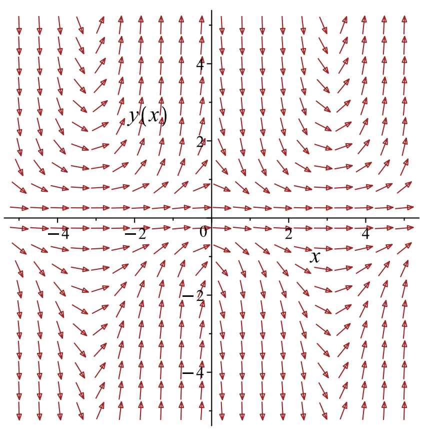
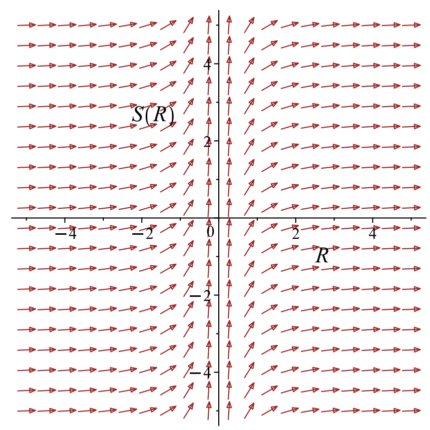
Which simplifies to

$$\mathrm{e}^{\cos(x)} = -\frac{1}{y} + c_1$$

Which gives

$$y = -\frac{1}{\mathrm{e}^{\cos(x)} - c_1}$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in x, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dx} = -y^2 \mathrm{e}^{\cos(x)} \sin(x)$ 	$R = y$ $S = \mathrm{e}^{\cos(x)}$	$\frac{dS}{dR} = \frac{1}{R^2}$ 

Summary

The solution(s) found are the following

$$y = -\frac{1}{\mathrm{e}^{\cos(x)} - c_1} \tag{1}$$

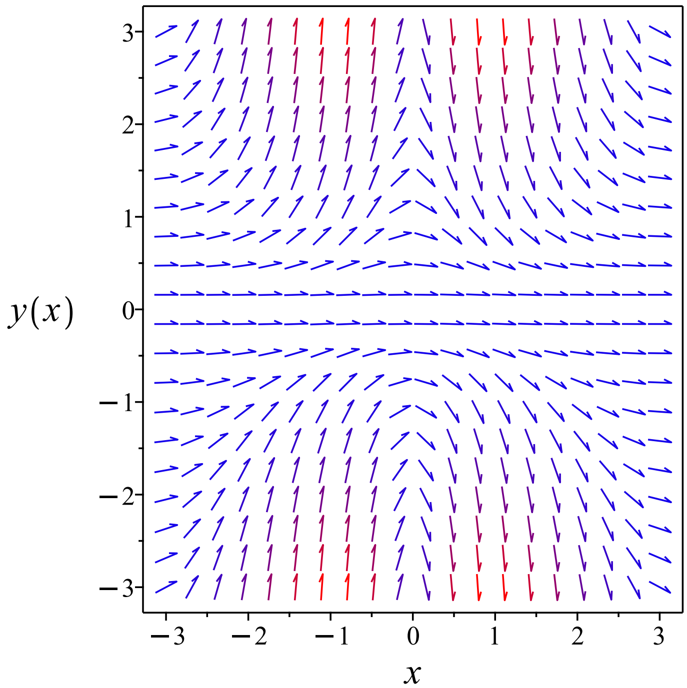


Figure 39: Slope field plot

Verification of solutions

$$y = -\frac{1}{e^{\cos(x)} - c_1}$$

Verified OK.

1.16.3 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x,y) + N(x,y)\frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x,y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y) = 0$$

Hence

$$\frac{\partial\phi}{\partial x} + \frac{\partial\phi}{\partial y}\frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$\begin{aligned}\frac{\partial \phi}{\partial x} &= M\\ \frac{\partial \phi}{\partial y} &= N\end{aligned}$$

But since $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x, y)$ but at least we know now that we can do that since the condition $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x, y)\,\mathrm{d}x + N(x, y)\,\mathrm{d}y = 0 \tag{1A}$$

Therefore

$$\begin{aligned}\left(-\frac{1}{y^2}\right)\mathrm{d}y &= \left(\mathrm{e}^{\cos(x)}\sin(x)\right)\mathrm{d}x\\ \left(-\mathrm{e}^{\cos(x)}\sin(x)\right)\mathrm{d}x + \left(-\frac{1}{y^2}\right)\mathrm{d}y &= 0 \qquad (2A)\end{aligned}$$

Comparing (1A) and (2A) shows that

$$\begin{aligned}M(x, y) &= -\mathrm{e}^{\cos(x)}\sin(x)\\ N(x, y) &= -\frac{1}{y^2}\end{aligned}$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Using result found above gives

$$\begin{aligned}\frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}\left(-\mathrm{e}^{\cos(x)}\sin(x)\right)\\ &= 0\end{aligned}$$

And

$$
\begin{aligned}
\frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}\left(-\frac{1}{y^2}\right)\\
&= 0
\end{aligned}
$$

Since $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$, then the ODE is <u>exact</u> The following equations are now set up to solve for the function $\phi(x, y)$

$$
\frac{\partial \phi}{\partial x} = M \tag{1}
$$

$$
\frac{\partial \phi}{\partial y} = N \tag{2}
$$

Integrating (1) w.r.t. x gives

$$
\begin{aligned}
\int \frac{\partial \phi}{\partial x}\,\mathrm{d}x &= \int M\,\mathrm{d}x\\
\int \frac{\partial \phi}{\partial x}\,\mathrm{d}x &= \int -\mathrm{e}^{\cos(x)}\sin(x)\,\mathrm{d}x
\end{aligned}
$$

$$
\phi = \mathrm{e}^{\cos(x)} + f(y) \tag{3}
$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$
\frac{\partial \phi}{\partial y} = 0 + f'(y) \tag{4}
$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = -\frac{1}{y^2}$. Therefore equation (4) becomes

$$
-\frac{1}{y^2} = 0 + f'(y) \tag{5}
$$

Solving equation (5) for $f'(y)$ gives

$$
f'(y) = -\frac{1}{y^2}
$$

Integrating the above w.r.t y gives

$$
\begin{aligned}
\int f'(y)\,\mathrm{d}y &= \int \left(-\frac{1}{y^2}\right)\mathrm{d}y\\
f(y) &= \frac{1}{y} + c_1
\end{aligned}
$$

Where c_1 is constant of integration. Substituting result found above for $f(y)$ into equation (3) gives ϕ

$$\phi = e^{\cos(x)} + \frac{1}{y} + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = e^{\cos(x)} + \frac{1}{y}$$

The solution becomes

$$y = -\frac{1}{e^{\cos(x)} - c_1}$$

Summary

The solution(s) found are the following

$$y = -\frac{1}{e^{\cos(x)} - c_1} \tag{1}$$

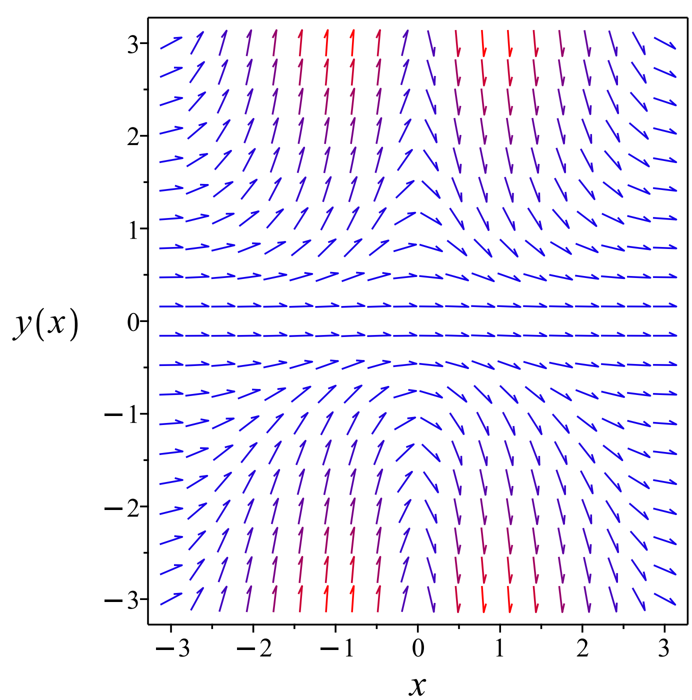


Figure 40: Slope field plot

Verification of solutions

$$y = -\frac{1}{e^{\cos(x)} - c_1}$$

Verified OK.

1.16.4 Solving as riccati ode

In canonical form the ODE is

$$\begin{aligned} y' &= F(x,y) \\ &= -y^2 e^{\cos(x)} \sin(x) \end{aligned}$$

This is a Riccati ODE. Comparing the ODE to solve

$$y' = -y^2 e^{\cos(x)} \sin(x)$$

With Riccati ODE standard form

$$y' = f_0(x) + f_1(x)y + f_2(x)y^2$$

Shows that $f_0(x) = 0$, $f_1(x) = 0$ and $f_2(x) = -e^{\cos(x)} \sin(x)$. Let

$$\begin{aligned} y &= \frac{-u'}{f_2 u} \\ &= \frac{-u'}{-e^{\cos(x)} \sin(x) u} \end{aligned} \tag{1}$$

Using the above substitution in the given ODE results (after some simplification)in a second order ODE to solve for $u(x)$ which is

$$f_2 u''(x) - (f_2' + f_1 f_2) u'(x) + f_2^2 f_0 u(x) = 0 \tag{2}$$

But

$$\begin{aligned} f_2' &= e^{\cos(x)} \sin(x)^2 - e^{\cos(x)} \cos(x) \\ f_1 f_2 &= 0 \\ f_2^2 f_0 &= 0 \end{aligned}$$

Substituting the above terms back in equation (2) gives

$$-e^{\cos(x)} \sin(x) u''(x) - \left(e^{\cos(x)} \sin(x)^2 - e^{\cos(x)} \cos(x)\right) u'(x) = 0$$

Solving the above ODE (this ode solved using Maple, not this program), gives

$$u(x) = c_1 + e^{\cos(x)} c_2$$

The above shows that

$$u'(x) = -e^{\cos(x)} \sin(x) c_2$$

Using the above in (1) gives the solution

$$y = -\frac{c_2}{c_1 + e^{\cos(x)} c_2}$$

Dividing both numerator and denominator by c_1 gives, after renaming the constant $\frac{c_2}{c_1} = c_3$ the following solution

$$y = -\frac{1}{c_3 + e^{\cos(x)}}$$

Summary

The solution(s) found are the following

$$y = -\frac{1}{c_3 + e^{\cos(x)}} \tag{1}$$

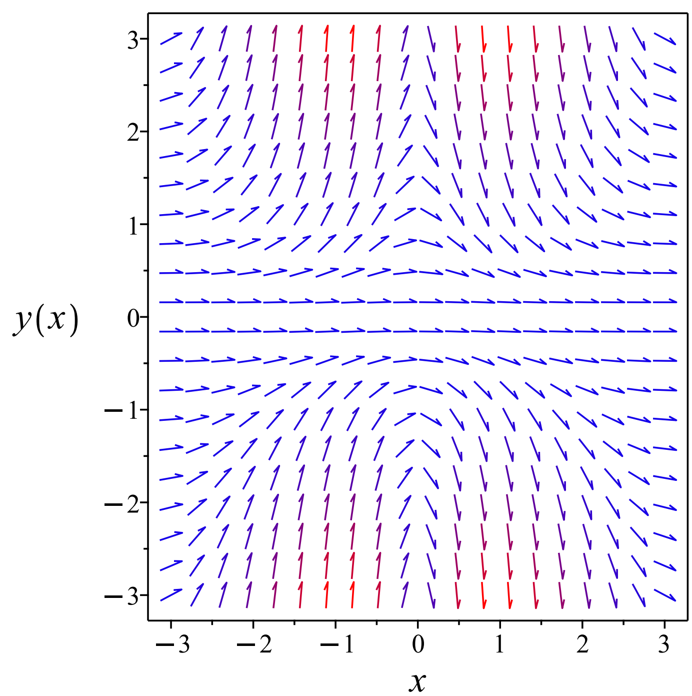


Figure 41: Slope field plot

Verification of solutions

$$y = -\frac{1}{c_3 + e^{\cos(x)}}$$

Verified OK.

1.16.5 Maple step by step solution

Let's solve

$y\, e^{\cos(x)} \sin(x) = -\frac{y'}{y}$

- Highest derivative means the order of the ODE is 1

  y'

- Separate variables

  $\frac{y'}{y^2} = -e^{\cos(x)} \sin(x)$

- Integrate both sides with respect to x

  $\int \frac{y'}{y^2} dx = \int -e^{\cos(x)} \sin(x)\, dx + c_1$

- Evaluate integral

  $-\frac{1}{y} = e^{\cos(x)} + c_1$

- Solve for y

  $y = -\frac{1}{e^{\cos(x)} + c_1}$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
<- Bernoulli successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 13

```
dsolve(1/y(x)*diff(y(x),x)+y(x)*exp(cos(x))*sin(x)=0,y(x), singsol=all)
```

$$y(x) = \frac{1}{-e^{\cos(x)} + c_1}$$

✓ Solution by Mathematica

Time used: 0.293 (sec). Leaf size: 21

```
DSolve[1/y[x]*y'[x]+y[x]*Exp[Cos[x]]*Sin[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to -\frac{1}{e^{\cos(x)} + c_1}$$
$$y(x) \to 0$$

1.17 problem 17

Internal problem ID [4928]
Internal file name [`OUTPUT/4421_Sunday_June_05_2022_01_18_30_PM_63187674/index.tex`]

Book: Fundamentals of Differential Equations. By Nagle, Saff and Snider. 9th edition. Boston. Pearson 2018.
Section: Chapter 2, First order differential equations. Section 2.2, Separable Equations. Exercises. page 46
Problem number: 17.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"exact", "riccati", "separable", "first_order_ode_lie_symmetry_lookup"**

Maple gives the following as the ode type

```
[_separable]
```

$$\boxed{y' - \left(1 + y^2\right) \tan\left(x\right) = 0}$$

With initial conditions

$$\left[y(0) = \sqrt{3}\right]$$

1.17.1 Existence and uniqueness analysis

This is non linear first order ODE. In canonical form it is written as

$$\begin{aligned} y' &= f(x,y) \\ &= (y^2+1)\tan(x) \end{aligned}$$

The x domain of $f(x,y)$ when $y=\sqrt{3}$ is

$$\left\{x<\frac{1}{2}\pi+\pi\_Z61\vee\frac{1}{2}\pi+\pi\_Z61<x\right\}$$

And the point $x_0=0$ is inside this domain. The y domain of $f(x,y)$ when $x=0$ is

$$\{-\infty<y<\infty\}$$

And the point $y_0=\sqrt{3}$ is inside this domain. Now we will look at the continuity of

$$\begin{aligned} \frac{\partial f}{\partial y} &= \frac{\partial}{\partial y}\left((y^2+1)\tan(x)\right) \\ &= 2y\tan(x) \end{aligned}$$

The x domain of $\frac{\partial f}{\partial y}$ when $y=\sqrt{3}$ is

$$\left\{x<\frac{1}{2}\pi+\pi\_Z61\vee\frac{1}{2}\pi+\pi\_Z61<x\right\}$$

And the point $x_0=0$ is inside this domain. The y domain of $\frac{\partial f}{\partial y}$ when $x=0$ is

$$\{-\infty<y<\infty\}$$

And the point $y_0=\sqrt{3}$ is inside this domain. Therefore solution exists and is unique.

1.17.2 Solving as separable ode

In canonical form the ODE is

$$\begin{aligned} y' &= F(x,y) \\ &= f(x)g(y) \\ &= (y^2+1)\tan(x) \end{aligned}$$

Where $f(x) = \tan(x)$ and $g(y) = y^2 + 1$. Integrating both sides gives

$$\frac{1}{y^2+1}\,dy = \tan(x)\,dx$$

$$\int \frac{1}{y^2+1}\,dy = \int \tan(x)\,dx$$

$$\arctan(y) = -\ln(\cos(x)) + c_1$$

Which results in

$$y = \tan(-\ln(\cos(x)) + c_1)$$

Initial conditions are used to solve for c_1. Substituting $x = 0$ and $y = \sqrt{3}$ in the above solution gives an equation to solve for the constant of integration.

$$\sqrt{3} = \tan(c_1)$$

$$c_1 = \frac{\pi}{3}$$

Substituting c_1 found above in the general solution gives

$$y = \frac{-i\sqrt{3}\,\cos(x)^{2i} - i\sqrt{3} - \cos(x)^{2i} + 1}{\sqrt{3}\,\cos(x)^{2i} - i\cos(x)^{2i} - \sqrt{3} - i}$$

Summary

The solution(s) found are the following

$$y = \frac{-i\sqrt{3}\,\cos(x)^{2i} - i\sqrt{3} - \cos(x)^{2i} + 1}{\sqrt{3}\,\cos(x)^{2i} - i\cos(x)^{2i} - \sqrt{3} - i} \tag{1}$$

Verification of solutions

$$y = \frac{-i\sqrt{3}\,\cos(x)^{2i} - i\sqrt{3} - \cos(x)^{2i} + 1}{\sqrt{3}\,\cos(x)^{2i} - i\cos(x)^{2i} - \sqrt{3} - i}$$

Verified OK.

1.17.3 Solving as first order ode lie symmetry lookup ode

Writing the ode as

$$
\begin{aligned}
y' &= (y^2+1)\tan(x) \\
y' &= \omega(x,y)
\end{aligned}
$$

The condition of Lie symmetry is the linearized PDE given by

$$
\eta_x + \omega(\eta_y - \xi_x) - \omega^2 \xi_y - \omega_x \xi - \omega_y \eta = 0 \tag{A}
$$

The type of this ode is known. It is of type `separable`. Therefore we do not need to solve the PDE (A), and can just use the lookup table shown below to find ξ, η

Table 36: Lie symmetry infinitesimal lookup table for known first order ODE's

ODE class	Form	ξ	η
linear ode	$y' = f(x)y(x) + g(x)$	0	$e^{\int f dx}$
separable ode	$y' = f(x)\, g(y)$	$\frac{1}{f}$	0
quadrature ode	$y' = f(x)$	0	1
quadrature ode	$y' = g(y)$	1	0
homogeneous ODEs of Class A	$y' = f\left(\frac{y}{x}\right)$	x	y
homogeneous ODEs of Class C	$y' = (a+bx+cy)^{\frac{n}{m}}$	1	$-\frac{b}{c}$
homogeneous class D	$y' = \frac{y}{x} + g(x)\, F\left(\frac{y}{x}\right)$	x^2	xy
First order special form ID 1	$y' = g(x)\, e^{h(x)+by} + f(x)$	$\frac{e^{-\int bf(x)dx-h(x)}}{g(x)}$	$\frac{f(x)e^{-\int bf(x)dx-h(x)}}{g(x)}$
polynomial type ode	$y' = \frac{a_1x+b_1y+c_1}{a_2x+b_2y+c_2}$	$\frac{a_1b_2x-a_2b_1x-b_1c_2+b_2c_1}{a_1b_2-a_2b_1}$	$\frac{a_1b_2y-a_2b_1y-a_1c_2-a_2c_1}{a_1b_2-a_2b_1}$
Bernoulli ode	$y' = f(x)\, y + g(x)\, y^n$	0	$e^{-\int (n-1)f(x)dx} y^n$
Reduced Riccati	$y' = f_1(x)\, y + f_2(x)\, y^2$	0	$e^{-\int f_1 dx}$

The above table shows that

$$\xi(x,y) = \frac{1}{\tan(x)}$$
$$\eta(x,y) = 0 \tag{A1}$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x,y) \to (R,S)$ where (R,S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y}\right) S(x,y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the canonical coordinates, where $S(R)$. Since $\eta = 0$ then in this special case

$$R = y$$

S is found from

$$\begin{aligned} S &= \int \frac{1}{\xi} dx \\ &= \int \frac{1}{\frac{1}{\tan(x)}} dx \end{aligned}$$

Which results in

$$S = -\ln(\cos(x))$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x,y) S_y}{R_x + \omega(x,y) R_y} \tag{2}$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x,y)$ is the right hand side of the original ode given by

$$\omega(x,y) = (y^2 + 1)\tan(x)$$

Evaluating all the partial derivatives gives

$$
\begin{aligned}
R_x &= 0 \\
R_y &= 1 \\
S_x &= \tan(x) \\
S_y &= 0
\end{aligned}
$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$
\frac{dS}{dR} = \frac{1}{y^2+1} \tag{2A}
$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$
\frac{dS}{dR} = \frac{1}{R^2+1}
$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$
S(R) = \arctan(R) + c_1 \tag{4}
$$

To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$
-\ln(\cos(x)) = \arctan(y) + c_1
$$

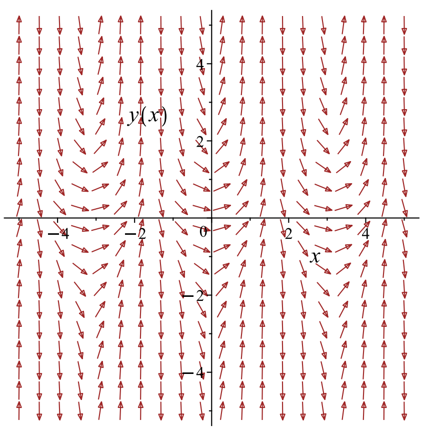
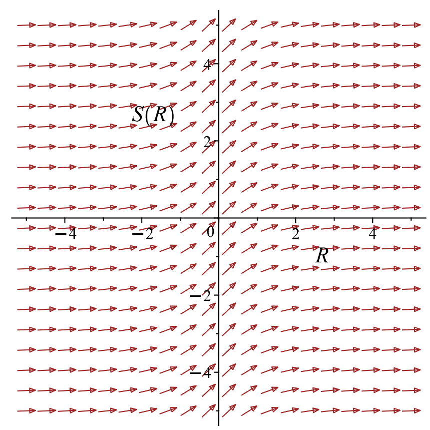
Which simplifies to

$$
-\ln(\cos(x)) = \arctan(y) + c_1
$$

Which gives

$$
y = -\tan(\ln(\cos(x)) + c_1)
$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in x, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dx} = (y^2+1)\tan(x)$ 	$R = y$ $S = -\ln(\cos(x))$	$\frac{dS}{dR} = \frac{1}{R^2+1}$ 

Initial conditions are used to solve for c_1. Substituting $x = 0$ and $y = \sqrt{3}$ in the above solution gives an equation to solve for the constant of integration.

$$\sqrt{3} = -\tan(c_1)$$

$$c_1 = -\frac{\pi}{3}$$

Substituting c_1 found above in the general solution gives

$$y = \frac{-i\sqrt{3}\,\cos(x)^{2i} - i\sqrt{3} - \cos(x)^{2i} + 1}{\sqrt{3}\,\cos(x)^{2i} - i\cos(x)^{2i} - \sqrt{3} - i}$$

Summary

The solution(s) found are the following

$$y = \frac{-i\sqrt{3}\,\cos(x)^{2i} - i\sqrt{3} - \cos(x)^{2i} + 1}{\sqrt{3}\,\cos(x)^{2i} - i\cos(x)^{2i} - \sqrt{3} - i} \tag{1}$$

Verification of solutions

$$y = \frac{-i\sqrt{3}\,\cos(x)^{2i} - i\sqrt{3} - \cos(x)^{2i} + 1}{\sqrt{3}\,\cos(x)^{2i} - i\cos(x)^{2i} - \sqrt{3} - i}$$

Verified OK.

1.17.4 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x,y)+N(x,y)\frac{dy}{dx}=0 \tag{A}$$

We assume there exists a function $\phi(x,y)=c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y)=0$$

Hence

$$\frac{\partial\phi}{\partial x}+\frac{\partial\phi}{\partial y}\frac{dy}{dx}=0 \tag{B}$$

Comparing (A,B) shows that

$$\begin{aligned}\frac{\partial\phi}{\partial x}&=M\\ \frac{\partial\phi}{\partial y}&=N\end{aligned}$$

But since $\frac{\partial^2\phi}{\partial x\partial y}=\frac{\partial^2\phi}{\partial y\partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y}=\frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x,y)$ but at least we know now that we can do that since the condition $\frac{\partial^2\phi}{\partial x\partial y}=\frac{\partial^2\phi}{\partial y\partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x,y)\,\mathrm{d}x+N(x,y)\,\mathrm{d}y=0 \tag{1A}$$

Therefore

$$\begin{aligned}\left(\frac{1}{y^2+1}\right)\mathrm{d}y&=(\tan(x))\,\mathrm{d}x\\ (-\tan(x))\,\mathrm{d}x+\left(\frac{1}{y^2+1}\right)\mathrm{d}y&=0\end{aligned} \tag{2A}$$

Comparing (1A) and (2A) shows that

$$\begin{aligned} M(x,y) &= -\tan(x) \\ N(x,y) &= \frac{1}{y^2+1} \end{aligned}$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Using result found above gives

$$\begin{aligned} \frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}(-\tan(x)) \\ &= 0 \end{aligned}$$

And

$$\begin{aligned} \frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}\left(\frac{1}{y^2+1}\right) \\ &= 0 \end{aligned}$$

Since $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$, then the ODE is exact The following equations are now set up to solve for the function $\phi(x,y)$

$$\frac{\partial \phi}{\partial x} = M \tag{1}$$

$$\frac{\partial \phi}{\partial y} = N \tag{2}$$

Integrating (1) w.r.t. x gives

$$\begin{aligned} \int \frac{\partial \phi}{\partial x} \, \mathrm{d}x &= \int M \, \mathrm{d}x \\ \int \frac{\partial \phi}{\partial x} \, \mathrm{d}x &= \int -\tan(x) \, \mathrm{d}x \\ \phi &= \ln(\cos(x)) + f(y) \end{aligned} \tag{3}$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$\frac{\partial \phi}{\partial y} = 0 + f'(y) \tag{4}$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = \frac{1}{y^2+1}$. Therefore equation (4) becomes

$$\frac{1}{y^2+1} = 0 + f'(y) \tag{5}$$

Solving equation (5) for $f'(y)$ gives

$$f'(y) = \frac{1}{y^2+1}$$

Integrating the above w.r.t y gives

$$\int f'(y)\,\mathrm{d}y = \int \left(\frac{1}{y^2+1}\right) \mathrm{d}y$$
$$f(y) = \arctan(y) + c_1$$

Where c_1 is constant of integration. Substituting result found above for $f(y)$ into equation (3) gives ϕ

$$\phi = \ln(\cos(x)) + \arctan(y) + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = \ln(\cos(x)) + \arctan(y)$$

The solution becomes

$$y = \tan(-\ln(\cos(x)) + c_1)$$

Initial conditions are used to solve for c_1. Substituting $x = 0$ and $y = \sqrt{3}$ in the above solution gives an equation to solve for the constant of integration.

$$\sqrt{3} = \tan(c_1)$$

$$c_1 = \frac{\pi}{3}$$

Substituting c_1 found above in the general solution gives

$$y = \frac{-i\sqrt{3}\,\cos(x)^{2i} - i\sqrt{3} - \cos(x)^{2i} + 1}{\sqrt{3}\,\cos(x)^{2i} - i\cos(x)^{2i} - \sqrt{3} - i}$$

Summary

The solution(s) found are the following

$$y = \frac{-i\sqrt{3}\, \cos(x)^{2i} - i\sqrt{3} - \cos(x)^{2i} + 1}{\sqrt{3}\, \cos(x)^{2i} - i\cos(x)^{2i} - \sqrt{3} - i} \tag{1}$$

Verification of solutions

$$y = \frac{-i\sqrt{3}\, \cos(x)^{2i} - i\sqrt{3} - \cos(x)^{2i} + 1}{\sqrt{3}\, \cos(x)^{2i} - i\cos(x)^{2i} - \sqrt{3} - i}$$

Verified OK.

1.17.5 Solving as riccati ode

In canonical form the ODE is

$$\begin{aligned} y' &= F(x, y) \\ &= (y^2 + 1) \tan(x) \end{aligned}$$

This is a Riccati ODE. Comparing the ODE to solve

$$y' = \tan(x)\, y^2 + \tan(x)$$

With Riccati ODE standard form

$$y' = f_0(x) + f_1(x)y + f_2(x)y^2$$

Shows that $f_0(x) = \tan(x)$, $f_1(x) = 0$ and $f_2(x) = \tan(x)$. Let

$$\begin{aligned} y &= \frac{-u'}{f_2 u} \\ &= \frac{-u'}{\tan(x)\, u} \end{aligned} \tag{1}$$

Using the above substitution in the given ODE results (after some simplification)in a second order ODE to solve for $u(x)$ which is

$$f_2 u''(x) - (f_2' + f_1 f_2)\, u'(x) + f_2^2 f_0 u(x) = 0 \tag{2}$$

But

$$\begin{aligned} f_2' &= 1 + \tan(x)^2 \\ f_1 f_2 &= 0 \\ f_2^2 f_0 &= \tan(x)^3 \end{aligned}$$

Substituting the above terms back in equation (2) gives

$$\tan(x) u''(x) - \left(1+\tan(x)^2\right) u'(x) + \tan(x)^3 u(x) = 0$$

Solving the above ODE (this ode solved using Maple, not this program), gives

$$u(x) = c_1 \cos(x)^{-i} + c_2 \cos(x)^{i}$$

The above shows that

$$u'(x) = i\left(c_1 \cos(x)^{-i} - c_2 \cos(x)^{i}\right) \tan(x)$$

Using the above in (1) gives the solution

$$y = -\frac{i\left(c_1 \cos(x)^{-i} - c_2 \cos(x)^{i}\right)}{c_1 \cos(x)^{-i} + c_2 \cos(x)^{i}}$$

Dividing both numerator and denominator by c_1 gives, after renaming the constant $\frac{c_2}{c_1} = c_3$ the following solution

$$y = \frac{i\left(-c_3 + \cos(x)^{2i}\right)}{\cos(x)^{2i} + c_3}$$

Initial conditions are used to solve for c_3. Substituting $x = 0$ and $y = \sqrt{3}$ in the above solution gives an equation to solve for the constant of integration.

$$\sqrt{3} = \frac{-c_3 i + i}{c_3 + 1}$$

$$c_3 = -\frac{-i + \sqrt{3}}{\sqrt{3} + i}$$

Substituting c_3 found above in the general solution gives

$$y = \frac{i\sqrt{3}\,\cos(x)^{2i} - \cos(x)^{2i} + i\sqrt{3} + 1}{\sqrt{3}\,\cos(x)^{2i} + i\cos(x)^{2i} - \sqrt{3} + i}$$

Summary
The solution(s) found are the following

$$y = \frac{i\sqrt{3}\,\cos(x)^{2i} - \cos(x)^{2i} + i\sqrt{3} + 1}{\sqrt{3}\,\cos(x)^{2i} + i\cos(x)^{2i} - \sqrt{3} + i} \tag{1}$$

Verification of solutions

$$y = \frac{i\sqrt{3}\,\cos(x)^{2i} - \cos(x)^{2i} + i\sqrt{3} + 1}{\sqrt{3}\,\cos(x)^{2i} + i\cos(x)^{2i} - \sqrt{3} + i}$$

Verified OK.

1.17.6 Maple step by step solution

Let's solve

$\left[y' - (1+y^2)\tan(x) = 0, y(0) = \sqrt{3}\right]$

- Highest derivative means the order of the ODE is 1

  y'

- Separate variables

  $\frac{y'}{1+y^2} = \tan(x)$

- Integrate both sides with respect to x

  $\int \frac{y'}{1+y^2} dx = \int \tan(x)\, dx + c_1$

- Evaluate integral

  $\arctan(y) = -\ln(\cos(x)) + c_1$

- Solve for y

  $y = \tan(-\ln(\cos(x)) + c_1)$

- Use initial condition $y(0) = \sqrt{3}$

  $\sqrt{3} = \tan(c_1)$

- Solve for c_1

  $c_1 = \frac{\pi}{3}$

- Substitute $c_1 = \frac{\pi}{3}$ into general solution and simplify

  $y = \cot\left(\ln(\cos(x)) + \frac{\pi}{6}\right)$

- Solution to the IVP

  $y = \cot\left(\ln(\cos(x)) + \frac{\pi}{6}\right)$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
<- separable successful`
```

✓ Solution by Maple

Time used: 0.079 (sec). Leaf size: 12

```
dsolve([diff(y(x),x)=(1+y(x)^2)*tan(x),y(0) = sqrt(3)],y(x), singsol=all)
```

$$y(x) = \cot\left(\frac{\pi}{6} + \ln\left(\cos\left(x\right)\right)\right)$$

✓ Solution by Mathematica

Time used: 0.255 (sec). Leaf size: 15

```
DSolve[{y'[x]==(1+y[x]^2)*Tan[x],{y[0]==Sqrt[3]}},y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \cot\left(\log(\cos(x)) + \frac{\pi}{6}\right)$$

1.18 problem 18

Internal problem ID [4929]
Internal file name [`OUTPUT/4422_Sunday_June_05_2022_01_18_42_PM_29759197/index.tex`]

Book: Fundamentals of Differential Equations. By Nagle, Saff and Snider. 9th edition. Boston. Pearson 2018.
Section: Chapter 2, First order differential equations. Section 2.2, Separable Equations. Exercises. page 46
Problem number: 18.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"exact", "linear", "separable", "first_order_ode_lie_symmetry_lookup"**

Maple gives the following as the ode type

`[_separable]`

$$\boxed{y' - x^3(1 - y) = 0}$$

With initial conditions

$$[y(0) = 3]$$

1.18.1 Existence and uniqueness analysis

This is a linear ODE. In canonical form it is written as

$$y' + p(x)y = q(x)$$

Where here

$$\begin{aligned} p(x) &= x^3 \\ q(x) &= x^3 \end{aligned}$$

Hence the ode is

$$y' + yx^3 = x^3$$

The domain of $p(x) = x^3$ is

$$\{-\infty < x < \infty\}$$

And the point $x_0 = 0$ is inside this domain. The domain of $q(x) = x^3$ is

$$\{-\infty < x < \infty\}$$

And the point $x_0 = 0$ is also inside this domain. Hence solution exists and is unique.

1.18.2 Solving as separable ode

In canonical form the ODE is

$$\begin{aligned} y' &= F(x, y) \\ &= f(x)g(y) \\ &= x^3(1 - y) \end{aligned}$$

Where $f(x) = x^3$ and $g(y) = 1 - y$. Integrating both sides gives

$$\begin{aligned} \frac{1}{1-y}\, dy &= x^3\, dx \\ \int \frac{1}{1-y}\, dy &= \int x^3\, dx \\ -\ln(y-1) &= \frac{x^4}{4} + c_1 \end{aligned}$$

Raising both side to exponential gives

$$\frac{1}{y-1} = \mathrm{e}^{\frac{x^4}{4}+c_1}$$

Which simplifies to

$$\frac{1}{y-1} = c_2 \mathrm{e}^{\frac{x^4}{4}}$$

Initial conditions are used to solve for c_1. Substituting $x = 0$ and $y = 3$ in the above solution gives an equation to solve for the constant of integration.

$$3 = \frac{\mathrm{e}^{-c_1}\mathrm{e}^{c_1}c_2 + \mathrm{e}^{-c_1}}{c_2}$$

$$c_1 = -\ln(2c_2)$$

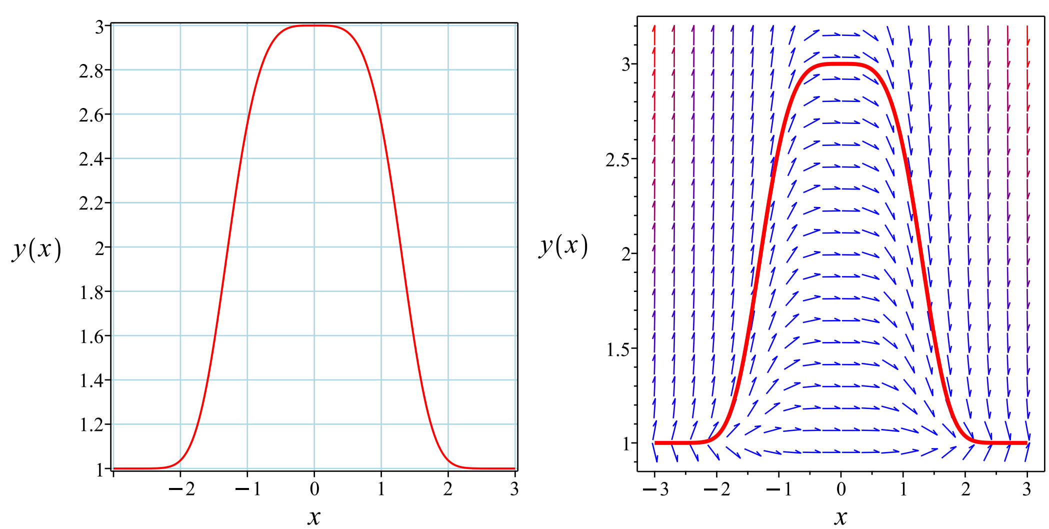
Substituting c_1 found above in the general solution gives

$$y = 1 + 2\,\mathrm{e}^{-\frac{x^4}{4}}$$

<u>Summary</u>

The solution(s) found are the following

$$y = 1 + 2\,\mathrm{e}^{-\frac{x^4}{4}} \tag{1}$$



(a) Solution plot

(b) Slope field plot

<u>Verification of solutions</u>

$$y = 1 + 2\,\mathrm{e}^{-\frac{x^4}{4}}$$

Verified OK.

1.18.3 Solving as linear ode

Entering Linear first order ODE solver. The integrating factor μ is

$$\mu = e^{\int x^3 dx} \\ = e^{\frac{x^4}{4}}$$

The ode becomes

$$\frac{d}{dx}(\mu y) = (\mu)\left(x^3\right) \\ \frac{d}{dx}\left(e^{\frac{x^4}{4}} y\right) = \left(e^{\frac{x^4}{4}}\right)\left(x^3\right) \\ d\left(e^{\frac{x^4}{4}} y\right) = \left(x^3 e^{\frac{x^4}{4}}\right) dx$$

Integrating gives

$$e^{\frac{x^4}{4}} y = \int x^3 e^{\frac{x^4}{4}} dx \\ e^{\frac{x^4}{4}} y = e^{\frac{x^4}{4}} + c_1$$

Dividing both sides by the integrating factor $\mu = e^{\frac{x^4}{4}}$ results in

$$y = e^{-\frac{x^4}{4}} e^{\frac{x^4}{4}} + c_1 e^{-\frac{x^4}{4}}$$

which simplifies to

$$y = 1 + c_1 e^{-\frac{x^4}{4}}$$

Initial conditions are used to solve for c_1. Substituting $x = 0$ and $y = 3$ in the above solution gives an equation to solve for the constant of integration.

$$3 = c_1 + 1$$

$$c_1 = 2$$

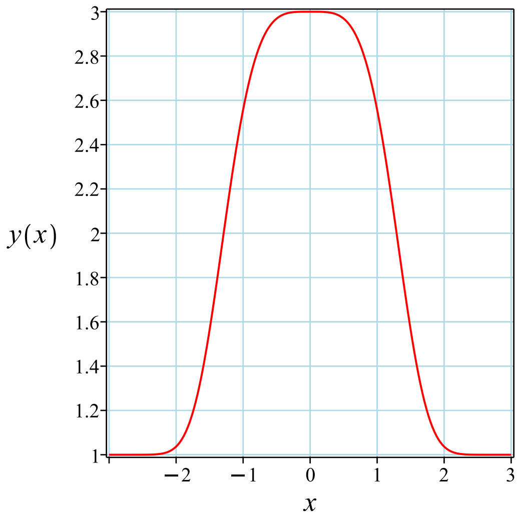
Substituting c_1 found above in the general solution gives

$$y = 1 + 2\,e^{-\frac{x^4}{4}}$$

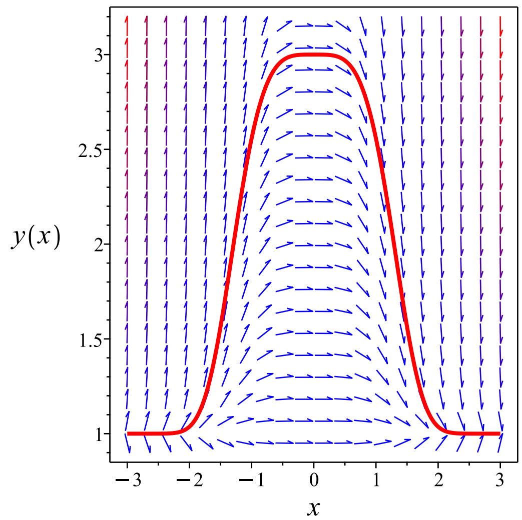
Summary

The solution(s) found are the following

$$y = 1 + 2\,e^{-\frac{x^4}{4}} \tag{1}$$



(a) Solution plot



(b) Slope field plot

Verification of solutions

$$y = 1 + 2\,\mathrm{e}^{-\frac{x^4}{4}}$$

Verified OK.

1.18.4 Solving as first order ode lie symmetry lookup ode

Writing the ode as

$$y' = -x^3(y-1)$$
$$y' = \omega(x, y)$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x + \omega(\eta_y - \xi_x) - \omega^2 \xi_y - \omega_x \xi - \omega_y \eta = 0 \tag{A}$$

The type of this ode is known. It is of type `linear`. Therefore we do not need to solve the PDE (A), and can just use the lookup table shown below to find ξ, η

Table 39: Lie symmetry infinitesimal lookup table for known first order ODE's

ODE class	Form	ξ	η
linear ode	$y' = f(x)y(x) + g(x)$	0	$e^{\int f dx}$
separable ode	$y' = f(x)\, g(y)$	$\frac{1}{f}$	0
quadrature ode	$y' = f(x)$	0	1
quadrature ode	$y' = g(y)$	1	0
homogeneous ODEs of Class A	$y' = f\left(\frac{y}{x}\right)$	x	y
homogeneous ODEs of Class C	$y' = (a + bx + cy)^{\frac{n}{m}}$	1	$-\frac{b}{c}$
homogeneous class D	$y' = \frac{y}{x} + g(x)\, F\left(\frac{y}{x}\right)$	x^2	xy
First order special form ID 1	$y' = g(x)\, e^{h(x)+by} + f(x)$	$\frac{e^{-\int bf(x)dx - h(x)}}{g(x)}$	$\frac{f(x)e^{-\int bf(x)dx - h(x)}}{g(x)}$
polynomial type ode	$y' = \frac{a_1x+b_1y+c_1}{a_2x+b_2y+c_2}$	$\frac{a_1b_2x - a_2b_1x - b_1c_2 + b_2c_1}{a_1b_2 - a_2b_1}$	$\frac{a_1b_2y - a_2b_1y - a_1c_2 - a_2c_1}{a_1b_2 - a_2b_1}$
Bernoulli ode	$y' = f(x)\, y + g(x)\, y^n$	0	$e^{-\int (n-1)f(x)dx} y^n$
Reduced Riccati	$y' = f_1(x)\, y + f_2(x)\, y^2$	0	$e^{-\int f_1 dx}$

The above table shows that

$$
\begin{aligned}
\xi(x, y) &= 0 \\
\eta(x, y) &= \mathrm{e}^{-\frac{x^4}{4}}
\end{aligned} \tag{A1}
$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x, y) \to (R, S)$ where (R, S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$
\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}
$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y}\right) S(x, y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the

canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$R = x$$

S is found from

$$\begin{aligned} S &= \int \frac{1}{\eta} dy \\ &= \int \frac{1}{\mathrm{e}^{-\frac{x^4}{4}}} dy \end{aligned}$$

Which results in

$$S = \mathrm{e}^{\frac{x^4}{4}} y$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x, y) S_y}{R_x + \omega(x, y) R_y} \tag{2}$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x, y)$ is the right hand side of the original ode given by

$$\omega(x, y) = -x^3(y - 1)$$

Evaluating all the partial derivatives gives

$$\begin{aligned} R_x &= 1 \\ R_y &= 0 \\ S_x &= x^3 \mathrm{e}^{\frac{x^4}{4}} y \\ S_y &= \mathrm{e}^{\frac{x^4}{4}} \end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = x^3 \mathrm{e}^{\frac{x^4}{4}} \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = R^3 \mathrm{e}^{\frac{R^4}{4}}$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by

integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$S(R) = \mathrm{e}^{\frac{R^4}{4}} + c_1 \tag{4}$$

To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$\mathrm{e}^{\frac{x^4}{4}} y = \mathrm{e}^{\frac{x^4}{4}} + c_1$$

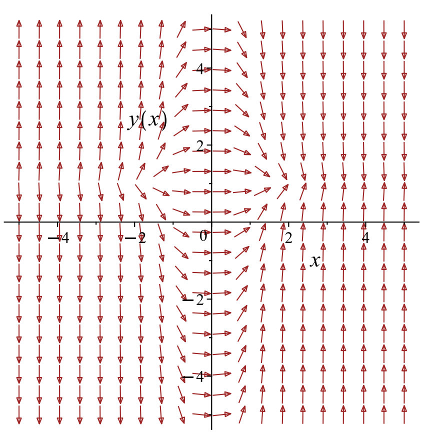
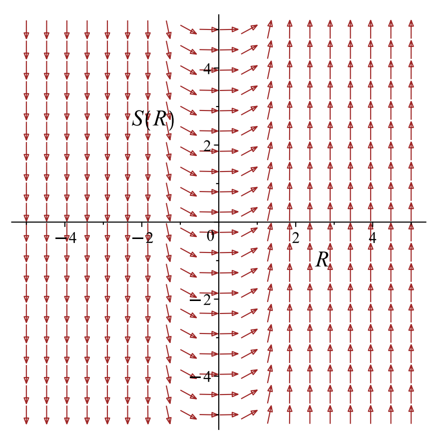
Which simplifies to

$$(y-1)\,\mathrm{e}^{\frac{x^4}{4}} - c_1 = 0$$

Which gives

$$y = \left(\mathrm{e}^{\frac{x^4}{4}} + c_1\right) \mathrm{e}^{-\frac{x^4}{4}}$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in x, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dx} = -x^3(y-1)$ 	$R = x$ $S = \mathrm{e}^{\frac{x^4}{4}} y$	$\frac{dS}{dR} = R^3 \mathrm{e}^{\frac{R^4}{4}}$ 

Initial conditions are used to solve for c_1. Substituting $x = 0$ and $y = 3$ in the above solution gives an equation to solve for the constant of integration.

$$3 = c_1 + 1$$

$$c_1 = 2$$

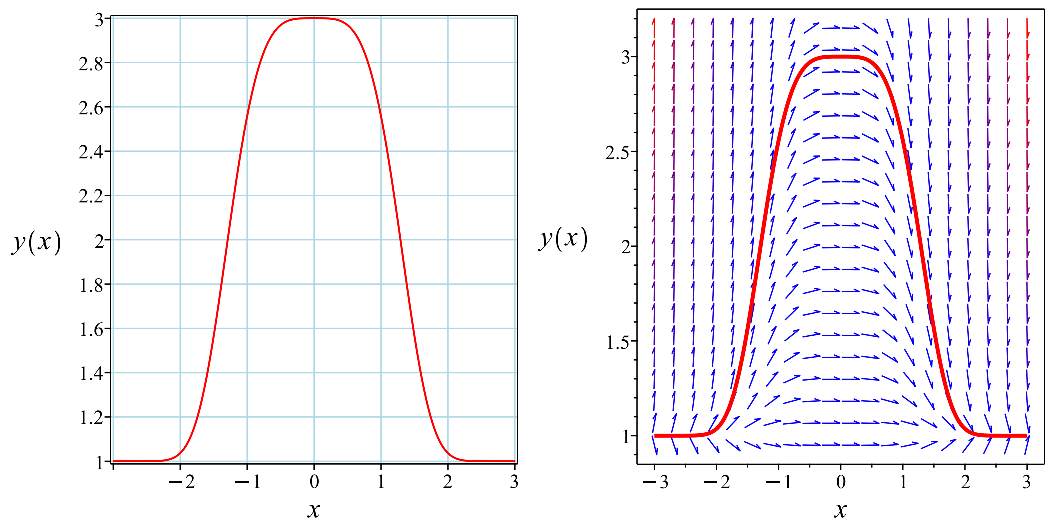
Substituting c_1 found above in the general solution gives

$$y = 1 + 2\,\mathrm{e}^{-\frac{x^4}{4}}$$

Summary

The solution(s) found are the following

$$y = 1 + 2\,\mathrm{e}^{-\frac{x^4}{4}} \tag{1}$$



(a) Solution plot (b) Slope field plot

Verification of solutions

$$y = 1 + 2\,\mathrm{e}^{-\frac{x^4}{4}}$$

Verified OK.

1.18.5 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x,y) + N(x,y)\frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x,y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y) = 0$$

Hence

$$\frac{\partial \phi}{\partial x}+\frac{\partial \phi}{\partial y}\frac{dy}{dx}=0 \tag{B}$$

Comparing (A,B) shows that

$$\begin{aligned}\frac{\partial \phi}{\partial x}&=M\\ \frac{\partial \phi}{\partial y}&=N\end{aligned}$$

But since $\frac{\partial^2 \phi}{\partial x \partial y}=\frac{\partial^2 \phi}{\partial y \partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y}=\frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x,y)$ but at least we know now that we can do that since the condition $\frac{\partial^2 \phi}{\partial x \partial y}=\frac{\partial^2 \phi}{\partial y \partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x,y)\,\mathrm{d}x+N(x,y)\,\mathrm{d}y=0 \tag{1A}$$

Therefore

$$\begin{aligned}\left(\frac{1}{1-y}\right)\mathrm{d}y&=\left(x^3\right)\mathrm{d}x\\ \left(-x^3\right)\mathrm{d}x+\left(\frac{1}{1-y}\right)\mathrm{d}y&=0 \end{aligned} \tag{2A}$$

Comparing (1A) and (2A) shows that

$$\begin{aligned}M(x,y)&=-x^3\\ N(x,y)&=\frac{1}{1-y}\end{aligned}$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y}=\frac{\partial N}{\partial x}$$

Using result found above gives

$$\begin{aligned}\frac{\partial M}{\partial y}&=\frac{\partial}{\partial y}\left(-x^3\right)\\ &=0\end{aligned}$$

And

$$\begin{aligned}\frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}\left(\frac{1}{1-y}\right)\\ &= 0\end{aligned}$$

Since $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$, then the ODE is <u>exact</u> The following equations are now set up to solve for the function $\phi(x, y)$

$$\frac{\partial \phi}{\partial x} = M \tag{1}$$

$$\frac{\partial \phi}{\partial y} = N \tag{2}$$

Integrating (1) w.r.t. x gives

$$\begin{aligned}\int \frac{\partial \phi}{\partial x}\, \mathrm{d}x &= \int M\, \mathrm{d}x\\ \int \frac{\partial \phi}{\partial x}\, \mathrm{d}x &= \int -x^3\, \mathrm{d}x\end{aligned}$$

$$\phi = -\frac{x^4}{4} + f(y) \tag{3}$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$\frac{\partial \phi}{\partial y} = 0 + f'(y) \tag{4}$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = \frac{1}{1-y}$. Therefore equation (4) becomes

$$\frac{1}{1-y} = 0 + f'(y) \tag{5}$$

Solving equation (5) for $f'(y)$ gives

$$f'(y) = -\frac{1}{y-1}$$

Integrating the above w.r.t y gives

$$\begin{aligned}\int f'(y)\, \mathrm{d}y &= \int \left(-\frac{1}{y-1}\right) \mathrm{d}y\\ f(y) &= -\ln(y-1) + c_1\end{aligned}$$

Where c_1 is constant of integration. Substituting result found above for $f(y)$ into equation (3) gives ϕ

$$\phi = -\frac{x^4}{4} - \ln(y-1) + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = -\frac{x^4}{4} - \ln(y-1)$$

The solution becomes

$$y = e^{-\frac{x^4}{4} - c_1} + 1$$

Initial conditions are used to solve for c_1. Substituting $x = 0$ and $y = 3$ in the above solution gives an equation to solve for the constant of integration.

$$3 = e^{-c_1} + 1$$

$$c_1 = -\ln(2)$$

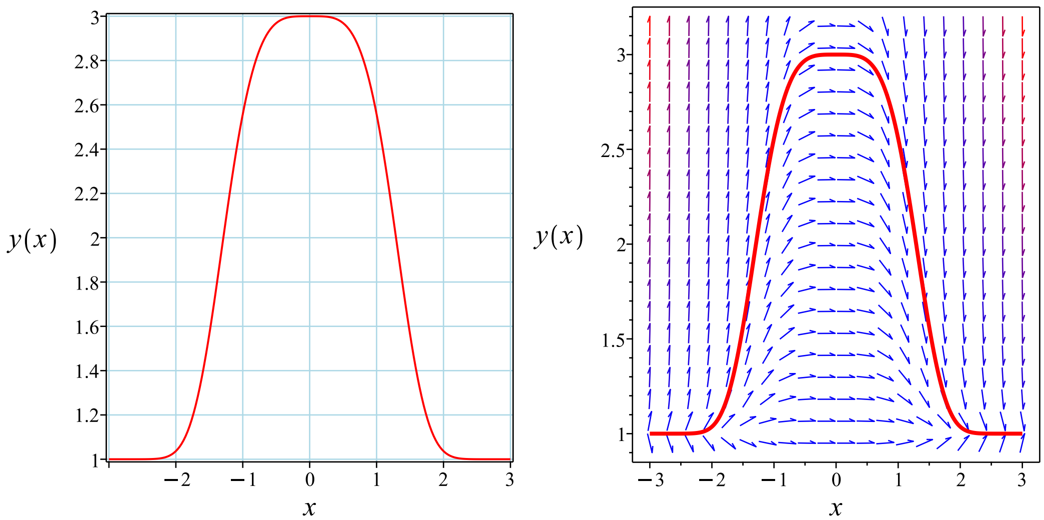
Substituting c_1 found above in the general solution gives

$$y = 1 + 2\,e^{-\frac{x^4}{4}}$$

Summary

The solution(s) found are the following

$$y = 1 + 2\,e^{-\frac{x^4}{4}} \tag{1}$$



(a) Solution plot (b) Slope field plot

<u>Verification of solutions</u>

$$y = 1 + 2\,e^{-\frac{x^4}{4}}$$

Verified OK.

1.18.6 Maple step by step solution

Let's solve

$[y' - x^3(1-y) = 0, y(0) = 3]$

- Highest derivative means the order of the ODE is 1

  y'

- Separate variables

  $\frac{y'}{1-y} = x^3$

- Integrate both sides with respect to x

  $\int \frac{y'}{1-y}dx = \int x^3 dx + c_1$

- Evaluate integral

  $-\ln(1-y) = \frac{x^4}{4} + c_1$

- Solve for y

  $y = -e^{-\frac{x^4}{4}-c_1} + 1$

- Use initial condition $y(0) = 3$

  $3 = -e^{-c_1} + 1$

- Solve for c_1

  $c_1 = -\ln(2) - I\pi$

- Substitute $c_1 = -\ln(2) - I\pi$ into general solution and simplify

  $y = 1 + 2\,e^{-\frac{x^4}{4}}$

- Solution to the IVP

  $y = 1 + 2\,e^{-\frac{x^4}{4}}$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
<- 1st order linear successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 14

```
dsolve([diff(y(x),x)=x^3*(1-y(x)),y(0) = 3],y(x), singsol=all)
```

$$y(x) = 1 + 2\,e^{-\frac{x^4}{4}}$$

✓ Solution by Mathematica

Time used: 0.044 (sec). Leaf size: 18

```
DSolve[{y'[x]==x^3*(1-y[x]),{y[0]==3}},y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to 2e^{-\frac{x^4}{4}} + 1$$

1.19 problem 19

Internal problem ID [4930]
Internal file name [`OUTPUT/4423_Sunday_June_05_2022_01_18_53_PM_20018817/index.tex`]

Book: Fundamentals of Differential Equations. By Nagle, Saff and Snider. 9th edition. Boston. Pearson 2018.
Section: Chapter 2, First order differential equations. Section 2.2, Separable Equations. Exercises. page 46
Problem number: 19.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"exact", "separable", "first_order_ode_lie_symmetry_lookup"**

Maple gives the following as the ode type

`[_separable]`

$$\boxed{\frac{y'}{2} - \sqrt{1+y}\,\cos(x) = 0}$$

With initial conditions

$$[y(\pi) = 0]$$

1.19.1 Existence and uniqueness analysis

This is non linear first order ODE. In canonical form it is written as

$$\begin{aligned} y' &= f(x,y) \\ &= 2\sqrt{1+y}\,\cos(x) \end{aligned}$$

The x domain of $f(x,y)$ when $y=0$ is

$$\{-\infty < x < \infty\}$$

And the point $x_0 = \pi$ is inside this domain. The y domain of $f(x, y)$ when $x = \pi$ is

$$\{-1 \leq y\}$$

And the point $y_0 = 0$ is inside this domain. Now we will look at the continuity of

$$\begin{aligned}
\frac{\partial f}{\partial y} &= \frac{\partial}{\partial y}\left(2\sqrt{1+y}\,\cos(x)\right) \\
&= \frac{\cos(x)}{\sqrt{1+y}}
\end{aligned}$$

The x domain of $\frac{\partial f}{\partial y}$ when $y = 0$ is

$$\{-\infty < x < \infty\}$$

And the point $x_0 = \pi$ is inside this domain. The y domain of $\frac{\partial f}{\partial y}$ when $x = \pi$ is

$$\{-1 < y\}$$

And the point $y_0 = 0$ is inside this domain. Therefore solution exists and is unique.

1.19.2 Solving as separable ode

In canonical form the ODE is

$$\begin{aligned}
y' &= F(x, y) \\
&= f(x)g(y) \\
&= 2\sqrt{1+y}\,\cos(x)
\end{aligned}$$

Where $f(x) = 2\cos(x)$ and $g(y) = \sqrt{1+y}$. Integrating both sides gives

$$\begin{aligned}
\frac{1}{\sqrt{1+y}}\,dy &= 2\cos(x)\,dx \\
\int \frac{1}{\sqrt{1+y}}\,dy &= \int 2\cos(x)\,dx \\
2\sqrt{1+y} &= 2\sin(x) + c_1
\end{aligned}$$

The solution is

$$2\sqrt{1+y} - 2\sin(x) - c_1 = 0$$

Initial conditions are used to solve for c_1. Substituting $x = \pi$ and $y = 0$ in the above solution gives an equation to solve for the constant of integration.

$$2 - c_1 = 0$$

$$c_1 = 2$$

Substituting c_1 found above in the general solution gives

$$2\sqrt{1+y} - 2\sin(x) - 2 = 0$$

Summary
The solution(s) found are the following

$$2\sqrt{1+y} - 2\sin(x) - 2 = 0 \tag{1}$$

Verification of solutions

$$2\sqrt{1+y} - 2\sin(x) - 2 = 0$$

Verified OK.

1.19.3 Solving as first order ode lie symmetry lookup ode

Writing the ode as

$$\begin{aligned} y' &= 2\sqrt{1+y}\,\cos(x) \\ y' &= \omega(x,y) \end{aligned}$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x + \omega(\eta_y - \xi_x) - \omega^2\xi_y - \omega_x\xi - \omega_y\eta = 0 \tag{A}$$

The type of this ode is known. It is of type `separable`. Therefore we do not need to solve the PDE (A), and can just use the lookup table shown below to find ξ, η

Table 42: Lie symmetry infinitesimal lookup table for known first order ODE's

ODE class	Form	ξ	η
linear ode	$y' = f(x)y(x) + g(x)$	0	$e^{\int f dx}$
separable ode	$y' = f(x)\, g(y)$	$\frac{1}{f}$	0
quadrature ode	$y' = f(x)$	0	1
quadrature ode	$y' = g(y)$	1	0
homogeneous ODEs of Class A	$y' = f\left(\frac{y}{x}\right)$	x	y
homogeneous ODEs of Class C	$y' = (a + bx + cy)^{\frac{n}{m}}$	1	$-\frac{b}{c}$
homogeneous class D	$y' = \frac{y}{x} + g(x)\, F\left(\frac{y}{x}\right)$	x^2	xy
First order special form ID 1	$y' = g(x)\, e^{h(x)+by} + f(x)$	$\frac{e^{-\int bf(x)dx - h(x)}}{g(x)}$	$\frac{f(x)e^{-\int bf(x)dx - h(x)}}{g(x)}$
polynomial type ode	$y' = \frac{a_1x+b_1y+c_1}{a_2x+b_2y+c_2}$	$\frac{a_1b_2x - a_2b_1x - b_1c_2 + b_2c_1}{a_1b_2 - a_2b_1}$	$\frac{a_1b_2y - a_2b_1y - a_1c_2 - a_2c_1}{a_1b_2 - a_2b_1}$
Bernoulli ode	$y' = f(x)\, y + g(x)\, y^n$	0	$e^{-\int (n-1)f(x)dx} y^n$
Reduced Riccati	$y' = f_1(x)\, y + f_2(x)\, y^2$	0	$e^{-\int f_1 dx}$

The above table shows that

$$
\begin{aligned}
\xi(x, y) &= \frac{1}{2\cos(x)} \\
\eta(x, y) &= 0
\end{aligned} \tag{A1}
$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x, y) \to (R, S)$ where (R, S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$
\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}
$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y}\right) S(x, y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the

canonical coordinates, where $S(R)$. Since $\eta = 0$ then in this special case

$$R = y$$

S is found from

$$\begin{aligned} S &= \int \frac{1}{\xi} dx \\ &= \int \frac{1}{\frac{1}{2\cos(x)}} dx \end{aligned}$$

Which results in

$$S = 2\sin(x)$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x,y) S_y}{R_x + \omega(x,y) R_y} \tag{2}$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x,y)$ is the right hand side of the original ode given by

$$\omega(x,y) = 2\sqrt{1+y}\,\cos(x)$$

Evaluating all the partial derivatives gives

$$\begin{aligned} R_x &= 0 \\ R_y &= 1 \\ S_x &= 2\cos(x) \\ S_y &= 0 \end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = \frac{1}{\sqrt{1+y}} \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = \frac{1}{\sqrt{1+R}}$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by

integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$S(R) = 2\sqrt{1+R} + c_1 \tag{4}$$

To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$2\sin(x) = 2\sqrt{1+y} + c_1$$

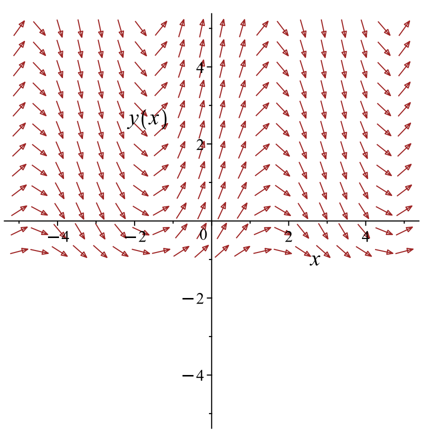
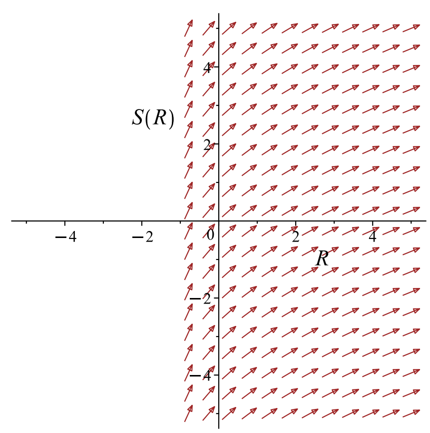
Which simplifies to

$$2\sin(x) = 2\sqrt{1+y} + c_1$$

Which gives

$$y = \frac{c_1^2}{4} - \sin(x)\, c_1 + \sin(x)^2 - 1$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in x, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dx} = 2\sqrt{1+y}\, \cos(x)$ 	$R = y$ $S = 2\sin(x)$	$\frac{dS}{dR} = \frac{1}{\sqrt{1+R}}$ 

Initial conditions are used to solve for c_1. Substituting $x = \pi$ and $y = 0$ in the above solution gives an equation to solve for the constant of integration.

$$0 = \frac{c_1^2}{4} - 1$$

$$
c_1 = -2
$$

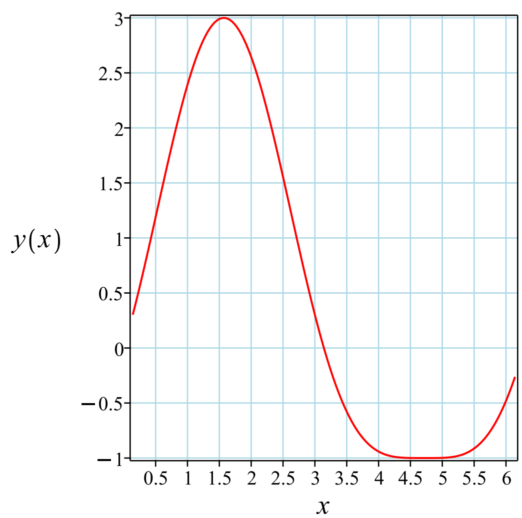
Substituting c_1 found above in the general solution gives

$$
y = \sin(x)^2 + 2\sin(x)
$$

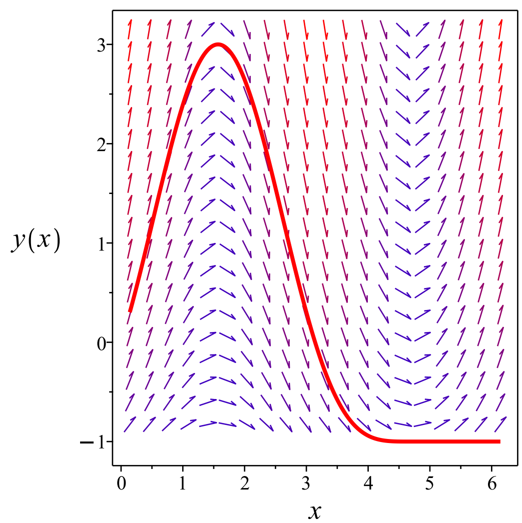
Summary

The solution(s) found are the following

$$
y = \sin(x)^2 + 2\sin(x) \tag{1}
$$



(a) Solution plot



(b) Slope field plot

Verification of solutions

$$
y = \sin(x)^2 + 2\sin(x)
$$

Verified OK.

1.19.4 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$
M(x,y) + N(x,y)\frac{dy}{dx} = 0 \tag{A}
$$

We assume there exists a function $\phi(x,y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$
\frac{d}{dx}\phi(x,y) = 0
$$

Hence

$$\frac{\partial \phi}{\partial x}+\frac{\partial \phi}{\partial y}\frac{dy}{dx}=0 \tag{B}$$

Comparing (A,B) shows that

$$\begin{aligned}\frac{\partial \phi}{\partial x} &= M\\ \frac{\partial \phi}{\partial y} &= N\end{aligned}$$

But since $\frac{\partial^2\phi}{\partial x\partial y}=\frac{\partial^2\phi}{\partial y\partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y}=\frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x,y)$ but at least we know now that we can do that since the condition $\frac{\partial^2\phi}{\partial x\partial y}=\frac{\partial^2\phi}{\partial y\partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x,y)\,\mathrm{d}x+N(x,y)\,\mathrm{d}y=0 \tag{1A}$$

Therefore

$$\begin{aligned}\left(\frac{1}{2\sqrt{1+y}}\right)\mathrm{d}y &= (\cos(x))\,\mathrm{d}x\\ (-\cos(x))\,\mathrm{d}x+\left(\frac{1}{2\sqrt{1+y}}\right)\mathrm{d}y &= 0 \qquad (2A)\end{aligned}$$

Comparing (1A) and (2A) shows that

$$\begin{aligned}M(x,y) &= -\cos(x)\\ N(x,y) &= \frac{1}{2\sqrt{1+y}}\end{aligned}$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y}=\frac{\partial N}{\partial x}$$

Using result found above gives

$$\begin{aligned}\frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}(-\cos(x))\\ &= 0\end{aligned}$$

And

$$
\begin{aligned}
\frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}\left(\frac{1}{2\sqrt{1+y}}\right) \\
&= 0
\end{aligned}
$$

Since $\frac{\partial M}{\partial y}=\frac{\partial N}{\partial x}$, then the ODE is exact The following equations are now set up to solve for the function $\phi(x,y)$

$$
\frac{\partial \phi}{\partial x} = M \tag{1}
$$

$$
\frac{\partial \phi}{\partial y} = N \tag{2}
$$

Integrating (1) w.r.t. x gives

$$
\begin{aligned}
\int \frac{\partial \phi}{\partial x}\,\mathrm{d}x &= \int M\,\mathrm{d}x \\
\int \frac{\partial \phi}{\partial x}\,\mathrm{d}x &= \int -\cos(x)\,\mathrm{d}x
\end{aligned}
$$

$$
\phi = -\sin(x) + f(y) \tag{3}
$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$
\frac{\partial \phi}{\partial y} = 0 + f'(y) \tag{4}
$$

But equation (2) says that $\frac{\partial \phi}{\partial y}=\frac{1}{2\sqrt{1+y}}$. Therefore equation (4) becomes

$$
\frac{1}{2\sqrt{1+y}} = 0 + f'(y) \tag{5}
$$

Solving equation (5) for $f'(y)$ gives

$$
f'(y) = \frac{1}{2\sqrt{1+y}}
$$

Integrating the above w.r.t y gives

$$
\begin{aligned}
\int f'(y)\,\mathrm{d}y &= \int \left(\frac{1}{2\sqrt{1+y}}\right)\mathrm{d}y \\
f(y) &= \sqrt{1+y} + c_1
\end{aligned}
$$

Where c_1 is constant of integration. Substituting result found above for $f(y)$ into equation (3) gives ϕ

$$\phi = \sqrt{1+y} - \sin(x) + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = \sqrt{1+y} - \sin(x)$$

The solution becomes

$$y = c_1^2 + 2\sin(x)\, c_1 + \sin(x)^2 - 1$$

Initial conditions are used to solve for c_1. Substituting $x = \pi$ and $y = 0$ in the above solution gives an equation to solve for the constant of integration.

$$0 = c_1^2 - 1$$

$$c_1 = -1$$

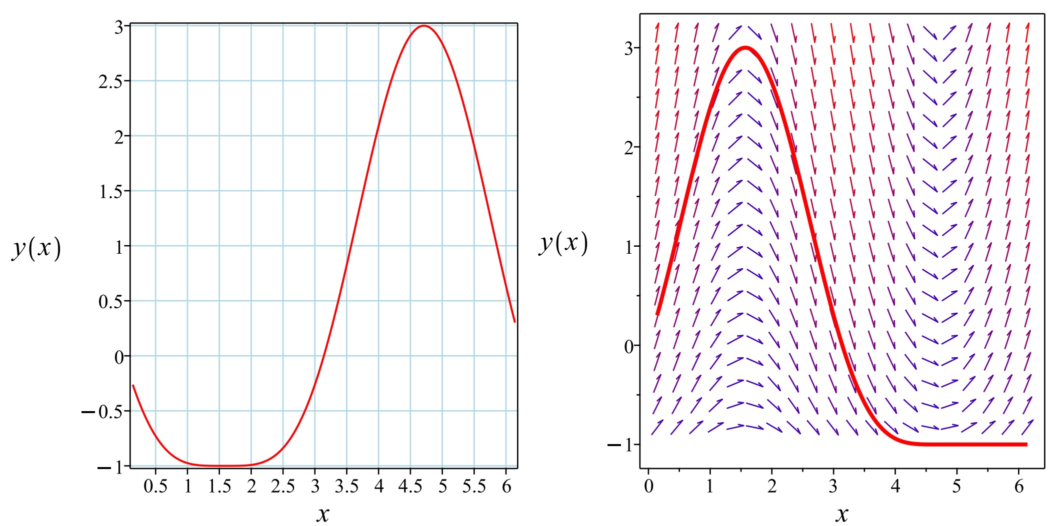
Substituting c_1 found above in the general solution gives

$$y = \sin(x)^2 - 2\sin(x)$$

Summary

The solution(s) found are the following

$$y = \sin(x)^2 - 2\sin(x) \tag{1}$$



(a) Solution plot

(b) Slope field plot

Verification of solutions

$$y = \sin(x)^2 - 2\sin(x)$$

Verified OK.

1.19.5 Maple step by step solution

Let's solve

$\left[\frac{y'}{2} - \sqrt{1+y}\,\cos(x) = 0, y(\pi) = 0\right]$

- Highest derivative means the order of the ODE is 1

  y'

- Separate variables

  $\frac{y'}{\sqrt{1+y}} = 2\cos(x)$

- Integrate both sides with respect to x

  $\int \frac{y'}{\sqrt{1+y}} dx = \int 2\cos(x)\, dx + c_1$

- Evaluate integral

  $2\sqrt{1+y} = 2\sin(x) + c_1$

- Solve for y

  $y = \frac{c_1^2}{4} + \sin(x)\, c_1 + \sin(x)^2 - 1$

- Use initial condition $y(\pi) = 0$

  $0 = \frac{c_1^2}{4} - 1$

- Solve for c_1

  $c_1 = (-2, 2)$

- Substitute $c_1 = (-2, 2)$ into general solution and simplify

  $y = \sin(x)(\sin(x) - 2)$

- Solution to the IVP

  $y = \sin(x)(\sin(x) - 2)$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
<- separable successful`
```

✓ Solution by Maple
Time used: 0.375 (sec). Leaf size: 11

```
dsolve([1/2*diff(y(x),x)=sqrt(1+y(x))*cos(x),y(Pi) = 0],y(x), singsol=all)
```

$$y(x) = \sin(x)(\sin(x) + 2)$$

✓ Solution by Mathematica
Time used: 0.152 (sec). Leaf size: 23

```
DSolve[{1/2*y'[x]==Sqrt[1+y[x]]*Cos[x],{y[Pi]==0}},y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to (\sin(x) - 2)\sin(x)$$
$$y(x) \to \sin(x)(\sin(x) + 2)$$

1.20 problem 20

Internal problem ID [4931]
Internal file name [`OUTPUT/4424_Sunday_June_05_2022_01_19_04_PM_71748754/index.tex`]

Book: Fundamentals of Differential Equations. By Nagle, Saff and Snider. 9th edition. Boston. Pearson 2018.
Section: Chapter 2, First order differential equations. Section 2.2, Separable Equations. Exercises. page 46
Problem number: 20.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"exact", "separable", "first_order_ode_lie_symmetry_lookup"**

Maple gives the following as the ode type

`[_separable]`

$$\boxed{x^2y' - \frac{4x^2 - x - 2}{(x+1)(1+y)} = 0}$$

With initial conditions

$$[y(1) = 1]$$

1.20.1 Existence and uniqueness analysis

This is non linear first order ODE. In canonical form it is written as

$$\begin{aligned} y' &= f(x,y) \\ &= \frac{4x^2 - x - 2}{(x+1)(1+y)x^2} \end{aligned}$$

The x domain of $f(x,y)$ when $y = 1$ is

$$\{-\infty \le x < -1, -1 < x < 0, 0 < x \le \infty\}$$

And the point $x_0 = 1$ is inside this domain. The y domain of $f(x,y)$ when $x = 1$ is

$$\{y < -1 \vee -1 < y\}$$

And the point $y_0 = 1$ is inside this domain. Now we will look at the continuity of

$$\begin{aligned} \frac{\partial f}{\partial y} &= \frac{\partial}{\partial y}\left(\frac{4x^2 - x - 2}{(x+1)(1+y)x^2}\right) \\ &= -\frac{4x^2 - x - 2}{(x+1)(1+y)^2 x^2} \end{aligned}$$

The x domain of $\frac{\partial f}{\partial y}$ when $y = 1$ is

$$\{-\infty \le x < -1, -1 < x < 0, 0 < x \le \infty\}$$

And the point $x_0 = 1$ is inside this domain. The y domain of $\frac{\partial f}{\partial y}$ when $x = 1$ is

$$\{y < -1 \vee -1 < y\}$$

And the point $y_0 = 1$ is inside this domain. Therefore solution exists and is unique.

1.20.2 Solving as separable ode

In canonical form the ODE is

$$\begin{aligned} y' &= F(x,y) \\ &= f(x)g(y) \\ &= \frac{4x^2 - x - 2}{(x+1)(1+y)x^2} \end{aligned}$$

Where $f(x) = \frac{4x^2-x-2}{(x+1)x^2}$ and $g(y) = \frac{1}{1+y}$. Integrating both sides gives

$$\frac{1}{\frac{1}{1+y}}\,dy = \frac{4x^2-x-2}{(x+1)\,x^2}\,dx$$

$$\int \frac{1}{\frac{1}{1+y}}\,dy = \int \frac{4x^2-x-2}{(x+1)\,x^2}\,dx$$

$$y + \frac{1}{2}y^2 = 3\ln(x+1) + \ln(x) + \frac{2}{x} + c_1$$

Which results in

$$y = \frac{-x + \sqrt{6\ln(x+1)\,x^2 + 2\ln(x)\,x^2 + 2c_1x^2 + x^2 + 4x}}{x}$$

$$y = -\frac{x + \sqrt{6\ln(x+1)\,x^2 + 2\ln(x)\,x^2 + 2c_1x^2 + x^2 + 4x}}{x}$$

Initial conditions are used to solve for c_1. Substituting $x = 1$ and $y = 1$ in the above solution gives an equation to solve for the constant of integration.

$$1 = -1 - \sqrt{6\ln(2) + 2c_1 + 5}$$

Warning: Unable to solve for constant of integration. Initial conditions are used to solve for c_1. Substituting $x = 1$ and $y = 1$ in the above solution gives an equation to solve for the constant of integration.

$$1 = -1 + \sqrt{6\ln(2) + 2c_1 + 5}$$

$$c_1 = -3\ln(2) - \frac{1}{2}$$

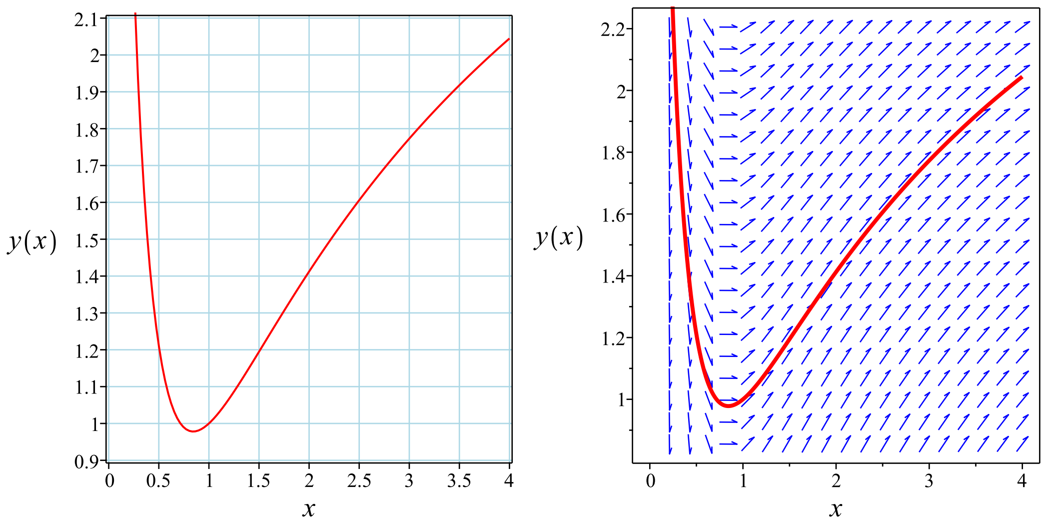
Substituting c_1 found above in the general solution gives

$$y = \frac{-x + \sqrt{2}\,\sqrt{3\ln(x+1)\,x^2 + \ln(x)\,x^2 - 3x^2\ln(2) + 2x}}{x}$$

Summary

The solution(s) found are the following

$$y = \frac{-x + \sqrt{2}\,\sqrt{3\ln(x+1)\,x^2 + \ln(x)\,x^2 - 3x^2\ln(2) + 2x}}{x} \tag{1}$$



(a) Solution plot (b) Slope field plot

Verification of solutions

$$y = \frac{-x + \sqrt{2}\,\sqrt{3\ln\left(x+1\right)x^2 + \ln\left(x\right)x^2 - 3x^2\ln\left(2\right) + 2x}}{x}$$

Verified OK.

1.20.3 Solving as first order ode lie symmetry lookup ode

Writing the ode as

$$y' = \frac{4x^2 - x - 2}{\left(x+1\right)\left(1+y\right)x^2}$$
$$y' = \omega(x, y)$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x + \omega(\eta_y - \xi_x) - \omega^2\xi_y - \omega_x\xi - \omega_y\eta = 0 \tag{A}$$

The type of this ode is known. It is of type `separable`. Therefore we do not need to solve the PDE (A), and can just use the lookup table shown below to find ξ, η

Table 45: Lie symmetry infinitesimal lookup table for known first order ODE's

ODE class	Form	ξ	η
linear ode	$y' = f(x)y(x) + g(x)$	0	$e^{\int f dx}$
separable ode	$y' = f(x)\, g(y)$	$\frac{1}{f}$	0
quadrature ode	$y' = f(x)$	0	1
quadrature ode	$y' = g(y)$	1	0
homogeneous ODEs of Class A	$y' = f\left(\frac{y}{x}\right)$	x	y
homogeneous ODEs of Class C	$y' = (a + bx + cy)^{\frac{n}{m}}$	1	$-\frac{b}{c}$
homogeneous class D	$y' = \frac{y}{x} + g(x)\, F\left(\frac{y}{x}\right)$	x^2	xy
First order special form ID 1	$y' = g(x)\, e^{h(x)+by} + f(x)$	$\frac{e^{-\int bf(x)dx - h(x)}}{g(x)}$	$\frac{f(x)e^{-\int bf(x)dx - h(x)}}{g(x)}$
polynomial type ode	$y' = \frac{a_1x+b_1y+c_1}{a_2x+b_2y+c_2}$	$\frac{a_1b_2x - a_2b_1x - b_1c_2 + b_2c_1}{a_1b_2 - a_2b_1}$	$\frac{a_1b_2y - a_2b_1y - a_1c_2 - a_2c_1}{a_1b_2 - a_2b_1}$
Bernoulli ode	$y' = f(x)\, y + g(x)\, y^n$	0	$e^{-\int (n-1)f(x)dx} y^n$
Reduced Riccati	$y' = f_1(x)\, y + f_2(x)\, y^2$	0	$e^{-\int f_1 dx}$

The above table shows that

$$
\begin{aligned}
\xi(x,y) &= \frac{(x+1)\,x^2}{4x^2 - x - 2} \\
\eta(x,y) &= 0
\end{aligned} \tag{A1}
$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x,y) \to (R,S)$ where (R,S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$
\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}
$$

The above comes from the requirements that $\left(\xi\frac{\partial}{\partial x} + \eta\frac{\partial}{\partial y}\right) S(x,y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the

canonical coordinates, where $S(R)$. Since $\eta = 0$ then in this special case

$$R = y$$

S is found from

$$\begin{aligned} S &= \int \frac{1}{\xi} dx \\ &= \int \frac{1}{\frac{(x+1)x^2}{4x^2-x-2}} dx \end{aligned}$$

Which results in

$$S = 3\ln(x+1) + \ln(x) + \frac{2}{x}$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x,y)S_y}{R_x + \omega(x,y)R_y} \tag{2}$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x,y)$ is the right hand side of the original ode given by

$$\omega(x,y) = \frac{4x^2 - x - 2}{(x+1)(1+y)x^2}$$

Evaluating all the partial derivatives gives

$$\begin{aligned} R_x &= 0 \\ R_y &= 1 \\ S_x &= \frac{4x^2 - x - 2}{(x+1)x^2} \\ S_y &= 0 \end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = 1 + y \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = 1 + R$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordinates R, S. Integrating the above gives

$$S(R) = \frac{1}{2}R^2 + R + c_1 \tag{4}$$

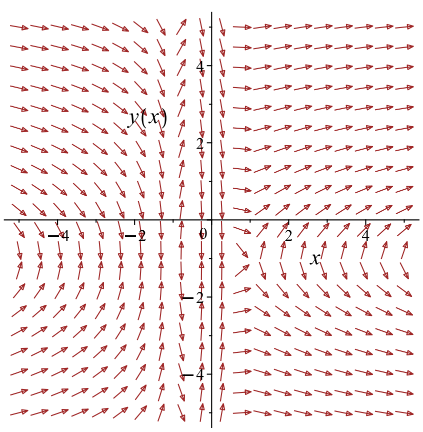
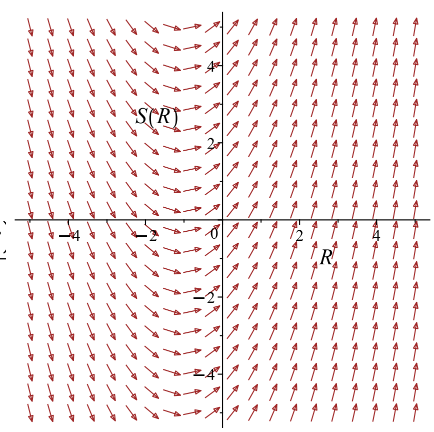
To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$\frac{3\ln(x+1)x + \ln(x)x + 2}{x} = \frac{y^2}{2} + y + c_1$$

Which simplifies to

$$\frac{3\ln(x+1)x + \ln(x)x + 2}{x} = \frac{y^2}{2} + y + c_1$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

<table>
<tr><th>Original ode in x, y coordinates</th><th>Canonical coordinates transformation</th><th>ODE in canonical coordinates (R, S)</th></tr>
<tr><td>$\frac{dy}{dx} = \frac{4x^2-x-2}{(x+1)(1+y)x^2}$

</td><td>$R = y$
$S = \frac{3\ln(x+1)x + \ln(x}{x}$</td><td>$\frac{dS}{dR} = 1 + R$

</td></tr>
</table>

Initial conditions are used to solve for c_1. Substituting $x = 1$ and $y = 1$ in the above solution gives an equation to solve for the constant of integration.

$$3\ln(2) + 2 = \frac{3}{2} + c_1$$

$$c_1 = 3\ln(2) + \frac{1}{2}$$

Substituting c_1 found above in the general solution gives

$$\frac{3\ln(x+1)x + \ln(x)x + 2}{x} = \frac{y^2}{2} + y + 3\ln(2) + \frac{1}{2}$$

The above simplifies to

$$-y^2x + 6\ln(x+1)x + 2\ln(x)x - 6\ln(2)x - 2xy - x + 4 = 0$$

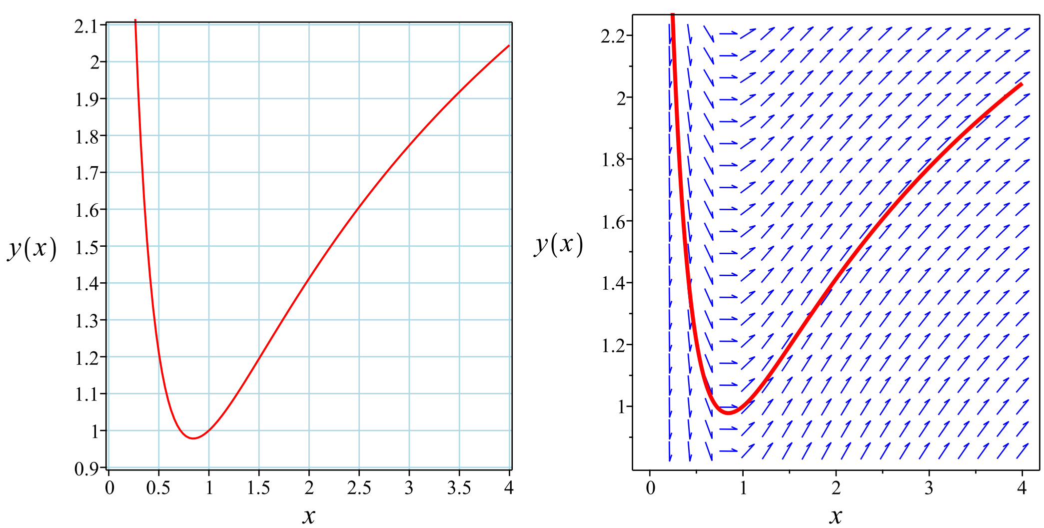
Solving for y from the above gives

$$y = \frac{-x + \sqrt{2}\sqrt{x(2 + 3\ln(x+1)x + \ln(x)x - 3\ln(2)x)}}{x}$$

Summary

The solution(s) found are the following

$$y = \frac{-x + \sqrt{2}\sqrt{x(2 + 3\ln(x+1)x + \ln(x)x - 3\ln(2)x)}}{x} \tag{1}$$



(a) Solution plot

(b) Slope field plot

Verification of solutions

$$y = \frac{-x + \sqrt{2}\sqrt{x(2 + 3\ln(x+1)x + \ln(x)x - 3\ln(2)x)}}{x}$$

Verified OK.

1.20.4 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x,y)+N(x,y)\frac{dy}{dx}=0 \tag{A}$$

We assume there exists a function $\phi(x,y)=c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y)=0$$

Hence

$$\frac{\partial\phi}{\partial x}+\frac{\partial\phi}{\partial y}\frac{dy}{dx}=0 \tag{B}$$

Comparing (A,B) shows that

$$\frac{\partial\phi}{\partial x}=M$$
$$\frac{\partial\phi}{\partial y}=N$$

But since $\frac{\partial^2\phi}{\partial x\partial y}=\frac{\partial^2\phi}{\partial y\partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y}=\frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x,y)$ but at least we know now that we can do that since the condition $\frac{\partial^2\phi}{\partial x\partial y}=\frac{\partial^2\phi}{\partial y\partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x,y)\,\mathrm{d}x+N(x,y)\,\mathrm{d}y=0 \tag{1A}$$

Therefore

$$(1+y)\,\mathrm{d}y=\left(\frac{4x^2-x-2}{(x+1)\,x^2}\right)\mathrm{d}x$$

$$\left(-\frac{4x^2-x-2}{(x+1)\,x^2}\right)\mathrm{d}x+(1+y)\,\mathrm{d}y=0 \tag{2A}$$

Comparing (1A) and (2A) shows that

$$
\begin{aligned}
M(x,y) &= -\frac{4x^2 - x - 2}{(x+1)\,x^2} \\
N(x,y) &= 1 + y
\end{aligned}
$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$
\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}
$$

Using result found above gives

$$
\begin{aligned}
\frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}\left(-\frac{4x^2 - x - 2}{(x+1)\,x^2}\right) \\
&= 0
\end{aligned}
$$

And

$$
\begin{aligned}
\frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}(1+y) \\
&= 0
\end{aligned}
$$

Since $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$, then the ODE is <u>exact</u> The following equations are now set up to solve for the function $\phi(x,y)$

$$
\begin{aligned}
\frac{\partial \phi}{\partial x} &= M \qquad (1) \\
\frac{\partial \phi}{\partial y} &= N \qquad (2)
\end{aligned}
$$

Integrating (1) w.r.t. x gives

$$
\begin{aligned}
\int \frac{\partial \phi}{\partial x}\,\mathrm{d}x &= \int M\,\mathrm{d}x \\
\int \frac{\partial \phi}{\partial x}\,\mathrm{d}x &= \int -\frac{4x^2 - x - 2}{(x+1)\,x^2}\,\mathrm{d}x \\
\phi &= -3\ln(x+1) - \ln(x) - \frac{2}{x} + f(y) \qquad (3)
\end{aligned}
$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$
\frac{\partial \phi}{\partial y} = 0 + f'(y) \qquad (4)
$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = 1 + y$. Therefore equation (4) becomes

$$1 + y = 0 + f'(y) \tag{5}$$

Solving equation (5) for $f'(y)$ gives

$$f'(y) = 1 + y$$

Integrating the above w.r.t y gives

$$\int f'(y) \, \mathrm{d}y = \int (1 + y) \, \mathrm{d}y$$

$$f(y) = y + \frac{1}{2} y^2 + c_1$$

Where c_1 is constant of integration. Substituting result found above for $f(y)$ into equation (3) gives ϕ

$$\phi = -3 \ln(x + 1) - \ln(x) - \frac{2}{x} + y + \frac{y^2}{2} + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = -3 \ln(x + 1) - \ln(x) - \frac{2}{x} + y + \frac{y^2}{2}$$

Initial conditions are used to solve for c_1. Substituting $x = 1$ and $y = 1$ in the above solution gives an equation to solve for the constant of integration.

$$-3 \ln(2) - \frac{1}{2} = c_1$$

$$c_1 = -3 \ln(2) - \frac{1}{2}$$

Substituting c_1 found above in the general solution gives

$$-3 \ln(x + 1) - \ln(x) - \frac{2}{x} + y + \frac{y^2}{2} = -3 \ln(2) - \frac{1}{2}$$

The above simplifies to

$$y^2x - 6\ln(x+1)\,x - 2\ln(x)\,x + 6\ln(2)\,x + 2xy + x - 4 = 0$$

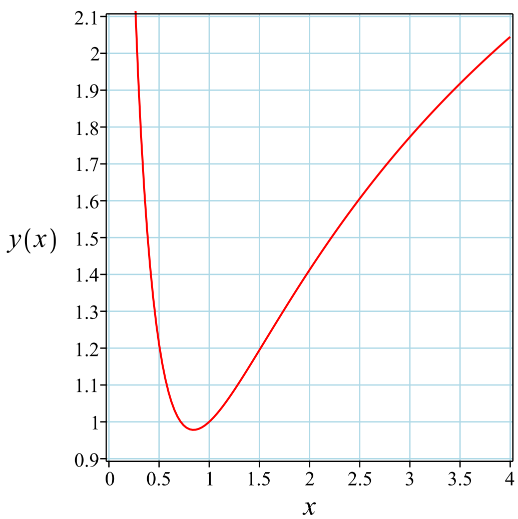
Solving for y from the above gives

$$y = \frac{-x + \sqrt{2}\,\sqrt{x\,(2 + 3\ln(x+1)\,x + \ln(x)\,x - 3\ln(2)\,x)}}{x}$$

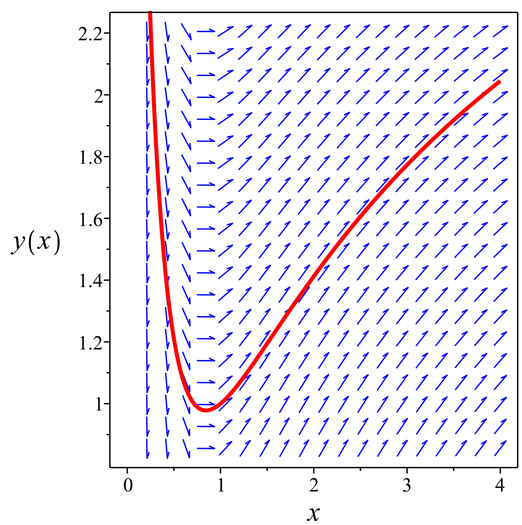
Summary

The solution(s) found are the following

$$y = \frac{-x + \sqrt{2}\,\sqrt{x\,(2 + 3\ln(x+1)\,x + \ln(x)\,x - 3\ln(2)\,x)}}{x} \tag{1}$$



(a) Solution plot



(b) Slope field plot

Verification of solutions

$$y = \frac{-x + \sqrt{2}\,\sqrt{x\,(2 + 3\ln(x+1)\,x + \ln(x)\,x - 3\ln(2)\,x)}}{x}$$

Verified OK.

1.20.5 Maple step by step solution

Let's solve

$\left[x^2 y' - \frac{4x^2-x-2}{(x+1)(1+y)} = 0, y(1) = 1\right]$

- Highest derivative means the order of the ODE is 1

  y'

- Separate variables

  $y'(1+y) = \frac{4x^2-x-2}{(x+1)x^2}$

- Integrate both sides with respect to x

  $\int y'(1+y)\,dx = \int \frac{4x^2-x-2}{(x+1)x^2}dx + c_1$

- Evaluate integral

  $\frac{y^2}{2} + y = 3\ln(x+1) + \ln(x) + \frac{2}{x} + c_1$

- Solve for y

  $\left\{y = \frac{-x+\sqrt{6\ln(x+1)x^2+2\ln(x)x^2+2c_1x^2+x^2+4x}}{x}, y = -\frac{x+\sqrt{6\ln(x+1)x^2+2\ln(x)x^2+2c_1x^2+x^2+4x}}{x}\right\}$

- Use initial condition $y(1) = 1$

  $1 = -1 + \sqrt{6\ln(2) + 2c_1 + 5}$

- Solve for c_1

  $c_1 = -3\ln(2) - \frac{1}{2}$

- Substitute $c_1 = -3\ln(2) - \frac{1}{2}$ into general solution and simplify

  $y = \frac{-x+\sqrt{2}\,\sqrt{x(2+3\ln(x+1)x+\ln(x)x-3\ln(2)x)}}{x}$

- Use initial condition $y(1) = 1$

  $1 = -1 - \sqrt{6\ln(2) + 2c_1 + 5}$

- Solution does not satisfy initial condition
- Solution to the IVP

  $y = \frac{-x+\sqrt{2}\,\sqrt{x(2+3\ln(x+1)x+\ln(x)x-3\ln(2)x)}}{x}$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
<- separable successful`
```

✓ Solution by Maple

Time used: 0.125 (sec). Leaf size: 38

```
dsolve([x^2*diff(y(x),x)=(4*x^2-x-2)/((x+1)*(y(x)+1)),y(1) = 1],y(x), singsol=all)
```

$$y(x) = \frac{-x + \sqrt{2}\,\sqrt{x\left(\ln\left(x\right)x - 3\ln\left(2\right)x + 3\ln\left(1+x\right)x + 2\right)}}{x}$$

✓ Solution by Mathematica

Time used: 0.291 (sec). Leaf size: 36

```
DSolve[{x^2*y'[x]==(4*x^2-x-2)/((x+1)*(y[x]+1)),{y[1]==1}},y[x],x,IncludeSingularSolutions -
```

$$y(x) \to \frac{\sqrt{2x\log(x) + 6x\log(x+1) - 6x\log(2) + 4}}{\sqrt{x}} - 1$$

1.21 problem 21

Internal problem ID [4932]
Internal file name [`OUTPUT/4425_Sunday_June_05_2022_01_19_12_PM_55937820/index.tex`]

Book: Fundamentals of Differential Equations. By Nagle, Saff and Snider. 9th edition. Boston. Pearson 2018.
Section: Chapter 2, First order differential equations. Section 2.2, Separable Equations. Exercises. page 46
Problem number: 21.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"exact", "separable", "first_order_ode_lie_symmetry_lookup"**

Maple gives the following as the ode type

`[_separable]`

$$\boxed{\frac{y'}{\theta}-\frac{y\sin\left(\theta\right)}{y^{2}+1}=0}$$

With initial conditions

$$[y(\pi)=1]$$

1.21.1 Existence and uniqueness analysis

This is non linear first order ODE. In canonical form it is written as

$$\begin{aligned}
y' &= f(\theta,y)\\
&= \frac{y\sin\left(\theta\right)\theta}{y^{2}+1}
\end{aligned}$$

The θ domain of $f(\theta, y)$ when $y = 1$ is

$$\{-\infty < \theta < \infty\}$$

And the point $\theta_0 = \pi$ is inside this domain. The y domain of $f(\theta, y)$ when $\theta = \pi$ is

$$\{-\infty < y < \infty\}$$

And the point $y_0 = 1$ is inside this domain. Now we will look at the continuity of

$$\begin{aligned}\frac{\partial f}{\partial y} &= \frac{\partial}{\partial y}\left(\frac{y \sin(\theta)\,\theta}{y^2+1}\right) \\ &= \frac{\sin(\theta)\,\theta}{y^2+1} - \frac{2y^2 \sin(\theta)\,\theta}{(y^2+1)^2}\end{aligned}$$

The θ domain of $\frac{\partial f}{\partial y}$ when $y = 1$ is

$$\{-\infty < \theta < \infty\}$$

And the point $\theta_0 = \pi$ is inside this domain. The y domain of $\frac{\partial f}{\partial y}$ when $\theta = \pi$ is

$$\{-\infty < y < \infty\}$$

And the point $y_0 = 1$ is inside this domain. Therefore solution exists and is unique.

1.21.2 Solving as separable ode

In canonical form the ODE is

$$\begin{aligned}y' &= F(\theta, y) \\ &= f(\theta) g(y) \\ &= \frac{y \sin(\theta)\,\theta}{y^2+1}\end{aligned}$$

Where $f(\theta) = \sin(\theta)\,\theta$ and $g(y) = \frac{y}{y^2+1}$. Integrating both sides gives

$$\frac{1}{\frac{y}{y^2+1}}\, dy = \sin(\theta)\,\theta\, d\theta$$

$$\int \frac{1}{\frac{y}{y^2+1}}\, dy = \int \sin(\theta)\,\theta\, d\theta$$

$$\frac{y^2}{2} + \ln(y) = \sin(\theta) - \cos(\theta)\,\theta + c_1$$

Which results in

$$y = \frac{1}{\sqrt{\frac{1}{\text{LambertW}\left(e^{-2\cos(\theta)\theta+2c_1+2\sin(\theta)}\right)}}}$$

Since c_1 is constant, then exponential powers of this constant are constants also, and these can be simplified to just c_1 in the above solution. The solution becomes

$$y = \frac{1}{\sqrt{\frac{1}{\text{LambertW}\left(c_1^2 e^{-2\cos(\theta)\theta+2\sin(\theta)}\right)}}}$$

Initial conditions are used to solve for c_1. Substituting $\theta = \pi$ and $y = 1$ in the above solution gives an equation to solve for the constant of integration.

$$1 = \frac{1}{\sqrt{\frac{1}{\text{LambertW}\left(e^{2\pi} c_1^2\right)}}}$$

$$c_1 = -e^{-2\pi}\sqrt{e^{2\pi} e}$$

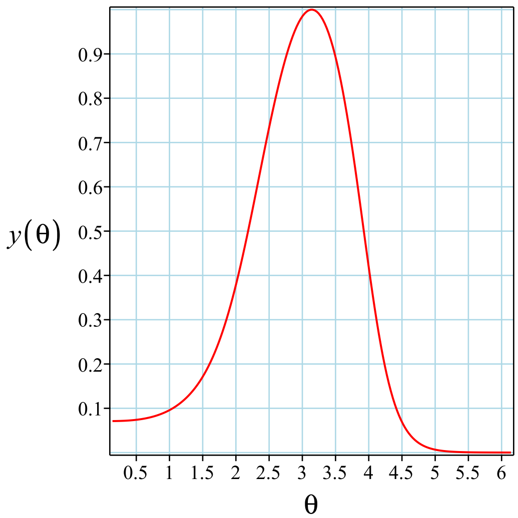
Substituting c_1 found above in the general solution gives

$$y = \frac{1}{\sqrt{\frac{1}{\text{LambertW}\left(e^{-2\pi+1-2\cos(\theta)\theta+2\sin(\theta)}\right)}}}$$

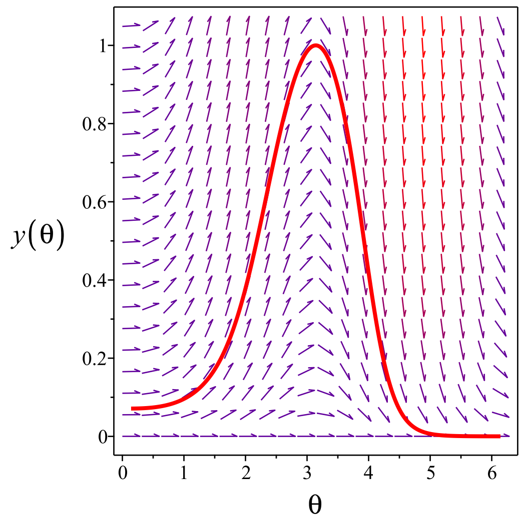
Summary

The solution(s) found are the following

$$y = \frac{1}{\sqrt{\frac{1}{\text{LambertW}\left(e^{-2\pi+1-2\cos(\theta)\theta+2\sin(\theta)}\right)}}} \tag{1}$$



(a) Solution plot



(b) Slope field plot

Verification of solutions

$$y = \frac{1}{\sqrt{\frac{1}{\operatorname{LambertW}\left(\mathrm{e}^{-2\pi+1-2\cos(\theta)\theta+2\sin(\theta)}\right)}}}$$

Verified OK.

1.21.3 Solving as first order ode lie symmetry lookup ode

Writing the ode as

$$y' = \frac{y\sin(\theta)\,\theta}{y^2+1}$$
$$y' = \omega(\theta, y)$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_\theta + \omega(\eta_y - \xi_\theta) - \omega^2\xi_y - \omega_\theta\xi - \omega_y\eta = 0 \tag{A}$$

The type of this ode is known. It is of type `separable`. Therefore we do not need to solve the PDE (A), and can just use the lookup table shown below to find ξ, η

Table 48: Lie symmetry infinitesimal lookup table for known first order ODE's

ODE class	Form	ξ	η
linear ode	$y' = f(x)y(x) + g(x)$	0	$e^{\int f dx}$
separable ode	$y' = f(x)\, g(y)$	$\frac{1}{f}$	0
quadrature ode	$y' = f(x)$	0	1
quadrature ode	$y' = g(y)$	1	0
homogeneous ODEs of Class A	$y' = f\left(\frac{y}{x}\right)$	x	y
homogeneous ODEs of Class C	$y' = (a + bx + cy)^{\frac{n}{m}}$	1	$-\frac{b}{c}$
homogeneous class D	$y' = \frac{y}{x} + g(x)\, F\left(\frac{y}{x}\right)$	x^2	xy
First order special form ID 1	$y' = g(x)\, e^{h(x)+by} + f(x)$	$\frac{e^{-\int bf(x)dx-h(x)}}{g(x)}$	$\frac{f(x)e^{-\int bf(x)dx-h(x)}}{g(x)}$
polynomial type ode	$y' = \frac{a_1x+b_1y+c_1}{a_2x+b_2y+c_2}$	$\frac{a_1b_2x-a_2b_1x-b_1c_2+b_2c_1}{a_1b_2-a_2b_1}$	$\frac{a_1b_2y-a_2b_1y-a_1c_2-a_2c_1}{a_1b_2-a_2b_1}$
Bernoulli ode	$y' = f(x)\, y + g(x)\, y^n$	0	$e^{-\int (n-1)f(x)dx} y^n$
Reduced Riccati	$y' = f_1(x)\, y + f_2(x)\, y^2$	0	$e^{-\int f_1 dx}$

The above table shows that

$$
\begin{aligned}
\xi(\theta, y) &= \frac{1}{\sin(\theta)\,\theta} \\
\eta(\theta, y) &= 0
\end{aligned} \tag{A1}
$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(\theta, y) \to (R, S)$ where (R, S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$
\frac{d\theta}{\xi} = \frac{dy}{\eta} = dS \tag{1}
$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial \theta} + \eta \frac{\partial}{\partial y}\right) S(\theta, y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the

canonical coordinates, where $S(R)$. Since $\eta = 0$ then in this special case

$$R = y$$

S is found from

$$\begin{aligned} S &= \int \frac{1}{\xi} d\theta \\ &= \int \frac{1}{\frac{1}{\sin(\theta)\theta}} d\theta \end{aligned}$$

Which results in

$$S = \sin(\theta) - \cos(\theta)\,\theta$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_\theta + \omega(\theta, y) S_y}{R_\theta + \omega(\theta, y) R_y} \tag{2}$$

Where in the above $R_\theta, R_y, S_\theta, S_y$ are all partial derivatives and $\omega(\theta, y)$ is the right hand side of the original ode given by

$$\omega(\theta, y) = \frac{y \sin(\theta)\,\theta}{y^2 + 1}$$

Evaluating all the partial derivatives gives

$$\begin{aligned} R_\theta &= 0 \\ R_y &= 1 \\ S_\theta &= \sin(\theta)\,\theta \\ S_y &= 0 \end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = \frac{y^2 + 1}{y} \tag{2A}$$

Unable to generate ode in canonical coordinates.

1.21.4 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x,y)+N(x,y)\frac{dy}{dx}=0 \tag{A}$$

We assume there exists a function $\phi(x,y)=c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y)=0$$

Hence

$$\frac{\partial\phi}{\partial x}+\frac{\partial\phi}{\partial y}\frac{dy}{dx}=0 \tag{B}$$

Comparing (A,B) shows that

$$\begin{aligned}\frac{\partial\phi}{\partial x}&=M\\ \frac{\partial\phi}{\partial y}&=N\end{aligned}$$

But since $\frac{\partial^2\phi}{\partial x\partial y}=\frac{\partial^2\phi}{\partial y\partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y}=\frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x,y)$ but at least we know now that we can do that since the condition $\frac{\partial^2\phi}{\partial x\partial y}=\frac{\partial^2\phi}{\partial y\partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(\theta,y)\,\mathrm{d}\theta+N(\theta,y)\,\mathrm{d}y=0 \tag{1A}$$

Therefore

$$\begin{aligned}\left(\frac{y^2+1}{y}\right)\mathrm{d}y&=(\sin(\theta)\,\theta)\,\mathrm{d}\theta\\ (-\sin(\theta)\,\theta)\,\mathrm{d}\theta+\left(\frac{y^2+1}{y}\right)\mathrm{d}y&=0 \end{aligned}\tag{2A}$$

Comparing (1A) and (2A) shows that

$$
\begin{aligned}
M(\theta, y) &= -\sin(\theta)\,\theta \\
N(\theta, y) &= \frac{y^2+1}{y}
\end{aligned}
$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$
\frac{\partial M}{\partial y} = \frac{\partial N}{\partial \theta}
$$

Using result found above gives

$$
\begin{aligned}
\frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}(-\sin(\theta)\,\theta) \\
&= 0
\end{aligned}
$$

And

$$
\begin{aligned}
\frac{\partial N}{\partial \theta} &= \frac{\partial}{\partial \theta}\left(\frac{y^2+1}{y}\right) \\
&= 0
\end{aligned}
$$

Since $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial \theta}$, then the ODE is <u>exact</u> The following equations are now set up to solve for the function $\phi(\theta, y)$

$$
\frac{\partial \phi}{\partial \theta} = M \tag{1}
$$

$$
\frac{\partial \phi}{\partial y} = N \tag{2}
$$

Integrating (1) w.r.t. θ gives

$$
\begin{aligned}
\int \frac{\partial \phi}{\partial \theta}\, \mathrm{d}\theta &= \int M\, \mathrm{d}\theta \\
\int \frac{\partial \phi}{\partial \theta}\, \mathrm{d}\theta &= \int -\sin(\theta)\,\theta\, \mathrm{d}\theta
\end{aligned}
$$

$$
\phi = -\sin(\theta) + \cos(\theta)\,\theta + f(y) \tag{3}
$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both θ and y. Taking derivative of equation (3) w.r.t y gives

$$
\frac{\partial \phi}{\partial y} = 0 + f'(y) \tag{4}
$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = \frac{y^2+1}{y}$. Therefore equation (4) becomes

$$\frac{y^2+1}{y} = 0 + f'(y) \tag{5}$$

Solving equation (5) for $f'(y)$ gives

$$f'(y) = \frac{y^2+1}{y}$$

Integrating the above w.r.t y gives

$$\int f'(y)\,\mathrm{d}y = \int \left(\frac{y^2+1}{y}\right) \mathrm{d}y$$

$$f(y) = \frac{y^2}{2} + \ln(y) + c_1$$

Where c_1 is constant of integration. Substituting result found above for $f(y)$ into equation (3) gives ϕ

$$\phi = -\sin(\theta) + \cos(\theta)\,\theta + \frac{y^2}{2} + \ln(y) + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = -\sin(\theta) + \cos(\theta)\,\theta + \frac{y^2}{2} + \ln(y)$$

The solution becomes

$$y = \frac{1}{\sqrt{\frac{1}{\mathrm{LambertW}\left(\mathrm{e}^{-2\cos(\theta)\theta+2c_1+2\sin(\theta)}\right)}}}$$

Initial conditions are used to solve for c_1. Substituting $\theta = \pi$ and $y = 1$ in the above solution gives an equation to solve for the constant of integration.

$$1 = \frac{1}{\sqrt{\frac{1}{\mathrm{LambertW}\left(\mathrm{e}^{2\pi+2c_1}\right)}}}$$

$$c_1 = -\pi + \frac{1}{2}$$

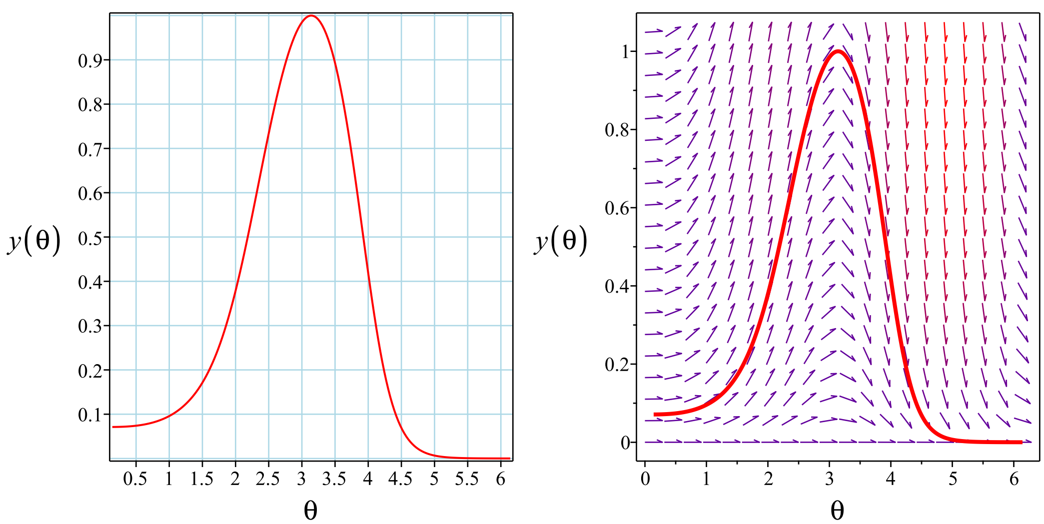
Substituting c_1 found above in the general solution gives

$$y = \frac{1}{\sqrt{\frac{1}{\operatorname{LambertW}\left(\mathrm{e}^{-2\pi+1-2\cos(\theta)\theta+2\sin(\theta)}\right)}}}$$

<u>Summary</u>

The solution(s) found are the following

$$y = \frac{1}{\sqrt{\frac{1}{\operatorname{LambertW}\left(\mathrm{e}^{-2\pi+1-2\cos(\theta)\theta+2\sin(\theta)}\right)}}} \tag{1}$$



(a) Solution plot

(b) Slope field plot

<u>Verification of solutions</u>

$$y = \frac{1}{\sqrt{\frac{1}{\operatorname{LambertW}\left(\mathrm{e}^{-2\pi+1-2\cos(\theta)\theta+2\sin(\theta)}\right)}}}$$

Verified OK.

1.21.5 Maple step by step solution

Let's solve

$\left[\frac{y'}{\theta}-\frac{y \sin(\theta)}{y^2+1}=0, y(\pi)=1\right]$

- Highest derivative means the order of the ODE is 1

  y'

- Separate variables

  $\frac{y'(y^2+1)}{y}=\sin(\theta)\,\theta$

- Integrate both sides with respect to θ

  $\int \frac{y'(y^2+1)}{y}d\theta=\int \sin(\theta)\,\theta d\theta+c_1$

- Evaluate integral

  $\frac{y^2}{2}+\ln(y)=\sin(\theta)-\cos(\theta)\,\theta+c_1$

- Solve for y

  $y=\frac{1}{\sqrt{\frac{1}{\mathit{LambertW}\left(e^{-2\cos(\theta)\theta+2c_1+2\sin(\theta)}\right)}}}$

- Use initial condition $y(\pi)=1$

  $1=\frac{1}{\sqrt{\frac{1}{\mathit{LambertW}\left(e^{2\pi+2c_1}\right)}}}$

- Solve for c_1

  $c_1=-\pi+\frac{1}{2}$

- Substitute $c_1=-\pi+\frac{1}{2}$ into general solution and simplify

  $y=\frac{1}{\sqrt{\frac{1}{\mathit{LambertW}\left(e^{-2\pi+1-2\cos(\theta)\theta+2\sin(\theta)}\right)}}}$

- Solution to the IVP

  $y=\frac{1}{\sqrt{\frac{1}{\mathit{LambertW}\left(e^{-2\pi+1-2\cos(\theta)\theta+2\sin(\theta)}\right)}}}$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
<- separable successful`
```

✓ Solution by Maple

Time used: 0.5 (sec). Leaf size: 35

```
dsolve([1/theta*diff(y(theta),theta)= y(theta)*sin(theta)/(y(theta)^2+1),y(Pi) = 1],y(theta)
```

$$y(\theta) = \frac{e^{-\cos(\theta)\theta+\sin(\theta)+\frac{1}{2}}}{\sqrt{\frac{e^{-2\cos(\theta)\theta+2\sin(\theta)+1}}{\operatorname{LambertW}\left(e^{-2\cos(\theta)\theta-2\pi+2\sin(\theta)+1}\right)}}}$$

✓ Solution by Mathematica

Time used: 3.744 (sec). Leaf size: 26

```
DSolve[{1/\[Theta]*y'[\[Theta]]== y[\[Theta]]*Sin[\[Theta]]/(y[\[Theta]]^2+1),{y[Pi]==1}},y[
```

$$y(\theta) \to \sqrt{W\left(e^{2\sin(\theta)-2\theta\cos(\theta)-2\pi+1}\right)}$$

1.22 problem 22

Internal problem ID [4933]
Internal file name [`OUTPUT/4426_Sunday_June_05_2022_01_19_22_PM_46408462/index.tex`]

Book: Fundamentals of Differential Equations. By Nagle, Saff and Snider. 9th edition. Boston. Pearson 2018.
Section: Chapter 2, First order differential equations. Section 2.2, Separable Equations. Exercises. page 46
Problem number: 22.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"exact", "separable", "differentialType", "first_order_ode_lie_symmetry_lookup"**

Maple gives the following as the ode type

```
[_separable]
```

$$\boxed{2y'y = -x^2}$$

With initial conditions

$$[y(0) = 2]$$

1.22.1 Existence and uniqueness analysis

This is non linear first order ODE. In canonical form it is written as

$$
\begin{aligned}
y' &= f(x,y) \\
&= -\frac{x^2}{2y}
\end{aligned}
$$

The x domain of $f(x,y)$ when $y=2$ is

$$
\{-\infty < x < \infty\}
$$

And the point $x_0 = 0$ is inside this domain. The y domain of $f(x,y)$ when $x = 0$ is

$$
\{-\infty < y < \infty\}
$$

And the point $y_0 = 2$ is inside this domain. Now we will look at the continuity of

$$
\begin{aligned}
\frac{\partial f}{\partial y} &= \frac{\partial}{\partial y}\left(-\frac{x^2}{2y}\right) \\
&= \frac{x^2}{2y^2}
\end{aligned}
$$

The x domain of $\frac{\partial f}{\partial y}$ when $y = 2$ is

$$
\{-\infty < x < \infty\}
$$

And the point $x_0 = 0$ is inside this domain. The y domain of $\frac{\partial f}{\partial y}$ when $x = 0$ is

$$
\{-\infty < y < \infty\}
$$

And the point $y_0 = 2$ is inside this domain. Therefore solution exists and is unique.

1.22.2 Solving as separable ode

In canonical form the ODE is

$$
\begin{aligned}
y' &= F(x,y) \\
&= f(x)g(y) \\
&= -\frac{x^2}{2y}
\end{aligned}
$$

Where $f(x) = -\frac{x^2}{2}$ and $g(y) = \frac{1}{y}$. Integrating both sides gives

$$\frac{1}{\frac{1}{y}} dy = -\frac{x^2}{2} dx$$

$$\int \frac{1}{\frac{1}{y}} dy = \int -\frac{x^2}{2} dx$$

$$\frac{y^2}{2} = -\frac{x^3}{6} + c_1$$

Which results in

$$y = \frac{\sqrt{-3x^3 + 18c_1}}{3}$$

$$y = -\frac{\sqrt{-3x^3 + 18c_1}}{3}$$

Initial conditions are used to solve for c_1. Substituting $x = 0$ and $y = 2$ in the above solution gives an equation to solve for the constant of integration.

$$2 = -\sqrt{2}\,\sqrt{c_1}$$

Warning: Unable to solve for constant of integration. Initial conditions are used to solve for c_1. Substituting $x = 0$ and $y = 2$ in the above solution gives an equation to solve for the constant of integration.

$$2 = \sqrt{2}\,\sqrt{c_1}$$

$$c_1 = 2$$

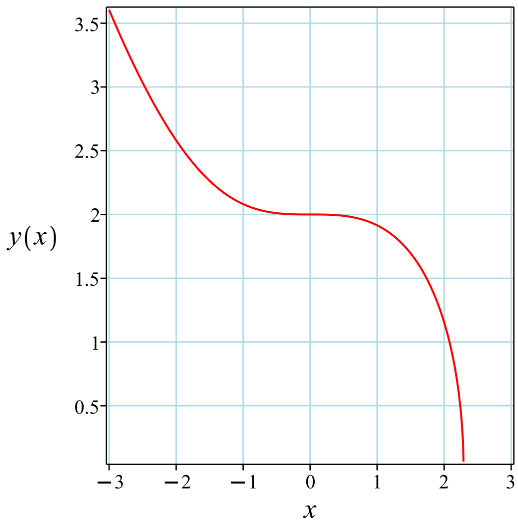
Substituting c_1 found above in the general solution gives

$$y = \frac{\sqrt{-3x^3 + 36}}{3}$$

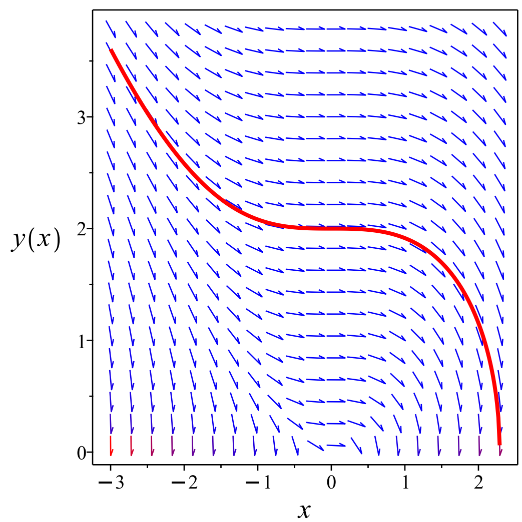
Summary

The solution(s) found are the following

$$y = \frac{\sqrt{-3x^3 + 36}}{3} \tag{1}$$



(a) Solution plot



(b) Slope field plot

Verification of solutions

$$y = \frac{\sqrt{-3x^3 + 36}}{3}$$

Verified OK.

1.22.3 Solving as differentialType ode

Writing the ode as

$$y' = -\frac{x^2}{2y} \tag{1}$$

Which becomes

$$(2y)\, dy = (-x^2)\, dx \tag{2}$$

But the RHS is complete differential because

$$(-x^2)\, dx = d\left(-\frac{x^3}{3}\right)$$

Hence (2) becomes

$$(2y)\, dy = d\left(-\frac{x^3}{3}\right)$$

Integrating both sides gives gives these solutions

$$y = \frac{\sqrt{-3x^3 + 9c_1}}{3} + c_1$$
$$y = -\frac{\sqrt{-3x^3 + 9c_1}}{3} + c_1$$

Initial conditions are used to solve for c_1. Substituting $x = 0$ and $y = 2$ in the above solution gives an equation to solve for the constant of integration.

$$2 = -\sqrt{c_1} + c_1$$

$$c_1 = 4$$

Substituting c_1 found above in the general solution gives

$$y = -\frac{\sqrt{-3x^3 + 36}}{3} + 4$$

Initial conditions are used to solve for c_1. Substituting $x = 0$ and $y = 2$ in the above solution gives an equation to solve for the constant of integration.

$$2 = \sqrt{c_1} + c_1$$

$$c_1 = 1$$

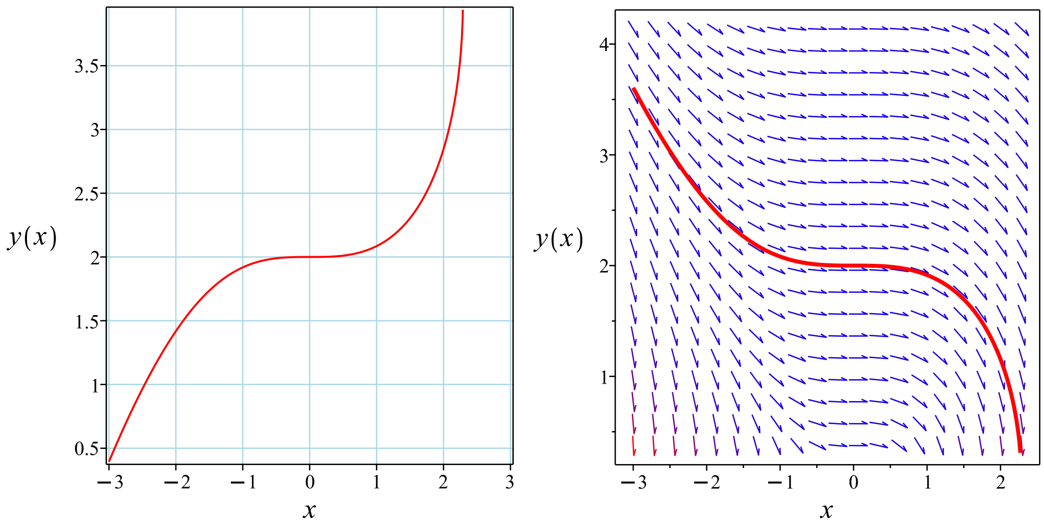
Substituting c_1 found above in the general solution gives

$$y = \frac{\sqrt{-3x^3 + 9}}{3} + 1$$

Summary

The solution(s) found are the following

$$y = \frac{\sqrt{-3x^3 + 9}}{3} + 1 \tag{1}$$
$$y = -\frac{\sqrt{-3x^3 + 36}}{3} + 4 \tag{2}$$



(a) Solution plot (b) Slope field plot

<u>Verification of solutions</u>

$$y = \frac{\sqrt{-3x^3+9}}{3} + 1$$

Verified OK.

$$y = -\frac{\sqrt{-3x^3+36}}{3} + 4$$

Verified OK.

1.22.4 Solving as first order ode lie symmetry lookup ode

Writing the ode as

$$y' = -\frac{x^2}{2y}$$
$$y' = \omega(x, y)$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x + \omega(\eta_y - \xi_x) - \omega^2\xi_y - \omega_x\xi - \omega_y\eta = 0 \tag{A}$$

The type of this ode is known. It is of type `separable`. Therefore we do not need to solve the PDE (A), and can just use the lookup table shown below to find ξ, η

Table 51: Lie symmetry infinitesimal lookup table for known first order ODE's

ODE class	Form	ξ	η
linear ode	$y' = f(x)y(x) + g(x)$	0	$e^{\int f dx}$
separable ode	$y' = f(x)\, g(y)$	$\frac{1}{f}$	0
quadrature ode	$y' = f(x)$	0	1
quadrature ode	$y' = g(y)$	1	0
homogeneous ODEs of Class A	$y' = f\left(\frac{y}{x}\right)$	x	y
homogeneous ODEs of Class C	$y' = (a + bx + cy)^{\frac{n}{m}}$	1	$-\frac{b}{c}$
homogeneous class D	$y' = \frac{y}{x} + g(x)\, F\left(\frac{y}{x}\right)$	x^2	xy
First order special form ID 1	$y' = g(x)\, e^{h(x)+by} + f(x)$	$\frac{e^{-\int bf(x)dx - h(x)}}{g(x)}$	$\frac{f(x)e^{-\int bf(x)dx - h(x)}}{g(x)}$
polynomial type ode	$y' = \frac{a_1x+b_1y+c_1}{a_2x+b_2y+c_2}$	$\frac{a_1b_2x - a_2b_1x - b_1c_2 + b_2c_1}{a_1b_2 - a_2b_1}$	$\frac{a_1b_2y - a_2b_1y - a_1c_2 - a_2c_1}{a_1b_2 - a_2b_1}$
Bernoulli ode	$y' = f(x)\, y + g(x)\, y^n$	0	$e^{-\int (n-1)f(x)dx} y^n$
Reduced Riccati	$y' = f_1(x)\, y + f_2(x)\, y^2$	0	$e^{-\int f_1 dx}$

The above table shows that

$$\xi(x, y) = -\frac{2}{x^2}$$
$$\eta(x, y) = 0 \tag{A1}$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x, y) \to (R, S)$ where (R, S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y}\right) S(x, y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the

canonical coordinates, where $S(R)$. Since $\eta = 0$ then in this special case

$$R = y$$

S is found from

$$\begin{aligned}S &= \int \frac{1}{\xi} dx \\ &= \int \frac{1}{-\frac{2}{x^2}} dx\end{aligned}$$

Which results in

$$S = -\frac{x^3}{6}$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x,y)S_y}{R_x + \omega(x,y)R_y} \tag{2}$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x,y)$ is the right hand side of the original ode given by

$$\omega(x,y) = -\frac{x^2}{2y}$$

Evaluating all the partial derivatives gives

$$\begin{aligned}R_x &= 0 \\ R_y &= 1 \\ S_x &= -\frac{x^2}{2} \\ S_y &= 0\end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = y \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = R$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordinates R, S. Integrating the above gives

$$S(R) = \frac{R^2}{2} + c_1 \tag{4}$$

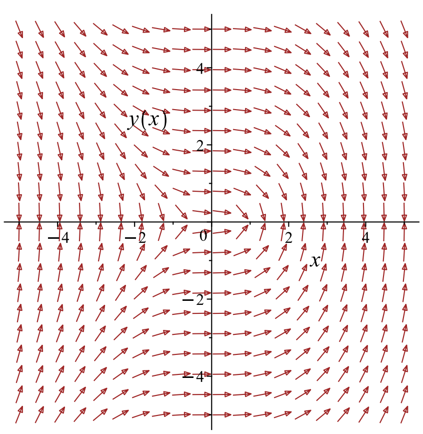
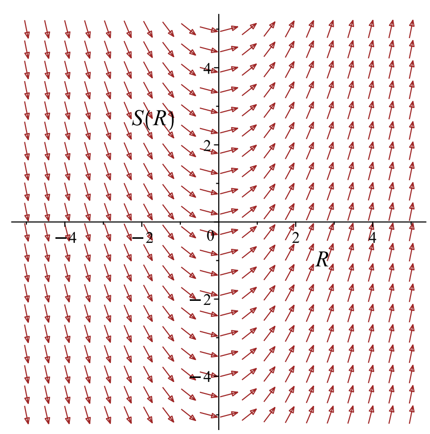
To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$-\frac{x^3}{6} = \frac{y^2}{2} + c_1$$

Which simplifies to

$$-\frac{x^3}{6} = \frac{y^2}{2} + c_1$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in x, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dx} = -\frac{x^2}{2y}$ 	$R = y$ $S = -\frac{x^3}{6}$	$\frac{dS}{dR} = R$ 

Initial conditions are used to solve for c_1. Substituting $x = 0$ and $y = 2$ in the above solution gives an equation to solve for the constant of integration.

$$0 = 2 + c_1$$

$$c_1 = -2$$

Substituting c_1 found above in the general solution gives

$$-\frac{x^3}{6} = \frac{y^2}{2} - 2$$

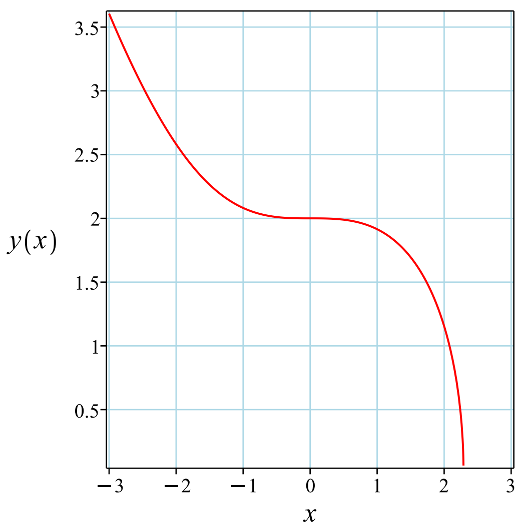
Solving for y from the above gives

$$y = \frac{\sqrt{-3x^3 + 36}}{3}$$

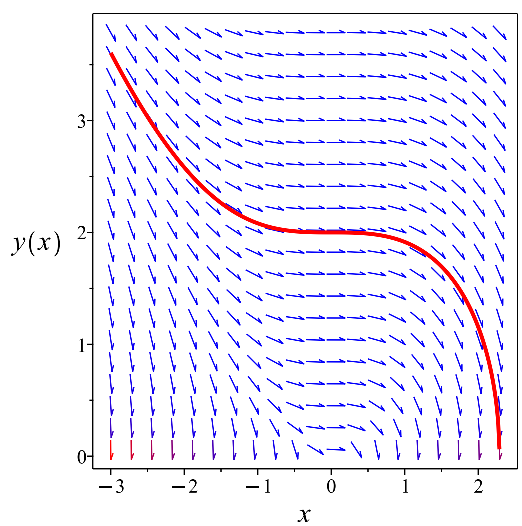
Summary

The solution(s) found are the following

$$y = \frac{\sqrt{-3x^3 + 36}}{3} \tag{1}$$



(a) Solution plot



(b) Slope field plot

Verification of solutions

$$y = \frac{\sqrt{-3x^3 + 36}}{3}$$

Verified OK.

1.22.5 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x,y)+N(x,y)\frac{dy}{dx}=0 \tag{A}$$

We assume there exists a function $\phi(x,y)=c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y)=0$$

Hence

$$\frac{\partial\phi}{\partial x}+\frac{\partial\phi}{\partial y}\frac{dy}{dx}=0 \tag{B}$$

Comparing (A,B) shows that

$$\begin{aligned}\frac{\partial\phi}{\partial x}&=M\\ \frac{\partial\phi}{\partial y}&=N\end{aligned}$$

But since $\frac{\partial^2\phi}{\partial x\partial y}=\frac{\partial^2\phi}{\partial y\partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y}=\frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x,y)$ but at least we know now that we can do that since the condition $\frac{\partial^2\phi}{\partial x\partial y}=\frac{\partial^2\phi}{\partial y\partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x,y)\,\mathrm{d}x+N(x,y)\,\mathrm{d}y=0 \tag{1A}$$

Therefore

$$\begin{aligned}(-2y)\,\mathrm{d}y&=\left(x^2\right)\mathrm{d}x\\ \left(-x^2\right)\mathrm{d}x+(-2y)\,\mathrm{d}y&=0 \end{aligned}\tag{2A}$$

Comparing (1A) and (2A) shows that

$$\begin{aligned}M(x,y)&=-x^2\\ N(x,y)&=-2y\end{aligned}$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Using result found above gives

$$\begin{aligned}\frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}(-x^2) \\ &= 0\end{aligned}$$

And

$$\begin{aligned}\frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}(-2y) \\ &= 0\end{aligned}$$

Since $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$, then the ODE is exact The following equations are now set up to solve for the function $\phi(x, y)$

$$\frac{\partial \phi}{\partial x} = M \tag{1}$$

$$\frac{\partial \phi}{\partial y} = N \tag{2}$$

Integrating (1) w.r.t. x gives

$$\int \frac{\partial \phi}{\partial x} \, \mathrm{d}x = \int M \, \mathrm{d}x$$

$$\int \frac{\partial \phi}{\partial x} \, \mathrm{d}x = \int -x^2 \, \mathrm{d}x$$

$$\phi = -\frac{x^3}{3} + f(y) \tag{3}$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$\frac{\partial \phi}{\partial y} = 0 + f'(y) \tag{4}$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = -2y$. Therefore equation (4) becomes

$$-2y = 0 + f'(y) \tag{5}$$

Solving equation (5) for $f'(y)$ gives

$$f'(y) = -2y$$

Integrating the above w.r.t y gives

$$\int f'(y)\,\mathrm{d}y = \int (-2y)\,\mathrm{d}y$$
$$f(y) = -y^2 + c_1$$

Where c_1 is constant of integration. Substituting result found above for $f(y)$ into equation (3) gives ϕ

$$\phi = -\frac{x^3}{3} - y^2 + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = -\frac{x^3}{3} - y^2$$

Initial conditions are used to solve for c_1. Substituting $x = 0$ and $y = 2$ in the above solution gives an equation to solve for the constant of integration.

$$-4 = c_1$$

$$c_1 = -4$$

Substituting c_1 found above in the general solution gives

$$-\frac{x^3}{3} - y^2 = -4$$

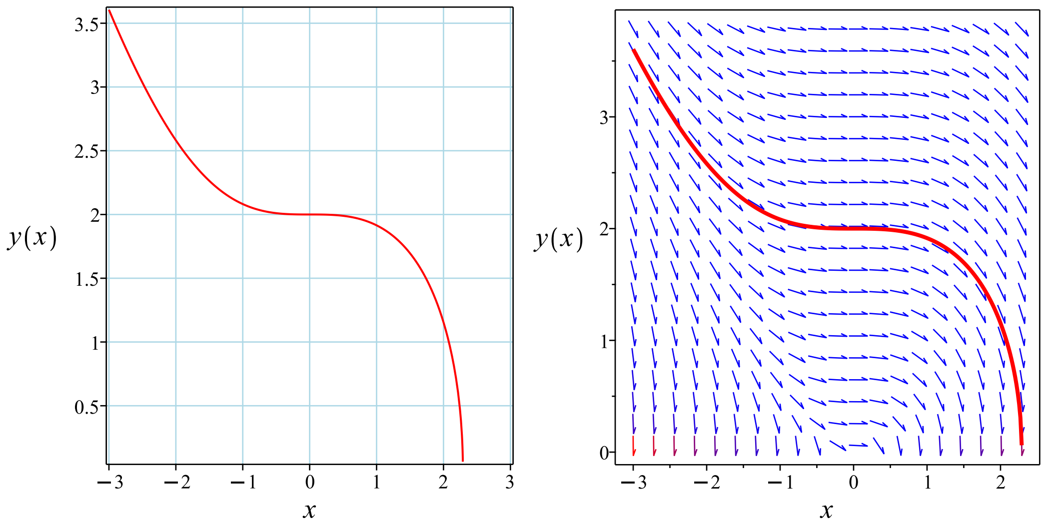
Solving for y from the above gives

$$y = \frac{\sqrt{-3x^3 + 36}}{3}$$

Summary

The solution(s) found are the following

$$y = \frac{\sqrt{-3x^3 + 36}}{3} \tag{1}$$



(a) Solution plot

(b) Slope field plot

<u>Verification of solutions</u>

$$y = \frac{\sqrt{-3x^3 + 36}}{3}$$

Verified OK.

1.22.6 Maple step by step solution

Let's solve

$[2y'y = -x^2, y(0) = 2]$

- Highest derivative means the order of the ODE is 1

  y'

- Integrate both sides with respect to x

  $\int 2y'ydx = \int -x^2dx + c_1$

- Evaluate integral

  $y^2 = -\frac{x^3}{3} + c_1$

- Solve for y

  $\left\{y = -\frac{\sqrt{-3x^3+9c_1}}{3}, y = \frac{\sqrt{-3x^3+9c_1}}{3}\right\}$

- Use initial condition $y(0) = 2$

$2 = -\frac{\sqrt{9}\,\sqrt{c_1}}{3}$

- Solution does not satisfy initial condition
- Use initial condition $y(0) = 2$

  $2 = \frac{\sqrt{9}\,\sqrt{c_1}}{3}$
- Solve for c_1

  $c_1 = 4$
- Substitute $c_1 = 4$ into general solution and simplify

  $y = \frac{\sqrt{-3x^3+36}}{3}$
- Solution to the IVP

  $y = \frac{\sqrt{-3x^3+36}}{3}$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
<- Bernoulli successful`
```

✓ Solution by Maple

Time used: 0.031 (sec). Leaf size: 15

```
dsolve([x^2+2*y(x)*diff(y(x),x)=0,y(0) = 2],y(x), singsol=all)
```

$$y(x) = \frac{\sqrt{-3x^3 + 36}}{3}$$

✓ Solution by Mathematica

Time used: 0.091 (sec). Leaf size: 18

```
DSolve[{x^2+2*y[x]*y'[x]==0,{y[0]==2}},y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \sqrt{4 - \frac{x^3}{3}}$$

1.23 problem 23

Internal problem ID [4934]
Internal file name [`OUTPUT/4427_Sunday_June_05_2022_01_19_33_PM_41942247/index.tex`]

Book: Fundamentals of Differential Equations. By Nagle, Saff and Snider. 9th edition. Boston. Pearson 2018.
Section: Chapter 2, First order differential equations. Section 2.2, Separable Equations. Exercises. page 46
Problem number: 23.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"exact", "separable", "first_order_ode_lie_symmetry_lookup"**

Maple gives the following as the ode type

`[_separable]`

$$\boxed{y' - 2t\cos(y)^2 = 0}$$

With initial conditions

$$\left[y(0) = \frac{\pi}{4}\right]$$

1.23.1 Existence and uniqueness analysis

This is non linear first order ODE. In canonical form it is written as

$$\begin{aligned} y' &= f(t,y) \\ &= 2t\cos(y)^2 \end{aligned}$$

The t domain of $f(t,y)$ when $y = \frac{\pi}{4}$ is

$$\{-\infty < t < \infty\}$$

And the point $t_0 = 0$ is inside this domain. The y domain of $f(t, y)$ when $t = 0$ is

$$\{-\infty < y < \infty\}$$

And the point $y_0 = \frac{\pi}{4}$ is inside this domain. Now we will look at the continuity of

$$\begin{aligned}\frac{\partial f}{\partial y} &= \frac{\partial}{\partial y}\left(2t\cos(y)^2\right)\\ &= -4t\cos(y)\sin(y)\end{aligned}$$

The t domain of $\frac{\partial f}{\partial y}$ when $y = \frac{\pi}{4}$ is

$$\{-\infty < t < \infty\}$$

And the point $t_0 = 0$ is inside this domain. The y domain of $\frac{\partial f}{\partial y}$ when $t = 0$ is

$$\{-\infty < y < \infty\}$$

And the point $y_0 = \frac{\pi}{4}$ is inside this domain. Therefore solution exists and is unique.

1.23.2 Solving as separable ode

In canonical form the ODE is

$$\begin{aligned}y' &= F(t, y)\\ &= f(t)g(y)\\ &= 2t\cos(y)^2\end{aligned}$$

Where $f(t) = 2t$ and $g(y) = \cos(y)^2$. Integrating both sides gives

$$\begin{aligned}\frac{1}{\cos(y)^2}\,dy &= 2t\,dt\\ \int \frac{1}{\cos(y)^2}\,dy &= \int 2t\,dt\\ \tan(y) &= t^2 + c_1\end{aligned}$$

Which results in

$$y = \arctan\left(t^2 + c_1\right)$$

Initial conditions are used to solve for c_1. Substituting $t = 0$ and $y = \frac{\pi}{4}$ in the above solution gives an equation to solve for the constant of integration.

$$\frac{\pi}{4} = \arctan(c_1)$$

$$c_1 = 1$$

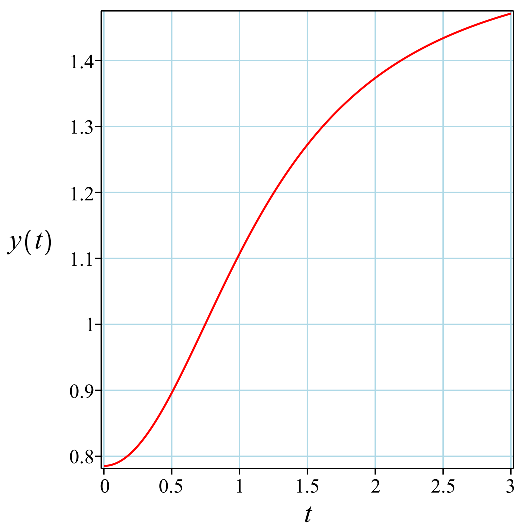
Substituting c_1 found above in the general solution gives

$$y = \arctan(t^2 + 1)$$

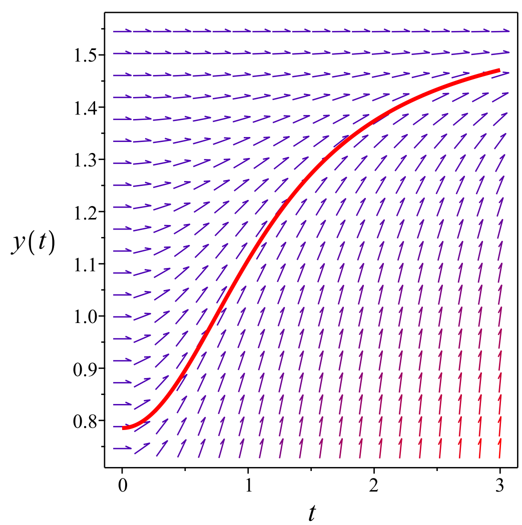
<u>Summary</u>

The solution(s) found are the following

$$y = \arctan(t^2 + 1) \tag{1}$$



(a) Solution plot



(b) Slope field plot

<u>Verification of solutions</u>

$$y = \arctan(t^2 + 1)$$

Verified OK.

1.23.3 Solving as first order ode lie symmetry lookup ode

Writing the ode as

$$y' = 2t\cos(y)^2$$
$$y' = \omega(t, y)$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_t + \omega(\eta_y - \xi_t) - \omega^2\xi_y - \omega_t\xi - \omega_y\eta = 0 \tag{A}$$

The type of this ode is known. It is of type `separable`. Therefore we do not need to solve the PDE (A), and can just use the lookup table shown below to find ξ, η

Table 54: Lie symmetry infinitesimal lookup table for known first order ODE's

ODE class	Form	ξ	η
linear ode	$y' = f(x)y(x) + g(x)$	0	$e^{\int f dx}$
separable ode	$y' = f(x)\,g(y)$	$\frac{1}{f}$	0
quadrature ode	$y' = f(x)$	0	1
quadrature ode	$y' = g(y)$	1	0
homogeneous ODEs of Class A	$y' = f\left(\frac{y}{x}\right)$	x	y
homogeneous ODEs of Class C	$y' = (a + bx + cy)^{\frac{n}{m}}$	1	$-\frac{b}{c}$
homogeneous class D	$y' = \frac{y}{x} + g(x)\,F\left(\frac{y}{x}\right)$	x^2	xy
First order special form ID 1	$y' = g(x)\,e^{h(x)+by} + f(x)$	$\frac{e^{-\int bf(x)dx-h(x)}}{g(x)}$	$\frac{f(x)e^{-\int bf(x)dx-h(x)}}{g(x)}$
polynomial type ode	$y' = \frac{a_1x+b_1y+c_1}{a_2x+b_2y+c_2}$	$\frac{a_1b_2x-a_2b_1x-b_1c_2+b_2c_1}{a_1b_2-a_2b_1}$	$\frac{a_1b_2y-a_2b_1y-a_1c_2-a_2c_1}{a_1b_2-a_2b_1}$
Bernoulli ode	$y' = f(x)\,y + g(x)\,y^n$	0	$e^{-\int(n-1)f(x)dx}y^n$
Reduced Riccati	$y' = f_1(x)\,y + f_2(x)\,y^2$	0	$e^{-\int f_1 dx}$

The above table shows that

$$
\begin{aligned}
\xi(t,y) &= \frac{1}{2t} \\
\eta(t,y) &= 0
\end{aligned} \tag{A1}
$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(t,y) \to (R,S)$ where (R,S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$
\frac{dt}{\xi} = \frac{dy}{\eta} = dS \tag{1}
$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial t} + \eta \frac{\partial}{\partial y}\right) S(t,y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the canonical coordinates, where $S(R)$. Since $\eta = 0$ then in this special case

$$
R = y
$$

S is found from

$$
\begin{aligned}
S &= \int \frac{1}{\xi} dt \\
&= \int \frac{1}{\frac{1}{2t}} dt
\end{aligned}
$$

Which results in

$$
S = t^2
$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$
\frac{dS}{dR} = \frac{S_t + \omega(t,y) S_y}{R_t + \omega(t,y) R_y} \tag{2}
$$

Where in the above R_t, R_y, S_t, S_y are all partial derivatives and $\omega(t,y)$ is the right hand side of the original ode given by

$$
\omega(t,y) = 2t \cos(y)^2
$$

Evaluating all the partial derivatives gives

$$
\begin{aligned}
R_t &= 0 \\
R_y &= 1 \\
S_t &= 2t \\
S_y &= 0
\end{aligned}
$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = \sec(y)^2 \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for t, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = \sec(R)^2$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$S(R) = \tan(R) + c_1 \tag{4}$$

To complete the solution, we just need to transform (4) back to t, y coordinates. This results in

$$t^2 = \tan(y) + c_1$$

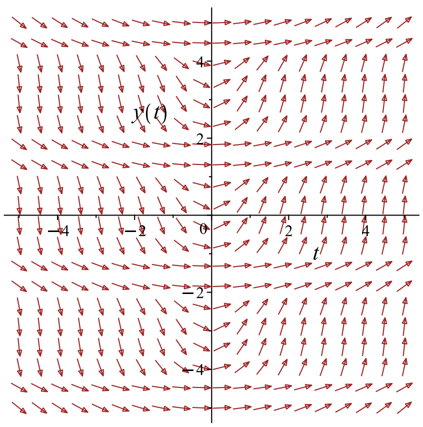
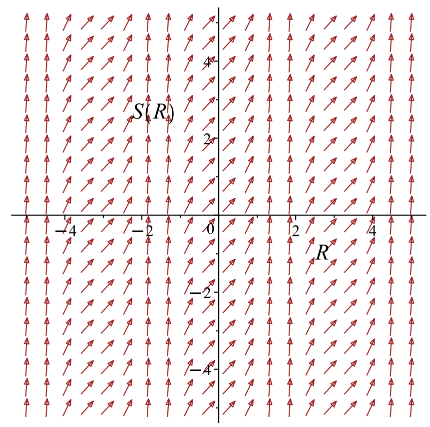
Which simplifies to

$$t^2 = \tan(y) + c_1$$

Which gives

$$y = -\arctan\left(-t^2 + c_1\right)$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in t, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dt} = 2t\cos(y)^2$ 	$R = y$ $S = t^2$	$\frac{dS}{dR} = \sec(R)^2$ 

Initial conditions are used to solve for c_1. Substituting $t = 0$ and $y = \frac{\pi}{4}$ in the above solution gives an equation to solve for the constant of integration.

$$\frac{\pi}{4} = -\arctan(c_1)$$

$$c_1 = -1$$

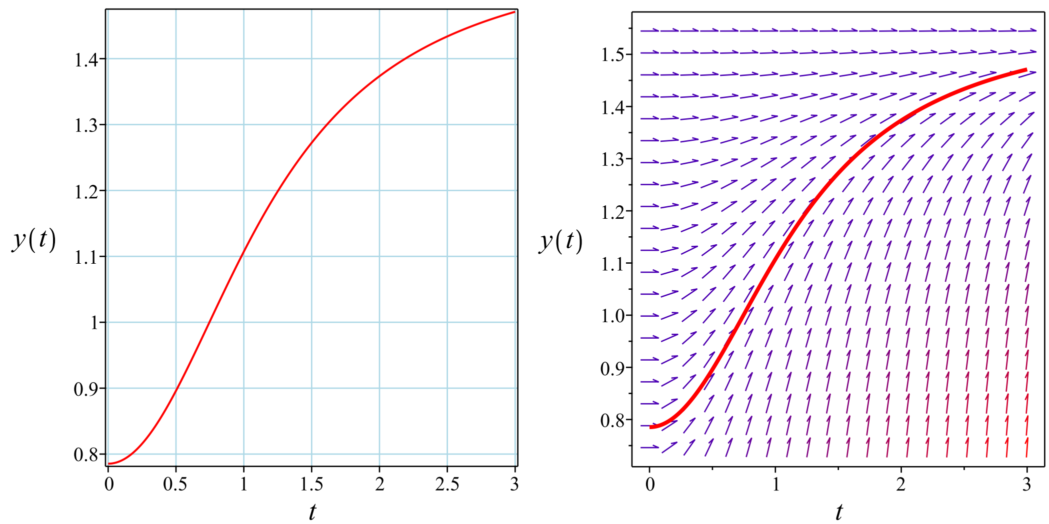
Substituting c_1 found above in the general solution gives

$$y = \arctan(t^2 + 1)$$

Summary

The solution(s) found are the following

$$y = \arctan(t^2 + 1) \tag{1}$$



(a) Solution plot (b) Slope field plot

<u>Verification of solutions</u>

$$y = \arctan\left(t^2 + 1\right)$$

Verified OK.

1.23.4 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x, y) + N(x, y)\frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x, y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x, y) = 0$$

Hence

$$\frac{\partial \phi}{\partial x} + \frac{\partial \phi}{\partial y}\frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$\begin{aligned}\frac{\partial \phi}{\partial x} &= M\\ \frac{\partial \phi}{\partial y} &= N\end{aligned}$$

But since $\frac{\partial^2\phi}{\partial x\partial y} = \frac{\partial^2\phi}{\partial y\partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x, y)$ but at least we know now that we can do that since the condition $\frac{\partial^2\phi}{\partial x\partial y} = \frac{\partial^2\phi}{\partial y\partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(t, y)\,\mathrm{d}t + N(t, y)\,\mathrm{d}y = 0 \tag{1A}$$

Therefore

$$\left(\frac{1}{2\cos(y)^2}\right)\mathrm{d}y = (t)\,\mathrm{d}t$$

$$(-t)\,\mathrm{d}t + \left(\frac{1}{2\cos(y)^2}\right)\mathrm{d}y = 0 \tag{2A}$$

Comparing (1A) and (2A) shows that

$$M(t, y) = -t$$
$$N(t, y) = \frac{1}{2\cos(y)^2}$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial t}$$

Using result found above gives

$$\frac{\partial M}{\partial y} = \frac{\partial}{\partial y}(-t)$$
$$= 0$$

And

$$\frac{\partial N}{\partial t} = \frac{\partial}{\partial t}\left(\frac{1}{2\cos(y)^2}\right)$$
$$= 0$$

Since $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial t}$, then the ODE is <u>exact</u> The following equations are now set up to solve for the function $\phi(t, y)$

$$\frac{\partial \phi}{\partial t} = M \tag{1}$$
$$\frac{\partial \phi}{\partial y} = N \tag{2}$$

Integrating (1) w.r.t. t gives

$$\int \frac{\partial \phi}{\partial t} \, \mathrm{d}t = \int M \, \mathrm{d}t$$
$$\int \frac{\partial \phi}{\partial t} \, \mathrm{d}t = \int -t \, \mathrm{d}t$$
$$\phi = -\frac{t^2}{2} + f(y) \tag{3}$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both t and y. Taking derivative of equation (3) w.r.t y gives

$$\frac{\partial \phi}{\partial y} = 0 + f'(y) \tag{4}$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = \frac{1}{2\cos(y)^2}$. Therefore equation (4) becomes

$$\frac{1}{2\cos(y)^2} = 0 + f'(y) \tag{5}$$

Solving equation (5) for $f'(y)$ gives

$$f'(y) = \frac{1}{2\cos(y)^2}$$

Integrating the above w.r.t y gives

$$\int f'(y) \, \mathrm{d}y = \int \left(\frac{\sec(y)^2}{2}\right) \mathrm{d}y$$
$$f(y) = \frac{\tan(y)}{2} + c_1$$

Where c_1 is constant of integration. Substituting result found above for $f(y)$ into equation (3) gives ϕ

$$\phi = -\frac{t^2}{2} + \frac{\tan(y)}{2} + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = -\frac{t^2}{2} + \frac{\tan(y)}{2}$$

Initial conditions are used to solve for c_1. Substituting $t = 0$ and $y = \frac{\pi}{4}$ in the above solution gives an equation to solve for the constant of integration.

$$\frac{1}{2} = c_1$$

$$c_1 = \frac{1}{2}$$

Substituting c_1 found above in the general solution gives

$$-\frac{t^2}{2} + \frac{\tan(y)}{2} = \frac{1}{2}$$

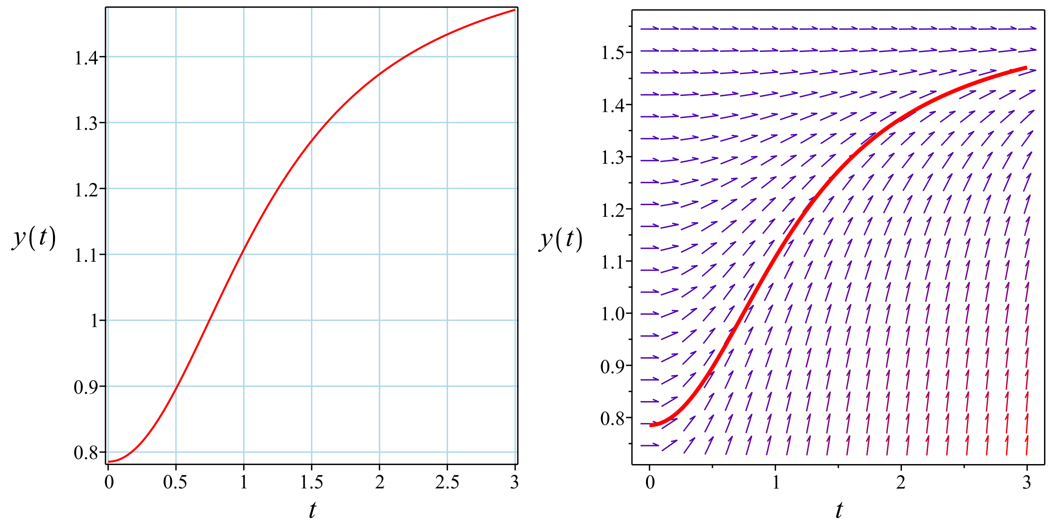
Solving for y from the above gives

$$y = \arctan(t^2 + 1)$$

Summary

The solution(s) found are the following

$$y = \arctan(t^2 + 1) \tag{1}$$



(a) Solution plot (b) Slope field plot

Verification of solutions

$$y = \arctan\left(t^2 + 1\right)$$

Verified OK.

1.23.5 Maple step by step solution

Let's solve

$\left[y' - 2t\cos\left(y\right)^2 = 0, y(0) = \frac{\pi}{4}\right]$

- Highest derivative means the order of the ODE is 1

  y'

- Separate variables

  $\frac{y'}{\cos(y)^2} = 2t$

- Integrate both sides with respect to t

  $\int \frac{y'}{\cos(y)^2}dt = \int 2tdt + c_1$

- Evaluate integral

  $\tan\left(y\right) = t^2 + c_1$

- Solve for y

  $y = \arctan\left(t^2 + c_1\right)$

- Use initial condition $y(0) = \frac{\pi}{4}$

  $\frac{\pi}{4} = \arctan(c_1)$
- Solve for c_1

  $c_1 = 1$
- Substitute $c_1 = 1$ into general solution and simplify

  $y = \arctan(t^2 + 1)$
- Solution to the IVP

  $y = \arctan(t^2 + 1)$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
<- separable successful`
```

✓ Solution by Maple

Time used: 0.094 (sec). Leaf size: 10

```
dsolve([diff(y(t),t)=2*t*cos(y(t))^2,y(0) = 1/4*Pi],y(t), singsol=all)
```

$$y(t) = \arctan(t^2 + 1)$$

✓ Solution by Mathematica

Time used: 0.428 (sec). Leaf size: 11

```
DSolve[{y'[t]==2*t*Cos[y[t]]^2,{y[0]==Pi/4}},y[t],t,IncludeSingularSolutions -> True]
```

$$y(t) \to \arctan(t^2 + 1)$$

1.24 problem 24

Internal problem ID [4935]
Internal file name [`OUTPUT/4428_Sunday_June_05_2022_01_19_45_PM_60921594/index.tex`]

Book: Fundamentals of Differential Equations. By Nagle, Saff and Snider. 9th edition. Boston. Pearson 2018.
Section: Chapter 2, First order differential equations. Section 2.2, Separable Equations. Exercises. page 46
Problem number: 24.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"exact", "separable", "first order special form ID 1", "first_order_ode_lie_symmetry_lookup"**

Maple gives the following as the ode type

`[_separable]`

$$\boxed{y' - 8x^3 \mathrm{e}^{-2y} = 0}$$

With initial conditions

$$[y(1) = 0]$$

1.24.1 Existence and uniqueness analysis

This is non linear first order ODE. In canonical form it is written as

$$\begin{aligned} y' &= f(x,y) \\ &= 8x^3 \mathrm{e}^{-2y} \end{aligned}$$

The x domain of $f(x,y)$ when $y=0$ is

$$\{-\infty < x < \infty\}$$

And the point $x_0 = 1$ is inside this domain. The y domain of $f(x,y)$ when $x=1$ is

$$\{-\infty < y < \infty\}$$

And the point $y_0 = 0$ is inside this domain. Now we will look at the continuity of

$$\begin{aligned} \frac{\partial f}{\partial y} &= \frac{\partial}{\partial y}\left(8x^3 \mathrm{e}^{-2y}\right) \\ &= -16x^3 \mathrm{e}^{-2y} \end{aligned}$$

The x domain of $\frac{\partial f}{\partial y}$ when $y=0$ is

$$\{-\infty < x < \infty\}$$

And the point $x_0 = 1$ is inside this domain. The y domain of $\frac{\partial f}{\partial y}$ when $x=1$ is

$$\{-\infty < y < \infty\}$$

And the point $y_0 = 0$ is inside this domain. Therefore solution exists and is unique.

1.24.2 Solving as separable ode

In canonical form the ODE is

$$\begin{aligned} y' &= F(x,y) \\ &= f(x)g(y) \\ &= 8x^3 \mathrm{e}^{-2y} \end{aligned}$$

Where $f(x) = 8x^3$ and $g(y) = \mathrm{e}^{-2y}$. Integrating both sides gives

$$\frac{1}{\mathrm{e}^{-2y}}\, dy = 8x^3\, dx$$

$$\int \frac{1}{\mathrm{e}^{-2y}}\, dy = \int 8x^3\, dx$$

$$\frac{\mathrm{e}^{2y}}{2} = 2x^4 + c_1$$

Which results in

$$y = -\frac{\ln\left(\frac{1}{4x^4+2c_1}\right)}{2}$$

Initial conditions are used to solve for c_1. Substituting $x = 1$ and $y = 0$ in the above solution gives an equation to solve for the constant of integration.

$$0 = \frac{\ln(2)}{2} - \frac{\ln\left(\frac{1}{2+c_1}\right)}{2}$$

$$c_1 = -\frac{3}{2}$$

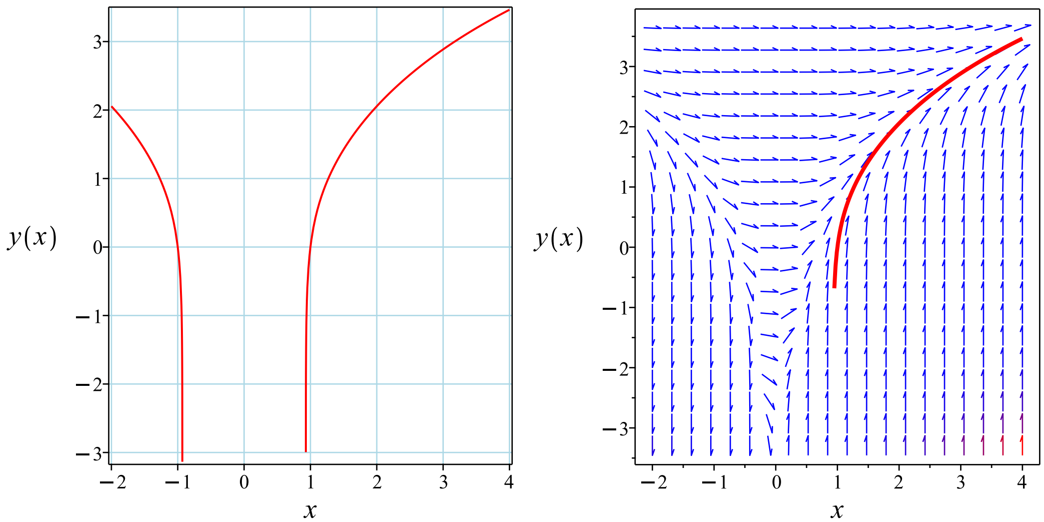
Substituting c_1 found above in the general solution gives

$$y = -\frac{\ln\left(\frac{1}{4x^4-3}\right)}{2}$$

Summary

The solution(s) found are the following

$$y = -\frac{\ln\left(\frac{1}{4x^4-3}\right)}{2} \tag{1}$$



(a) Solution plot

(b) Slope field plot

<u>Verification of solutions</u>

$$y = -\frac{\ln\left(\frac{1}{4x^4-3}\right)}{2}$$

Verified OK.

1.24.3 Solving as first order special form ID 1 ode

Writing the ode as

$$y' = 8x^3 e^{-2y} \tag{1}$$

And using the substitution $u = e^{2y}$ then

$$u' = 2y' e^{2y}$$

The above shows that

$$\begin{aligned} y' &= \frac{u'(x)\, e^{-2y}}{2} \\ &= \frac{u'(x)}{2u} \end{aligned}$$

Substituting this in (1) gives

$$\frac{u'(x)}{2u} = \frac{8x^3}{u}$$

The above simplifies to

$$u'(x) = 16x^3 \tag{2}$$

Now ode (2) is solved for $u(x)$ Integrating both sides gives

$$\begin{aligned} u(x) &= \int 16x^3 \,\mathrm{d}x \\ &= 4x^4 + c_1 \end{aligned}$$

Substituting the solution found for $u(x)$ in $u = \mathrm{e}^{2y}$ gives

$$\begin{aligned} y &= \frac{\ln(u(x))}{2} \\ &= \frac{\ln(4x^4 + c_1)}{2} \\ &= \frac{\ln(4x^4 + c_1)}{2} \end{aligned}$$

Initial conditions are used to solve for c_1. Substituting $x = 1$ and $y = 0$ in the above solution gives an equation to solve for the constant of integration.

$$0 = \frac{\ln(4 + c_1)}{2}$$

$$c_1 = -3$$

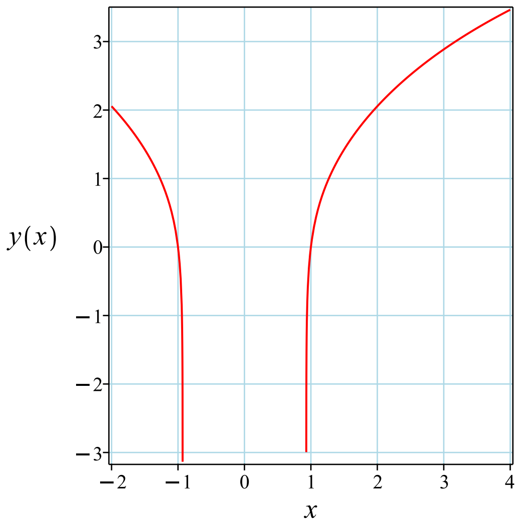
Substituting c_1 found above in the general solution gives

$$y = \frac{\ln(4x^4 - 3)}{2}$$

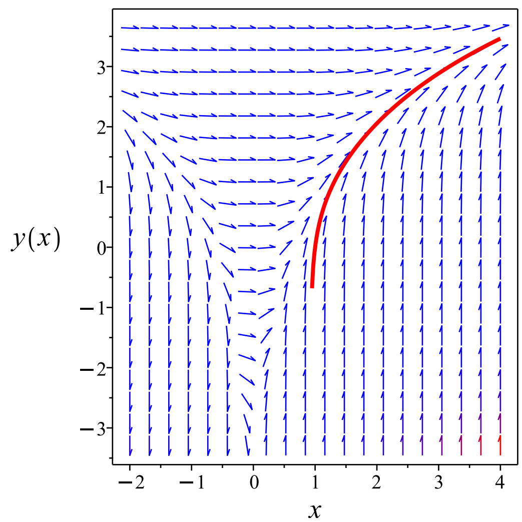
Summary

The solution(s) found are the following

$$y = \frac{\ln(4x^4 - 3)}{2} \tag{1}$$



(a) Solution plot



(b) Slope field plot

<u>Verification of solutions</u>

$$y = \frac{\ln(4x^4 - 3)}{2}$$

Verified OK.

1.24.4 Solving as first order ode lie symmetry lookup ode

Writing the ode as

$$y' = 8x^3 e^{-2y}$$
$$y' = \omega(x, y)$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x + \omega(\eta_y - \xi_x) - \omega^2 \xi_y - \omega_x \xi - \omega_y \eta = 0 \tag{A}$$

The type of this ode is known. It is of type `separable`. Therefore we do not need to solve the PDE (A), and can just use the lookup table shown below to find ξ, η

Table 57: Lie symmetry infinitesimal lookup table for known first order ODE's

ODE class	Form	ξ	η
linear ode	$y' = f(x)y(x) + g(x)$	0	$e^{\int f dx}$
separable ode	$y' = f(x)\, g(y)$	$\frac{1}{f}$	0
quadrature ode	$y' = f(x)$	0	1
quadrature ode	$y' = g(y)$	1	0
homogeneous ODEs of Class A	$y' = f\left(\frac{y}{x}\right)$	x	y
homogeneous ODEs of Class C	$y' = (a + bx + cy)^{\frac{n}{m}}$	1	$-\frac{b}{c}$
homogeneous class D	$y' = \frac{y}{x} + g(x)\, F\left(\frac{y}{x}\right)$	x^2	xy
First order special form ID 1	$y' = g(x)\, e^{h(x)+by} + f(x)$	$\frac{e^{-\int bf(x)dx - h(x)}}{g(x)}$	$\frac{f(x)e^{-\int bf(x)dx - h(x)}}{g(x)}$
polynomial type ode	$y' = \frac{a_1x+b_1y+c_1}{a_2x+b_2y+c_2}$	$\frac{a_1b_2x - a_2b_1x - b_1c_2 + b_2c_1}{a_1b_2 - a_2b_1}$	$\frac{a_1b_2y - a_2b_1y - a_1c_2 - a_2c_1}{a_1b_2 - a_2b_1}$
Bernoulli ode	$y' = f(x)\, y + g(x)\, y^n$	0	$e^{-\int (n-1)f(x)dx} y^n$
Reduced Riccati	$y' = f_1(x)\, y + f_2(x)\, y^2$	0	$e^{-\int f_1 dx}$

The above table shows that

$$
\begin{aligned}
\xi(x, y) &= \frac{1}{8x^3} \\
\eta(x, y) &= 0 \qquad \text{(A1)}
\end{aligned}
$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x, y) \to (R, S)$ where (R, S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$
\frac{dx}{\xi} = \frac{dy}{\eta} = dS \qquad (1)
$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y}\right) S(x, y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the

canonical coordinates, where $S(R)$. Since $\eta = 0$ then in this special case

$$R = y$$

S is found from

$$\begin{aligned} S &= \int \frac{1}{\xi} dx \\ &= \int \frac{1}{\frac{1}{8x^3}} dx \end{aligned}$$

Which results in

$$S = 2x^4$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x,y) S_y}{R_x + \omega(x,y) R_y} \tag{2}$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x,y)$ is the right hand side of the original ode given by

$$\omega(x,y) = 8x^3 \mathrm{e}^{-2y}$$

Evaluating all the partial derivatives gives

$$\begin{aligned} R_x &= 0 \\ R_y &= 1 \\ S_x &= 8x^3 \\ S_y &= 0 \end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = \mathrm{e}^{2y} \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = \mathrm{e}^{2R}$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by

integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$S(R) = \frac{e^{2R}}{2} + c_1 \tag{4}$$

To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$2x^4 = \frac{e^{2y}}{2} + c_1$$

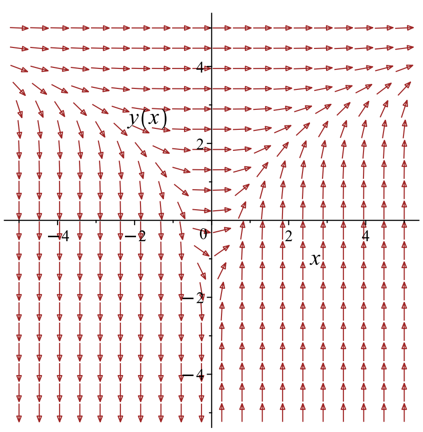
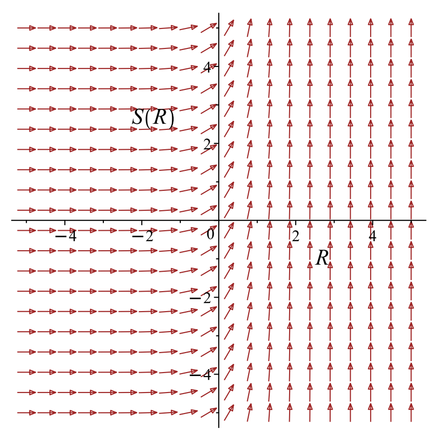
Which simplifies to

$$2x^4 = \frac{e^{2y}}{2} + c_1$$

Which gives

$$y = \frac{\ln(4x^4 - 2c_1)}{2}$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in x, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dx} = 8x^3 e^{-2y}$ 	$R = y$ $S = 2x^4$	$\frac{dS}{dR} = e^{2R}$ 

Initial conditions are used to solve for c_1. Substituting $x = 1$ and $y = 0$ in the above solution gives an equation to solve for the constant of integration.

$$0 = \frac{\ln(2)}{2} + \frac{\ln(2 - c_1)}{2}$$

$$c_1 = \frac{3}{2}$$

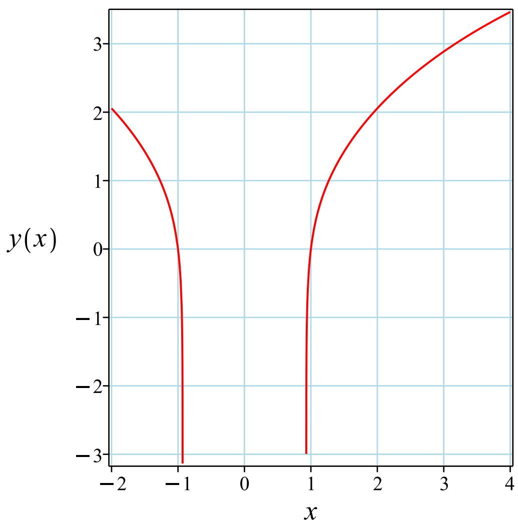
Substituting c_1 found above in the general solution gives

$$y = \frac{\ln(4x^4 - 3)}{2}$$

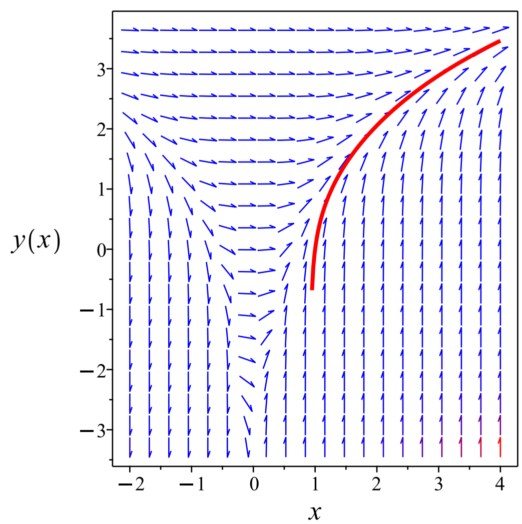
Summary

The solution(s) found are the following

$$y = \frac{\ln(4x^4 - 3)}{2} \tag{1}$$



(a) Solution plot



(b) Slope field plot

Verification of solutions

$$y = \frac{\ln(4x^4 - 3)}{2}$$

Verified OK.

1.24.5 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x,y)+N(x,y)\frac{dy}{dx}=0 \tag{A}$$

We assume there exists a function $\phi(x,y)=c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y)=0$$

Hence

$$\frac{\partial\phi}{\partial x}+\frac{\partial\phi}{\partial y}\frac{dy}{dx}=0 \tag{B}$$

Comparing (A,B) shows that

$$\begin{aligned}\frac{\partial\phi}{\partial x}&=M\\ \frac{\partial\phi}{\partial y}&=N\end{aligned}$$

But since $\frac{\partial^2\phi}{\partial x\partial y}=\frac{\partial^2\phi}{\partial y\partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y}=\frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x,y)$ but at least we know now that we can do that since the condition $\frac{\partial^2\phi}{\partial x\partial y}=\frac{\partial^2\phi}{\partial y\partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x,y)\,\mathrm{d}x+N(x,y)\,\mathrm{d}y=0 \tag{1A}$$

Therefore

$$\begin{aligned}\left(\frac{\mathrm{e}^{2y}}{8}\right)\mathrm{d}y&=\left(x^3\right)\mathrm{d}x\\ \left(-x^3\right)\mathrm{d}x+\left(\frac{\mathrm{e}^{2y}}{8}\right)\mathrm{d}y&=0 \end{aligned} \tag{2A}$$

Comparing (1A) and (2A) shows that

$$
\begin{aligned}
M(x,y) &= -x^3 \\
N(x,y) &= \frac{e^{2y}}{8}
\end{aligned}
$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$
\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}
$$

Using result found above gives

$$
\begin{aligned}
\frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}(-x^3) \\
&= 0
\end{aligned}
$$

And

$$
\begin{aligned}
\frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}\left(\frac{e^{2y}}{8}\right) \\
&= 0
\end{aligned}
$$

Since $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$, then the ODE is exact The following equations are now set up to solve for the function $\phi(x,y)$

$$
\frac{\partial \phi}{\partial x} = M \tag{1}
$$

$$
\frac{\partial \phi}{\partial y} = N \tag{2}
$$

Integrating (1) w.r.t. x gives

$$
\begin{aligned}
\int \frac{\partial \phi}{\partial x} \, dx &= \int M \, dx \\
\int \frac{\partial \phi}{\partial x} \, dx &= \int -x^3 \, dx
\end{aligned}
$$

$$
\phi = -\frac{x^4}{4} + f(y) \tag{3}
$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$
\frac{\partial \phi}{\partial y} = 0 + f'(y) \tag{4}
$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = \frac{e^{2y}}{8}$. Therefore equation (4) becomes

$$\frac{e^{2y}}{8} = 0 + f'(y) \tag{5}$$

Solving equation (5) for $f'(y)$ gives

$$\begin{aligned} f'(y) &= \frac{e^{2y}}{8} \\ &= \frac{e^{2y}}{8} \end{aligned}$$

Integrating the above w.r.t y results in

$$\begin{aligned} \int f'(y)\,\mathrm{d}y &= \int \left(\frac{e^{2y}}{8}\right) \mathrm{d}y \\ f(y) &= \frac{e^{2y}}{16} + c_1 \end{aligned}$$

Where c_1 is constant of integration. Substituting result found above for $f(y)$ into equation (3) gives ϕ

$$\phi = -\frac{x^4}{4} + \frac{e^{2y}}{16} + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = -\frac{x^4}{4} + \frac{e^{2y}}{16}$$

The solution becomes

$$y = \frac{\ln(4x^4 + 16c_1)}{2}$$

Initial conditions are used to solve for c_1. Substituting $x = 1$ and $y = 0$ in the above solution gives an equation to solve for the constant of integration.

$$0 = \ln(2) + \frac{\ln(1 + 4c_1)}{2}$$

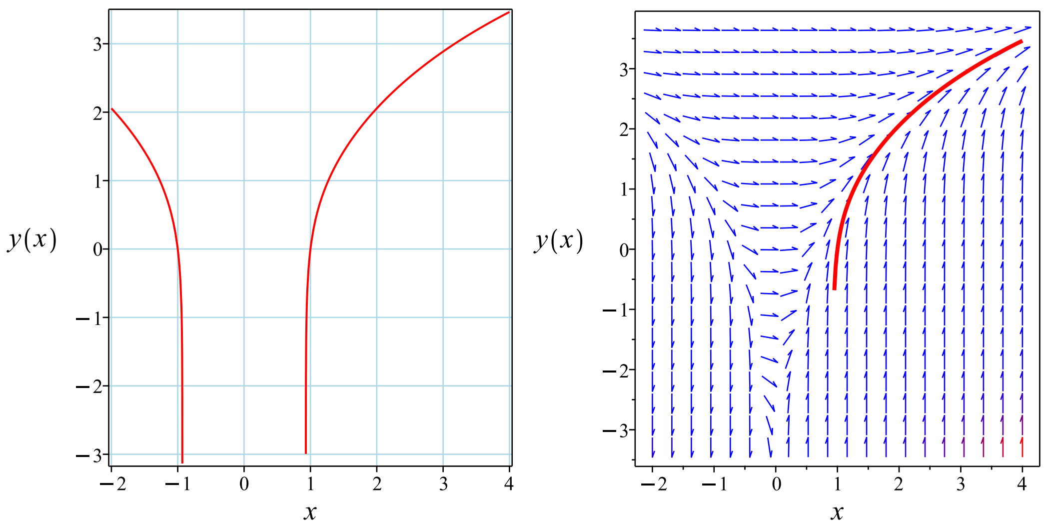
$$c_1 = -\frac{3}{16}$$

Substituting c_1 found above in the general solution gives

$$y = \frac{\ln(4x^4 - 3)}{2}$$

Summary
The solution(s) found are the following

$$y = \frac{\ln(4x^4 - 3)}{2} \tag{1}$$



(a) Solution plot

(b) Slope field plot

Verification of solutions

$$y = \frac{\ln(4x^4 - 3)}{2}$$

Verified OK.

1.24.6 Maple step by step solution

Let's solve

$[y' - 8x^3 \mathrm{e}^{-2y} = 0, y(1) = 0]$

- Highest derivative means the order of the ODE is 1

  y'

- Separate variables

  $\frac{y'}{\mathrm{e}^{-2y}} = 8x^3$

- Integrate both sides with respect to x

  $\int \frac{y'}{\mathrm{e}^{-2y}} dx = \int 8x^3 dx + c_1$

- Evaluate integral

  $\frac{1}{2\,\mathrm{e}^{-2y}} = 2x^4 + c_1$

- Solve for y

  $y = -\frac{\ln\left(\frac{1}{2(2x^4+c_1)}\right)}{2}$

- Use initial condition $y(1) = 0$

  $0 = -\frac{\ln\left(\frac{1}{2(2+c_1)}\right)}{2}$

- Solve for c_1

  $c_1 = -\frac{3}{2}$

- Substitute $c_1 = -\frac{3}{2}$ into general solution and simplify

  $y = -\frac{\ln\left(\frac{1}{4x^4-3}\right)}{2}$

- Solution to the IVP

  $y = -\frac{\ln\left(\frac{1}{4x^4-3}\right)}{2}$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
<- separable successful`
```

✓ Solution by Maple

Time used: 0.047 (sec). Leaf size: 14

```
dsolve([diff(y(x),x)=8*x^3*exp(-2*y(x)),y(1) = 0],y(x), singsol=all)
```

$$y(x) = \frac{\ln(4x^4 - 3)}{2}$$

✓ Solution by Mathematica

Time used: 0.346 (sec). Leaf size: 17

```
DSolve[{y'[x]==8*x^3*Exp[-2*y[x]],{y[1]==0}},y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \frac{1}{2}\log(4x^4 - 3)$$

1.25 problem 25

Internal problem ID [4936]
Internal file name [`OUTPUT/4429_Sunday_June_05_2022_01_19_56_PM_90702917/index.tex`]

Book: Fundamentals of Differential Equations. By Nagle, Saff and Snider. 9th edition. Boston. Pearson 2018.
Section: Chapter 2, First order differential equations. Section 2.2, Separable Equations. Exercises. page 46
Problem number: 25.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"exact", "linear", "separable", "first_order_ode_lie_symmetry_lookup"**

Maple gives the following as the ode type

```
[_separable]
```

$$\boxed{y' - x^2(1+y) = 0}$$

With initial conditions

$$[y(0) = 3]$$

1.25.1 Existence and uniqueness analysis

This is a linear ODE. In canonical form it is written as

$$y' + p(x)y = q(x)$$

Where here

$$\begin{aligned} p(x) &= -x^2 \\ q(x) &= x^2 \end{aligned}$$

Hence the ode is

$$y' - yx^2 = x^2$$

The domain of $p(x) = -x^2$ is

$$\{-\infty < x < \infty\}$$

And the point $x_0 = 0$ is inside this domain. The domain of $q(x) = x^2$ is

$$\{-\infty < x < \infty\}$$

And the point $x_0 = 0$ is also inside this domain. Hence solution exists and is unique.

1.25.2 Solving as separable ode

In canonical form the ODE is

$$\begin{aligned} y' &= F(x,y) \\ &= f(x)g(y) \\ &= x^2(1+y) \end{aligned}$$

Where $f(x) = x^2$ and $g(y) = 1 + y$. Integrating both sides gives

$$\frac{1}{1+y}\,dy = x^2\,dx$$

$$\int \frac{1}{1+y}\,dy = \int x^2\,dx$$

$$\ln(1+y) = \frac{x^3}{3} + c_1$$

Raising both side to exponential gives

$$1 + y = e^{\frac{x^3}{3} + c_1}$$

Which simplifies to

$$1 + y = c_2 e^{\frac{x^3}{3}}$$

Initial conditions are used to solve for c_1. Substituting $x = 0$ and $y = 3$ in the above solution gives an equation to solve for the constant of integration.

$$3 = c_2 e^{c_1} - 1$$

$$c_1 = \ln\left(\frac{4}{c_2}\right)$$

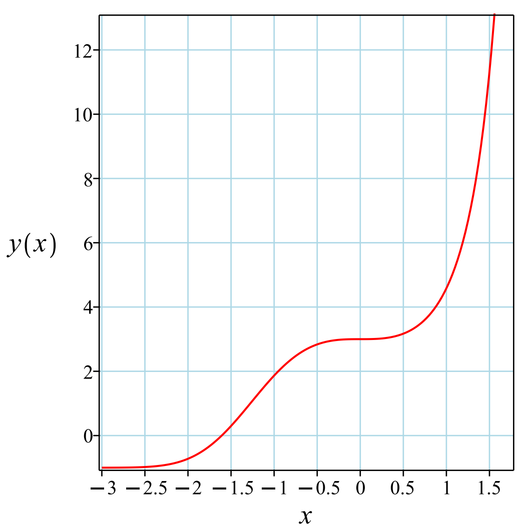
Substituting c_1 found above in the general solution gives

$$y = 4\,e^{\frac{x^3}{3}} - 1$$

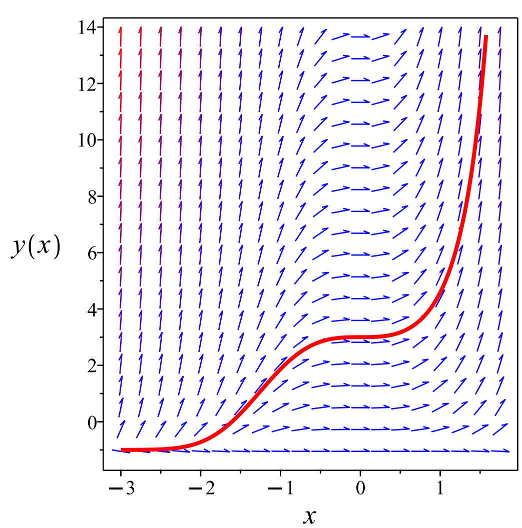
Summary

The solution(s) found are the following

$$y = 4\,e^{\frac{x^3}{3}} - 1 \tag{1}$$



(a) Solution plot



(b) Slope field plot

Verification of solutions

$$y = 4\,e^{\frac{x^3}{3}} - 1$$

Verified OK.

1.25.3 Solving as linear ode

Entering Linear first order ODE solver. The integrating factor μ is

$$\begin{aligned}\mu &= \mathrm{e}^{\int -x^2 dx} \\ &= \mathrm{e}^{-\frac{x^3}{3}}\end{aligned}$$

The ode becomes

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}x}(\mu y) &= (\mu)\left(x^2\right) \\ \frac{\mathrm{d}}{\mathrm{d}x}\left(\mathrm{e}^{-\frac{x^3}{3}} y\right) &= \left(\mathrm{e}^{-\frac{x^3}{3}}\right)\left(x^2\right) \\ \mathrm{d}\left(\mathrm{e}^{-\frac{x^3}{3}} y\right) &= \left(x^2 \mathrm{e}^{-\frac{x^3}{3}}\right) \mathrm{d}x\end{aligned}$$

Integrating gives

$$\begin{aligned}\mathrm{e}^{-\frac{x^3}{3}} y &= \int x^2 \mathrm{e}^{-\frac{x^3}{3}} \mathrm{d}x \\ \mathrm{e}^{-\frac{x^3}{3}} y &= -\mathrm{e}^{-\frac{x^3}{3}} + c_1\end{aligned}$$

Dividing both sides by the integrating factor $\mu = \mathrm{e}^{-\frac{x^3}{3}}$ results in

$$y = -\mathrm{e}^{\frac{x^3}{3}} \mathrm{e}^{-\frac{x^3}{3}} + c_1 \mathrm{e}^{\frac{x^3}{3}}$$

which simplifies to

$$y = -1 + c_1 \mathrm{e}^{\frac{x^3}{3}}$$

Initial conditions are used to solve for c_1. Substituting $x = 0$ and $y = 3$ in the above solution gives an equation to solve for the constant of integration.

$$3 = -1 + c_1$$

$$c_1 = 4$$

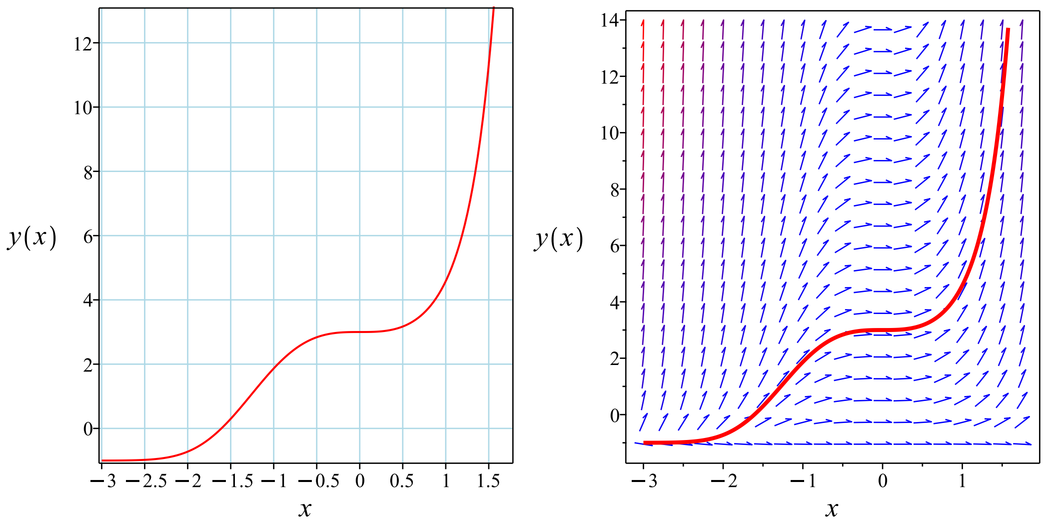
Substituting c_1 found above in the general solution gives

$$y = 4\,\mathrm{e}^{\frac{x^3}{3}} - 1$$

Summary

The solution(s) found are the following

$$y = 4\,\mathrm{e}^{\frac{x^3}{3}} - 1 \tag{1}$$



(a) Solution plot (b) Slope field plot

<u>Verification of solutions</u>

$$y = 4\,\mathrm{e}^{\frac{x^3}{3}} - 1$$

Verified OK.

1.25.4 Solving as first order ode lie symmetry lookup ode

Writing the ode as

$$y' = x^2(1+y)$$
$$y' = \omega(x,y)$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x + \omega(\eta_y - \xi_x) - \omega^2\xi_y - \omega_x\xi - \omega_y\eta = 0 \tag{A}$$

The type of this ode is known. It is of type `linear`. Therefore we do not need to solve the PDE (A), and can just use the lookup table shown below to find ξ, η

Table 60: Lie symmetry infinitesimal lookup table for known first order ODE's

ODE class	Form	ξ	η
linear ode	$y' = f(x)y(x) + g(x)$	0	$e^{\int f dx}$
separable ode	$y' = f(x)\, g(y)$	$\frac{1}{f}$	0
quadrature ode	$y' = f(x)$	0	1
quadrature ode	$y' = g(y)$	1	0
homogeneous ODEs of Class A	$y' = f\left(\frac{y}{x}\right)$	x	y
homogeneous ODEs of Class C	$y' = (a + bx + cy)^{\frac{n}{m}}$	1	$-\frac{b}{c}$
homogeneous class D	$y' = \frac{y}{x} + g(x)\, F\left(\frac{y}{x}\right)$	x^2	xy
First order special form ID 1	$y' = g(x)\, e^{h(x)+by} + f(x)$	$\frac{e^{-\int bf(x)dx - h(x)}}{g(x)}$	$\frac{f(x)e^{-\int bf(x)dx - h(x)}}{g(x)}$
polynomial type ode	$y' = \frac{a_1x+b_1y+c_1}{a_2x+b_2y+c_2}$	$\frac{a_1b_2x-a_2b_1x-b_1c_2+b_2c_1}{a_1b_2-a_2b_1}$	$\frac{a_1b_2y-a_2b_1y-a_1c_2-a_2c_1}{a_1b_2-a_2b_1}$
Bernoulli ode	$y' = f(x)\, y + g(x)\, y^n$	0	$e^{-\int (n-1)f(x)dx} y^n$
Reduced Riccati	$y' = f_1(x)\, y + f_2(x)\, y^2$	0	$e^{-\int f_1 dx}$

The above table shows that

$$
\begin{aligned}
\xi(x,y) &= 0 \\
\eta(x,y) &= \mathrm{e}^{\frac{x^3}{3}}
\end{aligned}
\tag{A1}
$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x,y) \to (R,S)$ where (R,S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$
\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}
$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y}\right) S(x,y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the

canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$R = x$$

S is found from

$$\begin{aligned} S &= \int \frac{1}{\eta} dy \\ &= \int \frac{1}{e^{\frac{x^3}{3}}} dy \end{aligned}$$

Which results in

$$S = e^{-\frac{x^3}{3}} y$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x, y) S_y}{R_x + \omega(x, y) R_y} \tag{2}$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x, y)$ is the right hand side of the original ode given by

$$\omega(x, y) = x^2(1 + y)$$

Evaluating all the partial derivatives gives

$$\begin{aligned} R_x &= 1 \\ R_y &= 0 \\ S_x &= -x^2 e^{-\frac{x^3}{3}} y \\ S_y &= e^{-\frac{x^3}{3}} \end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = x^2 e^{-\frac{x^3}{3}} \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = R^2 e^{-\frac{R^3}{3}}$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by

integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$S(R) = -\mathrm{e}^{-\frac{R^3}{3}} + c_1 \tag{4}$$

To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$\mathrm{e}^{-\frac{x^3}{3}} y = -\mathrm{e}^{-\frac{x^3}{3}} + c_1$$

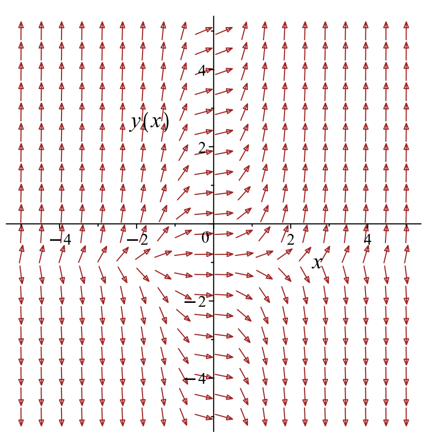
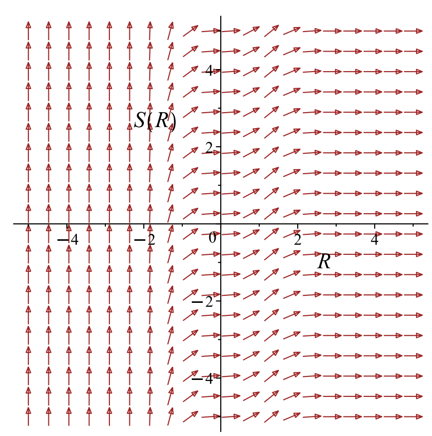
Which simplifies to

$$\mathrm{e}^{-\frac{x^3}{3}} y = -\mathrm{e}^{-\frac{x^3}{3}} + c_1$$

Which gives

$$y = -\left(\mathrm{e}^{-\frac{x^3}{3}} - c_1\right) \mathrm{e}^{\frac{x^3}{3}}$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in x, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dx} = x^2(1 + y)$ 	$R = x$ $S = \mathrm{e}^{-\frac{x^3}{3}} y$	$\frac{dS}{dR} = R^2 \mathrm{e}^{-\frac{R^3}{3}}$ 

Initial conditions are used to solve for c_1. Substituting $x = 0$ and $y = 3$ in the above solution gives an equation to solve for the constant of integration.

$$3 = -1 + c_1$$

$$
c_1 = 4
$$

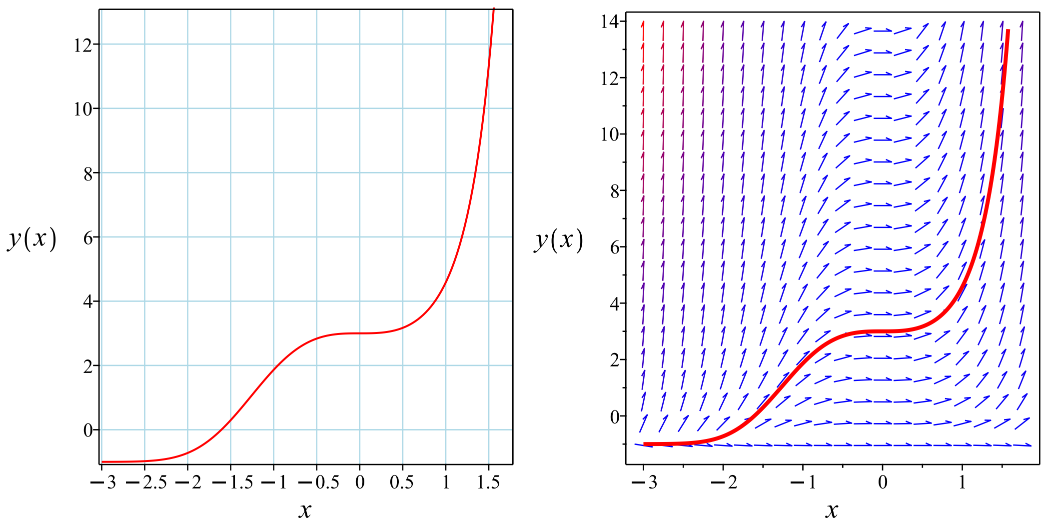
Substituting c_1 found above in the general solution gives

$$
y = 4\,\mathrm{e}^{\frac{x^3}{3}} - 1
$$

Summary

The solution(s) found are the following

$$
y = 4\,\mathrm{e}^{\frac{x^3}{3}} - 1 \tag{1}
$$



(a) Solution plot

(b) Slope field plot

Verification of solutions

$$
y = 4\,\mathrm{e}^{\frac{x^3}{3}} - 1
$$

Verified OK.

1.25.5 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$
M(x, y) + N(x, y)\frac{dy}{dx} = 0 \tag{A}
$$

We assume there exists a function $\phi(x, y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$
\frac{d}{dx}\phi(x, y) = 0
$$

Hence

$$\frac{\partial \phi}{\partial x}+\frac{\partial \phi}{\partial y}\frac{dy}{dx}=0 \tag{B}$$

Comparing (A,B) shows that

$$\begin{aligned}\frac{\partial \phi}{\partial x}&=M\\ \frac{\partial \phi}{\partial y}&=N\end{aligned}$$

But since $\frac{\partial^2\phi}{\partial x\partial y}=\frac{\partial^2\phi}{\partial y\partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y}=\frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x,y)$ but at least we know now that we can do that since the condition $\frac{\partial^2\phi}{\partial x\partial y}=\frac{\partial^2\phi}{\partial y\partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x,y)\,\mathrm{d}x+N(x,y)\,\mathrm{d}y=0 \tag{1A}$$

Therefore

$$\begin{aligned}\left(\frac{1}{1+y}\right)\mathrm{d}y&=\left(x^2\right)\mathrm{d}x\\ \left(-x^2\right)\mathrm{d}x+\left(\frac{1}{1+y}\right)\mathrm{d}y&=0 \end{aligned}\tag{2A}$$

Comparing (1A) and (2A) shows that

$$\begin{aligned}M(x,y)&=-x^2\\ N(x,y)&=\frac{1}{1+y}\end{aligned}$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y}=\frac{\partial N}{\partial x}$$

Using result found above gives

$$\begin{aligned}\frac{\partial M}{\partial y}&=\frac{\partial}{\partial y}\left(-x^2\right)\\ &=0\end{aligned}$$

And

$$
\begin{aligned}
\frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}\left(\frac{1}{1+y}\right) \\
&= 0
\end{aligned}
$$

Since $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$, then the ODE is exact The following equations are now set up to solve for the function $\phi(x, y)$

$$
\begin{aligned}
\frac{\partial \phi}{\partial x} &= M \qquad (1) \\
\frac{\partial \phi}{\partial y} &= N \qquad (2)
\end{aligned}
$$

Integrating (1) w.r.t. x gives

$$
\begin{aligned}
\int \frac{\partial \phi}{\partial x} \, \mathrm{d}x &= \int M \, \mathrm{d}x \\
\int \frac{\partial \phi}{\partial x} \, \mathrm{d}x &= \int -x^2 \, \mathrm{d}x \\
\phi &= -\frac{x^3}{3} + f(y) \qquad (3)
\end{aligned}
$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$
\frac{\partial \phi}{\partial y} = 0 + f'(y) \qquad (4)
$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = \frac{1}{1+y}$. Therefore equation (4) becomes

$$
\frac{1}{1+y} = 0 + f'(y) \qquad (5)
$$

Solving equation (5) for $f'(y)$ gives

$$
f'(y) = \frac{1}{1+y}
$$

Integrating the above w.r.t y gives

$$
\begin{aligned}
\int f'(y) \, \mathrm{d}y &= \int \left(\frac{1}{1+y}\right) \mathrm{d}y \\
f(y) &= \ln(1+y) + c_1
\end{aligned}
$$

Where c_1 is constant of integration. Substituting result found above for $f(y)$ into equation (3) gives ϕ

$$\phi = -\frac{x^3}{3} + \ln(1+y) + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = -\frac{x^3}{3} + \ln(1+y)$$

The solution becomes

$$y = e^{\frac{x^3}{3}+c_1} - 1$$

Initial conditions are used to solve for c_1. Substituting $x = 0$ and $y = 3$ in the above solution gives an equation to solve for the constant of integration.

$$3 = e^{c_1} - 1$$

$$c_1 = 2\ln(2)$$

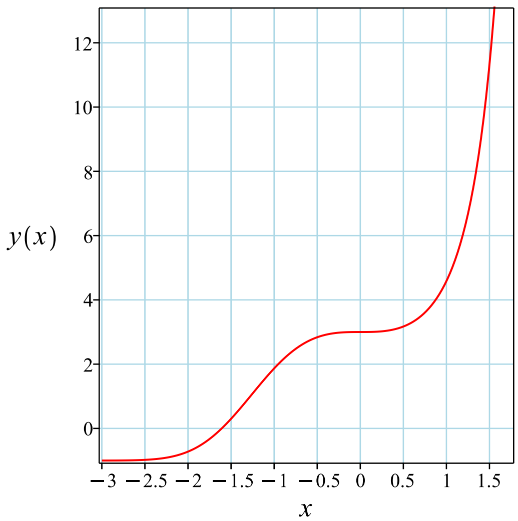
Substituting c_1 found above in the general solution gives

$$y = 4\,e^{\frac{x^3}{3}} - 1$$

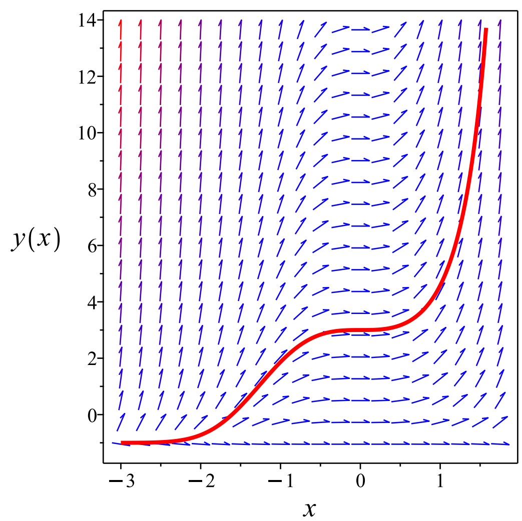
Summary

The solution(s) found are the following

$$y = 4\,e^{\frac{x^3}{3}} - 1 \tag{1}$$



(a) Solution plot



(b) Slope field plot

Verification of solutions

$$y = 4\,\mathrm{e}^{\frac{x^3}{3}} - 1$$

Verified OK.

1.25.6 Maple step by step solution

Let's solve

$[y' - x^2(1+y) = 0, y(0) = 3]$

- Highest derivative means the order of the ODE is 1

  y'

- Separate variables

  $\frac{y'}{1+y} = x^2$

- Integrate both sides with respect to x

  $\int \frac{y'}{1+y} dx = \int x^2 dx + c_1$

- Evaluate integral

  $\ln(1+y) = \frac{x^3}{3} + c_1$

- Solve for y

  $y = \mathrm{e}^{\frac{x^3}{3} + c_1} - 1$

- Use initial condition $y(0) = 3$

  $3 = e^{c_1} - 1$
- Solve for c_1

  $c_1 = 2\ln(2)$
- Substitute $c_1 = 2\ln(2)$ into general solution and simplify

  $y = 4\,e^{\frac{x^3}{3}} - 1$
- Solution to the IVP

  $y = 4\,e^{\frac{x^3}{3}} - 1$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
<- 1st order linear successful`
```

✓ Solution by Maple
Time used: 0.016 (sec). Leaf size: 14

```
dsolve([diff(y(x),x)=x^2*(1+y(x)),y(0) = 3],y(x), singsol=all)
```

$$y(x) = -1 + 4\,e^{\frac{x^3}{3}}$$

✓ Solution by Mathematica
Time used: 0.05 (sec). Leaf size: 18

```
DSolve[{y'[x]==x^2*(1+y[x]),{y[0]==3}},y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to 4e^{\frac{x^3}{3}} - 1$$

1.26 problem 26

Internal problem ID [4937]
Internal file name [`OUTPUT/4430_Sunday_June_05_2022_01_20_07_PM_35813157/index.tex`]

Book: Fundamentals of Differential Equations. By Nagle, Saff and Snider. 9th edition. Boston. Pearson 2018.
Section: Chapter 2, First order differential equations. Section 2.2, Separable Equations. Exercises. page 46
Problem number: 26.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"exact", "separable", "first_order_ode_lie_symmetry_lookup"**

Maple gives the following as the ode type

```
[_separable]
```

$$\boxed{\sqrt{y}+(x+1)\,y'=0}$$

With initial conditions

$$[y(0)=1]$$

1.26.1 Existence and uniqueness analysis

This is non linear first order ODE. In canonical form it is written as

$$\begin{aligned} y' &= f(x,y) \\ &= -\frac{\sqrt{y}}{x+1} \end{aligned}$$

The x domain of $f(x,y)$ when $y=1$ is

$$\{x<-1\vee -1<x\}$$

And the point $x_0 = 0$ is inside this domain. The y domain of $f(x,y)$ when $x = 0$ is

$$\{0 \leq y\}$$

And the point $y_0 = 1$ is inside this domain. Now we will look at the continuity of

$$\begin{aligned}\frac{\partial f}{\partial y} &= \frac{\partial}{\partial y}\left(-\frac{\sqrt{y}}{x+1}\right)\\ &= -\frac{1}{2\sqrt{y}\,(x+1)}\end{aligned}$$

The x domain of $\frac{\partial f}{\partial y}$ when $y = 1$ is

$$\{x < -1 \vee -1 < x\}$$

And the point $x_0 = 0$ is inside this domain. The y domain of $\frac{\partial f}{\partial y}$ when $x = 0$ is

$$\{0 < y\}$$

And the point $y_0 = 1$ is inside this domain. Therefore solution exists and is unique.

1.26.2 Solving as separable ode

In canonical form the ODE is

$$\begin{aligned}y' &= F(x,y)\\ &= f(x)g(y)\\ &= -\frac{\sqrt{y}}{x+1}\end{aligned}$$

Where $f(x) = -\frac{1}{x+1}$ and $g(y) = \sqrt{y}$. Integrating both sides gives

$$\begin{aligned}\frac{1}{\sqrt{y}}\,dy &= -\frac{1}{x+1}\,dx\\ \int \frac{1}{\sqrt{y}}\,dy &= \int -\frac{1}{x+1}\,dx\\ 2\sqrt{y} &= -\ln(x+1) + c_1\end{aligned}$$

The solution is

$$2\sqrt{y} + \ln(x+1) - c_1 = 0$$

Initial conditions are used to solve for c_1. Substituting $x = 0$ and $y = 1$ in the above solution gives an equation to solve for the constant of integration.

$$2 - c_1 = 0$$

$$c_1 = 2$$

Substituting c_1 found above in the general solution gives

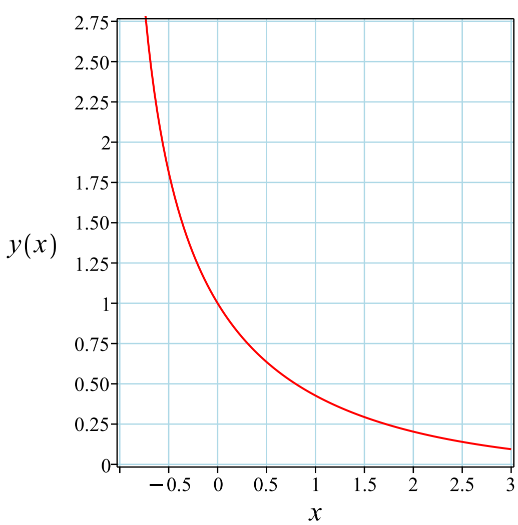
$$2\sqrt{y} + \ln(x+1) - 2 = 0$$

Solving for y from the above gives

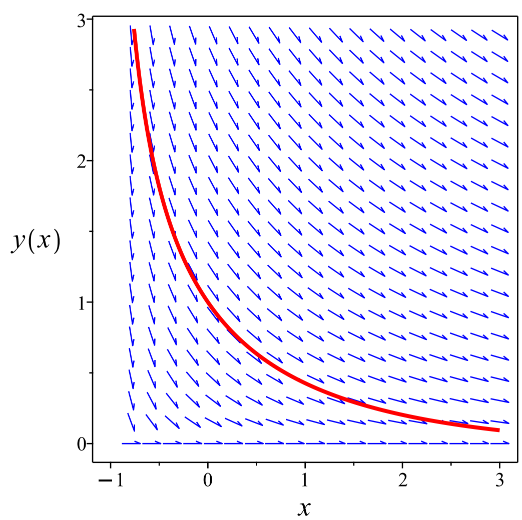
$$y = \frac{(\ln(x+1) - 2)^2}{4}$$

Summary
The solution(s) found are the following

$$y = \frac{(\ln(x+1) - 2)^2}{4} \tag{1}$$



(a) Solution plot



(b) Slope field plot

Verification of solutions

$$y = \frac{(\ln(x+1) - 2)^2}{4}$$

Verified OK.

1.26.3 Solving as first order ode lie symmetry lookup ode

Writing the ode as

$$y' = -\frac{\sqrt{y}}{x+1}$$
$$y' = \omega(x,y)$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x + \omega(\eta_y - \xi_x) - \omega^2 \xi_y - \omega_x \xi - \omega_y \eta = 0 \tag{A}$$

The type of this ode is known. It is of type `separable`. Therefore we do not need to solve the PDE (A), and can just use the lookup table shown below to find ξ, η

Table 63: Lie symmetry infinitesimal lookup table for known first order ODE's

ODE class	Form	ξ	η
linear ode	$y' = f(x)y(x) + g(x)$	0	$e^{\int f dx}$
separable ode	$y' = f(x)\, g(y)$	$\frac{1}{f}$	0
quadrature ode	$y' = f(x)$	0	1
quadrature ode	$y' = g(y)$	1	0
homogeneous ODEs of Class A	$y' = f\left(\frac{y}{x}\right)$	x	y
homogeneous ODEs of Class C	$y' = (a + bx + cy)^{\frac{n}{m}}$	1	$-\frac{b}{c}$
homogeneous class D	$y' = \frac{y}{x} + g(x)\, F\left(\frac{y}{x}\right)$	x^2	xy
First order special form ID 1	$y' = g(x)\, e^{h(x)+by} + f(x)$	$\frac{e^{-\int bf(x)dx - h(x)}}{g(x)}$	$\frac{f(x)e^{-\int bf(x)dx - h(x)}}{g(x)}$
polynomial type ode	$y' = \frac{a_1x+b_1y+c_1}{a_2x+b_2y+c_2}$	$\frac{a_1b_2x - a_2b_1x - b_1c_2 + b_2c_1}{a_1b_2 - a_2b_1}$	$\frac{a_1b_2y - a_2b_1y - a_1c_2 - a_2c_1}{a_1b_2 - a_2b_1}$
Bernoulli ode	$y' = f(x)\, y + g(x)\, y^n$	0	$e^{-\int (n-1)f(x)dx} y^n$
Reduced Riccati	$y' = f_1(x)\, y + f_2(x)\, y^2$	0	$e^{-\int f_1 dx}$

The above table shows that

$$\begin{aligned}\xi(x,y) &= -1 - x \\ \eta(x,y) &= 0\end{aligned} \tag{A1}$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x,y) \to (R,S)$ where (R,S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y}\right) S(x,y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the canonical coordinates, where $S(R)$. Since $\eta = 0$ then in this special case

$$R = y$$

S is found from

$$\begin{aligned}S &= \int \frac{1}{\xi} dx \\ &= \int \frac{1}{-1 - x} dx\end{aligned}$$

Which results in

$$S = -\ln(-1 - x)$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x,y) S_y}{R_x + \omega(x,y) R_y} \tag{2}$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x,y)$ is the right hand side of the original ode given by

$$\omega(x,y) = -\frac{\sqrt{y}}{x + 1}$$

Evaluating all the partial derivatives gives

$$
\begin{aligned}
R_x &= 0 \\
R_y &= 1 \\
S_x &= \frac{1}{-1-x} \\
S_y &= 0
\end{aligned}
$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$
\frac{dS}{dR} = \frac{1}{\sqrt{y}} \tag{2A}
$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$
\frac{dS}{dR} = \frac{1}{\sqrt{R}}
$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$
S(R) = 2\sqrt{R} + c_1 \tag{4}
$$

To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$
-\ln(-1-x) = 2\sqrt{y} + c_1
$$

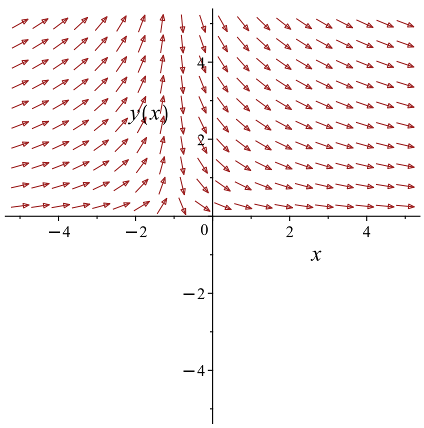
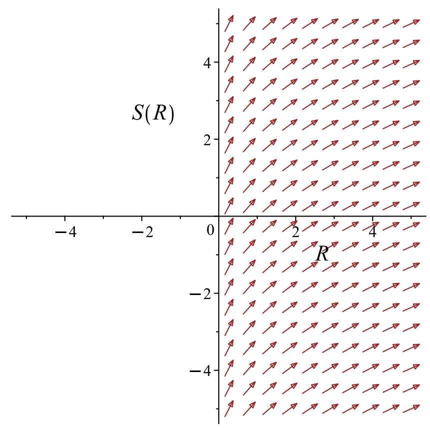
Which simplifies to

$$
-\ln(-1-x) = 2\sqrt{y} + c_1
$$

Which gives

$$
y = \frac{\ln(-1-x)^2}{4} + \frac{\ln(-1-x)\, c_1}{2} + \frac{c_1^2}{4}
$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in x, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dx} = -\frac{\sqrt{y}}{x+1}$ 	$R = y$ $S = -\ln(-1-x)$	$\frac{dS}{dR} = \frac{1}{\sqrt{R}}$ 

Initial conditions are used to solve for c_1. Substituting $x = 0$ and $y = 1$ in the above solution gives an equation to solve for the constant of integration.

$$1 = -\frac{1}{4}\pi^2 + \frac{1}{2}i\pi c_1 + \frac{1}{4}c_1^2$$

$$c_1 = -i\pi - 2$$

Substituting c_1 found above in the general solution gives

$$y = \frac{\ln(-1-x)^2}{4} - \frac{i\ln(-1-x)\pi}{2} - \ln(-1-x) - \frac{\pi^2}{4} + i\pi + 1$$

Summary

The solution(s) found are the following

$$y = \frac{\ln(-1-x)^2}{4} - \frac{i\ln(-1-x)\pi}{2} - \ln(-1-x) - \frac{\pi^2}{4} + i\pi + 1 \tag{1}$$

Verification of solutions

$$y = \frac{\ln(-1-x)^2}{4} - \frac{i\ln(-1-x)\pi}{2} - \ln(-1-x) - \frac{\pi^2}{4} + i\pi + 1$$

Verified OK.

1.26.4 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x,y)+N(x,y)\frac{dy}{dx}=0 \tag{A}$$

We assume there exists a function $\phi(x,y)=c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y)=0$$

Hence

$$\frac{\partial\phi}{\partial x}+\frac{\partial\phi}{\partial y}\frac{dy}{dx}=0 \tag{B}$$

Comparing (A,B) shows that

$$\begin{aligned}\frac{\partial\phi}{\partial x}&=M\\ \frac{\partial\phi}{\partial y}&=N\end{aligned}$$

But since $\frac{\partial^2\phi}{\partial x\partial y}=\frac{\partial^2\phi}{\partial y\partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y}=\frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x,y)$ but at least we know now that we can do that since the condition $\frac{\partial^2\phi}{\partial x\partial y}=\frac{\partial^2\phi}{\partial y\partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x,y)\,\mathrm{d}x+N(x,y)\,\mathrm{d}y=0 \tag{1A}$$

Therefore

$$\begin{aligned}\left(-\frac{1}{\sqrt{y}}\right)\mathrm{d}y&=\left(\frac{1}{x+1}\right)\mathrm{d}x\\ \left(-\frac{1}{x+1}\right)\mathrm{d}x+\left(-\frac{1}{\sqrt{y}}\right)\mathrm{d}y&=0\end{aligned} \tag{2A}$$

Comparing (1A) and (2A) shows that

$$\begin{aligned} M(x,y) &= -\frac{1}{x+1} \\ N(x,y) &= -\frac{1}{\sqrt{y}} \end{aligned}$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Using result found above gives

$$\begin{aligned} \frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}\left(-\frac{1}{x+1}\right) \\ &= 0 \end{aligned}$$

And

$$\begin{aligned} \frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}\left(-\frac{1}{\sqrt{y}}\right) \\ &= 0 \end{aligned}$$

Since $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$, then the ODE is <u>exact</u> The following equations are now set up to solve for the function $\phi(x,y)$

$$\frac{\partial \phi}{\partial x} = M \tag{1}$$

$$\frac{\partial \phi}{\partial y} = N \tag{2}$$

Integrating (1) w.r.t. x gives

$$\begin{aligned} \int \frac{\partial \phi}{\partial x} \,\mathrm{d}x &= \int M \,\mathrm{d}x \\ \int \frac{\partial \phi}{\partial x} \,\mathrm{d}x &= \int -\frac{1}{x+1} \,\mathrm{d}x \\ \phi &= -\ln(x+1) + f(y) \end{aligned} \tag{3}$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$\frac{\partial \phi}{\partial y} = 0 + f'(y) \tag{4}$$

But equation (2) says that $\frac{\partial\phi}{\partial y} = -\frac{1}{\sqrt{y}}$. Therefore equation (4) becomes

$$-\frac{1}{\sqrt{y}} = 0 + f'(y) \tag{5}$$

Solving equation (5) for $f'(y)$ gives

$$f'(y) = -\frac{1}{\sqrt{y}}$$

Integrating the above w.r.t y gives

$$\int f'(y)\,\mathrm{d}y = \int \left(-\frac{1}{\sqrt{y}}\right) \mathrm{d}y$$
$$f(y) = -2\sqrt{y} + c_1$$

Where c_1 is constant of integration. Substituting result found above for $f(y)$ into equation (3) gives ϕ

$$\phi = -\ln(x+1) - 2\sqrt{y} + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = -\ln(x+1) - 2\sqrt{y}$$

The solution becomes

$$y = \frac{\ln(x+1)^2}{4} + \frac{\ln(x+1)\,c_1}{2} + \frac{c_1^2}{4}$$

Initial conditions are used to solve for c_1. Substituting $x = 0$ and $y = 1$ in the above solution gives an equation to solve for the constant of integration.

$$1 = \frac{c_1^2}{4}$$

$$c_1 = -2$$

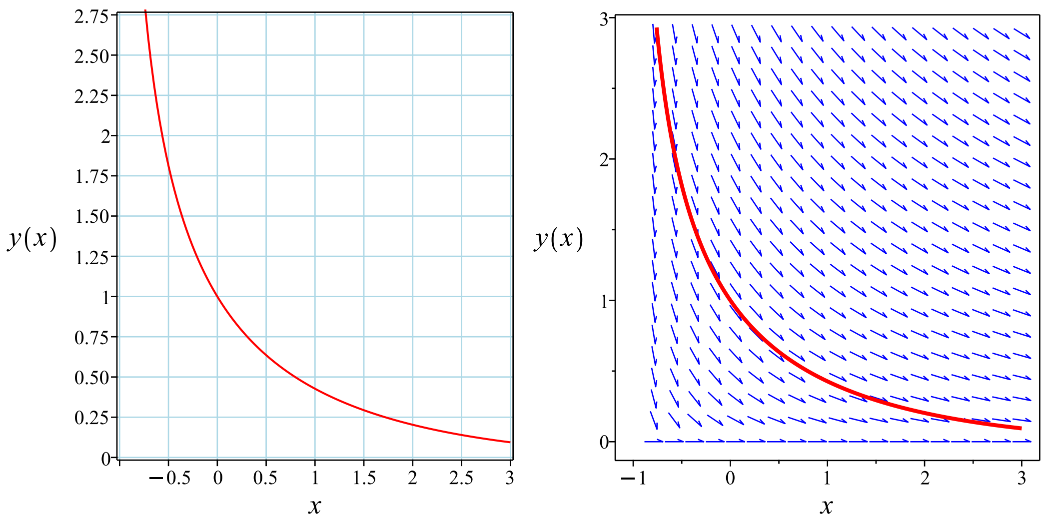
Substituting c_1 found above in the general solution gives

$$y = \frac{\ln(x+1)^2}{4} - \ln(x+1) + 1$$

Summary

The solution(s) found are the following

$$y = \frac{\ln(x+1)^2}{4} - \ln(x+1) + 1 \tag{1}$$



(a) Solution plot

(b) Slope field plot

Verification of solutions

$$y = \frac{\ln(x+1)^2}{4} - \ln(x+1) + 1$$

Verified OK.

1.26.5 Maple step by step solution

Let's solve

$\left[\sqrt{y} + (x+1)\, y' = 0, y(0) = 1\right]$

- Highest derivative means the order of the ODE is 1

  y'

- Separate variables

$\frac{y'}{\sqrt{y}} = -\frac{1}{x+1}$

- Integrate both sides with respect to x

  $\int \frac{y'}{\sqrt{y}}dx = \int -\frac{1}{x+1}dx + c_1$

- Evaluate integral

  $2\sqrt{y} = -\ln(x+1) + c_1$

- Solve for y

  $y = \frac{\ln(x+1)^2}{4} - \frac{\ln(x+1)c_1}{2} + \frac{c_1^2}{4}$

- Use initial condition $y(0) = 1$

  $1 = \frac{c_1^2}{4}$

- Solve for c_1

  $c_1 = (-2, 2)$

- Substitute $c_1 = (-2, 2)$ into general solution and simplify

  $y = \frac{(\ln(x+1)+2)^2}{4}$

- Solution to the IVP

  $y = \frac{(\ln(x+1)+2)^2}{4}$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
<- Bernoulli successful`
```

✓ Solution by Maple

Time used: 0.062 (sec). Leaf size: 14

```
dsolve([sqrt(y(x))+(1+x)*diff(y(x),x)=0,y(0) = 1],y(x), singsol=all)
```

$$y(x) = \frac{(\ln(1+x) - 2)^2}{4}$$

✓ Solution by Mathematica

Time used: 0.155 (sec). Leaf size: 33

```
DSolve[{Sqrt[y[x]]+(1+x)*y'[x]==0,{y[0]==1}},y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \frac{1}{4}(\log(x+1)-2)^2$$
$$y(x) \to \frac{1}{4}(\log(x+1)+2)^2$$

1.27 problem 27 part(a)

Internal problem ID [4938]
Internal file name [`OUTPUT/4431_Sunday_June_05_2022_01_20_44_PM_76753102/index.tex`]

Book: Fundamentals of Differential Equations. By Nagle, Saff and Snider. 9th edition. Boston. Pearson 2018.
Section: Chapter 2, First order differential equations. Section 2.2, Separable Equations. Exercises. page 46
Problem number: 27 part(a).
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"quadrature"**

Maple gives the following as the ode type

`[_quadrature]`

$$\boxed{y' = e^{x^2}}$$

With initial conditions

$$[y(0) = 0]$$

1.27.1 Existence and uniqueness analysis

This is a linear ODE. In canonical form it is written as

$$y' + p(x)y = q(x)$$

Where here

$$\begin{aligned} p(x) &= 0 \\ q(x) &= e^{x^2} \end{aligned}$$

Hence the ode is

$$y' = e^{x^2}$$

The domain of $p(x) = 0$ is

$$\{-\infty < x < \infty\}$$

And the point $x_0 = 0$ is inside this domain. The domain of $q(x) = e^{x^2}$ is

$$\{-\infty < x < \infty\}$$

And the point $x_0 = 0$ is also inside this domain. Hence solution exists and is unique.

1.27.2 Solving as quadrature ode

Integrating both sides gives

$$\begin{aligned} y &= \int e^{x^2} \, dx \\ &= \frac{\sqrt{\pi}\, \operatorname{erfi}(x)}{2} + c_1 \end{aligned}$$

Initial conditions are used to solve for c_1. Substituting $x = 0$ and $y = 0$ in the above solution gives an equation to solve for the constant of integration.

$$0 = c_1$$

$$c_1 = 0$$

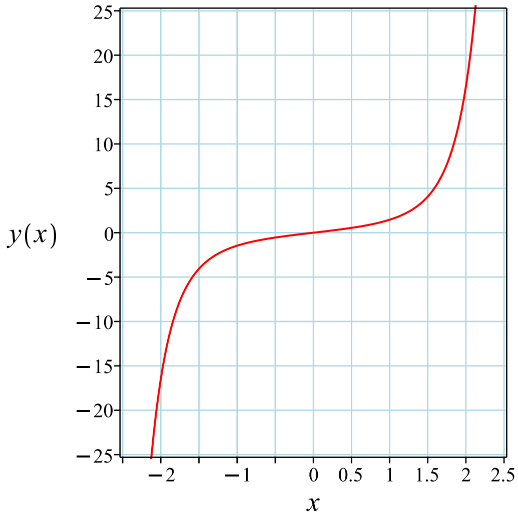
Substituting c_1 found above in the general solution gives

$$y = \frac{\sqrt{\pi}\, \operatorname{erfi}(x)}{2}$$

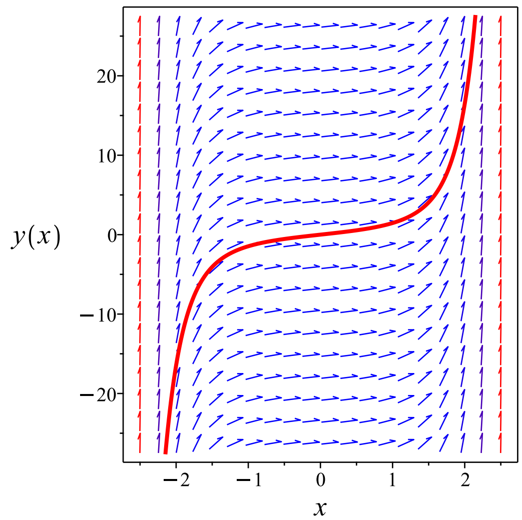
Summary

The solution(s) found are the following

$$y = \frac{\sqrt{\pi}\, \operatorname{erfi}(x)}{2} \tag{1}$$



(a) Solution plot



(b) Slope field plot

Verification of solutions

$$y = \frac{\sqrt{\pi}\,\operatorname{erfi}(x)}{2}$$

Verified OK.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
<- quadrature successful`
```

✓ Solution by Maple

Time used: 0.015 (sec). Leaf size: 11

```
dsolve([diff(y(x),x)=exp(x^2),y(0) = 0],y(x), singsol=all)
```

$$y(x) = \frac{\sqrt{\pi}\, \operatorname{erfi}(x)}{2}$$

✓ Solution by Mathematica

Time used: 0.004 (sec). Leaf size: 16

```
DSolve[{y'[x]==Exp[x^2],{y[0]==0}},y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \frac{1}{2}\sqrt{\pi}\text{erfi}(x)$$

1.28 problem 27 part(b)

Internal problem ID [4939]
Internal file name [`OUTPUT/4432_Sunday_June_05_2022_01_20_54_PM_93662734/index.tex`]

Book: Fundamentals of Differential Equations. By Nagle, Saff and Snider. 9th edition. Boston. Pearson 2018.
Section: Chapter 2, First order differential equations. Section 2.2, Separable Equations. Exercises. page 46
Problem number: 27 part(b).
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"exact", "separable", "first_order_ode_lie_symmetry_lookup"**

Maple gives the following as the ode type

`[_separable]`

$$\boxed{y' - \frac{e^{x^2}}{y^2} = 0}$$

With initial conditions

$$[y(0) = 1]$$

1.28.1 Existence and uniqueness analysis

This is non linear first order ODE. In canonical form it is written as

$$\begin{aligned} y' &= f(x, y) \\ &= \frac{e^{x^2}}{y^2} \end{aligned}$$

The x domain of $f(x, y)$ when $y = 1$ is

$$\{-\infty < x < \infty\}$$

And the point $x_0 = 0$ is inside this domain. The y domain of $f(x,y)$ when $x = 0$ is

$$\{y < 0 \lor 0 < y\}$$

And the point $y_0 = 1$ is inside this domain. Now we will look at the continuity of

$$\begin{aligned}\frac{\partial f}{\partial y} &= \frac{\partial}{\partial y}\left(\frac{e^{x^2}}{y^2}\right) \\ &= -\frac{2\,e^{x^2}}{y^3}\end{aligned}$$

The x domain of $\frac{\partial f}{\partial y}$ when $y = 1$ is

$$\{-\infty < x < \infty\}$$

And the point $x_0 = 0$ is inside this domain. The y domain of $\frac{\partial f}{\partial y}$ when $x = 0$ is

$$\{y < 0 \lor 0 < y\}$$

And the point $y_0 = 1$ is inside this domain. Therefore solution exists and is unique.

1.28.2 Solving as separable ode

In canonical form the ODE is

$$\begin{aligned}y' &= F(x,y) \\ &= f(x)g(y) \\ &= \frac{e^{x^2}}{y^2}\end{aligned}$$

Where $f(x) = e^{x^2}$ and $g(y) = \frac{1}{y^2}$. Integrating both sides gives

$$\begin{aligned}\frac{1}{\frac{1}{y^2}}\,dy &= e^{x^2}\,dx \\ \int \frac{1}{\frac{1}{y^2}}\,dy &= \int e^{x^2}\,dx \\ \frac{y^3}{3} &= \frac{\sqrt{\pi}\,\operatorname{erfi}(x)}{2} + c_1\end{aligned}$$

Which results in

$$y = \frac{\left(12\sqrt{\pi}\,\operatorname{erfi}\left(x\right)+24c_1\right)^{\frac{1}{3}}}{2}$$

$$y = -\frac{\left(12\sqrt{\pi}\,\operatorname{erfi}\left(x\right)+24c_1\right)^{\frac{1}{3}}}{4}+\frac{i\sqrt{3}\left(12\sqrt{\pi}\,\operatorname{erfi}\left(x\right)+24c_1\right)^{\frac{1}{3}}}{4}$$

$$y = -\frac{\left(12\sqrt{\pi}\,\operatorname{erfi}\left(x\right)+24c_1\right)^{\frac{1}{3}}}{4}-\frac{i\sqrt{3}\left(12\sqrt{\pi}\,\operatorname{erfi}\left(x\right)+24c_1\right)^{\frac{1}{3}}}{4}$$

Initial conditions are used to solve for c_1. Substituting $x = 0$ and $y = 1$ in the above solution gives an equation to solve for the constant of integration.

$$1 = -\frac{i c_1^{\frac{1}{3}} 24^{\frac{1}{3}}\sqrt{3}}{4}-\frac{c_1^{\frac{1}{3}} 24^{\frac{1}{3}}}{4}$$

Warning: Unable to solve for constant of integration. Initial conditions are used to solve for c_1. Substituting $x = 0$ and $y = 1$ in the above solution gives an equation to solve for the constant of integration.

$$1 = \frac{i c_1^{\frac{1}{3}} 24^{\frac{1}{3}}\sqrt{3}}{4}-\frac{c_1^{\frac{1}{3}} 24^{\frac{1}{3}}}{4}$$

Warning: Unable to solve for constant of integration. Initial conditions are used to solve for c_1. Substituting $x = 0$ and $y = 1$ in the above solution gives an equation to solve for the constant of integration.

$$1 = \frac{c_1^{\frac{1}{3}} 24^{\frac{1}{3}}}{2}$$

$$c_1 = \frac{1}{3}$$

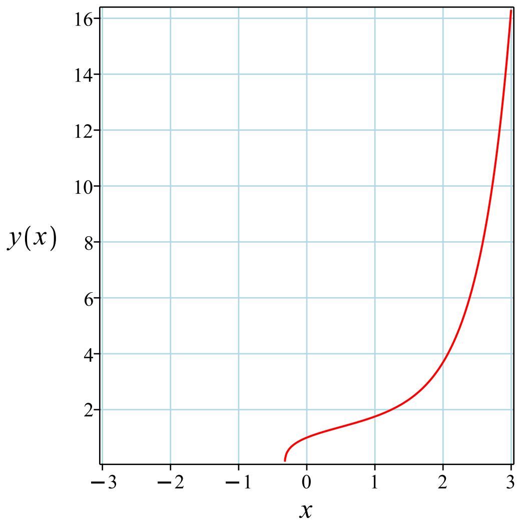
Substituting c_1 found above in the general solution gives

$$y = \frac{\left(12\sqrt{\pi}\,\operatorname{erfi}\left(x\right)+8\right)^{\frac{1}{3}}}{2}$$

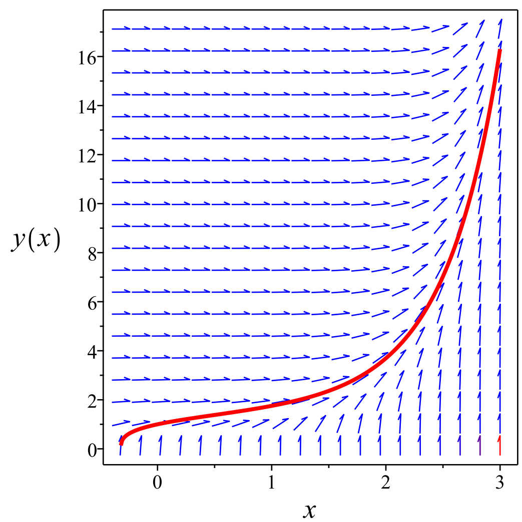
Summary

The solution(s) found are the following

$$y = \frac{\left(12\sqrt{\pi}\,\operatorname{erfi}\left(x\right)+8\right)^{\frac{1}{3}}}{2} \tag{1}$$



(a) Solution plot



(b) Slope field plot

<u>Verification of solutions</u>

$$y = \frac{\left(12\sqrt{\pi}\,\operatorname{erfi}(x) + 8\right)^{\frac{1}{3}}}{2}$$

Verified OK.

1.28.3 Solving as first order ode lie symmetry lookup ode

Writing the ode as

$$y' = \frac{e^{x^2}}{y^2}$$
$$y' = \omega(x, y)$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x + \omega(\eta_y - \xi_x) - \omega^2 \xi_y - \omega_x \xi - \omega_y \eta = 0 \tag{A}$$

The type of this ode is known. It is of type `separable`. Therefore we do not need to solve the PDE (A), and can just use the lookup table shown below to find ξ, η

Table 66: Lie symmetry infinitesimal lookup table for known first order ODE's

ODE class	Form	ξ	η
linear ode	$y' = f(x)y(x) + g(x)$	0	$e^{\int f dx}$
separable ode	$y' = f(x)\, g(y)$	$\frac{1}{f}$	0
quadrature ode	$y' = f(x)$	0	1
quadrature ode	$y' = g(y)$	1	0
homogeneous ODEs of Class A	$y' = f\left(\frac{y}{x}\right)$	x	y
homogeneous ODEs of Class C	$y' = (a + bx + cy)^{\frac{n}{m}}$	1	$-\frac{b}{c}$
homogeneous class D	$y' = \frac{y}{x} + g(x)\, F\left(\frac{y}{x}\right)$	x^2	xy
First order special form ID 1	$y' = g(x)\, e^{h(x)+by} + f(x)$	$\frac{e^{-\int bf(x)dx - h(x)}}{g(x)}$	$\frac{f(x)e^{-\int bf(x)dx - h(x)}}{g(x)}$
polynomial type ode	$y' = \frac{a_1x+b_1y+c_1}{a_2x+b_2y+c_2}$	$\frac{a_1b_2x - a_2b_1x - b_1c_2 + b_2c_1}{a_1b_2 - a_2b_1}$	$\frac{a_1b_2y - a_2b_1y - a_1c_2 - a_2c_1}{a_1b_2 - a_2b_1}$
Bernoulli ode	$y' = f(x)\, y + g(x)\, y^n$	0	$e^{-\int (n-1)f(x)dx} y^n$
Reduced Riccati	$y' = f_1(x)\, y + f_2(x)\, y^2$	0	$e^{-\int f_1 dx}$

The above table shows that

$$
\begin{aligned}
\xi(x, y) &= \mathrm{e}^{-x^2} \\
\eta(x, y) &= 0
\end{aligned}
\tag{A1}
$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x, y) \to (R, S)$ where (R, S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$
\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}
$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y}\right) S(x, y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the

canonical coordinates, where $S(R)$. Since $\eta = 0$ then in this special case

$$R = y$$

S is found from

$$\begin{aligned}
S &= \int \frac{1}{\xi} dx \\
&= \int \frac{1}{e^{-x^2}} dx
\end{aligned}$$

Which results in

$$S = \frac{\sqrt{\pi}\, \operatorname{erfi}(x)}{2}$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x,y) S_y}{R_x + \omega(x,y) R_y} \tag{2}$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x,y)$ is the right hand side of the original ode given by

$$\omega(x,y) = \frac{e^{x^2}}{y^2}$$

Evaluating all the partial derivatives gives

$$\begin{aligned}
R_x &= 0 \\
R_y &= 1 \\
S_x &= e^{x^2} \\
S_y &= 0
\end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = y^2 \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = R^2$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordinates R, S. Integrating the above gives

$$S(R) = \frac{R^3}{3} + c_1 \tag{4}$$

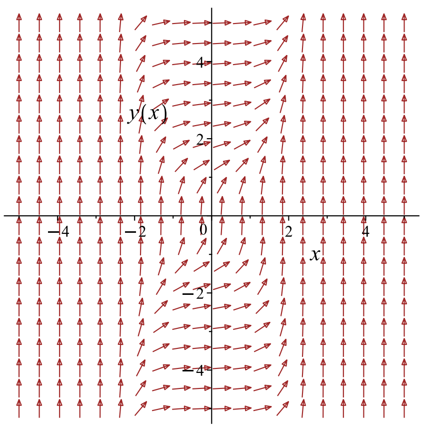
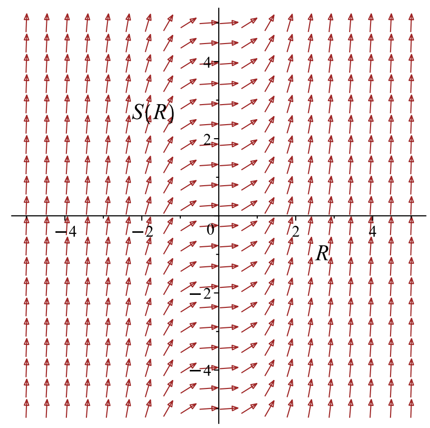
To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$\frac{\sqrt{\pi}\, \operatorname{erfi}(x)}{2} = \frac{y^3}{3} + c_1$$

Which simplifies to

$$\frac{\sqrt{\pi}\, \operatorname{erfi}(x)}{2} = \frac{y^3}{3} + c_1$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in x, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dx} = \frac{e^{x^2}}{y^2}$ 	$R = y$ $S = \frac{\sqrt{\pi}\, \operatorname{erfi}(x)}{2}$	$\frac{dS}{dR} = R^2$ 

Initial conditions are used to solve for c_1. Substituting $x = 0$ and $y = 1$ in the above solution gives an equation to solve for the constant of integration.

$$0 = \frac{1}{3} + c_1$$

$$
c_1 = -\frac{1}{3}
$$

Substituting c_1 found above in the general solution gives

$$
\frac{\sqrt{\pi}\,\operatorname{erfi}(x)}{2} = \frac{y^3}{3} - \frac{1}{3}
$$

<u>Summary</u>

The solution(s) found are the following

$$
\frac{\sqrt{\pi}\,\operatorname{erfi}(x)}{2} = \frac{y^3}{3} - \frac{1}{3} \tag{1}
$$

<u>Verification of solutions</u>

$$
\frac{\sqrt{\pi}\,\operatorname{erfi}(x)}{2} = \frac{y^3}{3} - \frac{1}{3}
$$

Verified OK.

1.28.4 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$
M(x,y) + N(x,y)\frac{dy}{dx} = 0 \tag{A}
$$

We assume there exists a function $\phi(x,y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$
\frac{d}{dx}\phi(x,y) = 0
$$

Hence

$$
\frac{\partial \phi}{\partial x} + \frac{\partial \phi}{\partial y}\frac{dy}{dx} = 0 \tag{B}
$$

Comparing (A,B) shows that

$$
\frac{\partial \phi}{\partial x} = M
$$

$$
\frac{\partial \phi}{\partial y} = N
$$

But since $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ then for the above to be valid, we require that

$$
\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}
$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x,y)$ but at least we know now that we can do that since the condition $\frac{\partial^2\phi}{\partial x\partial y} = \frac{\partial^2\phi}{\partial y\partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x,y)\,\mathrm{d}x + N(x,y)\,\mathrm{d}y = 0 \tag{1A}$$

Therefore

$$\begin{aligned}(y^2)\,\mathrm{d}y &= \left(\mathrm{e}^{x^2}\right)\mathrm{d}x\\ \left(-\mathrm{e}^{x^2}\right)\mathrm{d}x + (y^2)\,\mathrm{d}y &= 0 \qquad (2A)\end{aligned}$$

Comparing (1A) and (2A) shows that

$$\begin{aligned}M(x,y) &= -\mathrm{e}^{x^2}\\ N(x,y) &= y^2\end{aligned}$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Using result found above gives

$$\begin{aligned}\frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}\left(-\mathrm{e}^{x^2}\right)\\ &= 0\end{aligned}$$

And

$$\begin{aligned}\frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}(y^2)\\ &= 0\end{aligned}$$

Since $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$, then the ODE is exact The following equations are now set up to solve for the function $\phi(x,y)$

$$\frac{\partial\phi}{\partial x} = M \tag{1}$$

$$\frac{\partial\phi}{\partial y} = N \tag{2}$$

Integrating (1) w.r.t. x gives

$$
\begin{aligned}
\int \frac{\partial \phi}{\partial x} \mathrm{d}x &= \int M \mathrm{d}x \\
\int \frac{\partial \phi}{\partial x} \mathrm{d}x &= \int -\mathrm{e}^{x^2} \mathrm{d}x \\
\phi &= -\frac{\sqrt{\pi}\, \mathrm{erfi}\,(x)}{2} + f(y) \qquad (3)
\end{aligned}
$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$
\frac{\partial \phi}{\partial y} = 0 + f'(y) \qquad (4)
$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = y^2$. Therefore equation (4) becomes

$$
y^2 = 0 + f'(y) \qquad (5)
$$

Solving equation (5) for $f'(y)$ gives

$$
f'(y) = y^2
$$

Integrating the above w.r.t y gives

$$
\begin{aligned}
\int f'(y) \mathrm{d}y &= \int (y^2) \mathrm{d}y \\
f(y) &= \frac{y^3}{3} + c_1
\end{aligned}
$$

Where c_1 is constant of integration. Substituting result found above for $f(y)$ into equation (3) gives ϕ

$$
\phi = -\frac{\sqrt{\pi}\, \mathrm{erfi}\,(x)}{2} + \frac{y^3}{3} + c_1
$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$
c_1 = -\frac{\sqrt{\pi}\, \mathrm{erfi}\,(x)}{2} + \frac{y^3}{3}
$$

Initial conditions are used to solve for c_1. Substituting $x = 0$ and $y = 1$ in the above solution gives an equation to solve for the constant of integration.

$$\frac{1}{3} = c_1$$

$$c_1 = \frac{1}{3}$$

Substituting c_1 found above in the general solution gives

$$-\frac{\sqrt{\pi}\,\operatorname{erfi}(x)}{2} + \frac{y^3}{3} = \frac{1}{3}$$

Summary

The solution(s) found are the following

$$-\frac{\sqrt{\pi}\,\operatorname{erfi}(x)}{2} + \frac{y^3}{3} = \frac{1}{3} \tag{1}$$

Verification of solutions

$$-\frac{\sqrt{\pi}\,\operatorname{erfi}(x)}{2} + \frac{y^3}{3} = \frac{1}{3}$$

Verified OK.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
<- Bernoulli successful`
```

✓ Solution by Maple

Time used: 0.094 (sec). Leaf size: 17

```
dsolve([diff(y(x),x)=exp(x^2)/y(x)^2,y(0) = 1],y(x), singsol=all)
```

$$y(x) = \frac{\left(8 + 12\sqrt{\pi}\,\operatorname{erfi}(x)\right)^{\frac{1}{3}}}{2}$$

✓ Solution by Mathematica

Time used: 0.317 (sec). Leaf size: 22

```
DSolve[{y'[x]==Exp[x^2]/y[x]^2,{y[0]==1}},y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \sqrt[3]{\frac{3}{2}\sqrt{\pi}\text{erfi}(x)+1}$$

1.29 problem 27 part(c)

Internal problem ID [4940]
Internal file name [`OUTPUT/4433_Sunday_June_05_2022_01_21_02_PM_36880442/index.tex`]

Book: Fundamentals of Differential Equations. By Nagle, Saff and Snider. 9th edition. Boston. Pearson 2018.
Section: Chapter 2, First order differential equations. Section 2.2, Separable Equations. Exercises. page 46
Problem number: 27 part(c).
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"exact", "riccati", "separable", "first_order_ode_lie_symmetry_lookup"**

Maple gives the following as the ode type

```
[_separable]
```

$$\boxed{y' - \sqrt{1+\sin(x)}\,(1+y^2) = 0}$$

With initial conditions

$$[y(0) = 1]$$

1.29.1 Existence and uniqueness analysis

This is non linear first order ODE. In canonical form it is written as

$$\begin{aligned}
y' &= f(x,y)\\
&= \sqrt{1+\sin(x)}\,(y^2+1)
\end{aligned}$$

The x domain of $f(x,y)$ when $y=1$ is

$$\left\{x < -\frac{1}{2}\pi + 2\pi\_Z68 \vee -\frac{1}{2}\pi + 2\pi\_Z68 < x\right\}$$

And the point $x_0=0$ is inside this domain. The y domain of $f(x,y)$ when $x=0$ is

$$\{-\infty < y < \infty\}$$

And the point $y_0=1$ is inside this domain. Now we will look at the continuity of

$$\begin{aligned}
\frac{\partial f}{\partial y} &= \frac{\partial}{\partial y}\left(\sqrt{1+\sin(x)}\,(y^2+1)\right)\\
&= 2\sqrt{1+\sin(x)}\,y
\end{aligned}$$

The x domain of $\frac{\partial f}{\partial y}$ when $y=1$ is

$$\left\{x < -\frac{1}{2}\pi + 2\pi\_Z68 \vee -\frac{1}{2}\pi + 2\pi\_Z68 < x\right\}$$

And the point $x_0=0$ is inside this domain. The y domain of $\frac{\partial f}{\partial y}$ when $x=0$ is

$$\{-\infty < y < \infty\}$$

And the point $y_0=1$ is inside this domain. Therefore solution exists and is unique.

1.29.2 Solving as separable ode

In canonical form the ODE is

$$\begin{aligned}
y' &= F(x,y)\\
&= f(x)g(y)\\
&= \sqrt{1+\sin(x)}\,(y^2+1)
\end{aligned}$$

Where $f(x) = \sqrt{1+\sin(x)}$ and $g(y) = y^2+1$. Integrating both sides gives

$$\frac{1}{y^2+1}\,dy = \sqrt{1+\sin(x)}\,dx$$

$$\int \frac{1}{y^2+1}\,dy = \int \sqrt{1+\sin(x)}\,dx$$

$$\arctan(y) = \frac{2(-1+\sin(x))\sqrt{1+\sin(x)}}{\cos(x)} + c_1$$

Which results in

$$y = \tan\left(\frac{2\sqrt{1+\sin(x)}\,\sin(x) + \cos(x)\,c_1 - 2\sqrt{1+\sin(x)}}{\cos(x)}\right)$$

Initial conditions are used to solve for c_1. Substituting $x = 0$ and $y = 1$ in the above solution gives an equation to solve for the constant of integration.

$$1 = \frac{\sin(-2+c_1)}{\cos(-2+c_1)}$$

$$c_1 = 2 + \frac{\pi}{4}$$

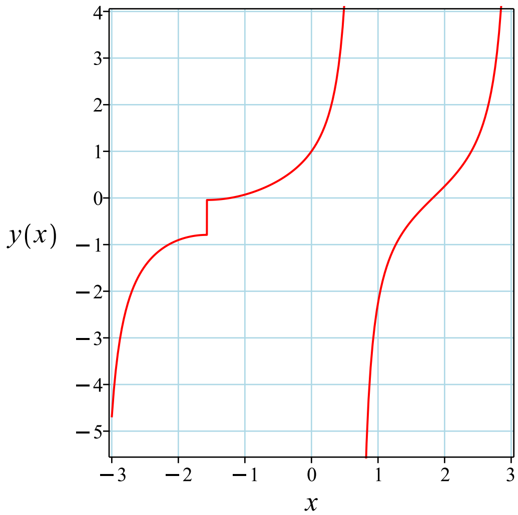
Substituting c_1 found above in the general solution gives

$$y = \frac{\cos\left(-2+2\sqrt{1+\sin(x)}\,\sec(x) - 2\sqrt{1+\sin(x)}\,\tan(x) + \frac{\pi}{4}\right)}{\sin\left(-2+2\sqrt{1+\sin(x)}\,\sec(x) - 2\sqrt{1+\sin(x)}\,\tan(x) + \frac{\pi}{4}\right)}$$

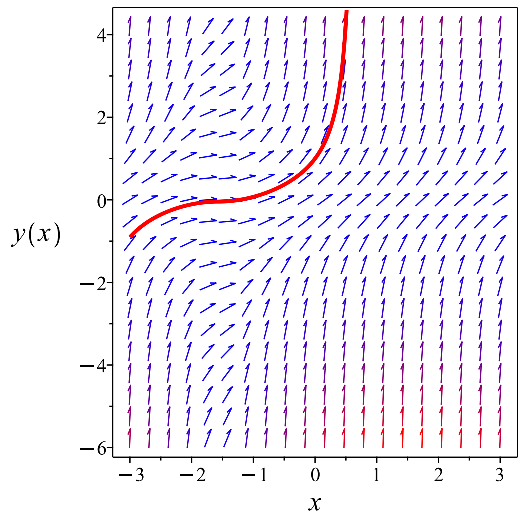
Summary

The solution(s) found are the following

$$y = \frac{\cos\left(-2+2\sqrt{1+\sin(x)}\,\sec(x) - 2\sqrt{1+\sin(x)}\,\tan(x) + \frac{\pi}{4}\right)}{\sin\left(-2+2\sqrt{1+\sin(x)}\,\sec(x) - 2\sqrt{1+\sin(x)}\,\tan(x) + \frac{\pi}{4}\right)} \tag{1}$$



(a) Solution plot



(b) Slope field plot

Verification of solutions

$$y = \frac{\cos\left(-2 + 2\sqrt{1+\sin(x)}\,\sec(x) - 2\sqrt{1+\sin(x)}\,\tan(x) + \frac{\pi}{4}\right)}{\sin\left(-2 + 2\sqrt{1+\sin(x)}\,\sec(x) - 2\sqrt{1+\sin(x)}\,\tan(x) + \frac{\pi}{4}\right)}$$

Verified OK.

1.29.3 Solving as first order ode lie symmetry lookup ode

Writing the ode as

$$y' = \sqrt{1+\sin(x)}\,(y^2+1)$$
$$y' = \omega(x,y)$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x + \omega(\eta_y - \xi_x) - \omega^2\xi_y - \omega_x\xi - \omega_y\eta = 0 \tag{A}$$

The type of this ode is known. It is of type `separable`. Therefore we do not need to solve the PDE (A), and can just use the lookup table shown below to find ξ, η

Table 68: Lie symmetry infinitesimal lookup table for known first order ODE's

ODE class	Form	ξ	η
linear ode	$y' = f(x)y(x) + g(x)$	0	$e^{\int f dx}$
separable ode	$y' = f(x)\, g(y)$	$\frac{1}{f}$	0
quadrature ode	$y' = f(x)$	0	1
quadrature ode	$y' = g(y)$	1	0
homogeneous ODEs of Class A	$y' = f\left(\frac{y}{x}\right)$	x	y
homogeneous ODEs of Class C	$y' = (a + bx + cy)^{\frac{n}{m}}$	1	$-\frac{b}{c}$
homogeneous class D	$y' = \frac{y}{x} + g(x)\, F\left(\frac{y}{x}\right)$	x^2	xy
First order special form ID 1	$y' = g(x)\, e^{h(x)+by} + f(x)$	$\frac{e^{-\int bf(x)dx - h(x)}}{g(x)}$	$\frac{f(x)e^{-\int bf(x)dx - h(x)}}{g(x)}$
polynomial type ode	$y' = \frac{a_1x+b_1y+c_1}{a_2x+b_2y+c_2}$	$\frac{a_1b_2x - a_2b_1x - b_1c_2 + b_2c_1}{a_1b_2 - a_2b_1}$	$\frac{a_1b_2y - a_2b_1y - a_1c_2 - a_2c_1}{a_1b_2 - a_2b_1}$
Bernoulli ode	$y' = f(x)\, y + g(x)\, y^n$	0	$e^{-\int (n-1)f(x)dx} y^n$
Reduced Riccati	$y' = f_1(x)\, y + f_2(x)\, y^2$	0	$e^{-\int f_1 dx}$

The above table shows that

$$
\begin{aligned}
\xi(x, y) &= \frac{1}{\sqrt{1 + \sin(x)}} \\
\eta(x, y) &= 0
\end{aligned} \tag{A1}
$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x, y) \to (R, S)$ where (R, S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$
\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}
$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y}\right) S(x, y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the

canonical coordinates, where $S(R)$. Since $\eta = 0$ then in this special case

$$R = y$$

S is found from

$$\begin{aligned} S &= \int \frac{1}{\xi} dx \\ &= \int \frac{1}{\frac{1}{\sqrt{1+\sin(x)}}} dx \end{aligned}$$

Which results in

$$S = \frac{2(-1 + \sin(x))\sqrt{1 + \sin(x)}}{\cos(x)}$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x, y)S_y}{R_x + \omega(x, y)R_y} \tag{2}$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x, y)$ is the right hand side of the original ode given by

$$\omega(x, y) = \sqrt{1 + \sin(x)}\,(y^2 + 1)$$

Evaluating all the partial derivatives gives

$$\begin{aligned} R_x &= 0 \\ R_y &= 1 \\ S_x &= -\frac{\sqrt{2}\left(\cos(x)^2 \csc\left(\frac{\pi}{4} + \frac{x}{2}\right) - 4\sin\left(\frac{\pi}{4} + \frac{x}{2}\right)\right)}{2 + 2\sin(x)} \\ S_y &= 0 \end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = \frac{\left(-\cos(x)^2 \csc\left(\frac{\pi}{4} + \frac{x}{2}\right)^4 + 4\csc\left(\frac{\pi}{4} + \frac{x}{2}\right)^2\right)\operatorname{csgn}\left(\sin\left(\frac{\pi}{4} + \frac{x}{2}\right)\right)}{4y^2 + 4} \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = \frac{\operatorname{RootOf}\left(\_Z^2 + S(R)^2 - 8\right)}{\sqrt{-S(R)^2 + 8}\,(R^2 + 1)}$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$\arctan(R) - \left(\int^{S(R)} \frac{\sqrt{-\_a^2+8}}{\operatorname{RootOf}\left(\_Z^2+\_a^2-8\right)} d\_a\right) + c_1 = 0 \tag{4}$$

Unable to solve for constant of integration due to RootOf in solution.

Summary

The solution(s) found are the following

$$\arctan(y) - \left(\int^{2\sqrt{1+\sin(x)}\,(\tan(x)-\sec(x))} \frac{\sqrt{-\_a^2+8}}{\operatorname{RootOf}\left(\_Z^2+\_a^2-8\right)} d\_a\right) + c_1 = 0 \tag{1}$$

Verification of solutions

$$\arctan(y) - \left(\int^{2\sqrt{1+\sin(x)}\,(\tan(x)-\sec(x))} \frac{\sqrt{-\_a^2+8}}{\operatorname{RootOf}\left(\_Z^2+\_a^2-8\right)} d\_a\right) + c_1 = 0$$

Warning, solution could not be verified

1.29.4 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x,y) + N(x,y)\frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x,y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y) = 0$$

Hence

$$\frac{\partial \phi}{\partial x} + \frac{\partial \phi}{\partial y}\frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$\begin{aligned}\frac{\partial \phi}{\partial x} &= M\\ \frac{\partial \phi}{\partial y} &= N\end{aligned}$$

But since $\frac{\partial^2\phi}{\partial x\partial y} = \frac{\partial^2\phi}{\partial y\partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x,y)$ but at least we know now that we can do that since the condition $\frac{\partial^2\phi}{\partial x\partial y} = \frac{\partial^2\phi}{\partial y\partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x,y)\,\mathrm{d}x + N(x,y)\,\mathrm{d}y = 0 \tag{1A}$$

Therefore

$$\begin{aligned}\left(\frac{1}{y^2+1}\right)\mathrm{d}y &= \left(\sqrt{1+\sin(x)}\right)\mathrm{d}x\\ \left(-\sqrt{1+\sin(x)}\right)\mathrm{d}x + \left(\frac{1}{y^2+1}\right)\mathrm{d}y &= 0 \end{aligned}\tag{2A}$$

Comparing (1A) and (2A) shows that

$$\begin{aligned}M(x,y) &= -\sqrt{1+\sin(x)}\\ N(x,y) &= \frac{1}{y^2+1}\end{aligned}$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Using result found above gives

$$\begin{aligned}\frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}\left(-\sqrt{1+\sin(x)}\right)\\ &= 0\end{aligned}$$

And

$$\begin{aligned}\frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}\left(\frac{1}{y^2+1}\right)\\ &= 0\end{aligned}$$

Since $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$, then the ODE is <u>exact</u> The following equations are now set up to solve for the function $\phi(x, y)$

$$
\begin{aligned}
\frac{\partial \phi}{\partial x} &= M \qquad (1) \\
\frac{\partial \phi}{\partial y} &= N \qquad (2)
\end{aligned}
$$

Integrating (1) w.r.t. x gives

$$
\begin{aligned}
\int \frac{\partial \phi}{\partial x} \mathrm{d}x &= \int M \mathrm{d}x \\
\int \frac{\partial \phi}{\partial x} \mathrm{d}x &= \int -\sqrt{1+\sin(x)} \mathrm{d}x \\
\phi &= -2\sqrt{1+\sin(x)} (\tan(x) - \sec(x)) + f(y) \qquad (3)
\end{aligned}
$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$
\frac{\partial \phi}{\partial y} = 0 + f'(y) \qquad (4)
$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = \frac{1}{y^2+1}$. Therefore equation (4) becomes

$$
\frac{1}{y^2+1} = 0 + f'(y) \qquad (5)
$$

Solving equation (5) for $f'(y)$ gives

$$
f'(y) = \frac{1}{y^2+1}
$$

Integrating the above w.r.t y gives

$$
\begin{aligned}
\int f'(y) \mathrm{d}y &= \int \left(\frac{1}{y^2+1}\right) \mathrm{d}y \\
f(y) &= \arctan(y) + c_1
\end{aligned}
$$

Where c_1 is constant of integration. Substituting result found above for $f(y)$ into equation (3) gives ϕ

$$\phi = -2\sqrt{1+\sin(x)}\,(\tan(x) - \sec(x)) + \arctan(y) + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = -2\sqrt{1+\sin(x)}\,(\tan(x) - \sec(x)) + \arctan(y)$$

The solution becomes

$$y = \tan\left(2\sqrt{1+\sin(x)}\,\tan(x) - 2\sqrt{1+\sin(x)}\,\sec(x) + c_1\right)$$

Initial conditions are used to solve for c_1. Substituting $x = 0$ and $y = 1$ in the above solution gives an equation to solve for the constant of integration.

$$1 = \frac{\sin(-2+c_1)}{\cos(-2+c_1)}$$

$$c_1 = 2 + \frac{\pi}{4}$$

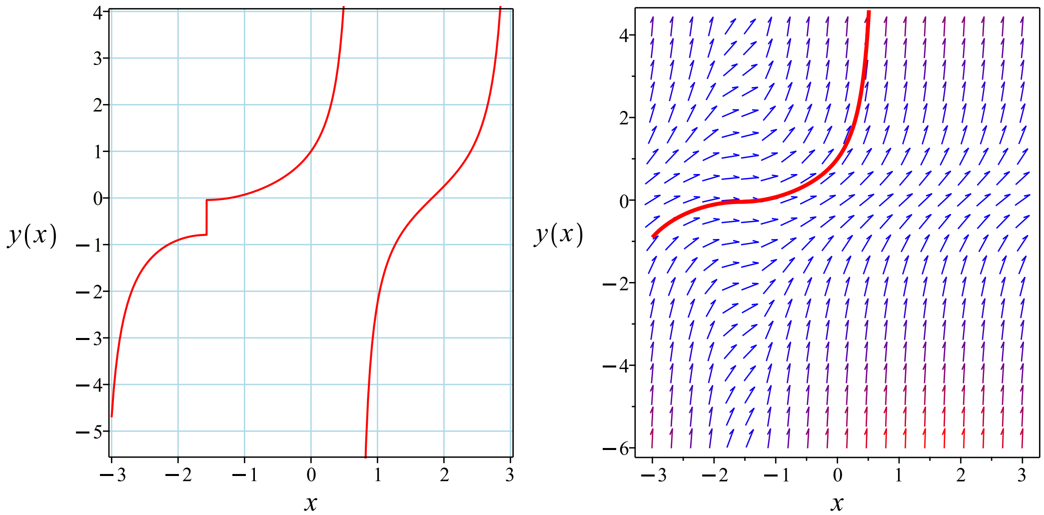
Substituting c_1 found above in the general solution gives

$$y = \frac{\cos\left(-2+2\sqrt{1+\sin(x)}\,\sec(x) - 2\sqrt{1+\sin(x)}\,\tan(x) + \frac{\pi}{4}\right)}{\sin\left(-2+2\sqrt{1+\sin(x)}\,\sec(x) - 2\sqrt{1+\sin(x)}\,\tan(x) + \frac{\pi}{4}\right)}$$

Summary

The solution(s) found are the following

$$y = \frac{\cos\left(-2+2\sqrt{1+\sin(x)}\,\sec(x) - 2\sqrt{1+\sin(x)}\,\tan(x) + \frac{\pi}{4}\right)}{\sin\left(-2+2\sqrt{1+\sin(x)}\,\sec(x) - 2\sqrt{1+\sin(x)}\,\tan(x) + \frac{\pi}{4}\right)} \tag{1}$$



(a) Solution plot (b) Slope field plot

Verification of solutions

$$y = \frac{\cos\left(-2+2\sqrt{1+\sin(x)}\,\sec(x)-2\sqrt{1+\sin(x)}\,\tan(x)+\frac{\pi}{4}\right)}{\sin\left(-2+2\sqrt{1+\sin(x)}\,\sec(x)-2\sqrt{1+\sin(x)}\,\tan(x)+\frac{\pi}{4}\right)}$$

Verified OK.

1.29.5 Solving as riccati ode

In canonical form the ODE is

$$\begin{aligned} y' &= F(x,y) \\ &= \sqrt{1+\sin(x)}\,(y^2+1) \end{aligned}$$

This is a Riccati ODE. Comparing the ODE to solve

$$y' = \sqrt{1+\sin(x)}\,y^2 + \sqrt{1+\sin(x)}$$

With Riccati ODE standard form

$$y' = f_0(x) + f_1(x)y + f_2(x)y^2$$

Shows that $f_0(x) = \sqrt{1+\sin(x)}$, $f_1(x) = 0$ and $f_2(x) = \sqrt{1+\sin(x)}$. Let

$$\begin{aligned} y &= \frac{-u'}{f_2 u} \\ &= \frac{-u'}{\sqrt{1+\sin(x)}\,u} \end{aligned} \tag{1}$$

Using the above substitution in the given ODE results (after some simplification)in a second order ODE to solve for $u(x)$ which is

$$f_2u''(x) - (f_2' + f_1f_2)\,u'(x) + f_2^2f_0u(x) = 0 \tag{2}$$

But

$$\begin{aligned}
f_2' &= \frac{\cos(x)}{2\sqrt{1+\sin(x)}} \\
f_1f_2 &= 0 \\
f_2^2f_0 &= (1+\sin(x))^{\frac{3}{2}}
\end{aligned}$$

Substituting the above terms back in equation (2) gives

$$\sqrt{1+\sin(x)}\,u''(x) - \frac{\cos(x)\,u'(x)}{2\sqrt{1+\sin(x)}} + (1+\sin(x))^{\frac{3}{2}}\,u(x) = 0$$

Solving the above ODE (this ode solved using Maple, not this program), gives Unable to solve. Terminating.

1.29.6 Maple step by step solution

Let's solve

$\left[y' - \sqrt{1+\sin(x)}\,(1+y^2) = 0, y(0) = 1\right]$

- Highest derivative means the order of the ODE is 1

  y'

- Separate variables

  $\frac{y'}{1+y^2} = \sqrt{1+\sin(x)}$

- Integrate both sides with respect to x

  $\int \frac{y'}{1+y^2}dx = \int \sqrt{1+\sin(x)}dx + c_1$

- Evaluate integral

  $\arctan(y) = \frac{2(-1+\sin(x))\sqrt{1+\sin(x)}}{\cos(x)} + c_1$

- Solve for y

  $y = \tan\left(\frac{2\sqrt{1+\sin(x)}\,\sin(x)+\cos(x)c_1-2\sqrt{1+\sin(x)}}{\cos(x)}\right)$

- Use initial condition $y(0) = 1$

$$1 = \tan(-2 + c_1)$$

- Solve for c_1

$$c_1 = 2 + \frac{\pi}{4}$$

- Substitute $c_1 = 2 + \frac{\pi}{4}$ into general solution and simplify

$$y = \cot\left(-2 + 2\sqrt{1 + \sin(x)}\,(\sec(x) - \tan(x)) + \frac{\pi}{4}\right)$$

- Solution to the IVP

$$y = \cot\left(-2 + 2\sqrt{1 + \sin(x)}\,(\sec(x) - \tan(x)) + \frac{\pi}{4}\right)$$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
<- separable successful`
```

✓ Solution by Maple

Time used: 0.297 (sec). Leaf size: 21

```
dsolve([diff(y(x),x)=sqrt(1+sin(x))*(1+y(x)^2),y(0) = 1],y(x), singsol=all)
```

$$y(x) = \tan\left(\sqrt{2}\left(\int_0^x \operatorname{csgn}\left(\sin\left(\frac{\pi}{4} + \frac{\_z1}{2}\right)\right)\sin\left(\frac{\pi}{4} + \frac{\_z1}{2}\right)d\_z1\right) + \frac{\pi}{4}\right)$$

✓ Solution by Mathematica

Time used: 0.305 (sec). Leaf size: 29

```
DSolve[{y'[x]==Sqrt[1+Sin[x]]*(1+y[x]^2),{y[0]==1}},y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \tan\left(\frac{1}{4}\left(8\sin\left(\frac{x}{2}\right) - 8\cos\left(\frac{x}{2}\right) + \pi + 8\right)\right)$$

1.30 problem 28

Internal problem ID [4941]
Internal file name [`OUTPUT/4434_Sunday_June_05_2022_02_56_35_PM_78282201/index.tex`]

Book: Fundamentals of Differential Equations. By Nagle, Saff and Snider. 9th edition. Boston. Pearson 2018.
Section: Chapter 2, First order differential equations. Section 2.2, Separable Equations. Exercises. page 46
Problem number: 28.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"exact", "linear", "separable", "homogeneousTypeD2", "first_order_ode_lie_symmetry_lookup"**

Maple gives the following as the ode type

`[_separable]`

$$\boxed{y' - 2y + 2ty = 0}$$

With initial conditions

$$[y(0) = 3]$$

1.30.1 Existence and uniqueness analysis

This is a linear ODE. In canonical form it is written as

$$y' + p(t)y = q(t)$$

Where here

$$\begin{aligned} p(t) &= 2t - 2 \\ q(t) &= 0 \end{aligned}$$

Hence the ode is

$$y' + (2t - 2)\, y = 0$$

The domain of $p(t) = 2t - 2$ is

$$\{-\infty < t < \infty\}$$

And the point $t_0 = 0$ is inside this domain. Hence solution exists and is unique.

1.30.2 Solving as separable ode

In canonical form the ODE is

$$\begin{aligned} y' &= F(t, y) \\ &= f(t)g(y) \\ &= (-2t + 2)\, y \end{aligned}$$

Where $f(t) = -2t + 2$ and $g(y) = y$. Integrating both sides gives

$$\begin{aligned} \frac{1}{y}\, dy &= -2t + 2\, dt \\ \int \frac{1}{y}\, dy &= \int -2t + 2\, dt \\ \ln(y) &= -t^2 + c_1 + 2t \\ y &= e^{-t^2 + c_1 + 2t} \\ &= c_1 e^{-t^2 + 2t} \end{aligned}$$

Initial conditions are used to solve for c_1. Substituting $t = 0$ and $y = 3$ in the above solution gives an equation to solve for the constant of integration.

$$3 = c_1$$

$$c_1 = 3$$

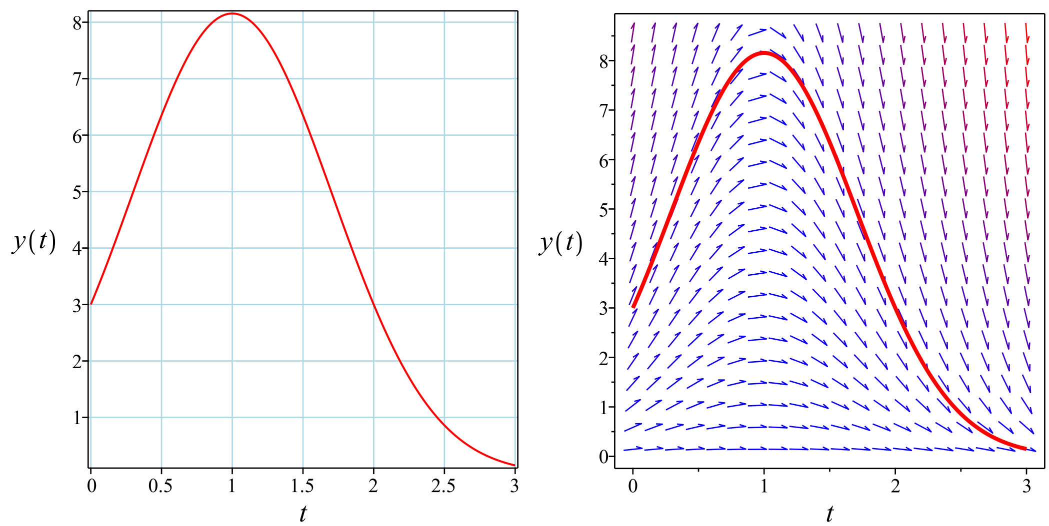
Substituting c_1 found above in the general solution gives

$$y = 3\,\mathrm{e}^{-t^2+2t}$$

Summary

The solution(s) found are the following

$$y = 3\,\mathrm{e}^{-t^2+2t} \tag{1}$$



(a) Solution plot (b) Slope field plot

Verification of solutions

$$y = 3\,\mathrm{e}^{-t^2+2t}$$

Verified OK.

1.30.3 Solving as linear ode

Entering Linear first order ODE solver. The integrating factor μ is

$$\mu = \mathrm{e}^{\int (2t-2)dt}$$
$$= \mathrm{e}^{t^2-2t}$$

Which simplifies to

$$\mu = \mathrm{e}^{t(-2+t)}$$

The ode becomes

$$\frac{d}{dt}\mu y = 0$$

$$\frac{d}{dt}\left(e^{t(-2+t)}y\right) = 0$$

Integrating gives

$$e^{t(-2+t)}y = c_1$$

Dividing both sides by the integrating factor $\mu = e^{t(-2+t)}$ results in

$$y = c_1 e^{-t(-2+t)}$$

Initial conditions are used to solve for c_1. Substituting $t = 0$ and $y = 3$ in the above solution gives an equation to solve for the constant of integration.

$$3 = c_1$$

$$c_1 = 3$$

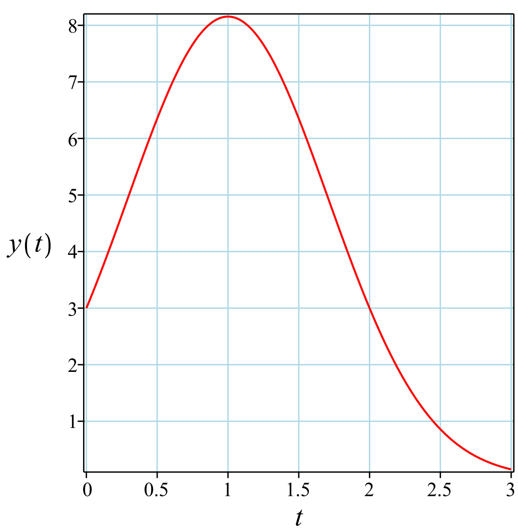
Substituting c_1 found above in the general solution gives

$$y = 3\,e^{-t(-2+t)}$$

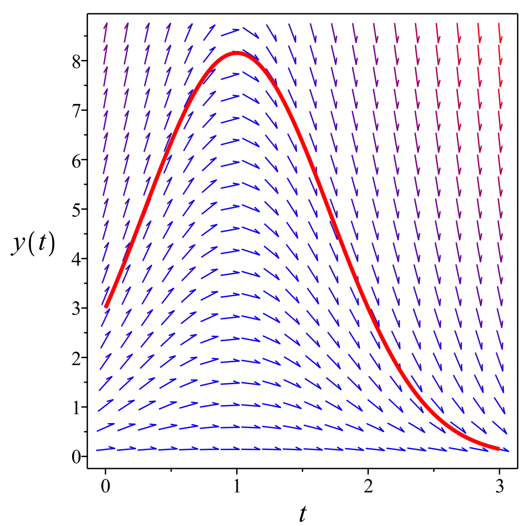
Summary

The solution(s) found are the following

$$y = 3\,e^{-t(-2+t)} \tag{1}$$



(a) Solution plot



(b) Slope field plot

Verification of solutions

$$y = 3\,\mathrm{e}^{-t(-2+t)}$$

Verified OK.

1.30.4 Solving as homogeneousTypeD2 ode

Using the change of variables $y = u(t)\,t$ on the above ode results in new ode in $u(t)$

$$u'(t)\,t + u(t) - 2u(t)\,t + 2t^2u(t) = 0$$

In canonical form the ODE is

$$\begin{aligned}
u' &= F(t,u) \\
&= f(t)g(u) \\
&= -\frac{u(2t^2-2t+1)}{t}
\end{aligned}$$

Where $f(t) = -\frac{2t^2-2t+1}{t}$ and $g(u) = u$. Integrating both sides gives

$$\begin{aligned}
\frac{1}{u}\,du &= -\frac{2t^2-2t+1}{t}\,dt \\
\int \frac{1}{u}\,du &= \int -\frac{2t^2-2t+1}{t}\,dt \\
\ln(u) &= -t^2+2t-\ln(t)+c_2 \\
u &= \mathrm{e}^{-t^2+2t-\ln(t)+c_2} \\
&= c_2\mathrm{e}^{-t^2+2t-\ln(t)}
\end{aligned}$$

Which simplifies to

$$u(t) = \frac{c_2\mathrm{e}^{-t^2}\mathrm{e}^{2t}}{t}$$

Therefore the solution y is

$$\begin{aligned}
y &= ut \\
&= c_2\mathrm{e}^{-t^2}\mathrm{e}^{2t}
\end{aligned}$$

Initial conditions are used to solve for c_2. Substituting $t = 0$ and $y = 3$ in the above solution gives an equation to solve for the constant of integration.

$$3 = c_2$$

$$c_2 = 3$$

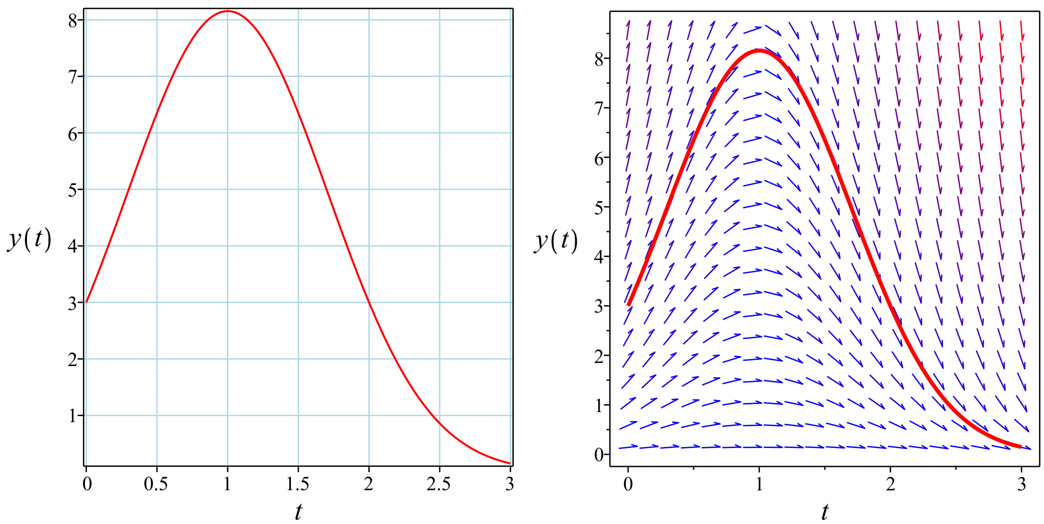
Substituting c_2 found above in the general solution gives

$$y = 3\,\mathrm{e}^{-t(-2+t)}$$

Summary

The solution(s) found are the following

$$y = 3\,\mathrm{e}^{-t(-2+t)} \tag{1}$$



(a) Solution plot (b) Slope field plot

Verification of solutions

$$y = 3\,\mathrm{e}^{-t(-2+t)}$$

Verified OK.

1.30.5 Solving as first order ode lie symmetry lookup ode

Writing the ode as

$$y' = -2ty + 2y$$
$$y' = \omega(t, y)$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_t + \omega(\eta_y - \xi_t) - \omega^2 \xi_y - \omega_t \xi - \omega_y \eta = 0 \tag{A}$$

The type of this ode is known. It is of type `linear`. Therefore we do not need to solve the PDE (A), and can just use the lookup table shown below to find ξ, η

Table 71: Lie symmetry infinitesimal lookup table for known first order ODE's

ODE class	Form	ξ	η
linear ode	$y' = f(x)y(x) + g(x)$	0	$e^{\int f dx}$
separable ode	$y' = f(x)\, g(y)$	$\frac{1}{f}$	0
quadrature ode	$y' = f(x)$	0	1
quadrature ode	$y' = g(y)$	1	0
homogeneous ODEs of Class A	$y' = f\left(\frac{y}{x}\right)$	x	y
homogeneous ODEs of Class C	$y' = (a + bx + cy)^{\frac{n}{m}}$	1	$-\frac{b}{c}$
homogeneous class D	$y' = \frac{y}{x} + g(x)\, F\left(\frac{y}{x}\right)$	x^2	xy
First order special form ID 1	$y' = g(x)\, e^{h(x)+by} + f(x)$	$\frac{e^{-\int bf(x)dx - h(x)}}{g(x)}$	$\frac{f(x)e^{-\int bf(x)dx - h(x)}}{g(x)}$
polynomial type ode	$y' = \frac{a_1x+b_1y+c_1}{a_2x+b_2y+c_2}$	$\frac{a_1b_2x - a_2b_1x - b_1c_2 + b_2c_1}{a_1b_2 - a_2b_1}$	$\frac{a_1b_2y - a_2b_1y - a_1c_2 - a_2c_1}{a_1b_2 - a_2b_1}$
Bernoulli ode	$y' = f(x)\, y + g(x)\, y^n$	0	$e^{-\int (n-1)f(x)dx} y^n$
Reduced Riccati	$y' = f_1(x)\, y + f_2(x)\, y^2$	0	$e^{-\int f_1 dx}$

The above table shows that

$$
\begin{aligned}
\xi(t, y) &= 0 \\
\eta(t, y) &= \mathrm{e}^{-t^2+2t}
\end{aligned} \tag{A1}
$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(t, y) \to (R, S)$ where (R, S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$
\frac{dt}{\xi} = \frac{dy}{\eta} = dS \tag{1}
$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial t} + \eta \frac{\partial}{\partial y}\right) S(t, y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$R = t$$

S is found from

$$\begin{aligned} S &= \int \frac{1}{\eta} dy \\ &= \int \frac{1}{\mathrm{e}^{-t^2+2t}} dy \end{aligned}$$

Which results in

$$S = \mathrm{e}^{t^2-2t} y$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_t + \omega(t, y) S_y}{R_t + \omega(t, y) R_y} \tag{2}$$

Where in the above R_t, R_y, S_t, S_y are all partial derivatives and $\omega(t, y)$ is the right hand side of the original ode given by

$$\omega(t, y) = -2ty + 2y$$

Evaluating all the partial derivatives gives

$$\begin{aligned} R_t &= 1 \\ R_y &= 0 \\ S_t &= 2(t-1)\,\mathrm{e}^{t(-2+t)} y \\ S_y &= \mathrm{e}^{t(-2+t)} \end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = 0 \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for t, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = 0$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordinates R, S. Integrating the above gives

$$S(R) = c_1 \tag{4}$$

To complete the solution, we just need to transform (4) back to t, y coordinates. This results in

$$e^{t(-2+t)} y = c_1$$

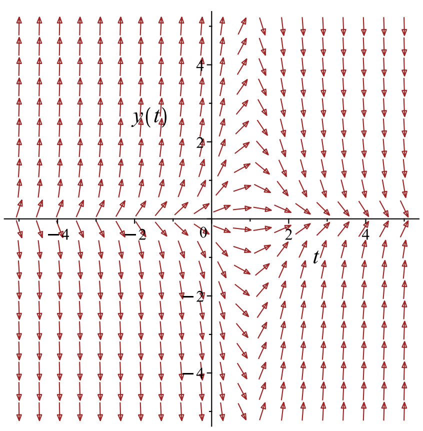
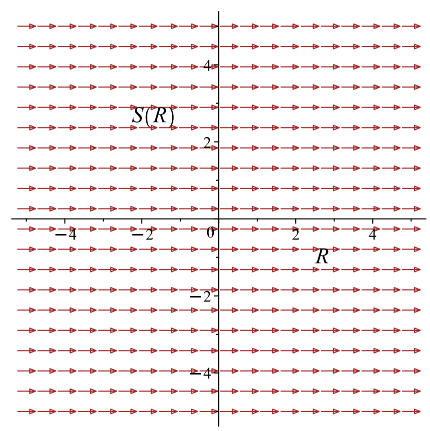
Which simplifies to

$$e^{t(-2+t)} y = c_1$$

Which gives

$$y = c_1 e^{-t(-2+t)}$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in t, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dt} = -2ty + 2y$ 	$R = t$ $S = e^{t(-2+t)} y$	$\frac{dS}{dR} = 0$ 

Initial conditions are used to solve for c_1. Substituting $t = 0$ and $y = 3$ in the above solution gives an equation to solve for the constant of integration.

$$3 = c_1$$

$$c_1 = 3$$

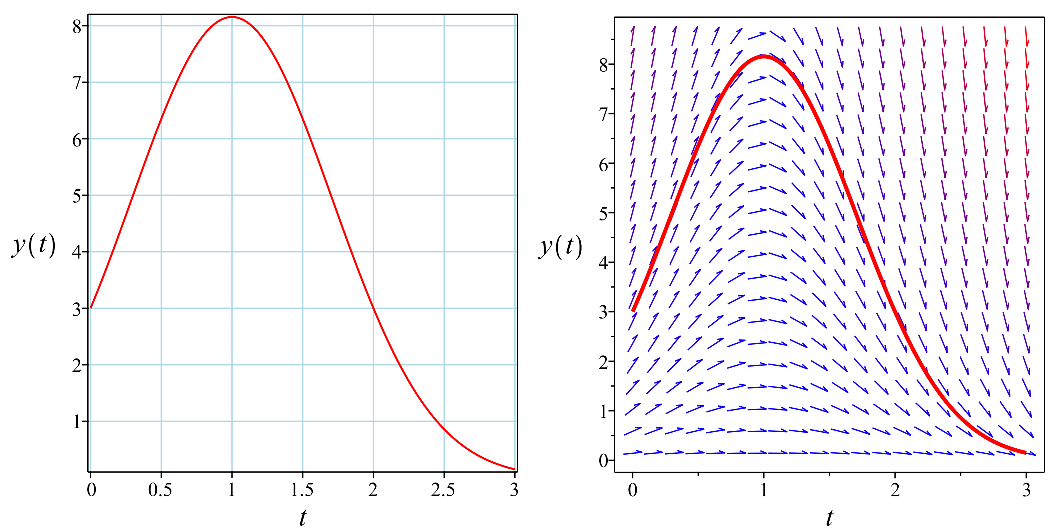
Substituting c_1 found above in the general solution gives

$$y = 3\,\mathrm{e}^{-t(-2+t)}$$

Summary

The solution(s) found are the following

$$y = 3\,\mathrm{e}^{-t(-2+t)} \tag{1}$$



(a) Solution plot

(b) Slope field plot

Verification of solutions

$$y = 3\,\mathrm{e}^{-t(-2+t)}$$

Verified OK.

1.30.6 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x,y) + N(x,y)\,\frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x,y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y) = 0$$

Hence

$$\frac{\partial \phi}{\partial x}+\frac{\partial \phi}{\partial y}\frac{dy}{dx}=0 \tag{B}$$

Comparing (A,B) shows that

$$\begin{aligned}\frac{\partial \phi}{\partial x}&=M\\ \frac{\partial \phi}{\partial y}&=N\end{aligned}$$

But since $\frac{\partial^2\phi}{\partial x\partial y}=\frac{\partial^2\phi}{\partial y\partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y}=\frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x,y)$ but at least we know now that we can do that since the condition $\frac{\partial^2\phi}{\partial x\partial y}=\frac{\partial^2\phi}{\partial y\partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(t,y)\,\mathrm{d}t+N(t,y)\,\mathrm{d}y=0 \tag{1A}$$

Therefore

$$\begin{aligned}\left(-\frac{1}{2y}\right)\mathrm{d}y&=(t-1)\,\mathrm{d}t\\ (-t+1)\,\mathrm{d}t+\left(-\frac{1}{2y}\right)\mathrm{d}y&=0 \qquad (2A)\end{aligned}$$

Comparing (1A) and (2A) shows that

$$\begin{aligned}M(t,y)&=-t+1\\ N(t,y)&=-\frac{1}{2y}\end{aligned}$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y}=\frac{\partial N}{\partial t}$$

Using result found above gives

$$\begin{aligned}\frac{\partial M}{\partial y}&=\frac{\partial}{\partial y}(-t+1)\\ &=0\end{aligned}$$

And

$$
\begin{aligned}
\frac{\partial N}{\partial t} &= \frac{\partial}{\partial t}\left(-\frac{1}{2y}\right) \\
&= 0
\end{aligned}
$$

Since $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial t}$, then the ODE is <u>exact</u> The following equations are now set up to solve for the function $\phi(t, y)$

$$
\frac{\partial \phi}{\partial t} = M \tag{1}
$$

$$
\frac{\partial \phi}{\partial y} = N \tag{2}
$$

Integrating (1) w.r.t. t gives

$$
\begin{aligned}
\int \frac{\partial \phi}{\partial t}\,\mathrm{d}t &= \int M\,\mathrm{d}t \\
\int \frac{\partial \phi}{\partial t}\,\mathrm{d}t &= \int -t + 1\,\mathrm{d}t
\end{aligned}
$$

$$
\phi = -\frac{1}{2}t^2 + t + f(y) \tag{3}
$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both t and y. Taking derivative of equation (3) w.r.t y gives

$$
\frac{\partial \phi}{\partial y} = 0 + f'(y) \tag{4}
$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = -\frac{1}{2y}$. Therefore equation (4) becomes

$$
-\frac{1}{2y} = 0 + f'(y) \tag{5}
$$

Solving equation (5) for $f'(y)$ gives

$$
f'(y) = -\frac{1}{2y}
$$

Integrating the above w.r.t y gives

$$
\begin{aligned}
\int f'(y)\,\mathrm{d}y &= \int \left(-\frac{1}{2y}\right)\mathrm{d}y \\
f(y) &= -\frac{\ln(y)}{2} + c_1
\end{aligned}
$$

Where c_1 is constant of integration. Substituting result found above for $f(y)$ into equation (3) gives ϕ

$$\phi = -\frac{t^2}{2} + t - \frac{\ln(y)}{2} + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = -\frac{t^2}{2} + t - \frac{\ln(y)}{2}$$

The solution becomes

$$y = \mathrm{e}^{-t^2 - 2c_1 + 2t}$$

Initial conditions are used to solve for c_1. Substituting $t = 0$ and $y = 3$ in the above solution gives an equation to solve for the constant of integration.

$$3 = \mathrm{e}^{-2c_1}$$

$$c_1 = -\frac{\ln(3)}{2}$$

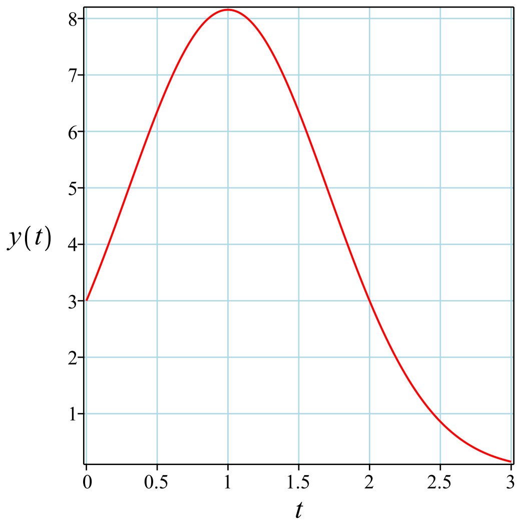
Substituting c_1 found above in the general solution gives

$$y = 3\,\mathrm{e}^{-t(-2+t)}$$

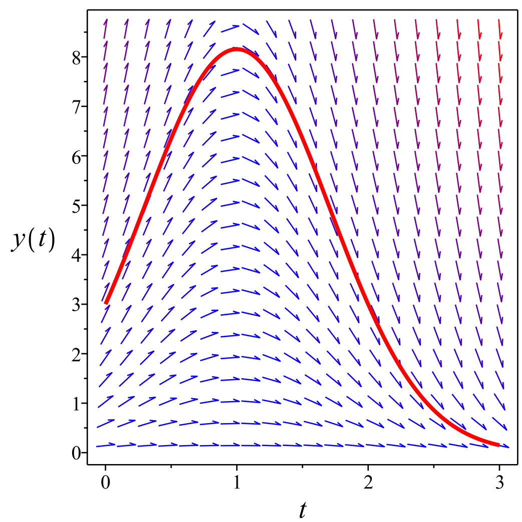
Summary

The solution(s) found are the following

$$y = 3\,\mathrm{e}^{-t(-2+t)} \tag{1}$$



(a) Solution plot



(b) Slope field plot

Verification of solutions

$$y = 3\,\mathrm{e}^{-t(-2+t)}$$

Verified OK.

1.30.7 Maple step by step solution

Let's solve

$[y' - 2y + 2ty = 0, y(0) = 3]$

- Highest derivative means the order of the ODE is 1

  y'

- Separate variables

  $\frac{y'}{y} = -2t + 2$

- Integrate both sides with respect to t

  $\int \frac{y'}{y} dt = \int (-2t + 2)\, dt + c_1$

- Evaluate integral

  $\ln(y) = -t^2 + c_1 + 2t$

- Solve for y

  $y = \mathrm{e}^{-t^2 + c_1 + 2t}$

- Use initial condition $y(0) = 3$

  $3 = e^{c_1}$
- Solve for c_1

  $c_1 = \ln(3)$
- Substitute $c_1 = \ln(3)$ into general solution and simplify

  $y = 3\,e^{-t(-2+t)}$
- Solution to the IVP

  $y = 3\,e^{-t(-2+t)}$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
<- 1st order linear successful`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 13

```
dsolve([diff(y(t),t)=2*y(t)-2*t*y(t),y(0) = 3],y(t), singsol=all)
```

$$y(t) = 3\,e^{-t(t-2)}$$

✓ Solution by Mathematica

Time used: 0.026 (sec). Leaf size: 15

```
DSolve[{y'[t]==2*y[t]-2*t*y[t],{y[0]==3}},y[t],t,IncludeSingularSolutions -> True]
```

$$y(t) \to 3e^{-((t-2)t)}$$

1.31 problem 29 part(a)

Internal problem ID [4942]
Internal file name [`OUTPUT/4435_Sunday_June_05_2022_02_56_36_PM_45107296/index.tex`]

Book: Fundamentals of Differential Equations. By Nagle, Saff and Snider. 9th edition. Boston. Pearson 2018.
Section: Chapter 2, First order differential equations. Section 2.2, Separable Equations. Exercises. page 46
Problem number: 29 part(a).
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"quadrature"**

Maple gives the following as the ode type

`[_quadrature]`

$$\boxed{y' - y^{\frac{1}{3}} = 0}$$

1.31.1 Solving as quadrature ode

Integrating both sides gives

$$\int \frac{1}{y^{\frac{1}{3}}} dy = \int dx$$

$$\frac{3y^{\frac{2}{3}}}{2} = x + c_1$$

Summary
The solution(s) found are the following

$$\frac{3y^{\frac{2}{3}}}{2} = x + c_1 \tag{1}$$

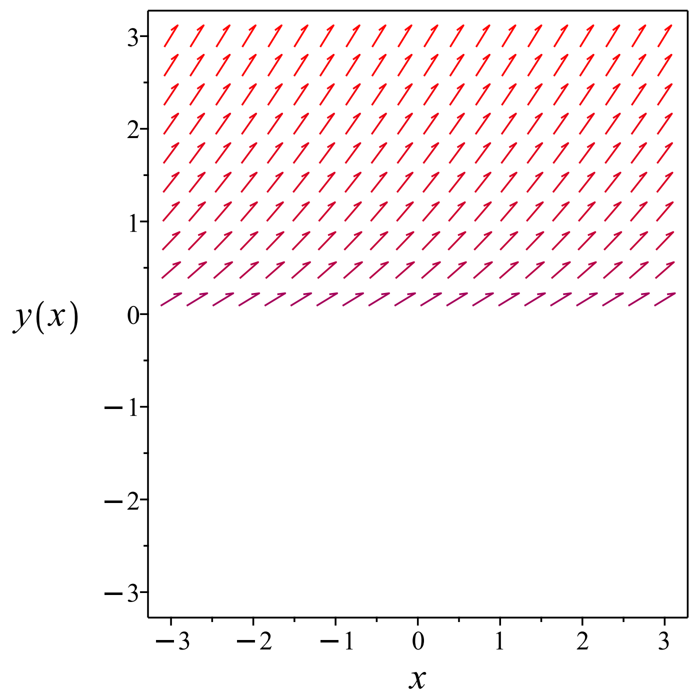


Figure 79: Slope field plot

Verification of solutions

$$\frac{3y^{\frac{2}{3}}}{2} = x + c_1$$

Verified OK.

1.31.2 Maple step by step solution

Let's solve

$y' - y^{\frac{1}{3}} = 0$

- Highest derivative means the order of the ODE is 1

  y'

- Separate variables

  $\frac{y'}{y^{\frac{1}{3}}} = 1$

- Integrate both sides with respect to x

  $\int \frac{y'}{y^{\frac{1}{3}}} dx = \int 1dx + c_1$

- Evaluate integral

  $$\frac{3y^{\frac{2}{3}}}{2} = x + c_1$$

- Solve for y

  $$y = \frac{(6x+6c_1)^{\frac{3}{2}}}{27}$$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
<- Bernoulli successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 14

```
dsolve(diff(y(x),x)=y(x)^(1/3),y(x), singsol=all)
```

$$y(x)^{\frac{2}{3}} - \frac{2x}{3} - c_1 = 0$$

✓ Solution by Mathematica

Time used: 0.165 (sec). Leaf size: 29

```
DSolve[y'[x]==y[x]^(1/3),y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \frac{2}{3}\sqrt{\frac{2}{3}}(x + c_1)^{3/2}$$
$$y(x) \to 0$$

1.32 problem 29 part(b)

Internal problem ID [4943]
Internal file name [`OUTPUT/4436_Sunday_June_05_2022_02_56_37_PM_19645502/index.tex`]

Book: Fundamentals of Differential Equations. By Nagle, Saff and Snider. 9th edition. Boston. Pearson 2018.
Section: Chapter 2, First order differential equations. Section 2.2, Separable Equations. Exercises. page 46
Problem number: 29 part(b).
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"quadrature"**

Maple gives the following as the ode type

`[_quadrature]`

$$\boxed{y' - y^{\frac{1}{3}} = 0}$$

With initial conditions

$$[y(0) = 0]$$

1.32.1 Existence and uniqueness analysis

This is non linear first order ODE. In canonical form it is written as

$$\begin{aligned} y' &= f(x, y) \\ &= y^{\frac{1}{3}} \end{aligned}$$

The y domain of $f(x, y)$ when $x = 0$ is

$$\{0 \leq y\}$$

And the point $y_0 = 0$ is inside this domain. Now we will look at the continuity of

$$\begin{aligned}\frac{\partial f}{\partial y} &= \frac{\partial}{\partial y}\left(y^{\frac{1}{3}}\right) \\ &= \frac{1}{3y^{\frac{2}{3}}}\end{aligned}$$

The y domain of $\frac{\partial f}{\partial y}$ when $x = 0$ is

$$\{0 < y\}$$

But the point $y_0 = 0$ is not inside this domain. Hence existence and uniqueness theorem does not apply. Solution exists but no guarantee that unique solution exists.

1.32.2 Solving as quadrature ode

Integrating both sides gives

$$\int \frac{1}{y^{\frac{1}{3}}} dy = \int dx$$

$$\frac{3y^{\frac{2}{3}}}{2} = x + c_1$$

Initial conditions are used to solve for c_1. Substituting $x = 0$ and $y = 0$ in the above solution gives an equation to solve for the constant of integration.

$$0 = c_1$$

$$c_1 = 0$$

Substituting c_1 found above in the general solution gives

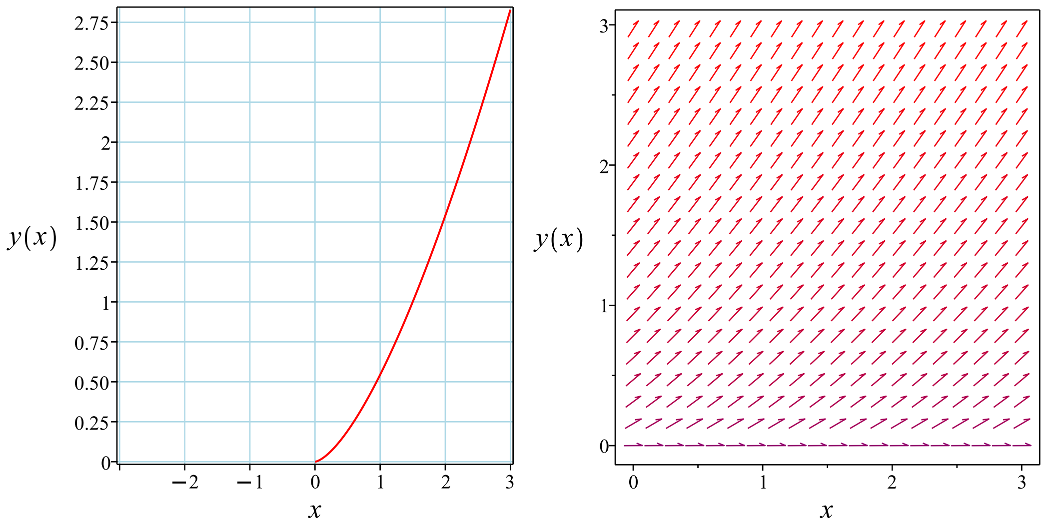
$$\frac{3y^{\frac{2}{3}}}{2} = x$$

Solving for y from the above gives

$$y = \frac{2x^{\frac{3}{2}}\sqrt{6}}{9}$$

Summary
The solution(s) found are the following

$$y = \frac{2x^{\frac{3}{2}}\sqrt{6}}{9} \tag{1}$$



(a) Solution plot (b) Slope field plot

<u>Verification of solutions</u>

$$y = \frac{2x^{\frac{3}{2}}\sqrt{6}}{9}$$

Verified OK.

1.32.3 Maple step by step solution

Let's solve

$\left[y' - y^{\frac{1}{3}} = 0, y(0) = 0\right]$

- Highest derivative means the order of the ODE is 1

  y'

- Separate variables

  $\frac{y'}{y^{\frac{1}{3}}} = 1$

- Integrate both sides with respect to x

  $\int \frac{y'}{y^{\frac{1}{3}}} dx = \int 1 dx + c_1$

- Evaluate integral

  $\frac{3y^{\frac{2}{3}}}{2} = x + c_1$

- Solve for y

$$y = \frac{(6x+6c_1)^{\frac{3}{2}}}{27}$$

- Use initial condition $y(0) = 0$

$$0 = \frac{2\sqrt{6}\, c_1^{\frac{3}{2}}}{9}$$

- Solve for c_1

$$c_1 = 0$$

- Substitute $c_1 = 0$ into general solution and simplify

$$y = \frac{2x^{\frac{3}{2}}\sqrt{6}}{9}$$

- Solution to the IVP

$$y = \frac{2x^{\frac{3}{2}}\sqrt{6}}{9}$$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
<- Bernoulli successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 5

```
dsolve([diff(y(x),x)=y(x)^(1/3),y(0) = 0],y(x), singsol=all)
```

$$y(x) = 0$$

✓ Solution by Mathematica

Time used: 0.004 (sec). Leaf size: 21

```
DSolve[{y'[x]==y[x]^(1/3),{y[0]==0}},y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \frac{2}{3}\sqrt{\frac{2}{3}}x^{3/2}$$

1.33 problem 30

Internal problem ID [4944]
Internal file name [`OUTPUT/4437_Sunday_June_05_2022_02_56_38_PM_1906324/index.tex`]

Book: Fundamentals of Differential Equations. By Nagle, Saff and Snider. 9th edition. Boston. Pearson 2018.
Section: Chapter 2, First order differential equations. Section 2.2, Separable Equations. Exercises. page 46
Problem number: 30.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"exact", "separable", "first_order_ode_lie_symmetry_lookup"**

Maple gives the following as the ode type

`[_separable]`

$$\boxed{y' - (x-3)(1+y)^{\frac{2}{3}} = 0}$$

1.33.1 Solving as separable ode

In canonical form the ODE is

$$\begin{aligned} y' &= F(x,y) \\ &= f(x)g(y) \\ &= (x-3)(1+y)^{\frac{2}{3}} \end{aligned}$$

Where $f(x) = x-3$ and $g(y) = (1+y)^{\frac{2}{3}}$. Integrating both sides gives

$$\frac{1}{(1+y)^{\frac{2}{3}}}\, dy = x - 3\, dx$$

$$\int \frac{1}{(1+y)^{\frac{2}{3}}}\, dy = \int x - 3\, dx$$

$$3(1+y)^{\frac{1}{3}} = \frac{1}{2}x^2 - 3x + c_1$$

The solution is

$$3(1+y)^{\frac{1}{3}} - \frac{x^2}{2} + 3x - c_1 = 0$$

Summary

The solution(s) found are the following

$$3(1+y)^{\frac{1}{3}} - \frac{x^2}{2} + 3x - c_1 = 0 \tag{1}$$

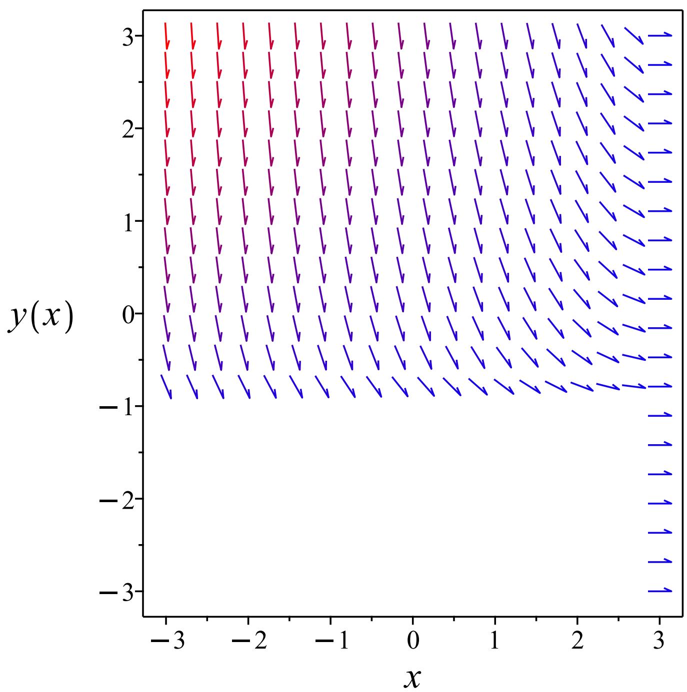


Figure 81: Slope field plot

Verification of solutions

$$3(1+y)^{\frac{1}{3}} - \frac{x^2}{2} + 3x - c_1 = 0$$

Verified OK.

1.33.2 Solving as first order ode lie symmetry lookup ode

Writing the ode as

$$
\begin{aligned}
y' &= (x-3)\,(1+y)^{\frac{2}{3}} \\
y' &= \omega(x,y)
\end{aligned}
$$

The condition of Lie symmetry is the linearized PDE given by

$$
\eta_x + \omega(\eta_y - \xi_x) - \omega^2 \xi_y - \omega_x \xi - \omega_y \eta = 0 \tag{A}
$$

The type of this ode is known. It is of type `separable`. Therefore we do not need to solve the PDE (A), and can just use the lookup table shown below to find ξ, η

Table 76: Lie symmetry infinitesimal lookup table for known first order ODE's

ODE class	Form	ξ	η
linear ode	$y' = f(x)y(x) + g(x)$	0	$e^{\int f dx}$
separable ode	$y' = f(x)\, g(y)$	$\frac{1}{f}$	0
quadrature ode	$y' = f(x)$	0	1
quadrature ode	$y' = g(y)$	1	0
homogeneous ODEs of Class A	$y' = f\left(\frac{y}{x}\right)$	x	y
homogeneous ODEs of Class C	$y' = (a+bx+cy)^{\frac{n}{m}}$	1	$-\frac{b}{c}$
homogeneous class D	$y' = \frac{y}{x} + g(x)\, F\left(\frac{y}{x}\right)$	x^2	xy
First order special form ID 1	$y' = g(x)\, e^{h(x)+by} + f(x)$	$\frac{e^{-\int bf(x)dx - h(x)}}{g(x)}$	$\frac{f(x)e^{-\int bf(x)dx - h(x)}}{g(x)}$
polynomial type ode	$y' = \frac{a_1x+b_1y+c_1}{a_2x+b_2y+c_2}$	$\frac{a_1b_2x - a_2b_1x - b_1c_2 + b_2c_1}{a_1b_2 - a_2b_1}$	$\frac{a_1b_2y - a_2b_1y - a_1c_2 - a_2c_1}{a_1b_2 - a_2b_1}$
Bernoulli ode	$y' = f(x)\, y + g(x)\, y^n$	0	$e^{-\int (n-1)f(x)dx} y^n$
Reduced Riccati	$y' = f_1(x)\, y + f_2(x)\, y^2$	0	$e^{-\int f_1 dx}$

The above table shows that

$$
\begin{aligned}
\xi(x,y) &= \frac{1}{x-3} \\
\eta(x,y) &= 0
\end{aligned} \tag{A1}
$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x,y) \to (R,S)$ where (R,S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$
\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}
$$

The above comes from the requirements that $\left(\xi\frac{\partial}{\partial x} + \eta\frac{\partial}{\partial y}\right) S(x,y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the canonical coordinates, where $S(R)$. Since $\eta = 0$ then in this special case

$$
R = y
$$

S is found from

$$
\begin{aligned}
S &= \int \frac{1}{\xi} dx \\
&= \int \frac{1}{\frac{1}{x-3}} dx
\end{aligned}
$$

Which results in

$$
S = \frac{1}{2}x^2 - 3x
$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$
\frac{dS}{dR} = \frac{S_x + \omega(x,y)S_y}{R_x + \omega(x,y)R_y} \tag{2}
$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x,y)$ is the right hand side of the original ode given by

$$
\omega(x,y) = (x-3)\,(1+y)^{\frac{2}{3}}
$$

Evaluating all the partial derivatives gives

$$
\begin{aligned}
R_x &= 0 \\
R_y &= 1 \\
S_x &= x-3 \\
S_y &= 0
\end{aligned}
$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = \frac{1}{(1+y)^{\frac{2}{3}}} \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = \frac{1}{(1+R)^{\frac{2}{3}}}$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$S(R) = 3(1+R)^{\frac{1}{3}} + c_1 \tag{4}$$

To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$\frac{1}{2}x^2 - 3x = 3(1+y)^{\frac{1}{3}} + c_1$$

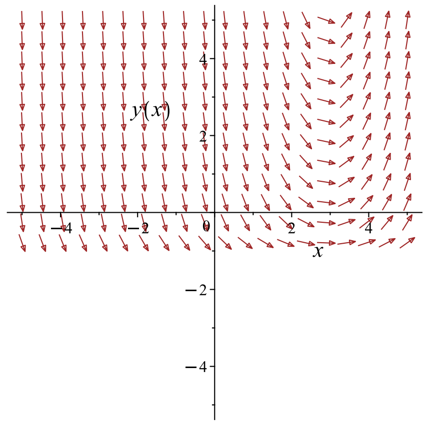
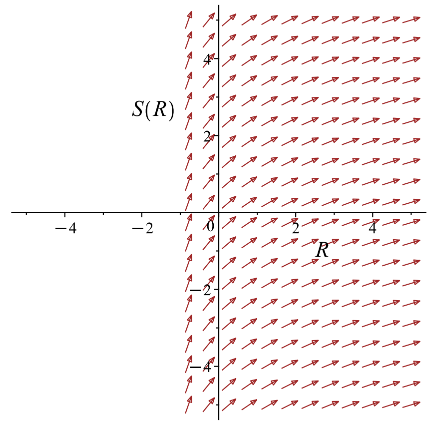
Which simplifies to

$$\frac{1}{2}x^2 - 3x = 3(1+y)^{\frac{1}{3}} + c_1$$

Which gives

$$y = \frac{1}{216}x^6 - \frac{1}{36}c_1x^4 - \frac{1}{12}x^5 + \frac{1}{18}x^2c_1^2 + \frac{1}{3}c_1x^3 + \frac{1}{2}x^4 - \frac{1}{27}c_1^3 - \frac{1}{3}c_1^2x - c_1x^2 - x^3 - 1$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in x, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dx} = (x-3)(1+y)^{\frac{2}{3}}$ 	$R = y$ $S = \frac{1}{2}x^2 - 3x$	$\frac{dS}{dR} = \frac{1}{(1+R)^{\frac{2}{3}}}$ 

Summary

The solution(s) found are the following

$$y = \frac{1}{216}x^6 - \frac{1}{36}c_1x^4 - \frac{1}{12}x^5 + \frac{1}{18}x^2c_1^2 + \frac{1}{3}c_1x^3 + \frac{1}{2}x^4 - \frac{1}{27}c_1^3 - \frac{1}{3}c_1^2x - c_1x^2 - x^3 \quad (1)$$

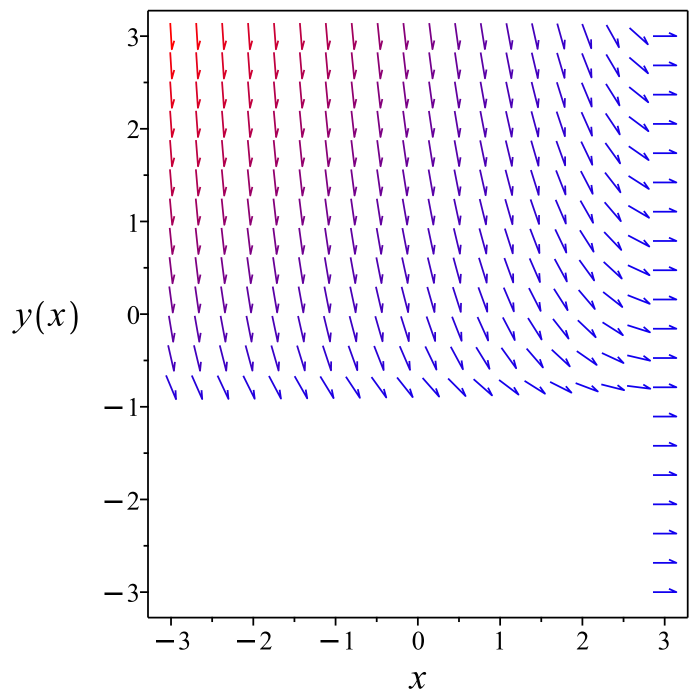


Figure 82: Slope field plot

Verification of solutions

$$y = \frac{1}{216}x^6 - \frac{1}{36}c_1x^4 - \frac{1}{12}x^5 + \frac{1}{18}x^2c_1^2 + \frac{1}{3}c_1x^3 + \frac{1}{2}x^4 - \frac{1}{27}c_1^3 - \frac{1}{3}c_1^2x - c_1x^2 - x^3 - 1$$

Verified OK.

1.33.3 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x,y) + N(x,y)\frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x,y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y) = 0$$

Hence

$$\frac{\partial\phi}{\partial x} + \frac{\partial\phi}{\partial y}\frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$\frac{\partial\phi}{\partial x}=M$$
$$\frac{\partial\phi}{\partial y}=N$$

But since $\frac{\partial^2\phi}{\partial x\partial y}=\frac{\partial^2\phi}{\partial y\partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y}=\frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x,y)$ but at least we know now that we can do that since the condition $\frac{\partial^2\phi}{\partial x\partial y}=\frac{\partial^2\phi}{\partial y\partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x,y)\,\mathrm{d}x+N(x,y)\,\mathrm{d}y=0 \tag{1A}$$

Therefore

$$\left(\frac{1}{(1+y)^{\frac{2}{3}}}\right)\mathrm{d}y=(x-3)\,\mathrm{d}x$$
$$(-x+3)\,\mathrm{d}x+\left(\frac{1}{(1+y)^{\frac{2}{3}}}\right)\mathrm{d}y=0 \tag{2A}$$

Comparing (1A) and (2A) shows that

$$M(x,y)=-x+3$$
$$N(x,y)=\frac{1}{(1+y)^{\frac{2}{3}}}$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y}=\frac{\partial N}{\partial x}$$

Using result found above gives

$$\begin{aligned}\frac{\partial M}{\partial y}&=\frac{\partial}{\partial y}(-x+3)\\&=0\end{aligned}$$

And

$$
\begin{aligned}
\frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}\left(\frac{1}{(1+y)^{\frac{2}{3}}}\right) \\
&= 0
\end{aligned}
$$

Since $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$, then the ODE is <u>exact</u> The following equations are now set up to solve for the function $\phi(x, y)$

$$
\frac{\partial \phi}{\partial x} = M \tag{1}
$$

$$
\frac{\partial \phi}{\partial y} = N \tag{2}
$$

Integrating (1) w.r.t. x gives

$$
\begin{aligned}
\int \frac{\partial \phi}{\partial x} \mathrm{d}x &= \int M \mathrm{d}x \\
\int \frac{\partial \phi}{\partial x} \mathrm{d}x &= \int -x + 3 \mathrm{d}x \\
\phi &= -\frac{1}{2}x^2 + 3x + f(y)
\end{aligned} \tag{3}
$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$
\frac{\partial \phi}{\partial y} = 0 + f'(y) \tag{4}
$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = \frac{1}{(1+y)^{\frac{2}{3}}}$. Therefore equation (4) becomes

$$
\frac{1}{(1+y)^{\frac{2}{3}}} = 0 + f'(y) \tag{5}
$$

Solving equation (5) for $f'(y)$ gives

$$
f'(y) = \frac{1}{(1+y)^{\frac{2}{3}}}
$$

Integrating the above w.r.t y gives

$$\int f'(y)\,\mathrm{d}y = \int \left(\frac{1}{(1+y)^{\frac{2}{3}}}\right) \mathrm{d}y$$

$$f(y) = 3(1+y)^{\frac{1}{3}} + c_1$$

Where c_1 is constant of integration. Substituting result found above for $f(y)$ into equation (3) gives ϕ

$$\phi = -\frac{x^2}{2} + 3x + 3(1+y)^{\frac{1}{3}} + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = -\frac{x^2}{2} + 3x + 3(1+y)^{\frac{1}{3}}$$

The solution becomes

$$y = \frac{1}{216}x^6 + \frac{1}{36}c_1x^4 - \frac{1}{12}x^5 + \frac{1}{18}x^2c_1^2 - \frac{1}{3}c_1x^3 + \frac{1}{2}x^4 + \frac{1}{27}c_1^3 - \frac{1}{3}c_1^2x + c_1x^2 - x^3 - 1$$

Summary

The solution(s) found are the following

$$y = \frac{1}{216}x^6 + \frac{1}{36}c_1x^4 - \frac{1}{12}x^5 + \frac{1}{18}x^2c_1^2 - \frac{1}{3}c_1x^3 + \frac{1}{2}x^4 + \frac{1}{27}c_1^3 - \frac{1}{3}c_1^2x + c_1x^2 - x^3 - 1 \quad (1)$$

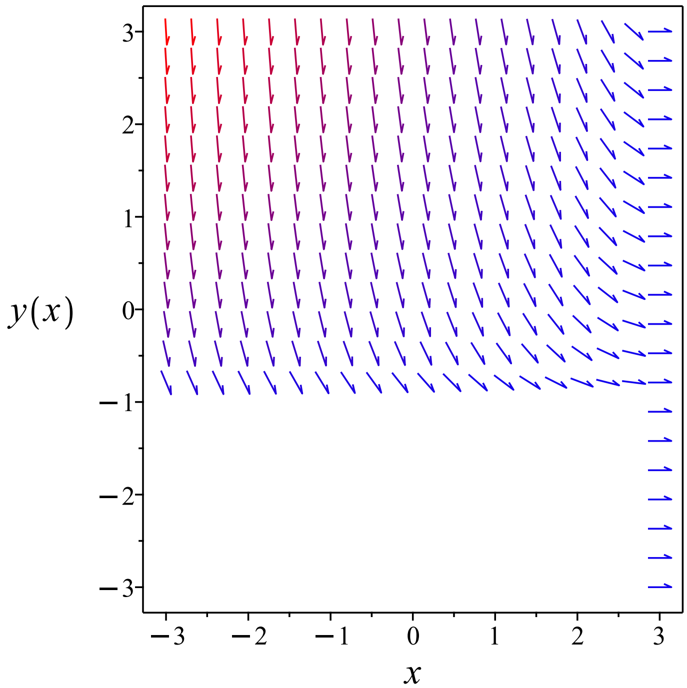


Figure 83: Slope field plot

Verification of solutions

$$y = \frac{1}{216}x^6 + \frac{1}{36}c_1x^4 - \frac{1}{12}x^5 + \frac{1}{18}x^2c_1^2 - \frac{1}{3}c_1x^3 + \frac{1}{2}x^4 + \frac{1}{27}c_1^3 - \frac{1}{3}c_1^2x + c_1x^2 - x^3 - 1$$

Verified OK.

1.33.4 Maple step by step solution

Let's solve

$y' - (x-3)(1+y)^{\frac{2}{3}} = 0$

- Highest derivative means the order of the ODE is 1

  y'

- Separate variables

  $\frac{y'}{(1+y)^{\frac{2}{3}}} = x - 3$

- Integrate both sides with respect to x

  $\int \frac{y'}{(1+y)^{\frac{2}{3}}}dx = \int (x-3)\,dx + c_1$

- Evaluate integral

  $3(1+y)^{\frac{1}{3}} = \frac{1}{2}x^2 - 3x + c_1$

- Solve for y

  $y = \frac{1}{216}x^6 + \frac{1}{36}c_1x^4 - \frac{1}{12}x^5 + \frac{1}{18}x^2c_1^2 - \frac{1}{3}c_1x^3 + \frac{1}{2}x^4 + \frac{1}{27}c_1^3 - \frac{1}{3}c_1^2x + c_1x^2 - x^3 - 1$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
<- separable successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 21

```
dsolve(diff(y(x),x)=(x-3)*(y(x)+1)^(2/3),y(x), singsol=all)
```

$$\frac{x^2}{2} - 3x - 3(y(x)+1)^{\frac{1}{3}} + c_1 = 0$$

✓ Solution by Mathematica

Time used: 0.197 (sec). Leaf size: 67

```
DSolve[y'[x]==(x-3)*(y[x]+1)^(2/3),y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \frac{1}{216}\left(x^6 - 18x^5 + 6(18+c_1)x^4 - 72(3+c_1)x^3 + 12c_1(18+c_1)x^2 - 72{c_1}^2x + 8\left(-27+{c_1}^3\right)\right)$$

$$y(x) \to -1$$

1.34 problem 31 part(a)

Internal problem ID [4945]
Internal file name [`OUTPUT/4438_Sunday_June_05_2022_02_56_39_PM_63821867/index.tex`]

Book: Fundamentals of Differential Equations. By Nagle, Saff and Snider. 9th edition. Boston. Pearson 2018.
Section: Chapter 2, First order differential equations. Section 2.2, Separable Equations. Exercises. page 46
Problem number: 31 part(a).
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"exact", "separable", "first_order_ode_lie_symmetry_lookup"**

Maple gives the following as the ode type

`[_separable]`

$$\boxed{y' - y^3 x = 0}$$

1.34.1 Solving as separable ode

In canonical form the ODE is

$$\begin{aligned} y' &= F(x,y) \\ &= f(x)g(y) \\ &= x\,y^3 \end{aligned}$$

Where $f(x) = x$ and $g(y) = y^3$. Integrating both sides gives

$$\frac{1}{y^3}\,dy = x\,dx$$

$$\int \frac{1}{y^3}\,dy = \int x\,dx$$

$$-\frac{1}{2y^2} = \frac{x^2}{2} + c_1$$

Which results in

$$y = -\frac{1}{\sqrt{-x^2 - 2c_1}}$$
$$y = \frac{1}{\sqrt{-x^2 - 2c_1}}$$

Summary
The solution(s) found are the following

$$y = -\frac{1}{\sqrt{-x^2 - 2c_1}} \tag{1}$$
$$y = \frac{1}{\sqrt{-x^2 - 2c_1}} \tag{2}$$

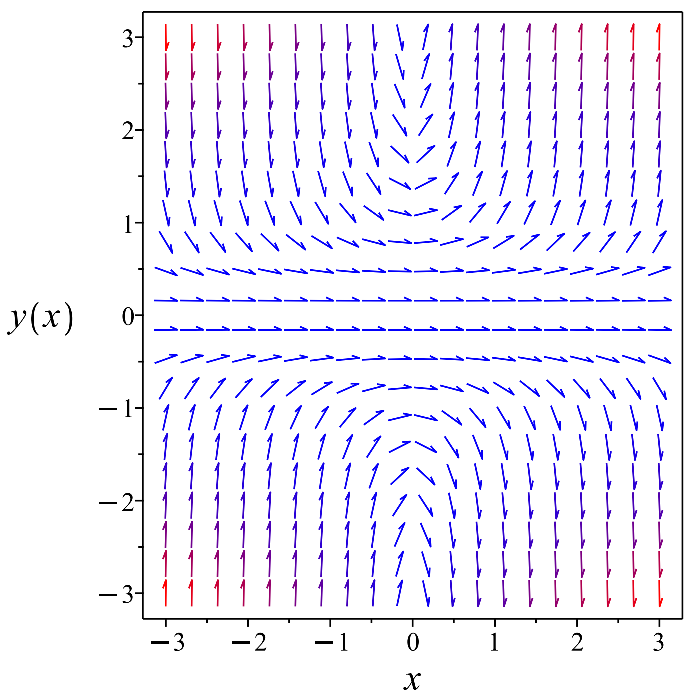


Figure 84: Slope field plot

Verification of solutions

$$y = -\frac{1}{\sqrt{-x^2 - 2c_1}}$$

Verified OK.

$$y = \frac{1}{\sqrt{-x^2 - 2c_1}}$$

Verified OK.

1.34.2 Solving as first order ode lie symmetry lookup ode

Writing the ode as

$$\begin{aligned} y' &= x\,y^3 \\ y' &= \omega(x, y) \end{aligned}$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x + \omega(\eta_y - \xi_x) - \omega^2 \xi_y - \omega_x \xi - \omega_y \eta = 0 \tag{A}$$

The type of this ode is known. It is of type `separable`. Therefore we do not need to solve the PDE (A), and can just use the lookup table shown below to find ξ, η

Table 79: Lie symmetry infinitesimal lookup table for known first order ODE's

ODE class	Form	ξ	η
linear ode	$y' = f(x)y(x) + g(x)$	0	$e^{\int f dx}$
separable ode	$y' = f(x)\, g(y)$	$\frac{1}{f}$	0
quadrature ode	$y' = f(x)$	0	1
quadrature ode	$y' = g(y)$	1	0
homogeneous ODEs of Class A	$y' = f\left(\frac{y}{x}\right)$	x	y
homogeneous ODEs of Class C	$y' = (a + bx + cy)^{\frac{n}{m}}$	1	$-\frac{b}{c}$
homogeneous class D	$y' = \frac{y}{x} + g(x)\, F\left(\frac{y}{x}\right)$	x^2	xy
First order special form ID 1	$y' = g(x)\, e^{h(x)+by} + f(x)$	$\frac{e^{-\int bf(x)dx - h(x)}}{g(x)}$	$\frac{f(x)e^{-\int bf(x)dx - h(x)}}{g(x)}$
polynomial type ode	$y' = \frac{a_1x+b_1y+c_1}{a_2x+b_2y+c_2}$	$\frac{a_1b_2x - a_2b_1x - b_1c_2 + b_2c_1}{a_1b_2 - a_2b_1}$	$\frac{a_1b_2y - a_2b_1y - a_1c_2 - a_2c_1}{a_1b_2 - a_2b_1}$
Bernoulli ode	$y' = f(x)\, y + g(x)\, y^n$	0	$e^{-\int (n-1)f(x)dx} y^n$
Reduced Riccati	$y' = f_1(x)\, y + f_2(x)\, y^2$	0	$e^{-\int f_1 dx}$

The above table shows that

$$\xi(x,y) = \frac{1}{x}$$
$$\eta(x,y) = 0 \tag{A1}$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x,y) \to (R,S)$ where (R,S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y}\right) S(x,y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the

canonical coordinates, where $S(R)$. Since $\eta = 0$ then in this special case

$$R = y$$

S is found from

$$\begin{aligned} S &= \int \frac{1}{\xi} dx \\ &= \int \frac{1}{\frac{1}{x}} dx \end{aligned}$$

Which results in

$$S = \frac{x^2}{2}$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x, y) S_y}{R_x + \omega(x, y) R_y} \tag{2}$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x, y)$ is the right hand side of the original ode given by

$$\omega(x, y) = x\, y^3$$

Evaluating all the partial derivatives gives

$$\begin{aligned} R_x &= 0 \\ R_y &= 1 \\ S_x &= x \\ S_y &= 0 \end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = \frac{1}{y^3} \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = \frac{1}{R^3}$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by

integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$S(R) = -\frac{1}{2R^2} + c_1 \tag{4}$$

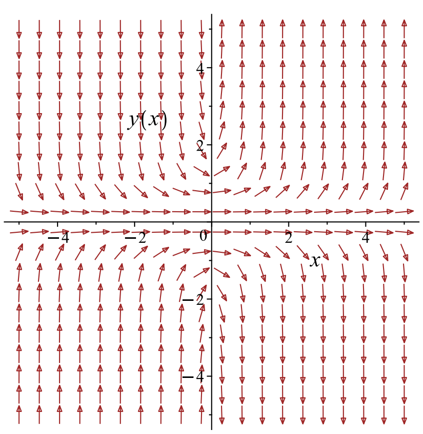
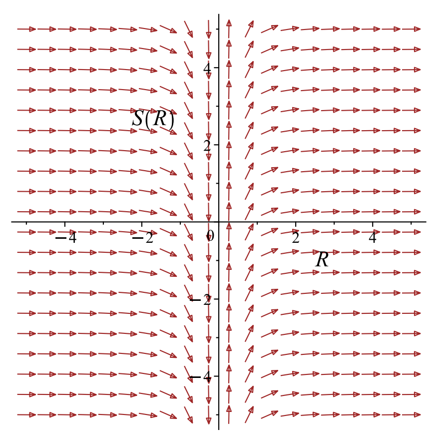
To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$\frac{x^2}{2} = -\frac{1}{2y^2} + c_1$$

Which simplifies to

$$\frac{x^2}{2} = -\frac{1}{2y^2} + c_1$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in x, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dx} = x\, y^3$ 	$R = y$ $S = \frac{x^2}{2}$	$\frac{dS}{dR} = \frac{1}{R^3}$ 

Summary

The solution(s) found are the following

$$\frac{x^2}{2} = -\frac{1}{2y^2} + c_1 \tag{1}$$

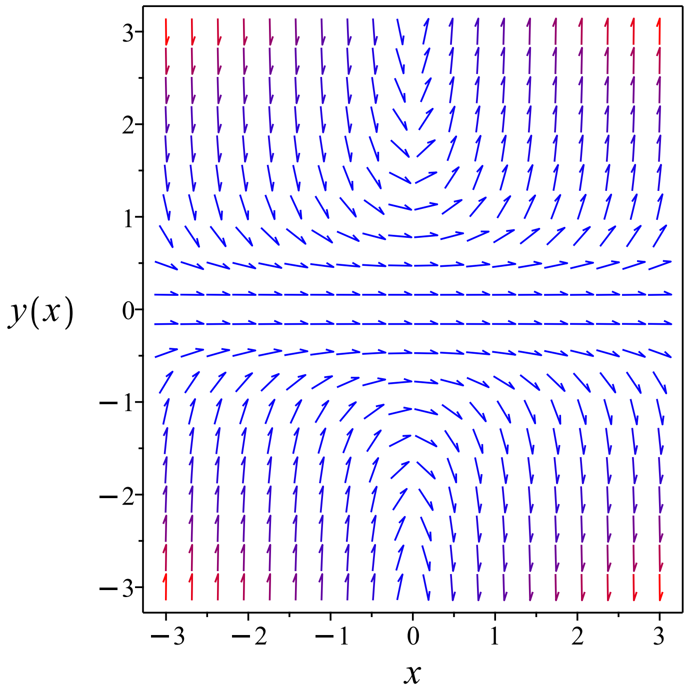


Figure 85: Slope field plot

Verification of solutions

$$\frac{x^2}{2} = -\frac{1}{2y^2} + c_1$$

Verified OK.

1.34.3 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x,y) + N(x,y)\frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x,y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y) = 0$$

Hence

$$\frac{\partial \phi}{\partial x} + \frac{\partial \phi}{\partial y}\frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$\frac{\partial \phi}{\partial x} = M$$
$$\frac{\partial \phi}{\partial y} = N$$

But since $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x, y)$ but at least we know now that we can do that since the condition $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x, y)\, \mathrm{d}x + N(x, y)\, \mathrm{d}y = 0 \tag{1A}$$

Therefore

$$\left(\frac{1}{y^3}\right) \mathrm{d}y = (x)\, \mathrm{d}x$$
$$(-x)\, \mathrm{d}x + \left(\frac{1}{y^3}\right) \mathrm{d}y = 0 \tag{2A}$$

Comparing (1A) and (2A) shows that

$$M(x, y) = -x$$
$$N(x, y) = \frac{1}{y^3}$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Using result found above gives

$$\begin{aligned}\frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}(-x) \\ &= 0\end{aligned}$$

And

$$
\begin{aligned}
\frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}\left(\frac{1}{y^3}\right) \\
&= 0
\end{aligned}
$$

Since $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$, then the ODE is exact The following equations are now set up to solve for the function $\phi(x, y)$

$$
\begin{aligned}
\frac{\partial \phi}{\partial x} &= M \quad (1) \\
\frac{\partial \phi}{\partial y} &= N \quad (2)
\end{aligned}
$$

Integrating (1) w.r.t. x gives

$$
\begin{aligned}
\int \frac{\partial \phi}{\partial x} \, \mathrm{d}x &= \int M \, \mathrm{d}x \\
\int \frac{\partial \phi}{\partial x} \, \mathrm{d}x &= \int -x \, \mathrm{d}x \\
\phi &= -\frac{x^2}{2} + f(y) \quad (3)
\end{aligned}
$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$
\frac{\partial \phi}{\partial y} = 0 + f'(y) \quad (4)
$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = \frac{1}{y^3}$. Therefore equation (4) becomes

$$
\frac{1}{y^3} = 0 + f'(y) \quad (5)
$$

Solving equation (5) for $f'(y)$ gives

$$
f'(y) = \frac{1}{y^3}
$$

Integrating the above w.r.t y gives

$$\int f'(y)\,\mathrm{d}y = \int \left(\frac{1}{y^3}\right)\mathrm{d}y$$
$$f(y) = -\frac{1}{2y^2} + c_1$$

Where c_1 is constant of integration. Substituting result found above for $f(y)$ into equation (3) gives ϕ

$$\phi = -\frac{x^2}{2} - \frac{1}{2y^2} + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = -\frac{x^2}{2} - \frac{1}{2y^2}$$

<u>Summary</u>
The solution(s) found are the following

$$-\frac{x^2}{2} - \frac{1}{2y^2} = c_1 \tag{1}$$

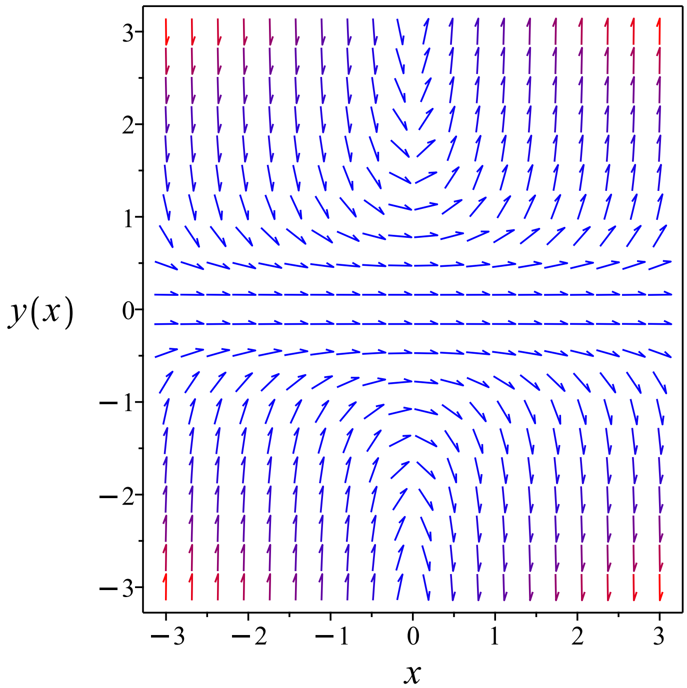


Figure 86: Slope field plot

<u>Verification of solutions</u>

$$-\frac{x^2}{2} - \frac{1}{2y^2} = c_1$$

Verified OK.

1.34.4 Maple step by step solution

Let's solve

$y' - y^3 x = 0$

- Highest derivative means the order of the ODE is 1

  y'

- Separate variables

  $\frac{y'}{y^3} = x$

- Integrate both sides with respect to x

  $\int \frac{y'}{y^3} dx = \int x dx + c_1$

- Evaluate integral

$$-\frac{1}{2y^2} = \frac{x^2}{2} + c_1$$

- Solve for y

$$\left\{y = \frac{1}{\sqrt{-x^2-2c_1}}, y = -\frac{1}{\sqrt{-x^2-2c_1}}\right\}$$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
<- Bernoulli successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 27

```
dsolve(diff(y(x),x)=x*y(x)^3,y(x), singsol=all)
```

$$y(x) = \frac{1}{\sqrt{-x^2+c_1}}$$
$$y(x) = -\frac{1}{\sqrt{-x^2+c_1}}$$

✓ Solution by Mathematica

Time used: 0.146 (sec). Leaf size: 44

```
DSolve[y'[x]==x*y[x]^3,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to -\frac{1}{\sqrt{-x^2-2c_1}}$$
$$y(x) \to \frac{1}{\sqrt{-x^2-2c_1}}$$
$$y(x) \to 0$$

1.35 problem 31 part(b.1)

Internal problem ID [4946]
Internal file name [`OUTPUT/4439_Sunday_June_05_2022_02_56_40_PM_98078028/index.tex`]

Book: Fundamentals of Differential Equations. By Nagle, Saff and Snider. 9th edition. Boston. Pearson 2018.
Section: Chapter 2, First order differential equations. Section 2.2, Separable Equations. Exercises. page 46
Problem number: 31 part(b.1).
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"exact", "separable", "first_order_ode_lie_symmetry_lookup"**

Maple gives the following as the ode type

`[_separable]`

$$\boxed{y' - y^3 x = 0}$$

With initial conditions

$$[y(0) = 1]$$

1.35.1 Existence and uniqueness analysis

This is non linear first order ODE. In canonical form it is written as

$$\begin{aligned} y' &= f(x, y) \\ &= x\, y^3 \end{aligned}$$

The x domain of $f(x, y)$ when $y = 1$ is

$$\{-\infty < x < \infty\}$$

And the point $x_0 = 0$ is inside this domain. The y domain of $f(x, y)$ when $x = 0$ is

$$\{-\infty < y < \infty\}$$

And the point $y_0 = 1$ is inside this domain. Now we will look at the continuity of

$$\begin{aligned}\frac{\partial f}{\partial y} &= \frac{\partial}{\partial y}(x\,y^3)\\ &= 3y^2x\end{aligned}$$

The x domain of $\frac{\partial f}{\partial y}$ when $y = 1$ is

$$\{-\infty < x < \infty\}$$

And the point $x_0 = 0$ is inside this domain. The y domain of $\frac{\partial f}{\partial y}$ when $x = 0$ is

$$\{-\infty < y < \infty\}$$

And the point $y_0 = 1$ is inside this domain. Therefore solution exists and is unique.

1.35.2 Solving as separable ode

In canonical form the ODE is

$$\begin{aligned}y' &= F(x, y)\\ &= f(x)g(y)\\ &= x\,y^3\end{aligned}$$

Where $f(x) = x$ and $g(y) = y^3$. Integrating both sides gives

$$\begin{aligned}\frac{1}{y^3}\,dy &= x\,dx\\ \int \frac{1}{y^3}\,dy &= \int x\,dx\\ -\frac{1}{2y^2} &= \frac{x^2}{2} + c_1\end{aligned}$$

Which results in

$$\begin{aligned}y &= -\frac{1}{\sqrt{-x^2 - 2c_1}}\\ y &= \frac{1}{\sqrt{-x^2 - 2c_1}}\end{aligned}$$

Initial conditions are used to solve for c_1. Substituting $x = 0$ and $y = 1$ in the above solution gives an equation to solve for the constant of integration.

$$1 = \frac{1}{\sqrt{-2c_1}}$$

$$c_1 = -\frac{1}{2}$$

Substituting c_1 found above in the general solution gives

$$y = \frac{1}{\sqrt{-x^2 + 1}}$$

Initial conditions are used to solve for c_1. Substituting $x = 0$ and $y = 1$ in the above solution gives an equation to solve for the constant of integration.

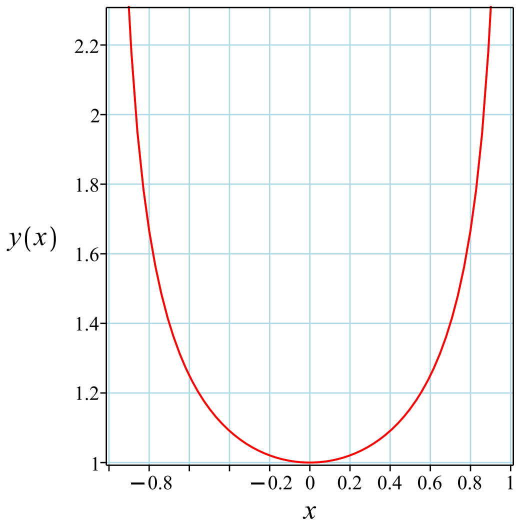
$$1 = -\frac{1}{\sqrt{-2c_1}}$$

Warning: Unable to solve for c_1. No particular solution can be found using given initial conditions for this solution. removing this solution as not valid.

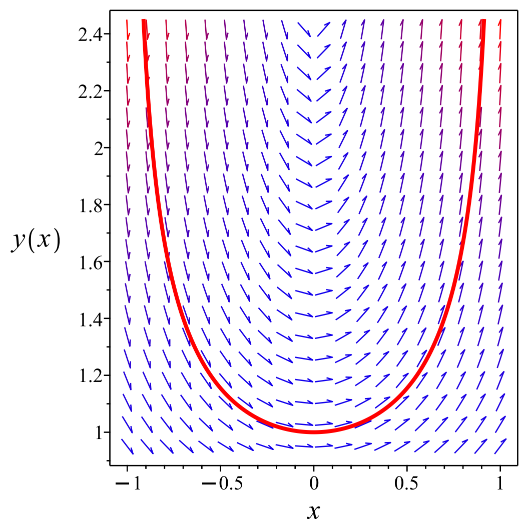
Summary

The solution(s) found are the following

$$y = -$$



(a) Solution plot



(b) Slope field plot

Verification of solutions

$$y = \frac{1}{\sqrt{-x^2+1}}$$

Verified OK.

1.35.3 Solving as first order ode lie symmetry lookup ode

Writing the ode as

$$\begin{aligned} y' &= x\,y^3 \\ y' &= \omega(x,y) \end{aligned}$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x + \omega(\eta_y - \xi_x) - \omega^2 \xi_y - \omega_x \xi - \omega_y \eta = 0 \tag{A}$$

The type of this ode is known. It is of type `separable`. Therefore we do not need to solve the PDE (A), and can just use the lookup table shown below to find ξ, η

Table 82: Lie symmetry infinitesimal lookup table for known first order ODE's

ODE class	Form	ξ	η
linear ode	$y' = f(x)y(x) + g(x)$	0	$e^{\int f dx}$
separable ode	$y' = f(x)\, g(y)$	$\frac{1}{f}$	0
quadrature ode	$y' = f(x)$	0	1
quadrature ode	$y' = g(y)$	1	0
homogeneous ODEs of Class A	$y' = f\left(\frac{y}{x}\right)$	x	y
homogeneous ODEs of Class C	$y' = (a + bx + cy)^{\frac{n}{m}}$	1	$-\frac{b}{c}$
homogeneous class D	$y' = \frac{y}{x} + g(x)\, F\left(\frac{y}{x}\right)$	x^2	xy
First order special form ID 1	$y' = g(x)\, e^{h(x)+by} + f(x)$	$\frac{e^{-\int bf(x)dx - h(x)}}{g(x)}$	$\frac{f(x)e^{-\int bf(x)dx - h(x)}}{g(x)}$
polynomial type ode	$y' = \frac{a_1x+b_1y+c_1}{a_2x+b_2y+c_2}$	$\frac{a_1b_2x - a_2b_1x - b_1c_2 + b_2c_1}{a_1b_2 - a_2b_1}$	$\frac{a_1b_2y - a_2b_1y - a_1c_2 - a_2c_1}{a_1b_2 - a_2b_1}$
Bernoulli ode	$y' = f(x)\, y + g(x)\, y^n$	0	$e^{-\int (n-1)f(x)dx} y^n$
Reduced Riccati	$y' = f_1(x)\, y + f_2(x)\, y^2$	0	$e^{-\int f_1 dx}$

The above table shows that

$$
\begin{aligned}
\xi(x,y) &= \frac{1}{x} \\
\eta(x,y) &= 0
\end{aligned} \tag{A1}
$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x,y) \to (R,S)$ where (R,S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$
\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}
$$

The above comes from the requirements that $\left(\xi\frac{\partial}{\partial x} + \eta\frac{\partial}{\partial y}\right) S(x,y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the

canonical coordinates, where $S(R)$. Since $\eta = 0$ then in this special case

$$R = y$$

S is found from

$$\begin{aligned} S &= \int \frac{1}{\xi} dx \\ &= \int \frac{1}{\frac{1}{x}} dx \end{aligned}$$

Which results in

$$S = \frac{x^2}{2}$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x, y) S_y}{R_x + \omega(x, y) R_y} \tag{2}$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x, y)$ is the right hand side of the original ode given by

$$\omega(x, y) = x\, y^3$$

Evaluating all the partial derivatives gives

$$\begin{aligned} R_x &= 0 \\ R_y &= 1 \\ S_x &= x \\ S_y &= 0 \end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = \frac{1}{y^3} \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = \frac{1}{R^3}$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by

integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$S(R) = -\frac{1}{2R^2} + c_1 \tag{4}$$

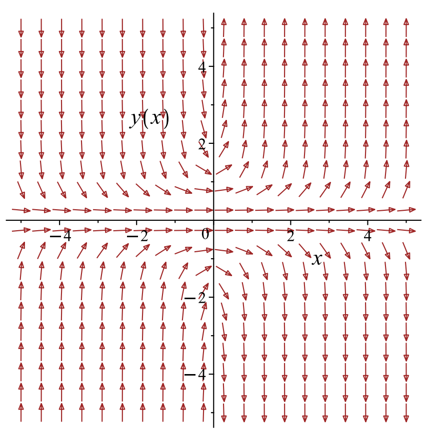
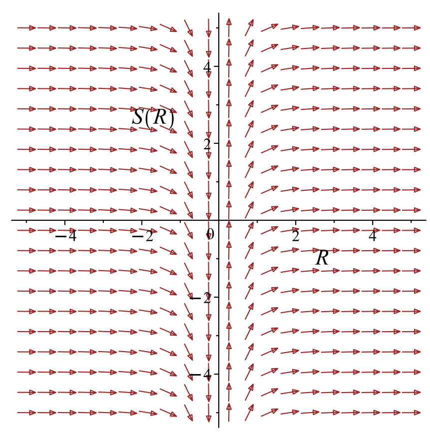
To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$\frac{x^2}{2} = -\frac{1}{2y^2} + c_1$$

Which simplifies to

$$\frac{x^2}{2} = -\frac{1}{2y^2} + c_1$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in x, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dx} = x\,y^3$ 	$R = y$ $S = \frac{x^2}{2}$	$\frac{dS}{dR} = \frac{1}{R^3}$ 

Initial conditions are used to solve for c_1. Substituting $x = 0$ and $y = 1$ in the above solution gives an equation to solve for the constant of integration.

$$0 = -\frac{1}{2} + c_1$$

$$c_1 = \frac{1}{2}$$

Substituting c_1 found above in the general solution gives

$$\frac{x^2}{2} = \frac{y^2 - 1}{2y^2}$$

The above simplifies to

$$y^2x^2 - y^2 + 1 = 0$$

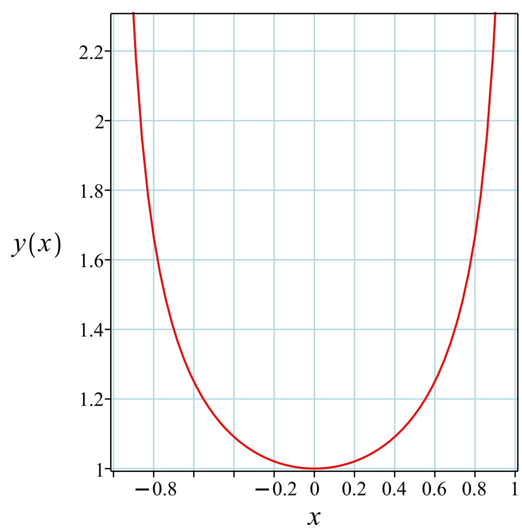
Solving for y from the above gives

$$y = \frac{1}{\sqrt{-x^2 + 1}}$$

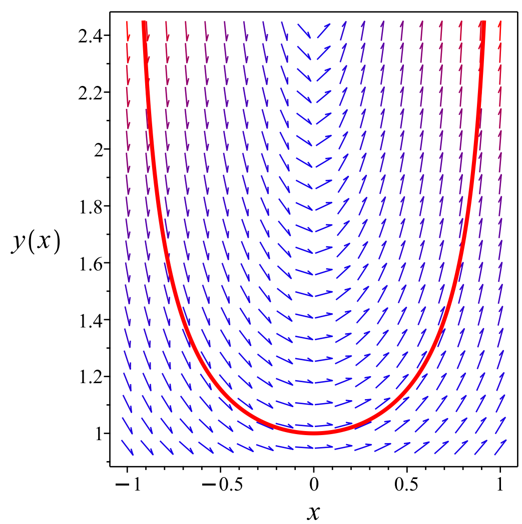
Summary

The solution(s) found are the following

$$y = \frac{1}{\sqrt{-x^2 + 1}} \tag{1}$$



(a) Solution plot



(b) Slope field plot

Verification of solutions

$$y = \frac{1}{\sqrt{-x^2 + 1}}$$

Verified OK.

1.35.4 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x,y)+N(x,y)\frac{dy}{dx}=0 \tag{A}$$

We assume there exists a function $\phi(x,y)=c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y)=0$$

Hence

$$\frac{\partial\phi}{\partial x}+\frac{\partial\phi}{\partial y}\frac{dy}{dx}=0 \tag{B}$$

Comparing (A,B) shows that

$$\begin{aligned}\frac{\partial\phi}{\partial x}&=M\\ \frac{\partial\phi}{\partial y}&=N\end{aligned}$$

But since $\frac{\partial^2\phi}{\partial x\partial y}=\frac{\partial^2\phi}{\partial y\partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y}=\frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x,y)$ but at least we know now that we can do that since the condition $\frac{\partial^2\phi}{\partial x\partial y}=\frac{\partial^2\phi}{\partial y\partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x,y)\,\mathrm{d}x+N(x,y)\,\mathrm{d}y=0 \tag{1A}$$

Therefore

$$\begin{aligned}\left(\frac{1}{y^3}\right)\mathrm{d}y&=(x)\,\mathrm{d}x\\ (-x)\,\mathrm{d}x+\left(\frac{1}{y^3}\right)\mathrm{d}y&=0 \qquad (2A)\end{aligned}$$

Comparing (1A) and (2A) shows that

$$
\begin{aligned}
M(x,y) &= -x \\
N(x,y) &= \frac{1}{y^3}
\end{aligned}
$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$
\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}
$$

Using result found above gives

$$
\begin{aligned}
\frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}(-x) \\
&= 0
\end{aligned}
$$

And

$$
\begin{aligned}
\frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}\left(\frac{1}{y^3}\right) \\
&= 0
\end{aligned}
$$

Since $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$, then the ODE is exact The following equations are now set up to solve for the function $\phi(x,y)$

$$
\frac{\partial \phi}{\partial x} = M \tag{1}
$$

$$
\frac{\partial \phi}{\partial y} = N \tag{2}
$$

Integrating (1) w.r.t. x gives

$$
\begin{aligned}
\int \frac{\partial \phi}{\partial x} \, \mathrm{d}x &= \int M \, \mathrm{d}x \\
\int \frac{\partial \phi}{\partial x} \, \mathrm{d}x &= \int -x \, \mathrm{d}x
\end{aligned}
$$

$$
\phi = -\frac{x^2}{2} + f(y) \tag{3}
$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$
\frac{\partial \phi}{\partial y} = 0 + f'(y) \tag{4}
$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = \frac{1}{y^3}$. Therefore equation (4) becomes

$$\frac{1}{y^3} = 0 + f'(y) \tag{5}$$

Solving equation (5) for $f'(y)$ gives

$$f'(y) = \frac{1}{y^3}$$

Integrating the above w.r.t y gives

$$\int f'(y)\,\mathrm{d}y = \int \left(\frac{1}{y^3}\right) \mathrm{d}y$$

$$f(y) = -\frac{1}{2y^2} + c_1$$

Where c_1 is constant of integration. Substituting result found above for $f(y)$ into equation (3) gives ϕ

$$\phi = -\frac{x^2}{2} - \frac{1}{2y^2} + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = -\frac{x^2}{2} - \frac{1}{2y^2}$$

Initial conditions are used to solve for c_1. Substituting $x = 0$ and $y = 1$ in the above solution gives an equation to solve for the constant of integration.

$$-\frac{1}{2} = c_1$$

$$c_1 = -\frac{1}{2}$$

Substituting c_1 found above in the general solution gives

$$-\frac{x^2}{2} - \frac{1}{2y^2} = -\frac{1}{2}$$

The above simplifies to

$$-y^2x^2 + y^2 - 1 = 0$$

Summary
The solution(s) found are the following

$$-y^2x^2 + y^2 - 1 = 0 \tag{1}$$

Verification of solutions

$$-y^2x^2 + y^2 - 1 = 0$$

Verified OK.

1.35.5 Maple step by step solution

Let's solve

$[y' - y^3x = 0, y(0) = 1]$

- Highest derivative means the order of the ODE is 1

  y'

- Separate variables

  $\frac{y'}{y^3} = x$

- Integrate both sides with respect to x

  $\int \frac{y'}{y^3} dx = \int x dx + c_1$

- Evaluate integral

  $-\frac{1}{2y^2} = \frac{x^2}{2} + c_1$

- Solve for y

  $\left\{y = \frac{1}{\sqrt{-x^2-2c_1}}, y = -\frac{1}{\sqrt{-x^2-2c_1}}\right\}$

- Use initial condition $y(0) = 1$

  $1 = \frac{1}{\sqrt{-2c_1}}$

- Solve for c_1

  $c_1 = -\frac{1}{2}$

- Substitute $c_1 = -\frac{1}{2}$ into general solution and simplify

  $y = \frac{1}{\sqrt{-x^2+1}}$

- Use initial condition $y(0) = 1$

  $1 = -\frac{1}{\sqrt{-2c_1}}$

- Solution does not satisfy initial condition

- Solution to the IVP

  $y = \frac{1}{\sqrt{-x^2+1}}$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
<- Bernoulli successful`
```

✓ Solution by Maple

Time used: 0.047 (sec). Leaf size: 13

```
dsolve([diff(y(x),x)=x*y(x)^3,y(0) = 1],y(x), singsol=all)
```

$$y(x) = \frac{1}{\sqrt{-x^2+1}}$$

✓ Solution by Mathematica

Time used: 0.096 (sec). Leaf size: 16

```
DSolve[{y'[x]==x*y[x]^3,{y[0]==1}},y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \frac{1}{\sqrt{1-x^2}}$$

1.36 problem 31 part(b.2)

Internal problem ID [4947]
Internal file name [`OUTPUT/4440_Sunday_June_05_2022_02_56_41_PM_88066967/index.tex`]

Book: Fundamentals of Differential Equations. By Nagle, Saff and Snider. 9th edition. Boston. Pearson 2018.
Section: Chapter 2, First order differential equations. Section 2.2, Separable Equations. Exercises. page 46
Problem number: 31 part(b.2).
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"exact", "separable", "first_order_ode_lie_symmetry_lookup"**

Maple gives the following as the ode type

`[_separable]`

$$\boxed{y' - y^3 x = 0}$$

With initial conditions

$$\left[y(0) = \frac{1}{2}\right]$$

1.36.1 Existence and uniqueness analysis

This is non linear first order ODE. In canonical form it is written as

$$\begin{aligned} y' &= f(x, y) \\ &= x\,y^3 \end{aligned}$$

The x domain of $f(x, y)$ when $y = \frac{1}{2}$ is

$$\{-\infty < x < \infty\}$$

And the point $x_0 = 0$ is inside this domain. The y domain of $f(x, y)$ when $x = 0$ is

$$\{-\infty < y < \infty\}$$

And the point $y_0 = \frac{1}{2}$ is inside this domain. Now we will look at the continuity of

$$\begin{aligned}\frac{\partial f}{\partial y} &= \frac{\partial}{\partial y}(x\,y^3)\\ &= 3y^2x\end{aligned}$$

The x domain of $\frac{\partial f}{\partial y}$ when $y = \frac{1}{2}$ is

$$\{-\infty < x < \infty\}$$

And the point $x_0 = 0$ is inside this domain. The y domain of $\frac{\partial f}{\partial y}$ when $x = 0$ is

$$\{-\infty < y < \infty\}$$

And the point $y_0 = \frac{1}{2}$ is inside this domain. Therefore solution exists and is unique.

1.36.2 Solving as separable ode

In canonical form the ODE is

$$\begin{aligned}y' &= F(x, y)\\ &= f(x)g(y)\\ &= x\,y^3\end{aligned}$$

Where $f(x) = x$ and $g(y) = y^3$. Integrating both sides gives

$$\begin{aligned}\frac{1}{y^3}\,dy &= x\,dx\\ \int \frac{1}{y^3}\,dy &= \int x\,dx\\ -\frac{1}{2y^2} &= \frac{x^2}{2} + c_1\end{aligned}$$

Which results in

$$\begin{aligned}y &= -\frac{1}{\sqrt{-x^2 - 2c_1}}\\ y &= \frac{1}{\sqrt{-x^2 - 2c_1}}\end{aligned}$$

Initial conditions are used to solve for c_1. Substituting $x = 0$ and $y = \frac{1}{2}$ in the above solution gives an equation to solve for the constant of integration.

$$\frac{1}{2} = \frac{1}{\sqrt{-2c_1}}$$

$$c_1 = -2$$

Substituting c_1 found above in the general solution gives

$$y = \frac{1}{\sqrt{-x^2 + 4}}$$

Initial conditions are used to solve for c_1. Substituting $x = 0$ and $y = \frac{1}{2}$ in the above solution gives an equation to solve for the constant of integration.

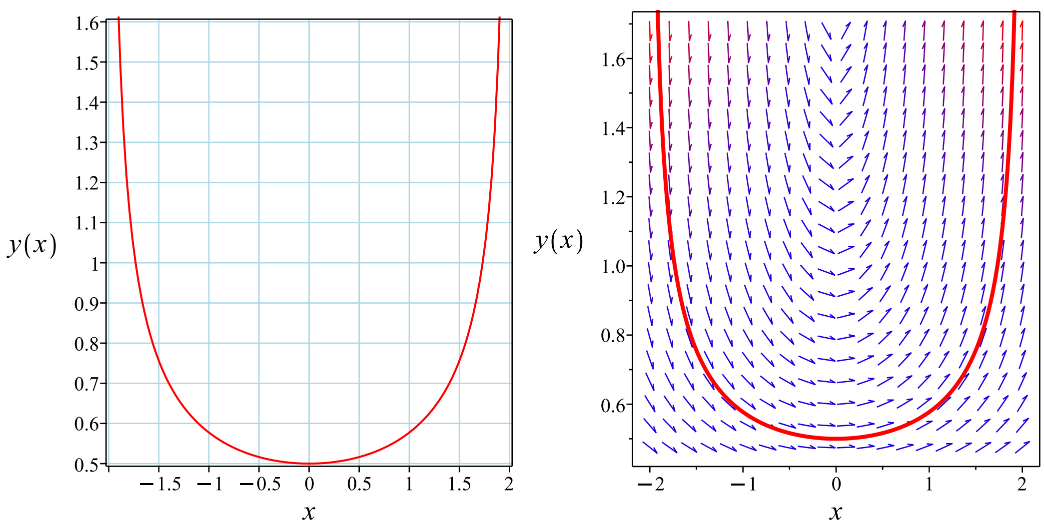
$$\frac{1}{2} = -\frac{1}{\sqrt{-2c_1}}$$

Warning: Unable to solve for c_1. No particular solution can be found using given initial conditions for this solution. removing this solution as not valid.

Summary

The solution(s) found are the following

$$y = -$$



(a) Solution plot

(b) Slope field plot

Verification of solutions

$$y = \frac{1}{\sqrt{-x^2 + 4}}$$

Verified OK.

1.36.3 Solving as first order ode lie symmetry lookup ode

Writing the ode as

$$y' = x\,y^3$$
$$y' = \omega(x,y)$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x + \omega(\eta_y - \xi_x) - \omega^2\xi_y - \omega_x\xi - \omega_y\eta = 0 \tag{A}$$

The type of this ode is known. It is of type `separable`. Therefore we do not need to solve the PDE (A), and can just use the lookup table shown below to find ξ, η

Table 85: Lie symmetry infinitesimal lookup table for known first order ODE's

ODE class	Form	ξ	η
linear ode	$y' = f(x)y(x) + g(x)$	0	$e^{\int f dx}$
separable ode	$y' = f(x)\,g(y)$	$\frac{1}{f}$	0
quadrature ode	$y' = f(x)$	0	1
quadrature ode	$y' = g(y)$	1	0
homogeneous ODEs of Class A	$y' = f\left(\frac{y}{x}\right)$	x	y
homogeneous ODEs of Class C	$y' = (a + bx + cy)^{\frac{n}{m}}$	1	$-\frac{b}{c}$
homogeneous class D	$y' = \frac{y}{x} + g(x)\,F\left(\frac{y}{x}\right)$	x^2	xy
First order special form ID 1	$y' = g(x)\,e^{h(x)+by} + f(x)$	$\frac{e^{-\int bf(x)dx - h(x)}}{g(x)}$	$\frac{f(x)e^{-\int bf(x)dx - h(x)}}{g(x)}$
polynomial type ode	$y' = \frac{a_1x+b_1y+c_1}{a_2x+b_2y+c_2}$	$\frac{a_1b_2x - a_2b_1x - b_1c_2 + b_2c_1}{a_1b_2 - a_2b_1}$	$\frac{a_1b_2y - a_2b_1y - a_1c_2 - a_2c_1}{a_1b_2 - a_2b_1}$
Bernoulli ode	$y' = f(x)\,y + g(x)\,y^n$	0	$e^{-\int (n-1)f(x)dx}y^n$
Reduced Riccati	$y' = f_1(x)\,y + f_2(x)\,y^2$	0	$e^{-\int f_1 dx}$

The above table shows that

$$
\begin{aligned}
\xi(x, y) &= \frac{1}{x} \\
\eta(x, y) &= 0
\end{aligned} \tag{A1}
$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x, y) \to (R, S)$ where (R, S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$
\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}
$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y}\right) S(x, y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the canonical coordinates, where $S(R)$. Since $\eta = 0$ then in this special case

$$
R = y
$$

S is found from

$$
\begin{aligned}
S &= \int \frac{1}{\xi} dx \\
&= \int \frac{1}{\frac{1}{x}} dx
\end{aligned}
$$

Which results in

$$
S = \frac{x^2}{2}
$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$
\frac{dS}{dR} = \frac{S_x + \omega(x, y) S_y}{R_x + \omega(x, y) R_y} \tag{2}
$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x, y)$ is the right hand side of the original ode given by

$$
\omega(x, y) = x\, y^3
$$

Evaluating all the partial derivatives gives

$$
\begin{aligned}
R_x &= 0 \\
R_y &= 1 \\
S_x &= x \\
S_y &= 0
\end{aligned}
$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$
\frac{dS}{dR} = \frac{1}{y^3} \tag{2A}
$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$
\frac{dS}{dR} = \frac{1}{R^3}
$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$
S(R) = -\frac{1}{2R^2} + c_1 \tag{4}
$$

To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$
\frac{x^2}{2} = -\frac{1}{2y^2} + c_1
$$

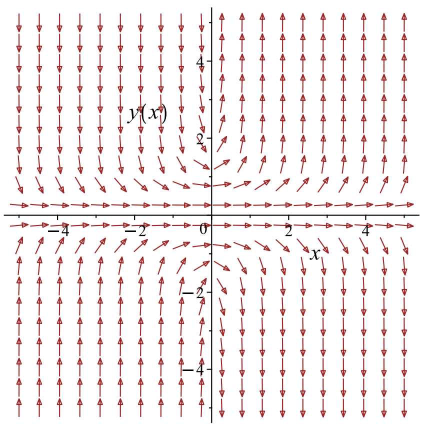
Which simplifies to

$$
\frac{x^2}{2} = -\frac{1}{2y^2} + c_1
$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

<table>
<tr><th>Original ode in x, y coordinates</th><th>Canonical coordinates transformation</th><th>ODE in canonical coordinates (R, S)</th></tr>
<tr><td>

$\frac{dy}{dx} = x\,y^3$



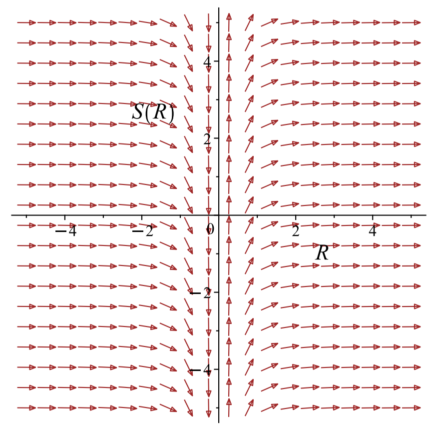
</td><td>

$R = y$

$S = \frac{x^2}{2}$

</td><td>

$\frac{dS}{dR} = \frac{1}{R^3}$



</td></tr>
</table>

Initial conditions are used to solve for c_1. Substituting $x = 0$ and $y = \frac{1}{2}$ in the above solution gives an equation to solve for the constant of integration.

$$0 = -2 + c_1$$

$$c_1 = 2$$

Substituting c_1 found above in the general solution gives

$$\frac{x^2}{2} = \frac{4y^2 - 1}{2y^2}$$

The above simplifies to

$$y^2x^2 - 4y^2 + 1 = 0$$

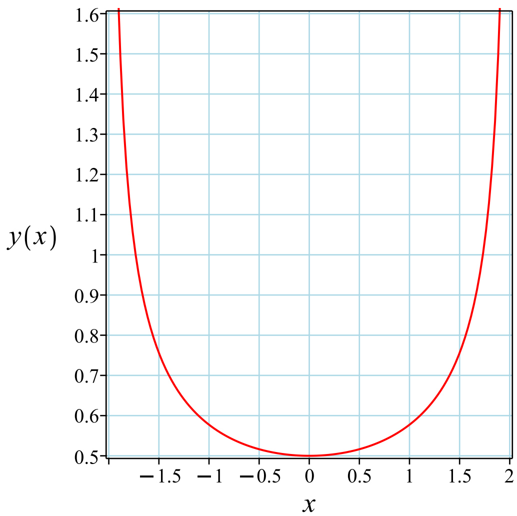
Solving for y from the above gives

$$y = \frac{1}{\sqrt{-x^2 + 4}}$$

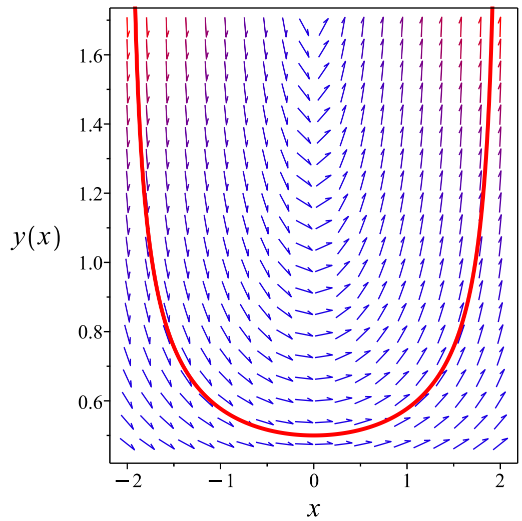
Summary

The solution(s) found are the following

$$y = \frac{1}{\sqrt{-x^2 + 4}} \tag{1}$$



(a) Solution plot



(b) Slope field plot

Verification of solutions

$$y = \frac{1}{\sqrt{-x^2 + 4}}$$

Verified OK.

1.36.4 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x, y) + N(x, y) \frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x, y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x, y) = 0$$

Hence

$$\frac{\partial \phi}{\partial x} + \frac{\partial \phi}{\partial y}\frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$\frac{\partial \phi}{\partial x} = M$$
$$\frac{\partial \phi}{\partial y} = N$$

But since $\frac{\partial^2\phi}{\partial x\partial y} = \frac{\partial^2\phi}{\partial y\partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x, y)$ but at least we know now that we can do that since the condition $\frac{\partial^2\phi}{\partial x\partial y} = \frac{\partial^2\phi}{\partial y\partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x, y)\,\mathrm{d}x + N(x, y)\,\mathrm{d}y = 0 \tag{1A}$$

Therefore

$$\left(\frac{1}{y^3}\right)\mathrm{d}y = (x)\,\mathrm{d}x$$
$$(-x)\,\mathrm{d}x + \left(\frac{1}{y^3}\right)\mathrm{d}y = 0 \tag{2A}$$

Comparing (1A) and (2A) shows that

$$M(x, y) = -x$$
$$N(x, y) = \frac{1}{y^3}$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Using result found above gives

$$\begin{aligned}\frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}(-x)\\ &= 0\end{aligned}$$

And

$$\begin{aligned}\frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}\left(\frac{1}{y^3}\right)\\ &= 0\end{aligned}$$

Since $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$, then the ODE is exact The following equations are now set up to solve for the function $\phi(x, y)$

$$\frac{\partial \phi}{\partial x} = M \tag{1}$$
$$\frac{\partial \phi}{\partial y} = N \tag{2}$$

Integrating (1) w.r.t. x gives

$$\int \frac{\partial \phi}{\partial x} \mathrm{d}x = \int M \mathrm{d}x$$
$$\int \frac{\partial \phi}{\partial x} \mathrm{d}x = \int -x \mathrm{d}x$$
$$\phi = -\frac{x^2}{2} + f(y) \tag{3}$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$\frac{\partial \phi}{\partial y} = 0 + f'(y) \tag{4}$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = \frac{1}{y^3}$. Therefore equation (4) becomes

$$\frac{1}{y^3} = 0 + f'(y) \tag{5}$$

Solving equation (5) for $f'(y)$ gives

$$f'(y) = \frac{1}{y^3}$$

Integrating the above w.r.t y gives

$$\int f'(y) \mathrm{d}y = \int \left(\frac{1}{y^3}\right) \mathrm{d}y$$
$$f(y) = -\frac{1}{2y^2} + c_1$$

Where c_1 is constant of integration. Substituting result found above for $f(y)$ into equation (3) gives ϕ

$$\phi = -\frac{x^2}{2} - \frac{1}{2y^2} + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = -\frac{x^2}{2} - \frac{1}{2y^2}$$

Initial conditions are used to solve for c_1. Substituting $x = 0$ and $y = \frac{1}{2}$ in the above solution gives an equation to solve for the constant of integration.

$$-2 = c_1$$

$$c_1 = -2$$

Substituting c_1 found above in the general solution gives

$$-\frac{x^2}{2} - \frac{1}{2y^2} = -2$$

The above simplifies to

$$-y^2x^2 + 4y^2 - 1 = 0$$

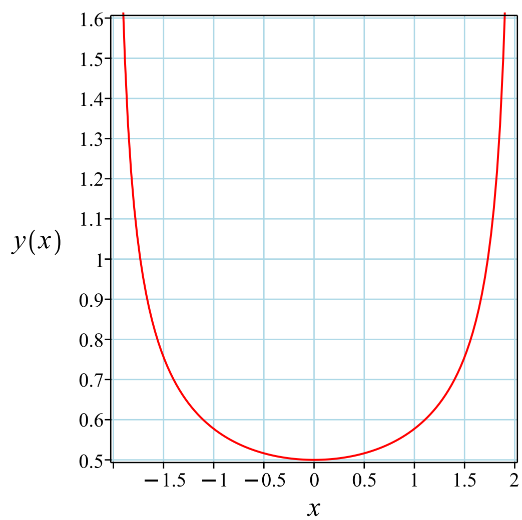
Solving for y from the above gives

$$y = \frac{1}{\sqrt{-x^2 + 4}}$$

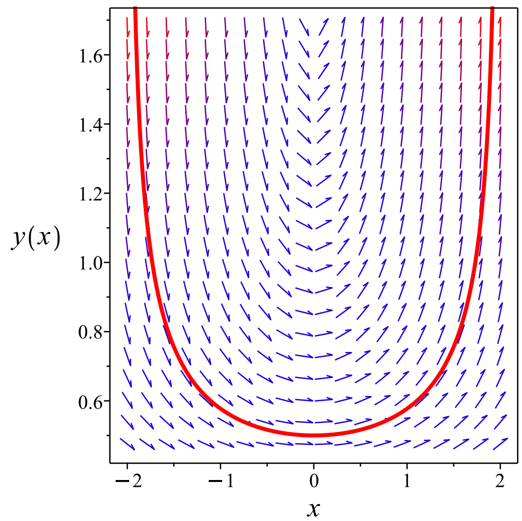
Summary

The solution(s) found are the following

$$y = \frac{1}{\sqrt{-x^2 + 4}} \tag{1}$$



(a) Solution plot



(b) Slope field plot

<u>Verification of solutions</u>

$$y = \frac{1}{\sqrt{-x^2 + 4}}$$

Verified OK.

1.36.5 Maple step by step solution

Let's solve

$\left[y' - y^3 x = 0, y(0) = \frac{1}{2}\right]$

- Highest derivative means the order of the ODE is 1

  y'

- Separate variables

  $\frac{y'}{y^3} = x$

- Integrate both sides with respect to x

  $\int \frac{y'}{y^3} dx = \int x dx + c_1$

- Evaluate integral

  $-\frac{1}{2y^2} = \frac{x^2}{2} + c_1$

- Solve for y

$$\left\{y=\frac{1}{\sqrt{-x^2-2c_1}}, y=-\frac{1}{\sqrt{-x^2-2c_1}}\right\}$$

- Use initial condition $y(0)=\frac{1}{2}$

  $\frac{1}{2}=\frac{1}{\sqrt{-2c_1}}$

- Solve for c_1

  $c_1=-2$

- Substitute $c_1=-2$ into general solution and simplify

  $y=\frac{1}{\sqrt{-x^2+4}}$

- Use initial condition $y(0)=\frac{1}{2}$

  $\frac{1}{2}=-\frac{1}{\sqrt{-2c_1}}$

- Solution does not satisfy initial condition

- Solution to the IVP

  $y=\frac{1}{\sqrt{-x^2+4}}$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
<- Bernoulli successful`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 13

```
dsolve([diff(y(x),x)=x*y(x)^3,y(0) = 1/2],y(x), singsol=all)
```

$$y(x)=\frac{1}{\sqrt{-x^2+4}}$$

✓ Solution by Mathematica

Time used: 0.095 (sec). Leaf size: 16

```
DSolve[{y'[x]==x*y[x]^3,{y[0]==1/2}},y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \frac{1}{\sqrt{4-x^2}}$$

1.37 problem 31 part(b.3)

Internal problem ID [4948]
Internal file name [`OUTPUT/4441_Sunday_June_05_2022_02_56_42_PM_79079768/index.tex`]

Book: Fundamentals of Differential Equations. By Nagle, Saff and Snider. 9th edition. Boston. Pearson 2018.
Section: Chapter 2, First order differential equations. Section 2.2, Separable Equations. Exercises. page 46
Problem number: 31 part(b.3).
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"exact", "separable", "first_order_ode_lie_symmetry_lookup"**

Maple gives the following as the ode type

`[_separable]`

$$\boxed{y' - y^3 x = 0}$$

With initial conditions

$$[y(0) = 2]$$

1.37.1 Existence and uniqueness analysis

This is non linear first order ODE. In canonical form it is written as

$$\begin{aligned} y' &= f(x,y) \\ &= x\,y^3 \end{aligned}$$

The x domain of $f(x,y)$ when $y = 2$ is

$$\{-\infty < x < \infty\}$$

And the point $x_0 = 0$ is inside this domain. The y domain of $f(x,y)$ when $x = 0$ is

$$\{-\infty < y < \infty\}$$

And the point $y_0 = 2$ is inside this domain. Now we will look at the continuity of

$$\begin{aligned}\frac{\partial f}{\partial y} &= \frac{\partial}{\partial y}(x\,y^3)\\ &= 3y^2x\end{aligned}$$

The x domain of $\frac{\partial f}{\partial y}$ when $y = 2$ is

$$\{-\infty < x < \infty\}$$

And the point $x_0 = 0$ is inside this domain. The y domain of $\frac{\partial f}{\partial y}$ when $x = 0$ is

$$\{-\infty < y < \infty\}$$

And the point $y_0 = 2$ is inside this domain. Therefore solution exists and is unique.

1.37.2 Solving as separable ode

In canonical form the ODE is

$$\begin{aligned}y' &= F(x,y)\\ &= f(x)g(y)\\ &= x\,y^3\end{aligned}$$

Where $f(x) = x$ and $g(y) = y^3$. Integrating both sides gives

$$\frac{1}{y^3}\,dy = x\,dx$$

$$\int \frac{1}{y^3}\,dy = \int x\,dx$$

$$-\frac{1}{2y^2} = \frac{x^2}{2} + c_1$$

Which results in

$$y = -\frac{1}{\sqrt{-x^2 - 2c_1}}$$

$$y = \frac{1}{\sqrt{-x^2 - 2c_1}}$$

Initial conditions are used to solve for c_1. Substituting $x = 0$ and $y = 2$ in the above solution gives an equation to solve for the constant of integration.

$$2 = \frac{1}{\sqrt{-2c_1}}$$

$$c_1 = -\frac{1}{8}$$

Substituting c_1 found above in the general solution gives

$$y = \frac{2}{\sqrt{-4x^2 + 1}}$$

Initial conditions are used to solve for c_1. Substituting $x = 0$ and $y = 2$ in the above solution gives an equation to solve for the constant of integration.

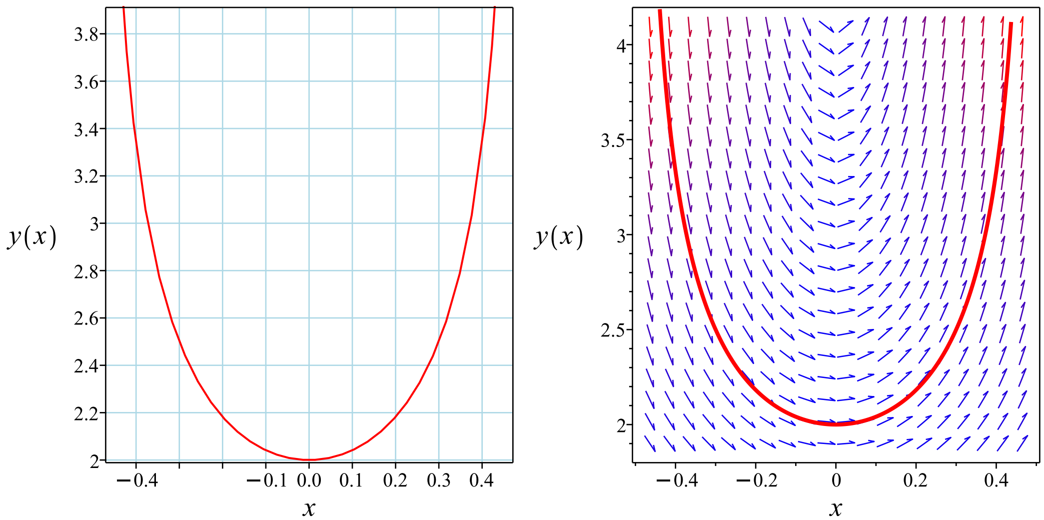
$$2 = -\frac{1}{\sqrt{-2c_1}}$$

Warning: Unable to solve for c_1. No particular solution can be found using given initial conditions for this solution. removing this solution as not valid.

Summary

The solution(s) found are the following

$$y = \frac{}{\sqrt{}}$$



(a) Solution plot (b) Slope field plot

Verification of solutions

$$y = \frac{2}{\sqrt{-4x^2 + 1}}$$

Verified OK.

1.37.3 Solving as first order ode lie symmetry lookup ode

Writing the ode as

$$\begin{aligned} y' &= x\, y^3 \\ y' &= \omega(x, y) \end{aligned}$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x + \omega(\eta_y - \xi_x) - \omega^2 \xi_y - \omega_x \xi - \omega_y \eta = 0 \tag{A}$$

The type of this ode is known. It is of type `separable`. Therefore we do not need to solve the PDE (A), and can just use the lookup table shown below to find ξ, η

Table 88: Lie symmetry infinitesimal lookup table for known first order ODE's

ODE class	Form	ξ	η
linear ode	$y' = f(x)y(x) + g(x)$	0	$e^{\int f dx}$
separable ode	$y' = f(x)\, g(y)$	$\frac{1}{f}$	0
quadrature ode	$y' = f(x)$	0	1
quadrature ode	$y' = g(y)$	1	0
homogeneous ODEs of Class A	$y' = f\left(\frac{y}{x}\right)$	x	y
homogeneous ODEs of Class C	$y' = (a + bx + cy)^{\frac{n}{m}}$	1	$-\frac{b}{c}$
homogeneous class D	$y' = \frac{y}{x} + g(x)\, F\left(\frac{y}{x}\right)$	x^2	xy
First order special form ID 1	$y' = g(x)\, e^{h(x)+by} + f(x)$	$\frac{e^{-\int b f(x) dx - h(x)}}{g(x)}$	$\frac{f(x) e^{-\int b f(x) dx - h(x)}}{g(x)}$
polynomial type ode	$y' = \frac{a_1 x + b_1 y + c_1}{a_2 x + b_2 y + c_2}$	$\frac{a_1 b_2 x - a_2 b_1 x - b_1 c_2 + b_2 c_1}{a_1 b_2 - a_2 b_1}$	$\frac{a_1 b_2 y - a_2 b_1 y - a_1 c_2 - a_2 c_1}{a_1 b_2 - a_2 b_1}$
Bernoulli ode	$y' = f(x)\, y + g(x)\, y^n$	0	$e^{-\int (n-1) f(x) dx} y^n$
Reduced Riccati	$y' = f_1(x)\, y + f_2(x)\, y^2$	0	$e^{-\int f_1 dx}$

The above table shows that

$$
\begin{aligned}
\xi(x, y) &= \frac{1}{x} \\
\eta(x, y) &= 0
\end{aligned}
\tag{A1}
$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x, y) \to (R, S)$ where (R, S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$
\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}
$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y}\right) S(x, y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the

canonical coordinates, where $S(R)$. Since $\eta = 0$ then in this special case

$$R = y$$

S is found from

$$\begin{aligned} S &= \int \frac{1}{\xi} dx \\ &= \int \frac{1}{\frac{1}{x}} dx \end{aligned}$$

Which results in

$$S = \frac{x^2}{2}$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x, y) S_y}{R_x + \omega(x, y) R_y} \tag{2}$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x, y)$ is the right hand side of the original ode given by

$$\omega(x, y) = x\, y^3$$

Evaluating all the partial derivatives gives

$$\begin{aligned} R_x &= 0 \\ R_y &= 1 \\ S_x &= x \\ S_y &= 0 \end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = \frac{1}{y^3} \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = \frac{1}{R^3}$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by

integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$S(R) = -\frac{1}{2R^2} + c_1 \tag{4}$$

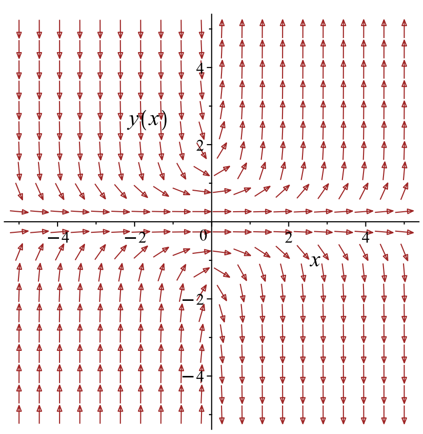
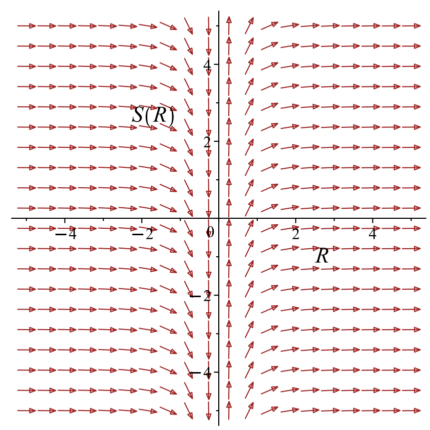
To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$\frac{x^2}{2} = -\frac{1}{2y^2} + c_1$$

Which simplifies to

$$\frac{x^2}{2} = -\frac{1}{2y^2} + c_1$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in x, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dx} = x\,y^3$ 	$R = y$ $S = \frac{x^2}{2}$	$\frac{dS}{dR} = \frac{1}{R^3}$ 

Initial conditions are used to solve for c_1. Substituting $x = 0$ and $y = 2$ in the above solution gives an equation to solve for the constant of integration.

$$0 = c_1 - \frac{1}{8}$$

$$c_1 = \frac{1}{8}$$

Substituting c_1 found above in the general solution gives

$$\frac{x^2}{2} = \frac{y^2 - 4}{8y^2}$$

The above simplifies to

$$4y^2x^2 - y^2 + 4 = 0$$

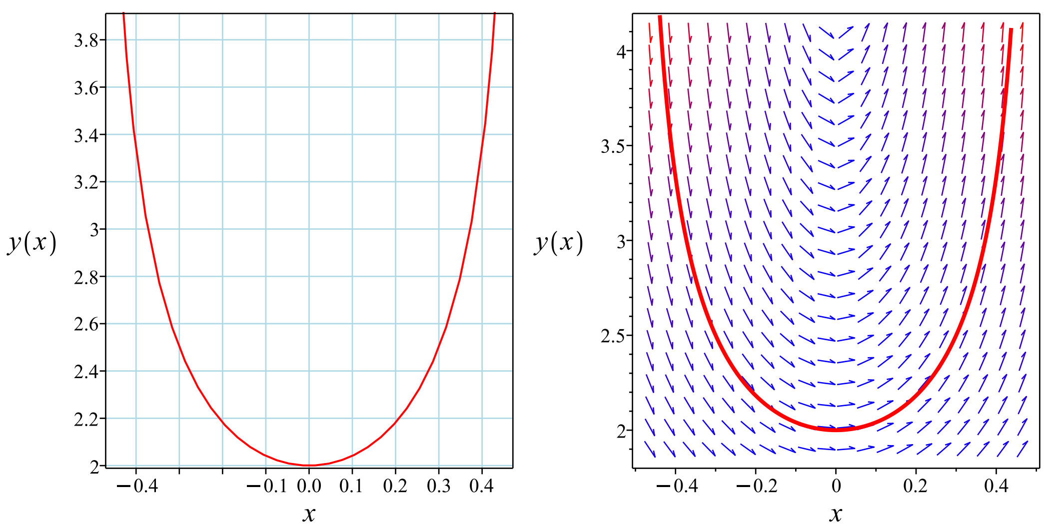
Solving for y from the above gives

$$y = \frac{2}{\sqrt{-4x^2 + 1}}$$

Summary

The solution(s) found are the following

$$y = \frac{2}{\sqrt{-4x^2 + 1}} \tag{1}$$



(a) Solution plot

(b) Slope field plot

Verification of solutions

$$y = \frac{2}{\sqrt{-4x^2 + 1}}$$

Verified OK.

1.37.4 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x,y) + N(x,y)\frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x,y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y) = 0$$

Hence

$$\frac{\partial\phi}{\partial x} + \frac{\partial\phi}{\partial y}\frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$\begin{aligned}\frac{\partial\phi}{\partial x} &= M\\ \frac{\partial\phi}{\partial y} &= N\end{aligned}$$

But since $\frac{\partial^2\phi}{\partial x\partial y} = \frac{\partial^2\phi}{\partial y\partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x,y)$ but at least we know now that we can do that since the condition $\frac{\partial^2\phi}{\partial x\partial y} = \frac{\partial^2\phi}{\partial y\partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x,y)\,\mathrm{d}x + N(x,y)\,\mathrm{d}y = 0 \tag{1A}$$

Therefore

$$\begin{aligned}\left(\frac{1}{y^3}\right)\mathrm{d}y &= (x)\,\mathrm{d}x\\ (-x)\,\mathrm{d}x + \left(\frac{1}{y^3}\right)\mathrm{d}y &= 0 \end{aligned}\tag{2A}$$

Comparing (1A) and (2A) shows that

$$
\begin{aligned}
M(x,y) &= -x \\
N(x,y) &= \frac{1}{y^3}
\end{aligned}
$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$
\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}
$$

Using result found above gives

$$
\begin{aligned}
\frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}(-x) \\
&= 0
\end{aligned}
$$

And

$$
\begin{aligned}
\frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}\left(\frac{1}{y^3}\right) \\
&= 0
\end{aligned}
$$

Since $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$, then the ODE is <u>exact</u> The following equations are now set up to solve for the function $\phi(x,y)$

$$
\frac{\partial \phi}{\partial x} = M \tag{1}
$$

$$
\frac{\partial \phi}{\partial y} = N \tag{2}
$$

Integrating (1) w.r.t. x gives

$$
\begin{aligned}
\int \frac{\partial \phi}{\partial x}\,\mathrm{d}x &= \int M\,\mathrm{d}x \\
\int \frac{\partial \phi}{\partial x}\,\mathrm{d}x &= \int -x\,\mathrm{d}x
\end{aligned}
$$

$$
\phi = -\frac{x^2}{2} + f(y) \tag{3}
$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$
\frac{\partial \phi}{\partial y} = 0 + f'(y) \tag{4}
$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = \frac{1}{y^3}$. Therefore equation (4) becomes

$$\frac{1}{y^3} = 0 + f'(y) \tag{5}$$

Solving equation (5) for $f'(y)$ gives

$$f'(y) = \frac{1}{y^3}$$

Integrating the above w.r.t y gives

$$\int f'(y) \, \mathrm{d}y = \int \left(\frac{1}{y^3}\right) \mathrm{d}y$$

$$f(y) = -\frac{1}{2y^2} + c_1$$

Where c_1 is constant of integration. Substituting result found above for $f(y)$ into equation (3) gives ϕ

$$\phi = -\frac{x^2}{2} - \frac{1}{2y^2} + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = -\frac{x^2}{2} - \frac{1}{2y^2}$$

Initial conditions are used to solve for c_1. Substituting $x = 0$ and $y = 2$ in the above solution gives an equation to solve for the constant of integration.

$$-\frac{1}{8} = c_1$$

$$c_1 = -\frac{1}{8}$$

Substituting c_1 found above in the general solution gives

$$-\frac{x^2}{2} - \frac{1}{2y^2} = -\frac{1}{8}$$

The above simplifies to

$$-4y^2x^2 + y^2 - 4 = 0$$

Summary
The solution(s) found are the following

$$-4y^2x^2 + y^2 - 4 = 0 \tag{1}$$

Verification of solutions

$$-4y^2x^2 + y^2 - 4 = 0$$

Verified OK.

1.37.5 Maple step by step solution

Let's solve

$[y' - y^3x = 0, y(0) = 2]$

- Highest derivative means the order of the ODE is 1

  y'

- Separate variables

  $\frac{y'}{y^3} = x$

- Integrate both sides with respect to x

  $\int \frac{y'}{y^3}dx = \int x dx + c_1$

- Evaluate integral

  $-\frac{1}{2y^2} = \frac{x^2}{2} + c_1$

- Solve for y

  $\left\{y = \frac{1}{\sqrt{-x^2-2c_1}}, y = -\frac{1}{\sqrt{-x^2-2c_1}}\right\}$

- Use initial condition $y(0) = 2$

  $2 = \frac{1}{\sqrt{-2c_1}}$

- Solve for c_1

  $c_1 = -\frac{1}{8}$

- Substitute $c_1 = -\frac{1}{8}$ into general solution and simplify

  $y = \frac{2}{\sqrt{-4x^2+1}}$

- Use initial condition $y(0) = 2$

  $2 = -\frac{1}{\sqrt{-2c_1}}$

- Solution does not satisfy initial condition

- Solution to the IVP

  $y = \frac{2}{\sqrt{-4x^2+1}}$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
<- Bernoulli successful`
```

✓ Solution by Maple

Time used: 0.015 (sec). Leaf size: 15

```
dsolve([diff(y(x),x)=x*y(x)^3,y(0) = 2],y(x), singsol=all)
```

$$y(x) = \frac{2}{\sqrt{-4x^2 + 1}}$$

✓ Solution by Mathematica

Time used: 0.097 (sec). Leaf size: 18

```
DSolve[{y'[x]==x*y[x]^3,{y[0]==2}},y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \frac{2}{\sqrt{1 - 4x^2}}$$

1.38 problem 32

Internal problem ID [4949]
Internal file name [`OUTPUT/4442_Sunday_June_05_2022_02_56_43_PM_76292468/index.tex`]

Book: Fundamentals of Differential Equations. By Nagle, Saff and Snider. 9th edition. Boston. Pearson 2018.
Section: Chapter 2, First order differential equations. Section 2.2, Separable Equations. Exercises. page 46
Problem number: 32.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"quadrature"**

Maple gives the following as the ode type

`[_quadrature]`

$$\boxed{y' - y^2 + 3y = 2}$$

With initial conditions

$$\left[y(0) = \frac{3}{2}\right]$$

1.38.1 Existence and uniqueness analysis

This is non linear first order ODE. In canonical form it is written as

$$\begin{aligned} y' &= f(x, y) \\ &= y^2 - 3y + 2 \end{aligned}$$

The y domain of $f(x, y)$ when $x = 0$ is

$$\{-\infty < y < \infty\}$$

And the point $y_0 = \frac{3}{2}$ is inside this domain. Now we will look at the continuity of

$$\begin{aligned}\frac{\partial f}{\partial y} &= \frac{\partial}{\partial y}\left(y^2 - 3y + 2\right) \\ &= 2y - 3\end{aligned}$$

The y domain of $\frac{\partial f}{\partial y}$ when $x = 0$ is

$$\{-\infty < y < \infty\}$$

And the point $y_0 = \frac{3}{2}$ is inside this domain. Therefore solution exists and is unique.

1.38.2 Solving as quadrature ode

Integrating both sides gives

$$\begin{aligned}\int \frac{1}{y^2 - 3y + 2} dy &= \int dx \\ -\ln(y-1) + \ln(y-2) &= x + c_1\end{aligned}$$

Raising both side to exponential gives

$$e^{-\ln(y-1)+\ln(y-2)} = e^{x+c_1}$$

Which simplifies to

$$\frac{y-2}{y-1} = c_2 e^x$$

Initial conditions are used to solve for c_2. Substituting $x = 0$ and $y = \frac{3}{2}$ in the above solution gives an equation to solve for the constant of integration.

$$\frac{3}{2} = \frac{c_2 - 2}{-1 + c_2}$$

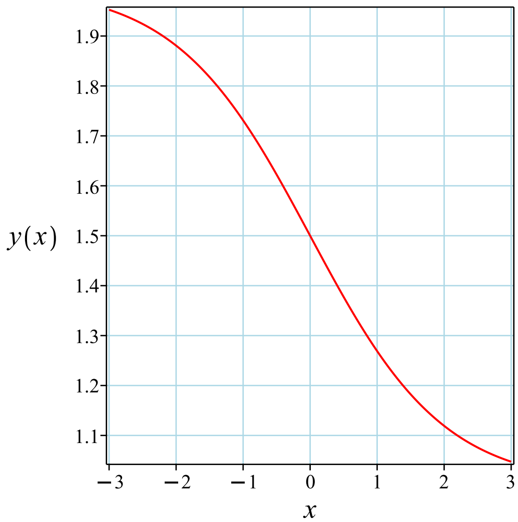
$$c_2 = -1$$

Substituting c_2 found above in the general solution gives

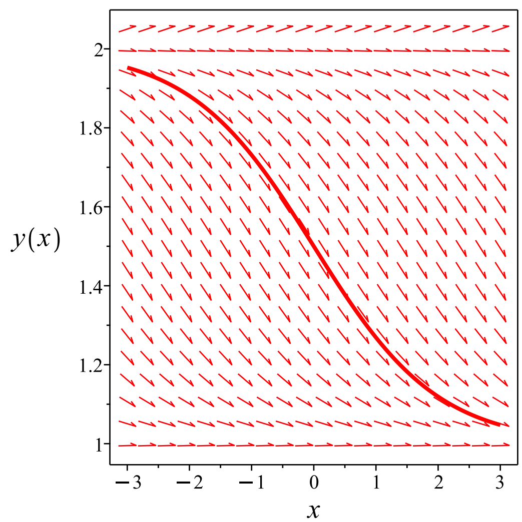
$$y = \frac{e^x + 2}{1 + e^x}$$

Summary
The solution(s) found are the following

$$y = \frac{e^x + 2}{1 + e^x} \tag{1}$$



(a) Solution plot



(b) Slope field plot

<u>Verification of solutions</u>

$$y = \frac{e^x + 2}{1 + e^x}$$

Verified OK.

1.38.3 Maple step by step solution

Let's solve

$\left[y' - y^2 + 3y = 2, y(0) = \frac{3}{2}\right]$

- Highest derivative means the order of the ODE is 1

  y'

- Separate variables

  $\frac{y'}{y^2-3y+2} = 1$

- Integrate both sides with respect to x

  $\int \frac{y'}{y^2-3y+2}dx = \int 1dx + c_1$

- Evaluate integral

  $-\ln(y-1) + \ln(y-2) = x + c_1$

- Solve for y

$y = \frac{-2+e^{x+c_1}}{e^{x+c_1}-1}$

- Use initial condition $y(0) = \frac{3}{2}$

  $\frac{3}{2} = \frac{-2+e^{c_1}}{e^{c_1}-1}$

- Solve for c_1

  $c_1 = I\pi$

- Substitute $c_1 = I\pi$ into general solution and simplify

  $y = \frac{e^x+2}{1+e^x}$

- Solution to the IVP

  $y = \frac{e^x+2}{1+e^x}$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
<- separable successful`
```

✓ Solution by Maple

Time used: 0.047 (sec). Leaf size: 15

```
dsolve([diff(y(x),x)=y(x)^2-3*y(x)+2,y(0) = 3/2],y(x), singsol=all)
```

$$y(x) = \frac{e^x + 2}{1 + e^x}$$

✓ Solution by Mathematica

Time used: 0.008 (sec). Leaf size: 18

```
DSolve[{y'[x]==y[x]^2-3*y[x]+2,{y[0]==3/2}},y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \frac{e^x + 2}{e^x + 1}$$

2 Chapter 2, First order differential equations. Section 2.3, Linear equations. Exercises. page 54

2.1 problem 1

Internal problem ID [4950]
Internal file name [`OUTPUT/4443_Sunday_June_05_2022_02_56_44_PM_9185702/index.tex`]

Book: Fundamentals of Differential Equations. By Nagle, Saff and Snider. 9th edition. Boston. Pearson 2018.
Section: Chapter 2, First order differential equations. Section 2.3, Linear equations. Exercises. page 54
Problem number: 1.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"linear", "exactWithIntegrationFactor", "first_order_ode_lie_symmetry_lookup"**

Maple gives the following as the ode type

```
[_linear]
```

$$\boxed{x^2y' - y = -\sin(x)}$$

2.1.1 Solving as linear ode

Entering Linear first order ODE solver. In canonical form a linear first order is

$$y' + p(x)y = q(x)$$

Where here

$$p(x) = -\frac{1}{x^2}$$
$$q(x) = -\frac{\sin(x)}{x^2}$$

Hence the ode is

$$y' - \frac{y}{x^2} = -\frac{\sin(x)}{x^2}$$

The integrating factor μ is

$$
\begin{aligned}
\mu &= \mathrm{e}^{\int -\frac{1}{x^2}dx} \\
&= \mathrm{e}^{\frac{1}{x}}
\end{aligned}
$$

The ode becomes

$$
\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}x}(\mu y) &= (\mu)\left(-\frac{\sin(x)}{x^2}\right) \\
\frac{\mathrm{d}}{\mathrm{d}x}\left(\mathrm{e}^{\frac{1}{x}}y\right) &= \left(\mathrm{e}^{\frac{1}{x}}\right)\left(-\frac{\sin(x)}{x^2}\right) \\
\mathrm{d}\left(\mathrm{e}^{\frac{1}{x}}y\right) &= \left(-\frac{\sin(x)\,\mathrm{e}^{\frac{1}{x}}}{x^2}\right)\,\mathrm{d}x
\end{aligned}
$$

Integrating gives

$$
\begin{aligned}
\mathrm{e}^{\frac{1}{x}}y &= \int -\frac{\sin(x)\,\mathrm{e}^{\frac{1}{x}}}{x^2}\,\mathrm{d}x \\
\mathrm{e}^{\frac{1}{x}}y &= \int -\frac{\sin(x)\,\mathrm{e}^{\frac{1}{x}}}{x^2}dx + c_1
\end{aligned}
$$

Dividing both sides by the integrating factor $\mu = \mathrm{e}^{\frac{1}{x}}$ results in

$$
y = \mathrm{e}^{-\frac{1}{x}}\left(\int -\frac{\sin(x)\,\mathrm{e}^{\frac{1}{x}}}{x^2}dx\right) + c_1\mathrm{e}^{-\frac{1}{x}}
$$

which simplifies to

$$
y = \mathrm{e}^{-\frac{1}{x}}\left(-\left(\int \frac{\sin(x)\,\mathrm{e}^{\frac{1}{x}}}{x^2}dx\right) + c_1\right)
$$

Summary
The solution(s) found are the following

$$
y = \mathrm{e}^{-\frac{1}{x}}\left(-\left(\int \frac{\sin(x)\,\mathrm{e}^{\frac{1}{x}}}{x^2}dx\right) + c_1\right) \tag{1}
$$

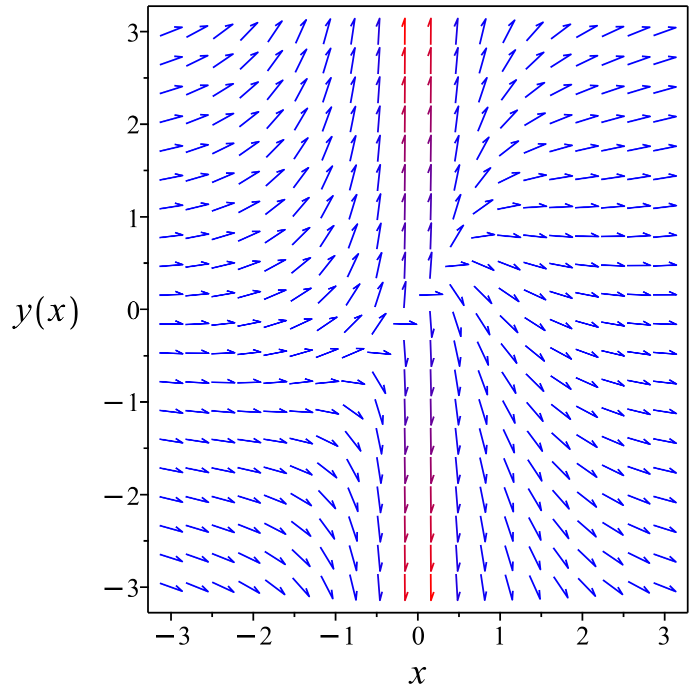


Figure 95: Slope field plot

Verification of solutions

$$y = \mathrm{e}^{-\frac{1}{x}} \left(-\left(\int \frac{\sin(x)\, \mathrm{e}^{\frac{1}{x}}}{x^2} dx \right) + c_1 \right)$$

Verified OK.

2.1.2 Solving as first order ode lie symmetry lookup ode

Writing the ode as

$$y' = -\frac{\sin(x) - y}{x^2}$$
$$y' = \omega(x, y)$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x + \omega(\eta_y - \xi_x) - \omega^2 \xi_y - \omega_x \xi - \omega_y \eta = 0 \tag{A}$$

The type of this ode is known. It is of type `linear`. Therefore we do not need to solve the PDE (A), and can just use the lookup table shown below to find ξ, η

Table 92: Lie symmetry infinitesimal lookup table for known first order ODE's

ODE class	Form	ξ	η
linear ode	$y' = f(x)y(x) + g(x)$	0	$e^{\int f dx}$
separable ode	$y' = f(x)\, g(y)$	$\frac{1}{f}$	0
quadrature ode	$y' = f(x)$	0	1
quadrature ode	$y' = g(y)$	1	0
homogeneous ODEs of Class A	$y' = f\left(\frac{y}{x}\right)$	x	y
homogeneous ODEs of Class C	$y' = (a + bx + cy)^{\frac{n}{m}}$	1	$-\frac{b}{c}$
homogeneous class D	$y' = \frac{y}{x} + g(x)\, F\left(\frac{y}{x}\right)$	x^2	xy
First order special form ID 1	$y' = g(x)\, e^{h(x)+by} + f(x)$	$\frac{e^{-\int bf(x)dx - h(x)}}{g(x)}$	$\frac{f(x)e^{-\int bf(x)dx - h(x)}}{g(x)}$
polynomial type ode	$y' = \frac{a_1x+b_1y+c_1}{a_2x+b_2y+c_2}$	$\frac{a_1b_2x - a_2b_1x - b_1c_2 + b_2c_1}{a_1b_2 - a_2b_1}$	$\frac{a_1b_2y - a_2b_1y - a_1c_2 - a_2c_1}{a_1b_2 - a_2b_1}$
Bernoulli ode	$y' = f(x)\, y + g(x)\, y^n$	0	$e^{-\int (n-1)f(x)dx} y^n$
Reduced Riccati	$y' = f_1(x)\, y + f_2(x)\, y^2$	0	$e^{-\int f_1 dx}$

The above table shows that

$$
\begin{aligned}
\xi(x, y) &= 0 \\
\eta(x, y) &= \mathrm{e}^{-\frac{1}{x}}
\end{aligned} \tag{A1}
$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x, y) \to (R, S)$ where (R, S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$
\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}
$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y}\right) S(x, y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the

canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$R = x$$

S is found from

$$\begin{aligned}
S &= \int \frac{1}{\eta} dy \\
&= \int \frac{1}{\mathrm{e}^{-\frac{1}{x}}} dy
\end{aligned}$$

Which results in

$$S = \mathrm{e}^{\frac{1}{x}} y$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x,y) S_y}{R_x + \omega(x,y) R_y} \tag{2}$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x,y)$ is the right hand side of the original ode given by

$$\omega(x,y) = -\frac{\sin(x) - y}{x^2}$$

Evaluating all the partial derivatives gives

$$\begin{aligned}
R_x &= 1 \\
R_y &= 0 \\
S_x &= -\frac{\mathrm{e}^{\frac{1}{x}} y}{x^2} \\
S_y &= \mathrm{e}^{\frac{1}{x}}
\end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = -\frac{\sin(x) \mathrm{e}^{\frac{1}{x}}}{x^2} \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = -\frac{\sin(R) \mathrm{e}^{\frac{1}{R}}}{R^2}$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordinates R, S. Integrating the above gives

$$S(R) = \int -\frac{\sin(R)\, e^{\frac{1}{R}}}{R^2} dR + c_1 \tag{4}$$

To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$e^{\frac{1}{x}} y = \int -\frac{\sin(x)\, e^{\frac{1}{x}}}{x^2} dx + c_1$$

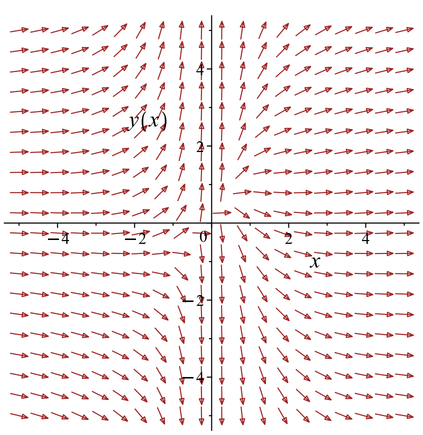
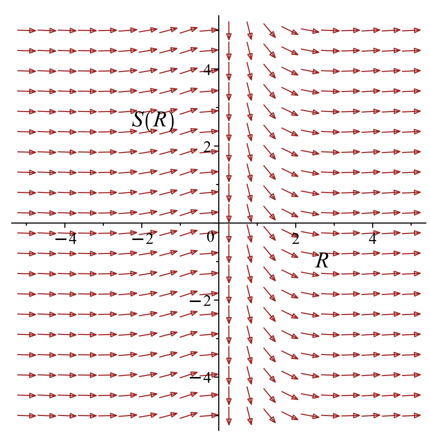
Which simplifies to

$$e^{\frac{1}{x}} y = \int -\frac{\sin(x)\, e^{\frac{1}{x}}}{x^2} dx + c_1$$

Which gives

$$y = \left(\int -\frac{\sin(x)\, e^{\frac{1}{x}}}{x^2} dx + c_1\right) e^{-\frac{1}{x}}$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in x, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dx} = -\frac{\sin(x)-y}{x^2}$ 	$R = x$ $S = e^{\frac{1}{x}} y$	$\frac{dS}{dR} = -\frac{\sin(R) e^{\frac{1}{R}}}{R^2}$ 

Summary

The solution(s) found are the following

$$y = \left(\int -\frac{\sin(x)\,e^{\frac{1}{x}}}{x^2}dx + c_1\right)e^{-\frac{1}{x}} \tag{1}$$

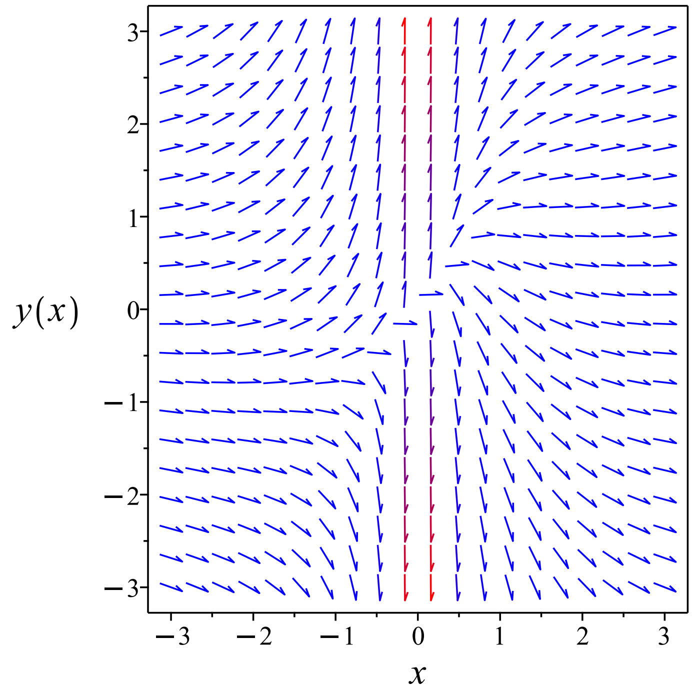


Figure 96: Slope field plot

Verification of solutions

$$y = \left(\int -\frac{\sin(x)\,e^{\frac{1}{x}}}{x^2}dx + c_1\right)e^{-\frac{1}{x}}$$

Verified OK.

2.1.3 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x,y) + N(x,y)\frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x,y) = c$ where c is constant, that satisfies the

ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y) = 0$$

Hence

$$\frac{\partial \phi}{\partial x} + \frac{\partial \phi}{\partial y}\frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$\begin{aligned}\frac{\partial \phi}{\partial x} &= M\\ \frac{\partial \phi}{\partial y} &= N\end{aligned}$$

But since $\frac{\partial^2\phi}{\partial x\partial y} = \frac{\partial^2\phi}{\partial y\partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x,y)$ but at least we know now that we can do that since the condition $\frac{\partial^2\phi}{\partial x\partial y} = \frac{\partial^2\phi}{\partial y\partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x,y)\,\mathrm{d}x + N(x,y)\,\mathrm{d}y = 0 \tag{1A}$$

Therefore

$$\begin{aligned}(x^2)\,\mathrm{d}y &= (-\sin(x) + y)\,\mathrm{d}x\\ (\sin(x) - y)\,\mathrm{d}x + (x^2)\,\mathrm{d}y &= 0 \qquad (2A)\end{aligned}$$

Comparing (1A) and (2A) shows that

$$\begin{aligned}M(x,y) &= \sin(x) - y\\ N(x,y) &= x^2\end{aligned}$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Using result found above gives

$$\begin{aligned}\frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}(\sin(x) - y) \\ &= -1\end{aligned}$$

And

$$\begin{aligned}\frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}(x^2) \\ &= 2x\end{aligned}$$

Since $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$, then the ODE is not exact. Since the ODE is not exact, we will try to find an integrating factor to make it exact. Let

$$\begin{aligned}A &= \frac{1}{N}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right) \\ &= \frac{1}{x^2}((-1) - (2x)) \\ &= \frac{-1 - 2x}{x^2}\end{aligned}$$

Since A does not depend on y, then it can be used to find an integrating factor. The integrating factor μ is

$$\begin{aligned}\mu &= e^{\int A \, \mathrm{d}x} \\ &= e^{\int \frac{-1-2x}{x^2} \, \mathrm{d}x}\end{aligned}$$

The result of integrating gives

$$\begin{aligned}\mu &= e^{-2\ln(x) + \frac{1}{x}} \\ &= \frac{\mathrm{e}^{\frac{1}{x}}}{x^2}\end{aligned}$$

M and N are multiplied by this integrating factor, giving new M and new N which are called $\overline{M}$ and $\overline{N}$ for now so not to confuse them with the original M and N.

$$\begin{aligned}\overline{M} &= \mu M \\ &= \frac{\mathrm{e}^{\frac{1}{x}}}{x^2}(\sin(x) - y) \\ &= \frac{\mathrm{e}^{\frac{1}{x}}(\sin(x) - y)}{x^2}\end{aligned}$$

And

$$
\begin{aligned}
\overline{N} &= \mu N \\
&= \frac{e^{\frac{1}{x}}}{x^2}(x^2) \\
&= e^{\frac{1}{x}}
\end{aligned}
$$

Now a modified ODE is ontained from the original ODE, which is exact and can be solved. The modified ODE is

$$
\begin{aligned}
\overline{M} + \overline{N}\frac{dy}{dx} &= 0 \\
\left(\frac{e^{\frac{1}{x}}(\sin(x) - y)}{x^2}\right) + \left(e^{\frac{1}{x}}\right)\frac{dy}{dx} &= 0
\end{aligned}
$$

The following equations are now set up to solve for the function $\phi(x, y)$

$$
\frac{\partial \phi}{\partial x} = \overline{M} \tag{1}
$$

$$
\frac{\partial \phi}{\partial y} = \overline{N} \tag{2}
$$

Integrating (1) w.r.t. x gives

$$
\begin{aligned}
\int \frac{\partial \phi}{\partial x} dx &= \int \overline{M} dx \\
\int \frac{\partial \phi}{\partial x} dx &= \int \frac{e^{\frac{1}{x}}(\sin(x) - y)}{x^2} dx \\
\phi &= \int^x \frac{e^{\frac{1}{\_a}}(\sin(\_a) - y)}{\_a^2} d\_a + f(y)
\end{aligned} \tag{3}
$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$
\frac{\partial \phi}{\partial y} = e^{\frac{1}{x}} + f'(y) \tag{4}
$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = e^{\frac{1}{x}}$. Therefore equation (4) becomes

$$
e^{\frac{1}{x}} = e^{\frac{1}{x}} + f'(y) \tag{5}
$$

Solving equation (5) for $f'(y)$ gives

$$f'(y) = 0$$

Therefore

$$f(y) = c_1$$

Where c_1 is constant of integration. Substituting this result for $f(y)$ into equation (3) gives ϕ

$$\phi = \int^x \frac{e^{\frac{1}{\_a}}(\sin(\_a) - y)}{\_a^2} d\_a + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = \int^x \frac{e^{\frac{1}{\_a}}(\sin(\_a) - y)}{\_a^2} d\_a$$

Summary

The solution(s) found are the following

$$\int^x \frac{e^{\frac{1}{\_a}}(\sin(\_a) - y)}{\_a^2} d\_a = c_1 \tag{1}$$

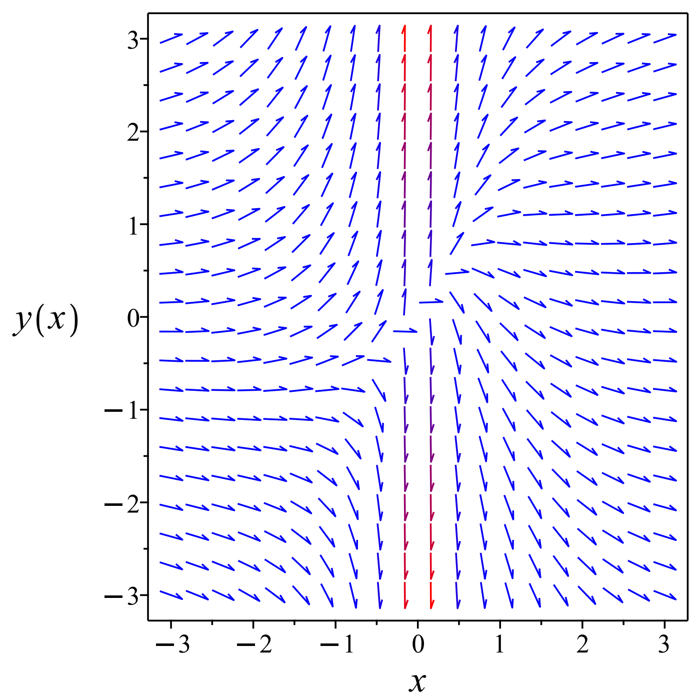


Figure 97: Slope field plot

Verification of solutions

$$\int^{x} \frac{\mathrm{e}^{\frac{1}{\_a}} \left(\sin \left(\_a\right)-y\right)}{\_a^{2}}d\_a = c_{1}$$

Verified OK.

2.1.4 Maple step by step solution

Let's solve

$x^{2} y'-y=-\sin \left(x \right)$

- Highest derivative means the order of the ODE is 1

  y'

- Isolate the derivative

  $y'=\frac{y}{x^{2}}-\frac{\sin \left(x \right)}{x^{2}}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE

  $y'-\frac{y}{x^{2}}=-\frac{\sin \left(x \right)}{x^{2}}$

- The ODE is linear; multiply by an integrating factor $\mu(x)$

  $\mu \left(x \right) \left(y'-\frac{y}{x^{2}}\right)=-\frac{\mu \left(x \right) \sin \left(x \right)}{x^{2}}$

- Assume the lhs of the ODE is the total derivative $\frac{d}{dx}(\mu(x)\, y)$

  $\mu \left(x \right) \left(y'-\frac{y}{x^{2}}\right)=\mu' \left(x \right) y +\mu \left(x \right) y'$

- Isolate $\mu'(x)$

  $\mu' \left(x \right)=-\frac{\mu \left(x \right)}{x^{2}}$

- Solve to find the integrating factor

  $\mu \left(x \right)=\mathrm{e}^{\frac{1}{x}}$

- Integrate both sides with respect to x

  $\int \left(\frac{d}{d x}\left(\mu \left(x \right) y \right)\right)d x =\int -\frac{\mu \left(x \right) \sin \left(x \right)}{x^{2}}d x +c_{1}$

- Evaluate the integral on the lhs

  $\mu \left(x \right) y =\int -\frac{\mu \left(x \right) \sin \left(x \right)}{x^{2}}d x +c_{1}$

- Solve for y

  $y =\frac{\int -\frac{\mu \left(x \right) \sin \left(x \right)}{x^{2}}d x +c_{1}}{\mu \left(x \right)}$

- Substitute $\mu \left(x \right)=\mathrm{e}^{\frac{1}{x}}$

$$y = \frac{\int -\frac{\sin(x)\,\mathrm{e}^{\frac{1}{x}}}{x^2}dx + c_1}{\mathrm{e}^{\frac{1}{x}}}$$

- Simplify

$$y = \left(-\left(\int \frac{\sin(x)\,\mathrm{e}^{\frac{1}{x}}}{x^2}dx\right) + c_1\right)\mathrm{e}^{-\frac{1}{x}}$$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
<- 1st order linear successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 27

```
dsolve(x^2*diff(y(x),x)+sin(x)-y(x)=0,y(x), singsol=all)
```

$$y(x) = \left(-\left(\int \frac{\sin(x)\,\mathrm{e}^{\frac{1}{x}}}{x^2}dx\right) + c_1\right)\mathrm{e}^{-\frac{1}{x}}$$

✓ Solution by Mathematica

Time used: 1.622 (sec). Leaf size: 38

```
DSolve[x^2*y'[x]+Sin[x]-y[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to e^{-1/x}\left(\int_1^x -\frac{e^{\frac{1}{K[1]}}\sin(K[1])}{K[1]^2}dK[1] + c_1\right)$$

2.2 problem 2

Internal problem ID [4951]
Internal file name [`OUTPUT/4444_Sunday_June_05_2022_02_56_45_PM_64370746/index.tex`]

Book: Fundamentals of Differential Equations. By Nagle, Saff and Snider. 9th edition. Boston. Pearson 2018.
Section: Chapter 2, First order differential equations. Section 2.3, Linear equations. Exercises. page 54
Problem number: 2.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"unknown"**

Maple gives the following as the ode type

```
[`y=_G(x,y')`]
```

Unable to solve or complete the solution.

$$\boxed{x' + xt - e^x = 0}$$

Unable to determine ODE type.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
trying homogeneous types:
trying Chini
differential order: 1; looking for linear symmetries
trying exact
Looking for potential symmetries
trying inverse_Riccati
trying an equivalence to an Abel ODE
differential order: 1; trying a linearization to 2nd order
--- trying a change of variables {x -> y(x), y(x) -> x}
differential order: 1; trying a linearization to 2nd order
trying 1st order ODE linearizable_by_differentiation
--- Trying Lie symmetry methods, 1st order ---
`, `-> Computing symmetries using: way = 3
`, `-> Computing symmetries using: way = 4
`, `-> Computing symmetries using: way = 5
trying symmetry patterns for 1st order ODEs
-> trying a symmetry pattern of the form [F(x)*G(y), 0]
-> trying a symmetry pattern of the form [0, F(x)*G(y)]
-> trying symmetry patterns of the forms [F(x),G(y)] and [G(y),F(x)]
-> trying a symmetry pattern of the form [F(x),G(x)]
-> trying a symmetry pattern of the form [F(y),G(y)]
-> trying a symmetry pattern of the form [F(x)+G(y), 0]
-> trying a symmetry pattern of the form [0, F(x)+G(y)]
-> trying a symmetry pattern of the form [F(x),G(x)*y+H(x)]
-> trying a symmetry pattern of conformal type`
```

✗ Solution by Maple

```
dsolve(diff(x(t),t)+x(t)*t=exp(x(t)),x(t), singsol=all)
```

No solution found

✗ Solution by Mathematica

Time used: 0.0 (sec). Leaf size: 0

```
DSolve[x'[t]+x[t]*t==Exp[x[t]],x[t],t,IncludeSingularSolutions -> True]
```

Not solved

2.3 problem 3

Internal problem ID [4952]
Internal file name [`OUTPUT/4445_Sunday_June_05_2022_02_56_46_PM_28211667/index.tex`]

Book: Fundamentals of Differential Equations. By Nagle, Saff and Snider. 9th edition. Boston. Pearson 2018.
Section: Chapter 2, First order differential equations. Section 2.3, Linear equations. Exercises. page 54
Problem number: 3.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"exact", "linear", "separable", "homogeneousTypeD2", "first_order_ode_lie_symmetry_lookup"**

Maple gives the following as the ode type

```
[_separable]
```

$$\boxed{(t^2+1)\,y'-ty+y=0}$$

2.3.1 Solving as separable ode

In canonical form the ODE is

$$\begin{aligned} y' &= F(t,y) \\ &= f(t)g(y) \\ &= \frac{y(t-1)}{t^2+1} \end{aligned}$$

Where $f(t) = \frac{t-1}{t^2+1}$ and $g(y) = y$. Integrating both sides gives

$$\begin{aligned}
\frac{1}{y}\,dy &= \frac{t-1}{t^2+1}\,dt \\
\int \frac{1}{y}\,dy &= \int \frac{t-1}{t^2+1}\,dt \\
\ln(y) &= \frac{\ln(t^2+1)}{2} - \arctan(t) + c_1 \\
y &= e^{\frac{\ln(t^2+1)}{2} - \arctan(t) + c_1} \\
&= c_1 e^{\frac{\ln(t^2+1)}{2} - \arctan(t)}
\end{aligned}$$

Which simplifies to

$$y = c_1\sqrt{t^2+1}\,e^{-\arctan(t)}$$

Summary

The solution(s) found are the following

$$y = c_1\sqrt{t^2+1}\,e^{-\arctan(t)} \tag{1}$$

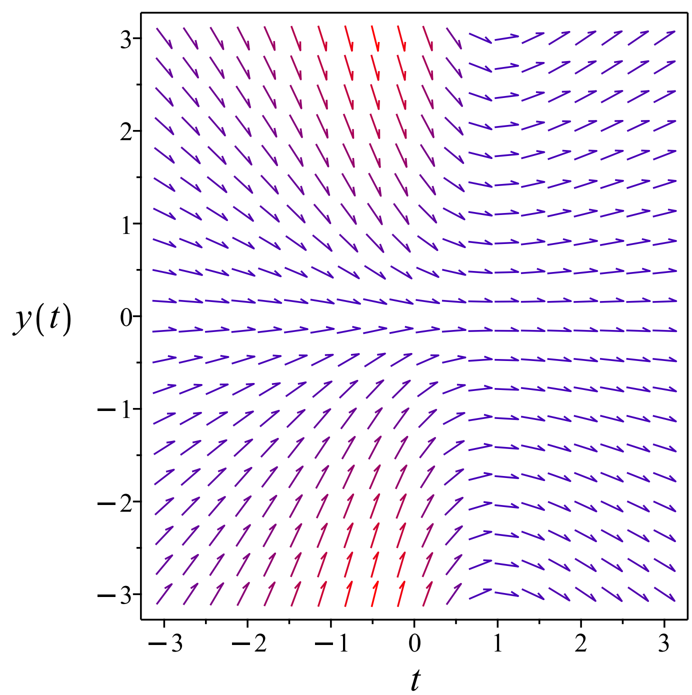


Figure 98: Slope field plot

Verification of solutions

$$y = c_1\sqrt{t^2+1}\,\mathrm{e}^{-\arctan(t)}$$

Verified OK.

2.3.2 Solving as linear ode

Entering Linear first order ODE solver. In canonical form a linear first order is

$$y' + p(t)y = q(t)$$

Where here

$$p(t) = -\frac{t-1}{t^2+1}$$
$$q(t) = 0$$

Hence the ode is

$$y' - \frac{y(t-1)}{t^2+1} = 0$$

The integrating factor μ is

$$\mu = \mathrm{e}^{\int -\frac{t-1}{t^2+1}dt}$$
$$= \mathrm{e}^{-\frac{\ln(t^2+1)}{2}+\arctan(t)}$$

Which simplifies to

$$\mu = \frac{\mathrm{e}^{\arctan(t)}}{\sqrt{t^2+1}}$$

The ode becomes

$$\frac{\mathrm{d}}{\mathrm{d}t}\mu y = 0$$
$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\mathrm{e}^{\arctan(t)}y}{\sqrt{t^2+1}}\right) = 0$$

Integrating gives

$$\frac{\mathrm{e}^{\arctan(t)}y}{\sqrt{t^2+1}} = c_1$$

Dividing both sides by the integrating factor $\mu = \frac{\mathrm{e}^{\arctan(t)}}{\sqrt{t^2+1}}$ results in

$$y = c_1\sqrt{t^2+1}\,\mathrm{e}^{-\arctan(t)}$$

Summary

The solution(s) found are the following

$$y = c_1 \sqrt{t^2 + 1}\, \mathrm{e}^{-\arctan(t)} \tag{1}$$

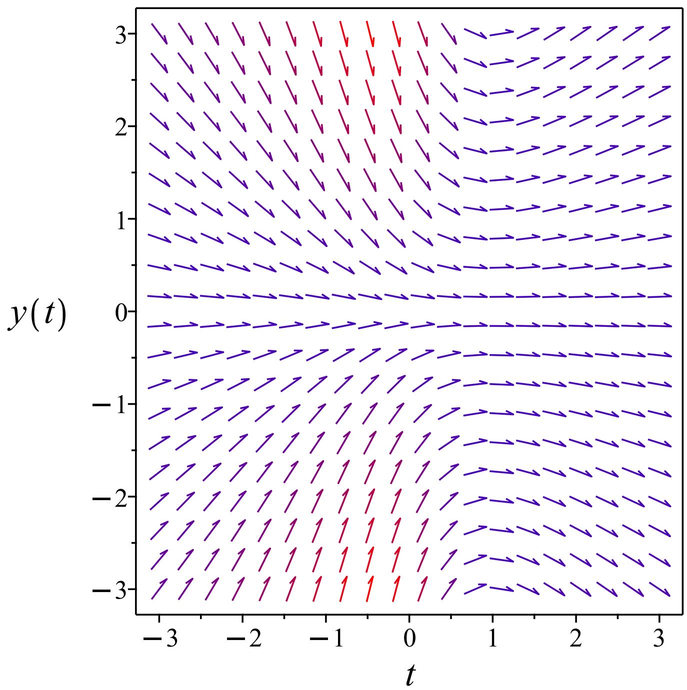


Figure 99: Slope field plot

Verification of solutions

$$y = c_1 \sqrt{t^2 + 1}\, \mathrm{e}^{-\arctan(t)}$$

Verified OK.

2.3.3 Solving as homogeneousTypeD2 ode

Using the change of variables $y = u(t)\, t$ on the above ode results in new ode in $u(t)$

$$(t^2 + 1)\,(u'(t)\, t + u(t)) - t^2 u(t) + u(t)\, t = 0$$

In canonical form the ODE is

$$\begin{aligned} u' &= F(t, u) \\ &= f(t) g(u) \\ &= -\frac{u(t + 1)}{(t^2 + 1)\, t} \end{aligned}$$

Where $f(t) = -\frac{t+1}{(t^2+1)t}$ and $g(u) = u$. Integrating both sides gives

$$
\begin{aligned}
\frac{1}{u}\,du &= -\frac{t+1}{(t^2+1)\,t}\,dt \\
\int \frac{1}{u}\,du &= \int -\frac{t+1}{(t^2+1)\,t}\,dt \\
\ln(u) &= \frac{\ln(t^2+1)}{2} - \arctan(t) - \ln(t) + c_2 \\
u &= \mathrm{e}^{\frac{\ln(t^2+1)}{2} - \arctan(t) - \ln(t) + c_2} \\
&= c_2 \mathrm{e}^{\frac{\ln(t^2+1)}{2} - \arctan(t) - \ln(t)}
\end{aligned}
$$

Which simplifies to

$$
u(t) = \frac{c_2\sqrt{t^2+1}\,\mathrm{e}^{-\arctan(t)}}{t}
$$

Therefore the solution y is

$$
\begin{aligned}
y &= ut \\
&= c_2\sqrt{t^2+1}\,\mathrm{e}^{-\arctan(t)}
\end{aligned}
$$

Summary

The solution(s) found are the following

$$
y = c_2\sqrt{t^2+1}\,\mathrm{e}^{-\arctan(t)} \tag{1}
$$

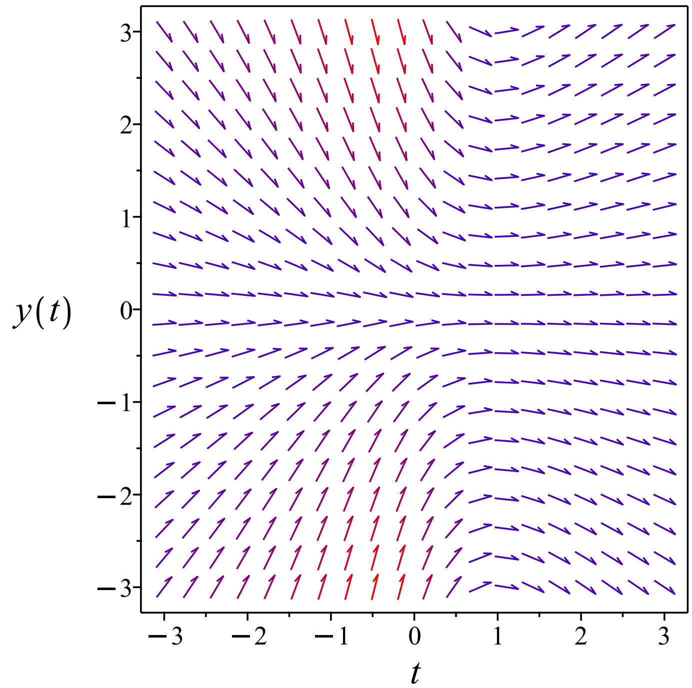


Figure 100: Slope field plot

Verification of solutions

$$y = c_2\sqrt{t^2+1}\,\mathrm{e}^{-\arctan(t)}$$

Verified OK.

2.3.4 Solving as first order ode lie symmetry lookup ode

Writing the ode as

$$y' = \frac{y(t-1)}{t^2+1}$$
$$y' = \omega(t,y)$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_t + \omega(\eta_y - \xi_t) - \omega^2\xi_y - \omega_t\xi - \omega_y\eta = 0 \tag{A}$$

The type of this ode is known. It is of type `linear`. Therefore we do not need to solve the PDE (A), and can just use the lookup table shown below to find ξ, η

Table 95: Lie symmetry infinitesimal lookup table for known first order ODE's

ODE class	Form	ξ	η
linear ode	$y' = f(x)y(x) + g(x)$	0	$e^{\int f dx}$
separable ode	$y' = f(x)\, g(y)$	$\frac{1}{f}$	0
quadrature ode	$y' = f(x)$	0	1
quadrature ode	$y' = g(y)$	1	0
homogeneous ODEs of Class A	$y' = f\left(\frac{y}{x}\right)$	x	y
homogeneous ODEs of Class C	$y' = (a + bx + cy)^{\frac{n}{m}}$	1	$-\frac{b}{c}$
homogeneous class D	$y' = \frac{y}{x} + g(x)\, F\left(\frac{y}{x}\right)$	x^2	xy
First order special form ID 1	$y' = g(x)\, e^{h(x)+by} + f(x)$	$\frac{e^{-\int bf(x)dx-h(x)}}{g(x)}$	$\frac{f(x)e^{-\int bf(x)dx-h(x)}}{g(x)}$
polynomial type ode	$y' = \frac{a_1x+b_1y+c_1}{a_2x+b_2y+c_2}$	$\frac{a_1b_2x-a_2b_1x-b_1c_2+b_2c_1}{a_1b_2-a_2b_1}$	$\frac{a_1b_2y-a_2b_1y-a_1c_2-a_2c_1}{a_1b_2-a_2b_1}$
Bernoulli ode	$y' = f(x)\, y + g(x)\, y^n$	0	$e^{-\int (n-1)f(x)dx} y^n$
Reduced Riccati	$y' = f_1(x)\, y + f_2(x)\, y^2$	0	$e^{-\int f_1 dx}$

The above table shows that

$$
\begin{aligned}
\xi(t, y) &= 0 \\
\eta(t, y) &= \mathrm{e}^{\frac{\ln(t^2+1)}{2} - \arctan(t)}
\end{aligned}
\tag{A1}
$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(t, y) \to (R, S)$ where (R, S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$
\frac{dt}{\xi} = \frac{dy}{\eta} = dS \tag{1}
$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial t} + \eta \frac{\partial}{\partial y}\right) S(t, y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the

canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$R = t$$

S is found from

$$\begin{aligned} S &= \int \frac{1}{\eta} dy \\ &= \int \frac{1}{\mathrm{e}^{\frac{\ln(t^2+1)}{2} - \arctan(t)}} dy \end{aligned}$$

Which results in

$$S = \mathrm{e}^{\ln\left(\frac{1}{\sqrt{t^2+1}}\right) + \arctan(t)} y$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_t + \omega(t, y) S_y}{R_t + \omega(t, y) R_y} \tag{2}$$

Where in the above R_t, R_y, S_t, S_y are all partial derivatives and $\omega(t, y)$ is the right hand side of the original ode given by

$$\omega(t, y) = \frac{y(t-1)}{t^2+1}$$

Evaluating all the partial derivatives gives

$$\begin{aligned} R_t &= 1 \\ R_y &= 0 \\ S_t &= \frac{(-t+1)\, \mathrm{e}^{\arctan(t)} y}{(t^2+1)^{\frac{3}{2}}} \\ S_y &= \frac{\mathrm{e}^{\arctan(t)}}{\sqrt{t^2+1}} \end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = 0 \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for t, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = 0$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordinates R, S. Integrating the above gives

$$S(R) = c_1 \tag{4}$$

To complete the solution, we just need to transform (4) back to t, y coordinates. This results in

$$\frac{e^{\arctan(t)} y}{\sqrt{t^2+1}} = c_1$$

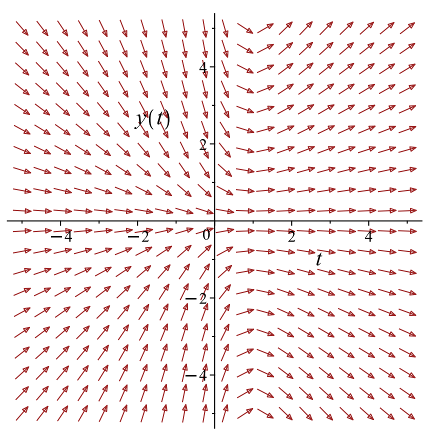
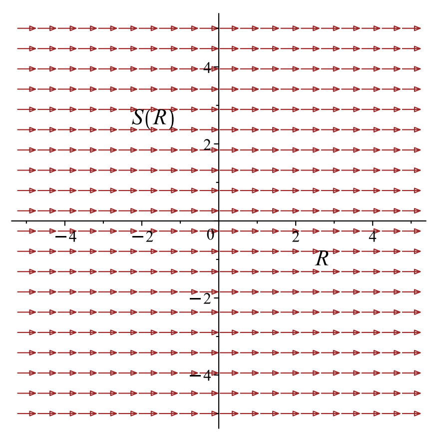
Which simplifies to

$$\frac{e^{\arctan(t)} y}{\sqrt{t^2+1}} = c_1$$

Which gives

$$y = c_1 \sqrt{t^2+1}\, e^{-\arctan(t)}$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in t, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dt} = \frac{y(t-1)}{t^2+1}$ 	$R = t$ $S = \frac{e^{\arctan(t)} y}{\sqrt{t^2+1}}$	$\frac{dS}{dR} = 0$ 

Summary

The solution(s) found are the following

$$y = c_1 \sqrt{t^2+1}\, \mathrm{e}^{-\arctan(t)} \tag{1}$$

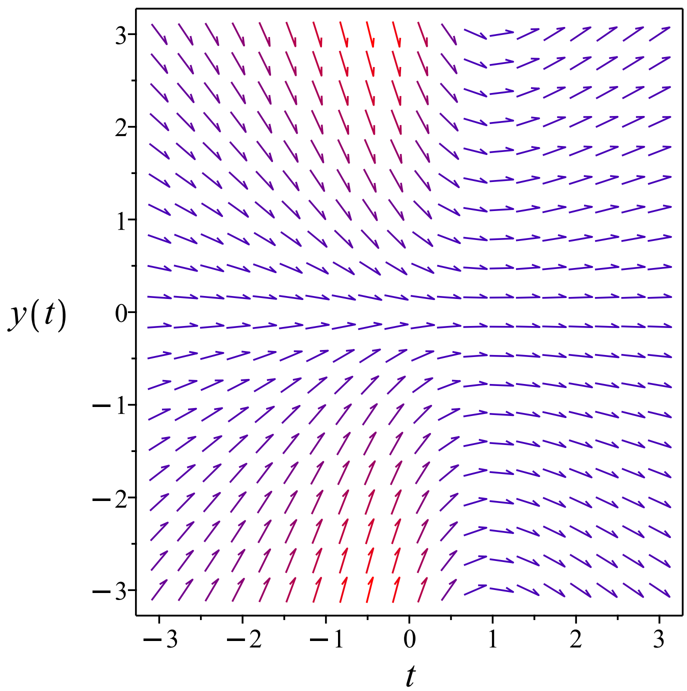


Figure 101: Slope field plot

Verification of solutions

$$y = c_1 \sqrt{t^2+1}\, \mathrm{e}^{-\arctan(t)}$$

Verified OK.

2.3.5 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x,y) + N(x,y)\frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x,y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y) = 0$$

Hence

$$\frac{\partial\phi}{\partial x}+\frac{\partial\phi}{\partial y}\frac{dy}{dx}=0 \tag{B}$$

Comparing (A,B) shows that

$$\begin{aligned}\frac{\partial\phi}{\partial x}&=M\\ \frac{\partial\phi}{\partial y}&=N\end{aligned}$$

But since $\frac{\partial^2\phi}{\partial x\partial y}=\frac{\partial^2\phi}{\partial y\partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y}=\frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x,y)$ but at least we know now that we can do that since the condition $\frac{\partial^2\phi}{\partial x\partial y}=\frac{\partial^2\phi}{\partial y\partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(t,y)\,\mathrm{d}t+N(t,y)\,\mathrm{d}y=0 \tag{1A}$$

Therefore

$$\begin{aligned}\left(\frac{1}{y}\right)\mathrm{d}y&=\left(\frac{t-1}{t^2+1}\right)\mathrm{d}t\\ \left(-\frac{t-1}{t^2+1}\right)\mathrm{d}t+\left(\frac{1}{y}\right)\mathrm{d}y&=0 \qquad (2A)\end{aligned}$$

Comparing (1A) and (2A) shows that

$$\begin{aligned}M(t,y)&=-\frac{t-1}{t^2+1}\\ N(t,y)&=\frac{1}{y}\end{aligned}$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y}=\frac{\partial N}{\partial t}$$

Using result found above gives

$$\begin{aligned}\frac{\partial M}{\partial y}&=\frac{\partial}{\partial y}\left(-\frac{t-1}{t^2+1}\right)\\ &=0\end{aligned}$$

And

$$
\begin{aligned}
\frac{\partial N}{\partial t} &= \frac{\partial}{\partial t}\left(\frac{1}{y}\right) \\
&= 0
\end{aligned}
$$

Since $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial t}$, then the ODE is exact The following equations are now set up to solve for the function $\phi(t, y)$

$$
\frac{\partial \phi}{\partial t} = M \tag{1}
$$

$$
\frac{\partial \phi}{\partial y} = N \tag{2}
$$

Integrating (1) w.r.t. t gives

$$
\begin{aligned}
\int \frac{\partial \phi}{\partial t} \,\mathrm{d}t &= \int M \,\mathrm{d}t \\
\int \frac{\partial \phi}{\partial t} \,\mathrm{d}t &= \int -\frac{t-1}{t^2+1} \,\mathrm{d}t \\
\phi &= -\frac{\ln(t^2+1)}{2} + \arctan(t) + f(y)
\end{aligned} \tag{3}
$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both t and y. Taking derivative of equation (3) w.r.t y gives

$$
\frac{\partial \phi}{\partial y} = 0 + f'(y) \tag{4}
$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = \frac{1}{y}$. Therefore equation (4) becomes

$$
\frac{1}{y} = 0 + f'(y) \tag{5}
$$

Solving equation (5) for $f'(y)$ gives

$$
f'(y) = \frac{1}{y}
$$

Integrating the above w.r.t y gives

$$
\begin{aligned}
\int f'(y) \,\mathrm{d}y &= \int \left(\frac{1}{y}\right) \mathrm{d}y \\
f(y) &= \ln(y) + c_1
\end{aligned}
$$

Where c_1 is constant of integration. Substituting result found above for $f(y)$ into equation (3) gives ϕ

$$\phi = -\frac{\ln(t^2+1)}{2} + \arctan(t) + \ln(y) + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = -\frac{\ln(t^2+1)}{2} + \arctan(t) + \ln(y)$$

Summary
The solution(s) found are the following

$$-\frac{\ln(t^2+1)}{2} + \arctan(t) + \ln(y) = c_1 \tag{1}$$

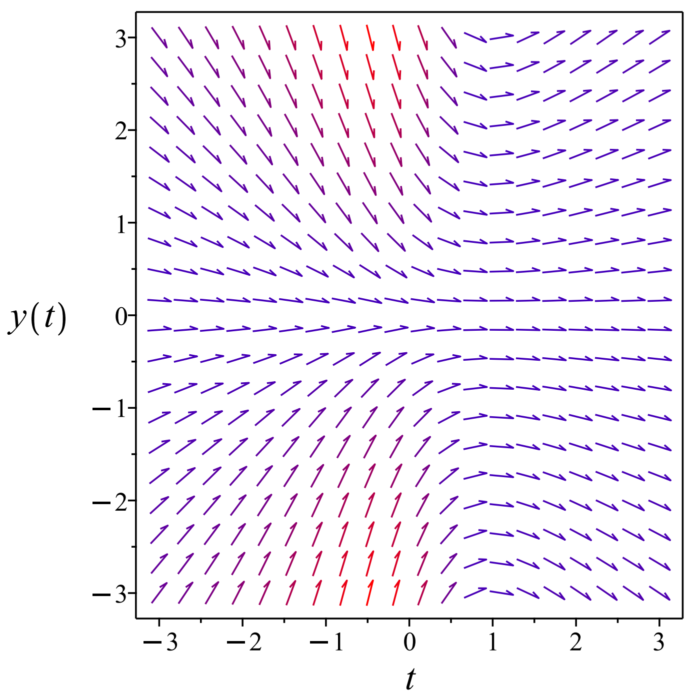


Figure 102: Slope field plot

Verification of solutions

$$-\frac{\ln(t^2+1)}{2} + \arctan(t) + \ln(y) = c_1$$

Verified OK.

2.3.6 Maple step by step solution

Let's solve

$(t^2+1)\, y' - ty + y = 0$

- Highest derivative means the order of the ODE is 1

  y'

- Separate variables

  $\frac{y'}{y} = \frac{t-1}{t^2+1}$

- Integrate both sides with respect to t

  $\int \frac{y'}{y} dt = \int \frac{t-1}{t^2+1} dt + c_1$

- Evaluate integral

  $\ln(y) = \frac{\ln(t^2+1)}{2} - \arctan(t) + c_1$

- Solve for y

  $y = e^{\frac{\ln(t^2+1)}{2} - \arctan(t) + c_1}$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
<- 1st order linear successful`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 18

```
dsolve((t^2+1)*diff(y(t),t)=y(t)*t-y(t),y(t), singsol=all)
```

$$y(t) = c_1 \sqrt{t^2+1}\, e^{-\arctan(t)}$$

✓ Solution by Mathematica

Time used: 0.04 (sec). Leaf size: 28

```
DSolve[(t^2+1)*y'[t]==y[t]*t-y[t],y[t],t,IncludeSingularSolutions -> True]
```

$$y(t) \to c_1 \sqrt{t^2+1} e^{-\arctan(t)}$$
$$y(t) \to 0$$

2.4 problem 4

Internal problem ID [4953]
Internal file name [`OUTPUT/4446_Sunday_June_05_2022_02_56_47_PM_9756762/index.tex`]

Book: Fundamentals of Differential Equations. By Nagle, Saff and Snider. 9th edition. Boston. Pearson 2018.
Section: Chapter 2, First order differential equations. Section 2.3, Linear equations. Exercises. page 54
Problem number: 4.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"linear", "exactWithIntegrationFactor", "first_order_ode_lie_symmetry_lookup"**

Maple gives the following as the ode type

`[_linear]`

$$\boxed{-\mathrm{e}^{t} y' - y \ln(t) = -3t}$$

2.4.1 Solving as linear ode

Entering Linear first order ODE solver. In canonical form a linear first order is

$$y' + p(t)y = q(t)$$

Where here

$$p(t) = \mathrm{e}^{-t} \ln(t)$$
$$q(t) = 3t\,\mathrm{e}^{-t}$$

Hence the ode is

$$y' + \mathrm{e}^{-t} \ln(t)\, y = 3t\,\mathrm{e}^{-t}$$

The integrating factor μ is

$$\mu = e^{\int e^{-t} \ln(t) dt}$$

The ode becomes

$$\begin{aligned}
\frac{d}{dt}(\mu y) &= (\mu) \left(3t\, e^{-t}\right) \\
\frac{d}{dt}\left(e^{\int e^{-t} \ln(t) dt} y\right) &= \left(e^{\int e^{-t} \ln(t) dt}\right) \left(3t\, e^{-t}\right) \\
d\left(e^{\int e^{-t} \ln(t) dt} y\right) &= \left(3t\, e^{-t+\int e^{-t} \ln(t) dt}\right) dt
\end{aligned}$$

Integrating gives

$$\begin{aligned}
e^{\int e^{-t} \ln(t) dt} y &= \int 3t\, e^{-t+\int e^{-t} \ln(t) dt} \, dt \\
e^{\int e^{-t} \ln(t) dt} y &= \int 3t\, e^{-t+\int e^{-t} \ln(t) dt} dt + c_1
\end{aligned}$$

Dividing both sides by the integrating factor $\mu = e^{\int e^{-t} \ln(t) dt}$ results in

$$y = e^{-\left(\int e^{-t} \ln(t) dt\right)} \left(\int 3t\, e^{-t+\int e^{-t} \ln(t) dt} dt\right) + c_1 e^{-\left(\int e^{-t} \ln(t) dt\right)}$$

which simplifies to

$$y = e^{-\left(\int e^{-t} \ln(t) dt\right)} \left(3\left(\int t\, e^{-t+\int e^{-t} \ln(t) dt} dt\right) + c_1\right)$$

Which can be simplified to become

$$y = e^{\int -e^{-t} \ln(t) dt} \left(3\left(\int t\, e^{-t} e^{\int e^{-t} \ln(t) dt} dt\right) + c_1\right)$$

Summary

The solution(s) found are the following

$$y = e^{\int -e^{-t} \ln(t) dt} \left(3\left(\int t\, e^{-t} e^{\int e^{-t} \ln(t) dt} dt\right) + c_1\right) \tag{1}$$

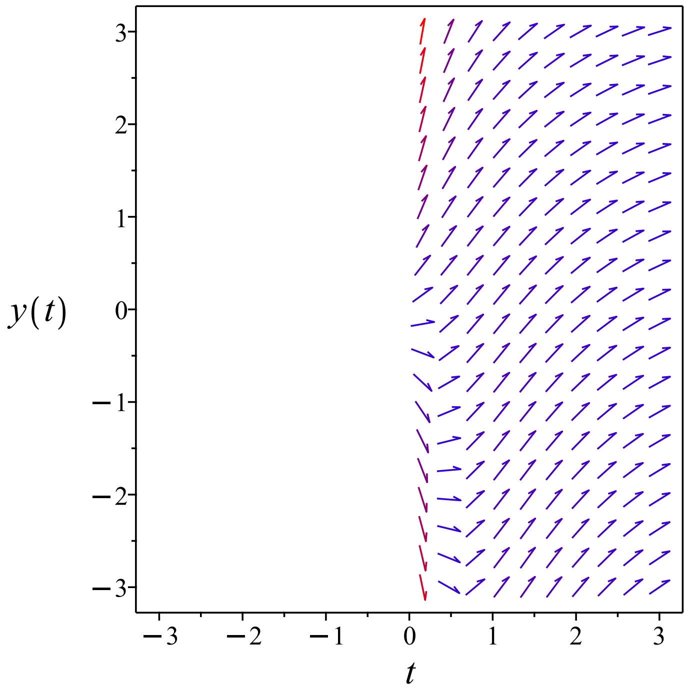


Figure 103: Slope field plot

Verification of solutions

$$y = e^{\int -e^{-t} \ln(t) dt} \left(3\left(\int t\, e^{-t} e^{\int e^{-t} \ln(t) dt} dt\right) + c_1\right)$$

Verified OK.

2.4.2 Solving as first order ode lie symmetry lookup ode

Writing the ode as

$$y' = -e^{-t}(-3t + y \ln(t))$$
$$y' = \omega(t, y)$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_t + \omega(\eta_y - \xi_t) - \omega^2 \xi_y - \omega_t \xi - \omega_y \eta = 0 \tag{A}$$

The type of this ode is known. It is of type `linear`. Therefore we do not need to solve the PDE (A), and can just use the lookup table shown below to find ξ, η

Table 98: Lie symmetry infinitesimal lookup table for known first order ODE's

ODE class	Form	ξ	η
linear ode	$y' = f(x)y(x) + g(x)$	0	$e^{\int f dx}$
separable ode	$y' = f(x)\, g(y)$	$\frac{1}{f}$	0
quadrature ode	$y' = f(x)$	0	1
quadrature ode	$y' = g(y)$	1	0
homogeneous ODEs of Class A	$y' = f\left(\frac{y}{x}\right)$	x	y
homogeneous ODEs of Class C	$y' = (a + bx + cy)^{\frac{n}{m}}$	1	$-\frac{b}{c}$
homogeneous class D	$y' = \frac{y}{x} + g(x)\, F\left(\frac{y}{x}\right)$	x^2	xy
First order special form ID 1	$y' = g(x)\, e^{h(x)+by} + f(x)$	$\frac{e^{-\int bf(x)dx-h(x)}}{g(x)}$	$\frac{f(x)e^{-\int bf(x)dx-h(x)}}{g(x)}$
polynomial type ode	$y' = \frac{a_1x+b_1y+c_1}{a_2x+b_2y+c_2}$	$\frac{a_1b_2x-a_2b_1x-b_1c_2+b_2c_1}{a_1b_2-a_2b_1}$	$\frac{a_1b_2y-a_2b_1y-a_1c_2-a_2c_1}{a_1b_2-a_2b_1}$
Bernoulli ode	$y' = f(x)\, y + g(x)\, y^n$	0	$e^{-\int (n-1)f(x)dx} y^n$
Reduced Riccati	$y' = f_1(x)\, y + f_2(x)\, y^2$	0	$e^{-\int f_1 dx}$

The above table shows that

$$
\begin{aligned}
\xi(t,y) &= 0 \\
\eta(t,y) &= \mathrm{e}^{\mathrm{e}^{-t}\ln(t)+\mathrm{expIntegral}_1(t)}
\end{aligned}
\tag{A1}
$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(t,y) \to (R,S)$ where (R,S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$
\frac{dt}{\xi} = \frac{dy}{\eta} = dS \tag{1}
$$

The above comes from the requirements that $\left(\xi\frac{\partial}{\partial t} + \eta\frac{\partial}{\partial y}\right) S(t,y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the

canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$R = t$$

S is found from

$$\begin{aligned}
S &= \int \frac{1}{\eta} dy \\
&= \int \frac{1}{e^{e^{-t} \ln(t) + \text{expIntegral}_1(t)}} dy
\end{aligned}$$

Which results in

$$S = e^{-e^{-t} \ln(t) - \text{expIntegral}_1(t)} y$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_t + \omega(t,y) S_y}{R_t + \omega(t,y) R_y} \tag{2}$$

Where in the above R_t, R_y, S_t, S_y are all partial derivatives and $\omega(t,y)$ is the right hand side of the original ode given by

$$\omega(t,y) = -e^{-t}(-3t + y \ln(t))$$

Evaluating all the partial derivatives gives

$$\begin{aligned}
R_t &= 1 \\
R_y &= 0 \\
S_t &= y \ln(t) \, e^{-t - \text{expIntegral}_1(t)} t^{-e^{-t}} \\
S_y &= t^{-e^{-t}} e^{-\text{expIntegral}_1(t)}
\end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = 3t^{1-e^{-t}} e^{-t - \text{expIntegral}_1(t)} \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for t, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = 3R^{1-e^{-R}} e^{-R - \text{expIntegral}_1(R)}$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by

integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$S(R) = \int 3R^{1-e^{-R}} e^{-R-\text{expIntegral}_1(R)} dR + c_1 \tag{4}$$

To complete the solution, we just need to transform (4) back to t, y coordinates. This results in

$$t^{-e^{-t}} e^{-\text{expIntegral}_1(t)} y = \int 3t^{1-e^{-t}} e^{-t-\text{expIntegral}_1(t)} dt + c_1$$

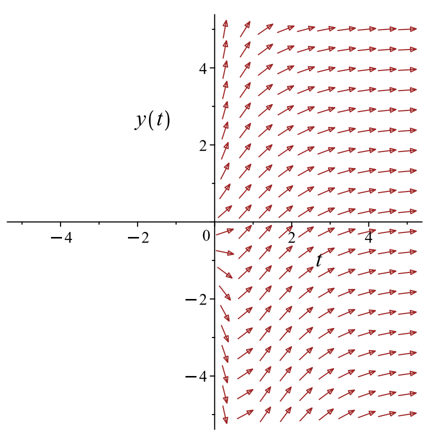
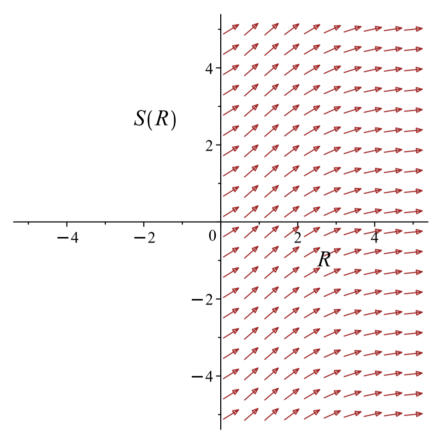
Which simplifies to

$$t^{-e^{-t}} e^{-\text{expIntegral}_1(t)} y = \int 3t^{1-e^{-t}} e^{-t-\text{expIntegral}_1(t)} dt + c_1$$

Which gives

$$y = \left(\int 3t^{1-e^{-t}} e^{-t-\text{expIntegral}_1(t)} dt + c_1\right) t^{e^{-t}} e^{\text{expIntegral}_1(t)}$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in t, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dt} = -e^{-t}(-3t + y\ln(t))$ 	$R = t$ $S = t^{-e^{-t}} e^{-\text{expIntegral}_1(t)} y$	$\frac{dS}{dR} = 3R^{1-e^{-R}} e^{-R-\text{expIntegral}_1(R)}$ 

Summary

The solution(s) found are the following

$$y = \left(\int 3t^{1-e^{-t}} e^{-t-\text{expIntegral}_1(t)} dt + c_1\right) t^{e^{-t}} e^{\text{expIntegral}_1(t)} \tag{1}$$

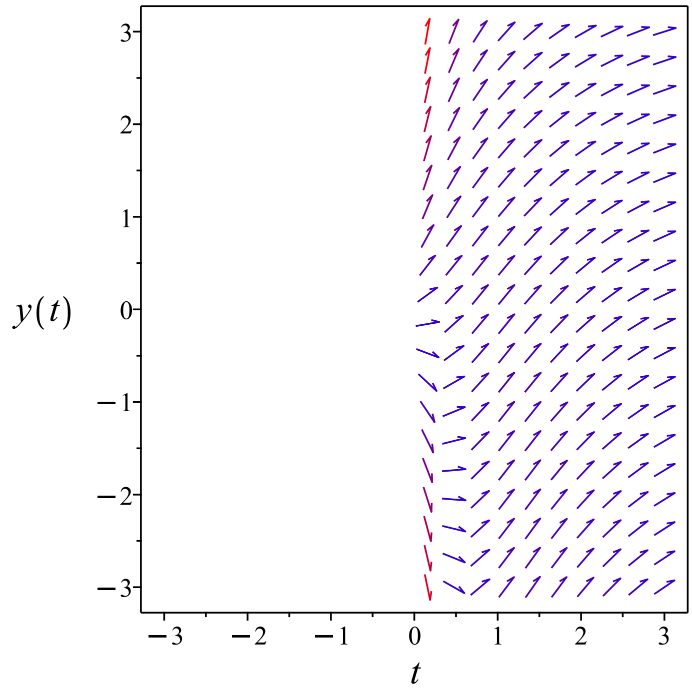


Figure 104: Slope field plot

Verification of solutions

$$y = \left(\int 3t^{1-\mathrm{e}^{-t}}\mathrm{e}^{-t-\mathrm{expIntegral}_1(t)}dt + c_1\right) t^{\mathrm{e}^{-t}}\mathrm{e}^{\mathrm{expIntegral}_1(t)}$$

Verified OK.

2.4.3 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x,y) + N(x,y)\frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x,y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y) = 0$$

Hence

$$\frac{\partial\phi}{\partial x} + \frac{\partial\phi}{\partial y}\frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$\begin{aligned}\frac{\partial \phi}{\partial x} &= M\\ \frac{\partial \phi}{\partial y} &= N\end{aligned}$$

But since $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x, y)$ but at least we know now that we can do that since the condition $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(t, y)\,\mathrm{d}t + N(t, y)\,\mathrm{d}y = 0 \tag{1A}$$

Therefore

$$\begin{aligned}\left(-\mathrm{e}^{t}\right)\mathrm{d}y &= \left(-3t + y\ln\left(t\right)\right)\mathrm{d}t\\ \left(3t - y\ln\left(t\right)\right)\mathrm{d}t + \left(-\mathrm{e}^{t}\right)\mathrm{d}y &= 0 \qquad (2A)\end{aligned}$$

Comparing (1A) and (2A) shows that

$$\begin{aligned}M(t, y) &= 3t - y\ln\left(t\right)\\ N(t, y) &= -\mathrm{e}^{t}\end{aligned}$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial t}$$

Using result found above gives

$$\begin{aligned}\frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}\left(3t - y\ln\left(t\right)\right)\\ &= -\ln\left(t\right)\end{aligned}$$

And

$$\begin{aligned}\frac{\partial N}{\partial t} &= \frac{\partial}{\partial t}\left(-\mathrm{e}^{t}\right)\\ &= -\mathrm{e}^{t}\end{aligned}$$

Since $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial t}$, then the ODE is not exact. Since the ODE is not exact, we will try to find an integrating factor to make it exact. Let

$$\begin{aligned} A &= \frac{1}{N}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial t}\right) \\ &= -\mathrm{e}^{-t}\left((-\ln(t)) - \left(-\mathrm{e}^{t}\right)\right) \\ &= \mathrm{e}^{-t}\ln(t) - 1 \end{aligned}$$

Since A does not depend on y, then it can be used to find an integrating factor. The integrating factor μ is

$$\begin{aligned} \mu &= e^{\int A\,\mathrm{d}t} \\ &= e^{\int \mathrm{e}^{-t}\ln(t)-1\,\mathrm{d}t} \end{aligned}$$

The result of integrating gives

$$\begin{aligned} \mu &= e^{-\mathrm{e}^{-t}\ln(t)-\mathrm{expIntegral}_1(t)-t} \\ &= \mathrm{e}^{-t-\mathrm{expIntegral}_1(t)}t^{-\mathrm{e}^{-t}} \end{aligned}$$

M and N are multiplied by this integrating factor, giving new M and new N which are called $\overline{M}$ and $\overline{N}$ for now so not to confuse them with the original M and N.

$$\begin{aligned} \overline{M} &= \mu M \\ &= \mathrm{e}^{-t-\mathrm{expIntegral}_1(t)}t^{-\mathrm{e}^{-t}}(3t - y\ln(t)) \\ &= (3t - y\ln(t))\,\mathrm{e}^{-t-\mathrm{expIntegral}_1(t)}t^{-\mathrm{e}^{-t}} \end{aligned}$$

And

$$\begin{aligned} \overline{N} &= \mu N \\ &= \mathrm{e}^{-t-\mathrm{expIntegral}_1(t)}t^{-\mathrm{e}^{-t}}\left(-\mathrm{e}^{t}\right) \\ &= -t^{-\mathrm{e}^{-t}}\mathrm{e}^{-\mathrm{expIntegral}_1(t)} \end{aligned}$$

Now a modified ODE is ontained from the original ODE, which is exact and can be solved. The modified ODE is

$$\begin{aligned} \overline{M} + \overline{N}\frac{\mathrm{d}y}{\mathrm{d}t} &= 0 \\ \left((3t - y\ln(t))\,\mathrm{e}^{-t-\mathrm{expIntegral}_1(t)}t^{-\mathrm{e}^{-t}}\right) + \left(-t^{-\mathrm{e}^{-t}}\mathrm{e}^{-\mathrm{expIntegral}_1(t)}\right)\frac{\mathrm{d}y}{\mathrm{d}t} &= 0 \end{aligned}$$

The following equations are now set up to solve for the function $\phi(t, y)$

$$\frac{\partial \phi}{\partial t} = \overline{M} \tag{1}$$

$$\frac{\partial \phi}{\partial y} = \overline{N} \tag{2}$$

Integrating (1) w.r.t. t gives

$$
\begin{aligned}
\int \frac{\partial \phi}{\partial t} \mathrm{d}t &= \int \overline{M} \mathrm{d}t \\
\int \frac{\partial \phi}{\partial t} \mathrm{d}t &= \int (3t - y \ln(t)) \mathrm{e}^{-t-\mathrm{expIntegral}_1(t)} t^{-\mathrm{e}^{-t}} \mathrm{d}t \\
\phi &= \int^t (3\_a - y \ln(\_a)) \mathrm{e}^{-\_a-\mathrm{expIntegral}_1(\_a)} \_a^{-\mathrm{e}^{-\_a}} d\_a + f(y) \qquad (3)
\end{aligned}
$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both t and y. Taking derivative of equation (3) w.r.t y gives

$$
\frac{\partial \phi}{\partial y} = -\left(\int^t \ln(\_a) \mathrm{e}^{-\_a-\mathrm{expIntegral}_1(\_a)} \_a^{-\mathrm{e}^{-\_a}} d\_a \right) + f'(y) \qquad (4)
$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = -t^{-\mathrm{e}^{-t}} \mathrm{e}^{-\mathrm{expIntegral}_1(t)}$. Therefore equation (4) becomes

$$
-t^{-\mathrm{e}^{-t}} \mathrm{e}^{-\mathrm{expIntegral}_1(t)} = -\left(\int^t \ln(\_a) \mathrm{e}^{-\_a-\mathrm{expIntegral}_1(\_a)} \_a^{-\mathrm{e}^{-\_a}} d\_a \right) + f'(y) \qquad (5)
$$

Solving equation (5) for $f'(y)$ gives

$$
f'(y) = -t^{-\mathrm{e}^{-t}} \mathrm{e}^{-\mathrm{expIntegral}_1(t)} + \int^t \ln(\_a) \mathrm{e}^{-\_a-\mathrm{expIntegral}_1(\_a)} \_a^{-\mathrm{e}^{-\_a}} d\_a
$$

Integrating the above w.r.t y gives

$$
\begin{aligned}
\int f'(y) \mathrm{d}y &= \int \left(-t^{-\mathrm{e}^{-t}} \mathrm{e}^{-\mathrm{expIntegral}_1(t)} + \int^t \ln(\_a) \mathrm{e}^{-\_a-\mathrm{expIntegral}_1(\_a)} \_a^{-\mathrm{e}^{-\_a}} d\_a \right) \mathrm{d}y \\
f(y) &= \left(-t^{-\mathrm{e}^{-t}} \mathrm{e}^{-\mathrm{expIntegral}_1(t)} + \int^t \ln(\_a) \mathrm{e}^{-\_a-\mathrm{expIntegral}_1(\_a)} \_a^{-\mathrm{e}^{-\_a}} d\_a \right) y + c_1
\end{aligned}
$$

Where c_1 is constant of integration. Substituting result found above for $f(y)$ into equation (3) gives ϕ

$$
\begin{aligned}
\phi = & \int^t (3\_a - y \ln(\_a)) \mathrm{e}^{-\_a-\mathrm{expIntegral}_1(\_a)} \_a^{-\mathrm{e}^{-\_a}} d\_a \\
& + \left(-t^{-\mathrm{e}^{-t}} \mathrm{e}^{-\mathrm{expIntegral}_1(t)} + \int^t \ln(\_a) \mathrm{e}^{-\_a-\mathrm{expIntegral}_1(\_a)} \_a^{-\mathrm{e}^{-\_a}} d\_a \right) y + c_1
\end{aligned}
$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = \int^t (3\_a - y \ln(\_a))\, \mathrm{e}^{-\_a - \mathrm{expIntegral}_1(\_a)}\, \_a^{-\mathrm{e}^{-\_a}} d\_a + \left(-t^{-\mathrm{e}^{-t}} \mathrm{e}^{-\,\mathrm{expIntegral}_1(t)} + \int^t \ln(\_a)\, \mathrm{e}^{-\_a - \mathrm{expIntegral}_1(\_a)}\, \_a^{-\mathrm{e}^{-\_a}} d\_a\right) y$$

<u>Summary</u>
The solution(s) found are the following

$$\int^t (3\_a - y \ln(\_a))\, \mathrm{e}^{-\_a - \mathrm{expIntegral}_1(\_a)}\, \_a^{-\mathrm{e}^{-\_a}} d\_a + \left(-t^{-\mathrm{e}^{-t}} \mathrm{e}^{-\,\mathrm{expIntegral}_1(t)} + \int^t \ln(\_a)\, \mathrm{e}^{-\_a - \mathrm{expIntegral}_1(\_a)}\, \_a^{-\mathrm{e}^{-\_a}} d\_a\right) y = c_1 \tag{1}$$

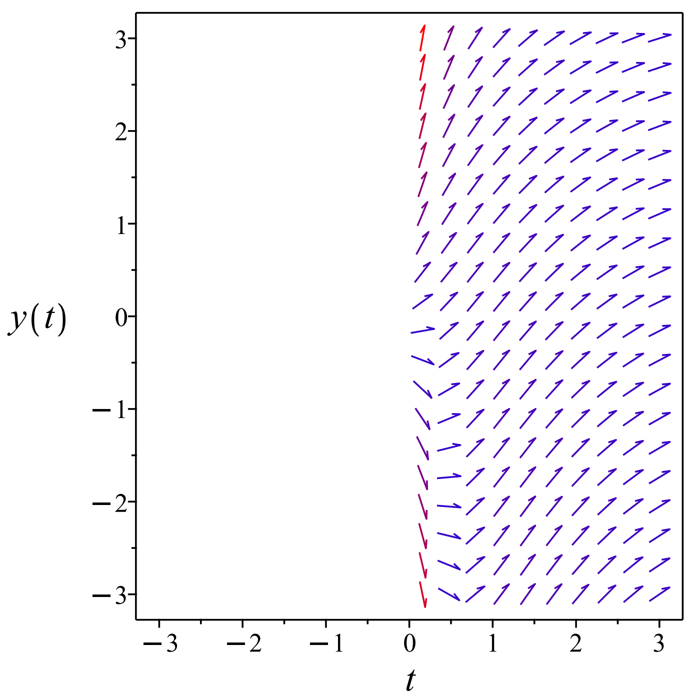


Figure 105: Slope field plot

Verification of solutions

$$\int^t \left(3\_a - y\ln\left(\_a\right)\right) e^{-\_a-\text{expIntegral}_1(\_a)}\_a^{-e^{-\_a}} d\_a + \left(-t^{-e^{-t}} e^{-\,\text{expIntegral}_1(t)} + \int^t \ln\left(\_a\right) e^{-\_a-\text{expIntegral}_1(\_a)}\_a^{-e^{-\_a}} d\_a\right) y = c_1$$

Verified OK.

2.4.4 Maple step by step solution

Let's solve

$-e^t y' - y\ln(t) = -3t$

- Highest derivative means the order of the ODE is 1

  y'

- Isolate the derivative

  $y' = -\frac{\ln(t)y}{e^t} + \frac{3t}{e^t}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE

  $y' + \frac{\ln(t)y}{e^t} = \frac{3t}{e^t}$

- The ODE is linear; multiply by an integrating factor $\mu(t)$

  $\mu(t)\left(y' + \frac{\ln(t)y}{e^t}\right) = \frac{3\mu(t)t}{e^t}$

- Assume the lhs of the ODE is the total derivative $\frac{d}{dt}(\mu(t)\, y)$

  $\mu(t)\left(y' + \frac{\ln(t)y}{e^t}\right) = \mu'(t)\, y + \mu(t)\, y'$

- Isolate $\mu'(t)$

  $\mu'(t) = \frac{\mu(t)\ln(t)}{e^t}$

- Solve to find the integrating factor

  $\mu(t) = \frac{e^{-\mathrm{Ei}_1(t)} e^{-t} e^t}{t^{e^{-t}}}$

- Integrate both sides with respect to t

  $\int \left(\frac{d}{dt}(\mu(t)\, y)\right) dt = \int \frac{3\mu(t)t}{e^t} dt + c_1$

- Evaluate the integral on the lhs

  $\mu(t)\, y = \int \frac{3\mu(t)t}{e^t} dt + c_1$

- Solve for y

$$y = \frac{\int \frac{3\mu(t)t}{e^t}dt + c_1}{\mu(t)}$$

- Substitute $\mu(t) = \frac{e^{-\mathrm{Ei}_1(t)e^{-t}}e^t}{t^{e^{-t}}}$

$$y = \frac{t^{e^{-t}}\left(\int \frac{3\,e^{-\mathrm{Ei}_1(t)}e^{-t}t}{t^{e^{-t}}}dt + c_1\right)}{e^{-\mathrm{Ei}_1(t)}e^{-t}e^t}$$

- Simplify

$$y = \left(3\left(\int t^{1-e^{-t}}e^{-t-\mathrm{Ei}_1(t)}dt\right) + c_1\right)t^{e^{-t}}e^{\mathrm{Ei}_1(t)}$$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
<- 1st order linear successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 42

```
dsolve(3*t=exp(t)*diff(y(t),t)+y(t)*ln(t),y(t), singsol=all)
```

$$y(t) = \left(3\left(\int t^{1-e^{-t}}e^{-t-\mathrm{expIntegral}_1(t)}dt\right) + c_1\right)t^{e^{-t}}e^{\mathrm{expIntegral}_1(t)}$$

✓ Solution by Mathematica

Time used: 0.237 (sec). Leaf size: 58

```
DSolve[3*t==Exp[t]*y'[t]+y[t]*Log[t],y[t],t,IncludeSingularSolutions -> True]
```

$$y(t) \to t^{e^{-t}}e^{-\mathrm{ExpIntegralEi}(-t)}\left(\int_1^t 3e^{\mathrm{ExpIntegralEi}(-K[1])-K[1]}K[1]^{1-e^{-K[1]}}dK[1] + c_1\right)$$

2.5 problem 5

Internal problem ID [4954]
Internal file name [`OUTPUT/4447_Sunday_June_05_2022_02_56_48_PM_20503039/index.tex`]

Book: Fundamentals of Differential Equations. By Nagle, Saff and Snider. 9th edition. Boston. Pearson 2018.
Section: Chapter 2, First order differential equations. Section 2.3, Linear equations. Exercises. page 54
Problem number: 5.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"unknown"**

Maple gives the following as the ode type

```
[[_Abel, `2nd type`, `class A`]]
```

Unable to solve or complete the solution.

$$\boxed{xx' + xt^2 = \sin(t)}$$

Unable to determine ODE type.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
trying homogeneous types:
trying Chini
differential order: 1; looking for linear symmetries
trying exact
trying Abel
Looking for potential symmetries
Looking for potential symmetries
Looking for potential symmetries
trying inverse_Riccati
trying an equivalence to an Abel ODE
differential order: 1; trying a linearization to 2nd order
--- trying a change of variables {x -> y(x), y(x) -> x}
differential order: 1; trying a linearization to 2nd order
trying 1st order ODE linearizable_by_differentiation
--- Trying Lie symmetry methods, 1st order ---
`, `-> Computing symmetries using: way = 3
`, `-> Computing symmetries using: way = 4
`, `-> Computing symmetries using: way = 2
trying symmetry patterns for 1st order ODEs
-> trying a symmetry pattern of the form [F(x)*G(y), 0]
-> trying a symmetry pattern of the form [0, F(x)*G(y)]
-> trying symmetry patterns of the forms [F(x),G(y)] and [G(y),F(x)]
`, `-> Computing symmetries using: way = HINT
   -> Calling odsolve with the ODE`, diff(y(x), x)+cos(x)*y(x)/sin(x), y(x)`      *** Sublev
      Methods for first order ODEs:
      --- Trying classification methods ---
      trying a quadrature
      trying 1st order linear
      <- 1st order linear successful
   -> Calling odsolve with the ODE`, diff(y(x), x)+2*y(x)/x, y(x)`      *** Sublevel 2 ***
      Methods for first order ODEs:
      --- Trying classification methods ---
      trying a quadrature
      trying 1st order linear
      <- 1st order linear successful
`, `-> Computing symmetries using: way = HINT
   -> Calling odsolve with the ODE`, diff(y(x), x) = 0, y(x)`      *** Sublevel 2 ***
```

✗ Solution by Maple

```
dsolve(x(t)*diff(x(t),t)+t^2*x(t)=sin(t),x(t), singsol=all)
```

No solution found

✗ Solution by Mathematica

Time used: 0.0 (sec). Leaf size: 0

```
DSolve[x[t]*x'[t]+t^2*x[t]==Sin[t],x[t],t,IncludeSingularSolutions -> True]
```

Not solved

2.6 problem 6

Internal problem ID [4955]
Internal file name [`OUTPUT/4448_Sunday_June_05_2022_02_56_49_PM_54920825/index.tex`]

Book: Fundamentals of Differential Equations. By Nagle, Saff and Snider. 9th edition. Boston. Pearson 2018.
Section: Chapter 2, First order differential equations. Section 2.3, Linear equations. Exercises. page 54
Problem number: 6.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"linear", "exactWithIntegrationFactor", "first_order_ode_lie_symmetry_lookup"**

Maple gives the following as the ode type

```
[[_linear, `class A`]]
```

$$\boxed{3r - r' = -\theta^3}$$

2.6.1 Solving as linear ode

Entering Linear first order ODE solver. In canonical form a linear first order is

$$r' + p(\theta)r = q(\theta)$$

Where here

$$p(\theta) = -3$$
$$q(\theta) = \theta^3$$

Hence the ode is

$$r' - 3r = \theta^3$$

The integrating factor μ is

$$\begin{aligned}\mu &= \mathrm{e}^{\int (-3)d\theta} \\ &= \mathrm{e}^{-3\theta}\end{aligned}$$

The ode becomes

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}\theta}(\mu r) &= (\mu)\left(\theta^3\right) \\ \frac{\mathrm{d}}{\mathrm{d}\theta}\left(\mathrm{e}^{-3\theta} r\right) &= \left(\mathrm{e}^{-3\theta}\right)\left(\theta^3\right) \\ \mathrm{d}\left(\mathrm{e}^{-3\theta} r\right) &= \left(\theta^3 \mathrm{e}^{-3\theta}\right)\,\mathrm{d}\theta\end{aligned}$$

Integrating gives

$$\begin{aligned}\mathrm{e}^{-3\theta} r &= \int \theta^3 \mathrm{e}^{-3\theta}\,\mathrm{d}\theta \\ \mathrm{e}^{-3\theta} r &= -\frac{(9\theta^3 + 9\theta^2 + 6\theta + 2)\,\mathrm{e}^{-3\theta}}{27} + c_1\end{aligned}$$

Dividing both sides by the integrating factor $\mu = \mathrm{e}^{-3\theta}$ results in

$$r = -\frac{\mathrm{e}^{3\theta}(9\theta^3 + 9\theta^2 + 6\theta + 2)\,\mathrm{e}^{-3\theta}}{27} + c_1 \mathrm{e}^{3\theta}$$

which simplifies to

$$r = -\frac{\theta^3}{3} - \frac{\theta^2}{3} - \frac{2\theta}{9} - \frac{2}{27} + c_1 \mathrm{e}^{3\theta}$$

Summary

The solution(s) found are the following

$$r = -\frac{\theta^3}{3} - \frac{\theta^2}{3} - \frac{2\theta}{9} - \frac{2}{27} + c_1 \mathrm{e}^{3\theta} \tag{1}$$

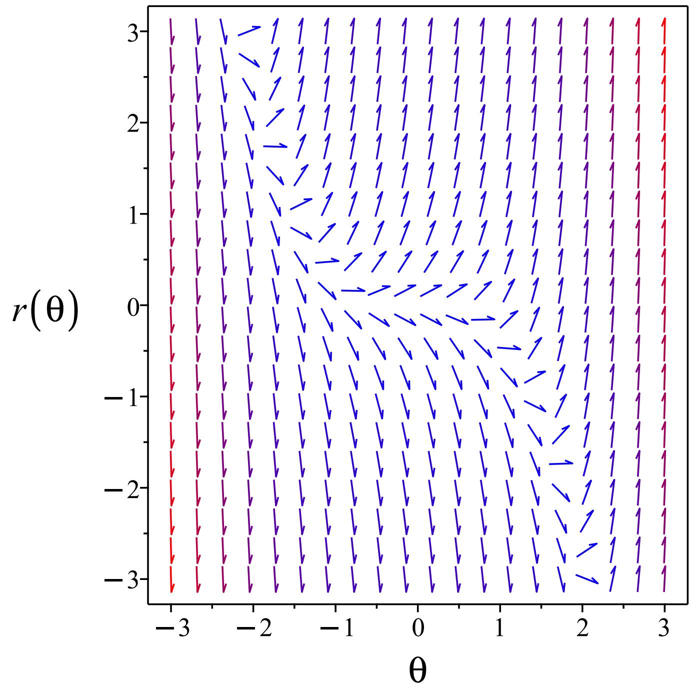


Figure 106: Slope field plot

Verification of solutions

$$r = -\frac{\theta^3}{3} - \frac{\theta^2}{3} - \frac{2\theta}{9} - \frac{2}{27} + c_1 e^{3\theta}$$

Verified OK.

2.6.2 Solving as first order ode lie symmetry lookup ode

Writing the ode as

$$r' = \theta^3 + 3r$$
$$r' = \omega(\theta, r)$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_\theta + \omega(\eta_r - \xi_\theta) - \omega^2 \xi_r - \omega_\theta \xi - \omega_r \eta = 0 \tag{A}$$

The type of this ode is known. It is of type `linear`. Therefore we do not need to solve the PDE (A), and can just use the lookup table shown below to find ξ, η

Table 101: Lie symmetry infinitesimal lookup table for known first order ODE's

ODE class	Form	ξ	η
linear ode	$y' = f(x)y(x) + g(x)$	0	$e^{\int f dx}$
separable ode	$y' = f(x)\, g(y)$	$\frac{1}{f}$	0
quadrature ode	$y' = f(x)$	0	1
quadrature ode	$y' = g(y)$	1	0
homogeneous ODEs of Class A	$y' = f\left(\frac{y}{x}\right)$	x	y
homogeneous ODEs of Class C	$y' = (a + bx + cy)^{\frac{n}{m}}$	1	$-\frac{b}{c}$
homogeneous class D	$y' = \frac{y}{x} + g(x)\, F\left(\frac{y}{x}\right)$	x^2	xy
First order special form ID 1	$y' = g(x)\, e^{h(x)+by} + f(x)$	$\frac{e^{-\int bf(x)dx-h(x)}}{g(x)}$	$\frac{f(x)e^{-\int bf(x)dx-h(x)}}{g(x)}$
polynomial type ode	$y' = \frac{a_1x+b_1y+c_1}{a_2x+b_2y+c_2}$	$\frac{a_1b_2x-a_2b_1x-b_1c_2+b_2c_1}{a_1b_2-a_2b_1}$	$\frac{a_1b_2y-a_2b_1y-a_1c_2-a_2c_1}{a_1b_2-a_2b_1}$
Bernoulli ode	$y' = f(x)\, y + g(x)\, y^n$	0	$e^{-\int (n-1)f(x)dx} y^n$
Reduced Riccati	$y' = f_1(x)\, y + f_2(x)\, y^2$	0	$e^{-\int f_1 dx}$

The above table shows that

$$
\begin{aligned}
\xi(\theta, r) &= 0 \\
\eta(\theta, r) &= \mathrm{e}^{3\theta}
\end{aligned} \tag{A1}
$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(\theta, r) \to (R, S)$ where (R, S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$
\frac{d\theta}{\xi} = \frac{dr}{\eta} = dS \tag{1}
$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial \theta} + \eta \frac{\partial}{\partial r}\right) S(\theta, r) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the

canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$R = \theta$$

S is found from

$$\begin{aligned} S &= \int \frac{1}{\eta} dy \\ &= \int \frac{1}{\mathrm{e}^{3\theta}} dy \end{aligned}$$

Which results in

$$S = \mathrm{e}^{-3\theta} r$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_\theta + \omega(\theta, r) S_r}{R_\theta + \omega(\theta, r) R_r} \tag{2}$$

Where in the above $R_\theta, R_r, S_\theta, S_r$ are all partial derivatives and $\omega(\theta, r)$ is the right hand side of the original ode given by

$$\omega(\theta, r) = \theta^3 + 3r$$

Evaluating all the partial derivatives gives

$$\begin{aligned} R_\theta &= 1 \\ R_r &= 0 \\ S_\theta &= -3\,\mathrm{e}^{-3\theta} r \\ S_r &= \mathrm{e}^{-3\theta} \end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = \theta^3 \mathrm{e}^{-3\theta} \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for θ, r in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = R^3 \mathrm{e}^{-3R}$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by

integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$S(R) = -\frac{(9R^3 + 9R^2 + 6R + 2)\,\mathrm{e}^{-3R}}{27} + c_1 \tag{4}$$

To complete the solution, we just need to transform (4) back to θ, r coordinates. This results in

$$\mathrm{e}^{-3\theta} r = -\frac{(9\theta^3 + 9\theta^2 + 6\theta + 2)\,\mathrm{e}^{-3\theta}}{27} + c_1$$

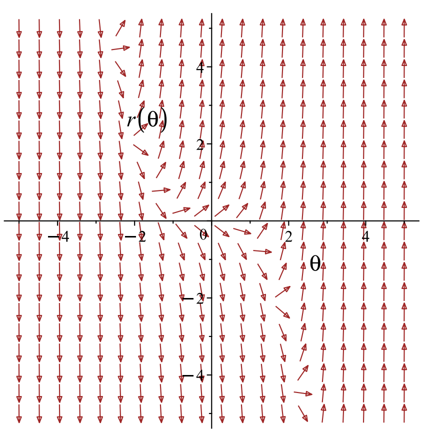
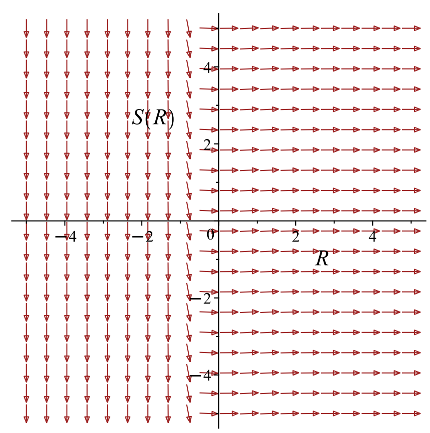
Which simplifies to

$$\mathrm{e}^{-3\theta} r = -\frac{(9\theta^3 + 9\theta^2 + 6\theta + 2)\,\mathrm{e}^{-3\theta}}{27} + c_1$$

Which gives

$$r = -\frac{\left(9\theta^3 \mathrm{e}^{-3\theta} + 9\theta^2 \mathrm{e}^{-3\theta} + 6\,\mathrm{e}^{-3\theta}\theta + 2\,\mathrm{e}^{-3\theta} - 27c_1\right)\mathrm{e}^{3\theta}}{27}$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in θ, r coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dr}{d\theta} = \theta^3 + 3r$ 	$R = \theta$ $S = \mathrm{e}^{-3\theta} r$	$\frac{dS}{dR} = R^3 \mathrm{e}^{-3R}$ 

Summary

The solution(s) found are the following

$$r = -\frac{\left(9\theta^3 \mathrm{e}^{-3\theta} + 9\theta^2 \mathrm{e}^{-3\theta} + 6\,\mathrm{e}^{-3\theta}\theta + 2\,\mathrm{e}^{-3\theta} - 27c_1\right)\mathrm{e}^{3\theta}}{27} \tag{1}$$

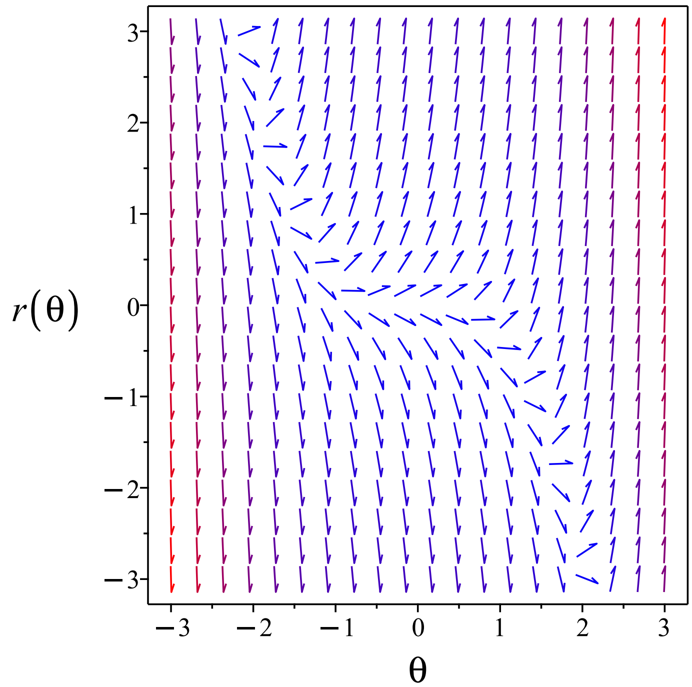


Figure 107: Slope field plot

Verification of solutions

$$r = -\frac{\left(9\theta^3 \mathrm{e}^{-3\theta} + 9\theta^2 \mathrm{e}^{-3\theta} + 6\,\mathrm{e}^{-3\theta}\theta + 2\,\mathrm{e}^{-3\theta} - 27c_1\right)\mathrm{e}^{3\theta}}{27}$$

Verified OK.

2.6.3 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x,y) + N(x,y)\frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x,y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y) = 0$$

Hence

$$\frac{\partial \phi}{\partial x} + \frac{\partial \phi}{\partial y}\frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$\frac{\partial \phi}{\partial x} = M$$
$$\frac{\partial \phi}{\partial y} = N$$

But since $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x,y)$ but at least we know now that we can do that since the condition $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(\theta,r)\,\mathrm{d}\theta + N(\theta,r)\,\mathrm{d}r = 0 \tag{1A}$$

Therefore

$$(-1)\,\mathrm{d}r = \left(-\theta^3 - 3r\right)\mathrm{d}\theta$$
$$\left(\theta^3 + 3r\right)\mathrm{d}\theta + (-1)\,\mathrm{d}r = 0 \tag{2A}$$

Comparing (1A) and (2A) shows that

$$M(\theta,r) = \theta^3 + 3r$$
$$N(\theta,r) = -1$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial r} = \frac{\partial N}{\partial \theta}$$

Using result found above gives

$$\begin{aligned}\frac{\partial M}{\partial r} &= \frac{\partial}{\partial r}\left(\theta^3 + 3r\right)\\ &= 3\end{aligned}$$

And

$$\begin{aligned}\frac{\partial N}{\partial \theta} &= \frac{\partial}{\partial \theta}(-1)\\ &= 0\end{aligned}$$

Since $\frac{\partial M}{\partial r} \neq \frac{\partial N}{\partial \theta}$, then the ODE is <u>not exact</u>. Since the ODE is not exact, we will try to find an integrating factor to make it exact. Let

$$\begin{aligned}
A &= \frac{1}{N}\left(\frac{\partial M}{\partial r} - \frac{\partial N}{\partial \theta}\right) \\
&= -1((3) - (0)) \\
&= -3
\end{aligned}$$

Since A does not depend on r, then it can be used to find an integrating factor. The integrating factor μ is

$$\begin{aligned}
\mu &= e^{\int A \, \mathrm{d}\theta} \\
&= e^{\int -3 \, \mathrm{d}\theta}
\end{aligned}$$

The result of integrating gives

$$\begin{aligned}
\mu &= e^{-3\theta} \\
&= \mathrm{e}^{-3\theta}
\end{aligned}$$

M and N are multiplied by this integrating factor, giving new M and new N which are called $\overline{M}$ and $\overline{N}$ for now so not to confuse them with the original M and N.

$$\begin{aligned}
\overline{M} &= \mu M \\
&= \mathrm{e}^{-3\theta}(\theta^3 + 3r) \\
&= \mathrm{e}^{-3\theta}(\theta^3 + 3r)
\end{aligned}$$

And

$$\begin{aligned}
\overline{N} &= \mu N \\
&= \mathrm{e}^{-3\theta}(-1) \\
&= -\mathrm{e}^{-3\theta}
\end{aligned}$$

Now a modified ODE is ontained from the original ODE, which is exact and can be solved. The modified ODE is

$$\begin{aligned}
\overline{M} + \overline{N}\frac{\mathrm{d}r}{\mathrm{d}\theta} &= 0 \\
\left(\mathrm{e}^{-3\theta}(\theta^3 + 3r)\right) + \left(-\mathrm{e}^{-3\theta}\right)\frac{\mathrm{d}r}{\mathrm{d}\theta} &= 0
\end{aligned}$$

The following equations are now set up to solve for the function $\phi(\theta, r)$

$$\frac{\partial \phi}{\partial \theta} = \overline{M} \tag{1}$$

$$\frac{\partial \phi}{\partial r} = \overline{N} \tag{2}$$

Integrating (1) w.r.t. θ gives

$$\int \frac{\partial \phi}{\partial \theta} \, d\theta = \int \overline{M} \, d\theta$$

$$\int \frac{\partial \phi}{\partial \theta} \, d\theta = \int e^{-3\theta} \left(\theta^3 + 3r\right) d\theta$$

$$\phi = -\frac{(9\theta^3 + 9\theta^2 + 27r + 6\theta + 2) \, e^{-3\theta}}{27} + f(r) \tag{3}$$

Where $f(r)$ is used for the constant of integration since ϕ is a function of both θ and r. Taking derivative of equation (3) w.r.t r gives

$$\frac{\partial \phi}{\partial r} = -e^{-3\theta} + f'(r) \tag{4}$$

But equation (2) says that $\frac{\partial \phi}{\partial r} = -e^{-3\theta}$. Therefore equation (4) becomes

$$-e^{-3\theta} = -e^{-3\theta} + f'(r) \tag{5}$$

Solving equation (5) for $f'(r)$ gives

$$f'(r) = 0$$

Therefore

$$f(r) = c_1$$

Where c_1 is constant of integration. Substituting this result for $f(r)$ into equation (3) gives ϕ

$$\phi = -\frac{(9\theta^3 + 9\theta^2 + 27r + 6\theta + 2) \, e^{-3\theta}}{27} + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = -\frac{(9\theta^3 + 9\theta^2 + 27r + 6\theta + 2) \, e^{-3\theta}}{27}$$

The solution becomes

$$r = -\frac{\left(9\theta^3 e^{-3\theta} + 9\theta^2 e^{-3\theta} + 6 \, e^{-3\theta}\theta + 2 \, e^{-3\theta} + 27c_1\right) e^{3\theta}}{27}$$

Summary

The solution(s) found are the following

$$r = -\frac{\left(9\theta^{3} \mathrm{e}^{-3\theta}+9\theta^{2} \mathrm{e}^{-3\theta}+6\,\mathrm{e}^{-3\theta} \theta +2\,\mathrm{e}^{-3\theta}+27c_{1}\right) \mathrm{e}^{3\theta}}{27} \tag{1}$$

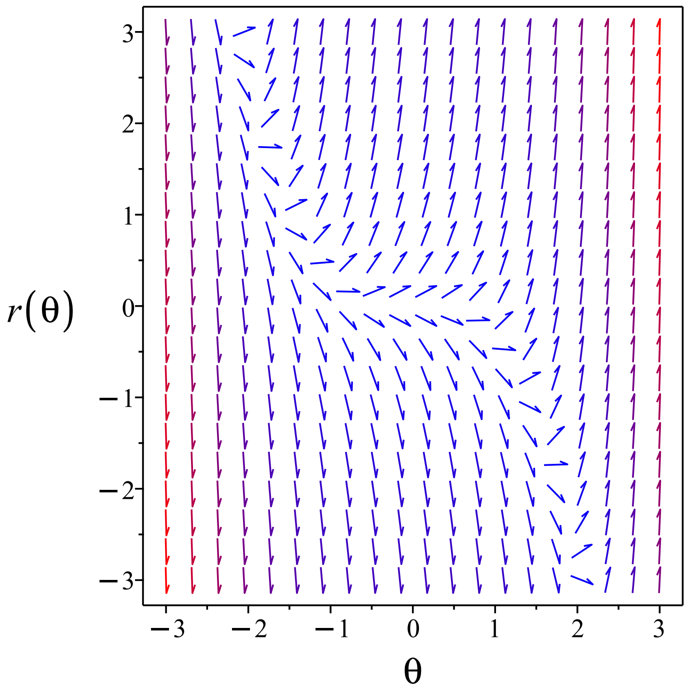


Figure 108: Slope field plot

Verification of solutions

$$r = -\frac{\left(9\theta^{3} \mathrm{e}^{-3\theta}+9\theta^{2} \mathrm{e}^{-3\theta}+6\,\mathrm{e}^{-3\theta} \theta +2\,\mathrm{e}^{-3\theta}+27c_{1}\right) \mathrm{e}^{3\theta}}{27}$$

Verified OK.

2.6.4 Maple step by step solution

Let's solve

$3r - r' = -\theta^{3}$

- Highest derivative means the order of the ODE is 1

  r'

- Isolate the derivative

$r' = 3r + \theta^3$

- Group terms with r on the lhs of the ODE and the rest on the rhs of the ODE

$r' - 3r = \theta^3$

- The ODE is linear; multiply by an integrating factor $\mu(\theta)$

$\mu(\theta)\left(r' - 3r\right) = \mu(\theta)\,\theta^3$

- Assume the lhs of the ODE is the total derivative $\frac{d}{d\theta}(\mu(\theta)\,r)$

$\mu(\theta)\left(r' - 3r\right) = \mu'(\theta)\,r + \mu(\theta)\,r'$

- Isolate $\mu'(\theta)$

$\mu'(\theta) = -3\mu(\theta)$

- Solve to find the integrating factor

$\mu(\theta) = \mathrm{e}^{-3\theta}$

- Integrate both sides with respect to θ

$\int \left(\frac{d}{d\theta}(\mu(\theta)\,r)\right) d\theta = \int \mu(\theta)\,\theta^3 d\theta + c_1$

- Evaluate the integral on the lhs

$\mu(\theta)\,r = \int \mu(\theta)\,\theta^3 d\theta + c_1$

- Solve for r

$r = \frac{\int \mu(\theta)\theta^3 d\theta + c_1}{\mu(\theta)}$

- Substitute $\mu(\theta) = \mathrm{e}^{-3\theta}$

$r = \frac{\int \theta^3 \mathrm{e}^{-3\theta} d\theta + c_1}{\mathrm{e}^{-3\theta}}$

- Evaluate the integrals on the rhs

$r = \frac{-\frac{\left(9\theta^3 + 9\theta^2 + 6\theta + 2\right)\mathrm{e}^{-3\theta}}{27} + c_1}{\mathrm{e}^{-3\theta}}$

- Simplify

$r = -\frac{\theta^3}{3} - \frac{\theta^2}{3} - \frac{2\theta}{9} - \frac{2}{27} + c_1 \mathrm{e}^{3\theta}$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
<- 1st order linear successful`
```

✓ Solution by Maple
Time used: 0.0 (sec). Leaf size: 25

```
dsolve(3*r(theta)=diff(r(theta),theta)-theta^3,r(theta), singsol=all)
```

$$r(\theta) = -\frac{\theta^2}{3} - \frac{\theta^3}{3} - \frac{2\theta}{9} - \frac{2}{27} + e^{3\theta} c_1$$

✓ Solution by Mathematica
Time used: 0.052 (sec). Leaf size: 33

```
DSolve[3*r[\[Theta]]==r'[\[Theta]]-\[Theta]^3,r[\[Theta]],\[Theta],IncludeSingularSolutions
```

$$r(\theta) \to \frac{1}{27}\left(-9\theta^3 - 9\theta^2 - 6\theta - 2\right) + c_1 e^{3\theta}$$

2.7 problem 7

Internal problem ID [4956]
Internal file name [`OUTPUT/4449_Sunday_June_05_2022_02_56_50_PM_52705372/index.tex`]

Book: Fundamentals of Differential Equations. By Nagle, Saff and Snider. 9th edition. Boston. Pearson 2018.
Section: Chapter 2, First order differential equations. Section 2.3, Linear equations. Exercises. page 54
Problem number: 7.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"linear", "exactWithIntegrationFactor", "first_order_ode_lie_symmetry_lookup"**

Maple gives the following as the ode type

```
[[_linear, `class A`]]
```

$$\boxed{y' - y = e^{3x}}$$

2.7.1 Solving as linear ode

Entering Linear first order ODE solver. In canonical form a linear first order is

$$y' + p(x)y = q(x)$$

Where here

$$p(x) = -1$$
$$q(x) = e^{3x}$$

Hence the ode is

$$y' - y = e^{3x}$$

The integrating factor μ is

$$\mu = e^{\int (-1)dx}$$
$$= e^{-x}$$

The ode becomes

$$\frac{d}{dx}(\mu y) = (\mu)\left(e^{3x}\right)$$
$$\frac{d}{dx}\left(e^{-x}y\right) = \left(e^{-x}\right)\left(e^{3x}\right)$$
$$d\left(e^{-x}y\right) = e^{2x}\, dx$$

Integrating gives

$$e^{-x}y = \int e^{2x}\, dx$$
$$e^{-x}y = \frac{e^{2x}}{2} + c_1$$

Dividing both sides by the integrating factor $\mu = e^{-x}$ results in

$$y = \frac{e^{x}e^{2x}}{2} + c_1 e^{x}$$

which simplifies to

$$y = \frac{e^{3x}}{2} + c_1 e^{x}$$

Summary
The solution(s) found are the following

$$y = \frac{e^{3x}}{2} + c_1 e^{x} \tag{1}$$

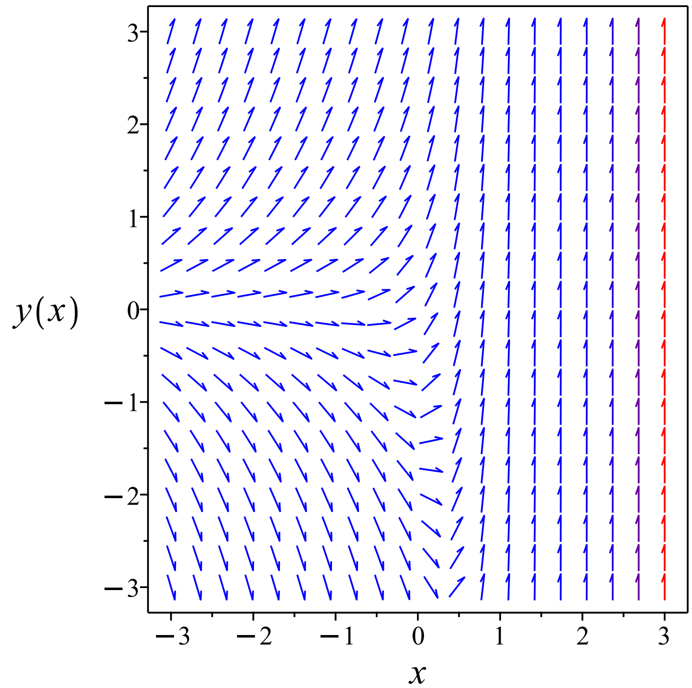


Figure 109: Slope field plot

Verification of solutions

$$y = \frac{e^{3x}}{2} + c_1 e^{x}$$

Verified OK.

2.7.2 Solving as first order ode lie symmetry lookup ode

Writing the ode as

$$y' = y + e^{3x}$$
$$y' = \omega(x, y)$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x + \omega(\eta_y - \xi_x) - \omega^2 \xi_y - \omega_x \xi - \omega_y \eta = 0 \tag{A}$$

The type of this ode is known. It is of type `linear`. Therefore we do not need to solve the PDE (A), and can just use the lookup table shown below to find ξ, η

Table 104: Lie symmetry infinitesimal lookup table for known first order ODE's

ODE class	Form	ξ	η
linear ode	$y' = f(x)y(x) + g(x)$	0	$e^{\int f dx}$
separable ode	$y' = f(x)\, g(y)$	$\frac{1}{f}$	0
quadrature ode	$y' = f(x)$	0	1
quadrature ode	$y' = g(y)$	1	0
homogeneous ODEs of Class A	$y' = f\left(\frac{y}{x}\right)$	x	y
homogeneous ODEs of Class C	$y' = (a + bx + cy)^{\frac{n}{m}}$	1	$-\frac{b}{c}$
homogeneous class D	$y' = \frac{y}{x} + g(x)\, F\left(\frac{y}{x}\right)$	x^2	xy
First order special form ID 1	$y' = g(x)\, e^{h(x)+by} + f(x)$	$\frac{e^{-\int b f(x) dx - h(x)}}{g(x)}$	$\frac{f(x) e^{-\int b f(x) dx - h(x)}}{g(x)}$
polynomial type ode	$y' = \frac{a_1 x + b_1 y + c_1}{a_2 x + b_2 y + c_2}$	$\frac{a_1 b_2 x - a_2 b_1 x - b_1 c_2 + b_2 c_1}{a_1 b_2 - a_2 b_1}$	$\frac{a_1 b_2 y - a_2 b_1 y - a_1 c_2 - a_2 c_1}{a_1 b_2 - a_2 b_1}$
Bernoulli ode	$y' = f(x)\, y + g(x)\, y^n$	0	$e^{-\int (n-1) f(x) dx} y^n$
Reduced Riccati	$y' = f_1(x)\, y + f_2(x)\, y^2$	0	$e^{-\int f_1 dx}$

The above table shows that

$$
\begin{aligned}
\xi(x, y) &= 0 \\
\eta(x, y) &= \mathrm{e}^{x}
\end{aligned}
\tag{A1}
$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x, y) \to (R, S)$ where (R, S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$
\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}
$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y}\right) S(x, y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the

canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$R = x$$

S is found from

$$\begin{aligned} S &= \int \frac{1}{\eta} dy \\ &= \int \frac{1}{e^{x}} dy \end{aligned}$$

Which results in

$$S = e^{-x} y$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x, y) S_y}{R_x + \omega(x, y) R_y} \tag{2}$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x, y)$ is the right hand side of the original ode given by

$$\omega(x, y) = y + e^{3x}$$

Evaluating all the partial derivatives gives

$$\begin{aligned} R_x &= 1 \\ R_y &= 0 \\ S_x &= -e^{-x} y \\ S_y &= e^{-x} \end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = e^{2x} \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = e^{2R}$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by

integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$S(R) = \frac{e^{2R}}{2} + c_1 \tag{4}$$

To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$y\, e^{-x} = \frac{e^{2x}}{2} + c_1$$

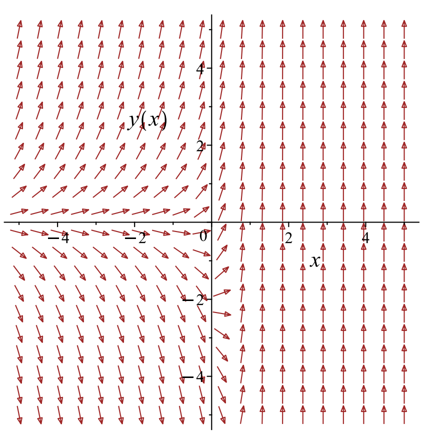
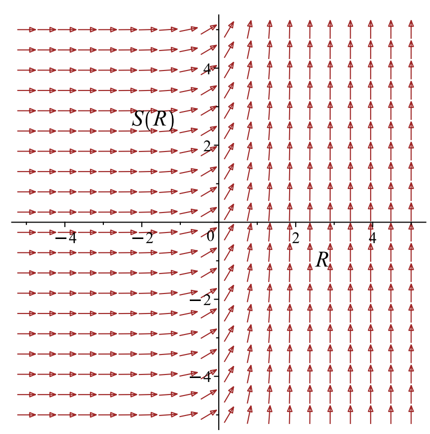
Which simplifies to

$$y\, e^{-x} = \frac{e^{2x}}{2} + c_1$$

Which gives

$$y = \frac{(e^{2x} + 2c_1)\, e^{x}}{2}$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in x, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dx} = y + e^{3x}$ 	$R = x$ $S = e^{-x} y$	$\frac{dS}{dR} = e^{2R}$ 

Summary
The solution(s) found are the following

$$y = \frac{(e^{2x} + 2c_1)\, e^{x}}{2} \tag{1}$$

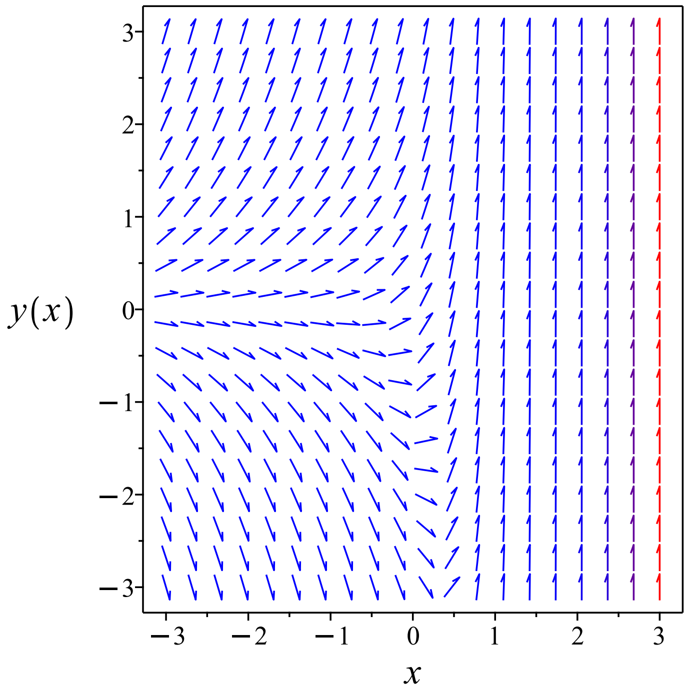


Figure 110: Slope field plot

Verification of solutions

$$y = \frac{(e^{2x} + 2c_1) e^{x}}{2}$$

Verified OK.

2.7.3 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x,y) + N(x,y)\frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x,y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y) = 0$$

Hence

$$\frac{\partial \phi}{\partial x} + \frac{\partial \phi}{\partial y}\frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$
\begin{aligned}
\frac{\partial \phi}{\partial x} &= M \\
\frac{\partial \phi}{\partial y} &= N
\end{aligned}
$$

But since $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ then for the above to be valid, we require that

$$
\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}
$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x, y)$ but at least we know now that we can do that since the condition $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$
M(x, y)\,\mathrm{d}x + N(x, y)\,\mathrm{d}y = 0 \tag{1A}
$$

Therefore

$$
\begin{aligned}
\mathrm{d}y &= \left(y + \mathrm{e}^{3x}\right) \mathrm{d}x \\
\left(-y - \mathrm{e}^{3x}\right) \mathrm{d}x + \mathrm{d}y &= 0 \qquad (2A)
\end{aligned}
$$

Comparing (1A) and (2A) shows that

$$
\begin{aligned}
M(x, y) &= -y - \mathrm{e}^{3x} \\
N(x, y) &= 1
\end{aligned}
$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$
\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}
$$

Using result found above gives

$$
\begin{aligned}
\frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}\left(-y - \mathrm{e}^{3x}\right) \\
&= -1
\end{aligned}
$$

And

$$
\begin{aligned}
\frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}(1) \\
&= 0
\end{aligned}
$$

Since $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$, then the ODE is <u>not exact</u>. Since the ODE is not exact, we will try to find an integrating factor to make it exact. Let

$$
\begin{aligned}
A &= \frac{1}{N}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right) \\
&= 1((-1) - (0)) \\
&= -1
\end{aligned}
$$

Since A does not depend on y, then it can be used to find an integrating factor. The integrating factor μ is

$$
\begin{aligned}
\mu &= e^{\int A \, \mathrm{d}x} \\
&= e^{\int -1 \, \mathrm{d}x}
\end{aligned}
$$

The result of integrating gives

$$
\begin{aligned}
\mu &= e^{-x} \\
&= \mathrm{e}^{-x}
\end{aligned}
$$

M and N are multiplied by this integrating factor, giving new M and new N which are called $\overline{M}$ and $\overline{N}$ for now so not to confuse them with the original M and N.

$$
\begin{aligned}
\overline{M} &= \mu M \\
&= \mathrm{e}^{-x}(-y - \mathrm{e}^{3x}) \\
&= -\mathrm{e}^{-x} y - \mathrm{e}^{2x}
\end{aligned}
$$

And

$$
\begin{aligned}
\overline{N} &= \mu N \\
&= \mathrm{e}^{-x}(1) \\
&= \mathrm{e}^{-x}
\end{aligned}
$$

Now a modified ODE is ontained from the original ODE, which is exact and can be solved. The modified ODE is

$$
\begin{aligned}
\overline{M} + \overline{N}\frac{\mathrm{d}y}{\mathrm{d}x} &= 0 \\
\left(-\mathrm{e}^{-x} y - \mathrm{e}^{2x}\right) + \left(\mathrm{e}^{-x}\right)\frac{\mathrm{d}y}{\mathrm{d}x} &= 0
\end{aligned}
$$

The following equations are now set up to solve for the function $\phi(x, y)$

$$
\frac{\partial \phi}{\partial x} = \overline{M} \tag{1}
$$

$$
\frac{\partial \phi}{\partial y} = \overline{N} \tag{2}
$$

Integrating (1) w.r.t. x gives

$$\int \frac{\partial \phi}{\partial x} \, \mathrm{d}x = \int \overline{M} \, \mathrm{d}x$$
$$\int \frac{\partial \phi}{\partial x} \, \mathrm{d}x = \int -\mathrm{e}^{-x} y - \mathrm{e}^{2x} \, \mathrm{d}x$$
$$\phi = \mathrm{e}^{-x} y - \frac{\mathrm{e}^{2x}}{2} + f(y) \tag{3}$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$\frac{\partial \phi}{\partial y} = \mathrm{e}^{-x} + f'(y) \tag{4}$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = \mathrm{e}^{-x}$. Therefore equation (4) becomes

$$\mathrm{e}^{-x} = \mathrm{e}^{-x} + f'(y) \tag{5}$$

Solving equation (5) for $f'(y)$ gives

$$f'(y) = 0$$

Therefore

$$f(y) = c_1$$

Where c_1 is constant of integration. Substituting this result for $f(y)$ into equation (3) gives ϕ

$$\phi = \mathrm{e}^{-x} y - \frac{\mathrm{e}^{2x}}{2} + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = \mathrm{e}^{-x} y - \frac{\mathrm{e}^{2x}}{2}$$

The solution becomes

$$y = \frac{(\mathrm{e}^{2x} + 2c_1) \, \mathrm{e}^{x}}{2}$$

Summary

The solution(s) found are the following

$$y = \frac{(e^{2x} + 2c_1) e^x}{2} \tag{1}$$

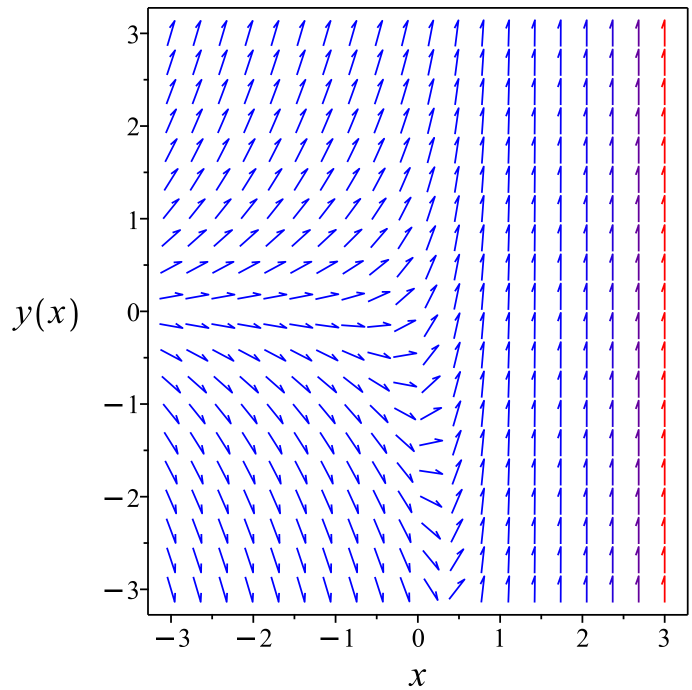


Figure 111: Slope field plot

Verification of solutions

$$y = \frac{(e^{2x} + 2c_1) e^x}{2}$$

Verified OK.

2.7.4 Maple step by step solution

Let's solve

$y' - y = e^{3x}$

- Highest derivative means the order of the ODE is 1

  y'

- Isolate the derivative

$y' = y + e^{3x}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE

$y' - y = e^{3x}$

- The ODE is linear; multiply by an integrating factor $\mu(x)$

$\mu(x)(y' - y) = \mu(x)e^{3x}$

- Assume the lhs of the ODE is the total derivative $\frac{d}{dx}(\mu(x)y)$

$\mu(x)(y' - y) = \mu'(x)y + \mu(x)y'$

- Isolate $\mu'(x)$

$\mu'(x) = -\mu(x)$

- Solve to find the integrating factor

$\mu(x) = e^{-x}$

- Integrate both sides with respect to x

$\int \left(\frac{d}{dx}(\mu(x)y)\right)dx = \int \mu(x)e^{3x}dx + c_1$

- Evaluate the integral on the lhs

$\mu(x)y = \int \mu(x)e^{3x}dx + c_1$

- Solve for y

$y = \frac{\int \mu(x)e^{3x}dx + c_1}{\mu(x)}$

- Substitute $\mu(x) = e^{-x}$

$y = \frac{\int e^{-x}e^{3x}dx + c_1}{e^{-x}}$

- Evaluate the integrals on the rhs

$y = \frac{\frac{e^{2x}}{2} + c_1}{e^{-x}}$

- Simplify

$y = \frac{(e^{2x} + 2c_1)e^{x}}{2}$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
<- 1st order linear successful`
```

✓ Solution by Maple

Time used: 0.015 (sec). Leaf size: 16

```
dsolve(diff(y(x),x)-y(x)-exp(3*x)=0,y(x), singsol=all)
```

$$y(x) = \frac{(e^{2x} + 2c_1) e^x}{2}$$

✓ Solution by Mathematica

Time used: 0.04 (sec). Leaf size: 21

```
DSolve[y'[x]-y[x]-Exp[3*x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \frac{e^{3x}}{2} + c_1 e^x$$

2.8 problem 8

Internal problem ID [4957]
Internal file name [`OUTPUT/4450_Sunday_June_05_2022_02_56_51_PM_99420712/index.tex`]

Book: Fundamentals of Differential Equations. By Nagle, Saff and Snider. 9th edition. Boston. Pearson 2018.
Section: Chapter 2, First order differential equations. Section 2.3, Linear equations. Exercises. page 54
Problem number: 8.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"linear", "homogeneousTypeD2", "exactWithIntegrationFactor", "first_order_ode_lie_symmetry_lookup"**

Maple gives the following as the ode type

`[_linear]`

$$\boxed{y' - \frac{y}{x} = 1 + 2x}$$

2.8.1 Solving as linear ode

Entering Linear first order ODE solver. In canonical form a linear first order is

$$y' + p(x)y = q(x)$$

Where here

$$p(x) = -\frac{1}{x}$$
$$q(x) = 1 + 2x$$

Hence the ode is

$$y' - \frac{y}{x} = 1 + 2x$$

The integrating factor μ is

$$\mu = e^{\int -\frac{1}{x}dx}$$
$$= \frac{1}{x}$$

The ode becomes

$$\frac{d}{dx}(\mu y) = (\mu)(1 + 2x)$$
$$\frac{d}{dx}\left(\frac{y}{x}\right) = \left(\frac{1}{x}\right)(1 + 2x)$$
$$d\left(\frac{y}{x}\right) = \left(\frac{1 + 2x}{x}\right) dx$$

Integrating gives

$$\frac{y}{x} = \int \frac{1 + 2x}{x} dx$$
$$\frac{y}{x} = 2x + \ln(x) + c_1$$

Dividing both sides by the integrating factor $\mu = \frac{1}{x}$ results in

$$y = x(2x + \ln(x)) + c_1 x$$

which simplifies to

$$y = x(2x + \ln(x) + c_1)$$

Summary

The solution(s) found are the following

$$y = x(2x + \ln(x) + c_1) \tag{1}$$

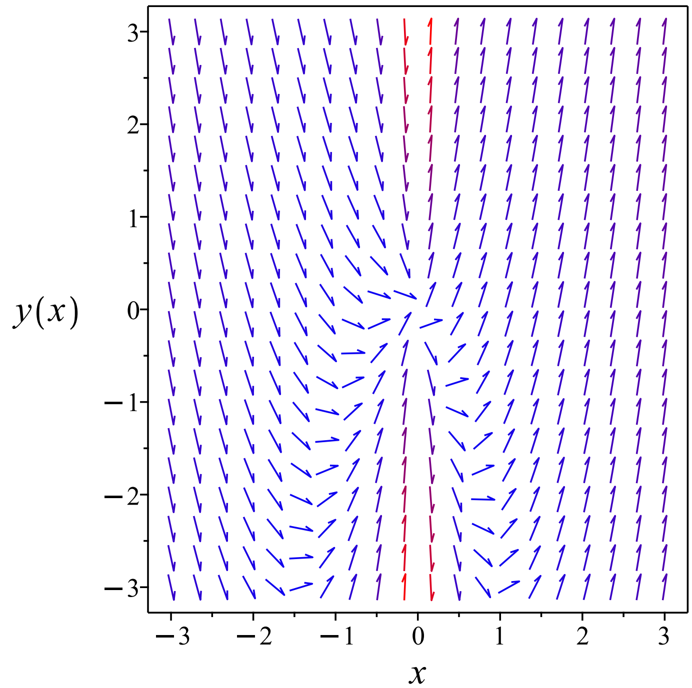


Figure 112: Slope field plot

Verification of solutions

$$y = x(2x + \ln(x) + c_1)$$

Verified OK.

2.8.2 Solving as homogeneousTypeD2 ode

Using the change of variables $y = u(x)\,x$ on the above ode results in new ode in $u(x)$

$$u'(x)\,x = 1 + 2x$$

Integrating both sides gives

$$\begin{aligned} u(x) &= \int \frac{1+2x}{x}\,\mathrm{d}x \\ &= 2x + \ln(x) + c_2 \end{aligned}$$

Therefore the solution y is

$$\begin{aligned} y &= xu \\ &= x(2x + \ln(x) + c_2) \end{aligned}$$

Summary

The solution(s) found are the following

$$y = x(2x + \ln(x) + c_2) \tag{1}$$

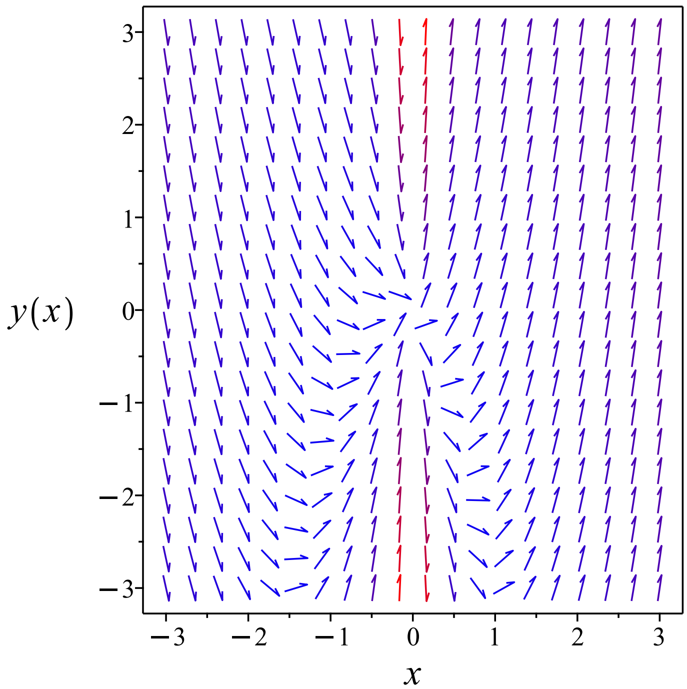


Figure 113: Slope field plot

Verification of solutions

$$y = x(2x + \ln(x) + c_2)$$

Verified OK.

2.8.3 Solving as first order ode lie symmetry lookup ode

Writing the ode as

$$y' = \frac{2x^2 + x + y}{x}$$
$$y' = \omega(x, y)$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x + \omega(\eta_y - \xi_x) - \omega^2 \xi_y - \omega_x \xi - \omega_y \eta = 0 \tag{A}$$

The type of this ode is known. It is of type `linear`. Therefore we do not need to solve the PDE (A), and can just use the lookup table shown below to find ξ, η

Table 107: Lie symmetry infinitesimal lookup table for known first order ODE's

ODE class	Form	ξ	η
linear ode	$y' = f(x)y(x) + g(x)$	0	$e^{\int f dx}$
separable ode	$y' = f(x)\, g(y)$	$\frac{1}{f}$	0
quadrature ode	$y' = f(x)$	0	1
quadrature ode	$y' = g(y)$	1	0
homogeneous ODEs of Class A	$y' = f\left(\frac{y}{x}\right)$	x	y
homogeneous ODEs of Class C	$y' = (a + bx + cy)^{\frac{n}{m}}$	1	$-\frac{b}{c}$
homogeneous class D	$y' = \frac{y}{x} + g(x)\, F\left(\frac{y}{x}\right)$	x^2	xy
First order special form ID 1	$y' = g(x)\, e^{h(x)+by} + f(x)$	$\frac{e^{-\int bf(x)dx-h(x)}}{g(x)}$	$\frac{f(x)e^{-\int bf(x)dx-h(x)}}{g(x)}$
polynomial type ode	$y' = \frac{a_1x+b_1y+c_1}{a_2x+b_2y+c_2}$	$\frac{a_1b_2x-a_2b_1x-b_1c_2+b_2c_1}{a_1b_2-a_2b_1}$	$\frac{a_1b_2y-a_2b_1y-a_1c_2-a_2c_1}{a_1b_2-a_2b_1}$
Bernoulli ode	$y' = f(x)\, y + g(x)\, y^n$	0	$e^{-\int (n-1)f(x)dx} y^n$
Reduced Riccati	$y' = f_1(x)\, y + f_2(x)\, y^2$	0	$e^{-\int f_1 dx}$

The above table shows that

$$
\begin{aligned}
\xi(x, y) &= 0 \\
\eta(x, y) &= x \qquad \text{(A1)}
\end{aligned}
$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x, y) \to (R, S)$ where (R, S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$
\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}
$$

The above comes from the requirements that $\left(\xi\frac{\partial}{\partial x}+\eta\frac{\partial}{\partial y}\right)S(x,y)=1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the canonical coordinates, where $S(R)$. Since $\xi=0$ then in this special case

$$R = x$$

S is found from

$$\begin{aligned}S &= \int \frac{1}{\eta} dy\\ &= \int \frac{1}{x} dy\end{aligned}$$

Which results in

$$S = \frac{y}{x}$$

Now that R,S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x,y)S_y}{R_x + \omega(x,y)R_y} \tag{2}$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x,y)$ is the right hand side of the original ode given by

$$\omega(x,y) = \frac{2x^2 + x + y}{x}$$

Evaluating all the partial derivatives gives

$$\begin{aligned}R_x &= 1\\ R_y &= 0\\ S_x &= -\frac{y}{x^2}\\ S_y &= \frac{1}{x}\end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = \frac{1+2x}{x} \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for x,y in terms of R,S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = \frac{1+2R}{R}$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$S(R) = 2R + \ln(R) + c_1 \tag{4}$$

To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$\frac{y}{x} = 2x + \ln(x) + c_1$$

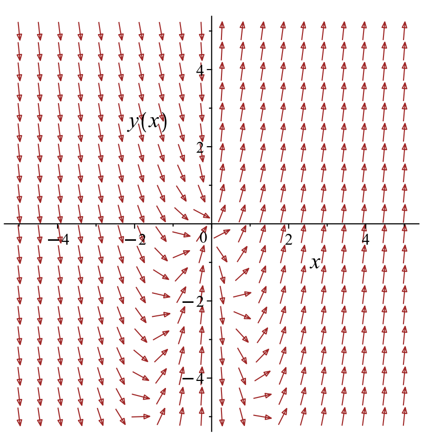
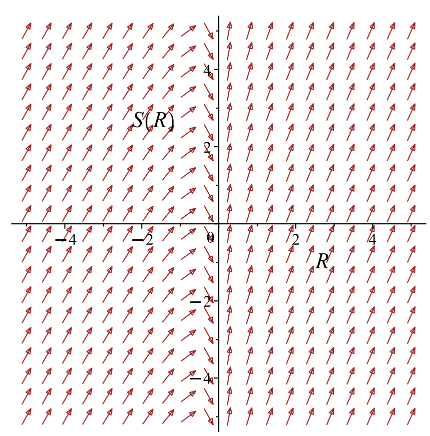
Which simplifies to

$$\frac{y}{x} = 2x + \ln(x) + c_1$$

Which gives

$$y = x(2x + \ln(x) + c_1)$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

<table>
<tr><th>Original ode in x, y coordinates</th><th>Canonical coordinates transformation</th><th>ODE in canonical coordinates (R, S)</th></tr>
<tr><td>$\frac{dy}{dx} = \frac{2x^2+x+y}{x}$

</td><td>$R = x$
$S = \frac{y}{x}$</td><td>$\frac{dS}{dR} = \frac{1+2R}{R}$

</td></tr>
</table>

Summary

The solution(s) found are the following

$$y = x(2x + \ln(x) + c_1) \tag{1}$$

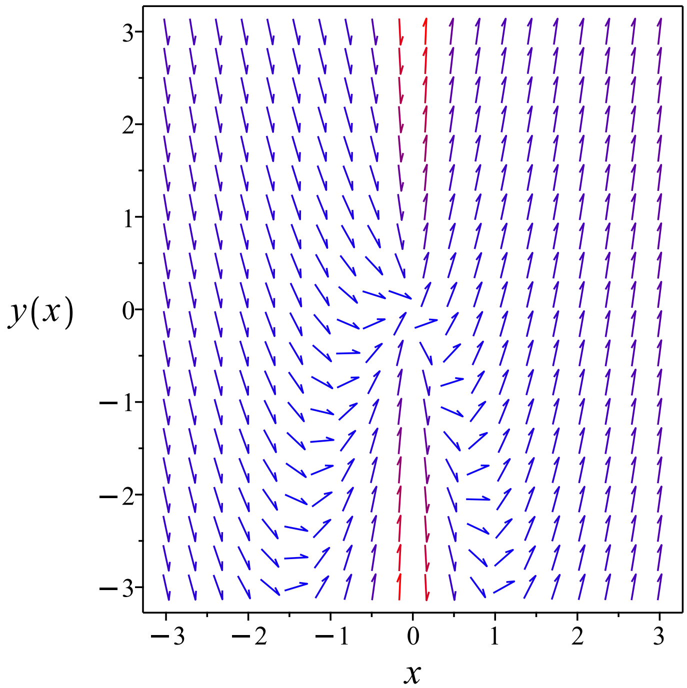


Figure 114: Slope field plot

Verification of solutions

$$y = x(2x + \ln(x) + c_1)$$

Verified OK.

2.8.4 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x, y) + N(x, y)\frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x, y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x, y) = 0$$

Hence

$$\frac{\partial\phi}{\partial x} + \frac{\partial\phi}{\partial y}\frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$
\begin{aligned}
\frac{\partial \phi}{\partial x} &= M \\
\frac{\partial \phi}{\partial y} &= N
\end{aligned}
$$

But since $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ then for the above to be valid, we require that

$$
\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}
$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x, y)$ but at least we know now that we can do that since the condition $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$
M(x, y) \, \mathrm{d}x + N(x, y) \, \mathrm{d}y = 0 \tag{1A}
$$

Therefore

$$
\begin{aligned}
\mathrm{d}y &= \left(\frac{y}{x} + 2x + 1\right) \mathrm{d}x \\
\left(-\frac{y}{x} - 2x - 1\right) \mathrm{d}x + \mathrm{d}y &= 0 \qquad \text{(2A)}
\end{aligned}
$$

Comparing (1A) and (2A) shows that

$$
\begin{aligned}
M(x, y) &= -\frac{y}{x} - 2x - 1 \\
N(x, y) &= 1
\end{aligned}
$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$
\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}
$$

Using result found above gives

$$
\begin{aligned}
\frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}\left(-\frac{y}{x} - 2x - 1\right) \\
&= -\frac{1}{x}
\end{aligned}
$$

And

$$
\begin{aligned}
\frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}(1) \\
&= 0
\end{aligned}
$$

Since $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$, then the ODE is not exact. Since the ODE is not exact, we will try to find an integrating factor to make it exact. Let

$$
\begin{aligned}
A &= \frac{1}{N}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right) \\
&= 1\left(\left(-\frac{1}{x}\right) - (0)\right) \\
&= -\frac{1}{x}
\end{aligned}
$$

Since A does not depend on y, then it can be used to find an integrating factor. The integrating factor μ is

$$
\begin{aligned}
\mu &= e^{\int A \, \mathrm{d}x} \\
&= e^{\int -\frac{1}{x} \, \mathrm{d}x}
\end{aligned}
$$

The result of integrating gives

$$
\begin{aligned}
\mu &= e^{-\ln(x)} \\
&= \frac{1}{x}
\end{aligned}
$$

M and N are multiplied by this integrating factor, giving new M and new N which are called $\overline{M}$ and $\overline{N}$ for now so not to confuse them with the original M and N.

$$
\begin{aligned}
\overline{M} &= \mu M \\
&= \frac{1}{x}\left(-\frac{y}{x} - 2x - 1\right) \\
&= \frac{-2x^2 - x - y}{x^2}
\end{aligned}
$$

And

$$
\begin{aligned}
\overline{N} &= \mu N \\
&= \frac{1}{x}(1) \\
&= \frac{1}{x}
\end{aligned}
$$

Now a modified ODE is ontained from the original ODE, which is exact and can be solved. The modified ODE is

$$\overline{M}+\overline{N}\frac{\mathrm{d}y}{\mathrm{d}x}=0$$
$$\left(\frac{-2x^2-x-y}{x^2}\right)+\left(\frac{1}{x}\right)\frac{\mathrm{d}y}{\mathrm{d}x}=0$$

The following equations are now set up to solve for the function $\phi(x,y)$

$$\frac{\partial\phi}{\partial x}=\overline{M} \tag{1}$$
$$\frac{\partial\phi}{\partial y}=\overline{N} \tag{2}$$

Integrating (1) w.r.t. x gives

$$\int\frac{\partial\phi}{\partial x}\,\mathrm{d}x=\int\overline{M}\,\mathrm{d}x$$
$$\int\frac{\partial\phi}{\partial x}\,\mathrm{d}x=\int\frac{-2x^2-x-y}{x^2}\,\mathrm{d}x$$
$$\phi=-2x-\ln(x)+\frac{y}{x}+f(y) \tag{3}$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$\frac{\partial\phi}{\partial y}=\frac{1}{x}+f'(y) \tag{4}$$

But equation (2) says that $\frac{\partial\phi}{\partial y}=\frac{1}{x}$. Therefore equation (4) becomes

$$\frac{1}{x}=\frac{1}{x}+f'(y) \tag{5}$$

Solving equation (5) for $f'(y)$ gives

$$f'(y)=0$$

Therefore

$$f(y)=c_1$$

Where c_1 is constant of integration. Substituting this result for $f(y)$ into equation (3) gives ϕ

$$\phi = -2x - \ln(x) + \frac{y}{x} + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = -2x - \ln(x) + \frac{y}{x}$$

The solution becomes

$$y = x(2x + \ln(x) + c_1)$$

Summary

The solution(s) found are the following

$$y = x(2x + \ln(x) + c_1) \tag{1}$$

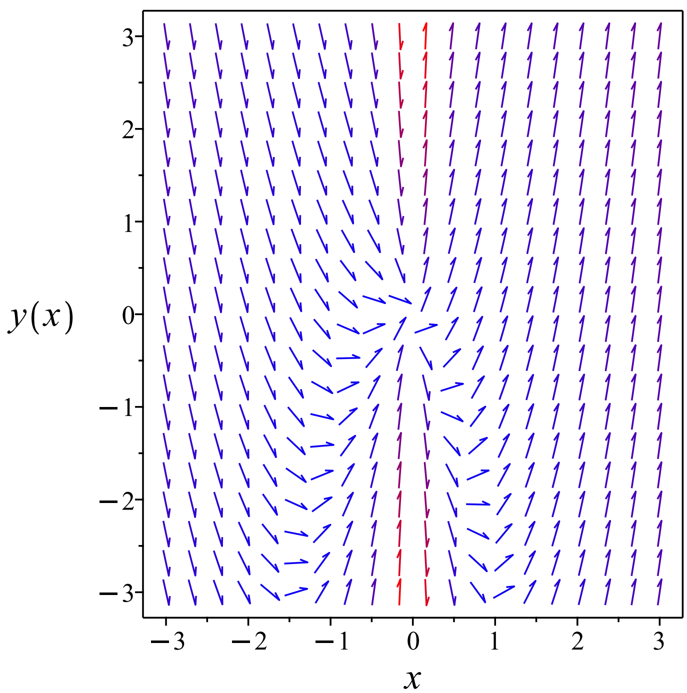


Figure 115: Slope field plot

Verification of solutions

$$y = x(2x + \ln(x) + c_1)$$

Verified OK.

2.8.5 Maple step by step solution

Let's solve

$y' - \frac{y}{x} = 1 + 2x$

- Highest derivative means the order of the ODE is 1

  y'

- Isolate the derivative

  $y' = \frac{y}{x} + 2x + 1$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE

  $y' - \frac{y}{x} = 1 + 2x$

- The ODE is linear; multiply by an integrating factor $\mu(x)$

  $\mu(x)\left(y' - \frac{y}{x}\right) = \mu(x)(1 + 2x)$

- Assume the lhs of the ODE is the total derivative $\frac{d}{dx}(\mu(x)\,y)$

  $\mu(x)\left(y' - \frac{y}{x}\right) = \mu'(x)\,y + \mu(x)\,y'$

- Isolate $\mu'(x)$

  $\mu'(x) = -\frac{\mu(x)}{x}$

- Solve to find the integrating factor

  $\mu(x) = \frac{1}{x}$

- Integrate both sides with respect to x

  $\int \left(\frac{d}{dx}(\mu(x)\,y)\right) dx = \int \mu(x)(1 + 2x)\,dx + c_1$

- Evaluate the integral on the lhs

  $\mu(x)\,y = \int \mu(x)(1 + 2x)\,dx + c_1$

- Solve for y

  $y = \frac{\int \mu(x)(1+2x)dx + c_1}{\mu(x)}$

- Substitute $\mu(x) = \frac{1}{x}$

  $y = x\left(\int \frac{1+2x}{x}dx + c_1\right)$

- Evaluate the integrals on the rhs

  $y = x(2x + \ln(x) + c_1)$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
<- 1st order linear successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 13

```
dsolve(diff(y(x),x)=y(x)/x+2*x+1,y(x), singsol=all)
```

$$y(x) = (2x + \ln(x) + c_1)\, x$$

✓ Solution by Mathematica

Time used: 0.025 (sec). Leaf size: 15

```
DSolve[y'[x]==y[x]/x+2*x+1,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to x(2x + \log(x) + c_1)$$

2.9 problem 9

Internal problem ID [4958]
Internal file name [`OUTPUT/4451_Sunday_June_05_2022_02_56_52_PM_93686659/index.tex`]

Book: Fundamentals of Differential Equations. By Nagle, Saff and Snider. 9th edition. Boston. Pearson 2018.
Section: Chapter 2, First order differential equations. Section 2.3, Linear equations. Exercises. page 54
Problem number: 9.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"linear", "exactWithIntegrationFactor", "first_order_ode_lie_symmetry_lookup"**

Maple gives the following as the ode type

`[_linear]`

$$\boxed{r' + \tan(\theta) r = \sec(\theta)}$$

2.9.1 Solving as linear ode

Entering Linear first order ODE solver. In canonical form a linear first order is

$$r' + p(\theta)r = q(\theta)$$

Where here

$$p(\theta) = \tan(\theta)$$
$$q(\theta) = \sec(\theta)$$

Hence the ode is

$$r' + \tan(\theta) r = \sec(\theta)$$

The integrating factor μ is

$$\mu = e^{\int \tan(\theta)d\theta}$$
$$= \frac{1}{\cos(\theta)}$$

Which simplifies to

$$\mu = \sec(\theta)$$

The ode becomes

$$\frac{d}{d\theta}(\mu r) = (\mu)(\sec(\theta))$$
$$\frac{d}{d\theta}(r\sec(\theta)) = (\sec(\theta))(\sec(\theta))$$
$$d(r\sec(\theta)) = \sec(\theta)^2\, d\theta$$

Integrating gives

$$r\sec(\theta) = \int \sec(\theta)^2\, d\theta$$
$$r\sec(\theta) = \tan(\theta) + c_1$$

Dividing both sides by the integrating factor $\mu = \sec(\theta)$ results in

$$r = \cos(\theta)\tan(\theta) + c_1\cos(\theta)$$

which simplifies to

$$r = c_1\cos(\theta) + \sin(\theta)$$

Summary

The solution(s) found are the following

$$r = c_1\cos(\theta) + \sin(\theta) \tag{1}$$

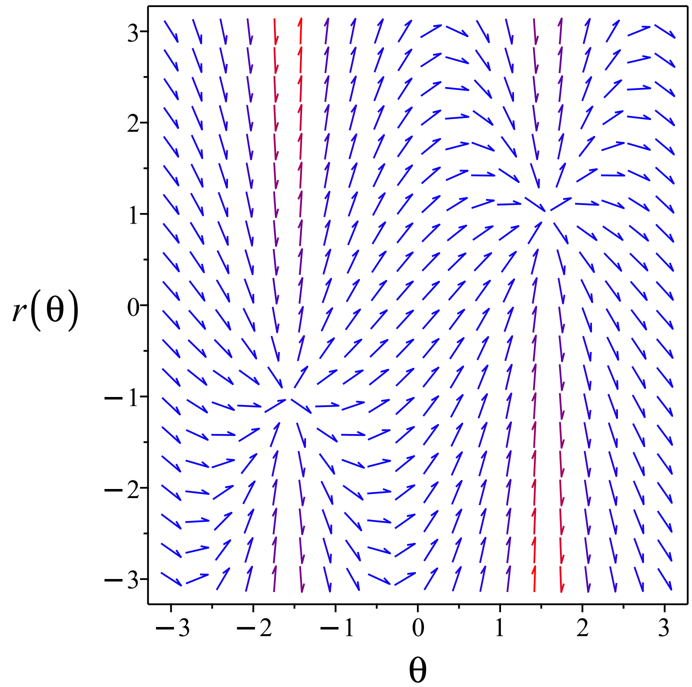


Figure 116: Slope field plot

Verification of solutions

$$r = c_1 \cos(\theta) + \sin(\theta)$$

Verified OK.

2.9.2 Solving as first order ode lie symmetry lookup ode

Writing the ode as

$$\begin{aligned} r' &= -\tan(\theta)\, r + \sec(\theta) \\ r' &= \omega(\theta, r) \end{aligned}$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_\theta + \omega(\eta_r - \xi_\theta) - \omega^2 \xi_r - \omega_\theta \xi - \omega_r \eta = 0 \tag{A}$$

The type of this ode is known. It is of type `linear`. Therefore we do not need to solve the PDE (A), and can just use the lookup table shown below to find ξ, η

Table 110: Lie symmetry infinitesimal lookup table for known first order ODE's

ODE class	Form	ξ	η
linear ode	$y' = f(x)y(x) + g(x)$	0	$e^{\int f dx}$
separable ode	$y' = f(x)\,g(y)$	$\frac{1}{f}$	0
quadrature ode	$y' = f(x)$	0	1
quadrature ode	$y' = g(y)$	1	0
homogeneous ODEs of Class A	$y' = f\left(\frac{y}{x}\right)$	x	y
homogeneous ODEs of Class C	$y' = (a + bx + cy)^{\frac{n}{m}}$	1	$-\frac{b}{c}$
homogeneous class D	$y' = \frac{y}{x} + g(x)\,F\left(\frac{y}{x}\right)$	x^2	xy
First order special form ID 1	$y' = g(x)\,e^{h(x)+by} + f(x)$	$\frac{e^{-\int bf(x)dx-h(x)}}{g(x)}$	$\frac{f(x)e^{-\int bf(x)dx-h(x)}}{g(x)}$
polynomial type ode	$y' = \frac{a_1x+b_1y+c_1}{a_2x+b_2y+c_2}$	$\frac{a_1b_2x-a_2b_1x-b_1c_2+b_2c_1}{a_1b_2-a_2b_1}$	$\frac{a_1b_2y-a_2b_1y-a_1c_2-a_2c_1}{a_1b_2-a_2b_1}$
Bernoulli ode	$y' = f(x)\,y + g(x)\,y^n$	0	$e^{-\int (n-1)f(x)dx}y^n$
Reduced Riccati	$y' = f_1(x)\,y + f_2(x)\,y^2$	0	$e^{-\int f_1 dx}$

The above table shows that

$$
\begin{aligned}
\xi(\theta, r) &= 0 \\
\eta(\theta, r) &= \cos(\theta) \qquad \text{(A1)}
\end{aligned}
$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(\theta, r) \to (R, S)$ where (R, S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$
\frac{d\theta}{\xi} = \frac{dr}{\eta} = dS \tag{1}
$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial \theta} + \eta \frac{\partial}{\partial r}\right) S(\theta, r) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the

canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$R = \theta$$

S is found from

$$\begin{aligned} S &= \int \frac{1}{\eta} dy \\ &= \int \frac{1}{\cos(\theta)} dy \end{aligned}$$

Which results in

$$S = \frac{r}{\cos(\theta)}$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_\theta + \omega(\theta, r) S_r}{R_\theta + \omega(\theta, r) R_r} \tag{2}$$

Where in the above $R_\theta, R_r, S_\theta, S_r$ are all partial derivatives and $\omega(\theta, r)$ is the right hand side of the original ode given by

$$\omega(\theta, r) = -\tan(\theta) r + \sec(\theta)$$

Evaluating all the partial derivatives gives

$$\begin{aligned} R_\theta &= 1 \\ R_r &= 0 \\ S_\theta &= r \sec(\theta) \tan(\theta) \\ S_r &= \sec(\theta) \end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = \sec(\theta)^2 \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for θ, r in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = \sec(R)^2$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by

integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$S(R) = \tan(R) + c_1 \tag{4}$$

To complete the solution, we just need to transform (4) back to θ, r coordinates. This results in

$$r \sec(\theta) = \tan(\theta) + c_1$$

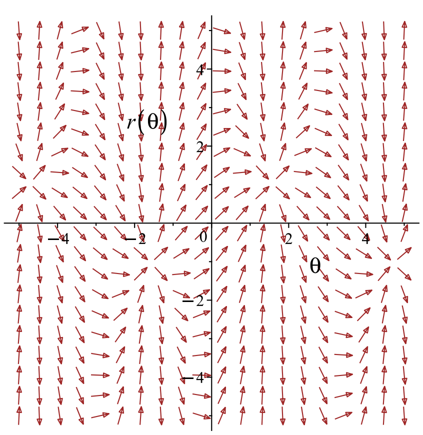
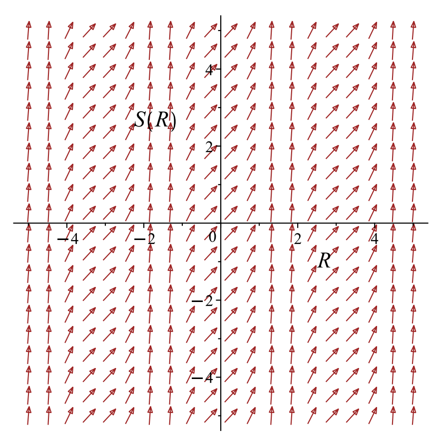
Which simplifies to

$$r \sec(\theta) = \tan(\theta) + c_1$$

Which gives

$$r = \frac{\tan(\theta) + c_1}{\sec(\theta)}$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in θ, r coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dr}{d\theta} = -\tan(\theta) r + \sec(\theta)$ 	$R = \theta$ $S = r\sec(\theta)$	$\frac{dS}{dR} = \sec(R)^2$ 

Summary
The solution(s) found are the following

$$r = \frac{\tan(\theta) + c_1}{\sec(\theta)} \tag{1}$$

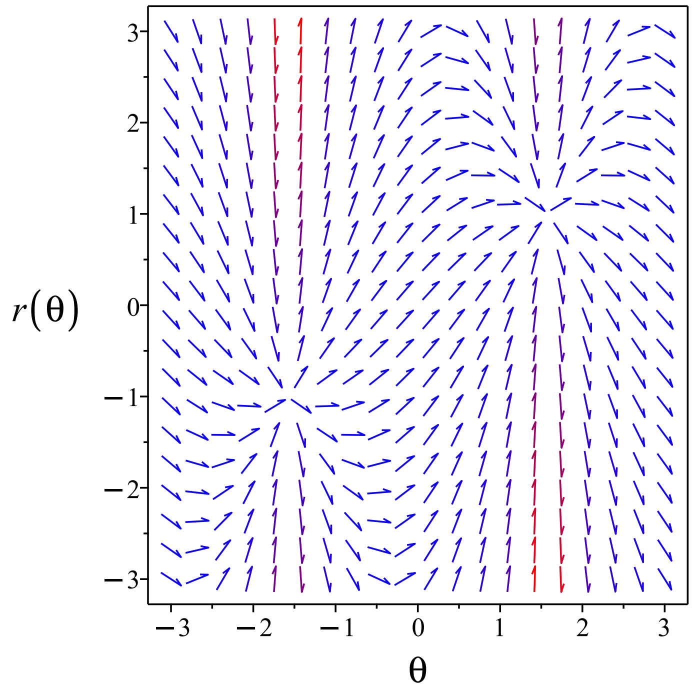


Figure 117: Slope field plot

Verification of solutions

$$r = \frac{\tan(\theta) + c_1}{\sec(\theta)}$$

Verified OK.

2.9.3 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x, y) + N(x, y)\frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x, y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x, y) = 0$$

Hence

$$\frac{\partial \phi}{\partial x} + \frac{\partial \phi}{\partial y}\frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$\begin{aligned}\frac{\partial \phi}{\partial x} &= M\\ \frac{\partial \phi}{\partial y} &= N\end{aligned}$$

But since $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x, y)$ but at least we know now that we can do that since the condition $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(\theta, r)\,\mathrm{d}\theta + N(\theta, r)\,\mathrm{d}r = 0 \tag{1A}$$

Therefore

$$\begin{aligned}\mathrm{d}r &= (-\tan(\theta)\, r + \sec(\theta))\,\mathrm{d}\theta\\ (\tan(\theta)\, r - \sec(\theta))\,\mathrm{d}\theta + \mathrm{d}r &= 0 \qquad (2A)\end{aligned}$$

Comparing (1A) and (2A) shows that

$$\begin{aligned}M(\theta, r) &= \tan(\theta)\, r - \sec(\theta)\\ N(\theta, r) &= 1\end{aligned}$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial r} = \frac{\partial N}{\partial \theta}$$

Using result found above gives

$$\begin{aligned}\frac{\partial M}{\partial r} &= \frac{\partial}{\partial r}(\tan(\theta)\, r - \sec(\theta))\\ &= \tan(\theta)\end{aligned}$$

And

$$\begin{aligned}\frac{\partial N}{\partial \theta} &= \frac{\partial}{\partial \theta}(1)\\ &= 0\end{aligned}$$

Since $\frac{\partial M}{\partial r} \neq \frac{\partial N}{\partial \theta}$, then the ODE is not exact. Since the ODE is not exact, we will try to find an integrating factor to make it exact. Let

$$
\begin{aligned}
A &= \frac{1}{N}\left(\frac{\partial M}{\partial r} - \frac{\partial N}{\partial \theta}\right) \\
&= 1((\tan(\theta)) - (0)) \\
&= \tan(\theta)
\end{aligned}
$$

Since A does not depend on r, then it can be used to find an integrating factor. The integrating factor μ is

$$
\begin{aligned}
\mu &= e^{\int A \, \mathrm{d}\theta} \\
&= e^{\int \tan(\theta) \, \mathrm{d}\theta}
\end{aligned}
$$

The result of integrating gives

$$
\begin{aligned}
\mu &= e^{-\ln(\cos(\theta))} \\
&= \sec(\theta)
\end{aligned}
$$

M and N are multiplied by this integrating factor, giving new M and new N which are called $\overline{M}$ and $\overline{N}$ for now so not to confuse them with the original M and N.

$$
\begin{aligned}
\overline{M} &= \mu M \\
&= \sec(\theta)\left(\tan(\theta) r - \sec(\theta)\right) \\
&= \left(\sin(\theta) r - 1\right) \sec(\theta)^2
\end{aligned}
$$

And

$$
\begin{aligned}
\overline{N} &= \mu N \\
&= \sec(\theta)(1) \\
&= \sec(\theta)
\end{aligned}
$$

Now a modified ODE is ontained from the original ODE, which is exact and can be solved. The modified ODE is

$$
\begin{aligned}
\overline{M} + \overline{N} \frac{\mathrm{d}r}{\mathrm{d}\theta} &= 0 \\
\left(\left(\sin(\theta) r - 1\right) \sec(\theta)^2\right) + \left(\sec(\theta)\right) \frac{\mathrm{d}r}{\mathrm{d}\theta} &= 0
\end{aligned}
$$

The following equations are now set up to solve for the function $\phi(\theta, r)$

$$
\frac{\partial \phi}{\partial \theta} = \overline{M} \tag{1}
$$

$$
\frac{\partial \phi}{\partial r} = \overline{N} \tag{2}
$$

Integrating (1) w.r.t. θ gives

$$\int \frac{\partial \phi}{\partial \theta} \,\mathrm{d}\theta = \int \overline{M} \,\mathrm{d}\theta$$

$$\int \frac{\partial \phi}{\partial \theta} \,\mathrm{d}\theta = \int (\sin(\theta)\, r - 1) \sec(\theta)^2 \,\mathrm{d}\theta$$

$$\phi = r \sec(\theta) - \tan(\theta) + f(r) \tag{3}$$

Where $f(r)$ is used for the constant of integration since ϕ is a function of both θ and r. Taking derivative of equation (3) w.r.t r gives

$$\frac{\partial \phi}{\partial r} = \sec(\theta) + f'(r) \tag{4}$$

But equation (2) says that $\frac{\partial \phi}{\partial r} = \sec(\theta)$. Therefore equation (4) becomes

$$\sec(\theta) = \sec(\theta) + f'(r) \tag{5}$$

Solving equation (5) for $f'(r)$ gives

$$f'(r) = 0$$

Therefore

$$f(r) = c_1$$

Where c_1 is constant of integration. Substituting this result for $f(r)$ into equation (3) gives ϕ

$$\phi = r \sec(\theta) - \tan(\theta) + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = r \sec(\theta) - \tan(\theta)$$

The solution becomes

$$r = \frac{\tan(\theta) + c_1}{\sec(\theta)}$$

Summary

The solution(s) found are the following

$$r = \frac{\tan(\theta) + c_1}{\sec(\theta)} \tag{1}$$

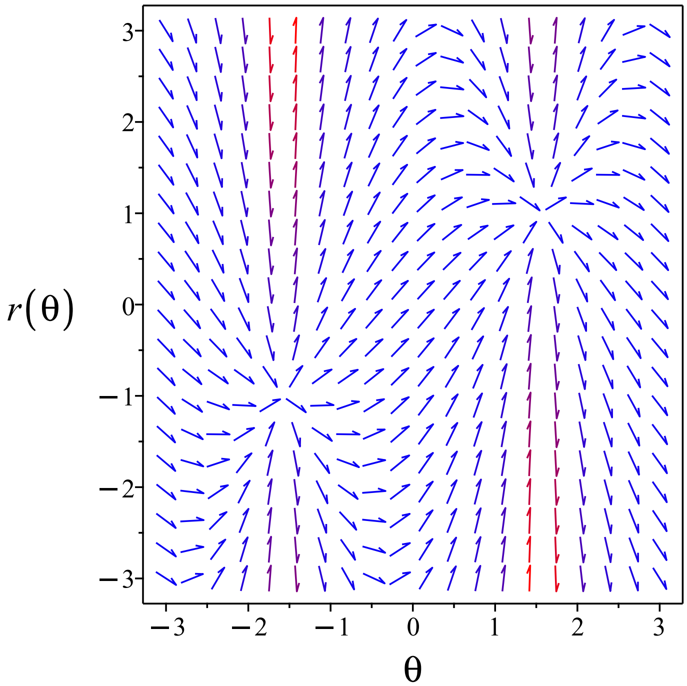


Figure 118: Slope field plot

Verification of solutions

$$r = \frac{\tan(\theta) + c_1}{\sec(\theta)}$$

Verified OK.

2.9.4 Maple step by step solution

Let's solve

$r' + \tan(\theta) r = \sec(\theta)$

- Highest derivative means the order of the ODE is 1

  r'

- Isolate the derivative

$r' = -\tan(\theta) r + \sec(\theta)$

- Group terms with r on the lhs of the ODE and the rest on the rhs of the ODE

  $r' + \tan(\theta) r = \sec(\theta)$

- The ODE is linear; multiply by an integrating factor $\mu(\theta)$

  $\mu(\theta) (r' + \tan(\theta) r) = \mu(\theta) \sec(\theta)$

- Assume the lhs of the ODE is the total derivative $\frac{d}{d\theta}(\mu(\theta) r)$

  $\mu(\theta) (r' + \tan(\theta) r) = \mu'(\theta) r + \mu(\theta) r'$

- Isolate $\mu'(\theta)$

  $\mu'(\theta) = \mu(\theta) \tan(\theta)$

- Solve to find the integrating factor

  $\mu(\theta) = \frac{1}{\cos(\theta)}$

- Integrate both sides with respect to θ

  $\int \left(\frac{d}{d\theta}(\mu(\theta) r)\right) d\theta = \int \mu(\theta) \sec(\theta) d\theta + c_1$

- Evaluate the integral on the lhs

  $\mu(\theta) r = \int \mu(\theta) \sec(\theta) d\theta + c_1$

- Solve for r

  $r = \frac{\int \mu(\theta) \sec(\theta) d\theta + c_1}{\mu(\theta)}$

- Substitute $\mu(\theta) = \frac{1}{\cos(\theta)}$

  $r = \cos(\theta) \left(\int \frac{\sec(\theta)}{\cos(\theta)} d\theta + c_1\right)$

- Evaluate the integrals on the rhs

  $r = \cos(\theta) (\tan(\theta) + c_1)$

- Simplify

  $r = c_1 \cos(\theta) + \sin(\theta)$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
<- 1st order linear successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 11

```
dsolve(diff(r(theta),theta)+r(theta)*tan(theta)=sec(theta),r(theta), singsol=all)
```

$$r(\theta) = \cos(\theta) c_1 + \sin(\theta)$$

✓ Solution by Mathematica

Time used: 0.048 (sec). Leaf size: 13

```
DSolve[r'[\[Theta]]+r[\[Theta]]*Tan[\[Theta]]==Sec[\[Theta]],r[\[Theta]],\[Theta],IncludeSin
```

$$r(\theta) \to \sin(\theta) + c_1 \cos(\theta)$$

2.10 problem 10

Internal problem ID [4959]
Internal file name [`OUTPUT/4452_Sunday_June_05_2022_02_56_53_PM_48604271/index.tex`]

Book: Fundamentals of Differential Equations. By Nagle, Saff and Snider. 9th edition. Boston. Pearson 2018.
Section: Chapter 2, First order differential equations. Section 2.3, Linear equations. Exercises. page 54
Problem number: 10.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"linear", "exactWithIntegrationFactor", "first_order_ode_lie_symmetry_lookup"**

Maple gives the following as the ode type

`[_linear]`

$$\boxed{xy' + 2y = \frac{1}{x^3}}$$

2.10.1 Solving as linear ode

Entering Linear first order ODE solver. In canonical form a linear first order is

$$y' + p(x)y = q(x)$$

Where here

$$p(x) = \frac{2}{x}$$
$$q(x) = \frac{1}{x^4}$$

Hence the ode is

$$y' + \frac{2y}{x} = \frac{1}{x^4}$$

The integrating factor μ is

$$\mu = e^{\int \frac{2}{x}dx}$$
$$= x^2$$

The ode becomes

$$\frac{d}{dx}(\mu y) = (\mu)\left(\frac{1}{x^4}\right)$$
$$\frac{d}{dx}(y\,x^2) = (x^2)\left(\frac{1}{x^4}\right)$$
$$d(y\,x^2) = \frac{1}{x^2}\,dx$$

Integrating gives

$$y\,x^2 = \int \frac{1}{x^2}\,dx$$
$$y\,x^2 = -\frac{1}{x} + c_1$$

Dividing both sides by the integrating factor $\mu = x^2$ results in

$$y = -\frac{1}{x^3} + \frac{c_1}{x^2}$$

which simplifies to

$$y = \frac{c_1 x - 1}{x^3}$$

Summary

The solution(s) found are the following

$$y = \frac{c_1 x - 1}{x^3} \tag{1}$$

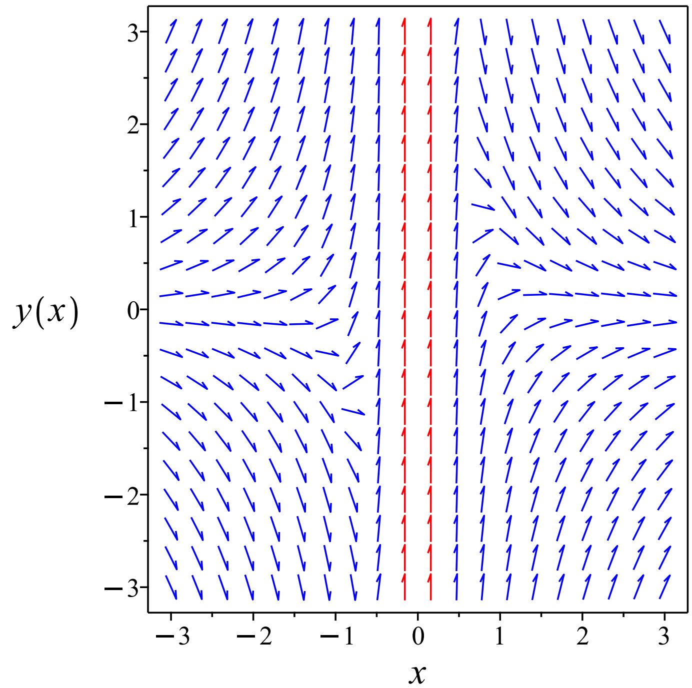


Figure 119: Slope field plot

Verification of solutions

$$y = \frac{c_1 x - 1}{x^3}$$

Verified OK.

2.10.2 Solving as first order ode lie symmetry lookup ode

Writing the ode as

$$y' = -\frac{2y\,x^3 - 1}{x^4}$$
$$y' = \omega(x, y)$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x + \omega(\eta_y - \xi_x) - \omega^2 \xi_y - \omega_x \xi - \omega_y \eta = 0 \tag{A}$$

The type of this ode is known. It is of type `linear`. Therefore we do not need to solve the PDE (A), and can just use the lookup table shown below to find ξ, η

Table 113: Lie symmetry infinitesimal lookup table for known first order ODE's

ODE class	Form	ξ	η
linear ode	$y' = f(x)y(x) + g(x)$	0	$e^{\int f dx}$
separable ode	$y' = f(x)\, g(y)$	$\frac{1}{f}$	0
quadrature ode	$y' = f(x)$	0	1
quadrature ode	$y' = g(y)$	1	0
homogeneous ODEs of Class A	$y' = f\left(\frac{y}{x}\right)$	x	y
homogeneous ODEs of Class C	$y' = (a + bx + cy)^{\frac{n}{m}}$	1	$-\frac{b}{c}$
homogeneous class D	$y' = \frac{y}{x} + g(x)\, F\left(\frac{y}{x}\right)$	x^2	xy
First order special form ID 1	$y' = g(x)\, e^{h(x)+by} + f(x)$	$\frac{e^{-\int bf(x)dx - h(x)}}{g(x)}$	$\frac{f(x)e^{-\int bf(x)dx - h(x)}}{g(x)}$
polynomial type ode	$y' = \frac{a_1x+b_1y+c_1}{a_2x+b_2y+c_2}$	$\frac{a_1b_2x - a_2b_1x - b_1c_2 + b_2c_1}{a_1b_2 - a_2b_1}$	$\frac{a_1b_2y - a_2b_1y - a_1c_2 - a_2c_1}{a_1b_2 - a_2b_1}$
Bernoulli ode	$y' = f(x)\, y + g(x)\, y^n$	0	$e^{-\int (n-1)f(x)dx} y^n$
Reduced Riccati	$y' = f_1(x)\, y + f_2(x)\, y^2$	0	$e^{-\int f_1 dx}$

The above table shows that

$$
\begin{aligned}
\xi(x, y) &= 0 \\
\eta(x, y) &= \frac{1}{x^2}
\end{aligned} \tag{A1}
$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x, y) \to (R, S)$ where (R, S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$
\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}
$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y}\right) S(x, y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the

canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$R = x$$

S is found from

$$\begin{aligned} S &= \int \frac{1}{\eta} dy \\ &= \int \frac{1}{\frac{1}{x^2}} dy \end{aligned}$$

Which results in

$$S = y\,x^2$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x, y) S_y}{R_x + \omega(x, y) R_y} \tag{2}$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x, y)$ is the right hand side of the original ode given by

$$\omega(x, y) = -\frac{2y\,x^3 - 1}{x^4}$$

Evaluating all the partial derivatives gives

$$\begin{aligned} R_x &= 1 \\ R_y &= 0 \\ S_x &= 2xy \\ S_y &= x^2 \end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = \frac{1}{x^2} \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = \frac{1}{R^2}$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by

integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$S(R) = -\frac{1}{R} + c_1 \tag{4}$$

To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$yx^2 = c_1 - \frac{1}{x}$$

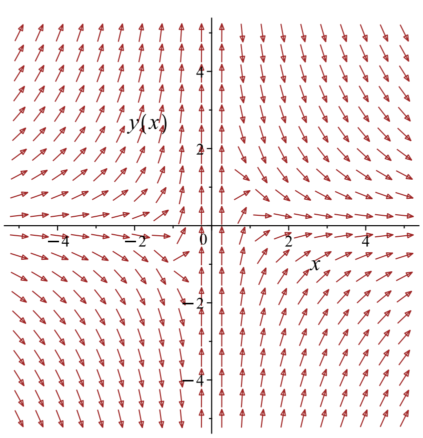
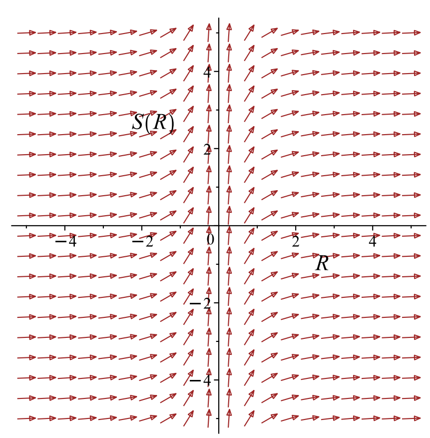
Which simplifies to

$$yx^2 = c_1 - \frac{1}{x}$$

Which gives

$$y = \frac{c_1 x - 1}{x^3}$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in x, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dx} = -\frac{2y\,x^3-1}{x^4}$ 	$R = x$ $S = y\,x^2$	$\frac{dS}{dR} = \frac{1}{R^2}$ 

Summary

The solution(s) found are the following

$$y = \frac{c_1 x - 1}{x^3} \tag{1}$$

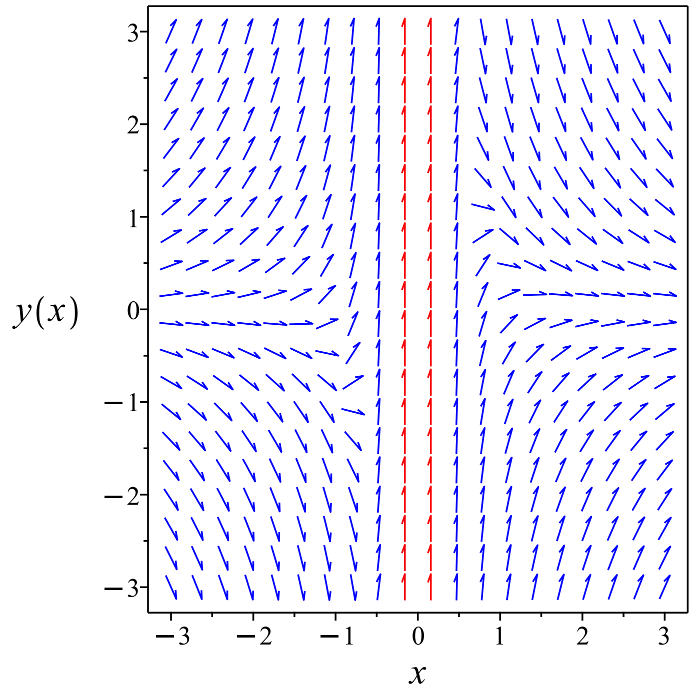


Figure 120: Slope field plot

<u>Verification of solutions</u>

$$y = \frac{c_1 x - 1}{x^3}$$

Verified OK.

2.10.3 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x,y) + N(x,y)\frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x,y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y) = 0$$

Hence

$$\frac{\partial \phi}{\partial x} + \frac{\partial \phi}{\partial y}\frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$
\begin{aligned}
\frac{\partial \phi}{\partial x} &= M \\
\frac{\partial \phi}{\partial y} &= N
\end{aligned}
$$

But since $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ then for the above to be valid, we require that

$$
\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}
$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x, y)$ but at least we know now that we can do that since the condition $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$
M(x, y)\,\mathrm{d}x + N(x, y)\,\mathrm{d}y = 0 \tag{1A}
$$

Therefore

$$
\begin{aligned}
(x)\,\mathrm{d}y &= \left(-2y + \frac{1}{x^3}\right)\mathrm{d}x \\
\left(2y - \frac{1}{x^3}\right)\mathrm{d}x + (x)\,\mathrm{d}y &= 0 \qquad (2A)
\end{aligned}
$$

Comparing (1A) and (2A) shows that

$$
\begin{aligned}
M(x, y) &= 2y - \frac{1}{x^3} \\
N(x, y) &= x
\end{aligned}
$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$
\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}
$$

Using result found above gives

$$
\begin{aligned}
\frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}\left(2y - \frac{1}{x^3}\right) \\
&= 2
\end{aligned}
$$

And

$$
\begin{aligned}
\frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}(x) \\
&= 1
\end{aligned}
$$

Since $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$, then the ODE is <u>not exact</u>. Since the ODE is not exact, we will try to find an integrating factor to make it exact. Let

$$
\begin{aligned}
A &= \frac{1}{N}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right) \\
&= \frac{1}{x}((2) - (1)) \\
&= \frac{1}{x}
\end{aligned}
$$

Since A does not depend on y, then it can be used to find an integrating factor. The integrating factor μ is

$$
\begin{aligned}
\mu &= e^{\int A \, \mathrm{d}x} \\
&= e^{\int \frac{1}{x} \, \mathrm{d}x}
\end{aligned}
$$

The result of integrating gives

$$
\begin{aligned}
\mu &= e^{\ln(x)} \\
&= x
\end{aligned}
$$

M and N are multiplied by this integrating factor, giving new M and new N which are called $\overline{M}$ and $\overline{N}$ for now so not to confuse them with the original M and N.

$$
\begin{aligned}
\overline{M} &= \mu M \\
&= x\left(2y - \frac{1}{x^3}\right) \\
&= \left(2y - \frac{1}{x^3}\right)x
\end{aligned}
$$

And

$$
\begin{aligned}
\overline{N} &= \mu N \\
&= x(x) \\
&= x^2
\end{aligned}
$$

Now a modified ODE is ontained from the original ODE, which is exact and can be solved. The modified ODE is

$$
\begin{aligned}
\overline{M} + \overline{N}\frac{\mathrm{d}y}{\mathrm{d}x} &= 0 \\
\left(\left(2y - \frac{1}{x^3}\right)x\right) + \left(x^2\right)\frac{\mathrm{d}y}{\mathrm{d}x} &= 0
\end{aligned}
$$

The following equations are now set up to solve for the function $\phi(x, y)$

$$
\frac{\partial \phi}{\partial x} = \overline{M} \tag{1}
$$

$$
\frac{\partial \phi}{\partial y} = \overline{N} \tag{2}
$$

Integrating (1) w.r.t. x gives

$$
\begin{aligned}
\int \frac{\partial \phi}{\partial x}\,\mathrm{d}x &= \int \overline{M}\,\mathrm{d}x \\
\int \frac{\partial \phi}{\partial x}\,\mathrm{d}x &= \int \left(2y - \frac{1}{x^3}\right)x\,\mathrm{d}x
\end{aligned}
$$

$$
\phi = y\,x^2 + \frac{1}{x} + f(y) \tag{3}
$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$
\frac{\partial \phi}{\partial y} = x^2 + f'(y) \tag{4}
$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = x^2$. Therefore equation (4) becomes

$$
x^2 = x^2 + f'(y) \tag{5}
$$

Solving equation (5) for $f'(y)$ gives

$$
f'(y) = 0
$$

Therefore

$$
f(y) = c_1
$$

Where c_1 is constant of integration. Substituting this result for $f(y)$ into equation (3) gives ϕ

$$\phi = y\,x^2 + \frac{1}{x} + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = y\,x^2 + \frac{1}{x}$$

The solution becomes

$$y = \frac{c_1 x - 1}{x^3}$$

Summary

The solution(s) found are the following

$$y = \frac{c_1 x - 1}{x^3} \tag{1}$$

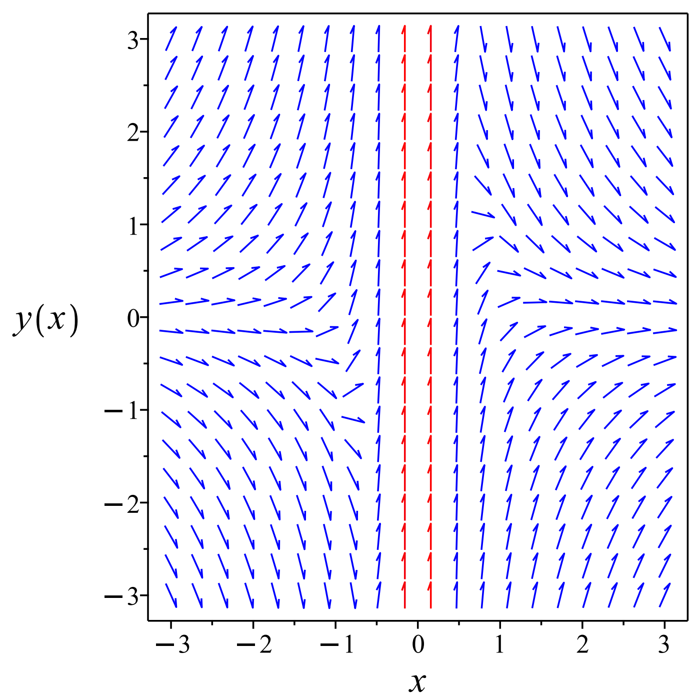


Figure 121: Slope field plot

Verification of solutions

$$y = \frac{c_1 x - 1}{x^3}$$

Verified OK.

2.10.4 Maple step by step solution

Let's solve

$xy' + 2y = \frac{1}{x^3}$

- Highest derivative means the order of the ODE is 1

  y'

- Isolate the derivative

  $y' = -\frac{2y}{x} + \frac{1}{x^4}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE

  $y' + \frac{2y}{x} = \frac{1}{x^4}$

- The ODE is linear; multiply by an integrating factor $\mu(x)$

  $\mu(x)\left(y' + \frac{2y}{x}\right) = \frac{\mu(x)}{x^4}$

- Assume the lhs of the ODE is the total derivative $\frac{d}{dx}(\mu(x)\, y)$

  $\mu(x)\left(y' + \frac{2y}{x}\right) = \mu'(x)\, y + \mu(x)\, y'$

- Isolate $\mu'(x)$

  $\mu'(x) = \frac{2\mu(x)}{x}$

- Solve to find the integrating factor

  $\mu(x) = x^2$

- Integrate both sides with respect to x

  $\int \left(\frac{d}{dx}(\mu(x)\, y)\right) dx = \int \frac{\mu(x)}{x^4} dx + c_1$

- Evaluate the integral on the lhs

  $\mu(x)\, y = \int \frac{\mu(x)}{x^4} dx + c_1$

- Solve for y

  $y = \frac{\int \frac{\mu(x)}{x^4} dx + c_1}{\mu(x)}$

- Substitute $\mu(x) = x^2$

$y = \frac{\int \frac{1}{x^2} dx + c_1}{x^2}$

- Evaluate the integrals on the rhs

  $y = \frac{c_1 - \frac{1}{x}}{x^2}$

- Simplify

  $y = \frac{c_1 x - 1}{x^3}$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
<- 1st order linear successful`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 13

```
dsolve(x*diff(y(x),x)+2*y(x)=1/x^3,y(x), singsol=all)
```

$$y(x) = \frac{c_1 x - 1}{x^3}$$

✓ Solution by Mathematica

Time used: 0.027 (sec). Leaf size: 15

```
DSolve[x*y'[x]+2*y[x]==1/x^3,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \frac{-1 + c_1 x}{x^3}$$

2.11 problem 11

Internal problem ID [4960]
Internal file name [`OUTPUT/4453_Sunday_June_05_2022_02_56_54_PM_18452267/index.tex`]

Book: Fundamentals of Differential Equations. By Nagle, Saff and Snider. 9th edition. Boston. Pearson 2018.
Section: Chapter 2, First order differential equations. Section 2.3, Linear equations. Exercises. page 54
Problem number: 11.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"linear", "exactWithIntegrationFactor", "first_order_ode_lie_symmetry_lookup"**

Maple gives the following as the ode type

```
[[_linear, `class A`]]
```

$$\boxed{y - y' = -t - 1}$$

2.11.1 Solving as linear ode

Entering Linear first order ODE solver. In canonical form a linear first order is

$$y' + p(t)y = q(t)$$

Where here

$$p(t) = -1$$
$$q(t) = t + 1$$

Hence the ode is

$$y' - y = t + 1$$

The integrating factor μ is

$$
\begin{aligned}
\mu &= \mathrm{e}^{\int (-1)dt} \\
&= \mathrm{e}^{-t}
\end{aligned}
$$

The ode becomes

$$
\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}(\mu y) &= (\mu)(t+1) \\
\frac{\mathrm{d}}{\mathrm{d}t}\left(\mathrm{e}^{-t} y\right) &= \left(\mathrm{e}^{-t}\right)(t+1) \\
\mathrm{d}\left(\mathrm{e}^{-t} y\right) &= \left((t+1)\,\mathrm{e}^{-t}\right)\,\mathrm{d}t
\end{aligned}
$$

Integrating gives

$$
\begin{aligned}
\mathrm{e}^{-t} y &= \int (t+1)\,\mathrm{e}^{-t}\,\mathrm{d}t \\
\mathrm{e}^{-t} y &= -(t+2)\,\mathrm{e}^{-t} + c_1
\end{aligned}
$$

Dividing both sides by the integrating factor $\mu = \mathrm{e}^{-t}$ results in

$$
y = -\mathrm{e}^{t}(t+2)\,\mathrm{e}^{-t} + c_1 \mathrm{e}^{t}
$$

which simplifies to

$$
y = -t - 2 + c_1 \mathrm{e}^{t}
$$

Summary

The solution(s) found are the following

$$
y = -t - 2 + c_1 \mathrm{e}^{t} \tag{1}
$$

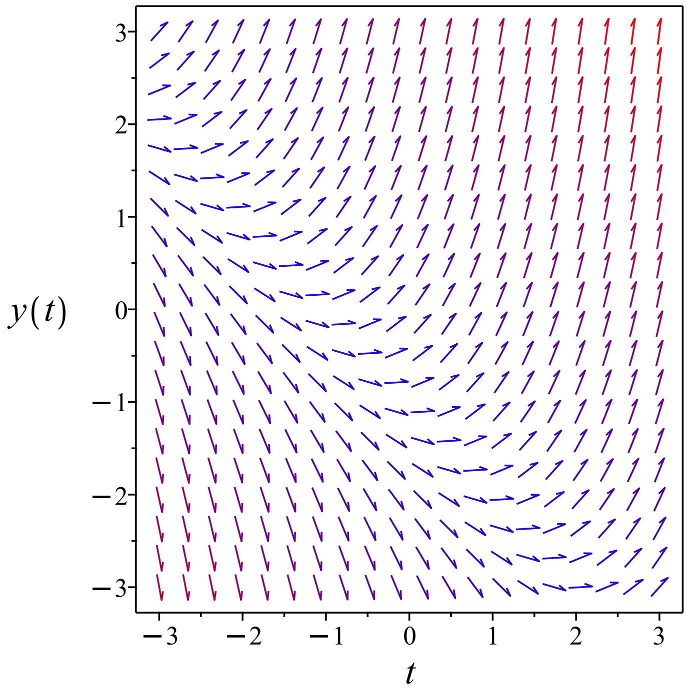


Figure 122: Slope field plot

Verification of solutions

$$y = -t - 2 + c_1 e^t$$

Verified OK.

2.11.2 Solving as first order ode lie symmetry lookup ode

Writing the ode as

$$y' = t + y + 1$$
$$y' = \omega(t, y)$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_t + \omega(\eta_y - \xi_t) - \omega^2 \xi_y - \omega_t \xi - \omega_y \eta = 0 \tag{A}$$

The type of this ode is known. It is of type `linear`. Therefore we do not need to solve the PDE (A), and can just use the lookup table shown below to find ξ, η

Table 116: Lie symmetry infinitesimal lookup table for known first order ODE's

ODE class	Form	ξ	η
linear ode	$y' = f(x)y(x) + g(x)$	0	$e^{\int f dx}$
separable ode	$y' = f(x)\, g(y)$	$\frac{1}{f}$	0
quadrature ode	$y' = f(x)$	0	1
quadrature ode	$y' = g(y)$	1	0
homogeneous ODEs of Class A	$y' = f\left(\frac{y}{x}\right)$	x	y
homogeneous ODEs of Class C	$y' = (a + bx + cy)^{\frac{n}{m}}$	1	$-\frac{b}{c}$
homogeneous class D	$y' = \frac{y}{x} + g(x)\, F\left(\frac{y}{x}\right)$	x^2	xy
First order special form ID 1	$y' = g(x)\, e^{h(x)+by} + f(x)$	$\frac{e^{-\int bf(x)dx - h(x)}}{g(x)}$	$\frac{f(x)e^{-\int bf(x)dx - h(x)}}{g(x)}$
polynomial type ode	$y' = \frac{a_1x+b_1y+c_1}{a_2x+b_2y+c_2}$	$\frac{a_1b_2x-a_2b_1x-b_1c_2+b_2c_1}{a_1b_2-a_2b_1}$	$\frac{a_1b_2y-a_2b_1y-a_1c_2-a_2c_1}{a_1b_2-a_2b_1}$
Bernoulli ode	$y' = f(x)\, y + g(x)\, y^n$	0	$e^{-\int (n-1)f(x)dx} y^n$
Reduced Riccati	$y' = f_1(x)\, y + f_2(x)\, y^2$	0	$e^{-\int f_1 dx}$

The above table shows that

$$
\begin{aligned}
\xi(t, y) &= 0 \\
\eta(t, y) &= \mathrm{e}^t
\end{aligned} \tag{A1}
$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(t, y) \to (R, S)$ where (R, S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$
\frac{dt}{\xi} = \frac{dy}{\eta} = dS \tag{1}
$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial t} + \eta \frac{\partial}{\partial y}\right) S(t, y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the

canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$R = t$$

S is found from

$$\begin{aligned} S &= \int \frac{1}{\eta} dy \\ &= \int \frac{1}{\mathrm{e}^{t}} dy \end{aligned}$$

Which results in

$$S = \mathrm{e}^{-t} y$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_t + \omega(t,y) S_y}{R_t + \omega(t,y) R_y} \tag{2}$$

Where in the above R_t, R_y, S_t, S_y are all partial derivatives and $\omega(t,y)$ is the right hand side of the original ode given by

$$\omega(t,y) = t + y + 1$$

Evaluating all the partial derivatives gives

$$\begin{aligned} R_t &= 1 \\ R_y &= 0 \\ S_t &= -\mathrm{e}^{-t} y \\ S_y &= \mathrm{e}^{-t} \end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = (t+1)\,\mathrm{e}^{-t} \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for t, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = (R+1)\,\mathrm{e}^{-R}$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by

integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$S(R) = -(R+2)\,\mathrm{e}^{-R} + c_1 \tag{4}$$

To complete the solution, we just need to transform (4) back to t, y coordinates. This results in

$$\mathrm{e}^{-t} y = -(t+2)\,\mathrm{e}^{-t} + c_1$$

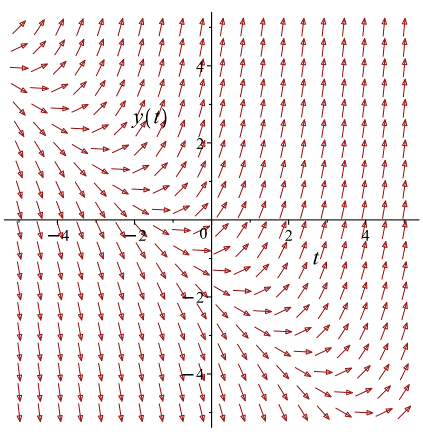
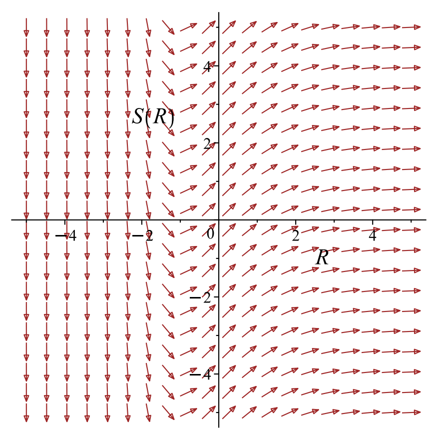
Which simplifies to

$$(t+y+2)\,\mathrm{e}^{-t} - c_1 = 0$$

Which gives

$$y = -\left(t\,\mathrm{e}^{-t} + 2\,\mathrm{e}^{-t} - c_1\right)\mathrm{e}^{t}$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in t, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dt} = t + y + 1$ 	$R = t$ $S = \mathrm{e}^{-t} y$	$\frac{dS}{dR} = (R+1)\,\mathrm{e}^{-R}$ 

Summary

The solution(s) found are the following

$$y = -\left(t\,\mathrm{e}^{-t} + 2\,\mathrm{e}^{-t} - c_1\right)\mathrm{e}^{t} \tag{1}$$

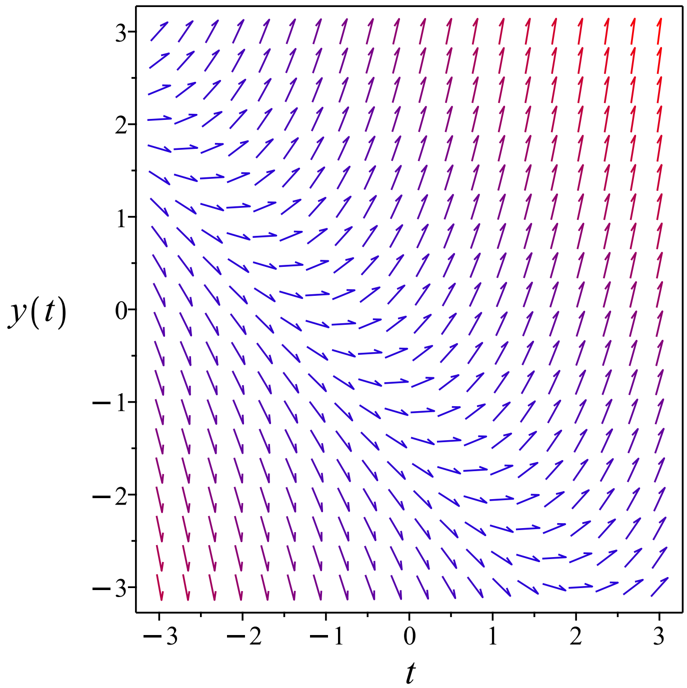


Figure 123: Slope field plot

Verification of solutions

$$y = -\left(t\,\mathrm{e}^{-t} + 2\,\mathrm{e}^{-t} - c_1\right)\mathrm{e}^{t}$$

Verified OK.

2.11.3 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x,y) + N(x,y)\frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x,y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y) = 0$$

Hence

$$\frac{\partial\phi}{\partial x} + \frac{\partial\phi}{\partial y}\frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$\frac{\partial \phi}{\partial x} = M$$
$$\frac{\partial \phi}{\partial y} = N$$

But since $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x, y)$ but at least we know now that we can do that since the condition $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(t, y)\,\mathrm{d}t + N(t, y)\,\mathrm{d}y = 0 \tag{1A}$$

Therefore

$$(-1)\,\mathrm{d}y = (-t - y - 1)\,\mathrm{d}t$$
$$(t + y + 1)\,\mathrm{d}t + (-1)\,\mathrm{d}y = 0 \tag{2A}$$

Comparing (1A) and (2A) shows that

$$M(t, y) = t + y + 1$$
$$N(t, y) = -1$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial t}$$

Using result found above gives

$$\frac{\partial M}{\partial y} = \frac{\partial}{\partial y}(t + y + 1)$$
$$= 1$$

And

$$\frac{\partial N}{\partial t} = \frac{\partial}{\partial t}(-1)$$
$$= 0$$

Since $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial t}$, then the ODE is <u>not exact</u>. Since the ODE is not exact, we will try to find an integrating factor to make it exact. Let

$$
\begin{aligned}
A &= \frac{1}{N}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial t}\right) \\
&= -1((1) - (0)) \\
&= -1
\end{aligned}
$$

Since A does not depend on y, then it can be used to find an integrating factor. The integrating factor μ is

$$
\begin{aligned}
\mu &= e^{\int A \, \mathrm{d}t} \\
&= e^{\int -1 \, \mathrm{d}t}
\end{aligned}
$$

The result of integrating gives

$$
\begin{aligned}
\mu &= e^{-t} \\
&= \mathrm{e}^{-t}
\end{aligned}
$$

M and N are multiplied by this integrating factor, giving new M and new N which are called $\overline{M}$ and $\overline{N}$ for now so not to confuse them with the original M and N.

$$
\begin{aligned}
\overline{M} &= \mu M \\
&= \mathrm{e}^{-t}(t + y + 1) \\
&= \mathrm{e}^{-t}(t + y + 1)
\end{aligned}
$$

And

$$
\begin{aligned}
\overline{N} &= \mu N \\
&= \mathrm{e}^{-t}(-1) \\
&= -\mathrm{e}^{-t}
\end{aligned}
$$

Now a modified ODE is ontained from the original ODE, which is exact and can be solved. The modified ODE is

$$
\begin{aligned}
\overline{M} + \overline{N}\frac{\mathrm{d}y}{\mathrm{d}t} &= 0 \\
\left(\mathrm{e}^{-t}(t + y + 1)\right) + \left(-\mathrm{e}^{-t}\right)\frac{\mathrm{d}y}{\mathrm{d}t} &= 0
\end{aligned}
$$

The following equations are now set up to solve for the function $\phi(t, y)$

$$
\frac{\partial \phi}{\partial t} = \overline{M} \tag{1}
$$

$$
\frac{\partial \phi}{\partial y} = \overline{N} \tag{2}
$$

Integrating (1) w.r.t. t gives

$$\int \frac{\partial \phi}{\partial t} \, \mathrm{d}t = \int \overline{M} \, \mathrm{d}t$$

$$\int \frac{\partial \phi}{\partial t} \, \mathrm{d}t = \int \mathrm{e}^{-t}(t+y+1) \, \mathrm{d}t$$

$$\phi = -(t+y+2)\,\mathrm{e}^{-t} + f(y) \tag{3}$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both t and y. Taking derivative of equation (3) w.r.t y gives

$$\frac{\partial \phi}{\partial y} = -\mathrm{e}^{-t} + f'(y) \tag{4}$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = -\mathrm{e}^{-t}$. Therefore equation (4) becomes

$$-\mathrm{e}^{-t} = -\mathrm{e}^{-t} + f'(y) \tag{5}$$

Solving equation (5) for $f'(y)$ gives

$$f'(y) = 0$$

Therefore

$$f(y) = c_1$$

Where c_1 is constant of integration. Substituting this result for $f(y)$ into equation (3) gives ϕ

$$\phi = -(t+y+2)\,\mathrm{e}^{-t} + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = -(t+y+2)\,\mathrm{e}^{-t}$$

The solution becomes

$$y = -\left(t\,\mathrm{e}^{-t} + 2\,\mathrm{e}^{-t} + c_1\right)\mathrm{e}^{t}$$

<u>Summary</u>

The solution(s) found are the following

$$y = -\left(t\,\mathrm{e}^{-t} + 2\,\mathrm{e}^{-t} + c_1\right)\mathrm{e}^{t} \tag{1}$$

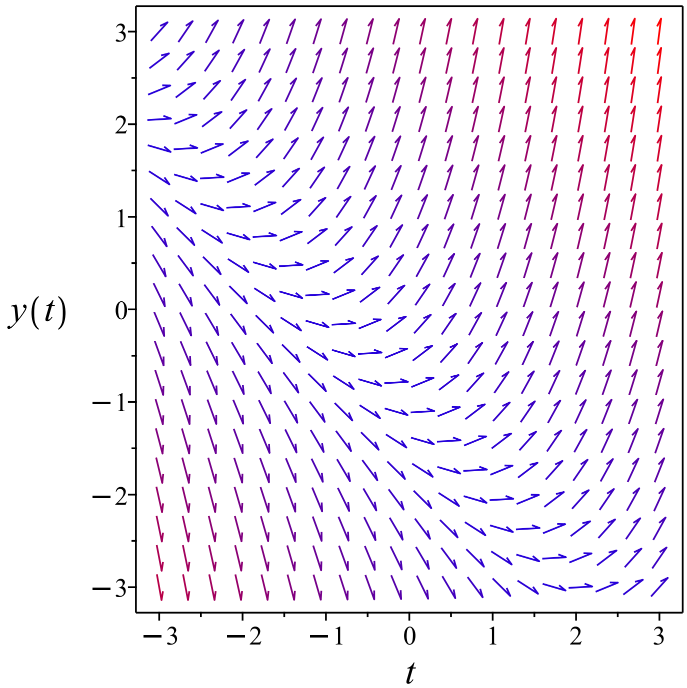


Figure 124: Slope field plot

Verification of solutions

$$y = -\left(t\,\mathrm{e}^{-t} + 2\,\mathrm{e}^{-t} + c_1\right)\mathrm{e}^{t}$$

Verified OK.

2.11.4 Maple step by step solution

Let's solve

$y - y' = -t - 1$

- Highest derivative means the order of the ODE is 1

  y'

- Isolate the derivative

  $y' = t + y + 1$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE

  $y' - y = t + 1$

- The ODE is linear; multiply by an integrating factor $\mu(t)$

  $\mu(t)\,(y' - y) = \mu(t)\,(t + 1)$

- Assume the lhs of the ODE is the total derivative $\frac{d}{dt}(\mu(t)\,y)$

  $\mu(t)\,(y'-y)=\mu'(t)\,y+\mu(t)\,y'$
- Isolate $\mu'(t)$

  $\mu'(t)=-\mu(t)$
- Solve to find the integrating factor

  $\mu(t)=\mathrm{e}^{-t}$
- Integrate both sides with respect to t

  $\int\left(\frac{d}{dt}(\mu(t)\,y)\right)dt=\int\mu(t)\,(t+1)\,dt+c_1$
- Evaluate the integral on the lhs

  $\mu(t)\,y=\int\mu(t)\,(t+1)\,dt+c_1$
- Solve for y

  $y=\frac{\int\mu(t)(t+1)dt+c_1}{\mu(t)}$
- Substitute $\mu(t)=\mathrm{e}^{-t}$

  $y=\frac{\int(t+1)\mathrm{e}^{-t}dt+c_1}{\mathrm{e}^{-t}}$
- Evaluate the integrals on the rhs

  $y=\frac{-(t+2)\mathrm{e}^{-t}+c_1}{\mathrm{e}^{-t}}$
- Simplify

  $y=-t-2+c_1\mathrm{e}^{t}$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
<- 1st order linear successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 13

```
dsolve((t+y(t)+1)-diff(y(t),t)=0,y(t), singsol=all)
```

$$y(t) = -t - 2 + c_1 e^t$$

✓ Solution by Mathematica

Time used: 0.049 (sec). Leaf size: 16

```
DSolve[(t+y[t]+1)-y'[t]==0,y[t],t,IncludeSingularSolutions -> True]
```

$$y(t) \to -t + c_1 e^t - 2$$

2.12 problem 12

Internal problem ID [4961]
Internal file name [`OUTPUT/4454_Sunday_June_05_2022_02_56_55_PM_45338768/index.tex`]

Book: Fundamentals of Differential Equations. By Nagle, Saff and Snider. 9th edition. Boston. Pearson 2018.
Section: Chapter 2, First order differential equations. Section 2.3, Linear equations. Exercises. page 54
Problem number: 12.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"linear", "exactWithIntegrationFactor", "first_order_ode_lie_symmetry_lookup"**

Maple gives the following as the ode type

```
[[_linear, `class A`]]
```

$$\boxed{y' + 4y = \mathrm{e}^{-4x} x^2}$$

2.12.1 Solving as linear ode

Entering Linear first order ODE solver. In canonical form a linear first order is

$$y' + p(x)y = q(x)$$

Where here

$$\begin{aligned} p(x) &= 4 \\ q(x) &= \mathrm{e}^{-4x} x^2 \end{aligned}$$

Hence the ode is

$$y' + 4y = \mathrm{e}^{-4x} x^2$$

The integrating factor μ is

$$
\begin{aligned}
\mu &= e^{\int 4dx} \\
&= e^{4x}
\end{aligned}
$$

The ode becomes

$$
\begin{aligned}
\frac{d}{dx}(\mu y) &= (\mu)\left(e^{-4x}x^2\right) \\
\frac{d}{dx}\left(e^{4x}y\right) &= \left(e^{4x}\right)\left(e^{-4x}x^2\right) \\
d\left(e^{4x}y\right) &= x^2\,dx
\end{aligned}
$$

Integrating gives

$$
\begin{aligned}
e^{4x}y &= \int x^2\,dx \\
e^{4x}y &= \frac{x^3}{3} + c_1
\end{aligned}
$$

Dividing both sides by the integrating factor $\mu = e^{4x}$ results in

$$
y = \frac{e^{-4x}x^3}{3} + c_1 e^{-4x}
$$

which simplifies to

$$
y = e^{-4x}\left(\frac{x^3}{3} + c_1\right)
$$

Summary

The solution(s) found are the following

$$
y = e^{-4x}\left(\frac{x^3}{3} + c_1\right) \tag{1}
$$

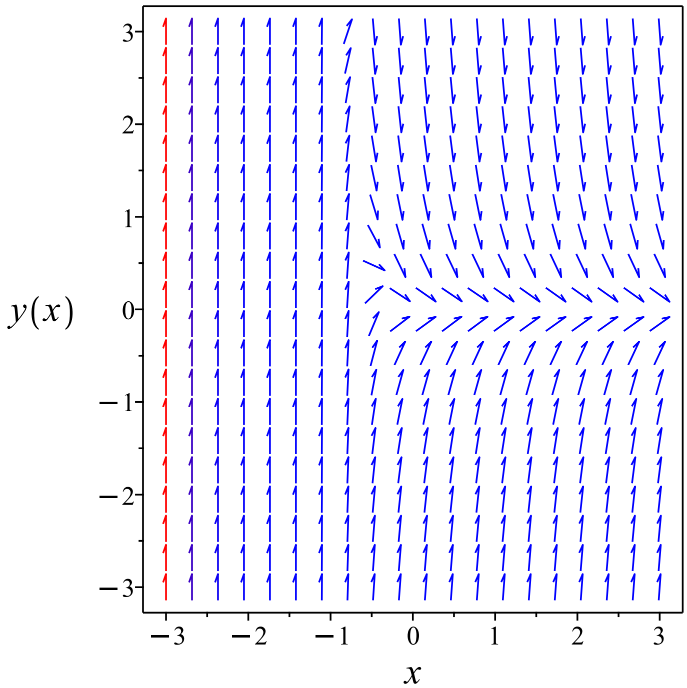


Figure 125: Slope field plot

Verification of solutions

$$y = e^{-4x}\left(\frac{x^3}{3} + c_1\right)$$

Verified OK.

2.12.2 Solving as first order ode lie symmetry lookup ode

Writing the ode as

$$y' = e^{-4x}x^2 - 4y$$
$$y' = \omega(x, y)$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x + \omega(\eta_y - \xi_x) - \omega^2\xi_y - \omega_x\xi - \omega_y\eta = 0 \tag{A}$$

The type of this ode is known. It is of type `linear`. Therefore we do not need to solve the PDE (A), and can just use the lookup table shown below to find ξ, η

Table 119: Lie symmetry infinitesimal lookup table for known first order ODE's

ODE class	Form	ξ	η
linear ode	$y' = f(x)y(x) + g(x)$	0	$e^{\int f dx}$
separable ode	$y' = f(x)\, g(y)$	$\frac{1}{f}$	0
quadrature ode	$y' = f(x)$	0	1
quadrature ode	$y' = g(y)$	1	0
homogeneous ODEs of Class A	$y' = f\left(\frac{y}{x}\right)$	x	y
homogeneous ODEs of Class C	$y' = (a + bx + cy)^{\frac{n}{m}}$	1	$-\frac{b}{c}$
homogeneous class D	$y' = \frac{y}{x} + g(x)\, F\left(\frac{y}{x}\right)$	x^2	xy
First order special form ID 1	$y' = g(x)\, e^{h(x)+by} + f(x)$	$\frac{e^{-\int bf(x)dx - h(x)}}{g(x)}$	$\frac{f(x)e^{-\int bf(x)dx - h(x)}}{g(x)}$
polynomial type ode	$y' = \frac{a_1x+b_1y+c_1}{a_2x+b_2y+c_2}$	$\frac{a_1b_2x - a_2b_1x - b_1c_2 + b_2c_1}{a_1b_2 - a_2b_1}$	$\frac{a_1b_2y - a_2b_1y - a_1c_2 - a_2c_1}{a_1b_2 - a_2b_1}$
Bernoulli ode	$y' = f(x)\, y + g(x)\, y^n$	0	$e^{-\int (n-1)f(x)dx} y^n$
Reduced Riccati	$y' = f_1(x)\, y + f_2(x)\, y^2$	0	$e^{-\int f_1 dx}$

The above table shows that

$$\begin{aligned} \xi(x, y) &= 0 \\ \eta(x, y) &= \mathrm{e}^{-4x} \end{aligned} \tag{A1}$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x, y) \to (R, S)$ where (R, S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y}\right) S(x, y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the

canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$R = x$$

S is found from

$$\begin{aligned} S &= \int \frac{1}{\eta} dy \\ &= \int \frac{1}{\mathrm{e}^{-4x}} dy \end{aligned}$$

Which results in

$$S = \mathrm{e}^{4x} y$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x, y) S_y}{R_x + \omega(x, y) R_y} \tag{2}$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x, y)$ is the right hand side of the original ode given by

$$\omega(x, y) = \mathrm{e}^{-4x} x^2 - 4y$$

Evaluating all the partial derivatives gives

$$\begin{aligned} R_x &= 1 \\ R_y &= 0 \\ S_x &= 4\,\mathrm{e}^{4x} y \\ S_y &= \mathrm{e}^{4x} \end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = x^2 \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = R^2$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by

integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$S(R) = \frac{R^3}{3} + c_1 \tag{4}$$

To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$e^{4x} y = \frac{x^3}{3} + c_1$$

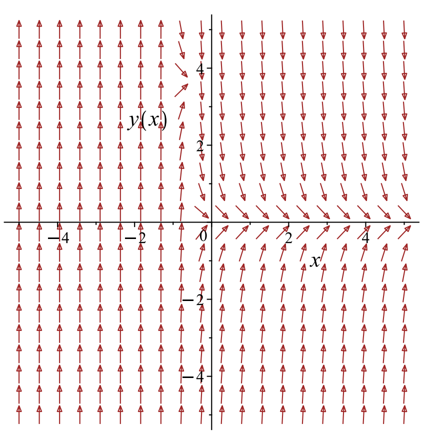
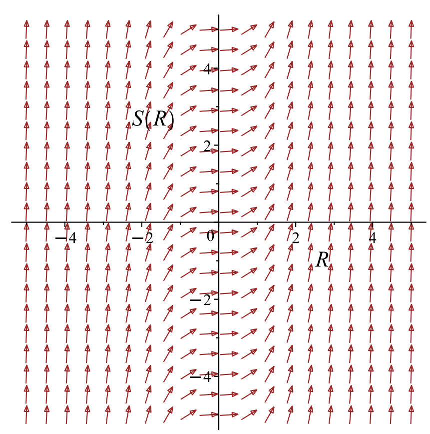
Which simplifies to

$$e^{4x} y = \frac{x^3}{3} + c_1$$

Which gives

$$y = \frac{e^{-4x}(x^3 + 3c_1)}{3}$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in x, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dx} = e^{-4x}x^2 - 4y$ 	$R = x$ $S = e^{4x} y$	$\frac{dS}{dR} = R^2$ 

Summary

The solution(s) found are the following

$$y = \frac{e^{-4x}(x^3 + 3c_1)}{3} \tag{1}$$

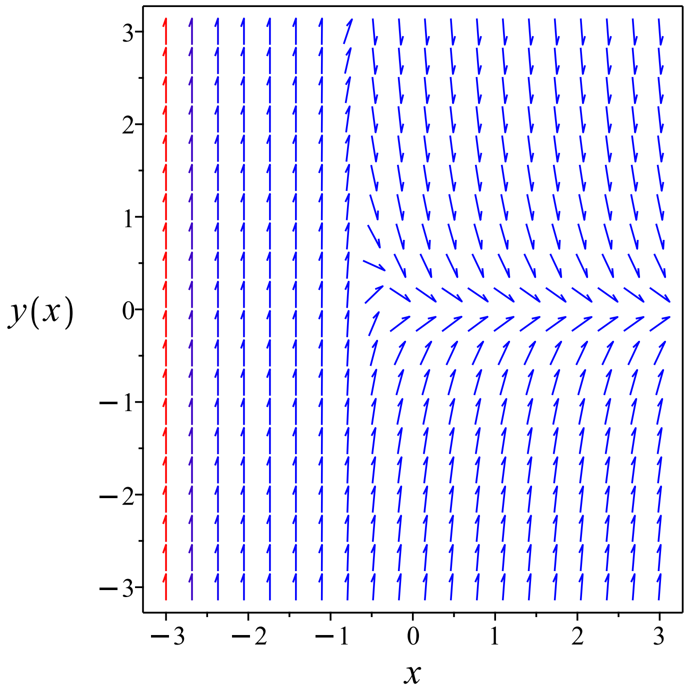


Figure 126: Slope field plot

Verification of solutions

$$y = \frac{e^{-4x}(x^3 + 3c_1)}{3}$$

Verified OK.

2.12.3 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x, y) + N(x, y)\frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x, y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x, y) = 0$$

Hence

$$\frac{\partial \phi}{\partial x} + \frac{\partial \phi}{\partial y}\frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$
\begin{aligned}
\frac{\partial \phi}{\partial x} &= M \\
\frac{\partial \phi}{\partial y} &= N
\end{aligned}
$$

But since $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ then for the above to be valid, we require that

$$
\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}
$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x, y)$ but at least we know now that we can do that since the condition $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$
M(x, y)\, \mathrm{d}x + N(x, y)\, \mathrm{d}y = 0 \tag{1A}
$$

Therefore

$$
\begin{aligned}
\mathrm{d}y &= \left(\mathrm{e}^{-4x} x^2 - 4y\right) \mathrm{d}x \\
\left(-\mathrm{e}^{-4x} x^2 + 4y\right) \mathrm{d}x + \mathrm{d}y &= 0 \qquad \text{(2A)}
\end{aligned}
$$

Comparing (1A) and (2A) shows that

$$
\begin{aligned}
M(x, y) &= -\mathrm{e}^{-4x} x^2 + 4y \\
N(x, y) &= 1
\end{aligned}
$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$
\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}
$$

Using result found above gives

$$
\begin{aligned}
\frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}\left(-\mathrm{e}^{-4x} x^2 + 4y\right) \\
&= 4
\end{aligned}
$$

And

$$
\begin{aligned}
\frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}(1) \\
&= 0
\end{aligned}
$$

Since $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$, then the ODE is not exact. Since the ODE is not exact, we will try to find an integrating factor to make it exact. Let

$$\begin{aligned} A &= \frac{1}{N}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right) \\ &= 1((4) - (0)) \\ &= 4 \end{aligned}$$

Since A does not depend on y, then it can be used to find an integrating factor. The integrating factor μ is

$$\begin{aligned} \mu &= e^{\int A \, \mathrm{d}x} \\ &= e^{\int 4 \, \mathrm{d}x} \end{aligned}$$

The result of integrating gives

$$\begin{aligned} \mu &= e^{4x} \\ &= \mathrm{e}^{4x} \end{aligned}$$

M and N are multiplied by this integrating factor, giving new M and new N which are called $\overline{M}$ and $\overline{N}$ for now so not to confuse them with the original M and N.

$$\begin{aligned} \overline{M} &= \mu M \\ &= \mathrm{e}^{4x}\left(-\mathrm{e}^{-4x}x^2 + 4y\right) \\ &= 4\,\mathrm{e}^{4x}y - x^2 \end{aligned}$$

And

$$\begin{aligned} \overline{N} &= \mu N \\ &= \mathrm{e}^{4x}(1) \\ &= \mathrm{e}^{4x} \end{aligned}$$

Now a modified ODE is ontained from the original ODE, which is exact and can be solved. The modified ODE is

$$\begin{aligned} \overline{M} + \overline{N}\frac{\mathrm{d}y}{\mathrm{d}x} &= 0 \\ \left(4\,\mathrm{e}^{4x}y - x^2\right) + \left(\mathrm{e}^{4x}\right)\frac{\mathrm{d}y}{\mathrm{d}x} &= 0 \end{aligned}$$

The following equations are now set up to solve for the function $\phi(x, y)$

$$\frac{\partial \phi}{\partial x} = \overline{M} \tag{1}$$

$$\frac{\partial \phi}{\partial y} = \overline{N} \tag{2}$$

Integrating (1) w.r.t. x gives

$$\int \frac{\partial \phi}{\partial x}\,\mathrm{d}x = \int \overline{M}\,\mathrm{d}x$$

$$\int \frac{\partial \phi}{\partial x}\,\mathrm{d}x = \int 4\,\mathrm{e}^{4x}y - x^2\,\mathrm{d}x$$

$$\phi = -\frac{x^3}{3} + \mathrm{e}^{4x}y + f(y) \tag{3}$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$\frac{\partial \phi}{\partial y} = \mathrm{e}^{4x} + f'(y) \tag{4}$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = \mathrm{e}^{4x}$. Therefore equation (4) becomes

$$\mathrm{e}^{4x} = \mathrm{e}^{4x} + f'(y) \tag{5}$$

Solving equation (5) for $f'(y)$ gives

$$f'(y) = 0$$

Therefore

$$f(y) = c_1$$

Where c_1 is constant of integration. Substituting this result for $f(y)$ into equation (3) gives ϕ

$$\phi = -\frac{x^3}{3} + \mathrm{e}^{4x}y + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = -\frac{x^3}{3} + \mathrm{e}^{4x}y$$

The solution becomes

$$y = \frac{\mathrm{e}^{-4x}(x^3 + 3c_1)}{3}$$

Summary

The solution(s) found are the following

$$y = \frac{e^{-4x}(x^3 + 3c_1)}{3} \tag{1}$$

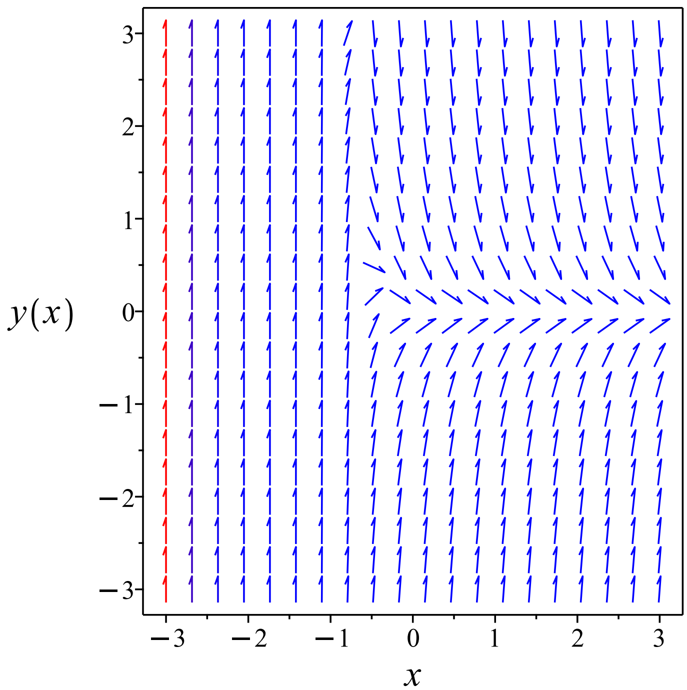


Figure 127: Slope field plot

Verification of solutions

$$y = \frac{e^{-4x}(x^3 + 3c_1)}{3}$$

Verified OK.

2.12.4 Maple step by step solution

Let's solve

$y' + 4y = e^{-4x}x^2$

- Highest derivative means the order of the ODE is 1

  y'

- Isolate the derivative

$$y' = \mathrm{e}^{-4x}x^2 - 4y$$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE

$$y' + 4y = \mathrm{e}^{-4x}x^2$$

- The ODE is linear; multiply by an integrating factor $\mu(x)$

$$\mu(x)\left(y' + 4y\right) = \mu(x)\,\mathrm{e}^{-4x}x^2$$

- Assume the lhs of the ODE is the total derivative $\frac{d}{dx}(\mu(x)\,y)$

$$\mu(x)\left(y' + 4y\right) = \mu'(x)\,y + \mu(x)\,y'$$

- Isolate $\mu'(x)$

$$\mu'(x) = 4\mu(x)$$

- Solve to find the integrating factor

$$\mu(x) = \mathrm{e}^{4x}$$

- Integrate both sides with respect to x

$$\int \left(\frac{d}{dx}(\mu(x)\,y)\right) dx = \int \mu(x)\,\mathrm{e}^{-4x}x^2 dx + c_1$$

- Evaluate the integral on the lhs

$$\mu(x)\,y = \int \mu(x)\,\mathrm{e}^{-4x}x^2 dx + c_1$$

- Solve for y

$$y = \frac{\int \mu(x)\mathrm{e}^{-4x}x^2 dx + c_1}{\mu(x)}$$

- Substitute $\mu(x) = \mathrm{e}^{4x}$

$$y = \frac{\int \mathrm{e}^{-4x}x^2\mathrm{e}^{4x} dx + c_1}{\mathrm{e}^{4x}}$$

- Evaluate the integrals on the rhs

$$y = \frac{\frac{x^3}{3} + c_1}{\mathrm{e}^{4x}}$$

- Simplify

$$y = \frac{\mathrm{e}^{-4x}\left(x^3 + 3c_1\right)}{3}$$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
<- 1st order linear successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 17

```
dsolve(diff(y(x),x)=x^2*exp(-4*x)-4*y(x),y(x), singsol=all)
```

$$y(x) = \frac{(x^3 + 3c_1)\,e^{-4x}}{3}$$

✓ Solution by Mathematica

Time used: 0.058 (sec). Leaf size: 22

```
DSolve[y'[x]==x^2*Exp[-4*x]-4*y[x],y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \frac{1}{3}e^{-4x}(x^3 + 3c_1)$$

2.13 problem 13

Internal problem ID [4962]
Internal file name [`OUTPUT/4455_Sunday_June_05_2022_02_56_56_PM_15748080/index.tex`]

Book: Fundamentals of Differential Equations. By Nagle, Saff and Snider. 9th edition. Boston. Pearson 2018.
Section: Chapter 2, First order differential equations. Section 2.3, Linear equations. Exercises. page 54
Problem number: 13.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"unknown"**

Maple gives the following as the ode type

```
[_rational, [_Abel, `2nd type`, `class C`]]
```

Unable to solve or complete the solution.

$$\boxed{y'y - 5y^3 = -2x}$$

Unable to determine ODE type.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
trying homogeneous types:
trying Chini
differential order: 1; looking for linear symmetries
trying exact
trying Abel
Looking for potential symmetries
Looking for potential symmetries
Looking for potential symmetries
trying inverse_Riccati
trying an equivalence to an Abel ODE
differential order: 1; trying a linearization to 2nd order
--- trying a change of variables {x -> y(x), y(x) -> x}
differential order: 1; trying a linearization to 2nd order
trying 1st order ODE linearizable_by_differentiation
--- Trying Lie symmetry methods, 1st order ---
`, `-> Computing symmetries using: way = 3
`, `-> Computing symmetries using: way = 4
`, `-> Computing symmetries using: way = 2
trying symmetry patterns for 1st order ODEs
-> trying a symmetry pattern of the form [F(x)*G(y), 0]
-> trying a symmetry pattern of the form [0, F(x)*G(y)]
-> trying symmetry patterns of the forms [F(x),G(y)] and [G(y),F(x)]
-> trying a symmetry pattern of the form [F(x),G(x)]
-> trying a symmetry pattern of the form [F(y),G(y)]
-> trying a symmetry pattern of the form [F(x)+G(y), 0]
-> trying a symmetry pattern of the form [0, F(x)+G(y)]
-> trying a symmetry pattern of the form [F(x),G(x)*y+H(x)]
-> trying a symmetry pattern of conformal type`
```

✗ Solution by Maple

```
dsolve(y(x)*diff(y(x),x)+2*x=5*y(x)^3,y(x), singsol=all)
```

No solution found

✗ Solution by Mathematica

Time used: 0.0 (sec). Leaf size: 0

```
DSolve[y[x]*y'[x]+2*x==5*y[x]^3,y[x],x,IncludeSingularSolutions -> True]
```

Not solved

2.14 problem 14

Internal problem ID [4963]
Internal file name [`OUTPUT/4456_Sunday_June_05_2022_02_56_57_PM_96989353/index.tex`]

Book: Fundamentals of Differential Equations. By Nagle, Saff and Snider. 9th edition. Boston. Pearson 2018.
Section: Chapter 2, First order differential equations. Section 2.3, Linear equations. Exercises. page 54
Problem number: 14.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"linear", "exactWithIntegrationFactor", "first_order_ode_lie_symmetry_lookup"**

Maple gives the following as the ode type

`[_linear]`

$$\boxed{xy' + 3y = -3x^2 + \frac{\sin(x)}{x}}$$

2.14.1 Solving as linear ode

Entering Linear first order ODE solver. In canonical form a linear first order is

$$y' + p(x)y = q(x)$$

Where here

$$p(x) = \frac{3}{x}$$
$$q(x) = \frac{-3x^3 + \sin(x)}{x^2}$$

Hence the ode is

$$y' + \frac{3y}{x} = \frac{-3x^3 + \sin(x)}{x^2}$$

The integrating factor μ is

$$\mu = e^{\int \frac{3}{x}dx}$$
$$= x^3$$

The ode becomes

$$\frac{d}{dx}(\mu y) = (\mu)\left(\frac{-3x^3 + \sin(x)}{x^2}\right)$$
$$\frac{d}{dx}(y\,x^3) = (x^3)\left(\frac{-3x^3 + \sin(x)}{x^2}\right)$$
$$d(y\,x^3) = (-3x^4 + \sin(x)\,x)\,dx$$

Integrating gives

$$y\,x^3 = \int -3x^4 + \sin(x)\,x\,dx$$
$$y\,x^3 = \sin(x) - \cos(x)\,x - \frac{3x^5}{5} + c_1$$

Dividing both sides by the integrating factor $\mu = x^3$ results in

$$y = \frac{\sin(x) - \cos(x)\,x - \frac{3x^5}{5}}{x^3} + \frac{c_1}{x^3}$$

which simplifies to

$$y = \frac{\sin(x) - \cos(x)\,x - \frac{3x^5}{5} + c_1}{x^3}$$

Summary

The solution(s) found are the following

$$y = \frac{\sin(x) - \cos(x)\,x - \frac{3x^5}{5} + c_1}{x^3} \tag{1}$$

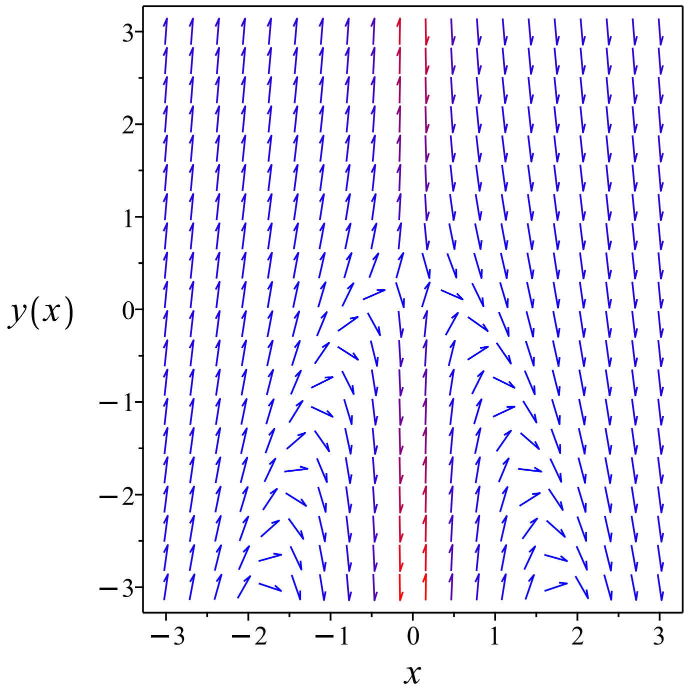


Figure 128: Slope field plot

Verification of solutions

$$y = \frac{\sin(x) - \cos(x)\, x - \frac{3x^5}{5} + c_1}{x^3}$$

Verified OK.

2.14.2 Solving as first order ode lie symmetry lookup ode

Writing the ode as

$$y' = \frac{-3x^3 - 3xy + \sin(x)}{x^2}$$
$$y' = \omega(x, y)$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x + \omega(\eta_y - \xi_x) - \omega^2 \xi_y - \omega_x \xi - \omega_y \eta = 0 \tag{A}$$

The type of this ode is known. It is of type `linear`. Therefore we do not need to solve the PDE (A), and can just use the lookup table shown below to find ξ, η

Table 122: Lie symmetry infinitesimal lookup table for known first order ODE's

ODE class	Form	ξ	η
linear ode	$y' = f(x)y(x) + g(x)$	0	$e^{\int f dx}$
separable ode	$y' = f(x)\,g(y)$	$\frac{1}{f}$	0
quadrature ode	$y' = f(x)$	0	1
quadrature ode	$y' = g(y)$	1	0
homogeneous ODEs of Class A	$y' = f\left(\frac{y}{x}\right)$	x	y
homogeneous ODEs of Class C	$y' = (a + bx + cy)^{\frac{n}{m}}$	1	$-\frac{b}{c}$
homogeneous class D	$y' = \frac{y}{x} + g(x)\,F\left(\frac{y}{x}\right)$	x^2	xy
First order special form ID 1	$y' = g(x)\,e^{h(x)+by} + f(x)$	$\frac{e^{-\int bf(x)dx - h(x)}}{g(x)}$	$\frac{f(x)e^{-\int bf(x)dx - h(x)}}{g(x)}$
polynomial type ode	$y' = \frac{a_1x+b_1y+c_1}{a_2x+b_2y+c_2}$	$\frac{a_1b_2x - a_2b_1x - b_1c_2 + b_2c_1}{a_1b_2 - a_2b_1}$	$\frac{a_1b_2y - a_2b_1y - a_1c_2 - a_2c_1}{a_1b_2 - a_2b_1}$
Bernoulli ode	$y' = f(x)\,y + g(x)\,y^n$	0	$e^{-\int(n-1)f(x)dx}y^n$
Reduced Riccati	$y' = f_1(x)\,y + f_2(x)\,y^2$	0	$e^{-\int f_1 dx}$

The above table shows that

$$\begin{aligned} \xi(x,y) &= 0 \\ \eta(x,y) &= \frac{1}{x^3} \end{aligned} \tag{A1}$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x, y) \to (R, S)$ where (R, S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y}\right) S(x,y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the

canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$R = x$$

S is found from

$$\begin{aligned} S &= \int \frac{1}{\eta} dy \\ &= \int \frac{1}{\frac{1}{x^3}} dy \end{aligned}$$

Which results in

$$S = y\, x^3$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x,y) S_y}{R_x + \omega(x,y) R_y} \tag{2}$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x,y)$ is the right hand side of the original ode given by

$$\omega(x,y) = \frac{-3x^3 - 3xy + \sin(x)}{x^2}$$

Evaluating all the partial derivatives gives

$$\begin{aligned} R_x &= 1 \\ R_y &= 0 \\ S_x &= 3y\, x^2 \\ S_y &= x^3 \end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = -3x^4 + \sin(x)\, x \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = -3R^4 + \sin(R)\, R$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by

integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$S(R) = \sin(R) - \cos(R) R - \frac{3R^5}{5} + c_1 \tag{4}$$

To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$y x^3 = \sin(x) - \cos(x) x - \frac{3x^5}{5} + c_1$$

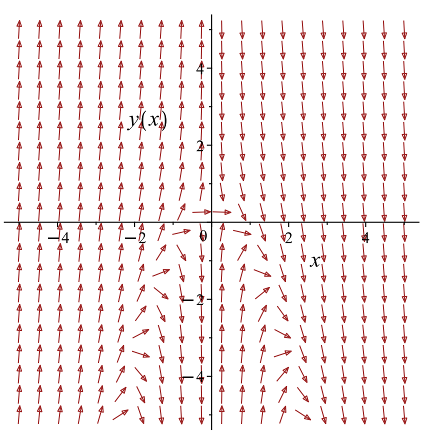
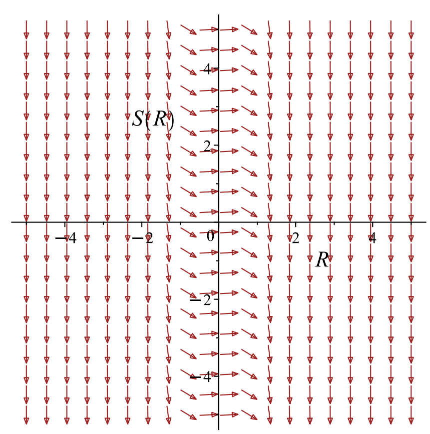
Which simplifies to

$$y x^3 = \sin(x) - \cos(x) x - \frac{3x^5}{5} + c_1$$

Which gives

$$y = \frac{-3x^5 - 5\cos(x) x + 5\sin(x) + 5c_1}{5x^3}$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in x, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dx} = \frac{-3x^3 - 3xy + \sin(x)}{x^2}$ 	$R = x$ $S = y\, x^3$	$\frac{dS}{dR} = -3R^4 + \sin(R) R$ 

Summary

The solution(s) found are the following

$$y = \frac{-3x^5 - 5\cos(x) x + 5\sin(x) + 5c_1}{5x^3} \tag{1}$$

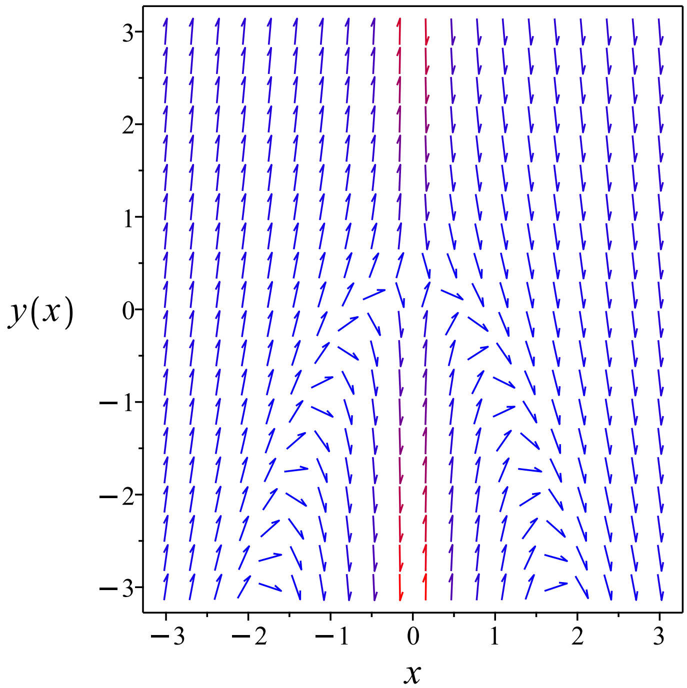


Figure 129: Slope field plot

Verification of solutions

$$y = \frac{-3x^5 - 5\cos(x)x + 5\sin(x) + 5c_1}{5x^3}$$

Verified OK.

2.14.3 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x,y) + N(x,y)\frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x,y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y) = 0$$

Hence

$$\frac{\partial \phi}{\partial x} + \frac{\partial \phi}{\partial y}\frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$\frac{\partial \phi}{\partial x} = M$$
$$\frac{\partial \phi}{\partial y} = N$$

But since $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x, y)$ but at least we know now that we can do that since the condition $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x, y)\,\mathrm{d}x + N(x, y)\,\mathrm{d}y = 0 \tag{1A}$$

Therefore

$$(x)\,\mathrm{d}y = \left(-3x^2 - 3y + \frac{\sin(x)}{x}\right)\mathrm{d}x$$

$$\left(3x^2 + 3y - \frac{\sin(x)}{x}\right)\mathrm{d}x + (x)\,\mathrm{d}y = 0 \tag{2A}$$

Comparing (1A) and (2A) shows that

$$M(x, y) = 3x^2 + 3y - \frac{\sin(x)}{x}$$
$$N(x, y) = x$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Using result found above gives

$$\begin{aligned}\frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}\left(3x^2 + 3y - \frac{\sin(x)}{x}\right) \\ &= 3\end{aligned}$$

And

$$
\begin{aligned}
\frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}(x) \\
&= 1
\end{aligned}
$$

Since $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$, then the ODE is <u>not exact</u>. Since the ODE is not exact, we will try to find an integrating factor to make it exact. Let

$$
\begin{aligned}
A &= \frac{1}{N}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right) \\
&= \frac{1}{x}((3) - (1)) \\
&= \frac{2}{x}
\end{aligned}
$$

Since A does not depend on y, then it can be used to find an integrating factor. The integrating factor μ is

$$
\begin{aligned}
\mu &= e^{\int A \, \mathrm{d}x} \\
&= e^{\int \frac{2}{x} \, \mathrm{d}x}
\end{aligned}
$$

The result of integrating gives

$$
\begin{aligned}
\mu &= e^{2\ln(x)} \\
&= x^2
\end{aligned}
$$

M and N are multiplied by this integrating factor, giving new M and new N which are called $\overline{M}$ and $\overline{N}$ for now so not to confuse them with the original M and N.

$$
\begin{aligned}
\overline{M} &= \mu M \\
&= x^2\left(3x^2 + 3y - \frac{\sin(x)}{x}\right) \\
&= -\sin(x)\, x + 3x^2(x^2 + y)
\end{aligned}
$$

And

$$
\begin{aligned}
\overline{N} &= \mu N \\
&= x^2(x) \\
&= x^3
\end{aligned}
$$

Now a modified ODE is ontained from the original ODE, which is exact and can be solved. The modified ODE is

$$
\begin{aligned}
\overline{M} + \overline{N}\frac{\mathrm{d}y}{\mathrm{d}x} &= 0 \\
\left(-\sin(x)\, x + 3x^2(x^2 + y)\right) + \left(x^3\right)\frac{\mathrm{d}y}{\mathrm{d}x} &= 0
\end{aligned}
$$

The following equations are now set up to solve for the function $\phi(x, y)$

$$\frac{\partial \phi}{\partial x} = \overline{M} \tag{1}$$
$$\frac{\partial \phi}{\partial y} = \overline{N} \tag{2}$$

Integrating (1) w.r.t. x gives

$$\int \frac{\partial \phi}{\partial x} \mathrm{d}x = \int \overline{M} \mathrm{d}x$$
$$\int \frac{\partial \phi}{\partial x} \mathrm{d}x = \int -\sin(x)\, x + 3x^2 (x^2 + y) \mathrm{d}x$$
$$\phi = \frac{3x^5}{5} + y\, x^3 - \sin(x) + \cos(x)\, x + f(y) \tag{3}$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$\frac{\partial \phi}{\partial y} = x^3 + f'(y) \tag{4}$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = x^3$. Therefore equation (4) becomes

$$x^3 = x^3 + f'(y) \tag{5}$$

Solving equation (5) for $f'(y)$ gives

$$f'(y) = 0$$

Therefore

$$f(y) = c_1$$

Where c_1 is constant of integration. Substituting this result for $f(y)$ into equation (3) gives ϕ

$$\phi = \frac{3x^5}{5} + y\, x^3 - \sin(x) + \cos(x)\, x + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = \frac{3x^5}{5} + y\, x^3 - \sin(x) + \cos(x)\, x$$

The solution becomes

$$y = \frac{-3x^5 - 5\cos(x)\,x + 5\sin(x) + 5c_1}{5x^3}$$

Summary

The solution(s) found are the following

$$y = \frac{-3x^5 - 5\cos(x)\,x + 5\sin(x) + 5c_1}{5x^3} \tag{1}$$

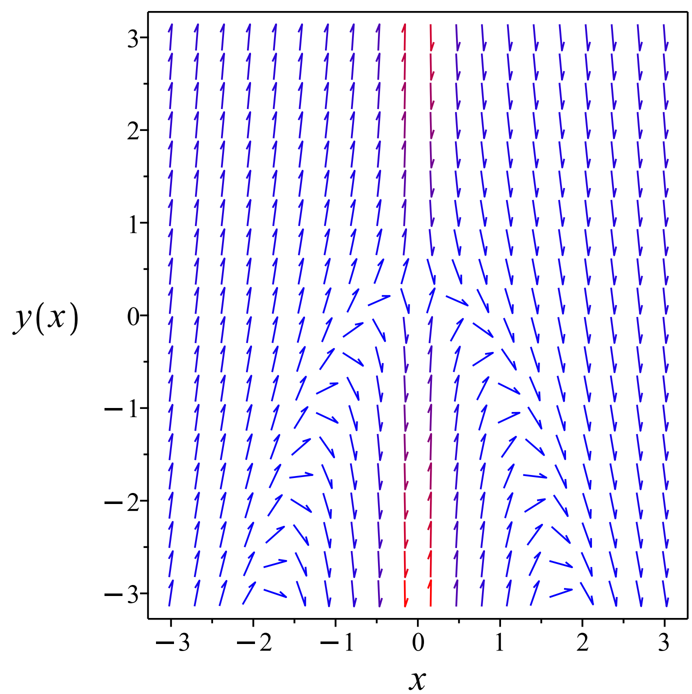


Figure 130: Slope field plot

Verification of solutions

$$y = \frac{-3x^5 - 5\cos(x)\,x + 5\sin(x) + 5c_1}{5x^3}$$

Verified OK.

2.14.4 Maple step by step solution

Let's solve

$xy' + 3y = -3x^2 + \frac{\sin(x)}{x}$

- Highest derivative means the order of the ODE is 1

  y'

- Isolate the derivative

  $y' = -\frac{3y}{x} + \frac{-3x^3+\sin(x)}{x^2}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE

  $y' + \frac{3y}{x} = \frac{-3x^3+\sin(x)}{x^2}$

- The ODE is linear; multiply by an integrating factor $\mu(x)$

  $\mu(x)\left(y' + \frac{3y}{x}\right) = \frac{\mu(x)(-3x^3+\sin(x))}{x^2}$

- Assume the lhs of the ODE is the total derivative $\frac{d}{dx}(\mu(x)\,y)$

  $\mu(x)\left(y' + \frac{3y}{x}\right) = \mu'(x)\,y + \mu(x)\,y'$

- Isolate $\mu'(x)$

  $\mu'(x) = \frac{3\mu(x)}{x}$

- Solve to find the integrating factor

  $\mu(x) = x^3$

- Integrate both sides with respect to x

  $\int \left(\frac{d}{dx}(\mu(x)\,y)\right) dx = \int \frac{\mu(x)(-3x^3+\sin(x))}{x^2} dx + c_1$

- Evaluate the integral on the lhs

  $\mu(x)\,y = \int \frac{\mu(x)(-3x^3+\sin(x))}{x^2} dx + c_1$

- Solve for y

  $y = \frac{\int \frac{\mu(x)\left(-3x^3+\sin(x)\right)}{x^2} dx + c_1}{\mu(x)}$

- Substitute $\mu(x) = x^3$

  $y = \frac{\int (-3x^3+\sin(x))x dx + c_1}{x^3}$

- Evaluate the integrals on the rhs

  $y = \frac{\sin(x) - \cos(x)x - \frac{3x^5}{5} + c_1}{x^3}$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
<- 1st order linear successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 22

```
dsolve(x*diff(y(x),x)+3*(y(x)+x^2)=sin(x)/x,y(x), singsol=all)
```

$$y(x) = \frac{-\frac{3x^5}{5} - x\cos(x) + \sin(x) + c_1}{x^3}$$

✓ Solution by Mathematica

Time used: 0.043 (sec). Leaf size: 31

```
DSolve[x*y'[x]+3*(y[x]+x^2)==Sin[x]/x,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \frac{-3x^5 + 5\sin(x) - 5x\cos(x) + 5c_1}{5x^3}$$

2.15 problem 15

Internal problem ID [4964]
Internal file name [`OUTPUT/4457_Sunday_June_05_2022_02_56_58_PM_53776950/index.tex`]

Book: Fundamentals of Differential Equations. By Nagle, Saff and Snider. 9th edition. Boston. Pearson 2018.
Section: Chapter 2, First order differential equations. Section 2.3, Linear equations. Exercises. page 54
Problem number: 15.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"exact", "linear", "separable", "first_order_ode_lie_symmetry_lookup"**

Maple gives the following as the ode type

`[_separable]`

$$\boxed{(x^2+1)\,y'+xy=x}$$

2.15.1 Solving as separable ode

In canonical form the ODE is

$$\begin{aligned} y' &= F(x,y) \\ &= f(x)g(y) \\ &= \frac{x(1-y)}{x^2+1} \end{aligned}$$

Where $f(x)=\frac{x}{x^2+1}$ and $g(y)=1-y$. Integrating both sides gives

$$\frac{1}{1-y}\,dy=\frac{x}{x^2+1}\,dx$$

$$\int \frac{1}{1-y}\,dy = \int \frac{x}{x^2+1}\,dx$$

$$-\ln(y-1) = \frac{\ln(x^2+1)}{2} + c_1$$

Raising both side to exponential gives

$$\frac{1}{y-1} = e^{\frac{\ln(x^2+1)}{2} + c_1}$$

Which simplifies to

$$\frac{1}{y-1} = c_2\sqrt{x^2+1}$$

Which simplifies to

$$y = \frac{\left(c_2\sqrt{x^2+1}\,e^{c_1}+1\right)e^{-c_1}}{c_2\sqrt{x^2+1}}$$

Summary

The solution(s) found are the following

$$y = \frac{\left(c_2\sqrt{x^2+1}\,e^{c_1}+1\right)e^{-c_1}}{c_2\sqrt{x^2+1}} \tag{1}$$

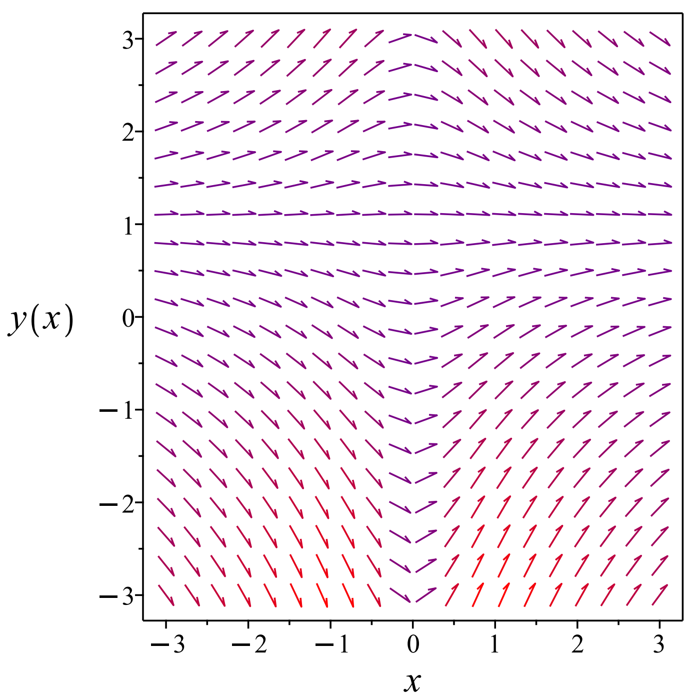


Figure 131: Slope field plot

Verification of solutions

$$y = \frac{\left(c_2\sqrt{x^2+1}\,\mathrm{e}^{c_1}+1\right)\mathrm{e}^{-c_1}}{c_2\sqrt{x^2+1}}$$

Verified OK.

2.15.2 Solving as linear ode

Entering Linear first order ODE solver. In canonical form a linear first order is

$$y' + p(x)y = q(x)$$

Where here

$$\begin{aligned}p(x) &= \frac{x}{x^2+1}\\ q(x) &= \frac{x}{x^2+1}\end{aligned}$$

Hence the ode is

$$y' + \frac{xy}{x^2+1} = \frac{x}{x^2+1}$$

The integrating factor μ is

$$\begin{aligned}\mu &= \mathrm{e}^{\int \frac{x}{x^2+1}dx}\\ &= \sqrt{x^2+1}\end{aligned}$$

The ode becomes

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}x}(\mu y) &= (\mu)\left(\frac{x}{x^2+1}\right)\\ \frac{\mathrm{d}}{\mathrm{d}x}\left(\sqrt{x^2+1}\,y\right) &= \left(\sqrt{x^2+1}\right)\left(\frac{x}{x^2+1}\right)\\ \mathrm{d}\left(\sqrt{x^2+1}\,y\right) &= \left(\frac{x}{\sqrt{x^2+1}}\right)\mathrm{d}x\end{aligned}$$

Integrating gives

$$\begin{aligned}\sqrt{x^2+1}\,y &= \int \frac{x}{\sqrt{x^2+1}}\,\mathrm{d}x\\ \sqrt{x^2+1}\,y &= \sqrt{x^2+1}+c_1\end{aligned}$$

Dividing both sides by the integrating factor $\mu = \sqrt{x^2+1}$ results in

$$y = 1 + \frac{c_1}{\sqrt{x^2+1}}$$

Summary

The solution(s) found are the following

$$y = 1 + \frac{c_1}{\sqrt{x^2 + 1}} \tag{1}$$

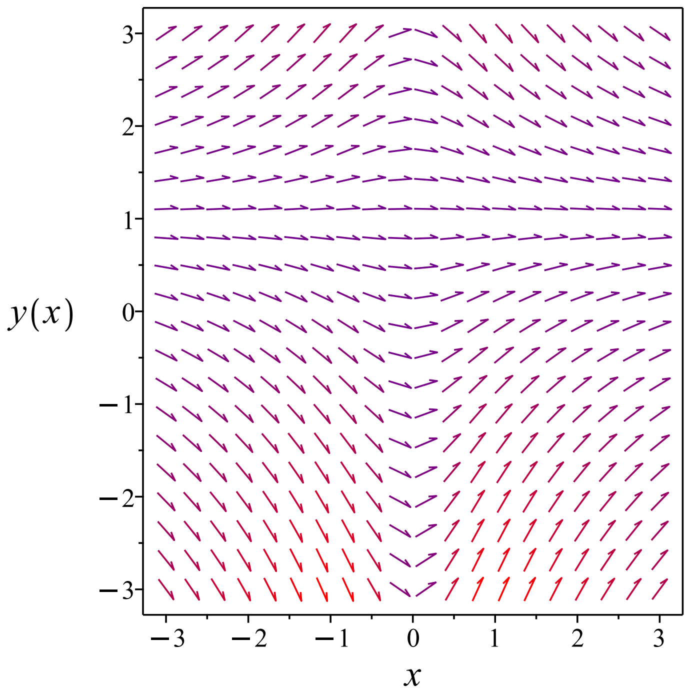


Figure 132: Slope field plot

Verification of solutions

$$y = 1 + \frac{c_1}{\sqrt{x^2 + 1}}$$

Verified OK.

2.15.3 Solving as first order ode lie symmetry lookup ode

Writing the ode as

$$y' = -\frac{x(y - 1)}{x^2 + 1}$$
$$y' = \omega(x, y)$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x + \omega(\eta_y - \xi_x) - \omega^2 \xi_y - \omega_x \xi - \omega_y \eta = 0 \tag{A}$$

The type of this ode is known. It is of type `linear`. Therefore we do not need to solve the PDE (A), and can just use the lookup table shown below to find ξ, η

Table 125: Lie symmetry infinitesimal lookup table for known first order ODE's

ODE class	Form	ξ	η
linear ode	$y' = f(x)y(x) + g(x)$	0	$e^{\int f dx}$
separable ode	$y' = f(x)\, g(y)$	$\frac{1}{f}$	0
quadrature ode	$y' = f(x)$	0	1
quadrature ode	$y' = g(y)$	1	0
homogeneous ODEs of Class A	$y' = f\left(\frac{y}{x}\right)$	x	y
homogeneous ODEs of Class C	$y' = (a + bx + cy)^{\frac{n}{m}}$	1	$-\frac{b}{c}$
homogeneous class D	$y' = \frac{y}{x} + g(x)\, F\left(\frac{y}{x}\right)$	x^2	xy
First order special form ID 1	$y' = g(x)\, e^{h(x)+by} + f(x)$	$\frac{e^{-\int bf(x)dx-h(x)}}{g(x)}$	$\frac{f(x)e^{-\int bf(x)dx-h(x)}}{g(x)}$
polynomial type ode	$y' = \frac{a_1x+b_1y+c_1}{a_2x+b_2y+c_2}$	$\frac{a_1b_2x-a_2b_1x-b_1c_2+b_2c_1}{a_1b_2-a_2b_1}$	$\frac{a_1b_2y-a_2b_1y-a_1c_2-a_2c_1}{a_1b_2-a_2b_1}$
Bernoulli ode	$y' = f(x)\, y + g(x)\, y^n$	0	$e^{-\int(n-1)f(x)dx} y^n$
Reduced Riccati	$y' = f_1(x)\, y + f_2(x)\, y^2$	0	$e^{-\int f_1 dx}$

The above table shows that

$$
\begin{aligned}
\xi(x, y) &= 0 \\
\eta(x, y) &= \frac{1}{\sqrt{x^2+1}}
\end{aligned}
\tag{A1}
$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x, y) \to (R, S)$ where (R, S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$
\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}
$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y}\right) S(x,y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$R = x$$

S is found from

$$\begin{aligned} S &= \int \frac{1}{\eta} dy \\ &= \int \frac{1}{\frac{1}{\sqrt{x^2+1}}} dy \end{aligned}$$

Which results in

$$S = \sqrt{x^2+1}\, y$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x,y) S_y}{R_x + \omega(x,y) R_y} \tag{2}$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x,y)$ is the right hand side of the original ode given by

$$\omega(x,y) = -\frac{x(y-1)}{x^2+1}$$

Evaluating all the partial derivatives gives

$$\begin{aligned} R_x &= 1 \\ R_y &= 0 \\ S_x &= \frac{yx}{\sqrt{x^2+1}} \\ S_y &= \sqrt{x^2+1} \end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = \frac{x}{\sqrt{x^2+1}} \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = \frac{R}{\sqrt{R^2+1}}$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordinates R, S. Integrating the above gives

$$S(R) = \sqrt{R^2 + 1} + c_1 \tag{4}$$

To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$\sqrt{x^2 + 1}\, y = \sqrt{x^2 + 1} + c_1$$

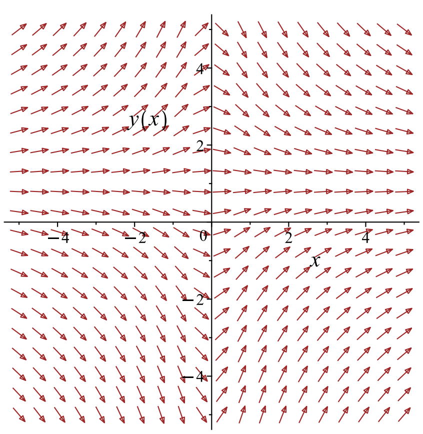
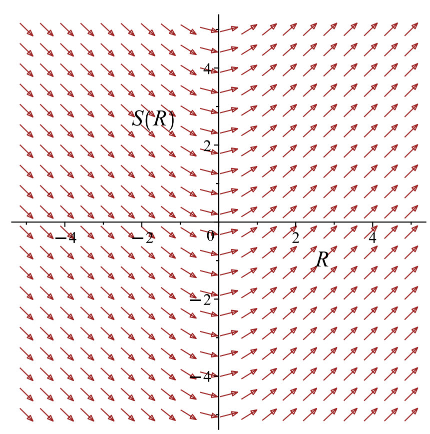
Which simplifies to

$$(y - 1)\sqrt{x^2 + 1} - c_1 = 0$$

Which gives

$$y = \frac{\sqrt{x^2 + 1} + c_1}{\sqrt{x^2 + 1}}$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in x, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dx} = -\frac{x(y-1)}{x^2+1}$ 	$R = x$ $S = \sqrt{x^2 + 1}\, y$	$\frac{dS}{dR} = \frac{R}{\sqrt{R^2+1}}$ 

Summary

The solution(s) found are the following

$$y = \frac{\sqrt{x^2+1}+c_1}{\sqrt{x^2+1}} \tag{1}$$

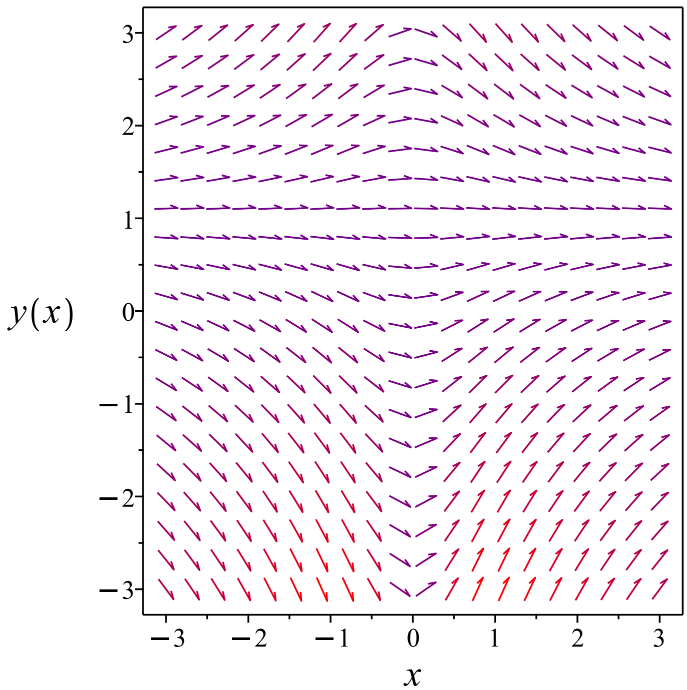


Figure 133: Slope field plot

Verification of solutions

$$y = \frac{\sqrt{x^2+1}+c_1}{\sqrt{x^2+1}}$$

Verified OK.

2.15.4 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x,y) + N(x,y)\frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x,y) = c$ where c is constant, that satisfies the

ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y) = 0$$

Hence

$$\frac{\partial \phi}{\partial x} + \frac{\partial \phi}{\partial y}\frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$\frac{\partial \phi}{\partial x} = M$$
$$\frac{\partial \phi}{\partial y} = N$$

But since $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x,y)$ but at least we know now that we can do that since the condition $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x,y)\,\mathrm{d}x + N(x,y)\,\mathrm{d}y = 0 \tag{1A}$$

Therefore

$$\left(\frac{1}{1-y}\right)\mathrm{d}y = \left(\frac{x}{x^2+1}\right)\mathrm{d}x$$
$$\left(-\frac{x}{x^2+1}\right)\mathrm{d}x + \left(\frac{1}{1-y}\right)\mathrm{d}y = 0 \tag{2A}$$

Comparing (1A) and (2A) shows that

$$M(x,y) = -\frac{x}{x^2+1}$$
$$N(x,y) = \frac{1}{1-y}$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Using result found above gives

$$\begin{aligned}\frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}\left(-\frac{x}{x^2+1}\right)\\ &= 0\end{aligned}$$

And

$$\begin{aligned}\frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}\left(\frac{1}{1-y}\right)\\ &= 0\end{aligned}$$

Since $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$, then the ODE is exact The following equations are now set up to solve for the function $\phi(x,y)$

$$\frac{\partial \phi}{\partial x} = M \tag{1}$$

$$\frac{\partial \phi}{\partial y} = N \tag{2}$$

Integrating (1) w.r.t. x gives

$$\int \frac{\partial \phi}{\partial x}\,\mathrm{d}x = \int M\,\mathrm{d}x$$

$$\int \frac{\partial \phi}{\partial x}\,\mathrm{d}x = \int -\frac{x}{x^2+1}\,\mathrm{d}x$$

$$\phi = -\frac{\ln(x^2+1)}{2} + f(y) \tag{3}$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$\frac{\partial \phi}{\partial y} = 0 + f'(y) \tag{4}$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = \frac{1}{1-y}$. Therefore equation (4) becomes

$$\frac{1}{1-y} = 0 + f'(y) \tag{5}$$

Solving equation (5) for $f'(y)$ gives

$$f'(y) = -\frac{1}{y-1}$$

Integrating the above w.r.t y gives

$$\int f'(y)\,\mathrm{d}y = \int \left(-\frac{1}{y-1}\right)\mathrm{d}y$$
$$f(y) = -\ln(y-1) + c_1$$

Where c_1 is constant of integration. Substituting result found above for $f(y)$ into equation (3) gives ϕ

$$\phi = -\frac{\ln(x^2+1)}{2} - \ln(y-1) + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = -\frac{\ln(x^2+1)}{2} - \ln(y-1)$$

The solution becomes

$$y = \mathrm{e}^{-\frac{\ln(x^2+1)}{2} - c_1} + 1$$

Summary

The solution(s) found are the following

$$y = \mathrm{e}^{-\frac{\ln(x^2+1)}{2} - c_1} + 1 \tag{1}$$

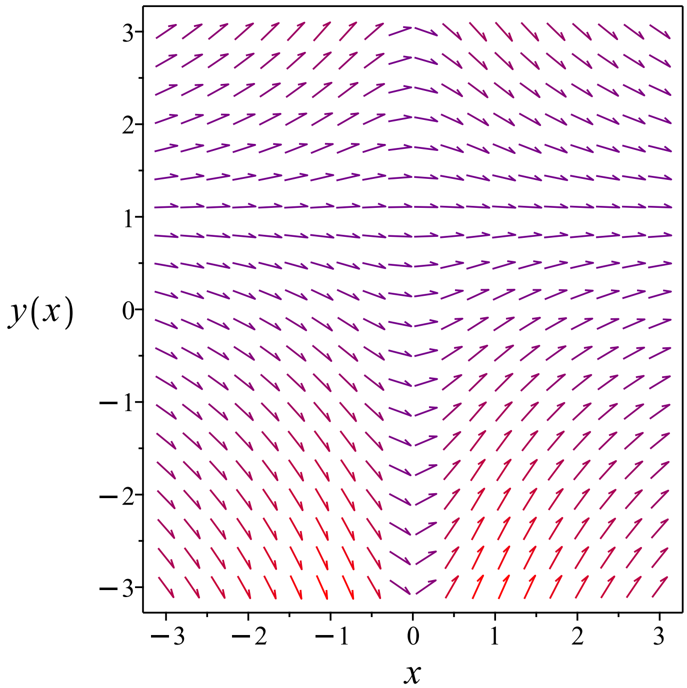


Figure 134: Slope field plot

Verification of solutions

$$y = e^{-\frac{\ln(x^2+1)}{2}-c_1} + 1$$

Verified OK.

2.15.5 Maple step by step solution

Let's solve

$(x^2+1)\,y' + xy = x$

- Highest derivative means the order of the ODE is 1

  y'

- Separate variables

  $\frac{y'}{y-1} = -\frac{x}{x^2+1}$

- Integrate both sides with respect to x

  $\int \frac{y'}{y-1}dx = \int -\frac{x}{x^2+1}dx + c_1$

- Evaluate integral

$$\ln(y-1) = -\frac{\ln(x^2+1)}{2} + c_1$$

- Solve for y

$$y = e^{-\frac{\ln(x^2+1)}{2}+c_1} + 1$$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
<- 1st order linear successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 15

```
dsolve((x^2+1)*diff(y(x),x)+x*y(x)-x=0,y(x), singsol=all)
```

$$y(x) = 1 + \frac{c_1}{\sqrt{x^2+1}}$$

✓ Solution by Mathematica

Time used: 0.033 (sec). Leaf size: 24

```
DSolve[(x^2+1)*y'[x]+x*y[x]-x==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to 1 + \frac{c_1}{\sqrt{x^2+1}}$$
$$y(x) \to 1$$

2.16 problem 16

Internal problem ID [4965]
Internal file name [`OUTPUT/4458_Sunday_June_05_2022_02_56_59_PM_50786068/index.tex`]

Book: Fundamentals of Differential Equations. By Nagle, Saff and Snider. 9th edition. Boston. Pearson 2018.
Section: Chapter 2, First order differential equations. Section 2.3, Linear equations. Exercises. page 54
Problem number: 16.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"linear", "exactWithIntegrationFactor", "first_order_ode_lie_symmetry_lookup"**

Maple gives the following as the ode type

`[_linear]`

$$\boxed{\left(-x^{2}+1\right) y'-y x^{2}=\left(x+1\right) \sqrt{-x^{2}+1}}$$

2.16.1 Solving as linear ode

Entering Linear first order ODE solver. In canonical form a linear first order is

$$y'+p(x)y=q(x)$$

Where here

$$p(x)=\frac{x^{2}}{x^{2}-1}$$
$$q(x)=-\frac{\sqrt{-x^{2}+1}}{x-1}$$

Hence the ode is

$$y'+\frac{x^{2}y}{x^{2}-1}=-\frac{\sqrt{-x^{2}+1}}{x-1}$$

The integrating factor μ is

$$
\begin{aligned}
\mu &= \mathrm{e}^{\int \frac{x^2}{x^2-1}dx} \\
&= \mathrm{e}^{x+\frac{\ln(x-1)}{2}-\frac{\ln(x+1)}{2}}
\end{aligned}
$$

Which simplifies to

$$
\mu = \frac{\sqrt{x-1}\,\mathrm{e}^{x}}{\sqrt{x+1}}
$$

The ode becomes

$$
\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}x}(\mu y) &= (\mu)\left(-\frac{\sqrt{-x^2+1}}{x-1}\right) \\
\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{\sqrt{x-1}\,\mathrm{e}^{x} y}{\sqrt{x+1}}\right) &= \left(\frac{\sqrt{x-1}\,\mathrm{e}^{x}}{\sqrt{x+1}}\right)\left(-\frac{\sqrt{-x^2+1}}{x-1}\right) \\
\mathrm{d}\left(\frac{\sqrt{x-1}\,\mathrm{e}^{x} y}{\sqrt{x+1}}\right) &= \left(-\frac{\sqrt{-x^2+1}\,\mathrm{e}^{x}}{\sqrt{x-1}\,\sqrt{x+1}}\right)\mathrm{d}x
\end{aligned}
$$

Integrating gives

$$
\begin{aligned}
\frac{\sqrt{x-1}\,\mathrm{e}^{x} y}{\sqrt{x+1}} &= \int -\frac{\sqrt{-x^2+1}\,\mathrm{e}^{x}}{\sqrt{x-1}\,\sqrt{x+1}}\,\mathrm{d}x \\
\frac{\sqrt{x-1}\,\mathrm{e}^{x} y}{\sqrt{x+1}} &= -\frac{\sqrt{-x^2+1}\,\mathrm{e}^{x}}{\sqrt{x-1}\,\sqrt{x+1}} + c_1
\end{aligned}
$$

Dividing both sides by the integrating factor $\mu = \frac{\sqrt{x-1}\,\mathrm{e}^{x}}{\sqrt{x+1}}$ results in

$$
y = -\frac{\mathrm{e}^{-x}\sqrt{-x^2+1}\,\mathrm{e}^{x}}{x-1} + \frac{c_1\sqrt{x+1}\,\mathrm{e}^{-x}}{\sqrt{x-1}}
$$

which simplifies to

$$
y = \frac{-\sqrt{-x^2+1}\,\sqrt{x-1} + c_1\sqrt{x+1}\,\mathrm{e}^{-x}(x-1)}{(x-1)^{\frac{3}{2}}}
$$

<u>Summary</u>
The solution(s) found are the following

$$
y = \frac{-\sqrt{-x^2+1}\,\sqrt{x-1} + c_1\sqrt{x+1}\,\mathrm{e}^{-x}(x-1)}{(x-1)^{\frac{3}{2}}} \tag{1}
$$

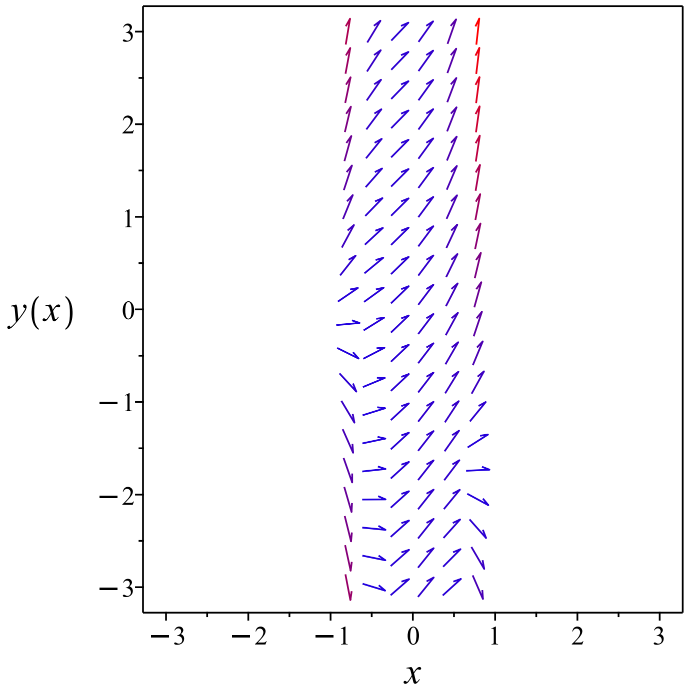


Figure 135: Slope field plot

Verification of solutions

$$y = \frac{-\sqrt{-x^2+1}\,\sqrt{x-1} + c_1\sqrt{x+1}\,\mathrm{e}^{-x}(x-1)}{(x-1)^{\frac{3}{2}}}$$

Verified OK.

2.16.2 Solving as first order ode lie symmetry lookup ode

Writing the ode as

$$y' = -\frac{y\,x^2 + \sqrt{-x^2+1}\,x + \sqrt{-x^2+1}}{x^2-1}$$
$$y' = \omega(x,y)$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x + \omega(\eta_y - \xi_x) - \omega^2\xi_y - \omega_x\xi - \omega_y\eta = 0 \tag{A}$$

The type of this ode is known. It is of type `linear`. Therefore we do not need to solve the PDE (A), and can just use the lookup table shown below to find ξ, η

Table 128: Lie symmetry infinitesimal lookup table for known first order ODE's

ODE class	Form	ξ	η
linear ode	$y' = f(x)y(x) + g(x)$	0	$e^{\int f dx}$
separable ode	$y' = f(x)\, g(y)$	$\frac{1}{f}$	0
quadrature ode	$y' = f(x)$	0	1
quadrature ode	$y' = g(y)$	1	0
homogeneous ODEs of Class A	$y' = f\left(\frac{y}{x}\right)$	x	y
homogeneous ODEs of Class C	$y' = (a + bx + cy)^{\frac{n}{m}}$	1	$-\frac{b}{c}$
homogeneous class D	$y' = \frac{y}{x} + g(x)\, F\left(\frac{y}{x}\right)$	x^2	xy
First order special form ID 1	$y' = g(x)\, e^{h(x)+by} + f(x)$	$\frac{e^{-\int bf(x)dx - h(x)}}{g(x)}$	$\frac{f(x)e^{-\int bf(x)dx - h(x)}}{g(x)}$
polynomial type ode	$y' = \frac{a_1x+b_1y+c_1}{a_2x+b_2y+c_2}$	$\frac{a_1b_2x - a_2b_1x - b_1c_2 + b_2c_1}{a_1b_2 - a_2b_1}$	$\frac{a_1b_2y - a_2b_1y - a_1c_2 - a_2c_1}{a_1b_2 - a_2b_1}$
Bernoulli ode	$y' = f(x)\, y + g(x)\, y^n$	0	$e^{-\int (n-1)f(x)dx} y^n$
Reduced Riccati	$y' = f_1(x)\, y + f_2(x)\, y^2$	0	$e^{-\int f_1 dx}$

The above table shows that

$$
\begin{aligned}
\xi(x, y) &= 0 \\
\eta(x, y) &= \mathrm{e}^{-x - \frac{\ln(x-1)}{2} + \frac{\ln(x+1)}{2}}
\end{aligned}
\tag{A1}
$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x, y) \to (R, S)$ where (R, S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$
\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}
$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y}\right) S(x, y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the

canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$R = x$$

S is found from

$$\begin{aligned}S &= \int \frac{1}{\eta} dy \\ &= \int \frac{1}{\mathrm{e}^{-x-\frac{\ln(x-1)}{2}+\frac{\ln(x+1)}{2}}} dy\end{aligned}$$

Which results in

$$S = \mathrm{e}^{x+\ln\left(\sqrt{x-1}\right)+\ln\left(\frac{1}{\sqrt{x+1}}\right)} y$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x,y) S_y}{R_x + \omega(x,y) R_y} \tag{2}$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x,y)$ is the right hand side of the original ode given by

$$\omega(x,y) = -\frac{y\,x^2 + \sqrt{-x^2+1}\,x + \sqrt{-x^2+1}}{x^2-1}$$

Evaluating all the partial derivatives gives

$$\begin{aligned}R_x &= 1 \\ R_y &= 0 \\ S_x &= \frac{\mathrm{e}^x y\,x^2}{\sqrt{x-1}\,(x+1)^{\frac{3}{2}}} \\ S_y &= \frac{\sqrt{x-1}\,\mathrm{e}^x}{\sqrt{x+1}}\end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = -\frac{\sqrt{-x^2+1}\,\mathrm{e}^x}{\sqrt{x-1}\,\sqrt{x+1}} \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = -\frac{\sqrt{-R^2+1}\,\mathrm{e}^R}{\sqrt{R-1}\,\sqrt{R+1}}$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$S(R) = -\frac{\sqrt{-R^2+1}\,\mathrm{e}^R}{\sqrt{R-1}\,\sqrt{R+1}} + c_1 \tag{4}$$

To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$\frac{\sqrt{x-1}\,\mathrm{e}^x y}{\sqrt{x+1}} = -\frac{\sqrt{-x^2+1}\,\mathrm{e}^x}{\sqrt{x-1}\,\sqrt{x+1}} + c_1$$

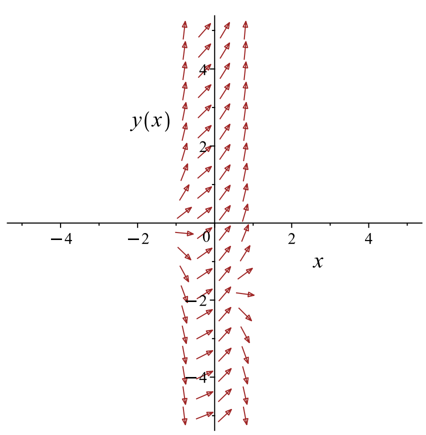
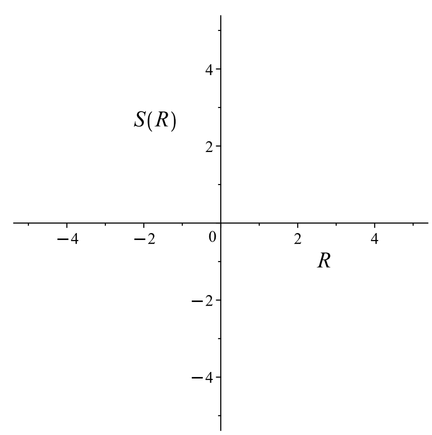
Which simplifies to

$$\frac{\sqrt{x-1}\,\mathrm{e}^x y}{\sqrt{x+1}} = -\frac{\sqrt{-x^2+1}\,\mathrm{e}^x}{\sqrt{x-1}\,\sqrt{x+1}} + c_1$$

Which gives

$$y = -\frac{\left(-c_1\sqrt{x-1}\,\sqrt{x+1} + \sqrt{-x^2+1}\,\mathrm{e}^x\right)\mathrm{e}^{-x}}{x-1}$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in x, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dx} = -\frac{y\,x^2+\sqrt{-x^2+1}\,x+\sqrt{-x^2+1}}{x^2-1}$ 	$R = x$ $S = \frac{\sqrt{x-1}\,\mathrm{e}^x y}{\sqrt{x+1}}$	$\frac{dS}{dR} = -\frac{\sqrt{-R^2+1}\,\mathrm{e}^R}{\sqrt{R-1}\,\sqrt{R+1}}$ 

Summary
The solution(s) found are the following

$$y = -\frac{\left(-c_1\sqrt{x-1}\,\sqrt{x+1}+\sqrt{-x^2+1}\,\mathrm{e}^{x}\right)\mathrm{e}^{-x}}{x-1} \tag{1}$$

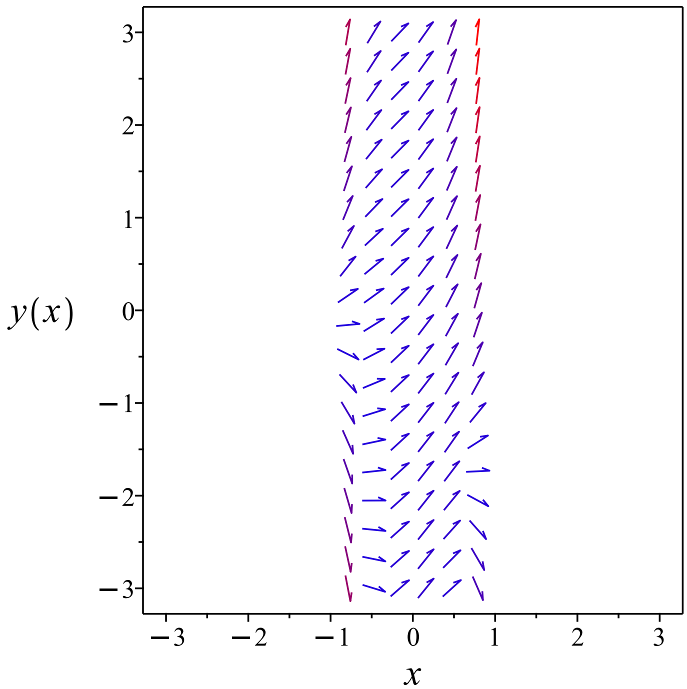


Figure 136: Slope field plot

Verification of solutions

$$y = -\frac{\left(-c_1\sqrt{x-1}\,\sqrt{x+1}+\sqrt{-x^2+1}\,\mathrm{e}^{x}\right)\mathrm{e}^{-x}}{x-1}$$

Verified OK.

2.16.3 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x,y)+N(x,y)\frac{dy}{dx}=0 \tag{A}$$

We assume there exists a function $\phi(x,y)=c$ where c is constant, that satisfies the

ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y) = 0$$

Hence

$$\frac{\partial \phi}{\partial x} + \frac{\partial \phi}{\partial y}\frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$\begin{aligned}\frac{\partial \phi}{\partial x} &= M\\ \frac{\partial \phi}{\partial y} &= N\end{aligned}$$

But since $\frac{\partial^2\phi}{\partial x\partial y} = \frac{\partial^2\phi}{\partial y\partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x,y)$ but at least we know now that we can do that since the condition $\frac{\partial^2\phi}{\partial x\partial y} = \frac{\partial^2\phi}{\partial y\partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x,y)\,\mathrm{d}x + N(x,y)\,\mathrm{d}y = 0 \tag{1A}$$

Therefore

$$\left(-x^2+1\right)\mathrm{d}y = \left(y\,x^2 + (x+1)\sqrt{-x^2+1}\right)\mathrm{d}x$$

$$\left(-y\,x^2 - (x+1)\sqrt{-x^2+1}\right)\mathrm{d}x + \left(-x^2+1\right)\mathrm{d}y = 0 \tag{2A}$$

Comparing (1A) and (2A) shows that

$$\begin{aligned}M(x,y) &= -y\,x^2 - (x+1)\sqrt{-x^2+1}\\ N(x,y) &= -x^2+1\end{aligned}$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Using result found above gives

$$
\begin{aligned}
\frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}\left(-y\,x^2 - (x+1)\sqrt{-x^2+1}\right) \\
&= -x^2
\end{aligned}
$$

And

$$
\begin{aligned}
\frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}\left(-x^2+1\right) \\
&= -2x
\end{aligned}
$$

Since $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$, then the ODE is <u>not exact</u>. Since the ODE is not exact, we will try to find an integrating factor to make it exact. Let

$$
\begin{aligned}
A &= \frac{1}{N}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right) \\
&= -\frac{1}{x^2-1}\left(\left(-x^2\right) - \left(-2x\right)\right) \\
&= \frac{x(-2+x)}{x^2-1}
\end{aligned}
$$

Since A does not depend on y, then it can be used to find an integrating factor. The integrating factor μ is

$$
\begin{aligned}
\mu &= e^{\int A \, \mathrm{d}x} \\
&= e^{\int \frac{x(-2+x)}{x^2-1} \, \mathrm{d}x}
\end{aligned}
$$

The result of integrating gives

$$
\begin{aligned}
\mu &= e^{x - \frac{\ln(x-1)}{2} - \frac{3\ln(x+1)}{2}} \\
&= \frac{\mathrm{e}^{x}}{\sqrt{x-1}\,(x+1)^{\frac{3}{2}}}
\end{aligned}
$$

M and N are multiplied by this integrating factor, giving new M and new N which are called $\overline{M}$ and $\overline{N}$ for now so not to confuse them with the original M and N.

$$
\begin{aligned}
\overline{M} &= \mu M \\
&= \frac{\mathrm{e}^{x}}{\sqrt{x-1}\,(x+1)^{\frac{3}{2}}}\left(-y\,x^2 - (x+1)\sqrt{-x^2+1}\right) \\
&= \frac{\left(-y\,x^2 - \sqrt{-x^2+1}\,x - \sqrt{-x^2+1}\right)\mathrm{e}^{x}}{\sqrt{x-1}\,(x+1)^{\frac{3}{2}}}
\end{aligned}
$$

And

$$\begin{aligned}\overline{N} &= \mu N \\ &= \frac{e^{x}}{\sqrt{x-1}\,(x+1)^{\frac{3}{2}}}(-x^{2}+1) \\ &= -\frac{\sqrt{x-1}\,e^{x}}{\sqrt{x+1}}\end{aligned}$$

Now a modified ODE is ontained from the original ODE, which is exact and can be solved. The modified ODE is

$$\overline{M} + \overline{N}\frac{dy}{dx} = 0$$

$$\left(\frac{\left(-y\,x^{2}-\sqrt{-x^{2}+1}\,x-\sqrt{-x^{2}+1}\right)e^{x}}{\sqrt{x-1}\,(x+1)^{\frac{3}{2}}}\right) + \left(-\frac{\sqrt{x-1}\,e^{x}}{\sqrt{x+1}}\right)\frac{dy}{dx} = 0$$

The following equations are now set up to solve for the function $\phi(x,y)$

$$\frac{\partial \phi}{\partial x} = \overline{M} \tag{1}$$

$$\frac{\partial \phi}{\partial y} = \overline{N} \tag{2}$$

Integrating (1) w.r.t. x gives

$$\int \frac{\partial \phi}{\partial x}\,dx = \int \overline{M}\,dx$$

$$\int \frac{\partial \phi}{\partial x}\,dx = \int \frac{\left(-y\,x^{2}-\sqrt{-x^{2}+1}\,x-\sqrt{-x^{2}+1}\right)e^{x}}{\sqrt{x-1}\,(x+1)^{\frac{3}{2}}}\,dx$$

$$\phi = \int^{x} \frac{\left(-y\_a^{2}-\sqrt{-\_a^{2}+1}\,\_a-\sqrt{-\_a^{2}+1}\right)e^{-a}}{\sqrt{\_a-1}\,(\_a+1)^{\frac{3}{2}}}d\_a + f(y) \tag{3}$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$\frac{\partial \phi}{\partial y} = -\frac{\sqrt{x-1}\,e^{x}}{\sqrt{x+1}} + f'(y) \tag{4}$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = -\frac{\sqrt{x-1}\,e^{x}}{\sqrt{x+1}}$. Therefore equation (4) becomes

$$-\frac{\sqrt{x-1}\,e^{x}}{\sqrt{x+1}} = -\frac{\sqrt{x-1}\,e^{x}}{\sqrt{x+1}} + f'(y) \tag{5}$$

Solving equation (5) for $f'(y)$ gives

$$f'(y) = 0$$

Therefore

$$f(y) = c_1$$

Where c_1 is constant of integration. Substituting this result for $f(y)$ into equation (3) gives ϕ

$$\phi = \int^x \frac{\left(-y\_a^2 - \sqrt{-\_a^2+1}\,\_a - \sqrt{-\_a^2+1}\right) e^{-\_a}}{\sqrt{\_a-1}\,(\_a+1)^{\frac{3}{2}}} d\_a + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = \int^x \frac{\left(-y\_a^2 - \sqrt{-\_a^2+1}\,\_a - \sqrt{-\_a^2+1}\right) e^{-\_a}}{\sqrt{\_a-1}\,(\_a+1)^{\frac{3}{2}}} d\_a$$

Summary
The solution(s) found are the following

$$\int^x \frac{\left(-y\_a^2 - \sqrt{-\_a^2+1}\,\_a - \sqrt{-\_a^2+1}\right) e^{-\_a}}{\sqrt{\_a-1}\,(\_a+1)^{\frac{3}{2}}} d\_a = c_1 \tag{1}$$

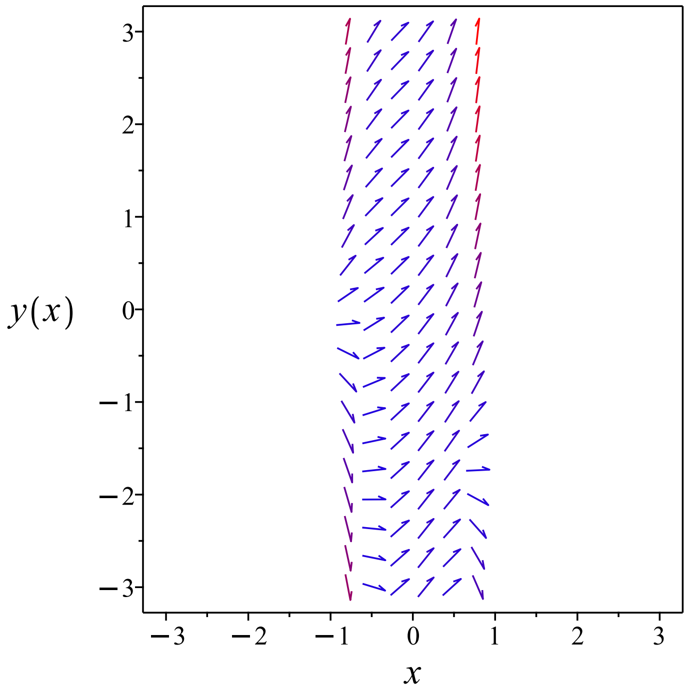


Figure 137: Slope field plot

Verification of solutions

$$\int^{x} \frac{\left(-y\_a^{2}-\sqrt{-\_a^{2}+1}\,\_a-\sqrt{-\_a^{2}+1}\right) \mathrm{e}^{-a}}{\sqrt{\_a-1}\,\left(\_a+1\right)^{\frac{3}{2}}}d\_a = c_1$$

Verified OK.

2.16.4 Maple step by step solution

Let's solve

$\left(-x^{2}+1\right) y'-y x^{2}=\left(x+1\right) \sqrt{-x^{2}+1}$

- Highest derivative means the order of the ODE is 1

  y'

- Isolate the derivative

  $y'=-\frac{x^{2} y}{x^{2}-1}+\frac{x+1}{\sqrt{-x^{2}+1}}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE

  $y'+\frac{x^{2} y}{x^{2}-1}=\frac{x+1}{\sqrt{-x^{2}+1}}$

- The ODE is linear; multiply by an integrating factor $\mu(x)$

$$\mu(x)\left(y'+\frac{x^2 y}{x^2-1}\right)=\frac{\mu(x)(x+1)}{\sqrt{-x^2+1}}$$

- Assume the lhs of the ODE is the total derivative $\frac{d}{dx}(\mu(x)\,y)$

$$\mu(x)\left(y'+\frac{x^2 y}{x^2-1}\right)=\mu'(x)\,y+\mu(x)\,y'$$

- Isolate $\mu'(x)$

$$\mu'(x)=\frac{\mu(x)x^2}{x^2-1}$$

- Solve to find the integrating factor

$$\mu(x)=\frac{\sqrt{x-1}\,\mathrm{e}^{x}}{\sqrt{x+1}}$$

- Integrate both sides with respect to x

$$\int\left(\frac{d}{dx}(\mu(x)\,y)\right)dx=\int\frac{\mu(x)(x+1)}{\sqrt{-x^2+1}}dx+c_1$$

- Evaluate the integral on the lhs

$$\mu(x)\,y=\int\frac{\mu(x)(x+1)}{\sqrt{-x^2+1}}dx+c_1$$

- Solve for y

$$y=\frac{\int\frac{\mu(x)(x+1)}{\sqrt{-x^2+1}}dx+c_1}{\mu(x)}$$

- Substitute $\mu(x)=\frac{\sqrt{x-1}\,\mathrm{e}^{x}}{\sqrt{x+1}}$

$$y=\frac{\sqrt{x+1}\left(\int\frac{\sqrt{x-1}\,\sqrt{x+1}\,\mathrm{e}^{x}}{\sqrt{-x^2+1}}dx+c_1\right)}{\sqrt{x-1}\,\mathrm{e}^{x}}$$

- Evaluate the integrals on the rhs

$$y=\frac{\sqrt{x+1}\left(\frac{\sqrt{x-1}\,\sqrt{x+1}\,\mathrm{e}^{x}}{\sqrt{-x^2+1}}+c_1\right)}{\sqrt{x-1}\,\mathrm{e}^{x}}$$

- Simplify

$$y=\frac{\sqrt{x+1}\left(\sqrt{x-1}\,\sqrt{x+1}+c_1\sqrt{-x^2+1}\,\mathrm{e}^{-x}\right)}{\sqrt{-x^2+1}\,\sqrt{x-1}}$$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
<- 1st order linear successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 52

```
dsolve((1-x^2)*diff(y(x),x)-x^2*y(x)=(1+x)*sqrt(1-x^2),y(x), singsol=all)
```

$$y(x) = \frac{\mathrm{e}^{-x}\sqrt{1+x}\,c_1\sqrt{-x^2+1}+\sqrt{x-1}\,x+\sqrt{x-1}}{\sqrt{-x^2+1}\,\sqrt{x-1}}$$

✓ Solution by Mathematica

Time used: 0.085 (sec). Leaf size: 33

```
DSolve[(1-x^2)*y'[x]-x^2*y[x]==(1+x)*Sqrt[1-x^2],y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \frac{e^{-x}\sqrt{x+1}(e^x+c_1)}{\sqrt{1-x}}$$

2.17 problem 17

Internal problem ID [4966]
Internal file name [`OUTPUT/4459_Sunday_June_05_2022_02_57_00_PM_19801828/index.tex`]

Book: Fundamentals of Differential Equations. By Nagle, Saff and Snider. 9th edition. Boston. Pearson 2018.
Section: Chapter 2, First order differential equations. Section 2.3, Linear equations. Exercises. page 54
Problem number: 17.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"linear", "homogeneousTypeD2", "exactWithIntegrationFactor", "first_order_ode_lie_symmetry_lookup"**

Maple gives the following as the ode type

`[_linear]`

$$\boxed{y' - \frac{y}{x} = \mathrm{e}^{x} x}$$

With initial conditions

$$[y(1) = \mathrm{e} - 1]$$

2.17.1 Existence and uniqueness analysis

This is a linear ODE. In canonical form it is written as

$$y' + p(x)y = q(x)$$

Where here

$$\begin{aligned} p(x) &= -\frac{1}{x} \\ q(x) &= \mathrm{e}^{x} x \end{aligned}$$

Hence the ode is

$$y' - \frac{y}{x} = \mathrm{e}^{x} x$$

The domain of $p(x) = -\frac{1}{x}$ is

$$\{x < 0 \lor 0 < x\}$$

And the point $x_0 = 1$ is inside this domain. The domain of $q(x) = \mathrm{e}^{x} x$ is

$$\{-\infty < x < \infty\}$$

And the point $x_0 = 1$ is also inside this domain. Hence solution exists and is unique.

2.17.2 Solving as linear ode

Entering Linear first order ODE solver. The integrating factor μ is

$$\begin{aligned} \mu &= \mathrm{e}^{\int -\frac{1}{x}dx} \\ &= \frac{1}{x} \end{aligned}$$

The ode becomes

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}x}(\mu y) &= (\mu)\left(\mathrm{e}^{x} x\right) \\ \frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{y}{x}\right) &= \left(\frac{1}{x}\right)\left(\mathrm{e}^{x} x\right) \\ \mathrm{d}\left(\frac{y}{x}\right) &= \mathrm{e}^{x}\,\mathrm{d}x \end{aligned}$$

Integrating gives

$$\frac{y}{x} = \int e^{x}\, dx$$

$$\frac{y}{x} = e^{x} + c_1$$

Dividing both sides by the integrating factor $\mu = \frac{1}{x}$ results in

$$y = e^{x} x + c_1 x$$

which simplifies to

$$y = x(e^{x} + c_1)$$

Initial conditions are used to solve for c_1. Substituting $x = 1$ and $y = e - 1$ in the above solution gives an equation to solve for the constant of integration.

$$e - 1 = e + c_1$$

$$c_1 = -1$$

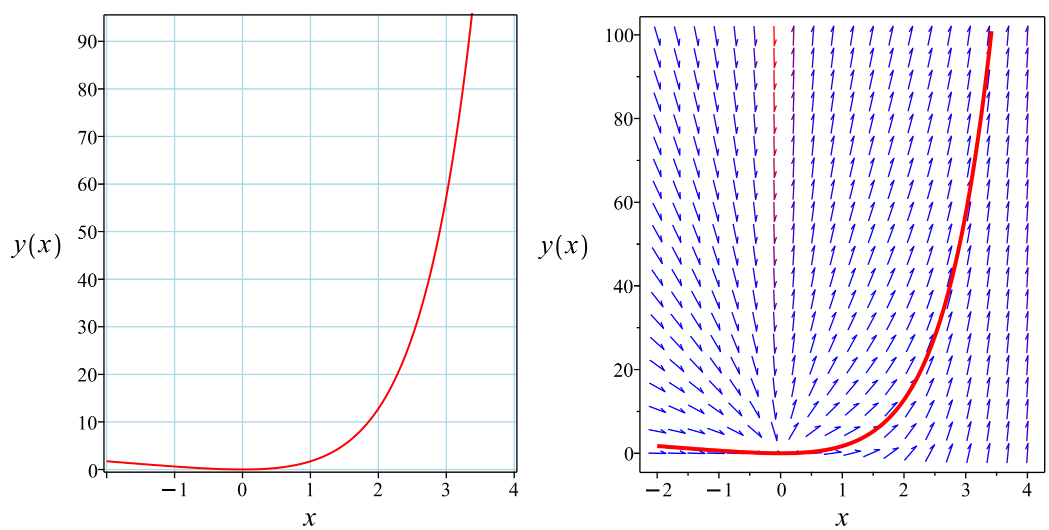
Substituting c_1 found above in the general solution gives

$$y = x(e^{x} - 1)$$

Summary

The solution(s) found are the following

$$y = x(e^{x} - 1) \tag{1}$$



(a) Solution plot

(b) Slope field plot

Verification of solutions

$$y = x(\mathrm{e}^x - 1)$$

Verified OK.

2.17.3 Solving as homogeneousTypeD2 ode

Using the change of variables $y = u(x)\,x$ on the above ode results in new ode in $u(x)$

$$u'(x)\,x = \mathrm{e}^x x$$

Integrating both sides gives

$$\begin{aligned} u(x) &= \int \mathrm{e}^x \,\mathrm{d}x \\ &= \mathrm{e}^x + c_2 \end{aligned}$$

Therefore the solution y is

$$\begin{aligned} y &= xu \\ &= x(\mathrm{e}^x + c_2) \end{aligned}$$

Initial conditions are used to solve for c_2. Substituting $x = 1$ and $y = \mathrm{e} - 1$ in the above solution gives an equation to solve for the constant of integration.

$$\mathrm{e} - 1 = \mathrm{e} + c_2$$

$$c_2 = -1$$

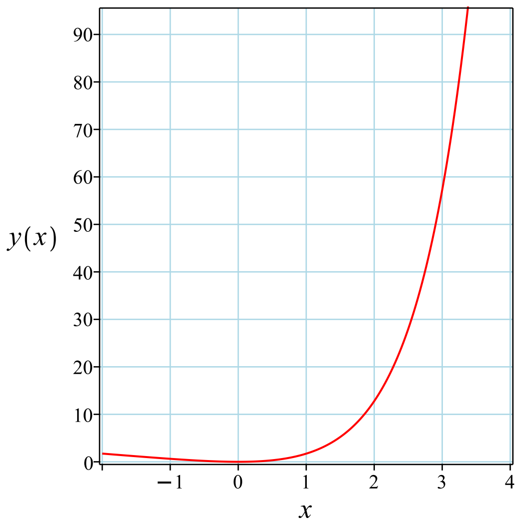
Substituting c_2 found above in the general solution gives

$$y = x(\mathrm{e}^x - 1)$$

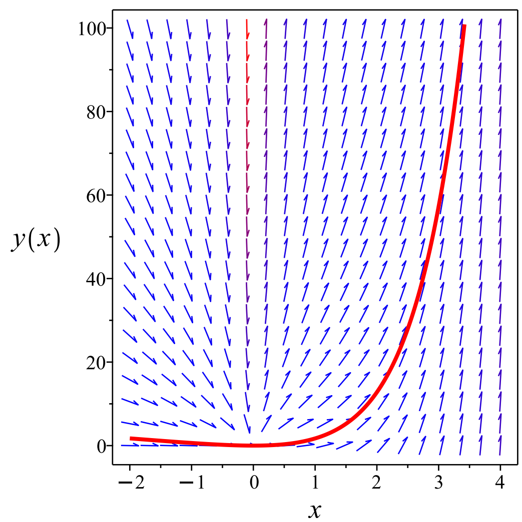
Summary

The solution(s) found are the following

$$y = x(\mathrm{e}^x - 1) \tag{1}$$



(a) Solution plot



(b) Slope field plot

Verification of solutions

$$y = x(e^{x} - 1)$$

Verified OK.

2.17.4 Solving as first order ode lie symmetry lookup ode

Writing the ode as

$$y' = \frac{x^2 e^{x} + y}{x}$$
$$y' = \omega(x, y)$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x + \omega(\eta_y - \xi_x) - \omega^2 \xi_y - \omega_x \xi - \omega_y \eta = 0 \tag{A}$$

The type of this ode is known. It is of type `linear`. Therefore we do not need to solve the PDE (A), and can just use the lookup table shown below to find ξ, η

Table 131: Lie symmetry infinitesimal lookup table for known first order ODE's

ODE class	Form	ξ	η
linear ode	$y' = f(x)y(x) + g(x)$	0	$e^{\int f dx}$
separable ode	$y' = f(x)\, g(y)$	$\frac{1}{f}$	0
quadrature ode	$y' = f(x)$	0	1
quadrature ode	$y' = g(y)$	1	0
homogeneous ODEs of Class A	$y' = f\left(\frac{y}{x}\right)$	x	y
homogeneous ODEs of Class C	$y' = (a + bx + cy)^{\frac{n}{m}}$	1	$-\frac{b}{c}$
homogeneous class D	$y' = \frac{y}{x} + g(x)\, F\left(\frac{y}{x}\right)$	x^2	xy
First order special form ID 1	$y' = g(x)\, e^{h(x)+by} + f(x)$	$\frac{e^{-\int bf(x)dx-h(x)}}{g(x)}$	$\frac{f(x)e^{-\int bf(x)dx-h(x)}}{g(x)}$
polynomial type ode	$y' = \frac{a_1x+b_1y+c_1}{a_2x+b_2y+c_2}$	$\frac{a_1b_2x-a_2b_1x-b_1c_2+b_2c_1}{a_1b_2-a_2b_1}$	$\frac{a_1b_2y-a_2b_1y-a_1c_2-a_2c_1}{a_1b_2-a_2b_1}$
Bernoulli ode	$y' = f(x)\, y + g(x)\, y^n$	0	$e^{-\int (n-1)f(x)dx} y^n$
Reduced Riccati	$y' = f_1(x)\, y + f_2(x)\, y^2$	0	$e^{-\int f_1 dx}$

The above table shows that

$$\begin{aligned} \xi(x,y) &= 0 \\ \eta(x,y) &= x \end{aligned} \tag{A1}$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x,y) \to (R,S)$ where (R,S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y}\right) S(x,y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the

canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$R = x$$

S is found from

$$\begin{aligned} S &= \int \frac{1}{\eta} dy \\ &= \int \frac{1}{x} dy \end{aligned}$$

Which results in

$$S = \frac{y}{x}$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x, y) S_y}{R_x + \omega(x, y) R_y} \tag{2}$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x, y)$ is the right hand side of the original ode given by

$$\omega(x, y) = \frac{x^2 \mathrm{e}^x + y}{x}$$

Evaluating all the partial derivatives gives

$$\begin{aligned} R_x &= 1 \\ R_y &= 0 \\ S_x &= -\frac{y}{x^2} \\ S_y &= \frac{1}{x} \end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = \mathrm{e}^x \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = \mathrm{e}^R$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordinates R, S. Integrating the above gives

$$S(R) = e^{R} + c_1 \tag{4}$$

To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$\frac{y}{x} = e^{x} + c_1$$

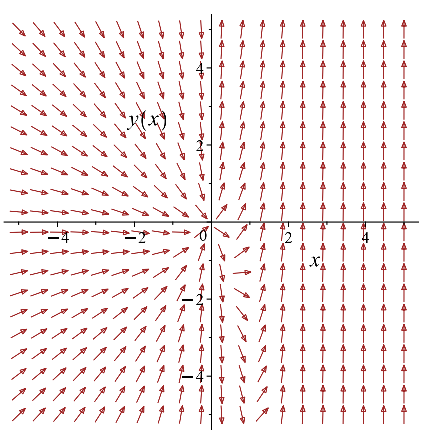
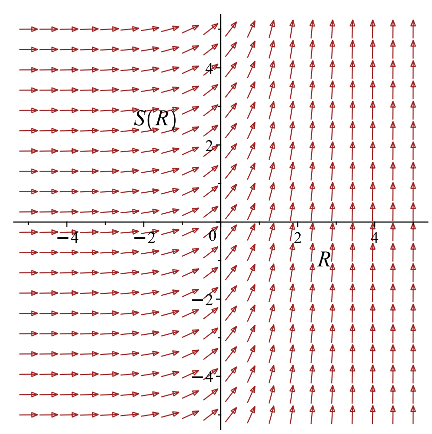
Which simplifies to

$$\frac{y}{x} = e^{x} + c_1$$

Which gives

$$y = x(e^{x} + c_1)$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in x, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dx} = \frac{x^2 e^x + y}{x}$ 	$R = x$ $S = \frac{y}{x}$	$\frac{dS}{dR} = e^{R}$ 

Initial conditions are used to solve for c_1. Substituting $x = 1$ and $y = e - 1$ in the above solution gives an equation to solve for the constant of integration.

$$e - 1 = e + c_1$$

$$c_1 = -1$$

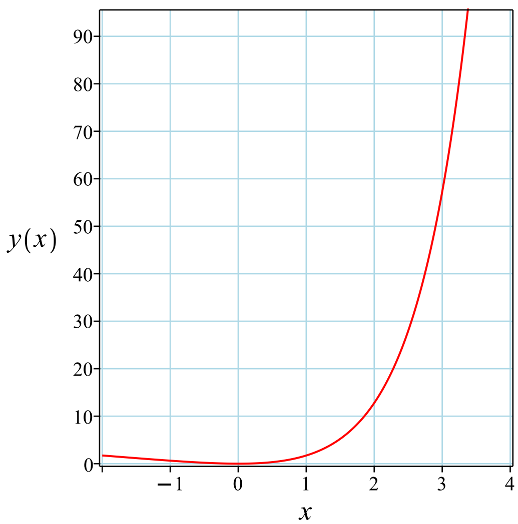
Substituting c_1 found above in the general solution gives

$$y = x(e^x - 1)$$

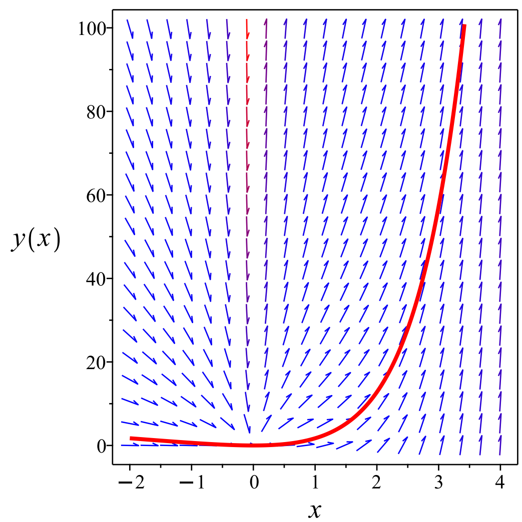
Summary

The solution(s) found are the following

$$y = x(e^x - 1) \tag{1}$$



(a) Solution plot



(b) Slope field plot

Verification of solutions

$$y = x(e^x - 1)$$

Verified OK.

2.17.5 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x, y) + N(x, y) \frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x, y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx} \phi(x, y) = 0$$

Hence

$$\frac{\partial \phi}{\partial x} + \frac{\partial \phi}{\partial y}\frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$\begin{aligned}\frac{\partial \phi}{\partial x} &= M\\ \frac{\partial \phi}{\partial y} &= N\end{aligned}$$

But since $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x, y)$ but at least we know now that we can do that since the condition $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x, y)\,\mathrm{d}x + N(x, y)\,\mathrm{d}y = 0 \tag{1A}$$

Therefore

$$\begin{aligned}\mathrm{d}y &= \left(\frac{y}{x} + \mathrm{e}^x x\right)\mathrm{d}x\\ \left(-\frac{y}{x} - \mathrm{e}^x x\right)\mathrm{d}x + \mathrm{d}y &= 0 \end{aligned} \tag{2A}$$

Comparing (1A) and (2A) shows that

$$\begin{aligned}M(x, y) &= -\frac{y}{x} - \mathrm{e}^x x\\ N(x, y) &= 1\end{aligned}$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Using result found above gives

$$\begin{aligned}\frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}\left(-\frac{y}{x} - \mathrm{e}^x x\right)\\ &= -\frac{1}{x}\end{aligned}$$

And

$$\begin{aligned}\frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}(1)\\ &= 0\end{aligned}$$

Since $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$, then the ODE is not exact. Since the ODE is not exact, we will try to find an integrating factor to make it exact. Let

$$\begin{aligned}A &= \frac{1}{N}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right)\\ &= 1\left(\left(-\frac{1}{x}\right) - (0)\right)\\ &= -\frac{1}{x}\end{aligned}$$

Since A does not depend on y, then it can be used to find an integrating factor. The integrating factor μ is

$$\begin{aligned}\mu &= e^{\int A \, \mathrm{d}x}\\ &= e^{\int -\frac{1}{x} \, \mathrm{d}x}\end{aligned}$$

The result of integrating gives

$$\begin{aligned}\mu &= e^{-\ln(x)}\\ &= \frac{1}{x}\end{aligned}$$

M and N are multiplied by this integrating factor, giving new M and new N which are called $\overline{M}$ and $\overline{N}$ for now so not to confuse them with the original M and N.

$$\begin{aligned}\overline{M} &= \mu M\\ &= \frac{1}{x}\left(-\frac{y}{x} - \mathrm{e}^{x} x\right)\\ &= \frac{-x^2\mathrm{e}^{x} - y}{x^2}\end{aligned}$$

And

$$\begin{aligned}\overline{N} &= \mu N\\ &= \frac{1}{x}(1)\\ &= \frac{1}{x}\end{aligned}$$

Now a modified ODE is ontained from the original ODE, which is exact and can be solved. The modified ODE is

$$\overline{M} + \overline{N}\frac{\mathrm{d}y}{\mathrm{d}x} = 0$$
$$\left(\frac{-x^2 \mathrm{e}^x - y}{x^2}\right) + \left(\frac{1}{x}\right)\frac{\mathrm{d}y}{\mathrm{d}x} = 0$$

The following equations are now set up to solve for the function $\phi(x, y)$

$$\frac{\partial \phi}{\partial x} = \overline{M} \tag{1}$$
$$\frac{\partial \phi}{\partial y} = \overline{N} \tag{2}$$

Integrating (1) w.r.t. x gives

$$\int \frac{\partial \phi}{\partial x}\,\mathrm{d}x = \int \overline{M}\,\mathrm{d}x$$
$$\int \frac{\partial \phi}{\partial x}\,\mathrm{d}x = \int \frac{-x^2 \mathrm{e}^x - y}{x^2}\,\mathrm{d}x$$
$$\phi = \frac{-\mathrm{e}^x x + y}{x} + f(y) \tag{3}$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$\frac{\partial \phi}{\partial y} = \frac{1}{x} + f'(y) \tag{4}$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = \frac{1}{x}$. Therefore equation (4) becomes

$$\frac{1}{x} = \frac{1}{x} + f'(y) \tag{5}$$

Solving equation (5) for $f'(y)$ gives

$$f'(y) = 0$$

Therefore

$$f(y) = c_1$$

Where c_1 is constant of integration. Substituting this result for $f(y)$ into equation (3) gives ϕ

$$\phi = \frac{-e^x x + y}{x} + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = \frac{-e^x x + y}{x}$$

The solution becomes

$$y = x(e^x + c_1)$$

Initial conditions are used to solve for c_1. Substituting $x = 1$ and $y = e - 1$ in the above solution gives an equation to solve for the constant of integration.

$$e - 1 = e + c_1$$

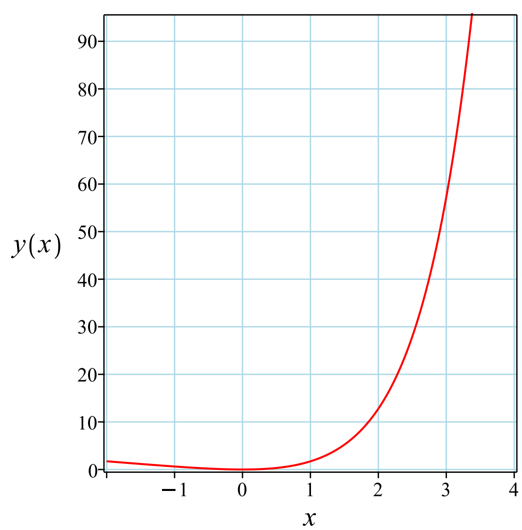
$$c_1 = -1$$

Substituting c_1 found above in the general solution gives

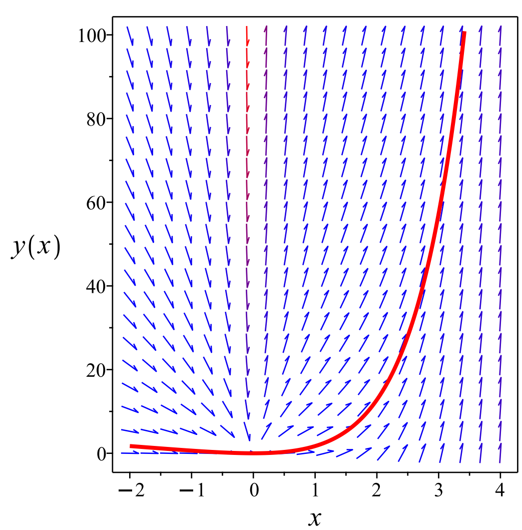
$$y = x(e^x - 1)$$

Summary
The solution(s) found are the following

$$y = x(e^x - 1) \tag{1}$$



(a) Solution plot



(b) Slope field plot

Verification of solutions

$$y = x(e^x - 1)$$

Verified OK.

2.17.6 Maple step by step solution

Let's solve

$\left[y' - \frac{y}{x} = e^x x, y(1) = e - 1\right]$

- Highest derivative means the order of the ODE is 1

  y'

- Isolate the derivative

  $y' = \frac{y}{x} + e^x x$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE

  $y' - \frac{y}{x} = e^x x$

- The ODE is linear; multiply by an integrating factor $\mu(x)$

  $\mu(x)\left(y' - \frac{y}{x}\right) = \mu(x) e^x x$

- Assume the lhs of the ODE is the total derivative $\frac{d}{dx}(\mu(x) y)$

  $\mu(x)\left(y' - \frac{y}{x}\right) = \mu'(x) y + \mu(x) y'$

- Isolate $\mu'(x)$

  $\mu'(x) = -\frac{\mu(x)}{x}$

- Solve to find the integrating factor

  $\mu(x) = \frac{1}{x}$

- Integrate both sides with respect to x

  $\int \left(\frac{d}{dx}(\mu(x) y)\right) dx = \int \mu(x) e^x x dx + c_1$

- Evaluate the integral on the lhs

  $\mu(x) y = \int \mu(x) e^x x dx + c_1$

- Solve for y

  $y = \frac{\int \mu(x) e^x x dx + c_1}{\mu(x)}$

- Substitute $\mu(x) = \frac{1}{x}$

  $y = x\left(\int e^x dx + c_1\right)$

- Evaluate the integrals on the rhs

  $y = x(e^x + c_1)$
- Use initial condition $y(1) = e - 1$

  $e - 1 = e + c_1$
- Solve for c_1

  $c_1 = -1$
- Substitute $c_1 = -1$ into general solution and simplify

  $y = x(e^x - 1)$
- Solution to the IVP

  $y = x(e^x - 1)$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
<- 1st order linear successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 10

```
dsolve([diff(y(x),x)-y(x)/x=x*exp(x),y(1) = -1+exp(1)],y(x), singsol=all)
```

$$y(x) = (e^x - 1)\, x$$

✓ Solution by Mathematica

Time used: 0.045 (sec). Leaf size: 12

```
DSolve[{y'[x]-y[x]/x==x*Exp[x],{y[1]==Exp[1]-1}},y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \rightarrow (e^x - 1)\, x$$

2.18 problem 18

Internal problem ID [4967]
Internal file name [`OUTPUT/4460_Sunday_June_05_2022_02_57_01_PM_10135229/index.tex`]

Book: Fundamentals of Differential Equations. By Nagle, Saff and Snider. 9th edition. Boston. Pearson 2018.
Section: Chapter 2, First order differential equations. Section 2.3, Linear equations. Exercises. page 54
Problem number: 18.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"linear", "exactWithIntegrationFactor", "first_order_ode_lie_symmetry_lookup"**

Maple gives the following as the ode type

```
[[_linear, `class A`]]
```

$$\boxed{y' + 4y = \mathrm{e}^{-x}}$$

With initial conditions

$$\left[y(0) = \frac{4}{3}\right]$$

2.18.1 Existence and uniqueness analysis

This is a linear ODE. In canonical form it is written as

$$y' + p(x)y = q(x)$$

Where here

$$\begin{aligned} p(x) &= 4 \\ q(x) &= \mathrm{e}^{-x} \end{aligned}$$

Hence the ode is

$$y' + 4y = e^{-x}$$

The domain of $p(x) = 4$ is

$$\{-\infty < x < \infty\}$$

And the point $x_0 = 0$ is inside this domain. The domain of $q(x) = e^{-x}$ is

$$\{-\infty < x < \infty\}$$

And the point $x_0 = 0$ is also inside this domain. Hence solution exists and is unique.

2.18.2 Solving as linear ode

Entering Linear first order ODE solver. The integrating factor μ is

$$\begin{aligned}\mu &= e^{\int 4dx} \\ &= e^{4x}\end{aligned}$$

The ode becomes

$$\begin{aligned}\frac{d}{dx}(\mu y) &= (\mu)\left(e^{-x}\right) \\ \frac{d}{dx}\left(e^{4x}y\right) &= \left(e^{4x}\right)\left(e^{-x}\right) \\ d\left(e^{4x}y\right) &= e^{3x}\, dx\end{aligned}$$

Integrating gives

$$\begin{aligned}e^{4x}y &= \int e^{3x}\, dx \\ e^{4x}y &= \frac{e^{3x}}{3} + c_1\end{aligned}$$

Dividing both sides by the integrating factor $\mu = e^{4x}$ results in

$$y = \frac{e^{-4x}e^{3x}}{3} + c_1 e^{-4x}$$

which simplifies to

$$y = \frac{e^{-x}}{3} + c_1 e^{-4x}$$

Initial conditions are used to solve for c_1. Substituting $x = 0$ and $y = \frac{4}{3}$ in the above solution gives an equation to solve for the constant of integration.

$$\frac{4}{3} = \frac{1}{3} + c_1$$

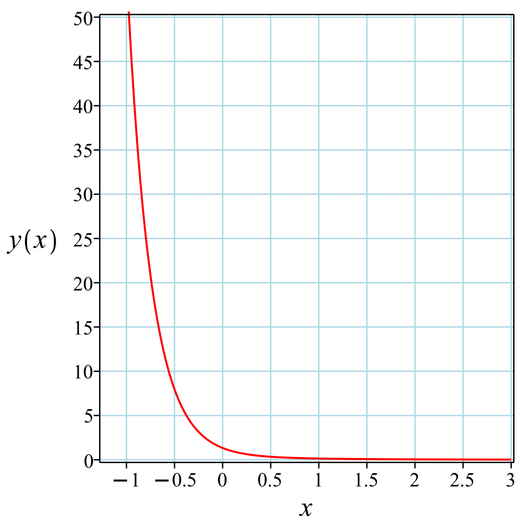
$$c_1 = 1$$

Substituting c_1 found above in the general solution gives

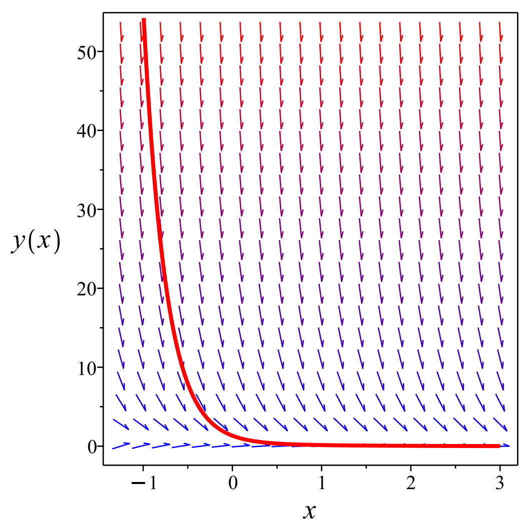
$$y = \frac{e^{-x}}{3} + e^{-4x}$$

<u>Summary</u>
The solution(s) found are the following

$$y = \frac{e^{-x}}{3} + e^{-4x} \tag{1}$$



(a) Solution plot



(b) Slope field plot

<u>Verification of solutions</u>

$$y = \frac{e^{-x}}{3} + e^{-4x}$$

Verified OK.

2.18.3 Solving as first order ode lie symmetry lookup ode

Writing the ode as

$$
\begin{aligned}
y' &= -4y + \mathrm{e}^{-x} \\
y' &= \omega(x, y)
\end{aligned}
$$

The condition of Lie symmetry is the linearized PDE given by

$$
\eta_x + \omega(\eta_y - \xi_x) - \omega^2 \xi_y - \omega_x \xi - \omega_y \eta = 0 \tag{A}
$$

The type of this ode is known. It is of type `linear`. Therefore we do not need to solve the PDE (A), and can just use the lookup table shown below to find ξ, η

Table 134: Lie symmetry infinitesimal lookup table for known first order ODE's

ODE class	Form	ξ	η
linear ode	$y' = f(x)y(x) + g(x)$	0	$e^{\int f dx}$
separable ode	$y' = f(x)\, g(y)$	$\frac{1}{f}$	0
quadrature ode	$y' = f(x)$	0	1
quadrature ode	$y' = g(y)$	1	0
homogeneous ODEs of Class A	$y' = f\left(\frac{y}{x}\right)$	x	y
homogeneous ODEs of Class C	$y' = (a + bx + cy)^{\frac{n}{m}}$	1	$-\frac{b}{c}$
homogeneous class D	$y' = \frac{y}{x} + g(x)\, F\left(\frac{y}{x}\right)$	x^2	xy
First order special form ID 1	$y' = g(x)\, e^{h(x)+by} + f(x)$	$\frac{e^{-\int bf(x)dx - h(x)}}{g(x)}$	$\frac{f(x)e^{-\int bf(x)dx - h(x)}}{g(x)}$
polynomial type ode	$y' = \frac{a_1x+b_1y+c_1}{a_2x+b_2y+c_2}$	$\frac{a_1b_2x - a_2b_1x - b_1c_2 + b_2c_1}{a_1b_2 - a_2b_1}$	$\frac{a_1b_2y - a_2b_1y - a_1c_2 - a_2c_1}{a_1b_2 - a_2b_1}$
Bernoulli ode	$y' = f(x)\, y + g(x)\, y^n$	0	$e^{-\int (n-1)f(x)dx} y^n$
Reduced Riccati	$y' = f_1(x)\, y + f_2(x)\, y^2$	0	$e^{-\int f_1 dx}$

The above table shows that

$$\begin{aligned} \xi(x, y) &= 0 \\ \eta(x, y) &= e^{-4x} \end{aligned} \tag{A1}$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x, y) \to (R, S)$ where (R, S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y}\right) S(x, y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$R = x$$

S is found from

$$\begin{aligned} S &= \int \frac{1}{\eta} dy \\ &= \int \frac{1}{e^{-4x}} dy \end{aligned}$$

Which results in

$$S = e^{4x} y$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x, y) S_y}{R_x + \omega(x, y) R_y} \tag{2}$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x, y)$ is the right hand side of the original ode given by

$$\omega(x, y) = -4y + e^{-x}$$

Evaluating all the partial derivatives gives

$$\begin{aligned} R_x &= 1 \\ R_y &= 0 \\ S_x &= 4 e^{4x} y \\ S_y &= e^{4x} \end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = e^{3x} \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = e^{3R}$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$S(R) = \frac{e^{3R}}{3} + c_1 \tag{4}$$

To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$e^{4x}y = \frac{e^{3x}}{3} + c_1$$

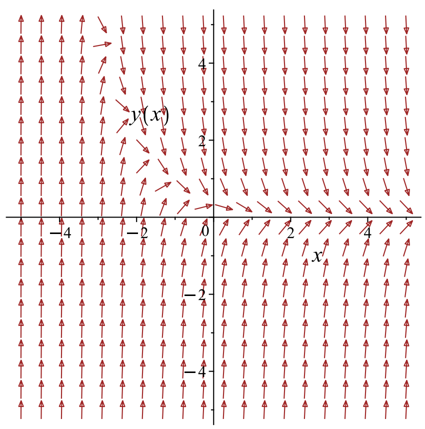
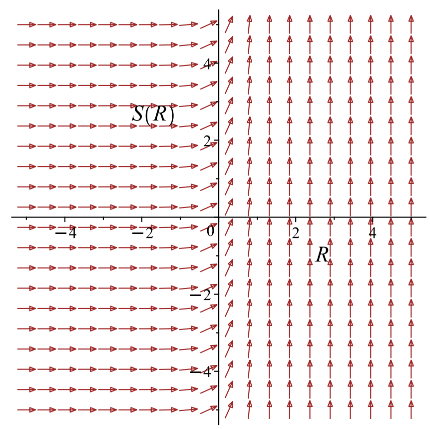
Which simplifies to

$$e^{4x}y = \frac{e^{3x}}{3} + c_1$$

Which gives

$$y = \frac{(e^{3x} + 3c_1)\, e^{-4x}}{3}$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in x, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dx} = -4y + e^{-x}$ 	$R = x$ $S = e^{4x} y$	$\frac{dS}{dR} = e^{3R}$ 

Initial conditions are used to solve for c_1. Substituting $x = 0$ and $y = \frac{4}{3}$ in the above solution gives an equation to solve for the constant of integration.

$$\frac{4}{3} = \frac{1}{3} + c_1$$

$$c_1 = 1$$

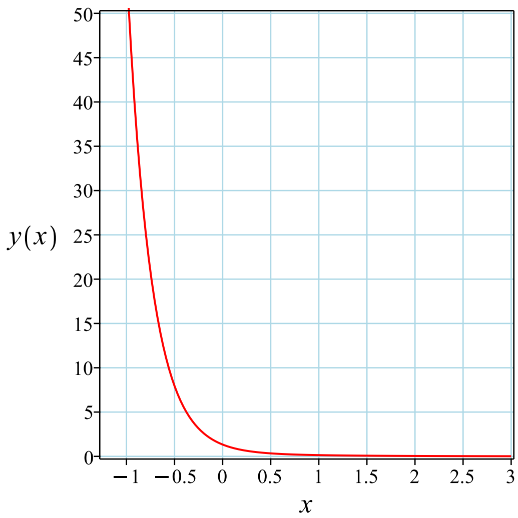
Substituting c_1 found above in the general solution gives

$$y = \frac{e^{-x}}{3} + e^{-4x}$$

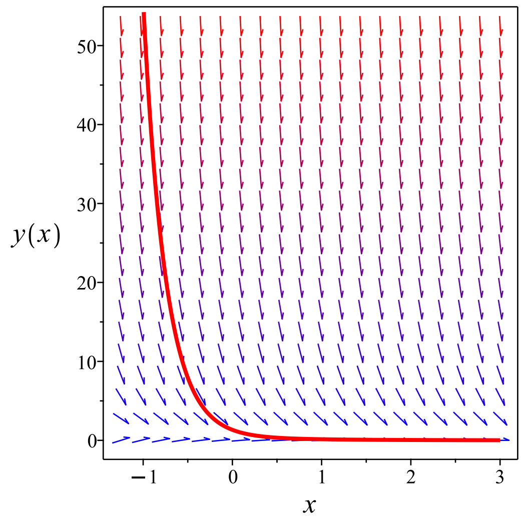
Summary

The solution(s) found are the following

$$y = \frac{e^{-x}}{3} + e^{-4x} \tag{1}$$



(a) Solution plot



(b) Slope field plot

<u>Verification of solutions</u>

$$y = \frac{e^{-x}}{3} + e^{-4x}$$

Verified OK.

2.18.4 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x,y) + N(x,y)\frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x,y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y) = 0$$

Hence

$$\frac{\partial \phi}{\partial x} + \frac{\partial \phi}{\partial y}\frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$\frac{\partial \phi}{\partial x} = M$$
$$\frac{\partial \phi}{\partial y} = N$$

But since $\frac{\partial^2\phi}{\partial x\partial y} = \frac{\partial^2\phi}{\partial y\partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x, y)$ but at least we know now that we can do that since the condition $\frac{\partial^2\phi}{\partial x\partial y} = \frac{\partial^2\phi}{\partial y\partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x, y)\,\mathrm{d}x + N(x, y)\,\mathrm{d}y = 0 \tag{1A}$$

Therefore

$$\begin{aligned}\mathrm{d}y &= \left(-4y + \mathrm{e}^{-x}\right)\mathrm{d}x\\ \left(4y - \mathrm{e}^{-x}\right)\mathrm{d}x + \mathrm{d}y &= 0 \qquad (2A)\end{aligned}$$

Comparing (1A) and (2A) shows that

$$\begin{aligned}M(x, y) &= 4y - \mathrm{e}^{-x}\\ N(x, y) &= 1\end{aligned}$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Using result found above gives

$$\begin{aligned}\frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}\left(4y - \mathrm{e}^{-x}\right)\\ &= 4\end{aligned}$$

And

$$\begin{aligned}\frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}(1)\\ &= 0\end{aligned}$$

Since $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$, then the ODE is <u>not exact</u>. Since the ODE is not exact, we will try to find an integrating factor to make it exact. Let

$$\begin{aligned}A &= \frac{1}{N}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right)\\ &= 1((4) - (0))\\ &= 4\end{aligned}$$

Since A does not depend on y, then it can be used to find an integrating factor. The integrating factor μ is

$$
\begin{aligned}
\mu &= e^{\int A \, \mathrm{d}x} \\
&= e^{\int 4 \, \mathrm{d}x}
\end{aligned}
$$

The result of integrating gives

$$
\begin{aligned}
\mu &= e^{4x} \\
&= \mathrm{e}^{4x}
\end{aligned}
$$

M and N are multiplied by this integrating factor, giving new M and new N which are called $\overline{M}$ and $\overline{N}$ for now so not to confuse them with the original M and N.

$$
\begin{aligned}
\overline{M} &= \mu M \\
&= \mathrm{e}^{4x}\left(4y - \mathrm{e}^{-x}\right) \\
&= \left(4y\,\mathrm{e}^{x} - 1\right)\mathrm{e}^{3x}
\end{aligned}
$$

And

$$
\begin{aligned}
\overline{N} &= \mu N \\
&= \mathrm{e}^{4x}(1) \\
&= \mathrm{e}^{4x}
\end{aligned}
$$

Now a modified ODE is ontained from the original ODE, which is exact and can be solved. The modified ODE is

$$
\begin{aligned}
\overline{M} + \overline{N}\frac{\mathrm{d}y}{\mathrm{d}x} &= 0 \\
\left(\left(4y\,\mathrm{e}^{x} - 1\right)\mathrm{e}^{3x}\right) + \left(\mathrm{e}^{4x}\right)\frac{\mathrm{d}y}{\mathrm{d}x} &= 0
\end{aligned}
$$

The following equations are now set up to solve for the function $\phi(x, y)$

$$
\frac{\partial \phi}{\partial x} = \overline{M} \tag{1}
$$

$$
\frac{\partial \phi}{\partial y} = \overline{N} \tag{2}
$$

Integrating (1) w.r.t. x gives

$$
\begin{aligned}
\int \frac{\partial \phi}{\partial x} \, \mathrm{d}x &= \int \overline{M} \, \mathrm{d}x \\
\int \frac{\partial \phi}{\partial x} \, \mathrm{d}x &= \int \left(4y\,\mathrm{e}^{x} - 1\right)\mathrm{e}^{3x} \, \mathrm{d}x
\end{aligned}
$$

$$
\phi = -\frac{\mathrm{e}^{3x}}{3} + \mathrm{e}^{4x}y + f(y) \tag{3}
$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$\frac{\partial \phi}{\partial y} = \mathrm{e}^{4x} + f'(y) \tag{4}$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = \mathrm{e}^{4x}$. Therefore equation (4) becomes

$$\mathrm{e}^{4x} = \mathrm{e}^{4x} + f'(y) \tag{5}$$

Solving equation (5) for $f'(y)$ gives

$$f'(y) = 0$$

Therefore

$$f(y) = c_1$$

Where c_1 is constant of integration. Substituting this result for $f(y)$ into equation (3) gives ϕ

$$\phi = -\frac{\mathrm{e}^{3x}}{3} + \mathrm{e}^{4x} y + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = -\frac{\mathrm{e}^{3x}}{3} + \mathrm{e}^{4x} y$$

The solution becomes

$$y = \frac{(\mathrm{e}^{3x} + 3c_1)\,\mathrm{e}^{-4x}}{3}$$

Initial conditions are used to solve for c_1. Substituting $x = 0$ and $y = \frac{4}{3}$ in the above solution gives an equation to solve for the constant of integration.

$$\frac{4}{3} = \frac{1}{3} + c_1$$

$$c_1 = 1$$

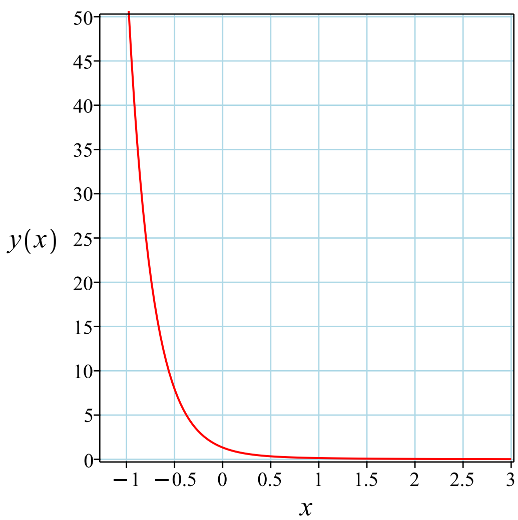
Substituting c_1 found above in the general solution gives

$$y = \frac{e^{-x}}{3} + e^{-4x}$$

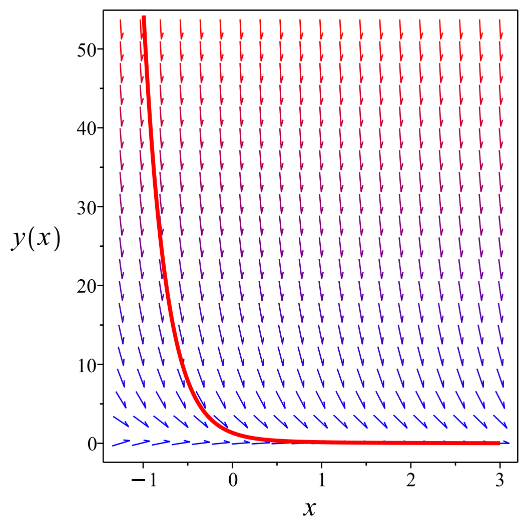
Summary

The solution(s) found are the following

$$y = \frac{e^{-x}}{3} + e^{-4x} \tag{1}$$



(a) Solution plot



(b) Slope field plot

Verification of solutions

$$y = \frac{e^{-x}}{3} + e^{-4x}$$

Verified OK.

2.18.5 Maple step by step solution

Let's solve

$\left[y' + 4y = e^{-x}, y(0) = \frac{4}{3}\right]$

- Highest derivative means the order of the ODE is 1

  y'

- Isolate the derivative

$y' = -4y + e^{-x}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE

  $y' + 4y = e^{-x}$

- The ODE is linear; multiply by an integrating factor $\mu(x)$

  $\mu(x)\left(y' + 4y\right) = \mu(x)\, e^{-x}$

- Assume the lhs of the ODE is the total derivative $\frac{d}{dx}\left(\mu(x)\, y\right)$

  $\mu(x)\left(y' + 4y\right) = \mu'(x)\, y + \mu(x)\, y'$

- Isolate $\mu'(x)$

  $\mu'(x) = 4\mu(x)$

- Solve to find the integrating factor

  $\mu(x) = e^{4x}$

- Integrate both sides with respect to x

  $\int \left(\frac{d}{dx}\left(\mu(x)\, y\right)\right) dx = \int \mu(x)\, e^{-x} dx + c_1$

- Evaluate the integral on the lhs

  $\mu(x)\, y = \int \mu(x)\, e^{-x} dx + c_1$

- Solve for y

  $y = \frac{\int \mu(x) e^{-x} dx + c_1}{\mu(x)}$

- Substitute $\mu(x) = e^{4x}$

  $y = \frac{\int e^{-x} e^{4x} dx + c_1}{e^{4x}}$

- Evaluate the integrals on the rhs

  $y = \frac{\frac{e^{3x}}{3} + c_1}{e^{4x}}$

- Simplify

  $y = \frac{\left(e^{3x} + 3c_1\right) e^{-4x}}{3}$

- Use initial condition $y(0) = \frac{4}{3}$

  $\frac{4}{3} = \frac{1}{3} + c_1$

- Solve for c_1

  $c_1 = 1$

- Substitute $c_1 = 1$ into general solution and simplify

  $y = \frac{\left(e^{3x} + 3\right) e^{-4x}}{3}$

- Solution to the IVP

  $y = \frac{(e^{3x}+3)e^{-4x}}{3}$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
<- 1st order linear successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 16

```
dsolve([diff(y(x),x)+4*y(x)-exp(-x)=0,y(0) = 4/3],y(x), singsol=all)
```

$$y(x) = \frac{(e^{3x}+3)\,e^{-4x}}{3}$$

✓ Solution by Mathematica

Time used: 0.049 (sec). Leaf size: 21

```
DSolve[{y'[x]+4*y[x]-Exp[-x]==0,{y[0]==4/3}},y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \rightarrow \frac{1}{3}e^{-4x}\left(e^{3x}+3\right)$$

2.19 problem 19

Internal problem ID [4968]
Internal file name [`OUTPUT/4461_Sunday_June_05_2022_02_57_03_PM_96782131/index.tex`]

Book: Fundamentals of Differential Equations. By Nagle, Saff and Snider. 9th edition. Boston. Pearson 2018.
Section: Chapter 2, First order differential equations. Section 2.3, Linear equations. Exercises. page 54
Problem number: 19.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"linear", "exactWithIntegrationFactor", "first_order_ode_lie_symmetry_lookup"**

Maple gives the following as the ode type

`[_linear]`

$$\boxed{t^2x' + 3xt = t^4 \ln(t) + 1}$$

With initial conditions

$$[x(1) = 0]$$

2.19.1 Existence and uniqueness analysis

This is a linear ODE. In canonical form it is written as

$$x' + p(t)x = q(t)$$

Where here

$$p(t) = \frac{3}{t}$$
$$q(t) = \frac{t^4 \ln(t) + 1}{t^2}$$

Hence the ode is

$$x' + \frac{3x}{t} = \frac{t^4 \ln(t) + 1}{t^2}$$

The domain of $p(t) = \frac{3}{t}$ is

$$\{t < 0 \lor 0 < t\}$$

And the point $t_0 = 1$ is inside this domain. The domain of $q(t) = \frac{t^4 \ln(t)+1}{t^2}$ is

$$\{0 < t\}$$

And the point $t_0 = 1$ is also inside this domain. Hence solution exists and is unique.

2.19.2 Solving as linear ode

Entering Linear first order ODE solver. The integrating factor μ is

$$\begin{aligned}\mu &= e^{\int \frac{3}{t} dt} \\ &= t^3\end{aligned}$$

The ode becomes

$$\begin{aligned}\frac{d}{dt}(\mu x) &= (\mu)\left(\frac{t^4 \ln(t) + 1}{t^2}\right) \\ \frac{d}{dt}(t^3 x) &= (t^3)\left(\frac{t^4 \ln(t) + 1}{t^2}\right) \\ d(t^3 x) &= \left((t^4 \ln(t) + 1)\, t\right)\, dt\end{aligned}$$

Integrating gives

$$\begin{aligned}t^3 x &= \int (t^4 \ln(t) + 1)\, t\, dt \\ t^3 x &= \frac{t^6 \ln(t)}{6} - \frac{t^6}{36} + \frac{t^2}{2} + c_1\end{aligned}$$

Dividing both sides by the integrating factor $\mu = t^3$ results in

$$x = \frac{\frac{t^6 \ln(t)}{6} - \frac{t^6}{36} + \frac{t^2}{2}}{t^3} + \frac{c_1}{t^3}$$

which simplifies to

$$x = \frac{6t^6 \ln(t) - t^6 + 18t^2 + 36c_1}{36t^3}$$

Initial conditions are used to solve for c_1. Substituting $t = 1$ and $x = 0$ in the above solution gives an equation to solve for the constant of integration.

$$0 = c_1 + \frac{17}{36}$$

$$c_1 = -\frac{17}{36}$$

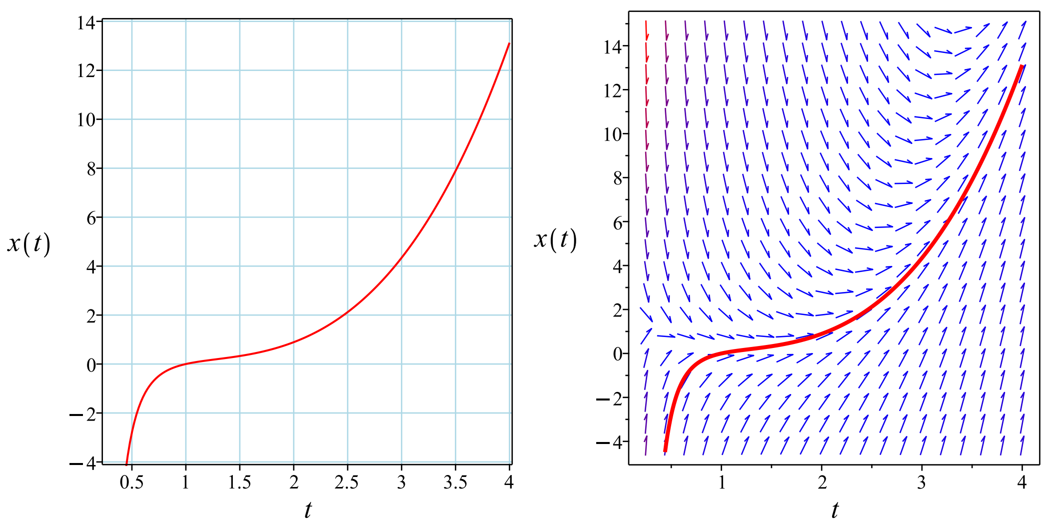
Substituting c_1 found above in the general solution gives

$$x = \frac{6t^6 \ln(t) - t^6 + 18t^2 - 17}{36t^3}$$

Summary

The solution(s) found are the following

$$x = \frac{6t^6 \ln(t) - t^6 + 18t^2 - 17}{36t^3} \tag{1}$$



(a) Solution plot (b) Slope field plot

Verification of solutions

$$x = \frac{6t^6 \ln(t) - t^6 + 18t^2 - 17}{36t^3}$$

Verified OK.

2.19.3 Solving as first order ode lie symmetry lookup ode

Writing the ode as

$$
\begin{aligned}
x' &= \frac{-3xt + t^4 \ln(t) + 1}{t^2} \\
x' &= \omega(t, x)
\end{aligned}
$$

The condition of Lie symmetry is the linearized PDE given by

$$
\eta_t + \omega(\eta_x - \xi_t) - \omega^2 \xi_x - \omega_t \xi - \omega_x \eta = 0 \tag{A}
$$

The type of this ode is known. It is of type `linear`. Therefore we do not need to solve the PDE (A), and can just use the lookup table shown below to find ξ, η

Table 137: Lie symmetry infinitesimal lookup table for known first order ODE's

ODE class	Form	ξ	η
linear ode	$y' = f(x)y(x) + g(x)$	0	$e^{\int f dx}$
separable ode	$y' = f(x)\,g(y)$	$\frac{1}{f}$	0
quadrature ode	$y' = f(x)$	0	1
quadrature ode	$y' = g(y)$	1	0
homogeneous ODEs of Class A	$y' = f\left(\frac{y}{x}\right)$	x	y
homogeneous ODEs of Class C	$y' = (a + bx + cy)^{\frac{n}{m}}$	1	$-\frac{b}{c}$
homogeneous class D	$y' = \frac{y}{x} + g(x)\,F\left(\frac{y}{x}\right)$	x^2	xy
First order special form ID 1	$y' = g(x)\,e^{h(x)+by} + f(x)$	$\frac{e^{-\int bf(x)dx - h(x)}}{g(x)}$	$\frac{f(x)e^{-\int bf(x)dx - h(x)}}{g(x)}$
polynomial type ode	$y' = \frac{a_1x+b_1y+c_1}{a_2x+b_2y+c_2}$	$\frac{a_1b_2x - a_2b_1x - b_1c_2 + b_2c_1}{a_1b_2 - a_2b_1}$	$\frac{a_1b_2y - a_2b_1y - a_1c_2 - a_2c_1}{a_1b_2 - a_2b_1}$
Bernoulli ode	$y' = f(x)\,y + g(x)\,y^n$	0	$e^{-\int (n-1)f(x)dx}y^n$
Reduced Riccati	$y' = f_1(x)\,y + f_2(x)\,y^2$	0	$e^{-\int f_1 dx}$

The above table shows that

$$\begin{aligned}\xi(t,x) &= 0\\ \eta(t,x) &= \frac{1}{t^3}\end{aligned}\tag{A1}$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(t,x) \to (R,S)$ where (R,S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$\frac{dt}{\xi} = \frac{dx}{\eta} = dS \tag{1}$$

The above comes from the requirements that $\left(\xi\frac{\partial}{\partial t} + \eta\frac{\partial}{\partial x}\right)S(t,x) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$R = t$$

S is found from

$$\begin{aligned}S &= \int \frac{1}{\eta} dy\\ &= \int \frac{1}{\frac{1}{t^3}} dy\end{aligned}$$

Which results in

$$S = t^3 x$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_t + \omega(t,x)S_x}{R_t + \omega(t,x)R_x} \tag{2}$$

Where in the above R_t, R_x, S_t, S_x are all partial derivatives and $\omega(t,x)$ is the right hand side of the original ode given by

$$\omega(t,x) = \frac{-3xt + t^4\ln(t) + 1}{t^2}$$

Evaluating all the partial derivatives gives

$$
\begin{aligned}
R_t &= 1 \\
R_x &= 0 \\
S_t &= 3t^2 x \\
S_x &= t^3
\end{aligned}
$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$
\frac{dS}{dR} = \left(t^4 \ln(t) + 1\right) t \tag{2A}
$$

We now need to express the RHS as function of R only. This is done by solving for t, x in terms of R, S from the result obtained earlier and simplifying. This gives

$$
\frac{dS}{dR} = \left(R^4 \ln(R) + 1\right) R
$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$
S(R) = \frac{R^6 \ln(R)}{6} - \frac{R^6}{36} + \frac{R^2}{2} + c_1 \tag{4}
$$

To complete the solution, we just need to transform (4) back to t, x coordinates. This results in

$$
t^3 x = \frac{t^6 \ln(t)}{6} - \frac{t^6}{36} + \frac{t^2}{2} + c_1
$$

Which simplifies to

$$
t^3 x = \frac{t^6 \ln(t)}{6} - \frac{t^6}{36} + \frac{t^2}{2} + c_1
$$

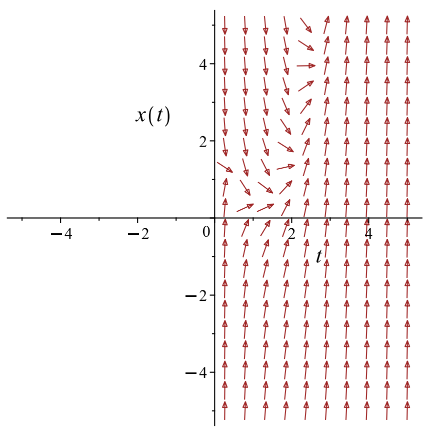
Which gives

$$
x = \frac{6t^6 \ln(t) - t^6 + 18t^2 + 36c_1}{36t^3}
$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

<table>
<tr><th>Original ode in t, x coordinates</th><th>Canonical coordinates transformation</th><th>ODE in canonical coordinates (R, S)</th></tr>
<tr><td>

$\frac{dx}{dt} = \frac{-3xt+t^4 \ln(t)+1}{t^2}$



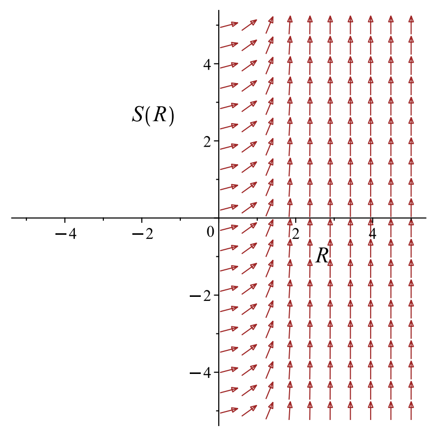
</td><td>

$R = t$

$S = t^3 x$

</td><td>

$\frac{dS}{dR} = (R^4 \ln(R) + 1) R$



</td></tr>
</table>

Initial conditions are used to solve for c_1. Substituting $t = 1$ and $x = 0$ in the above solution gives an equation to solve for the constant of integration.

$$0 = c_1 + \frac{17}{36}$$

$$c_1 = -\frac{17}{36}$$

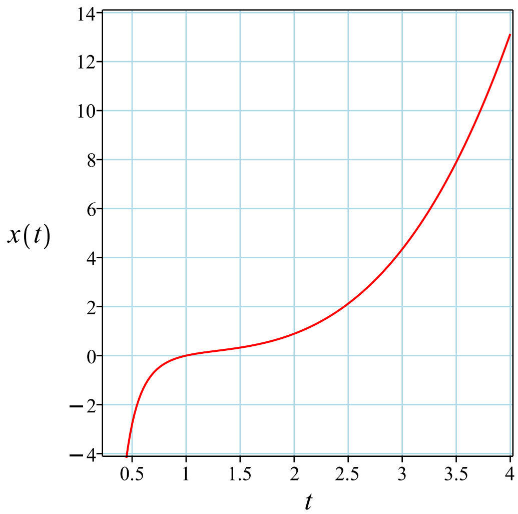
Substituting c_1 found above in the general solution gives

$$x = \frac{6t^6 \ln(t) - t^6 + 18t^2 - 17}{36t^3}$$

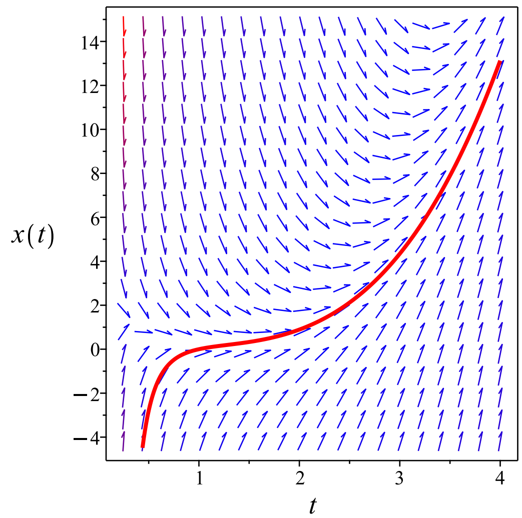
Summary

The solution(s) found are the following

$$x = \frac{6t^6 \ln(t) - t^6 + 18t^2 - 17}{36t^3} \tag{1}$$



(a) Solution plot



(b) Slope field plot

Verification of solutions

$$x = \frac{6t^6 \ln(t) - t^6 + 18t^2 - 17}{36t^3}$$

Verified OK.

2.19.4 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x, y) + N(x, y)\frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x, y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x, y) = 0$$

Hence

$$\frac{\partial \phi}{\partial x} + \frac{\partial \phi}{\partial y}\frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$\frac{\partial \phi}{\partial x} = M$$
$$\frac{\partial \phi}{\partial y} = N$$

But since $\frac{\partial^2\phi}{\partial x\partial y} = \frac{\partial^2\phi}{\partial y\partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x,y)$ but at least we know now that we can do that since the condition $\frac{\partial^2\phi}{\partial x\partial y} = \frac{\partial^2\phi}{\partial y\partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(t,x)\,\mathrm{d}t + N(t,x)\,\mathrm{d}x = 0 \tag{1A}$$

Therefore

$$\begin{aligned}
(t^2)\,\mathrm{d}x &= \left(-3xt + t^4\ln(t) + 1\right)\mathrm{d}t \\
\left(3xt - t^4\ln(t) - 1\right)\mathrm{d}t + (t^2)\,\mathrm{d}x &= 0 \qquad (2A)
\end{aligned}$$

Comparing (1A) and (2A) shows that

$$\begin{aligned}
M(t,x) &= 3xt - t^4\ln(t) - 1 \\
N(t,x) &= t^2
\end{aligned}$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial x} = \frac{\partial N}{\partial t}$$

Using result found above gives

$$\begin{aligned}
\frac{\partial M}{\partial x} &= \frac{\partial}{\partial x}\left(3xt - t^4\ln(t) - 1\right) \\
&= 3t
\end{aligned}$$

And

$$\begin{aligned}
\frac{\partial N}{\partial t} &= \frac{\partial}{\partial t}(t^2) \\
&= 2t
\end{aligned}$$

Since $\frac{\partial M}{\partial x} \neq \frac{\partial N}{\partial t}$, then the ODE is not exact. Since the ODE is not exact, we will try to find an integrating factor to make it exact. Let

$$\begin{aligned}
A &= \frac{1}{N}\left(\frac{\partial M}{\partial x} - \frac{\partial N}{\partial t}\right) \\
&= \frac{1}{t^2}((3t) - (2t)) \\
&= \frac{1}{t}
\end{aligned}$$

Since A does not depend on x, then it can be used to find an integrating factor. The integrating factor μ is

$$\begin{aligned}\mu &= e^{\int A\,\mathrm{d}t} \\ &= e^{\int \frac{1}{t}\,\mathrm{d}t}\end{aligned}$$

The result of integrating gives

$$\begin{aligned}\mu &= e^{\ln(t)} \\ &= t\end{aligned}$$

M and N are multiplied by this integrating factor, giving new M and new N which are called $\overline{M}$ and $\overline{N}$ for now so not to confuse them with the original M and N.

$$\begin{aligned}\overline{M} &= \mu M \\ &= t\left(3xt - t^4\ln(t) - 1\right) \\ &= \left(3xt - t^4\ln(t) - 1\right)t\end{aligned}$$

And

$$\begin{aligned}\overline{N} &= \mu N \\ &= t\left(t^2\right) \\ &= t^3\end{aligned}$$

Now a modified ODE is ontained from the original ODE, which is exact and can be solved. The modified ODE is

$$\begin{aligned}\overline{M} + \overline{N}\frac{\mathrm{d}x}{\mathrm{d}t} &= 0 \\ \left(\left(3xt - t^4\ln(t) - 1\right)t\right) + \left(t^3\right)\frac{\mathrm{d}x}{\mathrm{d}t} &= 0\end{aligned}$$

The following equations are now set up to solve for the function $\phi(t, x)$

$$\frac{\partial \phi}{\partial t} = \overline{M} \tag{1}$$

$$\frac{\partial \phi}{\partial x} = \overline{N} \tag{2}$$

Integrating (1) w.r.t. t gives

$$\begin{aligned}\int \frac{\partial \phi}{\partial t}\,\mathrm{d}t &= \int \overline{M}\,\mathrm{d}t \\ \int \frac{\partial \phi}{\partial t}\,\mathrm{d}t &= \int \left(3xt - t^4\ln(t) - 1\right)t\,\mathrm{d}t\end{aligned}$$

$$\phi = -\frac{\left(t^4\ln(t) - \frac{t^4}{6} - 6xt + 3\right)t^2}{6} + f(x) \tag{3}$$

Where $f(x)$ is used for the constant of integration since ϕ is a function of both t and x. Taking derivative of equation (3) w.r.t x gives

$$\frac{\partial \phi}{\partial x} = t^3 + f'(x) \tag{4}$$

But equation (2) says that $\frac{\partial \phi}{\partial x} = t^3$. Therefore equation (4) becomes

$$t^3 = t^3 + f'(x) \tag{5}$$

Solving equation (5) for $f'(x)$ gives

$$f'(x) = 0$$

Therefore

$$f(x) = c_1$$

Where c_1 is constant of integration. Substituting this result for $f(x)$ into equation (3) gives ϕ

$$\phi = -\frac{\left(t^4 \ln(t) - \frac{t^4}{6} - 6xt + 3\right) t^2}{6} + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = -\frac{\left(t^4 \ln(t) - \frac{t^4}{6} - 6xt + 3\right) t^2}{6}$$

The solution becomes

$$x = \frac{6t^6 \ln(t) - t^6 + 18t^2 + 36c_1}{36t^3}$$

Initial conditions are used to solve for c_1. Substituting $t = 1$ and $x = 0$ in the above solution gives an equation to solve for the constant of integration.

$$0 = c_1 + \frac{17}{36}$$

$$c_1 = -\frac{17}{36}$$

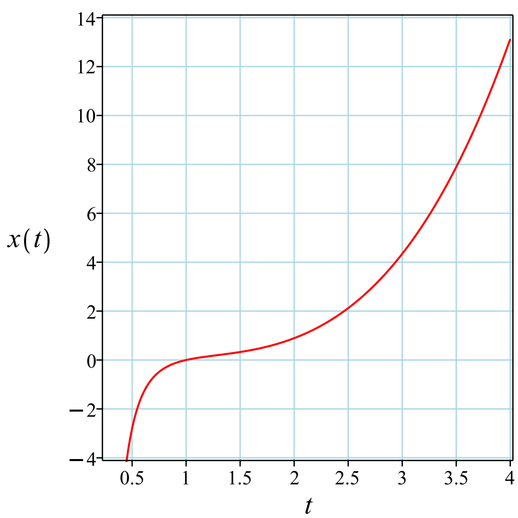
Substituting c_1 found above in the general solution gives

$$x = \frac{6t^6 \ln(t) - t^6 + 18t^2 - 17}{36t^3}$$

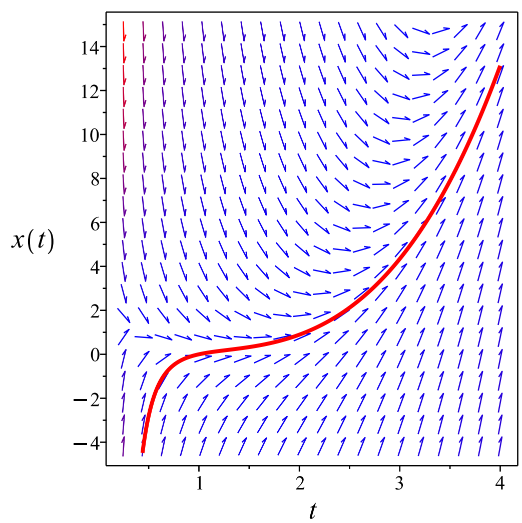
Summary

The solution(s) found are the following

$$x = \frac{6t^6 \ln(t) - t^6 + 18t^2 - 17}{36t^3} \tag{1}$$



(a) Solution plot



(b) Slope field plot

Verification of solutions

$$x = \frac{6t^6 \ln(t) - t^6 + 18t^2 - 17}{36t^3}$$

Verified OK.

2.19.5 Maple step by step solution

Let's solve

$[t^2x' + 3xt = t^4 \ln(t) + 1, x(1) = 0]$

- Highest derivative means the order of the ODE is 1

  x'

- Isolate the derivative

$x' = -\frac{3x}{t} + \frac{t^4 \ln(t)+1}{t^2}$

- Group terms with x on the lhs of the ODE and the rest on the rhs of the ODE

  $x' + \frac{3x}{t} = \frac{t^4 \ln(t)+1}{t^2}$
- The ODE is linear; multiply by an integrating factor $\mu(t)$

  $\mu(t)\left(x' + \frac{3x}{t}\right) = \frac{\mu(t)(t^4 \ln(t)+1)}{t^2}$
- Assume the lhs of the ODE is the total derivative $\frac{d}{dt}(\mu(t)\, x)$

  $\mu(t)\left(x' + \frac{3x}{t}\right) = \mu'(t)\, x + \mu(t)\, x'$
- Isolate $\mu'(t)$

  $\mu'(t) = \frac{3\mu(t)}{t}$
- Solve to find the integrating factor

  $\mu(t) = t^3$
- Integrate both sides with respect to t

  $\int \left(\frac{d}{dt}(\mu(t)\, x)\right) dt = \int \frac{\mu(t)(t^4 \ln(t)+1)}{t^2} dt + c_1$
- Evaluate the integral on the lhs

  $\mu(t)\, x = \int \frac{\mu(t)(t^4 \ln(t)+1)}{t^2} dt + c_1$
- Solve for x

  $x = \frac{\int \frac{\mu(t)\left(t^4 \ln(t)+1\right)}{t^2} dt + c_1}{\mu(t)}$
- Substitute $\mu(t) = t^3$

  $x = \frac{\int (t^4 \ln(t)+1)\, t dt + c_1}{t^3}$
- Evaluate the integrals on the rhs

  $x = \frac{\frac{t^6 \ln(t)}{6} - \frac{t^6}{36} + \frac{t^2}{2} + c_1}{t^3}$
- Simplify

  $x = \frac{6t^6 \ln(t) - t^6 + 18t^2 + 36c_1}{36t^3}$
- Use initial condition $x(1) = 0$

  $0 = c_1 + \frac{17}{36}$
- Solve for c_1

  $c_1 = -\frac{17}{36}$
- Substitute $c_1 = -\frac{17}{36}$ into general solution and simplify

$x = \frac{6t^6 \ln(t) - t^6 + 18t^2 - 17}{36t^3}$

- Solution to the IVP

  $x = \frac{6t^6 \ln(t) - t^6 + 18t^2 - 17}{36t^3}$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
<- 1st order linear successful`
```

✓ Solution by Maple

Time used: 0.015 (sec). Leaf size: 28

```
dsolve([t^2*diff(x(t),t)+3*t*x(t)=t^4*ln(t)+1,x(1) = 0],x(t), singsol=all)
```

$$x(t) = \frac{6t^6 \ln(t) - t^6 + 18t^2 - 17}{36t^3}$$

✓ Solution by Mathematica

Time used: 0.038 (sec). Leaf size: 29

```
DSolve[{t^2*x'[t]+3*t*x[t]==t^4*Log[t]+1,{x[1]==0}},x[t],t,IncludeSingularSolutions -> True]
```

$$x(t) \to -\frac{t^6 - 6t^6 \log(t) - 18t^2 + 17}{36t^3}$$

2.20 problem 20

Internal problem ID [4969]
Internal file name [`OUTPUT/4462_Sunday_June_05_2022_02_57_04_PM_5269354/index.tex`]

Book: Fundamentals of Differential Equations. By Nagle, Saff and Snider. 9th edition. Boston. Pearson 2018.
Section: Chapter 2, First order differential equations. Section 2.3, Linear equations. Exercises. page 54
Problem number: 20.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"linear", "exactWithIntegrationFactor", "first_order_ode_lie_symmetry_lookup"**

Maple gives the following as the ode type

`[_linear]`

$$\boxed{y' + \frac{3y}{x} = 3x - 2}$$

With initial conditions

$$[y(1) = 1]$$

2.20.1 Existence and uniqueness analysis

This is a linear ODE. In canonical form it is written as

$$y' + p(x)y = q(x)$$

Where here

$$p(x) = \frac{3}{x}$$
$$q(x) = 3x - 2$$

Hence the ode is

$$y' + \frac{3y}{x} = 3x - 2$$

The domain of $p(x) = \frac{3}{x}$ is

$$\{x < 0 \lor 0 < x\}$$

And the point $x_0 = 1$ is inside this domain. The domain of $q(x) = 3x - 2$ is

$$\{-\infty < x < \infty\}$$

And the point $x_0 = 1$ is also inside this domain. Hence solution exists and is unique.

2.20.2 Solving as linear ode

Entering Linear first order ODE solver. The integrating factor μ is

$$\begin{aligned}\mu &= \mathrm{e}^{\int \frac{3}{x}dx} \\ &= x^3\end{aligned}$$

The ode becomes

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}x}(\mu y) &= (\mu)(3x - 2) \\ \frac{\mathrm{d}}{\mathrm{d}x}\left(y\,x^3\right) &= \left(x^3\right)(3x - 2) \\ \mathrm{d}\left(y\,x^3\right) &= \left(3x^4 - 2x^3\right)\,\mathrm{d}x\end{aligned}$$

Integrating gives

$$\begin{aligned}y\,x^3 &= \int 3x^4 - 2x^3\,\mathrm{d}x \\ y\,x^3 &= \frac{3}{5}x^5 - \frac{1}{2}x^4 + c_1\end{aligned}$$

Dividing both sides by the integrating factor $\mu = x^3$ results in

$$y = \frac{\frac{3}{5}x^5 - \frac{1}{2}x^4}{x^3} + \frac{c_1}{x^3}$$

which simplifies to

$$y = \frac{6x^5 - 5x^4 + 10c_1}{10x^3}$$

Initial conditions are used to solve for c_1. Substituting $x = 1$ and $y = 1$ in the above solution gives an equation to solve for the constant of integration.

$$1 = c_1 + \frac{1}{10}$$

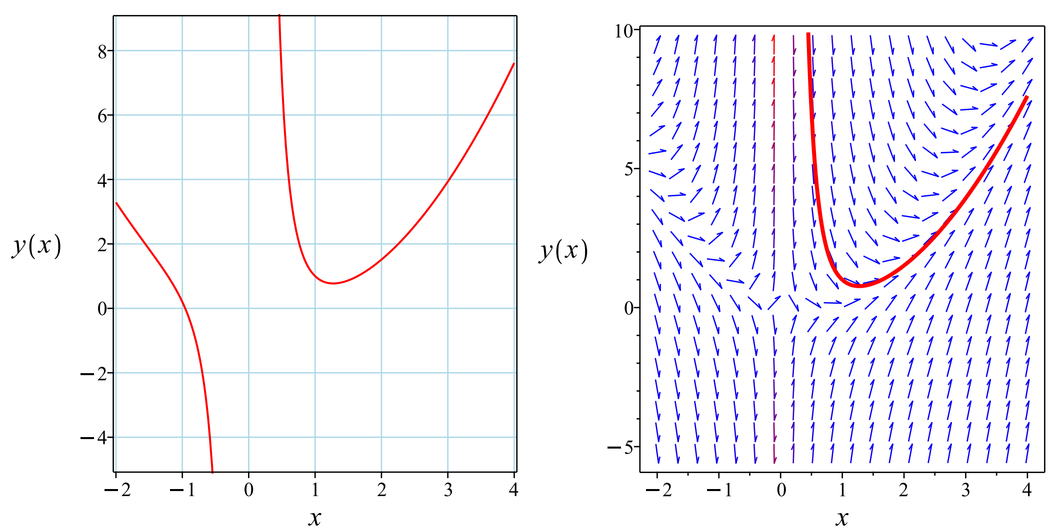
$$c_1 = \frac{9}{10}$$

Substituting c_1 found above in the general solution gives

$$y = \frac{6x^5 - 5x^4 + 9}{10x^3}$$

Summary
The solution(s) found are the following

$$y = \frac{6x^5 - 5x^4 + 9}{10x^3} \tag{1}$$



(a) Solution plot (b) Slope field plot

Verification of solutions

$$y = \frac{6x^5 - 5x^4 + 9}{10x^3}$$

Verified OK.

2.20.3 Solving as first order ode lie symmetry lookup ode

Writing the ode as

$$
\begin{aligned}
y' &= -\frac{-3x^2 + 2x + 3y}{x} \\
y' &= \omega(x, y)
\end{aligned}
$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x + \omega(\eta_y - \xi_x) - \omega^2 \xi_y - \omega_x \xi - \omega_y \eta = 0 \tag{A}$$

The type of this ode is known. It is of type `linear`. Therefore we do not need to solve the PDE (A), and can just use the lookup table shown below to find ξ, η

Table 140: Lie symmetry infinitesimal lookup table for known first order ODE's

ODE class	Form	ξ	η
linear ode	$y' = f(x)y(x) + g(x)$	0	$e^{\int f dx}$
separable ode	$y' = f(x)\, g(y)$	$\frac{1}{f}$	0
quadrature ode	$y' = f(x)$	0	1
quadrature ode	$y' = g(y)$	1	0
homogeneous ODEs of Class A	$y' = f\left(\frac{y}{x}\right)$	x	y
homogeneous ODEs of Class C	$y' = (a + bx + cy)^{\frac{n}{m}}$	1	$-\frac{b}{c}$
homogeneous class D	$y' = \frac{y}{x} + g(x)\, F\left(\frac{y}{x}\right)$	x^2	xy
First order special form ID 1	$y' = g(x)\, e^{h(x)+by} + f(x)$	$\frac{e^{-\int bf(x)dx - h(x)}}{g(x)}$	$\frac{f(x)e^{-\int bf(x)dx - h(x)}}{g(x)}$
polynomial type ode	$y' = \frac{a_1x+b_1y+c_1}{a_2x+b_2y+c_2}$	$\frac{a_1b_2x - a_2b_1x - b_1c_2 + b_2c_1}{a_1b_2 - a_2b_1}$	$\frac{a_1b_2y - a_2b_1y - a_1c_2 - a_2c_1}{a_1b_2 - a_2b_1}$
Bernoulli ode	$y' = f(x)\, y + g(x)\, y^n$	0	$e^{-\int (n-1)f(x)dx} y^n$
Reduced Riccati	$y' = f_1(x)\, y + f_2(x)\, y^2$	0	$e^{-\int f_1 dx}$

The above table shows that

$$\begin{aligned}\xi(x,y) &= 0 \\ \eta(x,y) &= \frac{1}{x^3}\end{aligned} \tag{A1}$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x, y) \to (R, S)$ where (R, S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y}\right) S(x,y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$R = x$$

S is found from

$$\begin{aligned}S &= \int \frac{1}{\eta} dy \\ &= \int \frac{1}{\frac{1}{x^3}} dy\end{aligned}$$

Which results in

$$S = y\,x^3$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x,y) S_y}{R_x + \omega(x,y) R_y} \tag{2}$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x, y)$ is the right hand side of the original ode given by

$$\omega(x,y) = -\frac{-3x^2 + 2x + 3y}{x}$$

Evaluating all the partial derivatives gives

$$
\begin{aligned}
R_x &= 1 \\
R_y &= 0 \\
S_x &= 3y\,x^2 \\
S_y &= x^3
\end{aligned}
$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$
\frac{dS}{dR} = 3x^4 - 2x^3 \tag{2A}
$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$
\frac{dS}{dR} = 3R^4 - 2R^3
$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$
S(R) = \frac{3}{5}R^5 - \frac{1}{2}R^4 + c_1 \tag{4}
$$

To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$
y x^3 = \frac{3}{5}x^5 - \frac{1}{2}x^4 + c_1
$$

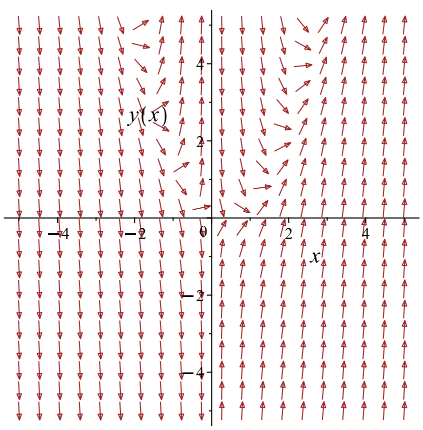
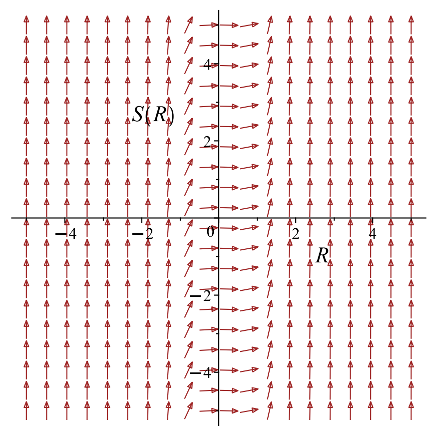
Which simplifies to

$$
y x^3 = \frac{3}{5}x^5 - \frac{1}{2}x^4 + c_1
$$

Which gives

$$
y = \frac{6x^5 - 5x^4 + 10c_1}{10x^3}
$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in x, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dx} = -\frac{-3x^2+2x+3y}{x}$ 	$R = x$ $S = y\,x^3$	$\frac{dS}{dR} = 3R^4 - 2R^3$ 

Initial conditions are used to solve for c_1. Substituting $x = 1$ and $y = 1$ in the above solution gives an equation to solve for the constant of integration.

$$1 = c_1 + \frac{1}{10}$$

$$c_1 = \frac{9}{10}$$

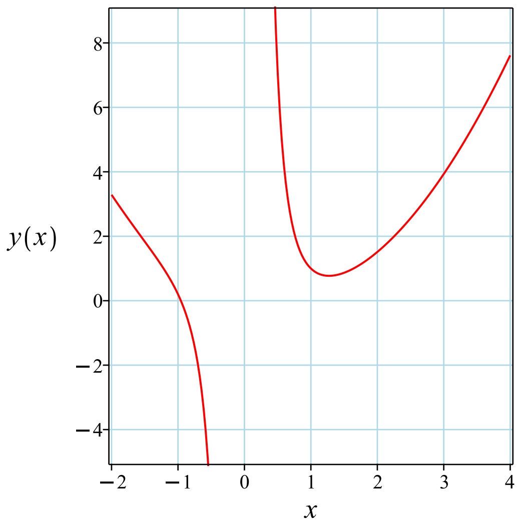
Substituting c_1 found above in the general solution gives

$$y = \frac{6x^5 - 5x^4 + 9}{10x^3}$$

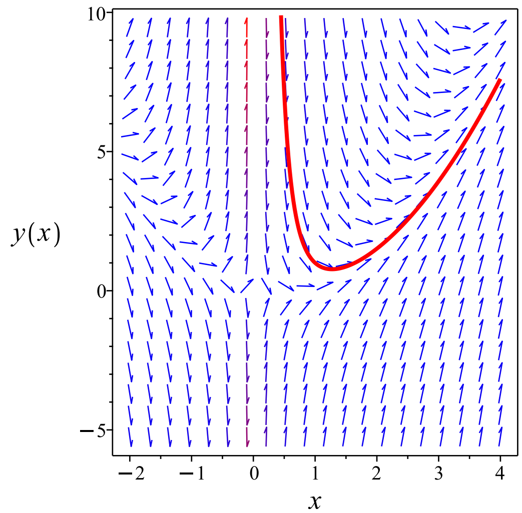
Summary

The solution(s) found are the following

$$y = \frac{6x^5 - 5x^4 + 9}{10x^3} \tag{1}$$



(a) Solution plot



(b) Slope field plot

Verification of solutions

$$y = \frac{6x^5 - 5x^4 + 9}{10x^3}$$

Verified OK.

2.20.4 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x, y) + N(x, y) \frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x, y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x, y) = 0$$

Hence

$$\frac{\partial \phi}{\partial x} + \frac{\partial \phi}{\partial y}\frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$\frac{\partial \phi}{\partial x} = M$$
$$\frac{\partial \phi}{\partial y} = N$$

But since $\frac{\partial^2\phi}{\partial x\partial y} = \frac{\partial^2\phi}{\partial y\partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x, y)$ but at least we know now that we can do that since the condition $\frac{\partial^2\phi}{\partial x\partial y} = \frac{\partial^2\phi}{\partial y\partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x, y)\,\mathrm{d}x + N(x, y)\,\mathrm{d}y = 0 \tag{1A}$$

Therefore

$$\mathrm{d}y = \left(-\frac{3y}{x} - 2 + 3x\right)\mathrm{d}x$$

$$\left(-3x + 2 + \frac{3y}{x}\right)\mathrm{d}x + \mathrm{d}y = 0 \tag{2A}$$

Comparing (1A) and (2A) shows that

$$\begin{aligned} M(x, y) &= -3x + 2 + \frac{3y}{x} \\ N(x, y) &= 1 \end{aligned}$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Using result found above gives

$$\begin{aligned} \frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}\left(-3x + 2 + \frac{3y}{x}\right) \\ &= \frac{3}{x} \end{aligned}$$

And

$$\begin{aligned} \frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}(1) \\ &= 0 \end{aligned}$$

Since $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$, then the ODE is <u>not exact</u>. Since the ODE is not exact, we will try to find an integrating factor to make it exact. Let

$$\begin{aligned}
A &= \frac{1}{N}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right) \\
&= 1\left(\left(\frac{3}{x}\right) - (0)\right) \\
&= \frac{3}{x}
\end{aligned}$$

Since A does not depend on y, then it can be used to find an integrating factor. The integrating factor μ is

$$\begin{aligned}
\mu &= e^{\int A \, \mathrm{d}x} \\
&= e^{\int \frac{3}{x} \, \mathrm{d}x}
\end{aligned}$$

The result of integrating gives

$$\begin{aligned}
\mu &= e^{3\ln(x)} \\
&= x^3
\end{aligned}$$

M and N are multiplied by this integrating factor, giving new M and new N which are called $\overline{M}$ and $\overline{N}$ for now so not to confuse them with the original M and N.

$$\begin{aligned}
\overline{M} &= \mu M \\
&= x^3\left(-3x + 2 + \frac{3y}{x}\right) \\
&= -3x^4 + 2x^3 + 3y\,x^2
\end{aligned}$$

And

$$\begin{aligned}
\overline{N} &= \mu N \\
&= x^3(1) \\
&= x^3
\end{aligned}$$

Now a modified ODE is ontained from the original ODE, which is exact and can be solved. The modified ODE is

$$\begin{aligned}
\overline{M} + \overline{N}\frac{\mathrm{d}y}{\mathrm{d}x} &= 0 \\
\left(-3x^4 + 2x^3 + 3y\,x^2\right) + \left(x^3\right)\frac{\mathrm{d}y}{\mathrm{d}x} &= 0
\end{aligned}$$

The following equations are now set up to solve for the function $\phi(x,y)$

$$\frac{\partial \phi}{\partial x} = \overline{M} \tag{1}$$
$$\frac{\partial \phi}{\partial y} = \overline{N} \tag{2}$$

Integrating (1) w.r.t. x gives

$$\begin{aligned}\int \frac{\partial \phi}{\partial x}\,\mathrm{d}x &= \int \overline{M}\,\mathrm{d}x\\ \int \frac{\partial \phi}{\partial x}\,\mathrm{d}x &= \int -3x^4 + 2x^3 + 3y\,x^2\,\mathrm{d}x\\ \phi &= -\frac{3}{5}x^5 + \frac{1}{2}x^4 + y\,x^3 + f(y) \qquad (3)\end{aligned}$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$\frac{\partial \phi}{\partial y} = x^3 + f'(y) \tag{4}$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = x^3$. Therefore equation (4) becomes

$$x^3 = x^3 + f'(y) \tag{5}$$

Solving equation (5) for $f'(y)$ gives

$$f'(y) = 0$$

Therefore

$$f(y) = c_1$$

Where c_1 is constant of integration. Substituting this result for $f(y)$ into equation (3) gives ϕ

$$\phi = -\frac{3}{5}x^5 + \frac{1}{2}x^4 + y\,x^3 + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = -\frac{3}{5}x^5 + \frac{1}{2}x^4 + y\,x^3$$

The solution becomes

$$y = \frac{6x^5 - 5x^4 + 10c_1}{10x^3}$$

Initial conditions are used to solve for c_1. Substituting $x = 1$ and $y = 1$ in the above solution gives an equation to solve for the constant of integration.

$$1 = c_1 + \frac{1}{10}$$

$$c_1 = \frac{9}{10}$$

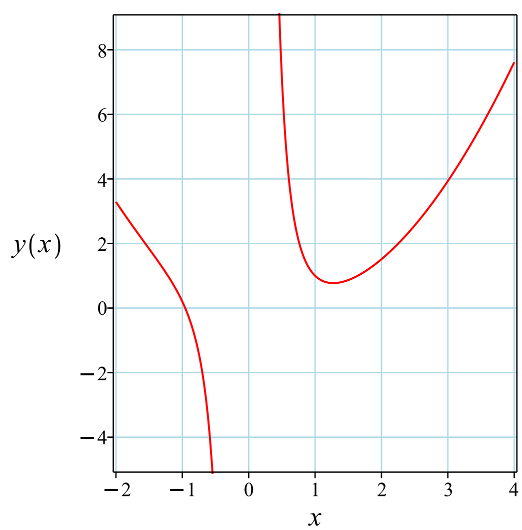
Substituting c_1 found above in the general solution gives

$$y = \frac{6x^5 - 5x^4 + 9}{10x^3}$$

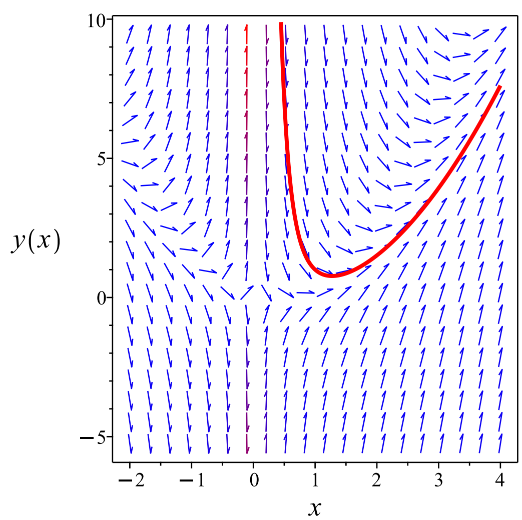
Summary

The solution(s) found are the following

$$y = \frac{6x^5 - 5x^4 + 9}{10x^3} \tag{1}$$



(a) Solution plot



(b) Slope field plot

Verification of solutions

$$y = \frac{6x^5 - 5x^4 + 9}{10x^3}$$

Verified OK.

2.20.5 Maple step by step solution

Let's solve

$\left[y' + \frac{3y}{x} = 3x - 2, y(1) = 1\right]$

- Highest derivative means the order of the ODE is 1

  y'

- Isolate the derivative

  $y' = -\frac{3y}{x} - 2 + 3x$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE

  $y' + \frac{3y}{x} = 3x - 2$

- The ODE is linear; multiply by an integrating factor $\mu(x)$

  $\mu(x)\left(y' + \frac{3y}{x}\right) = \mu(x)(3x - 2)$

- Assume the lhs of the ODE is the total derivative $\frac{d}{dx}(\mu(x)\,y)$

  $\mu(x)\left(y' + \frac{3y}{x}\right) = \mu'(x)\,y + \mu(x)\,y'$

- Isolate $\mu'(x)$

  $\mu'(x) = \frac{3\mu(x)}{x}$

- Solve to find the integrating factor

  $\mu(x) = x^3$

- Integrate both sides with respect to x

  $\int\left(\frac{d}{dx}(\mu(x)\,y)\right)dx = \int \mu(x)(3x - 2)\,dx + c_1$

- Evaluate the integral on the lhs

  $\mu(x)\,y = \int \mu(x)(3x - 2)\,dx + c_1$

- Solve for y

  $y = \frac{\int \mu(x)(3x-2)dx + c_1}{\mu(x)}$

- Substitute $\mu(x) = x^3$

  $y = \frac{\int (3x-2)x^3 dx + c_1}{x^3}$

- Evaluate the integrals on the rhs

  $y = \frac{\frac{3}{5}x^5 - \frac{1}{2}x^4 + c_1}{x^3}$

- Use initial condition $y(1) = 1$

$1 = c_1 + \frac{1}{10}$

- Solve for c_1

  $c_1 = \frac{9}{10}$

- Substitute $c_1 = \frac{9}{10}$ into general solution and simplify

  $y = \frac{6x^5-5x^4+9}{10x^3}$

- Solution to the IVP

  $y = \frac{6x^5-5x^4+9}{10x^3}$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
<- 1st order linear successful`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 18

```
dsolve([diff(y(x),x)+3*y(x)/x+2=3*x,y(1) = 1],y(x), singsol=all)
```

$$y(x) = \frac{3x^2}{5} - \frac{x}{2} + \frac{9}{10x^3}$$

✓ Solution by Mathematica

Time used: 0.029 (sec). Leaf size: 24

```
DSolve[{y'[x]+3*y[x]/x+2==3*x,{y[1]==1}},y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \frac{6x^5 - 5x^4 + 9}{10x^3}$$

2.21 problem 21

Internal problem ID [4970]
Internal file name [`OUTPUT/4463_Sunday_June_05_2022_02_57_05_PM_67035084/index.tex`]

Book: Fundamentals of Differential Equations. By Nagle, Saff and Snider. 9th edition. Boston. Pearson 2018.
Section: Chapter 2, First order differential equations. Section 2.3, Linear equations. Exercises. page 54
Problem number: 21.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"linear", "exactWithIntegrationFactor", "first_order_ode_lie_symmetry_lookup"**

Maple gives the following as the ode type

`[_linear]`

$$\boxed{\cos(x) y' + \sin(x) y = 2 \cos(x)^2 x}$$

With initial conditions

$$\left[y\left(\frac{\pi}{4}\right) = -\frac{15\sqrt{2}\,\pi^2}{32}\right]$$

2.21.1 Existence and uniqueness analysis

This is a linear ODE. In canonical form it is written as

$$y' + p(x)y = q(x)$$

Where here

$$\begin{aligned} p(x) &= \tan(x) \\ q(x) &= 2\cos(x)\, x \end{aligned}$$

Hence the ode is

$$y' + y\tan(x) = 2\cos(x)\, x$$

The domain of $p(x) = \tan(x)$ is

$$\left\{x < \frac{1}{2}\pi + \pi\_Z85 \vee \frac{1}{2}\pi + \pi\_Z85 < x\right\}$$

And the point $x_0 = \frac{\pi}{4}$ is inside this domain. The domain of $q(x) = 2\cos(x)\, x$ is

$$\{-\infty < x < \infty\}$$

And the point $x_0 = \frac{\pi}{4}$ is also inside this domain. Hence solution exists and is unique.

2.21.2 Solving as linear ode

Entering Linear first order ODE solver. The integrating factor μ is

$$\begin{aligned} \mu &= e^{\int \tan(x)dx} \\ &= \frac{1}{\cos(x)} \end{aligned}$$

Which simplifies to

$$\mu = \sec(x)$$

The ode becomes

$$\begin{aligned} \frac{d}{dx}(\mu y) &= (\mu)\,(2\cos(x)\, x) \\ \frac{d}{dx}(y\sec(x)) &= (\sec(x))\,(2\cos(x)\, x) \\ d(y\sec(x)) &= (2x)\, dx \end{aligned}$$

Integrating gives

$$y \sec(x) = \int 2x \, dx$$
$$y \sec(x) = x^2 + c_1$$

Dividing both sides by the integrating factor $\mu = \sec(x)$ results in

$$y = \cos(x) x^2 + \cos(x) c_1$$

which simplifies to

$$y = \cos(x) \left(x^2 + c_1\right)$$

Initial conditions are used to solve for c_1. Substituting $x = \frac{\pi}{4}$ and $y = -\frac{15\sqrt{2}\,\pi^2}{32}$ in the above solution gives an equation to solve for the constant of integration.

$$-\frac{15\sqrt{2}\,\pi^2}{32} = \frac{\sqrt{2}\,\pi^2}{32} + \frac{c_1\sqrt{2}}{2}$$

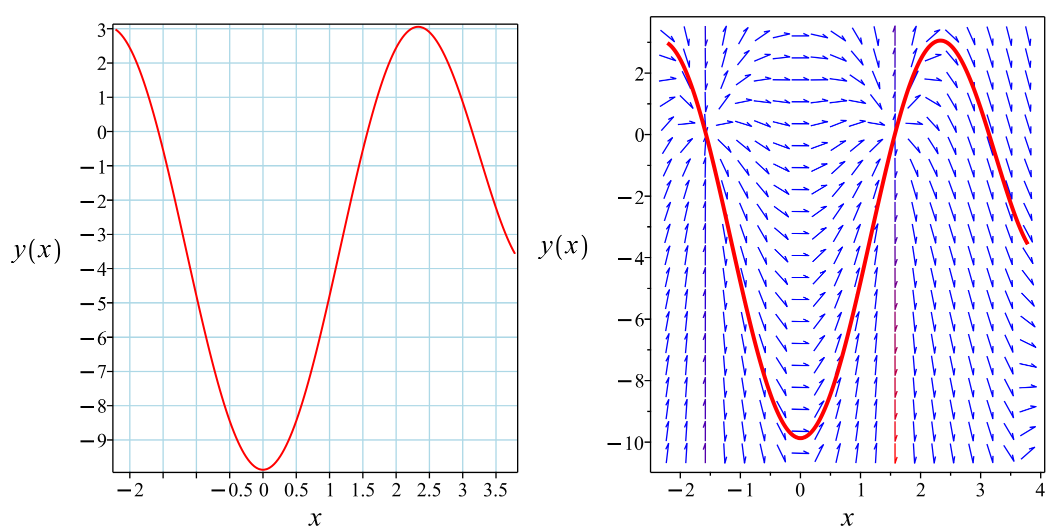
$$c_1 = -\pi^2$$

Substituting c_1 found above in the general solution gives

$$y = -\cos(x)\pi^2 + \cos(x) x^2$$

Summary
The solution(s) found are the following

$$y = -\cos(x)\pi^2 + \cos(x) x^2 \tag{1}$$



(a) Solution plot

(b) Slope field plot

Verification of solutions

$$y = -\cos(x)\,\pi^2 + \cos(x)\,x^2$$

Verified OK.

2.21.3 Solving as first order ode lie symmetry lookup ode

Writing the ode as

$$y' = -\frac{-2\cos(x)^2 x + y\sin(x)}{\cos(x)}$$
$$y' = \omega(x, y)$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x + \omega(\eta_y - \xi_x) - \omega^2 \xi_y - \omega_x \xi - \omega_y \eta = 0 \tag{A}$$

The type of this ode is known. It is of type `linear`. Therefore we do not need to solve the PDE (A), and can just use the lookup table shown below to find ξ, η

Table 143: Lie symmetry infinitesimal lookup table for known first order ODE's

ODE class	Form	ξ	η
linear ode	$y' = f(x)y(x) + g(x)$	0	$e^{\int f dx}$
separable ode	$y' = f(x)\, g(y)$	$\frac{1}{f}$	0
quadrature ode	$y' = f(x)$	0	1
quadrature ode	$y' = g(y)$	1	0
homogeneous ODEs of Class A	$y' = f\left(\frac{y}{x}\right)$	x	y
homogeneous ODEs of Class C	$y' = (a + bx + cy)^{\frac{n}{m}}$	1	$-\frac{b}{c}$
homogeneous class D	$y' = \frac{y}{x} + g(x)\, F\left(\frac{y}{x}\right)$	x^2	xy
First order special form ID 1	$y' = g(x)\, e^{h(x)+by} + f(x)$	$\frac{e^{-\int bf(x)dx-h(x)}}{g(x)}$	$\frac{f(x)e^{-\int bf(x)dx-h(x)}}{g(x)}$
polynomial type ode	$y' = \frac{a_1x+b_1y+c_1}{a_2x+b_2y+c_2}$	$\frac{a_1b_2x-a_2b_1x-b_1c_2+b_2c_1}{a_1b_2-a_2b_1}$	$\frac{a_1b_2y-a_2b_1y-a_1c_2-a_2c_1}{a_1b_2-a_2b_1}$
Bernoulli ode	$y' = f(x)\, y + g(x)\, y^n$	0	$e^{-\int (n-1)f(x)dx} y^n$
Reduced Riccati	$y' = f_1(x)\, y + f_2(x)\, y^2$	0	$e^{-\int f_1 dx}$

The above table shows that

$$\begin{aligned}\xi(x,y) &= 0\\ \eta(x,y) &= \cos(x)\end{aligned} \tag{A1}$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x,y) \to (R,S)$ where (R,S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}$$

The above comes from the requirements that $\left(\xi\frac{\partial}{\partial x} + \eta\frac{\partial}{\partial y}\right) S(x,y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the

canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$R = x$$

S is found from

$$\begin{aligned} S &= \int \frac{1}{\eta} dy \\ &= \int \frac{1}{\cos(x)} dy \end{aligned}$$

Which results in

$$S = \frac{y}{\cos(x)}$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x,y)S_y}{R_x + \omega(x,y)R_y} \tag{2}$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x,y)$ is the right hand side of the original ode given by

$$\omega(x,y) = -\frac{-2\cos(x)^2 x + y\sin(x)}{\cos(x)}$$

Evaluating all the partial derivatives gives

$$\begin{aligned} R_x &= 1 \\ R_y &= 0 \\ S_x &= y\sec(x)\tan(x) \\ S_y &= \sec(x) \end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = 2x \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = 2R$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordinates R, S. Integrating the above gives

$$S(R) = R^2 + c_1 \tag{4}$$

To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$y \sec(x) = x^2 + c_1$$

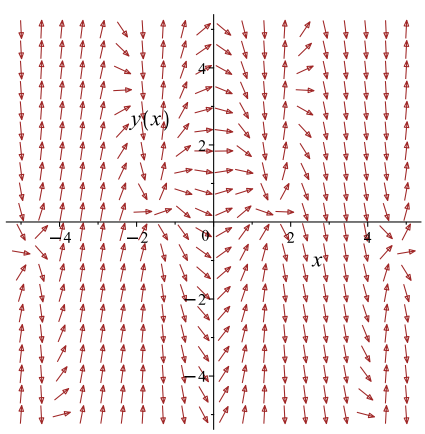
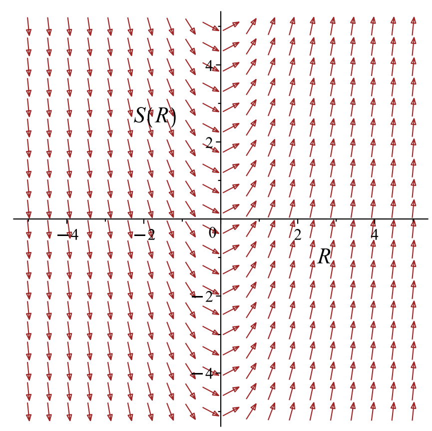
Which simplifies to

$$y \sec(x) = x^2 + c_1$$

Which gives

$$y = \frac{x^2 + c_1}{\sec(x)}$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in x, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dx} = -\frac{-2\cos(x)^2 x + y\sin(x)}{\cos(x)}$ 	$R = x$ $S = y\sec(x)$	$\frac{dS}{dR} = 2R$ 

Initial conditions are used to solve for c_1. Substituting $x = \frac{\pi}{4}$ and $y = -\frac{15\sqrt{2}\,\pi^2}{32}$ in the above solution gives an equation to solve for the constant of integration.

$$-\frac{15\sqrt{2}\,\pi^2}{32} = \frac{\sqrt{2}\,\pi^2}{32} + \frac{c_1\sqrt{2}}{2}$$

$$c_1 = -\pi^2$$

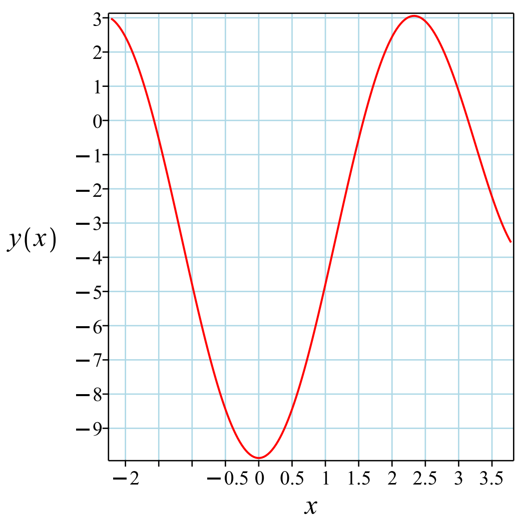
Substituting c_1 found above in the general solution gives

$$y = -\cos(x)\,\pi^2 + \cos(x)\,x^2$$

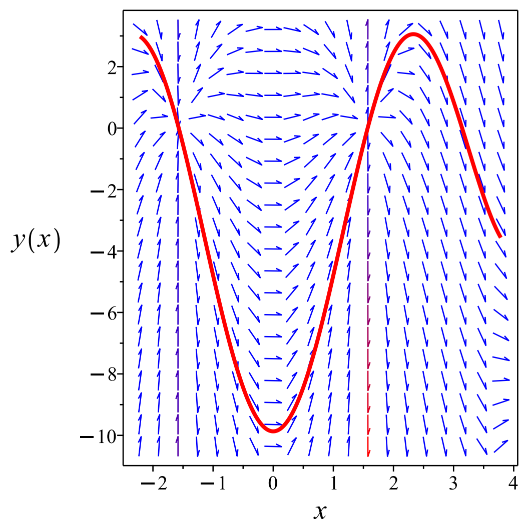
Summary

The solution(s) found are the following

$$y = -\cos(x)\,\pi^2 + \cos(x)\,x^2 \tag{1}$$



(a) Solution plot



(b) Slope field plot

Verification of solutions

$$y = -\cos(x)\,\pi^2 + \cos(x)\,x^2$$

Verified OK.

2.21.4 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x,y) + N(x,y)\frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x,y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y) = 0$$

Hence

$$\frac{\partial \phi}{\partial x}+\frac{\partial \phi}{\partial y}\frac{dy}{dx}=0 \tag{B}$$

Comparing (A,B) shows that

$$\begin{aligned}\frac{\partial \phi}{\partial x}&=M\\ \frac{\partial \phi}{\partial y}&=N\end{aligned}$$

But since $\frac{\partial^2\phi}{\partial x\partial y}=\frac{\partial^2\phi}{\partial y\partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y}=\frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x,y)$ but at least we know now that we can do that since the condition $\frac{\partial^2\phi}{\partial x\partial y}=\frac{\partial^2\phi}{\partial y\partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x,y)\,\mathrm{d}x+N(x,y)\,\mathrm{d}y=0 \tag{1A}$$

Therefore

$$\begin{aligned}(\cos(x))\,\mathrm{d}y&=\left(-y\sin(x)+2\cos(x)^2 x\right)\mathrm{d}x\\ \left(-2\cos(x)^2 x+y\sin(x)\right)\mathrm{d}x+(\cos(x))\,\mathrm{d}y&=0\end{aligned} \tag{2A}$$

Comparing (1A) and (2A) shows that

$$\begin{aligned}M(x,y)&=-2\cos(x)^2 x+y\sin(x)\\ N(x,y)&=\cos(x)\end{aligned}$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y}=\frac{\partial N}{\partial x}$$

Using result found above gives

$$\begin{aligned}\frac{\partial M}{\partial y}&=\frac{\partial}{\partial y}\left(-2\cos(x)^2 x+y\sin(x)\right)\\ &=\sin(x)\end{aligned}$$

And

$$
\begin{aligned}
\frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}(\cos(x)) \\
&= -\sin(x)
\end{aligned}
$$

Since $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$, then the ODE is <u>not exact</u>. Since the ODE is not exact, we will try to find an integrating factor to make it exact. Let

$$
\begin{aligned}
A &= \frac{1}{N}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right) \\
&= \sec(x)\left((\sin(x)) - (-\sin(x))\right) \\
&= 2\tan(x)
\end{aligned}
$$

Since A does not depend on y, then it can be used to find an integrating factor. The integrating factor μ is

$$
\begin{aligned}
\mu &= e^{\int A\,\mathrm{d}x} \\
&= e^{\int 2\tan(x)\,\mathrm{d}x}
\end{aligned}
$$

The result of integrating gives

$$
\begin{aligned}
\mu &= e^{-2\ln(\cos(x))} \\
&= \sec(x)^2
\end{aligned}
$$

M and N are multiplied by this integrating factor, giving new M and new N which are called $\overline{M}$ and $\overline{N}$ for now so not to confuse them with the original M and N.

$$
\begin{aligned}
\overline{M} &= \mu M \\
&= \sec(x)^2\left(-2\cos(x)^2 x + y\sin(x)\right) \\
&= -2x + y\sec(x)\tan(x)
\end{aligned}
$$

And

$$
\begin{aligned}
\overline{N} &= \mu N \\
&= \sec(x)^2(\cos(x)) \\
&= \sec(x)
\end{aligned}
$$

Now a modified ODE is ontained from the original ODE, which is exact and can be solved. The modified ODE is

$$
\begin{aligned}
\overline{M} + \overline{N}\frac{\mathrm{d}y}{\mathrm{d}x} &= 0 \\
(-2x + y\sec(x)\tan(x)) + (\sec(x))\frac{\mathrm{d}y}{\mathrm{d}x} &= 0
\end{aligned}
$$

The following equations are now set up to solve for the function $\phi(x, y)$

$$\frac{\partial \phi}{\partial x} = \overline{M} \tag{1}$$
$$\frac{\partial \phi}{\partial y} = \overline{N} \tag{2}$$

Integrating (1) w.r.t. x gives

$$\int \frac{\partial \phi}{\partial x} \, \mathrm{d}x = \int \overline{M} \, \mathrm{d}x$$
$$\int \frac{\partial \phi}{\partial x} \, \mathrm{d}x = \int -2x + y \sec(x) \tan(x) \, \mathrm{d}x$$
$$\phi = y \sec(x) - x^2 + f(y) \tag{3}$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$\frac{\partial \phi}{\partial y} = \sec(x) + f'(y) \tag{4}$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = \sec(x)$. Therefore equation (4) becomes

$$\sec(x) = \sec(x) + f'(y) \tag{5}$$

Solving equation (5) for $f'(y)$ gives

$$f'(y) = 0$$

Therefore

$$f(y) = c_1$$

Where c_1 is constant of integration. Substituting this result for $f(y)$ into equation (3) gives ϕ

$$\phi = y \sec(x) - x^2 + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = y \sec(x) - x^2$$

The solution becomes

$$y = \frac{x^2 + c_1}{\sec(x)}$$

Initial conditions are used to solve for c_1. Substituting $x = \frac{\pi}{4}$ and $y = -\frac{15\sqrt{2}\,\pi^2}{32}$ in the above solution gives an equation to solve for the constant of integration.

$$-\frac{15\sqrt{2}\,\pi^2}{32} = \frac{\sqrt{2}\,\pi^2}{32} + \frac{c_1\sqrt{2}}{2}$$

$$c_1 = -\pi^2$$

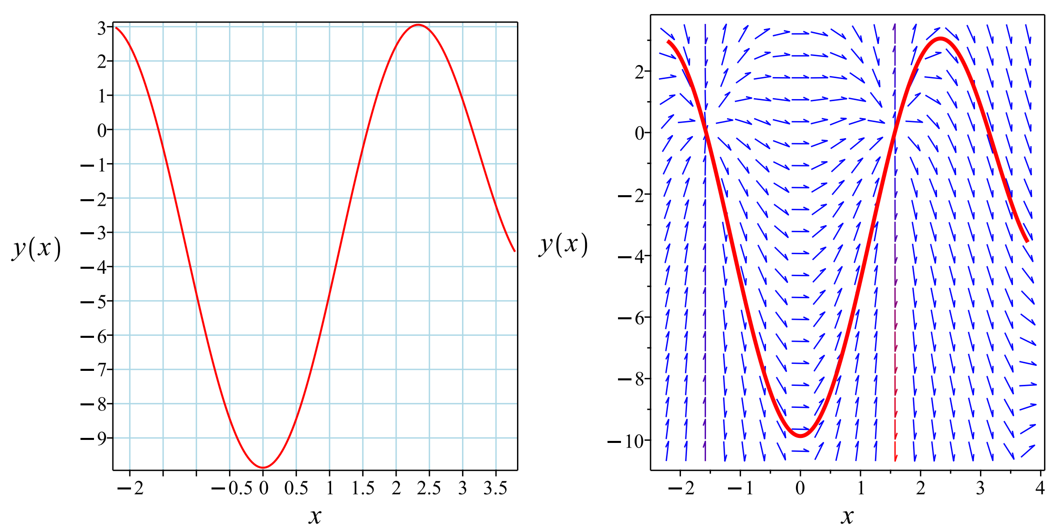
Substituting c_1 found above in the general solution gives

$$y = -\cos(x)\,\pi^2 + \cos(x)\,x^2$$

Summary

The solution(s) found are the following

$$y = -\cos(x)\,\pi^2 + \cos(x)\,x^2 \tag{1}$$



(a) Solution plot (b) Slope field plot

Verification of solutions

$$y = -\cos(x)\,\pi^2 + \cos(x)\,x^2$$

Verified OK.

2.21.5 Maple step by step solution

Let's solve

$\left[\cos(x) y' + \sin(x) y = 2\cos(x)^2 x, y\left(\frac{\pi}{4}\right) = -\frac{15\sqrt{2}\,\pi^2}{32}\right]$

- Highest derivative means the order of the ODE is 1

  y'

- Isolate the derivative

  $y' = -\frac{\sin(x)y}{\cos(x)} + 2\cos(x) x$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE

  $y' + \frac{\sin(x)y}{\cos(x)} = 2\cos(x) x$

- The ODE is linear; multiply by an integrating factor $\mu(x)$

  $\mu(x)\left(y' + \frac{\sin(x)y}{\cos(x)}\right) = 2\mu(x)\cos(x) x$

- Assume the lhs of the ODE is the total derivative $\frac{d}{dx}(\mu(x) y)$

  $\mu(x)\left(y' + \frac{\sin(x)y}{\cos(x)}\right) = \mu'(x) y + \mu(x) y'$

- Isolate $\mu'(x)$

  $\mu'(x) = \frac{\mu(x)\sin(x)}{\cos(x)}$

- Solve to find the integrating factor

  $\mu(x) = \frac{1}{\cos(x)}$

- Integrate both sides with respect to x

  $\int \left(\frac{d}{dx}(\mu(x) y)\right) dx = \int 2\mu(x)\cos(x) x dx + c_1$

- Evaluate the integral on the lhs

  $\mu(x) y = \int 2\mu(x)\cos(x) x dx + c_1$

- Solve for y

  $y = \frac{\int 2\mu(x)\cos(x)xdx + c_1}{\mu(x)}$

- Substitute $\mu(x) = \frac{1}{\cos(x)}$

  $y = \cos(x)\left(\int 2x dx + c_1\right)$

- Evaluate the integrals on the rhs

  $y = \cos(x)(x^2 + c_1)$

- Use initial condition $y\left(\frac{\pi}{4}\right) = -\frac{15\sqrt{2}\,\pi^2}{32}$

  $$-\frac{15\sqrt{2}\,\pi^2}{32} = \frac{\sqrt{2}\left(\frac{\pi^2}{16}+c_1\right)}{2}$$

- Solve for c_1

  $$c_1 = -\pi^2$$

- Substitute $c_1 = -\pi^2$ into general solution and simplify

  $$y = (-\pi^2 + x^2)\cos(x)$$

- Solution to the IVP

  $$y = (-\pi^2 + x^2)\cos(x)$$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
<- 1st order linear successful`
```

✓ Solution by Maple

Time used: 0.015 (sec). Leaf size: 16

```
dsolve([cos(x)*diff(y(x),x)+y(x)*sin(x)=2*x*cos(x)^2,y(1/4*Pi) = -15/32*sqrt(2)*Pi^2],y(x),
```

$$y(x) = \left(-\pi^2 + x^2\right)\cos(x)$$

✓ Solution by Mathematica

Time used: 0.071 (sec). Leaf size: 17

```
DSolve[{Cos[x]*y'[x]+y[x]*Sin[x]==2*x*Cos[x]^2,{y[Pi/4]==-15*Sqrt[2]*Pi^2/32}},y[x],x,Includ
```

$$y(x) \to \left(x^2 - \pi^2\right)\cos(x)$$

2.22 problem 22

Internal problem ID [4971]
Internal file name [`OUTPUT/4464_Sunday_June_05_2022_02_57_07_PM_32710655/index.tex`]

Book: Fundamentals of Differential Equations. By Nagle, Saff and Snider. 9th edition. Boston. Pearson 2018.
Section: Chapter 2, First order differential equations. Section 2.3, Linear equations. Exercises. page 54
Problem number: 22.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"exact", "linear", "first_order_ode_lie_symmetry_lookup"**

Maple gives the following as the ode type

`[_linear]`

$$\boxed{\sin(x)\, y' + y\cos(x) = \sin(x)\, x}$$

With initial conditions

$$\left[y\left(\frac{\pi}{2}\right) = 2\right]$$

2.22.1 Existence and uniqueness analysis

This is a linear ODE. In canonical form it is written as

$$y' + p(x)y = q(x)$$

Where here

$$p(x) = \cot(x)$$
$$q(x) = x$$

Hence the ode is

$$y' + \cot(x)\, y = x$$

The domain of $p(x) = \cot(x)$ is

$$\{x < \pi\_Z86 \lor \pi\_Z86 < x\}$$

And the point $x_0 = \frac{\pi}{2}$ is inside this domain. The domain of $q(x) = x$ is

$$\{-\infty < x < \infty\}$$

And the point $x_0 = \frac{\pi}{2}$ is also inside this domain. Hence solution exists and is unique.

2.22.2 Solving as linear ode

Entering Linear first order ODE solver. The integrating factor μ is

$$\begin{aligned} \mu &= e^{\int \cot(x)dx} \\ &= \sin(x) \end{aligned}$$

The ode becomes

$$\begin{aligned} \frac{d}{dx}(\mu y) &= (\mu)(x) \\ \frac{d}{dx}(y\sin(x)) &= (\sin(x))(x) \\ d(y\sin(x)) &= (\sin(x)\, x)\; dx \end{aligned}$$

Integrating gives

$$\begin{aligned} y\sin(x) &= \int \sin(x)\, x\, dx \\ y\sin(x) &= -\cos(x)\, x + \sin(x) + c_1 \end{aligned}$$

Dividing both sides by the integrating factor $\mu = \sin(x)$ results in

$$y = \csc(x)\,(-\cos(x)\, x + \sin(x)) + c_1 \csc(x)$$

which simplifies to

$$y = -x\cot(x) + 1 + c_1 \csc(x)$$

Initial conditions are used to solve for c_1. Substituting $x = \frac{\pi}{2}$ and $y = 2$ in the above solution gives an equation to solve for the constant of integration.

$$2 = c_1 + 1$$

$$c_1 = 1$$

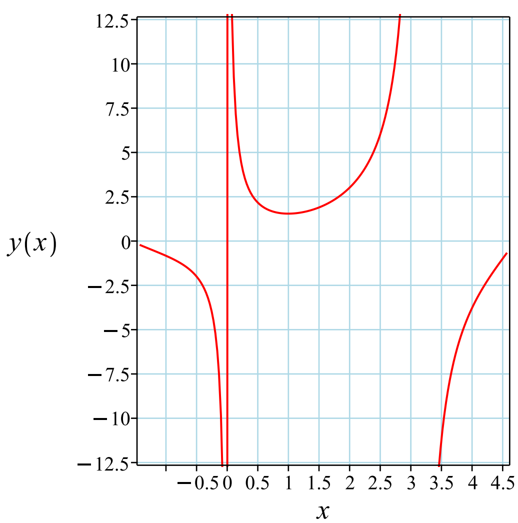
Substituting c_1 found above in the general solution gives

$$y = -x\cot(x) + 1 + \csc(x)$$

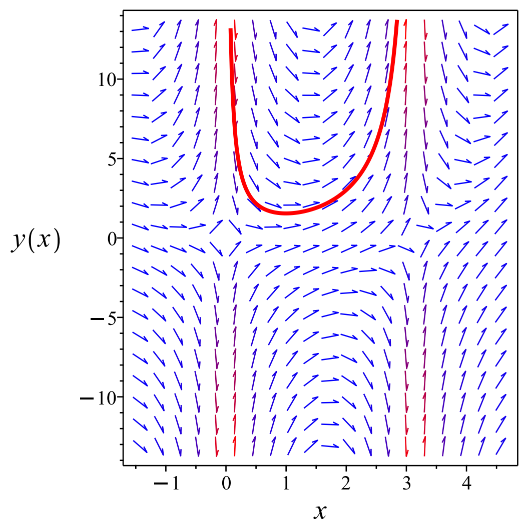
Summary

The solution(s) found are the following

$$y = -x\cot(x) + 1 + \csc(x) \tag{1}$$



(a) Solution plot



(b) Slope field plot

Verification of solutions

$$y = -x\cot(x) + 1 + \csc(x)$$

Verified OK.

2.22.3 Solving as first order ode lie symmetry lookup ode

Writing the ode as

$$y' = \frac{-y\cos(x) + \sin(x)\,x}{\sin(x)}$$
$$y' = \omega(x, y)$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x + \omega(\eta_y - \xi_x) - \omega^2\xi_y - \omega_x\xi - \omega_y\eta = 0 \tag{A}$$

The type of this ode is known. It is of type `linear`. Therefore we do not need to solve the PDE (A), and can just use the lookup table shown below to find ξ, η

Table 146: Lie symmetry infinitesimal lookup table for known first order ODE's

ODE class	Form	ξ	η
linear ode	$y' = f(x)y(x) + g(x)$	0	$e^{\int f dx}$
separable ode	$y' = f(x)\,g(y)$	$\frac{1}{f}$	0
quadrature ode	$y' = f(x)$	0	1
quadrature ode	$y' = g(y)$	1	0
homogeneous ODEs of Class A	$y' = f\left(\frac{y}{x}\right)$	x	y
homogeneous ODEs of Class C	$y' = (a + bx + cy)^{\frac{n}{m}}$	1	$-\frac{b}{c}$
homogeneous class D	$y' = \frac{y}{x} + g(x)\,F\left(\frac{y}{x}\right)$	x^2	xy
First order special form ID 1	$y' = g(x)\,e^{h(x)+by} + f(x)$	$\frac{e^{-\int bf(x)dx - h(x)}}{g(x)}$	$\frac{f(x)e^{-\int bf(x)dx - h(x)}}{g(x)}$
polynomial type ode	$y' = \frac{a_1x+b_1y+c_1}{a_2x+b_2y+c_2}$	$\frac{a_1b_2x-a_2b_1x-b_1c_2+b_2c_1}{a_1b_2-a_2b_1}$	$\frac{a_1b_2y-a_2b_1y-a_1c_2-a_2c_1}{a_1b_2-a_2b_1}$
Bernoulli ode	$y' = f(x)\,y + g(x)\,y^n$	0	$e^{-\int(n-1)f(x)dx}y^n$
Reduced Riccati	$y' = f_1(x)\,y + f_2(x)\,y^2$	0	$e^{-\int f_1 dx}$

The above table shows that

$$
\begin{aligned}
\xi(x,y) &= 0 \\
\eta(x,y) &= \frac{1}{\sin\left(x\right)}
\end{aligned}
\tag{A1}
$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x,y) \to (R,S)$ where (R,S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$
\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}
$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y}\right) S(x,y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$
R = x
$$

S is found from

$$
\begin{aligned}
S &= \int \frac{1}{\eta} dy \\
&= \int \frac{1}{\frac{1}{\sin(x)}} dy
\end{aligned}
$$

Which results in

$$
S = y \sin\left(x\right)
$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$
\frac{dS}{dR} = \frac{S_x + \omega(x,y) S_y}{R_x + \omega(x,y) R_y} \tag{2}
$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x,y)$ is the right hand side of the original ode given by

$$
\omega(x,y) = \frac{-y\cos\left(x\right) + \sin\left(x\right) x}{\sin\left(x\right)}
$$

Evaluating all the partial derivatives gives

$$
\begin{aligned}
R_x &= 1 \\
R_y &= 0 \\
S_x &= y\cos(x) \\
S_y &= \sin(x)
\end{aligned}
$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$
\frac{dS}{dR} = \sin(x)\, x \tag{2A}
$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$
\frac{dS}{dR} = \sin(R)\, R
$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$
S(R) = \sin(R) - \cos(R)\, R + c_1 \tag{4}
$$

To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$
\sin(x)\, y = -\cos(x)\, x + \sin(x) + c_1
$$

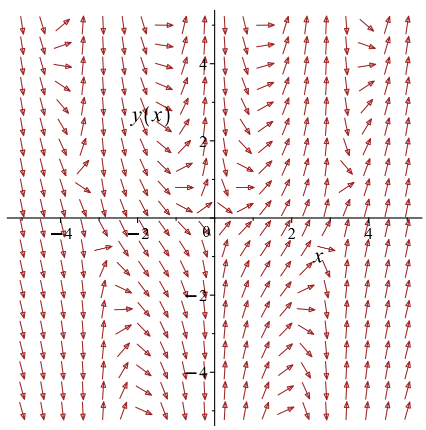
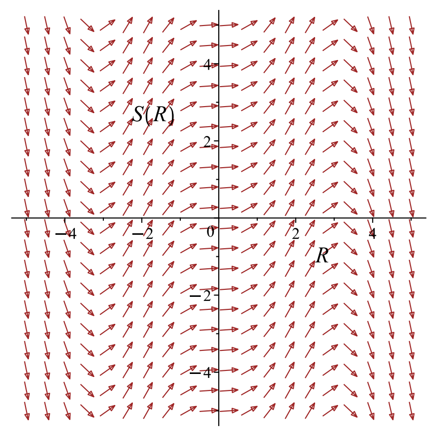
Which simplifies to

$$
\sin(x)\, y = -\cos(x)\, x + \sin(x) + c_1
$$

Which gives

$$
y = \frac{-\cos(x)\, x + \sin(x) + c_1}{\sin(x)}
$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in x, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dx} = \frac{-y\cos(x)+\sin(x)x}{\sin(x)}$ 	$R = x$ $S = y\sin(x)$	$\frac{dS}{dR} = \sin(R) R$ 

Initial conditions are used to solve for c_1. Substituting $x = \frac{\pi}{2}$ and $y = 2$ in the above solution gives an equation to solve for the constant of integration.

$$2 = c_1 + 1$$

$$c_1 = 1$$

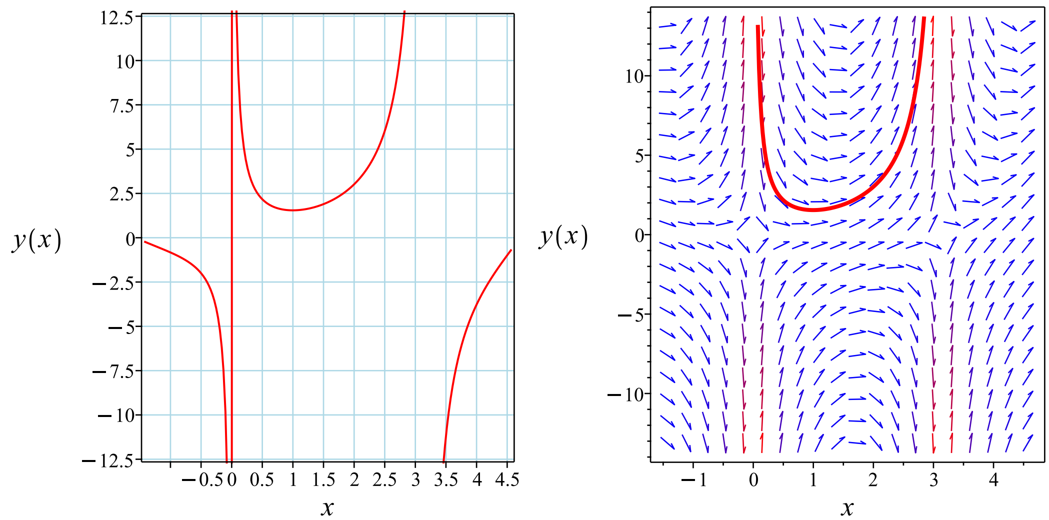
Substituting c_1 found above in the general solution gives

$$y = -x\cot(x) + 1 + \csc(x)$$

Summary

The solution(s) found are the following

$$y = -x\cot(x) + 1 + \csc(x) \tag{1}$$



(a) Solution plot

(b) Slope field plot

Verification of solutions

$$y = -x\cot(x) + 1 + \csc(x)$$

Verified OK.

2.22.4 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x,y) + N(x,y)\frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x,y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y) = 0$$

Hence

$$\frac{\partial\phi}{\partial x} + \frac{\partial\phi}{\partial y}\frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$\frac{\partial\phi}{\partial x} = M$$
$$\frac{\partial\phi}{\partial y} = N$$

But since $\frac{\partial^2\phi}{\partial x\partial y} = \frac{\partial^2\phi}{\partial y\partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x,y)$ but at least we know now that we can do that since the condition $\frac{\partial^2\phi}{\partial x\partial y} = \frac{\partial^2\phi}{\partial y\partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x,y)\,\mathrm{d}x + N(x,y)\,\mathrm{d}y = 0 \tag{1A}$$

Therefore

$$\begin{aligned}(\sin(x))\,\mathrm{d}y &= (-y\cos(x) + \sin(x)\,x)\,\mathrm{d}x\\ (y\cos(x) - \sin(x)\,x)\,\mathrm{d}x + (\sin(x))\,\mathrm{d}y &= 0 \end{aligned} \tag{2A}$$

Comparing (1A) and (2A) shows that

$$\begin{aligned}M(x,y) &= y\cos(x) - \sin(x)\,x\\ N(x,y) &= \sin(x)\end{aligned}$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Using result found above gives

$$\begin{aligned}\frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}(y\cos(x) - \sin(x)\,x)\\ &= \cos(x)\end{aligned}$$

And

$$\begin{aligned}\frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}(\sin(x))\\ &= \cos(x)\end{aligned}$$

Since $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$, then the ODE is exact The following equations are now set up to solve for the function $\phi(x,y)$

$$\frac{\partial \phi}{\partial x} = M \tag{1}$$

$$\frac{\partial \phi}{\partial y} = N \tag{2}$$

Integrating (1) w.r.t. x gives

$$\int \frac{\partial \phi}{\partial x} \, dx = \int M \, dx$$

$$\int \frac{\partial \phi}{\partial x} \, dx = \int y \cos(x) - \sin(x) \, x \, dx$$

$$\phi = (y - 1) \sin(x) + \cos(x) \, x + f(y) \tag{3}$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$\frac{\partial \phi}{\partial y} = \sin(x) + f'(y) \tag{4}$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = \sin(x)$. Therefore equation (4) becomes

$$\sin(x) = \sin(x) + f'(y) \tag{5}$$

Solving equation (5) for $f'(y)$ gives

$$f'(y) = 0$$

Therefore

$$f(y) = c_1$$

Where c_1 is constant of integration. Substituting this result for $f(y)$ into equation (3) gives ϕ

$$\phi = (y - 1) \sin(x) + \cos(x) \, x + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = (y - 1) \sin(x) + \cos(x) \, x$$

The solution becomes

$$y = \frac{-\cos(x) \, x + \sin(x) + c_1}{\sin(x)}$$

Initial conditions are used to solve for c_1. Substituting $x = \frac{\pi}{2}$ and $y = 2$ in the above solution gives an equation to solve for the constant of integration.

$$2 = c_1 + 1$$

$$c_1 = 1$$

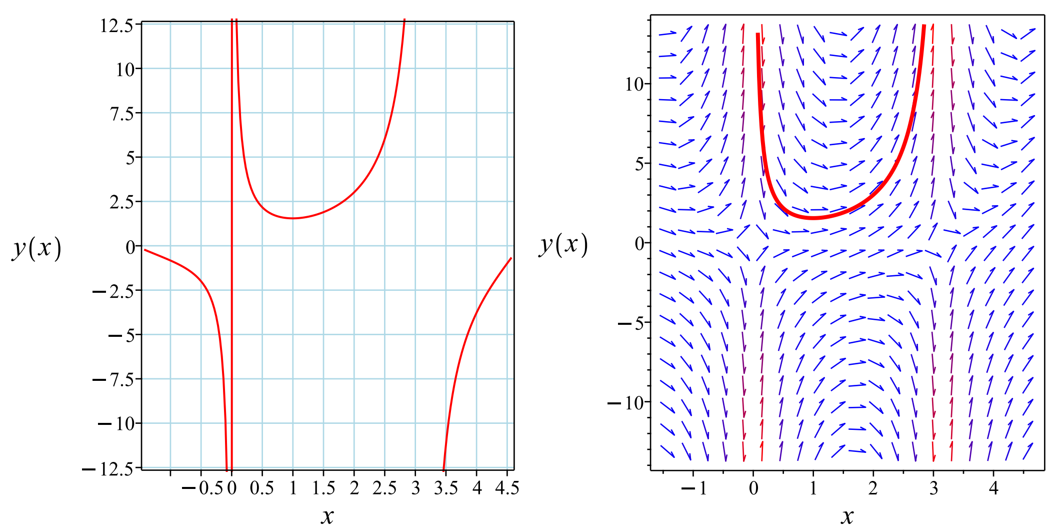
Substituting c_1 found above in the general solution gives

$$y = -x \cot(x) + 1 + \csc(x)$$

Summary

The solution(s) found are the following

$$y = -x \cot(x) + 1 + \csc(x) \tag{1}$$



(a) Solution plot

(b) Slope field plot

Verification of solutions

$$y = -x \cot(x) + 1 + \csc(x)$$

Verified OK.

2.22.5 Maple step by step solution

Let's solve

$\left[\sin(x)\, y' + y\cos(x) = \sin(x)\, x, y\left(\frac{\pi}{2}\right) = 2\right]$

- Highest derivative means the order of the ODE is 1

  y'

- Isolate the derivative

  $y' = -\frac{\cos(x) y}{\sin(x)} + x$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE

  $y' + \frac{\cos(x) y}{\sin(x)} = x$

- The ODE is linear; multiply by an integrating factor $\mu(x)$

  $\mu(x)\left(y' + \frac{\cos(x) y}{\sin(x)}\right) = \mu(x)\, x$

- Assume the lhs of the ODE is the total derivative $\frac{d}{dx}(\mu(x)\, y)$

  $\mu(x)\left(y' + \frac{\cos(x) y}{\sin(x)}\right) = \mu'(x)\, y + \mu(x)\, y'$

- Isolate $\mu'(x)$

  $\mu'(x) = \frac{\mu(x)\cos(x)}{\sin(x)}$

- Solve to find the integrating factor

  $\mu(x) = \sin(x)$

- Integrate both sides with respect to x

  $\int \left(\frac{d}{dx}(\mu(x)\, y)\right) dx = \int \mu(x)\, x dx + c_1$

- Evaluate the integral on the lhs

  $\mu(x)\, y = \int \mu(x)\, x dx + c_1$

- Solve for y

  $y = \frac{\int \mu(x) x dx + c_1}{\mu(x)}$

- Substitute $\mu(x) = \sin(x)$

  $y = \frac{\int \sin(x) x dx + c_1}{\sin(x)}$

- Evaluate the integrals on the rhs

  $y = \frac{-\cos(x) x + \sin(x) + c_1}{\sin(x)}$

- Simplify

$$y = -x\cot(x) + 1 + c_1 \csc(x)$$

- Use initial condition $y\left(\frac{\pi}{2}\right) = 2$

$$2 = c_1 + 1$$

- Solve for c_1

$$c_1 = 1$$

- Substitute $c_1 = 1$ into general solution and simplify

$$y = -x\cot(x) + 1 + \csc(x)$$

- Solution to the IVP

$$y = -x\cot(x) + 1 + \csc(x)$$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
<- 1st order linear successful`
```

✓ Solution by Maple

Time used: 0.015 (sec). Leaf size: 13

```
dsolve([sin(x)*diff(y(x),x)+y(x)*cos(x)=x*sin(x),y(1/2*Pi) = 2],y(x), singsol=all)
```

$$y(x) = -\cot(x)\, x + 1 + \csc(x)$$

✓ Solution by Mathematica

Time used: 0.05 (sec). Leaf size: 14

```
DSolve[{Sin[x]*y'[x]+y[x]*Cos[x]==x*Sin[x],{y[Pi/2]==2}},y[x],x,IncludeSingularSolutions ->
```

$$y(x) \to -x\cot(x) + \csc(x) + 1$$

2.23 problem 27

Internal problem ID [4972]
Internal file name [`OUTPUT/4465_Sunday_June_05_2022_02_57_08_PM_90153216/index.tex`]

Book: Fundamentals of Differential Equations. By Nagle, Saff and Snider. 9th edition. Boston. Pearson 2018.
Section: Chapter 2, First order differential equations. Section 2.3, Linear equations. Exercises. page 54
Problem number: 27.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"linear", "exactWithIntegrationFactor", "first_order_ode_lie_symmetry_lookup"**

Maple gives the following as the ode type

`[_linear]`

$$\boxed{y' + y\sqrt{1+\sin(x)^2} = x}$$

With initial conditions

$$[y(0) = 2]$$

2.23.1 Existence and uniqueness analysis

This is a linear ODE. In canonical form it is written as

$$y' + p(x)y = q(x)$$

Where here

$$p(x) = \frac{\sqrt{2}\,\sqrt{3-\cos(2x)}}{2}$$
$$q(x) = x$$

Hence the ode is

$$y' + \frac{\sqrt{2}\, \sqrt{3-\cos\left(2x\right)}\, y}{2} = x$$

The domain of $p(x) = \frac{\sqrt{2}\, \sqrt{3-\cos(2x)}}{2}$ is

$$\{-\infty < x < \infty\}$$

And the point $x_0 = 0$ is inside this domain. The domain of $q(x) = x$ is

$$\{-\infty < x < \infty\}$$

And the point $x_0 = 0$ is also inside this domain. Hence solution exists and is unique.

2.23.2 Solving as linear ode

Entering Linear first order ODE solver. The integrating factor μ is

$$\mu = e^{\int_0^x \frac{\sqrt{2}\, \sqrt{3-\cos(2\_a)}}{2} d\_a}$$

The ode becomes

$$\begin{aligned}
\frac{d}{dx}(\mu y) &= (\mu)\,(x) \\
\frac{d}{dx}\left(e^{\int_0^x \frac{\sqrt{2}\, \sqrt{3-\cos(2_a)}}{2} d_a} y\right) &= \left(e^{\int_0^x \frac{\sqrt{2}\, \sqrt{3-\cos(2_a)}}{2} d_a}\right)(x) \\
d\left(e^{\int_0^x \frac{\sqrt{2}\, \sqrt{3-\cos(2_a)}}{2} d_a} y\right) &= \left(x\, e^{\frac{\sqrt{2}\left(\int_0^x \sqrt{3-\cos(2_a)}d_a\right)}{2}}\right) dx
\end{aligned}$$

Integrating gives

$$\begin{aligned}
e^{\int_0^x \frac{\sqrt{2}\, \sqrt{3-\cos(2_a)}}{2} d_a} y &= \int x\, e^{\frac{\sqrt{2}\left(\int_0^x \sqrt{3-\cos(2_a)}d_a\right)}{2}} dx \\
e^{\int_0^x \frac{\sqrt{2}\, \sqrt{3-\cos(2_a)}}{2} d_a} y &= \int_0^x _a\, e^{\frac{\sqrt{2}\left(\int_0^{_a} \sqrt{3-\cos(2_a)}d_a\right)}{2}} d_a + c_1
\end{aligned}$$

Dividing both sides by the integrating factor $\mu = e^{\int_0^x \frac{\sqrt{2}\, \sqrt{3-\cos(2\_a)}}{2} d\_a}$ results in

$$y = e^{-\frac{\sqrt{2}\left(\int_0^x \sqrt{3-\cos(2\_a)}d\_a\right)}{2}} \left(\int_0^x \_a\, e^{\frac{\sqrt{2}\left(\int_0^{\_a} \sqrt{3-\cos(2\_a)}d\_a\right)}{2}} d\_a\right) + c_1 e^{-\frac{\sqrt{2}\left(\int_0^x \sqrt{3-\cos(2\_a)}d\_a\right)}{2}}$$

which simplifies to

$$y = e^{-\frac{\sqrt{2}\left(\int_0^x \sqrt{3-\cos(2\_a)}d\_a\right)}{2}} \left(\int_0^x \_a\, e^{\frac{\sqrt{2}\left(\int_0^{\_a} \sqrt{3-\cos(2\_a)}d\_a\right)}{2}} d\_a + c_1\right)$$

Initial conditions are used to solve for c_1. Substituting $x = 0$ and $y = 2$ in the above solution gives an equation to solve for the constant of integration.

$$2 = \mathrm{e}^{-\frac{\sqrt{2}\left(\int_0^0 \sqrt{3-\cos(2\_a)}d\_a\right)}{2}} \left(\int_0^0 \_a\,\mathrm{e}^{\frac{\sqrt{2}\left(\int_0^{\_a} \sqrt{3-\cos(2\_a)}d\_a\right)}{2}} d\_a + c_1\right)$$

$$c_1 = 2$$

Substituting c_1 found above in the general solution gives

$$y = \mathrm{e}^{-\frac{\sqrt{2}\left(\int_0^x \sqrt{3-\cos(2\_a)}d\_a\right)}{2}} \left(\int_0^x \_a\,\mathrm{e}^{\frac{\sqrt{2}\left(\int_0^{\_a} \sqrt{3-\cos(2\_a)}d\_a\right)}{2}} d\_a\right) + 2\,\mathrm{e}^{-\frac{\sqrt{2}\left(\int_0^x \sqrt{3-\cos(2\_a)}d\_a\right)}{2}}$$

Summary

The solution(s) found are the following

$$y = \mathrm{e}^{-\frac{\sqrt{2}\left(\int_0^x \sqrt{3-\cos(2\_a)}d\_a\right)}{2}} \left(\int_0^x \_a\,\mathrm{e}^{\frac{\sqrt{2}\left(\int_0^{\_a} \sqrt{3-\cos(2\_a)}d\_a\right)}{2}} d\_a\right) + 2\,\mathrm{e}^{-\frac{\sqrt{2}\left(\int_0^x \sqrt{3-\cos(2\_a)}d\_a\right)}{2}} \tag{1}$$

Verification of solutions

$$y = \mathrm{e}^{-\frac{\sqrt{2}\left(\int_0^x \sqrt{3-\cos(2\_a)}d\_a\right)}{2}} \left(\int_0^x \_a\,\mathrm{e}^{\frac{\sqrt{2}\left(\int_0^{\_a} \sqrt{3-\cos(2\_a)}d\_a\right)}{2}} d\_a\right) + 2\,\mathrm{e}^{-\frac{\sqrt{2}\left(\int_0^x \sqrt{3-\cos(2\_a)}d\_a\right)}{2}}$$

Verified OK.

2.23.3 Solving as first order ode lie symmetry lookup ode

Writing the ode as

$$y' = -y\sqrt{1+\sin(x)^2} + x$$
$$y' = \omega(x, y)$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x + \omega(\eta_y - \xi_x) - \omega^2\xi_y - \omega_x\xi - \omega_y\eta = 0 \tag{A}$$

The type of this ode is known. It is of type `linear`. Therefore we do not need to solve the PDE (A), and can just use the lookup table shown below to find ξ, η

Table 149: Lie symmetry infinitesimal lookup table for known first order ODE's

ODE class	Form	ξ	η
linear ode	$y' = f(x)y(x) + g(x)$	0	$e^{\int f dx}$
separable ode	$y' = f(x)\, g(y)$	$\frac{1}{f}$	0
quadrature ode	$y' = f(x)$	0	1
quadrature ode	$y' = g(y)$	1	0
homogeneous ODEs of Class A	$y' = f\left(\frac{y}{x}\right)$	x	y
homogeneous ODEs of Class C	$y' = (a + bx + cy)^{\frac{n}{m}}$	1	$-\frac{b}{c}$
homogeneous class D	$y' = \frac{y}{x} + g(x)\, F\left(\frac{y}{x}\right)$	x^2	xy
First order special form ID 1	$y' = g(x)\, e^{h(x)+by} + f(x)$	$\frac{e^{-\int bf(x)dx - h(x)}}{g(x)}$	$\frac{f(x)e^{-\int bf(x)dx - h(x)}}{g(x)}$
polynomial type ode	$y' = \frac{a_1x+b_1y+c_1}{a_2x+b_2y+c_2}$	$\frac{a_1b_2x - a_2b_1x - b_1c_2 + b_2c_1}{a_1b_2 - a_2b_1}$	$\frac{a_1b_2y - a_2b_1y - a_1c_2 - a_2c_1}{a_1b_2 - a_2b_1}$
Bernoulli ode	$y' = f(x)\, y + g(x)\, y^n$	0	$e^{-\int (n-1)f(x)dx} y^n$
Reduced Riccati	$y' = f_1(x)\, y + f_2(x)\, y^2$	0	$e^{-\int f_1 dx}$

The above table shows that

$$
\begin{aligned}
\xi(x,y) &= 0 \\
\eta(x,y) &= \mathrm{e}^{\frac{\sqrt{2}\,\sqrt{-\left(\cos(x)^2-2\right)\sin(x)^2}\,\sqrt{\frac{1}{2}-\frac{\cos(2x)}{2}}\;\mathrm{EllipticE}\left(\cos(x), \frac{\sqrt{2}}{2}\right)}{\sqrt{\sin(x)^4+\sin(x)^2}\,\sin(x)}}
\end{aligned} \tag{A1}
$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x, y) \to (R, S)$ where (R, S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$
\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}
$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y}\right) S(x,y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the

canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$R = x$$

S is found from

$$\begin{aligned} S &= \int \frac{1}{\eta} dy \\ &= \int \frac{1}{\mathrm{e}^{\frac{\sqrt{2}\sqrt{-\left(\cos(x)^2-2\right)\sin(x)^2}\sqrt{\frac{1}{2}-\frac{\cos(2x)}{2}}\,\mathrm{EllipticE}\left(\cos(x),\frac{\sqrt{2}}{2}\right)}{\sqrt{\sin(x)^4+\sin(x)^2}\,\sin(x)}}} dy \end{aligned}$$

Which results in

$$S = \mathrm{e}^{-\frac{\mathrm{EllipticE}\left(\cos(x),\frac{\sqrt{2}}{2}\right)\sqrt{1-\cos(2x)}}{\sin(x)}} y$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x,y) S_y}{R_x + \omega(x,y) R_y} \tag{2}$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x,y)$ is the right hand side of the original ode given by

$$\omega(x,y) = -y\sqrt{1+\sin(x)^2} + x$$

Evaluating all the partial derivatives gives

$$\begin{aligned} R_x &= 1 \\ R_y &= 0 \\ S_x &= \frac{y\sqrt{2}\,\sqrt{3-\cos(2x)}\,\mathrm{e}^{-\sqrt{2}\,\mathrm{EllipticE}\left(\cos(x),\frac{\sqrt{2}}{2}\right)}}{2} \\ S_y &= \mathrm{e}^{-\sqrt{2}\,\mathrm{EllipticE}\left(\cos(x),\frac{\sqrt{2}}{2}\right)} \end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = \mathrm{e}^{-\sqrt{2}\,\mathrm{EllipticE}\left(\cos(x),\frac{\sqrt{2}}{2}\right)} x \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = \mathrm{e}^{-\sqrt{2}\,\mathrm{EllipticE}\left(\cos(R),\frac{\sqrt{2}}{2}\right)} R$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordinates R, S. Integrating the above gives

$$S(R) = \int \mathrm{e}^{-\sqrt{2}\,\text{EllipticE}\left(\cos(R), \frac{\sqrt{2}}{2}\right)} R dR + c_1 \tag{4}$$

To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$y\,\mathrm{e}^{-\sqrt{2}\,\text{EllipticE}\left(\cos(x), \frac{\sqrt{2}}{2}\right)} = \int \mathrm{e}^{-\sqrt{2}\,\text{EllipticE}\left(\cos(x), \frac{\sqrt{2}}{2}\right)} x dx + c_1$$

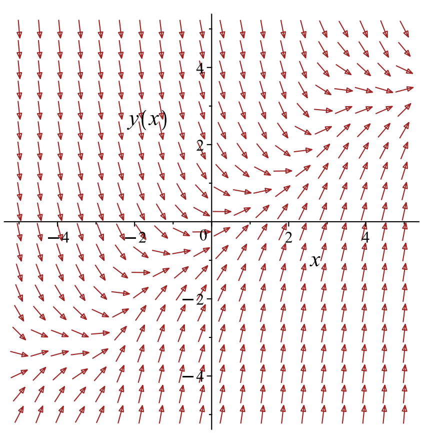
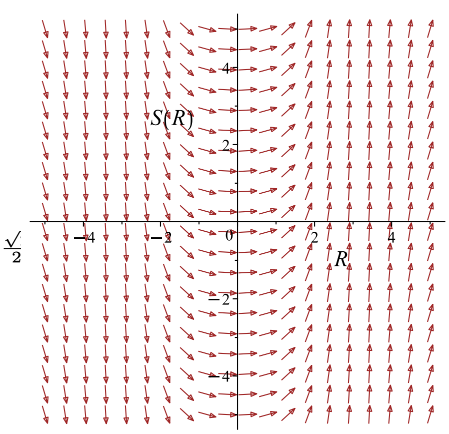
Which simplifies to

$$y\,\mathrm{e}^{-\sqrt{2}\,\text{EllipticE}\left(\cos(x), \frac{\sqrt{2}}{2}\right)} = \int \mathrm{e}^{-\sqrt{2}\,\text{EllipticE}\left(\cos(x), \frac{\sqrt{2}}{2}\right)} x dx + c_1$$

Which gives

$$y = \left(\int \mathrm{e}^{-\sqrt{2}\,\text{EllipticE}\left(\cos(x), \frac{\sqrt{2}}{2}\right)} x dx + c_1\right) \mathrm{e}^{\sqrt{2}\,\text{EllipticE}\left(\cos(x), \frac{\sqrt{2}}{2}\right)}$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in x, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dx} = -y\sqrt{1+\sin(x)^2} + x$ 	$R = x$ $S = y\,\mathrm{e}^{-\sqrt{2}\,\text{EllipticE}\left(\cos(x), \frac{\sqrt{2}}{2}\right)}$	$\frac{dS}{dR} = \mathrm{e}^{-\sqrt{2}\,\text{EllipticE}\left(\cos(R), \frac{\sqrt{2}}{2}\right)} R$ 

Initial conditions are used to solve for c_1. Substituting $x = 0$ and $y = 2$ in the above solution gives an equation to solve for the constant of integration.

$$2 = \left(\int^{0} \mathrm{e}^{-\sqrt{2}\, \mathrm{EllipticE}\left(\cos(\_a), \frac{\sqrt{2}}{2}\right)} \_ad\_a + c_1\right) \mathrm{e}^{\sqrt{2}\, \mathrm{EllipticE}\left(\frac{\sqrt{2}}{2}\right)}$$

$$c_1 = -\left(\mathrm{e}^{\sqrt{2}\, \mathrm{EllipticE}\left(\frac{\sqrt{2}}{2}\right)} \left(\int^{0} \mathrm{e}^{-\sqrt{2}\, \mathrm{EllipticE}\left(\cos(\_a), \frac{\sqrt{2}}{2}\right)} \_ad\_a\right) - 2\right) \mathrm{e}^{-\sqrt{2}\, \mathrm{EllipticE}\left(\frac{\sqrt{2}}{2}\right)}$$

Substituting c_1 found above in the general solution gives

$$y = \left(\int \mathrm{e}^{-\sqrt{2}\, \mathrm{EllipticE}\left(\cos(x), \frac{\sqrt{2}}{2}\right)} x dx - \left(\mathrm{e}^{\sqrt{2}\, \mathrm{EllipticE}\left(\frac{\sqrt{2}}{2}\right)} \left(\int^{0} \mathrm{e}^{-\sqrt{2}\, \mathrm{EllipticE}\left(\cos(\_a), \frac{\sqrt{2}}{2}\right)} \_ad\_a\right) - 2\right) \mathrm{e}^{-\sqrt{2}\, \mathrm{Ellip}}$$

Summary

The solution(s) found are the following

$$y = \left(\int \mathrm{e}^{-\sqrt{2}\, \mathrm{EllipticE}\left(\cos(x), \frac{\sqrt{2}}{2}\right)} x dx - \left(\mathrm{e}^{\sqrt{2}\, \mathrm{EllipticE}\left(\frac{\sqrt{2}}{2}\right)} \left(\int^{0} \mathrm{e}^{-\sqrt{2}\, \mathrm{EllipticE}\left(\cos(\_a), \frac{\sqrt{2}}{2}\right)} \_ad\_a\right) - 2\right) \mathrm{e}^{-\sqrt{2}\, \mathrm{EllipticE}\left(\frac{\sqrt{2}}{2}\right)}\right) \mathrm{e}^{\sqrt{2}\, \mathrm{EllipticE}\left(\cos(x), \frac{\sqrt{2}}{2}\right)} \tag{1}$$

Verification of solutions

$$y = \left(\int \mathrm{e}^{-\sqrt{2}\, \mathrm{EllipticE}\left(\cos(x), \frac{\sqrt{2}}{2}\right)} x dx - \left(\mathrm{e}^{\sqrt{2}\, \mathrm{EllipticE}\left(\frac{\sqrt{2}}{2}\right)} \left(\int^{0} \mathrm{e}^{-\sqrt{2}\, \mathrm{EllipticE}\left(\cos(\_a), \frac{\sqrt{2}}{2}\right)} \_ad\_a\right) - 2\right) \mathrm{e}^{-\sqrt{2}\, \mathrm{EllipticE}\left(\frac{\sqrt{2}}{2}\right)}\right) \mathrm{e}^{\sqrt{2}\, \mathrm{EllipticE}\left(\cos(x), \frac{\sqrt{2}}{2}\right)}$$

Verified OK.

2.23.4 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x, y) + N(x, y) \frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x, y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x, y) = 0$$

Hence

$$\frac{\partial \phi}{\partial x} + \frac{\partial \phi}{\partial y}\frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$\begin{aligned}\frac{\partial \phi}{\partial x} &= M\\ \frac{\partial \phi}{\partial y} &= N\end{aligned}$$

But since $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x, y)$ but at least we know now that we can do that since the condition $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x, y)\,\mathrm{d}x + N(x, y)\,\mathrm{d}y = 0 \tag{1A}$$

Therefore

$$\mathrm{d}y = \left(-y\sqrt{1+\sin(x)^2} + x\right)\mathrm{d}x$$

$$\left(y\sqrt{1+\sin(x)^2} - x\right)\mathrm{d}x + \mathrm{d}y = 0 \tag{2A}$$

Comparing (1A) and (2A) shows that

$$\begin{aligned}M(x, y) &= y\sqrt{1+\sin(x)^2} - x\\ N(x, y) &= 1\end{aligned}$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Using result found above gives

$$\begin{aligned}\frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}\left(y\sqrt{1+\sin(x)^2} - x\right)\\ &= \frac{\sqrt{2}\,\sqrt{3-\cos(2x)}}{2}\end{aligned}$$

And

$$
\begin{aligned}
\frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}(1) \\
&= 0
\end{aligned}
$$

Since $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$, then the ODE is not exact. Since the ODE is not exact, we will try to find an integrating factor to make it exact. Let

$$
\begin{aligned}
A &= \frac{1}{N}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right) \\
&= 1\left(\left(\sqrt{1+\sin(x)^2}\right) - (0)\right) \\
&= \frac{\sqrt{2}\,\sqrt{3-\cos(2x)}}{2}
\end{aligned}
$$

Since A does not depend on y, then it can be used to find an integrating factor. The integrating factor μ is

$$
\begin{aligned}
\mu &= e^{\int A\,\mathrm{d}x} \\
&= e^{\int \frac{\sqrt{2}\,\sqrt{3-\cos(2x)}}{2}\,\mathrm{d}x}
\end{aligned}
$$

The result of integrating gives

$$
\begin{aligned}
\mu &= e^{-\frac{\sqrt{2}\,\sqrt{-\left(\cos(x)^2-2\right)\sin(x)^2}\,\sqrt{\frac{1}{2}-\frac{\cos(2x)}{2}}\,\mathrm{EllipticE}\left(\cos(x),\frac{\sqrt{2}}{2}\right)}{\sqrt{\sin(x)^4+\sin(x)^2}\,\sin(x)}} \\
&= \mathrm{e}^{-\sqrt{2}\,\mathrm{csgn}(\sin(x))\,\mathrm{EllipticE}\left(\cos(x),\frac{\sqrt{2}}{2}\right)}
\end{aligned}
$$

M and N are multiplied by this integrating factor, giving new M and new N which are called $\overline{M}$ and $\overline{N}$ for now so not to confuse them with the original M and N.

$$
\begin{aligned}
\overline{M} &= \mu M \\
&= \mathrm{e}^{-\sqrt{2}\,\mathrm{csgn}(\sin(x))\,\mathrm{EllipticE}\left(\cos(x),\frac{\sqrt{2}}{2}\right)}\left(y\sqrt{1+\sin(x)^2} - x\right) \\
&= -\left(-y\sqrt{-\cos(x)^2+2} + x\right)\mathrm{e}^{-\sqrt{2}\,\mathrm{csgn}(\sin(x))\,\mathrm{EllipticE}\left(\cos(x),\frac{\sqrt{2}}{2}\right)}
\end{aligned}
$$

And

$$
\begin{aligned}
\overline{N} &= \mu N \\
&= \mathrm{e}^{-\sqrt{2}\,\mathrm{csgn}(\sin(x))\,\mathrm{EllipticE}\left(\cos(x),\frac{\sqrt{2}}{2}\right)}(1) \\
&= \mathrm{e}^{-\sqrt{2}\,\mathrm{csgn}(\sin(x))\,\mathrm{EllipticE}\left(\cos(x),\frac{\sqrt{2}}{2}\right)}
\end{aligned}
$$

Now a modified ODE is ontained from the original ODE, which is exact and can be solved. The modified ODE is

$$\overline{M} + \overline{N}\frac{dy}{dx} = \left(-\left(-y\sqrt{-\cos(x)^2+2}+x\right)e^{-\sqrt{2}\,\mathrm{csgn}(\sin(x))\,\mathrm{EllipticE}\left(\cos(x),\frac{\sqrt{2}}{2}\right)}\right) + \left(e^{-\sqrt{2}\,\mathrm{csgn}(\sin(x))\,\mathrm{EllipticE}\left(\cos(x),\frac{\sqrt{2}}{2}\right)}\right)\frac{dy}{dx} =$$

The following equations are now set up to solve for the function $\phi(x,y)$

$$\frac{\partial \phi}{\partial x} = \overline{M} \tag{1}$$

$$\frac{\partial \phi}{\partial y} = \overline{N} \tag{2}$$

Integrating (1) w.r.t. x gives

$$\int \frac{\partial \phi}{\partial x}\,dx = \int \overline{M}\,dx$$

$$\int \frac{\partial \phi}{\partial x}\,dx = \int -\left(-y\sqrt{-\cos(x)^2+2}+x\right)e^{-\sqrt{2}\,\mathrm{csgn}(\sin(x))\,\mathrm{EllipticE}\left(\cos(x),\frac{\sqrt{2}}{2}\right)}\,dx$$

$$\phi = \int_0^x -\left(-y\sqrt{-\cos(\_a)^2+2}+\_a\right)e^{-\sqrt{2}\,\mathrm{csgn}(\sin(\_a))\,\mathrm{EllipticE}\left(\cos(\_a),\frac{\sqrt{2}}{2}\right)}d\_a + f(y) \tag{3}$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$\frac{\partial \phi}{\partial y} = \int_0^x \sqrt{-\cos(\_a)^2+2}\,e^{-\sqrt{2}\,\mathrm{csgn}(\sin(\_a))\,\mathrm{EllipticE}\left(\cos(\_a),\frac{\sqrt{2}}{2}\right)}d\_a + f'(y) \tag{4}$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = e^{-\sqrt{2}\,\mathrm{csgn}(\sin(x))\,\mathrm{EllipticE}\left(\cos(x),\frac{\sqrt{2}}{2}\right)}$. Therefore equation (4) becomes

$$\begin{aligned} &e^{-\sqrt{2}\,\mathrm{csgn}(\sin(x))\,\mathrm{EllipticE}\left(\cos(x),\frac{\sqrt{2}}{2}\right)} \\ &= \int_0^x \sqrt{-\cos(\_a)^2+2}\,e^{-\sqrt{2}\,\mathrm{csgn}(\sin(\_a))\,\mathrm{EllipticE}\left(\cos(\_a),\frac{\sqrt{2}}{2}\right)}d\_a + f'(y) \end{aligned} \tag{5}$$

Solving equation (5) for $f'(y)$ gives

$$\begin{aligned} f'(y) = &-\left(\int_0^x \sqrt{-\cos(\_a)^2+2}\,e^{-\sqrt{2}\,\mathrm{csgn}(\sin(\_a))\,\mathrm{EllipticE}\left(\cos(\_a),\frac{\sqrt{2}}{2}\right)}d\_a\right) \\ &+ e^{-\sqrt{2}\,\mathrm{csgn}(\sin(x))\,\mathrm{EllipticE}\left(\cos(x),\frac{\sqrt{2}}{2}\right)} \end{aligned}$$

Integrating the above w.r.t y gives

$$\int f'(y)\,\mathrm{d}y = \int \left(-\left(\int_0^x \sqrt{-\cos(\_a)^2+2}\,\mathrm{e}^{-\sqrt{2}\ \mathrm{csgn}(\sin(\_a))\ \mathrm{EllipticE}\left(\cos(\_a),\frac{\sqrt{2}}{2}\right)}d\_a\right) + \mathrm{e}^{-\sqrt{2}\ \mathrm{csgn}(\sin(x))\ \mathrm{EllipticE}\left(\cos(x),\frac{\sqrt{2}}{2}\right)}\right)\mathrm{d}y$$

$$f(y) = \left(-\left(\int_0^x \sqrt{-\cos(\_a)^2+2}\,\mathrm{e}^{-\sqrt{2}\ \mathrm{csgn}(\sin(\_a))\ \mathrm{EllipticE}\left(\cos(\_a),\frac{\sqrt{2}}{2}\right)}d\_a\right) + \mathrm{e}^{-\sqrt{2}\ \mathrm{csgn}(\sin(x))\ \mathrm{EllipticE}\left(\cos(x),\frac{\sqrt{2}}{2}\right)}\right)y + c_1$$

Assuming $0 < \sin(\_a)$ then

$$f(y) = \left(-\left(\int_0^x \sqrt{-\cos(\_a)^2+2}\,\mathrm{e}^{-\sqrt{2}\ \mathrm{csgn}(\sin(\_a))\ \mathrm{EllipticE}\left(\cos(\_a),\frac{\sqrt{2}}{2}\right)}d\_a\right) + \mathrm{e}^{-\sqrt{2}\ \mathrm{csgn}(\sin(x))\ \mathrm{EllipticE}\left(\cos(x),\frac{\sqrt{}}{2}\right.}$$

Assuming $0 < \sin(x)$ then

$$f(y) = \left(-\left(\int_0^x \sqrt{-\cos(\_a)^2+2}\,\mathrm{e}^{-\sqrt{2}\ \mathrm{csgn}(\sin(\_a))\ \mathrm{EllipticE}\left(\cos(\_a),\frac{\sqrt{2}}{2}\right)}d\_a\right) + \mathrm{e}^{-\sqrt{2}\ \mathrm{EllipticE}\left(\cos(x),\frac{\sqrt{2}}{2}\right)}\right)y+c_1$$

Where c_1 is constant of integration. Substituting result found above for $f(y)$ into equation (3) gives ϕ

$$\phi = \int_0^x -\left(-y\sqrt{-\cos(\_a)^2+2}+\_a\right)\mathrm{e}^{-\sqrt{2}\ \mathrm{csgn}(\sin(\_a))\ \mathrm{EllipticE}\left(\cos(\_a),\frac{\sqrt{2}}{2}\right)}d\_a + \left(-\left(\int_0^x \sqrt{-\cos(\_a)^2+2}\,\mathrm{e}^{-\sqrt{2}\ \mathrm{csgn}(\sin(\_a))\ \mathrm{EllipticE}\left(\cos(\_a),\frac{\sqrt{2}}{2}\right)}d\_a\right) + \mathrm{e}^{-\sqrt{2}\ \mathrm{EllipticE}\left(\cos(x),\frac{\sqrt{2}}{2}\right)}\right)y + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = \int_0^x -\left(-y\sqrt{-\cos(\_a)^2+2}+\_a\right)\mathrm{e}^{-\sqrt{2}\ \mathrm{csgn}(\sin(\_a))\ \mathrm{EllipticE}\left(\cos(\_a),\frac{\sqrt{2}}{2}\right)}d\_a + \left(-\left(\int_0^x \sqrt{-\cos(\_a)^2+2}\,\mathrm{e}^{-\sqrt{2}\ \mathrm{csgn}(\sin(\_a))\ \mathrm{EllipticE}\left(\cos(\_a),\frac{\sqrt{2}}{2}\right)}d\_a\right) + \mathrm{e}^{-\sqrt{2}\ \mathrm{EllipticE}\left(\cos(x),\frac{\sqrt{2}}{2}\right)}\right)y$$

The solution becomes

$$y = \left(\int_0^x \mathrm{e}^{-\sqrt{2}\ \mathrm{csgn}(\sin(\_a))\ \mathrm{EllipticE}\left(\cos(\_a), \frac{\sqrt{2}}{2}\right)} \_ad\_a + c_1\right) \mathrm{e}^{\sqrt{2}\ \mathrm{EllipticE}\left(\cos(x), \frac{\sqrt{2}}{2}\right)}$$

Simplifying the solution $y = \left(\int_0^x \mathrm{e}^{-\sqrt{2}\ \mathrm{csgn}(\sin(\_a))\ \mathrm{EllipticE}\left(\cos(\_a), \frac{\sqrt{2}}{2}\right)} \_ad\_a + c_1\right) \mathrm{e}^{\sqrt{2}\ \mathrm{EllipticE}\left(\cos(x), \frac{\sqrt{2}}{2}\right)}$ to $y = \left(\int_0^x \mathrm{e}^{-\sqrt{2}\ \mathrm{EllipticE}\left(\cos(\_a), \frac{\sqrt{2}}{2}\right)} \_ad\_a + c_1\right) \mathrm{e}^{\sqrt{2}\ \mathrm{EllipticE}\left(\cos(x), \frac{\sqrt{2}}{2}\right)}$ Initial conditions are used to solve for c_1. Substituting $x = 0$ and $y = 2$ in the above solution gives an equation to solve for the constant of integration.

$$2 = c_1 \mathrm{e}^{\sqrt{2}\ \mathrm{EllipticE}\left(\frac{\sqrt{2}}{2}\right)}$$

$$c_1 = 2\,\mathrm{e}^{-\sqrt{2}\ \mathrm{EllipticE}\left(\frac{\sqrt{2}}{2}\right)}$$

Substituting c_1 found above in the general solution gives

$$y = \mathrm{e}^{\sqrt{2}\ \mathrm{EllipticE}\left(\cos(x), \frac{\sqrt{2}}{2}\right)} \left(\int_0^x \mathrm{e}^{-\sqrt{2}\ \mathrm{EllipticE}\left(\cos(\_a), \frac{\sqrt{2}}{2}\right)} \_ad\_a\right) + 2\,\mathrm{e}^{-\sqrt{2}\ \left(\mathrm{EllipticE}\left(\frac{\sqrt{2}}{2}\right) - \mathrm{EllipticE}\left(\cos(x), \frac{\sqrt{2}}{2}\right)\right)}$$

Summary

The solution(s) found are the following

$$y = \mathrm{e}^{\sqrt{2}\ \mathrm{EllipticE}\left(\cos(x), \frac{\sqrt{2}}{2}\right)} \left(\int_0^x \mathrm{e}^{-\sqrt{2}\ \mathrm{EllipticE}\left(\cos(\_a), \frac{\sqrt{2}}{2}\right)} \_ad\_a\right) + 2\,\mathrm{e}^{-\sqrt{2}\ \left(\mathrm{EllipticE}\left(\frac{\sqrt{2}}{2}\right) - \mathrm{EllipticE}\left(\cos(x), \frac{\sqrt{2}}{2}\right)\right)} \tag{1}$$

Verification of solutions

$$y = \mathrm{e}^{\sqrt{2}\ \mathrm{EllipticE}\left(\cos(x), \frac{\sqrt{2}}{2}\right)} \left(\int_0^x \mathrm{e}^{-\sqrt{2}\ \mathrm{EllipticE}\left(\cos(\_a), \frac{\sqrt{2}}{2}\right)} \_ad\_a\right) + 2\,\mathrm{e}^{-\sqrt{2}\ \left(\mathrm{EllipticE}\left(\frac{\sqrt{2}}{2}\right) - \mathrm{EllipticE}\left(\cos(x), \frac{\sqrt{2}}{2}\right)\right)}$$

Verified OK. `{sin(_a)::positive, sin(x)::positive}`

2.23.5 Maple step by step solution

Let's solve

$$\left[y' + y\sqrt{1+\sin(x)^2} = x, y(0) = 2\right]$$

- Highest derivative means the order of the ODE is 1

y'

- Isolate the derivative

$y' = -y\sqrt{1+\sin(x)^2} + x$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE

$y' + y\sqrt{1+\sin(x)^2} = x$

- The ODE is linear; multiply by an integrating factor $\mu(x)$

$\mu(x)\left(y' + y\sqrt{1+\sin(x)^2}\right) = \mu(x)\,x$

- Assume the lhs of the ODE is the total derivative $\frac{d}{dx}(\mu(x)\,y)$

$\mu(x)\left(y' + y\sqrt{1+\sin(x)^2}\right) = \mu'(x)\,y + \mu(x)\,y'$

- Isolate $\mu'(x)$

$\mu'(x) = \mu(x)\sqrt{1+\sin(x)^2}$

- Solve to find the integrating factor

$\mu(x) = \mathrm{e}^{\frac{\sqrt{\left(1+\sin(x)^2\right)\cos(x)^2}\,\sqrt{\cos(x)^2}\,\mathit{EllipticE}(\sin(x),\mathrm{I})}{\sqrt{-(-1+\sin(x))(1+\sin(x))\left(1+\sin(x)^2\right)}\,\cos(x)}}$

- Integrate both sides with respect to x

$\int\left(\frac{d}{dx}(\mu(x)\,y)\right)dx = \int\mu(x)\,x dx + c_1$

- Evaluate the integral on the lhs

$\mu(x)\,y = \int\mu(x)\,x dx + c_1$

- Solve for y

$y = \frac{\int\mu(x)x dx + c_1}{\mu(x)}$

- Substitute $\mu(x) = \mathrm{e}^{\frac{\sqrt{\left(1+\sin(x)^2\right)\cos(x)^2}\,\sqrt{\cos(x)^2}\,\mathit{EllipticE}(\sin(x),\mathrm{I})}{\sqrt{-(-1+\sin(x))(1+\sin(x))\left(1+\sin(x)^2\right)}\,\cos(x)}}$

$$y = \frac{\int e^{\frac{\sqrt{\left(1+\sin(x)^2\right)\cos(x)^2}\,\sqrt{\cos(x)^2}\,EllipticE(\sin(x),I)}{\sqrt{-(-1+\sin(x))(1+\sin(x))\left(1+\sin(x)^2\right)}\,\cos(x)}} x dx + c_1}{e^{\frac{\sqrt{\left(1+\sin(x)^2\right)\cos(x)^2}\,\sqrt{\cos(x)^2}\,EllipticE(\sin(x),I)}{\sqrt{-(-1+\sin(x))(1+\sin(x))\left(1+\sin(x)^2\right)}\,\cos(x)}}}$$

- Simplify
$$y = \left(\int e^{\mathrm{csgn}(\cos(x))EllipticE(\sin(x),I)} x dx + c_1\right) e^{-\mathrm{csgn}(\cos(x))EllipticE(\sin(x),I)}$$
- Use initial condition $y(0) = 2$
$$2 = \int^0 e^{\mathrm{csgn}(\cos(\_a))EllipticE(\sin(\_a),I)}\_ad\_a + c_1$$
- Solve for c_1
$$c_1 = -\left(\int^0 e^{\mathrm{csgn}(\cos(\_a))EllipticE(\sin(\_a),I)}\_ad\_a\right) + 2$$
- Substitute $c_1 = -\left(\int^0 e^{\mathrm{csgn}(\cos(\_a))EllipticE(\sin(\_a),I)}\_ad\_a\right) + 2$ into general solution and simplify
$$y = \left(\int e^{\mathrm{csgn}(\cos(x))EllipticE(\sin(x),I)} x dx - \left(\int^0 e^{\mathrm{csgn}(\cos(\_a))EllipticE(\sin(\_a),I)}\_ad\_a\right) + 2\right) e^{-\mathrm{csgn}(\cos(x))Ellip}$$
- Solution to the IVP
$$y = \left(\int e^{\mathrm{csgn}(\cos(x))EllipticE(\sin(x),I)} x dx - \left(\int^0 e^{\mathrm{csgn}(\cos(\_a))EllipticE(\sin(\_a),I)}\_ad\_a\right) + 2\right) e^{-\mathrm{csgn}(\cos(x))Ellip}$$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
<- 1st order linear successful`
```

✓ Solution by Maple

Time used: 0.172 (sec). Leaf size: 36

```
dsolve([diff(y(x),x)+y(x)*sqrt(1+sin(x)^2)=x,y(0) = 2],y(x), singsol=all)
```

$$y(x) = e^{-\mathrm{csgn}(\cos(x))\,\mathrm{EllipticE}(\sin(x),i)}\left(\int_0^x \_z1\, e^{\mathrm{csgn}(\cos(\_z1))\,\mathrm{EllipticE}(\sin(\_z1),i)} d\_z1 + 2\right)$$

✓ Solution by Mathematica

Time used: 0.209 (sec). Leaf size: 31

```
DSolve[{y'[x]+y[x]*Sqrt[1+Sin[x]^2]==x,{y[0]==2}},y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to e^{-E(x|-1)}\left(\int_0^x e^{E(K[1]|-1)}K[1]dK[1] + 2\right)$$

2.24 problem 29

Internal problem ID [4973]
Internal file name [`OUTPUT/4466_Sunday_June_05_2022_02_57_15_PM_65389889/index.tex`]

Book: Fundamentals of Differential Equations. By Nagle, Saff and Snider. 9th edition. Boston. Pearson 2018.
Section: Chapter 2, First order differential equations. Section 2.3, Linear equations. Exercises. page 54
Problem number: 29.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"exactWithIntegrationFactor", "first_order_ode_lie_symmetry_calculated"**

Maple gives the following as the ode type

```
[[_1st_order, _with_exponential_symmetries]]
```

$$\boxed{\left(e^{4y}+2x\right)y'=1}$$

2.24.1 Solving as first order ode lie symmetry calculated ode

Writing the ode as

$$y'=\frac{1}{e^{4y}+2x}$$
$$y'=\omega(x,y)$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x+\omega(\eta_y-\xi_x)-\omega^2\xi_y-\omega_x\xi-\omega_y\eta=0 \tag{A}$$

The type of this ode is not in the lookup table. To determine ξ,η then (A) is solved using ansatz. Making bivariate polynomials of degree 1 to use as anstaz gives

$$\xi=xa_2+ya_3+a_1 \tag{1E}$$
$$\eta=xb_2+yb_3+b_1 \tag{2E}$$

Where the unknown coefficients are

$$\{a_1, a_2, a_3, b_1, b_2, b_3\}$$

Substituting equations (1E,2E) and ω into (A) gives

$$b_2 + \frac{b_3 - a_2}{\mathrm{e}^{4y} + 2x} - \frac{a_3}{\left(\mathrm{e}^{4y} + 2x\right)^2} + \frac{2xa_2 + 2ya_3 + 2a_1}{\left(\mathrm{e}^{4y} + 2x\right)^2} + \frac{4\,\mathrm{e}^{4y}(xb_2 + yb_3 + b_1)}{\left(\mathrm{e}^{4y} + 2x\right)^2} = 0 \quad \text{(5E)}$$

Putting the above in normal form gives

$$\frac{\mathrm{e}^{8y}b_2 + 8\,\mathrm{e}^{4y}xb_2 + 4\,\mathrm{e}^{4y}yb_3 + 4x^2b_2 - a_2\mathrm{e}^{4y} + 4\,\mathrm{e}^{4y}b_1 + b_3\mathrm{e}^{4y} + 2b_3x + 2ya_3 + 2a_1 - a_3}{\left(\mathrm{e}^{4y} + 2x\right)^2} = 0$$

Setting the numerator to zero gives

$$\mathrm{e}^{8y}b_2 + 8\,\mathrm{e}^{4y}xb_2 + 4\,\mathrm{e}^{4y}yb_3 + 4x^2b_2 - a_2\mathrm{e}^{4y} + 4\,\mathrm{e}^{4y}b_1 + b_3\mathrm{e}^{4y} + 2b_3x + 2ya_3 + 2a_1 - a_3 = 0 \quad \text{(6E)}$$

Simplifying the above gives

$$\mathrm{e}^{8y}b_2 + 8\,\mathrm{e}^{4y}xb_2 + 4\,\mathrm{e}^{4y}yb_3 + 4x^2b_2 - a_2\mathrm{e}^{4y} + 4\,\mathrm{e}^{4y}b_1 + b_3\mathrm{e}^{4y} + 2b_3x + 2ya_3 + 2a_1 - a_3 = 0 \quad \text{(6E)}$$

Looking at the above PDE shows the following are all the terms with $\{x, y\}$ in them.

$$\{x, y, \mathrm{e}^{4y}, \mathrm{e}^{8y}\}$$

The following substitution is now made to be able to collect on all terms with $\{x, y\}$ in them

$$\{x = v_1, y = v_2, \mathrm{e}^{4y} = v_3, \mathrm{e}^{8y} = v_4\}$$

The above PDE (6E) now becomes

$$4v_1^2b_2 + 8v_3v_1b_2 + 4v_3v_2b_3 - a_2v_3 + 2v_2a_3 + 4v_3b_1 + v_4b_2 + 2b_3v_1 + b_3v_3 + 2a_1 - a_3 = 0 \quad \text{(7E)}$$

Collecting the above on the terms v_i introduced, and these are

$$\{v_1, v_2, v_3, v_4\}$$

Equation (7E) now becomes

$$4v_1^2b_2+8v_3v_1b_2+2b_3v_1+4v_3v_2b_3+2v_2a_3+(-a_2+4b_1+b_3)\,v_3+v_4b_2+2a_1-a_3=0 \quad (8E)$$

Setting each coefficients in (8E) to zero gives the following equations to solve

$$\begin{aligned} b_2 &= 0 \\ 2a_3 &= 0 \\ 4b_2 &= 0 \\ 8b_2 &= 0 \\ 2b_3 &= 0 \\ 4b_3 &= 0 \\ 2a_1 - a_3 &= 0 \\ -a_2 + 4b_1 + b_3 &= 0 \end{aligned}$$

Solving the above equations for the unknowns gives

$$\begin{aligned} a_1 &= 0 \\ a_2 &= 4b_1 \\ a_3 &= 0 \\ b_1 &= b_1 \\ b_2 &= 0 \\ b_3 &= 0 \end{aligned}$$

Substituting the above solution in the anstaz (1E,2E) (using 1 as arbitrary value for any unknown in the RHS) gives

$$\begin{aligned} \xi &= 4x \\ \eta &= 1 \end{aligned}$$

Shifting is now applied to make $\xi = 0$ in order to simplify the rest of the computation

$$\begin{aligned} \eta &= \eta - \omega(x,y)\,\xi \\ &= 1 - \left(\frac{1}{e^{4y}+2x}\right)(4x) \\ &= \frac{-2x+e^{4y}}{e^{4y}+2x} \\ \xi &= 0 \end{aligned}$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x, y) \to (R, S)$ where (R, S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y}\right) S(x, y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$R = x$$

S is found from

$$\begin{aligned} S &= \int \frac{1}{\eta} dy \\ &= \int \frac{1}{\frac{-2x+e^{4y}}{e^{4y}+2x}} dy \end{aligned}$$

Which results in

$$S = -\ln\left(e^{y}\right) + \frac{\ln\left(-2x + e^{4y}\right)}{2}$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x, y) S_y}{R_x + \omega(x, y) R_y} \tag{2}$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x, y)$ is the right hand side of the original ode given by

$$\omega(x, y) = \frac{1}{e^{4y} + 2x}$$

Evaluating all the partial derivatives gives

$$\begin{aligned} R_x &= 1 \\ R_y &= 0 \\ S_x &= \frac{1}{2x - e^{4y}} \\ S_y &= \frac{e^{4y} + 2x}{-2x + e^{4y}} \end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = 0 \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = 0$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$S(R) = c_1 \tag{4}$$

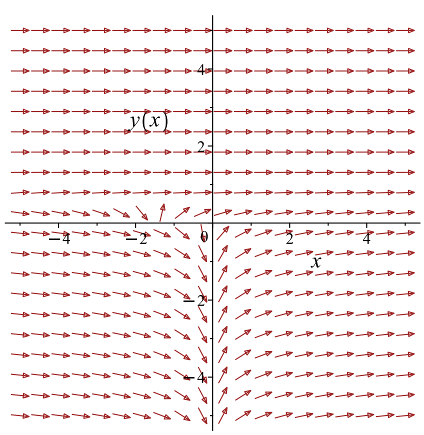
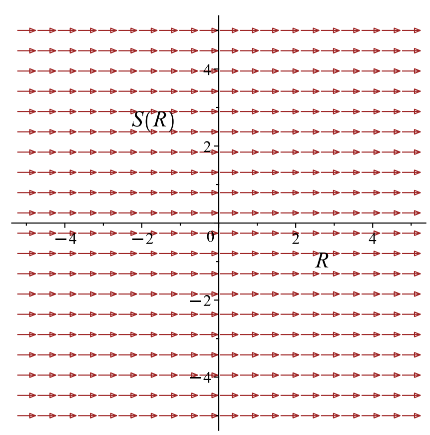
To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$-y + \frac{\ln(-2x + e^{4y})}{2} = c_1$$

Which simplifies to

$$-y + \frac{\ln(-2x + e^{4y})}{2} = c_1$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in x, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dx} = \frac{1}{e^{4y} + 2x}$ 	$R = x$ $S = -y + \frac{\ln(-2x + e^{4y})}{2}$	$\frac{dS}{dR} = 0$ 

Summary
The solution(s) found are the following

$$-y+\frac{\ln\left(-2x+e^{4y}\right)}{2}=c_1 \tag{1}$$

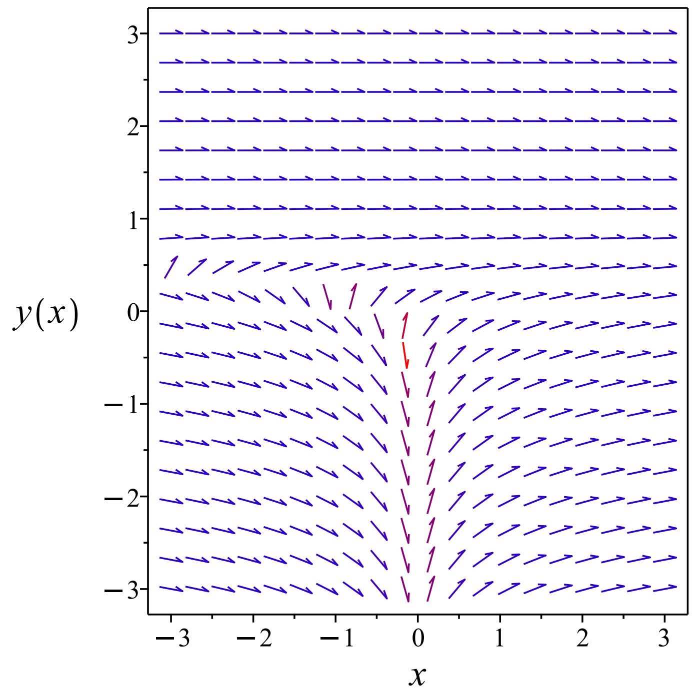


Figure 157: Slope field plot

Verification of solutions

$$-y+\frac{\ln\left(-2x+e^{4y}\right)}{2}=c_1$$

Verified OK.

2.24.2 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x,y)+N(x,y)\frac{dy}{dx}=0 \tag{A}$$

We assume there exists a function $\phi(x,y)=c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y)=0$$

Hence

$$\frac{\partial\phi}{\partial x}+\frac{\partial\phi}{\partial y}\frac{dy}{dx}=0 \tag{B}$$

Comparing (A,B) shows that

$$\begin{aligned}\frac{\partial\phi}{\partial x}&=M\\ \frac{\partial\phi}{\partial y}&=N\end{aligned}$$

But since $\frac{\partial^2\phi}{\partial x\partial y}=\frac{\partial^2\phi}{\partial y\partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y}=\frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x,y)$ but at least we know now that we can do that since the condition $\frac{\partial^2\phi}{\partial x\partial y}=\frac{\partial^2\phi}{\partial y\partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x,y)\,\mathrm{d}x+N(x,y)\,\mathrm{d}y=0 \tag{1A}$$

Therefore

$$\begin{aligned}\left(\mathrm{e}^{4y}+2x\right)\mathrm{d}y&=\mathrm{d}x\\ -\mathrm{d}x+\left(\mathrm{e}^{4y}+2x\right)\mathrm{d}y&=0 \qquad (2A)\end{aligned}$$

Comparing (1A) and (2A) shows that

$$\begin{aligned}M(x,y)&=-1\\ N(x,y)&=\mathrm{e}^{4y}+2x\end{aligned}$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y}=\frac{\partial N}{\partial x}$$

Using result found above gives

$$\begin{aligned}\frac{\partial M}{\partial y}&=\frac{\partial}{\partial y}(-1)\\ &=0\end{aligned}$$

And

$$
\begin{aligned}
\frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}(\mathrm{e}^{4y} + 2x) \\
&= 2
\end{aligned}
$$

Since $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$, then the ODE is <u>not exact</u>. Since the ODE is not exact, we will try to find an integrating factor to make it exact. Let

$$
\begin{aligned}
A &= \frac{1}{N}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right) \\
&= \frac{1}{\mathrm{e}^{4y} + 2x}((0) - (2)) \\
&= -\frac{2}{\mathrm{e}^{4y} + 2x}
\end{aligned}
$$

Since A depends on y, it can not be used to obtain an integrating factor. We will now try a second method to find an integrating factor. Let

$$
\begin{aligned}
B &= \frac{1}{M}\left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) \\
&= -1((2) - (0)) \\
&= -2
\end{aligned}
$$

Since B does not depend on x, it can be used to obtain an integrating factor. Let the integrating factor be μ. Then

$$
\begin{aligned}
\mu &= e^{\int B \, \mathrm{d}y} \\
&= e^{\int -2 \, \mathrm{d}y}
\end{aligned}
$$

The result of integrating gives

$$
\begin{aligned}
\mu &= e^{-2y} \\
&= \mathrm{e}^{-2y}
\end{aligned}
$$

M and N are now multiplied by this integrating factor, giving new M and new N which are called $\overline{M}$ and $\overline{N}$ so not to confuse them with the original M and N.

$$
\begin{aligned}
\overline{M} &= \mu M \\
&= \mathrm{e}^{-2y}(-1) \\
&= -\mathrm{e}^{-2y}
\end{aligned}
$$

And

$$
\begin{aligned}
\overline{N} &= \mu N \\
&= \mathrm{e}^{-2y}\left(\mathrm{e}^{4y} + 2x\right) \\
&= \mathrm{e}^{2y} + 2x\,\mathrm{e}^{-2y}
\end{aligned}
$$

So now a modified ODE is obtained from the original ODE which will be exact and can be solved using the standard method. The modified ODE is

$$
\begin{aligned}
\overline{M} + \overline{N}\frac{\mathrm{d}y}{\mathrm{d}x} &= 0 \\
\left(-\mathrm{e}^{-2y}\right) + \left(\mathrm{e}^{2y} + 2x\,\mathrm{e}^{-2y}\right)\frac{\mathrm{d}y}{\mathrm{d}x} &= 0
\end{aligned}
$$

The following equations are now set up to solve for the function $\phi(x, y)$

$$
\frac{\partial \phi}{\partial x} = \overline{M} \tag{1}
$$

$$
\frac{\partial \phi}{\partial y} = \overline{N} \tag{2}
$$

Integrating (1) w.r.t. x gives

$$
\begin{aligned}
\int \frac{\partial \phi}{\partial x}\,\mathrm{d}x &= \int \overline{M}\,\mathrm{d}x \\
\int \frac{\partial \phi}{\partial x}\,\mathrm{d}x &= \int -\mathrm{e}^{-2y}\,\mathrm{d}x \\
\phi &= -x\,\mathrm{e}^{-2y} + f(y) \qquad (3)
\end{aligned}
$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$
\frac{\partial \phi}{\partial y} = 2x\,\mathrm{e}^{-2y} + f'(y) \tag{4}
$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = \mathrm{e}^{2y} + 2x\,\mathrm{e}^{-2y}$. Therefore equation (4) becomes

$$
\mathrm{e}^{2y} + 2x\,\mathrm{e}^{-2y} = 2x\,\mathrm{e}^{-2y} + f'(y) \tag{5}
$$

Solving equation (5) for $f'(y)$ gives

$$
f'(y) = \mathrm{e}^{2y}
$$

Integrating the above w.r.t y gives

$$\int f'(y)\,\mathrm{d}y = \int \left(\mathrm{e}^{2y}\right) \mathrm{d}y$$

$$f(y) = \frac{\mathrm{e}^{2y}}{2} + c_1$$

Where c_1 is constant of integration. Substituting result found above for $f(y)$ into equation (3) gives ϕ

$$\phi = -x\,\mathrm{e}^{-2y} + \frac{\mathrm{e}^{2y}}{2} + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = -x\,\mathrm{e}^{-2y} + \frac{\mathrm{e}^{2y}}{2}$$

Summary

The solution(s) found are the following

$$-x\,\mathrm{e}^{-2y} + \frac{\mathrm{e}^{2y}}{2} = c_1 \tag{1}$$

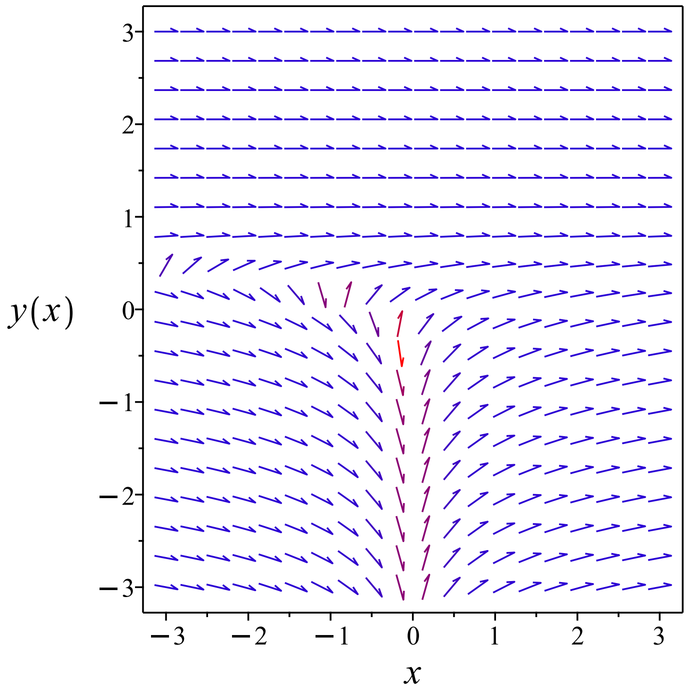


Figure 158: Slope field plot

Verification of solutions

$$-x\,\mathrm{e}^{-2y} + \frac{\mathrm{e}^{2y}}{2} = c_1$$

Verified OK.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
<- 1st order linear successful
<- inverse linear successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 41

```
dsolve((exp(4*y(x)) + 2*x)*diff(y(x),x)-1=0,y(x), singsol=all)
```

$$y(x) = \frac{\ln\left(-c_1 - \sqrt{c_1^2 + 2x}\right)}{2}$$
$$y(x) = \frac{\ln\left(-c_1 + \sqrt{c_1^2 + 2x}\right)}{2}$$

✓ Solution by Mathematica

Time used: 0.159 (sec). Leaf size: 113

```
DSolve[(Exp[4*y[x]]+2*x)*y'[x]-1==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \log\left(-\sqrt{-\sqrt{2x + c_1{}^2} - c_1}\right)$$
$$y(x) \to \frac{1}{2}\log\left(-\sqrt{2x + c_1{}^2} - c_1\right)$$
$$y(x) \to \log\left(-\sqrt{\sqrt{2x + c_1{}^2} - c_1}\right)$$
$$y(x) \to \frac{1}{2}\log\left(\sqrt{2x + c_1{}^2} - c_1\right)$$

2.25 problem 30

Internal problem ID [4974]
Internal file name [`OUTPUT/4467_Sunday_June_05_2022_02_57_16_PM_29877559/index.tex`]

Book: Fundamentals of Differential Equations. By Nagle, Saff and Snider. 9th edition. Boston. Pearson 2018.
Section: Chapter 2, First order differential equations. Section 2.3, Linear equations. Exercises. page 54
Problem number: 30.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"bernoulli", "exactWithIntegrationFactor", "first_order_ode_lie_symmetry_lookup"**

Maple gives the following as the ode type

```
[_rational, _Bernoulli]
```

$$\boxed{y' + 2y - \frac{x}{y^2} = 0}$$

2.25.1 Solving as first order ode lie symmetry lookup ode

Writing the ode as

$$y' = -\frac{2y^3 - x}{y^2}$$
$$y' = \omega(x, y)$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x + \omega(\eta_y - \xi_x) - \omega^2\xi_y - \omega_x\xi - \omega_y\eta = 0 \tag{A}$$

The type of this ode is known. It is of type `Bernoulli`. Therefore we do not need to solve the PDE (A), and can just use the lookup table shown below to find ξ, η

Table 152: Lie symmetry infinitesimal lookup table for known first order ODE's

ODE class	Form	ξ	η
linear ode	$y' = f(x)y(x) + g(x)$	0	$e^{\int f dx}$
separable ode	$y' = f(x)\, g(y)$	$\frac{1}{f}$	0
quadrature ode	$y' = f(x)$	0	1
quadrature ode	$y' = g(y)$	1	0
homogeneous ODEs of Class A	$y' = f\left(\frac{y}{x}\right)$	x	y
homogeneous ODEs of Class C	$y' = (a + bx + cy)^{\frac{n}{m}}$	1	$-\frac{b}{c}$
homogeneous class D	$y' = \frac{y}{x} + g(x)\, F\left(\frac{y}{x}\right)$	x^2	xy
First order special form ID 1	$y' = g(x)\, e^{h(x)+by} + f(x)$	$\frac{e^{-\int bf(x)dx-h(x)}}{g(x)}$	$\frac{f(x)e^{-\int bf(x)dx-h(x)}}{g(x)}$
polynomial type ode	$y' = \frac{a_1x+b_1y+c_1}{a_2x+b_2y+c_2}$	$\frac{a_1b_2x-a_2b_1x-b_1c_2+b_2c_1}{a_1b_2-a_2b_1}$	$\frac{a_1b_2y-a_2b_1y-a_1c_2-a_2c_1}{a_1b_2-a_2b_1}$
Bernoulli ode	$y' = f(x)\, y + g(x)\, y^n$	0	$e^{-\int (n-1)f(x)dx} y^n$
Reduced Riccati	$y' = f_1(x)\, y + f_2(x)\, y^2$	0	$e^{-\int f_1 dx}$

The above table shows that

$$\xi(x,y) = 0$$
$$\eta(x,y) = \frac{e^{-6x}}{y^2} \tag{A1}$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x,y) \to (R,S)$ where (R,S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y}\right) S(x,y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the

canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$R = x$$

S is found from

$$\begin{aligned} S &= \int \frac{1}{\eta} dy \\ &= \int \frac{1}{\frac{e^{-6x}}{y^2}} dy \end{aligned}$$

Which results in

$$S = \frac{y^3 e^{6x}}{3}$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x, y) S_y}{R_x + \omega(x, y) R_y} \tag{2}$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x, y)$ is the right hand side of the original ode given by

$$\omega(x, y) = -\frac{2y^3 - x}{y^2}$$

Evaluating all the partial derivatives gives

$$\begin{aligned} R_x &= 1 \\ R_y &= 0 \\ S_x &= 2y^3 e^{6x} \\ S_y &= y^2 e^{6x} \end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = e^{6x} x \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = e^{6R} R$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordinates R, S. Integrating the above gives

$$S(R) = \frac{(6R - 1)\,e^{6R}}{36} + c_1 \tag{4}$$

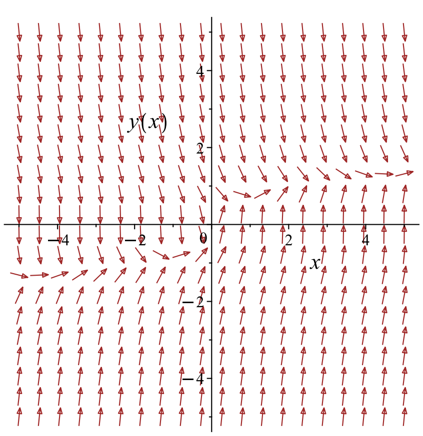
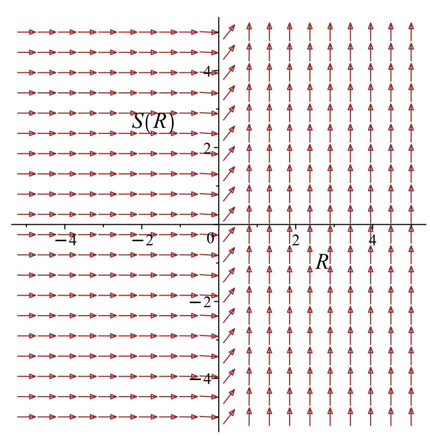
To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$\frac{y^3 e^{6x}}{3} = \frac{(-1 + 6x)\,e^{6x}}{36} + c_1$$

Which simplifies to

$$\frac{(12y^3 - 6x + 1)\,e^{6x}}{36} - c_1 = 0$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

<table>
<tr><th>Original ode in x, y coordinates</th><th>Canonical coordinates transformation</th><th>ODE in canonical coordinates (R, S)</th></tr>
<tr><td>$\frac{dy}{dx} = -\frac{2y^3 - x}{y^2}$

</td><td>$R = x$
$S = \frac{y^3 e^{6x}}{3}$</td><td>$\frac{dS}{dR} = e^{6R} R$

</td></tr>
</table>

Summary
The solution(s) found are the following

$$\frac{(12y^3 - 6x + 1)\,e^{6x}}{36} - c_1 = 0 \tag{1}$$

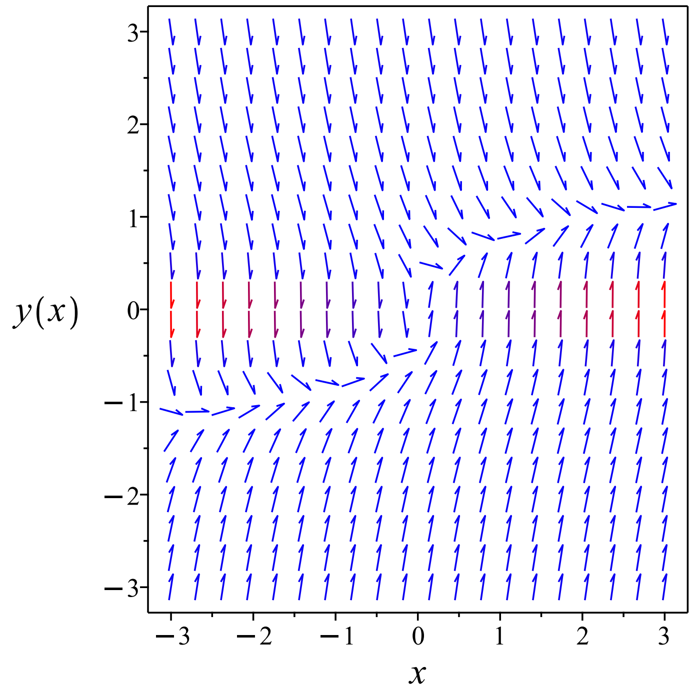


Figure 159: Slope field plot

Verification of solutions

$$\frac{(12y^3 - 6x + 1)\,\mathrm{e}^{6x}}{36} - c_1 = 0$$

Verified OK.

2.25.2 Solving as bernoulli ode

In canonical form, the ODE is

$$\begin{aligned} y' &= F(x,y) \\ &= -\frac{2y^3 - x}{y^2} \end{aligned}$$

This is a Bernoulli ODE.

$$y' = -2y + x\frac{1}{y^2} \tag{1}$$

The standard Bernoulli ODE has the form

$$y' = f_0(x)y + f_1(x)y^n \tag{2}$$

The first step is to divide the above equation by y^n which gives

$$\frac{y'}{y^n} = f_0(x)y^{1-n} + f_1(x) \tag{3}$$

The next step is use the substitution $w = y^{1-n}$ in equation (3) which generates a new ODE in $w(x)$ which will be linear and can be easily solved using an integrating factor. Backsubstitution then gives the solution $y(x)$ which is what we want.

This method is now applied to the ODE at hand. Comparing the ODE (1) With (2) Shows that

$$\begin{aligned} f_0(x) &= -2 \\ f_1(x) &= x \\ n &= -2 \end{aligned}$$

Dividing both sides of ODE (1) by $y^n = \frac{1}{y^2}$ gives

$$y'y^2 = -2y^3 + x \tag{4}$$

Let

$$\begin{aligned} w &= y^{1-n} \\ &= y^3 \end{aligned} \tag{5}$$

Taking derivative of equation (5) w.r.t x gives

$$w' = 3y^2y' \tag{6}$$

Substituting equations (5) and (6) into equation (4) gives

$$\begin{aligned} \frac{w'(x)}{3} &= -2w(x) + x \\ w' &= -6w + 3x \end{aligned} \tag{7}$$

The above now is a linear ODE in $w(x)$ which is now solved.

Entering Linear first order ODE solver. In canonical form a linear first order is

$$w'(x) + p(x)w(x) = q(x)$$

Where here

$$\begin{aligned} p(x) &= 6 \\ q(x) &= 3x \end{aligned}$$

Hence the ode is

$$w'(x) + 6w(x) = 3x$$

The integrating factor μ is

$$\mu = e^{\int 6dx} \\ = e^{6x}$$

The ode becomes

$$\frac{d}{dx}(\mu w) = (\mu)(3x)$$
$$\frac{d}{dx}(e^{6x} w) = (e^{6x})(3x)$$
$$d(e^{6x} w) = (3 e^{6x} x)\, dx$$

Integrating gives

$$e^{6x} w = \int 3 e^{6x} x \, dx$$
$$e^{6x} w = \frac{(-1+6x)\, e^{6x}}{12} + c_1$$

Dividing both sides by the integrating factor $\mu = e^{6x}$ results in

$$w(x) = \frac{e^{-6x}(-1+6x)\, e^{6x}}{12} + c_1 e^{-6x}$$

which simplifies to

$$w(x) = -\frac{1}{12} + \frac{x}{2} + c_1 e^{-6x}$$

Replacing w in the above by y^3 using equation (5) gives the final solution.

$$y^3 = -\frac{1}{12} + \frac{x}{2} + c_1 e^{-6x}$$

Solving for y gives

$$y(x) = \frac{(-18 + 108x + 216c_1 e^{-6x})^{\frac{1}{3}}}{6}$$
$$y(x) = \frac{(-18 + 108x + 216c_1 e^{-6x})^{\frac{1}{3}} (i\sqrt{3} - 1)}{12}$$
$$y(x) = -\frac{(-18 + 108x + 216c_1 e^{-6x})^{\frac{1}{3}} (1 + i\sqrt{3})}{12}$$

Summary

The solution(s) found are the following

$$y = \frac{\left(-18+108x+216c_1 e^{-6x}\right)^{\frac{1}{3}}}{6} \tag{1}$$

$$y = \frac{\left(-18+108x+216c_1 e^{-6x}\right)^{\frac{1}{3}} \left(i\sqrt{3}-1\right)}{12} \tag{2}$$

$$y = -\frac{\left(-18+108x+216c_1 e^{-6x}\right)^{\frac{1}{3}} \left(1+i\sqrt{3}\right)}{12} \tag{3}$$

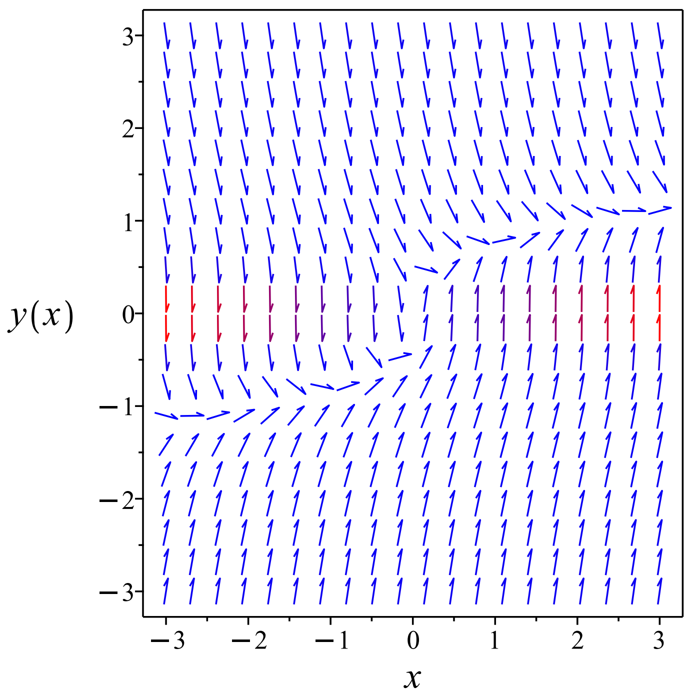


Figure 160: Slope field plot

Verification of solutions

$$y = \frac{\left(-18+108x+216c_1 e^{-6x}\right)^{\frac{1}{3}}}{6}$$

Verified OK.

$$y = \frac{\left(-18+108x+216c_1 e^{-6x}\right)^{\frac{1}{3}} \left(i\sqrt{3}-1\right)}{12}$$

Verified OK.

$$y = -\frac{\left(-18+108x+216c_1 e^{-6x}\right)^{\frac{1}{3}} \left(1+i\sqrt{3}\right)}{12}$$

Verified OK.

2.25.3 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x,y) + N(x,y)\frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x,y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y) = 0$$

Hence

$$\frac{\partial \phi}{\partial x} + \frac{\partial \phi}{\partial y}\frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$\frac{\partial \phi}{\partial x} = M$$
$$\frac{\partial \phi}{\partial y} = N$$

But since $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x,y)$ but at least we know now that we can do that since the condition

$\frac{\partial^2\phi}{\partial x\partial y} = \frac{\partial^2\phi}{\partial y\partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x,y)\,\mathrm{d}x + N(x,y)\,\mathrm{d}y = 0 \tag{1A}$$

Therefore

$$\begin{aligned}(y^2)\,\mathrm{d}y &= (-2y^3 + x)\,\mathrm{d}x \\ (2y^3 - x)\,\mathrm{d}x + (y^2)\,\mathrm{d}y &= 0 \end{aligned} \tag{2A}$$

Comparing (1A) and (2A) shows that

$$\begin{aligned}M(x,y) &= 2y^3 - x \\ N(x,y) &= y^2\end{aligned}$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Using result found above gives

$$\begin{aligned}\frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}(2y^3 - x) \\ &= 6y^2\end{aligned}$$

And

$$\begin{aligned}\frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}(y^2) \\ &= 0\end{aligned}$$

Since $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$, then the ODE is <u>not exact</u>. Since the ODE is not exact, we will try to find an integrating factor to make it exact. Let

$$\begin{aligned}A &= \frac{1}{N}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right) \\ &= \frac{1}{y^2}\left((6y^2) - (0)\right) \\ &= 6\end{aligned}$$

Since A does not depend on y, then it can be used to find an integrating factor. The integrating factor μ is

$$\begin{aligned}\mu &= e^{\int A \, \mathrm{d}x} \\ &= e^{\int 6 \, \mathrm{d}x}\end{aligned}$$

The result of integrating gives

$$\begin{aligned}\mu &= e^{6x} \\ &= \mathrm{e}^{6x}\end{aligned}$$

M and N are multiplied by this integrating factor, giving new M and new N which are called $\overline{M}$ and $\overline{N}$ for now so not to confuse them with the original M and N.

$$\begin{aligned}\overline{M} &= \mu M \\ &= \mathrm{e}^{6x}(2y^3 - x) \\ &= -\mathrm{e}^{6x}(-2y^3 + x)\end{aligned}$$

And

$$\begin{aligned}\overline{N} &= \mu N \\ &= \mathrm{e}^{6x}(y^2) \\ &= y^2 \mathrm{e}^{6x}\end{aligned}$$

Now a modified ODE is ontained from the original ODE, which is exact and can be solved. The modified ODE is

$$\begin{aligned}\overline{M} + \overline{N} \frac{\mathrm{d}y}{\mathrm{d}x} &= 0 \\ \left(-\mathrm{e}^{6x}(-2y^3 + x)\right) + \left(y^2 \mathrm{e}^{6x}\right) \frac{\mathrm{d}y}{\mathrm{d}x} &= 0\end{aligned}$$

The following equations are now set up to solve for the function $\phi(x, y)$

$$\frac{\partial \phi}{\partial x} = \overline{M} \tag{1}$$

$$\frac{\partial \phi}{\partial y} = \overline{N} \tag{2}$$

Integrating (1) w.r.t. x gives

$$\begin{aligned}\int \frac{\partial \phi}{\partial x} \mathrm{d}x &= \int \overline{M} \, \mathrm{d}x \\ \int \frac{\partial \phi}{\partial x} \mathrm{d}x &= \int -\mathrm{e}^{6x}(-2y^3 + x) \, \mathrm{d}x\end{aligned}$$

$$\phi = -\frac{\mathrm{e}^{6x}\left(-2y^3 + x - \frac{1}{6}\right)}{6} + f(y) \tag{3}$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$\frac{\partial \phi}{\partial y} = y^2 e^{6x} + f'(y) \tag{4}$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = y^2 e^{6x}$. Therefore equation (4) becomes

$$y^2 e^{6x} = y^2 e^{6x} + f'(y) \tag{5}$$

Solving equation (5) for $f'(y)$ gives

$$f'(y) = 0$$

Therefore

$$f(y) = c_1$$

Where c_1 is constant of integration. Substituting this result for $f(y)$ into equation (3) gives ϕ

$$\phi = -\frac{e^{6x}\left(-2y^3 + x - \frac{1}{6}\right)}{6} + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = -\frac{e^{6x}\left(-2y^3 + x - \frac{1}{6}\right)}{6}$$

Summary

The solution(s) found are the following

$$-\frac{e^{6x}\left(-2y^3 + x - \frac{1}{6}\right)}{6} = c_1 \tag{1}$$

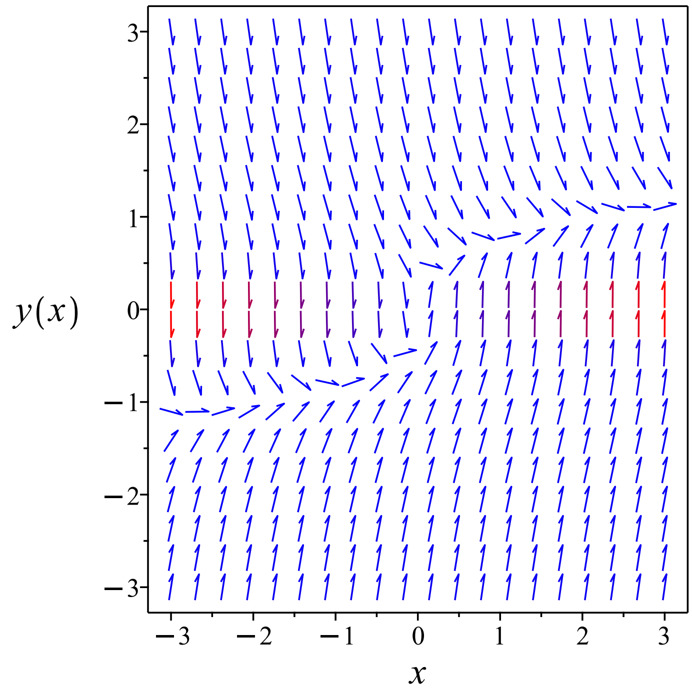


Figure 161: Slope field plot

Verification of solutions

$$-\frac{e^{6x}\left(-2y^3+x-\frac{1}{6}\right)}{6}=c_1$$

Verified OK.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
<- Bernoulli successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 74

```
dsolve(diff(y(x),x)+2*y(x)=x*y(x)^(-2),y(x), singsol=all)
```

$$y(x) = \frac{\left(-18+216\,\mathrm{e}^{-6x}c_1+108x\right)^{\frac{1}{3}}}{6}$$
$$y(x) = -\frac{\left(-18+216\,\mathrm{e}^{-6x}c_1+108x\right)^{\frac{1}{3}}\left(1+i\sqrt{3}\right)}{12}$$
$$y(x) = \frac{\left(-18+216\,\mathrm{e}^{-6x}c_1+108x\right)^{\frac{1}{3}}\left(i\sqrt{3}-1\right)}{12}$$

✓ Solution by Mathematica

Time used: 5.146 (sec). Leaf size: 99

```
DSolve[y'[x]+2*y[x]==x*y[x]^(-2),y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to -\frac{\sqrt[3]{-\frac{1}{3}}\sqrt[3]{6x+12c_1e^{-6x}-1}}{2^{2/3}}$$
$$y(x) \to \frac{\sqrt[3]{2x+4c_1e^{-6x}-\frac{1}{3}}}{2^{2/3}}$$
$$y(x) \to \left(-\frac{1}{2}\right)^{2/3}\sqrt[3]{2x+4c_1e^{-6x}-\frac{1}{3}}$$

2.26 problem 36 part(b)

Internal problem ID [4975]
Internal file name [`OUTPUT/4468_Sunday_June_05_2022_02_57_18_PM_12999692/index.tex`]

Book: Fundamentals of Differential Equations. By Nagle, Saff and Snider. 9th edition. Boston. Pearson 2018.
Section: Chapter 2, First order differential equations. Section 2.3, Linear equations. Exercises. page 54
Problem number: 36 part(b).
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"linear", "exactWithIntegrationFactor", "first_order_ode_lie_symmetry_lookup"**

Maple gives the following as the ode type

```
[_linear]
```

$$\boxed{y' + \frac{3y}{x} = x^2}$$

2.26.1 Solving as linear ode

Entering Linear first order ODE solver. In canonical form a linear first order is

$$y' + p(x)y = q(x)$$

Where here

$$p(x) = \frac{3}{x}$$
$$q(x) = x^2$$

Hence the ode is

$$y' + \frac{3y}{x} = x^2$$

The integrating factor μ is

$$
\begin{aligned}
\mu &= e^{\int \frac{3}{x}dx} \\
&= x^3
\end{aligned}
$$

The ode becomes

$$
\begin{aligned}
\frac{d}{dx}(\mu y) &= (\mu) \left(x^2\right) \\
\frac{d}{dx}\left(y\, x^3\right) &= \left(x^3\right) \left(x^2\right) \\
d\left(y\, x^3\right) &= x^5\, dx
\end{aligned}
$$

Integrating gives

$$
\begin{aligned}
y\, x^3 &= \int x^5\, dx \\
y\, x^3 &= \frac{x^6}{6} + c_1
\end{aligned}
$$

Dividing both sides by the integrating factor $\mu = x^3$ results in

$$
y = \frac{x^3}{6} + \frac{c_1}{x^3}
$$

Summary

The solution(s) found are the following

$$
y = \frac{x^3}{6} + \frac{c_1}{x^3} \tag{1}
$$

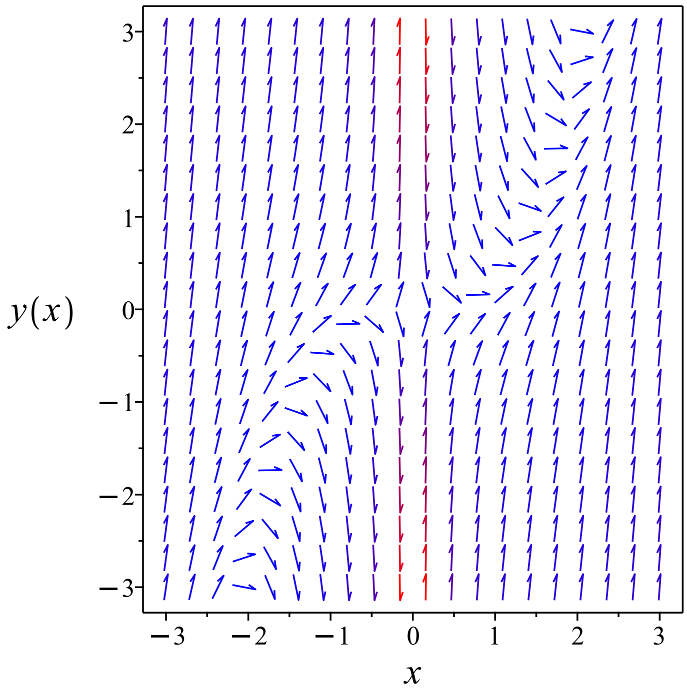


Figure 162: Slope field plot

Verification of solutions

$$y = \frac{x^3}{6} + \frac{c_1}{x^3}$$

Verified OK.

2.26.2 Solving as first order ode lie symmetry lookup ode

Writing the ode as

$$\begin{aligned} y' &= -\frac{-x^3 + 3y}{x} \\ y' &= \omega(x, y) \end{aligned}$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x + \omega(\eta_y - \xi_x) - \omega^2 \xi_y - \omega_x \xi - \omega_y \eta = 0 \tag{A}$$

The type of this ode is known. It is of type `linear`. Therefore we do not need to solve the PDE (A), and can just use the lookup table shown below to find ξ, η

Table 154: Lie symmetry infinitesimal lookup table for known first order ODE's

ODE class	Form	ξ	η
linear ode	$y' = f(x)y(x) + g(x)$	0	$e^{\int f dx}$
separable ode	$y' = f(x)\, g(y)$	$\frac{1}{f}$	0
quadrature ode	$y' = f(x)$	0	1
quadrature ode	$y' = g(y)$	1	0
homogeneous ODEs of Class A	$y' = f\left(\frac{y}{x}\right)$	x	y
homogeneous ODEs of Class C	$y' = (a + bx + cy)^{\frac{n}{m}}$	1	$-\frac{b}{c}$
homogeneous class D	$y' = \frac{y}{x} + g(x)\, F\left(\frac{y}{x}\right)$	x^2	xy
First order special form ID 1	$y' = g(x)\, e^{h(x)+by} + f(x)$	$\frac{e^{-\int bf(x)dx - h(x)}}{g(x)}$	$\frac{f(x)e^{-\int bf(x)dx - h(x)}}{g(x)}$
polynomial type ode	$y' = \frac{a_1x+b_1y+c_1}{a_2x+b_2y+c_2}$	$\frac{a_1b_2x - a_2b_1x - b_1c_2 + b_2c_1}{a_1b_2 - a_2b_1}$	$\frac{a_1b_2y - a_2b_1y - a_1c_2 - a_2c_1}{a_1b_2 - a_2b_1}$
Bernoulli ode	$y' = f(x)\, y + g(x)\, y^n$	0	$e^{-\int (n-1)f(x)dx} y^n$
Reduced Riccati	$y' = f_1(x)\, y + f_2(x)\, y^2$	0	$e^{-\int f_1 dx}$

The above table shows that

$$
\begin{aligned}
\xi(x,y) &= 0 \\
\eta(x,y) &= \frac{1}{x^3}
\end{aligned} \tag{A1}
$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x,y) \to (R,S)$ where (R,S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$
\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}
$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y}\right) S(x,y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the

canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$R = x$$

S is found from

$$\begin{aligned}S &= \int \frac{1}{\eta} dy \\ &= \int \frac{1}{\frac{1}{x^3}} dy\end{aligned}$$

Which results in

$$S = y\,x^3$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x,y) S_y}{R_x + \omega(x,y) R_y} \tag{2}$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x,y)$ is the right hand side of the original ode given by

$$\omega(x,y) = -\frac{-x^3 + 3y}{x}$$

Evaluating all the partial derivatives gives

$$\begin{aligned}R_x &= 1 \\ R_y &= 0 \\ S_x &= 3y\,x^2 \\ S_y &= x^3\end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = x^5 \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = R^5$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by

integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$S(R) = \frac{R^6}{6} + c_1 \tag{4}$$

To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$y x^3 = \frac{x^6}{6} + c_1$$

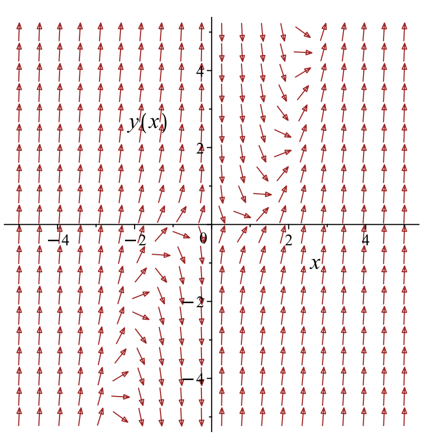
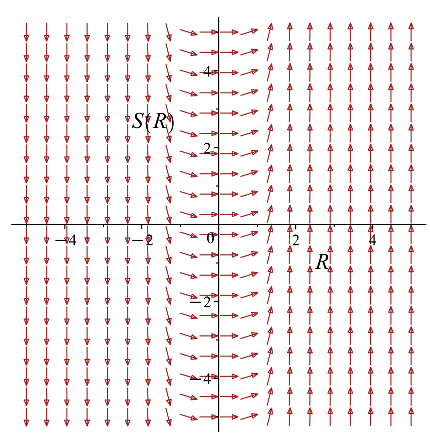
Which simplifies to

$$y x^3 = \frac{x^6}{6} + c_1$$

Which gives

$$y = \frac{x^6 + 6c_1}{6x^3}$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in x, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dx} = -\frac{-x^3+3y}{x}$ 	$R = x$ $S = y\, x^3$	$\frac{dS}{dR} = R^5$ 

Summary

The solution(s) found are the following

$$y = \frac{x^6 + 6c_1}{6x^3} \tag{1}$$

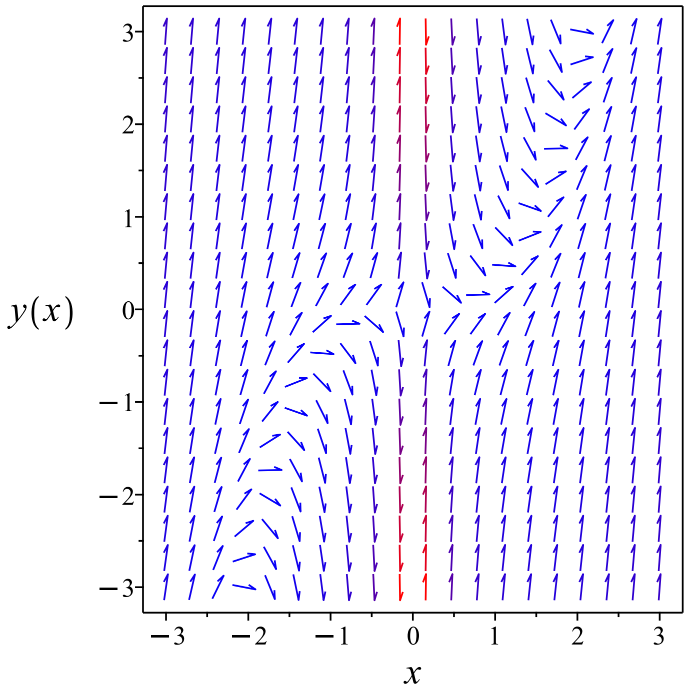


Figure 163: Slope field plot

Verification of solutions

$$y = \frac{x^6 + 6c_1}{6x^3}$$

Verified OK.

2.26.3 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x, y) + N(x, y)\frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x, y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x, y) = 0$$

Hence

$$\frac{\partial \phi}{\partial x} + \frac{\partial \phi}{\partial y}\frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$\begin{aligned}\frac{\partial \phi}{\partial x} &= M\\ \frac{\partial \phi}{\partial y} &= N\end{aligned}$$

But since $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x, y)$ but at least we know now that we can do that since the condition $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x, y)\, \mathrm{d}x + N(x, y)\, \mathrm{d}y = 0 \tag{1A}$$

Therefore

$$\begin{aligned}\mathrm{d}y &= \left(-\frac{3y}{x} + x^2\right) \mathrm{d}x\\ \left(-x^2 + \frac{3y}{x}\right) \mathrm{d}x + \mathrm{d}y &= 0 \qquad (2\mathrm{A})\end{aligned}$$

Comparing (1A) and (2A) shows that

$$\begin{aligned}M(x, y) &= -x^2 + \frac{3y}{x}\\ N(x, y) &= 1\end{aligned}$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Using result found above gives

$$\begin{aligned}\frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}\left(-x^2 + \frac{3y}{x}\right)\\ &= \frac{3}{x}\end{aligned}$$

And

$$\begin{aligned}\frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}(1)\\ &= 0\end{aligned}$$

Since $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$, then the ODE is not exact. Since the ODE is not exact, we will try to find an integrating factor to make it exact. Let

$$\begin{aligned}A &= \frac{1}{N}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right)\\ &= 1\left(\left(\frac{3}{x}\right) - (0)\right)\\ &= \frac{3}{x}\end{aligned}$$

Since A does not depend on y, then it can be used to find an integrating factor. The integrating factor μ is

$$\begin{aligned}\mu &= e^{\int A \, \mathrm{d}x}\\ &= e^{\int \frac{3}{x} \, \mathrm{d}x}\end{aligned}$$

The result of integrating gives

$$\begin{aligned}\mu &= e^{3\ln(x)}\\ &= x^3\end{aligned}$$

M and N are multiplied by this integrating factor, giving new M and new N which are called $\overline{M}$ and $\overline{N}$ for now so not to confuse them with the original M and N.

$$\begin{aligned}\overline{M} &= \mu M\\ &= x^3\left(-x^2 + \frac{3y}{x}\right)\\ &= -x^2(x^3 - 3y)\end{aligned}$$

And

$$\begin{aligned}\overline{N} &= \mu N\\ &= x^3(1)\\ &= x^3\end{aligned}$$

Now a modified ODE is ontained from the original ODE, which is exact and can be solved. The modified ODE is

$$\overline{M} + \overline{N}\frac{\mathrm{d}y}{\mathrm{d}x} = 0$$
$$\left(-x^2\left(x^3 - 3y\right)\right) + \left(x^3\right)\frac{\mathrm{d}y}{\mathrm{d}x} = 0$$

The following equations are now set up to solve for the function $\phi(x, y)$

$$\frac{\partial \phi}{\partial x} = \overline{M} \tag{1}$$
$$\frac{\partial \phi}{\partial y} = \overline{N} \tag{2}$$

Integrating (1) w.r.t. x gives

$$\int \frac{\partial \phi}{\partial x}\,\mathrm{d}x = \int \overline{M}\,\mathrm{d}x$$
$$\int \frac{\partial \phi}{\partial x}\,\mathrm{d}x = \int -x^2\left(x^3 - 3y\right)\mathrm{d}x$$
$$\phi = -\frac{\left(x^3 - 3y\right)^2}{6} + f(y) \tag{3}$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$\frac{\partial \phi}{\partial y} = x^3 - 3y + f'(y) \tag{4}$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = x^3$. Therefore equation (4) becomes

$$x^3 = x^3 - 3y + f'(y) \tag{5}$$

Solving equation (5) for $f'(y)$ gives

$$f'(y) = 3y$$

Integrating the above w.r.t y gives

$$\int f'(y)\,\mathrm{d}y = \int (3y)\,\mathrm{d}y$$
$$f(y) = \frac{3y^2}{2} + c_1$$

Where c_1 is constant of integration. Substituting result found above for $f(y)$ into equation (3) gives ϕ

$$\phi = -\frac{(x^3 - 3y)^2}{6} + \frac{3y^2}{2} + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = -\frac{(x^3 - 3y)^2}{6} + \frac{3y^2}{2}$$

The solution becomes

$$y = \frac{x^6 + 6c_1}{6x^3}$$

Summary

The solution(s) found are the following

$$y = \frac{x^6 + 6c_1}{6x^3} \tag{1}$$

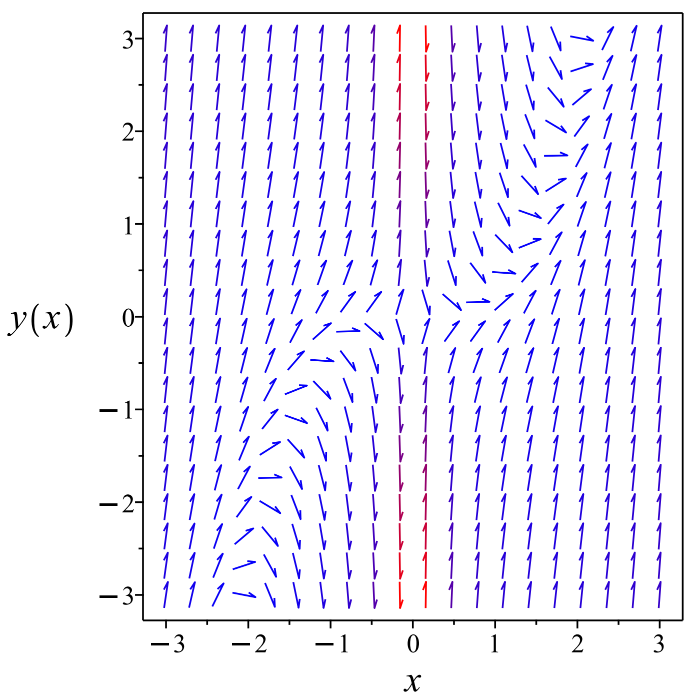


Figure 164: Slope field plot

Verification of solutions

$$y = \frac{x^6 + 6c_1}{6x^3}$$

Verified OK.

2.26.4 Maple step by step solution

Let's solve

$y' + \frac{3y}{x} = x^2$

- Highest derivative means the order of the ODE is 1

  y'

- Isolate the derivative

  $y' = -\frac{3y}{x} + x^2$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE

  $y' + \frac{3y}{x} = x^2$

- The ODE is linear; multiply by an integrating factor $\mu(x)$

  $\mu(x)\left(y' + \frac{3y}{x}\right) = \mu(x)\, x^2$

- Assume the lhs of the ODE is the total derivative $\frac{d}{dx}(\mu(x)\, y)$

  $\mu(x)\left(y' + \frac{3y}{x}\right) = \mu'(x)\, y + \mu(x)\, y'$

- Isolate $\mu'(x)$

  $\mu'(x) = \frac{3\mu(x)}{x}$

- Solve to find the integrating factor

  $\mu(x) = x^3$

- Integrate both sides with respect to x

  $\int \left(\frac{d}{dx}(\mu(x)\, y)\right) dx = \int \mu(x)\, x^2 dx + c_1$

- Evaluate the integral on the lhs

  $\mu(x)\, y = \int \mu(x)\, x^2 dx + c_1$

- Solve for y

  $y = \frac{\int \mu(x) x^2 dx + c_1}{\mu(x)}$

- Substitute $\mu(x) = x^3$

$y = \frac{\int x^5 dx + c_1}{x^3}$

- Evaluate the integrals on the rhs

  $y = \frac{\frac{x^6}{6} + c_1}{x^3}$

- Simplify

  $y = \frac{x^6 + 6c_1}{6x^3}$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
<- 1st order linear successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 16

```
dsolve(diff(y(x),x)+3/x*y(x)=x^2,y(x), singsol=all)
```

$$y(x) = \frac{x^6 + 6c_1}{6x^3}$$

✓ Solution by Mathematica

Time used: 0.026 (sec). Leaf size: 19

```
DSolve[y'[x]+3/x*y[x]==x^2,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \frac{x^3}{6} + \frac{c_1}{x^3}$$

2.27 problem 37

Internal problem ID [4976]
Internal file name [`OUTPUT/4469_Sunday_June_05_2022_02_57_19_PM_19163397/index.tex`]

Book: Fundamentals of Differential Equations. By Nagle, Saff and Snider. 9th edition. Boston. Pearson 2018.
Section: Chapter 2, First order differential equations. Section 2.3, Linear equations. Exercises. page 54
Problem number: 37.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"linear", "exactWithIntegrationFactor", "first_order_ode_lie_symmetry_lookup"**

Maple gives the following as the ode type

```
[[_linear, `class A`]]
```

$$\boxed{x' + kx = \alpha - \beta \cos\left(\frac{\pi t}{12}\right)}$$

With initial conditions

$$[x(0) = x_0]$$

2.27.1 Existence and uniqueness analysis

This is a linear ODE. In canonical form it is written as

$$x' + p(t)x = q(t)$$

Where here

$$\begin{aligned} p(t) &= k \\ q(t) &= \alpha - \beta \cos\left(\frac{\pi t}{12}\right) \end{aligned}$$

Hence the ode is

$$x' + kx = \alpha - \beta \cos\left(\frac{\pi t}{12}\right)$$

The domain of $p(t) = k$ is

$$\{-\infty < t < \infty\}$$

And the point $t_0 = 0$ is inside this domain. The domain of $q(t) = \alpha - \beta \cos\left(\frac{\pi t}{12}\right)$ is

$$\{-\infty < t < \infty\}$$

And the point $t_0 = 0$ is also inside this domain. Hence solution exists and is unique.

2.27.2 Solving as linear ode

Entering Linear first order ODE solver. The integrating factor μ is

$$\begin{aligned} \mu &= \mathrm{e}^{\int k dt} \\ &= \mathrm{e}^{kt} \end{aligned}$$

The ode becomes

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t}(\mu x) &= (\mu)\left(\alpha - \beta \cos\left(\frac{\pi t}{12}\right)\right) \\ \frac{\mathrm{d}}{\mathrm{d}t}\left(\mathrm{e}^{kt} x\right) &= \left(\mathrm{e}^{kt}\right)\left(\alpha - \beta \cos\left(\frac{\pi t}{12}\right)\right) \\ \mathrm{d}\left(\mathrm{e}^{kt} x\right) &= \left(-\left(-\alpha + \beta \cos\left(\frac{\pi t}{12}\right)\right)\mathrm{e}^{kt}\right) \mathrm{d}t \end{aligned}$$

Integrating gives

$$
\begin{aligned}
e^{kt}x &= \int -\left(-\alpha+\beta\cos\left(\frac{\pi t}{12}\right)\right) e^{kt}\, dt \\
e^{kt}x &= \frac{\alpha\, e^{kt}}{k} - \beta\left(\frac{k\, e^{kt}\cos\left(\frac{\pi t}{12}\right)}{k^2+\frac{\pi^2}{144}} + \frac{\pi\, e^{kt}\sin\left(\frac{\pi t}{12}\right)}{12k^2+\frac{\pi^2}{12}}\right) + c_1
\end{aligned}
$$

Dividing both sides by the integrating factor $\mu = e^{kt}$ results in

$$
x = e^{-kt}\left(\frac{\alpha\, e^{kt}}{k} - \beta\left(\frac{k\, e^{kt}\cos\left(\frac{\pi t}{12}\right)}{k^2+\frac{\pi^2}{144}} + \frac{\pi\, e^{kt}\sin\left(\frac{\pi t}{12}\right)}{12k^2+\frac{\pi^2}{12}}\right)\right) + c_1 e^{-kt}
$$

which simplifies to

$$
x = \frac{-144\beta\, k^2\cos\left(\frac{\pi t}{12}\right) - 12\pi\beta k\sin\left(\frac{\pi t}{12}\right) + 144\left(e^{-kt}c_1 k+\alpha\right)\left(k^2+\frac{\pi^2}{144}\right)}{\pi^2 k + 144k^3}
$$

Initial conditions are used to solve for c_1. Substituting $t = 0$ and $x = x_0$ in the above solution gives an equation to solve for the constant of integration.

$$
x_0 = \frac{\pi^2 c_1 k + 144c_1 k^3 + \pi^2\alpha + 144\alpha\, k^2 - 144\beta\, k^2}{\pi^2 k + 144k^3}
$$

$$
c_1 = -\frac{-\pi^2 k x_0 - 144k^3 x_0 + \pi^2\alpha + 144\alpha\, k^2 - 144\beta\, k^2}{k\left(\pi^2+144k^2\right)}
$$

Substituting c_1 found above in the general solution gives

$$
x = \frac{\pi^2 e^{-kt} k x_0 + 144\, e^{-kt} k^3 x_0 - 144\beta\, k^2\cos\left(\frac{\pi t}{12}\right) - \pi^2 e^{-kt}\alpha - 12\pi\beta k\sin\left(\frac{\pi t}{12}\right) - 144\, e^{-kt}\alpha\, k^2 + 144\, e^{-kt}\beta\, k}{\pi^2 k + 144k^3}
$$

Summary

The solution(s) found are the following

$$
x = \frac{\pi^2 e^{-kt} k x_0 + 144\, e^{-kt} k^3 x_0 - 144\beta\, k^2\cos\left(\frac{\pi t}{12}\right) - \pi^2 e^{-kt}\alpha - 12\pi\beta k\sin\left(\frac{\pi t}{12}\right) - 144\, e^{-kt}\alpha\, k^2 + 144\, e^{-kt}\beta\, k^2}{\pi^2 k + 144k^3} \tag{1}
$$

Verification of solutions

$$
x = \frac{\pi^2 e^{-kt} k x_0 + 144\, e^{-kt} k^3 x_0 - 144\beta\, k^2\cos\left(\frac{\pi t}{12}\right) - \pi^2 e^{-kt}\alpha - 12\pi\beta k\sin\left(\frac{\pi t}{12}\right) - 144\, e^{-kt}\alpha\, k^2 + 144\, e^{-kt}\beta\, k^2}{\pi^2 k + 144k^3}
$$

Verified OK.

2.27.3 Solving as first order ode lie symmetry lookup ode

Writing the ode as

$$
\begin{aligned}
x' &= \alpha - \beta\cos\left(\frac{\pi t}{12}\right) - kx \\
x' &= \omega(t,x)
\end{aligned}
$$

The condition of Lie symmetry is the linearized PDE given by

$$
\eta_t + \omega(\eta_x - \xi_t) - \omega^2\xi_x - \omega_t\xi - \omega_x\eta = 0 \tag{A}
$$

The type of this ode is known. It is of type `linear`. Therefore we do not need to solve the PDE (A), and can just use the lookup table shown below to find ξ, η

Table 157: Lie symmetry infinitesimal lookup table for known first order ODE's

ODE class	Form	ξ	η
linear ode	$y' = f(x)y(x) + g(x)$	0	$e^{\int f dx}$
separable ode	$y' = f(x)\, g(y)$	$\frac{1}{f}$	0
quadrature ode	$y' = f(x)$	0	1
quadrature ode	$y' = g(y)$	1	0
homogeneous ODEs of Class A	$y' = f\left(\frac{y}{x}\right)$	x	y
homogeneous ODEs of Class C	$y' = (a + bx + cy)^{\frac{n}{m}}$	1	$-\frac{b}{c}$
homogeneous class D	$y' = \frac{y}{x} + g(x)\, F\left(\frac{y}{x}\right)$	x^2	xy
First order special form ID 1	$y' = g(x)\, e^{h(x)+by} + f(x)$	$\frac{e^{-\int bf(x)dx - h(x)}}{g(x)}$	$\frac{f(x)e^{-\int bf(x)dx - h(x)}}{g(x)}$
polynomial type ode	$y' = \frac{a_1x+b_1y+c_1}{a_2x+b_2y+c_2}$	$\frac{a_1b_2x - a_2b_1x - b_1c_2 + b_2c_1}{a_1b_2 - a_2b_1}$	$\frac{a_1b_2y - a_2b_1y - a_1c_2 - a_2c_1}{a_1b_2 - a_2b_1}$
Bernoulli ode	$y' = f(x)\, y + g(x)\, y^n$	0	$e^{-\int (n-1)f(x)dx} y^n$
Reduced Riccati	$y' = f_1(x)\, y + f_2(x)\, y^2$	0	$e^{-\int f_1 dx}$

The above table shows that

$$\begin{aligned}\xi(t, x) &= 0 \\ \eta(t, x) &= \mathrm{e}^{-kt}\end{aligned} \tag{A1}$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(t, x) \to (R, S)$ where (R, S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$\frac{dt}{\xi} = \frac{dx}{\eta} = dS \tag{1}$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial t} + \eta \frac{\partial}{\partial x}\right) S(t, x) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$R = t$$

S is found from

$$\begin{aligned}S &= \int \frac{1}{\eta} dy \\ &= \int \frac{1}{\mathrm{e}^{-kt}} dy\end{aligned}$$

Which results in

$$S = \mathrm{e}^{kt} x$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_t + \omega(t, x) S_x}{R_t + \omega(t, x) R_x} \tag{2}$$

Where in the above R_t, R_x, S_t, S_x are all partial derivatives and $\omega(t, x)$ is the right hand side of the original ode given by

$$\omega(t, x) = \alpha - \beta \cos\left(\frac{\pi t}{12}\right) - kx$$

Evaluating all the partial derivatives gives

$$\begin{aligned}R_t &= 1 \\ R_x &= 0 \\ S_t &= k\,\mathrm{e}^{kt} x \\ S_x &= \mathrm{e}^{kt}\end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = -\left(-\alpha + \beta \cos\left(\frac{\pi t}{12}\right)\right) e^{kt} \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for t, x in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = -\left(-\alpha + \beta \cos\left(\frac{\pi R}{12}\right)\right) e^{kR}$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$S(R) = \frac{\alpha\, e^{kR}}{k} - \beta\left(\frac{k\, e^{kR} \cos\left(\frac{\pi R}{12}\right)}{k^2 + \frac{\pi^2}{144}} + \frac{\pi\, e^{kR} \sin\left(\frac{\pi R}{12}\right)}{12k^2 + \frac{\pi^2}{12}}\right) + c_1 \tag{4}$$

To complete the solution, we just need to transform (4) back to t, x coordinates. This results in

$$e^{kt} x = \frac{\alpha\, e^{kt}}{k} - \beta\left(\frac{k\, e^{kt} \cos\left(\frac{\pi t}{12}\right)}{k^2 + \frac{\pi^2}{144}} + \frac{\pi\, e^{kt} \sin\left(\frac{\pi t}{12}\right)}{12k^2 + \frac{\pi^2}{12}}\right) + c_1$$

Which simplifies to

$$e^{kt} x = \frac{\alpha\, e^{kt}}{k} - \beta\left(\frac{k\, e^{kt} \cos\left(\frac{\pi t}{12}\right)}{k^2 + \frac{\pi^2}{144}} + \frac{\pi\, e^{kt} \sin\left(\frac{\pi t}{12}\right)}{12k^2 + \frac{\pi^2}{12}}\right) + c_1$$

Which gives

$$x = -\frac{e^{-kt}\left(144 \cos\left(\frac{\pi t}{12}\right) e^{kt} \beta\, k^2 + 12\pi \sin\left(\frac{\pi t}{12}\right) e^{kt} \beta k - \pi^2 e^{kt} \alpha - \pi^2 c_1 k - 144\, e^{kt} \alpha\, k^2 - 144 c_1 k^3\right)}{k\left(\pi^2 + 144k^2\right)}$$

Initial conditions are used to solve for c_1. Substituting $t = 0$ and $x = x_0$ in the above solution gives an equation to solve for the constant of integration.

$$x_0 = \frac{\pi^2 c_1 k + 144 c_1 k^3 + \pi^2 \alpha + 144\alpha\, k^2 - 144\beta\, k^2}{\pi^2 k + 144k^3}$$

$$c_1 = -\frac{-\pi^2 k x_0 - 144k^3 x_0 + \pi^2 \alpha + 144\alpha\, k^2 - 144\beta\, k^2}{k\left(\pi^2 + 144k^2\right)}$$

Substituting c_1 found above in the general solution gives

$$x = \frac{-144\cos\left(\frac{\pi t}{12}\right) \mathrm{e}^{kt}\mathrm{e}^{-kt}\beta\, k^2 - 12\pi\sin\left(\frac{\pi t}{12}\right) \mathrm{e}^{kt}\mathrm{e}^{-kt}\beta k + \mathrm{e}^{-kt}\pi^2\mathrm{e}^{kt}\alpha + \pi^2\mathrm{e}^{-kt}kx_0 + 144\,\mathrm{e}^{-kt}\mathrm{e}^{kt}\alpha\, k^2 + 144\,\mathrm{e}}{\pi^2 k + 144k^3}$$

Summary

The solution(s) found are the following

$$x = \frac{-144\cos\left(\frac{\pi t}{12}\right) \mathrm{e}^{kt}\mathrm{e}^{-kt}\beta\, k^2 - 12\pi\sin\left(\frac{\pi t}{12}\right) \mathrm{e}^{kt}\mathrm{e}^{-kt}\beta k + \mathrm{e}^{-kt}\pi^2\mathrm{e}^{kt}\alpha + \pi^2\mathrm{e}^{-kt}kx_0 + 144\,\mathrm{e}^{-kt}\mathrm{e}^{kt}\alpha\, k^2 + 144\,\mathrm{e}^{-}}{\pi^2 k + 144k^3} \tag{1}$$

Verification of solutions

$$x = \frac{-144\cos\left(\frac{\pi t}{12}\right) \mathrm{e}^{kt}\mathrm{e}^{-kt}\beta\, k^2 - 12\pi\sin\left(\frac{\pi t}{12}\right) \mathrm{e}^{kt}\mathrm{e}^{-kt}\beta k + \mathrm{e}^{-kt}\pi^2\mathrm{e}^{kt}\alpha + \pi^2\mathrm{e}^{-kt}kx_0 + 144\,\mathrm{e}^{-kt}\mathrm{e}^{kt}\alpha\, k^2 + 144\,\mathrm{e}^{-}}{\pi^2 k + 144k^3}$$

Verified OK.

2.27.4 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x,y) + N(x,y)\frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x,y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y) = 0$$

Hence

$$\frac{\partial \phi}{\partial x} + \frac{\partial \phi}{\partial y}\frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$\begin{aligned}\frac{\partial \phi}{\partial x} &= M\\ \frac{\partial \phi}{\partial y} &= N\end{aligned}$$

But since $\frac{\partial^2\phi}{\partial x\partial y} = \frac{\partial^2\phi}{\partial y\partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x,y)$ but at least we know now that we can do that since the condition $\frac{\partial^2\phi}{\partial x\partial y} = \frac{\partial^2\phi}{\partial y\partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(t,x)\,\mathrm{d}t + N(t,x)\,\mathrm{d}x = 0 \tag{1A}$$

Therefore

$$\mathrm{d}x = \left(\alpha - \beta\cos\left(\frac{\pi t}{12}\right) - kx\right)\mathrm{d}t$$

$$\left(\beta\cos\left(\frac{\pi t}{12}\right) + kx - \alpha\right)\mathrm{d}t + \mathrm{d}x = 0 \tag{2A}$$

Comparing (1A) and (2A) shows that

$$\begin{aligned} M(t,x) &= \beta\cos\left(\frac{\pi t}{12}\right) + kx - \alpha \\ N(t,x) &= 1 \end{aligned}$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial x} = \frac{\partial N}{\partial t}$$

Using result found above gives

$$\begin{aligned} \frac{\partial M}{\partial x} &= \frac{\partial}{\partial x}\left(\beta\cos\left(\frac{\pi t}{12}\right) + kx - \alpha\right) \\ &= k \end{aligned}$$

And

$$\begin{aligned} \frac{\partial N}{\partial t} &= \frac{\partial}{\partial t}(1) \\ &= 0 \end{aligned}$$

Since $\frac{\partial M}{\partial x} \neq \frac{\partial N}{\partial t}$, then the ODE is <u>not exact</u>. Since the ODE is not exact, we will try to find an integrating factor to make it exact. Let

$$\begin{aligned} A &= \frac{1}{N}\left(\frac{\partial M}{\partial x} - \frac{\partial N}{\partial t}\right) \\ &= 1((k) - (0)) \\ &= k \end{aligned}$$

Since A does not depend on x, then it can be used to find an integrating factor. The integrating factor μ is

$$
\begin{aligned}
\mu &= e^{\int A \,\mathrm{d}t} \\
&= e^{\int k \,\mathrm{d}t}
\end{aligned}
$$

The result of integrating gives

$$
\begin{aligned}
\mu &= e^{kt} \\
&= \mathrm{e}^{kt}
\end{aligned}
$$

M and N are multiplied by this integrating factor, giving new M and new N which are called $\overline{M}$ and $\overline{N}$ for now so not to confuse them with the original M and N.

$$
\begin{aligned}
\overline{M} &= \mu M \\
&= \mathrm{e}^{kt}\left(\beta \cos\left(\frac{\pi t}{12}\right) + kx - \alpha\right) \\
&= \left(\beta \cos\left(\frac{\pi t}{12}\right) + kx - \alpha\right)\mathrm{e}^{kt}
\end{aligned}
$$

And

$$
\begin{aligned}
\overline{N} &= \mu N \\
&= \mathrm{e}^{kt}(1) \\
&= \mathrm{e}^{kt}
\end{aligned}
$$

Now a modified ODE is ontained from the original ODE, which is exact and can be solved. The modified ODE is

$$
\begin{aligned}
\overline{M} + \overline{N}\frac{\mathrm{d}x}{\mathrm{d}t} &= 0 \\
\left(\left(\beta \cos\left(\frac{\pi t}{12}\right) + kx - \alpha\right)\mathrm{e}^{kt}\right) + \left(\mathrm{e}^{kt}\right)\frac{\mathrm{d}x}{\mathrm{d}t} &= 0
\end{aligned}
$$

The following equations are now set up to solve for the function $\phi(t, x)$

$$
\frac{\partial \phi}{\partial t} = \overline{M} \tag{1}
$$

$$
\frac{\partial \phi}{\partial x} = \overline{N} \tag{2}
$$

Integrating (1) w.r.t. t gives

$$
\begin{aligned}
\int \frac{\partial \phi}{\partial t} \,\mathrm{d}t &= \int \overline{M} \,\mathrm{d}t \\
\int \frac{\partial \phi}{\partial t} \,\mathrm{d}t &= \int \left(\beta \cos\left(\frac{\pi t}{12}\right) + kx - \alpha\right) \mathrm{e}^{kt} \,\mathrm{d}t \\
\phi &= \frac{144\left(\beta\, k^2 \cos\left(\frac{\pi t}{12}\right) + \frac{\pi \beta k \sin\left(\frac{\pi t}{12}\right)}{12} + \left(k^2 + \frac{\pi^2}{144}\right)(kx - \alpha)\right) \mathrm{e}^{kt}}{\pi^2 k + 144k^3} + f(x) \qquad (3)
\end{aligned}
$$

Where $f(x)$ is used for the constant of integration since ϕ is a function of both t and x. Taking derivative of equation (3) w.r.t x gives

$$
\begin{aligned}
\frac{\partial \phi}{\partial x} &= \frac{144\left(k^2 + \frac{\pi^2}{144}\right) k\, \mathrm{e}^{kt}}{\pi^2 k + 144k^3} + f'(x) \qquad (4) \\
&= \mathrm{e}^{kt} + f'(x)
\end{aligned}
$$

But equation (2) says that $\frac{\partial \phi}{\partial x} = \mathrm{e}^{kt}$. Therefore equation (4) becomes

$$
\mathrm{e}^{kt} = \mathrm{e}^{kt} + f'(x) \qquad (5)
$$

Solving equation (5) for $f'(x)$ gives

$$
f'(x) = 0
$$

Therefore

$$
f(x) = c_1
$$

Where c_1 is constant of integration. Substituting this result for $f(x)$ into equation (3) gives ϕ

$$
\phi = \frac{144\left(\beta\, k^2 \cos\left(\frac{\pi t}{12}\right) + \frac{\pi \beta k \sin\left(\frac{\pi t}{12}\right)}{12} + \left(k^2 + \frac{\pi^2}{144}\right)(kx - \alpha)\right) \mathrm{e}^{kt}}{\pi^2 k + 144k^3} + c_1
$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$
c_1 = \frac{144\left(\beta\, k^2 \cos\left(\frac{\pi t}{12}\right) + \frac{\pi \beta k \sin\left(\frac{\pi t}{12}\right)}{12} + \left(k^2 + \frac{\pi^2}{144}\right)(kx - \alpha)\right) \mathrm{e}^{kt}}{\pi^2 k + 144k^3}
$$

The solution becomes

$$x = -\frac{e^{-kt}\left(144\cos\left(\frac{\pi t}{12}\right) e^{kt}\beta k^{2}+12\pi \sin\left(\frac{\pi t}{12}\right) e^{kt}\beta k-\pi^{2} e^{kt}\alpha -\pi^{2} c_{1} k-144 e^{kt}\alpha k^{2}-144 c_{1} k^{3}\right)}{k\left(\pi^{2}+144 k^{2}\right)}$$

Initial conditions are used to solve for c_1. Substituting $t = 0$ and $x = x_0$ in the above solution gives an equation to solve for the constant of integration.

$$x_0 = \frac{\pi^{2} c_{1} k+144 c_{1} k^{3}+\pi^{2}\alpha +144\alpha k^{2}-144\beta k^{2}}{\pi^{2} k+144 k^{3}}$$

$$c_1 = -\frac{-\pi^{2} k x_{0}-144 k^{3} x_{0}+\pi^{2}\alpha +144\alpha k^{2}-144\beta k^{2}}{k\left(\pi^{2}+144 k^{2}\right)}$$

Substituting c_1 found above in the general solution gives

$$x = \frac{-144\cos\left(\frac{\pi t}{12}\right) e^{kt} e^{-kt}\beta k^{2}-12\pi \sin\left(\frac{\pi t}{12}\right) e^{kt} e^{-kt}\beta k+e^{-kt}\pi^{2} e^{kt}\alpha +\pi^{2} e^{-kt} k x_{0}+144 e^{-kt} e^{kt}\alpha k^{2}+144 e}{\pi^{2} k+144 k^{3}}$$

Summary

The solution(s) found are the following

$$x = \frac{-144\cos\left(\frac{\pi t}{12}\right) e^{kt} e^{-kt}\beta k^{2}-12\pi \sin\left(\frac{\pi t}{12}\right) e^{kt} e^{-kt}\beta k+e^{-kt}\pi^{2} e^{kt}\alpha +\pi^{2} e^{-kt} k x_{0}+144 e^{-kt} e^{kt}\alpha k^{2}+144 e^{-}}{\pi^{2} k+144 k^{3}} \quad (1)$$

Verification of solutions

$$x = \frac{-144\cos\left(\frac{\pi t}{12}\right) e^{kt} e^{-kt}\beta k^{2}-12\pi \sin\left(\frac{\pi t}{12}\right) e^{kt} e^{-kt}\beta k+e^{-kt}\pi^{2} e^{kt}\alpha +\pi^{2} e^{-kt} k x_{0}+144 e^{-kt} e^{kt}\alpha k^{2}+144 e^{-}}{\pi^{2} k+144 k^{3}}$$

Verified OK.

2.27.5 Maple step by step solution

Let's solve

$\left[x'+kx=\alpha -\beta \cos\left(\frac{\pi t}{12}\right), x(0)=x_0\right]$

- Highest derivative means the order of the ODE is 1

  x'

- Isolate the derivative

$$x' = \alpha - \beta \cos\left(\frac{\pi t}{12}\right) - kx$$

- Group terms with x on the lhs of the ODE and the rest on the rhs of the ODE

$$x' + kx = \alpha - \beta \cos\left(\frac{\pi t}{12}\right)$$

- The ODE is linear; multiply by an integrating factor $\mu(t)$

$$\mu(t)\left(x' + kx\right) = \mu(t)\left(\alpha - \beta \cos\left(\frac{\pi t}{12}\right)\right)$$

- Assume the lhs of the ODE is the total derivative $\frac{d}{dt}(\mu(t)\, x)$

$$\mu(t)\left(x' + kx\right) = \mu'(t)\, x + \mu(t)\, x'$$

- Isolate $\mu'(t)$

$$\mu'(t) = \mu(t)\, k$$

- Solve to find the integrating factor

$$\mu(t) = \mathrm{e}^{kt}$$

- Integrate both sides with respect to t

$$\int \left(\frac{d}{dt}(\mu(t)\, x)\right) dt = \int \mu(t)\left(\alpha - \beta \cos\left(\frac{\pi t}{12}\right)\right) dt + c_1$$

- Evaluate the integral on the lhs

$$\mu(t)\, x = \int \mu(t)\left(\alpha - \beta \cos\left(\frac{\pi t}{12}\right)\right) dt + c_1$$

- Solve for x

$$x = \frac{\int \mu(t)\left(\alpha - \beta \cos\left(\frac{\pi t}{12}\right)\right) dt + c_1}{\mu(t)}$$

- Substitute $\mu(t) = \mathrm{e}^{kt}$

$$x = \frac{\int \left(\alpha - \beta \cos\left(\frac{\pi t}{12}\right)\right) \mathrm{e}^{kt} dt + c_1}{\mathrm{e}^{kt}}$$

- Evaluate the integrals on the rhs

$$x = \frac{\frac{\alpha\, \mathrm{e}^{kt}}{k} - \beta\left(\frac{k\, \mathrm{e}^{kt} \cos\left(\frac{\pi t}{12}\right)}{k^2 + \frac{\pi^2}{144}} + \frac{\pi\, \mathrm{e}^{kt} \sin\left(\frac{\pi t}{12}\right)}{12\left(k^2 + \frac{\pi^2}{144}\right)}\right) + c_1}{\mathrm{e}^{kt}}$$

- Simplify

$$x = \frac{-144\beta\, k^2 \cos\left(\frac{\pi t}{12}\right) - 12\pi\beta k \sin\left(\frac{\pi t}{12}\right) + 144\left(\mathrm{e}^{-kt} c_1 k + \alpha\right)\left(k^2 + \frac{\pi^2}{144}\right)}{\pi^2 k + 144k^3}$$

- Use initial condition $x(0) = x_0$

$$x_0 = \frac{-144\beta\, k^2 + 144(c_1 k + \alpha)\left(k^2 + \frac{\pi^2}{144}\right)}{\pi^2 k + 144k^3}$$

- Solve for c_1

$$c_1 = -\frac{-\pi^2 k x_0 - 144k^3 x_0 + \pi^2\alpha + 144\alpha\, k^2 - 144\beta\, k^2}{k(\pi^2 + 144k^2)}$$

- Substitute $c_1 = -\frac{-\pi^2 k x_0 - 144k^3 x_0 + \pi^2 \alpha + 144\alpha k^2 - 144\beta k^2}{k(\pi^2 + 144k^2)}$ into general solution and simplify

$$x = \frac{-144\beta k^2 \cos\left(\frac{\pi t}{12}\right) - 12\pi\beta k \sin\left(\frac{\pi t}{12}\right) + \left(144k^3 x_0 + 144(\beta - \alpha) k^2 + \pi^2 k x_0 - \pi^2 \alpha\right) e^{-kt} + 144\alpha k^2 + \pi^2 \alpha}{\pi^2 k + 144k^3}$$

- Solution to the IVP

$$x = \frac{-144\beta k^2 \cos\left(\frac{\pi t}{12}\right) - 12\pi\beta k \sin\left(\frac{\pi t}{12}\right) + \left(144k^3 x_0 + 144(\beta - \alpha) k^2 + \pi^2 k x_0 - \pi^2 \alpha\right) e^{-kt} + 144\alpha k^2 + \pi^2 \alpha}{\pi^2 k + 144k^3}$$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
<- 1st order linear successful`
```

✓ Solution by Maple

Time used: 0.047 (sec). Leaf size: 86

```
dsolve([diff(x(t),t)=alpha-beta*cos(Pi*t/12)-k*x(t),x(0) = x__0],x(t), singsol=all)
```

$$x(t) = \frac{-144\cos\left(\frac{\pi t}{12}\right)\beta k^2 - 12\sin\left(\frac{\pi t}{12}\right)\pi\beta k + \left(144k^3 x_0 + 144(\beta - \alpha) k^2 + \pi^2 k x_0 - \pi^2 \alpha\right) e^{-kt} + 144\alpha k^2 + \pi^2 \alpha}{\pi^2 k + 144k^3}$$

✓ Solution by Mathematica

Time used: 0.291 (sec). Leaf size: 64

```
DSolve[{x'[t]==\[Alpha]-\[Beta]*Cos[Pi*t/12]-k*x[t],{}},x[t],t,IncludeSingularSolutions -> T
```

$$x(t) \to -\frac{12\pi\beta \sin\left(\frac{\pi t}{12}\right)}{144k^2 + \pi^2} - \frac{144\beta k \cos\left(\frac{\pi t}{12}\right)}{144k^2 + \pi^2} + \frac{\alpha}{k} + c_1 e^{-kt}$$

2.28 problem 40

Internal problem ID [4977]
Internal file name [`OUTPUT/4470_Sunday_June_05_2022_02_57_20_PM_31826978/index.tex`]

Book: Fundamentals of Differential Equations. By Nagle, Saff and Snider. 9th edition. Boston. Pearson 2018.
Section: Chapter 2, First order differential equations. Section 2.3, Linear equations. Exercises. page 54
Problem number: 40.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"quadrature"**

Maple gives the following as the ode type

`[_quadrature]`

$$\boxed{u' - \alpha(1-u) + \beta u = 0}$$

2.28.1 Solving as quadrature ode

Integrating both sides gives

$$\int \frac{1}{-\alpha u - \beta u + \alpha} du = \int dt$$
$$\frac{\ln\left(\left(-\alpha-\beta\right)u+\alpha\right)}{-\alpha-\beta} = t + c_1$$

Raising both side to exponential gives

$$\mathrm{e}^{\frac{\ln\left(\left(-\alpha-\beta\right)u+\alpha\right)}{-\alpha-\beta}} = \mathrm{e}^{t+c_1}$$

Which simplifies to

$$(-\alpha u - \beta u + \alpha)^{-\frac{1}{\alpha+\beta}} = c_2 \mathrm{e}^{t}$$

Summary

The solution(s) found are the following

$$u = \frac{\left(c_2 \mathrm{e}^{t}\right)^{-\alpha-\beta} - \alpha}{-\alpha-\beta} \tag{1}$$

Verification of solutions

$$u = \frac{(c_2 e^t)^{-\alpha-\beta} - \alpha}{-\alpha - \beta}$$

Verified OK.

2.28.2 Maple step by step solution

Let's solve

$u' - \alpha(1-u) + \beta u = 0$

- Highest derivative means the order of the ODE is 1

  u'

- Separate variables

  $\frac{u'}{\alpha(1-u)-\beta u} = 1$

- Integrate both sides with respect to t

  $\int \frac{u'}{\alpha(1-u)-\beta u} dt = \int 1 dt + c_1$

- Evaluate integral

  $\frac{\ln((-\alpha-\beta)u+\alpha)}{-\alpha-\beta} = t + c_1$

- Solve for u

  $u = -\frac{e^{-c_1\alpha-c_1\beta-t\alpha-t\beta}-\alpha}{\alpha+\beta}$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
<- 1st order linear successful`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 24

```
dsolve(diff(u(t),t)=alpha*(1-u(t))-beta*u(t),u(t), singsol=all)
```

$$u(t) = \frac{c_1(\alpha+\beta)\,\mathrm{e}^{-(\alpha+\beta)t}+\alpha}{\alpha+\beta}$$

✓ Solution by Mathematica

Time used: 0.043 (sec). Leaf size: 35

```
DSolve[u'[t]==\[Alpha]*(1-u[t])-\[Beta]*u[t],u[t],t,IncludeSingularSolutions -> True]
```

$$u(t) \to \frac{\alpha}{\alpha+\beta}+c_1e^{-t(\alpha+\beta)}$$
$$u(t) \to \frac{\alpha}{\alpha+\beta}$$

3 Chapter 2, First order differential equations. Section 2.4, Exact equations. Exercises. page 64

3.1 problem 1

Internal problem ID [4978]
Internal file name [`OUTPUT/4471_Sunday_June_05_2022_02_57_21_PM_33931657/index.tex`]

Book: Fundamentals of Differential Equations. By Nagle, Saff and Snider. 9th edition. Boston. Pearson 2018.
Section: Chapter 2, First order differential equations. Section 2.4, Exact equations. Exercises. page 64
Problem number: 1.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"linear", "homogeneousTypeD2", "exactWithIntegrationFactor", "first_order_ode_lie_symmetry_lookup"**

Maple gives the following as the ode type

`[_linear]`

$$\boxed{yx^2 - y'x^3 = -x^4 \cos(x)}$$

3.1.1 Solving as linear ode

Entering Linear first order ODE solver. In canonical form a linear first order is

$$y' + p(x)y = q(x)$$

Where here

$$p(x) = -\frac{1}{x}$$
$$q(x) = \cos(x)\, x$$

Hence the ode is

$$y' - \frac{y}{x} = \cos(x)\, x$$

The integrating factor μ is

$$
\begin{aligned}
\mu &= \mathrm{e}^{\int -\frac{1}{x}dx} \\
&= \frac{1}{x}
\end{aligned}
$$

The ode becomes

$$
\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}x}(\mu y) &= (\mu)\left(\cos\left(x\right)x\right) \\
\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{y}{x}\right) &= \left(\frac{1}{x}\right)\left(\cos\left(x\right)x\right) \\
\mathrm{d}\left(\frac{y}{x}\right) &= \cos\left(x\right)\,\mathrm{d}x
\end{aligned}
$$

Integrating gives

$$
\begin{aligned}
\frac{y}{x} &= \int \cos\left(x\right)\,\mathrm{d}x \\
\frac{y}{x} &= \sin\left(x\right) + c_1
\end{aligned}
$$

Dividing both sides by the integrating factor $\mu = \frac{1}{x}$ results in

$$
y = \sin\left(x\right)x + c_1 x
$$

which simplifies to

$$
y = x(\sin\left(x\right) + c_1)
$$

Summary

The solution(s) found are the following

$$
y = x(\sin\left(x\right) + c_1) \tag{1}
$$

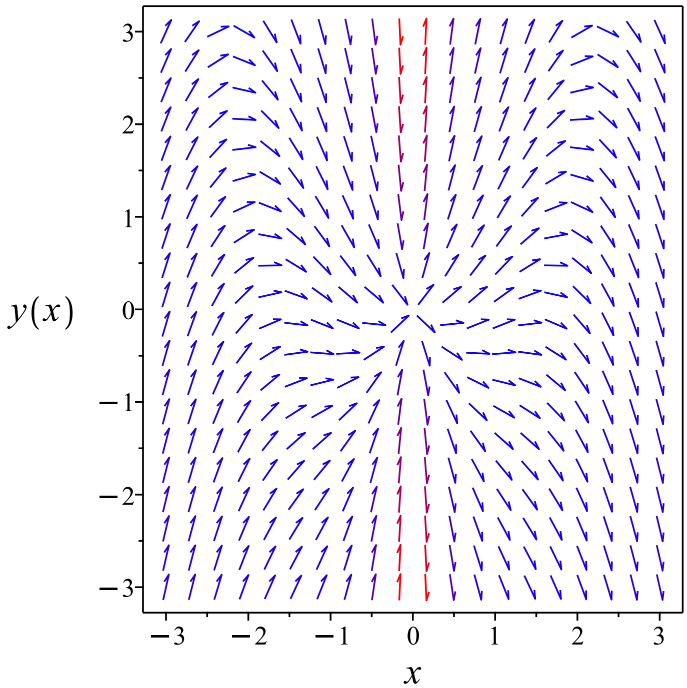


Figure 165: Slope field plot

Verification of solutions

$$y = x(\sin(x) + c_1)$$

Verified OK.

3.1.2 Solving as homogeneousTypeD2 ode

Using the change of variables $y = u(x)\,x$ on the above ode results in new ode in $u(x)$

$$u(x)\,x^3 - (u'(x)\,x + u(x))\,x^3 = -x^4\cos(x)$$

Integrating both sides gives

$$\begin{aligned} u(x) &= \int \cos(x)\,\mathrm{d}x \\ &= \sin(x) + c_2 \end{aligned}$$

Therefore the solution y is

$$\begin{aligned} y &= xu \\ &= x(\sin(x) + c_2) \end{aligned}$$

Summary

The solution(s) found are the following

$$y = x(\sin(x) + c_2) \tag{1}$$

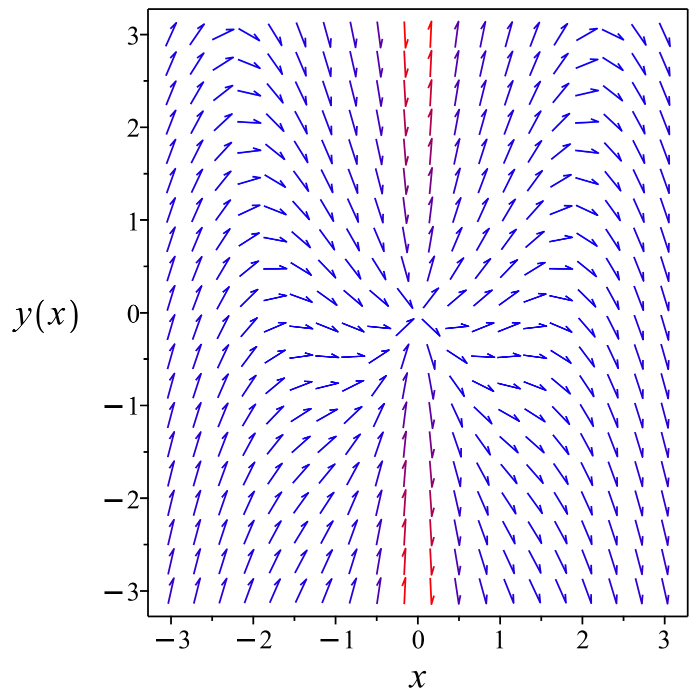


Figure 166: Slope field plot

Verification of solutions

$$y = x(\sin(x) + c_2)$$

Verified OK.

3.1.3 Solving as first order ode lie symmetry lookup ode

Writing the ode as

$$y' = \frac{\cos(x)\, x^2 + y}{x}$$
$$y' = \omega(x, y)$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x + \omega(\eta_y - \xi_x) - \omega^2 \xi_y - \omega_x \xi - \omega_y \eta = 0 \tag{A}$$

The type of this ode is known. It is of type `linear`. Therefore we do not need to solve the PDE (A), and can just use the lookup table shown below to find ξ, η

Table 161: Lie symmetry infinitesimal lookup table for known first order ODE's

ODE class	Form	ξ	η
linear ode	$y' = f(x)y(x) + g(x)$	0	$e^{\int f dx}$
separable ode	$y' = f(x)\, g(y)$	$\frac{1}{f}$	0
quadrature ode	$y' = f(x)$	0	1
quadrature ode	$y' = g(y)$	1	0
homogeneous ODEs of Class A	$y' = f\left(\frac{y}{x}\right)$	x	y
homogeneous ODEs of Class C	$y' = (a + bx + cy)^{\frac{n}{m}}$	1	$-\frac{b}{c}$
homogeneous class D	$y' = \frac{y}{x} + g(x)\, F\left(\frac{y}{x}\right)$	x^2	xy
First order special form ID 1	$y' = g(x)\, e^{h(x)+by} + f(x)$	$\frac{e^{-\int bf(x)dx - h(x)}}{g(x)}$	$\frac{f(x)e^{-\int bf(x)dx - h(x)}}{g(x)}$
polynomial type ode	$y' = \frac{a_1x+b_1y+c_1}{a_2x+b_2y+c_2}$	$\frac{a_1b_2x - a_2b_1x - b_1c_2 + b_2c_1}{a_1b_2 - a_2b_1}$	$\frac{a_1b_2y - a_2b_1y - a_1c_2 - a_2c_1}{a_1b_2 - a_2b_1}$
Bernoulli ode	$y' = f(x)\, y + g(x)\, y^n$	0	$e^{-\int (n-1)f(x)dx} y^n$
Reduced Riccati	$y' = f_1(x)\, y + f_2(x)\, y^2$	0	$e^{-\int f_1 dx}$

The above table shows that

$$
\begin{aligned}
\xi(x, y) &= 0 \\
\eta(x, y) &= x
\end{aligned} \tag{A1}
$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x, y) \to (R, S)$ where (R, S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$
\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}
$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y}\right) S(x, y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$R = x$$

S is found from

$$\begin{aligned} S &= \int \frac{1}{\eta} dy \\ &= \int \frac{1}{x} dy \end{aligned}$$

Which results in

$$S = \frac{y}{x}$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x, y) S_y}{R_x + \omega(x, y) R_y} \tag{2}$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x, y)$ is the right hand side of the original ode given by

$$\omega(x, y) = \frac{\cos(x) x^2 + y}{x}$$

Evaluating all the partial derivatives gives

$$\begin{aligned} R_x &= 1 \\ R_y &= 0 \\ S_x &= -\frac{y}{x^2} \\ S_y &= \frac{1}{x} \end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = \cos(x) \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = \cos(R)$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordinates R, S. Integrating the above gives

$$S(R) = \sin(R) + c_1 \tag{4}$$

To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$\frac{y}{x} = \sin(x) + c_1$$

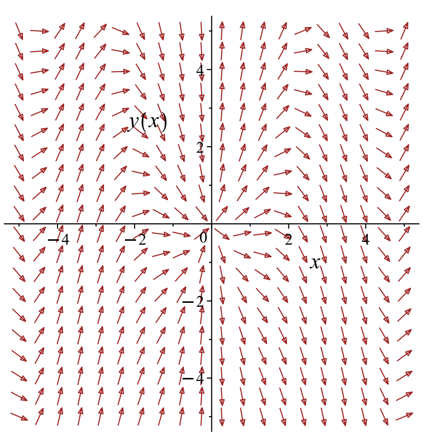
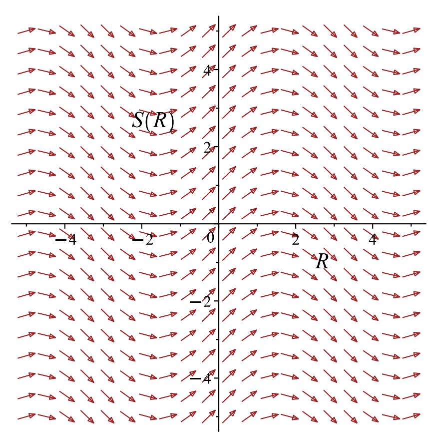
Which simplifies to

$$\frac{y}{x} = \sin(x) + c_1$$

Which gives

$$y = x(\sin(x) + c_1)$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in x, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dx} = \frac{\cos(x)x^2+y}{x}$ 	$R = x$ $S = \frac{y}{x}$	$\frac{dS}{dR} = \cos(R)$ 

Summary
The solution(s) found are the following

$$y = x(\sin(x) + c_1) \tag{1}$$

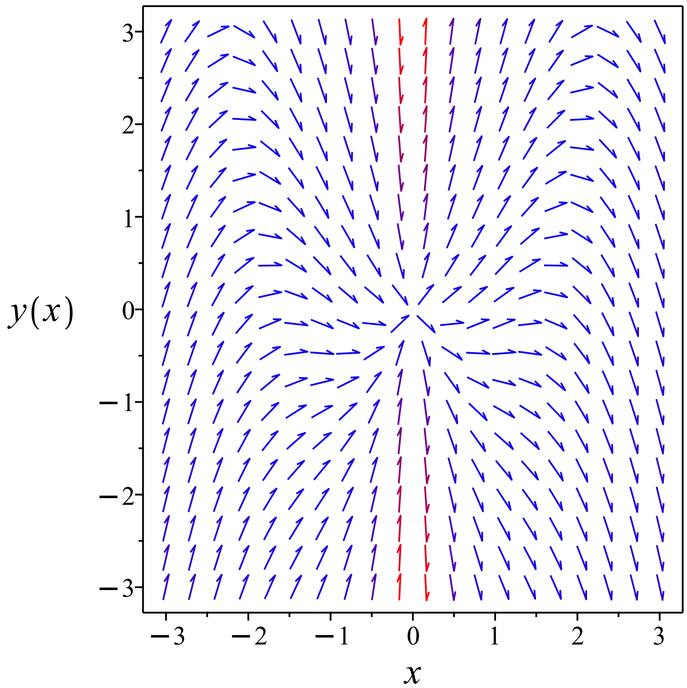


Figure 167: Slope field plot

Verification of solutions

$$y = x(\sin(x) + c_1)$$

Verified OK.

3.1.4 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x, y) + N(x, y)\frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x, y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x, y) = 0$$

Hence

$$\frac{\partial \phi}{\partial x} + \frac{\partial \phi}{\partial y}\frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$
\begin{aligned}
\frac{\partial \phi}{\partial x} &= M \\
\frac{\partial \phi}{\partial y} &= N
\end{aligned}
$$

But since $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ then for the above to be valid, we require that

$$
\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}
$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x, y)$ but at least we know now that we can do that since the condition $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$
M(x, y) \, \mathrm{d}x + N(x, y) \, \mathrm{d}y = 0 \tag{1A}
$$

Therefore

$$
\begin{aligned}
\left(-x^3\right) \mathrm{d}y &= \left(-y\, x^2 - x^4 \cos(x)\right) \mathrm{d}x \\
\left(y\, x^2 + x^4 \cos(x)\right) \mathrm{d}x + \left(-x^3\right) \mathrm{d}y &= 0 \qquad (2A)
\end{aligned}
$$

Comparing (1A) and (2A) shows that

$$
\begin{aligned}
M(x, y) &= y\, x^2 + x^4 \cos(x) \\
N(x, y) &= -x^3
\end{aligned}
$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$
\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}
$$

Using result found above gives

$$
\begin{aligned}
\frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}\left(y\, x^2 + x^4 \cos(x)\right) \\
&= x^2
\end{aligned}
$$

And

$$
\begin{aligned}
\frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}\left(-x^3\right) \\
&= -3x^2
\end{aligned}
$$

Since $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$, then the ODE is <u>not exact</u>. Since the ODE is not exact, we will try to find an integrating factor to make it exact. Let

$$\begin{aligned} A &= \frac{1}{N}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right) \\ &= -\frac{1}{x^3}\left(\left(x^2\right) - \left(-3x^2\right)\right) \\ &= -\frac{4}{x} \end{aligned}$$

Since A does not depend on y, then it can be used to find an integrating factor. The integrating factor μ is

$$\begin{aligned} \mu &= e^{\int A \,\mathrm{d}x} \\ &= e^{\int -\frac{4}{x} \,\mathrm{d}x} \end{aligned}$$

The result of integrating gives

$$\begin{aligned} \mu &= e^{-4\ln(x)} \\ &= \frac{1}{x^4} \end{aligned}$$

M and N are multiplied by this integrating factor, giving new M and new N which are called $\overline{M}$ and $\overline{N}$ for now so not to confuse them with the original M and N.

$$\begin{aligned} \overline{M} &= \mu M \\ &= \frac{1}{x^4}\left(y\,x^2 + x^4\cos(x)\right) \\ &= \frac{\cos(x)\,x^2 + y}{x^2} \end{aligned}$$

And

$$\begin{aligned} \overline{N} &= \mu N \\ &= \frac{1}{x^4}\left(-x^3\right) \\ &= -\frac{1}{x} \end{aligned}$$

Now a modified ODE is ontained from the original ODE, which is exact and can be solved. The modified ODE is

$$\begin{aligned} \overline{M} + \overline{N}\frac{\mathrm{d}y}{\mathrm{d}x} &= 0 \\ \left(\frac{\cos(x)\,x^2 + y}{x^2}\right) + \left(-\frac{1}{x}\right)\frac{\mathrm{d}y}{\mathrm{d}x} &= 0 \end{aligned}$$

The following equations are now set up to solve for the function $\phi(x,y)$

$$\frac{\partial \phi}{\partial x} = \overline{M} \tag{1}$$
$$\frac{\partial \phi}{\partial y} = \overline{N} \tag{2}$$

Integrating (1) w.r.t. x gives

$$\int \frac{\partial \phi}{\partial x} \,\mathrm{d}x = \int \overline{M} \,\mathrm{d}x$$
$$\int \frac{\partial \phi}{\partial x} \,\mathrm{d}x = \int \frac{\cos(x)\, x^2 + y}{x^2} \,\mathrm{d}x$$
$$\phi = \sin(x) - \frac{y}{x} + f(y) \tag{3}$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$\frac{\partial \phi}{\partial y} = -\frac{1}{x} + f'(y) \tag{4}$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = -\frac{1}{x}$. Therefore equation (4) becomes

$$-\frac{1}{x} = -\frac{1}{x} + f'(y) \tag{5}$$

Solving equation (5) for $f'(y)$ gives

$$f'(y) = 0$$

Therefore

$$f(y) = c_1$$

Where c_1 is constant of integration. Substituting this result for $f(y)$ into equation (3) gives ϕ

$$\phi = \sin(x) - \frac{y}{x} + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = \sin(x) - \frac{y}{x}$$

The solution becomes

$$y = (-c_1 + \sin(x))\,x$$

Summary

The solution(s) found are the following

$$y = (-c_1 + \sin(x))\,x \tag{1}$$

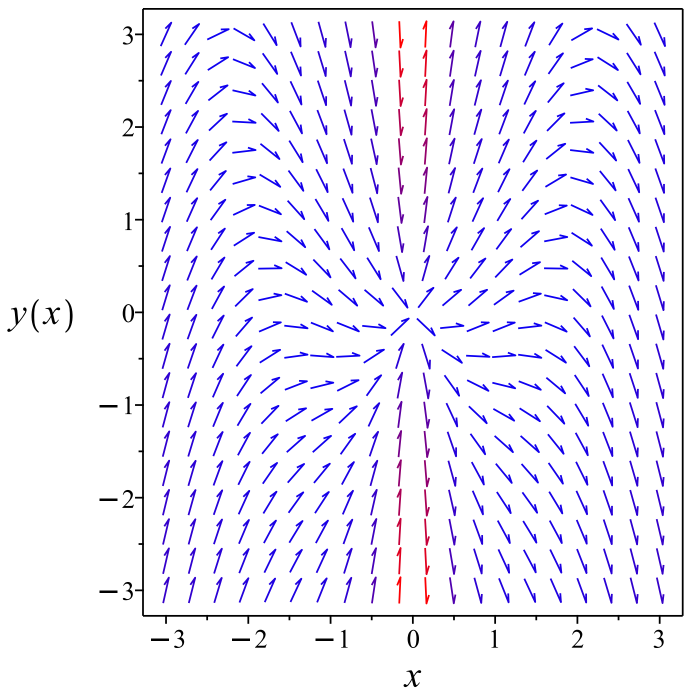


Figure 168: Slope field plot

Verification of solutions

$$y = (-c_1 + \sin(x))\,x$$

Verified OK.

3.1.5 Maple step by step solution

Let's solve

$yx^2 - y'x^3 = -x^4 \cos(x)$

- Highest derivative means the order of the ODE is 1

  y'

- Isolate the derivative

  $y' = \frac{y}{x} + \cos(x)\, x$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE

  $y' - \frac{y}{x} = \cos(x)\, x$

- The ODE is linear; multiply by an integrating factor $\mu(x)$

  $\mu(x) \left(y' - \frac{y}{x}\right) = \mu(x) \cos(x)\, x$

- Assume the lhs of the ODE is the total derivative $\frac{d}{dx}(\mu(x)\, y)$

  $\mu(x) \left(y' - \frac{y}{x}\right) = \mu'(x)\, y + \mu(x)\, y'$

- Isolate $\mu'(x)$

  $\mu'(x) = -\frac{\mu(x)}{x}$

- Solve to find the integrating factor

  $\mu(x) = \frac{1}{x}$

- Integrate both sides with respect to x

  $\int \left(\frac{d}{dx}(\mu(x)\, y)\right) dx = \int \mu(x) \cos(x)\, x dx + c_1$

- Evaluate the integral on the lhs

  $\mu(x)\, y = \int \mu(x) \cos(x)\, x dx + c_1$

- Solve for y

  $y = \frac{\int \mu(x) \cos(x) x dx + c_1}{\mu(x)}$

- Substitute $\mu(x) = \frac{1}{x}$

  $y = x\left(\int \cos(x)\, dx + c_1\right)$

- Evaluate the integrals on the rhs

  $y = x(\sin(x) + c_1)$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
<- 1st order linear successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 10

```
dsolve((x^2*y(x)+x^4*cos(x))-x^3*diff(y(x),x)=0,y(x), singsol=all)
```

$$y(x) = (\sin(x) + c_1)\, x$$

✓ Solution by Mathematica

Time used: 0.037 (sec). Leaf size: 12

```
DSolve[(x^2*y[x]+x^4*Cos[x])-x^3*y'[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to x(\sin(x) + c_1)$$

3.2 problem 2

Internal problem ID [4979]
Internal file name [`OUTPUT/4472_Sunday_June_05_2022_02_57_22_PM_19377599/index.tex`]

Book: Fundamentals of Differential Equations. By Nagle, Saff and Snider. 9th edition. Boston. Pearson 2018.
Section: Chapter 2, First order differential equations. Section 2.4, Exact equations. Exercises. page 64
Problem number: 2.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"linear", "exactWithIntegrationFactor", "first_order_ode_lie_symmetry_lookup"**

Maple gives the following as the ode type

`[_linear]`

$$\boxed{xy' - 2y = -x^{\frac{10}{3}}}$$

3.2.1 Solving as linear ode

Entering Linear first order ODE solver. In canonical form a linear first order is

$$y' + p(x)y = q(x)$$

Where here

$$p(x) = -\frac{2}{x}$$
$$q(x) = -x^{\frac{7}{3}}$$

Hence the ode is

$$y' - \frac{2y}{x} = -x^{\frac{7}{3}}$$

The integrating factor μ is

$$
\begin{aligned}
\mu &= \mathrm{e}^{\int -\frac{2}{x}dx} \\
&= \frac{1}{x^2}
\end{aligned}
$$

The ode becomes

$$
\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}x}(\mu y) &= (\mu)\left(-x^{\frac{7}{3}}\right) \\
\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{y}{x^2}\right) &= \left(\frac{1}{x^2}\right)\left(-x^{\frac{7}{3}}\right) \\
\mathrm{d}\left(\frac{y}{x^2}\right) &= \left(-x^{\frac{1}{3}}\right)\, \mathrm{d}x
\end{aligned}
$$

Integrating gives

$$
\begin{aligned}
\frac{y}{x^2} &= \int -x^{\frac{1}{3}}\, \mathrm{d}x \\
\frac{y}{x^2} &= -\frac{3x^{\frac{4}{3}}}{4} + c_1
\end{aligned}
$$

Dividing both sides by the integrating factor $\mu = \frac{1}{x^2}$ results in

$$
y = -\frac{3x^{\frac{10}{3}}}{4} + c_1 x^2
$$

<u>Summary</u>
The solution(s) found are the following

$$
y = -\frac{3x^{\frac{10}{3}}}{4} + c_1 x^2 \tag{1}
$$

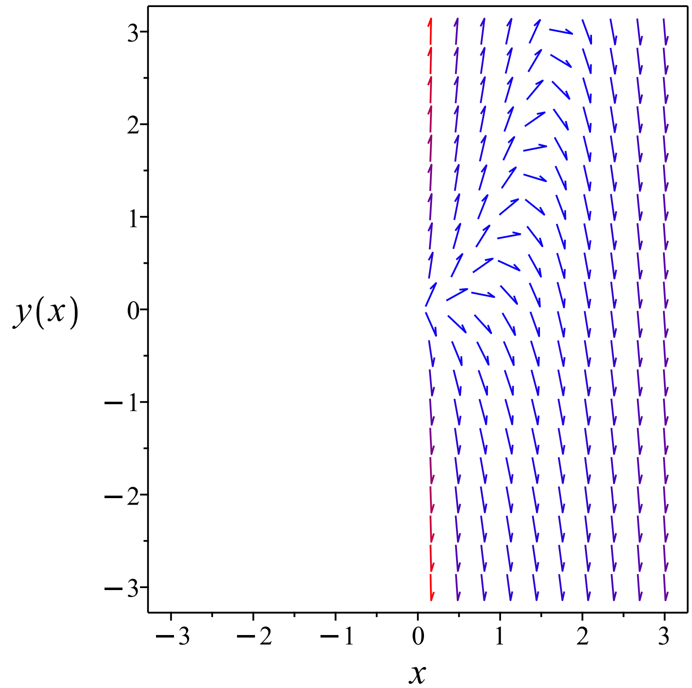


Figure 169: Slope field plot

Verification of solutions

$$y = -\frac{3x^{\frac{10}{3}}}{4} + c_1 x^2$$

Verified OK.

3.2.2 Solving as first order ode lie symmetry lookup ode

Writing the ode as

$$y' = -\frac{x^{\frac{10}{3}} - 2y}{x}$$
$$y' = \omega(x, y)$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x + \omega(\eta_y - \xi_x) - \omega^2 \xi_y - \omega_x \xi - \omega_y \eta = 0 \tag{A}$$

The type of this ode is known. It is of type `linear`. Therefore we do not need to solve the PDE (A), and can just use the lookup table shown below to find ξ, η

Table 164: Lie symmetry infinitesimal lookup table for known first order ODE's

ODE class	Form	ξ	η
linear ode	$y' = f(x)y(x) + g(x)$	0	$e^{\int f dx}$
separable ode	$y' = f(x)\, g(y)$	$\frac{1}{f}$	0
quadrature ode	$y' = f(x)$	0	1
quadrature ode	$y' = g(y)$	1	0
homogeneous ODEs of Class A	$y' = f\left(\frac{y}{x}\right)$	x	y
homogeneous ODEs of Class C	$y' = (a + bx + cy)^{\frac{n}{m}}$	1	$-\frac{b}{c}$
homogeneous class D	$y' = \frac{y}{x} + g(x)\, F\left(\frac{y}{x}\right)$	x^2	xy
First order special form ID 1	$y' = g(x)\, e^{h(x)+by} + f(x)$	$\frac{e^{-\int bf(x)dx-h(x)}}{g(x)}$	$\frac{f(x)e^{-\int bf(x)dx-h(x)}}{g(x)}$
polynomial type ode	$y' = \frac{a_1x+b_1y+c_1}{a_2x+b_2y+c_2}$	$\frac{a_1b_2x-a_2b_1x-b_1c_2+b_2c_1}{a_1b_2-a_2b_1}$	$\frac{a_1b_2y-a_2b_1y-a_1c_2-a_2c_1}{a_1b_2-a_2b_1}$
Bernoulli ode	$y' = f(x)\, y + g(x)\, y^n$	0	$e^{-\int (n-1)f(x)dx} y^n$
Reduced Riccati	$y' = f_1(x)\, y + f_2(x)\, y^2$	0	$e^{-\int f_1 dx}$

The above table shows that

$$
\begin{aligned}
\xi(x, y) &= 0 \\
\eta(x, y) &= x^2
\end{aligned} \tag{A1}
$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x, y) \to (R, S)$ where (R, S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$
\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}
$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y}\right) S(x, y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the

canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$R = x$$

S is found from

$$\begin{aligned} S &= \int \frac{1}{\eta} dy \\ &= \int \frac{1}{x^2} dy \end{aligned}$$

Which results in

$$S = \frac{y}{x^2}$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x, y) S_y}{R_x + \omega(x, y) R_y} \tag{2}$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x, y)$ is the right hand side of the original ode given by

$$\omega(x, y) = -\frac{x^{\frac{10}{3}} - 2y}{x}$$

Evaluating all the partial derivatives gives

$$\begin{aligned} R_x &= 1 \\ R_y &= 0 \\ S_x &= -\frac{2y}{x^3} \\ S_y &= \frac{1}{x^2} \end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = -x^{\frac{1}{3}} \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = -R^{\frac{1}{3}}$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordinates R, S. Integrating the above gives

$$S(R) = -\frac{3R^{\frac{4}{3}}}{4} + c_1 \tag{4}$$

To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$\frac{y}{x^2} = -\frac{3x^{\frac{4}{3}}}{4} + c_1$$

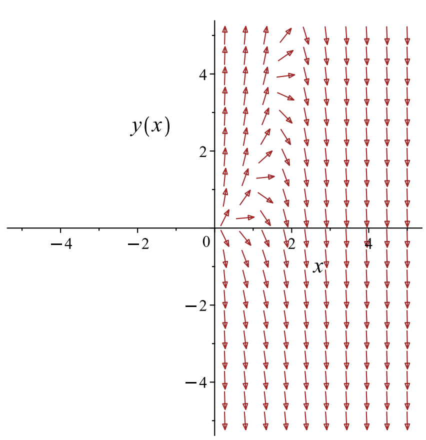
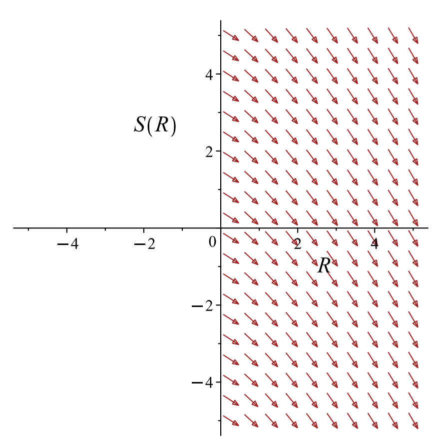
Which simplifies to

$$\frac{y}{x^2} = -\frac{3x^{\frac{4}{3}}}{4} + c_1$$

Which gives

$$y = -\frac{\left(3x^{\frac{4}{3}} - 4c_1\right) x^2}{4}$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in x, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dx} = -\frac{x^{\frac{10}{3}} - 2y}{x}$ 	$R = x$ $S = \frac{y}{x^2}$	$\frac{dS}{dR} = -R^{\frac{1}{3}}$ 

Summary
The solution(s) found are the following

$$y = -\frac{\left(3x^{\frac{4}{3}} - 4c_1\right)x^2}{4} \tag{1}$$

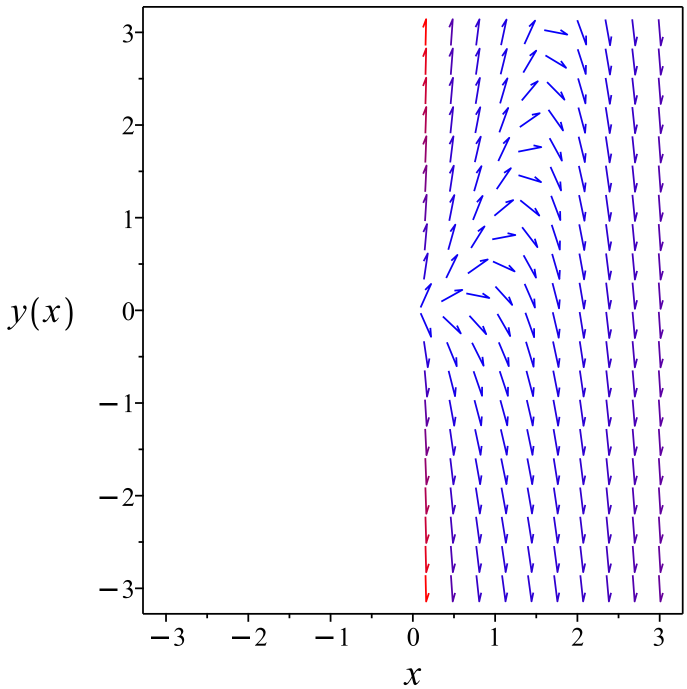


Figure 170: Slope field plot

Verification of solutions

$$y = -\frac{\left(3x^{\frac{4}{3}} - 4c_1\right)x^2}{4}$$

Verified OK.

3.2.3 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x,y) + N(x,y)\frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x,y) = c$ where c is constant, that satisfies the

ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y)=0$$

Hence

$$\frac{\partial\phi}{\partial x}+\frac{\partial\phi}{\partial y}\frac{dy}{dx}=0 \tag{B}$$

Comparing (A,B) shows that

$$\begin{aligned}\frac{\partial\phi}{\partial x}&=M\\ \frac{\partial\phi}{\partial y}&=N\end{aligned}$$

But since $\frac{\partial^2\phi}{\partial x\partial y}=\frac{\partial^2\phi}{\partial y\partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y}=\frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x,y)$ but at least we know now that we can do that since the condition $\frac{\partial^2\phi}{\partial x\partial y}=\frac{\partial^2\phi}{\partial y\partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x,y)\,\mathrm{d}x+N(x,y)\,\mathrm{d}y=0 \tag{1A}$$

Therefore

$$\begin{aligned}(x)\,\mathrm{d}y&=\left(-x^{\frac{10}{3}}+2y\right)\mathrm{d}x\\ \left(x^{\frac{10}{3}}-2y\right)\mathrm{d}x+(x)\,\mathrm{d}y&=0 \qquad \text{(2A)}\end{aligned}$$

Comparing (1A) and (2A) shows that

$$\begin{aligned}M(x,y)&=x^{\frac{10}{3}}-2y\\ N(x,y)&=x\end{aligned}$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y}=\frac{\partial N}{\partial x}$$

Using result found above gives

$$\begin{aligned}\frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}\left(x^{\frac{10}{3}} - 2y\right)\\ &= -2\end{aligned}$$

And

$$\begin{aligned}\frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}(x)\\ &= 1\end{aligned}$$

Since $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$, then the ODE is not exact. Since the ODE is not exact, we will try to find an integrating factor to make it exact. Let

$$\begin{aligned}A &= \frac{1}{N}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right)\\ &= \frac{1}{x}((-2) - (1))\\ &= -\frac{3}{x}\end{aligned}$$

Since A does not depend on y, then it can be used to find an integrating factor. The integrating factor μ is

$$\begin{aligned}\mu &= e^{\int A \,\mathrm{d}x}\\ &= e^{\int -\frac{3}{x} \,\mathrm{d}x}\end{aligned}$$

The result of integrating gives

$$\begin{aligned}\mu &= e^{-3\ln(x)}\\ &= \frac{1}{x^3}\end{aligned}$$

M and N are multiplied by this integrating factor, giving new M and new N which are called $\overline{M}$ and $\overline{N}$ for now so not to confuse them with the original M and N.

$$\begin{aligned}\overline{M} &= \mu M\\ &= \frac{1}{x^3}\left(x^{\frac{10}{3}} - 2y\right)\\ &= \frac{x^{\frac{10}{3}} - 2y}{x^3}\end{aligned}$$

And

$$
\begin{aligned}
\overline{N} &= \mu N \\
&= \frac{1}{x^3}(x) \\
&= \frac{1}{x^2}
\end{aligned}
$$

Now a modified ODE is ontained from the original ODE, which is exact and can be solved. The modified ODE is

$$
\begin{aligned}
\overline{M} + \overline{N}\frac{\mathrm{d}y}{\mathrm{d}x} &= 0 \\
\left(\frac{x^{\frac{10}{3}} - 2y}{x^3}\right) + \left(\frac{1}{x^2}\right)\frac{\mathrm{d}y}{\mathrm{d}x} &= 0
\end{aligned}
$$

The following equations are now set up to solve for the function $\phi(x, y)$

$$
\frac{\partial \phi}{\partial x} = \overline{M} \tag{1}
$$

$$
\frac{\partial \phi}{\partial y} = \overline{N} \tag{2}
$$

Integrating (1) w.r.t. x gives

$$
\begin{aligned}
\int \frac{\partial \phi}{\partial x} \mathrm{d}x &= \int \overline{M} \mathrm{d}x \\
\int \frac{\partial \phi}{\partial x} \mathrm{d}x &= \int \frac{x^{\frac{10}{3}} - 2y}{x^3} \mathrm{d}x
\end{aligned}
$$

$$
\phi = \frac{3x^{\frac{4}{3}}}{4} + \frac{y}{x^2} + f(y) \tag{3}
$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$
\frac{\partial \phi}{\partial y} = \frac{1}{x^2} + f'(y) \tag{4}
$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = \frac{1}{x^2}$. Therefore equation (4) becomes

$$
\frac{1}{x^2} = \frac{1}{x^2} + f'(y) \tag{5}
$$

Solving equation (5) for $f'(y)$ gives

$$f'(y) = 0$$

Therefore

$$f(y) = c_1$$

Where c_1 is constant of integration. Substituting this result for $f(y)$ into equation (3) gives ϕ

$$\phi = \frac{3x^{\frac{4}{3}}}{4} + \frac{y}{x^2} + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = \frac{3x^{\frac{4}{3}}}{4} + \frac{y}{x^2}$$

The solution becomes

$$y = -\frac{\left(3x^{\frac{4}{3}} - 4c_1\right) x^2}{4}$$

Summary

The solution(s) found are the following

$$y = -\frac{\left(3x^{\frac{4}{3}} - 4c_1\right) x^2}{4} \tag{1}$$

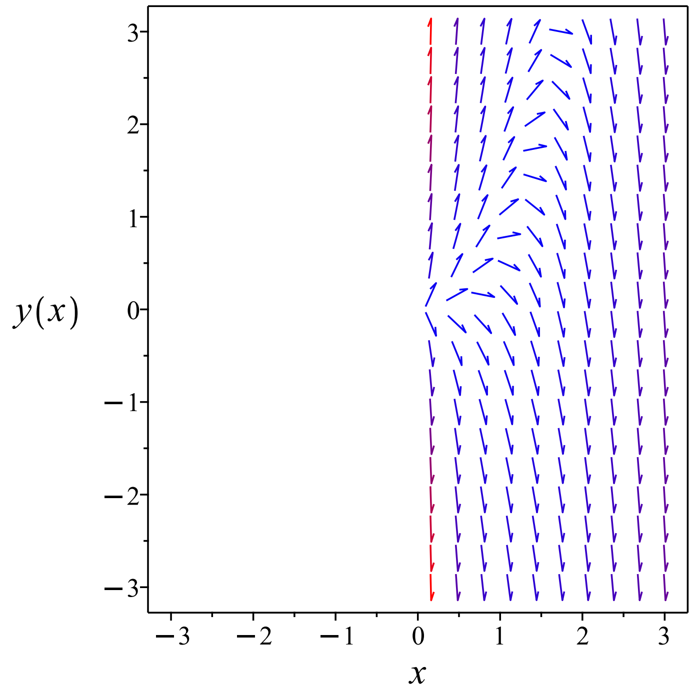


Figure 171: Slope field plot

Verification of solutions

$$y = -\frac{\left(3x^{\frac{4}{3}} - 4c_1\right)x^2}{4}$$

Verified OK.

3.2.4 Maple step by step solution

Let's solve

$xy' - 2y = -x^{\frac{10}{3}}$

- Highest derivative means the order of the ODE is 1

  y'

- Isolate the derivative

  $y' = \frac{2y}{x} - x^{\frac{7}{3}}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE

  $y' - \frac{2y}{x} = -x^{\frac{7}{3}}$

- The ODE is linear; multiply by an integrating factor $\mu(x)$

$$\mu(x)\left(y' - \frac{2y}{x}\right) = -\mu(x)\, x^{\frac{7}{3}}$$

- Assume the lhs of the ODE is the total derivative $\frac{d}{dx}(\mu(x)\, y)$

$$\mu(x)\left(y' - \frac{2y}{x}\right) = \mu'(x)\, y + \mu(x)\, y'$$

- Isolate $\mu'(x)$

$$\mu'(x) = -\frac{2\mu(x)}{x}$$

- Solve to find the integrating factor

$$\mu(x) = \frac{1}{x^2}$$

- Integrate both sides with respect to x

$$\int \left(\frac{d}{dx}(\mu(x)\, y)\right) dx = \int -\mu(x)\, x^{\frac{7}{3}} dx + c_1$$

- Evaluate the integral on the lhs

$$\mu(x)\, y = \int -\mu(x)\, x^{\frac{7}{3}} dx + c_1$$

- Solve for y

$$y = \frac{\int -\mu(x) x^{\frac{7}{3}} dx + c_1}{\mu(x)}$$

- Substitute $\mu(x) = \frac{1}{x^2}$

$$y = x^2 \left(\int -x^{\frac{1}{3}} dx + c_1\right)$$

- Evaluate the integrals on the rhs

$$y = \left(-\frac{3x^{\frac{4}{3}}}{4} + c_1\right) x^2$$

- Simplify

$$y = -\frac{\left(3x^{\frac{4}{3}} - 4c_1\right) x^2}{4}$$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
<- 1st order linear successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 18

```
dsolve((x^(10/3)-2*y(x))+x*diff(y(x),x)=0,y(x), singsol=all)
```

$$y(x) = -\frac{\left(3x^{\frac{4}{3}} - 4c_1\right) x^2}{4}$$

✓ Solution by Mathematica

Time used: 0.032 (sec). Leaf size: 21

```
DSolve[(x^(10/3)-2*y[x])+x*y'[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to -\frac{3x^{10/3}}{4} + c_1 x^2$$

3.3 problem 3

Internal problem ID [4980]
Internal file name [`OUTPUT/4473_Sunday_June_05_2022_02_57_23_PM_78588279/index.tex`]

Book: Fundamentals of Differential Equations. By Nagle, Saff and Snider. 9th edition. Boston. Pearson 2018.
Section: Chapter 2, First order differential equations. Section 2.4, Exact equations. Exercises. page 64
Problem number: 3.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"exact", "separable", "first_order_ode_lie_symmetry_lookup"**

Maple gives the following as the ode type

`[_separable]`

$$\boxed{\sqrt{-2y-y^2}+\left(-x^2+2x+3\right)y'=0}$$

3.3.1 Solving as separable ode

In canonical form the ODE is

$$\begin{aligned} y' &= F(x,y)\\ &= f(x)g(y)\\ &= \frac{\sqrt{-y^2-2y}}{x^2-2x-3} \end{aligned}$$

Where $f(x)=\frac{1}{x^2-2x-3}$ and $g(y)=\sqrt{-y^2-2y}$. Integrating both sides gives

$$\frac{1}{\sqrt{-y^2-2y}}\,dy=\frac{1}{x^2-2x-3}\,dx$$

$$\int \frac{1}{\sqrt{-y^2-2y}}\,dy = \int \frac{1}{x^2-2x-3}\,dx$$

$$\arcsin(1+y) = -\frac{\ln(x+1)}{4} + \frac{\ln(x-3)}{4} + c_1$$

Which results in

$$y = -1 + \sin\left(-\frac{\ln(x+1)}{4} + \frac{\ln(x-3)}{4} + c_1\right)$$

Summary

The solution(s) found are the following

$$y = -1 + \sin\left(-\frac{\ln(x+1)}{4} + \frac{\ln(x-3)}{4} + c_1\right) \tag{1}$$

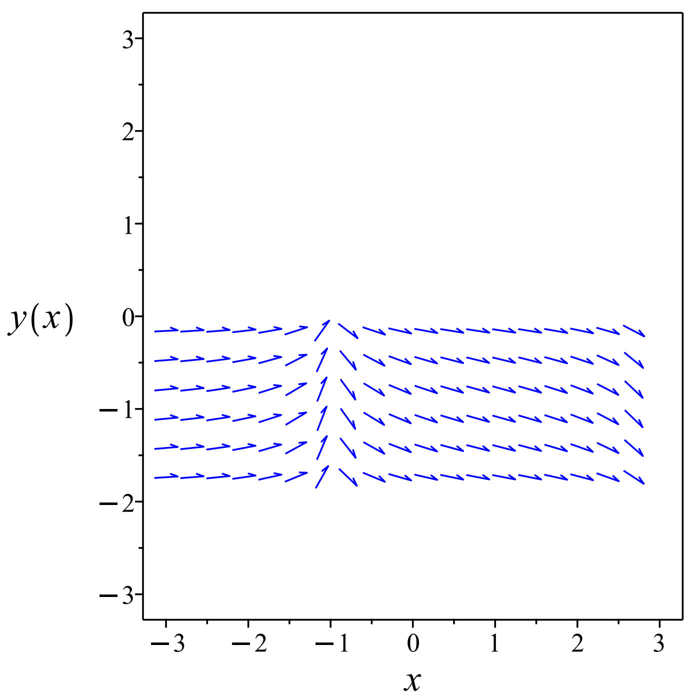


Figure 172: Slope field plot

Verification of solutions

$$y = -1 + \sin\left(-\frac{\ln(x+1)}{4} + \frac{\ln(x-3)}{4} + c_1\right)$$

Verified OK.

3.3.2 Solving as first order ode lie symmetry lookup ode

Writing the ode as

$$
\begin{aligned}
y' &= \frac{\sqrt{-y^2-2y}}{x^2-2x-3} \\
y' &= \omega(x,y)
\end{aligned}
$$

The condition of Lie symmetry is the linearized PDE given by

$$
\eta_x + \omega(\eta_y - \xi_x) - \omega^2 \xi_y - \omega_x \xi - \omega_y \eta = 0 \tag{A}
$$

The type of this ode is known. It is of type `separable`. Therefore we do not need to solve the PDE (A), and can just use the lookup table shown below to find ξ, η

Table 167: Lie symmetry infinitesimal lookup table for known first order ODE's

ODE class	Form	ξ	η
linear ode	$y' = f(x)y(x) + g(x)$	0	$e^{\int f dx}$
separable ode	$y' = f(x)\, g(y)$	$\frac{1}{f}$	0
quadrature ode	$y' = f(x)$	0	1
quadrature ode	$y' = g(y)$	1	0
homogeneous ODEs of Class A	$y' = f\left(\frac{y}{x}\right)$	x	y
homogeneous ODEs of Class C	$y' = (a+bx+cy)^{\frac{n}{m}}$	1	$-\frac{b}{c}$
homogeneous class D	$y' = \frac{y}{x} + g(x)\, F\left(\frac{y}{x}\right)$	x^2	xy
First order special form ID 1	$y' = g(x)\, e^{h(x)+by} + f(x)$	$\frac{e^{-\int b f(x) dx - h(x)}}{g(x)}$	$\frac{f(x) e^{-\int b f(x) dx - h(x)}}{g(x)}$
polynomial type ode	$y' = \frac{a_1 x + b_1 y + c_1}{a_2 x + b_2 y + c_2}$	$\frac{a_1 b_2 x - a_2 b_1 x - b_1 c_2 + b_2 c_1}{a_1 b_2 - a_2 b_1}$	$\frac{a_1 b_2 y - a_2 b_1 y - a_1 c_2 - a_2 c_1}{a_1 b_2 - a_2 b_1}$
Bernoulli ode	$y' = f(x)\, y + g(x)\, y^n$	0	$e^{-\int (n-1) f(x) dx} y^n$
Reduced Riccati	$y' = f_1(x)\, y + f_2(x)\, y^2$	0	$e^{-\int f_1 dx}$

The above table shows that

$$
\begin{aligned}
\xi(x,y) &= x^2 - 2x - 3 \\
\eta(x,y) &= 0
\end{aligned}
\tag{A1}
$$

The next step is to determine the canonical coordinates R,S. The canonical coordinates map $(x,y) \to (R,S)$ where (R,S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$
\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}
$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y}\right) S(x,y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the canonical coordinates, where $S(R)$. Since $\eta = 0$ then in this special case

$$
R = y
$$

S is found from

$$
\begin{aligned}
S &= \int \frac{1}{\xi} dx \\
&= \int \frac{1}{x^2 - 2x - 3} dx
\end{aligned}
$$

Which results in

$$
S = -\frac{\ln(x+1)}{4} + \frac{\ln(x-3)}{4}
$$

Now that R,S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$
\frac{dS}{dR} = \frac{S_x + \omega(x,y) S_y}{R_x + \omega(x,y) R_y} \tag{2}
$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x,y)$ is the right hand side of the original ode given by

$$
\omega(x,y) = \frac{\sqrt{-y^2 - 2y}}{x^2 - 2x - 3}
$$

Evaluating all the partial derivatives gives

$$\begin{aligned}
R_x &= 0 \\
R_y &= 1 \\
S_x &= \frac{1}{(x+1)(x-3)} \\
S_y &= 0
\end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = \frac{1}{\sqrt{-y\,(y+2)}} \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = \frac{1}{\sqrt{-R\,(R+2)}}$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$S(R) = \arcsin(R+1) + c_1 \tag{4}$$

To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$-\frac{\ln(x+1)}{4} + \frac{\ln(x-3)}{4} = \arcsin(1+y) + c_1$$

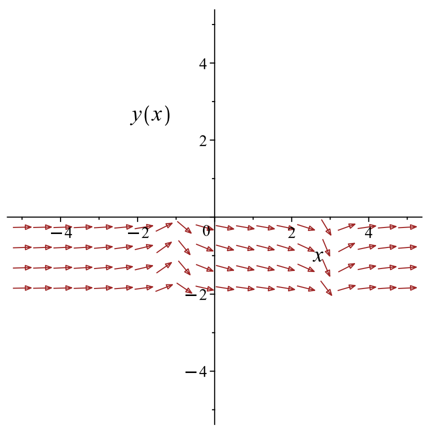
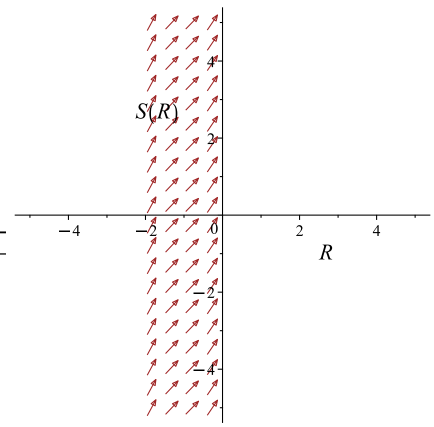
Which simplifies to

$$-\frac{\ln(x+1)}{4} + \frac{\ln(x-3)}{4} = \arcsin(1+y) + c_1$$

Which gives

$$y = -1 - \sin\left(\frac{\ln(x+1)}{4} - \frac{\ln(x-3)}{4} + c_1\right)$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in x, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dx} = \frac{\sqrt{-y^2-2y}}{x^2-2x-3}$ 	$R = y$ $S = -\frac{\ln(x+1)}{4} + \frac{\ln(x -}{4}$	$\frac{dS}{dR} = \frac{1}{\sqrt{-R(R+2)}}$ 

Summary

The solution(s) found are the following

$$y = -1 - \sin\left(\frac{\ln(x+1)}{4} - \frac{\ln(x-3)}{4} + c_1\right) \tag{1}$$

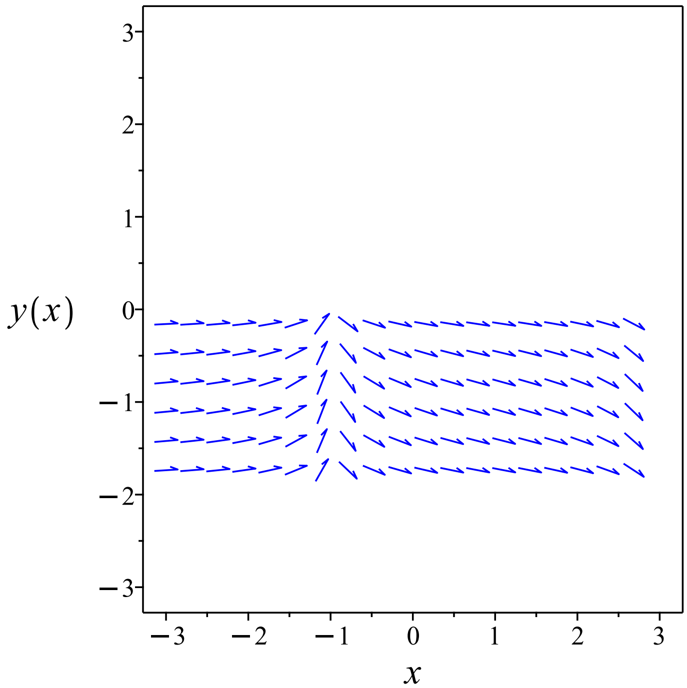


Figure 173: Slope field plot

Verification of solutions

$$y = -1 - \sin\left(\frac{\ln(x+1)}{4} - \frac{\ln(x-3)}{4} + c_1\right)$$

Verified OK.

3.3.3 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x,y) + N(x,y)\frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x,y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y) = 0$$

Hence

$$\frac{\partial \phi}{\partial x} + \frac{\partial \phi}{\partial y}\frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$\frac{\partial \phi}{\partial x} = M$$
$$\frac{\partial \phi}{\partial y} = N$$

But since $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x,y)$ but at least we know now that we can do that since the condition $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x,y)\,\mathrm{d}x + N(x,y)\,\mathrm{d}y = 0 \tag{1A}$$

Therefore

$$\left(\frac{1}{\sqrt{-y^2-2y}}\right)\mathrm{d}y = \left(\frac{1}{x^2-2x-3}\right)\mathrm{d}x$$
$$\left(-\frac{1}{x^2-2x-3}\right)\mathrm{d}x + \left(\frac{1}{\sqrt{-y^2-2y}}\right)\mathrm{d}y = 0 \tag{2A}$$

Comparing (1A) and (2A) shows that

$$M(x,y) = -\frac{1}{x^2-2x-3}$$
$$N(x,y) = \frac{1}{\sqrt{-y^2-2y}}$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Using result found above gives

$$\begin{aligned}\frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}\left(-\frac{1}{x^2-2x-3}\right)\\ &= 0\end{aligned}$$

And

$$
\begin{aligned}
\frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}\left(\frac{1}{\sqrt{-y^2-2y}}\right) \\
&= 0
\end{aligned}
$$

Since $\frac{\partial M}{\partial y}=\frac{\partial N}{\partial x}$, then the ODE is exact The following equations are now set up to solve for the function $\phi(x,y)$

$$
\frac{\partial \phi}{\partial x} = M \tag{1}
$$

$$
\frac{\partial \phi}{\partial y} = N \tag{2}
$$

Integrating (1) w.r.t. x gives

$$
\begin{aligned}
\int \frac{\partial \phi}{\partial x} \, \mathrm{d}x &= \int M \, \mathrm{d}x \\
\int \frac{\partial \phi}{\partial x} \, \mathrm{d}x &= \int -\frac{1}{x^2-2x-3} \, \mathrm{d}x
\end{aligned}
$$

$$
\phi = \frac{\ln(x+1)}{4} - \frac{\ln(x-3)}{4} + f(y) \tag{3}
$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$
\frac{\partial \phi}{\partial y} = 0 + f'(y) \tag{4}
$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = \frac{1}{\sqrt{-y^2-2y}}$. Therefore equation (4) becomes

$$
\frac{1}{\sqrt{-y^2-2y}} = 0 + f'(y) \tag{5}
$$

Solving equation (5) for $f'(y)$ gives

$$
\begin{aligned}
f'(y) &= \frac{1}{\sqrt{-y^2-2y}} \\
&= \frac{1}{\sqrt{-y\,(y+2)}}
\end{aligned}
$$

Integrating the above w.r.t y results in

$$\int f'(y)\,\mathrm{d}y = \int \left(\frac{1}{\sqrt{-y\,(y+2)}}\right) \mathrm{d}y$$

$$f(y) = \arcsin\left(1+y\right) + c_1$$

Where c_1 is constant of integration. Substituting result found above for $f(y)$ into equation (3) gives ϕ

$$\phi = \frac{\ln\left(x+1\right)}{4} - \frac{\ln\left(x-3\right)}{4} + \arcsin\left(1+y\right) + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = \frac{\ln\left(x+1\right)}{4} - \frac{\ln\left(x-3\right)}{4} + \arcsin\left(1+y\right)$$

The solution becomes

$$y = -1 + \sin\left(-\frac{\ln\left(x+1\right)}{4} + \frac{\ln\left(x-3\right)}{4} + c_1\right)$$

Summary

The solution(s) found are the following

$$y = -1 + \sin\left(-\frac{\ln\left(x+1\right)}{4} + \frac{\ln\left(x-3\right)}{4} + c_1\right) \tag{1}$$

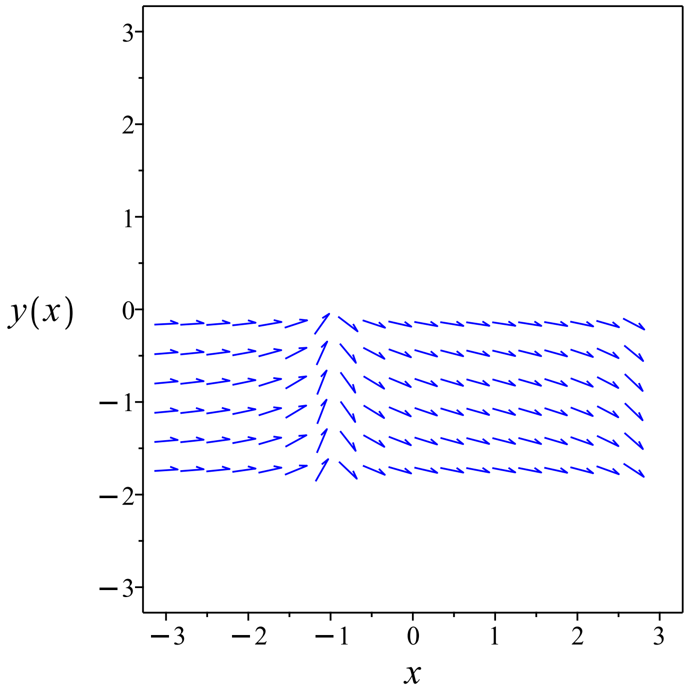


Figure 174: Slope field plot

Verification of solutions

$$y = -1 + \sin\left(-\frac{\ln(x+1)}{4} + \frac{\ln(x-3)}{4} + c_1\right)$$

Verified OK.

3.3.4 Maple step by step solution

Let's solve

$\sqrt{-2y - y^2} + (-x^2 + 2x + 3)\, y' = 0$

- Highest derivative means the order of the ODE is 1

  y'

- Separate variables

  $\frac{y'}{\sqrt{-2y-y^2}} = -\frac{1}{-x^2+2x+3}$

- Integrate both sides with respect to x

  $\int \frac{y'}{\sqrt{-2y-y^2}} dx = \int -\frac{1}{-x^2+2x+3} dx + c_1$

- Evaluate integral

$$\arcsin(1+y) = -\frac{\ln(x+1)}{4} + \frac{\ln(x-3)}{4} + c_1$$

- Solve for y

$$y = -1 + \sin\left(-\frac{\ln(x+1)}{4} + \frac{\ln(x-3)}{4} + c_1\right)$$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
<- separable successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 21

```
dsolve(sqrt(-2*y(x)-y(x)^2)+(3+2*x-x^2)*diff(y(x),x)=0,y(x), singsol=all)
```

$$y(x) = -1 + \sin\left(\frac{\ln(x-3)}{4} - \frac{\ln(1+x)}{4} + c_1\right)$$

✓ Solution by Mathematica

Time used: 60.208 (sec). Leaf size: 385

```
DSolve[Sqrt[-2*y[x]-y[x]^2]+(3+2*x-x^2)*y'[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to -1 - \frac{1}{4}\sqrt{8 - e^{-4ic_1}\left(-x^2+2x+3\right)^{-i}\sqrt{e^{4ic_1}\left(-x^2+2x+3\right)^{i}\left((x+1)^i+16e^{4ic_1}(3-x)^i\right)^2}}$$

$$y(x) \to \frac{1}{4}\left(-4 + \sqrt{8 - e^{-4ic_1}\left(-x^2+2x+3\right)^{-i}\sqrt{e^{4ic_1}\left(-x^2+2x+3\right)^{i}\left((x+1)^i+16e^{4ic_1}(3-x)^i\right)^2}}\right)$$

$$y(x) \to \frac{1}{4}\left(-4 - \sqrt{8 + e^{-4ic_1}\left(-x^2+2x+3\right)^{-i}\sqrt{e^{4ic_1}\left(-x^2+2x+3\right)^{i}\left((x+1)^i+16e^{4ic_1}(3-x)^i\right)^2}}\right)$$

$$y(x) \to \frac{1}{4}\left(-4 + \sqrt{8 + e^{-4ic_1}\left(-x^2+2x+3\right)^{-i}\sqrt{e^{4ic_1}\left(-x^2+2x+3\right)^{i}\left((x+1)^i+16e^{4ic_1}(3-x)^i\right)^2}}\right)$$

3.4 problem 4

Internal problem ID [4981]
Internal file name [`OUTPUT/4474_Sunday_June_05_2022_02_57_24_PM_57829874/index.tex`]

Book: Fundamentals of Differential Equations. By Nagle, Saff and Snider. 9th edition. Boston. Pearson 2018.
Section: Chapter 2, First order differential equations. Section 2.4, Exact equations. Exercises. page 64
Problem number: 4.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"exact"**

Maple gives the following as the ode type

`[_exact]`

$$\boxed{y\,\mathrm{e}^{xy} + (x\,\mathrm{e}^{xy} - 2y)\,y' = -2x}$$

3.4.1 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x,y) + N(x,y)\,\frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x,y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y) = 0$$

Hence

$$\frac{\partial\phi}{\partial x} + \frac{\partial\phi}{\partial y}\frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$
\begin{aligned}
\frac{\partial \phi}{\partial x} &= M \\
\frac{\partial \phi}{\partial y} &= N
\end{aligned}
$$

But since $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ then for the above to be valid, we require that

$$
\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}
$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x, y)$ but at least we know now that we can do that since the condition $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$
M(x, y) \, \mathrm{d}x + N(x, y) \, \mathrm{d}y = 0 \tag{1A}
$$

Therefore

$$
\begin{aligned}
(x \, \mathrm{e}^{xy} - 2y) \, \mathrm{d}y &= (-y \, \mathrm{e}^{xy} - 2x) \, \mathrm{d}x \\
(y \, \mathrm{e}^{xy} + 2x) \, \mathrm{d}x + (x \, \mathrm{e}^{xy} - 2y) \, \mathrm{d}y &= 0 \qquad (2A)
\end{aligned}
$$

Comparing (1A) and (2A) shows that

$$
\begin{aligned}
M(x, y) &= y \, \mathrm{e}^{xy} + 2x \\
N(x, y) &= x \, \mathrm{e}^{xy} - 2y
\end{aligned}
$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$
\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}
$$

Using result found above gives

$$
\begin{aligned}
\frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}(y \, \mathrm{e}^{xy} + 2x) \\
&= \mathrm{e}^{xy}(xy + 1)
\end{aligned}
$$

And

$$
\begin{aligned}
\frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}(x \, \mathrm{e}^{xy} - 2y) \\
&= \mathrm{e}^{xy}(xy + 1)
\end{aligned}
$$

Since $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$, then the ODE is exact The following equations are now set up to solve for the function $\phi(x, y)$

$$\frac{\partial \phi}{\partial x} = M \tag{1}$$

$$\frac{\partial \phi}{\partial y} = N \tag{2}$$

Integrating (1) w.r.t. x gives

$$\int \frac{\partial \phi}{\partial x} \, dx = \int M \, dx$$

$$\int \frac{\partial \phi}{\partial x} \, dx = \int y \, e^{xy} + 2x \, dx$$

$$\phi = e^{xy} + x^2 + f(y) \tag{3}$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$\frac{\partial \phi}{\partial y} = x \, e^{xy} + f'(y) \tag{4}$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = x \, e^{xy} - 2y$. Therefore equation (4) becomes

$$x \, e^{xy} - 2y = x \, e^{xy} + f'(y) \tag{5}$$

Solving equation (5) for $f'(y)$ gives

$$f'(y) = -2y$$

Integrating the above w.r.t y gives

$$\int f'(y) \, dy = \int (-2y) \, dy$$

$$f(y) = -y^2 + c_1$$

Where c_1 is constant of integration. Substituting result found above for $f(y)$ into equation (3) gives ϕ

$$\phi = e^{xy} + x^2 - y^2 + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = e^{xy} + x^2 - y^2$$

Summary

The solution(s) found are the following

$$e^{xy} + x^2 - y^2 = c_1 \tag{1}$$

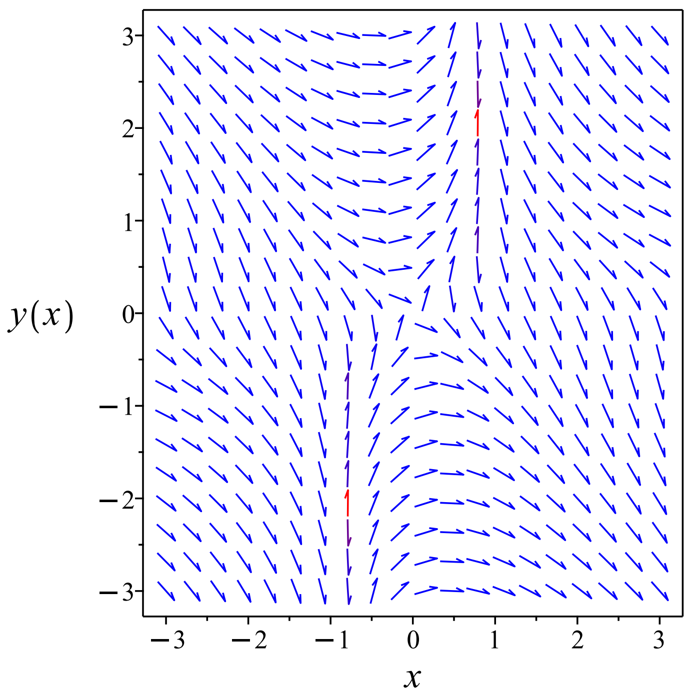


Figure 175: Slope field plot

Verification of solutions

$$e^{xy} + x^2 - y^2 = c_1$$

Verified OK.

3.4.2 Maple step by step solution

Let's solve

$y\,e^{xy} + (x\,e^{xy} - 2y)\,y' = -2x$

- Highest derivative means the order of the ODE is 1

  y'

□ Check if ODE is exact

  ◦ ODE is exact if the lhs is the total derivative of a C^2 function

    $F'(x,y) = 0$

  ◦ Compute derivative of lhs

    $F'(x,y) + \left(\frac{\partial}{\partial y}F(x,y)\right) y' = 0$

  ◦ Evaluate derivatives

    $e^{xy} + xy\,e^{xy} = e^{xy} + xy\,e^{xy}$

  ◦ Condition met, ODE is exact

- Exact ODE implies solution will be of this form

  $\left[F(x,y) = c_1, M(x,y) = F'(x,y)\,, N(x,y) = \frac{\partial}{\partial y}F(x,y)\right]$

- Solve for $F(x,y)$ by integrating $M(x,y)$ with respect to x

  $F(x,y) = \int (y\,e^{xy} + 2x)\,dx + f_1(y)$

- Evaluate integral

  $F(x,y) = e^{xy} + x^2 + f_1(y)$

- Take derivative of $F(x,y)$ with respect to y

  $N(x,y) = \frac{\partial}{\partial y}F(x,y)$

- Compute derivative

  $x\,e^{xy} - 2y = x\,e^{xy} + \frac{d}{dy}f_1(y)$

- Isolate for $\frac{d}{dy}f_1(y)$

  $\frac{d}{dy}f_1(y) = -2y$

- Solve for $f_1(y)$

  $f_1(y) = -y^2$

- Substitute $f_1(y)$ into equation for $F(x,y)$

$$F(x,y) = e^{xy} + x^2 - y^2$$

- Substitute $F(x,y)$ into the solution of the ODE

$$e^{xy} + x^2 - y^2 = c_1$$

- Solve for y

$$y = RootOf\left(-e^{x\_Z} - x^2 + \_Z^2 + c_1\right)$$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
trying homogeneous types:
trying Chini
differential order: 1; looking for linear symmetries
trying exact
<- exact successful`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 19

```
dsolve((y(x)*exp(x*y(x))+2*x)+(x*exp(x*y(x))-2*y(x))*diff(y(x),x)=0,y(x), singsol=all)
```

$$e^{xy(x)} + x^2 - y(x)^2 + c_1 = 0$$

✓ Solution by Mathematica

Time used: 0.268 (sec). Leaf size: 22

```
DSolve[(y[x]*Exp[x*y[x]]+2*x)+(x*Exp[x*y[x]]-2*y[x])*y'[x]==0,y[x],x,IncludeSingularSolution
```

$$\text{Solve}\left[x^2 + e^{xy(x)} - y(x)^2 = c_1, y(x)\right]$$

3.5 problem 5

Internal problem ID [4982]
Internal file name [`OUTPUT/4475_Sunday_June_05_2022_02_57_25_PM_28284387/index.tex`]

Book: Fundamentals of Differential Equations. By Nagle, Saff and Snider. 9th edition. Boston. Pearson 2018.
Section: Chapter 2, First order differential equations. Section 2.4, Exact equations. Exercises. page 64
Problem number: 5.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"exact"**

Maple gives the following as the ode type

`[_separable]`

$$\boxed{y' + xy = 0}$$

3.5.1 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x, y) + N(x, y)\frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x, y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x, y) = 0$$

Hence

$$\frac{\partial\phi}{\partial x} + \frac{\partial\phi}{\partial y}\frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$\frac{\partial \phi}{\partial x} = M$$
$$\frac{\partial \phi}{\partial y} = N$$

But since $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x, y)$ but at least we know now that we can do that since the condition $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x, y)\,\mathrm{d}x + N(x, y)\,\mathrm{d}y = 0 \tag{1A}$$

Therefore

$$\left(-\frac{1}{y}\right) \mathrm{d}y = (x)\,\mathrm{d}x$$
$$(-x)\,\mathrm{d}x + \left(-\frac{1}{y}\right) \mathrm{d}y = 0 \tag{2A}$$

Comparing (1A) and (2A) shows that

$$M(x, y) = -x$$
$$N(x, y) = -\frac{1}{y}$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Using result found above gives

$$\begin{aligned}\frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}(-x) \\ &= 0\end{aligned}$$

And

$$
\begin{aligned}
\frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}\left(-\frac{1}{y}\right) \\
&= 0
\end{aligned}
$$

Since $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$, then the ODE is <u>exact</u> The following equations are now set up to solve for the function $\phi(x, y)$

$$
\frac{\partial \phi}{\partial x} = M \tag{1}
$$

$$
\frac{\partial \phi}{\partial y} = N \tag{2}
$$

Integrating (1) w.r.t. x gives

$$
\begin{aligned}
\int \frac{\partial \phi}{\partial x} \,\mathrm{d}x &= \int M \,\mathrm{d}x \\
\int \frac{\partial \phi}{\partial x} \,\mathrm{d}x &= \int -x \,\mathrm{d}x
\end{aligned}
$$

$$
\phi = -\frac{x^2}{2} + f(y) \tag{3}
$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$
\frac{\partial \phi}{\partial y} = 0 + f'(y) \tag{4}
$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = -\frac{1}{y}$. Therefore equation (4) becomes

$$
-\frac{1}{y} = 0 + f'(y) \tag{5}
$$

Solving equation (5) for $f'(y)$ gives

$$
f'(y) = -\frac{1}{y}
$$

Integrating the above w.r.t y gives

$$
\begin{aligned}
\int f'(y) \,\mathrm{d}y &= \int \left(-\frac{1}{y}\right) \mathrm{d}y \\
f(y) &= -\ln(y) + c_1
\end{aligned}
$$

Where c_1 is constant of integration. Substituting result found above for $f(y)$ into equation (3) gives ϕ

$$\phi = -\frac{x^2}{2} - \ln(y) + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = -\frac{x^2}{2} - \ln(y)$$

The solution becomes

$$y = e^{-\frac{x^2}{2}-c_1}$$

Summary
The solution(s) found are the following

$$y = e^{-\frac{x^2}{2}-c_1} \tag{1}$$

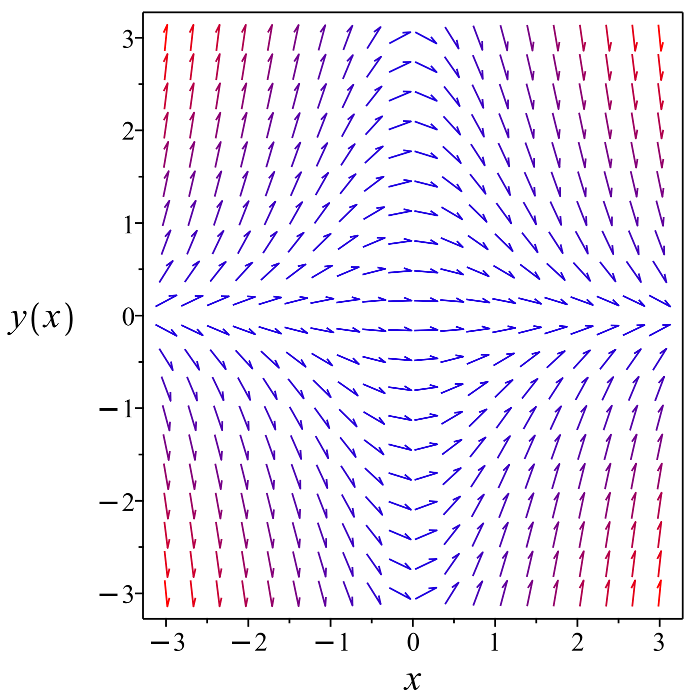


Figure 176: Slope field plot

Verification of solutions

$$y = \mathrm{e}^{-\frac{x^2}{2}-c_1}$$

Verified OK.

3.5.2 Maple step by step solution

Let's solve

$y' + xy = 0$

- Highest derivative means the order of the ODE is 1

  y'

- Separate variables

  $\frac{y'}{y} = -x$

- Integrate both sides with respect to x

  $\int \frac{y'}{y}dx = \int -xdx + c_1$

- Evaluate integral

  $\ln(y) = -\frac{x^2}{2} + c_1$

- Solve for y

  $y = \mathrm{e}^{-\frac{x^2}{2}+c_1}$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
<- 1st order linear successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 12

```
dsolve(x*y(x)+diff(y(x),x)=0,y(x), singsol=all)
```

$$y(x) = c_1\mathrm{e}^{-\frac{x^2}{2}}$$

✓ Solution by Mathematica

Time used: 0.024 (sec). Leaf size: 22

```
DSolve[x*y[x]+y'[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to c_1 e^{-\frac{x^2}{2}}$$
$$y(x) \to 0$$

3.6 problem 6

Internal problem ID [4983]
Internal file name [`OUTPUT/4476_Sunday_June_05_2022_02_57_26_PM_45094693/index.tex`]

Book: Fundamentals of Differential Equations. By Nagle, Saff and Snider. 9th edition. Boston. Pearson 2018.
Section: Chapter 2, First order differential equations. Section 2.4, Exact equations. Exercises. page 64
Problem number: 6.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"exact"**

Maple gives the following as the ode type

```
[_exact, [_1st_order, `_with_symmetry_[F(x)*G(y),0]`]]
```

$$\boxed{y^2 + (2xy + \cos(y))\, y' = 0}$$

3.6.1 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x,y) + N(x,y)\frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x,y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y) = 0$$

Hence

$$\frac{\partial\phi}{\partial x} + \frac{\partial\phi}{\partial y}\frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$\frac{\partial \phi}{\partial x} = M$$
$$\frac{\partial \phi}{\partial y} = N$$

But since $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x, y)$ but at least we know now that we can do that since the condition $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x, y)\, \mathrm{d}x + N(x, y)\, \mathrm{d}y = 0 \tag{1A}$$

Therefore

$$\begin{aligned}(2xy + \cos(y))\, \mathrm{d}y &= \left(-y^2\right) \mathrm{d}x \\ \left(y^2\right) \mathrm{d}x + (2xy + \cos(y))\, \mathrm{d}y &= 0 \qquad \text{(2A)}\end{aligned}$$

Comparing (1A) and (2A) shows that

$$\begin{aligned}M(x, y) &= y^2 \\ N(x, y) &= 2xy + \cos(y)\end{aligned}$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Using result found above gives

$$\begin{aligned}\frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}\left(y^2\right) \\ &= 2y\end{aligned}$$

And

$$\begin{aligned}\frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}(2xy + \cos(y)) \\ &= 2y\end{aligned}$$

Since $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$, then the ODE is exact The following equations are now set up to solve for the function $\phi(x,y)$

$$\frac{\partial \phi}{\partial x} = M \tag{1}$$
$$\frac{\partial \phi}{\partial y} = N \tag{2}$$

Integrating (1) w.r.t. x gives

$$\int \frac{\partial \phi}{\partial x} \, \mathrm{d}x = \int M \, \mathrm{d}x$$
$$\int \frac{\partial \phi}{\partial x} \, \mathrm{d}x = \int y^2 \, \mathrm{d}x$$
$$\phi = y^2 x + f(y) \tag{3}$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$\frac{\partial \phi}{\partial y} = 2xy + f'(y) \tag{4}$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = 2xy + \cos(y)$. Therefore equation (4) becomes

$$2xy + \cos(y) = 2xy + f'(y) \tag{5}$$

Solving equation (5) for $f'(y)$ gives

$$f'(y) = \cos(y)$$

Integrating the above w.r.t y gives

$$\int f'(y) \, \mathrm{d}y = \int (\cos(y)) \, \mathrm{d}y$$
$$f(y) = \sin(y) + c_1$$

Where c_1 is constant of integration. Substituting result found above for $f(y)$ into equation (3) gives ϕ

$$\phi = y^2 x + \sin(y) + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = y^2 x + \sin(y)$$

Summary

The solution(s) found are the following

$$xy^2 + \sin(y) = c_1 \tag{1}$$

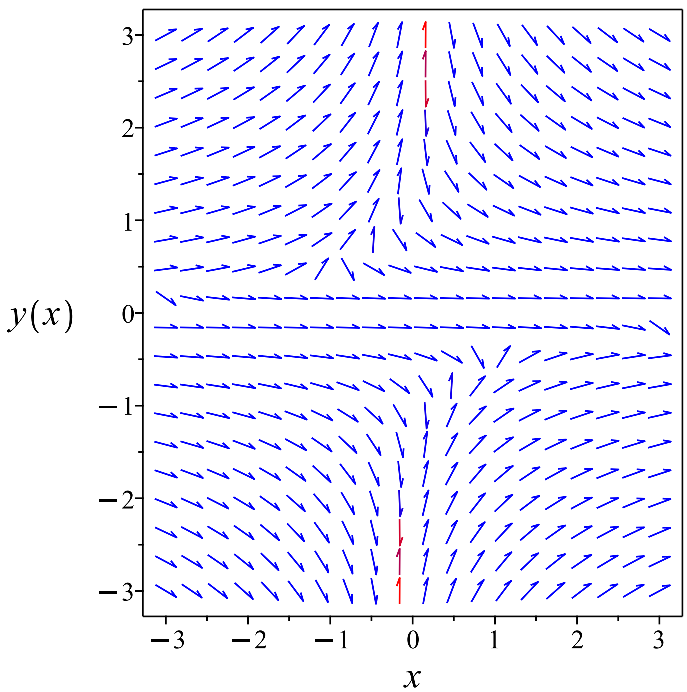


Figure 177: Slope field plot

Verification of solutions

$$xy^2 + \sin(y) = c_1$$

Verified OK.

3.6.2 Maple step by step solution

Let's solve

$y^2 + (2xy + \cos(y))\, y' = 0$

- Highest derivative means the order of the ODE is 1

  y'

□ Check if ODE is exact

  ◦ ODE is exact if the lhs is the total derivative of a C^2 function

    $F'(x, y) = 0$

  ◦ Compute derivative of lhs

    $F'(x, y) + \left(\frac{\partial}{\partial y} F(x, y)\right) y' = 0$

  ◦ Evaluate derivatives

    $2y = 2y$

  ◦ Condition met, ODE is exact

- Exact ODE implies solution will be of this form

  $\left[F(x, y) = c_1, M(x, y) = F'(x, y), N(x, y) = \frac{\partial}{\partial y} F(x, y)\right]$

- Solve for $F(x, y)$ by integrating $M(x, y)$ with respect to x

  $F(x, y) = \int y^2 dx + f_1(y)$

- Evaluate integral

  $F(x, y) = y^2 x + f_1(y)$

- Take derivative of $F(x, y)$ with respect to y

  $N(x, y) = \frac{\partial}{\partial y} F(x, y)$

- Compute derivative

  $2xy + \cos(y) = 2xy + \frac{d}{dy} f_1(y)$

- Isolate for $\frac{d}{dy} f_1(y)$

  $\frac{d}{dy} f_1(y) = \cos(y)$

- Solve for $f_1(y)$

  $f_1(y) = \sin(y)$

- Substitute $f_1(y)$ into equation for $F(x, y)$

$$F(x,y) = y^2x + \sin(y)$$

- Substitute $F(x,y)$ into the solution of the ODE

$$y^2x + \sin(y) = c_1$$

- Solve for y

$$y = RootOf\left(-\_Z^2x + c_1 - \sin(\_Z)\right)$$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
<- 1st order linear successful
<- inverse linear successful`
```

✓ Solution by Maple

Time used: 0.031 (sec). Leaf size: 17

```
dsolve(y(x)^2+(2*x*y(x)+cos(y(x)))*diff(y(x),x)=0,y(x), singsol=all)
```

$$x + \frac{\sin(y(x)) - c_1}{y(x)^2} = 0$$

✓ Solution by Mathematica

Time used: 0.151 (sec). Leaf size: 22

```
DSolve[y[x]^2+(2*x*y[x]+Cos[y[x]])*y'[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$\text{Solve}\left[x = -\frac{\sin(y(x))}{y(x)^2} + \frac{c_1}{y(x)^2}, y(x)\right]$$

3.7 problem 7

Internal problem ID [4984]
Internal file name [`OUTPUT/4477_Sunday_June_05_2022_02_57_27_PM_38929072/index.tex`]

Book: Fundamentals of Differential Equations. By Nagle, Saff and Snider. 9th edition. Boston. Pearson 2018.
Section: Chapter 2, First order differential equations. Section 2.4, Exact equations. Exercises. page 64
Problem number: 7.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"exact"**

Maple gives the following as the ode type

`[_exact]`

$$\boxed{y \cos(xy) + (x \cos(xy) - 2y) y' = -2x}$$

3.7.1 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x,y) + N(x,y)\frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x,y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y) = 0$$

Hence

$$\frac{\partial \phi}{\partial x} + \frac{\partial \phi}{\partial y}\frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$\frac{\partial \phi}{\partial x} = M$$
$$\frac{\partial \phi}{\partial y} = N$$

But since $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x, y)$ but at least we know now that we can do that since the condition $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x, y)\, \mathrm{d}x + N(x, y)\, \mathrm{d}y = 0 \tag{1A}$$

Therefore

$$\begin{aligned}
(x \cos(xy) - 2y)\, \mathrm{d}y &= (-2x - y \cos(xy))\, \mathrm{d}x \\
(2x + y \cos(xy))\, \mathrm{d}x + (x \cos(xy) - 2y)\, \mathrm{d}y &= 0 \qquad (2A)
\end{aligned}$$

Comparing (1A) and (2A) shows that

$$\begin{aligned}
M(x, y) &= 2x + y \cos(xy) \\
N(x, y) &= x \cos(xy) - 2y
\end{aligned}$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Using result found above gives

$$\begin{aligned}
\frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}(2x + y \cos(xy)) \\
&= \cos(xy) - xy \sin(xy)
\end{aligned}$$

And

$$\begin{aligned}
\frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}(x \cos(xy) - 2y) \\
&= \cos(xy) - xy \sin(xy)
\end{aligned}$$

Since $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$, then the ODE is exact The following equations are now set up to solve for the function $\phi(x, y)$

$$\frac{\partial \phi}{\partial x} = M \tag{1}$$

$$\frac{\partial \phi}{\partial y} = N \tag{2}$$

Integrating (1) w.r.t. x gives

$$\int \frac{\partial \phi}{\partial x} \, \mathrm{d}x = \int M \, \mathrm{d}x$$

$$\int \frac{\partial \phi}{\partial x} \, \mathrm{d}x = \int 2x + y \cos(xy) \, \mathrm{d}x$$

$$\phi = \sin(xy) + x^2 + f(y) \tag{3}$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$\frac{\partial \phi}{\partial y} = x \cos(xy) + f'(y) \tag{4}$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = x \cos(xy) - 2y$. Therefore equation (4) becomes

$$x \cos(xy) - 2y = x \cos(xy) + f'(y) \tag{5}$$

Solving equation (5) for $f'(y)$ gives

$$f'(y) = -2y$$

Integrating the above w.r.t y gives

$$\int f'(y) \, \mathrm{d}y = \int (-2y) \, \mathrm{d}y$$

$$f(y) = -y^2 + c_1$$

Where c_1 is constant of integration. Substituting result found above for $f(y)$ into equation (3) gives ϕ

$$\phi = \sin(xy) + x^2 - y^2 + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = \sin(xy) + x^2 - y^2$$

Summary
The solution(s) found are the following

$$\sin(xy) + x^2 - y^2 = c_1 \tag{1}$$

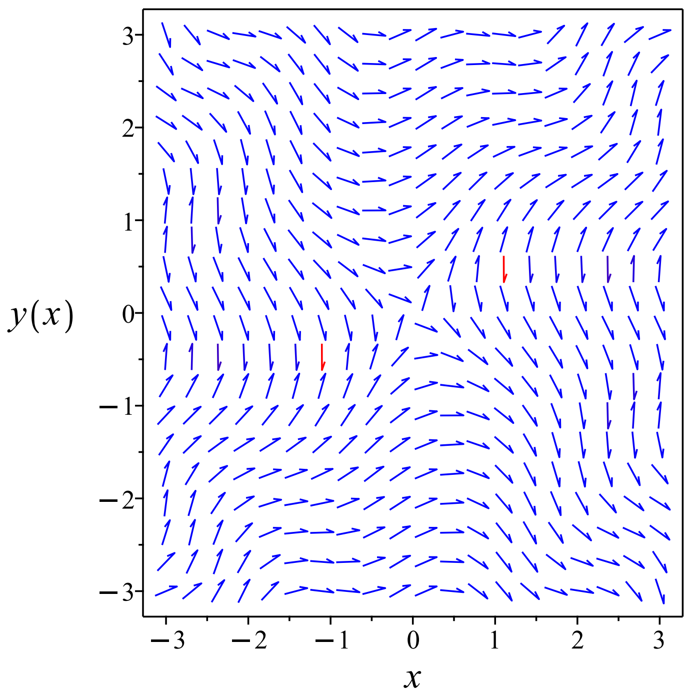


Figure 178: Slope field plot

Verification of solutions

$$\sin(xy) + x^2 - y^2 = c_1$$

Verified OK.

3.7.2 Maple step by step solution

Let's solve

$y\cos(xy) + (x\cos(xy) - 2y)\,y' = -2x$

- Highest derivative means the order of the ODE is 1

  y'

□ Check if ODE is exact

  - ODE is exact if the lhs is the total derivative of a C^2 function

    $F'(x,y) = 0$

  - Compute derivative of lhs

    $F'(x,y) + \left(\frac{\partial}{\partial y}F(x,y)\right)y' = 0$

  - Evaluate derivatives

    $\cos(xy) - xy\sin(xy) = \cos(xy) - xy\sin(xy)$

  - Condition met, ODE is exact

- Exact ODE implies solution will be of this form

  $\left[F(x,y) = c_1, M(x,y) = F'(x,y), N(x,y) = \frac{\partial}{\partial y}F(x,y)\right]$

- Solve for $F(x,y)$ by integrating $M(x,y)$ with respect to x

  $F(x,y) = \int (2x + y\cos(xy))\,dx + f_1(y)$

- Evaluate integral

  $F(x,y) = \sin(xy) + x^2 + f_1(y)$

- Take derivative of $F(x,y)$ with respect to y

  $N(x,y) = \frac{\partial}{\partial y}F(x,y)$

- Compute derivative

  $x\cos(xy) - 2y = x\cos(xy) + \frac{d}{dy}f_1(y)$

- Isolate for $\frac{d}{dy}f_1(y)$

  $\frac{d}{dy}f_1(y) = -2y$

- Solve for $f_1(y)$

  $f_1(y) = -y^2$

- Substitute $f_1(y)$ into equation for $F(x,y)$

$$F(x,y) = \sin(xy) + x^2 - y^2$$

- Substitute $F(x,y)$ into the solution of the ODE

$$\sin(xy) + x^2 - y^2 = c_1$$

- Solve for y

$$y = \frac{RootOf\left(-x^4 - \sin\left(\_Z\right)x^2 + c_1 x^2 + \_Z^2\right)}{x}$$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
trying homogeneous types:
trying Chini
differential order: 1; looking for linear symmetries
trying exact
<- exact successful`
```

✓ Solution by Maple

Time used: 0.032 (sec). Leaf size: 29

```
dsolve((2*x+y(x)*cos(x*y(x)))+(x*cos(x*y(x))-2*y(x))*diff(y(x),x)=0,y(x), singsol=all)
```

$$y(x) = \frac{\operatorname{RootOf}\left(x^4 + \sin\left(\_Z\right)x^2 + c_1 x^2 - \_Z^2\right)}{x}$$

✓ Solution by Mathematica

Time used: 0.186 (sec). Leaf size: 21

```
DSolve[(2*x+y[x]*Cos[x*y[x]])+(x*Cos[x*y[x]]-2*y[x])*y'[x]==0,y[x],x,IncludeSingularSolution
```

$$\text{Solve}\left[x^2 - y(x)^2 + \sin(xy(x)) = c_1, y(x)\right]$$

3.8 problem 8

Internal problem ID [4985]
Internal file name [`OUTPUT/4478_Sunday_June_05_2022_02_57_28_PM_96616221/index.tex`]

Book: Fundamentals of Differential Equations. By Nagle, Saff and Snider. 9th edition. Boston. Pearson 2018.
Section: Chapter 2, First order differential equations. Section 2.4, Exact equations. Exercises. page 64
Problem number: 8.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"exactWithIntegrationFactor"**

Maple gives the following as the ode type

`[_linear]`

$$\boxed{\theta r' + 3r = \theta + 1}$$

3.8.1 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x,y) + N(x,y)\,\frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x,y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y) = 0$$

Hence

$$\frac{\partial \phi}{\partial x} + \frac{\partial \phi}{\partial y}\frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$\begin{aligned}\frac{\partial \phi}{\partial x} &= M \\ \frac{\partial \phi}{\partial y} &= N\end{aligned}$$

But since $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x, y)$ but at least we know now that we can do that since the condition $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(\theta, r)\,\mathrm{d}\theta + N(\theta, r)\,\mathrm{d}r = 0 \tag{1A}$$

Therefore

$$\begin{aligned}(\theta)\,\mathrm{d}r &= (-3r + \theta + 1)\,\mathrm{d}\theta \\ (3r - \theta - 1)\,\mathrm{d}\theta + (\theta)\,\mathrm{d}r &= 0 \qquad (2A)\end{aligned}$$

Comparing (1A) and (2A) shows that

$$\begin{aligned}M(\theta, r) &= 3r - \theta - 1 \\ N(\theta, r) &= \theta\end{aligned}$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial r} = \frac{\partial N}{\partial \theta}$$

Using result found above gives

$$\begin{aligned}\frac{\partial M}{\partial r} &= \frac{\partial}{\partial r}(3r - \theta - 1) \\ &= 3\end{aligned}$$

And

$$\begin{aligned}\frac{\partial N}{\partial \theta} &= \frac{\partial}{\partial \theta}(\theta) \\ &= 1\end{aligned}$$

Since $\frac{\partial M}{\partial r} \neq \frac{\partial N}{\partial \theta}$, then the ODE is not exact. Since the ODE is not exact, we will try to find an integrating factor to make it exact. Let

$$\begin{aligned} A &= \frac{1}{N}\left(\frac{\partial M}{\partial r} - \frac{\partial N}{\partial \theta}\right) \\ &= \frac{1}{\theta}((3) - (1)) \\ &= \frac{2}{\theta} \end{aligned}$$

Since A does not depend on r, then it can be used to find an integrating factor. The integrating factor μ is

$$\begin{aligned} \mu &= e^{\int A \, \mathrm{d}\theta} \\ &= e^{\int \frac{2}{\theta} \, \mathrm{d}\theta} \end{aligned}$$

The result of integrating gives

$$\begin{aligned} \mu &= e^{2\ln(\theta)} \\ &= \theta^2 \end{aligned}$$

M and N are multiplied by this integrating factor, giving new M and new N which are called $\overline{M}$ and $\overline{N}$ for now so not to confuse them with the original M and N.

$$\begin{aligned} \overline{M} &= \mu M \\ &= \theta^2(3r - \theta - 1) \\ &= \theta^2(3r - \theta - 1) \end{aligned}$$

And

$$\begin{aligned} \overline{N} &= \mu N \\ &= \theta^2(\theta) \\ &= \theta^3 \end{aligned}$$

Now a modified ODE is ontained from the original ODE, which is exact and can be solved. The modified ODE is

$$\begin{aligned} \overline{M} + \overline{N}\frac{\mathrm{d}r}{\mathrm{d}\theta} &= 0 \\ \left(\theta^2(3r - \theta - 1)\right) + \left(\theta^3\right)\frac{\mathrm{d}r}{\mathrm{d}\theta} &= 0 \end{aligned}$$

The following equations are now set up to solve for the function $\phi(\theta, r)$

$$\frac{\partial \phi}{\partial \theta} = \overline{M} \tag{1}$$

$$\frac{\partial \phi}{\partial r} = \overline{N} \tag{2}$$

Integrating (1) w.r.t. θ gives

$$\int \frac{\partial \phi}{\partial \theta} \,\mathrm{d}\theta = \int \overline{M} \,\mathrm{d}\theta$$

$$\int \frac{\partial \phi}{\partial \theta} \,\mathrm{d}\theta = \int \theta^2(3r - \theta - 1) \,\mathrm{d}\theta$$

$$\phi = \frac{\theta^3(-3\theta + 12r - 4)}{12} + f(r) \tag{3}$$

Where $f(r)$ is used for the constant of integration since ϕ is a function of both θ and r. Taking derivative of equation (3) w.r.t r gives

$$\frac{\partial \phi}{\partial r} = \theta^3 + f'(r) \tag{4}$$

But equation (2) says that $\frac{\partial \phi}{\partial r} = \theta^3$. Therefore equation (4) becomes

$$\theta^3 = \theta^3 + f'(r) \tag{5}$$

Solving equation (5) for $f'(r)$ gives

$$f'(r) = 0$$

Therefore

$$f(r) = c_1$$

Where c_1 is constant of integration. Substituting this result for $f(r)$ into equation (3) gives ϕ

$$\phi = \frac{\theta^3(-3\theta + 12r - 4)}{12} + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = \frac{\theta^3(-3\theta + 12r - 4)}{12}$$

The solution becomes

$$r = \frac{3\theta^4 + 4\theta^3 + 12c_1}{12\theta^3}$$

Summary
The solution(s) found are the following

$$r = \frac{3\theta^4 + 4\theta^3 + 12c_1}{12\theta^3} \tag{1}$$

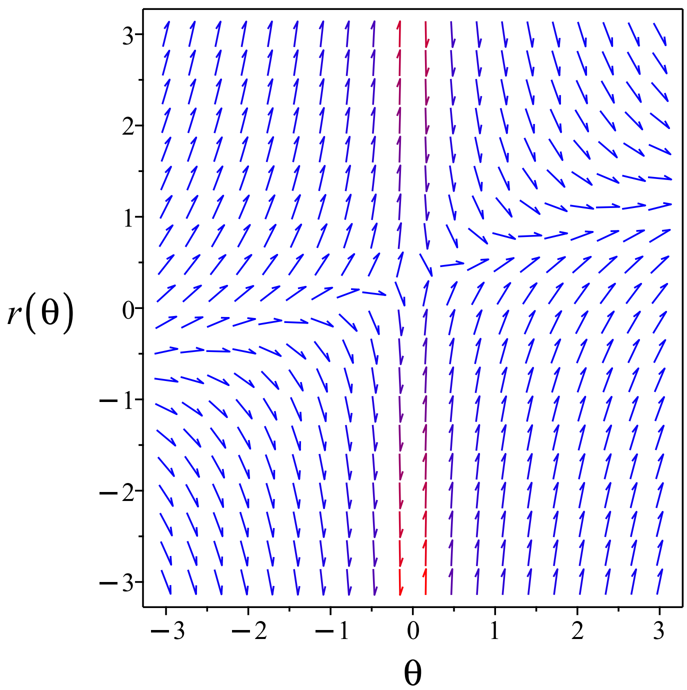


Figure 179: Slope field plot

Verification of solutions

$$r = \frac{3\theta^4 + 4\theta^3 + 12c_1}{12\theta^3}$$

Verified OK.

3.8.2 Maple step by step solution

Let's solve

$\theta r' + 3r = \theta + 1$

- Highest derivative means the order of the ODE is 1

  r'

- Isolate the derivative

  $r' = -\frac{3r}{\theta} + \frac{\theta+1}{\theta}$

- Group terms with r on the lhs of the ODE and the rest on the rhs of the ODE

  $r' + \frac{3r}{\theta} = \frac{\theta+1}{\theta}$

- The ODE is linear; multiply by an integrating factor $\mu(\theta)$

  $\mu(\theta)\left(r' + \frac{3r}{\theta}\right) = \frac{\mu(\theta)(\theta+1)}{\theta}$

- Assume the lhs of the ODE is the total derivative $\frac{d}{d\theta}(\mu(\theta)\,r)$

  $\mu(\theta)\left(r' + \frac{3r}{\theta}\right) = \mu'(\theta)\,r + \mu(\theta)\,r'$

- Isolate $\mu'(\theta)$

  $\mu'(\theta) = \frac{3\mu(\theta)}{\theta}$

- Solve to find the integrating factor

  $\mu(\theta) = \theta^3$

- Integrate both sides with respect to θ

  $\int \left(\frac{d}{d\theta}(\mu(\theta)\,r)\right) d\theta = \int \frac{\mu(\theta)(\theta+1)}{\theta} d\theta + c_1$

- Evaluate the integral on the lhs

  $\mu(\theta)\,r = \int \frac{\mu(\theta)(\theta+1)}{\theta} d\theta + c_1$

- Solve for r

  $r = \frac{\int \frac{\mu(\theta)(\theta+1)}{\theta} d\theta + c_1}{\mu(\theta)}$

- Substitute $\mu(\theta) = \theta^3$

  $r = \frac{\int \theta^2(\theta+1)d\theta + c_1}{\theta^3}$

- Evaluate the integrals on the rhs

  $r = \frac{c_1 + \frac{1}{4}\theta^4 + \frac{1}{3}\theta^3}{\theta^3}$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
<- 1st order linear successful`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 14

```
dsolve(theta*diff(r(theta),theta)+(3*r(theta)-theta-1)=0,r(theta), singsol=all)
```

$$r(\theta) = \frac{\theta}{4} + \frac{1}{3} + \frac{c_1}{\theta^3}$$

✓ Solution by Mathematica

Time used: 0.027 (sec). Leaf size: 20

```
DSolve[\[Theta]*r'[\[Theta]]+(3*r[\[Theta]]-\[Theta]-1)==0,r[\[Theta]],\[Theta],IncludeSingu
```

$$r(\theta) \to \frac{c_1}{\theta^3} + \frac{\theta}{4} + \frac{1}{3}$$

3.9 problem 9

Internal problem ID [4986]
Internal file name [`OUTPUT/4479_Sunday_June_05_2022_02_57_29_PM_94944544/index.tex`]

Book: Fundamentals of Differential Equations. By Nagle, Saff and Snider. 9th edition. Boston. Pearson 2018.
Section: Chapter 2, First order differential equations. Section 2.4, Exact equations. Exercises. page 64
Problem number: 9.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"exact"**

Maple gives the following as the ode type

`[_linear]`

$$\boxed{2xy + \left(x^2 - 1\right) y' = -3}$$

3.9.1 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x, y) + N(x, y)\frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x, y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x, y) = 0$$

Hence

$$\frac{\partial\phi}{\partial x} + \frac{\partial\phi}{\partial y}\frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$\frac{\partial \phi}{\partial x} = M$$
$$\frac{\partial \phi}{\partial y} = N$$

But since $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x,y)$ but at least we know now that we can do that since the condition $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x,y)\,\mathrm{d}x + N(x,y)\,\mathrm{d}y = 0 \tag{1A}$$

Therefore

$$\begin{aligned}(x^2 - 1)\,\mathrm{d}y &= (-2xy - 3)\,\mathrm{d}x \\ (2xy + 3)\,\mathrm{d}x + (x^2 - 1)\,\mathrm{d}y &= 0 \end{aligned} \tag{2A}$$

Comparing (1A) and (2A) shows that

$$M(x,y) = 2xy + 3$$
$$N(x,y) = x^2 - 1$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Using result found above gives

$$\begin{aligned}\frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}(2xy + 3) \\ &= 2x\end{aligned}$$

And

$$\begin{aligned}\frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}(x^2 - 1) \\ &= 2x\end{aligned}$$

Since $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$, then the ODE is exact The following equations are now set up to solve for the function $\phi(x, y)$

$$\frac{\partial \phi}{\partial x} = M \tag{1}$$

$$\frac{\partial \phi}{\partial y} = N \tag{2}$$

Integrating (1) w.r.t. x gives

$$\int \frac{\partial \phi}{\partial x} \, \mathrm{d}x = \int M \, \mathrm{d}x$$

$$\int \frac{\partial \phi}{\partial x} \, \mathrm{d}x = \int 2xy + 3 \, \mathrm{d}x$$

$$\phi = y\,x^2 + 3x + f(y) \tag{3}$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$\frac{\partial \phi}{\partial y} = x^2 + f'(y) \tag{4}$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = x^2 - 1$. Therefore equation (4) becomes

$$x^2 - 1 = x^2 + f'(y) \tag{5}$$

Solving equation (5) for $f'(y)$ gives

$$f'(y) = -1$$

Integrating the above w.r.t y gives

$$\int f'(y) \, \mathrm{d}y = \int (-1) \, \mathrm{d}y$$

$$f(y) = -y + c_1$$

Where c_1 is constant of integration. Substituting result found above for $f(y)$ into equation (3) gives ϕ

$$\phi = y\,x^2 + 3x - y + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = y\,x^2 + 3x - y$$

The solution becomes

$$y = \frac{c_1 - 3x}{x^2 - 1}$$

Summary

The solution(s) found are the following

$$y = \frac{c_1 - 3x}{x^2 - 1} \tag{1}$$

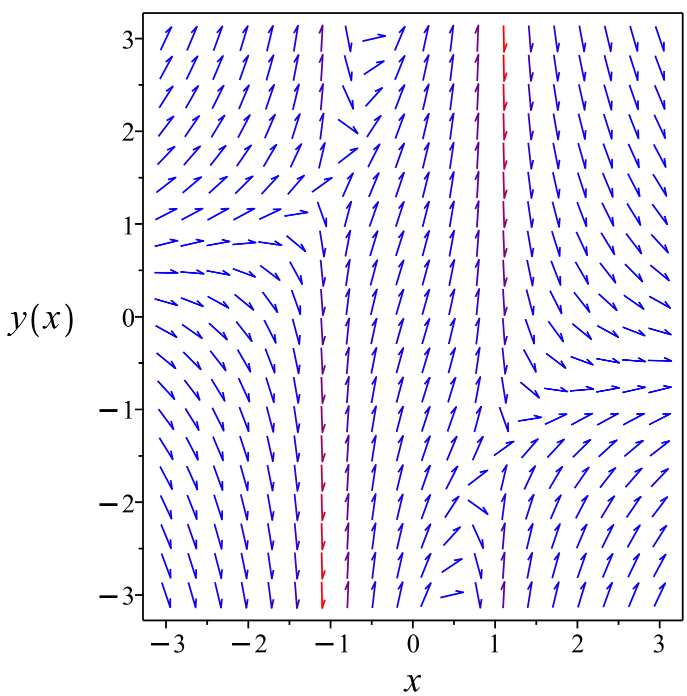


Figure 180: Slope field plot

Verification of solutions

$$y = \frac{c_1 - 3x}{x^2 - 1}$$

Verified OK.

3.9.2 Maple step by step solution

Let's solve

$2xy + (x^2 - 1)\, y' = -3$

- Highest derivative means the order of the ODE is 1

  y'

- Isolate the derivative

  $y' = -\frac{2xy}{x^2-1} - \frac{3}{x^2-1}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE

  $y' + \frac{2xy}{x^2-1} = -\frac{3}{x^2-1}$

- The ODE is linear; multiply by an integrating factor $\mu(x)$

  $\mu(x)\left(y' + \frac{2xy}{x^2-1}\right) = -\frac{3\mu(x)}{x^2-1}$

- Assume the lhs of the ODE is the total derivative $\frac{d}{dx}(\mu(x)\, y)$

  $\mu(x)\left(y' + \frac{2xy}{x^2-1}\right) = \mu'(x)\, y + \mu(x)\, y'$

- Isolate $\mu'(x)$

  $\mu'(x) = \frac{2\mu(x)x}{x^2-1}$

- Solve to find the integrating factor

  $\mu(x) = (x-1)\,(x+1)$

- Integrate both sides with respect to x

  $\int \left(\frac{d}{dx}(\mu(x)\, y)\right) dx = \int -\frac{3\mu(x)}{x^2-1} dx + c_1$

- Evaluate the integral on the lhs

  $\mu(x)\, y = \int -\frac{3\mu(x)}{x^2-1} dx + c_1$

- Solve for y

  $y = \frac{\int -\frac{3\mu(x)}{x^2-1} dx + c_1}{\mu(x)}$

- Substitute $\mu(x) = (x-1)\,(x+1)$

  $y = \frac{\int -\frac{3(x-1)(x+1)}{x^2-1} dx + c_1}{(x-1)(x+1)}$

- Evaluate the integrals on the rhs

  $y = \frac{c_1 - 3x}{(x-1)(x+1)}$

- Simplify

$$y = \frac{c_1 - 3x}{x^2 - 1}$$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
<- 1st order linear successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 17

```
dsolve((2*x*y(x)+3)+(x^2-1)*diff(y(x),x)=0,y(x), singsol=all)
```

$$y(x) = \frac{-3x + c_1}{x^2 - 1}$$

✓ Solution by Mathematica

Time used: 0.029 (sec). Leaf size: 19

```
DSolve[(2*x*y[x]+3)+(x^2-1)*y'[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \frac{-3x + c_1}{x^2 - 1}$$

3.10 problem 10

Internal problem ID [4987]
Internal file name [`OUTPUT/4480_Sunday_June_05_2022_02_57_30_PM_70723789/index.tex`]

Book: Fundamentals of Differential Equations. By Nagle, Saff and Snider. 9th edition. Boston. Pearson 2018.
Section: Chapter 2, First order differential equations. Section 2.4, Exact equations. Exercises. page 64
Problem number: 10.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"exact"**

Maple gives the following as the ode type

```
[[_homogeneous, `class A`], _exact, _rational, [_Abel, `2nd
   type`, `class A`]]
```

$$\boxed{y + (x - 2y)\, y' = -2x}$$

3.10.1 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x, y) + N(x, y)\,\frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x, y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x, y) = 0$$

Hence

$$\frac{\partial \phi}{\partial x} + \frac{\partial \phi}{\partial y}\frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$\frac{\partial \phi}{\partial x} = M$$
$$\frac{\partial \phi}{\partial y} = N$$

But since $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x,y)$ but at least we know now that we can do that since the condition $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x,y)\,\mathrm{d}x + N(x,y)\,\mathrm{d}y = 0 \tag{1A}$$

Therefore

$$(x-2y)\,\mathrm{d}y = (-y-2x)\,\mathrm{d}x$$
$$(y+2x)\,\mathrm{d}x + (x-2y)\,\mathrm{d}y = 0 \tag{2A}$$

Comparing (1A) and (2A) shows that

$$M(x,y) = y + 2x$$
$$N(x,y) = x - 2y$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Using result found above gives

$$\begin{aligned}\frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}(y+2x) \\ &= 1\end{aligned}$$

And

$$\begin{aligned}\frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}(x-2y) \\ &= 1\end{aligned}$$

Since $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$, then the ODE is exact The following equations are now set up to solve for the function $\phi(x, y)$

$$\frac{\partial \phi}{\partial x} = M \tag{1}$$
$$\frac{\partial \phi}{\partial y} = N \tag{2}$$

Integrating (1) w.r.t. x gives

$$\int \frac{\partial \phi}{\partial x} \, \mathrm{d}x = \int M \, \mathrm{d}x$$
$$\int \frac{\partial \phi}{\partial x} \, \mathrm{d}x = \int y + 2x \, \mathrm{d}x$$
$$\phi = x(x + y) + f(y) \tag{3}$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$\frac{\partial \phi}{\partial y} = x + f'(y) \tag{4}$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = x - 2y$. Therefore equation (4) becomes

$$x - 2y = x + f'(y) \tag{5}$$

Solving equation (5) for $f'(y)$ gives

$$f'(y) = -2y$$

Integrating the above w.r.t y gives

$$\int f'(y) \, \mathrm{d}y = \int (-2y) \, \mathrm{d}y$$
$$f(y) = -y^2 + c_1$$

Where c_1 is constant of integration. Substituting result found above for $f(y)$ into equation (3) gives ϕ

$$\phi = x(x + y) - y^2 + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = x(x+y) - y^2$$

Summary

The solution(s) found are the following

$$x(x+y) - y^2 = c_1 \tag{1}$$

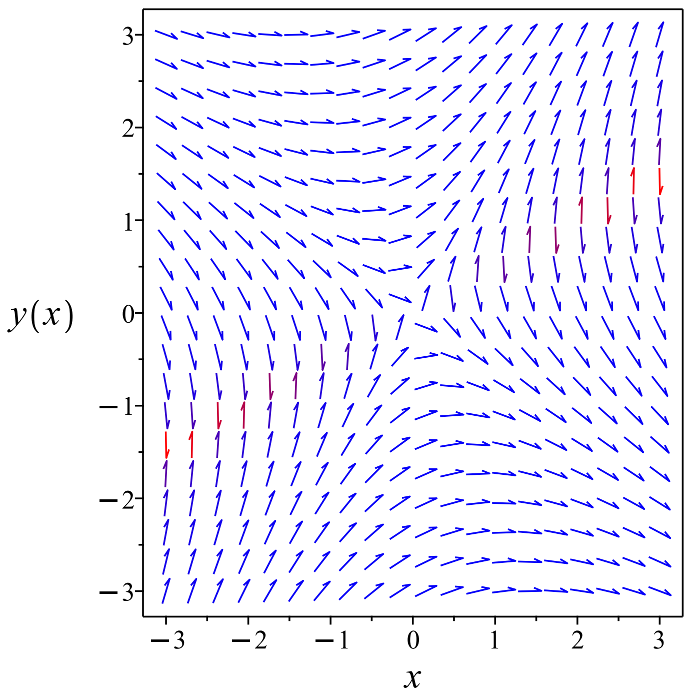


Figure 181: Slope field plot

Verification of solutions

$$x(x+y) - y^2 = c_1$$

Verified OK.

3.10.2 Maple step by step solution

Let's solve

$y + (x - 2y)\, y' = -2x$

- Highest derivative means the order of the ODE is 1

  y'

- ☐ Check if ODE is exact

  - ODE is exact if the lhs is the total derivative of a C^2 function

    $F'(x, y) = 0$

  - Compute derivative of lhs

    $F'(x, y) + \left(\frac{\partial}{\partial y} F(x, y)\right) y' = 0$

  - Evaluate derivatives

    $1 = 1$

  - Condition met, ODE is exact

- Exact ODE implies solution will be of this form

  $\left[F(x, y) = c_1, M(x, y) = F'(x, y), N(x, y) = \frac{\partial}{\partial y} F(x, y)\right]$

- Solve for $F(x, y)$ by integrating $M(x, y)$ with respect to x

  $F(x, y) = \int (y + 2x)\, dx + f_1(y)$

- Evaluate integral

  $F(x, y) = x^2 + xy + f_1(y)$

- Take derivative of $F(x, y)$ with respect to y

  $N(x, y) = \frac{\partial}{\partial y} F(x, y)$

- Compute derivative

  $x - 2y = x + \frac{d}{dy} f_1(y)$

- Isolate for $\frac{d}{dy} f_1(y)$

  $\frac{d}{dy} f_1(y) = -2y$

- Solve for $f_1(y)$

  $f_1(y) = -y^2$

- Substitute $f_1(y)$ into equation for $F(x, y)$

$$F(x,y) = x^2 + xy - y^2$$

- Substitute $F(x,y)$ into the solution of the ODE

$$x^2 + xy - y^2 = c_1$$

- Solve for y

$$\left\{y = \frac{x}{2} - \frac{\sqrt{5x^2-4c_1}}{2}, y = \frac{x}{2} + \frac{\sqrt{5x^2-4c_1}}{2}\right\}$$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
trying homogeneous types:
trying homogeneous D
<- homogeneous successful`
```

✓ Solution by Maple

Time used: 0.031 (sec). Leaf size: 51

```
dsolve((2*x+y(x))+(x-2*y(x))*diff(y(x),x)=0,y(x), singsol=all)
```

$$y(x) = \frac{c_1 x - \sqrt{5c_1^2x^2 + 4}}{2c_1}$$
$$y(x) = \frac{c_1 x + \sqrt{5c_1^2x^2 + 4}}{2c_1}$$

✓ Solution by Mathematica

Time used: 0.452 (sec). Leaf size: 102

```
DSolve[(2*x+y[x])+(x-2*y[x])*y'[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \frac{1}{2}\left(x - \sqrt{5x^2 - 4e^{c_1}}\right)$$
$$y(x) \to \frac{1}{2}\left(x + \sqrt{5x^2 - 4e^{c_1}}\right)$$
$$y(x) \to \frac{1}{2}\left(x - \sqrt{5}\sqrt{x^2}\right)$$
$$y(x) \to \frac{1}{2}\left(\sqrt{5}\sqrt{x^2} + x\right)$$

3.11 problem 11

Internal problem ID [4988]
Internal file name [`OUTPUT/4481_Sunday_June_05_2022_02_57_31_PM_54767562/index.tex`]

Book: Fundamentals of Differential Equations. By Nagle, Saff and Snider. 9th edition. Boston. Pearson 2018.
Section: Chapter 2, First order differential equations. Section 2.4, Exact equations. Exercises. page 64
Problem number: 11.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"exact"**

Maple gives the following as the ode type

`[_exact]`

$$\mathrm{e}^{x} \sin \left(y\right)+\left(\mathrm{e}^{x} \cos \left(y\right)+\frac{1}{3 y^{\frac{2}{3}}}\right) y^{\prime}=3 x^{2}$$

3.11.1 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x,y)+N(x,y)\frac{dy}{dx}=0 \tag{A}$$

We assume there exists a function $\phi(x,y)=c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y)=0$$

Hence

$$\frac{\partial\phi}{\partial x}+\frac{\partial\phi}{\partial y}\frac{dy}{dx}=0 \tag{B}$$

Comparing (A,B) shows that

$$\begin{aligned}\frac{\partial\phi}{\partial x} &= M\\ \frac{\partial\phi}{\partial y} &= N\end{aligned}$$

But since $\frac{\partial^2\phi}{\partial x\partial y} = \frac{\partial^2\phi}{\partial y\partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x, y)$ but at least we know now that we can do that since the condition $\frac{\partial^2\phi}{\partial x\partial y} = \frac{\partial^2\phi}{\partial y\partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x, y)\,\mathrm{d}x + N(x, y)\,\mathrm{d}y = 0 \tag{1A}$$

Therefore

$$\begin{aligned}\left(\mathrm{e}^x \cos(y) + \frac{1}{3y^{\frac{2}{3}}}\right)\mathrm{d}y &= \left(-\mathrm{e}^x \sin(y) + 3x^2\right)\mathrm{d}x\\ \left(\mathrm{e}^x \sin(y) - 3x^2\right)\mathrm{d}x + \left(\mathrm{e}^x \cos(y) + \frac{1}{3y^{\frac{2}{3}}}\right)\mathrm{d}y &= 0 \qquad (2A)\end{aligned}$$

Comparing (1A) and (2A) shows that

$$\begin{aligned}M(x, y) &= \mathrm{e}^x \sin(y) - 3x^2\\ N(x, y) &= \mathrm{e}^x \cos(y) + \frac{1}{3y^{\frac{2}{3}}}\end{aligned}$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Using result found above gives

$$\begin{aligned}\frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}\left(\mathrm{e}^x \sin(y) - 3x^2\right)\\ &= \mathrm{e}^x \cos(y)\end{aligned}$$

And

$$
\begin{aligned}
\frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}\left(\mathrm{e}^{x} \cos(y) + \frac{1}{3 y^{\frac{2}{3}}}\right) \\
&= \mathrm{e}^{x} \cos(y)
\end{aligned}
$$

Since $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$, then the ODE is exact The following equations are now set up to solve for the function $\phi(x,y)$

$$
\frac{\partial \phi}{\partial x} = M \tag{1}
$$

$$
\frac{\partial \phi}{\partial y} = N \tag{2}
$$

Integrating (1) w.r.t. x gives

$$
\begin{aligned}
\int \frac{\partial \phi}{\partial x} \mathrm{d}x &= \int M \mathrm{d}x \\
\int \frac{\partial \phi}{\partial x} \mathrm{d}x &= \int \mathrm{e}^{x} \sin(y) - 3x^{2} \mathrm{d}x
\end{aligned}
$$

$$
\phi = \mathrm{e}^{x} \sin(y) - x^{3} + f(y) \tag{3}
$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$
\frac{\partial \phi}{\partial y} = \mathrm{e}^{x} \cos(y) + f'(y) \tag{4}
$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = \mathrm{e}^{x} \cos(y) + \frac{1}{3 y^{\frac{2}{3}}}$. Therefore equation (4) becomes

$$
\mathrm{e}^{x} \cos(y) + \frac{1}{3 y^{\frac{2}{3}}} = \mathrm{e}^{x} \cos(y) + f'(y) \tag{5}
$$

Solving equation (5) for $f'(y)$ gives

$$
f'(y) = \frac{1}{3 y^{\frac{2}{3}}}
$$

Integrating the above w.r.t y gives

$$
\begin{aligned}
\int f'(y) \mathrm{d}y &= \int \left(\frac{1}{3 y^{\frac{2}{3}}}\right) \mathrm{d}y \\
f(y) &= y^{\frac{1}{3}} + c_{1}
\end{aligned}
$$

Where c_1 is constant of integration. Substituting result found above for $f(y)$ into equation (3) gives ϕ

$$\phi = e^x \sin(y) - x^3 + y^{\frac{1}{3}} + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = e^x \sin(y) - x^3 + y^{\frac{1}{3}}$$

Summary

The solution(s) found are the following

$$e^x \sin(y) - x^3 + y^{\frac{1}{3}} = c_1 \tag{1}$$

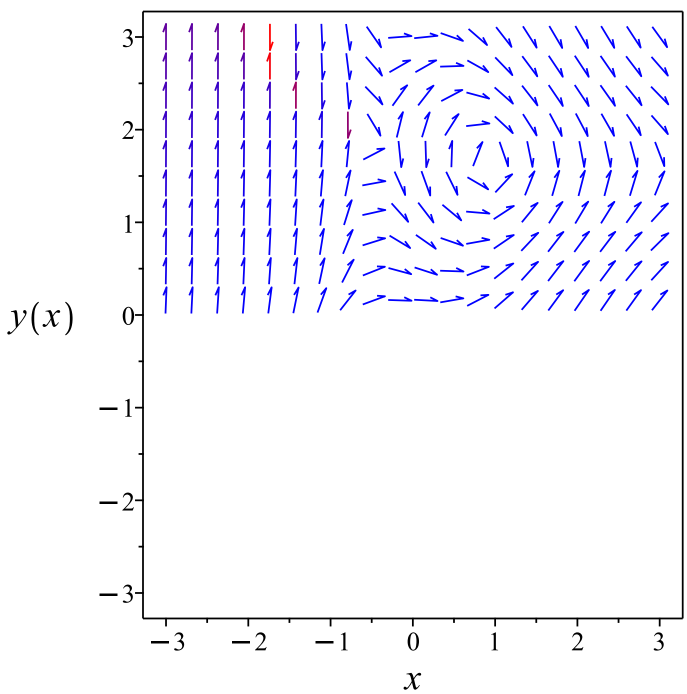


Figure 182: Slope field plot

Verification of solutions

$$e^x \sin(y) - x^3 + y^{\frac{1}{3}} = c_1$$

Verified OK.

3.11.2 Maple step by step solution

Let's solve

$e^x \sin(y) + \left(e^x \cos(y) + \frac{1}{3y^{\frac{2}{3}}}\right) y' = 3x^2$

- Highest derivative means the order of the ODE is 1

  y'

□ Check if ODE is exact

  ◦ ODE is exact if the lhs is the total derivative of a C^2 function

    $F'(x, y) = 0$

  ◦ Compute derivative of lhs

    $F'(x, y) + \left(\frac{\partial}{\partial y} F(x, y)\right) y' = 0$

  ◦ Evaluate derivatives

    $e^x \cos(y) = e^x \cos(y)$

  ◦ Condition met, ODE is exact

- Exact ODE implies solution will be of this form

  $\left[F(x, y) = c_1, M(x, y) = F'(x, y), N(x, y) = \frac{\partial}{\partial y} F(x, y)\right]$

- Solve for $F(x, y)$ by integrating $M(x, y)$ with respect to x

  $F(x, y) = \int (e^x \sin(y) - 3x^2)\, dx + f_1(y)$

- Evaluate integral

  $F(x, y) = e^x \sin(y) - x^3 + f_1(y)$

- Take derivative of $F(x, y)$ with respect to y

  $N(x, y) = \frac{\partial}{\partial y} F(x, y)$

- Compute derivative

  $e^x \cos(y) + \frac{1}{3y^{\frac{2}{3}}} = e^x \cos(y) + \frac{d}{dy} f_1(y)$

- Isolate for $\frac{d}{dy} f_1(y)$

  $\frac{d}{dy} f_1(y) = \frac{1}{3y^{\frac{2}{3}}}$

- Solve for $f_1(y)$

  $f_1(y) = y^{\frac{1}{3}}$

- Substitute $f_1(y)$ into equation for $F(x,y)$

  $F(x,y) = \mathrm{e}^x \sin(y) - x^3 + y^{\frac{1}{3}}$
- Substitute $F(x,y)$ into the solution of the ODE

  $\mathrm{e}^x \sin(y) - x^3 + y^{\frac{1}{3}} = c_1$
- Solve for y

  $y = RootOf\left(x^9 - 3\,\mathrm{e}^x \sin(\_Z)\, x^6 + 3(\mathrm{e}^x)^2 \sin(\_Z)^2\, x^3 + 3c_1 x^6 - (\mathrm{e}^x)^3 \sin(\_Z)^3 - 6\,\mathrm{e}^x \sin(\_Z)\right.$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
trying homogeneous types:
trying Chini
differential order: 1; looking for linear symmetries
trying exact`Warning: persistent store makes readlib obsolete |G:/public_html/my_notes/solvi
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 20

```
dsolve((exp(x)*sin(y(x))-3*x^2)+(exp(x)*cos(y(x))+y(x)^(-2/3)/3)*diff(y(x),x)=0,y(x), singso
```

$$\mathrm{e}^x \sin(y(x)) - x^3 + y(x)^{\frac{1}{3}} + c_1 = 0$$

✓ Solution by Mathematica

Time used: 0.416 (sec). Leaf size: 28

```
DSolve[(Exp[x]*Sin[y[x]]-3*x^2)+(Exp[x]*Cos[y[x]]+y[x]^(-2/3)/3)*y'[x]==0,y[x],x,IncludeSing
```

$$\text{Solve}\left[-3x^3 + 3\sqrt[3]{y(x)} + 3e^x \sin(y(x)) = c_1, y(x)\right]$$

3.12 problem 12

Internal problem ID [4989]
Internal file name [`OUTPUT/4482_Sunday_June_05_2022_02_57_34_PM_29878301/index.tex`]

Book: Fundamentals of Differential Equations. By Nagle, Saff and Snider. 9th edition. Boston. Pearson 2018.
Section: Chapter 2, First order differential equations. Section 2.4, Exact equations. Exercises. page 64
Problem number: 12.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"exact"**

Maple gives the following as the ode type

`[_exact]`

$$\boxed{\cos(x)\cos(y) - (\sin(x)\sin(y) + 2y)\, y' = -2x}$$

3.12.1 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x,y) + N(x,y)\frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x,y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y) = 0$$

Hence

$$\frac{\partial\phi}{\partial x} + \frac{\partial\phi}{\partial y}\frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$\frac{\partial \phi}{\partial x} = M$$
$$\frac{\partial \phi}{\partial y} = N$$

But since $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x, y)$ but at least we know now that we can do that since the condition $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x, y)\,\mathrm{d}x + N(x, y)\,\mathrm{d}y = 0 \tag{1A}$$

Therefore

$$(-\sin(x)\sin(y) - 2y)\,\mathrm{d}y = (-\cos(x)\cos(y) - 2x)\,\mathrm{d}x$$
$$(\cos(x)\cos(y) + 2x)\,\mathrm{d}x + (-\sin(x)\sin(y) - 2y)\,\mathrm{d}y = 0 \tag{2A}$$

Comparing (1A) and (2A) shows that

$$M(x, y) = \cos(x)\cos(y) + 2x$$
$$N(x, y) = -\sin(x)\sin(y) - 2y$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Using result found above gives

$$\begin{aligned}\frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}(\cos(x)\cos(y) + 2x) \\ &= -\cos(x)\sin(y)\end{aligned}$$

And

$$\begin{aligned}\frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}(-\sin(x)\sin(y) - 2y) \\ &= -\cos(x)\sin(y)\end{aligned}$$

Since $\frac{\partial M}{\partial y}=\frac{\partial N}{\partial x}$, then the ODE is exact The following equations are now set up to solve for the function $\phi(x,y)$

$$\frac{\partial \phi}{\partial x} = M \tag{1}$$
$$\frac{\partial \phi}{\partial y} = N \tag{2}$$

Integrating (1) w.r.t. x gives

$$\int \frac{\partial \phi}{\partial x} \, \mathrm{d}x = \int M \, \mathrm{d}x$$
$$\int \frac{\partial \phi}{\partial x} \, \mathrm{d}x = \int \cos(x)\cos(y) + 2x \, \mathrm{d}x$$
$$\phi = \sin(x)\cos(y) + x^2 + f(y) \tag{3}$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$\frac{\partial \phi}{\partial y} = -\sin(x)\sin(y) + f'(y) \tag{4}$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = -\sin(x)\sin(y) - 2y$. Therefore equation (4) becomes

$$-\sin(x)\sin(y) - 2y = -\sin(x)\sin(y) + f'(y) \tag{5}$$

Solving equation (5) for $f'(y)$ gives

$$f'(y) = -2y$$

Integrating the above w.r.t y gives

$$\int f'(y) \, \mathrm{d}y = \int (-2y) \, \mathrm{d}y$$
$$f(y) = -y^2 + c_1$$

Where c_1 is constant of integration. Substituting result found above for $f(y)$ into equation (3) gives ϕ

$$\phi = \sin(x)\cos(y) + x^2 - y^2 + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = \sin(x)\cos(y) + x^2 - y^2$$

Summary

The solution(s) found are the following

$$\sin(x)\cos(y) + x^2 - y^2 = c_1 \tag{1}$$

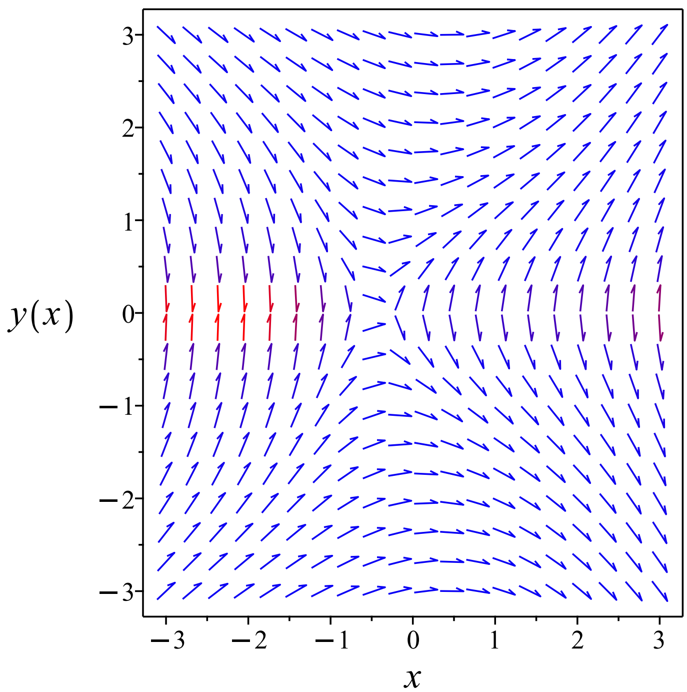


Figure 183: Slope field plot

Verification of solutions

$$\sin(x)\cos(y) + x^2 - y^2 = c_1$$

Verified OK.

3.12.2 Maple step by step solution

Let's solve

$\cos(x)\cos(y) - (\sin(x)\sin(y) + 2y)\, y' = -2x$

- Highest derivative means the order of the ODE is 1

  y'

- □ Check if ODE is exact
  - ◦ ODE is exact if the lhs is the total derivative of a C^2 function

    $F'(x, y) = 0$
  - ◦ Compute derivative of lhs

    $F'(x, y) + \left(\frac{\partial}{\partial y}F(x, y)\right) y' = 0$
  - ◦ Evaluate derivatives

    $-\cos(x)\sin(y) = -\cos(x)\sin(y)$
  - ◦ Condition met, ODE is exact

- Exact ODE implies solution will be of this form

  $\left[F(x, y) = c_1, M(x, y) = F'(x, y), N(x, y) = \frac{\partial}{\partial y}F(x, y)\right]$

- Solve for $F(x, y)$ by integrating $M(x, y)$ with respect to x

  $F(x, y) = \int (\cos(x)\cos(y) + 2x)\, dx + f_1(y)$

- Evaluate integral

  $F(x, y) = \sin(x)\cos(y) + x^2 + f_1(y)$

- Take derivative of $F(x, y)$ with respect to y

  $N(x, y) = \frac{\partial}{\partial y}F(x, y)$

- Compute derivative

  $-\sin(x)\sin(y) - 2y = -\sin(x)\sin(y) + \frac{d}{dy}f_1(y)$

- Isolate for $\frac{d}{dy}f_1(y)$

  $\frac{d}{dy}f_1(y) = -2y$

- Solve for $f_1(y)$

  $f_1(y) = -y^2$

- Substitute $f_1(y)$ into equation for $F(x, y)$

$$F(x,y) = \sin(x)\cos(y) + x^2 - y^2$$

- Substitute $F(x,y)$ into the solution of the ODE

$$\sin(x)\cos(y) + x^2 - y^2 = c_1$$

- Solve for y

$$y = RootOf\left(\_Z^2 - x^2 - \sin(x)\cos(\_Z) + c_1\right)$$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
trying homogeneous types:
trying Chini
differential order: 1; looking for linear symmetries
trying exact
<- exact successful`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 20

```
dsolve((cos(x)*cos(y(x))+2*x)-(sin(x)*sin(y(x))+2*y(x))*diff(y(x),x)=0,y(x), singsol=all)
```

$$\sin(x)\cos(y(x)) + x^2 - y(x)^2 + c_1 = 0$$

✓ Solution by Mathematica

Time used: 0.289 (sec). Leaf size: 25

```
DSolve[(Cos[x]*Cos[y[x]]+2*x)-(Sin[x]*Sin[y[x]]+2*y[x])*y'[x]==0,y[x],x,IncludeSingularSolut
```

$$\text{Solve}\left[-2x^2 + 2y(x)^2 - 2\sin(x)\cos(y(x)) = c_1, y(x)\right]$$

3.13 problem 13

Internal problem ID [4990]
Internal file name [`OUTPUT/4483_Sunday_June_05_2022_02_57_40_PM_9826243/index.tex`]

Book: Fundamentals of Differential Equations. By Nagle, Saff and Snider. 9th edition. Boston. Pearson 2018.
Section: Chapter 2, First order differential equations. Section 2.4, Exact equations. Exercises. page 64
Problem number: 13.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"exact"**

Maple gives the following as the ode type

`[_linear]`

$$\boxed{e^{t}(y-t)+\left(1+e^{t}\right)y'=0}$$

3.13.1 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x,y)+N(x,y)\frac{dy}{dx}=0 \tag{A}$$

We assume there exists a function $\phi(x,y)=c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y)=0$$

Hence

$$\frac{\partial\phi}{\partial x}+\frac{\partial\phi}{\partial y}\frac{dy}{dx}=0 \tag{B}$$

Comparing (A,B) shows that

$$\begin{aligned}\frac{\partial \phi}{\partial x} &= M \\ \frac{\partial \phi}{\partial y} &= N\end{aligned}$$

But since $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x, y)$ but at least we know now that we can do that since the condition $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(t, y) \, \mathrm{d}t + N(t, y) \, \mathrm{d}y = 0 \tag{1A}$$

Therefore

$$\begin{aligned}\left(1 + \mathrm{e}^t\right) \mathrm{d}y &= \left(-\mathrm{e}^t (y - t)\right) \mathrm{d}t \\ \left(\mathrm{e}^t (y - t)\right) \mathrm{d}t + \left(1 + \mathrm{e}^t\right) \mathrm{d}y &= 0 \qquad (2\mathrm{A})\end{aligned}$$

Comparing (1A) and (2A) shows that

$$\begin{aligned}M(t, y) &= \mathrm{e}^t (y - t) \\ N(t, y) &= 1 + \mathrm{e}^t\end{aligned}$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial t}$$

Using result found above gives

$$\begin{aligned}\frac{\partial M}{\partial y} &= \frac{\partial}{\partial y} \left(\mathrm{e}^t (y - t)\right) \\ &= \mathrm{e}^t\end{aligned}$$

And

$$\begin{aligned}\frac{\partial N}{\partial t} &= \frac{\partial}{\partial t} \left(1 + \mathrm{e}^t\right) \\ &= \mathrm{e}^t\end{aligned}$$

Since $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial t}$, then the ODE is exact The following equations are now set up to solve for the function $\phi(t, y)$

$$\frac{\partial \phi}{\partial t} = M \tag{1}$$

$$\frac{\partial \phi}{\partial y} = N \tag{2}$$

Integrating (1) w.r.t. t gives

$$\int \frac{\partial \phi}{\partial t} \, \mathrm{d}t = \int M \, \mathrm{d}t$$

$$\int \frac{\partial \phi}{\partial t} \, \mathrm{d}t = \int \mathrm{e}^t (y - t) \, \mathrm{d}t$$

$$\phi = -\mathrm{e}^t (t - 1 - y) + f(y) \tag{3}$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both t and y. Taking derivative of equation (3) w.r.t y gives

$$\frac{\partial \phi}{\partial y} = \mathrm{e}^t + f'(y) \tag{4}$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = 1 + \mathrm{e}^t$. Therefore equation (4) becomes

$$1 + \mathrm{e}^t = \mathrm{e}^t + f'(y) \tag{5}$$

Solving equation (5) for $f'(y)$ gives

$$f'(y) = 1$$

Integrating the above w.r.t y gives

$$\int f'(y) \, \mathrm{d}y = \int (1) \, \mathrm{d}y$$

$$f(y) = y + c_1$$

Where c_1 is constant of integration. Substituting result found above for $f(y)$ into equation (3) gives ϕ

$$\phi = -\mathrm{e}^t (t - 1 - y) + y + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = -e^t(t - 1 - y) + y$$

The solution becomes

$$y = \frac{t\,e^t - e^t + c_1}{1 + e^t}$$

Summary

The solution(s) found are the following

$$y = \frac{t\,e^t - e^t + c_1}{1 + e^t} \tag{1}$$

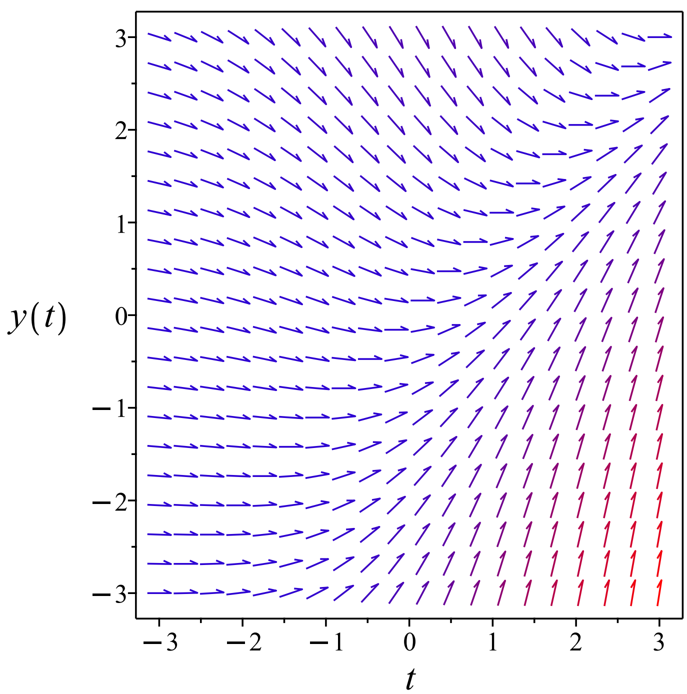


Figure 184: Slope field plot

Verification of solutions

$$y = \frac{t\,e^t - e^t + c_1}{1 + e^t}$$

Verified OK.

3.13.2 Maple step by step solution

Let's solve

$e^t(y-t)+(1+e^t)\, y'=0$

- Highest derivative means the order of the ODE is 1

  y'

- Isolate the derivative

  $y'=-\frac{e^t y}{1+e^t}+\frac{e^t t}{1+e^t}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE

  $y'+\frac{e^t y}{1+e^t}=\frac{e^t t}{1+e^t}$

- The ODE is linear; multiply by an integrating factor $\mu(t)$

  $\mu(t)\left(y'+\frac{e^t y}{1+e^t}\right)=\frac{\mu(t)e^t t}{1+e^t}$

- Assume the lhs of the ODE is the total derivative $\frac{d}{dt}(\mu(t)\, y)$

  $\mu(t)\left(y'+\frac{e^t y}{1+e^t}\right)=\mu'(t)\, y+\mu(t)\, y'$

- Isolate $\mu'(t)$

  $\mu'(t)=\frac{\mu(t)e^t}{1+e^t}$

- Solve to find the integrating factor

  $\mu(t)=1+e^t$

- Integrate both sides with respect to t

  $\int\left(\frac{d}{dt}(\mu(t)\, y)\right)dt=\int\frac{\mu(t)e^t t}{1+e^t}dt+c_1$

- Evaluate the integral on the lhs

  $\mu(t)\, y=\int\frac{\mu(t)e^t t}{1+e^t}dt+c_1$

- Solve for y

  $y=\frac{\int\frac{\mu(t)e^t t}{1+e^t}dt+c_1}{\mu(t)}$

- Substitute $\mu(t)=1+e^t$

  $y=\frac{\int t\, e^t dt+c_1}{1+e^t}$

- Evaluate the integrals on the rhs

  $y=\frac{(t-1)e^t+c_1}{1+e^t}$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
<- 1st order linear successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 19

```
dsolve(exp(t)*(y(t)-t)+(1+exp(t))*diff(y(t),t)=0,y(t), singsol=all)
```

$$y(t) = \frac{(t-1)\,\mathrm{e}^{t}+c_1}{1+\mathrm{e}^{t}}$$

✓ Solution by Mathematica

Time used: 0.071 (sec). Leaf size: 23

```
DSolve[Exp[t]*(y[t]-t)+(1+Exp[t])*y'[t]==0,y[t],t,IncludeSingularSolutions -> True]
```

$$y(t) \to \frac{e^t(t-1)+c_1}{e^t+1}$$

3.14 problem 14

Internal problem ID [4991]
Internal file name [`OUTPUT/4484_Sunday_June_05_2022_02_57_41_PM_54083834/index.tex`]

Book: Fundamentals of Differential Equations. By Nagle, Saff and Snider. 9th edition. Boston. Pearson 2018.
Section: Chapter 2, First order differential equations. Section 2.4, Exact equations. Exercises. page 64
Problem number: 14.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"exact"**

Maple gives the following as the ode type

`[_separable]`

$$\boxed{\ln(y) = -\frac{ty'}{y} - 1}$$

3.14.1 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x,y) + N(x,y)\frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x,y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y) = 0$$

Hence

$$\frac{\partial\phi}{\partial x} + \frac{\partial\phi}{\partial y}\frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$
\begin{aligned}
\frac{\partial \phi}{\partial x} &= M \\
\frac{\partial \phi}{\partial y} &= N
\end{aligned}
$$

But since $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ then for the above to be valid, we require that

$$
\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}
$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x, y)$ but at least we know now that we can do that since the condition $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$
M(t, y)\,\mathrm{d}t + N(t, y)\,\mathrm{d}y = 0 \tag{1A}
$$

Therefore

$$
\begin{aligned}
\left(-\frac{1}{y(\ln(y)+1)}\right)\mathrm{d}y &= \left(\frac{1}{t}\right)\mathrm{d}t \\
\left(-\frac{1}{t}\right)\mathrm{d}t + \left(-\frac{1}{y(\ln(y)+1)}\right)\mathrm{d}y &= 0 \qquad \text{(2A)}
\end{aligned}
$$

Comparing (1A) and (2A) shows that

$$
\begin{aligned}
M(t, y) &= -\frac{1}{t} \\
N(t, y) &= -\frac{1}{y(\ln(y)+1)}
\end{aligned}
$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$
\frac{\partial M}{\partial y} = \frac{\partial N}{\partial t}
$$

Using result found above gives

$$
\begin{aligned}
\frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}\left(-\frac{1}{t}\right) \\
&= 0
\end{aligned}
$$

And

$$
\begin{aligned}
\frac{\partial N}{\partial t} &= \frac{\partial}{\partial t}\left(-\frac{1}{y\left(\ln\left(y\right)+1\right)}\right) \\
&= 0
\end{aligned}
$$

Since $\frac{\partial M}{\partial y}=\frac{\partial N}{\partial t}$, then the ODE is exact The following equations are now set up to solve for the function $\phi(t,y)$

$$
\frac{\partial \phi}{\partial t} = M \tag{1}
$$

$$
\frac{\partial \phi}{\partial y} = N \tag{2}
$$

Integrating (1) w.r.t. t gives

$$
\begin{aligned}
\int \frac{\partial \phi}{\partial t}\,\mathrm{d}t &= \int M\,\mathrm{d}t \\
\int \frac{\partial \phi}{\partial t}\,\mathrm{d}t &= \int -\frac{1}{t}\,\mathrm{d}t
\end{aligned}
$$

$$
\phi = -\ln\left(t\right) + f(y) \tag{3}
$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both t and y. Taking derivative of equation (3) w.r.t y gives

$$
\frac{\partial \phi}{\partial y} = 0 + f'(y) \tag{4}
$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = -\frac{1}{y(\ln(y)+1)}$. Therefore equation (4) becomes

$$
-\frac{1}{y\left(\ln\left(y\right)+1\right)} = 0 + f'(y) \tag{5}
$$

Solving equation (5) for $f'(y)$ gives

$$
f'(y) = -\frac{1}{y\left(\ln\left(y\right)+1\right)}
$$

Integrating the above w.r.t y gives

$$
\begin{aligned}
\int f'(y)\,\mathrm{d}y &= \int \left(-\frac{1}{y\left(\ln\left(y\right)+1\right)}\right)\mathrm{d}y \\
f(y) &= -\ln\left(\ln\left(y\right)+1\right) + c_1
\end{aligned}
$$

Where c_1 is constant of integration. Substituting result found above for $f(y)$ into equation (3) gives ϕ

$$\phi = -\ln(t) - \ln(\ln(y) + 1) + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = -\ln(t) - \ln(\ln(y) + 1)$$

The solution becomes

$$y = e^{-\frac{(t\,e^{c_1}-1)e^{-c_1}}{t}}$$

Summary

The solution(s) found are the following

$$y = e^{-\frac{(t\,e^{c_1}-1)e^{-c_1}}{t}} \tag{1}$$

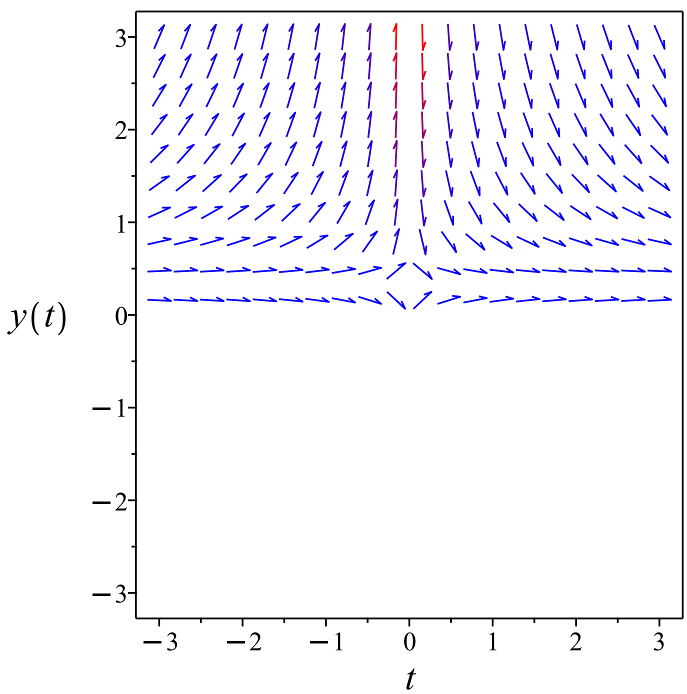


Figure 185: Slope field plot

Verification of solutions

$$y = e^{-\frac{(t\,e^{c_1}-1)e^{-c_1}}{t}}$$

Verified OK.

3.14.2 Maple step by step solution

Let's solve

$\ln(y) = -\frac{ty'}{y} - 1$

- Highest derivative means the order of the ODE is 1

  y'

- Integrate both sides with respect to t

  $\int \ln(y)\, dt = \int \left(-\frac{ty'}{y} - 1\right) dt + c_1$

- Cannot compute integral

  $\int \ln(y)\, dt = \int \left(-\frac{ty'}{y} - 1\right) dt + c_1$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
<- separable successful`Warning: persistent store makes readlib obsolete |G:/public_html/my_
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 18

```
dsolve((t/y(t))*diff(y(t),t)+(1+ln(y(t)))=0,y(t), singsol=all)
```

$$y(t) = e^{\frac{-c_1 t + 1}{t c_1}}$$

✓ Solution by Mathematica

Time used: 0.256 (sec). Leaf size: 24

```
DSolve[(t/y[t])*y'[t]+(1+Log[y[t]])==0,y[t],t,IncludeSingularSolutions -> True]
```

$$y(t) \to e^{-1+\frac{e^{c_1}}{t}}$$
$$y(t) \to \frac{1}{e}$$

3.15 problem 15

Internal problem ID [4992]
Internal file name [`OUTPUT/4485_Sunday_June_05_2022_02_57_42_PM_34058302/index.tex`]

Book: Fundamentals of Differential Equations. By Nagle, Saff and Snider. 9th edition. Boston. Pearson 2018.
Section: Chapter 2, First order differential equations. Section 2.4, Exact equations. Exercises. page 64
Problem number: 15.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"exact"**

Maple gives the following as the ode type

`[_linear]`

$$\boxed{\cos(\theta) r' - r \sin(\theta) = -\mathrm{e}^{\theta}}$$

3.15.1 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x, y) + N(x, y) \frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x, y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x, y) = 0$$

Hence

$$\frac{\partial \phi}{\partial x} + \frac{\partial \phi}{\partial y}\frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$\frac{\partial \phi}{\partial x} = M$$
$$\frac{\partial \phi}{\partial y} = N$$

But since $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x, y)$ but at least we know now that we can do that since the condition $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(\theta, r)\, \mathrm{d}\theta + N(\theta, r)\, \mathrm{d}r = 0 \tag{1A}$$

Therefore

$$(\cos(\theta))\, \mathrm{d}r = \left(\sin(\theta)\, r - \mathrm{e}^{\theta}\right) \mathrm{d}\theta$$
$$\left(-\sin(\theta)\, r + \mathrm{e}^{\theta}\right) \mathrm{d}\theta + (\cos(\theta))\, \mathrm{d}r = 0 \tag{2A}$$

Comparing (1A) and (2A) shows that

$$M(\theta, r) = -\sin(\theta)\, r + \mathrm{e}^{\theta}$$
$$N(\theta, r) = \cos(\theta)$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial r} = \frac{\partial N}{\partial \theta}$$

Using result found above gives

$$\begin{aligned} \frac{\partial M}{\partial r} &= \frac{\partial}{\partial r}\left(-\sin(\theta)\, r + \mathrm{e}^{\theta}\right) \\ &= -\sin(\theta) \end{aligned}$$

And

$$\begin{aligned} \frac{\partial N}{\partial \theta} &= \frac{\partial}{\partial \theta}(\cos(\theta)) \\ &= -\sin(\theta) \end{aligned}$$

Since $\frac{\partial M}{\partial r} = \frac{\partial N}{\partial \theta}$, then the ODE is <u>exact</u> The following equations are now set up to solve for the function $\phi(\theta, r)$

$$\frac{\partial \phi}{\partial \theta} = M \tag{1}$$
$$\frac{\partial \phi}{\partial r} = N \tag{2}$$

Integrating (1) w.r.t. θ gives

$$\int \frac{\partial \phi}{\partial \theta} \, \mathrm{d}\theta = \int M \, \mathrm{d}\theta$$
$$\int \frac{\partial \phi}{\partial \theta} \, \mathrm{d}\theta = \int -\sin(\theta) r + \mathrm{e}^{\theta} \, \mathrm{d}\theta$$
$$\phi = \cos(\theta) r + \mathrm{e}^{\theta} + f(r) \tag{3}$$

Where $f(r)$ is used for the constant of integration since ϕ is a function of both θ and r. Taking derivative of equation (3) w.r.t r gives

$$\frac{\partial \phi}{\partial r} = \cos(\theta) + f'(r) \tag{4}$$

But equation (2) says that $\frac{\partial \phi}{\partial r} = \cos(\theta)$. Therefore equation (4) becomes

$$\cos(\theta) = \cos(\theta) + f'(r) \tag{5}$$

Solving equation (5) for $f'(r)$ gives

$$f'(r) = 0$$

Therefore

$$f(r) = c_1$$

Where c_1 is constant of integration. Substituting this result for $f(r)$ into equation (3) gives ϕ

$$\phi = \cos(\theta) r + \mathrm{e}^{\theta} + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = \cos(\theta) r + \mathrm{e}^{\theta}$$

The solution becomes

$$r = -\frac{e^{\theta} - c_1}{\cos(\theta)}$$

Summary

The solution(s) found are the following

$$r = -\frac{e^{\theta} - c_1}{\cos(\theta)} \tag{1}$$

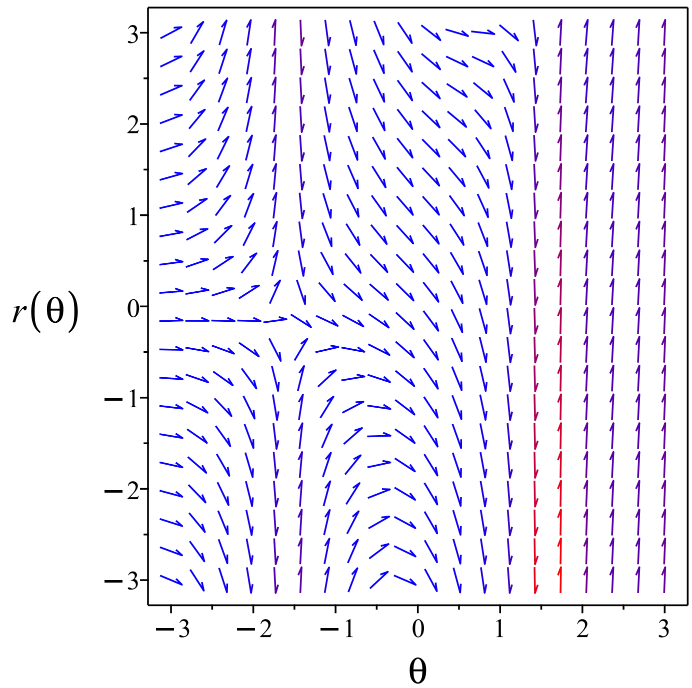


Figure 186: Slope field plot

Verification of solutions

$$r = -\frac{e^{\theta} - c_1}{\cos(\theta)}$$

Verified OK.

3.15.2 Maple step by step solution

Let's solve

$\cos(\theta) r' - r \sin(\theta) = -e^{\theta}$

- Highest derivative means the order of the ODE is 1

  r'

- Isolate the derivative

  $r' = \frac{\sin(\theta) r}{\cos(\theta)} - \frac{e^{\theta}}{\cos(\theta)}$

- Group terms with r on the lhs of the ODE and the rest on the rhs of the ODE

  $r' - \frac{\sin(\theta) r}{\cos(\theta)} = -\frac{e^{\theta}}{\cos(\theta)}$

- The ODE is linear; multiply by an integrating factor $\mu(\theta)$

  $\mu(\theta)\left(r' - \frac{\sin(\theta) r}{\cos(\theta)}\right) = -\frac{\mu(\theta) e^{\theta}}{\cos(\theta)}$

- Assume the lhs of the ODE is the total derivative $\frac{d}{d\theta}(\mu(\theta) r)$

  $\mu(\theta)\left(r' - \frac{\sin(\theta) r}{\cos(\theta)}\right) = \mu'(\theta) r + \mu(\theta) r'$

- Isolate $\mu'(\theta)$

  $\mu'(\theta) = -\frac{\mu(\theta)\sin(\theta)}{\cos(\theta)}$

- Solve to find the integrating factor

  $\mu(\theta) = \cos(\theta)$

- Integrate both sides with respect to θ

  $\int \left(\frac{d}{d\theta}(\mu(\theta) r)\right) d\theta = \int -\frac{\mu(\theta) e^{\theta}}{\cos(\theta)} d\theta + c_1$

- Evaluate the integral on the lhs

  $\mu(\theta) r = \int -\frac{\mu(\theta) e^{\theta}}{\cos(\theta)} d\theta + c_1$

- Solve for r

  $r = \frac{\int -\frac{\mu(\theta) e^{\theta}}{\cos(\theta)} d\theta + c_1}{\mu(\theta)}$

- Substitute $\mu(\theta) = \cos(\theta)$

  $r = \frac{\int -e^{\theta} d\theta + c_1}{\cos(\theta)}$

- Evaluate the integrals on the rhs

  $r = \frac{-e^{\theta} + c_1}{\cos(\theta)}$

- Simplify

  $$r = \left(-\mathrm{e}^{\theta} + c_1\right) \sec\left(\theta\right)$$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
<- 1st order linear successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 13

```
dsolve(cos(theta)*diff(r(theta),theta)-(r(theta)*sin(theta)-exp(theta))=0,r(theta), singsol=
```

$$r(\theta) = \left(-\mathrm{e}^{\theta} + c_1\right) \sec\left(\theta\right)$$

✓ Solution by Mathematica

Time used: 0.055 (sec). Leaf size: 16

```
DSolve[Cos[\[Theta]]*r'[\[Theta]]-(r[\[Theta]]*Sin[\[Theta]]-Exp[\[Theta]])==0,r[\[Theta]],\
```

$$r(\theta) \to \left(-e^{\theta} + c_1\right) \sec(\theta)$$

3.16 problem 16

Internal problem ID [4993]
Internal file name [`OUTPUT/4486_Sunday_June_05_2022_02_57_43_PM_99481650/index.tex`]

Book: Fundamentals of Differential Equations. By Nagle, Saff and Snider. 9th edition. Boston. Pearson 2018.
Section: Chapter 2, First order differential equations. Section 2.4, Exact equations. Exercises. page 64
Problem number: 16.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"exact"**

Maple gives the following as the ode type

```
[_exact]
```

$$\boxed{y\,\mathrm{e}^{xy}-\frac{1}{y}+\left(x\,\mathrm{e}^{xy}+\frac{x}{y^2}\right)y'=0}$$

3.16.1 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x,y)+N(x,y)\frac{dy}{dx}=0 \tag{A}$$

We assume there exists a function $\phi(x,y)=c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y)=0$$

Hence

$$\frac{\partial\phi}{\partial x}+\frac{\partial\phi}{\partial y}\frac{dy}{dx}=0 \tag{B}$$

Comparing (A,B) shows that

$$
\begin{aligned}
\frac{\partial \phi}{\partial x} &= M \\
\frac{\partial \phi}{\partial y} &= N
\end{aligned}
$$

But since $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ then for the above to be valid, we require that

$$
\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}
$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x, y)$ but at least we know now that we can do that since the condition $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$
M(x, y)\, \mathrm{d}x + N(x, y)\, \mathrm{d}y = 0 \tag{1A}
$$

Therefore

$$
\begin{aligned}
\left(x\, \mathrm{e}^{xy} + \frac{x}{y^2}\right) \mathrm{d}y &= \left(-y\, \mathrm{e}^{xy} + \frac{1}{y}\right) \mathrm{d}x \\
\left(y\, \mathrm{e}^{xy} - \frac{1}{y}\right) \mathrm{d}x + \left(x\, \mathrm{e}^{xy} + \frac{x}{y^2}\right) \mathrm{d}y &= 0 \qquad \text{(2A)}
\end{aligned}
$$

Comparing (1A) and (2A) shows that

$$
\begin{aligned}
M(x, y) &= y\, \mathrm{e}^{xy} - \frac{1}{y} \\
N(x, y) &= x\, \mathrm{e}^{xy} + \frac{x}{y^2}
\end{aligned}
$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$
\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}
$$

Using result found above gives

$$
\begin{aligned}
\frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}\left(y\, \mathrm{e}^{xy} - \frac{1}{y}\right) \\
&= \frac{1 + (x\, y^3 + y^2)\, \mathrm{e}^{xy}}{y^2}
\end{aligned}
$$

And

$$\frac{\partial N}{\partial x} = \frac{\partial}{\partial x}\left(x\,\mathrm{e}^{xy} + \frac{x}{y^2}\right)$$
$$= \frac{1 + (x\,y^3 + y^2)\,\mathrm{e}^{xy}}{y^2}$$

Since $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$, then the ODE is <u>exact</u> The following equations are now set up to solve for the function $\phi(x, y)$

$$\frac{\partial \phi}{\partial x} = M \tag{1}$$
$$\frac{\partial \phi}{\partial y} = N \tag{2}$$

Integrating (1) w.r.t. x gives

$$\int \frac{\partial \phi}{\partial x}\,\mathrm{d}x = \int M\,\mathrm{d}x$$
$$\int \frac{\partial \phi}{\partial x}\,\mathrm{d}x = \int y\,\mathrm{e}^{xy} - \frac{1}{y}\,\mathrm{d}x$$
$$\phi = \frac{y\,\mathrm{e}^{xy} - x}{y} + f(y) \tag{3}$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$\frac{\partial \phi}{\partial y} = \frac{\mathrm{e}^{xy} + xy\,\mathrm{e}^{xy}}{y} - \frac{y\,\mathrm{e}^{xy} - x}{y^2} + f'(y) \tag{4}$$
$$= \frac{x(y^2\mathrm{e}^{xy} + 1)}{y^2} + f'(y)$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = x\,\mathrm{e}^{xy} + \frac{x}{y^2}$. Therefore equation (4) becomes

$$x\,\mathrm{e}^{xy} + \frac{x}{y^2} = \frac{x(y^2\mathrm{e}^{xy} + 1)}{y^2} + f'(y) \tag{5}$$

Solving equation (5) for $f'(y)$ gives

$$f'(y) = 0$$

Therefore

$$f(y) = c_1$$

Where c_1 is constant of integration. Substituting this result for $f(y)$ into equation (3) gives ϕ

$$\phi = \frac{y\,e^{xy} - x}{y} + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = \frac{y\,e^{xy} - x}{y}$$

Summary

The solution(s) found are the following

$$\frac{y\,e^{xy} - x}{y} = c_1 \tag{1}$$

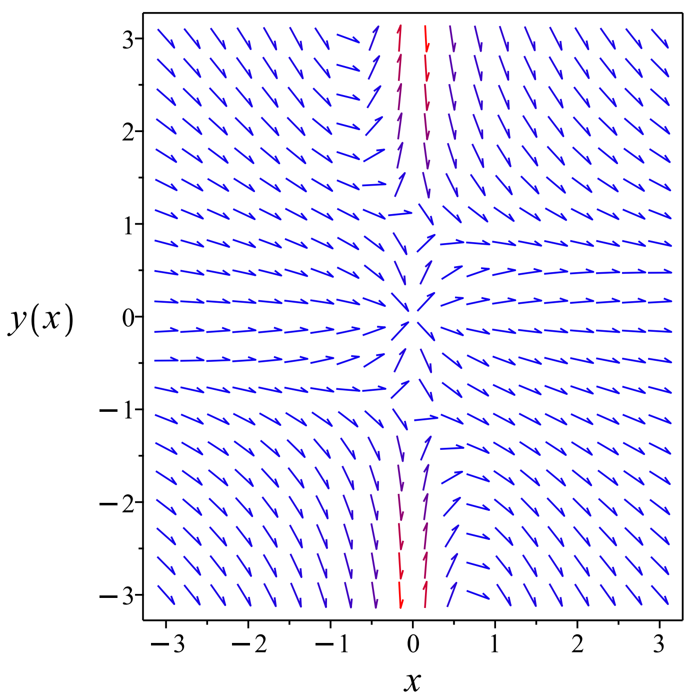


Figure 187: Slope field plot

Verification of solutions

$$\frac{y\,e^{xy} - x}{y} = c_1$$

Verified OK.

3.16.2 Maple step by step solution

Let's solve

$y\,e^{xy} - \frac{1}{y} + \left(x\,e^{xy} + \frac{x}{y^2}\right) y' = 0$

- Highest derivative means the order of the ODE is 1

  y'

- □ Check if ODE is exact
  - ∘ ODE is exact if the lhs is the total derivative of a C^2 function

    $F'(x, y) = 0$
  - ∘ Compute derivative of lhs

    $F'(x, y) + \left(\frac{\partial}{\partial y}F(x, y)\right) y' = 0$
  - ∘ Evaluate derivatives

    $e^{xy} + xy\,e^{xy} + \frac{1}{y^2} = e^{xy} + xy\,e^{xy} + \frac{1}{y^2}$
  - ∘ Condition met, ODE is exact
- Exact ODE implies solution will be of this form

  $\left[F(x, y) = c_1, M(x, y) = F'(x, y), N(x, y) = \frac{\partial}{\partial y}F(x, y)\right]$
- Solve for $F(x, y)$ by integrating $M(x, y)$ with respect to x

  $F(x, y) = \int \left(y\,e^{xy} - \frac{1}{y}\right) dx + f_1(y)$
- Evaluate integral

  $F(x, y) = e^{xy} - \frac{x}{y} + f_1(y)$
- Take derivative of $F(x, y)$ with respect to y

  $N(x, y) = \frac{\partial}{\partial y}F(x, y)$
- Compute derivative

  $x\,e^{xy} + \frac{x}{y^2} = x\,e^{xy} + \frac{x}{y^2} + \frac{d}{dy}f_1(y)$

- Isolate for $\frac{d}{dy}f_1(y)$

  $\frac{d}{dy}f_1(y) = 0$
- Solve for $f_1(y)$

  $f_1(y) = 0$
- Substitute $f_1(y)$ into equation for $F(x, y)$

  $F(x, y) = e^{xy} - \frac{x}{y}$
- Substitute $F(x, y)$ into the solution of the ODE

  $e^{xy} - \frac{x}{y} = c_1$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
trying homogeneous types:
trying Chini
differential order: 1; looking for linear symmetries
trying exact
<- exact successful`
```

✓ Solution by Maple

Time used: 0.015 (sec). Leaf size: 24

```
dsolve((y(x)*exp(x*y(x))-1/y(x))+(x*exp(x*y(x))+x/y(x)^2)*diff(y(x),x)=0,y(x), singsol=all)
```

$$\frac{y(x)\, e^{xy(x)} + c_1 y(x) - x}{y(x)} = 0$$

✓ Solution by Mathematica

Time used: 0.2 (sec). Leaf size: 20

```
DSolve[(y[x]*Exp[x*y[x]]-1/y[x])+(x*Exp[x*y[x]]+x/y[x]^2)*y'[x]==0,y[x],x,IncludeSingularSol
```

$$\text{Solve}\left[e^{xy(x)}-\frac{x}{y(x)}=c_1,y(x)\right]$$

3.17 problem 17

Internal problem ID [4994]
Internal file name [`OUTPUT/4487_Sunday_June_05_2022_02_57_43_PM_34237608/index.tex`]

Book: Fundamentals of Differential Equations. By Nagle, Saff and Snider. 9th edition. Boston. Pearson 2018.
Section: Chapter 2, First order differential equations. Section 2.4, Exact equations. Exercises. page 64
Problem number: 17.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"exact"**

Maple gives the following as the ode type

```
[[_homogeneous, `class G`], _rational]
```

$$\boxed{\frac{1}{y}-\left(3y-\frac{x}{y^2}\right)y'=0}$$

3.17.1 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x,y)+N(x,y)\frac{dy}{dx}=0 \tag{A}$$

We assume there exists a function $\phi(x,y)=c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y)=0$$

Hence

$$\frac{\partial\phi}{\partial x}+\frac{\partial\phi}{\partial y}\frac{dy}{dx}=0 \tag{B}$$

Comparing (A,B) shows that

$$\frac{\partial \phi}{\partial x} = M$$
$$\frac{\partial \phi}{\partial y} = N$$

But since $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x, y)$ but at least we know now that we can do that since the condition $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x, y)\,\mathrm{d}x + N(x, y)\,\mathrm{d}y = 0 \tag{1A}$$

Therefore

$$\left(3y^3 - x\right) \mathrm{d}y = (y)\,\mathrm{d}x$$
$$(-y)\,\mathrm{d}x + \left(3y^3 - x\right) \mathrm{d}y = 0 \tag{2A}$$

Comparing (1A) and (2A) shows that

$$M(x, y) = -y$$
$$N(x, y) = 3y^3 - x$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Using result found above gives

$$\frac{\partial M}{\partial y} = \frac{\partial}{\partial y}(-y)$$
$$= -1$$

And

$$\frac{\partial N}{\partial x} = \frac{\partial}{\partial x}\left(3y^3 - x\right)$$
$$= -1$$

Since $\frac{\partial M}{\partial y}=\frac{\partial N}{\partial x}$, then the ODE is <u>exact</u> The following equations are now set up to solve for the function $\phi(x,y)$

$$\frac{\partial \phi}{\partial x}=M \tag{1}$$

$$\frac{\partial \phi}{\partial y}=N \tag{2}$$

Integrating (1) w.r.t. x gives

$$\int \frac{\partial \phi}{\partial x}\,\mathrm{d}x=\int M\,\mathrm{d}x$$

$$\int \frac{\partial \phi}{\partial x}\,\mathrm{d}x=\int -y\,\mathrm{d}x$$

$$\phi=-xy+f(y) \tag{3}$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$\frac{\partial \phi}{\partial y}=-x+f'(y) \tag{4}$$

But equation (2) says that $\frac{\partial \phi}{\partial y}=3y^3-x$. Therefore equation (4) becomes

$$3y^3-x=-x+f'(y) \tag{5}$$

Solving equation (5) for $f'(y)$ gives

$$f'(y)=3y^3$$

Integrating the above w.r.t y gives

$$\int f'(y)\,\mathrm{d}y=\int (3y^3)\,\mathrm{d}y$$

$$f(y)=\frac{3y^4}{4}+c_1$$

Where c_1 is constant of integration. Substituting result found above for $f(y)$ into equation (3) gives ϕ

$$\phi=-xy+\frac{3}{4}y^4+c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = -xy + \frac{3}{4}y^4$$

Summary

The solution(s) found are the following

$$-xy + \frac{3y^4}{4} = c_1 \tag{1}$$

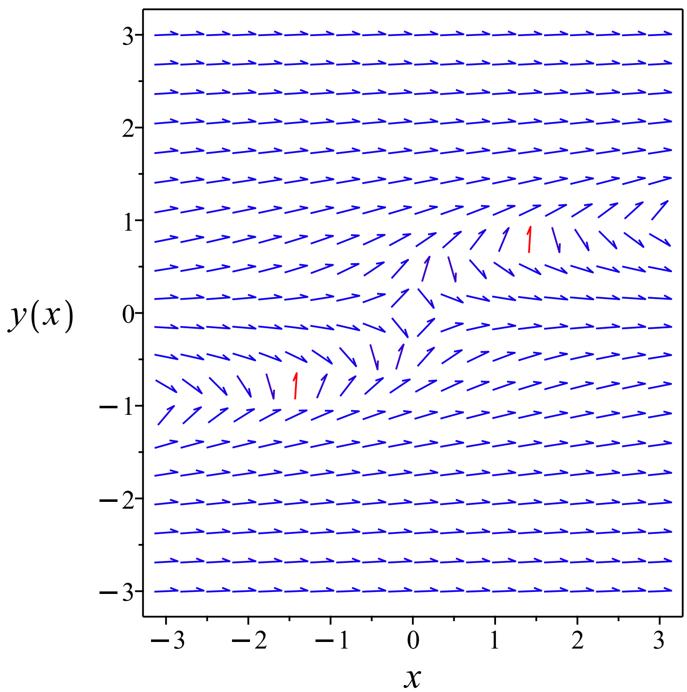


Figure 188: Slope field plot

Verification of solutions

$$-xy + \frac{3y^4}{4} = c_1$$

Verified OK.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
<- 1st order linear successful
<- inverse linear successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 18

```
dsolve(1/y(x)-(3*y(x)-x/y(x)^2)*diff(y(x),x)=0,y(x), singsol=all)
```

$$-\frac{c_1}{y(x)} + x - \frac{3y(x)^3}{4} = 0$$

✓ Solution by Mathematica

Time used: 32.895 (sec). Leaf size: 870

```
DSolve[1/y[x]-(3*y[x]-x/y[x]^2)*y'[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to -\frac{\sqrt{\frac{4c_1}{\sqrt[3]{3x^2-\sqrt{9x^4-64{c_1}^3}}}+\sqrt[3]{3x^2-\sqrt{9x^4-64{c_1}^3}}}+\sqrt{-\frac{2\sqrt{6}x}{\sqrt{\frac{4c_1}{\sqrt[3]{3x^2-\sqrt{9x^4-64{c_1}^3}}}+\sqrt[3]{3x^2-\sqrt{9x^4 \cdots}}}}\cdots}}{\sqrt{6}}$$

$$y(x) \to \frac{\sqrt{-\frac{2\sqrt{6}x}{\sqrt{\frac{4c_1}{\sqrt[3]{3x^2-\sqrt{9x^4-64{c_1}^3}}}+\sqrt[3]{3x^2-\sqrt{9x^4-64{c_1}^3}}}}-\sqrt[3]{3x^2-\sqrt{9x^4-64{c_1}^3}}-\frac{4c_1}{\sqrt[3]{3x^2-\sqrt{9x^4-6 \cdots}}}\cdots}}{\sqrt{6}}$$

$$y(x) \to \frac{\sqrt{\frac{4c_1}{\sqrt[3]{3x^2-\sqrt{9x^4-64{c_1}^3}}}+\sqrt[3]{3x^2-\sqrt{9x^4-64{c_1}^3}}}-\sqrt{\frac{2\sqrt{6}x}{\sqrt{\frac{4c_1}{\sqrt[3]{3x^2-\sqrt{9x^4-64{c_1}^3}}}+\sqrt[3]{3x^2-\sqrt{9x^4-\cdots}}}}\cdots}}{\sqrt{6}}$$

$$y(x) \to \frac{\sqrt{\frac{4c_1}{\sqrt[3]{3x^2-\sqrt{9x^4-64{c_1}^3}}}+\sqrt[3]{3x^2-\sqrt{9x^4-64{c_1}^3}}}+\sqrt{\frac{2\sqrt{6}x}{\sqrt{\frac{4c_1}{\sqrt[3]{3x^2-\sqrt{9x^4-64{c_1}^3}}}+\sqrt[3]{3x^2-\sqrt{9x^4-\cdots}}}}\cdots}}{\sqrt{6}}$$

3.18 problem 18

Internal problem ID [4995]
Internal file name [`OUTPUT/4488_Sunday_June_05_2022_02_57_44_PM_99596253/index.tex`]

Book: Fundamentals of Differential Equations. By Nagle, Saff and Snider. 9th edition. Boston. Pearson 2018.
Section: Chapter 2, First order differential equations. Section 2.4, Exact equations. Exercises. page 64
Problem number: 18.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"unknown"**

Maple gives the following as the ode type

```
[`y=_G(x,y')`]
```

Unable to solve or complete the solution.

$$\boxed{y^2 - \cos(x+y) - (2xy - \cos(x+y) - e^y)\,y' = -2x}$$

Unable to determine ODE type.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
trying homogeneous types:
trying Chini
differential order: 1; looking for linear symmetries
trying exact
Looking for potential symmetries
trying inverse_Riccati
trying an equivalence to an Abel ODE
differential order: 1; trying a linearization to 2nd order
--- trying a change of variables {x -> y(x), y(x) -> x}
differential order: 1; trying a linearization to 2nd order
trying 1st order ODE linearizable_by_differentiation
--- Trying Lie symmetry methods, 1st order ---
`, `-> Computing symmetries using: way = 3
`, `-> Computing symmetries using: way = 4
`, `-> Computing symmetries using: way = 5
trying symmetry patterns for 1st order ODEs
-> trying a symmetry pattern of the form [F(x)*G(y), 0]
-> trying a symmetry pattern of the form [0, F(x)*G(y)]
-> trying symmetry patterns of the forms [F(x),G(y)] and [G(y),F(x)]
-> trying a symmetry pattern of the form [F(x),G(x)]
-> trying a symmetry pattern of the form [F(y),G(y)]
-> trying a symmetry pattern of the form [F(x)+G(y), 0]
-> trying a symmetry pattern of the form [0, F(x)+G(y)]
-> trying a symmetry pattern of the form [F(x),G(x)*y+H(x)]
-> trying a symmetry pattern of conformal type`
```

✗ Solution by Maple

```
dsolve((2*x+y(x)^2-cos(x+y(x)))-(2*x*y(x)-cos(x+y(x))-exp(y(x)))*diff(y(x),x)=0,y(x), singso
```

No solution found

✗ Solution by Mathematica

Time used: 0.0 (sec). Leaf size: 0

```
DSolve[(2*x+y[x]^2-Cos[x+y[x]])-(2*x*y[x]-Cos[x+y[x]]-Exp[y[x]])*y'[x]==0,y[x],x,IncludeSing
```

Not solved

4 Chapter 2, First order differential equations. Review problems. page 79

4.1 problem 1

Internal problem ID [4996]
Internal file name [`OUTPUT/4489_Sunday_June_05_2022_02_59_15_PM_14995085/index.tex`]

Book: Fundamentals of Differential Equations. By Nagle, Saff and Snider. 9th edition. Boston. Pearson 2018.
Section: Chapter 2, First order differential equations. Review problems. page 79
Problem number: 1.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"exact", "separable", "first_order_ode_lie_symmetry_lookup"**

Maple gives the following as the ode type

`[_separable]`

$$\boxed{y' - \frac{e^{x+y}}{y-1} = 0}$$

4.1.1 Solving as separable ode

In canonical form the ODE is

$$\begin{aligned} y' &= F(x,y) \\ &= f(x)g(y) \\ &= \frac{e^x e^y}{y-1} \end{aligned}$$

Where $f(x) = e^x$ and $g(y) = \frac{e^y}{y-1}$. Integrating both sides gives

$$\frac{1}{\frac{e^y}{y-1}}\, dy = e^x\, dx$$

$$\int \frac{1}{\frac{e^{y}}{y-1}}dy = \int e^{x}dx$$
$$-y\,e^{-y} = e^{x} + c_1$$

Which results in

$$y = -\operatorname{LambertW}\left(e^{x} + c_1\right)$$

Summary

The solution(s) found are the following

$$y = -\operatorname{LambertW}\left(e^{x} + c_1\right) \tag{1}$$

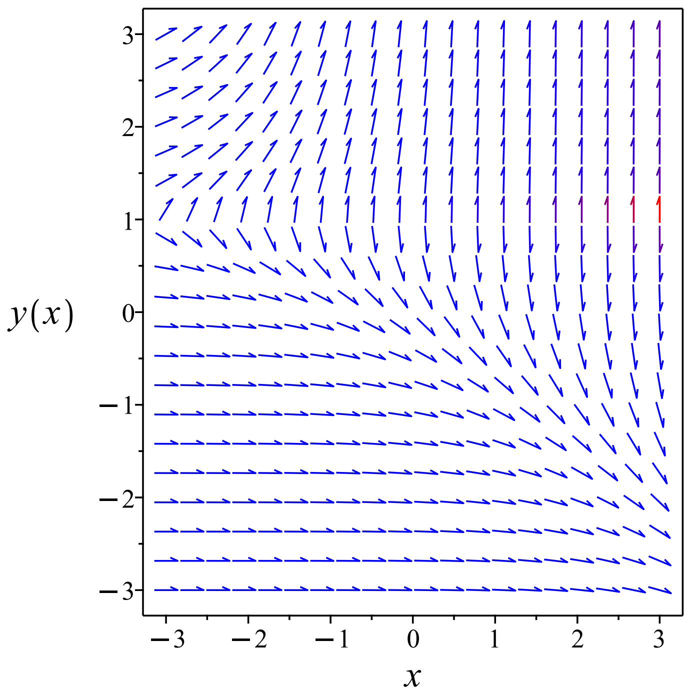


Figure 189: Slope field plot

Verification of solutions

$$y = -\operatorname{LambertW}\left(e^{x} + c_1\right)$$

Verified OK.

4.1.2 Solving as first order ode lie symmetry lookup ode

Writing the ode as

$$
\begin{aligned}
y' &= \frac{e^{x+y}}{y-1} \\
y' &= \omega(x,y)
\end{aligned}
$$

The condition of Lie symmetry is the linearized PDE given by

$$
\eta_x + \omega(\eta_y - \xi_x) - \omega^2 \xi_y - \omega_x \xi - \omega_y \eta = 0 \tag{A}
$$

The type of this ode is known. It is of type `separable`. Therefore we do not need to solve the PDE (A), and can just use the lookup table shown below to find ξ, η

Table 183: Lie symmetry infinitesimal lookup table for known first order ODE's

ODE class	Form	ξ	η
linear ode	$y' = f(x)y(x) + g(x)$	0	$e^{\int f dx}$
separable ode	$y' = f(x)\, g(y)$	$\frac{1}{f}$	0
quadrature ode	$y' = f(x)$	0	1
quadrature ode	$y' = g(y)$	1	0
homogeneous ODEs of Class A	$y' = f\left(\frac{y}{x}\right)$	x	y
homogeneous ODEs of Class C	$y' = (a + bx + cy)^{\frac{n}{m}}$	1	$-\frac{b}{c}$
homogeneous class D	$y' = \frac{y}{x} + g(x)\, F\left(\frac{y}{x}\right)$	x^2	xy
First order special form ID 1	$y' = g(x)\, e^{h(x)+by} + f(x)$	$\frac{e^{-\int b f(x) dx - h(x)}}{g(x)}$	$\frac{f(x) e^{-\int b f(x) dx - h(x)}}{g(x)}$
polynomial type ode	$y' = \frac{a_1 x + b_1 y + c_1}{a_2 x + b_2 y + c_2}$	$\frac{a_1 b_2 x - a_2 b_1 x - b_1 c_2 + b_2 c_1}{a_1 b_2 - a_2 b_1}$	$\frac{a_1 b_2 y - a_2 b_1 y - a_1 c_2 - a_2 c_1}{a_1 b_2 - a_2 b_1}$
Bernoulli ode	$y' = f(x)\, y + g(x)\, y^n$	0	$e^{-\int (n-1) f(x) dx} y^n$
Reduced Riccati	$y' = f_1(x)\, y + f_2(x)\, y^2$	0	$e^{-\int f_1 dx}$

The above table shows that

$$\begin{aligned}\xi(x,y) &= \mathrm{e}^{-x}\\ \eta(x,y) &= 0\end{aligned} \tag{A1}$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x,y) \to (R,S)$ where (R,S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}$$

The above comes from the requirements that $\left(\xi\frac{\partial}{\partial x} + \eta\frac{\partial}{\partial y}\right) S(x,y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the canonical coordinates, where $S(R)$. Since $\eta = 0$ then in this special case

$$R = y$$

S is found from

$$\begin{aligned}S &= \int \frac{1}{\xi} dx\\ &= \int \frac{1}{\mathrm{e}^{-x}} dx\end{aligned}$$

Which results in

$$S = \mathrm{e}^{x}$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x,y)S_y}{R_x + \omega(x,y)R_y} \tag{2}$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x,y)$ is the right hand side of the original ode given by

$$\omega(x,y) = \frac{\mathrm{e}^{x+y}}{y-1}$$

Evaluating all the partial derivatives gives

$$\begin{aligned}R_x &= 0\\ R_y &= 1\\ S_x &= \mathrm{e}^{x}\\ S_y &= 0\end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = (y-1)\,\mathrm{e}^{-y} \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = (R-1)\,\mathrm{e}^{-R}$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$S(R) = -\mathrm{e}^{-R} R + c_1 \tag{4}$$

To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$\mathrm{e}^{x} = -y\,\mathrm{e}^{-y} + c_1$$

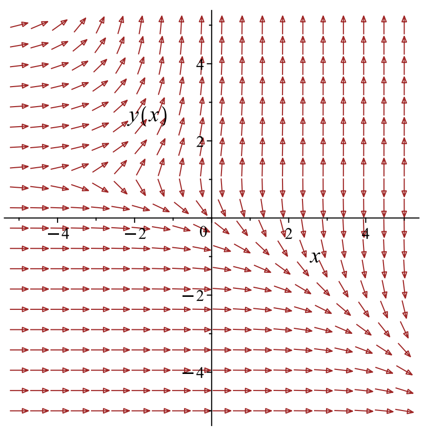
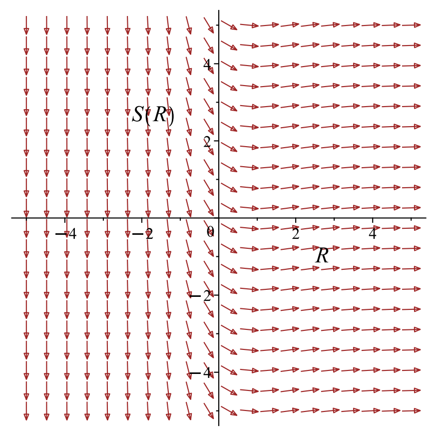
Which simplifies to

$$\mathrm{e}^{x} = -y\,\mathrm{e}^{-y} + c_1$$

Which gives

$$y = -\operatorname{LambertW}\left(\mathrm{e}^{x} - c_1\right)$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in x, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dx} = \frac{e^{x+y}}{y-1}$ 	$R = y$ $S = e^{x}$	$\frac{dS}{dR} = (R-1)\, e^{-R}$ 

Summary
The solution(s) found are the following

$$y = -\operatorname{LambertW}\left(e^{x} - c_1\right) \tag{1}$$

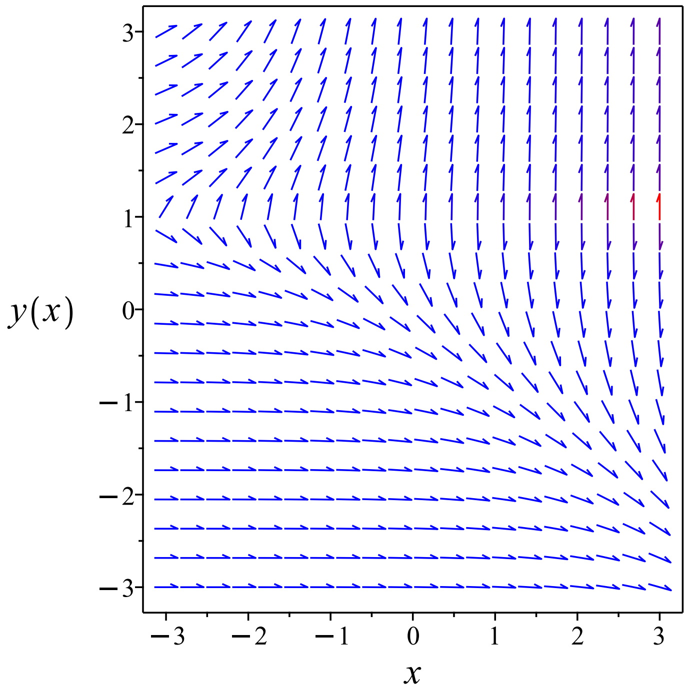


Figure 190: Slope field plot

Verification of solutions

$$y = -\text{LambertW}\left(e^{x} - c_1\right)$$

Verified OK.

4.1.3 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x, y) + N(x, y)\frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x, y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x, y) = 0$$

Hence

$$\frac{\partial \phi}{\partial x} + \frac{\partial \phi}{\partial y}\frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$\frac{\partial \phi}{\partial x} = M$$
$$\frac{\partial \phi}{\partial y} = N$$

But since $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x, y)$ but at least we know now that we can do that since the condition $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x, y)\,\mathrm{d}x + N(x, y)\,\mathrm{d}y = 0 \tag{1A}$$

Therefore

$$\left((y-1)\,\mathrm{e}^{-y}\right)\mathrm{d}y = (\mathrm{e}^{x})\,\mathrm{d}x$$
$$(-\mathrm{e}^{x})\,\mathrm{d}x + \left((y-1)\,\mathrm{e}^{-y}\right)\mathrm{d}y = 0 \tag{2A}$$

Comparing (1A) and (2A) shows that

$$M(x, y) = -\mathrm{e}^{x}$$
$$N(x, y) = (y-1)\,\mathrm{e}^{-y}$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Using result found above gives

$$\begin{aligned}\frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}(-\mathrm{e}^{x}) \\ &= 0\end{aligned}$$

And

$$\begin{aligned}\frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}\left((y-1)\,\mathrm{e}^{-y}\right) \\ &= 0\end{aligned}$$

Since $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$, then the ODE is exact The following equations are now set up to solve for the function $\phi(x, y)$

$$\frac{\partial \phi}{\partial x} = M \tag{1}$$

$$\frac{\partial \phi}{\partial y} = N \tag{2}$$

Integrating (1) w.r.t. x gives

$$\int \frac{\partial \phi}{\partial x} \, \mathrm{d}x = \int M \, \mathrm{d}x$$

$$\int \frac{\partial \phi}{\partial x} \, \mathrm{d}x = \int -\mathrm{e}^{x} \, \mathrm{d}x$$

$$\phi = -\mathrm{e}^{x} + f(y) \tag{3}$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$\frac{\partial \phi}{\partial y} = 0 + f'(y) \tag{4}$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = (y-1)\,\mathrm{e}^{-y}$. Therefore equation (4) becomes

$$(y-1)\,\mathrm{e}^{-y} = 0 + f'(y) \tag{5}$$

Solving equation (5) for $f'(y)$ gives

$$f'(y) = (y-1)\,\mathrm{e}^{-y}$$

Integrating the above w.r.t y gives

$$\int f'(y) \, \mathrm{d}y = \int \left((y-1)\,\mathrm{e}^{-y}\right) \mathrm{d}y$$

$$f(y) = -y\,\mathrm{e}^{-y} + c_1$$

Where c_1 is constant of integration. Substituting result found above for $f(y)$ into equation (3) gives ϕ

$$\phi = -\mathrm{e}^{x} - y\,\mathrm{e}^{-y} + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = -e^x - y\,e^{-y}$$

The solution becomes

$$y = -\operatorname{LambertW}(e^x + c_1)$$

Summary

The solution(s) found are the following

$$y = -\operatorname{LambertW}(e^x + c_1) \tag{1}$$

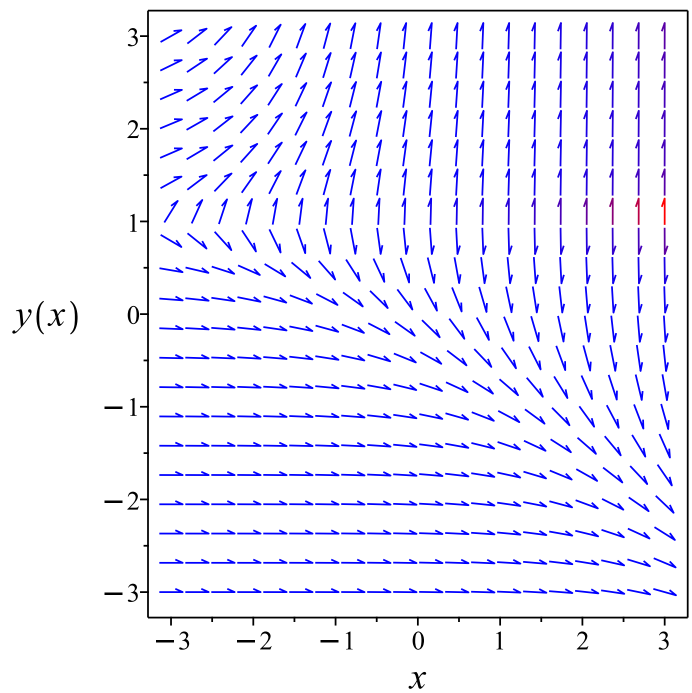


Figure 191: Slope field plot

Verification of solutions

$$y = -\operatorname{LambertW}(e^x + c_1)$$

Verified OK.

4.1.4 Maple step by step solution

Let's solve

$y' - \frac{e^{x+y}}{y-1} = 0$

- Highest derivative means the order of the ODE is 1

  y'

- Separate variables

  $\frac{y'(y-1)}{e^y} = e^x$

- Integrate both sides with respect to x

  $\int \frac{y'(y-1)}{e^y} dx = \int e^x dx + c_1$

- Evaluate integral

  $-\frac{y}{e^y} = e^x + c_1$

- Solve for y

  $y = -\mathit{LambertW}(e^x + c_1)$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
<- separable successful`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 11

```
dsolve(diff(y(x),x)=exp(x+y(x))/(y(x)-1),y(x), singsol=all)
```

$$y(x) = -\operatorname{LambertW}(c_1 + e^x)$$

✓ Solution by Mathematica

Time used: 60.144 (sec). Leaf size: 14

```
DSolve[y'[x]==Exp[x+y[x]]/(y[x]-1),y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to -W(e^x + c_1)$$

4.2 problem 2

Internal problem ID [4997]
Internal file name [`OUTPUT/4490_Sunday_June_05_2022_02_59_16_PM_46149629/index.tex`]

Book: Fundamentals of Differential Equations. By Nagle, Saff and Snider. 9th edition. Boston. Pearson 2018.
Section: Chapter 2, First order differential equations. Review problems. page 79
Problem number: 2.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"linear", "exactWithIntegrationFactor", "first_order_ode_lie_symmetry_lookup"**

Maple gives the following as the ode type

```
[[_linear, `class A`]]
```

$$\boxed{y' - 4y = 32x^2}$$

4.2.1 Solving as linear ode

Entering Linear first order ODE solver. In canonical form a linear first order is

$$y' + p(x)y = q(x)$$

Where here

$$p(x) = -4$$
$$q(x) = 32x^2$$

Hence the ode is

$$y' - 4y = 32x^2$$

The integrating factor μ is

$$\begin{aligned}\mu &= \mathrm{e}^{\int (-4)dx} \\ &= \mathrm{e}^{-4x}\end{aligned}$$

The ode becomes

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}x}(\mu y) &= (\mu)\left(32x^2\right) \\ \frac{\mathrm{d}}{\mathrm{d}x}\left(\mathrm{e}^{-4x} y\right) &= \left(\mathrm{e}^{-4x}\right)\left(32x^2\right) \\ \mathrm{d}\left(\mathrm{e}^{-4x} y\right) &= \left(32\,\mathrm{e}^{-4x} x^2\right)\, \mathrm{d}x\end{aligned}$$

Integrating gives

$$\begin{aligned}\mathrm{e}^{-4x} y &= \int 32\,\mathrm{e}^{-4x} x^2\, \mathrm{d}x \\ \mathrm{e}^{-4x} y &= -\left(8x^2 + 4x + 1\right) \mathrm{e}^{-4x} + c_1\end{aligned}$$

Dividing both sides by the integrating factor $\mu = \mathrm{e}^{-4x}$ results in

$$y = -\mathrm{e}^{4x}\left(8x^2 + 4x + 1\right) \mathrm{e}^{-4x} + c_1 \mathrm{e}^{4x}$$

which simplifies to

$$y = -8x^2 - 4x - 1 + c_1 \mathrm{e}^{4x}$$

Summary

The solution(s) found are the following

$$y = -8x^2 - 4x - 1 + c_1 \mathrm{e}^{4x} \tag{1}$$

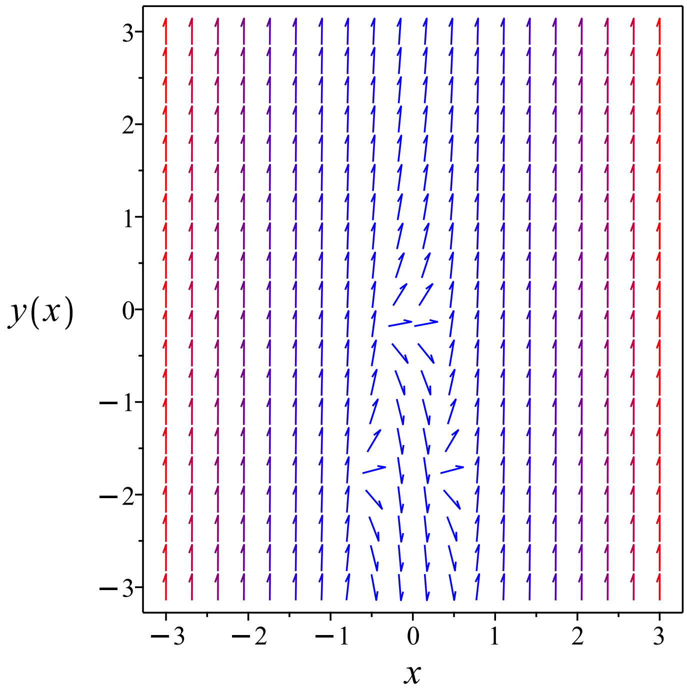


Figure 192: Slope field plot

Verification of solutions

$$y = -8x^2 - 4x - 1 + c_1 e^{4x}$$

Verified OK.

4.2.2 Solving as first order ode lie symmetry lookup ode

Writing the ode as

$$y' = 32x^2 + 4y$$
$$y' = \omega(x, y)$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x + \omega(\eta_y - \xi_x) - \omega^2 \xi_y - \omega_x \xi - \omega_y \eta = 0 \tag{A}$$

The type of this ode is known. It is of type `linear`. Therefore we do not need to solve the PDE (A), and can just use the lookup table shown below to find ξ, η

Table 186: Lie symmetry infinitesimal lookup table for known first order ODE's

ODE class	Form	ξ	η
linear ode	$y' = f(x)y(x) + g(x)$	0	$e^{\int f dx}$
separable ode	$y' = f(x)\, g(y)$	$\frac{1}{f}$	0
quadrature ode	$y' = f(x)$	0	1
quadrature ode	$y' = g(y)$	1	0
homogeneous ODEs of Class A	$y' = f\left(\frac{y}{x}\right)$	x	y
homogeneous ODEs of Class C	$y' = (a + bx + cy)^{\frac{n}{m}}$	1	$-\frac{b}{c}$
homogeneous class D	$y' = \frac{y}{x} + g(x)\, F\left(\frac{y}{x}\right)$	x^2	xy
First order special form ID 1	$y' = g(x)\, e^{h(x)+by} + f(x)$	$\frac{e^{-\int bf(x)dx-h(x)}}{g(x)}$	$\frac{f(x)e^{-\int bf(x)dx-h(x)}}{g(x)}$
polynomial type ode	$y' = \frac{a_1x+b_1y+c_1}{a_2x+b_2y+c_2}$	$\frac{a_1b_2x-a_2b_1x-b_1c_2+b_2c_1}{a_1b_2-a_2b_1}$	$\frac{a_1b_2y-a_2b_1y-a_1c_2-a_2c_1}{a_1b_2-a_2b_1}$
Bernoulli ode	$y' = f(x)\, y + g(x)\, y^n$	0	$e^{-\int (n-1)f(x)dx} y^n$
Reduced Riccati	$y' = f_1(x)\, y + f_2(x)\, y^2$	0	$e^{-\int f_1 dx}$

The above table shows that

$$\begin{aligned} \xi(x,y) &= 0 \\ \eta(x,y) &= e^{4x} \end{aligned} \tag{A1}$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x,y) \to (R,S)$ where (R,S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y}\right) S(x,y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the

canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$R = x$$

S is found from

$$\begin{aligned} S &= \int \frac{1}{\eta} dy \\ &= \int \frac{1}{e^{4x}} dy \end{aligned}$$

Which results in

$$S = e^{-4x} y$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x,y) S_y}{R_x + \omega(x,y) R_y} \tag{2}$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x, y)$ is the right hand side of the original ode given by

$$\omega(x,y) = 32x^2 + 4y$$

Evaluating all the partial derivatives gives

$$\begin{aligned} R_x &= 1 \\ R_y &= 0 \\ S_x &= -4\,e^{-4x} y \\ S_y &= e^{-4x} \end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = 32\,e^{-4x} x^2 \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = 32\,e^{-4R} R^2$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by

integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$S(R) = -\left(8R^2 + 4R + 1\right) e^{-4R} + c_1 \tag{4}$$

To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$e^{-4x} y = -\left(8x^2 + 4x + 1\right) e^{-4x} + c_1$$

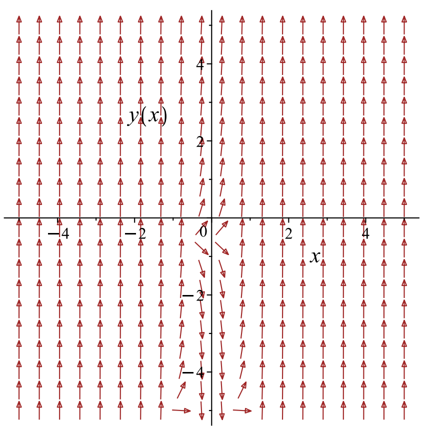
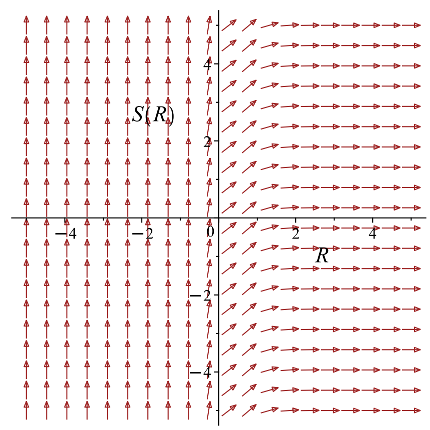
Which simplifies to

$$\left(8x^2 + 4x + y + 1\right) e^{-4x} - c_1 = 0$$

Which gives

$$y = -\left(8\,e^{-4x} x^2 + 4\,e^{-4x} x + e^{-4x} - c_1\right) e^{4x}$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in x, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dx} = 32x^2 + 4y$ 	$R = x$ $S = e^{-4x} y$	$\frac{dS}{dR} = 32\,e^{-4R} R^2$ 

Summary
The solution(s) found are the following

$$y = -\left(8\,e^{-4x} x^2 + 4\,e^{-4x} x + e^{-4x} - c_1\right) e^{4x} \tag{1}$$

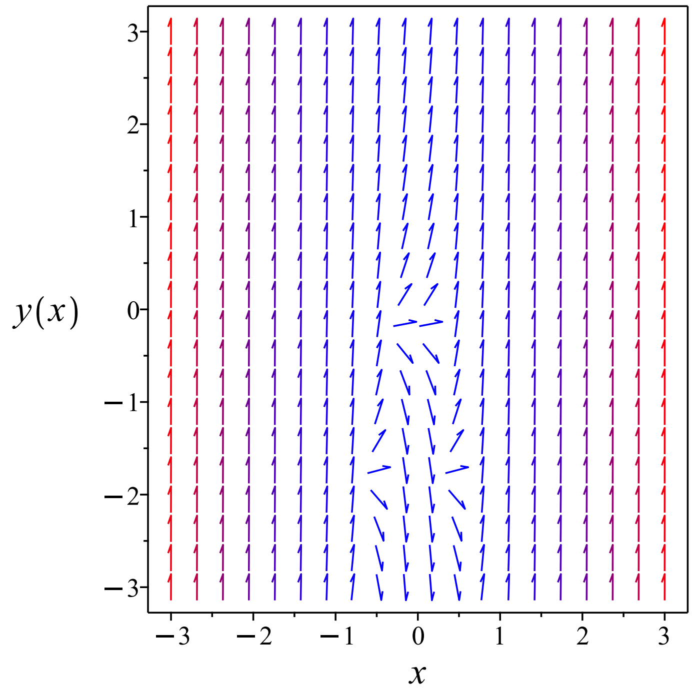


Figure 193: Slope field plot

Verification of solutions

$$y = -\left(8\,\mathrm{e}^{-4x}x^2 + 4\,\mathrm{e}^{-4x}x + \mathrm{e}^{-4x} - c_1\right)\mathrm{e}^{4x}$$

Verified OK.

4.2.3 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x,y) + N(x,y)\,\frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x,y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y) = 0$$

Hence

$$\frac{\partial\phi}{\partial x} + \frac{\partial\phi}{\partial y}\frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$
\begin{aligned}
\frac{\partial \phi}{\partial x} &= M \\
\frac{\partial \phi}{\partial y} &= N
\end{aligned}
$$

But since $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ then for the above to be valid, we require that

$$
\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}
$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x,y)$ but at least we know now that we can do that since the condition $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$
M(x,y)\,\mathrm{d}x + N(x,y)\,\mathrm{d}y = 0 \tag{1A}
$$

Therefore

$$
\begin{aligned}
\mathrm{d}y &= \left(32x^2 + 4y\right)\mathrm{d}x \\
\left(-32x^2 - 4y\right)\mathrm{d}x + \mathrm{d}y &= 0 \qquad (2A)
\end{aligned}
$$

Comparing (1A) and (2A) shows that

$$
\begin{aligned}
M(x,y) &= -32x^2 - 4y \\
N(x,y) &= 1
\end{aligned}
$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$
\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}
$$

Using result found above gives

$$
\begin{aligned}
\frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}\left(-32x^2 - 4y\right) \\
&= -4
\end{aligned}
$$

And

$$
\begin{aligned}
\frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}(1) \\
&= 0
\end{aligned}
$$

Since $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$, then the ODE is not exact. Since the ODE is not exact, we will try to find an integrating factor to make it exact. Let

$$
\begin{aligned}
A &= \frac{1}{N}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right) \\
&= 1((-4) - (0)) \\
&= -4
\end{aligned}
$$

Since A does not depend on y, then it can be used to find an integrating factor. The integrating factor μ is

$$
\begin{aligned}
\mu &= e^{\int A \, \mathrm{d}x} \\
&= e^{\int -4 \, \mathrm{d}x}
\end{aligned}
$$

The result of integrating gives

$$
\begin{aligned}
\mu &= e^{-4x} \\
&= \mathrm{e}^{-4x}
\end{aligned}
$$

M and N are multiplied by this integrating factor, giving new M and new N which are called $\overline{M}$ and $\overline{N}$ for now so not to confuse them with the original M and N.

$$
\begin{aligned}
\overline{M} &= \mu M \\
&= \mathrm{e}^{-4x}\left(-32x^2 - 4y\right) \\
&= \left(-32x^2 - 4y\right) \mathrm{e}^{-4x}
\end{aligned}
$$

And

$$
\begin{aligned}
\overline{N} &= \mu N \\
&= \mathrm{e}^{-4x}(1) \\
&= \mathrm{e}^{-4x}
\end{aligned}
$$

Now a modified ODE is ontained from the original ODE, which is exact and can be solved. The modified ODE is

$$
\begin{aligned}
\overline{M} + \overline{N}\frac{\mathrm{d}y}{\mathrm{d}x} &= 0 \\
\left(\left(-32x^2 - 4y\right) \mathrm{e}^{-4x}\right) + \left(\mathrm{e}^{-4x}\right) \frac{\mathrm{d}y}{\mathrm{d}x} &= 0
\end{aligned}
$$

The following equations are now set up to solve for the function $\phi(x, y)$

$$
\frac{\partial \phi}{\partial x} = \overline{M} \tag{1}
$$

$$
\frac{\partial \phi}{\partial y} = \overline{N} \tag{2}
$$

Integrating (1) w.r.t. x gives

$$\int \frac{\partial \phi}{\partial x}\,\mathrm{d}x = \int \overline{M}\,\mathrm{d}x$$

$$\int \frac{\partial \phi}{\partial x}\,\mathrm{d}x = \int \left(-32x^2 - 4y\right)\mathrm{e}^{-4x}\,\mathrm{d}x$$

$$\phi = \left(8x^2 + 4x + y + 1\right)\mathrm{e}^{-4x} + f(y) \tag{3}$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$\frac{\partial \phi}{\partial y} = \mathrm{e}^{-4x} + f'(y) \tag{4}$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = \mathrm{e}^{-4x}$. Therefore equation (4) becomes

$$\mathrm{e}^{-4x} = \mathrm{e}^{-4x} + f'(y) \tag{5}$$

Solving equation (5) for $f'(y)$ gives

$$f'(y) = 0$$

Therefore

$$f(y) = c_1$$

Where c_1 is constant of integration. Substituting this result for $f(y)$ into equation (3) gives ϕ

$$\phi = \left(8x^2 + 4x + y + 1\right)\mathrm{e}^{-4x} + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = \left(8x^2 + 4x + y + 1\right)\mathrm{e}^{-4x}$$

The solution becomes

$$y = -\left(8\,\mathrm{e}^{-4x}x^2 + 4\,\mathrm{e}^{-4x}x + \mathrm{e}^{-4x} - c_1\right)\mathrm{e}^{4x}$$

Summary
The solution(s) found are the following

$$y = -\left(8\,e^{-4x}x^2 + 4\,e^{-4x}x + e^{-4x} - c_1\right)e^{4x} \tag{1}$$

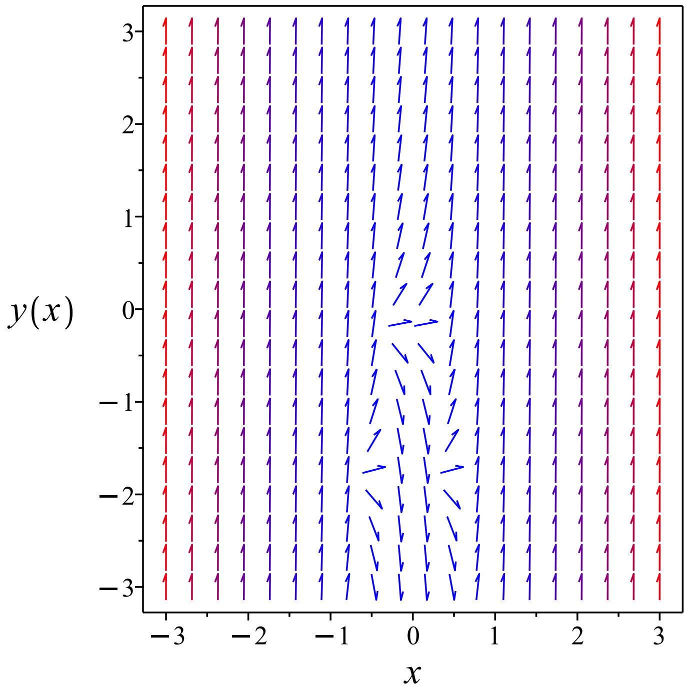


Figure 194: Slope field plot

Verification of solutions

$$y = -\left(8\,e^{-4x}x^2 + 4\,e^{-4x}x + e^{-4x} - c_1\right)e^{4x}$$

Verified OK.

4.2.4 Maple step by step solution

Let's solve

$y' - 4y = 32x^2$

- Highest derivative means the order of the ODE is 1

  y'

- Isolate the derivative

  $y' = 4y + 32x^2$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE

  $y' - 4y = 32x^2$
- The ODE is linear; multiply by an integrating factor $\mu(x)$

  $\mu(x)(y' - 4y) = 32\mu(x) x^2$
- Assume the lhs of the ODE is the total derivative $\frac{d}{dx}(\mu(x) y)$

  $\mu(x)(y' - 4y) = \mu'(x) y + \mu(x) y'$
- Isolate $\mu'(x)$

  $\mu'(x) = -4\mu(x)$
- Solve to find the integrating factor

  $\mu(x) = e^{-4x}$
- Integrate both sides with respect to x

  $\int \left(\frac{d}{dx}(\mu(x) y)\right) dx = \int 32\mu(x) x^2 dx + c_1$
- Evaluate the integral on the lhs

  $\mu(x) y = \int 32\mu(x) x^2 dx + c_1$
- Solve for y

  $y = \frac{\int 32\mu(x)x^2 dx + c_1}{\mu(x)}$
- Substitute $\mu(x) = e^{-4x}$

  $y = \frac{\int 32 e^{-4x} x^2 dx + c_1}{e^{-4x}}$
- Evaluate the integrals on the rhs

  $y = \frac{-(8x^2+4x+1)e^{-4x} + c_1}{e^{-4x}}$
- Simplify

  $y = -8x^2 - 4x - 1 + c_1 e^{4x}$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
<- 1st order linear successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 20

```
dsolve(diff(y(x),x)-4*y(x)=32*x^2,y(x), singsol=all)
```

$$y(x) = -8x^2 - 4x - 1 + e^{4x}c_1$$

✓ Solution by Mathematica

Time used: 0.05 (sec). Leaf size: 23

```
DSolve[y'[x]-4*y[x]==32*x^2,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to -8x^2 - 4x + c_1e^{4x} - 1$$

4.3 problem 3

Internal problem ID [4998]
Internal file name [`OUTPUT/4491_Sunday_June_05_2022_02_59_17_PM_80853047/index.tex`]

Book: Fundamentals of Differential Equations. By Nagle, Saff and Snider. 9th edition. Boston. Pearson 2018.
Section: Chapter 2, First order differential equations. Review problems. page 79
Problem number: 3.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"exact"**

Maple gives the following as the ode type

`[_exact, _rational]`

$$\boxed{\left(x^2-\frac{2}{y^3}\right)y'+2xy=3x^2}$$

4.3.1 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x,y)+N(x,y)\frac{dy}{dx}=0 \tag{A}$$

We assume there exists a function $\phi(x,y)=c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y)=0$$

Hence

$$\frac{\partial\phi}{\partial x}+\frac{\partial\phi}{\partial y}\frac{dy}{dx}=0 \tag{B}$$

Comparing (A,B) shows that

$$
\begin{aligned}
\frac{\partial \phi}{\partial x} &= M \\
\frac{\partial \phi}{\partial y} &= N
\end{aligned}
$$

But since $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ then for the above to be valid, we require that

$$
\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}
$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x, y)$ but at least we know now that we can do that since the condition $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$
M(x, y) \, \mathrm{d}x + N(x, y) \, \mathrm{d}y = 0 \tag{1A}
$$

Therefore

$$
\begin{aligned}
\left(x^2 - \frac{2}{y^3}\right) \mathrm{d}y &= \left(3x^2 - 2xy\right) \mathrm{d}x \\
\left(-3x^2 + 2xy\right) \mathrm{d}x + \left(x^2 - \frac{2}{y^3}\right) \mathrm{d}y &= 0 \qquad \text{(2A)}
\end{aligned}
$$

Comparing (1A) and (2A) shows that

$$
\begin{aligned}
M(x, y) &= -3x^2 + 2xy \\
N(x, y) &= x^2 - \frac{2}{y^3}
\end{aligned}
$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$
\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}
$$

Using result found above gives

$$
\begin{aligned}
\frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}\left(-3x^2 + 2xy\right) \\
&= 2x
\end{aligned}
$$

And

$$
\begin{aligned}
\frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}\left(x^2 - \frac{2}{y^3}\right) \\
&= 2x
\end{aligned}
$$

Since $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$, then the ODE is exact The following equations are now set up to solve for the function $\phi(x, y)$

$$
\frac{\partial \phi}{\partial x} = M \tag{1}
$$

$$
\frac{\partial \phi}{\partial y} = N \tag{2}
$$

Integrating (1) w.r.t. x gives

$$
\begin{aligned}
\int \frac{\partial \phi}{\partial x} \,\mathrm{d}x &= \int M \,\mathrm{d}x \\
\int \frac{\partial \phi}{\partial x} \,\mathrm{d}x &= \int -3x^2 + 2xy \,\mathrm{d}x
\end{aligned}
$$

$$
\phi = -x^2(x - y) + f(y) \tag{3}
$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$
\frac{\partial \phi}{\partial y} = x^2 + f'(y) \tag{4}
$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = x^2 - \frac{2}{y^3}$. Therefore equation (4) becomes

$$
x^2 - \frac{2}{y^3} = x^2 + f'(y) \tag{5}
$$

Solving equation (5) for $f'(y)$ gives

$$
f'(y) = -\frac{2}{y^3}
$$

Integrating the above w.r.t y gives

$$
\begin{aligned}
\int f'(y) \,\mathrm{d}y &= \int \left(-\frac{2}{y^3}\right) \mathrm{d}y \\
f(y) &= \frac{1}{y^2} + c_1
\end{aligned}
$$

Where c_1 is constant of integration. Substituting result found above for $f(y)$ into equation (3) gives ϕ

$$\phi = -x^2(x-y) + \frac{1}{y^2} + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = -x^2(x-y) + \frac{1}{y^2}$$

Summary

The solution(s) found are the following

$$-x^2(x-y) + \frac{1}{y^2} = c_1 \tag{1}$$

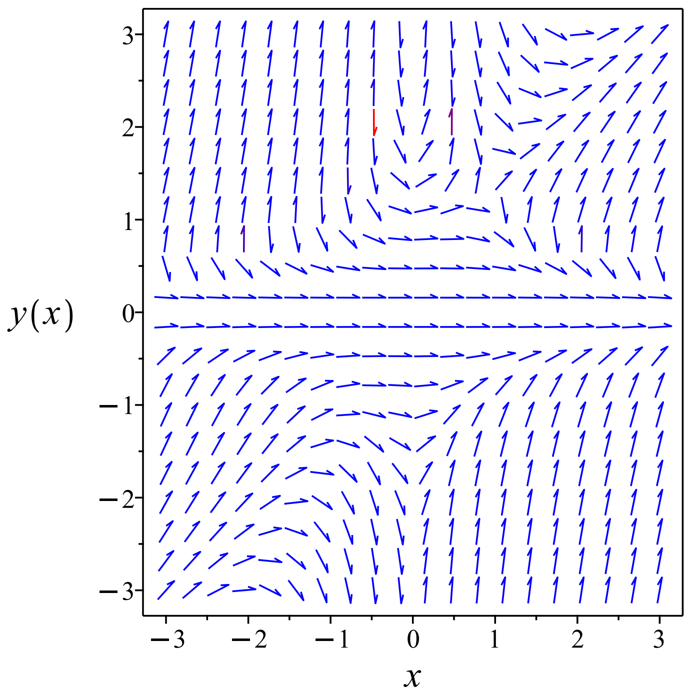


Figure 195: Slope field plot

Verification of solutions

$$-x^2(x-y) + \frac{1}{y^2} = c_1$$

Verified OK.

4.3.2 Maple step by step solution

Let's solve

$\left(x^2-\frac{2}{y^3}\right)y'+2xy=3x^2$

- Highest derivative means the order of the ODE is 1

  y'

□ Check if ODE is exact

  ◦ ODE is exact if the lhs is the total derivative of a C^2 function

    $F'(x,y)=0$

  ◦ Compute derivative of lhs

    $F'(x,y)+\left(\frac{\partial}{\partial y}F(x,y)\right)y'=0$

  ◦ Evaluate derivatives

    $2x=2x$

  ◦ Condition met, ODE is exact

- Exact ODE implies solution will be of this form

  $\left[F(x,y)=c_1, M(x,y)=F'(x,y), N(x,y)=\frac{\partial}{\partial y}F(x,y)\right]$

- Solve for $F(x,y)$ by integrating $M(x,y)$ with respect to x

  $F(x,y)=\int(-3x^2+2xy)\,dx+f_1(y)$

- Evaluate integral

  $F(x,y)=-x^3+y\,x^2+f_1(y)$

- Take derivative of $F(x,y)$ with respect to y

  $N(x,y)=\frac{\partial}{\partial y}F(x,y)$

- Compute derivative

  $x^2-\frac{2}{y^3}=x^2+\frac{d}{dy}f_1(y)$

- Isolate for $\frac{d}{dy}f_1(y)$

  $\frac{d}{dy}f_1(y)=-\frac{2}{y^3}$

- Solve for $f_1(y)$

  $f_1(y)=\frac{1}{y^2}$

- Substitute $f_1(y)$ into equation for $F(x,y)$

$$F(x,y) = -x^3 + y\,x^2 + \frac{1}{y^2}$$

- Substitute $F(x,y)$ into the solution of the ODE

$$-x^3 + y\,x^2 + \frac{1}{y^2} = c_1$$

- Solve for y

$$\left\{ y = \frac{\left(8x^9+24c_1x^6+24c_1^2x^3-108x^4+12\sqrt{-12x^9-36c_1x^6-36c_1^2x^3+81x^4-12c_1^3}\,x^2+8c_1^3\right)^{\frac{1}{3}}}{6x^2} + \frac{}{3x^2\left(8x^9+24c_1x^6+24c_1^2x^3-108x\right.} \right.$$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
trying homogeneous types:
trying Chini
differential order: 1; looking for linear symmetries
trying exact
<- exact successful`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 693

```
dsolve((x^2-2*y(x)^(-3))*diff(y(x),x)+(2*x*y(x)-3*x^2)=0,y(x), singsol=all)
```

$$y(x) = \frac{\frac{\left(8x^9-24c_1x^6+24c_1^2x^3+12\sqrt{3}\,\sqrt{-4x^9+12c_1x^6-12c_1^2x^3+27x^4+4c_1^3}\,x^2-108x^4-8c_1^3\right)^{\frac{1}{3}}}{2}+\frac{2\left(-x^3+c_1\right)^2}{\left(8x^9-24c_1x^6+24c_1^2x^3+12\sqrt{3}\,\sqrt{-4x^9+12c_1x^6-1}\right.}}{3x^2}$$

$$y(x) = \frac{\frac{\left(-i\sqrt{3}-1\right)\left(8x^9-24c_1x^6+24c_1^2x^3+12\sqrt{3}\,\sqrt{-4x^9+12c_1x^6-12c_1^2x^3+27x^4+4c_1^3}\,x^2-108x^4-8c_1^3\right)^{\frac{2}{3}}}{4}+\left(\left(8x^9-24c_1x^6+24c_1^2x^3+\right.\right.}{3\left(8x^9-24c_1x^6+24c_1^2x^3+12\sqrt{3}\,\sqrt{-4x^9+12c_1x\right.}$$

$$y(x) = \frac{\frac{\left(i\sqrt{3}-1\right)\left(8x^9-24c_1x^6+24c_1^2x^3+12\sqrt{3}\,\sqrt{-4x^9+12c_1x^6-12c_1^2x^3+27x^4+4c_1^3}\,x^2-108x^4-8c_1^3\right)^{\frac{2}{3}}}{4}+\left(x^3-c_1\right)\left(\left(8x^9-24c_1x^6+2\right.\right.}{3\left(8x^9-24c_1x^6+24c_1^2x^3+12\sqrt{3}\,\sqrt{-4x^9+12c_1\right.}$$

✓ Solution by Mathematica

Time used: 13.843 (sec). Leaf size: 676

```
DSolve[(x^2-2*y[x]^(-3))*y'[x]+(2*x*y[x]-3*x^2)==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \frac{2(x^3+c_1)+\frac{2(x^3+c_1)^2}{\sqrt[3]{x^9+3c_1x^6-\frac{27x^4}{2}+3c_1{}^2x^3+\frac{3}{2}\sqrt{3}\sqrt{-x^4\left(4x^9+12c_1x^6-27x^4+12c_1{}^2x^3+4c_1{}^3\right)}+c}}}{6x}$$

$$y(x) \to \frac{4(x^3+c_1)-\frac{2i\left(\sqrt{3}-i\right)(x^3+c_1)^2}{\sqrt[3]{x^9+3c_1x^6-\frac{27x^4}{2}+3c_1{}^2x^3+\frac{3}{2}\sqrt{3}\sqrt{-x^4\left(4x^9+12c_1x^6-27x^4+12c_1{}^2x^3+4c_1{}^3\right)}+c}}}{}$$

$$y(x) \to \frac{4(x^3+c_1)+\frac{2i\left(\sqrt{3}+i\right)(x^3+c_1)^2}{\sqrt[3]{x^9+3c_1x^6-\frac{27x^4}{2}+3c_1{}^2x^3+\frac{3}{2}\sqrt{3}\sqrt{-x^4\left(4x^9+12c_1x^6-27x^4+12c_1{}^2x^3+4c_1{}^3\right)}+c}}}{}$$

$$y(x) \to 0$$

4.4 problem 4

Internal problem ID [4999]
Internal file name [`OUTPUT/4492_Sunday_June_05_2022_02_59_19_PM_84428908/index.tex`]

Book: Fundamentals of Differential Equations. By Nagle, Saff and Snider. 9th edition. Boston. Pearson 2018.
Section: Chapter 2, First order differential equations. Review problems. page 79
Problem number: 4.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"linear", "exactWithIntegrationFactor", "first_order_ode_lie_symmetry_lookup"**

Maple gives the following as the ode type

```
[_linear]
```

$$\boxed{y' + \frac{3y}{x} = x^2 - 4x + 3}$$

4.4.1 Solving as linear ode

Entering Linear first order ODE solver. In canonical form a linear first order is

$$y' + p(x)y = q(x)$$

Where here

$$\begin{aligned} p(x) &= \frac{3}{x} \\ q(x) &= x^2 - 4x + 3 \end{aligned}$$

Hence the ode is

$$y' + \frac{3y}{x} = x^2 - 4x + 3$$

The integrating factor μ is

$$
\begin{aligned}
\mu &= e^{\int \frac{3}{x}dx} \\
&= x^3
\end{aligned}
$$

The ode becomes

$$
\begin{aligned}
\frac{d}{dx}(\mu y) &= (\mu)\left(x^2-4x+3\right) \\
\frac{d}{dx}\left(y\,x^3\right) &= \left(x^3\right)\left(x^2-4x+3\right) \\
d\left(y\,x^3\right) &= \left(x^5-4x^4+3x^3\right)\,dx
\end{aligned}
$$

Integrating gives

$$
\begin{aligned}
y\,x^3 &= \int x^5-4x^4+3x^3\,dx \\
y\,x^3 &= \frac{1}{6}x^6-\frac{4}{5}x^5+\frac{3}{4}x^4+c_1
\end{aligned}
$$

Dividing both sides by the integrating factor $\mu = x^3$ results in

$$
y = \frac{\frac{1}{6}x^6-\frac{4}{5}x^5+\frac{3}{4}x^4}{x^3}+\frac{c_1}{x^3}
$$

which simplifies to

$$
y = \frac{10x^6-48x^5+45x^4+60c_1}{60x^3}
$$

Summary

The solution(s) found are the following

$$
y = \frac{10x^6-48x^5+45x^4+60c_1}{60x^3} \tag{1}
$$

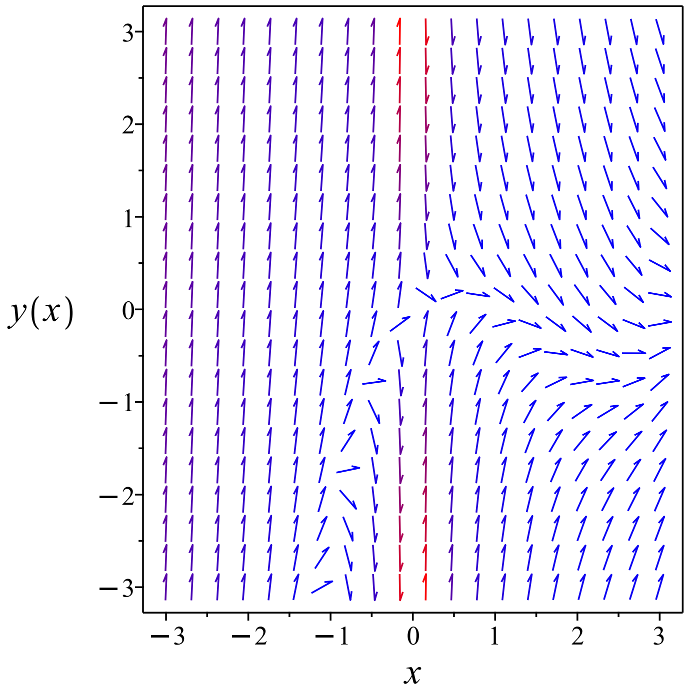


Figure 196: Slope field plot

Verification of solutions

$$y = \frac{10x^6 - 48x^5 + 45x^4 + 60c_1}{60x^3}$$

Verified OK.

4.4.2 Solving as first order ode lie symmetry lookup ode

Writing the ode as

$$y' = -\frac{-x^3 + 4x^2 - 3x + 3y}{x}$$
$$y' = \omega(x, y)$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x + \omega(\eta_y - \xi_x) - \omega^2 \xi_y - \omega_x \xi - \omega_y \eta = 0 \tag{A}$$

The type of this ode is known. It is of type `linear`. Therefore we do not need to solve the PDE (A), and can just use the lookup table shown below to find ξ, η

Table 190: Lie symmetry infinitesimal lookup table for known first order ODE's

ODE class	Form	ξ	η
linear ode	$y' = f(x)y(x) + g(x)$	0	$e^{\int f dx}$
separable ode	$y' = f(x)\, g(y)$	$\frac{1}{f}$	0
quadrature ode	$y' = f(x)$	0	1
quadrature ode	$y' = g(y)$	1	0
homogeneous ODEs of Class A	$y' = f\left(\frac{y}{x}\right)$	x	y
homogeneous ODEs of Class C	$y' = (a + bx + cy)^{\frac{n}{m}}$	1	$-\frac{b}{c}$
homogeneous class D	$y' = \frac{y}{x} + g(x)\, F\left(\frac{y}{x}\right)$	x^2	xy
First order special form ID 1	$y' = g(x)\, e^{h(x)+by} + f(x)$	$\frac{e^{-\int bf(x)dx - h(x)}}{g(x)}$	$\frac{f(x)e^{-\int bf(x)dx - h(x)}}{g(x)}$
polynomial type ode	$y' = \frac{a_1x+b_1y+c_1}{a_2x+b_2y+c_2}$	$\frac{a_1b_2x - a_2b_1x - b_1c_2 + b_2c_1}{a_1b_2 - a_2b_1}$	$\frac{a_1b_2y - a_2b_1y - a_1c_2 - a_2c_1}{a_1b_2 - a_2b_1}$
Bernoulli ode	$y' = f(x)\, y + g(x)\, y^n$	0	$e^{-\int (n-1)f(x)dx} y^n$
Reduced Riccati	$y' = f_1(x)\, y + f_2(x)\, y^2$	0	$e^{-\int f_1 dx}$

The above table shows that

$$
\begin{aligned}
\xi(x, y) &= 0 \\
\eta(x, y) &= \frac{1}{x^3}
\end{aligned}
\tag{A1}
$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x, y) \to (R, S)$ where (R, S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$
\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}
$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y}\right) S(x, y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the

canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$R = x$$

S is found from

$$\begin{aligned} S &= \int \frac{1}{\eta} dy \\ &= \int \frac{1}{\frac{1}{x^3}} dy \end{aligned}$$

Which results in

$$S = y\, x^3$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x, y) S_y}{R_x + \omega(x, y) R_y} \tag{2}$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x, y)$ is the right hand side of the original ode given by

$$\omega(x, y) = -\frac{-x^3 + 4x^2 - 3x + 3y}{x}$$

Evaluating all the partial derivatives gives

$$\begin{aligned} R_x &= 1 \\ R_y &= 0 \\ S_x &= 3y\, x^2 \\ S_y &= x^3 \end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = x^5 - 4x^4 + 3x^3 \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = R^5 - 4R^4 + 3R^3$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by

integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$S(R) = \frac{1}{6}R^6 - \frac{4}{5}R^5 + \frac{3}{4}R^4 + c_1 \tag{4}$$

To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$yx^3 = \frac{1}{6}x^6 - \frac{4}{5}x^5 + \frac{3}{4}x^4 + c_1$$

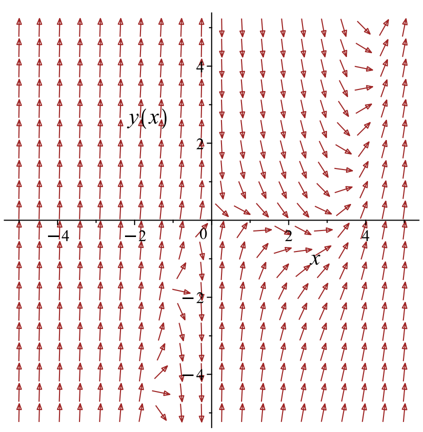
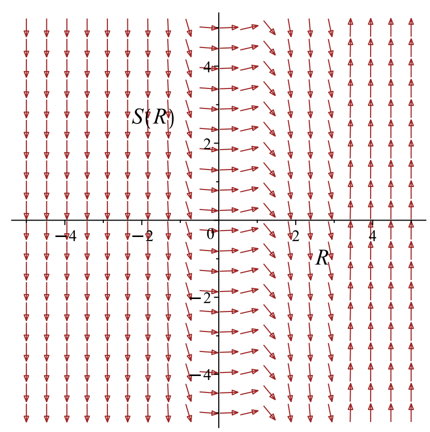
Which simplifies to

$$yx^3 = \frac{1}{6}x^6 - \frac{4}{5}x^5 + \frac{3}{4}x^4 + c_1$$

Which gives

$$y = \frac{10x^6 - 48x^5 + 45x^4 + 60c_1}{60x^3}$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in x, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dx} = -\frac{-x^3+4x^2-3x+3y}{x}$ 	$R = x$ $S = y\,x^3$	$\frac{dS}{dR} = R^5 - 4R^4 + 3R^3$ 

Summary
The solution(s) found are the following

$$y = \frac{10x^6 - 48x^5 + 45x^4 + 60c_1}{60x^3} \tag{1}$$

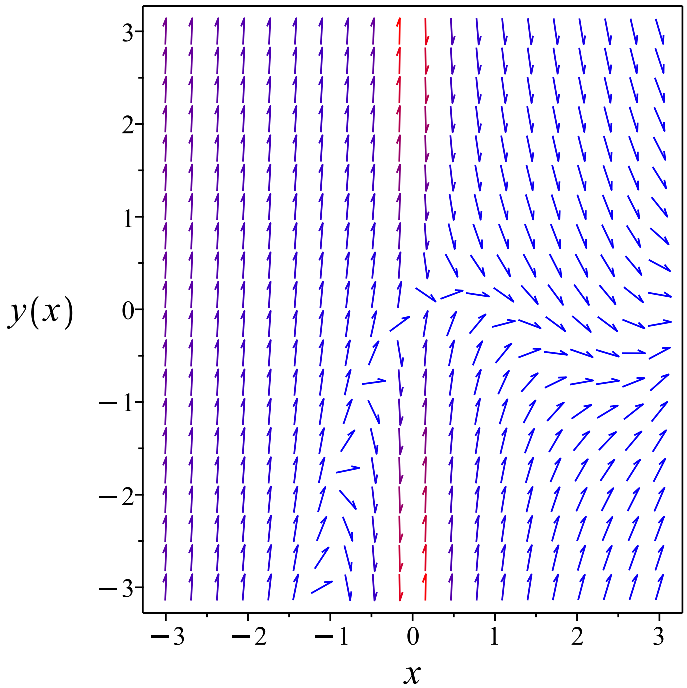


Figure 197: Slope field plot

Verification of solutions

$$y = \frac{10x^6 - 48x^5 + 45x^4 + 60c_1}{60x^3}$$

Verified OK.

4.4.3 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x,y) + N(x,y)\frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x,y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y) = 0$$

Hence

$$\frac{\partial \phi}{\partial x} + \frac{\partial \phi}{\partial y}\frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$\begin{aligned}\frac{\partial \phi}{\partial x} &= M \\ \frac{\partial \phi}{\partial y} &= N\end{aligned}$$

But since $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x,y)$ but at least we know now that we can do that since the condition $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x,y)\,\mathrm{d}x + N(x,y)\,\mathrm{d}y = 0 \tag{1A}$$

Therefore

$$\begin{aligned}\mathrm{d}y &= \left(-\frac{3y}{x} + x^2 - 4x + 3\right)\mathrm{d}x \\ \left(-x^2 + 4x - 3 + \frac{3y}{x}\right)\mathrm{d}x + \mathrm{d}y &= 0 \qquad (2A)\end{aligned}$$

Comparing (1A) and (2A) shows that

$$\begin{aligned}M(x,y) &= -x^2 + 4x - 3 + \frac{3y}{x} \\ N(x,y) &= 1\end{aligned}$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Using result found above gives

$$\begin{aligned}\frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}\left(-x^2 + 4x - 3 + \frac{3y}{x}\right) \\ &= \frac{3}{x}\end{aligned}$$

And

$$
\begin{aligned}
\frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}(1) \\
&= 0
\end{aligned}
$$

Since $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$, then the ODE is <u>not exact</u>. Since the ODE is not exact, we will try to find an integrating factor to make it exact. Let

$$
\begin{aligned}
A &= \frac{1}{N}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right) \\
&= 1\left(\left(\frac{3}{x}\right) - (0)\right) \\
&= \frac{3}{x}
\end{aligned}
$$

Since A does not depend on y, then it can be used to find an integrating factor. The integrating factor μ is

$$
\begin{aligned}
\mu &= e^{\int A \, \mathrm{d}x} \\
&= e^{\int \frac{3}{x} \, \mathrm{d}x}
\end{aligned}
$$

The result of integrating gives

$$
\begin{aligned}
\mu &= e^{3\ln(x)} \\
&= x^3
\end{aligned}
$$

M and N are multiplied by this integrating factor, giving new M and new N which are called $\overline{M}$ and $\overline{N}$ for now so not to confuse them with the original M and N.

$$
\begin{aligned}
\overline{M} &= \mu M \\
&= x^3\left(-x^2 + 4x - 3 + \frac{3y}{x}\right) \\
&= -x^2\left(x^3 - 4x^2 + 3x - 3y\right)
\end{aligned}
$$

And

$$
\begin{aligned}
\overline{N} &= \mu N \\
&= x^3(1) \\
&= x^3
\end{aligned}
$$

Now a modified ODE is ontained from the original ODE, which is exact and can be solved. The modified ODE is

$$\overline{M} + \overline{N}\frac{\mathrm{d}y}{\mathrm{d}x} = 0$$
$$\left(-x^2\left(x^3 - 4x^2 + 3x - 3y\right)\right) + \left(x^3\right)\frac{\mathrm{d}y}{\mathrm{d}x} = 0$$

The following equations are now set up to solve for the function $\phi(x, y)$

$$\frac{\partial \phi}{\partial x} = \overline{M} \tag{1}$$
$$\frac{\partial \phi}{\partial y} = \overline{N} \tag{2}$$

Integrating (1) w.r.t. x gives

$$\int \frac{\partial \phi}{\partial x}\,\mathrm{d}x = \int \overline{M}\,\mathrm{d}x$$
$$\int \frac{\partial \phi}{\partial x}\,\mathrm{d}x = \int -x^2\left(x^3 - 4x^2 + 3x - 3y\right)\mathrm{d}x$$
$$\phi = -\frac{1}{6}x^6 + \frac{4}{5}x^5 - \frac{3}{4}x^4 + y\,x^3 + f(y) \tag{3}$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$\frac{\partial \phi}{\partial y} = x^3 + f'(y) \tag{4}$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = x^3$. Therefore equation (4) becomes

$$x^3 = x^3 + f'(y) \tag{5}$$

Solving equation (5) for $f'(y)$ gives

$$f'(y) = 0$$

Therefore

$$f(y) = c_1$$

Where c_1 is constant of integration. Substituting this result for $f(y)$ into equation (3) gives ϕ

$$\phi = -\frac{1}{6}x^6 + \frac{4}{5}x^5 - \frac{3}{4}x^4 + y\,x^3 + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = -\frac{1}{6}x^6 + \frac{4}{5}x^5 - \frac{3}{4}x^4 + y\,x^3$$

The solution becomes

$$y = \frac{10x^6 - 48x^5 + 45x^4 + 60c_1}{60x^3}$$

Summary

The solution(s) found are the following

$$y = \frac{10x^6 - 48x^5 + 45x^4 + 60c_1}{60x^3} \tag{1}$$

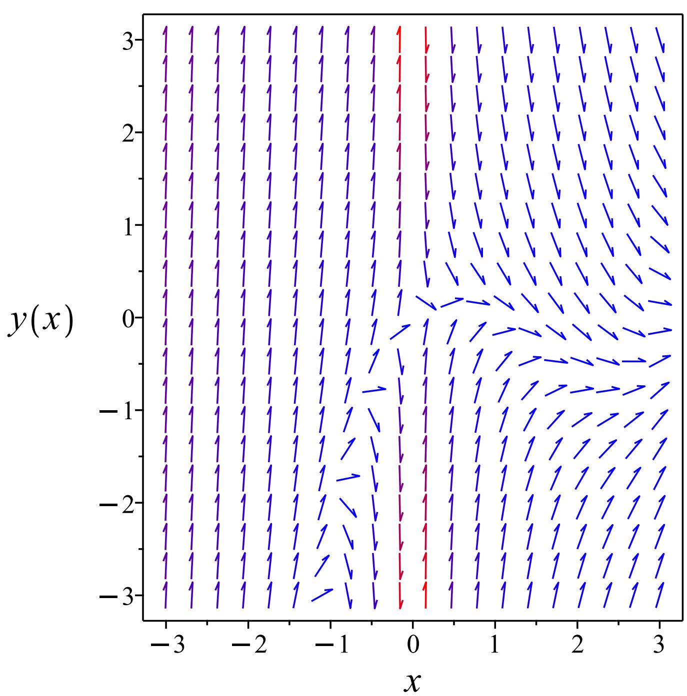


Figure 198: Slope field plot

Verification of solutions

$$y = \frac{10x^6 - 48x^5 + 45x^4 + 60c_1}{60x^3}$$

Verified OK.

4.4.4 Maple step by step solution

Let's solve

$y' + \frac{3y}{x} = x^2 - 4x + 3$

- Highest derivative means the order of the ODE is 1

  y'
- Isolate the derivative

  $y' = -\frac{3y}{x} + x^2 - 4x + 3$
- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE

  $y' + \frac{3y}{x} = x^2 - 4x + 3$
- The ODE is linear; multiply by an integrating factor $\mu(x)$

  $\mu(x)\left(y' + \frac{3y}{x}\right) = \mu(x)\left(x^2 - 4x + 3\right)$
- Assume the lhs of the ODE is the total derivative $\frac{d}{dx}(\mu(x)\, y)$

  $\mu(x)\left(y' + \frac{3y}{x}\right) = \mu'(x)\, y + \mu(x)\, y'$
- Isolate $\mu'(x)$

  $\mu'(x) = \frac{3\mu(x)}{x}$
- Solve to find the integrating factor

  $\mu(x) = x^3$
- Integrate both sides with respect to x

  $\int \left(\frac{d}{dx}(\mu(x)\, y)\right) dx = \int \mu(x)\left(x^2 - 4x + 3\right) dx + c_1$
- Evaluate the integral on the lhs

  $\mu(x)\, y = \int \mu(x)\left(x^2 - 4x + 3\right) dx + c_1$
- Solve for y

  $y = \frac{\int \mu(x)\left(x^2 - 4x + 3\right) dx + c_1}{\mu(x)}$
- Substitute $\mu(x) = x^3$

$$y = \frac{\int (x^2-4x+3)x^3 dx + c_1}{x^3}$$

- Evaluate the integrals on the rhs

$$y = \frac{\frac{1}{6}x^6 - \frac{4}{5}x^5 + \frac{3}{4}x^4 + c_1}{x^3}$$

- Simplify

$$y = \frac{10x^6 - 48x^5 + 45x^4 + 60c_1}{60x^3}$$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
<- 1st order linear successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 23

```
dsolve(diff(y(x),x)+3*y(x)/x=x^2-4*x+3,y(x), singsol=all)
```

$$y(x) = \frac{x^3}{6} - \frac{4x^2}{5} + \frac{3x}{4} + \frac{c_1}{x^3}$$

✓ Solution by Mathematica

Time used: 0.029 (sec). Leaf size: 31

```
DSolve[y'[x]+3*y[x]/x==x^2-4*x+3,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \frac{x^3}{6} + \frac{c_1}{x^3} - \frac{4x^2}{5} + \frac{3x}{4}$$

4.5 problem 6

Internal problem ID [5000]
Internal file name [`OUTPUT/4493_Sunday_June_05_2022_02_59_20_PM_66378432/index.tex`]

Book: Fundamentals of Differential Equations. By Nagle, Saff and Snider. 9th edition. Boston. Pearson 2018.
Section: Chapter 2, First order differential equations. Review problems. page 79
Problem number: 6.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"exact", "separable", "first_order_ode_lie_symmetry_lookup"**

Maple gives the following as the ode type

```
[_separable]
```

$$\boxed{2y^3x - \left(-x^2+1\right)y' = 0}$$

4.5.1 Solving as separable ode

In canonical form the ODE is

$$\begin{aligned} y' &= F(x,y) \\ &= f(x)g(y) \\ &= -\frac{2x\,y^3}{x^2-1} \end{aligned}$$

Where $f(x) = -\frac{2x}{x^2-1}$ and $g(y) = y^3$. Integrating both sides gives

$$\begin{aligned} \frac{1}{y^3}\,dy &= -\frac{2x}{x^2-1}\,dx \\ \int \frac{1}{y^3}\,dy &= \int -\frac{2x}{x^2-1}\,dx \end{aligned}$$

$$-\frac{1}{2y^2} = -\ln(x-1) - \ln(x+1) + c_1$$

Which results in

$$y = -\frac{1}{\sqrt{-2c_1 + 2\ln(x-1) + 2\ln(x+1)}}$$

$$y = \frac{1}{\sqrt{-2c_1 + 2\ln(x-1) + 2\ln(x+1)}}$$

Summary

The solution(s) found are the following

$$y = -\frac{1}{\sqrt{-2c_1 + 2\ln(x-1) + 2\ln(x+1)}} \tag{1}$$

$$y = \frac{1}{\sqrt{-2c_1 + 2\ln(x-1) + 2\ln(x+1)}} \tag{2}$$

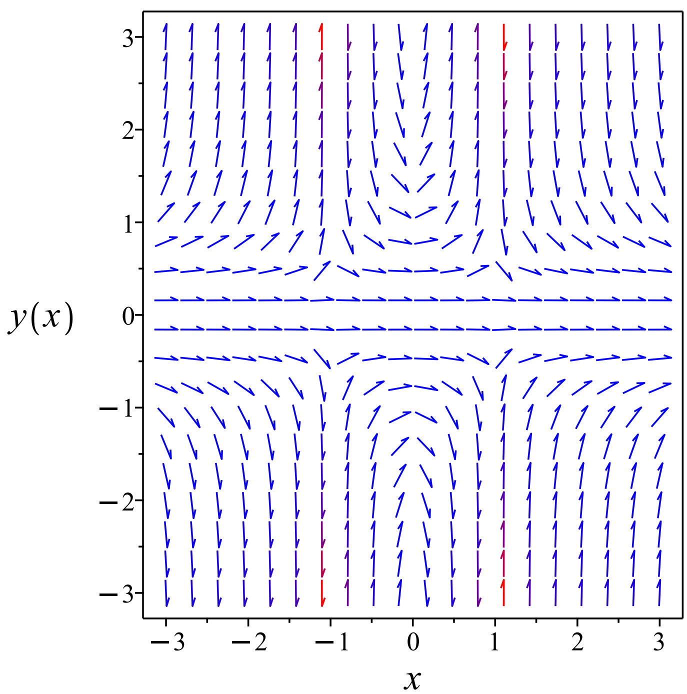


Figure 199: Slope field plot

Verification of solutions

$$y = -\frac{1}{\sqrt{-2c_1 + 2\ln(x-1) + 2\ln(x+1)}}$$

Verified OK.

$$y = \frac{1}{\sqrt{-2c_1 + 2\ln(x-1) + 2\ln(x+1)}}$$

Verified OK.

4.5.2 Solving as first order ode lie symmetry lookup ode

Writing the ode as

$$y' = -\frac{2x\,y^3}{x^2-1}$$
$$y' = \omega(x,y)$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x + \omega(\eta_y - \xi_x) - \omega^2\xi_y - \omega_x\xi - \omega_y\eta = 0 \tag{A}$$

The type of this ode is known. It is of type `separable`. Therefore we do not need to solve the PDE (A), and can just use the lookup table shown below to find ξ, η

Table 193: Lie symmetry infinitesimal lookup table for known first order ODE's

ODE class	Form	ξ	η
linear ode	$y' = f(x)y(x) + g(x)$	0	$e^{\int f dx}$
separable ode	$y' = f(x)\, g(y)$	$\frac{1}{f}$	0
quadrature ode	$y' = f(x)$	0	1
quadrature ode	$y' = g(y)$	1	0
homogeneous ODEs of Class A	$y' = f\left(\frac{y}{x}\right)$	x	y
homogeneous ODEs of Class C	$y' = (a + bx + cy)^{\frac{n}{m}}$	1	$-\frac{b}{c}$
homogeneous class D	$y' = \frac{y}{x} + g(x)\, F\left(\frac{y}{x}\right)$	x^2	xy
First order special form ID 1	$y' = g(x)\, e^{h(x)+by} + f(x)$	$\frac{e^{-\int bf(x)dx - h(x)}}{g(x)}$	$\frac{f(x)e^{-\int bf(x)dx - h(x)}}{g(x)}$
polynomial type ode	$y' = \frac{a_1x+b_1y+c_1}{a_2x+b_2y+c_2}$	$\frac{a_1b_2x - a_2b_1x - b_1c_2 + b_2c_1}{a_1b_2 - a_2b_1}$	$\frac{a_1b_2y - a_2b_1y - a_1c_2 - a_2c_1}{a_1b_2 - a_2b_1}$
Bernoulli ode	$y' = f(x)\, y + g(x)\, y^n$	0	$e^{-\int (n-1)f(x)dx} y^n$
Reduced Riccati	$y' = f_1(x)\, y + f_2(x)\, y^2$	0	$e^{-\int f_1 dx}$

The above table shows that

$$
\begin{aligned}
\xi(x, y) &= -\frac{x^2 - 1}{2x} \\
\eta(x, y) &= 0
\end{aligned} \tag{A1}
$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x, y) \to (R, S)$ where (R, S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$
\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}
$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y}\right) S(x, y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the

canonical coordinates, where $S(R)$. Since $\eta = 0$ then in this special case

$$R = y$$

S is found from

$$\begin{aligned} S &= \int \frac{1}{\xi} dx \\ &= \int \frac{1}{-\frac{x^2-1}{2x}} dx \end{aligned}$$

Which results in

$$S = -\ln(x-1) - \ln(x+1)$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x,y) S_y}{R_x + \omega(x,y) R_y} \tag{2}$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x,y)$ is the right hand side of the original ode given by

$$\omega(x,y) = -\frac{2x\,y^3}{x^2-1}$$

Evaluating all the partial derivatives gives

$$\begin{aligned} R_x &= 0 \\ R_y &= 1 \\ S_x &= -\frac{2x}{x^2-1} \\ S_y &= 0 \end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = \frac{1}{y^3} \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = \frac{1}{R^3}$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordinates R, S. Integrating the above gives

$$S(R) = -\frac{1}{2R^2} + c_1 \tag{4}$$

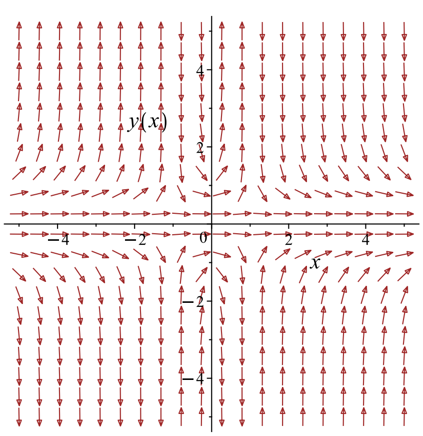
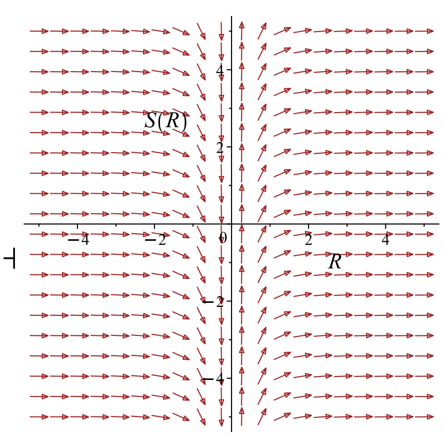
To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$-\ln(x-1) - \ln(x+1) = -\frac{1}{2y^2} + c_1$$

Which simplifies to

$$-\ln(x-1) - \ln(x+1) = -\frac{1}{2y^2} + c_1$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in x, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dx} = -\frac{2x\,y^3}{x^2-1}$ 	$R = y$ $S = -\ln(x-1) - \ln(x +$	$\frac{dS}{dR} = \frac{1}{R^3}$ 

Summary
The solution(s) found are the following

$$-\ln(x-1) - \ln(x+1) = -\frac{1}{2y^2} + c_1 \tag{1}$$

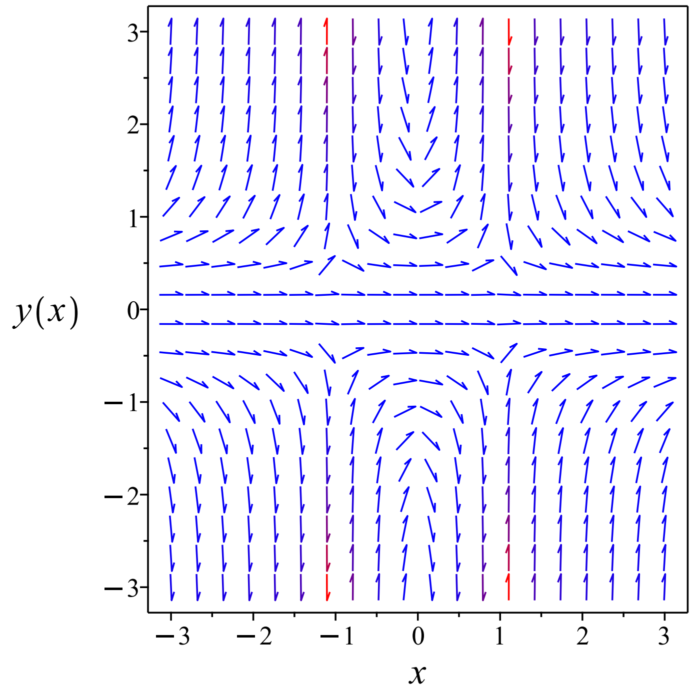


Figure 200: Slope field plot

<u>Verification of solutions</u>

$$
-\ln(x-1)-\ln(x+1)=-\frac{1}{2y^2}+c_1
$$

Verified OK.

4.5.3 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$
M(x,y)+N(x,y)\frac{dy}{dx}=0 \tag{A}
$$

We assume there exists a function $\phi(x,y)=c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$
\frac{d}{dx}\phi(x,y)=0
$$

Hence

$$
\frac{\partial\phi}{\partial x}+\frac{\partial\phi}{\partial y}\frac{dy}{dx}=0 \tag{B}
$$

Comparing (A,B) shows that

$$
\begin{aligned}
\frac{\partial \phi}{\partial x} &= M \\
\frac{\partial \phi}{\partial y} &= N
\end{aligned}
$$

But since $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ then for the above to be valid, we require that

$$
\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}
$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x, y)$ but at least we know now that we can do that since the condition $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$
M(x, y)\,\mathrm{d}x + N(x, y)\,\mathrm{d}y = 0 \tag{1A}
$$

Therefore

$$
\begin{aligned}
\left(-\frac{1}{2y^3}\right) \mathrm{d}y &= \left(\frac{x}{x^2-1}\right) \mathrm{d}x \\
\left(-\frac{x}{x^2-1}\right) \mathrm{d}x + \left(-\frac{1}{2y^3}\right) \mathrm{d}y &= 0 \qquad \text{(2A)}
\end{aligned}
$$

Comparing (1A) and (2A) shows that

$$
\begin{aligned}
M(x, y) &= -\frac{x}{x^2-1} \\
N(x, y) &= -\frac{1}{2y^3}
\end{aligned}
$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$
\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}
$$

Using result found above gives

$$
\begin{aligned}
\frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}\left(-\frac{x}{x^2-1}\right) \\
&= 0
\end{aligned}
$$

And

$$
\begin{aligned}
\frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}\left(-\frac{1}{2y^3}\right) \\
&= 0
\end{aligned}
$$

Since $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$, then the ODE is exact The following equations are now set up to solve for the function $\phi(x,y)$

$$
\frac{\partial \phi}{\partial x} = M \tag{1}
$$

$$
\frac{\partial \phi}{\partial y} = N \tag{2}
$$

Integrating (1) w.r.t. x gives

$$
\begin{aligned}
\int \frac{\partial \phi}{\partial x}\,\mathrm{d}x &= \int M\,\mathrm{d}x \\
\int \frac{\partial \phi}{\partial x}\,\mathrm{d}x &= \int -\frac{x}{x^2-1}\,\mathrm{d}x
\end{aligned}
$$

$$
\phi = -\frac{\ln(x-1)}{2} - \frac{\ln(x+1)}{2} + f(y) \tag{3}
$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$
\frac{\partial \phi}{\partial y} = 0 + f'(y) \tag{4}
$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = -\frac{1}{2y^3}$. Therefore equation (4) becomes

$$
-\frac{1}{2y^3} = 0 + f'(y) \tag{5}
$$

Solving equation (5) for $f'(y)$ gives

$$
f'(y) = -\frac{1}{2y^3}
$$

Integrating the above w.r.t y gives

$$\int f'(y)\,\mathrm{d}y = \int \left(-\frac{1}{2y^3}\right)\mathrm{d}y$$

$$f(y) = \frac{1}{4y^2} + c_1$$

Where c_1 is constant of integration. Substituting result found above for $f(y)$ into equation (3) gives ϕ

$$\phi = -\frac{\ln(x-1)}{2} - \frac{\ln(x+1)}{2} + \frac{1}{4y^2} + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = -\frac{\ln(x-1)}{2} - \frac{\ln(x+1)}{2} + \frac{1}{4y^2}$$

Summary

The solution(s) found are the following

$$-\frac{\ln(x-1)}{2} - \frac{\ln(x+1)}{2} + \frac{1}{4y^2} = c_1 \tag{1}$$

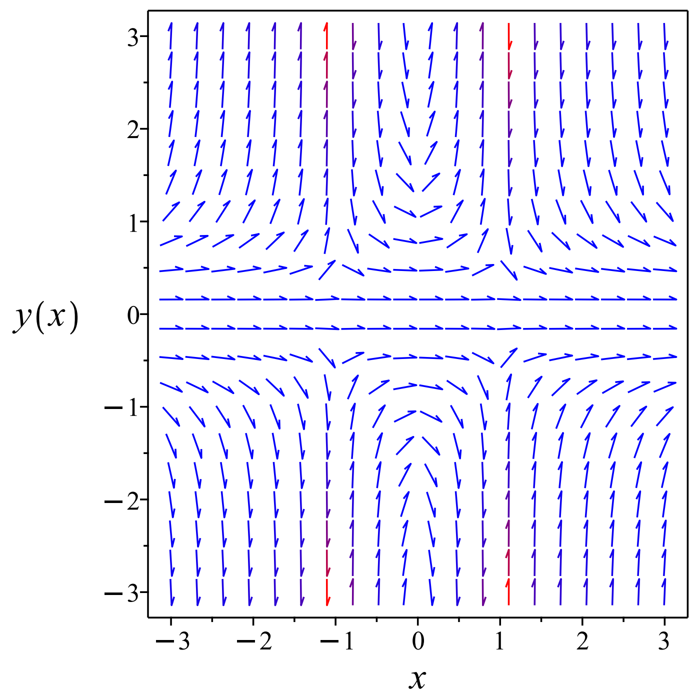


Figure 201: Slope field plot

Verification of solutions

$$-\frac{\ln(x-1)}{2}-\frac{\ln(x+1)}{2}+\frac{1}{4y^2}=c_1$$

Verified OK.

4.5.4 Maple step by step solution

Let's solve

$2y^3x-(-x^2+1)\,y'=0$

- Highest derivative means the order of the ODE is 1

  y'

- Separate variables

  $\frac{y'}{y^3}=\frac{2x}{-x^2+1}$

- Integrate both sides with respect to x

  $\int \frac{y'}{y^3}dx=\int \frac{2x}{-x^2+1}dx+c_1$

- Evaluate integral

$$-\frac{1}{2y^2} = -\ln(x-1) - \ln(x+1) + c_1$$

- Solve for y

$$\left\{y = \frac{1}{\sqrt{-2c_1 + 2\ln(x-1) + 2\ln(x+1)}}, y = -\frac{1}{\sqrt{-2c_1 + 2\ln(x-1) + 2\ln(x+1)}}\right\}$$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
<- Bernoulli successful`
```

✓ Solution by Maple

Time used: 0.015 (sec). Leaf size: 41

```
dsolve(2*x*y(x)^3-(1-x^2)*diff(y(x),x)=0,y(x), singsol=all)
```

$$y(x) = \frac{1}{\sqrt{c_1 + 2\ln(x-1) + 2\ln(x+1)}}$$
$$y(x) = -\frac{1}{\sqrt{c_1 + 2\ln(x-1) + 2\ln(x+1)}}$$

✓ Solution by Mathematica

Time used: 0.202 (sec). Leaf size: 57

```
DSolve[2*x*y[x]^3-(1-x^2)*y'[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to -\frac{1}{\sqrt{2}\sqrt{\log(x^2-1) - c_1}}$$
$$y(x) \to \frac{1}{\sqrt{2}\sqrt{\log(x^2-1) - c_1}}$$
$$y(x) \to 0$$

4.6 problem 7

Internal problem ID [5001]
Internal file name [`OUTPUT/4494_Sunday_June_05_2022_02_59_22_PM_41397006/index.tex`]

Book: Fundamentals of Differential Equations. By Nagle, Saff and Snider. 9th edition. Boston. Pearson 2018.
Section: Chapter 2, First order differential equations. Review problems. page 79
Problem number: 7.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"exact", "separable", "first_order_ode_lie_symmetry_lookup"**

Maple gives the following as the ode type

`[_separable]`

$$\boxed{t^3y^2 + \frac{t^4y'}{y^6} = 0}$$

4.6.1 Solving as separable ode

In canonical form the ODE is

$$\begin{aligned} y' &= F(t,y) \\ &= f(t)g(y) \\ &= -\frac{y^8}{t} \end{aligned}$$

Where $f(t) = -\frac{1}{t}$ and $g(y) = y^8$. Integrating both sides gives

$$\frac{1}{y^8}\,dy = -\frac{1}{t}\,dt$$

$$\int \frac{1}{y^8}\,dy = \int -\frac{1}{t}\,dt$$

$$-\frac{1}{7y^7} = -\ln(t) + c_1$$

The solution is

$$-\frac{1}{7y^7} + \ln(t) - c_1 = 0$$

Summary

The solution(s) found are the following

$$-\frac{1}{7y^7} + \ln(t) - c_1 = 0 \tag{1}$$

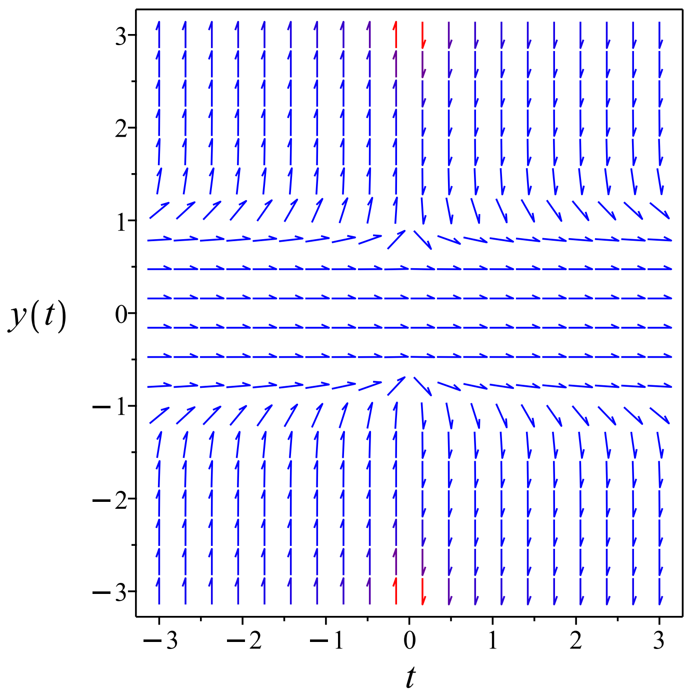


Figure 202: Slope field plot

Verification of solutions

$$-\frac{1}{7y^7} + \ln(t) - c_1 = 0$$

Verified OK.

4.6.2 Solving as first order ode lie symmetry lookup ode

Writing the ode as

$$
\begin{aligned}
y' &= -\frac{y^8}{t} \\
y' &= \omega(t, y)
\end{aligned}
$$

The condition of Lie symmetry is the linearized PDE given by

$$
\eta_t + \omega(\eta_y - \xi_t) - \omega^2 \xi_y - \omega_t \xi - \omega_y \eta = 0 \tag{A}
$$

The type of this ode is known. It is of type `separable`. Therefore we do not need to solve the PDE (A), and can just use the lookup table shown below to find ξ, η

Table 196: Lie symmetry infinitesimal lookup table for known first order ODE's

ODE class	Form	ξ	η
linear ode	$y' = f(x)y(x) + g(x)$	0	$e^{\int f dx}$
separable ode	$y' = f(x)\, g(y)$	$\frac{1}{f}$	0
quadrature ode	$y' = f(x)$	0	1
quadrature ode	$y' = g(y)$	1	0
homogeneous ODEs of Class A	$y' = f\left(\frac{y}{x}\right)$	x	y
homogeneous ODEs of Class C	$y' = (a + bx + cy)^{\frac{n}{m}}$	1	$-\frac{b}{c}$
homogeneous class D	$y' = \frac{y}{x} + g(x)\, F\left(\frac{y}{x}\right)$	x^2	xy
First order special form ID 1	$y' = g(x)\, e^{h(x)+by} + f(x)$	$\frac{e^{-\int bf(x)dx - h(x)}}{g(x)}$	$\frac{f(x)e^{-\int bf(x)dx - h(x)}}{g(x)}$
polynomial type ode	$y' = \frac{a_1x+b_1y+c_1}{a_2x+b_2y+c_2}$	$\frac{a_1b_2x - a_2b_1x - b_1c_2 + b_2c_1}{a_1b_2 - a_2b_1}$	$\frac{a_1b_2y - a_2b_1y - a_1c_2 - a_2c_1}{a_1b_2 - a_2b_1}$
Bernoulli ode	$y' = f(x)\, y + g(x)\, y^n$	0	$e^{-\int (n-1)f(x)dx} y^n$
Reduced Riccati	$y' = f_1(x)\, y + f_2(x)\, y^2$	0	$e^{-\int f_1 dx}$

The above table shows that

$$
\begin{aligned}
\xi(t,y) &= -t \\
\eta(t,y) &= 0
\end{aligned}
\tag{A1}
$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(t,y) \to (R,S)$ where (R,S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$
\frac{dt}{\xi} = \frac{dy}{\eta} = dS \tag{1}
$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial t} + \eta \frac{\partial}{\partial y}\right) S(t,y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the canonical coordinates, where $S(R)$. Since $\eta = 0$ then in this special case

$$
R = y
$$

S is found from

$$
\begin{aligned}
S &= \int \frac{1}{\xi} dt \\
&= \int \frac{1}{-t} dt
\end{aligned}
$$

Which results in

$$
S = -\ln(t)
$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$
\frac{dS}{dR} = \frac{S_t + \omega(t,y) S_y}{R_t + \omega(t,y) R_y} \tag{2}
$$

Where in the above R_t, R_y, S_t, S_y are all partial derivatives and $\omega(t,y)$ is the right hand side of the original ode given by

$$
\omega(t,y) = -\frac{y^8}{t}
$$

Evaluating all the partial derivatives gives

$$
\begin{aligned}
R_t &= 0 \\
R_y &= 1 \\
S_t &= -\frac{1}{t} \\
S_y &= 0
\end{aligned}
$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = \frac{1}{y^8} \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for t, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = \frac{1}{R^8}$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$S(R) = -\frac{1}{7R^7} + c_1 \tag{4}$$

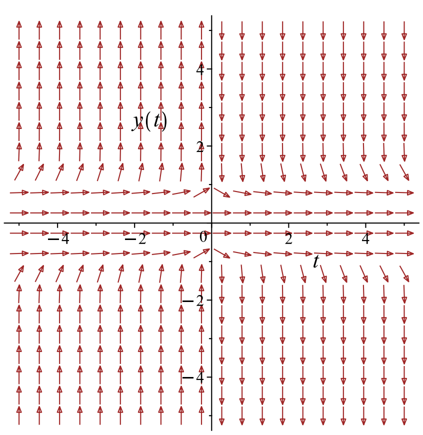
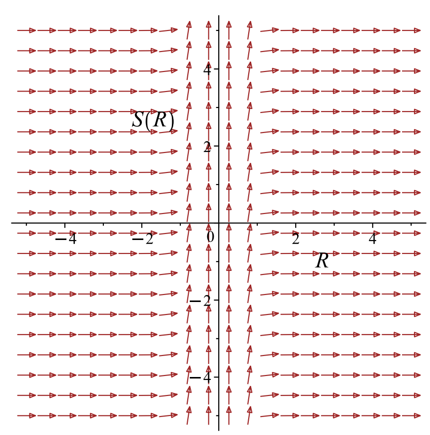
To complete the solution, we just need to transform (4) back to t, y coordinates. This results in

$$-\ln(t) = -\frac{1}{7y^7} + c_1$$

Which simplifies to

$$-\ln(t) = -\frac{1}{7y^7} + c_1$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in t, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dt} = -\frac{y^8}{t}$ 	$R = y$ $S = -\ln(t)$	$\frac{dS}{dR} = \frac{1}{R^8}$ 

Summary
The solution(s) found are the following

$$-\ln(t) = -\frac{1}{7y^7} + c_1 \tag{1}$$

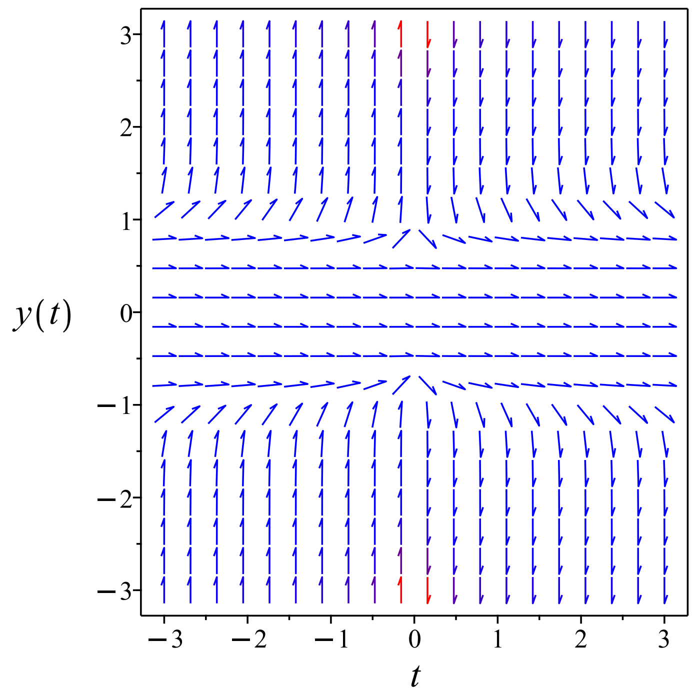


Figure 203: Slope field plot

Verification of solutions

$$-\ln(t) = -\frac{1}{7y^7} + c_1$$

Verified OK.

4.6.3 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x, y) + N(x, y)\frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x, y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x, y) = 0$$

Hence

$$\frac{\partial \phi}{\partial x}+\frac{\partial \phi}{\partial y}\frac{dy}{dx}=0 \tag{B}$$

Comparing (A,B) shows that

$$\begin{aligned}\frac{\partial \phi}{\partial x}&=M\\ \frac{\partial \phi}{\partial y}&=N\end{aligned}$$

But since $\frac{\partial^2\phi}{\partial x\partial y}=\frac{\partial^2\phi}{\partial y\partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y}=\frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x,y)$ but at least we know now that we can do that since the condition $\frac{\partial^2\phi}{\partial x\partial y}=\frac{\partial^2\phi}{\partial y\partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(t,y)\,\mathrm{d}t+N(t,y)\,\mathrm{d}y=0 \tag{1A}$$

Therefore

$$\begin{aligned}\left(-\frac{1}{y^8}\right)\mathrm{d}y&=\left(\frac{1}{t}\right)\mathrm{d}t\\ \left(-\frac{1}{t}\right)\mathrm{d}t+\left(-\frac{1}{y^8}\right)\mathrm{d}y&=0 \end{aligned}\tag{2A}$$

Comparing (1A) and (2A) shows that

$$\begin{aligned}M(t,y)&=-\frac{1}{t}\\ N(t,y)&=-\frac{1}{y^8}\end{aligned}$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y}=\frac{\partial N}{\partial t}$$

Using result found above gives

$$\begin{aligned}\frac{\partial M}{\partial y}&=\frac{\partial}{\partial y}\left(-\frac{1}{t}\right)\\ &=0\end{aligned}$$

And

$$
\begin{aligned}
\frac{\partial N}{\partial t} &= \frac{\partial}{\partial t}\left(-\frac{1}{y^8}\right) \\
&= 0
\end{aligned}
$$

Since $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial t}$, then the ODE is exact The following equations are now set up to solve for the function $\phi(t, y)$

$$
\frac{\partial \phi}{\partial t} = M \tag{1}
$$

$$
\frac{\partial \phi}{\partial y} = N \tag{2}
$$

Integrating (1) w.r.t. t gives

$$
\begin{aligned}
\int \frac{\partial \phi}{\partial t} \,\mathrm{d}t &= \int M \,\mathrm{d}t \\
\int \frac{\partial \phi}{\partial t} \,\mathrm{d}t &= \int -\frac{1}{t} \,\mathrm{d}t
\end{aligned}
$$

$$
\phi = -\ln(t) + f(y) \tag{3}
$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both t and y. Taking derivative of equation (3) w.r.t y gives

$$
\frac{\partial \phi}{\partial y} = 0 + f'(y) \tag{4}
$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = -\frac{1}{y^8}$. Therefore equation (4) becomes

$$
-\frac{1}{y^8} = 0 + f'(y) \tag{5}
$$

Solving equation (5) for $f'(y)$ gives

$$
f'(y) = -\frac{1}{y^8}
$$

Integrating the above w.r.t y gives

$$
\begin{aligned}
\int f'(y) \,\mathrm{d}y &= \int \left(-\frac{1}{y^8}\right) \mathrm{d}y \\
f(y) &= \frac{1}{7y^7} + c_1
\end{aligned}
$$

Where c_1 is constant of integration. Substituting result found above for $f(y)$ into equation (3) gives ϕ

$$\phi = -\ln(t) + \frac{1}{7y^7} + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = -\ln(t) + \frac{1}{7y^7}$$

Summary

The solution(s) found are the following

$$\frac{1}{7y^7} - \ln(t) = c_1 \tag{1}$$

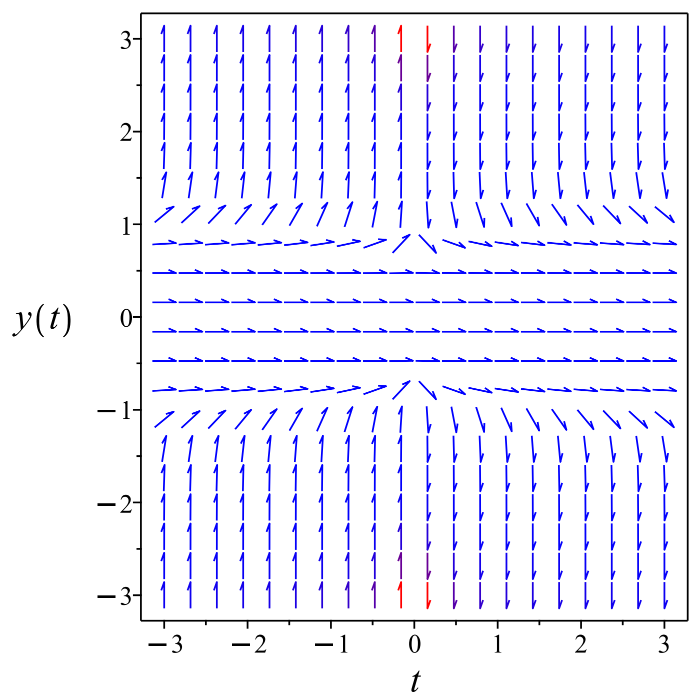


Figure 204: Slope field plot

Verification of solutions

$$\frac{1}{7y^7} - \ln(t) = c_1$$

Verified OK.

4.6.4 Maple step by step solution

Let's solve

$t^3 y^2 + \frac{t^4 y'}{y^6} = 0$

- Highest derivative means the order of the ODE is 1

  y'

- Separate variables

  $\frac{y'}{y^8} = -\frac{1}{t}$

- Integrate both sides with respect to t

  $\int \frac{y'}{y^8} dt = \int -\frac{1}{t} dt + c_1$

- Evaluate integral

  $-\frac{1}{7y^7} = -\ln(t) + c_1$

- Solve for y

  $y = \frac{7^{\frac{6}{7}}}{7(-c_1 + \ln(t))^{\frac{1}{7}}}$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
<- Bernoulli successful`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 105

```
dsolve(t^3*y(t)^2+t^4/(y(t)^6)*diff(y(t),t)=0,y(t), singsol=all)
```

$$y(t) = \frac{1}{(c_1 + 7\ln(t))^{\frac{1}{7}}}$$
$$y(t) = -\frac{(-1)^{\frac{1}{7}}}{(c_1 + 7\ln(t))^{\frac{1}{7}}}$$
$$y(t) = \frac{(-1)^{\frac{6}{7}}}{(c_1 + 7\ln(t))^{\frac{1}{7}}}$$
$$y(t) = -\frac{(-1)^{\frac{5}{7}}}{(c_1 + 7\ln(t))^{\frac{1}{7}}}$$
$$y(t) = \frac{(-1)^{\frac{2}{7}}}{(c_1 + 7\ln(t))^{\frac{1}{7}}}$$
$$y(t) = -\frac{(-1)^{\frac{3}{7}}}{(c_1 + 7\ln(t))^{\frac{1}{7}}}$$
$$y(t) = \frac{(-1)^{\frac{4}{7}}}{(c_1 + 7\ln(t))^{\frac{1}{7}}}$$

✓ Solution by Mathematica

Time used: 0.182 (sec). Leaf size: 183

```
DSolve[t^3*y[t]^2+t^4/(y[t]^6)*y'[t]==0,y[t],t,IncludeSingularSolutions -> True]
```

$$y(t) \to -\frac{\sqrt[7]{-\frac{1}{7}}}{\sqrt[7]{\log(t) - c_1}}$$
$$y(t) \to \frac{1}{\sqrt[7]{7}\sqrt[7]{\log(t) - c_1}}$$
$$y(t) \to \frac{(-1)^{2/7}}{\sqrt[7]{7}\sqrt[7]{\log(t) - c_1}}$$
$$y(t) \to -\frac{(-1)^{3/7}}{\sqrt[7]{7}\sqrt[7]{\log(t) - c_1}}$$
$$y(t) \to \frac{(-1)^{4/7}}{\sqrt[7]{7}\sqrt[7]{\log(t) - c_1}}$$
$$y(t) \to -\frac{(-1)^{5/7}}{\sqrt[7]{7}\sqrt[7]{\log(t) - c_1}}$$
$$y(t) \to \frac{(-1)^{6/7}}{\sqrt[7]{7}\sqrt[7]{\log(t) - c_1}}$$
$$y(t) \to 0$$

5 Chapter 8, Series solutions of differential equations. Section 8.3. page 443

5.1 problem 1

Internal problem ID [5002]
Internal file name [`OUTPUT/4495_Sunday_June_05_2022_02_59_23_PM_39274974/index.tex`]

Book: Fundamentals of Differential Equations. By Nagle, Saff and Snider. 9th edition. Boston. Pearson 2018.
Section: Chapter 8, Series solutions of differential equations. Section 8.3. page 443
Problem number: 1.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second order series method. Ordinary point", "second order series method. Taylor series method"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

$$\boxed{(x+1)\,y''-x^2y'+3y=0}$$

With the expansion point for the power series method at $x=0$.

Solving ode using Taylor series method. This gives review on how the Taylor series method works for solving second order ode.

Let

$$y''=f(x,y,y')$$

Assuming expansion is at $x_0=0$ (we can always shift the actual expansion point to 0 by change of variables) and assuming $f(x,y,y')$ is analytic at x_0 which must be the case for an ordinary point. Let initial conditions be $y(x_0)=y_0$ and $y'(x_0)=y'_0$. Using Taylor series gives

$$\begin{aligned}
y(x) &= y(x_0)+(x-x_0)\,y'(x_0)+\frac{(x-x_0)^2}{2}y''(x_0)+\frac{(x-x_0)^3}{3!}y'''(x_0)+\cdots\\
&= y_0+xy'_0+\frac{x^2}{2}\,f|_{x_0,y_0,y'_0}+\frac{x^3}{3!}\,f'|_{x_0,y_0,y'_0}+\cdots\\
&= y_0+xy'_0+\sum_{n=0}^{\infty}\frac{x^{n+2}}{(n+2)\,!}\,\frac{d^nf}{dx^n}\bigg|_{x_0,y_0,y'_0}
\end{aligned}$$

But

$$
\begin{aligned}
\frac{df}{dx} &= \frac{\partial f}{\partial x}\frac{dx}{dx} + \frac{\partial f}{\partial y}\frac{dy}{dx} + \frac{\partial f}{\partial y'}\frac{dy'}{dx} && (1) \\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}y' + \frac{\partial f}{\partial y'}y'' && (223) \\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}y' + \frac{\partial f}{\partial y'}f && (224) \\
\frac{d^2 f}{dx^2} &= \frac{d}{dx}\left(\frac{df}{dx}\right) \\
&= \frac{\partial}{\partial x}\left(\frac{df}{dx}\right) + \frac{\partial}{\partial y}\left(\frac{df}{dx}\right)y' + \frac{\partial}{\partial y'}\left(\frac{df}{dx}\right)f && (2) \\
\frac{d^3 f}{dx^3} &= \frac{d}{dx}\left(\frac{d^2 f}{dx^2}\right) \\
&= \frac{\partial}{\partial x}\left(\frac{d^2 f}{dx^2}\right) + \left(\frac{\partial}{\partial y}\frac{d^2 f}{dx^2}\right)y' + \frac{\partial}{\partial y'}\left(\frac{d^2 f}{dx^2}\right)f && (3) \\
&\vdots
\end{aligned}
$$

And so on. Hence if we name $F_0 = f(x, y, y')$ then the above can be written as

$$\begin{aligned}
F_0 &= f(x, y, y') \qquad (4)\\
F_1 &= \frac{df}{dx}\\
&= \frac{dF_0}{dx}\\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}y' + \frac{\partial f}{\partial y'}y''\\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}y' + \frac{\partial f}{\partial y'}f \qquad (5)\\
&= \frac{\partial F_0}{\partial x} + \frac{\partial F_0}{\partial y}y' + \frac{\partial F_0}{\partial y'}F_0\\
F_2 &= \frac{d}{dx}\left(\frac{d}{dx}f\right)\\
&= \frac{d}{dx}(F_1)\\
&= \frac{\partial}{\partial x}F_1 + \left(\frac{\partial F_1}{\partial y}\right)y' + \left(\frac{\partial F_1}{\partial y'}\right)y''\\
&= \frac{\partial}{\partial x}F_1 + \left(\frac{\partial F_1}{\partial y}\right)y' + \left(\frac{\partial F_1}{\partial y'}\right)F_0\\
&\vdots\\
F_n &= \frac{d}{dx}(F_{n-1})\\
&= \frac{\partial}{\partial x}F_{n-1} + \left(\frac{\partial F_{n-1}}{\partial y}\right)y' + \left(\frac{\partial F_{n-1}}{\partial y'}\right)y''\\
&= \frac{\partial}{\partial x}F_{n-1} + \left(\frac{\partial F_{n-1}}{\partial y}\right)y' + \left(\frac{\partial F_{n-1}}{\partial y'}\right)F_0 \qquad (6)
\end{aligned}$$

Therefore (6) can be used from now on along with

$$y(x) = y_0 + xy_0' + \sum_{n=0}^{\infty} \frac{x^{n+2}}{(n+2)!} F_n|_{x_0,y_0,y_0'} \qquad (7)$$

To find $y(x)$ series solution around $x = 0$. Hence

$$
\begin{aligned}
F_0 &= \frac{x^2 y' - 3y}{x+1} \\
F_1 &= \frac{dF_0}{dx} \\
&= \frac{\partial F_0}{\partial x} + \frac{\partial F_0}{\partial y} y' + \frac{\partial F_0}{\partial y'} F_0 \\
&= \frac{(x^4 + x^2 - x - 3)\, y' + (-3x^2 + 3)\, y}{(x+1)^2} \\
F_2 &= \frac{dF_1}{dx} \\
&= \frac{\partial F_1}{\partial x} + \frac{\partial F_1}{\partial y} y' + \frac{\partial F_1}{\partial y'} F_1 \\
&= \frac{(x^5 - x^4 + 4x^3 - 4x^2 - 2x + 8)\, y' - 3y(x^3 - x^2 + 2x - 1)}{(x+1)^2} \\
F_3 &= \frac{dF_2}{dx} \\
&= \frac{\partial F_2}{\partial x} + \frac{\partial F_2}{\partial y} y' + \frac{\partial F_2}{\partial y'} F_2 \\
&= \frac{(x^7 - x^6 + 7x^5 - 4x^4 - 2x^3 + 17x^2 - 9x - 15)\, y' - 3y(x^5 - x^4 + 5x^3 - x^2 - 6x + 12)}{(x+1)^3} \\
F_4 &= \frac{dF_3}{dx} \\
&= \frac{\partial F_3}{\partial x} + \frac{\partial F_3}{\partial y} y' + \frac{\partial F_3}{\partial y'} F_3 \\
&= \frac{x(x+1)\,(x^7 - 2x^6 + 13x^5 - 16x^4 + 22x^3 + 14x^2 - 51x + 34)\, y' - 3y(x^7 - x^6 + 9x^5 - 6x^3 + 33x^2 +}{(x+1)^4}
\end{aligned}
$$

And so on. Evaluating all the above at initial conditions $x = 0$ and $y(0) = y(0)$ and $y'(0) = y'(0)$ gives

$$
\begin{aligned}
F_0 &= -3y(0) \\
F_1 &= 3y(0) - 3y'(0) \\
F_2 &= 3y(0) + 8y'(0) \\
F_3 &= -36y(0) - 15y'(0) \\
F_4 &= 171y(0)
\end{aligned}
$$

Substituting all the above in (7) and simplifying gives the solution as

$$
y = \left(1 - \frac{3}{2}x^2 + \frac{1}{2}x^3 + \frac{1}{8}x^4 - \frac{3}{10}x^5 + \frac{19}{80}x^6\right) y(0) + \left(x - \frac{1}{2}x^3 + \frac{1}{3}x^4 - \frac{1}{8}x^5\right) y'(0) + O(x^6)
$$

Since the expansion point $x = 0$ is an ordinary, we can also solve this using standard power series The ode is normalized to be

$$(x+1)\, y'' - x^2 y' + 3y = 0$$

Let the solution be represented as power series of the form

$$y = \sum_{n=0}^{\infty} a_n x^n$$

Then

$$y' = \sum_{n=1}^{\infty} n a_n x^{n-1}$$
$$y'' = \sum_{n=2}^{\infty} n(n-1)\, a_n x^{n-2}$$

Substituting the above back into the ode gives

$$(x+1)\left(\sum_{n=2}^{\infty} n(n-1)\, a_n x^{n-2}\right) - x^2\left(\sum_{n=1}^{\infty} n a_n x^{n-1}\right) + 3\left(\sum_{n=0}^{\infty} a_n x^n\right) = 0 \tag{1}$$

Which simplifies to

$$\left(\sum_{n=2}^{\infty} n\, x^{n-1} a_n (n-1)\right) + \left(\sum_{n=2}^{\infty} n(n-1)\, a_n x^{n-2}\right) + \sum_{n=1}^{\infty} \left(-n\, x^{1+n} a_n\right) + \left(\sum_{n=0}^{\infty} 3 a_n x^n\right) = 0 \tag{2}$$

The next step is to make all powers of x be n in each summation term. Going over each summation term above with power of x in it which is not already x^n and adjusting the power and the corresponding index gives

$$\sum_{n=2}^{\infty} n\, x^{n-1} a_n (n-1) = \sum_{n=1}^{\infty} (1+n)\, a_{1+n} n\, x^n$$

$$\sum_{n=2}^{\infty} n(n-1)\, a_n x^{n-2} = \sum_{n=0}^{\infty} (n+2)\, a_{n+2} (1+n)\, x^n$$

$$\sum_{n=1}^{\infty} \left(-n\, x^{1+n} a_n\right) = \sum_{n=2}^{\infty} \left(-(n-1)\, a_{n-1} x^n\right)$$

Substituting all the above in Eq (2) gives the following equation where now all powers of x are the same and equal to n.

$$
\left(\sum_{n=1}^{\infty} (1+n)\, a_{1+n} n\, x^{n}\right) + \left(\sum_{n=0}^{\infty} (n+2)\, a_{n+2} (1+n)\, x^{n}\right) \\
+ \sum_{n\,=2}^{\infty} \left(-(n-1)\, a_{n-1} x^{n}\right) + \left(\sum_{n=0}^{\infty} 3 a_{n} x^{n}\right) = 0 \tag{3}
$$

$n = 0$ gives

$$2a_2 + 3a_0 = 0$$

$$a_2 = -\frac{3a_0}{2}$$

$n = 1$ gives

$$2a_2 + 6a_3 + 3a_1 = 0$$

Which after substituting earlier equations, simplifies to

$$a_3 = \frac{a_0}{2} - \frac{a_1}{2}$$

For $2 \leq n$, the recurrence equation is

$$(1+n)\, a_{1+n} n + (n+2)\, a_{n+2} (1+n) - (n-1)\, a_{n-1} + 3a_n = 0 \tag{4}$$

Solving for a_{n+2}, gives

$$
\begin{aligned}
a_{n+2} &= -\frac{n^2 a_{1+n} + n a_{1+n} - n a_{n-1} + 3a_n + a_{n-1}}{(n+2)\,(1+n)} \\
&= -\frac{3a_n}{(n+2)\,(1+n)} - \frac{(n^2+n)\, a_{1+n}}{(n+2)\,(1+n)} - \frac{(-n+1)\, a_{n-1}}{(n+2)\,(1+n)}
\end{aligned} \tag{5}
$$

For $n = 2$ the recurrence equation gives

$$6a_3 + 12a_4 - a_1 + 3a_2 = 0$$

Which after substituting the earlier terms found becomes

$$a_4 = \frac{a_0}{8} + \frac{a_1}{3}$$

For $n = 3$ the recurrence equation gives

$$12a_4 + 20a_5 - 2a_2 + 3a_3 = 0$$

Which after substituting the earlier terms found becomes

$$a_5 = -\frac{3a_0}{10} - \frac{a_1}{8}$$

For $n = 4$ the recurrence equation gives

$$20a_5 + 30a_6 - 3a_3 + 3a_4 = 0$$

Which after substituting the earlier terms found becomes

$$a_6 = \frac{19a_0}{80}$$

For $n = 5$ the recurrence equation gives

$$30a_6 + 42a_7 - 4a_4 + 3a_5 = 0$$

Which after substituting the earlier terms found becomes

$$a_7 = -\frac{229a_0}{1680} + \frac{41a_1}{1008}$$

And so on. Therefore the solution is

$$\begin{aligned} y &= \sum_{n=0}^{\infty} a_n x^n \\ &= a_3 x^3 + a_2 x^2 + a_1 x + a_0 + \dots \end{aligned}$$

Substituting the values for a_n found above, the solution becomes

$$y = a_0 + a_1 x - \frac{3a_0 x^2}{2} + \left(\frac{a_0}{2} - \frac{a_1}{2}\right) x^3 + \left(\frac{a_0}{8} + \frac{a_1}{3}\right) x^4 + \left(-\frac{3a_0}{10} - \frac{a_1}{8}\right) x^5 + \dots$$

Collecting terms, the solution becomes

$$y = \left(1-\frac{3}{2}x^{2}+\frac{1}{2}x^{3}+\frac{1}{8}x^{4}-\frac{3}{10}x^{5}\right)a_{0}+\left(x-\frac{1}{2}x^{3}+\frac{1}{3}x^{4}-\frac{1}{8}x^{5}\right)a_{1}+O\left(x^{6}\right) \quad (3)$$

At $x=0$ the solution above becomes

$$y = \left(1-\frac{3}{2}x^{2}+\frac{1}{2}x^{3}+\frac{1}{8}x^{4}-\frac{3}{10}x^{5}\right)c_{1}+\left(x-\frac{1}{2}x^{3}+\frac{1}{3}x^{4}-\frac{1}{8}x^{5}\right)c_{2}+O\left(x^{6}\right)$$

Summary

The solution(s) found are the following

$$\begin{aligned} y &= \left(1-\frac{3}{2}x^{2}+\frac{1}{2}x^{3}+\frac{1}{8}x^{4}-\frac{3}{10}x^{5}+\frac{19}{80}x^{6}\right)y(0) \\ &\quad+\left(x-\frac{1}{2}x^{3}+\frac{1}{3}x^{4}-\frac{1}{8}x^{5}\right)y'(0)+O\left(x^{6}\right) \end{aligned} \quad (1)$$

$$y = \left(1-\frac{3}{2}x^{2}+\frac{1}{2}x^{3}+\frac{1}{8}x^{4}-\frac{3}{10}x^{5}\right)c_{1}+\left(x-\frac{1}{2}x^{3}+\frac{1}{3}x^{4}-\frac{1}{8}x^{5}\right)c_{2}+O\left(x^{6}\right) \quad (2)$$

Verification of solutions

$$y = \left(1-\frac{3}{2}x^{2}+\frac{1}{2}x^{3}+\frac{1}{8}x^{4}-\frac{3}{10}x^{5}+\frac{19}{80}x^{6}\right)y(0)+\left(x-\frac{1}{2}x^{3}+\frac{1}{3}x^{4}-\frac{1}{8}x^{5}\right)y'(0)+O\left(x^{6}\right)$$

Verified OK.

$$y = \left(1-\frac{3}{2}x^{2}+\frac{1}{2}x^{3}+\frac{1}{8}x^{4}-\frac{3}{10}x^{5}\right)c_{1}+\left(x-\frac{1}{2}x^{3}+\frac{1}{3}x^{4}-\frac{1}{8}x^{5}\right)c_{2}+O\left(x^{6}\right)$$

Verified OK.

5.1.1 Maple step by step solution

Let's solve

$(x+1)\,y''-x^{2}y'+3y=0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y''=-\frac{3y}{x+1}+\frac{x^{2}y'}{x+1}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y'' - \frac{x^2 y'}{x+1} + \frac{3y}{x+1} = 0$

☐ Check to see if $x_0 = -1$ is a regular singular point

  - Define functions

    $\left[P_2(x) = -\frac{x^2}{x+1}, P_3(x) = \frac{3}{x+1}\right]$

  - $(x+1) \cdot P_2(x)$ is analytic at $x = -1$

    $\left((x+1) \cdot P_2(x)\right)\bigg|_{x=-1} = -1$

  - $(x+1)^2 \cdot P_3(x)$ is analytic at $x = -1$

    $\left((x+1)^2 \cdot P_3(x)\right)\bigg|_{x=-1} = 0$

  - $x = -1$is a regular singular point

    Check to see if $x_0 = -1$ is a regular singular point

    $x_0 = -1$

- Multiply by denominators

  $(x+1)\, y'' - x^2 y' + 3y = 0$

- Change variables using $x = u - 1$ so that the regular singular point is at $u = 0$

  $u\left(\frac{d^2}{du^2}y(u)\right) + (-u^2 + 2u - 1)\left(\frac{d}{du}y(u)\right) + 3y(u) = 0$

- Assume series solution for $y(u)$

  $y(u) = \sum_{k=0}^{\infty} a_k u^{k+r}$

☐ Rewrite ODE with series expansions

  - Convert $u^m \cdot \left(\frac{d}{du}y(u)\right)$ to series expansion for $m = 0..2$

    $u^m \cdot \left(\frac{d}{du}y(u)\right) = \sum_{k=0}^{\infty} a_k (k+r)\, u^{k+r-1+m}$

  - Shift index using $k- >k+1-m$

    $u^m \cdot \left(\frac{d}{du}y(u)\right) = \sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\, u^{k+r}$

  - Convert $u \cdot \left(\frac{d^2}{du^2}y(u)\right)$ to series expansion

    $u \cdot \left(\frac{d^2}{du^2}y(u)\right) = \sum_{k=0}^{\infty} a_k (k+r)(k+r-1)\, u^{k+r-1}$

- Shift index using $k- >k+1$

  $u \cdot \left(\frac{d^2}{du^2}y(u)\right) = \sum_{k=-1}^{\infty} a_{k+1}(k+1+r)(k+r)\, u^{k+r}$

  Rewrite ODE with series expansions

  $a_0 r(-2+r)\, u^{-1+r} + (a_1(1+r)(-1+r) + a_0(3+2r))\, u^r + \left(\sum_{k=1}^{\infty} (a_{k+1}(k+1+r)(k+r-1) + \right.$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $r(-2+r) = 0$

- Values of r that satisfy the indicial equation

  $r \in \{0, 2\}$

- Each term must be 0

  $a_1(1+r)(-1+r) + a_0(3+2r) = 0$

- Each term in the series must be 0, giving the recursion relation

  $a_{k+1}(k+1+r)(k+r-1) + (2a_k - a_{k-1})\, k + (2a_k - a_{k-1})\, r + 3a_k + a_{k-1} = 0$

- Shift index using $k- >k+1$

  $a_{k+2}(k+2+r)(k+r) + (2a_{k+1} - a_k)(k+1) + (2a_{k+1} - a_k)\, r + 3a_{k+1} + a_k = 0$

- Recursion relation that defines series solution to ODE

  $a_{k+2} = \frac{a_k k - 2k a_{k+1} + a_k r - 2r a_{k+1} - 5a_{k+1}}{(k+2+r)(k+r)}$

- Recursion relation for $r = 0$

  $a_{k+2} = \frac{a_k k - 2k a_{k+1} - 5a_{k+1}}{(k+2)k}$

- Series not valid for $r = 0$, division by 0 in the recursion relation at $k = 0$

  $a_{k+2} = \frac{a_k k - 2k a_{k+1} - 5a_{k+1}}{(k+2)k}$

- Recursion relation for $r = 2$

  $a_{k+2} = \frac{a_k k - 2k a_{k+1} + 2a_k - 9a_{k+1}}{(k+4)(k+2)}$

- Solution for $r = 2$

  $\left[y(u) = \sum_{k=0}^{\infty} a_k u^{k+2}, a_{k+2} = \frac{a_k k - 2k a_{k+1} + 2a_k - 9a_{k+1}}{(k+4)(k+2)}, 3a_1 + 7a_0 = 0\right]$

- Revert the change of variables $u = x+1$

  $\left[y = \sum_{k=0}^{\infty} a_k (x+1)^{k+2}, a_{k+2} = \frac{a_k k - 2k a_{k+1} + 2a_k - 9a_{k+1}}{(k+4)(k+2)}, 3a_1 + 7a_0 = 0\right]$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
<- No Liouvillian solutions exists
-> Trying a solution in terms of special functions:
   -> Bessel
   -> elliptic
   -> Legendre
   -> Kummer
      -> hyper3: Equivalence to 1F1 under a power @ Moebius
   -> hypergeometric
      -> heuristic approach
      -> hyper3: Equivalence to 2F1, 1F1 or 0F1 under a power @ Moebius
   -> Mathieu
      -> Equivalence to the rational form of Mathieu ODE under a power @ Moebius
trying a solution in terms of MeijerG functions
-> Heun: Equivalence to the GHE or one of its 4 confluent cases under a power @ Moebius
<- Heun successful: received ODE is equivalent to the  HeunB  ODE, case  c = 0 `
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 49

```
Order:=6;
dsolve((x+1)*diff(y(x),x$2)-x^2*diff(y(x),x)+3*y(x)=0,y(x),type='series',x=0);
```

$$y(x) = \left(1 - \frac{3}{2}x^2 + \frac{1}{2}x^3 + \frac{1}{8}x^4 - \frac{3}{10}x^5\right) y(0) + \left(x - \frac{1}{2}x^3 + \frac{1}{3}x^4 - \frac{1}{8}x^5\right) D(y)(0) + O(x^6)$$

✓ Solution by Mathematica

Time used: 0.001 (sec). Leaf size: 63

```
AsymptoticDSolveValue[(x+1)*y''[x]-x^2*y'[x]+3*y[x]==0,y[x],{x,0,5}]
```

$$y(x) \to c_2\left(-\frac{x^5}{8}+\frac{x^4}{3}-\frac{x^3}{2}+x\right)+c_1\left(-\frac{3x^5}{10}+\frac{x^4}{8}+\frac{x^3}{2}-\frac{3x^2}{2}+1\right)$$

5.2 problem 2

Internal problem ID [5003]
Internal file name [`OUTPUT/4496_Sunday_June_05_2022_02_59_24_PM_16781494/index.tex`]

Book: Fundamentals of Differential Equations. By Nagle, Saff and Snider. 9th edition. Boston. Pearson 2018.
Section: Chapter 8, Series solutions of differential equations. Section 8.3. page 443
Problem number: 2.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second order series method. Irregular singular point"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

Unable to solve or complete the solution.

$$\boxed{x^2y'' + 3y' - xy = 0}$$

With the expansion point for the power series method at $x = 0$.

The type of the expansion point is first determined. This is done on the homogeneous part of the ODE.

$$x^2y'' + 3y' - xy = 0$$

The following is summary of singularities for the above ode. Writing the ode as

$$y'' + p(x)y' + q(x)y = 0$$

Where

$$p(x) = \frac{3}{x^2}$$
$$q(x) = -\frac{1}{x}$$

Table 200: Table $p(x), q(x)$ singularites.

$p(x) = \frac{3}{x^2}$	
singularity	type
$x = 0$	"irregular"

$q(x) = -\frac{1}{x}$	
singularity	type
$x = 0$	"regular"

Combining everything together gives the following summary of singularities for the ode as

Regular singular points : []

Irregular singular points : $[0, \infty]$

Since $x = 0$ is not an ordinary point, then we will now check if it is a regular singular point. Unable to solve since $x = 0$ is not regular singular point. Terminating.

<u>Verification of solutions</u> N/A

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
<- No Liouvillian solutions exists
-> Trying a solution in terms of special functions:
   -> Bessel
   -> elliptic
   -> Legendre
   -> Kummer
      -> hyper3: Equivalence to 1F1 under a power @ Moebius
   -> hypergeometric
      -> heuristic approach
      -> hyper3: Equivalence to 2F1, 1F1 or 0F1 under a power @ Moebius
   -> Mathieu
      -> Equivalence to the rational form of Mathieu ODE under a power @ Moebius
trying a solution in terms of MeijerG functions
-> Heun: Equivalence to the GHE or one of its 4 confluent cases under a power @ Moebius
-> trying a solution of the form r0(x) * Y + r1(x) * Y where Y = exp(int(r(x), dx)) * 2F1([a
   trying a symmetry of the form [xi=0, eta=F(x)]
   trying differential order: 2; exact nonlinear
   trying symmetries linear in x and y(x)
   trying to convert to a linear ODE with constant coefficients
   trying 2nd order, integrating factor of the form mu(x,y)
   -> Trying a solution in terms of special functions:
      -> Bessel
      -> elliptic
      -> Legendre
      -> Kummer
         -> hyper3: Equivalence to 1F1 under a power @ Moebius
      -> hypergeometric
         -> heuristic approach
         -> hyper3: Equivalence to 2F1, 1F1 or 0F1 under a power @ Moebius
      -> Mathieu
         -> Equivalence to the rational form of Mathieu ODE under a power @ Moebius
      trying 2nd order exact linear
      trying symmetries linear in x and y(x)
      trying to convert to a linear ODE with constant coefficients
      trying to convert to an ODE of Bessel type
   trying to convert to an ODE of Bessel type
   -> trying reduction of order to Riccati
```

✗ Solution by Maple

```
Order:=6;
dsolve(x^2*diff(y(x),x$2)+3*diff(y(x),x)-x*y(x)=0,y(x),type='series',x=0);
```

No solution found

✓ Solution by Mathematica

Time used: 0.026 (sec). Leaf size: 85

```
AsymptoticDSolveValue[x^2*y''[x]+3*y'[x]-x*y[x]==0,y[x],{x,0,5}]
```

$$y(x) \to c_2 e^{3/x} \left(\frac{3001x^5}{1620} + \frac{613x^4}{648} + \frac{16x^3}{27} + \frac{x^2}{2} + \frac{2x}{3} + 1 \right) x^2 + c_1 \left(-\frac{23x^5}{810} + \frac{7x^4}{216} - \frac{x^3}{27} + \frac{x^2}{6} + 1 \right)$$

5.3 problem 3

Internal problem ID [5004]
Internal file name [`OUTPUT/4497_Sunday_June_05_2022_02_59_25_PM_52561784/index.tex`]

Book: Fundamentals of Differential Equations. By Nagle, Saff and Snider. 9th edition. Boston. Pearson 2018.
Section: Chapter 8, Series solutions of differential equations. Section 8.3. page 443
Problem number: 3.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second order series method. Ordinary point", "second order series method. Taylor series method"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

$$\boxed{(x^2-2)\,y''+2y'+\sin(x)\,y=0}$$

With the expansion point for the power series method at $x=0$.

Solving ode using Taylor series method. This gives review on how the Taylor series method works for solving second order ode.

Let

$$y''=f(x,y,y')$$

Assuming expansion is at $x_0=0$ (we can always shift the actual expansion point to 0 by change of variables) and assuming $f(x,y,y')$ is analytic at x_0 which must be the case for an ordinary point. Let initial conditions be $y(x_0)=y_0$ and $y'(x_0)=y'_0$. Using Taylor series gives

$$\begin{aligned}
y(x) &= y(x_0)+(x-x_0)\,y'(x_0)+\frac{(x-x_0)^2}{2}y''(x_0)+\frac{(x-x_0)^3}{3!}y'''(x_0)+\cdots\\
&= y_0+xy'_0+\frac{x^2}{2}\,f|_{x_0,y_0,y'_0}+\frac{x^3}{3!}\,f'|_{x_0,y_0,y'_0}+\cdots\\
&= y_0+xy'_0+\sum_{n=0}^{\infty}\frac{x^{n+2}}{(n+2)!}\,\frac{d^nf}{dx^n}\bigg|_{x_0,y_0,y'_0}
\end{aligned}$$

But

$$
\begin{aligned}
\frac{df}{dx} &= \frac{\partial f}{\partial x}\frac{dx}{dx} + \frac{\partial f}{\partial y}\frac{dy}{dx} + \frac{\partial f}{\partial y'}\frac{dy'}{dx} && (1)\\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}y' + \frac{\partial f}{\partial y'}y'' && (226)\\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}y' + \frac{\partial f}{\partial y'}f && (227)\\
\frac{d^2 f}{dx^2} &= \frac{d}{dx}\left(\frac{df}{dx}\right)\\
&= \frac{\partial}{\partial x}\left(\frac{df}{dx}\right) + \frac{\partial}{\partial y}\left(\frac{df}{dx}\right)y' + \frac{\partial}{\partial y'}\left(\frac{df}{dx}\right)f && (2)\\
\frac{d^3 f}{dx^3} &= \frac{d}{dx}\left(\frac{d^2 f}{dx^2}\right)\\
&= \frac{\partial}{\partial x}\left(\frac{d^2 f}{dx^2}\right) + \left(\frac{\partial}{\partial y}\frac{d^2 f}{dx^2}\right)y' + \frac{\partial}{\partial y'}\left(\frac{d^2 f}{dx^2}\right)f && (3)\\
&\vdots
\end{aligned}
$$

And so on. Hence if we name $F_0 = f(x, y, y')$ then the above can be written as

$$
\begin{aligned}
F_0 &= f(x, y, y') \qquad (4) \\
F_1 &= \frac{df}{dx} \\
&= \frac{dF_0}{dx} \\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}y' + \frac{\partial f}{\partial y'}y'' \\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}y' + \frac{\partial f}{\partial y'}f \qquad (5) \\
&= \frac{\partial F_0}{\partial x} + \frac{\partial F_0}{\partial y}y' + \frac{\partial F_0}{\partial y'}F_0 \\
F_2 &= \frac{d}{dx}\left(\frac{d}{dx}f\right) \\
&= \frac{d}{dx}(F_1) \\
&= \frac{\partial}{\partial x}F_1 + \left(\frac{\partial F_1}{\partial y}\right)y' + \left(\frac{\partial F_1}{\partial y'}\right)y'' \\
&= \frac{\partial}{\partial x}F_1 + \left(\frac{\partial F_1}{\partial y}\right)y' + \left(\frac{\partial F_1}{\partial y'}\right)F_0 \\
&\vdots \\
F_n &= \frac{d}{dx}(F_{n-1}) \\
&= \frac{\partial}{\partial x}F_{n-1} + \left(\frac{\partial F_{n-1}}{\partial y}\right)y' + \left(\frac{\partial F_{n-1}}{\partial y'}\right)y'' \\
&= \frac{\partial}{\partial x}F_{n-1} + \left(\frac{\partial F_{n-1}}{\partial y}\right)y' + \left(\frac{\partial F_{n-1}}{\partial y'}\right)F_0 \qquad (6)
\end{aligned}
$$

Therefore (6) can be used from now on along with

$$
y(x) = y_0 + xy'_0 + \sum_{n=0}^{\infty} \frac{x^{n+2}}{(n+2)!} F_n|_{x_0,y_0,y'_0} \qquad (7)
$$

To find $y(x)$ series solution around $x = 0$. Hence

$$
\begin{aligned}
F_0 &= -\frac{\sin(x)\, y + 2y'}{x^2 - 2} \\
F_1 &= \frac{dF_0}{dx} \\
&= \frac{\partial F_0}{\partial x} + \frac{\partial F_0}{\partial y} y' + \frac{\partial F_0}{\partial y'} F_0 \\
&= \frac{(-x^2 \sin(x) + 4x + 2\sin(x) + 4)\, y' - y((-2 - 2x)\sin(x) + (x^2 - 2)\cos(x))}{(x^2 - 2)^2} \\
F_2 &= \frac{dF_1}{dx} \\
&= \frac{\partial F_1}{\partial x} + \frac{\partial F_1}{\partial y} y' + \frac{\partial F_1}{\partial y'} F_1 \\
&= \frac{2\left(-8 - (x^2 - 2)^2 \cos(x) + 2(x^3 + x^2 - 2x - 2)\sin(x) - 6x^2 - 12x\right) y' + y\left((x^2 - 2)\sin(x)^2 + (x^4 \cdots\right.}{(x^2 - 2)^3} \\
F_3 &= \frac{dF_2}{dx} \\
&= \frac{\partial F_2}{\partial x} + \frac{\partial F_2}{\partial y} y' + \frac{\partial F_2}{\partial y'} F_2 \\
&= \frac{\left(\sin(x)^2 x^4 + 3(x^2 - 2)(x^4 - 10x^2 - 12x - 4)\sin(x) + (4x^2 - 4)\cos(x)^2 + 12\left(x + \frac{5}{6}\right)(x^2 - 2)^2 \cos \cdots\right.}{\cdots} \\
F_4 &= \frac{dF_3}{dx} \\
&= \frac{\partial F_3}{\partial x} + \frac{\partial F_3}{\partial y} y' + \frac{\partial F_3}{\partial y'} F_3 \\
&= \frac{\left((-12x^5 - 6x^4 + 24x^3 + 16x^2 - 32x - 16)\sin(x)^2 - 24\left(-\frac{(x^2-2)^2 \cos(x)}{4} + x^5 + \frac{3x^4}{4} - 8x^3 - 15x^2 - \cdots\right.\right.}{\cdots}
\end{aligned}
$$

And so on. Evaluating all the above at initial conditions $x = 0$ and $y(0) = y(0)$ and

$y'(0) = y'(0)$ gives

$$
\begin{aligned}
F_0 &= y'(0) \\
F_1 &= \frac{y(0)}{2} + y'(0) \\
F_2 &= \frac{y(0)}{2} + 3y'(0) \\
F_3 &= \frac{15y'(0)}{2} + \frac{3y(0)}{2} \\
F_4 &= \frac{11y(0)}{2} + \frac{51y'(0)}{2}
\end{aligned}
$$

Substituting all the above in (7) and simplifying gives the solution as

$$
\begin{aligned}
y = &\left(1 + \frac{1}{12}x^3 + \frac{1}{48}x^4 + \frac{1}{80}x^5 + \frac{11}{1440}x^6\right) y(0) \\
&+ \left(x + \frac{1}{2}x^2 + \frac{1}{6}x^3 + \frac{1}{8}x^4 + \frac{1}{16}x^5 + \frac{17}{480}x^6\right) y'(0) + O(x^6)
\end{aligned}
$$

Since the expansion point $x = 0$ is an ordinary, we can also solve this using standard power series The ode is normalized to be

$$
(x^2 - 2)\, y'' + 2y' + \sin(x)\, y = 0
$$

Let the solution be represented as power series of the form

$$
y = \sum_{n=0}^{\infty} a_n x^n
$$

Then

$$
\begin{aligned}
y' &= \sum_{n=1}^{\infty} n a_n x^{n-1} \\
y'' &= \sum_{n=2}^{\infty} n(n-1)\, a_n x^{n-2}
\end{aligned}
$$

Substituting the above back into the ode gives

$$
(x^2 - 2)\left(\sum_{n=2}^{\infty} n(n-1)\, a_n x^{n-2}\right) + 2\left(\sum_{n=1}^{\infty} n a_n x^{n-1}\right) + \sin(x)\left(\sum_{n=0}^{\infty} a_n x^n\right) = 0 \quad (1)
$$

Expanding $\sin(x)$ as Taylor series around $x = 0$ and keeping only the first 6 terms gives

$$
\begin{aligned}
\sin(x) &= x - \frac{1}{6}x^3 + \frac{1}{120}x^5 - \frac{1}{5040}x^7 + \dots \\
&= x - \frac{1}{6}x^3 + \frac{1}{120}x^5 - \frac{1}{5040}x^7
\end{aligned}
$$

Hence the ODE in Eq (1) becomes

$$
\begin{aligned}
&(x^2 - 2)\left(\sum_{n=2}^{\infty} n(n-1)\, a_n x^{n-2}\right) + 2\left(\sum_{n=1}^{\infty} n a_n x^{n-1}\right) \\
&+ \left(x - \frac{1}{6}x^3 + \frac{1}{120}x^5 - \frac{1}{5040}x^7\right)\left(\sum_{n=0}^{\infty} a_n x^n\right) = 0
\end{aligned}
$$

Expanding the third term in (1) gives

$$
\begin{aligned}
&(x^2 - 2)\left(\sum_{n=2}^{\infty} n(n-1)\, a_n x^{n-2}\right) + 2\left(\sum_{n=1}^{\infty} n a_n x^{n-1}\right) + x \cdot \left(\sum_{n=0}^{\infty} a_n x^n\right) \\
&- \frac{x^3}{6} \cdot \left(\sum_{n=0}^{\infty} a_n x^n\right) + \frac{x^5}{120} \cdot \left(\sum_{n=0}^{\infty} a_n x^n\right) - \frac{x^7}{5040} \cdot \left(\sum_{n=0}^{\infty} a_n x^n\right) = 0
\end{aligned}
$$

Which simplifies to

$$
\begin{aligned}
&\left(\sum_{n=2}^{\infty} x^n a_n n(n-1)\right) + \sum_{n=2}^{\infty} \left(-2n(n-1)\, a_n x^{n-2}\right) + \left(\sum_{n=1}^{\infty} 2n a_n x^{n-1}\right) \\
&+ \left(\sum_{n=0}^{\infty} x^{1+n} a_n\right) + \sum_{n=0}^{\infty} \left(-\frac{x^{n+3} a_n}{6}\right) + \left(\sum_{n=0}^{\infty} \frac{x^{n+5} a_n}{120}\right) + \sum_{n=0}^{\infty} \left(-\frac{x^{n+7} a_n}{5040}\right) = 0
\end{aligned} \tag{2}
$$

The next step is to make all powers of x be n in each summation term. Going over each summation term above with power of x in it which is not already x^n and adjusting the power and the corresponding index gives

$$
\sum_{n=2}^{\infty} \left(-2n(n-1)\, a_n x^{n-2}\right) = \sum_{n=0}^{\infty} \left(-2(n+2)\, a_{n+2}(1+n)\, x^n\right)
$$

$$
\sum_{n=1}^{\infty} 2n a_n x^{n-1} = \sum_{n=0}^{\infty} 2(1+n)\, a_{1+n} x^n
$$

$$\sum_{n=0}^{\infty} x^{1+n} a_n = \sum_{n=1}^{\infty} a_{n-1} x^n$$

$$\sum_{n=0}^{\infty} \left(-\frac{x^{n+3} a_n}{6}\right) = \sum_{n=3}^{\infty} \left(-\frac{a_{n-3} x^n}{6}\right)$$

$$\sum_{n=0}^{\infty} \frac{x^{n+5} a_n}{120} = \sum_{n=5}^{\infty} \frac{a_{n-5} x^n}{120}$$

$$\sum_{n=0}^{\infty} \left(-\frac{x^{n+7} a_n}{5040}\right) = \sum_{n=7}^{\infty} \left(-\frac{a_{n-7} x^n}{5040}\right)$$

Substituting all the above in Eq (2) gives the following equation where now all powers of x are the same and equal to n.

$$\left(\sum_{n=2}^{\infty} x^n a_n n(n-1)\right) + \sum_{n=0}^{\infty} \left(-2(n+2) a_{n+2}(1+n) x^n\right) + \left(\sum_{n=0}^{\infty} 2(1+n) a_{1+n} x^n\right) + \left(\sum_{n=1}^{\infty} a_{n-1} x^n\right) + \sum_{n=3}^{\infty} \left(-\frac{a_{n-3} x^n}{6}\right) + \left(\sum_{n=5}^{\infty} \frac{a_{n-5} x^n}{120}\right) + \sum_{n=7}^{\infty} \left(-\frac{a_{n-7} x^n}{5040}\right) = 0 \tag{3}$$

$n = 0$ gives

$$-4a_2 + 2a_1 = 0$$

$$a_2 = \frac{a_1}{2}$$

$n = 1$ gives

$$-12a_3 + 4a_2 + a_0 = 0$$

Which after substituting earlier equations, simplifies to

$$a_3 = \frac{a_0}{12} + \frac{a_1}{6}$$

$n = 2$ gives

$$2a_2 - 24a_4 + 6a_3 + a_1 = 0$$

Which after substituting earlier equations, simplifies to

$$a_4 = \frac{a_0}{48} + \frac{a_1}{8}$$

$n = 3$ gives

$$6a_3 - 40a_5 + 8a_4 + a_2 - \frac{a_0}{6} = 0$$

Which after substituting earlier equations, simplifies to

$$a_5 = \frac{a_0}{80} + \frac{a_1}{16}$$

$n = 4$ gives

$$12a_4 - 60a_6 + 10a_5 + a_3 - \frac{a_1}{6} = 0$$

Which after substituting earlier equations, simplifies to

$$a_6 = \frac{11a_0}{1440} + \frac{17a_1}{480}$$

$n = 5$ gives

$$20a_5 - 84a_7 + 12a_6 + a_4 - \frac{a_2}{6} + \frac{a_0}{120} = 0$$

Which after substituting earlier equations, simplifies to

$$a_7 = \frac{89a_0}{20160} + \frac{103a_1}{5040}$$

For $7 \leq n$, the recurrence equation is

$$na_n(n-1) - 2(n+2)\, a_{n+2}(1+n) + 2(1+n)\, a_{1+n} + a_{n-1} - \frac{a_{n-3}}{6} + \frac{a_{n-5}}{120} - \frac{a_{n-7}}{5040} = 0 \quad (4)$$

Solving for a_{n+2}, gives

$$a_{n+2} = \frac{5040n^2 a_n - 5040na_n + 10080na_{1+n} + 10080a_{1+n} - a_{n-7} + 42a_{n-5} - 840a_{n-3} + 5040a_{n-1}}{10080\,(n+2)\,(1+n)}$$

$$(5) \quad = \frac{(5040n^2 - 5040n)\, a_n}{10080\,(n+2)\,(1+n)} + \frac{(10080n + 10080)\, a_{1+n}}{10080\,(n+2)\,(1+n)} - \frac{a_{n-7}}{10080\,(n+2)\,(1+n)} + \frac{a_{n-5}}{240\,(n+2)\,(1+n)} - \frac{a_{n-3}}{12\,(n+2)\,(1+n)} + \frac{a_{n-1}}{2\,(n+2)\,(1+n)}$$

And so on. Therefore the solution is

$$
\begin{aligned}
y &= \sum_{n=0}^{\infty} a_n x^n \\
&= a_3 x^3 + a_2 x^2 + a_1 x + a_0 + \dots
\end{aligned}
$$

Substituting the values for a_n found above, the solution becomes

$$
y = a_0 + a_1 x + \frac{a_1 x^2}{2} + \left(\frac{a_0}{12} + \frac{a_1}{6}\right) x^3 + \left(\frac{a_0}{48} + \frac{a_1}{8}\right) x^4 + \left(\frac{a_0}{80} + \frac{a_1}{16}\right) x^5 + \dots
$$

Collecting terms, the solution becomes

$$
y = \left(1 + \frac{1}{12}x^3 + \frac{1}{48}x^4 + \frac{1}{80}x^5\right) a_0 + \left(x + \frac{1}{2}x^2 + \frac{1}{6}x^3 + \frac{1}{8}x^4 + \frac{1}{16}x^5\right) a_1 + O(x^6) \quad (3)
$$

At $x = 0$ the solution above becomes

$$
y = \left(1 + \frac{1}{12}x^3 + \frac{1}{48}x^4 + \frac{1}{80}x^5\right) c_1 + \left(x + \frac{1}{2}x^2 + \frac{1}{6}x^3 + \frac{1}{8}x^4 + \frac{1}{16}x^5\right) c_2 + O(x^6)
$$

Summary

The solution(s) found are the following

$$
\begin{aligned}
y = {} & \left(1 + \frac{1}{12}x^3 + \frac{1}{48}x^4 + \frac{1}{80}x^5 + \frac{11}{1440}x^6\right) y(0) \\
& + \left(x + \frac{1}{2}x^2 + \frac{1}{6}x^3 + \frac{1}{8}x^4 + \frac{1}{16}x^5 + \frac{17}{480}x^6\right) y'(0) + O(x^6)
\end{aligned} \quad (1)
$$

$$
y = \left(1 + \frac{1}{12}x^3 + \frac{1}{48}x^4 + \frac{1}{80}x^5\right) c_1 + \left(x + \frac{1}{2}x^2 + \frac{1}{6}x^3 + \frac{1}{8}x^4 + \frac{1}{16}x^5\right) c_2 + O(x^6) \quad (2)
$$

Verification of solutions

$$
\begin{aligned}
y = {} & \left(1 + \frac{1}{12}x^3 + \frac{1}{48}x^4 + \frac{1}{80}x^5 + \frac{11}{1440}x^6\right) y(0) \\
& + \left(x + \frac{1}{2}x^2 + \frac{1}{6}x^3 + \frac{1}{8}x^4 + \frac{1}{16}x^5 + \frac{17}{480}x^6\right) y'(0) + O(x^6)
\end{aligned}
$$

Verified OK.

$$
y = \left(1 + \frac{1}{12}x^3 + \frac{1}{48}x^4 + \frac{1}{80}x^5\right) c_1 + \left(x + \frac{1}{2}x^2 + \frac{1}{6}x^3 + \frac{1}{8}x^4 + \frac{1}{16}x^5\right) c_2 + O(x^6)
$$

Verified OK.

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Heun: Equivalence to the GHE or one of its 4 confluent cases under a power @ Moebius
-> trying a solution of the form r0(x) * Y + r1(x) * Y where Y = exp(int(r(x), dx)) * 2F1([a
-> Trying changes of variables to rationalize or make the ODE simpler
   trying a symmetry of the form [xi=0, eta=F(x)]
   checking if the LODE is missing y
   -> Heun: Equivalence to the GHE or one of its 4 confluent cases under a power @ Moebius
   -> trying a solution of the form r0(x) * Y + r1(x) * Y where Y = exp(int(r(x), dx)) * 2F1
      trying a symmetry of the form [xi=0, eta=F(x)]
      trying 2nd order exact linear
      trying symmetries linear in x and y(x)
      trying to convert to a linear ODE with constant coefficients
      -> trying with_periodic_functions in the coefficients
         --- Trying Lie symmetry methods, 2nd order ---
         `, `-> Computing symmetries using: way = 5
   trying a symmetry of the form [xi=0, eta=F(x)]
   checking if the LODE is missing y
   -> Heun: Equivalence to the GHE or one of its 4 confluent cases under a power @ Moebius
   -> trying a solution of the form r0(x) * Y + r1(x) * Y where Y = exp(int(r(x), dx)) * 2F1
      trying a symmetry of the form [xi=0, eta=F(x)]
      trying 2nd order exact linear
      trying symmetries linear in x and y(x)
      trying to convert to a linear ODE with constant coefficients
      -> trying with_periodic_functions in the coefficients
         --- Trying Lie symmetry methods, 2nd order ---
         `, `-> Computing symmetries using: way = 5
<- unable to find a useful change of variables
   trying a symmetry of the form [xi=0, eta=F(x)]
   trying differential order: 2; exact nonlinear
   trying symmetries linear in x and y(x)
   trying to convert to a linear ODE with constant coefficients
   trying 2nd order, integrating factor of the form mu(x,y)
   trying a symmetry of the form [xi=0, eta=F(x)]
   checking if the LODE is missing y
   -> Heun: Equivalence to the GHE or one of its 4 confluent cases under a power @ Moebius
   -> trying a solution of the form r0(x) * Y + r1(x) * Y where Y = exp(int(r(x), dx)) * 2F1
   -> Trying changes of variables to rationalize or make the ODE simpler
      trying a symmetry of the form [xi=0, eta=F(x)]
      checking if the LODE is missing y
      -> Heun: Equivalence to the GHE or one of its 4 confluent cases under a power @ Moebiu
      -> trying a solution of the form r0(x) * Y + r1(x) * Y where Y = exp(int(r(x), dx)) *
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 49

```
Order:=6;
dsolve((x^2-2)*diff(y(x),x$2)+2*diff(y(x),x)+sin(x)*y(x)=0,y(x),type='series',x=0);
```

$$y(x) = \left(1+\frac{1}{12}x^3+\frac{1}{48}x^4+\frac{1}{80}x^5\right)y(0)+\left(x+\frac{1}{2}x^2+\frac{1}{6}x^3+\frac{1}{8}x^4+\frac{1}{16}x^5\right)D(y)(0)+O(x^6)$$

✓ Solution by Mathematica

Time used: 0.001 (sec). Leaf size: 63

```
AsymptoticDSolveValue[(x^2-2)*y''[x]+2*y'[x]+Sin[x]*y[x]==0,y[x],{x,0,5}]
```

$$y(x) \to c_1\left(\frac{x^5}{80}+\frac{x^4}{48}+\frac{x^3}{12}+1\right)+c_2\left(\frac{x^5}{16}+\frac{x^4}{8}+\frac{x^3}{6}+\frac{x^2}{2}+x\right)$$

5.4 problem 4

Internal problem ID [5005]
Internal file name [`OUTPUT/4498_Sunday_June_05_2022_02_59_29_PM_41978586/index.tex`]

Book: Fundamentals of Differential Equations. By Nagle, Saff and Snider. 9th edition. Boston. Pearson 2018.
Section: Chapter 8, Series solutions of differential equations. Section 8.3. page 443
Problem number: 4.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second order series method. Regular singular point. Difference is integer"**

Maple gives the following as the ode type

```
[[_Emden, _Fowler]]
```

$$\boxed{(x^2+x)\,y''+3y'-6xy=0}$$

With the expansion point for the power series method at $x=0$.

The type of the expansion point is first determined. This is done on the homogeneous part of the ODE.

$$(x^2+x)\,y''+3y'-6xy=0$$

The following is summary of singularities for the above ode. Writing the ode as

$$y''+p(x)y'+q(x)y=0$$

Where

$$p(x)=\frac{3}{x\,(x+1)}$$
$$q(x)=-\frac{6}{x+1}$$

Table 201: Table $p(x), q(x)$ singularites.

$p(x) = \frac{3}{x(x+1)}$	
singularity	type
$x = -1$	"regular"
$x = 0$	"regular"

$q(x) = -\frac{6}{x+1}$	
singularity	type
$x = -1$	"regular"

Combining everything together gives the following summary of singularities for the ode as

Regular singular points : $[-1, 0]$

Irregular singular points : $[\infty]$

Since $x = 0$ is regular singular point, then Frobenius power series is used. The ode is normalized to be

$$y''x(x+1) + 3y' - 6xy = 0$$

Let the solution be represented as Frobenius power series of the form

$$y = \sum_{n=0}^{\infty} a_n x^{n+r}$$

Then

$$y' = \sum_{n=0}^{\infty} (n+r)\, a_n x^{n+r-1}$$

$$y'' = \sum_{n=0}^{\infty} (n+r)\,(n+r-1)\, a_n x^{n+r-2}$$

Substituting the above back into the ode gives

$$\left(\sum_{n=0}^{\infty} (n+r)\,(n+r-1)\, a_n x^{n+r-2}\right) x(x+1) + 3\left(\sum_{n=0}^{\infty} (n+r)\, a_n x^{n+r-1}\right) - 6x\left(\sum_{n=0}^{\infty} a_n x^{n+r}\right) = 0 \tag{1}$$

Which simplifies to

$$
\left(\sum_{n=0}^{\infty} x^{n+r} a_n (n+r)(n+r-1)\right) + \left(\sum_{n=0}^{\infty} x^{n+r-1} a_n (n+r)(n+r-1)\right) \\
+ \left(\sum_{n=0}^{\infty} 3(n+r) a_n x^{n+r-1}\right) + \sum_{n=0}^{\infty} \left(-6x^{1+n+r} a_n\right) = 0 \tag{2A}
$$

The next step is to make all powers of x be $n+r-1$ in each summation term. Going over each summation term above with power of x in it which is not already x^{n+r-1} and adjusting the power and the corresponding index gives

$$
\sum_{n=0}^{\infty} x^{n+r} a_n (n+r)(n+r-1) = \sum_{n=1}^{\infty} a_{n-1}(n+r-1)(n+r-2)\, x^{n+r-1}
$$

$$
\sum_{n=0}^{\infty} \left(-6x^{1+n+r} a_n\right) = \sum_{n=2}^{\infty} \left(-6a_{n-2} x^{n+r-1}\right)
$$

Substituting all the above in Eq (2A) gives the following equation where now all powers of x are the same and equal to $n+r-1$.

$$
\left(\sum_{n=1}^{\infty} a_{n-1}(n+r-1)(n+r-2)\, x^{n+r-1}\right) + \left(\sum_{n=0}^{\infty} x^{n+r-1} a_n (n+r)(n+r-1)\right) \\
+ \left(\sum_{n=0}^{\infty} 3(n+r) a_n x^{n+r-1}\right) + \sum_{n=2}^{\infty} \left(-6a_{n-2} x^{n+r-1}\right) = 0 \tag{2B}
$$

The indicial equation is obtained from $n=0$. From Eq (2B) this gives

$$
x^{n+r-1} a_n (n+r)(n+r-1) + 3(n+r) a_n x^{n+r-1} = 0
$$

When $n=0$ the above becomes

$$
x^{-1+r} a_0 r(-1+r) + 3r a_0 x^{-1+r} = 0
$$

Or

$$
\left(x^{-1+r} r(-1+r) + 3r\, x^{-1+r}\right) a_0 = 0
$$

Since $a_0 \neq 0$ then the above simplifies to

$$
r\, x^{-1+r}(2+r) = 0
$$

Since the above is true for all x then the indicial equation becomes

$$r(2+r) = 0$$

Solving for r gives the roots of the indicial equation as

$$\begin{aligned} r_1 &= 0 \\ r_2 &= -2 \end{aligned}$$

Since $a_0 \neq 0$ then the indicial equation becomes

$$r\,x^{-1+r}(2+r) = 0$$

Solving for r gives the roots of the indicial equation as Since $r_1 - r_2 = 2$ is an integer, then we can construct two linearly independent solutions

$$\begin{aligned} y_1(x) &= x^{r_1}\left(\sum_{n=0}^{\infty} a_n x^n\right) \\ y_2(x) &= C y_1(x)\ln(x) + x^{r_2}\left(\sum_{n=0}^{\infty} b_n x^n\right) \end{aligned}$$

Or

$$\begin{aligned} y_1(x) &= \sum_{n=0}^{\infty} a_n x^n \\ y_2(x) &= C y_1(x)\ln(x) + \frac{\sum_{n=0}^{\infty} b_n x^n}{x^2} \end{aligned}$$

Or

$$\begin{aligned} y_1(x) &= \sum_{n=0}^{\infty} a_n x^n \\ y_2(x) &= C y_1(x)\ln(x) + \left(\sum_{n=0}^{\infty} b_n x^{n-2}\right) \end{aligned}$$

Where C above can be zero. We start by finding y_1. Eq (2B) derived above is now used to find all a_n coefficients. The case $n = 0$ is skipped since it was used to find the roots of the indicial equation. a_0 is arbitrary and taken as $a_0 = 1$. Substituting $n = 1$ in Eq. (2B) gives

$$a_1 = -\frac{r(-1+r)}{r^2+4r+3}$$

For $2 \leq n$ the recursive equation is

$$a_{n-1}(n+r-1)(n+r-2)+a_n(n+r)(n+r-1)+3a_n(n+r)-6a_{n-2}=0 \quad (3)$$

Solving for a_n from recursive equation (4) gives

$$a_n = -\frac{n^2a_{n-1}+2nra_{n-1}+r^2a_{n-1}-3na_{n-1}-3ra_{n-1}-6a_{n-2}+2a_{n-1}}{n^2+2nr+r^2+2n+2r} \quad (4)$$

Which for the root $r=0$ becomes

$$a_n = \frac{-n^2a_{n-1}+3na_{n-1}+6a_{n-2}-2a_{n-1}}{n(n+2)} \quad (5)$$

At this point, it is a good idea to keep track of a_n in a table both before substituting $r=0$ and after as more terms are found using the above recursive equation.

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$-\frac{r(-1+r)}{r^2+4r+3}$	0

For $n=2$, using the above recursive equation gives

$$a_2 = \frac{r^3-r^2+6r+18}{r^3+9r^2+26r+24}$$

Which for the root $r=0$ becomes

$$a_2 = \frac{3}{4}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$-\frac{r(-1+r)}{r^2+4r+3}$	0
a_2	$\frac{r^3-r^2+6r+18}{r^3+9r^2+26r+24}$	$\frac{3}{4}$

For $n=3$, using the above recursive equation gives

$$a_3 = \frac{-r^5-r^4-11r^3-47r^2-18r-18}{(r+5)(r+3)^2(1+r)(r+4)}$$

Which for the root $r = 0$ becomes

$$a_3 = -\frac{1}{10}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$-\frac{r(-1+r)}{r^2+4r+3}$	0
a_2	$\frac{r^3-r^2+6r+18}{r^3+9r^2+26r+24}$	$\frac{3}{4}$
a_3	$\frac{-r^5-r^4-11r^3-47r^2-18r-18}{(r+5)(r+3)^2(1+r)(r+4)}$	$-\frac{1}{10}$

For $n = 4$, using the above recursive equation gives

$$a_4 = \frac{r^7 + 5r^6 + 25r^5 + 125r^4 + 280r^3 + 572r^2 + 972r + 612}{(r+6)(r+4)^2(2+r)(r+3)(1+r)(r+5)}$$

Which for the root $r = 0$ becomes

$$a_4 = \frac{17}{80}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$-\frac{r(-1+r)}{r^2+4r+3}$	0
a_2	$\frac{r^3-r^2+6r+18}{r^3+9r^2+26r+24}$	$\frac{3}{4}$
a_3	$\frac{-r^5-r^4-11r^3-47r^2-18r-18}{(r+5)(r+3)^2(1+r)(r+4)}$	$-\frac{1}{10}$
a_4	$\frac{r^7+5r^6+25r^5+125r^4+280r^3+572r^2+972r+612}{(r+6)(r+4)^2(2+r)(r+3)(1+r)(r+5)}$	$\frac{17}{80}$

For $n = 5$, using the above recursive equation gives

$$a_5 = \frac{-r^9 - 11r^8 - 70r^7 - 374r^6 - 1441r^5 - 4259r^4 - 10080r^3 - 15948r^2 - 14580r - 6804}{(r+7)(r+5)^2(1+r)(r+3)^2(2+r)(r+4)(r+6)}$$

Which for the root $r = 0$ becomes

$$a_5 = -\frac{9}{100}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$-\frac{r(-1+r)}{r^2+4r+3}$	0
a_2	$\frac{r^3-r^2+6r+18}{r^3+9r^2+26r+24}$	$\frac{3}{4}$
a_3	$\frac{-r^5-r^4-11r^3-47r^2-18r-18}{(r+5)(r+3)^2(1+r)(r+4)}$	$-\frac{1}{10}$
a_4	$\frac{r^7+5r^6+25r^5+125r^4+280r^3+572r^2+972r+612}{(r+6)(r+4)^2(2+r)(r+3)(1+r)(r+5)}$	$\frac{17}{80}$
a_5	$\frac{-r^9-11r^8-70r^7-374r^6-1441r^5-4259r^4-10080r^3-15948r^2-14580r-6804}{(r+7)(r+5)^2(1+r)(r+3)^2(2+r)(r+4)(r+6)}$	$-\frac{9}{100}$

Using the above table, then the solution $y_1(x)$ is

$$\begin{aligned}
y_1(x) &= a_0 + a_1x + a_2x^2 + a_3x^3 + a_4x^4 + a_5x^5 + a_6x^6\dots \\
&= 1 + \frac{3x^2}{4} - \frac{x^3}{10} + \frac{17x^4}{80} - \frac{9x^5}{100} + O(x^6)
\end{aligned}$$

Now the second solution $y_2(x)$ is found. Let

$$r_1 - r_2 = N$$

Where N is positive integer which is the difference between the two roots. r_1 is taken as the larger root. Hence for this problem we have $N = 2$. Now we need to determine if C is zero or not. This is done by finding $\lim_{r\to r_2} a_2(r)$. If this limit exists, then $C = 0$, else we need to keep the log term and $C \neq 0$. The above table shows that

$$\begin{aligned}
a_N &= a_2 \\
&= \frac{r^3 - r^2 + 6r + 18}{r^3 + 9r^2 + 26r + 24}
\end{aligned}$$

Therefore

$$\begin{aligned}
\lim_{r\to r_2} \frac{r^3 - r^2 + 6r + 18}{r^3 + 9r^2 + 26r + 24} &= \lim_{r\to -2} \frac{r^3 - r^2 + 6r + 18}{r^3 + 9r^2 + 26r + 24} \\
&= \mathit{undefined}
\end{aligned}$$

Since the limit does not exist then the log term is needed. Therefore the second solution has the form

$$y_2(x) = Cy_1(x)\ln(x) + \left(\sum_{n=0}^{\infty} b_n x^{n+r_2}\right)$$

Therefore

$$\frac{d}{dx}y_2(x) = Cy_1'(x)\ln(x) + \frac{Cy_1(x)}{x} + \left(\sum_{n=0}^{\infty} \frac{b_n x^{n+r_2}(n+r_2)}{x}\right)$$
$$= Cy_1'(x)\ln(x) + \frac{Cy_1(x)}{x} + \left(\sum_{n=0}^{\infty} x^{-1+n+r_2} b_n(n+r_2)\right)$$
$$\frac{d^2}{dx^2}y_2(x) = Cy_1''(x)\ln(x) + \frac{2Cy_1'(x)}{x} - \frac{Cy_1(x)}{x^2} + \sum_{n=0}^{\infty}\left(\frac{b_n x^{n+r_2}(n+r_2)^2}{x^2} - \frac{b_n x^{n+r_2}(n+r_2)}{x^2}\right)$$
$$= Cy_1''(x)\ln(x) + \frac{2Cy_1'(x)}{x} - \frac{Cy_1(x)}{x^2} + \left(\sum_{n=0}^{\infty} x^{-2+n+r_2} b_n(n+r_2)(-1+n+r_2)\right)$$

Substituting these back into the given ode $y''x(x+1) + 3y' - 6xy = 0$ gives

$$\left(Cy_1''(x)\ln(x) + \frac{2Cy_1'(x)}{x} - \frac{Cy_1(x)}{x^2} + \sum_{n=0}^{\infty}\left(\frac{b_n x^{n+r_2}(n+r_2)^2}{x^2} - \frac{b_n x^{n+r_2}(n+r_2)}{x^2}\right)\right)x(x+1) + 3Cy_1'(x)\ln(x) + \frac{3Cy_1(x)}{x} + 3\left(\sum_{n=0}^{\infty}\frac{b_n x^{n+r_2}(n+r_2)}{x}\right) - 6x\left(Cy_1(x)\ln(x) + \left(\sum_{n=0}^{\infty} b_n x^{n+r_2}\right)\right) = 0$$

Which can be written as

$$\left(\left(y_1''(x)\,x(x+1) + 3y_1'(x) - 6y_1(x)\,x\right)\ln(x) + \left(\frac{2y_1'(x)}{x} - \frac{y_1(x)}{x^2}\right)x(x+1) + \frac{3y_1(x)}{x}\right)C + \left(\sum_{n=0}^{\infty}\left(\frac{b_n x^{n+r_2}(n+r_2)^2}{x^2} - \frac{b_n x^{n+r_2}(n+r_2)}{x^2}\right)\right)x(x+1) + 3\left(\sum_{n=0}^{\infty}\frac{b_n x^{n+r_2}(n+r_2)}{x}\right) - 6x\left(\sum_{n=0}^{\infty} b_n x^{n+r_2}\right) = 0 \tag{7}$$

But since $y_1(x)$ is a solution to the ode, then

$$y_1''(x)\,x(x+1) + 3y_1'(x) - 6y_1(x)\,x = 0$$

Eq (7) simplifes to

$$\left(\left(\frac{2y_1'(x)}{x}-\frac{y_1(x)}{x^2}\right)x(x+1)+\frac{3y_1(x)}{x}\right)C \\ +\left(\sum_{n=0}^{\infty}\left(\frac{b_n x^{n+r_2}(n+r_2)^2}{x^2}-\frac{b_n x^{n+r_2}(n+r_2)}{x^2}\right)\right)x(x+1) \\ +3\left(\sum_{n=0}^{\infty}\frac{b_n x^{n+r_2}(n+r_2)}{x}\right)-6x\left(\sum_{n=0}^{\infty}b_n x^{n+r_2}\right)=0 \tag{8}$$

Substituting $y_1=\sum_{n=0}^{\infty}a_n x^{n+r_1}$ into the above gives

$$\frac{\left(2x(x+1)\left(\sum_{n=0}^{\infty}x^{-1+n+r_1}a_n(n+r_1)\right)-(-2+x)\left(\sum_{n=0}^{\infty}a_n x^{n+r_1}\right)\right)C}{x} \\ +\frac{\left(\sum_{n=0}^{\infty}x^{-2+n+r_2}b_n(n+r_2)(-1+n+r_2)\right)x^2(x+1)+3\left(\sum_{n=0}^{\infty}x^{-1+n+r_2}b_n(n+r_2)\right)x-6x^2\left(\sum_{n=0}^{\infty}b_n x^{n+r_2}\right)}{x} \\ =0 \tag{9}$$

Since $r_1=0$ and $r_2=-2$ then the above becomes

$$\frac{\left(2x(x+1)\left(\sum_{n=0}^{\infty}x^{n-1}a_n n\right)-(-2+x)\left(\sum_{n=0}^{\infty}a_n x^{n}\right)\right)C}{x} \\ +\frac{\left(\sum_{n=0}^{\infty}x^{-4+n}b_n(n-2)(-3+n)\right)x^2(x+1)+3\left(\sum_{n=0}^{\infty}x^{-3+n}b_n(n-2)\right)x-6x^2\left(\sum_{n=0}^{\infty}b_n x^{n-2}\right)}{x} \\ =0 \tag{10}$$

Which simplifies to

$$
\begin{aligned}
&\left(\sum_{n=0}^{\infty} 2C\, x^{n} a_{n} n\right)+\left(\sum_{n=0}^{\infty} 2Cn\, x^{n-1} a_{n}\right)+\left(\sum_{n=0}^{\infty} 2C\, x^{n-1} a_{n}\right) \\
&+\sum_{n=0}^{\infty}\left(-C a_{n} x^{n}\right)+\left(\sum_{n=0}^{\infty} x^{n-2} b_{n}\left(n^{2}-5n+6\right)\right) \\
&\quad+\left(\sum_{n=0}^{\infty} x^{-3+n} b_{n}\left(n^{2}-5n+6\right)\right) \\
&\quad+\left(\sum_{n=0}^{\infty} 3x^{-3+n} b_{n}(n-2)\right)+\sum_{n=0}^{\infty}\left(-6x^{n-1} b_{n}\right)=0
\end{aligned} \tag{2A}
$$

The next step is to make all powers of x be $-3+n$ in each summation term. Going over each summation term above with power of x in it which is not already x^{-3+n} and adjusting the power and the corresponding index gives

$$
\begin{aligned}
\sum_{n=0}^{\infty} 2C\, x^{n} a_{n} n &= \sum_{n=3}^{\infty} 2C(-3+n)\, a_{-3+n} x^{-3+n} \\
\sum_{n=0}^{\infty} 2Cn\, x^{n-1} a_{n} &= \sum_{n=2}^{\infty} 2C(n-2)\, a_{n-2} x^{-3+n} \\
\sum_{n=0}^{\infty} 2C\, x^{n-1} a_{n} &= \sum_{n=2}^{\infty} 2C a_{n-2} x^{-3+n} \\
\sum_{n=0}^{\infty}\left(-C a_{n} x^{n}\right) &= \sum_{n=3}^{\infty}\left(-C a_{-3+n} x^{-3+n}\right) \\
\sum_{n=0}^{\infty} x^{n-2} b_{n}\left(n^{2}-5n+6\right) &= \sum_{n=1}^{\infty} b_{n-1}\left((n-1)^{2}-5n+11\right) x^{-3+n} \\
\sum_{n=0}^{\infty}\left(-6x^{n-1} b_{n}\right) &= \sum_{n=2}^{\infty}\left(-6b_{n-2} x^{-3+n}\right)
\end{aligned}
$$

Substituting all the above in Eq (2A) gives the following equation where now all powers of x are the same and equal to $-3+n$.

$$\left(\sum_{n=3}^{\infty} 2C(-3+n)\, a_{-3+n} x^{-3+n}\right) + \left(\sum_{n=2}^{\infty} 2C(n-2)\, a_{n-2} x^{-3+n}\right) + \left(\sum_{n=2}^{\infty} 2C a_{n-2} x^{-3+n}\right) + \sum_{n=3}^{\infty} \left(-C a_{-3+n} x^{-3+n}\right) + \left(\sum_{n=1}^{\infty} b_{n-1}\left((n-1)^2-5n+11\right) x^{-3+n}\right) + \left(\sum_{n=0}^{\infty} x^{-3+n} b_n \left(n^2-5n+6\right)\right) + \left(\sum_{n=0}^{\infty} 3x^{-3+n} b_n (n-2)\right) + \sum_{n=2}^{\infty} \left(-6 b_{n-2} x^{-3+n}\right) = 0 \tag{2B}$$

For $n=0$ in Eq. (2B), we choose arbitray value for b_0 as $b_0=1$. For $n=1$, Eq (2B) gives

$$6b_0 - b_1 = 0$$

Which when replacing the above values found already for b_n and the values found earlier for a_n and for C, gives

$$6 - b_1 = 0$$

Solving the above for b_1 gives

$$b_1 = 6$$

For $n=N$, where $N=2$ which is the difference between the two roots, we are free to choose $b_2=0$. Hence for $n=2$, Eq (2B) gives

$$2C + 6 = 0$$

Which is solved for C. Solving for C gives

$$C = -3$$

For $n=3$, Eq (2B) gives

$$(-a_0 + 4a_1)\, C - 6b_1 + 3b_3 = 0$$

Which when replacing the above values found already for b_n and the values found earlier for a_n and for C, gives

$$-33 + 3b_3 = 0$$

Solving the above for b_3 gives

$$b_3 = 11$$

For $n = 4$, Eq (2B) gives

$$(a_1 + 6a_2) C - 6b_2 + 8b_4 = 0$$

Which when replacing the above values found already for b_n and the values found earlier for a_n and for C, gives

$$-\frac{27}{2} + 8b_4 = 0$$

Solving the above for b_4 gives

$$b_4 = \frac{27}{16}$$

For $n = 5$, Eq (2B) gives

$$(3a_2 + 8a_3) C - 6b_3 + 2b_4 + 15b_5 = 0$$

Which when replacing the above values found already for b_n and the values found earlier for a_n and for C, gives

$$-\frac{2679}{40} + 15b_5 = 0$$

Solving the above for b_5 gives

$$b_5 = \frac{893}{200}$$

Now that we found all b_n and C, we can calculate the second solution from

$$y_2(x) = C y_1(x) \ln(x) + \left(\sum_{n=0}^{\infty} b_n x^{n+r_2}\right)$$

Using the above value found for $C = -3$ and all b_n, then the second solution becomes

$$y_2(x) = (-3)\left(1 + \frac{3x^2}{4} - \frac{x^3}{10} + \frac{17x^4}{80} - \frac{9x^5}{100} + O(x^6)\right)\ln(x) + \frac{1 + 6x + 11x^3 + \frac{27x^4}{16} + \frac{893x^5}{200} + O(x^6)}{x^2}$$

Therefore the homogeneous solution is

$$\begin{aligned}
y_h(x) &= c_1 y_1(x) + c_2 y_2(x) \\
&= c_1\left(1 + \frac{3x^2}{4} - \frac{x^3}{10} + \frac{17x^4}{80} - \frac{9x^5}{100} + O(x^6)\right) \\
&+ c_2\left((-3)\left(1 + \frac{3x^2}{4} - \frac{x^3}{10} + \frac{17x^4}{80} - \frac{9x^5}{100} + O(x^6)\right)\ln(x) \right.\\
&\left. + \frac{1 + 6x + 11x^3 + \frac{27x^4}{16} + \frac{893x^5}{200} + O(x^6)}{x^2}\right)
\end{aligned}$$

Hence the final solution is

$$
\begin{aligned}
y &= y_h \\
&= c_1\left(1+\frac{3x^2}{4}-\frac{x^3}{10}+\frac{17x^4}{80}-\frac{9x^5}{100}+O(x^6)\right) \\
&\quad + c_2\left(\left(-3-\frac{9x^2}{4}+\frac{3x^3}{10}-\frac{51x^4}{80}+\frac{27x^5}{100}-3O(x^6)\right)\ln(x)\right. \\
&\quad \left. + \frac{1+6x+11x^3+\frac{27x^4}{16}+\frac{893x^5}{200}+O(x^6)}{x^2}\right)
\end{aligned}
$$

Summary

The solution(s) found are the following

$$
\begin{aligned}
y &= c_1\left(1+\frac{3x^2}{4}-\frac{x^3}{10}+\frac{17x^4}{80}-\frac{9x^5}{100}+O(x^6)\right) \\
&\quad + c_2\left(\left(-3-\frac{9x^2}{4}+\frac{3x^3}{10}-\frac{51x^4}{80}+\frac{27x^5}{100}-3O(x^6)\right)\ln(x)\right. \\
&\quad \left. + \frac{1+6x+11x^3+\frac{27x^4}{16}+\frac{893x^5}{200}+O(x^6)}{x^2}\right)
\end{aligned} \tag{1}
$$

Verification of solutions

$$
\begin{aligned}
y &= c_1\left(1+\frac{3x^2}{4}-\frac{x^3}{10}+\frac{17x^4}{80}-\frac{9x^5}{100}+O(x^6)\right) \\
&\quad + c_2\left(\left(-3-\frac{9x^2}{4}+\frac{3x^3}{10}-\frac{51x^4}{80}+\frac{27x^5}{100}-3O(x^6)\right)\ln(x)\right. \\
&\quad \left. + \frac{1+6x+11x^3+\frac{27x^4}{16}+\frac{893x^5}{200}+O(x^6)}{x^2}\right)
\end{aligned}
$$

Verified OK.

5.4.1 Maple step by step solution

Let's solve

$y''x(x+1)+3y'-6xy=0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y''=\frac{6y}{x+1}-\frac{3y'}{x(x+1)}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y''-\frac{6y}{x+1}+\frac{3y'}{x(x+1)}=0$

□ Check to see if x_0 is a regular singular point

  ◦ Define functions

  $\left[P_2(x)=\frac{3}{x(x+1)}, P_3(x)=-\frac{6}{x+1}\right]$

  ◦ $(x+1)\cdot P_2(x)$ is analytic at $x=-1$

  $\left((x+1)\cdot P_2(x)\right)\Big|_{x=-1}=-3$

  ◦ $(x+1)^2\cdot P_3(x)$ is analytic at $x=-1$

  $\left((x+1)^2\cdot P_3(x)\right)\Big|_{x=-1}=0$

  ◦ $x=-1$is a regular singular point

  Check to see if x_0 is a regular singular point

  $x_0=-1$

- Multiply by denominators

  $y''x(x+1)+3y'-6xy=0$

- Change variables using $x=u-1$ so that the regular singular point is at $u=0$

  $(u^2-u)\left(\frac{d^2}{du^2}y(u)\right)+3\frac{d}{du}y(u)+(-6u+6)\,y(u)=0$

- Assume series solution for $y(u)$

  $y(u)=\sum_{k=0}^{\infty}a_k u^{k+r}$

□ Rewrite ODE with series expansions

- Convert $u^m \cdot y(u)$ to series expansion for $m = 0..1$

  $$u^m \cdot y(u) = \sum_{k=0}^{\infty} a_k u^{k+r+m}$$

- Shift index using $k->k-m$

  $$u^m \cdot y(u) = \sum_{k=m}^{\infty} a_{k-m} u^{k+r}$$

- Convert $\frac{d}{du}y(u)$ to series expansion

  $$\frac{d}{du}y(u) = \sum_{k=0}^{\infty} a_k(k+r)\, u^{k+r-1}$$

- Shift index using $k->k+1$

  $$\frac{d}{du}y(u) = \sum_{k=-1}^{\infty} a_{k+1}(k+1+r)\, u^{k+r}$$

- Convert $u^m \cdot \left(\frac{d^2}{du^2}y(u)\right)$ to series expansion for $m = 1..2$

  $$u^m \cdot \left(\frac{d^2}{du^2}y(u)\right) = \sum_{k=0}^{\infty} a_k(k+r)\,(k+r-1)\, u^{k+r-2+m}$$

- Shift index using $k->k+2-m$

  $$u^m \cdot \left(\frac{d^2}{du^2}y(u)\right) = \sum_{k=-2+m}^{\infty} a_{k+2-m}(k+2-m+r)\,(k+1-m+r)\, u^{k+r}$$

  Rewrite ODE with series expansions

  $$-a_0 r(-4+r)\, u^{-1+r} + (-a_1(1+r)\,(-3+r) + a_0(r^2-r+6))\, u^r + \left(\sum_{k=1}^{\infty} (-a_{k+1}(k+1+r)\,(k-\right.$$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $$-r(-4+r) = 0$$

- Values of r that satisfy the indicial equation

  $$r \in \{0, 4\}$$

- Each term must be 0

  $$-a_1(1+r)\,(-3+r) + a_0(r^2-r+6) = 0$$

- Each term in the series must be 0, giving the recursion relation

  $$-a_{k+1}(k+1+r)\,(k-3+r) + (k^2+(2r-1)\,k+r^2-r+6)\, a_k - 6a_{k-1} = 0$$

- Shift index using $k->k+1$

  $$-a_{k+2}(k+2+r)\,(k-2+r) + \left((k+1)^2+(2r-1)\,(k+1)+r^2-r+6\right) a_{k+1} - 6a_k = 0$$

- Recursion relation that defines series solution to ODE

$$a_{k+2} = \frac{k^2 a_{k+1}+2kra_{k+1}+r^2 a_{k+1}+ka_{k+1}+ra_{k+1}-6a_k+6a_{k+1}}{(k+2+r)(k-2+r)}$$

- Recursion relation for $r = 0$

$$a_{k+2} = \frac{k^2 a_{k+1}+ka_{k+1}-6a_k+6a_{k+1}}{(k+2)(k-2)}$$

- Series not valid for $r = 0$, division by 0 in the recursion relation at $k = 2$

$$a_{k+2} = \frac{k^2 a_{k+1}+ka_{k+1}-6a_k+6a_{k+1}}{(k+2)(k-2)}$$

- Recursion relation for $r = 4$

$$a_{k+2} = \frac{k^2 a_{k+1}+9ka_{k+1}-6a_k+26a_{k+1}}{(k+6)(k+2)}$$

- Solution for $r = 4$

$$\left[y(u) = \sum_{k=0}^{\infty} a_k u^{k+4}, a_{k+2} = \frac{k^2 a_{k+1}+9ka_{k+1}-6a_k+26a_{k+1}}{(k+6)(k+2)}, -5a_1 + 18a_0 = 0\right]$$

- Revert the change of variables $u = x + 1$

$$\left[y = \sum_{k=0}^{\infty} a_k (x+1)^{k+4}, a_{k+2} = \frac{k^2 a_{k+1}+9ka_{k+1}-6a_k+26a_{k+1}}{(k+6)(k+2)}, -5a_1 + 18a_0 = 0\right]$$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
<- No Liouvillian solutions exists
-> Trying a solution in terms of special functions:
   -> Bessel
   -> elliptic
   -> Legendre
   -> Kummer
      -> hyper3: Equivalence to 1F1 under a power @ Moebius
   -> hypergeometric
      -> heuristic approach
      -> hyper3: Equivalence to 2F1, 1F1 or 0F1 under a power @ Moebius
   -> Mathieu
      -> Equivalence to the rational form of Mathieu ODE under a power @ Moebius
trying a solution in terms of MeijerG functions
-> Heun: Equivalence to the GHE or one of its 4 confluent cases under a power @ Moebius
<- Heun successful: received ODE is equivalent to the  HeunC  ODE, case  a <> 0, e <> 0, c =
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 60

```
Order:=6;
dsolve((x^2+x)*diff(y(x),x$2)+3*diff(y(x),x)-6*x*y(x)=0,y(x),type='series',x=0);
```

$$y(x) = \frac{c_1\left(1+\frac{3}{4}x^2-\frac{1}{10}x^3+\frac{17}{80}x^4-\frac{9}{100}x^5+\mathrm{O}\left(x^6\right)\right)x^2+c_2\left(\ln\left(x\right)\left(6x^2+\frac{9}{2}x^4-\frac{3}{5}x^5+\mathrm{O}\left(x^6\right)\right)+\left(-2-12x-\right.\right.}{x^2}$$

✓ Solution by Mathematica

Time used: 0.038 (sec). Leaf size: 73

```
AsymptoticDSolveValue[(x^2+x)*y''[x]+2*y'[x]-6*x*y[x]==0,y[x],{x,0,5}]
```

$$y(x) \to c_2\left(\frac{7x^4}{20} - \frac{x^3}{6} + x^2 + 1\right) + c_1\left(\frac{1}{3}\left(x^3 - 6x^2 - 6\right)\log(x) + \frac{7x^4 + 240x^3 + 72x^2 + 180x + 36}{36x}\right)$$

5.5 problem 5

Internal problem ID [5006]
Internal file name [`OUTPUT/4499_Sunday_June_05_2022_02_59_31_PM_27314559/index.tex`]

Book: Fundamentals of Differential Equations. By Nagle, Saff and Snider. 9th edition. Boston. Pearson 2018.
Section: Chapter 8, Series solutions of differential equations. Section 8.3. page 443
Problem number: 5.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second order series method. Ordinary point", "second order series method. Taylor series method"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

$$\boxed{\left(t^{2}-t-2\right) x''+\left(t+1\right) x'-\left(-2+t\right) x=0}$$

With the expansion point for the power series method at $t = 0$.

Solving ode using Taylor series method. This gives review on how the Taylor series method works for solving second order ode.

Let

$$y''=f(x,y,y')$$

Assuming expansion is at $x_0 = 0$ (we can always shift the actual expansion point to 0 by change of variables) and assuming $f(x,y,y')$ is analytic at x_0 which must be the case for an ordinary point. Let initial conditions be $y(x_0) = y_0$ and $y'(x_0) = y'_0$. Using Taylor series gives

$$\begin{aligned}
y(x) &= y(x_0)+(x-x_0)\,y'(x_0)+\frac{(x-x_0)^2}{2}y''(x_0)+\frac{(x-x_0)^3}{3!}y'''(x_0)+\cdots \\
&= y_0+xy'_0+\frac{x^2}{2}\,f|_{x_0,y_0,y'_0}+\frac{x^3}{3!}\,f'|_{x_0,y_0,y'_0}+\cdots \\
&= y_0+xy'_0+\sum_{n=0}^{\infty}\frac{x^{n+2}}{(n+2)\,!}\,\frac{d^nf}{dx^n}\bigg|_{x_0,y_0,y'_0}
\end{aligned}$$

But

$$\frac{df}{dx} = \frac{\partial f}{\partial x}\frac{dx}{dx} + \frac{\partial f}{\partial y}\frac{dy}{dx} + \frac{\partial f}{\partial y'}\frac{dy'}{dx} \tag{1}$$

$$= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}y' + \frac{\partial f}{\partial y'}y'' \tag{230}$$

$$= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}y' + \frac{\partial f}{\partial y'}f \tag{231}$$

$$\frac{d^2 f}{dx^2} = \frac{d}{dx}\left(\frac{df}{dx}\right)$$

$$= \frac{\partial}{\partial x}\left(\frac{df}{dx}\right) + \frac{\partial}{\partial y}\left(\frac{df}{dx}\right)y' + \frac{\partial}{\partial y'}\left(\frac{df}{dx}\right)f \tag{2}$$

$$\frac{d^3 f}{dx^3} = \frac{d}{dx}\left(\frac{d^2 f}{dx^2}\right)$$

$$= \frac{\partial}{\partial x}\left(\frac{d^2 f}{dx^2}\right) + \left(\frac{\partial}{\partial y}\frac{d^2 f}{dx^2}\right)y' + \frac{\partial}{\partial y'}\left(\frac{d^2 f}{dx^2}\right)f \tag{3}$$

$$\vdots$$

And so on. Hence if we name $F_0 = f(x, y, y')$ then the above can be written as

$$
\begin{aligned}
F_0 &= f(x, y, y') \qquad (4)\\
F_1 &= \frac{df}{dx}\\
&= \frac{dF_0}{dx}\\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}y' + \frac{\partial f}{\partial y'}y''\\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}y' + \frac{\partial f}{\partial y'}f \qquad (5)\\
&= \frac{\partial F_0}{\partial x} + \frac{\partial F_0}{\partial y}y' + \frac{\partial F_0}{\partial y'}F_0\\
F_2 &= \frac{d}{dx}\left(\frac{d}{dx}f\right)\\
&= \frac{d}{dx}(F_1)\\
&= \frac{\partial}{\partial x}F_1 + \left(\frac{\partial F_1}{\partial y}\right)y' + \left(\frac{\partial F_1}{\partial y'}\right)y''\\
&= \frac{\partial}{\partial x}F_1 + \left(\frac{\partial F_1}{\partial y}\right)y' + \left(\frac{\partial F_1}{\partial y'}\right)F_0\\
&\vdots\\
F_n &= \frac{d}{dx}(F_{n-1})\\
&= \frac{\partial}{\partial x}F_{n-1} + \left(\frac{\partial F_{n-1}}{\partial y}\right)y' + \left(\frac{\partial F_{n-1}}{\partial y'}\right)y''\\
&= \frac{\partial}{\partial x}F_{n-1} + \left(\frac{\partial F_{n-1}}{\partial y}\right)y' + \left(\frac{\partial F_{n-1}}{\partial y'}\right)F_0 \qquad (6)
\end{aligned}
$$

Therefore (6) can be used from now on along with

$$
y(x) = y_0 + xy_0' + \sum_{n=0}^{\infty} \frac{x^{n+2}}{(n+2)!} F_n|_{x_0,y_0,y_0'} \qquad (7)
$$

To find $y(x)$ series solution around $x = 0$. Hence

$$
\begin{aligned}
F_0 &= \frac{xt - x't - 2x - x'}{t^2 - t - 2} \\
F_1 &= \frac{dF_0}{dt} \\
&= \frac{\partial F_0}{\partial t} + \frac{\partial F_0}{\partial x} x' + \frac{\partial F_0}{\partial x'} F_0 \\
&= \frac{(t^3 - t^2 + 4t + 6)\, x' + (-2t^2 + 5t - 2)\, x}{(t^2 - t - 2)^2} \\
F_2 &= \frac{dF_1}{dt} \\
&= \frac{\partial F_1}{\partial t} + \frac{\partial F_1}{\partial x} x' + \frac{\partial F_1}{\partial x'} F_1 \\
&= \frac{(-4t^4 + 8t^3 - 24t^2 - 34t + 2)\, x' + x(-2 + t)\,(t^3 + 3t^2 - 3t + 13)}{(t^2 - t - 2)^3} \\
F_3 &= \frac{dF_2}{dt} \\
&= \frac{\partial F_2}{\partial t} + \frac{\partial F_2}{\partial x} x' + \frac{\partial F_2}{\partial x'} F_2 \\
&= \frac{(t + 1)\,(t^5 + 11t^4 - 47t^3 + 217t^2 - 88t + 124)\, x' - 6x(-2 + t)\,(t^4 + 2t^2 + 20t - 10)}{(t^2 - t - 2)^4} \\
F_4 &= \frac{dF_3}{dt} \\
&= \frac{\partial F_3}{\partial t} + \frac{\partial F_3}{\partial x} x' + \frac{\partial F_3}{\partial x'} F_3 \\
&= \frac{(-9t^7 - 33t^6 + 132t^5 - 1212t^4 - 195t^3 - 1407t^2 - 2040t + 540)\, x' + x(-2 + t)\,(t^6 + 30t^5 - 42t^4 +}{(t^2 - t - 2)^5}
\end{aligned}
$$

And so on. Evaluating all the above at initial conditions $t = 0$ and $x(0) = x(0)$ and $x'(0) = x'(0)$ gives

$$
\begin{aligned}
F_0 &= x(0) + \frac{x'(0)}{2} \\
F_1 &= -\frac{x(0)}{2} + \frac{3x'(0)}{2} \\
F_2 &= \frac{13x(0)}{4} - \frac{x'(0)}{4} \\
F_3 &= -\frac{15x(0)}{2} + \frac{31x'(0)}{4} \\
F_4 &= \frac{83x(0)}{2} - \frac{135x'(0)}{8}
\end{aligned}
$$

Substituting all the above in (7) and simplifying gives the solution as

$$x = \left(1+\frac{1}{2}t^2-\frac{1}{12}t^3+\frac{13}{96}t^4-\frac{1}{16}t^5+\frac{83}{1440}t^6\right)x(0) + \left(t+\frac{1}{4}t^2+\frac{1}{4}t^3-\frac{1}{96}t^4+\frac{31}{480}t^5-\frac{3}{128}t^6\right)x'(0)+O(t^6)$$

Since the expansion point $t = 0$ is an ordinary, we can also solve this using standard power series The ode is normalized to be

$$\left(t^2-t-2\right)x''+(t+1)\,x'+(2-t)\,x=0$$

Let the solution be represented as power series of the form

$$x=\sum_{n=0}^{\infty} a_n t^n$$

Then

$$x'=\sum_{n=1}^{\infty} n a_n t^{n-1}$$
$$x''=\sum_{n=2}^{\infty} n(n-1)\,a_n t^{n-2}$$

Substituting the above back into the ode gives

$$\left(t^2-t-2\right)\left(\sum_{n=2}^{\infty} n(n-1)\,a_n t^{n-2}\right)+(t+1)\left(\sum_{n=1}^{\infty} n a_n t^{n-1}\right)+(2-t)\left(\sum_{n=0}^{\infty} a_n t^n\right)=0 \tag{1}$$

Which simplifies to

$$\left(\sum_{n=2}^{\infty} t^n a_n n(n-1)\right)+\sum_{n=2}^{\infty}\left(-n\,t^{n-1}a_n(n-1)\right)+\sum_{n=2}^{\infty}\left(-2n(n-1)\,a_n t^{n-2}\right) + \left(\sum_{n=1}^{\infty} n a_n t^n\right)+\left(\sum_{n=1}^{\infty} n a_n t^{n-1}\right)+\left(\sum_{n=0}^{\infty} 2a_n t^n\right)+\sum_{n=0}^{\infty}\left(-t^{1+n}a_n\right)=0 \tag{2}$$

The next step is to make all powers of t be n in each summation term. Going over each summation term above with power of t in it which is not already t^n and adjusting the

power and the corresponding index gives

$$\sum_{n=2}^{\infty}\left(-n\,t^{n-1} a_n (n-1)\right) = \sum_{n=1}^{\infty}\left(-(1+n)\,a_{1+n} n\,t^n\right)$$

$$\sum_{n=2}^{\infty}\left(-2n(n-1)\,a_n t^{n-2}\right) = \sum_{n=0}^{\infty}\left(-2(n+2)\,a_{n+2}(1+n)\,t^n\right)$$

$$\sum_{n=1}^{\infty} n a_n t^{n-1} = \sum_{n=0}^{\infty}(1+n)\,a_{1+n} t^n$$

$$\sum_{n=0}^{\infty}\left(-t^{1+n} a_n\right) = \sum_{n=1}^{\infty}\left(-a_{n-1} t^n\right)$$

Substituting all the above in Eq (2) gives the following equation where now all powers of t are the same and equal to n.

$$\left(\sum_{n=2}^{\infty} t^n a_n n(n-1)\right) + \sum_{n=1}^{\infty}\left(-(1+n)\,a_{1+n} n\,t^n\right) + \sum_{n=0}^{\infty}\left(-2(n+2)\,a_{n+2}(1+n)\,t^n\right) + \left(\sum_{n=1}^{\infty} n a_n t^n\right) + \left(\sum_{n=0}^{\infty}(1+n)\,a_{1+n} t^n\right) + \left(\sum_{n=0}^{\infty} 2a_n t^n\right) + \sum_{n=1}^{\infty}\left(-a_{n-1} t^n\right) = 0 \tag{3}$$

$n = 0$ gives

$$-4a_2 + a_1 + 2a_0 = 0$$

$$a_2 = \frac{a_0}{2} + \frac{a_1}{4}$$

$n = 1$ gives

$$-12a_3 + 3a_1 - a_0 = 0$$

Which after substituting earlier equations, simplifies to

$$a_3 = -\frac{a_0}{12} + \frac{a_1}{4}$$

For $2 \leq n$, the recurrence equation is

$$n a_n (n-1) - (1+n)\,a_{1+n} n - 2(n+2)\,a_{n+2}(1+n) + n a_n + (1+n)\,a_{1+n} + 2a_n - a_{n-1} = 0 \tag{4}$$

Solving for a_{n+2}, gives

$$
\begin{aligned}
a_{n+2} &= \frac{n^2 a_n - n^2 a_{1+n} + 2a_n + a_{1+n} - a_{n-1}}{2\left(n+2\right)\left(1+n\right)} \\
&= \frac{\left(n^2+2\right) a_n}{2\left(n+2\right)\left(1+n\right)} + \frac{\left(-n^2+1\right) a_{1+n}}{2\left(n+2\right)\left(1+n\right)} - \frac{a_{n-1}}{2\left(n+2\right)\left(1+n\right)}
\end{aligned} \tag{5}
$$

For $n = 2$ the recurrence equation gives

$$6a_2 - 3a_3 - 24a_4 - a_1 = 0$$

Which after substituting the earlier terms found becomes

$$a_4 = \frac{13a_0}{96} - \frac{a_1}{96}$$

For $n = 3$ the recurrence equation gives

$$11a_3 - 8a_4 - 40a_5 - a_2 = 0$$

Which after substituting the earlier terms found becomes

$$a_5 = -\frac{a_0}{16} + \frac{31a_1}{480}$$

For $n = 4$ the recurrence equation gives

$$18a_4 - 15a_5 - 60a_6 - a_3 = 0$$

Which after substituting the earlier terms found becomes

$$a_6 = \frac{83a_0}{1440} - \frac{3a_1}{128}$$

For $n = 5$ the recurrence equation gives

$$27a_5 - 24a_6 - 84a_7 - a_4 = 0$$

Which after substituting the earlier terms found becomes

$$a_7 = -\frac{171a_0}{4480} + \frac{139a_1}{5040}$$

And so on. Therefore the solution is

$$x = \sum_{n=0}^{\infty} a_n t^n$$
$$= a_3 t^3 + a_2 t^2 + a_1 t + a_0 + \dots$$

Substituting the values for a_n found above, the solution becomes

$$x = a_0 + a_1 t + \left(\frac{a_0}{2} + \frac{a_1}{4}\right) t^2 + \left(-\frac{a_0}{12} + \frac{a_1}{4}\right) t^3 + \left(\frac{13a_0}{96} - \frac{a_1}{96}\right) t^4 + \left(-\frac{a_0}{16} + \frac{31a_1}{480}\right) t^5 + \dots$$

Collecting terms, the solution becomes

$$x = \left(1 + \frac{1}{2}t^2 - \frac{1}{12}t^3 + \frac{13}{96}t^4 - \frac{1}{16}t^5\right) a_0 + \left(t + \frac{1}{4}t^2 + \frac{1}{4}t^3 - \frac{1}{96}t^4 + \frac{31}{480}t^5\right) a_1 + O(t^6) \quad (3)$$

At $t = 0$ the solution above becomes

$$x = \left(1 + \frac{1}{2}t^2 - \frac{1}{12}t^3 + \frac{13}{96}t^4 - \frac{1}{16}t^5\right) c_1 + \left(t + \frac{1}{4}t^2 + \frac{1}{4}t^3 - \frac{1}{96}t^4 + \frac{31}{480}t^5\right) c_2 + O(t^6)$$

Summary

The solution(s) found are the following

$$x = \left(1 + \frac{1}{2}t^2 - \frac{1}{12}t^3 + \frac{13}{96}t^4 - \frac{1}{16}t^5 + \frac{83}{1440}t^6\right) x(0) + \left(t + \frac{1}{4}t^2 + \frac{1}{4}t^3 - \frac{1}{96}t^4 + \frac{31}{480}t^5 - \frac{3}{128}t^6\right) x'(0) + O(t^6) \quad (1)$$

$$x = \left(1 + \frac{1}{2}t^2 - \frac{1}{12}t^3 + \frac{13}{96}t^4 - \frac{1}{16}t^5\right) c_1 + \left(t + \frac{1}{4}t^2 + \frac{1}{4}t^3 - \frac{1}{96}t^4 + \frac{31}{480}t^5\right) c_2 + O(t^6) \quad (2)$$

Verification of solutions

$$x = \left(1 + \frac{1}{2}t^2 - \frac{1}{12}t^3 + \frac{13}{96}t^4 - \frac{1}{16}t^5 + \frac{83}{1440}t^6\right) x(0) + \left(t + \frac{1}{4}t^2 + \frac{1}{4}t^3 - \frac{1}{96}t^4 + \frac{31}{480}t^5 - \frac{3}{128}t^6\right) x'(0) + O(t^6)$$

Verified OK.

$$x = \left(1 + \frac{1}{2}t^2 - \frac{1}{12}t^3 + \frac{13}{96}t^4 - \frac{1}{16}t^5\right) c_1 + \left(t + \frac{1}{4}t^2 + \frac{1}{4}t^3 - \frac{1}{96}t^4 + \frac{31}{480}t^5\right) c_2 + O(t^6)$$

Verified OK.

5.5.1 Maple step by step solution

Let's solve

$(t^2 - t - 2)\, x'' + (t+1)\, x' + (2-t)\, x = 0$

- Highest derivative means the order of the ODE is 2

  x''

- Isolate 2nd derivative

  $x'' = \frac{x}{t+1} - \frac{x'}{-2+t}$

- Group terms with x on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $x'' + \frac{x'}{-2+t} - \frac{x}{t+1} = 0$

□ Check to see if t_0 is a regular singular point

  ◦ Define functions

  $\left[P_2(t) = \frac{1}{-2+t}, P_3(t) = -\frac{1}{t+1}\right]$

  ◦ $(t+1) \cdot P_2(t)$ is analytic at $t = -1$

  $\left((t+1) \cdot P_2(t)\right)\Big|_{t=-1} = 0$

  ◦ $(t+1)^2 \cdot P_3(t)$ is analytic at $t = -1$

  $\left((t+1)^2 \cdot P_3(t)\right)\Big|_{t=-1} = 0$

  ◦ $t = -1$is a regular singular point

  Check to see if t_0 is a regular singular point

  $t_0 = -1$

- Multiply by denominators

  $x''(-2+t)\,(t+1) + (t+1)\, x' + (2-t)\, x = 0$

- Change variables using $t = u - 1$ so that the regular singular point is at $u = 0$

  $(u^2 - 3u)\left(\frac{d^2}{du^2} x(u)\right) + u\left(\frac{d}{du} x(u)\right) + (3-u)\, x(u) = 0$

- Assume series solution for $x(u)$

  $x(u) = \sum_{k=0}^{\infty} a_k u^{k+r}$

□ Rewrite ODE with series expansions

  ◦ Convert $u^m \cdot x(u)$ to series expansion for $m = 0..1$

$$u^m \cdot x(u) = \sum_{k=0}^{\infty} a_k u^{k+r+m}$$

- Shift index using $k- >k-m$

  $$u^m \cdot x(u) = \sum_{k=m}^{\infty} a_{k-m} u^{k+r}$$

- Convert $u \cdot \left(\frac{d}{du}x(u)\right)$ to series expansion

  $$u \cdot \left(\frac{d}{du}x(u)\right) = \sum_{k=0}^{\infty} a_k(k+r)\, u^{k+r}$$

- Convert $u^m \cdot \left(\frac{d^2}{du^2}x(u)\right)$ to series expansion for $m = 1..2$

  $$u^m \cdot \left(\frac{d^2}{du^2}x(u)\right) = \sum_{k=0}^{\infty} a_k(k+r)\,(k+r-1)\, u^{k+r-2+m}$$

- Shift index using $k- >k+2-m$

  $$u^m \cdot \left(\frac{d^2}{du^2}x(u)\right) = \sum_{k=-2+m}^{\infty} a_{k+2-m}(k+2-m+r)\,(k+1-m+r)\, u^{k+r}$$

  Rewrite ODE with series expansions

  $$-3a_0r(-1+r)\, u^{-1+r} + (-3a_1(1+r)\, r + a_0(r^2+3))\, u^r + \left(\sum_{k=1}^{\infty} (-3a_{k+1}(k+1+r)\,(k+r) + a_k($$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $$-3r(-1+r) = 0$$

- Values of r that satisfy the indicial equation

  $$r \in \{0, 1\}$$

- Each term must be 0

  $$-3a_1(1+r)\, r + a_0(r^2+3) = 0$$

- Each term in the series must be 0, giving the recursion relation

  $$-3a_{k+1}(k+1+r)\,(k+r) + k^2a_k + 2kra_k + r^2a_k + 3a_k - a_{k-1} = 0$$

- Shift index using $k- >k+1$

  $$-3a_{k+2}(k+2+r)\,(k+1+r) + (k+1)^2\, a_{k+1} + 2(k+1)\, ra_{k+1} + r^2a_{k+1} + 3a_{k+1} - a_k = 0$$

- Recursion relation that defines series solution to ODE

  $$a_{k+2} = \frac{k^2a_{k+1}+2kra_{k+1}+r^2a_{k+1}+2ka_{k+1}+2ra_{k+1}-a_k+4a_{k+1}}{3(k+2+r)(k+1+r)}$$

- Recursion relation for $r = 0$

  $$a_{k+2} = \frac{k^2a_{k+1}+2ka_{k+1}-a_k+4a_{k+1}}{3(k+2)(k+1)}$$

- Solution for $r = 0$

$$\left[x(u) = \sum_{k=0}^{\infty} a_k u^k, a_{k+2} = \frac{k^2 a_{k+1} + 2k a_{k+1} - a_k + 4a_{k+1}}{3(k+2)(k+1)}, 3a_0 = 0\right]$$

- Revert the change of variables $u = t + 1$

$$\left[x = \sum_{k=0}^{\infty} a_k (t+1)^k, a_{k+2} = \frac{k^2 a_{k+1} + 2k a_{k+1} - a_k + 4a_{k+1}}{3(k+2)(k+1)}, 3a_0 = 0\right]$$

- Recursion relation for $r = 1$

$$a_{k+2} = \frac{k^2 a_{k+1} + 4k a_{k+1} - a_k + 7a_{k+1}}{3(k+3)(k+2)}$$

- Solution for $r = 1$

$$\left[x(u) = \sum_{k=0}^{\infty} a_k u^{k+1}, a_{k+2} = \frac{k^2 a_{k+1} + 4k a_{k+1} - a_k + 7a_{k+1}}{3(k+3)(k+2)}, -6a_1 + 4a_0 = 0\right]$$

- Revert the change of variables $u = t + 1$

$$\left[x = \sum_{k=0}^{\infty} a_k (t+1)^{k+1}, a_{k+2} = \frac{k^2 a_{k+1} + 4k a_{k+1} - a_k + 7a_{k+1}}{3(k+3)(k+2)}, -6a_1 + 4a_0 = 0\right]$$

- Combine solutions and rename parameters

$$\left[x = \left(\sum_{k=0}^{\infty} a_k (t+1)^k\right) + \left(\sum_{k=0}^{\infty} b_k (t+1)^{k+1}\right), a_{k+2} = \frac{k^2 a_{k+1} + 2k a_{k+1} - a_k + 4a_{k+1}}{3(k+2)(k+1)}, 3a_0 = 0, b_{k+2} = \frac{k^2 b_{k}}{}\right.$$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
<- No Liouvillian solutions exists
-> Trying a solution in terms of special functions:
   -> Bessel
   -> elliptic
   -> Legendre
   -> Kummer
      -> hyper3: Equivalence to 1F1 under a power @ Moebius
   -> hypergeometric
      -> heuristic approach
      -> hyper3: Equivalence to 2F1, 1F1 or 0F1 under a power @ Moebius
   -> Mathieu
      -> Equivalence to the rational form of Mathieu ODE under a power @ Moebius
trying a solution in terms of MeijerG functions
-> Heun: Equivalence to the GHE or one of its 4 confluent cases under a power @ Moebius
<- Heun successful: received ODE is equivalent to the  HeunC  ODE, case  a <> 0, e <> 0, c =
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 54

```
Order:=6;
dsolve((t^2-t-2)*diff(x(t),t$2)+(t+1)*diff(x(t),t)-(t-2)*x(t)=0,x(t),type='series',t=0);
```

$$x(t) = \left(1+\frac{1}{2}t^2-\frac{1}{12}t^3+\frac{13}{96}t^4-\frac{1}{16}t^5\right)x(0) + \left(t+\frac{1}{4}t^2+\frac{1}{4}t^3-\frac{1}{96}t^4+\frac{31}{480}t^5\right)D(x)(0)+O(t^6)$$

✓ Solution by Mathematica

Time used: 0.001 (sec). Leaf size: 70

```
AsymptoticDSolveValue[(t^2-t-2)*x''[t]+(t+1)*x'[t]-(t-2)*x[t]==0,x[t],{t,0,5}]
```

$$x(t) \to c_1\left(-\frac{t^5}{16}+\frac{13t^4}{96}-\frac{t^3}{12}+\frac{t^2}{2}+1\right)+c_2\left(\frac{31t^5}{480}-\frac{t^4}{96}+\frac{t^3}{4}+\frac{t^2}{4}+t\right)$$

5.6 problem 6

Internal problem ID [5007]
Internal file name [`OUTPUT/4500_Sunday_June_05_2022_02_59_33_PM_46350752/index.tex`]

Book: Fundamentals of Differential Equations. By Nagle, Saff and Snider. 9th edition. Boston. Pearson 2018.
Section: Chapter 8, Series solutions of differential equations. Section 8.3. page 443
Problem number: 6.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second order series method. Ordinary point", "second order series method. Taylor series method"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

$$\boxed{\left(x^{2}-1\right) y''+\left(1-x\right) y'+\left(x^{2}-2x+1\right) y=0}$$

With the expansion point for the power series method at $x=0$.

Solving ode using Taylor series method. This gives review on how the Taylor series method works for solving second order ode.

Let

$$y''=f(x,y,y')$$

Assuming expansion is at $x_0=0$ (we can always shift the actual expansion point to 0 by change of variables) and assuming $f(x,y,y')$ is analytic at x_0 which must be the case for an ordinary point. Let initial conditions be $y(x_0)=y_0$ and $y'(x_0)=y'_0$. Using Taylor series gives

$$\begin{aligned} y(x) &= y(x_0)+(x-x_0)\,y'(x_0)+\frac{(x-x_0)^2}{2}y''(x_0)+\frac{(x-x_0)^3}{3!}y'''(x_0)+\cdots \\ &= y_0+xy'_0+\frac{x^2}{2}f|_{x_0,y_0,y'_0}+\frac{x^3}{3!}f'|_{x_0,y_0,y'_0}+\cdots \\ &= y_0+xy'_0+\sum_{n=0}^{\infty}\frac{x^{n+2}}{(n+2)!}\left.\frac{d^n f}{dx^n}\right|_{x_0,y_0,y'_0} \end{aligned}$$

But

$$\frac{df}{dx} = \frac{\partial f}{\partial x}\frac{dx}{dx} + \frac{\partial f}{\partial y}\frac{dy}{dx} + \frac{\partial f}{\partial y'}\frac{dy'}{dx} \tag{1}$$

$$= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}y' + \frac{\partial f}{\partial y'}y'' \tag{233}$$

$$= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}y' + \frac{\partial f}{\partial y'}f \tag{234}$$

$$\frac{d^2 f}{dx^2} = \frac{d}{dx}\left(\frac{df}{dx}\right)$$

$$= \frac{\partial}{\partial x}\left(\frac{df}{dx}\right) + \frac{\partial}{\partial y}\left(\frac{df}{dx}\right) y' + \frac{\partial}{\partial y'}\left(\frac{df}{dx}\right) f \tag{2}$$

$$\frac{d^3 f}{dx^3} = \frac{d}{dx}\left(\frac{d^2 f}{dx^2}\right)$$

$$= \frac{\partial}{\partial x}\left(\frac{d^2 f}{dx^2}\right) + \left(\frac{\partial}{\partial y}\frac{d^2 f}{dx^2}\right) y' + \frac{\partial}{\partial y'}\left(\frac{d^2 f}{dx^2}\right) f \tag{3}$$

$$\vdots$$

And so on. Hence if we name $F_0 = f(x,y,y')$ then the above can be written as

$$
\begin{aligned}
F_0 &= f(x,y,y') \qquad (4)\\
F_1 &= \frac{df}{dx}\\
&= \frac{dF_0}{dx}\\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}y' + \frac{\partial f}{\partial y'}y''\\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}y' + \frac{\partial f}{\partial y'}f \qquad (5)\\
&= \frac{\partial F_0}{\partial x} + \frac{\partial F_0}{\partial y}y' + \frac{\partial F_0}{\partial y'}F_0\\
F_2 &= \frac{d}{dx}\left(\frac{d}{dx}f\right)\\
&= \frac{d}{dx}(F_1)\\
&= \frac{\partial}{\partial x}F_1 + \left(\frac{\partial F_1}{\partial y}\right)y' + \left(\frac{\partial F_1}{\partial y'}\right)y''\\
&= \frac{\partial}{\partial x}F_1 + \left(\frac{\partial F_1}{\partial y}\right)y' + \left(\frac{\partial F_1}{\partial y'}\right)F_0\\
&\vdots\\
F_n &= \frac{d}{dx}(F_{n-1})\\
&= \frac{\partial}{\partial x}F_{n-1} + \left(\frac{\partial F_{n-1}}{\partial y}\right)y' + \left(\frac{\partial F_{n-1}}{\partial y'}\right)y''\\
&= \frac{\partial}{\partial x}F_{n-1} + \left(\frac{\partial F_{n-1}}{\partial y}\right)y' + \left(\frac{\partial F_{n-1}}{\partial y'}\right)F_0 \qquad (6)
\end{aligned}
$$

Therefore (6) can be used from now on along with

$$
y(x) = y_0 + xy_0' + \sum_{n=0}^{\infty} \frac{x^{n+2}}{(n+2)!} F_n|_{x_0,y_0,y_0'} \tag{7}
$$

To find $y(x)$ series solution around $x = 0$. Hence

$$
\begin{aligned}
F_0 &= -\frac{xy - y - y'}{x+1} \\
F_1 &= \frac{dF_0}{dx} \\
&= \frac{\partial F_0}{\partial x} + \frac{\partial F_0}{\partial y} y' + \frac{\partial F_0}{\partial y'} F_0 \\
&= \frac{-xy' - y + y'}{x+1} \\
F_2 &= \frac{dF_1}{dx} \\
&= \frac{\partial F_1}{\partial x} + \frac{\partial F_1}{\partial y} y' + \frac{\partial F_1}{\partial y'} F_1 \\
&= \frac{(-2-2x)\, y' + y(x^2 - 2x + 2)}{(x+1)^2} \\
F_3 &= \frac{dF_2}{dx} \\
&= \frac{\partial F_2}{\partial x} + \frac{\partial F_2}{\partial y} y' + \frac{\partial F_2}{\partial y'} F_2 \\
&= \frac{(x^3 - x^2 + 2)\, y' + 2y(x^2 + 2x - 4)}{(x+1)^3} \\
F_4 &= \frac{dF_3}{dx} \\
&= \frac{\partial F_3}{\partial x} + \frac{\partial F_3}{\partial y} y' + \frac{\partial F_3}{\partial y'} F_3 \\
&= \frac{3(x^3 + 3x^2 - 2x - 4)\, y' - y(x^4 - 2x^3 + 3x^2 + 6x - 30)}{(x+1)^4}
\end{aligned}
$$

And so on. Evaluating all the above at initial conditions $x = 0$ and $y(0) = y(0)$ and $y'(0) = y'(0)$ gives

$$
\begin{aligned}
F_0 &= y(0) + y'(0) \\
F_1 &= -y(0) + y'(0) \\
F_2 &= 2y(0) - 2y'(0) \\
F_3 &= -8y(0) + 2y'(0) \\
F_4 &= 30y(0) - 12y'(0)
\end{aligned}
$$

Substituting all the above in (7) and simplifying gives the solution as

$$
\begin{aligned}
y = &\left(1+\frac{1}{2}x^{2}-\frac{1}{6}x^{3}+\frac{1}{12}x^{4}-\frac{1}{15}x^{5}+\frac{1}{24}x^{6}\right) y(0) \\
&+\left(x+\frac{1}{2}x^{2}+\frac{1}{6}x^{3}-\frac{1}{12}x^{4}+\frac{1}{60}x^{5}-\frac{1}{60}x^{6}\right) y'(0)+O(x^{6})
\end{aligned}
$$

Since the expansion point $x = 0$ is an ordinary, we can also solve this using standard power series The ode is normalized to be

$$
(x^{2}-1) y''+(1-x) y'+(x^{2}-2x+1) y = 0
$$

Let the solution be represented as power series of the form

$$
y = \sum_{n=0}^{\infty} a_{n} x^{n}
$$

Then

$$
\begin{aligned}
y' &= \sum_{n=1}^{\infty} n a_{n} x^{n-1} \\
y'' &= \sum_{n=2}^{\infty} n(n-1) a_{n} x^{n-2}
\end{aligned}
$$

Substituting the above back into the ode gives

$$
(x^{2}-1)\left(\sum_{n=2}^{\infty} n(n-1) a_{n} x^{n-2}\right)+(1-x)\left(\sum_{n=1}^{\infty} n a_{n} x^{n-1}\right)+(x^{2}-2x+1)\left(\sum_{n=0}^{\infty} a_{n} x^{n}\right) = 0 \tag{1}
$$

Which simplifies to

$$
\begin{aligned}
&\left(\sum_{n=2}^{\infty} x^{n} a_{n} n(n-1)\right)+\sum_{n=2}^{\infty}\left(-n(n-1) a_{n} x^{n-2}\right)+\left(\sum_{n=1}^{\infty} n a_{n} x^{n-1}\right) \\
&+\sum_{n=1}^{\infty}\left(-n a_{n} x^{n}\right)+\left(\sum_{n=0}^{\infty} x^{n+2} a_{n}\right)+\sum_{n=0}^{\infty}\left(-2x^{1+n} a_{n}\right)+\left(\sum_{n=0}^{\infty} a_{n} x^{n}\right) = 0
\end{aligned} \tag{2}
$$

The next step is to make all powers of x be n in each summation term. Going over each summation term above with power of x in it which is not already x^{n} and adjusting the

power and the corresponding index gives

$$\sum_{n=2}^{\infty} \left(-n(n-1)\, a_n x^{n-2}\right) = \sum_{n=0}^{\infty} \left(-(n+2)\, a_{n+2}(1+n)\, x^n\right)$$

$$\sum_{n=1}^{\infty} n a_n x^{n-1} = \sum_{n=0}^{\infty} (1+n)\, a_{1+n} x^n$$

$$\sum_{n=0}^{\infty} x^{n+2} a_n = \sum_{n=2}^{\infty} a_{n-2} x^n$$

$$\sum_{n=0}^{\infty} \left(-2x^{1+n} a_n\right) = \sum_{n=1}^{\infty} \left(-2a_{n-1} x^n\right)$$

Substituting all the above in Eq (2) gives the following equation where now all powers of x are the same and equal to n.

$$\left(\sum_{n=2}^{\infty} x^n a_n n(n-1)\right) + \sum_{n=0}^{\infty} \left(-(n+2)\, a_{n+2}(1+n)\, x^n\right) + \left(\sum_{n=0}^{\infty} (1+n)\, a_{1+n} x^n\right) + \sum_{n=1}^{\infty} (-n a_n x^n) + \left(\sum_{n=2}^{\infty} a_{n-2} x^n\right) + \sum_{n=1}^{\infty} (-2a_{n-1} x^n) + \left(\sum_{n=0}^{\infty} a_n x^n\right) = 0 \tag{3}$$

$n = 0$ gives

$$-2a_2 + a_1 + a_0 = 0$$

$$a_2 = \frac{a_0}{2} + \frac{a_1}{2}$$

$n = 1$ gives

$$-6a_3 + 2a_2 - 2a_0 = 0$$

Which after substituting earlier equations, simplifies to

$$a_3 = -\frac{a_0}{6} + \frac{a_1}{6}$$

For $2 \leq n$, the recurrence equation is

$$n a_n(n-1) - (n+2)\, a_{n+2}(1+n) + (1+n)\, a_{1+n} - n a_n + a_{n-2} - 2a_{n-1} + a_n = 0 \tag{4}$$

Solving for a_{n+2}, gives

$$
\begin{aligned}
a_{n+2} &= \frac{n^2 a_n - 2n a_n + n a_{1+n} + a_n + a_{1+n} + a_{n-2} - 2a_{n-1}}{(n+2)(1+n)} \\
&= \frac{(n^2-2n+1) a_n}{(n+2)(1+n)} + \frac{a_{1+n}}{n+2} + \frac{a_{n-2}}{(n+2)(1+n)} - \frac{2a_{n-1}}{(n+2)(1+n)}
\end{aligned} \tag{5}
$$

For $n = 2$ the recurrence equation gives

$$a_2 - 12a_4 + 3a_3 + a_0 - 2a_1 = 0$$

Which after substituting the earlier terms found becomes

$$a_4 = \frac{a_0}{12} - \frac{a_1}{12}$$

For $n = 3$ the recurrence equation gives

$$4a_3 - 20a_5 + 4a_4 + a_1 - 2a_2 = 0$$

Which after substituting the earlier terms found becomes

$$a_5 = -\frac{a_0}{15} + \frac{a_1}{60}$$

For $n = 4$ the recurrence equation gives

$$9a_4 - 30a_6 + 5a_5 + a_2 - 2a_3 = 0$$

Which after substituting the earlier terms found becomes

$$a_6 = \frac{a_0}{24} - \frac{a_1}{60}$$

For $n = 5$ the recurrence equation gives

$$16a_5 - 42a_7 + 6a_6 + a_3 - 2a_4 = 0$$

Which after substituting the earlier terms found becomes

$$a_7 = -\frac{23a_0}{840} + \frac{a_1}{84}$$

And so on. Therefore the solution is

$$\begin{aligned} y &= \sum_{n=0}^{\infty} a_n x^n \\ &= a_3 x^3 + a_2 x^2 + a_1 x + a_0 + \dots \end{aligned}$$

Substituting the values for a_n found above, the solution becomes

$$y = a_0 + a_1 x + \left(\frac{a_0}{2} + \frac{a_1}{2}\right) x^2 + \left(-\frac{a_0}{6} + \frac{a_1}{6}\right) x^3 + \left(\frac{a_0}{12} - \frac{a_1}{12}\right) x^4 + \left(-\frac{a_0}{15} + \frac{a_1}{60}\right) x^5 + \dots$$

Collecting terms, the solution becomes

$$y = \left(1 + \frac{1}{2}x^2 - \frac{1}{6}x^3 + \frac{1}{12}x^4 - \frac{1}{15}x^5\right) a_0 + \left(x + \frac{1}{2}x^2 + \frac{1}{6}x^3 - \frac{1}{12}x^4 + \frac{1}{60}x^5\right) a_1 + O(x^6) \tag{3}$$

At $x = 0$ the solution above becomes

$$y = \left(1 + \frac{1}{2}x^2 - \frac{1}{6}x^3 + \frac{1}{12}x^4 - \frac{1}{15}x^5\right) c_1 + \left(x + \frac{1}{2}x^2 + \frac{1}{6}x^3 - \frac{1}{12}x^4 + \frac{1}{60}x^5\right) c_2 + O(x^6)$$

Summary

The solution(s) found are the following

$$\begin{aligned} y = &\left(1 + \frac{1}{2}x^2 - \frac{1}{6}x^3 + \frac{1}{12}x^4 - \frac{1}{15}x^5 + \frac{1}{24}x^6\right) y(0) \\ &+ \left(x + \frac{1}{2}x^2 + \frac{1}{6}x^3 - \frac{1}{12}x^4 + \frac{1}{60}x^5 - \frac{1}{60}x^6\right) y'(0) + O(x^6) \end{aligned} \tag{1}$$

$$y = \left(1 + \frac{1}{2}x^2 - \frac{1}{6}x^3 + \frac{1}{12}x^4 - \frac{1}{15}x^5\right) c_1 + \left(x + \frac{1}{2}x^2 + \frac{1}{6}x^3 - \frac{1}{12}x^4 + \frac{1}{60}x^5\right) c_2 + O(x^6) \tag{2}$$

Verification of solutions

$$\begin{aligned} y = &\left(1 + \frac{1}{2}x^2 - \frac{1}{6}x^3 + \frac{1}{12}x^4 - \frac{1}{15}x^5 + \frac{1}{24}x^6\right) y(0) \\ &+ \left(x + \frac{1}{2}x^2 + \frac{1}{6}x^3 - \frac{1}{12}x^4 + \frac{1}{60}x^5 - \frac{1}{60}x^6\right) y'(0) + O(x^6) \end{aligned}$$

Verified OK.

$$y = \left(1 + \frac{1}{2}x^2 - \frac{1}{6}x^3 + \frac{1}{12}x^4 - \frac{1}{15}x^5\right) c_1 + \left(x + \frac{1}{2}x^2 + \frac{1}{6}x^3 - \frac{1}{12}x^4 + \frac{1}{60}x^5\right) c_2 + O(x^6)$$

Verified OK.

5.6.1 Maple step by step solution

Let's solve

$(x^2-1)\,y''+(1-x)\,y'+(x^2-2x+1)\,y=0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y''=-\frac{(x-1)y}{x+1}+\frac{y'}{x+1}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y''-\frac{y'}{x+1}+\frac{(x-1)y}{x+1}=0$

□ Check to see if $x_0=-1$ is a regular singular point

- ◦ Define functions

  $\left[P_2(x)=-\frac{1}{x+1}, P_3(x)=\frac{x-1}{x+1}\right]$

- ◦ $(x+1)\cdot P_2(x)$ is analytic at $x=-1$

  $\left((x+1)\cdot P_2(x)\right)\Big|_{x=-1}=-1$

- ◦ $(x+1)^2\cdot P_3(x)$ is analytic at $x=-1$

  $\left((x+1)^2\cdot P_3(x)\right)\Big|_{x=-1}=0$

- ◦ $x=-1$is a regular singular point

  Check to see if $x_0=-1$ is a regular singular point

  $x_0=-1$

- Multiply by denominators

  $(x+1)\,y''-y'+(x-1)\,y=0$

- Change variables using $x=u-1$ so that the regular singular point is at $u=0$

  $u\left(\frac{d^2}{du^2}y(u)\right)-\frac{d}{du}y(u)+(u-2)\,y(u)=0$

- Assume series solution for $y(u)$

  $y(u)=\sum_{k=0}^{\infty}a_k u^{k+r}$

□ Rewrite ODE with series expansions

- ◦ Convert $u^m\cdot y(u)$ to series expansion for $m=0..1$

$u^m \cdot y(u) = \sum_{k=0}^{\infty} a_k u^{k+r+m}$

- Shift index using $k->k-m$

  $u^m \cdot y(u) = \sum_{k=m}^{\infty} a_{k-m} u^{k+r}$

- Convert $\frac{d}{du}y(u)$ to series expansion

  $\frac{d}{du}y(u) = \sum_{k=0}^{\infty} a_k (k+r)\, u^{k+r-1}$

- Shift index using $k->k+1$

  $\frac{d}{du}y(u) = \sum_{k=-1}^{\infty} a_{k+1}(k+1+r)\, u^{k+r}$

- Convert $u \cdot \left(\frac{d^2}{du^2}y(u)\right)$ to series expansion

  $u \cdot \left(\frac{d^2}{du^2}y(u)\right) = \sum_{k=0}^{\infty} a_k (k+r)(k+r-1)\, u^{k+r-1}$

- Shift index using $k->k+1$

  $u \cdot \left(\frac{d^2}{du^2}y(u)\right) = \sum_{k=-1}^{\infty} a_{k+1}(k+1+r)(k+r)\, u^{k+r}$

  Rewrite ODE with series expansions

  $a_0 r(-2+r)\, u^{-1+r} + (a_1(1+r)(-1+r) - 2a_0)\, u^r + \left(\sum_{k=1}^{\infty} (a_{k+1}(k+1+r)(k+r-1) - 2a_k + a$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $r(-2+r) = 0$

- Values of r that satisfy the indicial equation

  $r \in \{0, 2\}$

- Each term must be 0

  $a_1(1+r)(-1+r) - 2a_0 = 0$

- Each term in the series must be 0, giving the recursion relation

  $a_{k+1}(k+1+r)(k+r-1) - 2a_k + a_{k-1} = 0$

- Shift index using $k->k+1$

  $a_{k+2}(k+2+r)(k+r) - 2a_{k+1} + a_k = 0$

- Recursion relation that defines series solution to ODE

  $a_{k+2} = -\frac{-2a_{k+1}+a_k}{(k+2+r)(k+r)}$

- Recursion relation for $r = 0$

  $a_{k+2} = -\frac{-2a_{k+1}+a_k}{(k+2)k}$

- Series not valid for $r = 0$, division by 0 in the recursion relation at $k = 0$

  $a_{k+2} = -\frac{-2a_{k+1}+a_k}{(k+2)k}$

- Recursion relation for $r = 2$

  $a_{k+2} = -\frac{-2a_{k+1}+a_k}{(k+4)(k+2)}$

- Solution for $r = 2$

  $\left[y(u) = \sum_{k=0}^{\infty} a_k u^{k+2}, a_{k+2} = -\frac{-2a_{k+1}+a_k}{(k+4)(k+2)}, 3a_1 - 2a_0 = 0\right]$

- Revert the change of variables $u = x + 1$

  $\left[y = \sum_{k=0}^{\infty} a_k (x+1)^{k+2}, a_{k+2} = -\frac{-2a_{k+1}+a_k}{(k+4)(k+2)}, 3a_1 - 2a_0 = 0\right]$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
<- No Liouvillian solutions exists
-> Trying a solution in terms of special functions:
   -> Bessel
   -> elliptic
   -> Legendre
   -> Kummer
      -> hyper3: Equivalence to 1F1 under a power @ Moebius
      <- hyper3 successful: received ODE is equivalent to the 1F1 ODE
   <- Kummer successful
<- special function solution successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 54

```
Order:=6;
dsolve((x^2-1)*diff(y(x),x$2)+(1-x)*diff(y(x),x)+(x^2-2*x+1)*y(x)=0,y(x),type='series',x=0);
```

$$y(x) = \left(1+\frac{1}{2}x^{2}-\frac{1}{6}x^{3}+\frac{1}{12}x^{4}-\frac{1}{15}x^{5}\right)y(0) + \left(x+\frac{1}{2}x^{2}+\frac{1}{6}x^{3}-\frac{1}{12}x^{4}+\frac{1}{60}x^{5}\right)D(y)(0)+O\left(x^{6}\right)$$

✓ Solution by Mathematica

Time used: 0.001 (sec). Leaf size: 70

```
AsymptoticDSolveValue[(x^2-1)*y''[x]+(1-x)*y'[x]+(x^2-2*x+1)*y[x]==0,y[x],{x,0,5}]
```

$$y(x) \to c_1\left(-\frac{x^5}{15}+\frac{x^4}{12}-\frac{x^3}{6}+\frac{x^2}{2}+1\right)+c_2\left(\frac{x^5}{60}-\frac{x^4}{12}+\frac{x^3}{6}+\frac{x^2}{2}+x\right)$$

5.7 problem 7

Internal problem ID [5008]
Internal file name [`OUTPUT/4501_Sunday_June_05_2022_02_59_34_PM_7114731/index.tex`]

Book: Fundamentals of Differential Equations. By Nagle, Saff and Snider. 9th edition. Boston. Pearson 2018.
Section: Chapter 8, Series solutions of differential equations. Section 8.3. page 443
Problem number: 7.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second order series method. Regular singular point. Difference is integer"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

$$\boxed{\sin(x) y'' + y\cos(x) = 0}$$

With the expansion point for the power series method at $x = 0$.

The type of the expansion point is first determined. This is done on the homogeneous part of the ODE.

$$\sin(x) y'' + y\cos(x) = 0$$

The following is summary of singularities for the above ode. Writing the ode as

$$y'' + p(x)y' + q(x)y = 0$$

Where

$$p(x) = 0$$
$$q(x) = \frac{\cos(x)}{\sin(x)}$$

Table 205: Table $p(x), q(x)$ singularites.

$p(x) = 0$	
singularity	type

$q(x) = \frac{\cos(x)}{\sin(x)}$	
singularity	type
$x = \pi Z$	"regular"

Combining everything together gives the following summary of singularities for the ode as

Regular singular points : $[\pi Z]$

Irregular singular points : $[\infty]$

Since $x = 0$ is regular singular point, then Frobenius power series is used. Let the solution be represented as Frobenius power series of the form

$$y = \sum_{n=0}^{\infty} a_n x^{n+r}$$

Then

$$y' = \sum_{n=0}^{\infty} (n+r) a_n x^{n+r-1}$$

$$y'' = \sum_{n=0}^{\infty} (n+r)(n+r-1) a_n x^{n+r-2}$$

Substituting the above back into the ode gives

$$\sin(x) \left(\sum_{n=0}^{\infty} (n+r)(n+r-1) a_n x^{n+r-2}\right) + \left(\sum_{n=0}^{\infty} a_n x^{n+r}\right) \cos(x) = 0 \tag{1}$$

Expanding $\sin(x)$ as Taylor series around $x = 0$ and keeping only the first 6 terms gives

$$\begin{aligned} \sin(x) &= x - \frac{1}{6}x^3 + \frac{1}{120}x^5 - \frac{1}{5040}x^7 + \dots \\ &= x - \frac{1}{6}x^3 + \frac{1}{120}x^5 - \frac{1}{5040}x^7 \end{aligned}$$

Expanding $\cos(x)$ as Taylor series around $x = 0$ and keeping only the first 6 terms gives

$$\begin{aligned}\cos(x) &= -\frac{1}{720}x^6 + \frac{1}{24}x^4 - \frac{1}{2}x^2 + 1 + \dots \\ &= -\frac{1}{720}x^6 + \frac{1}{24}x^4 - \frac{1}{2}x^2 + 1\end{aligned}$$

Which simplifies to

$$\begin{aligned}&\sum_{n=0}^{\infty}\left(-\frac{x^{n+r+5}a_n(n+r)(n+r-1)}{5040}\right) \\ &+\left(\sum_{n=0}^{\infty}\frac{x^{n+r+3}a_n(n+r)(n+r-1)}{120}\right) \\ &+\sum_{n=0}^{\infty}\left(-\frac{x^{1+n+r}a_n(n+r)(n+r-1)}{6}\right) \\ &+\left(\sum_{n=0}^{\infty}x^{n+r-1}a_n(n+r)(n+r-1)\right)+\sum_{n=0}^{\infty}\left(-\frac{x^{n+r+6}a_n}{720}\right) \\ &+\left(\sum_{n=0}^{\infty}\frac{x^{n+r+4}a_n}{24}\right)+\sum_{n=0}^{\infty}\left(-\frac{x^{n+r+2}a_n}{2}\right)+\left(\sum_{n=0}^{\infty}a_n x^{n+r}\right)=0\end{aligned} \tag{2A}$$

The next step is to make all powers of x be $n+r-1$ in each summation term. Going over each summation term above with power of x in it which is not already x^{n+r-1} and adjusting the power and the corresponding index gives

$$\sum_{n=0}^{\infty}\left(-\frac{x^{n+r+5}a_n(n+r)(n+r-1)}{5040}\right)=\sum_{n=6}^{\infty}\left(-\frac{a_{n-6}(n+r-6)(n+r-7)x^{n+r-1}}{5040}\right)$$

$$\sum_{n=0}^{\infty}\frac{x^{n+r+3}a_n(n+r)(n+r-1)}{120}=\sum_{n=4}^{\infty}\frac{a_{n-4}(-4+n+r)(n+r-5)x^{n+r-1}}{120}$$

$$\sum_{n=0}^{\infty}\left(-\frac{x^{1+n+r}a_n(n+r)(n+r-1)}{6}\right)=\sum_{n=2}^{\infty}\left(-\frac{a_{n-2}(n+r-2)(n+r-3)x^{n+r-1}}{6}\right)$$

$$\sum_{n=0}^{\infty}\left(-\frac{x^{n+r+6}a_n}{720}\right)=\sum_{n=7}^{\infty}\left(-\frac{a_{n-7}x^{n+r-1}}{720}\right)$$

$$\sum_{n=0}^{\infty}\frac{x^{n+r+4}a_n}{24}=\sum_{n=5}^{\infty}\frac{a_{n-5}x^{n+r-1}}{24}$$

$$\sum_{n=0}^{\infty}\left(-\frac{x^{n+r+2}a_n}{2}\right)=\sum_{n=3}^{\infty}\left(-\frac{a_{n-3}x^{n+r-1}}{2}\right)$$

$$\sum_{n=0}^{\infty}a_n x^{n+r}=\sum_{n=1}^{\infty}a_{n-1}x^{n+r-1}$$

Substituting all the above in Eq (2A) gives the following equation where now all powers of x are the same and equal to $n+r-1$.

$$\begin{aligned}
&\sum_{n=6}^{\infty}\left(-\frac{a_{n-6}(n+r-6)(n+r-7)x^{n+r-1}}{5040}\right)\\
&+\left(\sum_{n=4}^{\infty}\frac{a_{n-4}(-4+n+r)(n+r-5)x^{n+r-1}}{120}\right)\\
&+\sum_{n=2}^{\infty}\left(-\frac{a_{n-2}(n+r-2)(n+r-3)x^{n+r-1}}{6}\right)\\
&+\left(\sum_{n=0}^{\infty}x^{n+r-1}a_n(n+r)(n+r-1)\right)+\sum_{n=7}^{\infty}\left(-\frac{a_{n-7}x^{n+r-1}}{720}\right)\\
&+\left(\sum_{n=5}^{\infty}\frac{a_{n-5}x^{n+r-1}}{24}\right)+\sum_{n=3}^{\infty}\left(-\frac{a_{n-3}x^{n+r-1}}{2}\right)+\left(\sum_{n=1}^{\infty}a_{n-1}x^{n+r-1}\right)=0
\end{aligned} \tag{2B}$$

The indicial equation is obtained from $n=0$. From Eq (2B) this gives

$$x^{n+r-1}a_n(n+r)(n+r-1)=0$$

When $n=0$ the above becomes

$$x^{-1+r}a_0 r(-1+r)=0$$

Or

$$x^{-1+r}a_0 r(-1+r)=0$$

Since $a_0\neq 0$ then the above simplifies to

$$x^{-1+r}r(-1+r)=0$$

Since the above is true for all x then the indicial equation becomes

$$r(-1+r)=0$$

Solving for r gives the roots of the indicial equation as

$$\begin{aligned}r_1 &= 1\\ r_2 &= 0\end{aligned}$$

Since $a_0 \neq 0$ then the indicial equation becomes

$$x^{-1+r}r(-1+r) = 0$$

Solving for r gives the roots of the indicial equation as Since $r_1 - r_2 = 1$ is an integer, then we can construct two linearly independent solutions

$$\begin{aligned}y_1(x) &= x^{r_1}\left(\sum_{n=0}^{\infty} a_n x^n\right)\\ y_2(x) &= Cy_1(x)\ln(x) + x^{r_2}\left(\sum_{n=0}^{\infty} b_n x^n\right)\end{aligned}$$

Or

$$\begin{aligned}y_1(x) &= x\left(\sum_{n=0}^{\infty} a_n x^n\right)\\ y_2(x) &= Cy_1(x)\ln(x) + \left(\sum_{n=0}^{\infty} b_n x^n\right)\end{aligned}$$

Or

$$\begin{aligned}y_1(x) &= \sum_{n=0}^{\infty} a_n x^{1+n}\\ y_2(x) &= Cy_1(x)\ln(x) + \left(\sum_{n=0}^{\infty} b_n x^n\right)\end{aligned}$$

Where C above can be zero. We start by finding y_1. Eq (2B) derived above is now used to find all a_n coefficients. The case $n = 0$ is skipped since it was used to find the roots of the indicial equation. a_0 is arbitrary and taken as $a_0 = 1$. Substituting $n = 1$ in Eq. (2B) gives

$$a_1 = -\frac{1}{r(1+r)}$$

Substituting $n = 2$ in Eq. (2B) gives

$$a_2 = \frac{r^4 - r^2 + 6}{6r(1+r)^2(2+r)}$$

Substituting $n = 3$ in Eq. (2B) gives

$$a_3 = \frac{r^4 + 8r^3 + 11r^2 + 4r - 6}{6r(1+r)^2(2+r)^2(3+r)}$$

Substituting $n = 4$ in Eq. (2B) gives

$$a_4 = \frac{7r^8 + 56r^7 + 154r^6 + 140r^5 - 257r^4 - 1636r^3 - 3504r^2 - 3600r - 1080}{360r(1+r)^2(2+r)^2(3+r)^2(4+r)}$$

Substituting $n = 5$ in Eq. (2B) gives

$$a_5 = \frac{7r^8 + 104r^7 + 578r^6 + 1556r^5 + 2303r^4 + 2756r^3 + 4312r^2 + 5664r + 3240}{120r(1+r)^2(2+r)^2(3+r)^2(4+r)^2(5+r)}$$

Substituting $n = 6$ in Eq. (2B) gives

$$a_6 = \frac{31r^{12} + 744r^{11} + 7595r^{10} + 42780r^9 + 142935r^8 + 280224r^7 + 311861r^6 + 305940r^5 + 536074r^4 - 2}{15120r(1+r)^2(2+r)^2(3+r)^2(4+r)^2(5+r)^2(6+r)}$$

For $7 \leq n$ the recursive equation is

$$\begin{aligned}
&-\frac{a_{n-6}(n+r-6)(n+r-7)}{5040} + \frac{a_{n-4}(-4+n+r)(n+r-5)}{120} \\
&-\frac{a_{n-2}(n+r-2)(n+r-3)}{6} + a_n(n+r)(n+r-1) \\
&-\frac{a_{n-7}}{720} + \frac{a_{n-5}}{24} - \frac{a_{n-3}}{2} + a_{n-1} = 0
\end{aligned} \tag{3}$$

Solving for a_n from recursive equation (4) gives

$$a_n = \frac{n^2a_{n-6} - 42n^2a_{n-4} + 840n^2a_{n-2} + 2nra_{n-6} - 84nra_{n-4} + 1680nra_{n-2} + r^2a_{n-6} - 42r^2a_{n-4} + 840r^2}{} \tag{4}$$

Which for the root $r = 1$ becomes

$$a_n = \frac{(a_{n-6} - 42a_{n-4} + 840a_{n-2})n^2 + (-11a_{n-6} + 294a_{n-4} - 2520a_{n-2})n + 7a_{n-7} + 30a_{n-6} - 210a_{n-5} -}{5040n(1+n)} \tag{5}$$

At this point, it is a good idea to keep track of a_n in a table both before substituting $r = 1$ and after as more terms are found using the above recursive equation.

n	$a_{n,r}$
a_0	1
a_1	$-\frac{1}{r(1+r)}$
a_2	$\frac{r^4-r^2+6}{6r(1+r)^2(2+r)}$
a_3	$\frac{r^4+8r^3+11r^2+4r-6}{6r(1+r)^2(2+r)^2(3+r)}$
a_4	$\frac{7r^8+56r^7+154r^6+140r^5-257r^4-1636r^3-3504r^2-3600r-1080}{360r(1+r)^2(2+r)^2(3+r)^2(4+r)}$
a_5	$\frac{7r^8+104r^7+578r^6+1556r^5+2303r^4+2756r^3+4312r^2+5664r+3240}{120r(1+r)^2(2+r)^2(3+r)^2(4+r)^2(5+r)}$
a_6	$\frac{31r^{12}+744r^{11}+7595r^{10}+42780r^9+142935r^8+280224r^7+311861r^6+305940r^5+536074r^4-24888r^3-2631456r^2-4415040r-2585520}{15120r(1+r)^2(2+r)^2(3+r)^2(4+r)^2(5+r)^2(6+r)}$

Using the above table, then the solution $y_1(x)$ is

$$\begin{aligned} y_1(x) &= x\left(a_0+a_1x+a_2x^2+a_3x^3+a_4x^4+a_5x^5+a_6x^6\dots\right) \\ &= x\left(1-\frac{x}{2}+\frac{x^2}{12}+\frac{x^3}{48}-\frac{3x^4}{320}+\frac{19x^5}{9600}-\frac{59x^6}{403200}+O(x^6)\right) \end{aligned}$$

Now the second solution $y_2(x)$ is found. Let

$$r_1 - r_2 = N$$

Where N is positive integer which is the difference between the two roots. r_1 is taken as the larger root. Hence for this problem we have $N=1$. Now we need to determine if C is zero or not. This is done by finding $\lim_{r\to r_2} a_1(r)$. If this limit exists, then $C=0$, else we need to keep the log term and $C\neq 0$. The above table shows that

$$\begin{aligned} a_N &= a_1 \\ &= -\frac{1}{r(1+r)} \end{aligned}$$

Therefore

$$\begin{aligned} \lim_{r\to r_2} -\frac{1}{r(1+r)} &= \lim_{r\to 0} -\frac{1}{r(1+r)} \\ &= \textit{undefined} \end{aligned}$$

Since the limit does not exist then the log term is needed. Therefore the second solution has the form

$$y_2(x) = Cy_1(x)\ln(x) + \left(\sum_{n=0}^{\infty} b_n x^{n+r_2}\right)$$

Therefore

$$
\begin{aligned}
\frac{d}{dx}y_2(x) &= Cy_1'(x)\ln(x) + \frac{Cy_1(x)}{x} + \left(\sum_{n=0}^{\infty} \frac{b_n x^{n+r_2}(n+r_2)}{x}\right) \\
&= Cy_1'(x)\ln(x) + \frac{Cy_1(x)}{x} + \left(\sum_{n=0}^{\infty} x^{-1+n+r_2} b_n(n+r_2)\right) \\
\frac{d^2}{dx^2}y_2(x) &= Cy_1''(x)\ln(x) + \frac{2Cy_1'(x)}{x} - \frac{Cy_1(x)}{x^2} \\
&\quad + \sum_{n=0}^{\infty}\left(\frac{b_n x^{n+r_2}(n+r_2)^2}{x^2} - \frac{b_n x^{n+r_2}(n+r_2)}{x^2}\right) \\
&= Cy_1''(x)\ln(x) + \frac{2Cy_1'(x)}{x} - \frac{Cy_1(x)}{x^2} + \left(\sum_{n=0}^{\infty} x^{-2+n+r_2} b_n(n+r_2)(-1+n+r_2)\right)
\end{aligned}
$$

Substituting these back into the given ode $\sin(x)\,y'' + y\cos(x) = 0$ gives

$$
\begin{aligned}
&\sin(x)\left(Cy_1''(x)\ln(x) + \frac{2Cy_1'(x)}{x} - \frac{Cy_1(x)}{x^2}\right. \\
&\quad \left. + \sum_{n=0}^{\infty}\left(\frac{b_n x^{n+r_2}(n+r_2)^2}{x^2} - \frac{b_n x^{n+r_2}(n+r_2)}{x^2}\right)\right) \\
&\quad + \left(Cy_1(x)\ln(x) + \left(\sum_{n=0}^{\infty} b_n x^{n+r_2}\right)\right)\cos(x) = 0
\end{aligned}
$$

Which can be written as

$$
\begin{aligned}
&\left((\cos(x)\,y_1(x) + \sin(x)\,y_1''(x))\ln(x) + \sin(x)\left(\frac{2y_1'(x)}{x} - \frac{y_1(x)}{x^2}\right)\right)C \\
&\quad + \left(\sum_{n=0}^{\infty} b_n x^{n+r_2}\right)\cos(x) \\
&\quad + \sin(x)\left(\sum_{n=0}^{\infty}\left(\frac{b_n x^{n+r_2}(n+r_2)^2}{x^2} - \frac{b_n x^{n+r_2}(n+r_2)}{x^2}\right)\right) = 0
\end{aligned}
\tag{7}
$$

But since $y_1(x)$ is a solution to the ode, then

$$
\cos(x)\,y_1(x) + \sin(x)\,y_1''(x) = 0
$$

Eq (7) simplifes to

$$\sin(x)\left(\frac{2y_1'(x)}{x}-\frac{y_1(x)}{x^2}\right)C+\left(\sum_{n=0}^{\infty} b_n x^{n+r_2}\right)\cos(x) \\ +\sin(x)\left(\sum_{n=0}^{\infty}\left(\frac{b_n x^{n+r_2}(n+r_2)^2}{x^2}-\frac{b_n x^{n+r_2}(n+r_2)}{x^2}\right)\right)=0 \tag{8}$$

Substituting $y_1=\sum_{n=0}^{\infty} a_n x^{n+r_1}$ into the above gives

$$\frac{\sin(x)\left(2\left(\sum_{n=0}^{\infty} x^{-1+n+r_1}a_n(n+r_1)\right)x-\left(\sum_{n=0}^{\infty} a_n x^{n+r_1}\right)\right)C}{x^2} \\ +\left(\sum_{n=0}^{\infty} b_n x^{n+r_2}\right)\cos(x)+\sin(x)\left(\sum_{n=0}^{\infty} x^{-2+n+r_2}b_n(n+r_2)(-1+n+r_2)\right)=0 \tag{9}$$

Since $r_1=1$ and $r_2=0$ then the above becomes

$$\frac{\sin(x)\left(2\left(\sum_{n=0}^{\infty} x^{n}a_n(1+n)\right)x-\left(\sum_{n=0}^{\infty} a_n x^{1+n}\right)\right)C}{x^2} \\ +\left(\sum_{n=0}^{\infty} b_n x^{n}\right)\cos(x)+\sin(x)\left(\sum_{n=0}^{\infty} x^{n-2}b_n n(n-1)\right)=0 \tag{10}$$

Expanding $\frac{2\sin(x)C}{x}$ as Taylor series around $x=0$ and keeping only the first 6 terms gives

$$\begin{aligned}\frac{2\sin(x)C}{x}&=2C-\frac{1}{3}C\,x^2+\frac{1}{60}C\,x^4-\frac{1}{2520}C\,x^6+\dots\\&=2C-\frac{1}{3}C\,x^2+\frac{1}{60}C\,x^4-\frac{1}{2520}C\,x^6\end{aligned}$$

Expanding $-\frac{\sin(x)C}{x}$ as Taylor series around $x=0$ and keeping only the first 6 terms gives

$$\begin{aligned}-\frac{\sin(x)C}{x}&=-C+\frac{1}{6}C\,x^2-\frac{1}{120}C\,x^4+\frac{1}{5040}C\,x^6+\dots\\&=-C+\frac{1}{6}C\,x^2-\frac{1}{120}C\,x^4+\frac{1}{5040}C\,x^6\end{aligned}$$

Expanding $\cos(x)$ as Taylor series around $x = 0$ and keeping only the first 6 terms gives

$$\begin{aligned}\cos(x) &= -\frac{1}{720}x^6 + \frac{1}{24}x^4 - \frac{1}{2}x^2 + 1 + \dots \\ &= -\frac{1}{720}x^6 + \frac{1}{24}x^4 - \frac{1}{2}x^2 + 1\end{aligned}$$

Expanding $\sin(x)$ as Taylor series around $x = 0$ and keeping only the first 6 terms gives

$$\begin{aligned}\sin(x) &= x - \frac{1}{6}x^3 + \frac{1}{120}x^5 - \frac{1}{5040}x^7 + \dots \\ &= x - \frac{1}{6}x^3 + \frac{1}{120}x^5 - \frac{1}{5040}x^7\end{aligned}$$

Which simplifies to

$$\begin{aligned}&\sum_{n=0}^{\infty}\left(-\frac{C\,x^{n+6}a_n(1+n)}{2520}\right) + \left(\sum_{n=0}^{\infty}\frac{C\,x^{n+4}a_n(1+n)}{60}\right) \\ &+\sum_{n=0}^{\infty}\left(-\frac{C\,x^{n+2}a_n(1+n)}{3}\right) + \left(\sum_{n=0}^{\infty}2a_n x^n C(1+n)\right) + \sum_{n=0}^{\infty}(-a_n x^n C) \\ &+\left(\sum_{n=0}^{\infty}\frac{C\,x^{n+2}a_n}{6}\right) + \sum_{n=0}^{\infty}\left(-\frac{C\,x^{n+4}a_n}{120}\right) + \left(\sum_{n=0}^{\infty}\frac{C\,x^{n+6}a_n}{5040}\right) \\ &+\sum_{n=0}^{\infty}\left(-\frac{x^{n+6}b_n}{720}\right) + \left(\sum_{n=0}^{\infty}\frac{x^{n+4}b_n}{24}\right) + \sum_{n=0}^{\infty}\left(-\frac{x^{n+2}b_n}{2}\right) \\ &+\left(\sum_{n=0}^{\infty}b_n x^n\right) + \sum_{n=0}^{\infty}\left(-\frac{n\,x^{n+5}b_n(n-1)}{5040}\right) + \left(\sum_{n=0}^{\infty}\frac{n\,x^{n+3}b_n(n-1)}{120}\right) \\ &+\sum_{n=0}^{\infty}\left(-\frac{n\,x^{1+n}b_n(n-1)}{6}\right) + \left(\sum_{n=0}^{\infty}n\,x^{n-1}b_n(n-1)\right) = 0\end{aligned} \tag{2A}$$

The next step is to make all powers of x be $n-1$ in each summation term. Going over each summation term above with power of x in it which is not already x^{n-1} and adjusting the power and the corresponding index gives

$$\sum_{n=0}^{\infty}\left(-\frac{C\,x^{n+6}a_n(1+n)}{2520}\right) = \sum_{n=7}^{\infty}\left(-\frac{Ca_{n-7}(n-6)\,x^{n-1}}{2520}\right)$$

$$\sum_{n=0}^{\infty}\frac{C\,x^{n+4}a_n(1+n)}{60} = \sum_{n=5}^{\infty}\frac{Ca_{n-5}(n-4)\,x^{n-1}}{60}$$

$$\sum_{n=0}^{\infty}\left(-\frac{C\,x^{n+2}a_n(1+n)}{3}\right)=\sum_{n=3}^{\infty}\left(-\frac{Ca_{n-3}(n-2)\,x^{n-1}}{3}\right)$$

$$\sum_{n=0}^{\infty}2a_nx^nC(1+n)=\sum_{n=1}^{\infty}2Ca_{n-1}n\,x^{n-1}$$

$$\sum_{n=0}^{\infty}(-a_nx^nC)=\sum_{n=1}^{\infty}(-Ca_{n-1}x^{n-1})$$

$$\sum_{n=0}^{\infty}\frac{C\,x^{n+2}a_n}{6}=\sum_{n=3}^{\infty}\frac{Ca_{n-3}x^{n-1}}{6}$$

$$\sum_{n=0}^{\infty}\left(-\frac{C\,x^{n+4}a_n}{120}\right)=\sum_{n=5}^{\infty}\left(-\frac{Ca_{n-5}x^{n-1}}{120}\right)$$

$$\sum_{n=0}^{\infty}\frac{C\,x^{n+6}a_n}{5040}=\sum_{n=7}^{\infty}\frac{Ca_{n-7}x^{n-1}}{5040}$$

$$\sum_{n=0}^{\infty}\left(-\frac{x^{n+6}b_n}{720}\right)=\sum_{n=7}^{\infty}\left(-\frac{b_{n-7}x^{n-1}}{720}\right)$$

$$\sum_{n=0}^{\infty}\frac{x^{n+4}b_n}{24}=\sum_{n=5}^{\infty}\frac{b_{n-5}x^{n-1}}{24}$$

$$\sum_{n=0}^{\infty}\left(-\frac{x^{n+2}b_n}{2}\right)=\sum_{n=3}^{\infty}\left(-\frac{b_{n-3}x^{n-1}}{2}\right)$$

$$\sum_{n=0}^{\infty}b_nx^n=\sum_{n=1}^{\infty}b_{n-1}x^{n-1}$$

$$\sum_{n=0}^{\infty}\left(-\frac{n\,x^{n+5}b_n(n-1)}{5040}\right)=\sum_{n=6}^{\infty}\left(-\frac{(n-6)\,b_{n-6}(n-7)\,x^{n-1}}{5040}\right)$$

$$\sum_{n=0}^{\infty}\frac{n\,x^{n+3}b_n(n-1)}{120}=\sum_{n=4}^{\infty}\frac{(n-4)\,b_{n-4}(n-5)\,x^{n-1}}{120}$$

$$\sum_{n=0}^{\infty}\left(-\frac{n\,x^{1+n}b_n(n-1)}{6}\right)=\sum_{n=2}^{\infty}\left(-\frac{(n-2)\,b_{n-2}(n-3)\,x^{n-1}}{6}\right)$$

Substituting all the above in Eq (2A) gives the following equation where now all powers of x are the same and equal to $n-1$.

$$\begin{aligned}
&\sum_{n=7}^{\infty}\left(-\frac{C a_{n-7}(n-6)\, x^{n-1}}{2520}\right)+\left(\sum_{n=5}^{\infty}\frac{C a_{n-5}(n-4)\, x^{n-1}}{60}\right)\\
&+\sum_{n=3}^{\infty}\left(-\frac{C a_{n-3}(n-2)\, x^{n-1}}{3}\right)+\left(\sum_{n=1}^{\infty}2 C a_{n-1} n\, x^{n-1}\right)\\
&+\sum_{n=1}^{\infty}\left(-C a_{n-1} x^{n-1}\right)+\left(\sum_{n=3}^{\infty}\frac{C a_{n-3} x^{n-1}}{6}\right)\\
&+\sum_{n=5}^{\infty}\left(-\frac{C a_{n-5} x^{n-1}}{120}\right)+\left(\sum_{n=7}^{\infty}\frac{C a_{n-7} x^{n-1}}{5040}\right)\\
&+\sum_{n=7}^{\infty}\left(-\frac{b_{n-7} x^{n-1}}{720}\right)+\left(\sum_{n=5}^{\infty}\frac{b_{n-5} x^{n-1}}{24}\right)+\sum_{n=3}^{\infty}\left(-\frac{b_{n-3} x^{n-1}}{2}\right)\\
&+\left(\sum_{n=1}^{\infty}b_{n-1} x^{n-1}\right)+\sum_{n=6}^{\infty}\left(-\frac{(n-6)\, b_{n-6}(n-7)\, x^{n-1}}{5040}\right)\\
&+\left(\sum_{n=4}^{\infty}\frac{(n-4)\, b_{n-4}(n-5)\, x^{n-1}}{120}\right)\\
&+\sum_{n=2}^{\infty}\left(-\frac{(n-2)\, b_{n-2}(n-3)\, x^{n-1}}{6}\right)+\left(\sum_{n=0}^{\infty}n\, x^{n-1} b_n(n-1)\right)=0
\end{aligned}\tag{2B}$$

For $n=0$ in Eq. (2B), we choose arbitray value for b_0 as $b_0=1$. For $n=N$, where $N=1$ which is the difference between the two roots, we are free to choose $b_1=0$. Hence for $n=1$, Eq (2B) gives

$$C+1=0$$

Which is solved for C. Solving for C gives

$$C=-1$$

For $n=2$, Eq (2B) gives

$$3 C a_1+b_1+2 b_2=0$$

Which when replacing the above values found already for b_n and the values found earlier for a_n and for C, gives

$$2 b_2+\frac{3}{2}=0$$

Solving the above for b_2 gives

$$b_2=-\frac{3}{4}$$

For $n = 3$, Eq (2B) gives

$$\frac{(-a_0 + 30a_2) C}{6} - \frac{b_0}{2} + b_2 + 6b_3 = 0$$

Which when replacing the above values found already for b_n and the values found earlier for a_n and for C, gives

$$-\frac{3}{2} + 6b_3 = 0$$

Solving the above for b_3 gives

$$b_3 = \frac{1}{4}$$

For $n = 4$, Eq (2B) gives

$$\frac{(-3a_1 + 42a_3) C}{6} - \frac{b_1}{2} - \frac{b_2}{3} + b_3 + 12b_4 = 0$$

Which when replacing the above values found already for b_n and the values found earlier for a_n and for C, gives

$$\frac{5}{48} + 12b_4 = 0$$

Solving the above for b_4 gives

$$b_4 = -\frac{5}{576}$$

For $n = 5$, Eq (2B) gives

$$\frac{(a_0 - 100a_2 + 1080a_4) C}{120} + \frac{b_0}{24} - \frac{b_2}{2} - b_3 + b_4 + 20b_5 = 0$$

Which when replacing the above values found already for b_n and the values found earlier for a_n and for C, gives

$$\frac{437}{1440} + 20b_5 = 0$$

Solving the above for b_5 gives

$$b_5 = -\frac{437}{28800}$$

Now that we found all b_n and C, we can calculate the second solution from

$$y_2(x) = C y_1(x) \ln(x) + \left(\sum_{n=0}^{\infty} b_n x^{n+r_2}\right)$$

Using the above value found for $C = -1$ and all b_n, then the second solution becomes

$$y_2(x) = (-1) \left(x\left(1 - \frac{x}{2} + \frac{x^2}{12} + \frac{x^3}{48} - \frac{3x^4}{320} + \frac{19x^5}{9600} - \frac{59x^6}{403200} + O(x^6)\right)\right) \ln(x)$$
$$+ 1 - \frac{3x^2}{4} + \frac{x^3}{4} - \frac{5x^4}{576} - \frac{437x^5}{28800} + O(x^6)$$

Therefore the homogeneous solution is

$$
\begin{aligned}
y_h(x) &= c_1 y_1(x) + c_2 y_2(x) \\
&= c_1 x \left(1 - \frac{x}{2} + \frac{x^2}{12} + \frac{x^3}{48} - \frac{3x^4}{320} + \frac{19x^5}{9600} - \frac{59x^6}{403200} + O(x^6)\right) \\
&\quad + c_2 \left((-1)\left(x\left(1 - \frac{x}{2} + \frac{x^2}{12} + \frac{x^3}{48} - \frac{3x^4}{320} + \frac{19x^5}{9600} - \frac{59x^6}{403200} + O(x^6)\right)\right)\ln(x) \right. \\
&\quad \left. + 1 - \frac{3x^2}{4} + \frac{x^3}{4} - \frac{5x^4}{576} - \frac{437x^5}{28800} + O(x^6)\right)
\end{aligned}
$$

Hence the final solution is

$$
\begin{aligned}
y &= y_h \\
&= c_1 x \left(1 - \frac{x}{2} + \frac{x^2}{12} + \frac{x^3}{48} - \frac{3x^4}{320} + \frac{19x^5}{9600} - \frac{59x^6}{403200} + O(x^6)\right) \\
&\quad + c_2 \left(-x\left(1 - \frac{x}{2} + \frac{x^2}{12} + \frac{x^3}{48} - \frac{3x^4}{320} + \frac{19x^5}{9600} - \frac{59x^6}{403200} + O(x^6)\right)\ln(x) + 1 - \frac{3x^2}{4} \right. \\
&\quad \left. + \frac{x^3}{4} - \frac{5x^4}{576} - \frac{437x^5}{28800} + O(x^6)\right)
\end{aligned}
$$

<u>Summary</u>
The solution(s) found are the following

$$
\begin{aligned}
y &= c_1 x \left(1 - \frac{x}{2} + \frac{x^2}{12} + \frac{x^3}{48} - \frac{3x^4}{320} + \frac{19x^5}{9600} - \frac{59x^6}{403200} + O(x^6)\right) \\
&\quad + c_2 \left(-x\left(1 - \frac{x}{2} + \frac{x^2}{12} + \frac{x^3}{48} - \frac{3x^4}{320} + \frac{19x^5}{9600} - \frac{59x^6}{403200} + O(x^6)\right)\ln(x) + 1 \right. \\
&\quad \left. - \frac{3x^2}{4} + \frac{x^3}{4} - \frac{5x^4}{576} - \frac{437x^5}{28800} + O(x^6)\right)
\end{aligned} \tag{1}
$$

<u>Verification of solutions</u>

$$
\begin{aligned}
y &= c_1 x \left(1 - \frac{x}{2} + \frac{x^2}{12} + \frac{x^3}{48} - \frac{3x^4}{320} + \frac{19x^5}{9600} - \frac{59x^6}{403200} + O(x^6)\right) \\
&\quad + c_2 \left(-x\left(1 - \frac{x}{2} + \frac{x^2}{12} + \frac{x^3}{48} - \frac{3x^4}{320} + \frac{19x^5}{9600} - \frac{59x^6}{403200} + O(x^6)\right)\ln(x) + 1 - \frac{3x^2}{4} \right. \\
&\quad \left. + \frac{x^3}{4} - \frac{5x^4}{576} - \frac{437x^5}{28800} + O(x^6)\right)
\end{aligned}
$$

Verified OK.

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Heun: Equivalence to the GHE or one of its 4 confluent cases under a power @ Moebius
-> trying a solution of the form r0(x) * Y + r1(x) * Y where Y = exp(int(r(x), dx)) * 2F1([a
-> Trying changes of variables to rationalize or make the ODE simpler
   trying a symmetry of the form [xi=0, eta=F(x)]
   checking if the LODE is missing y
   -> Trying a solution in terms of special functions:
      -> Bessel
      -> elliptic
      -> Legendre
      -> Whittaker
         -> hyper3: Equivalence to 1F1 under a power @ Moebius
      -> hypergeometric
         -> heuristic approach
         -> hyper3: Equivalence to 2F1, 1F1 or 0F1 under a power @ Moebius
      -> Mathieu
         -> Equivalence to the rational form of Mathieu ODE under a power @ Moebius
   -> Heun: Equivalence to the GHE or one of its 4 confluent cases under a power @ Moebius
   -> Heun: Equivalence to the GHE or one of its 4 confluent cases under a power @ Moebius
   -> trying a solution of the form r0(x) * Y + r1(x) * Y where Y = exp(int(r(x), dx)) * 2F1
      trying a symmetry of the form [xi=0, eta=F(x)]
      trying 2nd order exact linear
      trying symmetries linear in x and y(x)
      trying to convert to a linear ODE with constant coefficients
      -> trying with_periodic_functions in the coefficients
         --- Trying Lie symmetry methods, 2nd order ---
         `, `-> Computing symmetries using: way = 5
   trying a symmetry of the form [xi=0, eta=F(x)]
   checking if the LODE is missing y
   -> Trying a solution in terms of special functions:
      -> Bessel
      -> elliptic
      -> Legendre
      -> Kummer
         -> hyper3: Equivalence to 1F1 under a power @ Moebius
      -> hypergeometric
         -> heuristic approach
         -> hyper3: Equivalence to 2F1, 1F1 or 0F1 under a power @ Moebius
      -> Mathieu
         -> Equivalence to the rational form of Mathieu ODE under a power @ Moebius
   -> Heun: Equivalence to the GHE or one of its 4 confluent cases under a power @ Moebius
   -> Heun: Equivalence to the GHE or one of its 4 confluent cases under a power @ Moebius
```

✓ Solution by Maple

Time used: 0.203 (sec). Leaf size: 58

```
Order:=6;
dsolve(sin(x)*diff(y(x),x$2)+cos(x)*y(x)=0,y(x),type='series',x=0);
```

$$y(x) = c_1 x\left(1-\frac{1}{2}x+\frac{1}{12}x^2+\frac{1}{48}x^3-\frac{3}{320}x^4+\frac{19}{9600}x^5+\mathrm{O}\left(x^6\right)\right) + c_2\left(\ln\left(x\right)\left(-x+\frac{1}{2}x^2-\frac{1}{12}x^3-\frac{1}{48}x^4+\frac{3}{320}x^5+\mathrm{O}\left(x^6\right)\right) + \left(1-\frac{3}{4}x^2+\frac{1}{4}x^3-\frac{5}{576}x^4-\frac{437}{28800}x^5+\mathrm{O}\left(x^6\right)\right)\right)$$

✓ Solution by Mathematica

Time used: 0.021 (sec). Leaf size: 85

```
AsymptoticDSolveValue[Sin[x]*y''[x]+Cos[x]*y[x]==0,y[x],{x,0,5}]
```

$$y(x) \to c_1\left(\frac{1}{576}\left(7x^4+192x^3-720x^2+576x+576\right)-\frac{1}{48}x\left(x^3+4x^2-24x+48\right)\log(x)\right) + c_2\left(-\frac{3x^5}{320}+\frac{x^4}{48}+\frac{x^3}{12}-\frac{x^2}{2}+x\right)$$

5.8 problem 8

Internal problem ID [5009]
Internal file name [`OUTPUT/4502_Sunday_June_05_2022_02_59_37_PM_28715982/index.tex`]

Book: Fundamentals of Differential Equations. By Nagle, Saff and Snider. 9th edition. Boston. Pearson 2018.
Section: Chapter 8, Series solutions of differential equations. Section 8.3. page 443
Problem number: 8.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second order series method. Ordinary point", "second order series method. Taylor series method"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

$$\boxed{e^{x} y'' - \left(x^{2}-1\right) y' + 2xy = 0}$$

With the expansion point for the power series method at $x = 0$.

Solving ode using Taylor series method. This gives review on how the Taylor series method works for solving second order ode.

Let

$$y'' = f(x, y, y')$$

Assuming expansion is at $x_0 = 0$ (we can always shift the actual expansion point to 0 by change of variables) and assuming $f(x, y, y')$ is analytic at x_0 which must be the case for an ordinary point. Let initial conditions be $y(x_0) = y_0$ and $y'(x_0) = y'_0$. Using Taylor series gives

$$\begin{aligned} y(x) &= y(x_0) + (x - x_0)\, y'(x_0) + \frac{(x - x_0)^2}{2} y''(x_0) + \frac{(x - x_0)^3}{3!} y'''(x_0) + \cdots \\ &= y_0 + x y'_0 + \frac{x^2}{2} f|_{x_0, y_0, y'_0} + \frac{x^3}{3!} f'|_{x_0, y_0, y'_0} + \cdots \\ &= y_0 + x y'_0 + \sum_{n=0}^{\infty} \frac{x^{n+2}}{(n+2)!} \frac{d^n f}{dx^n}\bigg|_{x_0, y_0, y'_0} \end{aligned}$$

But

$$\begin{aligned}
\frac{df}{dx} &= \frac{\partial f}{\partial x}\frac{dx}{dx} + \frac{\partial f}{\partial y}\frac{dy}{dx} + \frac{\partial f}{\partial y'}\frac{dy'}{dx} && (1)\\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}y' + \frac{\partial f}{\partial y'}y'' && (237)\\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}y' + \frac{\partial f}{\partial y'}f && (238)\\
\frac{d^2 f}{dx^2} &= \frac{d}{dx}\left(\frac{df}{dx}\right)\\
&= \frac{\partial}{\partial x}\left(\frac{df}{dx}\right) + \frac{\partial}{\partial y}\left(\frac{df}{dx}\right)y' + \frac{\partial}{\partial y'}\left(\frac{df}{dx}\right)f && (2)\\
\frac{d^3 f}{dx^3} &= \frac{d}{dx}\left(\frac{d^2 f}{dx^2}\right)\\
&= \frac{\partial}{\partial x}\left(\frac{d^2 f}{dx^2}\right) + \left(\frac{\partial}{\partial y}\frac{d^2 f}{dx^2}\right)y' + \frac{\partial}{\partial y'}\left(\frac{d^2 f}{dx^2}\right)f && (3)\\
&\vdots
\end{aligned}$$

And so on. Hence if we name $F_0 = f(x, y, y')$ then the above can be written as

$$
\begin{aligned}
F_0 &= f(x,y,y') \qquad (4)\\
F_1 &= \frac{df}{dx}\\
&= \frac{dF_0}{dx}\\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}y' + \frac{\partial f}{\partial y'}y''\\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}y' + \frac{\partial f}{\partial y'}f \qquad (5)\\
&= \frac{\partial F_0}{\partial x} + \frac{\partial F_0}{\partial y}y' + \frac{\partial F_0}{\partial y'}F_0\\
F_2 &= \frac{d}{dx}\left(\frac{d}{dx}f\right)\\
&= \frac{d}{dx}(F_1)\\
&= \frac{\partial}{\partial x}F_1 + \left(\frac{\partial F_1}{\partial y}\right)y' + \left(\frac{\partial F_1}{\partial y'}\right)y''\\
&= \frac{\partial}{\partial x}F_1 + \left(\frac{\partial F_1}{\partial y}\right)y' + \left(\frac{\partial F_1}{\partial y'}\right)F_0\\
&\vdots\\
F_n &= \frac{d}{dx}(F_{n-1})\\
&= \frac{\partial}{\partial x}F_{n-1} + \left(\frac{\partial F_{n-1}}{\partial y}\right)y' + \left(\frac{\partial F_{n-1}}{\partial y'}\right)y''\\
&= \frac{\partial}{\partial x}F_{n-1} + \left(\frac{\partial F_{n-1}}{\partial y}\right)y' + \left(\frac{\partial F_{n-1}}{\partial y'}\right)F_0 \qquad (6)
\end{aligned}
$$

Therefore (6) can be used from now on along with

$$
y(x) = y_0 + xy_0' + \sum_{n=0}^{\infty} \frac{x^{n+2}}{(n+2)!} F_n|_{x_0,y_0,y_0'} \qquad (7)
$$

To find $y(x)$ series solution around $x=0$. Hence

$$
\begin{aligned}
F_0 &= \left(x^2y' - 2xy - y'\right) \mathrm{e}^{-x}\\
F_1 &= \frac{dF_0}{dx}\\
&= \frac{\partial F_0}{\partial x} + \frac{\partial F_0}{\partial y} y' + \frac{\partial F_0}{\partial y'} F_0\\
&= \mathrm{e}^{-x}\left((x+1)\left(\left(x^2-1\right)y' - 2xy\right)\mathrm{e}^{-x} + (-1-x)\,y' + 2y\right)(x-1)\\
F_2 &= \frac{dF_1}{dx}\\
&= \frac{\partial F_1}{\partial x} + \frac{\partial F_1}{\partial y} y' + \frac{\partial F_1}{\partial y'} F_1\\
&= \left(\left(-3x^4 + 2x^3 + 6x^2 - 2x - 3\right)y' + 6y\left(x^3 - x^2 - x + \frac{1}{3}\right)\right)\mathrm{e}^{-2x} + (x-1)^2\,(x+1)^2\left(\left(x^2-1\right)y' -\right.\\
F_3 &= \frac{dF_2}{dx}\\
&= \frac{\partial F_2}{\partial x} + \frac{\partial F_2}{\partial y} y' + \frac{\partial F_2}{\partial y'} F_2\\
&= -6(x-1)\,(x+1)\left(\left(x^4 - x^3 - 2x^2 + x + 1\right)y' - 2y\left(x^3 - \frac{7}{6}x^2 - x + \frac{1}{6}\right)\right)\mathrm{e}^{-3x} + (x-1)^3\,(x+1)^3\\
F_4 &= \frac{dF_3}{dx}\\
&= \frac{\partial F_3}{\partial x} + \frac{\partial F_3}{\partial y} y' + \frac{\partial F_3}{\partial y'} F_3\\
&= -10(x-1)^2\,(x+1)^2\left(\left(x^4 - \frac{6}{5}x^3 - 2x^2 + \frac{6}{5}x + 1\right)y' - 2\left(x^3 - \frac{13}{10}x^2 - x + \frac{1}{10}\right)y\right)\mathrm{e}^{-4x} + \left(\left(25x\right.\right.
\end{aligned}
$$

And so on. Evaluating all the above at initial conditions $x=0$ and $y(0)=y(0)$ and $y'(0)=y'(0)$ gives

$$
\begin{aligned}
F_0 &= -y'(0)\\
F_1 &= 2y'(0) - 2y(0)\\
F_2 &= 6y(0) - 7y'(0)\\
F_3 &= -18y(0) + 23y'(0)\\
F_4 &= 54y(0) - 72y'(0)
\end{aligned}
$$

Substituting all the above in (7) and simplifying gives the solution as

$$
\begin{aligned}
y = {} &\left(1 - \frac{1}{3}x^3 + \frac{1}{4}x^4 - \frac{3}{20}x^5 + \frac{3}{40}x^6\right) y(0)\\
&+ \left(x - \frac{1}{2}x^2 + \frac{1}{3}x^3 - \frac{7}{24}x^4 + \frac{23}{120}x^5 - \frac{1}{10}x^6\right) y'(0) + O\left(x^6\right)
\end{aligned}
$$

Since the expansion point $x = 0$ is an ordinary, we can also solve this using standard power series Let the solution be represented as power series of the form

$$y = \sum_{n=0}^{\infty} a_n x^n$$

Then

$$y' = \sum_{n=1}^{\infty} n a_n x^{n-1}$$
$$y'' = \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2}$$

Substituting the above back into the ode gives

$$\sum_{n=2}^{\infty} n(n-1) a_n x^{n-2} = \left(x^2 \left(\sum_{n=1}^{\infty} n a_n x^{n-1}\right) - 2x \left(\sum_{n=0}^{\infty} a_n x^n\right) - \left(\sum_{n=1}^{\infty} n a_n x^{n-1}\right)\right) e^{-x} \tag{1}$$

Expanding e^x as Taylor series around $x = 0$ and keeping only the first 6 terms gives

$$e^x = 1 + x + \frac{1}{2}x^2 + \frac{1}{6}x^3 + \frac{1}{24}x^4 + \frac{1}{120}x^5 + \frac{1}{720}x^6 + \dots$$
$$= 1 + x + \frac{1}{2}x^2 + \frac{1}{6}x^3 + \frac{1}{24}x^4 + \frac{1}{120}x^5 + \frac{1}{720}x^6$$

Hence the ODE in Eq (1) becomes

$$\left(1 + x + \frac{1}{2}x^2 + \frac{1}{6}x^3 + \frac{1}{24}x^4 + \frac{1}{120}x^5 + \frac{1}{720}x^6\right) \left(\sum_{n=2}^{\infty} n(n-1) a_n x^{n-2}\right) + (-x^2 + 1) \left(\sum_{n=1}^{\infty} n a_n x^{n-1}\right) + 2x \left(\sum_{n=0}^{\infty} a_n x^n\right) = 0$$

Expanding the first term in (1) gives

$$1 \cdot \left(\sum_{n=2}^{\infty} n(n-1) a_n x^{n-2}\right) + x \cdot \left(\sum_{n=2}^{\infty} n(n-1) a_n x^{n-2}\right) + \frac{x^2}{2} \cdot \left(\sum_{n=2}^{\infty} n(n-1) a_n x^{n-2}\right) + \frac{x^3}{6} \cdot \left(\sum_{n=2}^{\infty} n(n-1) a_n x^{n-2}\right) + \frac{x^4}{24} \cdot \left(\sum_{n=2}^{\infty} n(n-1) a_n x^{n-2}\right) + \frac{x^5}{120} \cdot \left(\sum_{n=2}^{\infty} n(n-1) a_n x^{n-2}\right) + \frac{x^6}{720} \cdot \left(\sum_{n=2}^{\infty} n(n-1) a_n x^{n-2}\right) + (-x^2 + 1) \left(\sum_{n=1}^{\infty} n a_n x^{n-1}\right) + 2x \left(\sum_{n=0}^{\infty} a_n x^n\right) = 0$$

Which simplifies to

$$\left(\sum_{n=2}^{\infty} \frac{n\,x^{n+4} a_n (n-1)}{720}\right) + \left(\sum_{n=2}^{\infty} \frac{n\,x^{n+3} a_n (n-1)}{120}\right) + \left(\sum_{n=2}^{\infty} \frac{n\,x^{n+2} a_n (n-1)}{24}\right) + \left(\sum_{n=2}^{\infty} \frac{n\,x^{1+n} a_n (n-1)}{6}\right) + \left(\sum_{n=2}^{\infty} \frac{n a_n x^{n} (n-1)}{2}\right) + \left(\sum_{n=2}^{\infty} n\,x^{n-1} a_n (n-1)\right) + \left(\sum_{n=2}^{\infty} n(n-1)\,a_n x^{n-2}\right) + \sum_{n=1}^{\infty} \left(-n\,x^{1+n} a_n\right) + \left(\sum_{n=1}^{\infty} n a_n x^{n-1}\right) + \left(\sum_{n=0}^{\infty} 2x^{1+n} a_n\right) = 0 \tag{2}$$

The next step is to make all powers of x be n in each summation term. Going over each summation term above with power of x in it which is not already x^n and adjusting the power and the corresponding index gives

$$\sum_{n=2}^{\infty} \frac{n\,x^{n+4} a_n (n-1)}{720} = \sum_{n=6}^{\infty} \frac{(n-4)\,a_{n-4} (n-5)\,x^{n}}{720}$$

$$\sum_{n=2}^{\infty} \frac{n\,x^{n+3} a_n (n-1)}{120} = \sum_{n=5}^{\infty} \frac{(n-3)\,a_{n-3} (n-4)\,x^{n}}{120}$$

$$\sum_{n=2}^{\infty} \frac{n\,x^{n+2} a_n (n-1)}{24} = \sum_{n=4}^{\infty} \frac{(n-2)\,a_{n-2} (n-3)\,x^{n}}{24}$$

$$\sum_{n=2}^{\infty} \frac{n\,x^{1+n} a_n (n-1)}{6} = \sum_{n=3}^{\infty} \frac{(n-1)\,a_{n-1} (n-2)\,x^{n}}{6}$$

$$\sum_{n=2}^{\infty} n\,x^{n-1} a_n (n-1) = \sum_{n=1}^{\infty} (1+n)\,a_{1+n} n\,x^{n}$$

$$\sum_{n=2}^{\infty} n(n-1)\,a_n x^{n-2} = \sum_{n=0}^{\infty} (n+2)\,a_{n+2} (1+n)\,x^{n}$$

$$\sum_{n=1}^{\infty} \left(-n\,x^{1+n} a_n\right) = \sum_{n=2}^{\infty} \left(-(n-1)\,a_{n-1} x^{n}\right)$$

$$\sum_{n=1}^{\infty} n a_n x^{n-1} = \sum_{n=0}^{\infty} (1+n)\,a_{1+n} x^{n}$$

$$\sum_{n=0}^{\infty} 2x^{1+n} a_n = \sum_{n=1}^{\infty} 2a_{n-1} x^n$$

Substituting all the above in Eq (2) gives the following equation where now all powers of x are the same and equal to n.

$$\begin{aligned}
&\left(\sum_{n=6}^{\infty} \frac{(n-4)\, a_{n-4}(n-5)\, x^n}{720}\right)+\left(\sum_{n=5}^{\infty} \frac{(n-3)\, a_{n-3}(n-4)\, x^n}{120}\right) \\
&+\left(\sum_{n=4}^{\infty} \frac{(n-2)\, a_{n-2}(n-3)\, x^n}{24}\right)+\left(\sum_{n=3}^{\infty} \frac{(n-1)\, a_{n-1}(n-2)\, x^n}{6}\right) \\
&+\left(\sum_{n=2}^{\infty} \frac{n a_n x^n (n-1)}{2}\right)+\left(\sum_{n=1}^{\infty} (1+n)\, a_{1+n} n\, x^n\right) \\
&+\left(\sum_{n=0}^{\infty} (n+2)\, a_{n+2}(1+n)\, x^n\right)+\sum_{n=2}^{\infty} \left(-(n-1)\, a_{n-1} x^n\right) \\
&\quad+\left(\sum_{n=0}^{\infty} (1+n)\, a_{1+n} x^n\right)+\left(\sum_{n=1}^{\infty} 2a_{n-1} x^n\right)=0
\end{aligned} \tag{3}$$

$n = 0$ gives

$$2a_2 + a_1 = 0$$

$$a_2 = -\frac{a_1}{2}$$

$n = 1$ gives

$$4a_2 + 6a_3 + 2a_0 = 0$$

Which after substituting earlier equations, simplifies to

$$a_3 = -\frac{a_0}{3} + \frac{a_1}{3}$$

$n = 2$ gives

$$a_2 + 9a_3 + 12a_4 + a_1 = 0$$

Which after substituting earlier equations, simplifies to

$$a_4 = \frac{a_0}{4} - \frac{7a_1}{24}$$

$n = 3$ gives

$$\frac{a_2}{3} + 3a_3 + 16a_4 + 20a_5 = 0$$

Which after substituting earlier equations, simplifies to

$$-\frac{23a_1}{6} + 3a_0 + 20a_5 = 0$$

Or

$$a_5 = -\frac{3a_0}{20} + \frac{23a_1}{120}$$

$n = 4$ gives

$$\frac{a_2}{12} + 6a_4 + 25a_5 + 30a_6 = 0$$

Which after substituting earlier equations, simplifies to

$$3a_1 - \frac{9a_0}{4} + 30a_6 = 0$$

Or

$$a_6 = \frac{3a_0}{40} - \frac{a_1}{10}$$

$n = 5$ gives

$$\frac{a_2}{60} + \frac{a_3}{4} + 10a_5 + 36a_6 + 42a_7 = 0$$

Which after substituting earlier equations, simplifies to

$$-\frac{193a_1}{120} + \frac{67a_0}{60} + 42a_7 = 0$$

Or

$$a_7 = -\frac{67a_0}{2520} + \frac{193a_1}{5040}$$

For $6 \leq n$, the recurrence equation is

$$\begin{aligned} &\frac{(n-4)\, a_{n-4}(n-5)}{720} + \frac{(n-3)\, a_{n-3}(n-4)}{120} + \frac{(n-2)\, a_{n-2}(n-3)}{24} \\ &+ \frac{(n-1)\, a_{n-1}(n-2)}{6} + \frac{n a_n(n-1)}{2} + (1+n)\, a_{1+n} n \\ &+ (n+2)\, a_{n+2}(1+n) - (n-1)\, a_{n-1} + (1+n)\, a_{1+n} + 2a_{n-1} = 0 \end{aligned} \tag{4}$$

Solving for a_{n+2}, gives

(5)

$$a_{n+2} =$$
$$-\frac{360n^2a_n + 720n^2a_{1+n} + n^2a_{n-4} + 6n^2a_{n-3} + 30n^2a_{n-2} + 120n^2a_{n-1} - 360na_n + 1440na_{1+n} - 9na_{n-}}{720\left(n+2\right)\left(1}$$

$$= -\frac{\left(360n^2 - 360n\right)a_n}{720\left(n+2\right)\left(1+n\right)} - \frac{\left(720n^2 + 1440n + 720\right)a_{1+n}}{720\left(n+2\right)\left(1+n\right)} - \frac{\left(n^2 - 9n + 20\right)a_{n-4}}{720\left(n+2\right)\left(1+n\right)} - \frac{\left(6n^2 - 42n + 72\right)a_{n-3}}{720\left(n+2\right)\left(1+n\right)} - \frac{\left(30n^2 - 150n + 180\right)a_{n-2}}{720\left(n+2\right)\left(1+n\right)} - \frac{\left(120n^2 - 1080n + 2400\right)a_{n-1}}{720\left(n+2\right)\left(1+n\right)}$$

And so on. Therefore the solution is

$$\begin{aligned} y &= \sum_{n=0}^{\infty} a_n x^n \\ &= a_3x^3 + a_2x^2 + a_1x + a_0 + \dots \end{aligned}$$

Substituting the values for a_n found above, the solution becomes

$$y = a_0 + a_1x - \frac{a_1x^2}{2} + \left(-\frac{a_0}{3} + \frac{a_1}{3}\right)x^3 + \left(\frac{a_0}{4} - \frac{7a_1}{24}\right)x^4 + \left(-\frac{3a_0}{20} + \frac{23a_1}{120}\right)x^5 + \dots$$

Collecting terms, the solution becomes

$$y = \left(1 - \frac{1}{3}x^3 + \frac{1}{4}x^4 - \frac{3}{20}x^5\right)a_0 + \left(x - \frac{1}{2}x^2 + \frac{1}{3}x^3 - \frac{7}{24}x^4 + \frac{23}{120}x^5\right)a_1 + O\left(x^6\right) \quad (3)$$

At $x = 0$ the solution above becomes

$$y = \left(1 - \frac{1}{3}x^3 + \frac{1}{4}x^4 - \frac{3}{20}x^5\right)c_1 + \left(x - \frac{1}{2}x^2 + \frac{1}{3}x^3 - \frac{7}{24}x^4 + \frac{23}{120}x^5\right)c_2 + O\left(x^6\right)$$

Summary

The solution(s) found are the following

$$\begin{aligned} y &= \left(1 - \frac{1}{3}x^3 + \frac{1}{4}x^4 - \frac{3}{20}x^5 + \frac{3}{40}x^6\right)y(0) \\ &\quad + \left(x - \frac{1}{2}x^2 + \frac{1}{3}x^3 - \frac{7}{24}x^4 + \frac{23}{120}x^5 - \frac{1}{10}x^6\right)y'(0) + O\left(x^6\right) \end{aligned} \quad (1)$$

$$y = \left(1 - \frac{1}{3}x^3 + \frac{1}{4}x^4 - \frac{3}{20}x^5\right)c_1 + \left(x - \frac{1}{2}x^2 + \frac{1}{3}x^3 - \frac{7}{24}x^4 + \frac{23}{120}x^5\right)c_2 + O\left(x^6\right) \quad (2)$$

Verification of solutions

$$
\begin{aligned}
y = {} & \left(1-\frac{1}{3}x^{3}+\frac{1}{4}x^{4}-\frac{3}{20}x^{5}+\frac{3}{40}x^{6}\right) y(0) \\
& + \left(x-\frac{1}{2}x^{2}+\frac{1}{3}x^{3}-\frac{7}{24}x^{4}+\frac{23}{120}x^{5}-\frac{1}{10}x^{6}\right) y'(0)+O\left(x^{6}\right)
\end{aligned}
$$

Verified OK.

$$
y=\left(1-\frac{1}{3}x^{3}+\frac{1}{4}x^{4}-\frac{3}{20}x^{5}\right) c_{1}+\left(x-\frac{1}{2}x^{2}+\frac{1}{3}x^{3}-\frac{7}{24}x^{4}+\frac{23}{120}x^{5}\right) c_{2}+O\left(x^{6}\right)
$$

Verified OK.

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Heun: Equivalence to the GHE or one of its 4 confluent cases under a power @ Moebius
-> trying a solution of the form r0(x) * Y + r1(x) * Y where Y = exp(int(r(x), dx)) * 2F1([a
-> Trying changes of variables to rationalize or make the ODE simpler
   trying a symmetry of the form [xi=0, eta=F(x)]
   checking if the LODE is missing y
   -> Heun: Equivalence to the GHE or one of its 4 confluent cases under a power @ Moebius
   -> trying a solution of the form r0(x) * Y + r1(x) * Y where Y = exp(int(r(x), dx)) * 2F1
      trying a symmetry of the form [xi=0, eta=F(x)]
      trying 2nd order exact linear
      trying symmetries linear in x and y(x)
      trying to convert to a linear ODE with constant coefficients
<- unable to find a useful change of variables
   trying a symmetry of the form [xi=0, eta=F(x)]
   trying differential order: 2; exact nonlinear
   trying symmetries linear in x and y(x)
   trying to convert to a linear ODE with constant coefficients
   trying 2nd order, integrating factor of the form mu(x,y)
   trying a symmetry of the form [xi=0, eta=F(x)]
   checking if the LODE is missing y
   -> Heun: Equivalence to the GHE or one of its 4 confluent cases under a power @ Moebius
   -> trying a solution of the form r0(x) * Y + r1(x) * Y where Y = exp(int(r(x), dx)) * 2F1
   -> Trying changes of variables to rationalize or make the ODE simpler
      trying a symmetry of the form [xi=0, eta=F(x)]
      checking if the LODE is missing y
      -> Heun: Equivalence to the GHE or one of its 4 confluent cases under a power @ Moebiu
      -> trying a solution of the form r0(x) * Y + r1(x) * Y where Y = exp(int(r(x), dx)) *
         trying a symmetry of the form [xi=0, eta=F(x)]
         trying 2nd order exact linear
         trying symmetries linear in x and y(x)
         trying to convert to a linear ODE with constant coefficients
   <- unable to find a useful change of variables
      trying a symmetry of the form [xi=0, eta=F(x)]
   trying to convert to an ODE of Bessel type
   -> trying reduction of order to Riccati
      trying Riccati sub-methods:
         trying Riccati_symmetries
         -> trying a symmetry pattern of the form [F(x)*G(y), 0]
         -> trying a symmetry pattern of the form [0, F(x)*G(y)]
         -> trying a symmetry pattern of the form [F(x),G(x)*y+H(x)]
--- Trying Lie symmetry methods, 2nd order ---
`, `-> Computing symmetries using: way = 3`[0, u]
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 49

```
Order:=6;
dsolve(exp(x)*diff(y(x),x$2)-(x^2-1)*diff(y(x),x)+2*x*y(x)=0,y(x),type='series',x=0);
```

$$y(x) = \left(1-\frac{1}{3}x^3+\frac{1}{4}x^4-\frac{3}{20}x^5\right)y(0)+\left(x-\frac{1}{2}x^2+\frac{1}{3}x^3-\frac{7}{24}x^4+\frac{23}{120}x^5\right)D(y)(0)+O(x^6)$$

✓ Solution by Mathematica

Time used: 0.002 (sec). Leaf size: 63

```
AsymptoticDSolveValue[Exp[x]*y''[x]-(x^2-1)*y'[x]+2*x*y[x]==0,y[x],{x,0,5}]
```

$$y(x) \to c_1\left(-\frac{3x^5}{20}+\frac{x^4}{4}-\frac{x^3}{3}+1\right)+c_2\left(\frac{23x^5}{120}-\frac{7x^4}{24}+\frac{x^3}{3}-\frac{x^2}{2}+x\right)$$

5.9 problem 9

Internal problem ID [5010]
Internal file name [`OUTPUT/4503_Sunday_June_05_2022_02_59_39_PM_55860714/index.tex`]

Book: Fundamentals of Differential Equations. By Nagle, Saff and Snider. 9th edition. Boston. Pearson 2018.
Section: Chapter 8, Series solutions of differential equations. Section 8.3. page 443
Problem number: 9.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second order series method. Irregular singular point"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

Unable to solve or complete the solution.

$$\boxed{\sin(x)\,y'' - \ln(x)\,y = 0}$$

With the expansion point for the power series method at $x = 0$.

The type of the expansion point is first determined. This is done on the homogeneous part of the ODE.

$$\sin(x)\,y'' - \ln(x)\,y = 0$$

The following is summary of singularities for the above ode. Writing the ode as

$$y'' + p(x)y' + q(x)y = 0$$

Where

$$\begin{aligned} p(x) &= 0 \\ q(x) &= -\frac{\ln(x)}{\sin(x)} \end{aligned}$$

Table 206: Table $p(x), q(x)$ singularites.

$p(x) = 0$	
singularity	type

$q(x) = -\frac{\ln(x)}{\sin(x)}$	
singularity	type
$x = 0$	"irregular"
$x = \pi Z$	"regular"

Combining everything together gives the following summary of singularities for the ode as

Regular singular points : $[\pi Z]$

Irregular singular points : $[0, \infty]$

Since $x = 0$ is not an ordinary point, then we will now check if it is a regular singular point. Unable to solve since $x = 0$ is not regular singular point. Terminating.

Verification of solutions N/A

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Heun: Equivalence to the GHE or one of its 4 confluent cases under a power @ Moebius
-> trying a solution of the form r0(x) * Y + r1(x) * Y where Y = exp(int(r(x), dx)) * 2F1([a
-> Trying changes of variables to rationalize or make the ODE simpler
   trying a symmetry of the form [xi=0, eta=F(x)]
   checking if the LODE is missing y
   -> Heun: Equivalence to the GHE or one of its 4 confluent cases under a power @ Moebius
   -> trying a solution of the form r0(x) * Y + r1(x) * Y where Y = exp(int(r(x), dx)) * 2F1
      trying a symmetry of the form [xi=0, eta=F(x)]
      trying 2nd order exact linear
      trying symmetries linear in x and y(x)
      trying to convert to a linear ODE with constant coefficients
      -> trying with_periodic_functions in the coefficients
         --- Trying Lie symmetry methods, 2nd order ---
         `, `-> Computing symmetries using: way = 5`[0, u]
```

✗ Solution by Maple

```
Order:=6;
dsolve(sin(x)*diff(y(x),x$2)-ln(x)*y(x)=0,y(x),type='series',x=0);
```

No solution found

✗ Solution by Mathematica

Time used: 0.0 (sec). Leaf size: 0

```
AsymptoticDSolveValue[Sin[x]*y''[x]-Log[x]*y[x]==0,y[x],{x,0,5}]
```

Not solved

5.10 problem 11

Internal problem ID [5011]
Internal file name [`OUTPUT/4504_Sunday_June_05_2022_02_59_43_PM_56838994/index.tex`]

Book: Fundamentals of Differential Equations. By Nagle, Saff and Snider. 9th edition. Boston. Pearson 2018.
Section: Chapter 8, Series solutions of differential equations. Section 8.3. page 443
Problem number: 11.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"first order ode series method. Ordinary point", "first order ode series method. Taylor series method"**

Maple gives the following as the ode type

```
[_separable]
```

$$\boxed{y' + (x+2)\,y = 0}$$

With the expansion point for the power series method at $x = 0$.

5.10.1 Solving as series ode

Solving ode using Taylor series method. This gives review on how the Taylor series method works for solving first order ode. Let

$$y' = f(x,y)$$

Where $f(x,y)$ is analytic at expansion point x_0. We can always shift to $x_0 = 0$ if x_0 is not zero. So from now we assume $x_0 = 0$. Assume also that $y(x_0) = y_0$. Using Taylor series

$$\begin{aligned}
y(x) &= y(x_0) + (x-x_0)\,y'(x_0) + \frac{(x-x_0)^2}{2}y''(x_0) + \frac{(x-x_0)^3}{3!}y''(x_0) + \cdots \\
&= y_0 + xf + \frac{x^2}{2}\frac{df}{dx}\bigg|_{x_0,y_0} + \frac{x^3}{3!}\frac{d^2f}{dx^2}\bigg|_{x_0,y_0} + \cdots \\
&= y_0 + \sum_{n=0}^{\infty}\frac{x^{n+1}}{(n+1)\,!}\frac{d^nf}{dx^n}\bigg|_{x_0,y_0}
\end{aligned}$$

But

$$
\begin{aligned}
\frac{df}{dx} &= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} f && (1) \\
\frac{d^2 f}{dx^2} &= \frac{d}{dx}\left(\frac{df}{dx}\right) \\
&= \frac{\partial}{\partial x}\left(\frac{df}{dx}\right) + \frac{\partial}{\partial y}\left(\frac{df}{dx}\right) f && (2) \\
\frac{d^3 f}{dx^3} &= \frac{d}{dx}\left(\frac{d^2 f}{dx^2}\right) \\
&= \frac{\partial}{\partial x}\left(\frac{d^2 f}{dx^2}\right) + \left(\frac{\partial}{\partial y}\frac{d^2 f}{dx^2}\right) f && (3) \\
&\vdots
\end{aligned}
$$

And so on. Hence if we name $F_0 = f(x, y)$ then the above can be written as

$$
\begin{aligned}
F_0 &= f(x, y) && (4) \\
F_n &= \frac{d}{dx}(F_{n-1}) \\
&= \frac{\partial}{\partial x} F_{n-1} + \left(\frac{\partial F_{n-1}}{\partial y}\right) F_0 && (5)
\end{aligned}
$$

For example, for $n = 1$ we see that

$$
\begin{aligned}
F_1 &= \frac{d}{dx}(F_0) \\
&= \frac{\partial}{\partial x} F_0 + \left(\frac{\partial F_0}{\partial y}\right) F_0 \\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} f
\end{aligned}
$$

Which is (1). And when $n = 2$

$$
\begin{aligned}
F_2 &= \frac{d}{dx}(F_1) \\
&= \frac{\partial}{\partial x} F_1 + \left(\frac{\partial F_1}{\partial y}\right) F_0 \\
&= \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} f\right) + \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} f\right) f \\
&= \frac{\partial}{\partial x}\left(\frac{df}{dx}\right) + \frac{\partial}{\partial y}\left(\frac{df}{dx}\right) f
\end{aligned}
$$

Which is (2) and so on. Therefore (4,5) can be used from now on along with

$$y(x) = y_0 + \sum_{n=0}^{\infty} \frac{x^{n+1}}{(n+1)\,!} F_n|_{x_0,y_0} \tag{6}$$

Hence

$$\begin{aligned}
F_0 &= -(x+2)\,y \\
F_1 &= \frac{dF_0}{dx} \\
&= \frac{\partial F_0}{\partial x} + \frac{\partial F_0}{\partial y} F_0 \\
&= y(x^2+4x+3) \\
F_2 &= \frac{dF_1}{dx} \\
&= \frac{\partial F_1}{\partial x} + \frac{\partial F_1}{\partial y} F_1 \\
&= (x+2)\,y(-x^2-4x-1) \\
F_3 &= \frac{dF_2}{dx} \\
&= \frac{\partial F_2}{\partial x} + \frac{\partial F_2}{\partial y} F_2 \\
&= y(x^4+8x^3+18x^2+8x-5) \\
F_4 &= \frac{dF_3}{dx} \\
&= \frac{\partial F_3}{\partial x} + \frac{\partial F_3}{\partial y} F_3 \\
&= (x+2)\,y(-x^4-8x^3-14x^2+8x+9)
\end{aligned}$$

And so on. Evaluating all the above at initial conditions $x(0) = 0$ and $y(0) = y(0)$ gives

$$\begin{aligned}
F_0 &= -2y(0) \\
F_1 &= 3y(0) \\
F_2 &= -2y(0) \\
F_3 &= -5y(0) \\
F_4 &= 18y(0)
\end{aligned}$$

Substituting all the above in (6) and simplifying gives the solution as

$$y = \left(1 - 2x + \frac{3}{2}x^2 - \frac{1}{3}x^3 - \frac{5}{24}x^4 + \frac{3}{20}x^5\right) y(0) + O(x^6)$$

Since $x = 0$ is also an ordinary point, then standard power series can also be used. Writing the ODE as

$$
\begin{aligned}
y' + q(x)y &= p(x) \\
y' + (x+2)\, y &= 0
\end{aligned}
$$

Where

$$
\begin{aligned}
q(x) &= x + 2 \\
p(x) &= 0
\end{aligned}
$$

Next, the type of the expansion point $x = 0$ is determined. This point can be an ordinary point, a regular singular point (also called removable singularity), or irregular singular point (also called non-removable singularity or essential singularity). When $x = 0$ is an ordinary point, then the standard power series is used. If the point is a regular singular point, Frobenius series is used instead. Irregular singular point requires more advanced methods (asymptotic methods) and is not supported now. Hopefully this will be added in the future. $x = 0$ is called an ordinary point $q(x)$ has a Taylor series expansion around the point $x = 0$. $x = 0$ is called a regular singular point if $q(x)$ is not not analytic at $x = 0$ but $xq(x)$ has Taylor series expansion. And finally, $x = 0$ is an irregular singular point if the point is not ordinary and not regular singular. This is the most complicated case. Now the expansion point $x = 0$ is checked to see if it is an ordinary point or not. Let the solution be represented as power series of the form

$$
y = \sum_{n=0}^{\infty} a_n x^n
$$

Then

$$
y' = \sum_{n=1}^{\infty} n a_n x^{n-1}
$$

Substituting the above back into the ode gives

$$
\left(\sum_{n=1}^{\infty} n a_n x^{n-1}\right) + (x+2)\left(\sum_{n=0}^{\infty} a_n x^n\right) = 0 \tag{1}
$$

Which simplifies to

$$
\left(\sum_{n=1}^{\infty} n a_n x^{n-1}\right) + \left(\sum_{n=0}^{\infty} x^{1+n} a_n\right) + \left(\sum_{n=0}^{\infty} 2 a_n x^n\right) = 0 \tag{2}
$$

The next step is to make all powers of x be n in each summation term. Going over each summation term above with power of x in it which is not already x^n and adjusting the power and the corresponding index gives

$$\sum_{n=1}^{\infty} n a_n x^{n-1} = \sum_{n=0}^{\infty} (1+n) a_{1+n} x^n$$
$$\sum_{n=0}^{\infty} x^{1+n} a_n = \sum_{n=1}^{\infty} a_{n-1} x^n$$

Substituting all the above in Eq (2) gives the following equation where now all powers of x are the same and equal to n.

$$\left(\sum_{n=0}^{\infty} (1+n) a_{1+n} x^n\right) + \left(\sum_{n=1}^{\infty} a_{n-1} x^n\right) + \left(\sum_{n=0}^{\infty} 2 a_n x^n\right) = 0 \tag{3}$$

$n = 0$ gives

$$a_1 + 2a_0 = 0$$
$$a_1 = -2a_0$$

For $1 \leq n$, the recurrence equation is

$$(1+n) a_{1+n} + a_{n-1} + 2a_n = 0 \tag{4}$$

Solving for a_{1+n}, gives

$$a_{1+n} = -\frac{a_{n-1} + 2a_n}{1+n} \tag{5}$$

For $n = 1$ the recurrence equation gives

$$2a_2 + a_0 + 2a_1 = 0$$

Which after substituting the earlier terms found becomes

$$a_2 = \frac{3a_0}{2}$$

For $n = 2$ the recurrence equation gives

$$3a_3 + a_1 + 2a_2 = 0$$

Which after substituting the earlier terms found becomes

$$a_3 = -\frac{a_0}{3}$$

For $n = 3$ the recurrence equation gives

$$4a_4 + a_2 + 2a_3 = 0$$

Which after substituting the earlier terms found becomes

$$a_4 = -\frac{5a_0}{24}$$

For $n = 4$ the recurrence equation gives

$$5a_5 + a_3 + 2a_4 = 0$$

Which after substituting the earlier terms found becomes

$$a_5 = \frac{3a_0}{20}$$

For $n = 5$ the recurrence equation gives

$$6a_6 + a_4 + 2a_5 = 0$$

Which after substituting the earlier terms found becomes

$$a_6 = -\frac{11a_0}{720}$$

And so on. Therefore the solution is

$$\begin{aligned} y &= \sum_{n=0}^{\infty} a_n x^n \\ &= a_3 x^3 + a_2 x^2 + a_1 x + a_0 + \dots \end{aligned}$$

Substituting the values for a_n found above, the solution becomes

$$y = a_0 - 2a_0 x + \frac{3}{2} a_0 x^2 - \frac{1}{3} a_0 x^3 - \frac{5}{24} a_0 x^4 + \frac{3}{20} a_0 x^5 + \dots$$

Collecting terms, the solution becomes

$$y = \left(1 - 2x + \frac{3}{2}x^2 - \frac{1}{3}x^3 - \frac{5}{24}x^4 + \frac{3}{20}x^5\right) a_0 + O(x^6) \tag{3}$$

Summary

The solution(s) found are the following

$$y = \left(1 - 2x + \frac{3}{2}x^2 - \frac{1}{3}x^3 - \frac{5}{24}x^4 + \frac{3}{20}x^5\right) y(0) + O(x^6) \tag{1}$$

$$y = \left(1 - 2x + \frac{3}{2}x^2 - \frac{1}{3}x^3 - \frac{5}{24}x^4 + \frac{3}{20}x^5\right) c_1 + O(x^6) \tag{2}$$

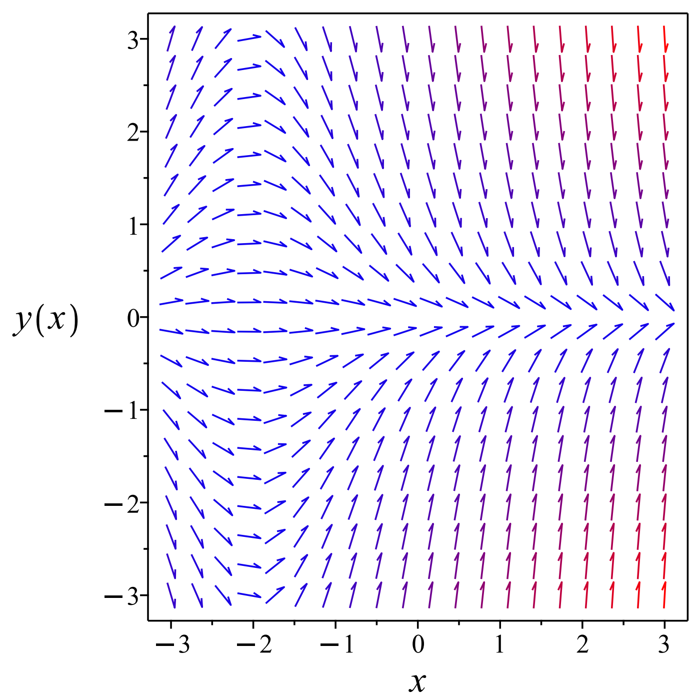


Figure 205: Slope field plot

Verification of solutions

$$y = \left(1 - 2x + \frac{3}{2}x^2 - \frac{1}{3}x^3 - \frac{5}{24}x^4 + \frac{3}{20}x^5\right) y(0) + O(x^6)$$

Verified OK.

$$y = \left(1 - 2x + \frac{3}{2}x^2 - \frac{1}{3}x^3 - \frac{5}{24}x^4 + \frac{3}{20}x^5\right) c_1 + O(x^6)$$

Verified OK.

5.10.2 Maple step by step solution

Let's solve

$y' + (x+2)\,y = 0$

- Highest derivative means the order of the ODE is 1

  y'

- Separate variables

  $\frac{y'}{y} = -x - 2$

- Integrate both sides with respect to x

  $\int \frac{y'}{y}dx = \int (-x-2)\,dx + c_1$

- Evaluate integral

  $\ln(y) = -\frac{1}{2}x^2 - 2x + c_1$

- Solve for y

  $y = \mathrm{e}^{-\frac{1}{2}x^2 - 2x + c_1}$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
<- 1st order linear successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 36

```
Order:=6;
dsolve(diff(y(x),x)+(x+2)*y(x)=0,y(x),type='series',x=0);
```

$$y(x) = \left(1 - 2x + \frac{3}{2}x^2 - \frac{1}{3}x^3 - \frac{5}{24}x^4 + \frac{3}{20}x^5\right) y(0) + O(x^6)$$

✓ Solution by Mathematica

Time used: 0.001 (sec). Leaf size: 39

```
AsymptoticDSolveValue[y'[x]+(x+2)*y[x]==0,y[x],{x,0,5}]
```

$$y(x) \to c_1 \left(\frac{3x^5}{20} - \frac{5x^4}{24} - \frac{x^3}{3} + \frac{3x^2}{2} - 2x + 1 \right)$$

5.11 problem 12

Internal problem ID [5012]
Internal file name [`OUTPUT/4505_Sunday_June_05_2022_02_59_44_PM_57457077/index.tex`]

Book: Fundamentals of Differential Equations. By Nagle, Saff and Snider. 9th edition. Boston. Pearson 2018.
Section: Chapter 8, Series solutions of differential equations. Section 8.3. page 443
Problem number: 12.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"first order ode series method. Ordinary point", "first order ode series method. Taylor series method"**

Maple gives the following as the ode type

```
[_quadrature]
```

$$\boxed{y' - y = 0}$$

With the expansion point for the power series method at $x = 0$.

5.11.1 Solving as series ode

Solving ode using Taylor series method. This gives review on how the Taylor series method works for solving first order ode. Let

$$y' = f(x, y)$$

Where $f(x, y)$ is analytic at expansion point x_0. We can always shift to $x_0 = 0$ if x_0 is not zero. So from now we assume $x_0 = 0$. Assume also that $y(x_0) = y_0$. Using Taylor series

$$\begin{aligned}
y(x) &= y(x_0) + (x - x_0)\, y'(x_0) + \frac{(x - x_0)^2}{2} y''(x_0) + \frac{(x - x_0)^3}{3!} y''(x_0) + \cdots \\
&= y_0 + xf + \frac{x^2}{2} \frac{df}{dx}\bigg|_{x_0,y_0} + \frac{x^3}{3!} \frac{d^2 f}{dx^2}\bigg|_{x_0,y_0} + \cdots \\
&= y_0 + \sum_{n=0}^{\infty} \frac{x^{n+1}}{(n+1)\,!} \frac{d^n f}{dx^n}\bigg|_{x_0,y_0}
\end{aligned}$$

But

$$\frac{df}{dx} = \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} f \tag{1}$$

$$\frac{d^2 f}{dx^2} = \frac{d}{dx}\left(\frac{df}{dx}\right)$$

$$= \frac{\partial}{\partial x}\left(\frac{df}{dx}\right) + \frac{\partial}{\partial y}\left(\frac{df}{dx}\right) f \tag{2}$$

$$\frac{d^3 f}{dx^3} = \frac{d}{dx}\left(\frac{d^2 f}{dx^2}\right)$$

$$= \frac{\partial}{\partial x}\left(\frac{d^2 f}{dx^2}\right) + \left(\frac{\partial}{\partial y}\frac{d^2 f}{dx^2}\right) f \tag{3}$$

$$\vdots$$

And so on. Hence if we name $F_0 = f(x,y)$ then the above can be written as

$$F_0 = f(x,y) \tag{4}$$

$$F_n = \frac{d}{dx}(F_{n-1})$$

$$= \frac{\partial}{\partial x} F_{n-1} + \left(\frac{\partial F_{n-1}}{\partial y}\right) F_0 \tag{5}$$

For example, for $n = 1$ we see that

$$\begin{aligned} F_1 &= \frac{d}{dx}(F_0) \\ &= \frac{\partial}{\partial x} F_0 + \left(\frac{\partial F_0}{\partial y}\right) F_0 \\ &= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} f \end{aligned}$$

Which is (1). And when $n = 2$

$$\begin{aligned} F_2 &= \frac{d}{dx}(F_1) \\ &= \frac{\partial}{\partial x} F_1 + \left(\frac{\partial F_1}{\partial y}\right) F_0 \\ &= \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} f\right) + \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} f\right) f \\ &= \frac{\partial}{\partial x}\left(\frac{df}{dx}\right) + \frac{\partial}{\partial y}\left(\frac{df}{dx}\right) f \end{aligned}$$

Which is (2) and so on. Therefore (4,5) can be used from now on along with

$$y(x) = y_0 + \sum_{n=0}^{\infty} \frac{x^{n+1}}{(n+1)\,!} F_n|_{x_0,y_0} \tag{6}$$

Hence

$$\begin{aligned}
F_0 &= y \\
F_1 &= \frac{dF_0}{dx} \\
&= \frac{\partial F_0}{\partial x} + \frac{\partial F_0}{\partial y} F_0 \\
&= y \\
F_2 &= \frac{dF_1}{dx} \\
&= \frac{\partial F_1}{\partial x} + \frac{\partial F_1}{\partial y} F_1 \\
&= y \\
F_3 &= \frac{dF_2}{dx} \\
&= \frac{\partial F_2}{\partial x} + \frac{\partial F_2}{\partial y} F_2 \\
&= y \\
F_4 &= \frac{dF_3}{dx} \\
&= \frac{\partial F_3}{\partial x} + \frac{\partial F_3}{\partial y} F_3 \\
&= y
\end{aligned}$$

And so on. Evaluating all the above at initial conditions $x(0) = 0$ and $y(0) = y(0)$ gives

$$\begin{aligned}
F_0 &= y(0) \\
F_1 &= y(0) \\
F_2 &= y(0) \\
F_3 &= y(0) \\
F_4 &= y(0)
\end{aligned}$$

Substituting all the above in (6) and simplifying gives the solution as

$$y = \left(1 + x + \frac{1}{2}x^2 + \frac{1}{6}x^3 + \frac{1}{24}x^4 + \frac{1}{120}x^5\right) y(0) + O(x^6)$$

Since $x = 0$ is also an ordinary point, then standard power series can also be used. Writing the ODE as

$$
\begin{aligned}
y' + q(x)y &= p(x) \\
y' - y &= 0
\end{aligned}
$$

Where

$$
\begin{aligned}
q(x) &= -1 \\
p(x) &= 0
\end{aligned}
$$

Next, the type of the expansion point $x = 0$ is determined. This point can be an ordinary point, a regular singular point (also called removable singularity), or irregular singular point (also called non-removable singularity or essential singularity). When $x = 0$ is an ordinary point, then the standard power series is used. If the point is a regular singular point, Frobenius series is used instead. Irregular singular point requires more advanced methods (asymptotic methods) and is not supported now. Hopefully this will be added in the future. $x = 0$ is called an ordinary point $q(x)$ has a Taylor series expansion around the point $x = 0$. $x = 0$ is called a regular singular point if $q(x)$ is not not analytic at $x = 0$ but $xq(x)$ has Taylor series expansion. And finally, $x = 0$ is an irregular singular point if the point is not ordinary and not regular singular. This is the most complicated case. Now the expansion point $x = 0$ is checked to see if it is an ordinary point or not. Let the solution be represented as power series of the form

$$
y = \sum_{n=0}^{\infty} a_n x^n
$$

Then

$$
y' = \sum_{n=1}^{\infty} n a_n x^{n-1}
$$

Substituting the above back into the ode gives

$$
\left(\sum_{n=1}^{\infty} n a_n x^{n-1}\right) - \left(\sum_{n=0}^{\infty} a_n x^n\right) = 0 \tag{1}
$$

Which simplifies to

$$
\left(\sum_{n=1}^{\infty} n a_n x^{n-1}\right) + \sum_{n=0}^{\infty} (-a_n x^n) = 0 \tag{2}
$$

The next step is to make all powers of x be n in each summation term. Going over each summation term above with power of x in it which is not already x^n and adjusting the power and the corresponding index gives

$$\sum_{n=1}^{\infty} n a_n x^{n-1} = \sum_{n=0}^{\infty} (n+1) a_{n+1} x^n$$

Substituting all the above in Eq (2) gives the following equation where now all powers of x are the same and equal to n.

$$\left(\sum_{n=0}^{\infty} (n+1) a_{n+1} x^n\right) + \sum_{n=0}^{\infty} (-a_n x^n) = 0 \tag{3}$$

For $0 \leq n$, the recurrence equation is

$$(n+1) a_{n+1} - a_n = 0 \tag{4}$$

Solving for a_{n+1}, gives

$$a_{n+1} = \frac{a_n}{n+1} \tag{5}$$

For $n = 0$ the recurrence equation gives

$$a_1 - a_0 = 0$$

Which after substituting the earlier terms found becomes

$$a_1 = a_0$$

For $n = 1$ the recurrence equation gives

$$2a_2 - a_1 = 0$$

Which after substituting the earlier terms found becomes

$$a_2 = \frac{a_0}{2}$$

For $n = 2$ the recurrence equation gives

$$3a_3 - a_2 = 0$$

Which after substituting the earlier terms found becomes

$$a_3 = \frac{a_0}{6}$$

For $n = 3$ the recurrence equation gives

$$4a_4 - a_3 = 0$$

Which after substituting the earlier terms found becomes

$$a_4 = \frac{a_0}{24}$$

For $n = 4$ the recurrence equation gives

$$5a_5 - a_4 = 0$$

Which after substituting the earlier terms found becomes

$$a_5 = \frac{a_0}{120}$$

For $n = 5$ the recurrence equation gives

$$6a_6 - a_5 = 0$$

Which after substituting the earlier terms found becomes

$$a_6 = \frac{a_0}{720}$$

And so on. Therefore the solution is

$$\begin{aligned} y &= \sum_{n=0}^{\infty} a_n x^n \\ &= a_3 x^3 + a_2 x^2 + a_1 x + a_0 + \dots \end{aligned}$$

Substituting the values for a_n found above, the solution becomes

$$y = a_0 + a_0 x + \frac{1}{2} a_0 x^2 + \frac{1}{6} a_0 x^3 + \frac{1}{24} a_0 x^4 + \frac{1}{120} a_0 x^5 + \dots$$

Collecting terms, the solution becomes

$$y = \left(1 + x + \frac{1}{2}x^2 + \frac{1}{6}x^3 + \frac{1}{24}x^4 + \frac{1}{120}x^5\right) a_0 + O(x^6) \tag{3}$$

Summary

The solution(s) found are the following

$$y = \left(1 + x + \frac{1}{2}x^2 + \frac{1}{6}x^3 + \frac{1}{24}x^4 + \frac{1}{120}x^5\right) y(0) + O(x^6) \tag{1}$$

$$y = \left(1 + x + \frac{1}{2}x^2 + \frac{1}{6}x^3 + \frac{1}{24}x^4 + \frac{1}{120}x^5\right) c_1 + O(x^6) \tag{2}$$

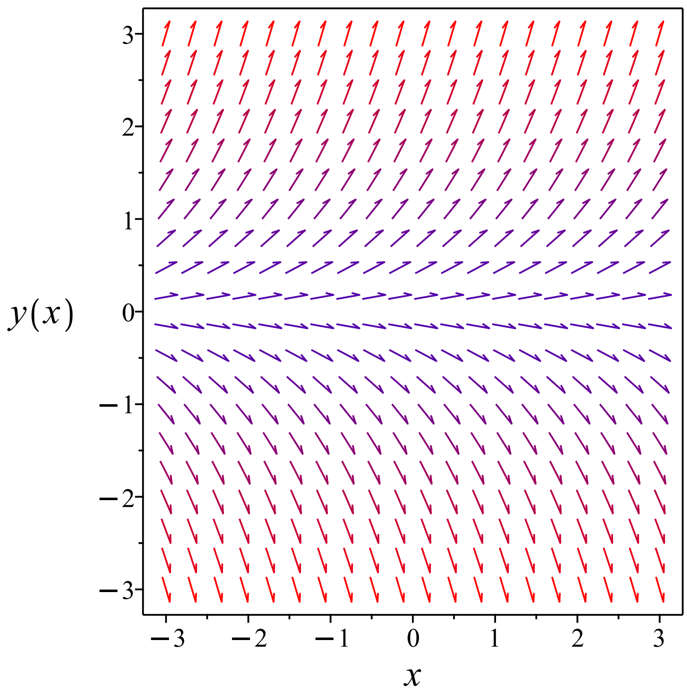


Figure 206: Slope field plot

Verification of solutions

$$y = \left(1 + x + \frac{1}{2}x^2 + \frac{1}{6}x^3 + \frac{1}{24}x^4 + \frac{1}{120}x^5\right) y(0) + O(x^6)$$

Verified OK.

$$y = \left(1 + x + \frac{1}{2}x^2 + \frac{1}{6}x^3 + \frac{1}{24}x^4 + \frac{1}{120}x^5\right) c_1 + O(x^6)$$

Verified OK.

5.11.2 Maple step by step solution

Let's solve

$y' - y = 0$

- Highest derivative means the order of the ODE is 1

  y'

- Separate variables

  $\frac{y'}{y} = 1$

- Integrate both sides with respect to x

  $\int \frac{y'}{y}dx = \int 1dx + c_1$

- Evaluate integral

  $\ln(y) = x + c_1$

- Solve for y

  $y = e^{x+c_1}$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
<- 1st order linear successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 34

```
Order:=6;
dsolve(diff(y(x),x)-y(x)=0,y(x),type='series',x=0);
```

$$y(x) = \left(1 + x + \frac{1}{2}x^2 + \frac{1}{6}x^3 + \frac{1}{24}x^4 + \frac{1}{120}x^5\right) y(0) + O(x^6)$$

✓ Solution by Mathematica

Time used: 0.001 (sec). Leaf size: 37

```
AsymptoticDSolveValue[y'[x]-y[x]==0,y[x],{x,0,5}]
```

$$y(x) \to c_1 \left(\frac{x^5}{120} + \frac{x^4}{24} + \frac{x^3}{6} + \frac{x^2}{2} + x + 1\right)$$

5.12 problem 13

Internal problem ID [5013]
Internal file name [OUTPUT/4506_Sunday_June_05_2022_02_59_45_PM_49606762/index.tex]

Book: Fundamentals of Differential Equations. By Nagle, Saff and Snider. 9th edition. Boston. Pearson 2018.
Section: Chapter 8, Series solutions of differential equations. Section 8.3. page 443
Problem number: 13.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"first order ode series method. Ordinary point", "first order ode series method. Taylor series method"**

Maple gives the following as the ode type

```
[_separable]
```

$$\boxed{z' - x^2 z = 0}$$

With the expansion point for the power series method at $x = 0$.

5.12.1 Solving as series ode

Solving ode using Taylor series method. This gives review on how the Taylor series method works for solving first order ode. Let

$$y' = f(x, y)$$

Where $f(x, y)$ is analytic at expansion point x_0. We can always shift to $x_0 = 0$ if x_0 is not zero. So from now we assume $x_0 = 0$. Assume also that $y(x_0) = y_0$. Using Taylor series

$$\begin{aligned} y(x) &= y(x_0) + (x - x_0)\, y'(x_0) + \frac{(x - x_0)^2}{2} y''(x_0) + \frac{(x - x_0)^3}{3!} y''(x_0) + \cdots \\ &= y_0 + xf + \frac{x^2}{2} \frac{df}{dx}\bigg|_{x_0, y_0} + \frac{x^3}{3!} \frac{d^2 f}{dx^2}\bigg|_{x_0, y_0} + \cdots \\ &= y_0 + \sum_{n=0}^{\infty} \frac{x^{n+1}}{(n+1)!} \frac{d^n f}{dx^n}\bigg|_{x_0, y_0} \end{aligned}$$

But

$$\frac{df}{dx} = \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} f \tag{1}$$

$$\frac{d^2 f}{dx^2} = \frac{d}{dx}\left(\frac{df}{dx}\right)$$

$$= \frac{\partial}{\partial x}\left(\frac{df}{dx}\right) + \frac{\partial}{\partial y}\left(\frac{df}{dx}\right) f \tag{2}$$

$$\frac{d^3 f}{dx^3} = \frac{d}{dx}\left(\frac{d^2 f}{dx^2}\right)$$

$$= \frac{\partial}{\partial x}\left(\frac{d^2 f}{dx^2}\right) + \left(\frac{\partial}{\partial y}\frac{d^2 f}{dx^2}\right) f \tag{3}$$

$$\vdots$$

And so on. Hence if we name $F_0 = f(x,y)$ then the above can be written as

$$F_0 = f(x,y) \tag{4}$$

$$F_n = \frac{d}{dx}(F_{n-1})$$

$$= \frac{\partial}{\partial x} F_{n-1} + \left(\frac{\partial F_{n-1}}{\partial y}\right) F_0 \tag{5}$$

For example, for $n = 1$ we see that

$$F_1 = \frac{d}{dx}(F_0)$$

$$= \frac{\partial}{\partial x} F_0 + \left(\frac{\partial F_0}{\partial y}\right) F_0$$

$$= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} f$$

Which is (1). And when $n = 2$

$$F_2 = \frac{d}{dx}(F_1)$$

$$= \frac{\partial}{\partial x} F_1 + \left(\frac{\partial F_1}{\partial y}\right) F_0$$

$$= \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} f\right) + \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} f\right) f$$

$$= \frac{\partial}{\partial x}\left(\frac{df}{dx}\right) + \frac{\partial}{\partial y}\left(\frac{df}{dx}\right) f$$

Which is (2) and so on. Therefore (4,5) can be used from now on along with

$$y(x) = y_0 + \sum_{n=0}^{\infty} \frac{x^{n+1}}{(n+1)!} F_n|_{x_0,y_0} \tag{6}$$

Hence

$$\begin{aligned}
F_0 &= x^2 z \\
F_1 &= \frac{dF_0}{dx} \\
&= \frac{\partial F_0}{\partial x} + \frac{\partial F_0}{\partial z} F_0 \\
&= xz(x^3 + 2) \\
F_2 &= \frac{dF_1}{dx} \\
&= \frac{\partial F_1}{\partial x} + \frac{\partial F_1}{\partial z} F_1 \\
&= z(x^6 + 6x^3 + 2) \\
F_3 &= \frac{dF_2}{dx} \\
&= \frac{\partial F_2}{\partial x} + \frac{\partial F_2}{\partial z} F_2 \\
&= x^2 z(x^6 + 12x^3 + 20) \\
F_4 &= \frac{dF_3}{dx} \\
&= \frac{\partial F_3}{\partial x} + \frac{\partial F_3}{\partial z} F_3 \\
&= xz(x^9 + 20x^6 + 80x^3 + 40)
\end{aligned}$$

And so on. Evaluating all the above at initial conditions $x(0) = 0$ and $z(0) = z(0)$ gives

$$\begin{aligned}
F_0 &= 0 \\
F_1 &= 0 \\
F_2 &= 2z(0) \\
F_3 &= 0 \\
F_4 &= 0
\end{aligned}$$

Substituting all the above in (6) and simplifying gives the solution as

$$z = \left(1 + \frac{x^3}{3}\right) z(0) + O(x^6)$$

Since $x = 0$ is also an ordinary point, then standard power series can also be used. Writing the ODE as

$$
\begin{aligned}
z' + q(x)z &= p(x) \\
z' - x^2 z &= 0
\end{aligned}
$$

Where

$$
\begin{aligned}
q(x) &= -x^2 \\
p(x) &= 0
\end{aligned}
$$

Next, the type of the expansion point $x = 0$ is determined. This point can be an ordinary point, a regular singular point (also called removable singularity), or irregular singular point (also called non-removable singularity or essential singularity). When $x = 0$ is an ordinary point, then the standard power series is used. If the point is a regular singular point, Frobenius series is used instead. Irregular singular point requires more advanced methods (asymptotic methods) and is not supported now. Hopefully this will be added in the future. $x = 0$ is called an ordinary point $q(x)$ has a Taylor series expansion around the point $x = 0$. $x = 0$ is called a regular singular point if $q(x)$ is not not analytic at $x = 0$ but $xq(x)$ has Taylor series expansion. And finally, $x = 0$ is an irregular singular point if the point is not ordinary and not regular singular. This is the most complicated case. Now the expansion point $x = 0$ is checked to see if it is an ordinary point or not. Let the solution be represented as power series of the form

$$
z = \sum_{n=0}^{\infty} a_n x^n
$$

Then

$$
z' = \sum_{n=1}^{\infty} n a_n x^{n-1}
$$

Substituting the above back into the ode gives

$$
\left(\sum_{n=1}^{\infty} n a_n x^{n-1}\right) - x^2 \left(\sum_{n=0}^{\infty} a_n x^n\right) = 0 \tag{1}
$$

Which simplifies to

$$
\left(\sum_{n=1}^{\infty} n a_n x^{n-1}\right) + \sum_{n=0}^{\infty} \left(-x^{n+2} a_n\right) = 0 \tag{2}
$$

The next step is to make all powers of x be n in each summation term. Going over each summation term above with power of x in it which is not already x^n and adjusting the power and the corresponding index gives

$$
\begin{aligned}
\sum_{n=1}^{\infty} n a_n x^{n-1} &= \sum_{n=0}^{\infty} (n+1) a_{n+1} x^n \\
\sum_{n=0}^{\infty} \left(-x^{n+2} a_n\right) &= \sum_{n=2}^{\infty} \left(-a_{n-2} x^n\right)
\end{aligned}
$$

Substituting all the above in Eq (2) gives the following equation where now all powers of x are the same and equal to n.

$$
\left(\sum_{n=0}^{\infty} (n+1) a_{n+1} x^n\right) + \sum_{n=2}^{\infty} \left(-a_{n-2} x^n\right) = 0 \tag{3}
$$

For $2 \leq n$, the recurrence equation is

$$
(n+1) a_{n+1} - a_{n-2} = 0 \tag{4}
$$

Solving for a_{n+1}, gives

$$
a_{n+1} = \frac{a_{n-2}}{n+1} \tag{5}
$$

For $n = 2$ the recurrence equation gives

$$
3a_3 - a_0 = 0
$$

Which after substituting the earlier terms found becomes

$$
a_3 = \frac{a_0}{3}
$$

For $n = 3$ the recurrence equation gives

$$
4a_4 - a_1 = 0
$$

Which after substituting the earlier terms found becomes

$$
a_4 = 0
$$

For $n = 4$ the recurrence equation gives

$$5a_5 - a_2 = 0$$

Which after substituting the earlier terms found becomes

$$a_5 = 0$$

For $n = 5$ the recurrence equation gives

$$6a_6 - a_3 = 0$$

Which after substituting the earlier terms found becomes

$$a_6 = \frac{a_0}{18}$$

And so on. Therefore the solution is

$$\begin{aligned} z &= \sum_{n=0}^{\infty} a_n x^n \\ &= a_3 x^3 + a_2 x^2 + a_1 x + a_0 + \dots \end{aligned}$$

Substituting the values for a_n found above, the solution becomes

$$z = a_0 + \frac{1}{3} a_0 x^3 + \dots$$

Collecting terms, the solution becomes

$$z = \left(1 + \frac{x^3}{3}\right) a_0 + O(x^6) \tag{3}$$

Summary
The solution(s) found are the following

$$z = \left(1 + \frac{x^3}{3}\right) z(0) + O(x^6) \tag{1}$$

$$z = \left(1 + \frac{x^3}{3}\right) c_1 + O(x^6) \tag{2}$$

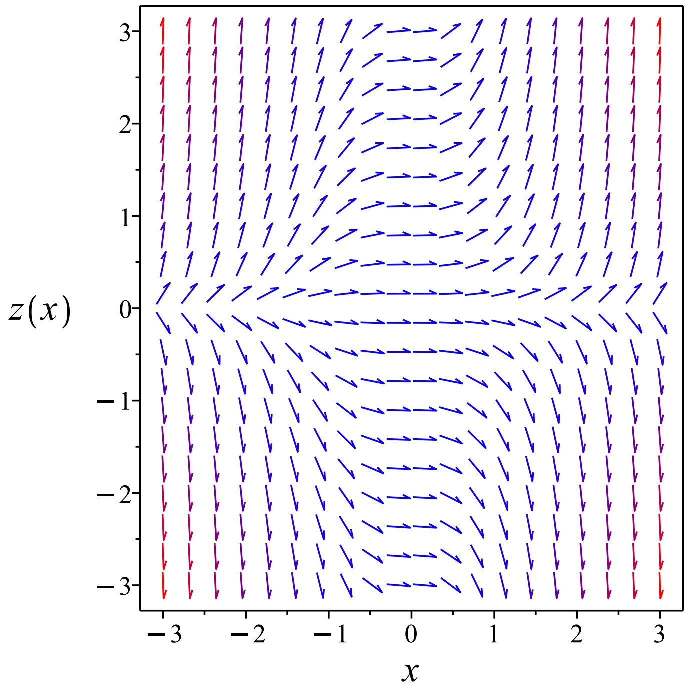


Figure 207: Slope field plot

Verification of solutions

$$z = \left(1 + \frac{x^3}{3}\right) z(0) + O(x^6)$$

Verified OK.

$$z = \left(1 + \frac{x^3}{3}\right) c_1 + O(x^6)$$

Verified OK.

5.12.2 Maple step by step solution

Let's solve

$z' - x^2 z = 0$

- Highest derivative means the order of the ODE is 1

  z'

- Separate variables

  $\frac{z'}{z} = x^2$

- Integrate both sides with respect to x

  $\int \frac{z'}{z} dx = \int x^2 dx + c_1$

- Evaluate integral

  $\ln(z) = \frac{x^3}{3} + c_1$

- Solve for z

  $z = e^{\frac{x^3}{3} + c_1}$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
<- 1st order linear successful`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 16

```
Order:=6;
dsolve(diff(z(x),x)-x^2*z(x)=0,z(x),type='series',x=0);
```

$$z(x) = \left(1 + \frac{x^3}{3}\right) z(0) + O(x^6)$$

✓ Solution by Mathematica

Time used: 0.001 (sec). Leaf size: 15

```
AsymptoticDSolveValue[z'[x]-x^2*z[x]==0,z[x],{x,0,5}]
```

$$z(x) \to c_1 \left(\frac{x^3}{3} + 1\right)$$

5.13 problem 14

Internal problem ID [5014]
Internal file name [`OUTPUT/4507_Sunday_June_05_2022_02_59_46_PM_68428205/index.tex`]

Book: Fundamentals of Differential Equations. By Nagle, Saff and Snider. 9th edition. Boston. Pearson 2018.
Section: Chapter 8, Series solutions of differential equations. Section 8.3. page 443
Problem number: 14.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second order series method. Ordinary point", "second order series method. Taylor series method"**

Maple gives the following as the ode type

```
[[_Emden, _Fowler]]
```

$$\boxed{(x^2+1)\,y''+y=0}$$

With the expansion point for the power series method at $x = 0$.

Solving ode using Taylor series method. This gives review on how the Taylor series method works for solving second order ode.

Let

$$y'' = f(x, y, y')$$

Assuming expansion is at $x_0 = 0$ (we can always shift the actual expansion point to 0 by change of variables) and assuming $f(x, y, y')$ is analytic at x_0 which must be the case for an ordinary point. Let initial conditions be $y(x_0) = y_0$ and $y'(x_0) = y'_0$. Using Taylor series gives

$$\begin{aligned} y(x) &= y(x_0) + (x - x_0)\,y'(x_0) + \frac{(x - x_0)^2}{2}y''(x_0) + \frac{(x - x_0)^3}{3!}y'''(x_0) + \cdots \\ &= y_0 + xy'_0 + \frac{x^2}{2}\,f|_{x_0,y_0,y'_0} + \frac{x^3}{3!}\,f'|_{x_0,y_0,y'_0} + \cdots \\ &= y_0 + xy'_0 + \sum_{n=0}^{\infty} \frac{x^{n+2}}{(n+2)\,!}\,\frac{d^n f}{dx^n}\bigg|_{x_0,y_0,y'_0} \end{aligned}$$

But

$$\begin{aligned}
\frac{df}{dx} &= \frac{\partial f}{\partial x}\frac{dx}{dx} + \frac{\partial f}{\partial y}\frac{dy}{dx} + \frac{\partial f}{\partial y'}\frac{dy'}{dx} && (1)\\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}y' + \frac{\partial f}{\partial y'}y'' && (243)\\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}y' + \frac{\partial f}{\partial y'}f && (244)\\
\frac{d^2 f}{dx^2} &= \frac{d}{dx}\left(\frac{df}{dx}\right)\\
&= \frac{\partial}{\partial x}\left(\frac{df}{dx}\right) + \frac{\partial}{\partial y}\left(\frac{df}{dx}\right)y' + \frac{\partial}{\partial y'}\left(\frac{df}{dx}\right)f && (2)\\
\frac{d^3 f}{dx^3} &= \frac{d}{dx}\left(\frac{d^2 f}{dx^2}\right)\\
&= \frac{\partial}{\partial x}\left(\frac{d^2 f}{dx^2}\right) + \left(\frac{\partial}{\partial y}\frac{d^2 f}{dx^2}\right)y' + \frac{\partial}{\partial y'}\left(\frac{d^2 f}{dx^2}\right)f && (3)\\
&\vdots
\end{aligned}$$

And so on. Hence if we name $F_0 = f(x, y, y')$ then the above can be written as

$$
\begin{aligned}
F_0 &= f(x, y, y') \qquad (4)\\
F_1 &= \frac{df}{dx}\\
&= \frac{dF_0}{dx}\\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}y' + \frac{\partial f}{\partial y'}y''\\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}y' + \frac{\partial f}{\partial y'}f \qquad (5)\\
&= \frac{\partial F_0}{\partial x} + \frac{\partial F_0}{\partial y}y' + \frac{\partial F_0}{\partial y'}F_0\\
F_2 &= \frac{d}{dx}\left(\frac{d}{dx}f\right)\\
&= \frac{d}{dx}(F_1)\\
&= \frac{\partial}{\partial x}F_1 + \left(\frac{\partial F_1}{\partial y}\right)y' + \left(\frac{\partial F_1}{\partial y'}\right)y''\\
&= \frac{\partial}{\partial x}F_1 + \left(\frac{\partial F_1}{\partial y}\right)y' + \left(\frac{\partial F_1}{\partial y'}\right)F_0\\
&\vdots\\
F_n &= \frac{d}{dx}(F_{n-1})\\
&= \frac{\partial}{\partial x}F_{n-1} + \left(\frac{\partial F_{n-1}}{\partial y}\right)y' + \left(\frac{\partial F_{n-1}}{\partial y'}\right)y''\\
&= \frac{\partial}{\partial x}F_{n-1} + \left(\frac{\partial F_{n-1}}{\partial y}\right)y' + \left(\frac{\partial F_{n-1}}{\partial y'}\right)F_0 \qquad (6)
\end{aligned}
$$

Therefore (6) can be used from now on along with

$$
y(x) = y_0 + xy_0' + \sum_{n=0}^{\infty} \frac{x^{n+2}}{(n+2)!} F_n|_{x_0, y_0, y_0'} \qquad (7)
$$

To find $y(x)$ series solution around $x=0$. Hence

$$
\begin{aligned}
F_0 &= -\frac{y}{x^2+1} \\
F_1 &= \frac{dF_0}{dx} \\
&= \frac{\partial F_0}{\partial x} + \frac{\partial F_0}{\partial y} y' + \frac{\partial F_0}{\partial y'} F_0 \\
&= \frac{-x^2 y' + 2xy - y'}{(x^2+1)^2} \\
F_2 &= \frac{dF_1}{dx} \\
&= \frac{\partial F_1}{\partial x} + \frac{\partial F_1}{\partial y} y' + \frac{\partial F_1}{\partial y'} F_1 \\
&= \frac{4y'x^3 - 5yx^2 + 4xy' + 3y}{(x^2+1)^3} \\
F_3 &= \frac{dF_2}{dx} \\
&= \frac{\partial F_2}{\partial x} + \frac{\partial F_2}{\partial y} y' + \frac{\partial F_2}{\partial y'} F_2 \\
&= \frac{(-17x^4 - 10x^2 + 7)\, y' + 16xy(x^2-2)}{(x^2+1)^4} \\
F_4 &= \frac{dF_3}{dx} \\
&= \frac{\partial F_3}{\partial x} + \frac{\partial F_3}{\partial y} y' + \frac{\partial F_3}{\partial y'} F_3 \\
&= \frac{(84x^5 - 24x^3 - 108x)\, y' + (-63x^4 + 282x^2 - 39)\, y}{(x^2+1)^5}
\end{aligned}
$$

And so on. Evaluating all the above at initial conditions $x=0$ and $y(0)=y(0)$ and $y'(0)=y'(0)$ gives

$$
\begin{aligned}
F_0 &= -y(0) \\
F_1 &= -y'(0) \\
F_2 &= 3y(0) \\
F_3 &= 7y'(0) \\
F_4 &= -39y(0)
\end{aligned}
$$

Substituting all the above in (7) and simplifying gives the solution as

$$
y = \left(1 - \frac{1}{2}x^2 + \frac{1}{8}x^4 - \frac{13}{240}x^6\right) y(0) + \left(x - \frac{1}{6}x^3 + \frac{7}{120}x^5\right) y'(0) + O(x^6)
$$

Since the expansion point $x = 0$ is an ordinary, we can also solve this using standard power series The ode is normalized to be

$$(x^2 + 1) y'' + y = 0$$

Let the solution be represented as power series of the form

$$y = \sum_{n=0}^{\infty} a_n x^n$$

Then

$$y' = \sum_{n=1}^{\infty} n a_n x^{n-1}$$
$$y'' = \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2}$$

Substituting the above back into the ode gives

$$(x^2 + 1) \left(\sum_{n=2}^{\infty} n(n-1) a_n x^{n-2}\right) + \left(\sum_{n=0}^{\infty} a_n x^n\right) = 0 \tag{1}$$

Which simplifies to

$$\left(\sum_{n=2}^{\infty} x^n a_n n(n-1)\right) + \left(\sum_{n=2}^{\infty} n(n-1) a_n x^{n-2}\right) + \left(\sum_{n=0}^{\infty} a_n x^n\right) = 0 \tag{2}$$

The next step is to make all powers of x be n in each summation term. Going over each summation term above with power of x in it which is not already x^n and adjusting the power and the corresponding index gives

$$\sum_{n=2}^{\infty} n(n-1) a_n x^{n-2} = \sum_{n=0}^{\infty} (n+2) a_{n+2}(n+1) x^n$$

Substituting all the above in Eq (2) gives the following equation where now all powers of x are the same and equal to n.

$$\left(\sum_{n=2}^{\infty} x^n a_n n(n-1)\right) + \left(\sum_{n=0}^{\infty} (n+2) a_{n+2}(n+1) x^n\right) + \left(\sum_{n=0}^{\infty} a_n x^n\right) = 0 \tag{3}$$

$n = 0$ gives

$$2a_2 + a_0 = 0$$

$$a_2 = -\frac{a_0}{2}$$

$n = 1$ gives

$$6a_3 + a_1 = 0$$

Which after substituting earlier equations, simplifies to

$$a_3 = -\frac{a_1}{6}$$

For $2 \leq n$, the recurrence equation is

$$na_n(n-1) + (n+2)\,a_{n+2}(n+1) + a_n = 0 \tag{4}$$

Solving for a_{n+2}, gives

$$a_{n+2} = -\frac{a_n(n^2 - n + 1)}{(n+2)\,(n+1)} \tag{5}$$

For $n = 2$ the recurrence equation gives

$$3a_2 + 12a_4 = 0$$

Which after substituting the earlier terms found becomes

$$a_4 = \frac{a_0}{8}$$

For $n = 3$ the recurrence equation gives

$$7a_3 + 20a_5 = 0$$

Which after substituting the earlier terms found becomes

$$a_5 = \frac{7a_1}{120}$$

For $n = 4$ the recurrence equation gives

$$13a_4 + 30a_6 = 0$$

Which after substituting the earlier terms found becomes

$$a_6 = -\frac{13a_0}{240}$$

For $n = 5$ the recurrence equation gives

$$21a_5 + 42a_7 = 0$$

Which after substituting the earlier terms found becomes

$$a_7 = -\frac{7a_1}{240}$$

And so on. Therefore the solution is

$$\begin{aligned} y &= \sum_{n=0}^{\infty} a_n x^n \\ &= a_3x^3 + a_2x^2 + a_1x + a_0 + \dots \end{aligned}$$

Substituting the values for a_n found above, the solution becomes

$$y = a_0 + a_1x - \frac{1}{2}a_0x^2 - \frac{1}{6}a_1x^3 + \frac{1}{8}a_0x^4 + \frac{7}{120}a_1x^5 + \dots$$

Collecting terms, the solution becomes

$$y = \left(1 - \frac{1}{2}x^2 + \frac{1}{8}x^4\right)a_0 + \left(x - \frac{1}{6}x^3 + \frac{7}{120}x^5\right)a_1 + O(x^6) \tag{3}$$

At $x = 0$ the solution above becomes

$$y = \left(1 - \frac{1}{2}x^2 + \frac{1}{8}x^4\right)c_1 + \left(x - \frac{1}{6}x^3 + \frac{7}{120}x^5\right)c_2 + O(x^6)$$

Summary

The solution(s) found are the following

$$y = \left(1 - \frac{1}{2}x^2 + \frac{1}{8}x^4 - \frac{13}{240}x^6\right)y(0) + \left(x - \frac{1}{6}x^3 + \frac{7}{120}x^5\right)y'(0) + O(x^6) \tag{1}$$

$$y = \left(1 - \frac{1}{2}x^2 + \frac{1}{8}x^4\right)c_1 + \left(x - \frac{1}{6}x^3 + \frac{7}{120}x^5\right)c_2 + O(x^6) \tag{2}$$

Verification of solutions

$$y = \left(1-\frac{1}{2}x^2+\frac{1}{8}x^4-\frac{13}{240}x^6\right)y(0)+\left(x-\frac{1}{6}x^3+\frac{7}{120}x^5\right)y'(0)+O(x^6)$$

Verified OK.

$$y = \left(1-\frac{1}{2}x^2+\frac{1}{8}x^4\right)c_1+\left(x-\frac{1}{6}x^3+\frac{7}{120}x^5\right)c_2+O(x^6)$$

Verified OK.

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
<- No Liouvillian solutions exists
-> Trying a solution in terms of special functions:
   -> Bessel
   -> elliptic
   -> Legendre
   -> Whittaker
      -> hyper3: Equivalence to 1F1 under a power @ Moebius
   -> hypergeometric
      -> heuristic approach
      -> hyper3: Equivalence to 2F1, 1F1 or 0F1 under a power @ Moebius
      <- hyper3 successful: received ODE is equivalent to the 2F1 ODE
   <- hypergeometric successful
<- special function solution successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 34

```
Order:=6;
dsolve((x^2+1)*diff(y(x),x$2)+y(x)=0,y(x),type='series',x=0);
```

$$y(x) = \left(1 - \frac{1}{2}x^2 + \frac{1}{8}x^4\right) y(0) + \left(x - \frac{1}{6}x^3 + \frac{7}{120}x^5\right) D(y)(0) + O(x^6)$$

✓ Solution by Mathematica

Time used: 0.001 (sec). Leaf size: 42

```
AsymptoticDSolveValue[(x^2+1)*y''[x]+y[x]==0,y[x],{x,0,5}]
```

$$y(x) \to c_2 \left(\frac{7x^5}{120} - \frac{x^3}{6} + x\right) + c_1 \left(\frac{x^4}{8} - \frac{x^2}{2} + 1\right)$$

5.14 problem 15

Internal problem ID [5015]
Internal file name [`OUTPUT/4508_Sunday_June_05_2022_02_59_47_PM_3126074/index.tex`]

Book: Fundamentals of Differential Equations. By Nagle, Saff and Snider. 9th edition. Boston. Pearson 2018.
Section: Chapter 8, Series solutions of differential equations. Section 8.3. page 443
Problem number: 15.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"exact linear second order ode", "second_order_integrable_as_is", "second order series method. Ordinary point", "second order series method. Taylor series method"**

Maple gives the following as the ode type

```
[[_2nd_order, _exact, _linear, _homogeneous]]
```

$$\boxed{y'' + (x-1)\,y' + y = 0}$$

With the expansion point for the power series method at $x = 0$.

Solving ode using Taylor series method. This gives review on how the Taylor series method works for solving second order ode.

Let

$$y'' = f(x, y, y')$$

Assuming expansion is at $x_0 = 0$ (we can always shift the actual expansion point to 0 by change of variables) and assuming $f(x, y, y')$ is analytic at x_0 which must be the case for an ordinary point. Let initial conditions be $y(x_0) = y_0$ and $y'(x_0) = y'_0$. Using Taylor series gives

$$\begin{aligned} y(x) &= y(x_0) + (x - x_0)\,y'(x_0) + \frac{(x-x_0)^2}{2}y''(x_0) + \frac{(x-x_0)^3}{3!}y'''(x_0) + \cdots \\ &= y_0 + xy'_0 + \frac{x^2}{2}f|_{x_0,y_0,y'_0} + \frac{x^3}{3!}f'|_{x_0,y_0,y'_0} + \cdots \\ &= y_0 + xy'_0 + \sum_{n=0}^{\infty}\frac{x^{n+2}}{(n+2)\,!}\left.\frac{d^n f}{dx^n}\right|_{x_0,y_0,y'_0} \end{aligned}$$

But

$$
\begin{aligned}
\frac{df}{dx} &= \frac{\partial f}{\partial x}\frac{dx}{dx} + \frac{\partial f}{\partial y}\frac{dy}{dx} + \frac{\partial f}{\partial y'}\frac{dy'}{dx} && (1)\\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}y' + \frac{\partial f}{\partial y'}y'' && (246)\\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}y' + \frac{\partial f}{\partial y'}f && (247)\\
\frac{d^2 f}{dx^2} &= \frac{d}{dx}\left(\frac{df}{dx}\right)\\
&= \frac{\partial}{\partial x}\left(\frac{df}{dx}\right) + \frac{\partial}{\partial y}\left(\frac{df}{dx}\right)y' + \frac{\partial}{\partial y'}\left(\frac{df}{dx}\right)f && (2)\\
\frac{d^3 f}{dx^3} &= \frac{d}{dx}\left(\frac{d^2 f}{dx^2}\right)\\
&= \frac{\partial}{\partial x}\left(\frac{d^2 f}{dx^2}\right) + \left(\frac{\partial}{\partial y}\frac{d^2 f}{dx^2}\right)y' + \frac{\partial}{\partial y'}\left(\frac{d^2 f}{dx^2}\right)f && (3)\\
&\vdots
\end{aligned}
$$

And so on. Hence if we name $F_0 = f(x,y,y')$ then the above can be written as

$$
\begin{aligned}
F_0 &= f(x,y,y') \qquad (4)\\
F_1 &= \frac{df}{dx}\\
&= \frac{dF_0}{dx}\\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}y' + \frac{\partial f}{\partial y'}y''\\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}y' + \frac{\partial f}{\partial y'}f \qquad (5)\\
&= \frac{\partial F_0}{\partial x} + \frac{\partial F_0}{\partial y}y' + \frac{\partial F_0}{\partial y'}F_0\\
F_2 &= \frac{d}{dx}\left(\frac{d}{dx}f\right)\\
&= \frac{d}{dx}(F_1)\\
&= \frac{\partial}{\partial x}F_1 + \left(\frac{\partial F_1}{\partial y}\right)y' + \left(\frac{\partial F_1}{\partial y'}\right)y''\\
&= \frac{\partial}{\partial x}F_1 + \left(\frac{\partial F_1}{\partial y}\right)y' + \left(\frac{\partial F_1}{\partial y'}\right)F_0\\
&\vdots\\
F_n &= \frac{d}{dx}(F_{n-1})\\
&= \frac{\partial}{\partial x}F_{n-1} + \left(\frac{\partial F_{n-1}}{\partial y}\right)y' + \left(\frac{\partial F_{n-1}}{\partial y'}\right)y''\\
&= \frac{\partial}{\partial x}F_{n-1} + \left(\frac{\partial F_{n-1}}{\partial y}\right)y' + \left(\frac{\partial F_{n-1}}{\partial y'}\right)F_0 \qquad (6)
\end{aligned}
$$

Therefore (6) can be used from now on along with

$$
y(x) = y_0 + xy_0' + \sum_{n=0}^{\infty} \frac{x^{n+2}}{(n+2)!} F_n|_{x_0,y_0,y_0'} \qquad (7)
$$

To find $y(x)$ series solution around $x = 0$. Hence

$$
\begin{aligned}
F_0 &= -xy' + y' - y \\
F_1 &= \frac{dF_0}{dx} \\
&= \frac{\partial F_0}{\partial x} + \frac{\partial F_0}{\partial y} y' + \frac{\partial F_0}{\partial y'} F_0 \\
&= y'\left(x^2 - 2x - 1\right) + (x - 1)\, y \\
F_2 &= \frac{dF_1}{dx} \\
&= \frac{\partial F_1}{\partial x} + \frac{\partial F_1}{\partial y} y' + \frac{\partial F_1}{\partial y'} F_1 \\
&= \left(-x^3 + 3x^2 + 2x - 4\right) y' - y\left(x^2 - 2x - 2\right) \\
F_3 &= \frac{dF_2}{dx} \\
&= \frac{\partial F_2}{\partial x} + \frac{\partial F_2}{\partial y} y' + \frac{\partial F_2}{\partial y'} F_2 \\
&= \left(x^4 - 4x^3 - 3x^2 + 14x\right) y' + y(x - 1)\left(x^2 - 2x - 6\right) \\
F_4 &= \frac{dF_3}{dx} \\
&= \frac{\partial F_3}{\partial x} + \frac{\partial F_3}{\partial y} y' + \frac{\partial F_3}{\partial y'} F_3 \\
&= \left(-x^5 + 5x^4 + 4x^3 - 32x^2 + 4x + 20\right) y' - y\left(x^4 - 4x^3 - 6x^2 + 20x + 4\right)
\end{aligned}
$$

And so on. Evaluating all the above at initial conditions $x = 0$ and $y(0) = y(0)$ and $y'(0) = y'(0)$ gives

$$
\begin{aligned}
F_0 &= -y(0) + y'(0) \\
F_1 &= -y'(0) - y(0) \\
F_2 &= -4y'(0) + 2y(0) \\
F_3 &= 6y(0) \\
F_4 &= 20y'(0) - 4y(0)
\end{aligned}
$$

Substituting all the above in (7) and simplifying gives the solution as

$$
\begin{aligned}
y = &\left(1 - \frac{1}{2}x^2 - \frac{1}{6}x^3 + \frac{1}{12}x^4 + \frac{1}{20}x^5 - \frac{1}{180}x^6\right) y(0) \\
&+ \left(x + \frac{1}{2}x^2 - \frac{1}{6}x^3 - \frac{1}{6}x^4 + \frac{1}{36}x^6\right) y'(0) + O\left(x^6\right)
\end{aligned}
$$

Since the expansion point $x = 0$ is an ordinary, we can also solve this using standard power series Let the solution be represented as power series of the form

$$y = \sum_{n=0}^{\infty} a_n x^n$$

Then

$$y' = \sum_{n=1}^{\infty} n a_n x^{n-1}$$
$$y'' = \sum_{n=2}^{\infty} n(n-1)\, a_n x^{n-2}$$

Substituting the above back into the ode gives

$$\sum_{n=2}^{\infty} n(n-1)\, a_n x^{n-2} = -x\left(\sum_{n=1}^{\infty} n a_n x^{n-1}\right) + \left(\sum_{n=1}^{\infty} n a_n x^{n-1}\right) - \left(\sum_{n=0}^{\infty} a_n x^n\right) \tag{1}$$

Which simplifies to

$$\left(\sum_{n=2}^{\infty} n(n-1)\, a_n x^{n-2}\right) + \left(\sum_{n=1}^{\infty} n a_n x^n\right) + \sum_{n=1}^{\infty} \left(-n a_n x^{n-1}\right) + \left(\sum_{n=0}^{\infty} a_n x^n\right) = 0 \tag{2}$$

The next step is to make all powers of x be n in each summation term. Going over each summation term above with power of x in it which is not already x^n and adjusting the power and the corresponding index gives

$$\sum_{n=2}^{\infty} n(n-1)\, a_n x^{n-2} = \sum_{n=0}^{\infty} (n+2)\, a_{n+2}(n+1)\, x^n$$
$$\sum_{n=1}^{\infty} \left(-n a_n x^{n-1}\right) = \sum_{n=0}^{\infty} \left(-(n+1)\, a_{n+1} x^n\right)$$

Substituting all the above in Eq (2) gives the following equation where now all powers of x are the same and equal to n.

$$\left(\sum_{n=0}^{\infty} (n+2)\, a_{n+2}(n+1)\, x^n\right) + \left(\sum_{n=1}^{\infty} n a_n x^n\right) + \sum_{n=0}^{\infty} \left(-(n+1)\, a_{n+1} x^n\right) + \left(\sum_{n=0}^{\infty} a_n x^n\right) = 0 \tag{3}$$

$n = 0$ gives

$$2a_2 - a_1 + a_0 = 0$$

$$a_2 = -\frac{a_0}{2} + \frac{a_1}{2}$$

For $1 \leq n$, the recurrence equation is

$$(n+2)\, a_{n+2}(n+1) + n a_n - (n+1)\, a_{n+1} + a_n = 0 \tag{4}$$

Solving for a_{n+2}, gives

$$
\begin{aligned}
a_{n+2} &= -\frac{a_n - a_{n+1}}{n+2} \\
&= -\frac{a_n}{n+2} + \frac{a_{n+1}}{n+2}
\end{aligned} \tag{5}
$$

For $n = 1$ the recurrence equation gives

$$6a_3 + 2a_1 - 2a_2 = 0$$

Which after substituting the earlier terms found becomes

$$a_3 = -\frac{a_1}{6} - \frac{a_0}{6}$$

For $n = 2$ the recurrence equation gives

$$12a_4 + 3a_2 - 3a_3 = 0$$

Which after substituting the earlier terms found becomes

$$a_4 = \frac{a_0}{12} - \frac{a_1}{6}$$

For $n = 3$ the recurrence equation gives

$$20a_5 + 4a_3 - 4a_4 = 0$$

Which after substituting the earlier terms found becomes

$$a_5 = \frac{a_0}{20}$$

For $n = 4$ the recurrence equation gives

$$30a_6 + 5a_4 - 5a_5 = 0$$

Which after substituting the earlier terms found becomes

$$a_6 = -\frac{a_0}{180} + \frac{a_1}{36}$$

For $n = 5$ the recurrence equation gives

$$42a_7 + 6a_5 - 6a_6 = 0$$

Which after substituting the earlier terms found becomes

$$a_7 = -\frac{a_0}{126} + \frac{a_1}{252}$$

And so on. Therefore the solution is

$$\begin{aligned} y &= \sum_{n=0}^{\infty} a_n x^n \\ &= a_3 x^3 + a_2 x^2 + a_1 x + a_0 + \dots \end{aligned}$$

Substituting the values for a_n found above, the solution becomes

$$y = a_0 + a_1 x + \left(-\frac{a_0}{2} + \frac{a_1}{2}\right) x^2 + \left(-\frac{a_1}{6} - \frac{a_0}{6}\right) x^3 + \left(\frac{a_0}{12} - \frac{a_1}{6}\right) x^4 + \frac{a_0 x^5}{20} + \dots$$

Collecting terms, the solution becomes

$$y = \left(1 - \frac{1}{2}x^2 - \frac{1}{6}x^3 + \frac{1}{12}x^4 + \frac{1}{20}x^5\right) a_0 + \left(x + \frac{1}{2}x^2 - \frac{1}{6}x^3 - \frac{1}{6}x^4\right) a_1 + O(x^6) \quad (3)$$

At $x = 0$ the solution above becomes

$$y = \left(1 - \frac{1}{2}x^2 - \frac{1}{6}x^3 + \frac{1}{12}x^4 + \frac{1}{20}x^5\right) c_1 + \left(x + \frac{1}{2}x^2 - \frac{1}{6}x^3 - \frac{1}{6}x^4\right) c_2 + O(x^6)$$

Summary

The solution(s) found are the following

$$
\begin{aligned}
y = &\left(1-\frac{1}{2}x^{2}-\frac{1}{6}x^{3}+\frac{1}{12}x^{4}+\frac{1}{20}x^{5}-\frac{1}{180}x^{6}\right) y(0) \\
&+\left(x+\frac{1}{2}x^{2}-\frac{1}{6}x^{3}-\frac{1}{6}x^{4}+\frac{1}{36}x^{6}\right) y'(0)+O\left(x^{6}\right)
\end{aligned} \tag{1}
$$

$$
y = \left(1-\frac{1}{2}x^{2}-\frac{1}{6}x^{3}+\frac{1}{12}x^{4}+\frac{1}{20}x^{5}\right) c_{1}+\left(x+\frac{1}{2}x^{2}-\frac{1}{6}x^{3}-\frac{1}{6}x^{4}\right) c_{2}+O\left(x^{6}\right) \tag{2}
$$

Verification of solutions

$$
\begin{aligned}
y = &\left(1-\frac{1}{2}x^{2}-\frac{1}{6}x^{3}+\frac{1}{12}x^{4}+\frac{1}{20}x^{5}-\frac{1}{180}x^{6}\right) y(0) \\
&+\left(x+\frac{1}{2}x^{2}-\frac{1}{6}x^{3}-\frac{1}{6}x^{4}+\frac{1}{36}x^{6}\right) y'(0)+O\left(x^{6}\right)
\end{aligned}
$$

Verified OK.

$$
y = \left(1-\frac{1}{2}x^{2}-\frac{1}{6}x^{3}+\frac{1}{12}x^{4}+\frac{1}{20}x^{5}\right) c_{1}+\left(x+\frac{1}{2}x^{2}-\frac{1}{6}x^{3}-\frac{1}{6}x^{4}\right) c_{2}+O\left(x^{6}\right)
$$

Verified OK.

5.14.1 Maple step by step solution

Let's solve

$y'' = -xy' + y' - y$

- Highest derivative means the order of the ODE is 2

  y''

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y'' + (x-1)\, y' + y = 0$

- Assume series solution for y

  $y = \sum_{k=0}^{\infty} a_k x^k$

□ Rewrite DE with series expansions

  ○ Convert $x^m \cdot y'$ to series expansion for $m = 0..1$

  $x^m \cdot y' = \sum_{k=\max(0,1-m)}^{\infty} a_k k\, x^{k-1+m}$

- Shift index using $k->k+1-m$

  $$x^m \cdot y' = \sum_{k=\max(0,1-m)+m-1}^{\infty} a_{k+1-m}(k+1-m)\, x^k$$

- Convert y'' to series expansion

  $$y'' = \sum_{k=2}^{\infty} a_k k(k-1)\, x^{k-2}$$

- Shift index using $k->k+2$

  $$y'' = \sum_{k=0}^{\infty} a_{k+2}(k+2)\,(k+1)\, x^k$$

  Rewrite DE with series expansions

  $$\sum_{k=0}^{\infty} \left(a_{k+2}(k+2)\,(k+1) - a_{k+1}(k+1) + a_k(k+1)\right) x^k = 0$$

- Each term in the series must be 0, giving the recursion relation

  $$(k+1)\left(a_{k+2}(k+2) - a_{k+1} + a_k\right) = 0$$

- Recursion relation that defines the series solution to the ODE

  $$\left[y = \sum_{k=0}^{\infty} a_k x^k, a_{k+2} = -\frac{-a_{k+1}+a_k}{k+2}\right]$$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
<- linear_1 successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 49

```
Order:=6;
dsolve(diff(y(x),x$2)+(x-1)*diff(y(x),x)+y(x)=0,y(x),type='series',x=0);
```

$$y(x) = \left(1 - \frac{1}{2}x^2 - \frac{1}{6}x^3 + \frac{1}{12}x^4 + \frac{1}{20}x^5\right) y(0) + \left(x + \frac{1}{2}x^2 - \frac{1}{6}x^3 - \frac{1}{6}x^4\right) D(y)(0) + O\left(x^6\right)$$

✓ Solution by Mathematica

Time used: 0.001 (sec). Leaf size: 63

```
AsymptoticDSolveValue[y''[x]+(x-1)*y'[x]+y[x]==0,y[x],{x,0,5}]
```

$$y(x) \to c_2\left(-\frac{x^4}{6} - \frac{x^3}{6} + \frac{x^2}{2} + x\right) + c_1\left(\frac{x^5}{20} + \frac{x^4}{12} - \frac{x^3}{6} - \frac{x^2}{2} + 1\right)$$

5.15 problem 16

Internal problem ID [5016]
Internal file name [`OUTPUT/4509_Sunday_June_05_2022_02_59_48_PM_6380777/index.tex`]

Book: Fundamentals of Differential Equations. By Nagle, Saff and Snider. 9th edition. Boston. Pearson 2018.
Section: Chapter 8, Series solutions of differential equations. Section 8.3. page 443
Problem number: 16.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second_order_linear_constant_coeff", "second order series method. Ordinary point", "linear_second_order_ode_solved_by_an_integrating_factor", "second order series method. Taylor series method"**

Maple gives the following as the ode type

`[[_2nd_order, _missing_x]]`

$$\boxed{y'' - 2y' + y = 0}$$

With the expansion point for the power series method at $x = 0$.

Solving ode using Taylor series method. This gives review on how the Taylor series method works for solving second order ode.

Let

$$y'' = f(x, y, y')$$

Assuming expansion is at $x_0 = 0$ (we can always shift the actual expansion point to 0 by change of variables) and assuming $f(x, y, y')$ is analytic at x_0 which must be the case for an ordinary point. Let initial conditions be $y(x_0) = y_0$ and $y'(x_0) = y'_0$. Using

Taylor series gives

$$\begin{aligned}
y(x) &= y(x_0) + (x - x_0)\, y'(x_0) + \frac{(x - x_0)^2}{2} y''(x_0) + \frac{(x - x_0)^3}{3!} y'''(x_0) + \cdots \\
&= y_0 + x y_0' + \frac{x^2}{2} f|_{x_0, y_0, y_0'} + \frac{x^3}{3!} f'|_{x_0, y_0, y_0'} + \cdots \\
&= y_0 + x y_0' + \sum_{n=0}^{\infty} \frac{x^{n+2}}{(n+2)\,!} \frac{d^n f}{dx^n}\bigg|_{x_0, y_0, y_0'}
\end{aligned}$$

But

$$\begin{aligned}
\frac{df}{dx} &= \frac{\partial f}{\partial x}\frac{dx}{dx} + \frac{\partial f}{\partial y}\frac{dy}{dx} + \frac{\partial f}{\partial y'}\frac{dy'}{dx} && (1)\\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} y' + \frac{\partial f}{\partial y'} y'' && (249)\\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} y' + \frac{\partial f}{\partial y'} f && (250)\\
\frac{d^2 f}{dx^2} &= \frac{d}{dx}\left(\frac{df}{dx}\right) \\
&= \frac{\partial}{\partial x}\left(\frac{df}{dx}\right) + \frac{\partial}{\partial y}\left(\frac{df}{dx}\right) y' + \frac{\partial}{\partial y'}\left(\frac{df}{dx}\right) f && (2)\\
\frac{d^3 f}{dx^3} &= \frac{d}{dx}\left(\frac{d^2 f}{dx^2}\right) \\
&= \frac{\partial}{\partial x}\left(\frac{d^2 f}{dx^2}\right) + \left(\frac{\partial}{\partial y}\frac{d^2 f}{dx^2}\right) y' + \frac{\partial}{\partial y'}\left(\frac{d^2 f}{dx^2}\right) f && (3)\\
&\vdots
\end{aligned}$$

And so on. Hence if we name $F_0 = f(x, y, y')$ then the above can be written as

$$\begin{aligned}
F_0 &= f(x, y, y') \qquad (4)\\
F_1 &= \frac{df}{dx}\\
&= \frac{dF_0}{dx}\\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} y' + \frac{\partial f}{\partial y'} y''\\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} y' + \frac{\partial f}{\partial y'} f \qquad (5)\\
&= \frac{\partial F_0}{\partial x} + \frac{\partial F_0}{\partial y} y' + \frac{\partial F_0}{\partial y'} F_0\\
F_2 &= \frac{d}{dx}\left(\frac{d}{dx} f\right)\\
&= \frac{d}{dx}(F_1)\\
&= \frac{\partial}{\partial x} F_1 + \left(\frac{\partial F_1}{\partial y}\right) y' + \left(\frac{\partial F_1}{\partial y'}\right) y''\\
&= \frac{\partial}{\partial x} F_1 + \left(\frac{\partial F_1}{\partial y}\right) y' + \left(\frac{\partial F_1}{\partial y'}\right) F_0\\
&\vdots\\
F_n &= \frac{d}{dx}(F_{n-1})\\
&= \frac{\partial}{\partial x} F_{n-1} + \left(\frac{\partial F_{n-1}}{\partial y}\right) y' + \left(\frac{\partial F_{n-1}}{\partial y'}\right) y''\\
&= \frac{\partial}{\partial x} F_{n-1} + \left(\frac{\partial F_{n-1}}{\partial y}\right) y' + \left(\frac{\partial F_{n-1}}{\partial y'}\right) F_0 \qquad (6)
\end{aligned}$$

Therefore (6) can be used from now on along with

$$y(x) = y_0 + x y_0' + \sum_{n=0}^{\infty} \frac{x^{n+2}}{(n+2)!} F_n\big|_{x_0, y_0, y_0'} \qquad (7)$$

To find $y(x)$ series solution around $x = 0$. Hence

$$
\begin{aligned}
F_0 &= 2y' - y \\
F_1 &= \frac{dF_0}{dx} \\
&= \frac{\partial F_0}{\partial x} + \frac{\partial F_0}{\partial y} y' + \frac{\partial F_0}{\partial y'} F_0 \\
&= 3y' - 2y \\
F_2 &= \frac{dF_1}{dx} \\
&= \frac{\partial F_1}{\partial x} + \frac{\partial F_1}{\partial y} y' + \frac{\partial F_1}{\partial y'} F_1 \\
&= 4y' - 3y \\
F_3 &= \frac{dF_2}{dx} \\
&= \frac{\partial F_2}{\partial x} + \frac{\partial F_2}{\partial y} y' + \frac{\partial F_2}{\partial y'} F_2 \\
&= 5y' - 4y \\
F_4 &= \frac{dF_3}{dx} \\
&= \frac{\partial F_3}{\partial x} + \frac{\partial F_3}{\partial y} y' + \frac{\partial F_3}{\partial y'} F_3 \\
&= 6y' - 5y
\end{aligned}
$$

And so on. Evaluating all the above at initial conditions $x = 0$ and $y(0) = y(0)$ and $y'(0) = y'(0)$ gives

$$
\begin{aligned}
F_0 &= 2y'(0) - y(0) \\
F_1 &= 3y'(0) - 2y(0) \\
F_2 &= 4y'(0) - 3y(0) \\
F_3 &= 5y'(0) - 4y(0) \\
F_4 &= 6y'(0) - 5y(0)
\end{aligned}
$$

Substituting all the above in (7) and simplifying gives the solution as

$$
\begin{aligned}
y = &\left(1 - \frac{1}{2}x^2 - \frac{1}{3}x^3 - \frac{1}{8}x^4 - \frac{1}{30}x^5 - \frac{1}{144}x^6\right) y(0) \\
&+ \left(x + x^2 + \frac{1}{2}x^3 + \frac{1}{6}x^4 + \frac{1}{24}x^5 + \frac{1}{120}x^6\right) y'(0) + O(x^6)
\end{aligned}
$$

Since the expansion point $x = 0$ is an ordinary, we can also solve this using standard power series Let the solution be represented as power series of the form

$$y = \sum_{n=0}^{\infty} a_n x^n$$

Then

$$y' = \sum_{n=1}^{\infty} n a_n x^{n-1}$$
$$y'' = \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2}$$

Substituting the above back into the ode gives

$$\sum_{n=2}^{\infty} n(n-1) a_n x^{n-2} = 2\left(\sum_{n=1}^{\infty} n a_n x^{n-1}\right) - \left(\sum_{n=0}^{\infty} a_n x^n\right) \tag{1}$$

Which simplifies to

$$\left(\sum_{n=2}^{\infty} n(n-1) a_n x^{n-2}\right) + \sum_{n=1}^{\infty} \left(-2n a_n x^{n-1}\right) + \left(\sum_{n=0}^{\infty} a_n x^n\right) = 0 \tag{2}$$

The next step is to make all powers of x be n in each summation term. Going over each summation term above with power of x in it which is not already x^n and adjusting the power and the corresponding index gives

$$\sum_{n=2}^{\infty} n(n-1) a_n x^{n-2} = \sum_{n=0}^{\infty} (n+2) a_{n+2}(n+1) x^n$$
$$\sum_{n=1}^{\infty} \left(-2n a_n x^{n-1}\right) = \sum_{n=0}^{\infty} \left(-2(n+1) a_{n+1} x^n\right)$$

Substituting all the above in Eq (2) gives the following equation where now all powers of x are the same and equal to n.

$$\left(\sum_{n=0}^{\infty} (n+2) a_{n+2}(n+1) x^n\right) + \sum_{n=0}^{\infty} \left(-2(n+1) a_{n+1} x^n\right) + \left(\sum_{n=0}^{\infty} a_n x^n\right) = 0 \tag{3}$$

For $0 \leq n$, the recurrence equation is

$$(n+2) a_{n+2}(n+1) - 2(n+1) a_{n+1} + a_n = 0 \tag{4}$$

Solving for a_{n+2}, gives

$$
\begin{aligned}
a_{n+2} &= \frac{2n a_{n+1} - a_n + 2a_{n+1}}{(n+2)(n+1)} \\
&= -\frac{a_n}{(n+2)(n+1)} + \frac{(2n+2)\, a_{n+1}}{(n+2)(n+1)}
\end{aligned} \tag{5}
$$

For $n = 0$ the recurrence equation gives

$$2a_2 - 2a_1 + a_0 = 0$$

Which after substituting the earlier terms found becomes

$$a_2 = a_1 - \frac{a_0}{2}$$

For $n = 1$ the recurrence equation gives

$$6a_3 - 4a_2 + a_1 = 0$$

Which after substituting the earlier terms found becomes

$$a_3 = \frac{a_1}{2} - \frac{a_0}{3}$$

For $n = 2$ the recurrence equation gives

$$12a_4 - 6a_3 + a_2 = 0$$

Which after substituting the earlier terms found becomes

$$a_4 = \frac{a_1}{6} - \frac{a_0}{8}$$

For $n = 3$ the recurrence equation gives

$$20a_5 - 8a_4 + a_3 = 0$$

Which after substituting the earlier terms found becomes

$$a_5 = \frac{a_1}{24} - \frac{a_0}{30}$$

For $n = 4$ the recurrence equation gives

$$30a_6 - 10a_5 + a_4 = 0$$

Which after substituting the earlier terms found becomes

$$a_6 = \frac{a_1}{120} - \frac{a_0}{144}$$

For $n = 5$ the recurrence equation gives

$$42a_7 - 12a_6 + a_5 = 0$$

Which after substituting the earlier terms found becomes

$$a_7 = \frac{a_1}{720} - \frac{a_0}{840}$$

And so on. Therefore the solution is

$$\begin{aligned} y &= \sum_{n=0}^{\infty} a_n x^n \\ &= a_3 x^3 + a_2 x^2 + a_1 x + a_0 + \dots \end{aligned}$$

Substituting the values for a_n found above, the solution becomes

$$y = a_0 + a_1 x + \left(a_1 - \frac{a_0}{2}\right) x^2 + \left(\frac{a_1}{2} - \frac{a_0}{3}\right) x^3 + \left(\frac{a_1}{6} - \frac{a_0}{8}\right) x^4 + \left(\frac{a_1}{24} - \frac{a_0}{30}\right) x^5 + \dots$$

Collecting terms, the solution becomes

$$y = \left(1 - \frac{1}{2}x^2 - \frac{1}{3}x^3 - \frac{1}{8}x^4 - \frac{1}{30}x^5\right) a_0 + \left(x + x^2 + \frac{1}{2}x^3 + \frac{1}{6}x^4 + \frac{1}{24}x^5\right) a_1 + O(x^6) \quad (3)$$

At $x = 0$ the solution above becomes

$$y = \left(1 - \frac{1}{2}x^2 - \frac{1}{3}x^3 - \frac{1}{8}x^4 - \frac{1}{30}x^5\right) c_1 + \left(x + x^2 + \frac{1}{2}x^3 + \frac{1}{6}x^4 + \frac{1}{24}x^5\right) c_2 + O(x^6)$$

Summary

The solution(s) found are the following

$$
\begin{aligned}
y = {} & \left(1-\frac{1}{2}x^{2}-\frac{1}{3}x^{3}-\frac{1}{8}x^{4}-\frac{1}{30}x^{5}-\frac{1}{144}x^{6}\right)y(0) \\
& + \left(x+x^{2}+\frac{1}{2}x^{3}+\frac{1}{6}x^{4}+\frac{1}{24}x^{5}+\frac{1}{120}x^{6}\right)y'(0)+O\left(x^{6}\right)
\end{aligned} \tag{1}
$$

$$
y = \left(1-\frac{1}{2}x^{2}-\frac{1}{3}x^{3}-\frac{1}{8}x^{4}-\frac{1}{30}x^{5}\right)c_{1}+\left(x+x^{2}+\frac{1}{2}x^{3}+\frac{1}{6}x^{4}+\frac{1}{24}x^{5}\right)c_{2}+O\left(x^{6}\right) \tag{2}
$$

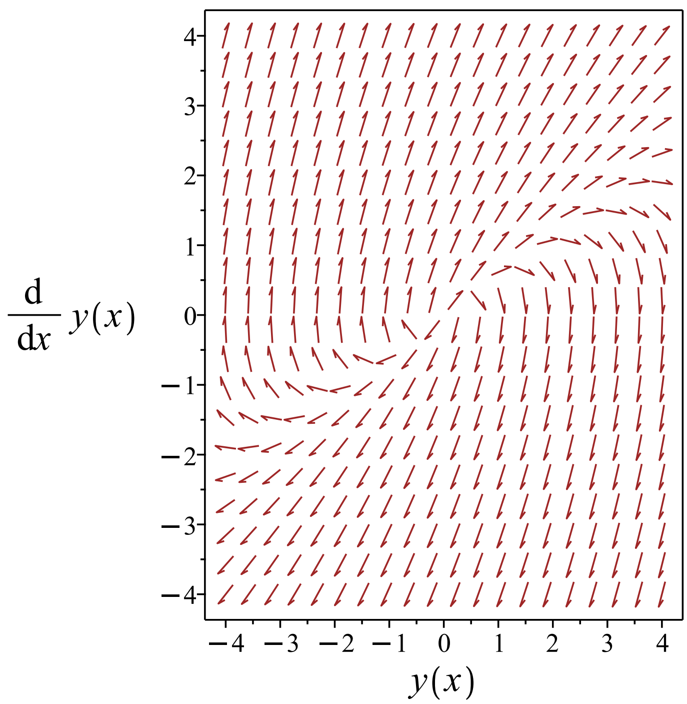


Figure 208: Slope field plot

Verification of solutions

$$
\begin{aligned}
y = {} & \left(1-\frac{1}{2}x^{2}-\frac{1}{3}x^{3}-\frac{1}{8}x^{4}-\frac{1}{30}x^{5}-\frac{1}{144}x^{6}\right)y(0) \\
& + \left(x+x^{2}+\frac{1}{2}x^{3}+\frac{1}{6}x^{4}+\frac{1}{24}x^{5}+\frac{1}{120}x^{6}\right)y'(0)+O\left(x^{6}\right)
\end{aligned}
$$

Verified OK.

$$
y = \left(1-\frac{1}{2}x^{2}-\frac{1}{3}x^{3}-\frac{1}{8}x^{4}-\frac{1}{30}x^{5}\right)c_{1}+\left(x+x^{2}+\frac{1}{2}x^{3}+\frac{1}{6}x^{4}+\frac{1}{24}x^{5}\right)c_{2}+O\left(x^{6}\right)
$$

Verified OK.

5.15.1 Maple step by step solution

Let's solve

$y'' = 2y' - y$

- Highest derivative means the order of the ODE is 2

  y''

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y'' - 2y' + y = 0$

- Characteristic polynomial of ODE

  $r^2 - 2r + 1 = 0$

- Factor the characteristic polynomial

  $(r-1)^2 = 0$

- Root of the characteristic polynomial

  $r = 1$

- 1st solution of the ODE

  $y_1(x) = e^x$

- Repeated root, multiply $y_1(x)$ by x to ensure linear independence

  $y_2(x) = e^x x$

- General solution of the ODE

  $y = c_1 y_1(x) + c_2 y_2(x)$

- Substitute in solutions

  $y = c_1 e^x + c_2 x\, e^x$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
<- constant coefficients successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 52

```
Order:=6;
dsolve(diff(y(x),x$2)-2*diff(y(x),x)+y(x)=0,y(x),type='series',x=0);
```

$$y(x) = \left(1 - \frac{1}{2}x^2 - \frac{1}{3}x^3 - \frac{1}{8}x^4 - \frac{1}{30}x^5\right) y(0) + \left(x + x^2 + \frac{1}{2}x^3 + \frac{1}{6}x^4 + \frac{1}{24}x^5\right) D(y)(0) + O(x^6)$$

✓ Solution by Mathematica

Time used: 0.001 (sec). Leaf size: 66

```
AsymptoticDSolveValue[y''[x]-2*y'[x]+y[x]==0,y[x],{x,0,5}]
```

$$y(x) \to c_1\left(-\frac{x^5}{30} - \frac{x^4}{8} - \frac{x^3}{3} - \frac{x^2}{2} + 1\right) + c_2\left(\frac{x^5}{24} + \frac{x^4}{6} + \frac{x^3}{2} + x^2 + x\right)$$

5.16 problem 17

Internal problem ID [5017]
Internal file name [`OUTPUT/4510_Sunday_June_05_2022_02_59_49_PM_56220328/index.tex`]

Book: Fundamentals of Differential Equations. By Nagle, Saff and Snider. 9th edition. Boston. Pearson 2018.
Section: Chapter 8, Series solutions of differential equations. Section 8.3. page 443
Problem number: 17.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second order series method. Ordinary point", "second order series method. Taylor series method"**

Maple gives the following as the ode type

`[_Lienard]`

$$\boxed{w'' - x^2 w' + w = 0}$$

With the expansion point for the power series method at $x = 0$.

Solving ode using Taylor series method. This gives review on how the Taylor series method works for solving second order ode.

Let

$$y'' = f(x, y, y')$$

Assuming expansion is at $x_0 = 0$ (we can always shift the actual expansion point to 0 by change of variables) and assuming $f(x, y, y')$ is analytic at x_0 which must be the case for an ordinary point. Let initial conditions be $y(x_0) = y_0$ and $y'(x_0) = y'_0$. Using Taylor series gives

$$\begin{aligned}
y(x) &= y(x_0) + (x - x_0)\, y'(x_0) + \frac{(x - x_0)^2}{2} y''(x_0) + \frac{(x - x_0)^3}{3!} y'''(x_0) + \cdots \\
&= y_0 + x y'_0 + \frac{x^2}{2} f|_{x_0, y_0, y'_0} + \frac{x^3}{3!} f'|_{x_0, y_0, y'_0} + \cdots \\
&= y_0 + x y'_0 + \sum_{n=0}^{\infty} \frac{x^{n+2}}{(n+2)!} \frac{d^n f}{dx^n}\bigg|_{x_0, y_0, y'_0}
\end{aligned}$$

But

$$\frac{df}{dx} = \frac{\partial f}{\partial x}\frac{dx}{dx} + \frac{\partial f}{\partial y}\frac{dy}{dx} + \frac{\partial f}{\partial y'}\frac{dy'}{dx} \tag{1}$$

$$= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}y' + \frac{\partial f}{\partial y'}y'' \tag{252}$$

$$= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}y' + \frac{\partial f}{\partial y'}f \tag{253}$$

$$\frac{d^2 f}{dx^2} = \frac{d}{dx}\left(\frac{df}{dx}\right)$$

$$= \frac{\partial}{\partial x}\left(\frac{df}{dx}\right) + \frac{\partial}{\partial y}\left(\frac{df}{dx}\right) y' + \frac{\partial}{\partial y'}\left(\frac{df}{dx}\right) f \tag{2}$$

$$\frac{d^3 f}{dx^3} = \frac{d}{dx}\left(\frac{d^2 f}{dx^2}\right)$$

$$= \frac{\partial}{\partial x}\left(\frac{d^2 f}{dx^2}\right) + \left(\frac{\partial}{\partial y}\frac{d^2 f}{dx^2}\right) y' + \frac{\partial}{\partial y'}\left(\frac{d^2 f}{dx^2}\right) f \tag{3}$$

$$\vdots$$

And so on. Hence if we name $F_0 = f(x,y,y')$ then the above can be written as

$$
\begin{aligned}
F_0 &= f(x,y,y') && (4)\\
F_1 &= \frac{df}{dx}\\
&= \frac{dF_0}{dx}\\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}y' + \frac{\partial f}{\partial y'}y''\\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}y' + \frac{\partial f}{\partial y'}f && (5)\\
&= \frac{\partial F_0}{\partial x} + \frac{\partial F_0}{\partial y}y' + \frac{\partial F_0}{\partial y'}F_0\\
F_2 &= \frac{d}{dx}\left(\frac{d}{dx}f\right)\\
&= \frac{d}{dx}(F_1)\\
&= \frac{\partial}{\partial x}F_1 + \left(\frac{\partial F_1}{\partial y}\right)y' + \left(\frac{\partial F_1}{\partial y'}\right)y''\\
&= \frac{\partial}{\partial x}F_1 + \left(\frac{\partial F_1}{\partial y}\right)y' + \left(\frac{\partial F_1}{\partial y'}\right)F_0\\
&\vdots\\
F_n &= \frac{d}{dx}(F_{n-1})\\
&= \frac{\partial}{\partial x}F_{n-1} + \left(\frac{\partial F_{n-1}}{\partial y}\right)y' + \left(\frac{\partial F_{n-1}}{\partial y'}\right)y''\\
&= \frac{\partial}{\partial x}F_{n-1} + \left(\frac{\partial F_{n-1}}{\partial y}\right)y' + \left(\frac{\partial F_{n-1}}{\partial y'}\right)F_0 && (6)
\end{aligned}
$$

Therefore (6) can be used from now on along with

$$
y(x) = y_0 + xy_0' + \sum_{n=0}^{\infty} \frac{x^{n+2}}{(n+2)!} F_n|_{x_0,y_0,y_0'} \tag{7}
$$

To find $y(x)$ series solution around $x = 0$. Hence

$$
\begin{aligned}
F_0 &= x^2 w' - w \\
F_1 &= \frac{dF_0}{dx} \\
&= \frac{\partial F_0}{\partial x} + \frac{\partial F_0}{\partial w} w' + \frac{\partial F_0}{\partial w'} F_0 \\
&= \left(x^4 + 2x - 1\right) w' - x^2 w \\
F_2 &= \frac{dF_1}{dx} \\
&= \frac{\partial F_1}{\partial x} + \frac{\partial F_1}{\partial w} w' + \frac{\partial F_1}{\partial w'} F_1 \\
&= \left(x^6 + 6x^3 - 2x^2 + 2\right) w' - w\left(x^4 + 4x - 1\right) \\
F_3 &= \frac{dF_2}{dx} \\
&= \frac{\partial F_2}{\partial x} + \frac{\partial F_2}{\partial w} w' + \frac{\partial F_2}{\partial w'} F_2 \\
&= \left(x^8 + 12x^5 - 3x^4 + 20x^2 - 8x + 1\right) w' - w\left(x^6 + 10x^3 - 2x^2 + 6\right) \\
F_4 &= \frac{dF_3}{dx} \\
&= \frac{\partial F_3}{\partial x} + \frac{\partial F_3}{\partial w} w' + \frac{\partial F_3}{\partial w'} F_3 \\
&= \left(x^{10} + 20x^7 - 4x^6 + 80x^4 - 30x^3 + 3x^2 + 40x - 14\right) w' - w\left(x^8 + 18x^5 - 3x^4 + 50x^2 - 12x + 1\right)
\end{aligned}
$$

And so on. Evaluating all the above at initial conditions $x = 0$ and $w(0) = w(0)$ and $w'(0) = w'(0)$ gives

$$
\begin{aligned}
F_0 &= -w(0) \\
F_1 &= -w'(0) \\
F_2 &= 2w'(0) + w(0) \\
F_3 &= w'(0) - 6w(0) \\
F_4 &= -14w'(0) - w(0)
\end{aligned}
$$

Substituting all the above in (7) and simplifying gives the solution as

$$
\begin{aligned}
w = &\left(1 - \frac{1}{2}x^2 + \frac{1}{24}x^4 - \frac{1}{20}x^5 - \frac{1}{720}x^6\right) w(0) \\
&+ \left(x - \frac{1}{6}x^3 + \frac{1}{12}x^4 + \frac{1}{120}x^5 - \frac{7}{360}x^6\right) w'(0) + O\left(x^6\right)
\end{aligned}
$$

Since the expansion point $x = 0$ is an ordinary, we can also solve this using standard power series Let the solution be represented as power series of the form

$$w = \sum_{n=0}^{\infty} a_n x^n$$

Then

$$w' = \sum_{n=1}^{\infty} n a_n x^{n-1}$$
$$w'' = \sum_{n=2}^{\infty} n(n-1)\, a_n x^{n-2}$$

Substituting the above back into the ode gives

$$\sum_{n=2}^{\infty} n(n-1)\, a_n x^{n-2} = x^2 \left(\sum_{n=1}^{\infty} n a_n x^{n-1}\right) - \left(\sum_{n=0}^{\infty} a_n x^n\right) \tag{1}$$

Which simplifies to

$$\left(\sum_{n=2}^{\infty} n(n-1)\, a_n x^{n-2}\right) + \sum_{n\,=1}^{\infty} \left(-n\, x^{1+n} a_n\right) + \left(\sum_{n=0}^{\infty} a_n x^n\right) = 0 \tag{2}$$

The next step is to make all powers of x be n in each summation term. Going over each summation term above with power of x in it which is not already x^n and adjusting the power and the corresponding index gives

$$\sum_{n\,=2}^{\infty} n(n-1)\, a_n x^{n-2} = \sum_{n=0}^{\infty} (n+2)\, a_{n+2}(1+n)\, x^n$$

$$\sum_{n\,=1}^{\infty} \left(-n\, x^{1+n} a_n\right) = \sum_{n=2}^{\infty} \left(-(n-1)\, a_{n-1} x^n\right)$$

Substituting all the above in Eq (2) gives the following equation where now all powers of x are the same and equal to n.

$$\left(\sum_{n=0}^{\infty} (n+2)\, a_{n+2}(1+n)\, x^n\right) + \sum_{n\,=2}^{\infty} \left(-(n-1)\, a_{n-1} x^n\right) + \left(\sum_{n=0}^{\infty} a_n x^n\right) = 0 \tag{3}$$

$n = 0$ gives

$$2a_2 + a_0 = 0$$

$$a_2 = -\frac{a_0}{2}$$

$n = 1$ gives

$$6a_3 + a_1 = 0$$

Which after substituting earlier equations, simplifies to

$$a_3 = -\frac{a_1}{6}$$

For $2 \leq n$, the recurrence equation is

$$(n+2)\, a_{n+2}(1+n) - (n-1)\, a_{n-1} + a_n = 0 \tag{4}$$

Solving for a_{n+2}, gives

$$\begin{aligned} a_{n+2} &= \frac{n a_{n-1} - a_n - a_{n-1}}{(n+2)\,(1+n)} \\ &= -\frac{a_n}{(n+2)\,(1+n)} + \frac{(n-1)\, a_{n-1}}{(n+2)\,(1+n)} \end{aligned} \tag{5}$$

For $n = 2$ the recurrence equation gives

$$12a_4 - a_1 + a_2 = 0$$

Which after substituting the earlier terms found becomes

$$a_4 = \frac{a_1}{12} + \frac{a_0}{24}$$

For $n = 3$ the recurrence equation gives

$$20a_5 - 2a_2 + a_3 = 0$$

Which after substituting the earlier terms found becomes

$$a_5 = -\frac{a_0}{20} + \frac{a_1}{120}$$

For $n = 4$ the recurrence equation gives

$$30a_6 - 3a_3 + a_4 = 0$$

Which after substituting the earlier terms found becomes

$$a_6 = -\frac{7a_1}{360} - \frac{a_0}{720}$$

For $n = 5$ the recurrence equation gives

$$42a_7 - 4a_4 + a_5 = 0$$

Which after substituting the earlier terms found becomes

$$a_7 = \frac{13a_1}{1680} + \frac{13a_0}{2520}$$

And so on. Therefore the solution is

$$\begin{aligned} w &= \sum_{n=0}^{\infty} a_n x^n \\ &= a_3 x^3 + a_2 x^2 + a_1 x + a_0 + \dots \end{aligned}$$

Substituting the values for a_n found above, the solution becomes

$$w = a_0 + a_1 x - \frac{a_0 x^2}{2} - \frac{a_1 x^3}{6} + \left(\frac{a_1}{12} + \frac{a_0}{24}\right) x^4 + \left(-\frac{a_0}{20} + \frac{a_1}{120}\right) x^5 + \dots$$

Collecting terms, the solution becomes

$$w = \left(1 - \frac{1}{2}x^2 + \frac{1}{24}x^4 - \frac{1}{20}x^5\right) a_0 + \left(x - \frac{1}{6}x^3 + \frac{1}{12}x^4 + \frac{1}{120}x^5\right) a_1 + O(x^6) \quad (3)$$

At $x = 0$ the solution above becomes

$$w = \left(1 - \frac{1}{2}x^2 + \frac{1}{24}x^4 - \frac{1}{20}x^5\right) c_1 + \left(x - \frac{1}{6}x^3 + \frac{1}{12}x^4 + \frac{1}{120}x^5\right) c_2 + O(x^6)$$

<u>Summary</u>
The solution(s) found are the following

$$\begin{aligned} w = &\left(1 - \frac{1}{2}x^2 + \frac{1}{24}x^4 - \frac{1}{20}x^5 - \frac{1}{720}x^6\right) w(0) \\ &+ \left(x - \frac{1}{6}x^3 + \frac{1}{12}x^4 + \frac{1}{120}x^5 - \frac{7}{360}x^6\right) w'(0) + O(x^6) \end{aligned} \quad (1)$$

$$w = \left(1 - \frac{1}{2}x^2 + \frac{1}{24}x^4 - \frac{1}{20}x^5\right) c_1 + \left(x - \frac{1}{6}x^3 + \frac{1}{12}x^4 + \frac{1}{120}x^5\right) c_2 + O(x^6) \quad (2)$$

Verification of solutions

$$w = \left(1-\frac{1}{2}x^{2}+\frac{1}{24}x^{4}-\frac{1}{20}x^{5}-\frac{1}{720}x^{6}\right) w(0) + \left(x-\frac{1}{6}x^{3}+\frac{1}{12}x^{4}+\frac{1}{120}x^{5}-\frac{7}{360}x^{6}\right) w'(0)+O\left(x^{6}\right)$$

Verified OK.

$$w = \left(1-\frac{1}{2}x^{2}+\frac{1}{24}x^{4}-\frac{1}{20}x^{5}\right) c_{1}+\left(x-\frac{1}{6}x^{3}+\frac{1}{12}x^{4}+\frac{1}{120}x^{5}\right) c_{2}+O\left(x^{6}\right)$$

Verified OK.

5.16.1 Maple step by step solution

Let's solve

$w'' = x^{2}w' - w$

- Highest derivative means the order of the ODE is 2

  w''

- Group terms with w on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $w'' - x^{2}w' + w = 0$

- Assume series solution for w

  $w = \sum_{k=0}^{\infty} a_{k}x^{k}$

□ Rewrite ODE with series expansions

  - Convert $x^{2}\cdot w'$ to series expansion

    $x^{2}\cdot w' = \sum_{k=0}^{\infty} a_{k}k\, x^{k+1}$

  - Shift index using $k->k-1$

    $x^{2}\cdot w' = \sum_{k=1}^{\infty} a_{k-1}(k-1)\, x^{k}$

  - Convert w'' to series expansion

    $w'' = \sum_{k=2}^{\infty} a_{k}k(k-1)\, x^{k-2}$

  - Shift index using $k->k+2$

$$w'' = \sum_{k=0}^{\infty} a_{k+2}(k+2)(k+1)x^k$$

Rewrite ODE with series expansions

$$2a_2 + a_0 + \left(\sum_{k=1}^{\infty} \left(a_{k+2}(k+2)(k+1) + a_k - a_{k-1}(k-1)\right) x^k\right) = 0$$

- Each term must be 0

$$2a_2 + a_0 = 0$$

- Each term in the series must be 0, giving the recursion relation

$$(k^2 + 3k + 2)\, a_{k+2} - a_{k-1}k + a_k + a_{k-1} = 0$$

- Shift index using $k- >k+1$

$$\left((k+1)^2 + 3k + 5\right) a_{k+3} - a_k(k+1) + a_{k+1} + a_k = 0$$

- Recursion relation that defines the series solution to the ODE

$$\left[w = \sum_{k=0}^{\infty} a_k x^k, a_{k+3} = \frac{a_k k - a_{k+1}}{k^2+5k+6}, 2a_2 + a_0 = 0\right]$$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
<- No Liouvillian solutions exists
-> Trying a solution in terms of special functions:
   -> Bessel
   -> elliptic
   -> Legendre
   -> Kummer
      -> hyper3: Equivalence to 1F1 under a power @ Moebius
   -> hypergeometric
      -> heuristic approach
      -> hyper3: Equivalence to 2F1, 1F1 or 0F1 under a power @ Moebius
   -> Mathieu
      -> Equivalence to the rational form of Mathieu ODE under a power @ Moebius
trying a solution in terms of MeijerG functions
-> Heun: Equivalence to the GHE or one of its 4 confluent cases under a power @ Moebius
<- Heun successful: received ODE is equivalent to the  HeunT  ODE, case  c = 0 `
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 44

```
Order:=6;
dsolve(diff(w(x),x$2)-x^2*diff(w(x),x)+w(x)=0,w(x),type='series',x=0);
```

$$w(x) = \left(1 - \frac{1}{2}x^2 + \frac{1}{24}x^4 - \frac{1}{20}x^5\right) w(0) + \left(x - \frac{1}{6}x^3 + \frac{1}{12}x^4 + \frac{1}{120}x^5\right) D(w)(0) + O(x^6)$$

✓ Solution by Mathematica

Time used: 0.001 (sec). Leaf size: 56

```
AsymptoticDSolveValue[w''[x]-x^2*w'[x]+w[x]==0,w[x],{x,0,5}]
```

$$w(x) \to c_2\left(\frac{x^5}{120}+\frac{x^4}{12}-\frac{x^3}{6}+x\right)+c_1\left(-\frac{x^5}{20}+\frac{x^4}{24}-\frac{x^2}{2}+1\right)$$

5.17 problem 18

Internal problem ID [5018]
Internal file name [OUTPUT/4511_Sunday_June_05_2022_02_59_50_PM_7680710/index.tex]

Book: Fundamentals of Differential Equations. By Nagle, Saff and Snider. 9th edition. Boston. Pearson 2018.
Section: Chapter 8, Series solutions of differential equations. Section 8.3. page 443
Problem number: 18.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second order series method. Ordinary point", "second_order_change_of_variable_on_y_method_2", "second order series method. Taylor series method", "second_order_ode_non_constant_coeff_transformation_on_B"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

$$\boxed{(2x-3)\,y''-xy'+y=0}$$

With the expansion point for the power series method at $x=0$.

Solving ode using Taylor series method. This gives review on how the Taylor series method works for solving second order ode.

Let

$$y''=f(x,y,y')$$

Assuming expansion is at $x_0=0$ (we can always shift the actual expansion point to 0 by change of variables) and assuming $f(x,y,y')$ is analytic at x_0 which must be the case for an ordinary point. Let initial conditions be $y(x_0)=y_0$ and $y'(x_0)=y_0'$. Using

Taylor series gives

$$
\begin{aligned}
y(x) &= y(x_0) + (x - x_0)\, y'(x_0) + \frac{(x-x_0)^2}{2} y''(x_0) + \frac{(x-x_0)^3}{3!} y'''(x_0) + \cdots \\
&= y_0 + x y_0' + \frac{x^2}{2}\, f|_{x_0,y_0,y_0'} + \frac{x^3}{3!}\, f'|_{x_0,y_0,y_0'} + \cdots \\
&= y_0 + x y_0' + \sum_{n=0}^{\infty} \frac{x^{n+2}}{(n+2)\,!} \left.\frac{d^n f}{dx^n}\right|_{x_0,y_0,y_0'}
\end{aligned}
$$

But

$$
\begin{aligned}
\frac{df}{dx} &= \frac{\partial f}{\partial x}\frac{dx}{dx} + \frac{\partial f}{\partial y}\frac{dy}{dx} + \frac{\partial f}{\partial y'}\frac{dy'}{dx} && (1) \\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} y' + \frac{\partial f}{\partial y'} y'' && (255) \\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} y' + \frac{\partial f}{\partial y'} f && (256) \\
\frac{d^2 f}{dx^2} &= \frac{d}{dx}\left(\frac{df}{dx}\right) \\
&= \frac{\partial}{\partial x}\left(\frac{df}{dx}\right) + \frac{\partial}{\partial y}\left(\frac{df}{dx}\right) y' + \frac{\partial}{\partial y'}\left(\frac{df}{dx}\right) f && (2) \\
\frac{d^3 f}{dx^3} &= \frac{d}{dx}\left(\frac{d^2 f}{dx^2}\right) \\
&= \frac{\partial}{\partial x}\left(\frac{d^2 f}{dx^2}\right) + \left(\frac{\partial}{\partial y}\frac{d^2 f}{dx^2}\right) y' + \frac{\partial}{\partial y'}\left(\frac{d^2 f}{dx^2}\right) f && (3) \\
&\vdots
\end{aligned}
$$

And so on. Hence if we name $F_0 = f(x,y,y')$ then the above can be written as

$$
\begin{aligned}
F_0 &= f(x,y,y') \qquad (4)\\
F_1 &= \frac{df}{dx}\\
&= \frac{dF_0}{dx}\\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}y' + \frac{\partial f}{\partial y'}y''\\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}y' + \frac{\partial f}{\partial y'}f \qquad (5)\\
&= \frac{\partial F_0}{\partial x} + \frac{\partial F_0}{\partial y}y' + \frac{\partial F_0}{\partial y'}F_0\\
F_2 &= \frac{d}{dx}\left(\frac{d}{dx}f\right)\\
&= \frac{d}{dx}(F_1)\\
&= \frac{\partial}{\partial x}F_1 + \left(\frac{\partial F_1}{\partial y}\right)y' + \left(\frac{\partial F_1}{\partial y'}\right)y''\\
&= \frac{\partial}{\partial x}F_1 + \left(\frac{\partial F_1}{\partial y}\right)y' + \left(\frac{\partial F_1}{\partial y'}\right)F_0\\
&\vdots\\
F_n &= \frac{d}{dx}(F_{n-1})\\
&= \frac{\partial}{\partial x}F_{n-1} + \left(\frac{\partial F_{n-1}}{\partial y}\right)y' + \left(\frac{\partial F_{n-1}}{\partial y'}\right)y''\\
&= \frac{\partial}{\partial x}F_{n-1} + \left(\frac{\partial F_{n-1}}{\partial y}\right)y' + \left(\frac{\partial F_{n-1}}{\partial y'}\right)F_0 \qquad (6)
\end{aligned}
$$

Therefore (6) can be used from now on along with

$$
y(x) = y_0 + xy_0' + \sum_{n=0}^{\infty}\frac{x^{n+2}}{(n+2)!}F_n|_{x_0,y_0,y_0'} \qquad (7)
$$

To find $y(x)$ series solution around $x = 0$. Hence

$$
\begin{aligned}
F_0 &= -\frac{-xy' + y}{2x - 3} \\
F_1 &= \frac{dF_0}{dx} \\
&= \frac{\partial F_0}{\partial x} + \frac{\partial F_0}{\partial y} y' + \frac{\partial F_0}{\partial y'} F_0 \\
&= \frac{(-2 + x)(-y + xy')}{(2x - 3)^2} \\
F_2 &= \frac{dF_1}{dx} \\
&= \frac{\partial F_1}{\partial x} + \frac{\partial F_1}{\partial y} y' + \frac{\partial F_1}{\partial y'} F_1 \\
&= \frac{(-y + xy')(x^2 - 4x + 5)}{(2x - 3)^3} \\
F_3 &= \frac{dF_2}{dx} \\
&= \frac{\partial F_2}{\partial x} + \frac{\partial F_2}{\partial y} y' + \frac{\partial F_2}{\partial y'} F_2 \\
&= \frac{(-y + xy')(x - 3)(x^2 - 3x + 6)}{(2x - 3)^4} \\
F_4 &= \frac{dF_3}{dx} \\
&= \frac{\partial F_3}{\partial x} + \frac{\partial F_3}{\partial y} y' + \frac{\partial F_3}{\partial y'} F_3 \\
&= \frac{(-y + xy')(x^4 - 8x^3 + 30x^2 - 72x + 99)}{(2x - 3)^5}
\end{aligned}
$$

And so on. Evaluating all the above at initial conditions $x = 0$ and $y(0) = y(0)$ and $y'(0) = y'(0)$ gives

$$
\begin{aligned}
F_0 &= \frac{y(0)}{3} \\
F_1 &= \frac{2y(0)}{9} \\
F_2 &= \frac{5y(0)}{27} \\
F_3 &= \frac{2y(0)}{9} \\
F_4 &= \frac{11y(0)}{27}
\end{aligned}
$$

Substituting all the above in (7) and simplifying gives the solution as

$$y = \left(1+\frac{1}{6}x^2+\frac{1}{27}x^3+\frac{5}{648}x^4+\frac{1}{540}x^5+\frac{11}{19440}x^6\right)y(0)+xy'(0)+O(x^6)$$

Since the expansion point $x = 0$ is an ordinary, we can also solve this using standard power series The ode is normalized to be

$$(2x-3)\,y''-xy'+y=0$$

Let the solution be represented as power series of the form

$$y=\sum_{n=0}^{\infty}a_n x^n$$

Then

$$y'=\sum_{n=1}^{\infty}n a_n x^{n-1}$$
$$y''=\sum_{n=2}^{\infty}n(n-1)\,a_n x^{n-2}$$

Substituting the above back into the ode gives

$$(2x-3)\left(\sum_{n=2}^{\infty}n(n-1)\,a_n x^{n-2}\right)-x\left(\sum_{n=1}^{\infty}n a_n x^{n-1}\right)+\left(\sum_{n=0}^{\infty}a_n x^n\right)=0 \tag{1}$$

Which simplifies to

$$\left(\sum_{n=2}^{\infty}2n\,x^{n-1}a_n(n-1)\right)+\sum_{n=2}^{\infty}\left(-3n(n-1)\,a_n x^{n-2}\right)+\sum_{n=1}^{\infty}\left(-n a_n x^n\right)+\left(\sum_{n=0}^{\infty}a_n x^n\right)=0 \tag{2}$$

The next step is to make all powers of x be n in each summation term. Going over each summation term above with power of x in it which is not already x^n and adjusting the power and the corresponding index gives

$$\sum_{n=2}^{\infty}2n\,x^{n-1}a_n(n-1)=\sum_{n=1}^{\infty}2(n+1)\,a_{n+1}n\,x^n$$

$$\sum_{n=2}^{\infty}\left(-3n(n-1)\,a_n x^{n-2}\right)=\sum_{n=0}^{\infty}\left(-3(n+2)\,a_{n+2}(n+1)\,x^n\right)$$

Substituting all the above in Eq (2) gives the following equation where now all powers of x are the same and equal to n.

$$\left(\sum_{n=1}^{\infty} 2(n+1)\, a_{n+1} n\, x^{n}\right) + \sum_{n=0}^{\infty} \left(-3(n+2)\, a_{n+2}(n+1)\, x^{n}\right) + \sum_{n=1}^{\infty} \left(-n a_{n} x^{n}\right) + \left(\sum_{n=0}^{\infty} a_{n} x^{n}\right) = 0 \tag{3}$$

$n = 0$ gives

$$-6a_2 + a_0 = 0$$

$$a_2 = \frac{a_0}{6}$$

For $1 \leq n$, the recurrence equation is

$$2(n+1)\, a_{n+1} n - 3(n+2)\, a_{n+2}(n+1) - n a_n + a_n = 0 \tag{4}$$

Solving for a_{n+2}, gives

$$\begin{aligned} a_{n+2} &= \frac{2n^2 a_{n+1} - n a_n + 2n a_{n+1} + a_n}{3\,(n+2)\,(n+1)} \\ &= \frac{(-n+1)\, a_n}{3\,(n+2)\,(n+1)} + \frac{(2n^2+2n)\, a_{n+1}}{3\,(n+2)\,(n+1)} \end{aligned} \tag{5}$$

For $n = 1$ the recurrence equation gives

$$4a_2 - 18a_3 = 0$$

Which after substituting the earlier terms found becomes

$$a_3 = \frac{a_0}{27}$$

For $n = 2$ the recurrence equation gives

$$12a_3 - 36a_4 - a_2 = 0$$

Which after substituting the earlier terms found becomes

$$a_4 = \frac{5a_0}{648}$$

For $n = 3$ the recurrence equation gives

$$24a_4 - 60a_5 - 2a_3 = 0$$

Which after substituting the earlier terms found becomes

$$a_5 = \frac{a_0}{540}$$

For $n = 4$ the recurrence equation gives

$$40a_5 - 90a_6 - 3a_4 = 0$$

Which after substituting the earlier terms found becomes

$$a_6 = \frac{11a_0}{19440}$$

For $n = 5$ the recurrence equation gives

$$60a_6 - 126a_7 - 4a_5 = 0$$

Which after substituting the earlier terms found becomes

$$a_7 = \frac{43a_0}{204120}$$

And so on. Therefore the solution is

$$\begin{aligned} y &= \sum_{n=0}^{\infty} a_n x^n \\ &= a_3 x^3 + a_2 x^2 + a_1 x + a_0 + \dots \end{aligned}$$

Substituting the values for a_n found above, the solution becomes

$$y = a_0 + a_1 x + \frac{1}{6} a_0 x^2 + \frac{1}{27} a_0 x^3 + \frac{5}{648} a_0 x^4 + \frac{1}{540} a_0 x^5 + \dots$$

Collecting terms, the solution becomes

$$y = \left(1 + \frac{1}{6}x^2 + \frac{1}{27}x^3 + \frac{5}{648}x^4 + \frac{1}{540}x^5\right) a_0 + a_1 x + O(x^6) \tag{3}$$

At $x = 0$ the solution above becomes

$$y = \left(1 + \frac{1}{6}x^2 + \frac{1}{27}x^3 + \frac{5}{648}x^4 + \frac{1}{540}x^5\right) c_1 + c_2 x + O(x^6)$$

Summary

The solution(s) found are the following

$$y = \left(1 + \frac{1}{6}x^2 + \frac{1}{27}x^3 + \frac{5}{648}x^4 + \frac{1}{540}x^5 + \frac{11}{19440}x^6\right) y(0) + xy'(0) + O(x^6) \tag{1}$$

$$y = \left(1 + \frac{1}{6}x^2 + \frac{1}{27}x^3 + \frac{5}{648}x^4 + \frac{1}{540}x^5\right) c_1 + c_2 x + O(x^6) \tag{2}$$

Verification of solutions

$$y = \left(1 + \frac{1}{6}x^2 + \frac{1}{27}x^3 + \frac{5}{648}x^4 + \frac{1}{540}x^5 + \frac{11}{19440}x^6\right) y(0) + xy'(0) + O(x^6)$$

Verified OK.

$$y = \left(1 + \frac{1}{6}x^2 + \frac{1}{27}x^3 + \frac{5}{648}x^4 + \frac{1}{540}x^5\right) c_1 + c_2 x + O(x^6)$$

Verified OK.

5.17.1 Maple step by step solution

Let's solve

$(2x - 3) y'' - xy' + y = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y'' = -\frac{y}{2x-3} + \frac{xy'}{2x-3}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y'' - \frac{xy'}{2x-3} + \frac{y}{2x-3} = 0$

□ Check to see if $x_0 = \frac{3}{2}$ is a regular singular point

- Define functions

  $\left[P_2(x) = -\frac{x}{2x-3}, P_3(x) = \frac{1}{2x-3}\right]$
- $\left(x - \frac{3}{2}\right) \cdot P_2(x)$ is analytic at $x = \frac{3}{2}$

  $\left(\left(x - \frac{3}{2}\right) \cdot P_2(x)\right)\Big|_{x=\frac{3}{2}} = -\frac{3}{4}$
- $\left(x - \frac{3}{2}\right)^2 \cdot P_3(x)$ is analytic at $x = \frac{3}{2}$

  $\left(\left(x - \frac{3}{2}\right)^2 \cdot P_3(x)\right)\Big|_{x=\frac{3}{2}} = 0$
- $x = \frac{3}{2}$is a regular singular point

  Check to see if $x_0 = \frac{3}{2}$ is a regular singular point

  $x_0 = \frac{3}{2}$

- Multiply by denominators

  $(2x - 3)\, y'' - xy' + y = 0$
- Change variables using $x = u + \frac{3}{2}$ so that the regular singular point is at $u = 0$

  $2u\left(\frac{d^2}{du^2}y(u)\right) + \left(-u - \frac{3}{2}\right)\left(\frac{d}{du}y(u)\right) + y(u) = 0$
- Assume series solution for $y(u)$

  $y(u) = \sum_{k=0}^{\infty} a_k u^{k+r}$

□ Rewrite ODE with series expansions

- Convert $u^m \cdot \left(\frac{d}{du}y(u)\right)$ to series expansion for $m = 0..1$

  $u^m \cdot \left(\frac{d}{du}y(u)\right) = \sum_{k=0}^{\infty} a_k(k + r)\, u^{k+r-1+m}$
- Shift index using $k- >k + 1 - m$

  $u^m \cdot \left(\frac{d}{du}y(u)\right) = \sum_{k=-1+m}^{\infty} a_{k+1-m}(k + 1 - m + r)\, u^{k+r}$
- Convert $u \cdot \left(\frac{d^2}{du^2}y(u)\right)$ to series expansion

  $u \cdot \left(\frac{d^2}{du^2}y(u)\right) = \sum_{k=0}^{\infty} a_k(k + r)\,(k + r - 1)\, u^{k+r-1}$
- Shift index using $k- >k + 1$

$$u \cdot \left(\frac{d^2}{du^2}y(u)\right) = \sum_{k=-1}^{\infty} a_{k+1}(k+1+r)(k+r)\,u^{k+r}$$

Rewrite ODE with series expansions

$$\frac{a_0 r(-7+4r)u^{-1+r}}{2} + \left(\sum_{k=0}^{\infty} \left(\frac{a_{k+1}(k+1+r)(4k-3+4r)}{2} - a_k(k+r-1)\right) u^{k+r}\right) = 0$$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $\frac{r(-7+4r)}{2} = 0$

- Values of r that satisfy the indicial equation

  $r \in \left\{0, \frac{7}{4}\right\}$

- Each term in the series must be 0, giving the recursion relation

  $2\left(k-\frac{3}{4}+r\right)(k+1+r)\,a_{k+1} - a_k(k+r-1) = 0$

- Recursion relation that defines series solution to ODE

  $a_{k+1} = \frac{2a_k(k+r-1)}{(4k-3+4r)(k+1+r)}$

- Recursion relation for $r = 0$; series terminates at $k = 1$

  $a_{k+1} = \frac{2a_k(k-1)}{(4k-3)(k+1)}$

- Apply recursion relation for $k = 0$

  $a_1 = \frac{2a_0}{3}$

- Terminating series solution of the ODE for $r = 0$. Use reduction of order to find the second li

  $y(u) = a_0 \cdot \left(1 + \frac{2u}{3}\right)$

- Revert the change of variables $u = x - \frac{3}{2}$

  $\left[y = \frac{2a_0 x}{3}\right]$

- Recursion relation for $r = \frac{7}{4}$

  $a_{k+1} = \frac{2a_k\left(k+\frac{3}{4}\right)}{(4k+4)\left(k+\frac{11}{4}\right)}$

- Solution for $r = \frac{7}{4}$

  $\left[y(u) = \sum_{k=0}^{\infty} a_k u^{k+\frac{7}{4}}, a_{k+1} = \frac{2a_k\left(k+\frac{3}{4}\right)}{(4k+4)\left(k+\frac{11}{4}\right)}\right]$

- Revert the change of variables $u = x - \frac{3}{2}$

  $\left[y = \sum_{k=0}^{\infty} a_k \left(x-\frac{3}{2}\right)^{k+\frac{7}{4}}, a_{k+1} = \frac{2a_k\left(k+\frac{3}{4}\right)}{(4k+4)\left(k+\frac{11}{4}\right)}\right]$

- Combine solutions and rename parameters

$$\left[y = \frac{2a_0 x}{3} + \left(\sum_{k=0}^{\infty} b_k \left(x - \frac{3}{2}\right)^{k+\frac{7}{4}}\right), b_{k+1} = \frac{2b_k \left(k+\frac{3}{4}\right)}{(4k+4)\left(k+\frac{11}{4}\right)}\right]$$

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Group is reducible, not completely reducible
   Solution has integrals. Trying a special function solution free of integrals...
   -> Trying a solution in terms of special functions:
      -> Bessel
      -> elliptic
      -> Legendre
      -> Kummer
         -> hyper3: Equivalence to 1F1 under a power @ Moebius
         <- hyper3 successful: received ODE is equivalent to the 1F1 ODE
      <- Kummer successful
   <- special function solution successful
      -> Trying to convert hypergeometric functions to elementary form...
      <- elementary form for at least one hypergeometric solution is achieved - returning wi
   <- Kovacics algorithm successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 34

```
Order:=6;
dsolve((2*x-3)*diff(y(x),x$2)-x*diff(y(x),x)+y(x)=0,y(x),type='series',x=0);
```

$$y(x) = \left(1 + \frac{1}{6}x^2 + \frac{1}{27}x^3 + \frac{5}{648}x^4 + \frac{1}{540}x^5\right) y(0) + D(y)(0)\,x + O\left(x^6\right)$$

✓ Solution by Mathematica

Time used: 0.001 (sec). Leaf size: 41

```
AsymptoticDSolveValue[(2*x-3)*y''[x]-x*y'[x]+y[x]==0,y[x],{x,0,5}]
```

$$y(x) \to c_1 \left(\frac{x^5}{540} + \frac{5x^4}{648} + \frac{x^3}{27} + \frac{x^2}{6} + 1 \right) + c_2 x$$

6 Chapter 8, Series solutions of differential equations. Section 8.4. page 449

6.1 problem 1

Internal problem ID [5019]
Internal file name [`OUTPUT/4512_Sunday_June_05_2022_02_59_51_PM_94225378/index.tex`]

Book: Fundamentals of Differential Equations. By Nagle, Saff and Snider. 9th edition. Boston. Pearson 2018.
Section: Chapter 8, Series solutions of differential equations. Section 8.4. page 449
Problem number: 1.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second order series method. Ordinary point", "second order series method. Taylor series method"**

Maple gives the following as the ode type

`[[_2nd_order, _with_linear_symmetries]]`

$$\boxed{(x+1)\,y''-3xy'+2y=0}$$

With the expansion point for the power series method at $x=1$.

The ode does not have its expansion point at $x=0$, therefore to simplify the computation of power series expansion, change of variable is made on the independent variable to shift the initial conditions and the expasion point back to zero. The new ode is then solved more easily since the expansion point is now at zero. The solution converted back to the original independent variable. Let

$$t=x-1$$

The ode is converted to be in terms of the new independent variable t. This results in

$$(t+2)\left(\frac{d^2}{dt^2}y(t)\right)-3(t+1)\left(\frac{d}{dt}y(t)\right)+2y(t)=0$$

With its expansion point and initial conditions now at $t=0$. The transformed ODE is now solved. Solving ode using Taylor series method. This gives review on how the Taylor series method works for solving second order ode.

Let

$$y'' = f(x, y, y')$$

Assuming expansion is at $x_0 = 0$ (we can always shift the actual expansion point to 0 by change of variables) and assuming $f(x, y, y')$ is analytic at x_0 which must be the case for an ordinary point. Let initial conditions be $y(x_0) = y_0$ and $y'(x_0) = y'_0$. Using Taylor series gives

$$\begin{aligned}
y(x) &= y(x_0) + (x - x_0)\, y'(x_0) + \frac{(x - x_0)^2}{2} y''(x_0) + \frac{(x - x_0)^3}{3!} y'''(x_0) + \cdots \\
&= y_0 + x y'_0 + \frac{x^2}{2} f|_{x_0, y_0, y'_0} + \frac{x^3}{3!} f'|_{x_0, y_0, y'_0} + \cdots \\
&= y_0 + x y'_0 + \sum_{n=0}^{\infty} \frac{x^{n+2}}{(n+2)!} \left.\frac{d^n f}{dx^n}\right|_{x_0, y_0, y'_0}
\end{aligned}$$

But

$$\begin{aligned}
\frac{df}{dx} &= \frac{\partial f}{\partial x}\frac{dx}{dx} + \frac{\partial f}{\partial y}\frac{dy}{dx} + \frac{\partial f}{\partial y'}\frac{dy'}{dx} && (1) \\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} y' + \frac{\partial f}{\partial y'} y'' && (258) \\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} y' + \frac{\partial f}{\partial y'} f && (259) \\
\frac{d^2 f}{dx^2} &= \frac{d}{dx}\left(\frac{df}{dx}\right) \\
&= \frac{\partial}{\partial x}\left(\frac{df}{dx}\right) + \frac{\partial}{\partial y}\left(\frac{df}{dx}\right) y' + \frac{\partial}{\partial y'}\left(\frac{df}{dx}\right) f && (2) \\
\frac{d^3 f}{dx^3} &= \frac{d}{dx}\left(\frac{d^2 f}{dx^2}\right) \\
&= \frac{\partial}{\partial x}\left(\frac{d^2 f}{dx^2}\right) + \left(\frac{\partial}{\partial y}\frac{d^2 f}{dx^2}\right) y' + \frac{\partial}{\partial y'}\left(\frac{d^2 f}{dx^2}\right) f && (3) \\
&\vdots
\end{aligned}$$

And so on. Hence if we name $F_0 = f(x,y,y')$ then the above can be written as

$$
\begin{aligned}
F_0 &= f(x,y,y') \qquad (4)\\
F_1 &= \frac{df}{dx}\\
&= \frac{dF_0}{dx}\\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}y' + \frac{\partial f}{\partial y'}y''\\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}y' + \frac{\partial f}{\partial y'}f \qquad (5)\\
&= \frac{\partial F_0}{\partial x} + \frac{\partial F_0}{\partial y}y' + \frac{\partial F_0}{\partial y'}F_0\\
F_2 &= \frac{d}{dx}\left(\frac{d}{dx}f\right)\\
&= \frac{d}{dx}(F_1)\\
&= \frac{\partial}{\partial x}F_1 + \left(\frac{\partial F_1}{\partial y}\right)y' + \left(\frac{\partial F_1}{\partial y'}\right)y''\\
&= \frac{\partial}{\partial x}F_1 + \left(\frac{\partial F_1}{\partial y}\right)y' + \left(\frac{\partial F_1}{\partial y'}\right)F_0\\
&\vdots\\
F_n &= \frac{d}{dx}(F_{n-1})\\
&= \frac{\partial}{\partial x}F_{n-1} + \left(\frac{\partial F_{n-1}}{\partial y}\right)y' + \left(\frac{\partial F_{n-1}}{\partial y'}\right)y''\\
&= \frac{\partial}{\partial x}F_{n-1} + \left(\frac{\partial F_{n-1}}{\partial y}\right)y' + \left(\frac{\partial F_{n-1}}{\partial y'}\right)F_0 \qquad (6)
\end{aligned}
$$

Therefore (6) can be used from now on along with

$$
y(x) = y_0 + xy_0' + \sum_{n=0}^{\infty}\frac{x^{n+2}}{(n+2)!}F_n|_{x_0,y_0,y_0'} \qquad (7)
$$

To find $y(x)$ series solution around $x=0$. Hence

$$
\begin{aligned}
F_0 &= -\frac{-3t\left(\frac{d}{dt}y(t)\right)+2y(t)-3\frac{d}{dt}y(t)}{t+2}\\
F_1 &= \frac{dF_0}{dt}\\
&= \frac{\partial F_0}{\partial t}+\frac{\partial F_0}{\partial y}\frac{d}{dt}y(t)+\frac{\partial F_0}{\partial \frac{d}{dt}y\left(t\right)}F_0\\
&= \frac{\left(9t^2+16t+8\right)\left(\frac{d}{dt}y(t)\right)+\left(-6t-4\right)y(t)}{\left(t+2\right)^2}\\
F_2 &= \frac{dF_1}{dt}\\
&= \frac{\partial F_1}{\partial t}+\frac{\partial F_1}{\partial y}\frac{d}{dt}y(t)+\frac{\partial F_1}{\partial \frac{d}{dt}y\left(t\right)}F_1\\
&= \frac{\left(27t^3+69t^2+76t+32\right)\left(\frac{d}{dt}y(t)\right)-18y(t)\left(t^2+\frac{13}{9}t+\frac{10}{9}\right)}{\left(t+2\right)^3}\\
F_3 &= \frac{dF_2}{dt}\\
&= \frac{\partial F_2}{\partial t}+\frac{\partial F_2}{\partial y}\frac{d}{dt}y(t)+\frac{\partial F_2}{\partial \frac{d}{dt}y\left(t\right)}F_2\\
&= \frac{\left(81t^4+270t^3+466t^2+376t+112\right)\left(\frac{d}{dt}y(t)\right)-54y(t)\left(t^3+\frac{20}{9}t^2+\frac{86}{27}t+\frac{28}{27}\right)}{\left(t+2\right)^4}\\
F_4 &= \frac{dF_3}{dt}\\
&= \frac{\partial F_3}{\partial t}+\frac{\partial F_3}{\partial y}\frac{d}{dt}y(t)+\frac{\partial F_3}{\partial \frac{d}{dt}y\left(t\right)}F_3\\
&= \frac{\left(243t^5+999t^4+2358t^3+2802t^2+1800t+528\right)\left(\frac{d}{dt}y(t)\right)-162y(t)\left(t^4+3t^3+\frac{508}{81}t^2+\frac{358}{81}t+\frac{172}{81}\right)}{\left(t+2\right)^5}
\end{aligned}
$$

And so on. Evaluating all the above at initial conditions $t=0$ and $y(0)=y(0)$ and $y'(0)=y'(0)$ gives

$$
\begin{aligned}
F_0 &= -y(0)+\frac{3y'(0)}{2}\\
F_1 &= 2y'(0)-y(0)\\
F_2 &= -\frac{5y(0)}{2}+4y'(0)\\
F_3 &= -\frac{7y(0)}{2}+7y'(0)\\
F_4 &= -\frac{43y(0)}{4}+\frac{33y'(0)}{2}
\end{aligned}
$$

Substituting all the above in (7) and simplifying gives the solution as

$$
\begin{aligned}
y(t) = &\left(1-\frac{1}{2}t^{2}-\frac{1}{6}t^{3}-\frac{5}{48}t^{4}-\frac{7}{240}t^{5}-\frac{43}{2880}t^{6}\right) y(0) \\
&+\left(t+\frac{3}{4}t^{2}+\frac{1}{3}t^{3}+\frac{1}{6}t^{4}+\frac{7}{120}t^{5}+\frac{11}{480}t^{6}\right) y'(0)+O\left(t^{6}\right)
\end{aligned}
$$

Since the expansion point $t = 0$ is an ordinary, we can also solve this using standard power series The ode is normalized to be

$$
(t+2)\left(\frac{d^{2}}{dt^{2}}y(t)\right)+(-3t-3)\left(\frac{d}{dt}y(t)\right)+2y(t) = 0
$$

Let the solution be represented as power series of the form

$$
y(t) = \sum_{n=0}^{\infty} a_{n} t^{n}
$$

Then

$$
\begin{aligned}
\frac{d}{dt}y(t) &= \sum_{n=1}^{\infty} n a_{n} t^{n-1} \\
\frac{d^{2}}{dt^{2}}y(t) &= \sum_{n=2}^{\infty} n(n-1)\, a_{n} t^{n-2}
\end{aligned}
$$

Substituting the above back into the ode gives

$$
(t+2)\left(\sum_{n=2}^{\infty} n(n-1)\, a_{n} t^{n-2}\right)+(-3t-3)\left(\sum_{n=1}^{\infty} n a_{n} t^{n-1}\right)+2\left(\sum_{n=0}^{\infty} a_{n} t^{n}\right) = 0 \tag{1}
$$

Which simplifies to

$$
\begin{aligned}
&\left(\sum_{n=2}^{\infty} n\, t^{n-1} a_{n}(n-1)\right)+\left(\sum_{n=2}^{\infty} 2n(n-1)\, a_{n} t^{n-2}\right) \\
&+\sum_{n=1}^{\infty}\left(-3n a_{n} t^{n}\right)+\sum_{n=1}^{\infty}\left(-3n a_{n} t^{n-1}\right)+\left(\sum_{n=0}^{\infty} 2a_{n} t^{n}\right) = 0
\end{aligned} \tag{2}
$$

The next step is to make all powers of t be n in each summation term. Going over each summation term above with power of t in it which is not already t^n and adjusting the

power and the corresponding index gives

$$\sum_{n=2}^{\infty} n\, t^{n-1} a_n (n-1) = \sum_{n=1}^{\infty} (n+1)\, a_{n+1} n\, t^n$$

$$\sum_{n=2}^{\infty} 2n(n-1)\, a_n t^{n-2} = \sum_{n=0}^{\infty} 2(n+2)\, a_{n+2}(n+1)\, t^n$$

$$\sum_{n=1}^{\infty} \left(-3n a_n t^{n-1}\right) = \sum_{n=0}^{\infty} \left(-3(n+1)\, a_{n+1} t^n\right)$$

Substituting all the above in Eq (2) gives the following equation where now all powers of t are the same and equal to n.

$$\left(\sum_{n=1}^{\infty} (n+1)\, a_{n+1} n\, t^n\right) + \left(\sum_{n=0}^{\infty} 2(n+2)\, a_{n+2}(n+1)\, t^n\right) + \sum_{n=1}^{\infty} \left(-3n a_n t^n\right) + \sum_{n=0}^{\infty} \left(-3(n+1)\, a_{n+1} t^n\right) + \left(\sum_{n=0}^{\infty} 2a_n t^n\right) = 0 \tag{3}$$

$n = 0$ gives

$$4a_2 - 3a_1 + 2a_0 = 0$$

$$a_2 = -\frac{a_0}{2} + \frac{3a_1}{4}$$

For $1 \leq n$, the recurrence equation is

$$(n+1)\, a_{n+1} n + 2(n+2)\, a_{n+2}(n+1) - 3n a_n - 3(n+1)\, a_{n+1} + 2a_n = 0 \tag{4}$$

Solving for a_{n+2}, gives

$$\begin{aligned} a_{n+2} &= -\frac{n^2 a_{n+1} - 3n a_n - 2n a_{n+1} + 2a_n - 3a_{n+1}}{2\,(n+2)\,(n+1)} \\ &= -\frac{(-3n+2)\, a_n}{2\,(n+2)\,(n+1)} - \frac{(n^2-2n-3)\, a_{n+1}}{2\,(n+2)\,(n+1)} \end{aligned} \tag{5}$$

For $n = 1$ the recurrence equation gives

$$-4a_2 + 12a_3 - a_1 = 0$$

Which after substituting the earlier terms found becomes

$$a_3 = -\frac{a_0}{6} + \frac{a_1}{3}$$

For $n = 2$ the recurrence equation gives

$$-3a_3 + 24a_4 - 4a_2 = 0$$

Which after substituting the earlier terms found becomes

$$a_4 = -\frac{5a_0}{48} + \frac{a_1}{6}$$

For $n = 3$ the recurrence equation gives

$$40a_5 - 7a_3 = 0$$

Which after substituting the earlier terms found becomes

$$a_5 = -\frac{7a_0}{240} + \frac{7a_1}{120}$$

For $n = 4$ the recurrence equation gives

$$5a_5 + 60a_6 - 10a_4 = 0$$

Which after substituting the earlier terms found becomes

$$a_6 = -\frac{43a_0}{2880} + \frac{11a_1}{480}$$

For $n = 5$ the recurrence equation gives

$$12a_6 + 84a_7 - 13a_5 = 0$$

Which after substituting the earlier terms found becomes

$$a_7 = -\frac{a_0}{420} + \frac{29a_1}{5040}$$

And so on. Therefore the solution is

$$
\begin{aligned}
y(t) &= \sum_{n=0}^{\infty} a_n t^n \\
&= a_3 t^3 + a_2 t^2 + a_1 t + a_0 + \dots
\end{aligned}
$$

Substituting the values for a_n found above, the solution becomes

$$
\begin{aligned}
y(t) = a_0 + a_1 t + \left(-\frac{a_0}{2} + \frac{3a_1}{4}\right) t^2 + \left(-\frac{a_0}{6} + \frac{a_1}{3}\right) t^3 \\
+ \left(-\frac{5a_0}{48} + \frac{a_1}{6}\right) t^4 + \left(-\frac{7a_0}{240} + \frac{7a_1}{120}\right) t^5 + \dots
\end{aligned}
$$

Collecting terms, the solution becomes

$$
y(t) = \left(1 - \frac{1}{2}t^2 - \frac{1}{6}t^3 - \frac{5}{48}t^4 - \frac{7}{240}t^5\right) a_0 + \left(t + \frac{3}{4}t^2 + \frac{1}{3}t^3 + \frac{1}{6}t^4 + \frac{7}{120}t^5\right) a_1 + O\left(t^6\right) \tag{3}
$$

At $t = 0$ the solution above becomes

$$
y(t) = \left(1 - \frac{1}{2}t^2 - \frac{1}{6}t^3 - \frac{5}{48}t^4 - \frac{7}{240}t^5\right) c_1 + \left(t + \frac{3}{4}t^2 + \frac{1}{3}t^3 + \frac{1}{6}t^4 + \frac{7}{120}t^5\right) c_2 + O\left(t^6\right)
$$

Replacing t in the above with the original independent variable $x$susing $t = x - 1$ results in

$$
\begin{aligned}
y = &\left(1 - \frac{(x-1)^2}{2} - \frac{(x-1)^3}{6} - \frac{5(x-1)^4}{48} - \frac{7(x-1)^5}{240} - \frac{43(x-1)^6}{2880}\right) y(1) \\
&+ \left(x - 1 + \frac{3(x-1)^2}{4} + \frac{(x-1)^3}{3} + \frac{(x-1)^4}{6} + \frac{7(x-1)^5}{120} + \frac{11(x-1)^6}{480}\right) y'(1) \\
&+ O\left((x-1)^6\right)
\end{aligned}
$$

Summary

The solution(s) found are the following

$$
\begin{aligned}
y = &\left(1 - \frac{(x-1)^2}{2} - \frac{(x-1)^3}{6} - \frac{5(x-1)^4}{48} - \frac{7(x-1)^5}{240} - \frac{43(x-1)^6}{2880}\right) y(1) \\
&+ \left(x - 1 + \frac{3(x-1)^2}{4} + \frac{(x-1)^3}{3} + \frac{(x-1)^4}{6} + \frac{7(x-1)^5}{120} + \frac{11(x-1)^6}{480}\right) y'(1) \\
&+ O\left((x-1)^6\right)
\end{aligned} \tag{1}
$$

Verification of solutions

$$y = \left(1-\frac{(x-1)^{2}}{2}-\frac{(x-1)^{3}}{6}-\frac{5(x-1)^{4}}{48}-\frac{7(x-1)^{5}}{240}-\frac{43(x-1)^{6}}{2880}\right)y(1) + \left(x-1+\frac{3(x-1)^{2}}{4}+\frac{(x-1)^{3}}{3}+\frac{(x-1)^{4}}{6}+\frac{7(x-1)^{5}}{120}+\frac{11(x-1)^{6}}{480}\right)y'(1) + O\left((x-1)^{6}\right)$$

Verified OK.

6.1.1 Maple step by step solution

Let's solve

$(x+1)\,y''-3xy'+2y=0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y''=-\frac{2y}{x+1}+\frac{3xy'}{x+1}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y''-\frac{3xy'}{x+1}+\frac{2y}{x+1}=0$

☐ Check to see if $x_0=-1$ is a regular singular point

  ◦ Define functions

    $\left[P_2(x)=-\frac{3x}{x+1}, P_3(x)=\frac{2}{x+1}\right]$

  ◦ $(x+1)\cdot P_2(x)$ is analytic at $x=-1$

    $\left((x+1)\cdot P_2(x)\right)\Big|_{x=-1}=3$

  ◦ $(x+1)^{2}\cdot P_3(x)$ is analytic at $x=-1$

    $\left((x+1)^{2}\cdot P_3(x)\right)\Big|_{x=-1}=0$

  ◦ $x=-1$is a regular singular point

  Check to see if $x_0=-1$ is a regular singular point

  $x_0=-1$

- Multiply by denominators

$(x+1)\,y'' - 3xy' + 2y = 0$

- Change variables using $x = u - 1$ so that the regular singular point is at $u = 0$

$u\left(\frac{d^2}{du^2}y(u)\right) + (-3u+3)\left(\frac{d}{du}y(u)\right) + 2y(u) = 0$

- Assume series solution for $y(u)$

$y(u) = \sum_{k=0}^{\infty} a_k u^{k+r}$

☐ Rewrite ODE with series expansions

  - Convert $u^m \cdot \left(\frac{d}{du}y(u)\right)$ to series expansion for $m = 0..1$

  $u^m \cdot \left(\frac{d}{du}y(u)\right) = \sum_{k=0}^{\infty} a_k(k+r)\,u^{k+r-1+m}$

  - Shift index using $k- > k+1-m$

  $u^m \cdot \left(\frac{d}{du}y(u)\right) = \sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\,u^{k+r}$

  - Convert $u \cdot \left(\frac{d^2}{du^2}y(u)\right)$ to series expansion

  $u \cdot \left(\frac{d^2}{du^2}y(u)\right) = \sum_{k=0}^{\infty} a_k(k+r)(k+r-1)\,u^{k+r-1}$

  - Shift index using $k- > k+1$

  $u \cdot \left(\frac{d^2}{du^2}y(u)\right) = \sum_{k=-1}^{\infty} a_{k+1}(k+1+r)(k+r)\,u^{k+r}$

Rewrite ODE with series expansions

$a_0 r(2+r)\,u^{-1+r} + \left(\sum_{k=0}^{\infty} \left(a_{k+1}(k+1+r)(k+3+r) - a_k(3k+3r-2)\right)u^{k+r}\right) = 0$

- $a_0$cannot be 0 by assumption, giving the indicial equation

$r(2+r) = 0$

- Values of r that satisfy the indicial equation

$r \in \{-2, 0\}$

- Each term in the series must be 0, giving the recursion relation

$a_{k+1}(k+1+r)(k+3+r) - 3\left(k+r-\frac{2}{3}\right)a_k = 0$

- Recursion relation that defines series solution to ODE

$a_{k+1} = \frac{(3k+3r-2)a_k}{(k+1+r)(k+3+r)}$

- Recursion relation for $r = -2$

$a_{k+1} = \frac{(3k-8)a_k}{(k-1)(k+1)}$

- Series not valid for $r = -2$, division by 0 in the recursion relation at $k = 1$

  $a_{k+1} = \frac{(3k-8)a_k}{(k-1)(k+1)}$

- Recursion relation for $r = 0$

  $a_{k+1} = \frac{(3k-2)a_k}{(k+1)(k+3)}$

- Solution for $r = 0$

  $\left[y(u) = \sum_{k=0}^{\infty} a_k u^k, a_{k+1} = \frac{(3k-2)a_k}{(k+1)(k+3)}\right]$

- Revert the change of variables $u = x + 1$

  $\left[y = \sum_{k=0}^{\infty} a_k (x+1)^k , a_{k+1} = \frac{(3k-2)a_k}{(k+1)(k+3)}\right]$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
<- No Liouvillian solutions exists
-> Trying a solution in terms of special functions:
   -> Bessel
   -> elliptic
   -> Legendre
   -> Kummer
      -> hyper3: Equivalence to 1F1 under a power @ Moebius
      <- hyper3 successful: received ODE is equivalent to the 1F1 ODE
   <- Kummer successful
<- special function solution successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 54

```
Order:=6;
dsolve((x+1)*diff(y(x),x$2)-3*x*diff(y(x),x)+2*y(x)=0,y(x),type='series',x=1);
```

$$y(x) = \left(1 - \frac{(x-1)^2}{2} - \frac{(x-1)^3}{6} - \frac{5(x-1)^4}{48} - \frac{7(x-1)^5}{240}\right) y(1) + \left(x - 1 + \frac{3(x-1)^2}{4} + \frac{(x-1)^3}{3} + \frac{(x-1)^4}{6} + \frac{7(x-1)^5}{120}\right) D(y)(1) + O(x^6)$$

✓ Solution by Mathematica

Time used: 0.001 (sec). Leaf size: 87

```
AsymptoticDSolveValue[(x+1)*y''[x]-3*x*y'[x]+2*y[x]==0,y[x],{x,1,5}]
```

$$y(x) \to c_1\left(-\frac{7}{240}(x-1)^5 - \frac{5}{48}(x-1)^4 - \frac{1}{6}(x-1)^3 - \frac{1}{2}(x-1)^2 + 1\right) + c_2\left(\frac{7}{120}(x-1)^5 + \frac{1}{6}(x-1)^4 + \frac{1}{3}(x-1)^3 + \frac{3}{4}(x-1)^2 + x - 1\right)$$

6.2 problem 2

Internal problem ID [5020]
Internal file name [OUTPUT/4513_Sunday_June_05_2022_02_59_52_PM_70745873/index.tex]

Book: Fundamentals of Differential Equations. By Nagle, Saff and Snider. 9th edition. Boston. Pearson 2018.
Section: Chapter 8, Series solutions of differential equations. Section 8.4. page 449
Problem number: 2.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second order series method. Ordinary point", "second order series method. Taylor series method"**

Maple gives the following as the ode type

`[_Hermite]`

$$\boxed{y'' - xy' - 3y = 0}$$

With the expansion point for the power series method at $x = 2$.

The ode does not have its expansion point at $x = 0$, therefore to simplify the computation of power series expansion, change of variable is made on the independent variable to shift the initial conditions and the expasion point back to zero. The new ode is then solved more easily since the expansion point is now at zero. The solution converted back to the original independent variable. Let

$$t = -2 + x$$

The ode is converted to be in terms of the new independent variable t. This results in

$$\frac{d^2}{dt^2}y(t) - (t+2)\left(\frac{d}{dt}y(t)\right) - 3y(t) = 0$$

With its expansion point and initial conditions now at $t = 0$. The transformed ODE is now solved. Solving ode using Taylor series method. This gives review on how the Taylor series method works for solving second order ode.

Let

$$y'' = f(x, y, y')$$

Assuming expansion is at $x_0 = 0$ (we can always shift the actual expansion point to 0 by change of variables) and assuming $f(x, y, y')$ is analytic at x_0 which must be the case for an ordinary point. Let initial conditions be $y(x_0) = y_0$ and $y'(x_0) = y'_0$. Using Taylor series gives

$$\begin{aligned}
y(x) &= y(x_0) + (x - x_0)\, y'(x_0) + \frac{(x - x_0)^2}{2} y''(x_0) + \frac{(x - x_0)^3}{3!} y'''(x_0) + \cdots \\
&= y_0 + x y'_0 + \frac{x^2}{2} f|_{x_0, y_0, y'_0} + \frac{x^3}{3!} f'|_{x_0, y_0, y'_0} + \cdots \\
&= y_0 + x y'_0 + \sum_{n=0}^{\infty} \frac{x^{n+2}}{(n+2)!} \frac{d^n f}{dx^n}\bigg|_{x_0, y_0, y'_0}
\end{aligned}$$

But

$$\begin{aligned}
\frac{df}{dx} &= \frac{\partial f}{\partial x}\frac{dx}{dx} + \frac{\partial f}{\partial y}\frac{dy}{dx} + \frac{\partial f}{\partial y'}\frac{dy'}{dx} && (1) \\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} y' + \frac{\partial f}{\partial y'} y'' && (261) \\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} y' + \frac{\partial f}{\partial y'} f && (262) \\
\frac{d^2 f}{dx^2} &= \frac{d}{dx}\left(\frac{df}{dx}\right) \\
&= \frac{\partial}{\partial x}\left(\frac{df}{dx}\right) + \frac{\partial}{\partial y}\left(\frac{df}{dx}\right) y' + \frac{\partial}{\partial y'}\left(\frac{df}{dx}\right) f && (2) \\
\frac{d^3 f}{dx^3} &= \frac{d}{dx}\left(\frac{d^2 f}{dx^2}\right) \\
&= \frac{\partial}{\partial x}\left(\frac{d^2 f}{dx^2}\right) + \left(\frac{\partial}{\partial y}\frac{d^2 f}{dx^2}\right) y' + \frac{\partial}{\partial y'}\left(\frac{d^2 f}{dx^2}\right) f && (3) \\
&\vdots
\end{aligned}$$

And so on. Hence if we name $F_0 = f(x,y,y')$ then the above can be written as

$$\begin{aligned}
F_0 &= f(x,y,y') \qquad (4)\\
F_1 &= \frac{df}{dx}\\
&= \frac{dF_0}{dx}\\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}y' + \frac{\partial f}{\partial y'}y''\\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}y' + \frac{\partial f}{\partial y'}f \qquad (5)\\
&= \frac{\partial F_0}{\partial x} + \frac{\partial F_0}{\partial y}y' + \frac{\partial F_0}{\partial y'}F_0\\
F_2 &= \frac{d}{dx}\left(\frac{d}{dx}f\right)\\
&= \frac{d}{dx}(F_1)\\
&= \frac{\partial}{\partial x}F_1 + \left(\frac{\partial F_1}{\partial y}\right)y' + \left(\frac{\partial F_1}{\partial y'}\right)y''\\
&= \frac{\partial}{\partial x}F_1 + \left(\frac{\partial F_1}{\partial y}\right)y' + \left(\frac{\partial F_1}{\partial y'}\right)F_0\\
&\vdots\\
F_n &= \frac{d}{dx}(F_{n-1})\\
&= \frac{\partial}{\partial x}F_{n-1} + \left(\frac{\partial F_{n-1}}{\partial y}\right)y' + \left(\frac{\partial F_{n-1}}{\partial y'}\right)y''\\
&= \frac{\partial}{\partial x}F_{n-1} + \left(\frac{\partial F_{n-1}}{\partial y}\right)y' + \left(\frac{\partial F_{n-1}}{\partial y'}\right)F_0 \qquad (6)
\end{aligned}$$

Therefore (6) can be used from now on along with

$$y(x) = y_0 + xy_0' + \sum_{n=0}^{\infty}\frac{x^{n+2}}{(n+2)!}F_n|_{x_0,y_0,y_0'} \qquad (7)$$

To find $y(x)$ series solution around $x = 0$. Hence

$$
\begin{aligned}
F_0 &= t\left(\frac{d}{dt}y(t)\right) + 2\frac{d}{dt}y(t) + 3y(t)\\
F_1 &= \frac{dF_0}{dt}\\
&= \frac{\partial F_0}{\partial t} + \frac{\partial F_0}{\partial y}\frac{d}{dt}y(t) + \frac{\partial F_0}{\partial \frac{d}{dt}y(t)}F_0\\
&= \left(t^2 + 4t + 8\right)\left(\frac{d}{dt}y(t)\right) + 3(t+2)\,y(t)\\
F_2 &= \frac{dF_1}{dt}\\
&= \frac{\partial F_1}{\partial t} + \frac{\partial F_1}{\partial y}\frac{d}{dt}y(t) + \frac{\partial F_1}{\partial \frac{d}{dt}y(t)}F_1\\
&= \left(t^3 + 6t^2 + 21t + 26\right)\left(\frac{d}{dt}y(t)\right) + 3y(t)\left(t^2 + 4t + 9\right)\\
F_3 &= \frac{dF_2}{dt}\\
&= \frac{\partial F_2}{\partial t} + \frac{\partial F_2}{\partial y}\frac{d}{dt}y(t) + \frac{\partial F_2}{\partial \frac{d}{dt}y(t)}F_2\\
&= \left(t^4 + 8t^3 + 39t^2 + 92t + 100\right)\left(\frac{d}{dt}y(t)\right) + 3y(t)\,(t+2)\left(t^2 + 4t + 15\right)\\
F_4 &= \frac{dF_3}{dt}\\
&= \frac{\partial F_3}{\partial t} + \frac{\partial F_3}{\partial y}\frac{d}{dt}y(t) + \frac{\partial F_3}{\partial \frac{d}{dt}y(t)}F_3\\
&= (t+2)\left(t^4 + 8t^3 + 46t^2 + 120t + 191\right)\left(\frac{d}{dt}y(t)\right) + 3y(t)\left(t^4 + 8t^3 + 42t^2 + 104t + 123\right)
\end{aligned}
$$

And so on. Evaluating all the above at initial conditions $t = 0$ and $y(0) = y(0)$ and $y'(0) = y'(0)$ gives

$$
\begin{aligned}
F_0 &= 3y(0) + 2y'(0)\\
F_1 &= 8y'(0) + 6y(0)\\
F_2 &= 26y'(0) + 27y(0)\\
F_3 &= 100y'(0) + 90y(0)\\
F_4 &= 382y'(0) + 369y(0)
\end{aligned}
$$

Substituting all the above in (7) and simplifying gives the solution as

$$
\begin{aligned}
y(t) = &\left(1+\frac{3}{2}t^{2}+t^{3}+\frac{9}{8}t^{4}+\frac{3}{4}t^{5}+\frac{41}{80}t^{6}\right)y(0)\\
&+\left(t+t^{2}+\frac{4}{3}t^{3}+\frac{13}{12}t^{4}+\frac{5}{6}t^{5}+\frac{191}{360}t^{6}\right)y'(0)+O(t^{6})
\end{aligned}
$$

Since the expansion point $t = 0$ is an ordinary, we can also solve this using standard power series Let the solution be represented as power series of the form

$$
y(t) = \sum_{n=0}^{\infty} a_n t^n
$$

Then

$$
\begin{aligned}
\frac{d}{dt}y(t) &= \sum_{n=1}^{\infty} n a_n t^{n-1}\\
\frac{d^2}{dt^2}y(t) &= \sum_{n=2}^{\infty} n(n-1)\,a_n t^{n-2}
\end{aligned}
$$

Substituting the above back into the ode gives

$$
\sum_{n=2}^{\infty} n(n-1)\,a_n t^{n-2} = t\left(\sum_{n=1}^{\infty} n a_n t^{n-1}\right)+2\left(\sum_{n=1}^{\infty} n a_n t^{n-1}\right)+3\left(\sum_{n=0}^{\infty} a_n t^n\right) \tag{1}
$$

Which simplifies to

$$
\left(\sum_{n=2}^{\infty} n(n-1)\,a_n t^{n-2}\right)+\sum_{n=1}^{\infty}(-n a_n t^n)+\sum_{n=1}^{\infty}\left(-2n a_n t^{n-1}\right)+\sum_{n=0}^{\infty}(-3a_n t^n) = 0 \tag{2}
$$

The next step is to make all powers of t be n in each summation term. Going over each summation term above with power of t in it which is not already t^n and adjusting the power and the corresponding index gives

$$
\begin{aligned}
\sum_{n=2}^{\infty} n(n-1)\,a_n t^{n-2} &= \sum_{n=0}^{\infty}(n+2)\,a_{n+2}(n+1)\,t^n\\
\sum_{n=1}^{\infty}\left(-2n a_n t^{n-1}\right) &= \sum_{n=0}^{\infty}\left(-2(n+1)\,a_{n+1}t^n\right)
\end{aligned}
$$

Substituting all the above in Eq (2) gives the following equation where now all powers of t are the same and equal to n.

$$\left(\sum_{n=0}^{\infty} (n+2)\, a_{n+2}(n+1)\, t^{n}\right) + \sum_{n=1}^{\infty} (-n a_n t^n) + \sum_{n=0}^{\infty} (-2(n+1)\, a_{n+1} t^n) + \sum_{n=0}^{\infty} (-3 a_n t^n) = 0 \tag{3}$$

$n = 0$ gives

$$2a_2 - 2a_1 - 3a_0 = 0$$

$$a_2 = \frac{3a_0}{2} + a_1$$

For $1 \leq n$, the recurrence equation is

$$(n+2)\, a_{n+2}(n+1) - n a_n - 2(n+1)\, a_{n+1} - 3a_n = 0 \tag{4}$$

Solving for a_{n+2}, gives

$$\begin{aligned} a_{n+2} &= \frac{n a_n + 2n a_{n+1} + 3a_n + 2a_{n+1}}{(n+2)\,(n+1)} \\ &= \frac{(n+3)\, a_n}{(n+2)\,(n+1)} + \frac{(2n+2)\, a_{n+1}}{(n+2)\,(n+1)} \end{aligned} \tag{5}$$

For $n = 1$ the recurrence equation gives

$$6a_3 - 4a_1 - 4a_2 = 0$$

Which after substituting the earlier terms found becomes

$$a_3 = \frac{4a_1}{3} + a_0$$

For $n = 2$ the recurrence equation gives

$$12a_4 - 5a_2 - 6a_3 = 0$$

Which after substituting the earlier terms found becomes

$$a_4 = \frac{9a_0}{8} + \frac{13a_1}{12}$$

For $n = 3$ the recurrence equation gives

$$20a_5 - 6a_3 - 8a_4 = 0$$

Which after substituting the earlier terms found becomes

$$a_5 = \frac{5a_1}{6} + \frac{3a_0}{4}$$

For $n = 4$ the recurrence equation gives

$$30a_6 - 7a_4 - 10a_5 = 0$$

Which after substituting the earlier terms found becomes

$$a_6 = \frac{41a_0}{80} + \frac{191a_1}{360}$$

For $n = 5$ the recurrence equation gives

$$42a_7 - 8a_5 - 12a_6 = 0$$

Which after substituting the earlier terms found becomes

$$a_7 = \frac{391a_1}{1260} + \frac{81a_0}{280}$$

And so on. Therefore the solution is

$$\begin{aligned} y(t) &= \sum_{n=0}^{\infty} a_n t^n \\ &= a_3 t^3 + a_2 t^2 + a_1 t + a_0 + \dots \end{aligned}$$

Substituting the values for a_n found above, the solution becomes

$$y(t) = a_0 + a_1 t + \left(\frac{3a_0}{2} + a_1\right) t^2 + \left(\frac{4a_1}{3} + a_0\right) t^3 + \left(\frac{9a_0}{8} + \frac{13a_1}{12}\right) t^4 + \left(\frac{5a_1}{6} + \frac{3a_0}{4}\right) t^5 + \dots$$

Collecting terms, the solution becomes

$$y(t) = \left(1+\frac{3}{2}t^{2}+t^{3}+\frac{9}{8}t^{4}+\frac{3}{4}t^{5}\right)a_{0}+\left(t+t^{2}+\frac{4}{3}t^{3}+\frac{13}{12}t^{4}+\frac{5}{6}t^{5}\right)a_{1}+O\left(t^{6}\right) \tag{3}$$

At $t = 0$ the solution above becomes

$$y(t) = \left(1+\frac{3}{2}t^{2}+t^{3}+\frac{9}{8}t^{4}+\frac{3}{4}t^{5}\right)c_{1}+\left(t+t^{2}+\frac{4}{3}t^{3}+\frac{13}{12}t^{4}+\frac{5}{6}t^{5}\right)c_{2}+O\left(t^{6}\right)$$

Replacing t in the above with the original independent variable $x$susing $t = -2+x$ results in

$$y = \left(1+\frac{3(-2+x)^{2}}{2}+(-2+x)^{3}+\frac{9(-2+x)^{4}}{8}+\frac{3(-2+x)^{5}}{4}+\frac{41(-2+x)^{6}}{80}\right)y(2) + \left(-2+x+(-2+x)^{2}+\frac{4(-2+x)^{3}}{3}+\frac{13(-2+x)^{4}}{12}+\frac{5(-2+x)^{5}}{6} + \frac{191(-2+x)^{6}}{360}\right)y'(2)+O\left((-2+x)^{6}\right)$$

Summary

The solution(s) found are the following

$$y = \left(1+\frac{3(-2+x)^{2}}{2}+(-2+x)^{3}+\frac{9(-2+x)^{4}}{8}+\frac{3(-2+x)^{5}}{4}+\frac{41(-2+x)^{6}}{80}\right)y(2) + \left(-2+x+(-2+x)^{2}+\frac{4(-2+x)^{3}}{3}+\frac{13(-2+x)^{4}}{12}+\frac{5(-2+x)^{5}}{6} + \frac{191(-2+x)^{6}}{360}\right)y'(2)+O\left((-2+x)^{6}\right) \tag{1}$$

Verification of solutions

$$y = \left(1+\frac{3(-2+x)^{2}}{2}+(-2+x)^{3}+\frac{9(-2+x)^{4}}{8}+\frac{3(-2+x)^{5}}{4}+\frac{41(-2+x)^{6}}{80}\right)y(2) + \left(-2+x+(-2+x)^{2}+\frac{4(-2+x)^{3}}{3}+\frac{13(-2+x)^{4}}{12}+\frac{5(-2+x)^{5}}{6} + \frac{191(-2+x)^{6}}{360}\right)y'(2)+O\left((-2+x)^{6}\right)$$

Verified OK.

6.2.1 Maple step by step solution

Let's solve

$y'' - xy' - 3y = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Assume series solution for y

  $y = \sum_{k=0}^{\infty} a_k x^k$

☐ Rewrite DE with series expansions

  ◦ Convert $x \cdot y'$ to series expansion

    $x \cdot y' = \sum_{k=0}^{\infty} a_k k\, x^k$

  ◦ Convert y'' to series expansion

    $y'' = \sum_{k=2}^{\infty} a_k k(k-1)\, x^{k-2}$

  ◦ Shift index using $k- > k+2$

    $y'' = \sum_{k=0}^{\infty} a_{k+2}(k+2)\,(k+1)\, x^k$

  Rewrite DE with series expansions

  $\sum_{k=0}^{\infty} \left(a_{k+2}(k+2)\,(k+1) - a_k(k+3)\right) x^k = 0$

- Each term in the series must be 0, giving the recursion relation

  $(k^2 + 3k + 2)\, a_{k+2} - a_k(k+3) = 0$

- Recursion relation that defines the series solution to the ODE

  $\left[y = \sum_{k=0}^{\infty} a_k x^k, a_{k+2} = \frac{a_k(k+3)}{k^2+3k+2}\right]$

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Group is reducible, not completely reducible
   Solution has integrals. Trying a special function solution free of integrals...
   -> Trying a solution in terms of special functions:
      -> Bessel
      -> elliptic
      -> Legendre
      <- Kummer successful
   <- special function solution successful
      -> Trying to convert hypergeometric functions to elementary form...
      <- elementary form could result into a too large expression - returning special functi
   <- Kovacic algorithm successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 50

```
Order:=6;
dsolve(diff(y(x),x$2)-x*diff(y(x),x)-3*y(x)=0,y(x),type='series',x=2);
```

$$y(x) = \left(1 + \frac{3(-2+x)^2}{2} + (-2+x)^3 + \frac{9(-2+x)^4}{8} + \frac{3(-2+x)^5}{4}\right) y(2) + \left(-2 + x + (-2+x)^2 + \frac{4(-2+x)^3}{3} + \frac{13(-2+x)^4}{12} + \frac{5(-2+x)^5}{6}\right) D(y)(2) + O(x^6)$$

✓ Solution by Mathematica

Time used: 0.001 (sec). Leaf size: 79

```
AsymptoticDSolveValue[y''[x]-x*y'[x]-3*y[x]==0,y[x],{x,2,5}]
```

$$
y(x) \to c_1\left(\frac{3}{4}(x-2)^5+\frac{9}{8}(x-2)^4+(x-2)^3+\frac{3}{2}(x-2)^2+1\right) + c_2\left(\frac{5}{6}(x-2)^5+\frac{13}{12}(x-2)^4+\frac{4}{3}(x-2)^3+(x-2)^2+x-2\right)
$$

6.3 problem 3

Internal problem ID [5021]
Internal file name [OUTPUT/4514_Sunday_June_05_2022_02_59_54_PM_18162344/index.tex]

Book: Fundamentals of Differential Equations. By Nagle, Saff and Snider. 9th edition. Boston. Pearson 2018.
Section: Chapter 8, Series solutions of differential equations. Section 8.4. page 449
Problem number: 3.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second order series method. Ordinary point", "second order series method. Taylor series method"**

Maple gives the following as the ode type

```
[[_Emden, _Fowler]]
```

$$\boxed{(x^2 + x + 1) y'' - 3y = 0}$$

With the expansion point for the power series method at $x = 1$.

The ode does not have its expansion point at $x = 0$, therefore to simplify the computation of power series expansion, change of variable is made on the independent variable to shift the initial conditions and the expasion point back to zero. The new ode is then solved more easily since the expansion point is now at zero. The solution converted back to the original independent variable. Let

$$t = x - 1$$

The ode is converted to be in terms of the new independent variable t. This results in

$$((t + 1)^2 + t + 2) \left(\frac{d^2}{dt^2} y(t)\right) - 3y(t) = 0$$

With its expansion point and initial conditions now at $t = 0$. The transformed ODE is now solved. Solving ode using Taylor series method. This gives review on how the Taylor series method works for solving second order ode.

Let

$$y'' = f(x, y, y')$$

Assuming expansion is at $x_0 = 0$ (we can always shift the actual expansion point to 0 by change of variables) and assuming $f(x, y, y')$ is analytic at x_0 which must be the case for an ordinary point. Let initial conditions be $y(x_0) = y_0$ and $y'(x_0) = y'_0$. Using Taylor series gives

$$\begin{aligned}
y(x) &= y(x_0) + (x - x_0)\, y'(x_0) + \frac{(x - x_0)^2}{2} y''(x_0) + \frac{(x - x_0)^3}{3!} y'''(x_0) + \cdots \\
&= y_0 + x y'_0 + \frac{x^2}{2} f|_{x_0, y_0, y'_0} + \frac{x^3}{3!} f'|_{x_0, y_0, y'_0} + \cdots \\
&= y_0 + x y'_0 + \sum_{n=0}^{\infty} \frac{x^{n+2}}{(n+2)!} \left.\frac{d^n f}{dx^n}\right|_{x_0, y_0, y'_0}
\end{aligned}$$

But

$$\begin{aligned}
\frac{df}{dx} &= \frac{\partial f}{\partial x}\frac{dx}{dx} + \frac{\partial f}{\partial y}\frac{dy}{dx} + \frac{\partial f}{\partial y'}\frac{dy'}{dx} && (1) \\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} y' + \frac{\partial f}{\partial y'} y'' && (264) \\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} y' + \frac{\partial f}{\partial y'} f && (265) \\
\frac{d^2 f}{dx^2} &= \frac{d}{dx}\left(\frac{df}{dx}\right) \\
&= \frac{\partial}{\partial x}\left(\frac{df}{dx}\right) + \frac{\partial}{\partial y}\left(\frac{df}{dx}\right) y' + \frac{\partial}{\partial y'}\left(\frac{df}{dx}\right) f && (2) \\
\frac{d^3 f}{dx^3} &= \frac{d}{dx}\left(\frac{d^2 f}{dx^2}\right) \\
&= \frac{\partial}{\partial x}\left(\frac{d^2 f}{dx^2}\right) + \left(\frac{\partial}{\partial y}\frac{d^2 f}{dx^2}\right) y' + \frac{\partial}{\partial y'}\left(\frac{d^2 f}{dx^2}\right) f && (3) \\
&\vdots
\end{aligned}$$

And so on. Hence if we name $F_0 = f(x,y,y')$ then the above can be written as

$$
\begin{aligned}
F_0 &= f(x,y,y') && (4)\\
F_1 &= \frac{df}{dx}\\
&= \frac{dF_0}{dx}\\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}y' + \frac{\partial f}{\partial y'}y''\\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}y' + \frac{\partial f}{\partial y'}f && (5)\\
&= \frac{\partial F_0}{\partial x} + \frac{\partial F_0}{\partial y}y' + \frac{\partial F_0}{\partial y'}F_0\\
F_2 &= \frac{d}{dx}\left(\frac{d}{dx}f\right)\\
&= \frac{d}{dx}(F_1)\\
&= \frac{\partial}{\partial x}F_1 + \left(\frac{\partial F_1}{\partial y}\right)y' + \left(\frac{\partial F_1}{\partial y'}\right)y''\\
&= \frac{\partial}{\partial x}F_1 + \left(\frac{\partial F_1}{\partial y}\right)y' + \left(\frac{\partial F_1}{\partial y'}\right)F_0\\
&\vdots\\
F_n &= \frac{d}{dx}(F_{n-1})\\
&= \frac{\partial}{\partial x}F_{n-1} + \left(\frac{\partial F_{n-1}}{\partial y}\right)y' + \left(\frac{\partial F_{n-1}}{\partial y'}\right)y''\\
&= \frac{\partial}{\partial x}F_{n-1} + \left(\frac{\partial F_{n-1}}{\partial y}\right)y' + \left(\frac{\partial F_{n-1}}{\partial y'}\right)F_0 && (6)
\end{aligned}
$$

Therefore (6) can be used from now on along with

$$y(x) = y_0 + xy'_0 + \sum_{n=0}^{\infty} \frac{x^{n+2}}{(n+2)!} F_n|_{x_0,y_0,y'_0} \tag{7}$$

To find $y(x)$ series solution around $x = 0$. Hence

$$
\begin{aligned}
F_0 &= \frac{3y(t)}{t^2 + 3t + 3} \\
F_1 &= \frac{dF_0}{dt} \\
&= \frac{\partial F_0}{\partial t} + \frac{\partial F_0}{\partial y}\frac{d}{dt}y(t) + \frac{\partial F_0}{\partial \frac{d}{dt}y(t)} F_0 \\
&= \frac{(3t^2 + 9t + 9)\left(\frac{d}{dt}y(t)\right) + (-6t - 9)\,y(t)}{(t^2 + 3t + 3)^2} \\
F_2 &= \frac{dF_1}{dt} \\
&= \frac{\partial F_1}{\partial t} + \frac{\partial F_1}{\partial y}\frac{d}{dt}y(t) + \frac{\partial F_1}{\partial \frac{d}{dt}y(t)} F_1 \\
&= \frac{(-12t^3 - 54t^2 - 90t - 54)\left(\frac{d}{dt}y(t)\right) + 27y(t)\left(t^2 + 3t + \frac{7}{3}\right)}{(t^2 + 3t + 3)^3} \\
F_3 &= \frac{dF_2}{dt} \\
&= \frac{\partial F_2}{\partial t} + \frac{\partial F_2}{\partial y}\frac{d}{dt}y(t) + \frac{\partial F_2}{\partial \frac{d}{dt}y(t)} F_2 \\
&= \frac{(63t^4 + 378t^3 + 891t^2 + 972t + 405)\left(\frac{d}{dt}y(t)\right) - 144\left(t + \frac{3}{2}\right)^3 y(t)}{(t^2 + 3t + 3)^4} \\
F_4 &= \frac{dF_3}{dt} \\
&= \frac{\partial F_3}{\partial t} + \frac{\partial F_3}{\partial y}\frac{d}{dt}y(t) + \frac{\partial F_3}{\partial \frac{d}{dt}y(t)} F_3 \\
&= \frac{(-396t^5 - 2970t^4 - 9072t^3 - 14094t^2 - 11016t - 3402)\left(\frac{d}{dt}y(t)\right) + 909\left(t^4 + 6t^3 + \frac{1341}{101}t^2 + \frac{1296}{101}t + \frac{4}{1}\right.}{(t^2 + 3t + 3)^5}
\end{aligned}
$$

And so on. Evaluating all the above at initial conditions $t = 0$ and $y(0) = y(0)$ and $y'(0) = y'(0)$ gives

$$
\begin{aligned}
F_0 &= y(0) \\
F_1 &= -y(0) + y'(0) \\
F_2 &= \frac{7y(0)}{3} - 2y'(0) \\
F_3 &= -6y(0) + 5y'(0) \\
F_4 &= 17y(0) - 14y'(0)
\end{aligned}
$$

Substituting all the above in (7) and simplifying gives the solution as

$$\begin{aligned}y(t) = &\left(1+\frac{1}{2}t^2-\frac{1}{6}t^3+\frac{7}{72}t^4-\frac{1}{20}t^5+\frac{17}{720}t^6\right)y(0)\\ &+\left(t+\frac{1}{6}t^3-\frac{1}{12}t^4+\frac{1}{24}t^5-\frac{7}{360}t^6\right)y'(0)+O(t^6)\end{aligned}$$

Since the expansion point $t = 0$ is an ordinary, we can also solve this using standard power series The ode is normalized to be

$$\left(\frac{d^2}{dt^2}y(t)\right)\left(t^2+3t+3\right)-3y(t) = 0$$

Let the solution be represented as power series of the form

$$y(t) = \sum_{n=0}^{\infty} a_n t^n$$

Then

$$\begin{aligned}\frac{d}{dt}y(t) &= \sum_{n=1}^{\infty} n a_n t^{n-1}\\ \frac{d^2}{dt^2}y(t) &= \sum_{n=2}^{\infty} n(n-1)\, a_n t^{n-2}\end{aligned}$$

Substituting the above back into the ode gives

$$\left(\sum_{n=2}^{\infty} n(n-1)\, a_n t^{n-2}\right)\left(t^2+3t+3\right)-3\left(\sum_{n=0}^{\infty} a_n t^n\right) = 0 \tag{1}$$

Which simplifies to

$$\begin{aligned}&\left(\sum_{n=2}^{\infty} t^n a_n n(n-1)\right)+\left(\sum_{n=2}^{\infty} 3n\, t^{n-1} a_n(n-1)\right)\\ &+\left(\sum_{n=2}^{\infty} 3n(n-1)\, a_n t^{n-2}\right)+\sum_{n=0}^{\infty}\left(-3a_n t^n\right) = 0\end{aligned} \tag{2}$$

The next step is to make all powers of t be n in each summation term. Going over each summation term above with power of t in it which is not already t^n and adjusting the

power and the corresponding index gives

$$\sum_{n=2}^{\infty} 3n\, t^{n-1} a_n (n-1) = \sum_{n=1}^{\infty} 3(n+1)\, a_{n+1} n\, t^n$$

$$\sum_{n=2}^{\infty} 3n(n-1)\, a_n t^{n-2} = \sum_{n=0}^{\infty} 3(n+2)\, a_{n+2}(n+1)\, t^n$$

Substituting all the above in Eq (2) gives the following equation where now all powers of t are the same and equal to n.

$$\left(\sum_{n=2}^{\infty} t^n a_n n(n-1)\right) + \left(\sum_{n=1}^{\infty} 3(n+1)\, a_{n+1} n\, t^n\right) + \left(\sum_{n=0}^{\infty} 3(n+2)\, a_{n+2}(n+1)\, t^n\right) + \sum_{n=0}^{\infty} (-3a_n t^n) = 0 \tag{3}$$

$n = 0$ gives

$$6a_2 - 3a_0 = 0$$

$$a_2 = \frac{a_0}{2}$$

$n = 1$ gives

$$6a_2 + 18a_3 - 3a_1 = 0$$

Which after substituting earlier equations, simplifies to

$$a_3 = -\frac{a_0}{6} + \frac{a_1}{6}$$

For $2 \leq n$, the recurrence equation is

$$na_n(n-1) + 3(n+1)\, a_{n+1} n + 3(n+2)\, a_{n+2}(n+1) - 3a_n = 0 \tag{4}$$

Solving for a_{n+2}, gives

$$\begin{aligned} a_{n+2} &= -\frac{n^2 a_n + 3n^2 a_{n+1} - na_n + 3na_{n+1} - 3a_n}{3\,(n+2)\,(n+1)} \\ &= -\frac{(n^2 - n - 3)\, a_n}{3\,(n+2)\,(n+1)} - \frac{(3n^2 + 3n)\, a_{n+1}}{3\,(n+2)\,(n+1)} \end{aligned} \tag{5}$$

For $n = 2$ the recurrence equation gives

$$-a_2 + 18a_3 + 36a_4 = 0$$

Which after substituting the earlier terms found becomes

$$a_4 = \frac{7a_0}{72} - \frac{a_1}{12}$$

For $n = 3$ the recurrence equation gives

$$3a_3 + 36a_4 + 60a_5 = 0$$

Which after substituting the earlier terms found becomes

$$a_5 = -\frac{a_0}{20} + \frac{a_1}{24}$$

For $n = 4$ the recurrence equation gives

$$9a_4 + 60a_5 + 90a_6 = 0$$

Which after substituting the earlier terms found becomes

$$a_6 = \frac{17a_0}{720} - \frac{7a_1}{360}$$

For $n = 5$ the recurrence equation gives

$$17a_5 + 90a_6 + 126a_7 = 0$$

Which after substituting the earlier terms found becomes

$$a_7 = -\frac{17a_0}{1680} + \frac{25a_1}{3024}$$

And so on. Therefore the solution is

$$\begin{aligned} y(t) &= \sum_{n=0}^{\infty} a_n t^n \\ &= a_3 t^3 + a_2 t^2 + a_1 t + a_0 + \dots \end{aligned}$$

Substituting the values for a_n found above, the solution becomes

$$y(t) = a_0 + a_1 t + \frac{a_0 t^2}{2} + \left(-\frac{a_0}{6} + \frac{a_1}{6}\right) t^3 + \left(\frac{7a_0}{72} - \frac{a_1}{12}\right) t^4 + \left(-\frac{a_0}{20} + \frac{a_1}{24}\right) t^5 + \dots$$

Collecting terms, the solution becomes

$$y(t) = \left(1 + \frac{1}{2}t^2 - \frac{1}{6}t^3 + \frac{7}{72}t^4 - \frac{1}{20}t^5\right) a_0 + \left(t + \frac{1}{6}t^3 - \frac{1}{12}t^4 + \frac{1}{24}t^5\right) a_1 + O(t^6) \quad (3)$$

At $t = 0$ the solution above becomes

$$y(t) = \left(1 + \frac{1}{2}t^2 - \frac{1}{6}t^3 + \frac{7}{72}t^4 - \frac{1}{20}t^5\right) c_1 + \left(t + \frac{1}{6}t^3 - \frac{1}{12}t^4 + \frac{1}{24}t^5\right) c_2 + O(t^6)$$

Replacing t in the above with the original independent variable $x s$using $t = x - 1$ results in

$$y = \left(1 + \frac{(x-1)^2}{2} - \frac{(x-1)^3}{6} + \frac{7(x-1)^4}{72} - \frac{(x-1)^5}{20} + \frac{17(x-1)^6}{720}\right) y(1) + \left(x - 1 + \frac{(x-1)^3}{6} - \frac{(x-1)^4}{12} + \frac{(x-1)^5}{24} - \frac{7(x-1)^6}{360}\right) y'(1) + O\left((x-1)^6\right)$$

Summary

The solution(s) found are the following

$$y = \left(1 + \frac{(x-1)^2}{2} - \frac{(x-1)^3}{6} + \frac{7(x-1)^4}{72} - \frac{(x-1)^5}{20} + \frac{17(x-1)^6}{720}\right) y(1) + \left(x - 1 + \frac{(x-1)^3}{6} - \frac{(x-1)^4}{12} + \frac{(x-1)^5}{24} - \frac{7(x-1)^6}{360}\right) y'(1) + O\left((x-1)^6\right) \quad (1)$$

Verification of solutions

$$y = \left(1 + \frac{(x-1)^2}{2} - \frac{(x-1)^3}{6} + \frac{7(x-1)^4}{72} - \frac{(x-1)^5}{20} + \frac{17(x-1)^6}{720}\right) y(1) + \left(x - 1 + \frac{(x-1)^3}{6} - \frac{(x-1)^4}{12} + \frac{(x-1)^5}{24} - \frac{7(x-1)^6}{360}\right) y'(1) + O\left((x-1)^6\right)$$

Verified OK.

6.3.1 Maple step by step solution

Let's solve

$(x^2+x+1)\,y''-3y=0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y''=\frac{3y}{x^2+x+1}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y''-\frac{3y}{x^2+x+1}=0$

□ Check to see if x_0 is a regular singular point

  ◦ Define functions

  $\left[P_2(x)=0, P_3(x)=-\frac{3}{x^2+x+1}\right]$

  ◦ $\left(x+\frac{1}{2}+\frac{\mathrm{I}\sqrt{3}}{2}\right)\cdot P_2(x)$ is analytic at $x=-\frac{1}{2}-\frac{\mathrm{I}\sqrt{3}}{2}$

  $\left(\left(x+\frac{1}{2}+\frac{\mathrm{I}\sqrt{3}}{2}\right)\cdot P_2(x)\right)\bigg|_{x=-\frac{1}{2}-\frac{\mathrm{I}\sqrt{3}}{2}}=0$

  ◦ $\left(x+\frac{1}{2}+\frac{\mathrm{I}\sqrt{3}}{2}\right)^2\cdot P_3(x)$ is analytic at $x=-\frac{1}{2}-\frac{\mathrm{I}\sqrt{3}}{2}$

  $\left(\left(x+\frac{1}{2}+\frac{\mathrm{I}\sqrt{3}}{2}\right)^2\cdot P_3(x)\right)\bigg|_{x=-\frac{1}{2}-\frac{\mathrm{I}\sqrt{3}}{2}}=0$

  ◦ $x=-\frac{1}{2}-\frac{\mathrm{I}\sqrt{3}}{2}$is a regular singular point

  Check to see if x_0 is a regular singular point

  $x_0=-\frac{1}{2}-\frac{\mathrm{I}\sqrt{3}}{2}$

- Multiply by denominators

  $(x^2+x+1)\,y''-3y=0$

- Change variables using $x=u-\frac{1}{2}-\frac{\mathrm{I}\sqrt{3}}{2}$ so that the regular singular point is at $u=0$

  $(u^2-\mathrm{I}u\sqrt{3})\left(\frac{d^2}{du^2}y(u)\right)-3y(u)=0$

- Assume series solution for $y(u)$

  $y(u)=\sum_{k=0}^{\infty} a_k u^{k+r}$

☐ Rewrite ODE with series expansions

- Convert $u^m \cdot \left(\frac{d^2}{du^2}y(u)\right)$ to series expansion for $m = 1..2$

$$u^m \cdot \left(\frac{d^2}{du^2}y(u)\right) = \sum_{k=0}^{\infty} a_k(k+r)(k+r-1)\,u^{k+r-2+m}$$

- Shift index using $k->k+2-m$

$$u^m \cdot \left(\frac{d^2}{du^2}y(u)\right) = \sum_{k=-2+m}^{\infty} a_{k+2-m}(k+2-m+r)(k+1-m+r)\,u^{k+r}$$

Rewrite ODE with series expansions

$$-\mathrm{I}\sqrt{3}\,r(r-1)\,a_0u^{r-1} + \left(\sum_{k=0}^{\infty}\left(-\mathrm{I}\sqrt{3}\,(k+1+r)(k+r)\,a_{k+1} + a_k(k^2+2kr+r^2-k-r-3)\right)u$$

- $a_0$cannot be 0 by assumption, giving the indicial equation

$$-\mathrm{I}\sqrt{3}\,r(r-1) = 0$$

- Values of r that satisfy the indicial equation

$$r \in \{0, 1\}$$

- Each term in the series must be 0, giving the recursion relation

$$-\mathrm{I}\sqrt{3}\,(k+1+r)(k+r)\,a_{k+1} + a_k(k^2+(2r-1)\,k+r^2-r-3) = 0$$

- Recursion relation that defines series solution to ODE

$$a_{k+1} = \frac{-\frac{\mathrm{I}}{3}a_k\,(k^2+2kr+r^2-k-r-3)\sqrt{3}}{k^2+2kr+r^2+k+r}$$

- Recursion relation for $r = 0$

$$a_{k+1} = \frac{-\frac{\mathrm{I}}{3}a_k\,(k^2-k-3)\sqrt{3}}{k^2+k}$$

- Solution for $r = 0$

$$\left[y(u) = \sum_{k=0}^{\infty} a_k u^k, a_{k+1} = \frac{-\frac{\mathrm{I}}{3}a_k\,(k^2-k-3)\sqrt{3}}{k^2+k}\right]$$

- Revert the change of variables $u = x + \frac{1}{2} + \frac{\mathrm{I}\sqrt{3}}{2}$

$$\left[y = \sum_{k=0}^{\infty} a_k\left(x+\frac{1}{2}+\frac{\mathrm{I}\sqrt{3}}{2}\right)^k, a_{k+1} = \frac{-\frac{\mathrm{I}}{3}a_k\,(k^2-k-3)\sqrt{3}}{k^2+k}\right]$$

- Recursion relation for $r = 1$

$$a_{k+1} = \frac{-\frac{\mathrm{I}}{3}a_k\,(k^2+k-3)\sqrt{3}}{k^2+3k+2}$$

- Solution for $r = 1$

$$\left[y(u) = \sum_{k=0}^{\infty} a_k u^{k+1}, a_{k+1} = \frac{-\frac{\mathrm{I}}{3}a_k\,(k^2+k-3)\sqrt{3}}{k^2+3k+2}\right]$$

- Revert the change of variables $u = x + \frac{1}{2} + \frac{I\sqrt{3}}{2}$

$$\left[y = \sum_{k=0}^{\infty} a_k \left(x + \frac{1}{2} + \frac{I\sqrt{3}}{2}\right)^{k+1}, a_{k+1} = \frac{-\frac{I}{3} a_k (k^2+k-3)\sqrt{3}}{k^2+3k+2}\right]$$

- Combine solutions and rename parameters

$$\left[y = \left(\sum_{k=0}^{\infty} a_k \left(x + \frac{1}{2} + \frac{I\sqrt{3}}{2}\right)^{k}\right) + \left(\sum_{k=0}^{\infty} b_k \left(x + \frac{1}{2} + \frac{I\sqrt{3}}{2}\right)^{k+1}\right), a_{k+1} = \frac{-\frac{I}{3} a_k (k^2-k-3)\sqrt{3}}{k^2+k}, b_{k+1} = \frac{-\frac{I}{3} b_k$$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
<- No Liouvillian solutions exists
-> Trying a solution in terms of special functions:
   -> Bessel
   -> elliptic
   -> Legendre
   -> Whittaker
      -> hyper3: Equivalence to 1F1 under a power @ Moebius
   -> hypergeometric
      -> heuristic approach
      -> hyper3: Equivalence to 2F1, 1F1 or 0F1 under a power @ Moebius
      <- hyper3 successful: received ODE is equivalent to the 2F1 ODE
   <- hypergeometric successful
<- special function solution successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 49

```
Order:=6;
dsolve((1+x+x^2)*diff(y(x),x$2)-3*y(x)=0,y(x),type='series',x=1);
```

$$y(x) = \left(1 + \frac{(x-1)^2}{2} - \frac{(x-1)^3}{6} + \frac{7(x-1)^4}{72} - \frac{(x-1)^5}{20}\right) y(1) + \left(x - 1 + \frac{(x-1)^3}{6} - \frac{(x-1)^4}{12} + \frac{(x-1)^5}{24}\right) D(y)(1) + O(x^6)$$

✓ Solution by Mathematica

Time used: 0.001 (sec). Leaf size: 78

```
AsymptoticDSolveValue[(1+x+x^2)*y''[x]-3*y[x]==0,y[x],{x,1,5}]
```

$$y(x) \to c_1 \left(-\frac{1}{20}(x-1)^5 + \frac{7}{72}(x-1)^4 - \frac{1}{6}(x-1)^3 + \frac{1}{2}(x-1)^2 + 1\right) + c_2 \left(\frac{1}{24}(x-1)^5 - \frac{1}{12}(x-1)^4 + \frac{1}{6}(x-1)^3 + x - 1\right)$$

6.4 problem 4

Internal problem ID [5022]
Internal file name [`OUTPUT/4515_Sunday_June_05_2022_02_59_56_PM_19870957/index.tex`]

Book: Fundamentals of Differential Equations. By Nagle, Saff and Snider. 9th edition. Boston. Pearson 2018.
Section: Chapter 8, Series solutions of differential equations. Section 8.4. page 449
Problem number: 4.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second order series method. Ordinary point", "second order series method. Taylor series method"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

$$\boxed{(x^2-5x+6)\, y''-3xy'-y=0}$$

With the expansion point for the power series method at $x=0$.

Solving ode using Taylor series method. This gives review on how the Taylor series method works for solving second order ode.

Let

$$y''=f(x,y,y')$$

Assuming expansion is at $x_0=0$ (we can always shift the actual expansion point to 0 by change of variables) and assuming $f(x,y,y')$ is analytic at x_0 which must be the case for an ordinary point. Let initial conditions be $y(x_0)=y_0$ and $y'(x_0)=y'_0$. Using Taylor series gives

$$\begin{aligned} y(x) &= y(x_0)+(x-x_0)\, y'(x_0)+\frac{(x-x_0)^2}{2}y''(x_0)+\frac{(x-x_0)^3}{3!}y'''(x_0)+\cdots \\ &= y_0+xy'_0+\frac{x^2}{2}f|_{x_0,y_0,y'_0}+\frac{x^3}{3!}f'|_{x_0,y_0,y'_0}+\cdots \\ &= y_0+xy'_0+\sum_{n=0}^{\infty}\frac{x^{n+2}}{(n+2)\,!}\frac{d^nf}{dx^n}\bigg|_{x_0,y_0,y'_0} \end{aligned}$$

But

$$\frac{df}{dx} = \frac{\partial f}{\partial x}\frac{dx}{dx} + \frac{\partial f}{\partial y}\frac{dy}{dx} + \frac{\partial f}{\partial y'}\frac{dy'}{dx} \tag{1}$$

$$= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}y' + \frac{\partial f}{\partial y'}y'' \tag{267}$$

$$= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}y' + \frac{\partial f}{\partial y'}f \tag{268}$$

$$\frac{d^2 f}{dx^2} = \frac{d}{dx}\left(\frac{df}{dx}\right)$$

$$= \frac{\partial}{\partial x}\left(\frac{df}{dx}\right) + \frac{\partial}{\partial y}\left(\frac{df}{dx}\right)y' + \frac{\partial}{\partial y'}\left(\frac{df}{dx}\right)f \tag{2}$$

$$\frac{d^3 f}{dx^3} = \frac{d}{dx}\left(\frac{d^2 f}{dx^2}\right)$$

$$= \frac{\partial}{\partial x}\left(\frac{d^2 f}{dx^2}\right) + \left(\frac{\partial}{\partial y}\frac{d^2 f}{dx^2}\right)y' + \frac{\partial}{\partial y'}\left(\frac{d^2 f}{dx^2}\right)f \tag{3}$$

$$\vdots$$

And so on. Hence if we name $F_0 = f(x, y, y')$ then the above can be written as

$$\begin{aligned}
F_0 &= f(x, y, y') \qquad (4)\\
F_1 &= \frac{df}{dx}\\
&= \frac{dF_0}{dx}\\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} y' + \frac{\partial f}{\partial y'} y''\\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} y' + \frac{\partial f}{\partial y'} f \qquad (5)\\
&= \frac{\partial F_0}{\partial x} + \frac{\partial F_0}{\partial y} y' + \frac{\partial F_0}{\partial y'} F_0\\
F_2 &= \frac{d}{dx}\left(\frac{d}{dx} f\right)\\
&= \frac{d}{dx}(F_1)\\
&= \frac{\partial}{\partial x} F_1 + \left(\frac{\partial F_1}{\partial y}\right) y' + \left(\frac{\partial F_1}{\partial y'}\right) y''\\
&= \frac{\partial}{\partial x} F_1 + \left(\frac{\partial F_1}{\partial y}\right) y' + \left(\frac{\partial F_1}{\partial y'}\right) F_0\\
&\vdots\\
F_n &= \frac{d}{dx}(F_{n-1})\\
&= \frac{\partial}{\partial x} F_{n-1} + \left(\frac{\partial F_{n-1}}{\partial y}\right) y' + \left(\frac{\partial F_{n-1}}{\partial y'}\right) y''\\
&= \frac{\partial}{\partial x} F_{n-1} + \left(\frac{\partial F_{n-1}}{\partial y}\right) y' + \left(\frac{\partial F_{n-1}}{\partial y'}\right) F_0 \qquad (6)
\end{aligned}$$

Therefore (6) can be used from now on along with

$$y(x) = y_0 + x y_0' + \sum_{n=0}^{\infty} \frac{x^{n+2}}{(n+2)!} F_n \big|_{x_0, y_0, y_0'} \qquad (7)$$

To find $y(x)$ series solution around $x = 0$. Hence

$$
\begin{aligned}
F_0 &= \frac{3xy' + y}{x^2 - 5x + 6} \\
F_1 &= \frac{dF_0}{dx} \\
&= \frac{\partial F_0}{\partial x} + \frac{\partial F_0}{\partial y} y' + \frac{\partial F_0}{\partial y'} F_0 \\
&= \frac{(7x^2 - 5x + 24)\, y' + y(5 + x)}{(x^2 - 5x + 6)^2} \\
F_2 &= \frac{dF_1}{dx} \\
&= \frac{\partial F_1}{\partial x} + \frac{\partial F_1}{\partial y} y' + \frac{\partial F_1}{\partial y'} F_1 \\
&= \frac{(8x^3 + 16x + 240)\, y' + 4y(x^2 - 5x + 20)}{(x^2 - 5x + 6)^3} \\
F_3 &= \frac{dF_2}{dx} \\
&= \frac{\partial F_2}{\partial x} + \frac{\partial F_2}{\partial y} y' + \frac{\partial F_2}{\partial y'} F_2 \\
&= \frac{(4x^4 - 40x^3 + 316x^2 - 1080x + 4176)\, y' - 8y(x^3 - 15x^2 + 77x - 165)}{(x^2 - 5x + 6)^4} \\
F_4 &= \frac{dF_3}{dx} \\
&= \frac{\partial F_3}{\partial x} + \frac{\partial F_3}{\partial y} y' + \frac{\partial F_3}{\partial y'} F_3 \\
&= \frac{(-12x^5 + 240x^4 - 2316x^3 + 11880x^2 - 43584x + 84960)\, y' + 44y\left(x^4 - \frac{200}{11}x^3 + \frac{1421}{11}x^2 - \frac{4860}{11}x + \frac{672}{11}\right.}{(x^2 - 5x + 6)^5}
\end{aligned}
$$

And so on. Evaluating all the above at initial conditions $x = 0$ and $y(0) = y(0)$ and $y'(0) = y'(0)$ gives

$$
\begin{aligned}
F_0 &= \frac{y(0)}{6} \\
F_1 &= \frac{5y(0)}{36} + \frac{2y'(0)}{3} \\
F_2 &= \frac{10y(0)}{27} + \frac{10y'(0)}{9} \\
F_3 &= \frac{55y(0)}{54} + \frac{29y'(0)}{9} \\
F_4 &= \frac{280y(0)}{81} + \frac{295y'(0)}{27}
\end{aligned}
$$

Substituting all the above in (7) and simplifying gives the solution as

$$
\begin{aligned}
y = & \left(1+\frac{1}{12}x^{2}+\frac{5}{216}x^{3}+\frac{5}{324}x^{4}+\frac{11}{1296}x^{5}+\frac{7}{1458}x^{6}\right) y(0) \\
& + \left(x+\frac{1}{9}x^{3}+\frac{5}{108}x^{4}+\frac{29}{1080}x^{5}+\frac{59}{3888}x^{6}\right) y'(0)+O\left(x^{6}\right)
\end{aligned}
$$

Since the expansion point $x = 0$ is an ordinary, we can also solve this using standard power series The ode is normalized to be

$$
\left(x^{2}-5x+6\right) y''-3xy'-y = 0
$$

Let the solution be represented as power series of the form

$$
y = \sum_{n=0}^{\infty} a_{n} x^{n}
$$

Then

$$
\begin{aligned}
y' &= \sum_{n=1}^{\infty} n a_{n} x^{n-1} \\
y'' &= \sum_{n=2}^{\infty} n(n-1)\, a_{n} x^{n-2}
\end{aligned}
$$

Substituting the above back into the ode gives

$$
\left(x^{2}-5x+6\right)\left(\sum_{n=2}^{\infty} n(n-1)\, a_{n} x^{n-2}\right)-3x\left(\sum_{n=1}^{\infty} n a_{n} x^{n-1}\right)-\left(\sum_{n=0}^{\infty} a_{n} x^{n}\right) = 0 \tag{1}
$$

Which simplifies to

$$
\begin{aligned}
& \left(\sum_{n=2}^{\infty} x^{n} a_{n} n(n-1)\right)+\sum_{n=2}^{\infty}\left(-5n\, x^{n-1} a_{n}(n-1)\right) \\
& + \left(\sum_{n=2}^{\infty} 6n(n-1)\, a_{n} x^{n-2}\right)+\sum_{n=1}^{\infty}\left(-3n a_{n} x^{n}\right)+\sum_{n=0}^{\infty}\left(-a_{n} x^{n}\right) = 0
\end{aligned} \tag{2}
$$

The next step is to make all powers of x be n in each summation term. Going over each summation term above with power of x in it which is not already x^{n} and adjusting the

power and the corresponding index gives

$$\sum_{n=2}^{\infty} \left(-5n\, x^{n-1} a_n (n-1)\right) = \sum_{n=1}^{\infty} \left(-5(n+1)\, a_{n+1} n\, x^n\right)$$

$$\sum_{n=2}^{\infty} 6n(n-1)\, a_n x^{n-2} = \sum_{n=0}^{\infty} 6(n+2)\, a_{n+2}(n+1)\, x^n$$

Substituting all the above in Eq (2) gives the following equation where now all powers of x are the same and equal to n.

$$\begin{aligned}\left(\sum_{n=2}^{\infty} x^n a_n n(n-1)\right) + \sum_{n=1}^{\infty} \left(-5(n+1)\, a_{n+1} n\, x^n\right) \\ + \left(\sum_{n=0}^{\infty} 6(n+2)\, a_{n+2}(n+1)\, x^n\right) + \sum_{n=1}^{\infty} \left(-3n a_n x^n\right) + \sum_{n=0}^{\infty} \left(-a_n x^n\right) = 0\end{aligned} \tag{3}$$

$n = 0$ gives

$$12a_2 - a_0 = 0$$

$$a_2 = \frac{a_0}{12}$$

$n = 1$ gives

$$-10a_2 + 36a_3 - 4a_1 = 0$$

Which after substituting earlier equations, simplifies to

$$a_3 = \frac{5a_0}{216} + \frac{a_1}{9}$$

For $2 \leq n$, the recurrence equation is

$$n a_n (n-1) - 5(n+1)\, a_{n+1} n + 6(n+2)\, a_{n+2}(n+1) - 3n a_n - a_n = 0 \tag{4}$$

Solving for a_{n+2}, gives

$$\begin{aligned} a_{n+2} &= -\frac{n^2 a_n - 5n^2 a_{n+1} - 4n a_n - 5n a_{n+1} - a_n}{6\,(n+2)\,(n+1)} \\ &= -\frac{(n^2 - 4n - 1)\, a_n}{6\,(n+2)\,(n+1)} - \frac{(-5n^2 - 5n)\, a_{n+1}}{6\,(n+2)\,(n+1)} \end{aligned} \tag{5}$$

For $n = 2$ the recurrence equation gives

$$-5a_2 - 30a_3 + 72a_4 = 0$$

Which after substituting the earlier terms found becomes

$$a_4 = \frac{5a_0}{324} + \frac{5a_1}{108}$$

For $n = 3$ the recurrence equation gives

$$-4a_3 - 60a_4 + 120a_5 = 0$$

Which after substituting the earlier terms found becomes

$$a_5 = \frac{11a_0}{1296} + \frac{29a_1}{1080}$$

For $n = 4$ the recurrence equation gives

$$-a_4 - 100a_5 + 180a_6 = 0$$

Which after substituting the earlier terms found becomes

$$a_6 = \frac{7a_0}{1458} + \frac{59a_1}{3888}$$

For $n = 5$ the recurrence equation gives

$$4a_5 - 150a_6 + 252a_7 = 0$$

Which after substituting the earlier terms found becomes

$$a_7 = \frac{667a_0}{244944} + \frac{7027a_1}{816480}$$

And so on. Therefore the solution is

$$\begin{aligned} y &= \sum_{n=0}^{\infty} a_n x^n \\ &= a_3x^3 + a_2x^2 + a_1x + a_0 + \dots \end{aligned}$$

Substituting the values for a_n found above, the solution becomes

$$y = a_0 + a_1 x + \frac{a_0 x^2}{12} + \left(\frac{5a_0}{216} + \frac{a_1}{9}\right) x^3 + \left(\frac{5a_0}{324} + \frac{5a_1}{108}\right) x^4 + \left(\frac{11a_0}{1296} + \frac{29a_1}{1080}\right) x^5 + \dots$$

Collecting terms, the solution becomes

$$\begin{aligned} y = &\left(1 + \frac{1}{12}x^2 + \frac{5}{216}x^3 + \frac{5}{324}x^4 + \frac{11}{1296}x^5\right) a_0 \\ &+ \left(x + \frac{1}{9}x^3 + \frac{5}{108}x^4 + \frac{29}{1080}x^5\right) a_1 + O(x^6) \end{aligned} \tag{3}$$

At $x = 0$ the solution above becomes

$$y = \left(1 + \frac{1}{12}x^2 + \frac{5}{216}x^3 + \frac{5}{324}x^4 + \frac{11}{1296}x^5\right) c_1 + \left(x + \frac{1}{9}x^3 + \frac{5}{108}x^4 + \frac{29}{1080}x^5\right) c_2 + O(x^6)$$

<u>Summary</u>
The solution(s) found are the following

$$\begin{aligned} y = &\left(1 + \frac{1}{12}x^2 + \frac{5}{216}x^3 + \frac{5}{324}x^4 + \frac{11}{1296}x^5 + \frac{7}{1458}x^6\right) y(0) \\ &+ \left(x + \frac{1}{9}x^3 + \frac{5}{108}x^4 + \frac{29}{1080}x^5 + \frac{59}{3888}x^6\right) y'(0) + O(x^6) \end{aligned} \tag{1}$$

$$\begin{aligned} y = &\left(1 + \frac{1}{12}x^2 + \frac{5}{216}x^3 + \frac{5}{324}x^4 + \frac{11}{1296}x^5\right) c_1 \\ &+ \left(x + \frac{1}{9}x^3 + \frac{5}{108}x^4 + \frac{29}{1080}x^5\right) c_2 + O(x^6) \end{aligned} \tag{2}$$

<u>Verification of solutions</u>

$$\begin{aligned} y = &\left(1 + \frac{1}{12}x^2 + \frac{5}{216}x^3 + \frac{5}{324}x^4 + \frac{11}{1296}x^5 + \frac{7}{1458}x^6\right) y(0) \\ &+ \left(x + \frac{1}{9}x^3 + \frac{5}{108}x^4 + \frac{29}{1080}x^5 + \frac{59}{3888}x^6\right) y'(0) + O(x^6) \end{aligned}$$

Verified OK.

$$y = \left(1 + \frac{1}{12}x^2 + \frac{5}{216}x^3 + \frac{5}{324}x^4 + \frac{11}{1296}x^5\right) c_1 + \left(x + \frac{1}{9}x^3 + \frac{5}{108}x^4 + \frac{29}{1080}x^5\right) c_2 + O(x^6)$$

Verified OK.

6.4.1 Maple step by step solution

Let's solve

$(x^2-5x+6)\,y''-3xy'-y=0$

- Highest derivative means the order of the ODE is 2

y''

- Isolate 2nd derivative

$y''=\frac{y}{x^2-5x+6}+\frac{3xy'}{x^2-5x+6}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

$y''-\frac{3xy'}{x^2-5x+6}-\frac{y}{x^2-5x+6}=0$

□ Check to see if x_0 is a regular singular point

  - Define functions

  $\left[P_2(x)=-\frac{3x}{x^2-5x+6},P_3(x)=-\frac{1}{x^2-5x+6}\right]$

  - $(-2+x)\cdot P_2(x)$ is analytic at $x=2$

  $\left((-2+x)\cdot P_2(x)\right)\bigg|_{x=2}=6$

  - $(-2+x)^2\cdot P_3(x)$ is analytic at $x=2$

  $\left((-2+x)^2\cdot P_3(x)\right)\bigg|_{x=2}=0$

  - $x=2$is a regular singular point

  Check to see if x_0 is a regular singular point

  $x_0=2$

- Multiply by denominators

$(x^2-5x+6)\,y''-3xy'-y=0$

- Change variables using $x=u+2$ so that the regular singular point is at $u=0$

$(u^2-u)\left(\frac{d^2}{du^2}y(u)\right)+(-3u-6)\left(\frac{d}{du}y(u)\right)-y(u)=0$

- Assume series solution for $y(u)$

$y(u)=\sum_{k=0}^{\infty}a_k u^{k+r}$

□ Rewrite ODE with series expansions

  - Convert $u^m\cdot\left(\frac{d}{du}y(u)\right)$ to series expansion for $m=0..1$

$$u^m \cdot \left(\frac{d}{du}y(u)\right) = \sum_{k=0}^{\infty} a_k(k+r)\, u^{k+r-1+m}$$

- Shift index using $k- >k+1-m$

$$u^m \cdot \left(\frac{d}{du}y(u)\right) = \sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\, u^{k+r}$$

- Convert $u^m \cdot \left(\frac{d^2}{du^2}y(u)\right)$ to series expansion for $m = 1..2$

$$u^m \cdot \left(\frac{d^2}{du^2}y(u)\right) = \sum_{k=0}^{\infty} a_k(k+r)(k+r-1)\, u^{k+r-2+m}$$

- Shift index using $k- >k+2-m$

$$u^m \cdot \left(\frac{d^2}{du^2}y(u)\right) = \sum_{k=-2+m}^{\infty} a_{k+2-m}(k+2-m+r)(k+1-m+r)\, u^{k+r}$$

Rewrite ODE with series expansions

$$-a_0 r(5+r)\, u^{-1+r} + \left(\sum_{k=0}^{\infty} \left(-a_{k+1}(k+1+r)(k+6+r) + a_k(k^2+2kr+r^2-4k-4r-1)\right) u^{k+}\right.$$

- $a_0$cannot be 0 by assumption, giving the indicial equation

$$-r(5+r) = 0$$

- Values of r that satisfy the indicial equation

$$r \in \{-5, 0\}$$

- Each term in the series must be 0, giving the recursion relation

$$-a_{k+1}(k+1+r)(k+6+r) + a_k(k^2+(2r-4)k+r^2-4r-1) = 0$$

- Recursion relation that defines series solution to ODE

$$a_{k+1} = \frac{a_k(k^2+2kr+r^2-4k-4r-1)}{(k+1+r)(k+6+r)}$$

- Recursion relation for $r = -5$

$$a_{k+1} = \frac{a_k(k^2-14k+44)}{(k-4)(k+1)}$$

- Series not valid for $r = -5$, division by 0 in the recursion relation at $k = 4$

$$a_{k+1} = \frac{a_k(k^2-14k+44)}{(k-4)(k+1)}$$

- Recursion relation for $r = 0$

$$a_{k+1} = \frac{a_k(k^2-4k-1)}{(k+1)(k+6)}$$

- Solution for $r = 0$

$$\left[y(u) = \sum_{k=0}^{\infty} a_k u^k, a_{k+1} = \frac{a_k(k^2-4k-1)}{(k+1)(k+6)}\right]$$

- Revert the change of variables $u = -2 + x$

$$\left[y = \sum_{k=0}^{\infty} a_k (-2+x)^k, a_{k+1} = \frac{a_k (k^2-4k-1)}{(k+1)(k+6)}\right]$$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
<- No Liouvillian solutions exists
-> Trying a solution in terms of special functions:
   -> Bessel
   -> elliptic
   -> Legendre
   -> Kummer
      -> hyper3: Equivalence to 1F1 under a power @ Moebius
   -> hypergeometric
      -> heuristic approach
      <- heuristic approach successful
      -> solution has integrals; searching for one without integrals...
         -> hyper3: Equivalence to 2F1, 1F1 or 0F1 under a power @ Moebius
         <- hyper3 successful: received ODE is equivalent to the 2F1 ODE
      <- hypergeometric solution without integrals succesful
   <- hypergeometric successful
<- special function solution successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 49

```
Order:=6;
dsolve((x^2-5*x+6)*diff(y(x),x$2)-3*x*diff(y(x),x)-y(x)=0,y(x),type='series',x=0);
```

$$y(x) = \left(1 + \frac{1}{12}x^2 + \frac{5}{216}x^3 + \frac{5}{324}x^4 + \frac{11}{1296}x^5\right) y(0) + \left(x + \frac{1}{9}x^3 + \frac{5}{108}x^4 + \frac{29}{1080}x^5\right) D(y)(0) + O(x^6)$$

✓ Solution by Mathematica

Time used: 0.001 (sec). Leaf size: 63

```
AsymptoticDSolveValue[(x^2-5*x+6)*y''[x]-3*x*y'[x]-y[x]==0,y[x],{x,0,5}]
```

$$y(x) \to c_2\left(\frac{29x^5}{1080} + \frac{5x^4}{108} + \frac{x^3}{9} + x\right) + c_1\left(\frac{11x^5}{1296} + \frac{5x^4}{324} + \frac{5x^3}{216} + \frac{x^2}{12} + 1\right)$$

6.5 problem 5

Internal problem ID [5023]
Internal file name [`OUTPUT/4516_Sunday_June_05_2022_02_59_57_PM_80071994/index.tex`]

Book: Fundamentals of Differential Equations. By Nagle, Saff and Snider. 9th edition. Boston. Pearson 2018.
Section: Chapter 8, Series solutions of differential equations. Section 8.4. page 449
Problem number: 5.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second order series method. Ordinary point", "second order series method. Taylor series method"**

Maple gives the following as the ode type

`[_Lienard]`

$$\boxed{y'' - \tan(x)\, y' + y = 0}$$

With the expansion point for the power series method at $x = 1$.

The ode does not have its expansion point at $x = 0$, therefore to simplify the computation of power series expansion, change of variable is made on the independent variable to shift the initial conditions and the expasion point back to zero. The new ode is then solved more easily since the expansion point is now at zero. The solution converted back to the original independent variable. Let

$$t = x - 1$$

The ode is converted to be in terms of the new independent variable t. This results in

$$\frac{d^2}{dt^2}y(t) - \tan(t+1)\left(\frac{d}{dt}y(t)\right) + y(t) = 0$$

With its expansion point and initial conditions now at $t = 0$. The transformed ODE is now solved. Solving ode using Taylor series method. This gives review on how the Taylor series method works for solving second order ode.

Let

$$y'' = f(x, y, y')$$

Assuming expansion is at $x_0 = 0$ (we can always shift the actual expansion point to 0 by change of variables) and assuming $f(x, y, y')$ is analytic at x_0 which must be the case for an ordinary point. Let initial conditions be $y(x_0) = y_0$ and $y'(x_0) = y'_0$. Using Taylor series gives

$$\begin{aligned} y(x) &= y(x_0) + (x - x_0)\, y'(x_0) + \frac{(x - x_0)^2}{2} y''(x_0) + \frac{(x - x_0)^3}{3!} y'''(x_0) + \cdots \\ &= y_0 + x y'_0 + \frac{x^2}{2} f|_{x_0, y_0, y'_0} + \frac{x^3}{3!} f'|_{x_0, y_0, y'_0} + \cdots \\ &= y_0 + x y'_0 + \sum_{n=0}^{\infty} \frac{x^{n+2}}{(n+2)\,!} \left.\frac{d^n f}{dx^n}\right|_{x_0, y_0, y'_0} \end{aligned}$$

But

$$\begin{aligned} \frac{df}{dx} &= \frac{\partial f}{\partial x}\frac{dx}{dx} + \frac{\partial f}{\partial y}\frac{dy}{dx} + \frac{\partial f}{\partial y'}\frac{dy'}{dx} && (1) \\ &= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} y' + \frac{\partial f}{\partial y'} y'' && (270) \\ &= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} y' + \frac{\partial f}{\partial y'} f && (271) \\ \frac{d^2 f}{dx^2} &= \frac{d}{dx}\left(\frac{df}{dx}\right) \\ &= \frac{\partial}{\partial x}\left(\frac{df}{dx}\right) + \frac{\partial}{\partial y}\left(\frac{df}{dx}\right) y' + \frac{\partial}{\partial y'}\left(\frac{df}{dx}\right) f && (2) \\ \frac{d^3 f}{dx^3} &= \frac{d}{dx}\left(\frac{d^2 f}{dx^2}\right) \\ &= \frac{\partial}{\partial x}\left(\frac{d^2 f}{dx^2}\right) + \left(\frac{\partial}{\partial y}\frac{d^2 f}{dx^2}\right) y' + \frac{\partial}{\partial y'}\left(\frac{d^2 f}{dx^2}\right) f && (3) \\ &\vdots \end{aligned}$$

And so on. Hence if we name $F_0 = f(x, y, y')$ then the above can be written as

$$\begin{aligned}
F_0 &= f(x,y,y') \qquad (4)\\
F_1 &= \frac{df}{dx}\\
&= \frac{dF_0}{dx}\\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}y' + \frac{\partial f}{\partial y'}y''\\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}y' + \frac{\partial f}{\partial y'}f \qquad (5)\\
&= \frac{\partial F_0}{\partial x} + \frac{\partial F_0}{\partial y}y' + \frac{\partial F_0}{\partial y'}F_0\\
F_2 &= \frac{d}{dx}\left(\frac{d}{dx}f\right)\\
&= \frac{d}{dx}(F_1)\\
&= \frac{\partial}{\partial x}F_1 + \left(\frac{\partial F_1}{\partial y}\right)y' + \left(\frac{\partial F_1}{\partial y'}\right)y''\\
&= \frac{\partial}{\partial x}F_1 + \left(\frac{\partial F_1}{\partial y}\right)y' + \left(\frac{\partial F_1}{\partial y'}\right)F_0\\
&\vdots\\
F_n &= \frac{d}{dx}(F_{n-1})\\
&= \frac{\partial}{\partial x}F_{n-1} + \left(\frac{\partial F_{n-1}}{\partial y}\right)y' + \left(\frac{\partial F_{n-1}}{\partial y'}\right)y''\\
&= \frac{\partial}{\partial x}F_{n-1} + \left(\frac{\partial F_{n-1}}{\partial y}\right)y' + \left(\frac{\partial F_{n-1}}{\partial y'}\right)F_0 \qquad (6)
\end{aligned}$$

Therefore (6) can be used from now on along with

$$y(x) = y_0 + xy_0' + \sum_{n=0}^{\infty} \frac{x^{n+2}}{(n+2)!} F_n|_{x_0,y_0,y_0'} \tag{7}$$

To find $y(x)$ series solution around $x=0$. Hence

$$
\begin{aligned}
F_0 &= \tan(t+1)\left(\frac{d}{dt}y(t)\right) - y(t)\\
F_1 &= \frac{dF_0}{dt}\\
&= \frac{\partial F_0}{\partial t} + \frac{\partial F_0}{\partial y}\frac{d}{dt}y(t) + \frac{\partial F_0}{\partial \frac{d}{dt}y(t)}F_0\\
&= -\tan(t+1)\left(-2\tan(t+1)\left(\frac{d}{dt}y(t)\right) + y(t)\right)\\
F_2 &= \frac{dF_1}{dt}\\
&= \frac{\partial F_1}{\partial t} + \frac{\partial F_1}{\partial y}\frac{d}{dt}y(t) + \frac{\partial F_1}{\partial \frac{d}{dt}y(t)}F_1\\
&= \left(6\tan(t+1)\left(\frac{d}{dt}y(t)\right) - 3y(t)\right)\sec(t+1)^2 - 3\tan(t+1)\left(\frac{d}{dt}y(t)\right) + 2y(t)\\
F_3 &= \frac{dF_2}{dt}\\
&= \frac{\partial F_2}{\partial t} + \frac{\partial F_2}{\partial y}\frac{d}{dt}y(t) + \frac{\partial F_2}{\partial \frac{d}{dt}y(t)}F_2\\
&= 24\left(\frac{d}{dt}y(t)\right)\sec(t+1)^4 + \left(-12\tan(t+1)\,y(t) - 27\frac{d}{dt}y(t)\right)\sec(t+1)^2 + 3\tan(t+1)\,y(t) + 5\frac{d}{dt}y\\
F_4 &= \frac{dF_3}{dt}\\
&= \frac{\partial F_3}{\partial t} + \frac{\partial F_3}{\partial y}\frac{d}{dt}y(t) + \frac{\partial F_3}{\partial \frac{d}{dt}y(t)}F_3\\
&= \left(8\left(\cos(t+1)^4 - \frac{93\cos(t+1)^2}{8} + 15\right)\left(\frac{d}{dt}y(t)\right)\tan(t+1) - 60y(t)\right)\sec(t+1)^4 + 54y(t)\sec(t
\end{aligned}
$$

And so on. Evaluating all the above at initial conditions $t=0$ and $y(0)=y(0)$ and $y'(0)=y'(0)$ gives

$$
\begin{aligned}
F_0 &= \tan(1)\,y'(0) - y(0)\\
F_1 &= 2y'(0)\tan(1)^2 - y(0)\tan(1)\\
F_2 &= 6\tan(1)\sec(1)^2\,y'(0) - 3\sec(1)^2\,y(0) - 3\tan(1)\,y'(0) + 2y(0)\\
F_3 &= 24\sec(1)^4\,y'(0) - 12\tan(1)\sec(1)^2\,y(0) - 27\sec(1)^2\,y'(0) + 3y(0)\tan(1) + 5y'(0)\\
F_4 &= 8\tan(1)\,y'(0) - 93\tan(1)\sec(1)^2\,y'(0) + 120\tan(1)\sec(1)^4\,y'(0) - 60\sec(1)^4\,y(0) + 54\sec(1)^2\,y(0
\end{aligned}
$$

Substituting all the above in (7) and simplifying gives the solution as

$$y(t) = \left(1 - \frac{t^2}{2} - \frac{t^3 \tan(1)}{6} - \frac{t^4 \sec(1)^2}{8} + \frac{t^4}{12} - \frac{t^5 \tan(1) \sec(1)^2}{10} + \frac{t^5 \tan(1)}{40} - \frac{t^6 \sec(1)^4}{12} + \frac{3t^6 \sec(1)^2}{40} - \frac{t^6}{144}\right) y(0)$$
$$+ \left(t + \frac{t^2 \tan(1)}{2} + \frac{t^3 \tan(1)^2}{3} + \frac{t^4 \tan(1) \sec(1)^2}{4} - \frac{t^4 \tan(1)}{8} + \frac{t^5 \sec(1)^4}{5} - \frac{9t^5 \sec(1)^2}{40} + \frac{t^5}{24} + \frac{t^6 \tan(1)}{90} - \frac{31t^6 \tan(1) \sec(1)^2}{240} + \frac{t^6 \tan(1) \sec(1)^4}{6}\right) y'(0)$$
$$+ O(t^6)$$

Since the expansion point $t = 0$ is an ordinary, we can also solve this using standard power series Let the solution be represented as power series of the form

$$y(t) = \sum_{n=0}^{\infty} a_n t^n$$

Then

$$\frac{d}{dt} y(t) = \sum_{n=1}^{\infty} n a_n t^{n-1}$$
$$\frac{d^2}{dt^2} y(t) = \sum_{n=2}^{\infty} n(n-1) a_n t^{n-2}$$

Substituting the above back into the ode gives

$$\sum_{n=2}^{\infty} n(n-1) a_n t^{n-2} = \tan(t+1) \left(\sum_{n=1}^{\infty} n a_n t^{n-1}\right) - \left(\sum_{n=0}^{\infty} a_n t^n\right) \tag{1}$$

Expanding $-\tan(t+1)$ as Taylor series around $t = 0$ and keeping only the first 6 terms gives

$$-\tan(t+1) = -\tan(1) + \left(-1 - \tan(1)^2\right) t - \left(1 + \tan(1)^2\right) \tan(1) t^2 + \left(-\frac{1}{3} - \frac{4\tan(1)^2}{3} - \tan(1)^4\right)$$
$$= -\tan(1) + \left(-1 - \tan(1)^2\right) t - \left(1 + \tan(1)^2\right) \tan(1) t^2 + \left(-\frac{1}{3} - \frac{4\tan(1)^2}{3} - \tan(1)^4\right)$$

Hence the ODE in Eq (1) becomes

$$\left(\sum_{n=2}^{\infty} n(n-1)\, a_n t^{n-2}\right)+\left(-\tan(1)+\left(-1-\tan(1)^2\right)t-\left(1+\tan(1)^2\right)\tan(1)\, t^2\right.$$
$$+\left(-\frac{1}{3}-\frac{4\tan(1)^2}{3}-\tan(1)^4\right)t^3+\left(-\frac{5\tan(1)^3}{3}-\frac{2\tan(1)}{3}-\tan(1)^5\right)t^4$$
$$+\left(-\frac{2}{15}-\frac{17\tan(1)^2}{15}-2\tan(1)^4-\tan(1)^6\right)t^5$$
$$\left.+\left(-\frac{77\tan(1)^3}{45}-\frac{17\tan(1)}{45}-\frac{7\tan(1)^5}{3}-\tan(1)^7\right)t^6\right)\left(\sum_{n=1}^{\infty} n a_n t^{n-1}\right)$$
$$+\left(\sum_{n=0}^{\infty} a_n t^n\right)=0$$

Expanding the second term in (1) gives

$$\left(\sum_{n=2}^{\infty} n(n-1)\, a_n t^{n-2}\right)+-\tan(1)\cdot\left(\sum_{n=1}^{\infty} n a_n t^{n-1}\right)$$
$$+\left(-1-\tan(1)^2\right)t\cdot\left(\sum_{n=1}^{\infty} n a_n t^{n-1}\right)-\left(1+\tan(1)^2\right)\tan(1)\, t^2$$
$$\cdot\left(\sum_{n=1}^{\infty} n a_n t^{n-1}\right)+\left(-\frac{1}{3}-\frac{4\tan(1)^2}{3}-\tan(1)^4\right)t^3\cdot\left(\sum_{n=1}^{\infty} n a_n t^{n-1}\right)$$
$$+\left(-\frac{5\tan(1)^3}{3}-\frac{2\tan(1)}{3}-\tan(1)^5\right)t^4\cdot\left(\sum_{n=1}^{\infty} n a_n t^{n-1}\right)$$
$$+\left(-\frac{2}{15}-\frac{17\tan(1)^2}{15}-2\tan(1)^4-\tan(1)^6\right)t^5\cdot\left(\sum_{n=1}^{\infty} n a_n t^{n-1}\right)$$
$$+\left(-\frac{77\tan(1)^3}{45}-\frac{17\tan(1)}{45}-\frac{7\tan(1)^5}{3}-\tan(1)^7\right)t^6$$
$$\cdot\left(\sum_{n=1}^{\infty} n a_n t^{n-1}\right)+\left(\sum_{n=0}^{\infty} a_n t^n\right)=0$$

Which simplifies to

$$
\begin{aligned}
&\left(\sum_{n=2}^{\infty} n(n-1)\, a_n t^{n-2}\right)\\
&+\left(\sum_{n=1}^{\infty} \frac{n\, t^{n+5} a_n \tan(1) \sec(1)^{6} \left(-\cos(4)+56\cos(2)-123\right)}{180}\right)\\
&+\sum_{n=1}^{\infty} \left(-a_n \left(\sec(1)^{4}-\sec(1)^{2}+\frac{2}{15}\right) \sec(1)^{2} n\, t^{n+4}\right)\\
&+\left(\sum_{n=1}^{\infty} \frac{n\, t^{n+3} a_n \tan(1) \sec(1)^{4} \left(-5+\cos(2)\right)}{6}\right)\\
&+\sum_{n=1}^{\infty} \left(-a_n \left(\sec(1)^{2}-\frac{2}{3}\right) \sec(1)^{2} n\, t^{n+2}\right)\\
&+\sum_{n=1}^{\infty} \left(-n\, t^{1+n} a_n \tan(1) \sec(1)^{2}\right)+\sum_{n=1}^{\infty} \left(-n a_n t^{n} \sec(1)^{2}\right)\\
&+\sum_{n=1}^{\infty} \left(-n\, t^{n-1} a_n \tan(1)\right)+\left(\sum_{n=0}^{\infty} a_n t^{n}\right)=0
\end{aligned}
\tag{2}
$$

The next step is to make all powers of t be n in each summation term. Going over each summation term above with power of t in it which is not already t^n and adjusting the power and the corresponding index gives

$$
\sum_{n=2}^{\infty} n(n-1)\, a_n t^{n-2}=\sum_{n=0}^{\infty} (n+2)\, a_{n+2}(1+n)\, t^{n}
$$

$$
\begin{aligned}
&\sum_{n=1}^{\infty} \frac{n\, t^{n+5} a_n \tan(1) \sec(1)^{6} \left(-\cos(4)+56\cos(2)-123\right)}{180}\\
&=\sum_{n=6}^{\infty} \frac{(n-5)\, a_{n-5} \sec(1)^{6} \left(-\cos(4)+56\cos(2)-123\right) \tan(1)\, t^{n}}{180}
\end{aligned}
$$

$$
\begin{aligned}
&\sum_{n=1}^{\infty} \left(-a_n \left(\sec(1)^{4}-\sec(1)^{2}+\frac{2}{15}\right) \sec(1)^{2} n\, t^{n+4}\right)\\
&=\sum_{n=5}^{\infty} \left(-(n-4)\, a_{n-4} \sec(1)^{2} \left(\sec(1)^{4}-\sec(1)^{2}+\frac{2}{15}\right) t^{n}\right)
\end{aligned}
$$

$$\sum_{n=1}^{\infty} \frac{n\,t^{n+3} a_n \tan(1)\sec(1)^4 (-5+\cos(2))}{6} = \sum_{n=4}^{\infty} \frac{(n-3)\,a_{n-3}\sec(1)^4(-5+\cos(2))\tan(1)\,t^n}{6}$$

$$\sum_{n=1}^{\infty} \left(-a_n\left(\sec(1)^2-\frac{2}{3}\right)\sec(1)^2\,n\,t^{n+2}\right) = \sum_{n=3}^{\infty} \left(-(n-2)\,a_{n-2}\sec(1)^2\left(\sec(1)^2-\frac{2}{3}\right)t^n\right)$$

$$\sum_{n=1}^{\infty} \left(-n\,t^{1+n} a_n \tan(1)\sec(1)^2\right) = \sum_{n=2}^{\infty} \left(-(n-1)\,a_{n-1}\sec(1)^2\tan(1)\,t^n\right)$$

$$\sum_{n=1}^{\infty} \left(-n\,t^{n-1} a_n \tan(1)\right) = \sum_{n=0}^{\infty} \left(-(1+n)\,a_{1+n}\tan(1)\,t^n\right)$$

Substituting all the above in Eq (2) gives the following equation where now all powers of t are the same and equal to n.

$$\begin{aligned}
&\left(\sum_{n=0}^{\infty} (n+2)\,a_{n+2}(1+n)\,t^n\right) \\
&+\left(\sum_{n=6}^{\infty} \frac{(n-5)\,a_{n-5}\sec(1)^6(-\cos(4)+56\cos(2)-123)\tan(1)\,t^n}{180}\right) \\
&+\sum_{n=5}^{\infty} \left(-(n-4)\,a_{n-4}\sec(1)^2\left(\sec(1)^4-\sec(1)^2+\frac{2}{15}\right)t^n\right) \\
&+\left(\sum_{n=4}^{\infty} \frac{(n-3)\,a_{n-3}\sec(1)^4(-5+\cos(2))\tan(1)\,t^n}{6}\right) \\
&+\sum_{n=3}^{\infty} \left(-(n-2)\,a_{n-2}\sec(1)^2\left(\sec(1)^2-\frac{2}{3}\right)t^n\right) \\
&+\sum_{n=2}^{\infty} \left(-(n-1)\,a_{n-1}\sec(1)^2\tan(1)\,t^n\right)+\sum_{n=1}^{\infty} \left(-n a_n t^n \sec(1)^2\right) \\
&+\sum_{n=0}^{\infty} \left(-(1+n)\,a_{1+n}\tan(1)\,t^n\right)+\left(\sum_{n=0}^{\infty} a_n t^n\right)=0
\end{aligned} \tag{3}$$

$n=0$ gives

$$2a_2 - a_1\tan(1) + a_0 = 0$$

$$
a_2 = -\frac{a_0}{2} + \frac{a_1 \tan(1)}{2}
$$

$n = 1$ gives

$$
6a_3 - a_1 \sec(1)^2 - 2a_2 \tan(1) + a_1 = 0
$$

Which after substituting earlier equations, simplifies to

$$
a_3 = -\frac{\tan(1)\, a_0}{6} + \frac{a_1 \tan(1)^2}{6} + \frac{a_1 \sec(1)^2}{6} - \frac{a_1}{6}
$$

$n = 2$ gives

$$
12a_4 - a_1 \sec(1)^2 \tan(1) - 2a_2 \sec(1)^2 - 3a_3 \tan(1) + a_2 = 0
$$

Which after substituting earlier equations, simplifies to

$$
a_4 = -\frac{\sec(1)^2 a_0}{12} - \frac{\tan(1)^2 a_0}{24} + \frac{a_0}{24} + \frac{5a_1 \sec(1)^2 \tan(1)}{24} + \frac{a_1 \tan(1)^3}{24} - \frac{a_1 \tan(1)}{12}
$$

$n = 3$ gives

$$
20a_5 - a_1 \sec(1)^2 \left(\sec(1)^2 - \frac{2}{3}\right) - 2a_2 \sec(1)^2 \tan(1) - 3a_3 \sec(1)^2 - 4a_4 \tan(1) + a_3 = 0
$$

Which after substituting earlier equations, simplifies to

$$
\begin{aligned}
a_5 = {} & -\frac{11 \sec(1)^2 \tan(1)\, a_0}{120} - \frac{\tan(1)^3 a_0}{120} + \frac{\tan(1)\, a_0}{60} + \frac{7 \sec(1)^2 \tan(1)^2 a_1}{60} \\
& + \frac{3a_1 \sec(1)^4}{40} - \frac{a_1 \sec(1)^2}{15} + \frac{a_1 \tan(1)^4}{120} - \frac{a_1 \tan(1)^2}{40} + \frac{a_1}{120}
\end{aligned}
$$

$n = 4$ gives

$$
\begin{aligned}
& 30a_6 + \frac{a_1 \sec(1)^4 (-5 + \cos(2)) \tan(1)}{6} - 2a_2 \sec(1)^2 \left(\sec(1)^2 - \frac{2}{3}\right) \\
& - 3a_3 \sec(1)^2 \tan(1) - 4a_4 \sec(1)^2 - 5a_5 \tan(1) + a_4 = 0
\end{aligned}
$$

Which after substituting earlier equations, simplifies to

$$
\begin{aligned}
a_6 = {} & \frac{\tan(1)^2 a_0}{240} - \frac{2\sec(1)^4 a_0}{45} - \frac{\tan(1)^4 a_0}{720} + \frac{11\sec(1)^2 a_0}{360} - \frac{a_0}{720} - \frac{3\sec(1)^2 \tan(1)^2 a_0}{80} \\
& + \frac{a_1 \tan(1)}{240} - \frac{a_1 \tan(1)^3}{180} + \frac{a_1 \tan(1)^5}{720} - \frac{a_1 \sec(1)^4 \tan(1)\cos(2)}{180} \\
& - \frac{49 a_1 \sec(1)^2 \tan(1)}{720} + \frac{17 a_1 \sec(1)^4 \tan(1)}{144} + \frac{\sec(1)^2 \tan(1)^3 a_1}{24}
\end{aligned}
$$

$n = 5$ gives

$$
\begin{aligned}
& 42a_7 - a_1 \sec(1)^2 \left(\sec(1)^4 - \sec(1)^2 + \frac{2}{15}\right) + \frac{a_2 \sec(1)^4 (-5 + \cos(2)) \tan(1)}{3} \\
& - 3a_3 \sec(1)^2 \left(\sec(1)^2 - \frac{2}{3}\right) - 4a_4 \sec(1)^2 \tan(1) - 5a_5 \sec(1)^2 - 6a_6 \tan(1) + a_5 = 0
\end{aligned}
$$

Which after substituting earlier equations, simplifies to

$$
\begin{aligned}
a_7 = {} & \frac{103\sec(1)^2 \tan(1) a_0}{5040} - \frac{41\sec(1)^4 \tan(1) a_0}{720} - \frac{13\sec(1)^2 \tan(1)^3 a_0}{1260} \\
& + \frac{\sec(1)^4 \tan(1) a_0 \cos(2)}{252} - \frac{\tan(1) a_0}{1680} + \frac{\tan(1)^3 a_0}{1260} - \frac{\tan(1)^5 a_0}{5040} \\
& - \frac{79\sec(1)^2 \tan(1)^2 a_1}{2520} + \frac{83\sec(1)^4 \tan(1)^2 a_1}{1008} + \frac{11\sec(1)^2 \tan(1)^4 a_1}{1008} \\
& - \frac{\sec(1)^4 \tan(1)^2 a_1 \cos(2)}{210} - \frac{a_1}{5040} + \frac{a_1 \tan(1)^2}{840} - \frac{a_1 \tan(1)^4}{1008} \\
& + \frac{a_1 \tan(1)^6}{5040} + \frac{5a_1 \sec(1)^6}{112} + \frac{23a_1 \sec(1)^2}{1680} - \frac{269a_1 \sec(1)^4}{5040}
\end{aligned}
$$

For $6 \le n$, the recurrence equation is

$$
\begin{aligned}
& (n+2)\, a_{n+2}(1+n) \\
& + \frac{(n-5)\, a_{n-5} \sec(1)^6 (-\cos(4) + 56\cos(2) - 123)\tan(1)}{180} \\
& - (n-4)\, a_{n-4} \sec(1)^2 \left(\sec(1)^4 - \sec(1)^2 + \frac{2}{15}\right) \\
& + \frac{(n-3)\, a_{n-3} \sec(1)^4 (-5 + \cos(2))\tan(1)}{6} \\
& - (n-2)\, a_{n-2} \sec(1)^2 \left(\sec(1)^2 - \frac{2}{3}\right) - (n-1)\, a_{n-1} \sec(1)^2 \tan(1) \\
& - n a_n \sec(1)^2 - (1+n)\, a_{1+n} \tan(1) + a_n = 0
\end{aligned}
\tag{4}
$$

Solving for a_{n+2}, gives

(5)

$$a_{n+2} = -\frac{-180 a_{n-1} \sec(1)^2 \tan(1) n - 280 \tan(1) \cos(2) \sec(1)^6 a_{n-5} + 5 \tan(1) \cos(4) \sec(1)^6 a_{n-5} - 123 \tan}{}$$

$$
\begin{aligned}
&= -\frac{\left(-180 n \sec(1)^2 + 180\right) a_n}{180 (n+2) (1+n)} - \frac{\left(-180 \tan(1) n - 180 \tan(1)\right) a_{1+n}}{180 (n+2) (1+n)} \\
&\quad - \frac{\left(56 \tan(1) \cos(2) \sec(1)^6 n - \tan(1) \cos(4) \sec(1)^6 n - 280 \tan(1) \sec(1)^6 \cos(2) + 5 \tan(1) \sec}{180 (n+2) (1+n)} \\
&\quad - \frac{\left(-180 \sec(1)^6 n + 720 \sec(1)^6 + 180 n \sec(1)^4 - 720 \sec(1)^4 - 24 n \sec(1)^2 + 96 \sec(1)^2\right) a_{n-4}}{180 (n+2) (1+n)} \\
&\quad - \frac{\left(30 \tan(1) \cos(2) \sec(1)^4 n - 90 \tan(1) \sec(1)^4 \cos(2) - 150 \tan(1) \sec(1)^4 n + 450 \tan(1) \sec(}{180 (n+2) (1+n)} \\
&\quad - \frac{\left(-180 n \sec(1)^4 + 360 \sec(1)^4 + 120 n \sec(1)^2 - 240 \sec(1)^2\right) a_{n-2}}{180 (n+2) (1+n)} \\
&\quad - \frac{\left(-180 \tan(1) \sec(1)^2 n + 180 \tan(1) \sec(1)^2\right) a_{n-1}}{180 (n+2) (1+n)}
\end{aligned}
$$

And so on. Therefore the solution is

$$
\begin{aligned}
y(t) &= \sum_{n=0}^{\infty} a_n t^n \\
&= a_3 t^3 + a_2 t^2 + a_1 t + a_0 + \dots
\end{aligned}
$$

Substituting the values for a_n found above, the solution becomes

$$
\begin{aligned}
y(t) = a_0 + a_1 t &+ \left(-\frac{a_0}{2} + \frac{a_1 \tan(1)}{2}\right) t^2 \\
&+ \left(-\frac{\tan(1) a_0}{6} + \frac{a_1 \tan(1)^2}{6} + \frac{a_1 \sec(1)^2}{6} - \frac{a_1}{6}\right) t^3 + \left(-\frac{\sec(1)^2 a_0}{12}\right. \\
&\quad \left. - \frac{\tan(1)^2 a_0}{24} + \frac{a_0}{24} + \frac{5 a_1 \sec(1)^2 \tan(1)}{24} + \frac{a_1 \tan(1)^3}{24} - \frac{a_1 \tan(1)}{12}\right) t^4 \\
&+ \left(-\frac{11 \sec(1)^2 \tan(1) a_0}{120} - \frac{\tan(1)^3 a_0}{120} + \frac{\tan(1) a_0}{60} + \frac{7 \sec(1)^2 \tan(1)^2 a_1}{60}\right. \\
&\quad \left. + \frac{3 a_1 \sec(1)^4}{40} - \frac{a_1 \sec(1)^2}{15} + \frac{a_1 \tan(1)^4}{120} - \frac{a_1 \tan(1)^2}{40} + \frac{a_1}{120}\right) t^5 + \dots
\end{aligned}
$$

Collecting terms, the solution becomes

$$
\begin{aligned}
y(t) = {} & \left(1-\frac{t^2}{2}-\frac{t^3 \tan(1)}{6}+\left(-\frac{\sec(1)^2}{12}-\frac{\tan(1)^2}{24}+\frac{1}{24}\right)t^4 \right. \\
& \left. +\left(-\frac{11\tan(1)\sec(1)^2}{120}-\frac{\tan(1)^3}{120}+\frac{\tan(1)}{60}\right)t^5\right)a_0+\left(t+\frac{t^2\tan(1)}{2} \right. \\
& +\left(\frac{\tan(1)^2}{6}+\frac{\sec(1)^2}{6}-\frac{1}{6}\right)t^3+\left(\frac{5\tan(1)\sec(1)^2}{24}+\frac{\tan(1)^3}{24}-\frac{\tan(1)}{12}\right)t^4 \\
& \left. +\left(\frac{7\sec(1)^2\tan(1)^2}{60}+\frac{3\sec(1)^4}{40}-\frac{\sec(1)^2}{15}+\frac{\tan(1)^4}{120}-\frac{\tan(1)^2}{40}+\frac{1}{120}\right)t^5\right)a_1 \\
& +O(t^6)
\end{aligned} \tag{3}
$$

At $t=0$ the solution above becomes

$$
\begin{aligned}
y(t) = {} & \left(1-\frac{t^2}{2}-\frac{t^3 \tan(1)}{6}+\left(-\frac{\sec(1)^2}{12}-\frac{\tan(1)^2}{24}+\frac{1}{24}\right)t^4 \right. \\
& \left. +\left(-\frac{11\tan(1)\sec(1)^2}{120}-\frac{\tan(1)^3}{120}+\frac{\tan(1)}{60}\right)t^5\right)c_1+\left(t+\frac{t^2\tan(1)}{2} \right. \\
& +\left(\frac{\tan(1)^2}{6}+\frac{\sec(1)^2}{6}-\frac{1}{6}\right)t^3+\left(\frac{5\tan(1)\sec(1)^2}{24}+\frac{\tan(1)^3}{24}-\frac{\tan(1)}{12}\right)t^4 \\
& \left. +\left(\frac{7\sec(1)^2\tan(1)^2}{60}+\frac{3\sec(1)^4}{40}-\frac{\sec(1)^2}{15}+\frac{\tan(1)^4}{120}-\frac{\tan(1)^2}{40}+\frac{1}{120}\right)t^5\right)c_2 \\
& +O(t^6)
\end{aligned}
$$

Replacing t in the above with the original independent variable $x$susing $t=x-1$ results in

$$y = \left(1 - \frac{(x-1)^2}{2} - \frac{(x-1)^3 \tan(1)}{6} - \frac{(x-1)^4 \sec(1)^2}{8} + \frac{(x-1)^4}{12} - \frac{(x-1)^5 \tan(1) \sec(1)^2}{10} + \frac{(x-1)^5 \tan(1)}{40} - \frac{(x-1)^6 \sec(1)^4}{12} + \frac{3(x-1)^6 \sec(1)^2}{40} - \frac{(x-1)^6}{144}\right) y(1) + \left(x - 1 + \frac{(x-1)^2 \tan(1)}{2} + \frac{(x-1)^3 \tan(1)^2}{3} + \frac{(x-1)^4 \tan(1) \sec(1)^2}{4} - \frac{(x-1)^4 \tan(1)}{8} + \frac{(x-1)^5 \sec(1)^4}{5} - \frac{9(x-1)^5 \sec(1)^2}{40} + \frac{(x-1)^5}{24} + \frac{(x-1)^6 \tan(1)}{90} - \frac{31(x-1)^6 \tan(1) \sec(1)^2}{240} + \frac{(x-1)^6 \tan(1) \sec(1)^4}{6}\right) y'(1) + O\left((x-1)^6\right)$$

Summary
The solution(s) found are the following

$$y = \left(1 - \frac{(x-1)^2}{2} - \frac{(x-1)^3 \tan(1)}{6} - \frac{(x-1)^4 \sec(1)^2}{8} + \frac{(x-1)^4}{12} - \frac{(x-1)^5 \tan(1) \sec(1)^2}{10} + \frac{(x-1)^5 \tan(1)}{40} - \frac{(x-1)^6 \sec(1)^4}{12} + \frac{3(x-1)^6 \sec(1)^2}{40} - \frac{(x-1)^6}{144}\right) y(1) + \left(x - 1 + \frac{(x-1)^2 \tan(1)}{2} + \frac{(x-1)^3 \tan(1)^2}{3} + \frac{(x-1)^4 \tan(1) \sec(1)^2}{4} - \frac{(x-1)^4 \tan(1)}{8} + \frac{(x-1)^5 \sec(1)^4}{5} - \frac{9(x-1)^5 \sec(1)^2}{40} + \frac{(x-1)^5}{24} + \frac{(x-1)^6 \tan(1)}{90} - \frac{31(x-1)^6 \tan(1) \sec(1)^2}{240} + \frac{(x-1)^6 \tan(1) \sec(1)^4}{6}\right) y'(1) + O\left((x-1)^6\right) \tag{1}$$

Verification of solutions

$$y = \left(1 - \frac{(x-1)^2}{2} - \frac{(x-1)^3 \tan(1)}{6} - \frac{(x-1)^4 \sec(1)^2}{8} + \frac{(x-1)^4}{12} - \frac{(x-1)^5 \tan(1) \sec(1)^2}{10} + \frac{(x-1)^5 \tan(1)}{40} - \frac{(x-1)^6 \sec(1)^4}{12} + \frac{3(x-1)^6 \sec(1)^2}{40} - \frac{(x-1)^6}{144}\right) y(1) + \left(x - 1 + \frac{(x-1)^2 \tan(1)}{2} + \frac{(x-1)^3 \tan(1)^2}{3} + \frac{(x-1)^4 \tan(1) \sec(1)^2}{4} - \frac{(x-1)^4 \tan(1)}{8} + \frac{(x-1)^5 \sec(1)^4}{5} - \frac{9(x-1)^5 \sec(1)^2}{40} + \frac{(x-1)^5}{24} + \frac{(x-1)^6 \tan(1)}{90} - \frac{31(x-1)^6 \tan(1) \sec(1)^2}{240} + \frac{(x-1)^6 \tan(1) \sec(1)^4}{6}\right) y'(1) + O\left((x-1)^6\right)$$

Verified OK.

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Heun: Equivalence to the GHE or one of its 4 confluent cases under a power @ Moebius
-> trying a solution of the form r0(x) * Y + r1(x) * Y where Y = exp(int(r(x), dx)) * 2F1([a
-> Trying changes of variables to rationalize or make the ODE simpler
   trying a quadrature
   checking if the LODE has constant coefficients
   checking if the LODE is of Euler type
   trying a symmetry of the form [xi=0, eta=F(x)]
   checking if the LODE is missing y
   -> Trying a Liouvillian solution using Kovacics algorithm
   <- No Liouvillian solutions exists
   -> Trying a solution in terms of special functions:
      -> Bessel
      -> elliptic
      -> Legendre
      <- Legendre successful
   <- special function solution successful
   Change of variables used:
      [x = arcsin(t)]
   Linear ODE actually solved:
      u(t)-2*t*diff(u(t),t)+(-t^2+1)*diff(diff(u(t),t),t) = 0
<- change of variables successful`
```

✓ Solution by Maple

Time used: 0.015 (sec). Leaf size: 106

```
Order:=6;
dsolve(diff(y(x),x$2)-tan(x)*diff(y(x),x)+y(x)=0,y(x),type='series',x=1);
```

$$y(x) = \left(1-\frac{(x-1)^2}{2}-\frac{\tan(1)(x-1)^3}{6}+\left(\frac{1}{12}-\frac{\sec(1)^2}{8}\right)(x-1)^4 + \frac{\tan(1)\left(1-4\sec(1)^2\right)(x-1)^5}{40}\right)y(1) + \left(x-1+\frac{\tan(1)(x-1)^2}{2}+\frac{\tan(1)^2(x-1)^3}{3}+\frac{\tan(1)\left(2\sec(1)^2-1\right)(x-1)^4}{8} + \frac{\left(5-27\sec(1)^2+24\sec(1)^4\right)(x-1)^5}{120}\right)D(y)(1)+O\left(x^6\right)$$

✓ Solution by Mathematica

Time used: 0.003 (sec). Leaf size: 442

```
AsymptoticDSolveValue[y''[x]-Tan[x]*y'[x]+y[x]==0,y[x],{x,1,5}]
```

$$
\begin{aligned}
y(x) \to\ & c_1\left(\frac{1}{24}(x-1)^4-\frac{1}{2}(x-1)^2+\frac{1}{20}(x-1)^5\left(-\tan^3(1)-\tan(1)\right)-\frac{1}{120}(x-1)^5\tan^3(1)\right.\\
&-\frac{1}{40}(x-1)^5\tan(1)\left(1+\tan^2(1)\right)+\frac{1}{60}(x-1)^5\tan(1)\left(-1-\tan^2(1)\right)\\
&+\frac{1}{12}(x-1)^4\left(-1-\tan^2(1)\right)-\frac{1}{24}(x-1)^4\tan^2(1)+\frac{1}{60}(x-1)^5\tan(1)\\
&\left.-\frac{1}{6}(x-1)^3\tan(1)+1\right)+c_2\left(\frac{1}{120}(x-1)^5-\frac{1}{6}(x-1)^3+x+\frac{1}{120}(x-1)^5\tan^4(1)\right.\\
&-\frac{1}{15}(x-1)^5\tan(1)\left(-\tan^3(1)-\tan(1)\right)-\frac{1}{12}(x-1)^4\left(-\tan^3(1)-\tan(1)\right)\\
&+\frac{1}{24}(x-1)^4\tan^3(1)-\frac{1}{40}(x-1)^5\left(-1-\tan^2(1)\right)\left(1+\tan^2(1)\right)\\
&+\frac{1}{40}(x-1)^5\tan^2(1)\left(1+\tan^2(1)\right)-\frac{1}{40}(x-1)^5\left(1+\tan^2(1)\right)\\
&-\frac{1}{40}(x-1)^5\tan^2(1)\left(-1-\tan^2(1)\right)+\frac{1}{120}(x-1)^5\left(-1-\tan^2(1)\right)\\
&-\frac{1}{40}(x-1)^5\tan^2(1)-\frac{1}{8}(x-1)^4\tan(1)\left(-1-\tan^2(1)\right)-\frac{1}{6}(x-1)^3\left(-1-\tan^2(1)\right)\\
&+\frac{1}{6}(x-1)^3\tan^2(1)-\frac{1}{60}(x-1)^5\left(-1-3\tan^4(1)-4\tan^2(1)\right)\\
&\left.-\frac{1}{12}(x-1)^4\tan(1)+\frac{1}{2}(x-1)^2\tan(1)-1\right)
\end{aligned}
$$

6.6 problem 6

Internal problem ID [5024]
Internal file name [`OUTPUT/4517_Sunday_June_05_2022_03_00_00_PM_50850701/index.tex`]

Book: Fundamentals of Differential Equations. By Nagle, Saff and Snider. 9th edition. Boston. Pearson 2018.
Section: Chapter 8, Series solutions of differential equations. Section 8.4. page 449
Problem number: 6.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second order series method. Ordinary point", "second order series method. Taylor series method"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

$$\boxed{(x^3+1)\,y''-xy'+2yx^2=0}$$

With the expansion point for the power series method at $x=1$.

The ode does not have its expansion point at $x=0$, therefore to simplify the computation of power series expansion, change of variable is made on the independent variable to shift the initial conditions and the expasion point back to zero. The new ode is then solved more easily since the expansion point is now at zero. The solution converted back to the original independent variable. Let

$$t=x-1$$

The ode is converted to be in terms of the new independent variable t. This results in

$$\left((t+1)^3+1\right)\left(\frac{d^2}{dt^2}y(t)\right)-(t+1)\left(\frac{d}{dt}y(t)\right)+2y(t)\,(t+1)^2=0$$

With its expansion point and initial conditions now at $t=0$. The transformed ODE is now solved. Solving ode using Taylor series method. This gives review on how the Taylor series method works for solving second order ode.

Let

$$y'' = f(x, y, y')$$

Assuming expansion is at $x_0 = 0$ (we can always shift the actual expansion point to 0 by change of variables) and assuming $f(x, y, y')$ is analytic at x_0 which must be the case for an ordinary point. Let initial conditions be $y(x_0) = y_0$ and $y'(x_0) = y'_0$. Using Taylor series gives

$$\begin{aligned} y(x) &= y(x_0) + (x - x_0)\, y'(x_0) + \frac{(x - x_0)^2}{2} y''(x_0) + \frac{(x - x_0)^3}{3!} y'''(x_0) + \cdots \\ &= y_0 + x y'_0 + \frac{x^2}{2} f|_{x_0, y_0, y'_0} + \frac{x^3}{3!} f'|_{x_0, y_0, y'_0} + \cdots \\ &= y_0 + x y'_0 + \sum_{n=0}^{\infty} \frac{x^{n+2}}{(n+2)!} \left.\frac{d^n f}{dx^n}\right|_{x_0, y_0, y'_0} \end{aligned}$$

But

$$\begin{aligned} \frac{df}{dx} &= \frac{\partial f}{\partial x}\frac{dx}{dx} + \frac{\partial f}{\partial y}\frac{dy}{dx} + \frac{\partial f}{\partial y'}\frac{dy'}{dx} && (1) \\ &= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} y' + \frac{\partial f}{\partial y'} y'' && (273) \\ &= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} y' + \frac{\partial f}{\partial y'} f && (274) \\ \frac{d^2 f}{dx^2} &= \frac{d}{dx}\left(\frac{df}{dx}\right) \\ &= \frac{\partial}{\partial x}\left(\frac{df}{dx}\right) + \frac{\partial}{\partial y}\left(\frac{df}{dx}\right) y' + \frac{\partial}{\partial y'}\left(\frac{df}{dx}\right) f && (2) \\ \frac{d^3 f}{dx^3} &= \frac{d}{dx}\left(\frac{d^2 f}{dx^2}\right) \\ &= \frac{\partial}{\partial x}\left(\frac{d^2 f}{dx^2}\right) + \left(\frac{\partial}{\partial y}\frac{d^2 f}{dx^2}\right) y' + \frac{\partial}{\partial y'}\left(\frac{d^2 f}{dx^2}\right) f && (3) \\ &\vdots \end{aligned}$$

And so on. Hence if we name $F_0 = f(x, y, y')$ then the above can be written as

$$
\begin{aligned}
F_0 &= f(x, y, y') && (4)\\
F_1 &= \frac{df}{dx}\\
&= \frac{dF_0}{dx}\\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}y' + \frac{\partial f}{\partial y'}y''\\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}y' + \frac{\partial f}{\partial y'}f && (5)\\
&= \frac{\partial F_0}{\partial x} + \frac{\partial F_0}{\partial y}y' + \frac{\partial F_0}{\partial y'}F_0\\
F_2 &= \frac{d}{dx}\left(\frac{d}{dx}f\right)\\
&= \frac{d}{dx}(F_1)\\
&= \frac{\partial}{\partial x}F_1 + \left(\frac{\partial F_1}{\partial y}\right)y' + \left(\frac{\partial F_1}{\partial y'}\right)y''\\
&= \frac{\partial}{\partial x}F_1 + \left(\frac{\partial F_1}{\partial y}\right)y' + \left(\frac{\partial F_1}{\partial y'}\right)F_0\\
&\vdots\\
F_n &= \frac{d}{dx}(F_{n-1})\\
&= \frac{\partial}{\partial x}F_{n-1} + \left(\frac{\partial F_{n-1}}{\partial y}\right)y' + \left(\frac{\partial F_{n-1}}{\partial y'}\right)y''\\
&= \frac{\partial}{\partial x}F_{n-1} + \left(\frac{\partial F_{n-1}}{\partial y}\right)y' + \left(\frac{\partial F_{n-1}}{\partial y'}\right)F_0 && (6)
\end{aligned}
$$

Therefore (6) can be used from now on along with

$$
y(x) = y_0 + xy_0' + \sum_{n=0}^{\infty} \frac{x^{n+2}}{(n+2)!} F_n|_{x_0, y_0, y_0'} \tag{7}
$$

To find $y(x)$ series solution around $x = 0$. Hence

$$
\begin{aligned}
F_0 &= -\frac{2y(t)\,t^2 + 4ty(t) - t\left(\frac{d}{dt}y(t)\right) + 2y(t) - \frac{d}{dt}y(t)}{t^3 + 3t^2 + 3t + 2} \\
F_1 &= \frac{dF_0}{dt} \\
&= \frac{\partial F_0}{\partial t} + \frac{\partial F_0}{\partial y}\frac{d}{dt}y(t) + \frac{\partial F_0}{\partial \frac{d}{dt}y\,(t)}F_0 \\
&= \frac{(-2t^5 - 10t^4 - 22t^3 - 27t^2 - 18t - 4)\left(\frac{d}{dt}y(t)\right) + 2y(t)\,(t+1)\,(t^3 + 2t^2 + t - 2)}{(t^3 + 3t^2 + 3t + 2)^2} \\
F_2 &= \frac{dF_1}{dt} \\
&= \frac{\partial F_1}{\partial t} + \frac{\partial F_1}{\partial y}\frac{d}{dt}y(t) + \frac{\partial F_1}{\partial \frac{d}{dt}y\,(t)}F_1 \\
&= \frac{(4t^7 + 24t^6 + 66t^5 + 100t^4 + 77t^3 + 3t^2 - 44t - 24)\left(\frac{d}{dt}y(t)\right) + 4\big(t^7 + 6t^6 + \frac{35}{2}t^5 + 33t^4 + 49t^3 + 53t}{(t^3 + 3t^2 + 3t + 2)^3} \\
F_3 &= \frac{dF_2}{dt} \\
&= \frac{\partial F_2}{\partial t} + \frac{\partial F_2}{\partial y}\frac{d}{dt}y(t) + \frac{\partial F_2}{\partial \frac{d}{dt}y\,(t)}F_2 \\
&= \frac{(4t^{10} + 28t^9 + 98t^8 + 234t^7 + 514t^6 + 1124t^5 + 2041t^4 + 2642t^3 + 2186t^2 + 984t + 160)\left(\frac{d}{dt}y(t)\right) - 1}{(t^3 + 3t^2 + 3t + 2)^4} \\
F_4 &= \frac{dF_3}{dt} \\
&= \frac{\partial F_3}{\partial t} + \frac{\partial F_3}{\partial y}\frac{d}{dt}y(t) + \frac{\partial F_3}{\partial \frac{d}{dt}y\,(t)}F_3 \\
&= \frac{(-24t^{12} - 228t^{11} - 1092t^{10} - 3462t^9 - 8702t^8 - 19300t^7 - 37108t^6 - 56717t^5 - 63301t^4 - 47186t^3}{}
\end{aligned}
$$

And so on. Evaluating all the above at initial conditions $t = 0$ and $y(0) = y(0)$ and $y'(0) = y'(0)$ gives

$$
\begin{aligned}
F_0 &= -y(0) + \frac{y'(0)}{2} \\
F_1 &= -y'(0) - y(0) \\
F_2 &= \frac{7y(0)}{2} - 3y'(0) \\
F_3 &= \frac{7y(0)}{2} + 10y'(0) \\
F_4 &= -\frac{187y(0)}{4} + 10y'(0)
\end{aligned}
$$

Substituting all the above in (7) and simplifying gives the solution as

$$
\begin{aligned}
y(t) = &\left(1-\frac{1}{2}t^{2}-\frac{1}{6}t^{3}+\frac{7}{48}t^{4}+\frac{7}{240}t^{5}-\frac{187}{2880}t^{6}\right) y(0) \\
&+\left(t+\frac{1}{4}t^{2}-\frac{1}{6}t^{3}-\frac{1}{8}t^{4}+\frac{1}{12}t^{5}+\frac{1}{72}t^{6}\right) y'(0)+O(t^{6})
\end{aligned}
$$

Since the expansion point $t = 0$ is an ordinary, we can also solve this using standard power series The ode is normalized to be

$$
\left(t^{3}+3t^{2}+3t+2\right)\left(\frac{d^{2}}{dt^{2}}y(t)\right)+(-t-1)\left(\frac{d}{dt}y(t)\right)+\left(2t^{2}+4t+2\right) y(t) = 0
$$

Let the solution be represented as power series of the form

$$
y(t) = \sum_{n=0}^{\infty} a_{n} t^{n}
$$

Then

$$
\begin{aligned}
\frac{d}{dt}y(t) &= \sum_{n=1}^{\infty} n a_{n} t^{n-1} \\
\frac{d^{2}}{dt^{2}}y(t) &= \sum_{n=2}^{\infty} n(n-1)\, a_{n} t^{n-2}
\end{aligned}
$$

Substituting the above back into the ode gives

$$
\left(t^{3}+3t^{2}+3t+2\right)\left(\sum_{n=2}^{\infty} n(n-1)\, a_{n} t^{n-2}\right)+(-t-1)\left(\sum_{n=1}^{\infty} n a_{n} t^{n-1}\right)+\left(2t^{2}+4t+2\right)\left(\sum_{n=0}^{\infty} a_{n} t^{n}\right) = 0 \tag{1}
$$

Which simplifies to

$$
\begin{aligned}
&\left(\sum_{n=2}^{\infty} n\, t^{1+n} a_{n}(n-1)\right)+\left(\sum_{n=2}^{\infty} 3t^{n} a_{n} n(n-1)\right)+\left(\sum_{n=2}^{\infty} 3n\, t^{n-1} a_{n}(n-1)\right) \\
&+\left(\sum_{n=2}^{\infty} 2n(n-1)\, a_{n} t^{n-2}\right)+\sum_{n=1}^{\infty}\left(-n a_{n} t^{n}\right)+\sum_{n=1}^{\infty}\left(-n a_{n} t^{n-1}\right) \\
&+\left(\sum_{n=0}^{\infty} 2t^{n+2} a_{n}\right)+\left(\sum_{n=0}^{\infty} 4t^{1+n} a_{n}\right)+\left(\sum_{n=0}^{\infty} 2a_{n} t^{n}\right) = 0
\end{aligned} \tag{2}
$$

The next step is to make all powers of t be n in each summation term. Going over each summation term above with power of t in it which is not already t^n and adjusting the power and the corresponding index gives

$$\sum_{n=2}^{\infty} n\, t^{1+n} a_n (n-1) = \sum_{n=3}^{\infty} (n-1)\, a_{n-1} (n-2)\, t^n$$

$$\sum_{n=2}^{\infty} 3n\, t^{n-1} a_n (n-1) = \sum_{n=1}^{\infty} 3(1+n)\, a_{1+n} n\, t^n$$

$$\sum_{n=2}^{\infty} 2n(n-1)\, a_n t^{n-2} = \sum_{n=0}^{\infty} 2(n+2)\, a_{n+2} (1+n)\, t^n$$

$$\sum_{n=1}^{\infty} \left(-n a_n t^{n-1}\right) = \sum_{n=0}^{\infty} \left(-(1+n)\, a_{1+n} t^n\right)$$

$$\sum_{n=0}^{\infty} 2t^{n+2} a_n = \sum_{n=2}^{\infty} 2a_{n-2} t^n$$

$$\sum_{n=0}^{\infty} 4t^{1+n} a_n = \sum_{n=1}^{\infty} 4a_{n-1} t^n$$

Substituting all the above in Eq (2) gives the following equation where now all powers of t are the same and equal to n.

$$\begin{aligned} &\left(\sum_{n=3}^{\infty} (n-1)\, a_{n-1} (n-2)\, t^n\right) + \left(\sum_{n=2}^{\infty} 3t^n a_n n(n-1)\right) \\ &+ \left(\sum_{n=1}^{\infty} 3(1+n)\, a_{1+n} n\, t^n\right) + \left(\sum_{n=0}^{\infty} 2(n+2)\, a_{n+2} (1+n)\, t^n\right) \\ &+ \sum_{n=1}^{\infty} (-n a_n t^n) + \sum_{n=0}^{\infty} \left(-(1+n)\, a_{1+n} t^n\right) + \left(\sum_{n=2}^{\infty} 2a_{n-2} t^n\right) \\ &+ \left(\sum_{n=1}^{\infty} 4a_{n-1} t^n\right) + \left(\sum_{n=0}^{\infty} 2a_n t^n\right) = 0 \end{aligned} \tag{3}$$

$n = 0$ gives

$$4a_2 - a_1 + 2a_0 = 0$$

$$a_2 = -\frac{a_0}{2} + \frac{a_1}{4}$$

$n = 1$ gives

$$4a_2 + 12a_3 + a_1 + 4a_0 = 0$$

Which after substituting earlier equations, simplifies to

$$a_3 = -\frac{a_0}{6} - \frac{a_1}{6}$$

$n = 2$ gives

$$6a_2 + 15a_3 + 24a_4 + 2a_0 + 4a_1 = 0$$

Which after substituting earlier equations, simplifies to

$$a_4 = \frac{7a_0}{48} - \frac{a_1}{8}$$

For $3 \leq n$, the recurrence equation is

$$\begin{aligned}(n-1)\, a_{n-1}(n-2) + 3na_n(n-1) + 3(1+n)\, a_{1+n}n \\ + 2(n+2)\, a_{n+2}(1+n) - na_n - (1+n)\, a_{1+n} + 2a_{n-2} + 4a_{n-1} + 2a_n = 0\end{aligned} \tag{4}$$

Solving for a_{n+2}, gives

$$\begin{aligned}a_{n+2} &= \\ &-\frac{3n^2a_n + 3n^2a_{1+n} + n^2a_{n-1} - 4na_n + 2na_{1+n} - 3na_{n-1} + 2a_n - a_{1+n} + 2a_{n-2} + 6a_{n-1}}{2\,(n+2)\,(1+n)} \\ &= -\frac{(3n^2-4n+2)\, a_n}{2\,(n+2)\,(1+n)} - \frac{(3n^2+2n-1)\, a_{1+n}}{2\,(n+2)\,(1+n)} - \frac{a_{n-2}}{(n+2)\,(1+n)} - \frac{(n^2-3n+6)\, a_{n-1}}{2\,(n+2)\,(1+n)}\end{aligned} \tag{5}$$

For $n = 3$ the recurrence equation gives

$$6a_2 + 17a_3 + 32a_4 + 40a_5 + 2a_1 = 0$$

Which after substituting the earlier terms found becomes

$$a_5 = \frac{7a_0}{240} + \frac{a_1}{12}$$

For $n = 4$ the recurrence equation gives

$$10a_3 + 34a_4 + 55a_5 + 60a_6 + 2a_2 = 0$$

Which after substituting the earlier terms found becomes

$$a_6 = -\frac{187a_0}{2880} + \frac{a_1}{72}$$

For $n = 5$ the recurrence equation gives

$$16a_4 + 57a_5 + 84a_6 + 84a_7 + 2a_3 = 0$$

Which after substituting the earlier terms found becomes

$$a_7 = \frac{43a_0}{2016} - \frac{43a_1}{1008}$$

And so on. Therefore the solution is

$$\begin{aligned} y(t) &= \sum_{n=0}^{\infty} a_n t^n \\ &= a_3 t^3 + a_2 t^2 + a_1 t + a_0 + \dots \end{aligned}$$

Substituting the values for a_n found above, the solution becomes

$$y(t) = a_0 + a_1 t + \left(-\frac{a_0}{2} + \frac{a_1}{4}\right) t^2 + \left(-\frac{a_0}{6} - \frac{a_1}{6}\right) t^3 + \left(\frac{7a_0}{48} - \frac{a_1}{8}\right) t^4 + \left(\frac{7a_0}{240} + \frac{a_1}{12}\right) t^5 + \dots$$

Collecting terms, the solution becomes

$$y(t) = \left(1 - \frac{1}{2}t^2 - \frac{1}{6}t^3 + \frac{7}{48}t^4 + \frac{7}{240}t^5\right) a_0 + \left(t + \frac{1}{4}t^2 - \frac{1}{6}t^3 - \frac{1}{8}t^4 + \frac{1}{12}t^5\right) a_1 + O(t^6) \quad (3)$$

At $t = 0$ the solution above becomes

$$y(t) = \left(1 - \frac{1}{2}t^2 - \frac{1}{6}t^3 + \frac{7}{48}t^4 + \frac{7}{240}t^5\right) c_1 + \left(t + \frac{1}{4}t^2 - \frac{1}{6}t^3 - \frac{1}{8}t^4 + \frac{1}{12}t^5\right) c_2 + O(t^6)$$

Replacing t in the above with the original independent variable $x$susing $t = x - 1$ results in

$$y = \left(1 - \frac{(x-1)^2}{2} - \frac{(x-1)^3}{6} + \frac{7(x-1)^4}{48} + \frac{7(x-1)^5}{240} - \frac{187(x-1)^6}{2880}\right) y(1) + \left(x - 1 + \frac{(x-1)^2}{4} - \frac{(x-1)^3}{6} - \frac{(x-1)^4}{8} + \frac{(x-1)^5}{12} + \frac{(x-1)^6}{72}\right) y'(1) + O\left((x-1)^6\right)$$

Summary
The solution(s) found are the following

$$y = \left(1 - \frac{(x-1)^2}{2} - \frac{(x-1)^3}{6} + \frac{7(x-1)^4}{48} + \frac{7(x-1)^5}{240} - \frac{187(x-1)^6}{2880}\right) y(1) + \left(x - 1 + \frac{(x-1)^2}{4} - \frac{(x-1)^3}{6} - \frac{(x-1)^4}{8} + \frac{(x-1)^5}{12} + \frac{(x-1)^6}{72}\right) y'(1) + O\left((x-1)^6\right) \tag{1}$$

Verification of solutions

$$y = \left(1 - \frac{(x-1)^2}{2} - \frac{(x-1)^3}{6} + \frac{7(x-1)^4}{48} + \frac{7(x-1)^5}{240} - \frac{187(x-1)^6}{2880}\right) y(1) + \left(x - 1 + \frac{(x-1)^2}{4} - \frac{(x-1)^3}{6} - \frac{(x-1)^4}{8} + \frac{(x-1)^5}{12} + \frac{(x-1)^6}{72}\right) y'(1) + O\left((x-1)^6\right)$$

Verified OK.

6.6.1 Maple step by step solution

Let's solve

$(x^3 + 1) y'' - xy' + 2yx^2 = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y'' = \frac{xy'}{x^3+1} - \frac{2x^2y}{x^3+1}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y'' - \frac{xy'}{x^3+1} + \frac{2x^2y}{x^3+1} = 0$

- ☐ Check to see if x_0 is a regular singular point
  - Define functions

    $\left[P_2(x) = -\frac{x}{x^3+1}, P_3(x) = \frac{2x^2}{x^3+1}\right]$
  - $(x+1)\cdot P_2(x)$ is analytic at $x=-1$

    $\left((x+1)\cdot P_2(x)\right)\Big|_{x=-1} = \frac{1}{3}$
  - $(x+1)^2\cdot P_3(x)$ is analytic at $x=-1$

    $\left((x+1)^2\cdot P_3(x)\right)\Big|_{x=-1} = 0$
  - $x=-1$is a regular singular point

    Check to see if x_0 is a regular singular point

    $x_0 = -1$
- Multiply by denominators

  $(x^3+1)\,y'' - xy' + 2yx^2 = 0$
- Change variables using $x = u-1$ so that the regular singular point is at $u = 0$

  $(u^3 - 3u^2 + 3u)\left(\frac{d^2}{du^2}y(u)\right) + (-u+1)\left(\frac{d}{du}y(u)\right) + (2u^2 - 4u + 2)\,y(u) = 0$
- Assume series solution for $y(u)$

  $y(u) = \sum_{k=0}^{\infty} a_k u^{k+r}$
- ☐ Rewrite ODE with series expansions
  - Convert $u^m\cdot y(u)$ to series expansion for $m = 0..2$

    $u^m\cdot y(u) = \sum_{k=0}^{\infty} a_k u^{k+r+m}$
  - Shift index using $k->k-m$

    $u^m\cdot y(u) = \sum_{k=m}^{\infty} a_{k-m} u^{k+r}$
  - Convert $u^m\cdot\left(\frac{d}{du}y(u)\right)$ to series expansion for $m = 0..1$

    $u^m\cdot\left(\frac{d}{du}y(u)\right) = \sum_{k=0}^{\infty} a_k(k+r)\,u^{k+r-1+m}$
  - Shift index using $k->k+1-m$

    $u^m\cdot\left(\frac{d}{du}y(u)\right) = \sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\,u^{k+r}$

- Convert $u^m \cdot \left(\frac{d^2}{du^2}y(u)\right)$ to series expansion for $m = 1..3$

  $u^m \cdot \left(\frac{d^2}{du^2}y(u)\right) = \sum_{k=0}^{\infty} a_k (k+r)(k+r-1)\, u^{k+r-2+m}$

- Shift index using $k->k+2-m$

  $u^m \cdot \left(\frac{d^2}{du^2}y(u)\right) = \sum_{k=-2+m}^{\infty} a_{k+2-m}(k+2-m+r)(k+1-m+r)\, u^{k+r}$

  Rewrite ODE with series expansions

  $a_0 r(-2+3r)\, u^{-1+r} + (a_1(1+r)(1+3r) - a_0(3r^2-2r-2))\, u^r + (a_2(2+r)(4+3r) - a_1(3r^2$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $r(-2+3r) = 0$

- Values of r that satisfy the indicial equation

  $r \in \left\{0, \frac{2}{3}\right\}$

- The coefficients of each power of u must be 0

  $[a_1(1+r)(1+3r) - a_0(3r^2-2r-2) = 0, a_2(2+r)(4+3r) - a_1(3r^2+4r-1) + a_0(r^2-r-$

- Solve for the dependent coefficient(s)

  $\left\{a_1 = \frac{a_0(3r^2-2r-2)}{3r^2+4r+1}, a_2 = \frac{a_0(6r^4+5r^3-2r^2+11r+6)}{9r^4+42r^3+67r^2+42r+8}\right\}$

- Each term in the series must be 0, giving the recursion relation

  $(-3a_k + a_{k-1} + 3a_{k+1})\, k^2 + ((-6a_k + 2a_{k-1} + 6a_{k+1})\, r + 2a_k - 3a_{k-1} + 4a_{k+1})\, k + (-3a_k + a_{k-}$

- Shift index using $k->k+2$

  $(-3a_{k+2} + a_{k+1} + 3a_{k+3})\,(k+2)^2 + ((-6a_{k+2} + 2a_{k+1} + 6a_{k+3})\, r + 2a_{k+2} - 3a_{k+1} + 4a_{k+3})\,(k+$

- Recursion relation that defines series solution to ODE

  $a_{k+3} = -\frac{k^2 a_{k+1} - 3k^2 a_{k+2} + 2kr a_{k+1} - 6kr a_{k+2} + r^2 a_{k+1} - 3r^2 a_{k+2} + k a_{k+1} - 10k a_{k+2} + r a_{k+1} - 10r a_{k+2} + 2a_k - 4a_{k+1} - 6a_{k+2}}{3k^2+6kr+3r^2+16k+16r+21}$

- Recursion relation for $r = 0$

  $a_{k+3} = -\frac{k^2 a_{k+1} - 3k^2 a_{k+2} + k a_{k+1} - 10k a_{k+2} + 2a_k - 4a_{k+1} - 6a_{k+2}}{3k^2+16k+21}$

- Solution for $r = 0$

  $\left[y(u) = \sum_{k=0}^{\infty} a_k u^k, a_{k+3} = -\frac{k^2 a_{k+1} - 3k^2 a_{k+2} + k a_{k+1} - 10k a_{k+2} + 2a_k - 4a_{k+1} - 6a_{k+2}}{3k^2+16k+21}, a_1 = -2a_0, a_2 = \frac{3a_0}{4}\right]$

- Revert the change of variables $u = x + 1$

$$\left[y = \sum_{k=0}^{\infty} a_k (x+1)^k, a_{k+3} = -\frac{k^2 a_{k+1} - 3k^2 a_{k+2} + k a_{k+1} - 10 k a_{k+2} + 2a_k - 4a_{k+1} - 6a_{k+2}}{3k^2+16k+21}, a_1 = -2a_0, a_2 = \frac{3a_0}{4}\right]$$

- Recursion relation for $r = \frac{2}{3}$

$$a_{k+3} = -\frac{k^2 a_{k+1} - 3k^2 a_{k+2} + \frac{7}{3} k a_{k+1} - 14 k a_{k+2} + 2a_k - \frac{26}{9} a_{k+1} - 14 a_{k+2}}{3k^2+20k+33}$$

- Solution for $r = \frac{2}{3}$

$$\left[y(u) = \sum_{k=0}^{\infty} a_k u^{k+\frac{2}{3}}, a_{k+3} = -\frac{k^2 a_{k+1} - 3k^2 a_{k+2} + \frac{7}{3} k a_{k+1} - 14 k a_{k+2} + 2a_k - \frac{26}{9} a_{k+1} - 14 a_{k+2}}{3k^2+20k+33}, a_1 = -\frac{2a_0}{5}, a_2 = \frac{17a}{90}\right.$$

- Revert the change of variables $u = x + 1$

$$\left[y = \sum_{k=0}^{\infty} a_k (x+1)^{k+\frac{2}{3}}, a_{k+3} = -\frac{k^2 a_{k+1} - 3k^2 a_{k+2} + \frac{7}{3} k a_{k+1} - 14 k a_{k+2} + 2a_k - \frac{26}{9} a_{k+1} - 14 a_{k+2}}{3k^2+20k+33}, a_1 = -\frac{2a_0}{5}, a_2 =\right.$$

- Combine solutions and rename parameters

$$\left[y = \left(\sum_{k=0}^{\infty} a_k (x+1)^k\right) + \left(\sum_{k=0}^{\infty} b_k (x+1)^{k+\frac{2}{3}}\right), a_{k+3} = -\frac{k^2 a_{k+1} - 3k^2 a_{k+2} + k a_{k+1} - 10 k a_{k+2} + 2a_k - 4a_{k+1} - 6a}{3k^2+16k+21}\right.$$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
<- No Liouvillian solutions exists
-> Trying a solution in terms of special functions:
   -> Bessel
   -> elliptic
   -> Legendre
   -> Kummer
      -> hyper3: Equivalence to 1F1 under a power @ Moebius
   -> hypergeometric
      -> heuristic approach
      -> hyper3: Equivalence to 2F1, 1F1 or 0F1 under a power @ Moebius
   -> Mathieu
      -> Equivalence to the rational form of Mathieu ODE under a power @ Moebius
trying a solution in terms of MeijerG functions
-> Heun: Equivalence to the GHE or one of its 4 confluent cases under a power @ Moebius
-> trying a solution of the form r0(x) * Y + r1(x) * Y where Y = exp(int(r(x), dx)) * 2F1([a
   trying a symmetry of the form [xi=0, eta=F(x)]
   trying differential order: 2; exact nonlinear
   trying symmetries linear in x and y(x)
   trying to convert to a linear ODE with constant coefficients
   trying 2nd order, integrating factor of the form mu(x,y)
   -> Trying a solution in terms of special functions:
      -> Bessel
      -> elliptic
      -> Legendre
      -> Whittaker
         -> hyper3: Equivalence to 1F1 under a power @ Moebius
      -> hypergeometric
         -> heuristic approach
         -> hyper3: Equivalence to 2F1, 1F1 or 0F1 under a power @ Moebius
      -> Mathieu
         -> Equivalence to the rational form of Mathieu ODE under a power @ Moebius
      trying 2nd order exact linear
      trying symmetries linear in x and y(x)
      trying to convert to a linear ODE with constant coefficients
      trying to convert to an ODE of Bessel type
   trying to convert to an ODE of Bessel type
   -> trying reduction of order to Riccati
```

✓ Solution by Maple

Time used: 0.015 (sec). Leaf size: 54

```
Order:=6;
dsolve((1+x^3)*diff(y(x),x$2)-x*diff(y(x),x)+2*x^2*y(x)=0,y(x),type='series',x=1);
```

$$y(x) = \left(1-\frac{(x-1)^2}{2}-\frac{(x-1)^3}{6}+\frac{7(x-1)^4}{48}+\frac{7(x-1)^5}{240}\right)y(1) + \left(x-1+\frac{(x-1)^2}{4}-\frac{(x-1)^3}{6}-\frac{(x-1)^4}{8}+\frac{(x-1)^5}{12}\right)D(y)(1)+O(x^6)$$

✓ Solution by Mathematica

Time used: 0.001 (sec). Leaf size: 78

```
AsymptoticDSolveValue[(1+x^3)*y''[x]-x*y'[x]+2*x*y[x]==0,y[x],{x,1,5}]
```

$$y(x) \to c_1\left(-\frac{1}{20}(x-1)^5+\frac{1}{8}(x-1)^4-\frac{1}{2}(x-1)^2+1\right) + c_2\left(\frac{19}{240}(x-1)^5-\frac{1}{24}(x-1)^4-\frac{1}{6}(x-1)^3+\frac{1}{4}(x-1)^2+x-1\right)$$

6.7 problem 7

Internal problem ID [5025]
Internal file name [`OUTPUT/4518_Sunday_June_05_2022_03_00_02_PM_26199806/index.tex`]

Book: Fundamentals of Differential Equations. By Nagle, Saff and Snider. 9th edition. Boston. Pearson 2018.
Section: Chapter 8, Series solutions of differential equations. Section 8.4. page 449
Problem number: 7.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"first order ode series method. Ordinary point", "first order ode series method. Taylor series method"**

Maple gives the following as the ode type

`[_separable]`

$$\boxed{y' + 2(x - 1)\, y = 0}$$

With the expansion point for the power series method at $x = 1$.

6.7.1 Solving as series ode

The ode does not have its expansion point at $x = 0$, therefore to simplify the computation of power series expansion, change of variable is made on the independent variable to shift the initial conditions and the expasion point back to zero. The new ode is then solved more easily since the expansion point is now at zero. The solution converted back to the original independent variable. Let

$$t = x - 1$$

The ode is converted to be in terms of the new independent variable t. This results in

$$\frac{d}{dt}y(t) + 2ty(t) = 0$$

With its expansion point and initial conditions now at $t = 0$. The transformed ODE is now solved.

Solving ode using Taylor series method. This gives review on how the Taylor series method works for solving first order ode. Let

$$y' = f(x, y)$$

Where $f(x, y)$ is analytic at expansion point x_0. We can always shift to $x_0 = 0$ if x_0 is not zero. So from now we assume $x_0 = 0$. Assume also that $y(x_0) = y_0$. Using Taylor series

$$\begin{aligned}
y(x) &= y(x_0) + (x - x_0)\, y'(x_0) + \frac{(x - x_0)^2}{2} y''(x_0) + \frac{(x - x_0)^3}{3!} y''(x_0) + \cdots \\
&= y_0 + xf + \frac{x^2}{2} \frac{df}{dx}\bigg|_{x_0, y_0} + \frac{x^3}{3!} \frac{d^2 f}{dx^2}\bigg|_{x_0, y_0} + \cdots \\
&= y_0 + \sum_{n=0}^{\infty} \frac{x^{n+1}}{(n+1)!} \frac{d^n f}{dx^n}\bigg|_{x_0, y_0}
\end{aligned}$$

But

$$\begin{aligned}
\frac{df}{dx} &= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} f \qquad (1) \\
\frac{d^2 f}{dx^2} &= \frac{d}{dx}\left(\frac{df}{dx}\right) \\
&= \frac{\partial}{\partial x}\left(\frac{df}{dx}\right) + \frac{\partial}{\partial y}\left(\frac{df}{dx}\right) f \qquad (2) \\
\frac{d^3 f}{dx^3} &= \frac{d}{dx}\left(\frac{d^2 f}{dx^2}\right) \\
&= \frac{\partial}{\partial x}\left(\frac{d^2 f}{dx^2}\right) + \left(\frac{\partial}{\partial y} \frac{d^2 f}{dx^2}\right) f \qquad (3) \\
&\vdots
\end{aligned}$$

And so on. Hence if we name $F_0 = f(x, y)$ then the above can be written as

$$\begin{aligned}
F_0 &= f(x, y) \qquad (4) \\
F_n &= \frac{d}{dx}(F_{n-1}) \\
&= \frac{\partial}{\partial x} F_{n-1} + \left(\frac{\partial F_{n-1}}{\partial y}\right) F_0 \qquad (5)
\end{aligned}$$

For example, for $n = 1$ we see that

$$\begin{aligned}
F_1 &= \frac{d}{dx}(F_0) \\
&= \frac{\partial}{\partial x}F_0 + \left(\frac{\partial F_0}{\partial y}\right)F_0 \\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}f
\end{aligned}$$

Which is (1). And when $n = 2$

$$\begin{aligned}
F_2 &= \frac{d}{dx}(F_1) \\
&= \frac{\partial}{\partial x}F_1 + \left(\frac{\partial F_1}{\partial y}\right)F_0 \\
&= \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}f\right) + \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}f\right)f \\
&= \frac{\partial}{\partial x}\left(\frac{df}{dx}\right) + \frac{\partial}{\partial y}\left(\frac{df}{dx}\right)f
\end{aligned}$$

Which is (2) and so on. Therefore (4,5) can be used from now on along with

$$y(x) = y_0 + \sum_{n=0}^{\infty} \frac{x^{n+1}}{(n+1)\,!}F_n|_{x_0,y_0} \tag{6}$$

Hence

$$
\begin{aligned}
F_0 &= -2ty(t) \\
F_1 &= \frac{dF_0}{dt} \\
&= \frac{\partial F_0}{\partial t} + \frac{\partial F_0}{\partial y} F_0 \\
&= \left(4t^2 - 2\right) y(t) \\
F_2 &= \frac{dF_1}{dt} \\
&= \frac{\partial F_1}{\partial t} + \frac{\partial F_1}{\partial y} F_1 \\
&= \left(-8t^3 + 12t\right) y(t) \\
F_3 &= \frac{dF_2}{dt} \\
&= \frac{\partial F_2}{\partial t} + \frac{\partial F_2}{\partial y} F_2 \\
&= 4y(t) \left(4t^4 - 12t^2 + 3\right) \\
F_4 &= \frac{dF_3}{dt} \\
&= \frac{\partial F_3}{\partial t} + \frac{\partial F_3}{\partial y} F_3 \\
&= -32y(t) \left(t^4 - 5t^2 + \frac{15}{4}\right) t
\end{aligned}
$$

And so on. Evaluating all the above at initial conditions $t(0) = 0$ and $y(0) = y(0)$ gives

$$
\begin{aligned}
F_0 &= 0 \\
F_1 &= -2y(0) \\
F_2 &= 0 \\
F_3 &= 12y(0) \\
F_4 &= 0
\end{aligned}
$$

Substituting all the above in (6) and simplifying gives the solution as

$$
y(t) = \left(1 - t^2 + \frac{1}{2}t^4\right) y(0) + O\left(t^6\right)
$$

Since $t = 0$ is also an ordinary point, then standard power series can also be used.

Writing the ODE as

$$\frac{d}{dt}y(t) + q(t)y(t) = p(t)$$
$$\frac{d}{dt}y(t) + 2ty(t) = 0$$

Where

$$q(t) = 2t$$
$$p(t) = 0$$

Next, the type of the expansion point $t = 0$ is determined. This point can be an ordinary point, a regular singular point (also called removable singularity), or irregular singular point (also called non-removable singularity or essential singularity). When $t = 0$ is an ordinary point, then the standard power series is used. If the point is a regular singular point, Frobenius series is used instead. Irregular singular point requires more advanced methods (asymptotic methods) and is not supported now. Hopefully this will be added in the future. $t = 0$ is called an ordinary point $q(t)$ has a Taylor series expansion around the point $t = 0$. $t = 0$ is called a regular singular point if $q(t)$ is not not analytic at $t = 0$ but $tq(t)$ has Taylor series expansion. And finally, $t = 0$ is an irregular singular point if the point is not ordinary and not regular singular. This is the most complicated case. Now the expansion point $t = 0$ is checked to see if it is an ordinary point or not. Let the solution be represented as power series of the form

$$y(t) = \sum_{n=0}^{\infty} a_n t^n$$

Then

$$\frac{d}{dt}y(t) = \sum_{n=1}^{\infty} n a_n t^{n-1}$$

Substituting the above back into the ode gives

$$\left(\sum_{n=1}^{\infty} n a_n t^{n-1}\right) + 2t\left(\sum_{n=0}^{\infty} a_n t^n\right) = 0 \tag{1}$$

Which simplifies to

$$\left(\sum_{n=1}^{\infty} n a_n t^{n-1}\right) + \left(\sum_{n=0}^{\infty} 2t^{1+n} a_n\right) = 0 \tag{2}$$

The next step is to make all powers of t be n in each summation term. Going over each summation term above with power of t in it which is not already t^n and adjusting the power and the corresponding index gives

$$\sum_{n=1}^{\infty} n a_n t^{n-1} = \sum_{n=0}^{\infty} (1+n) a_{1+n} t^n$$
$$\sum_{n=0}^{\infty} 2t^{1+n} a_n = \sum_{n=1}^{\infty} 2a_{n-1} t^n$$

Substituting all the above in Eq (2) gives the following equation where now all powers of t are the same and equal to n.

$$\left(\sum_{n=0}^{\infty} (1+n) a_{1+n} t^n\right) + \left(\sum_{n=1}^{\infty} 2a_{n-1} t^n\right) = 0 \tag{3}$$

For $1 \leq n$, the recurrence equation is

$$(1+n) a_{1+n} + 2a_{n-1} = 0 \tag{4}$$

Solving for a_{1+n}, gives

$$a_{1+n} = -\frac{2a_{n-1}}{1+n} \tag{5}$$

For $n = 1$ the recurrence equation gives

$$2a_2 + 2a_0 = 0$$

Which after substituting the earlier terms found becomes

$$a_2 = -a_0$$

For $n = 2$ the recurrence equation gives

$$3a_3 + 2a_1 = 0$$

Which after substituting the earlier terms found becomes

$$a_3 = 0$$

For $n = 3$ the recurrence equation gives

$$4a_4 + 2a_2 = 0$$

Which after substituting the earlier terms found becomes

$$a_4 = \frac{a_0}{2}$$

For $n = 4$ the recurrence equation gives

$$5a_5 + 2a_3 = 0$$

Which after substituting the earlier terms found becomes

$$a_5 = 0$$

For $n = 5$ the recurrence equation gives

$$6a_6 + 2a_4 = 0$$

Which after substituting the earlier terms found becomes

$$a_6 = -\frac{a_0}{6}$$

And so on. Therefore the solution is

$$\begin{aligned} y(t) &= \sum_{n=0}^{\infty} a_n t^n \\ &= a_3 t^3 + a_2 t^2 + a_1 t + a_0 + \dots \end{aligned}$$

Substituting the values for a_n found above, the solution becomes

$$y(t) = a_0 - a_0 t^2 + \frac{1}{2} a_0 t^4 + \dots$$

Collecting terms, the solution becomes

$$y(t) = \left(1 - t^2 + \frac{1}{2} t^4\right) a_0 + O(t^6) \tag{3}$$

Replacing t in the above with the original independent variable xs using $t = x - 1$ results in

$$y = \left(1 - (x-1)^2 + \frac{(x-1)^4}{2}\right) y(1) + O\big((x-1)^6\big)$$

Summary

The solution(s) found are the following

$$y = \left(1 - (x-1)^2 + \frac{(x-1)^4}{2}\right) y(1) + O\big((x-1)^6\big) \tag{1}$$

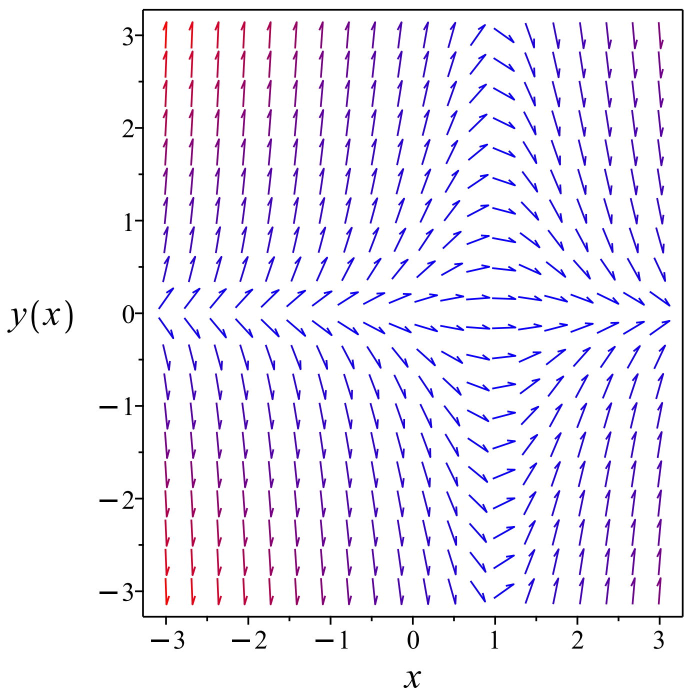


Figure 209: Slope field plot

Verification of solutions

$$y = \left(1 - (x-1)^2 + \frac{(x-1)^4}{2}\right) y(1) + O\big((x-1)^6\big)$$

Verified OK.

6.7.2 Maple step by step solution

Let's solve

$y' + 2(x-1)\, y = 0$

- Highest derivative means the order of the ODE is 1

  y'

- Separate variables

  $\frac{y'}{y} = -2x + 2$

- Integrate both sides with respect to x

  $\int \frac{y'}{y} dx = \int (-2x+2)\, dx + c_1$

- Evaluate integral

  $\ln(y) = -x^2 + c_1 + 2x$

- Solve for y

  $y = e^{-x^2+c_1+2x}$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
<- 1st order linear successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 21

```
Order:=6;
dsolve(diff(y(x),x)+2*(x-1)*y(x)=0,y(x),type='series',x=1);
```

$$y(x) = \left(1 - (x-1)^2 + \frac{(x-1)^4}{2}\right) y(1) + O(x^6)$$

✓ Solution by Mathematica

Time used: 0.001 (sec). Leaf size: 24

```
AsymptoticDSolveValue[y'[x]+2*(x-1)*y[x]==0,y[x],{x,1,5}]
```

$$y(x) \to c_1 \left(\frac{1}{2}(x-1)^4 - (x-1)^2 + 1 \right)$$

6.8 problem 8

Internal problem ID [5026]
Internal file name [`OUTPUT/4519_Sunday_June_05_2022_03_00_03_PM_91206933/index.tex`]

Book: Fundamentals of Differential Equations. By Nagle, Saff and Snider. 9th edition. Boston. Pearson 2018.
Section: Chapter 8, Series solutions of differential equations. Section 8.4. page 449
Problem number: 8.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"first order ode series method. Ordinary point", "first order ode series method. Taylor series method"**

Maple gives the following as the ode type

```
[_separable]
```

$$\boxed{y' - 2xy = 0}$$

With the expansion point for the power series method at $x = 1$.

6.8.1 Solving as series ode

The ode does not have its expansion point at $x = 0$, therefore to simplify the computation of power series expansion, change of variable is made on the independent variable to shift the initial conditions and the expasion point back to zero. The new ode is then solved more easily since the expansion point is now at zero. The solution converted back to the original independent variable. Let

$$t = x - 1$$

The ode is converted to be in terms of the new independent variable t. This results in

$$\frac{d}{dt}y(t) - 2(t+1)\,y(t) = 0$$

With its expansion point and initial conditions now at $t = 0$. The transformed ODE is now solved.

Solving ode using Taylor series method. This gives review on how the Taylor series method works for solving first order ode. Let

$$y' = f(x, y)$$

Where $f(x, y)$ is analytic at expansion point x_0. We can always shift to $x_0 = 0$ if x_0 is not zero. So from now we assume $x_0 = 0$. Assume also that $y(x_0) = y_0$. Using Taylor series

$$\begin{aligned}
y(x) &= y(x_0) + (x - x_0)\, y'(x_0) + \frac{(x - x_0)^2}{2} y''(x_0) + \frac{(x - x_0)^3}{3!} y''(x_0) + \cdots \\
&= y_0 + xf + \frac{x^2}{2} \frac{df}{dx}\bigg|_{x_0, y_0} + \frac{x^3}{3!} \frac{d^2 f}{dx^2}\bigg|_{x_0, y_0} + \cdots \\
&= y_0 + \sum_{n=0}^{\infty} \frac{x^{n+1}}{(n+1)\,!} \frac{d^n f}{dx^n}\bigg|_{x_0, y_0}
\end{aligned}$$

But

$$\begin{aligned}
\frac{df}{dx} &= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} f && (1) \\
\frac{d^2 f}{dx^2} &= \frac{d}{dx}\left(\frac{df}{dx}\right) \\
&= \frac{\partial}{\partial x}\left(\frac{df}{dx}\right) + \frac{\partial}{\partial y}\left(\frac{df}{dx}\right) f && (2) \\
\frac{d^3 f}{dx^3} &= \frac{d}{dx}\left(\frac{d^2 f}{dx^2}\right) \\
&= \frac{\partial}{\partial x}\left(\frac{d^2 f}{dx^2}\right) + \left(\frac{\partial}{\partial y} \frac{d^2 f}{dx^2}\right) f && (3) \\
&\vdots
\end{aligned}$$

And so on. Hence if we name $F_0 = f(x, y)$ then the above can be written as

$$\begin{aligned}
F_0 &= f(x, y) && (4) \\
F_n &= \frac{d}{dx}(F_{n-1}) \\
&= \frac{\partial}{\partial x} F_{n-1} + \left(\frac{\partial F_{n-1}}{\partial y}\right) F_0 && (5)
\end{aligned}$$

For example, for $n = 1$ we see that

$$\begin{aligned}
F_1 &= \frac{d}{dx}(F_0) \\
&= \frac{\partial}{\partial x} F_0 + \left(\frac{\partial F_0}{\partial y} \right) F_0 \\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} f
\end{aligned}$$

Which is (1). And when $n = 2$

$$\begin{aligned}
F_2 &= \frac{d}{dx}(F_1) \\
&= \frac{\partial}{\partial x} F_1 + \left(\frac{\partial F_1}{\partial y} \right) F_0 \\
&= \frac{\partial}{\partial x} \left(\frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} f \right) + \frac{\partial}{\partial y} \left(\frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} f \right) f \\
&= \frac{\partial}{\partial x} \left(\frac{df}{dx} \right) + \frac{\partial}{\partial y} \left(\frac{df}{dx} \right) f
\end{aligned}$$

Which is (2) and so on. Therefore (4,5) can be used from now on along with

$$y(x) = y_0 + \sum_{n=0}^{\infty} \frac{x^{n+1}}{(n+1)\,!} F_n|_{x_0,y_0} \tag{6}$$

Hence

$$
\begin{aligned}
F_0 &= 2(t+1)\,y(t) \\
F_1 &= \frac{dF_0}{dt} \\
&= \frac{\partial F_0}{\partial t} + \frac{\partial F_0}{\partial y} F_0 \\
&= \left(4t^2+8t+6\right) y(t) \\
F_2 &= \frac{dF_1}{dt} \\
&= \frac{\partial F_1}{\partial t} + \frac{\partial F_1}{\partial y} F_1 \\
&= 8y(t)\left(t^2+2t+\frac{5}{2}\right)(t+1) \\
F_3 &= \frac{dF_2}{dt} \\
&= \frac{\partial F_2}{\partial t} + \frac{\partial F_2}{\partial y} F_2 \\
&= 16y(t)\left(t^4+4t^3+9t^2+10t+\frac{19}{4}\right) \\
F_4 &= \frac{dF_3}{dt} \\
&= \frac{\partial F_3}{\partial t} + \frac{\partial F_3}{\partial y} F_3 \\
&= 32y(t)\,(t+1)\left(t^4+4t^3+11t^2+14t+\frac{39}{4}\right)
\end{aligned}
$$

And so on. Evaluating all the above at initial conditions $t(0)=0$ and $y(0)=y(0)$ gives

$$
\begin{aligned}
F_0 &= 2y(0) \\
F_1 &= 6y(0) \\
F_2 &= 20y(0) \\
F_3 &= 76y(0) \\
F_4 &= 312y(0)
\end{aligned}
$$

Substituting all the above in (6) and simplifying gives the solution as

$$
y(t) = \left(1+2t+3t^2+\frac{10}{3}t^3+\frac{19}{6}t^4+\frac{13}{5}t^5\right) y(0) + O\left(t^6\right)
$$

Since $t = 0$ is also an ordinary point, then standard power series can also be used. Writing the ODE as

$$\frac{d}{dt}y(t) + q(t)y(t) = p(t)$$
$$\frac{d}{dt}y(t) + (-2t - 2)\, y(t) = 0$$

Where

$$q(t) = -2t - 2$$
$$p(t) = 0$$

Next, the type of the expansion point $t = 0$ is determined. This point can be an ordinary point, a regular singular point (also called removable singularity), or irregular singular point (also called non-removable singularity or essential singularity). When $t = 0$ is an ordinary point, then the standard power series is used. If the point is a regular singular point, Frobenius series is used instead. Irregular singular point requires more advanced methods (asymptotic methods) and is not supported now. Hopefully this will be added in the future. $t = 0$ is called an ordinary point $q(t)$ has a Taylor series expansion around the point $t = 0$. $t = 0$ is called a regular singular point if $q(t)$ is not not analytic at $t = 0$ but $tq(t)$ has Taylor series expansion. And finally, $t = 0$ is an irregular singular point if the point is not ordinary and not regular singular. This is the most complicated case. Now the expansion point $t = 0$ is checked to see if it is an ordinary point or not. Let the solution be represented as power series of the form

$$y(t) = \sum_{n=0}^{\infty} a_n t^n$$

Then

$$\frac{d}{dt}y(t) = \sum_{n=1}^{\infty} n a_n t^{n-1}$$

Substituting the above back into the ode gives

$$\left(\sum_{n=1}^{\infty} n a_n t^{n-1}\right) + (-2t - 2)\left(\sum_{n=0}^{\infty} a_n t^n\right) = 0 \tag{1}$$

Which simplifies to

$$\left(\sum_{n=1}^{\infty} n a_n t^{n-1}\right) + \sum_{n=0}^{\infty} \left(-2t^{1+n} a_n\right) + \sum_{n=0}^{\infty} \left(-2 a_n t^n\right) = 0 \tag{2}$$

The next step is to make all powers of t be n in each summation term. Going over each summation term above with power of t in it which is not already t^n and adjusting the power and the corresponding index gives

$$\sum_{n=1}^{\infty} n a_n t^{n-1} = \sum_{n=0}^{\infty} (1+n)\, a_{1+n} t^n$$
$$\sum_{n=0}^{\infty} \left(-2t^{1+n} a_n\right) = \sum_{n=1}^{\infty} \left(-2a_{n-1} t^n\right)$$

Substituting all the above in Eq (2) gives the following equation where now all powers of t are the same and equal to n.

$$\left(\sum_{n=0}^{\infty} (1+n)\, a_{1+n} t^n\right) + \sum_{n=1}^{\infty} \left(-2a_{n-1} t^n\right) + \sum_{n=0}^{\infty} \left(-2a_n t^n\right) = 0 \tag{3}$$

$n = 0$ gives

$$a_1 - 2a_0 = 0$$
$$a_1 = 2a_0$$

For $1 \leq n$, the recurrence equation is

$$(1+n)\, a_{1+n} - 2a_{n-1} - 2a_n = 0 \tag{4}$$

Solving for a_{1+n}, gives

$$a_{1+n} = \frac{2a_{n-1} + 2a_n}{1+n} \tag{5}$$

For $n = 1$ the recurrence equation gives

$$2a_2 - 2a_0 - 2a_1 = 0$$

Which after substituting the earlier terms found becomes

$$a_2 = 3a_0$$

For $n = 2$ the recurrence equation gives

$$3a_3 - 2a_1 - 2a_2 = 0$$

Which after substituting the earlier terms found becomes

$$a_3 = \frac{10a_0}{3}$$

For $n = 3$ the recurrence equation gives

$$4a_4 - 2a_2 - 2a_3 = 0$$

Which after substituting the earlier terms found becomes

$$a_4 = \frac{19a_0}{6}$$

For $n = 4$ the recurrence equation gives

$$5a_5 - 2a_3 - 2a_4 = 0$$

Which after substituting the earlier terms found becomes

$$a_5 = \frac{13a_0}{5}$$

For $n = 5$ the recurrence equation gives

$$6a_6 - 2a_4 - 2a_5 = 0$$

Which after substituting the earlier terms found becomes

$$a_6 = \frac{173a_0}{90}$$

And so on. Therefore the solution is

$$\begin{aligned} y(t) &= \sum_{n=0}^{\infty} a_n t^n \\ &= a_3 t^3 + a_2 t^2 + a_1 t + a_0 + \dots \end{aligned}$$

Substituting the values for a_n found above, the solution becomes

$$y(t) = a_0 + 2a_0 t + 3a_0 t^2 + \frac{10}{3} a_0 t^3 + \frac{19}{6} a_0 t^4 + \frac{13}{5} a_0 t^5 + \dots$$

Collecting terms, the solution becomes

$$y(t) = \left(1 + 2t + 3t^2 + \frac{10}{3}t^3 + \frac{19}{6}t^4 + \frac{13}{5}t^5\right) a_0 + O\left(t^6\right) \tag{3}$$

Replacing t in the above with the original independent variable xs using $t = x - 1$ results in

$$y = \left(-1 + 2x + 3(x-1)^2 + \frac{10(x-1)^3}{3} + \frac{19(x-1)^4}{6} + \frac{13(x-1)^5}{5}\right) y(1) + O\left((x-1)^6\right)$$

Summary

The solution(s) found are the following

$$y = \left(-1 + 2x + 3(x-1)^2 + \frac{10(x-1)^3}{3} + \frac{19(x-1)^4}{6} + \frac{13(x-1)^5}{5}\right) y(1) + O\left((x-1)^6\right) \tag{1}$$

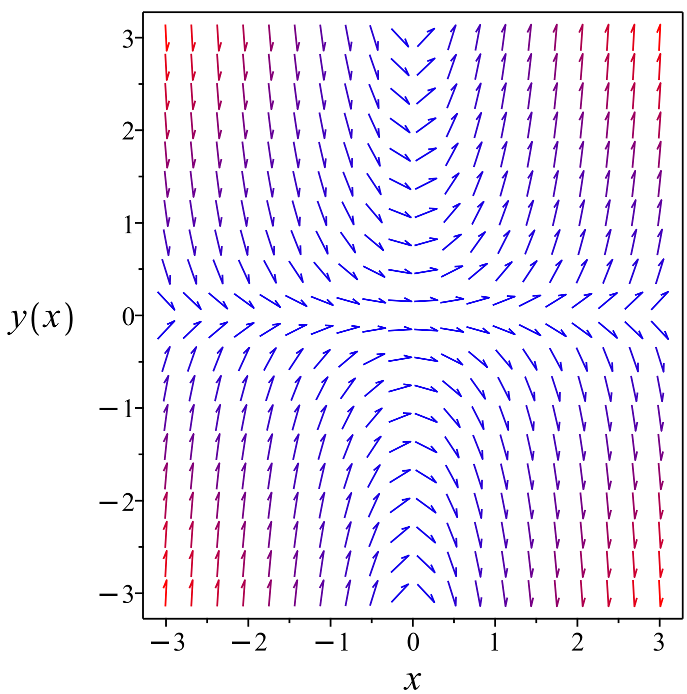


Figure 210: Slope field plot

Verification of solutions

$$y = \left(-1 + 2x + 3(x-1)^2 + \frac{10(x-1)^3}{3} + \frac{19(x-1)^4}{6} + \frac{13(x-1)^5}{5}\right) y(1) + O\left((x-1)^6\right)$$

Verified OK.

6.8.2 Maple step by step solution

Let's solve

$y' - 2xy = 0$

- Highest derivative means the order of the ODE is 1

  y'

- Separate variables

  $\frac{y'}{y} = 2x$

- Integrate both sides with respect to x

  $\int \frac{y'}{y} dx = \int 2x dx + c_1$

- Evaluate integral

  $\ln(y) = x^2 + c_1$

- Solve for y

  $y = e^{x^2 + c_1}$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
<- 1st order linear successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 36

```
Order:=6;
dsolve(diff(y(x),x)-2*x*y(x)=0,y(x),type='series',x=1);
```

$$y(x) = \left(-1 + 2x + 3(x-1)^2 + \frac{10(x-1)^3}{3} + \frac{19(x-1)^4}{6} + \frac{13(x-1)^5}{5}\right) y(1) + O\left(x^6\right)$$

✓ Solution by Mathematica

Time used: 0.001 (sec). Leaf size: 47

```
AsymptoticDSolveValue[y'[x]-2*x*y[x]==0,y[x],{x,1,5}]
```

$$y(x) \to c_1 \left(\frac{13}{5}(x-1)^5 + \frac{19}{6}(x-1)^4 + \frac{10}{3}(x-1)^3 + 3(x-1)^2 + 2(x-1) + 1\right)$$

6.9 problem 9

Internal problem ID [5027]
Internal file name [`OUTPUT/4520_Sunday_June_05_2022_03_00_04_PM_52749534/index.tex`]

Book: Fundamentals of Differential Equations. By Nagle, Saff and Snider. 9th edition. Boston. Pearson 2018.
Section: Chapter 8, Series solutions of differential equations. Section 8.4. page 449
Problem number: 9.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second order series method. Ordinary point", "second order series method. Taylor series method"**

Maple gives the following as the ode type

`[[_2nd_order, _with_linear_symmetries]]`

$$\boxed{(x^2-2x)\,y''+2y=0}$$

With the expansion point for the power series method at $x=1$.

The ode does not have its expansion point at $x=0$, therefore to simplify the computation of power series expansion, change of variable is made on the independent variable to shift the initial conditions and the expasion point back to zero. The new ode is then solved more easily since the expansion point is now at zero. The solution converted back to the original independent variable. Let

$$t=x-1$$

The ode is converted to be in terms of the new independent variable t. This results in

$$\left((t+1)^2-2t-2\right)\left(\frac{d^2}{dt^2}y(t)\right)+2y(t)=0$$

With its expansion point and initial conditions now at $t=0$. The transformed ODE is now solved. Solving ode using Taylor series method. This gives review on how the Taylor series method works for solving second order ode.

Let

$$y'' = f(x, y, y')$$

Assuming expansion is at $x_0 = 0$ (we can always shift the actual expansion point to 0 by change of variables) and assuming $f(x, y, y')$ is analytic at x_0 which must be the case for an ordinary point. Let initial conditions be $y(x_0) = y_0$ and $y'(x_0) = y'_0$. Using Taylor series gives

$$\begin{aligned} y(x) &= y(x_0) + (x - x_0)\, y'(x_0) + \frac{(x - x_0)^2}{2} y''(x_0) + \frac{(x - x_0)^3}{3!} y'''(x_0) + \cdots \\ &= y_0 + x y'_0 + \frac{x^2}{2} f|_{x_0, y_0, y'_0} + \frac{x^3}{3!} f'|_{x_0, y_0, y'_0} + \cdots \\ &= y_0 + x y'_0 + \sum_{n=0}^{\infty} \frac{x^{n+2}}{(n+2)!} \frac{d^n f}{dx^n}\bigg|_{x_0, y_0, y'_0} \end{aligned}$$

But

$$\begin{aligned} \frac{df}{dx} &= \frac{\partial f}{\partial x}\frac{dx}{dx} + \frac{\partial f}{\partial y}\frac{dy}{dx} + \frac{\partial f}{\partial y'}\frac{dy'}{dx} && (1) \\ &= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} y' + \frac{\partial f}{\partial y'} y'' && (278) \\ &= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} y' + \frac{\partial f}{\partial y'} f && (279) \\ \frac{d^2 f}{dx^2} &= \frac{d}{dx}\left(\frac{df}{dx}\right) \\ &= \frac{\partial}{\partial x}\left(\frac{df}{dx}\right) + \frac{\partial}{\partial y}\left(\frac{df}{dx}\right) y' + \frac{\partial}{\partial y'}\left(\frac{df}{dx}\right) f && (2) \\ \frac{d^3 f}{dx^3} &= \frac{d}{dx}\left(\frac{d^2 f}{dx^2}\right) \\ &= \frac{\partial}{\partial x}\left(\frac{d^2 f}{dx^2}\right) + \left(\frac{\partial}{\partial y}\frac{d^2 f}{dx^2}\right) y' + \frac{\partial}{\partial y'}\left(\frac{d^2 f}{dx^2}\right) f && (3) \\ &\vdots \end{aligned}$$

And so on. Hence if we name $F_0 = f(x,y,y')$ then the above can be written as

$$
\begin{aligned}
F_0 &= f(x,y,y') \qquad (4)\\
F_1 &= \frac{df}{dx}\\
&= \frac{dF_0}{dx}\\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}y' + \frac{\partial f}{\partial y'}y''\\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}y' + \frac{\partial f}{\partial y'}f \qquad (5)\\
&= \frac{\partial F_0}{\partial x} + \frac{\partial F_0}{\partial y}y' + \frac{\partial F_0}{\partial y'}F_0\\
F_2 &= \frac{d}{dx}\left(\frac{d}{dx}f\right)\\
&= \frac{d}{dx}(F_1)\\
&= \frac{\partial}{\partial x}F_1 + \left(\frac{\partial F_1}{\partial y}\right)y' + \left(\frac{\partial F_1}{\partial y'}\right)y''\\
&= \frac{\partial}{\partial x}F_1 + \left(\frac{\partial F_1}{\partial y}\right)y' + \left(\frac{\partial F_1}{\partial y'}\right)F_0\\
&\vdots\\
F_n &= \frac{d}{dx}(F_{n-1})\\
&= \frac{\partial}{\partial x}F_{n-1} + \left(\frac{\partial F_{n-1}}{\partial y}\right)y' + \left(\frac{\partial F_{n-1}}{\partial y'}\right)y''\\
&= \frac{\partial}{\partial x}F_{n-1} + \left(\frac{\partial F_{n-1}}{\partial y}\right)y' + \left(\frac{\partial F_{n-1}}{\partial y'}\right)F_0 \qquad (6)
\end{aligned}
$$

Therefore (6) can be used from now on along with

$$
y(x) = y_0 + xy_0' + \sum_{n=0}^{\infty}\frac{x^{n+2}}{(n+2)!}F_n\big|_{x_0,y_0,y_0'} \tag{7}
$$

To find $y(x)$ series solution around $x = 0$. Hence

$$
\begin{aligned}
F_0 &= -\frac{2y(t)}{t^2-1} \\
F_1 &= \frac{dF_0}{dt} \\
&= \frac{\partial F_0}{\partial t} + \frac{\partial F_0}{\partial y}\frac{d}{dt}y(t) + \frac{\partial F_0}{\partial \frac{d}{dt}y(t)} F_0 \\
&= \frac{-2\left(\frac{d}{dt}y(t)\right)t^2 + 4ty(t) + 2\frac{d}{dt}y(t)}{\left(t^2-1\right)^2} \\
F_2 &= \frac{dF_1}{dt} \\
&= \frac{\partial F_1}{\partial t} + \frac{\partial F_1}{\partial y}\frac{d}{dt}y(t) + \frac{\partial F_1}{\partial \frac{d}{dt}y(t)} F_1 \\
&= \frac{8\left(\frac{d}{dt}y(t)\right)t^3 - 8y(t)\,t^2 - 8t\left(\frac{d}{dt}y(t)\right) - 8y(t)}{\left(t^2-1\right)^3} \\
F_3 &= \frac{dF_2}{dt} \\
&= \frac{\partial F_2}{\partial t} + \frac{\partial F_2}{\partial y}\frac{d}{dt}y(t) + \frac{\partial F_2}{\partial \frac{d}{dt}y(t)} F_2 \\
&= \frac{\left(-32t^4+16t^2+16\right)\left(\frac{d}{dt}y(t)\right) + 16ty(t)\left(t^2+5\right)}{\left(t^2-1\right)^4} \\
F_4 &= \frac{dF_3}{dt} \\
&= \frac{\partial F_3}{\partial t} + \frac{\partial F_3}{\partial y}\frac{d}{dt}y(t) + \frac{\partial F_3}{\partial \frac{d}{dt}y(t)} F_3 \\
&= \frac{\left(144t^5+96t^3-240t\right)\left(\frac{d}{dt}y(t)\right) - 16y(t)\left(t^4+40t^2+7\right)}{\left(t^2-1\right)^5}
\end{aligned}
$$

And so on. Evaluating all the above at initial conditions $t = 0$ and $y(0) = y(0)$ and $y'(0) = y'(0)$ gives

$$
\begin{aligned}
F_0 &= 2y(0) \\
F_1 &= 2y'(0) \\
F_2 &= 8y(0) \\
F_3 &= 16y'(0) \\
F_4 &= 112y(0)
\end{aligned}
$$

Substituting all the above in (7) and simplifying gives the solution as

$$y(t) = \left(1 + t^2 + \frac{1}{3}t^4 + \frac{7}{45}t^6\right) y(0) + \left(t + \frac{1}{3}t^3 + \frac{2}{15}t^5\right) y'(0) + O(t^6)$$

Since the expansion point $t = 0$ is an ordinary, we can also solve this using standard power series The ode is normalized to be

$$\left(\frac{d^2}{dt^2}y(t)\right)(t^2 - 1) + 2y(t) = 0$$

Let the solution be represented as power series of the form

$$y(t) = \sum_{n=0}^{\infty} a_n t^n$$

Then

$$\frac{d}{dt}y(t) = \sum_{n=1}^{\infty} n a_n t^{n-1}$$

$$\frac{d^2}{dt^2}y(t) = \sum_{n=2}^{\infty} n(n-1)\, a_n t^{n-2}$$

Substituting the above back into the ode gives

$$\left(\sum_{n=2}^{\infty} n(n-1)\, a_n t^{n-2}\right)(t^2 - 1) + 2\left(\sum_{n=0}^{\infty} a_n t^n\right) = 0 \tag{1}$$

Which simplifies to

$$\left(\sum_{n=2}^{\infty} t^n a_n n(n-1)\right) + \sum_{n=2}^{\infty} \left(-n(n-1)\, a_n t^{n-2}\right) + \left(\sum_{n=0}^{\infty} 2a_n t^n\right) = 0 \tag{2}$$

The next step is to make all powers of t be n in each summation term. Going over each summation term above with power of t in it which is not already t^n and adjusting the power and the corresponding index gives

$$\sum_{n=2}^{\infty} \left(-n(n-1)\, a_n t^{n-2}\right) = \sum_{n=0}^{\infty} \left(-(n+2)\, a_{n+2}(n+1)\, t^n\right)$$

Substituting all the above in Eq (2) gives the following equation where now all powers of t are the same and equal to n.

$$\left(\sum_{n=2}^{\infty} t^n a_n n(n-1)\right) + \sum_{n=0}^{\infty} \left(-(n+2) a_{n+2}(n+1) t^n\right) + \left(\sum_{n=0}^{\infty} 2a_n t^n\right) = 0 \tag{3}$$

$n = 0$ gives

$$-2a_2 + 2a_0 = 0$$

$$a_2 = a_0$$

$n = 1$ gives

$$-6a_3 + 2a_1 = 0$$

Which after substituting earlier equations, simplifies to

$$a_3 = \frac{a_1}{3}$$

For $2 \leq n$, the recurrence equation is

$$na_n(n-1) - (n+2) a_{n+2}(n+1) + 2a_n = 0 \tag{4}$$

Solving for a_{n+2}, gives

$$a_{n+2} = \frac{a_n(n^2 - n + 2)}{(n+2)(n+1)} \tag{5}$$

For $n = 2$ the recurrence equation gives

$$4a_2 - 12a_4 = 0$$

Which after substituting the earlier terms found becomes

$$a_4 = \frac{a_0}{3}$$

For $n = 3$ the recurrence equation gives

$$8a_3 - 20a_5 = 0$$

Which after substituting the earlier terms found becomes

$$a_5 = \frac{2a_1}{15}$$

For $n = 4$ the recurrence equation gives

$$14a_4 - 30a_6 = 0$$

Which after substituting the earlier terms found becomes

$$a_6 = \frac{7a_0}{45}$$

For $n = 5$ the recurrence equation gives

$$22a_5 - 42a_7 = 0$$

Which after substituting the earlier terms found becomes

$$a_7 = \frac{22a_1}{315}$$

And so on. Therefore the solution is

$$\begin{aligned} y(t) &= \sum_{n=0}^{\infty} a_n t^n \\ &= a_3 t^3 + a_2 t^2 + a_1 t + a_0 + \dots \end{aligned}$$

Substituting the values for a_n found above, the solution becomes

$$y(t) = a_0 + a_1 t + a_0 t^2 + \frac{1}{3} a_1 t^3 + \frac{1}{3} a_0 t^4 + \frac{2}{15} a_1 t^5 + \dots$$

Collecting terms, the solution becomes

$$y(t) = \left(1 + t^2 + \frac{1}{3} t^4\right) a_0 + \left(t + \frac{1}{3} t^3 + \frac{2}{15} t^5\right) a_1 + O(t^6) \tag{3}$$

At $t = 0$ the solution above becomes

$$y(t) = \left(1 + t^2 + \frac{1}{3} t^4\right) c_1 + \left(t + \frac{1}{3} t^3 + \frac{2}{15} t^5\right) c_2 + O(t^6)$$

Replacing t in the above with the original independent variable x using $t = x-1$ results in

$$y = \left(1 + (x-1)^2 + \frac{(x-1)^4}{3} + \frac{7(x-1)^6}{45}\right) y(1) + \left(x - 1 + \frac{(x-1)^3}{3} + \frac{2(x-1)^5}{15}\right) y'(1) + O\left((x-1)^6\right)$$

Summary

The solution(s) found are the following

$$y = \left(1 + (x-1)^2 + \frac{(x-1)^4}{3} + \frac{7(x-1)^6}{45}\right) y(1) + \left(x - 1 + \frac{(x-1)^3}{3} + \frac{2(x-1)^5}{15}\right) y'(1) + O\left((x-1)^6\right) \tag{1}$$

Verification of solutions

$$y = \left(1 + (x-1)^2 + \frac{(x-1)^4}{3} + \frac{7(x-1)^6}{45}\right) y(1) + \left(x - 1 + \frac{(x-1)^3}{3} + \frac{2(x-1)^5}{15}\right) y'(1) + O\left((x-1)^6\right)$$

Verified OK.

6.9.1 Maple step by step solution

Let's solve

$(x^2 - 2x)\, y'' + 2y = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y'' = -\frac{2y}{x(-2+x)}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y'' + \frac{2y}{x(-2+x)} = 0$

□ Check to see if x_0 is a regular singular point

- Define functions

  $\left[P_2(x) = 0, P_3(x) = \frac{2}{x(-2+x)}\right]$

- $x \cdot P_2(x)$ is analytic at $x = 0$

  $\left(x \cdot P_2(x)\right)\bigg|_{x=0} = 0$

- $x^2 \cdot P_3(x)$ is analytic at $x = 0$

  $\left(x^2 \cdot P_3(x)\right)\bigg|_{x=0} = 0$

- $x = 0$is a regular singular point

  Check to see if x_0 is a regular singular point

  $x_0 = 0$

- Multiply by denominators

  $y'' x(-2+x) + 2y = 0$

- Assume series solution for y

  $y = \sum_{k=0}^{\infty} a_k x^{k+r}$

□ Rewrite ODE with series expansions

- Convert $x^m \cdot y''$ to series expansion for $m = 1..2$

  $x^m \cdot y'' = \sum_{k=0}^{\infty} a_k (k+r)(k+r-1)\, x^{k+r-2+m}$

- Shift index using $k->k+2-m$

  $x^m \cdot y'' = \sum_{k=-2+m}^{\infty} a_{k+2-m}(k+2-m+r)(k+1-m+r)\, x^{k+r}$

  Rewrite ODE with series expansions

  $-2a_0 r(-1+r)\, x^{-1+r} + \left(\sum_{k=0}^{\infty} \left(-2a_{k+1}(k+1+r)(k+r) + a_k(k^2+2kr+r^2-k-r+2)\right) x^{k+r}\right)$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $-2r(-1+r) = 0$

- Values of r that satisfy the indicial equation

  $r \in \{0, 1\}$

- Each term in the series must be 0, giving the recursion relation

$$\left(k^2 + (2r - 1)\,k + r^2 - r + 2\right) a_k - 2a_{k+1}(k + 1 + r)\,(k + r) = 0$$

- Recursion relation that defines series solution to ODE

$$a_{k+1} = \frac{(k^2+2kr+r^2-k-r+2)\,a_k}{2(k+1+r)(k+r)}$$

- Recursion relation for $r = 0$

$$a_{k+1} = \frac{(k^2-k+2)\,a_k}{2(k+1)k}$$

- Solution for $r = 0$

$$\left[y = \sum_{k=0}^{\infty} a_k x^k, a_{k+1} = \frac{(k^2-k+2)\,a_k}{2(k+1)k}\right]$$

- Recursion relation for $r = 1$

$$a_{k+1} = \frac{(k^2+k+2)\,a_k}{2(k+2)(k+1)}$$

- Solution for $r = 1$

$$\left[y = \sum_{k=0}^{\infty} a_k x^{k+1}, a_{k+1} = \frac{(k^2+k+2)\,a_k}{2(k+2)(k+1)}\right]$$

- Combine solutions and rename parameters

$$\left[y = \left(\sum_{k=0}^{\infty} a_k x^k\right) + \left(\sum_{k=0}^{\infty} b_k x^{k+1}\right), a_{k+1} = \frac{(k^2-k+2)\,a_k}{2(k+1)k}, b_{k+1} = \frac{(k^2+k+2)\,b_k}{2(k+2)(k+1)}\right]$$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
<- No Liouvillian solutions exists
-> Trying a solution in terms of special functions:
   -> Bessel
   -> elliptic
   -> Legendre
   -> Whittaker
      -> hyper3: Equivalence to 1F1 under a power @ Moebius
   -> hypergeometric
      -> heuristic approach
      -> hyper3: Equivalence to 2F1, 1F1 or 0F1 under a power @ Moebius
      <- hyper3 successful: received ODE is equivalent to the 2F1 ODE
   <- hypergeometric successful
<- special function solution successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 32

```
Order:=6;
dsolve((x^2-2*x)*diff(y(x),x$2)+2*y(x)=0,y(x),type='series',x=1);
```

$$y(x) = \left(1+(x-1)^2+\frac{(x-1)^4}{3}\right)y(1)+\left(x-1+\frac{(x-1)^3}{3}+\frac{2(x-1)^5}{15}\right)D(y)(1)+O(x^6)$$

✓ Solution by Mathematica

Time used: 0.001 (sec). Leaf size: 47

```
AsymptoticDSolveValue[(x^2-2*x)*y''[x]+2*y[x]==0,y[x],{x,1,5}]
```

$$y(x) \to c_1 \left(\frac{1}{3}(x-1)^4 + (x-1)^2 + 1 \right) + c_2 \left(\frac{2}{15}(x-1)^5 + \frac{1}{3}(x-1)^3 + x - 1 \right)$$

6.10 problem 10

Internal problem ID [5028]
Internal file name [`OUTPUT/4521_Sunday_June_05_2022_03_00_06_PM_44423711/index.tex`]

Book: Fundamentals of Differential Equations. By Nagle, Saff and Snider. 9th edition. Boston. Pearson 2018.
Section: Chapter 8, Series solutions of differential equations. Section 8.4. page 449
Problem number: 10.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second order series method. Ordinary point", "second_order_change_of_variable_on_x_method_1", "second order series method. Taylor series method"**

Maple gives the following as the ode type

```
[[_Emden, _Fowler]]
```

$$\boxed{x^2y'' - xy' + 2y = 0}$$

With the expansion point for the power series method at $x = 2$.

The ode does not have its expansion point at $x = 0$, therefore to simplify the computation of power series expansion, change of variable is made on the independent variable to shift the initial conditions and the expasion point back to zero. The new ode is then solved more easily since the expansion point is now at zero. The solution converted back to the original independent variable. Let

$$t = -2 + x$$

The ode is converted to be in terms of the new independent variable t. This results in

$$(t+2)^2 \left(\frac{d^2}{dt^2} y(t)\right) - (t+2) \left(\frac{d}{dt} y(t)\right) + 2y(t) = 0$$

With its expansion point and initial conditions now at $t = 0$. The transformed ODE is now solved. Solving ode using Taylor series method. This gives review on how the Taylor series method works for solving second order ode.

Let

$$y'' = f(x, y, y')$$

Assuming expansion is at $x_0 = 0$ (we can always shift the actual expansion point to 0 by change of variables) and assuming $f(x, y, y')$ is analytic at x_0 which must be the case for an ordinary point. Let initial conditions be $y(x_0) = y_0$ and $y'(x_0) = y'_0$. Using Taylor series gives

$$\begin{aligned}
y(x) &= y(x_0) + (x - x_0)\, y'(x_0) + \frac{(x - x_0)^2}{2} y''(x_0) + \frac{(x - x_0)^3}{3!} y'''(x_0) + \cdots \\
&= y_0 + x y'_0 + \frac{x^2}{2} f|_{x_0, y_0, y'_0} + \frac{x^3}{3!} f'|_{x_0, y_0, y'_0} + \cdots \\
&= y_0 + x y'_0 + \sum_{n=0}^{\infty} \frac{x^{n+2}}{(n+2)!} \left. \frac{d^n f}{dx^n} \right|_{x_0, y_0, y'_0}
\end{aligned}$$

But

$$\begin{aligned}
\frac{df}{dx} &= \frac{\partial f}{\partial x}\frac{dx}{dx} + \frac{\partial f}{\partial y}\frac{dy}{dx} + \frac{\partial f}{\partial y'}\frac{dy'}{dx} && (1) \\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} y' + \frac{\partial f}{\partial y'} y'' && (281) \\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} y' + \frac{\partial f}{\partial y'} f && (282) \\
\frac{d^2 f}{dx^2} &= \frac{d}{dx}\left(\frac{df}{dx}\right) \\
&= \frac{\partial}{\partial x}\left(\frac{df}{dx}\right) + \frac{\partial}{\partial y}\left(\frac{df}{dx}\right) y' + \frac{\partial}{\partial y'}\left(\frac{df}{dx}\right) f && (2) \\
\frac{d^3 f}{dx^3} &= \frac{d}{dx}\left(\frac{d^2 f}{dx^2}\right) \\
&= \frac{\partial}{\partial x}\left(\frac{d^2 f}{dx^2}\right) + \left(\frac{\partial}{\partial y}\frac{d^2 f}{dx^2}\right) y' + \frac{\partial}{\partial y'}\left(\frac{d^2 f}{dx^2}\right) f && (3) \\
&\ \ \vdots
\end{aligned}$$

And so on. Hence if we name $F_0 = f(x,y,y')$ then the above can be written as

$$
\begin{aligned}
F_0 &= f(x,y,y') \qquad (4)\\
F_1 &= \frac{df}{dx}\\
&= \frac{dF_0}{dx}\\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}y' + \frac{\partial f}{\partial y'}y''\\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}y' + \frac{\partial f}{\partial y'}f \qquad (5)\\
&= \frac{\partial F_0}{\partial x} + \frac{\partial F_0}{\partial y}y' + \frac{\partial F_0}{\partial y'}F_0\\
F_2 &= \frac{d}{dx}\left(\frac{d}{dx}f\right)\\
&= \frac{d}{dx}(F_1)\\
&= \frac{\partial}{\partial x}F_1 + \left(\frac{\partial F_1}{\partial y}\right)y' + \left(\frac{\partial F_1}{\partial y'}\right)y''\\
&= \frac{\partial}{\partial x}F_1 + \left(\frac{\partial F_1}{\partial y}\right)y' + \left(\frac{\partial F_1}{\partial y'}\right)F_0\\
&\vdots\\
F_n &= \frac{d}{dx}(F_{n-1})\\
&= \frac{\partial}{\partial x}F_{n-1} + \left(\frac{\partial F_{n-1}}{\partial y}\right)y' + \left(\frac{\partial F_{n-1}}{\partial y'}\right)y''\\
&= \frac{\partial}{\partial x}F_{n-1} + \left(\frac{\partial F_{n-1}}{\partial y}\right)y' + \left(\frac{\partial F_{n-1}}{\partial y'}\right)F_0 \qquad (6)
\end{aligned}
$$

Therefore (6) can be used from now on along with

$$
y(x) = y_0 + xy_0' + \sum_{n=0}^{\infty} \frac{x^{n+2}}{(n+2)!} F_n|_{x_0,y_0,y_0'} \qquad (7)
$$

To find $y(x)$ series solution around $x=0$. Hence

$$
\begin{aligned}
F_0 &= \frac{t\left(\frac{d}{dt}y(t)\right) + 2\frac{d}{dt}y(t) - 2y(t)}{(t+2)^2} \\
F_1 &= \frac{dF_0}{dt} \\
&= \frac{\partial F_0}{\partial t} + \frac{\partial F_0}{\partial y}\frac{d}{dt}y(t) + \frac{\partial F_0}{\partial \frac{d}{dt}y\left(t\right)} F_0 \\
&= \frac{(-2t-4)\left(\frac{d}{dt}y(t)\right) + 2y(t)}{(t+2)^3} \\
F_2 &= \frac{dF_1}{dt} \\
&= \frac{\partial F_1}{\partial t} + \frac{\partial F_1}{\partial y}\frac{d}{dt}y(t) + \frac{\partial F_1}{\partial \frac{d}{dt}y\left(t\right)} F_1 \\
&= \frac{(4t+8)\left(\frac{d}{dt}y(t)\right) - 2y(t)}{(t+2)^4} \\
F_3 &= \frac{dF_2}{dt} \\
&= \frac{\partial F_2}{\partial t} + \frac{\partial F_2}{\partial y}\frac{d}{dt}y(t) + \frac{\partial F_2}{\partial \frac{d}{dt}y\left(t\right)} F_2 \\
&= -\frac{10\left(\frac{d}{dt}y(t)\right)}{(t+2)^4} \\
F_4 &= \frac{dF_3}{dt} \\
&= \frac{\partial F_3}{\partial t} + \frac{\partial F_3}{\partial y}\frac{d}{dt}y(t) + \frac{\partial F_3}{\partial \frac{d}{dt}y\left(t\right)} F_3 \\
&= \frac{(30t+60)\left(\frac{d}{dt}y(t)\right) + 20y(t)}{(t+2)^6}
\end{aligned}
$$

And so on. Evaluating all the above at initial conditions $t=0$ and $y(0)=y(0)$ and

$y'(0) = y'(0)$ gives

$$
\begin{aligned}
F_0 &= -\frac{y(0)}{2} + \frac{y'(0)}{2} \\
F_1 &= \frac{y(0)}{4} - \frac{y'(0)}{2} \\
F_2 &= -\frac{y(0)}{8} + \frac{y'(0)}{2} \\
F_3 &= -\frac{5y'(0)}{8} \\
F_4 &= \frac{5y(0)}{16} + \frac{15y'(0)}{16}
\end{aligned}
$$

Substituting all the above in (7) and simplifying gives the solution as

$$
\begin{aligned}
y(t) = {} & \left(1 - \frac{1}{4}t^2 + \frac{1}{24}t^3 - \frac{1}{192}t^4 + \frac{1}{2304}t^6\right) y(0) \\
& + \left(t + \frac{1}{4}t^2 - \frac{1}{12}t^3 + \frac{1}{48}t^4 - \frac{1}{192}t^5 + \frac{1}{768}t^6\right) y'(0) + O(t^6)
\end{aligned}
$$

Since the expansion point $t = 0$ is an ordinary, we can also solve this using standard power series The ode is normalized to be

$$
(t^2 + 4t + 4)\left(\frac{d^2}{dt^2}y(t)\right) + (-t - 2)\left(\frac{d}{dt}y(t)\right) + 2y(t) = 0
$$

Let the solution be represented as power series of the form

$$
y(t) = \sum_{n=0}^{\infty} a_n t^n
$$

Then

$$
\begin{aligned}
\frac{d}{dt}y(t) &= \sum_{n=1}^{\infty} n a_n t^{n-1} \\
\frac{d^2}{dt^2}y(t) &= \sum_{n=2}^{\infty} n(n-1)\, a_n t^{n-2}
\end{aligned}
$$

Substituting the above back into the ode gives

$$
(t^2 + 4t + 4)\left(\sum_{n=2}^{\infty} n(n-1)\, a_n t^{n-2}\right) + (-t - 2)\left(\sum_{n=1}^{\infty} n a_n t^{n-1}\right) + 2\left(\sum_{n=0}^{\infty} a_n t^n\right) = 0 \tag{1}
$$

Which simplifies to

$$\left(\sum_{n=2}^{\infty} t^n a_n n(n-1)\right) + \left(\sum_{n=2}^{\infty} 4n\, t^{n-1} a_n (n-1)\right) + \left(\sum_{n=2}^{\infty} 4n(n-1)\, a_n t^{n-2}\right) + \sum_{n=1}^{\infty} (-n a_n t^n) + \sum_{n=1}^{\infty} (-2n a_n t^{n-1}) + \left(\sum_{n=0}^{\infty} 2 a_n t^n\right) = 0 \tag{2}$$

The next step is to make all powers of t be n in each summation term. Going over each summation term above with power of t in it which is not already t^n and adjusting the power and the corresponding index gives

$$\sum_{n=2}^{\infty} 4n\, t^{n-1} a_n (n-1) = \sum_{n=1}^{\infty} 4(n+1)\, a_{n+1} n\, t^n$$

$$\sum_{n=2}^{\infty} 4n(n-1)\, a_n t^{n-2} = \sum_{n=0}^{\infty} 4(n+2)\, a_{n+2} (n+1)\, t^n$$

$$\sum_{n=1}^{\infty} (-2n a_n t^{n-1}) = \sum_{n=0}^{\infty} (-2(n+1)\, a_{n+1} t^n)$$

Substituting all the above in Eq (2) gives the following equation where now all powers of t are the same and equal to n.

$$\left(\sum_{n=2}^{\infty} t^n a_n n(n-1)\right) + \left(\sum_{n=1}^{\infty} 4(n+1)\, a_{n+1} n\, t^n\right) + \left(\sum_{n=0}^{\infty} 4(n+2)\, a_{n+2} (n+1)\, t^n\right) + \sum_{n=1}^{\infty} (-n a_n t^n) + \sum_{n=0}^{\infty} (-2(n+1)\, a_{n+1} t^n) + \left(\sum_{n=0}^{\infty} 2 a_n t^n\right) = 0 \tag{3}$$

$n = 0$ gives

$$8a_2 - 2a_1 + 2a_0 = 0$$

$$a_2 = -\frac{a_0}{4} + \frac{a_1}{4}$$

$n = 1$ gives

$$4a_2 + 24a_3 + a_1 = 0$$

Which after substituting earlier equations, simplifies to

$$a_3 = \frac{a_0}{24} - \frac{a_1}{12}$$

For $2 \leq n$, the recurrence equation is

$$n a_n (n-1) + 4(n+1) a_{n+1} n + 4(n+2) a_{n+2} (n+1) - n a_n - 2(n+1) a_{n+1} + 2a_n = 0 \quad (4)$$

Solving for a_{n+2}, gives

$$\begin{aligned} a_{n+2} &= -\frac{n^2 a_n + 4n^2 a_{n+1} - 2n a_n + 2n a_{n+1} + 2a_n - 2a_{n+1}}{4\,(n+2)\,(n+1)} \\ &= -\frac{(n^2 - 2n + 2)\, a_n}{4\,(n+2)\,(n+1)} - \frac{(4n^2 + 2n - 2)\, a_{n+1}}{4\,(n+2)\,(n+1)} \end{aligned} \quad (5)$$

For $n = 2$ the recurrence equation gives

$$2a_2 + 18a_3 + 48a_4 = 0$$

Which after substituting the earlier terms found becomes

$$a_4 = -\frac{a_0}{192} + \frac{a_1}{48}$$

For $n = 3$ the recurrence equation gives

$$5a_3 + 40a_4 + 80a_5 = 0$$

Which after substituting the earlier terms found becomes

$$a_5 = -\frac{a_1}{192}$$

For $n = 4$ the recurrence equation gives

$$10a_4 + 70a_5 + 120a_6 = 0$$

Which after substituting the earlier terms found becomes

$$a_6 = \frac{a_0}{2304} + \frac{a_1}{768}$$

For $n = 5$ the recurrence equation gives

$$17a_5 + 108a_6 + 168a_7 = 0$$

Which after substituting the earlier terms found becomes

$$a_7 = -\frac{5a_1}{16128} - \frac{a_0}{3584}$$

And so on. Therefore the solution is

$$\begin{aligned} y(t) &= \sum_{n=0}^{\infty} a_n t^n \\ &= a_3 t^3 + a_2 t^2 + a_1 t + a_0 + \dots \end{aligned}$$

Substituting the values for a_n found above, the solution becomes

$$y(t) = a_0 + a_1 t + \left(-\frac{a_0}{4} + \frac{a_1}{4}\right) t^2 + \left(\frac{a_0}{24} - \frac{a_1}{12}\right) t^3 + \left(-\frac{a_0}{192} + \frac{a_1}{48}\right) t^4 - \frac{a_1 t^5}{192} + \dots$$

Collecting terms, the solution becomes

$$y(t) = \left(1 - \frac{1}{4}t^2 + \frac{1}{24}t^3 - \frac{1}{192}t^4\right) a_0 + \left(t + \frac{1}{4}t^2 - \frac{1}{12}t^3 + \frac{1}{48}t^4 - \frac{1}{192}t^5\right) a_1 + O(t^6) \quad (3)$$

At $t = 0$ the solution above becomes

$$y(t) = \left(1 - \frac{1}{4}t^2 + \frac{1}{24}t^3 - \frac{1}{192}t^4\right) c_1 + \left(t + \frac{1}{4}t^2 - \frac{1}{12}t^3 + \frac{1}{48}t^4 - \frac{1}{192}t^5\right) c_2 + O(t^6)$$

Replacing t in the above with the original independent variable $x$susing $t = -2 + x$ results in

$$\begin{aligned} y = &\left(1 - \frac{(-2+x)^2}{4} + \frac{(-2+x)^3}{24} - \frac{(-2+x)^4}{192} + \frac{(-2+x)^6}{2304}\right) y(2) \\ &+ \left(-2 + x + \frac{(-2+x)^2}{4} - \frac{(-2+x)^3}{12} + \frac{(-2+x)^4}{48} - \frac{(-2+x)^5}{192} + \frac{(-2+x)^6}{768}\right) y'(2) \\ &+ O\left((-2+x)^6\right) \end{aligned}$$

Summary

The solution(s) found are the following

$$
\begin{aligned}
y = {} & \left(1-\frac{\left(-2+x\right)^{2}}{4}+\frac{\left(-2+x\right)^{3}}{24}-\frac{\left(-2+x\right)^{4}}{192}+\frac{\left(-2+x\right)^{6}}{2304}\right) y(2) \\
& + \left(-2+x+\frac{\left(-2+x\right)^{2}}{4}-\frac{\left(-2+x\right)^{3}}{12}+\frac{\left(-2+x\right)^{4}}{48}-\frac{\left(-2+x\right)^{5}}{192} \right. \\
& \left. +\frac{\left(-2+x\right)^{6}}{768}\right) y'(2)+O\left(\left(-2+x\right)^{6}\right)
\end{aligned} \tag{1}
$$

Verification of solutions

$$
\begin{aligned}
y = {} & \left(1-\frac{\left(-2+x\right)^{2}}{4}+\frac{\left(-2+x\right)^{3}}{24}-\frac{\left(-2+x\right)^{4}}{192}+\frac{\left(-2+x\right)^{6}}{2304}\right) y(2) \\
& + \left(-2+x+\frac{\left(-2+x\right)^{2}}{4}-\frac{\left(-2+x\right)^{3}}{12}+\frac{\left(-2+x\right)^{4}}{48}-\frac{\left(-2+x\right)^{5}}{192}+\frac{\left(-2+x\right)^{6}}{768}\right) y'(2) \\
& + O\left(\left(-2+x\right)^{6}\right)
\end{aligned}
$$

Verified OK.

6.10.1 Maple step by step solution

Let's solve

$x^2y'' - xy' + 2y = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y'' = \frac{y'}{x} - \frac{2y}{x^2}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y'' - \frac{y'}{x} + \frac{2y}{x^2} = 0$

- Multiply by denominators of the ODE

  $x^2y'' - xy' + 2y = 0$

- Make a change of variables

  $t = \ln(x)$

☐ Substitute the change of variables back into the ODE

- Calculate the 1st derivative of y with respect to x , using the chain rule

  $y' = \left(\frac{d}{dt}y(t)\right) t'(x)$

- Compute derivative

  $y' = \frac{\frac{d}{dt}y(t)}{x}$

- Calculate the 2nd derivative of y with respect to x , using the chain rule

  $y'' = \left(\frac{d^2}{dt^2}y(t)\right) t'(x)^2 + t''(x) \left(\frac{d}{dt}y(t)\right)$

- Compute derivative

  $y'' = \frac{\frac{d^2}{dt^2}y(t)}{x^2} - \frac{\frac{d}{dt}y(t)}{x^2}$

  Substitute the change of variables back into the ODE

  $x^2\left(\frac{\frac{d^2}{dt^2}y(t)}{x^2} - \frac{\frac{d}{dt}y(t)}{x^2}\right) - \frac{d}{dt}y(t) + 2y(t) = 0$

- Simplify

  $\frac{d^2}{dt^2}y(t) - 2\frac{d}{dt}y(t) + 2y(t) = 0$

- Characteristic polynomial of ODE

  $r^2 - 2r + 2 = 0$

- Use quadratic formula to solve for r

  $r = \frac{2\pm(\sqrt{-4})}{2}$

- Roots of the characteristic polynomial

  $r = (1 - \mathrm{I}, 1 + \mathrm{I})$

- 1st solution of the ODE

  $y_1(t) = \mathrm{e}^t \cos(t)$

- 2nd solution of the ODE

  $y_2(t) = \mathrm{e}^t \sin(t)$

- General solution of the ODE

  $y(t) = c_1 y_1(t) + c_2 y_2(t)$

- Substitute in solutions

  $y(t) = c_1 \mathrm{e}^t \cos(t) + c_2 \mathrm{e}^t \sin(t)$

- Change variables back using $t = \ln(x)$

  $y = c_1 x \cos(\ln(x)) + c_2 x \sin(\ln(x))$

- Simplify

  $$y = x(c_1 \cos(\ln(x)) + c_2 \sin(\ln(x)))$$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
<- LODE of Euler type successful`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 49

```
Order:=6;
dsolve(x^2*diff(y(x),x$2)-x*diff(y(x),x)+2*y(x)=0,y(x),type='series',x=2);
```

$$y(x) = \left(1 - \frac{(-2+x)^2}{4} + \frac{(-2+x)^3}{24} - \frac{(-2+x)^4}{192}\right) y(2) + \left(-2+x+\frac{(-2+x)^2}{4} - \frac{(-2+x)^3}{12} + \frac{(-2+x)^4}{48} - \frac{(-2+x)^5}{192}\right) D(y)(2) + O(x^6)$$

✓ Solution by Mathematica

Time used: 0.001 (sec). Leaf size: 78

```
AsymptoticDSolveValue[x^2*y''[x]-x*y'[x]+2*y[x]==0,y[x],{x,2,5}]
```

$$y(x) \to c_1\left(-\frac{1}{192}(x-2)^4 + \frac{1}{24}(x-2)^3 - \frac{1}{4}(x-2)^2 + 1\right) + c_2\left(-\frac{1}{192}(x-2)^5 + \frac{1}{48}(x-2)^4 - \frac{1}{12}(x-2)^3 + \frac{1}{4}(x-2)^2 + x - 2\right)$$

6.11 problem 11

Internal problem ID [5029]
Internal file name [`OUTPUT/4522_Sunday_June_05_2022_03_00_07_PM_845541/index.tex`]

Book: Fundamentals of Differential Equations. By Nagle, Saff and Snider. 9th edition. Boston. Pearson 2018.
Section: Chapter 8, Series solutions of differential equations. Section 8.4. page 449
Problem number: 11.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second order series method. Ordinary point", "second order series method. Taylor series method"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

$$\boxed{x^2 y'' - y' + y = 0}$$

With the expansion point for the power series method at $x = 2$.

The ode does not have its expansion point at $x = 0$, therefore to simplify the computation of power series expansion, change of variable is made on the independent variable to shift the initial conditions and the expasion point back to zero. The new ode is then solved more easily since the expansion point is now at zero. The solution converted back to the original independent variable. Let

$$t = -2 + x$$

The ode is converted to be in terms of the new independent variable t. This results in

$$(t+2)^2 \left(\frac{d^2}{dt^2} y(t)\right) - \frac{d}{dt} y(t) + y(t) = 0$$

With its expansion point and initial conditions now at $t = 0$. The transformed ODE is now solved. Solving ode using Taylor series method. This gives review on how the Taylor series method works for solving second order ode.

Let

$$y'' = f(x, y, y')$$

Assuming expansion is at $x_0 = 0$ (we can always shift the actual expansion point to 0 by change of variables) and assuming $f(x, y, y')$ is analytic at x_0 which must be the case for an ordinary point. Let initial conditions be $y(x_0) = y_0$ and $y'(x_0) = y'_0$. Using Taylor series gives

$$\begin{aligned}
y(x) &= y(x_0) + (x - x_0)\, y'(x_0) + \frac{(x - x_0)^2}{2} y''(x_0) + \frac{(x - x_0)^3}{3!} y'''(x_0) + \cdots \\
&= y_0 + x y'_0 + \frac{x^2}{2} f|_{x_0, y_0, y'_0} + \frac{x^3}{3!} f'|_{x_0, y_0, y'_0} + \cdots \\
&= y_0 + x y'_0 + \sum_{n=0}^{\infty} \frac{x^{n+2}}{(n+2)\,!} \frac{d^n f}{dx^n}\bigg|_{x_0, y_0, y'_0}
\end{aligned}$$

But

$$\begin{aligned}
\frac{df}{dx} &= \frac{\partial f}{\partial x}\frac{dx}{dx} + \frac{\partial f}{\partial y}\frac{dy}{dx} + \frac{\partial f}{\partial y'}\frac{dy'}{dx} && (1) \\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} y' + \frac{\partial f}{\partial y'} y'' && (284) \\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} y' + \frac{\partial f}{\partial y'} f && (285) \\
\frac{d^2 f}{dx^2} &= \frac{d}{dx}\left(\frac{df}{dx}\right) \\
&= \frac{\partial}{\partial x}\left(\frac{df}{dx}\right) + \frac{\partial}{\partial y}\left(\frac{df}{dx}\right) y' + \frac{\partial}{\partial y'}\left(\frac{df}{dx}\right) f && (2) \\
\frac{d^3 f}{dx^3} &= \frac{d}{dx}\left(\frac{d^2 f}{dx^2}\right) \\
&= \frac{\partial}{\partial x}\left(\frac{d^2 f}{dx^2}\right) + \left(\frac{\partial}{\partial y}\frac{d^2 f}{dx^2}\right) y' + \frac{\partial}{\partial y'}\left(\frac{d^2 f}{dx^2}\right) f && (3) \\
&\vdots
\end{aligned}$$

And so on. Hence if we name $F_0 = f(x,y,y')$ then the above can be written as

$$\begin{aligned}
F_0 &= f(x,y,y') \qquad (4)\\
F_1 &= \frac{df}{dx}\\
&= \frac{dF_0}{dx}\\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}y' + \frac{\partial f}{\partial y'}y''\\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}y' + \frac{\partial f}{\partial y'}f \qquad (5)\\
&= \frac{\partial F_0}{\partial x} + \frac{\partial F_0}{\partial y}y' + \frac{\partial F_0}{\partial y'}F_0\\
F_2 &= \frac{d}{dx}\left(\frac{d}{dx}f\right)\\
&= \frac{d}{dx}(F_1)\\
&= \frac{\partial}{\partial x}F_1 + \left(\frac{\partial F_1}{\partial y}\right)y' + \left(\frac{\partial F_1}{\partial y'}\right)y''\\
&= \frac{\partial}{\partial x}F_1 + \left(\frac{\partial F_1}{\partial y}\right)y' + \left(\frac{\partial F_1}{\partial y'}\right)F_0\\
&\vdots\\
F_n &= \frac{d}{dx}(F_{n-1})\\
&= \frac{\partial}{\partial x}F_{n-1} + \left(\frac{\partial F_{n-1}}{\partial y}\right)y' + \left(\frac{\partial F_{n-1}}{\partial y'}\right)y''\\
&= \frac{\partial}{\partial x}F_{n-1} + \left(\frac{\partial F_{n-1}}{\partial y}\right)y' + \left(\frac{\partial F_{n-1}}{\partial y'}\right)F_0 \qquad (6)
\end{aligned}$$

Therefore (6) can be used from now on along with

$$y(x) = y_0 + xy_0' + \sum_{n=0}^{\infty} \frac{x^{n+2}}{(n+2)!} F_n|_{x_0,y_0,y_0'} \qquad (7)$$

To find $y(x)$ series solution around $x = 0$. Hence

$$
\begin{aligned}
F_0 &= \frac{\frac{d}{dt}y(t) - y(t)}{(t+2)^2} \\
F_1 &= \frac{dF_0}{dt} \\
&= \frac{\partial F_0}{\partial t} + \frac{\partial F_0}{\partial y}\frac{d}{dt}y(t) + \frac{\partial F_0}{\partial \frac{d}{dt}y(t)}F_0 \\
&= \frac{(-t^2 - 6t - 7)\left(\frac{d}{dt}y(t)\right) + y(t)(2t+3)}{(t+2)^4} \\
F_2 &= \frac{dF_1}{dt} \\
&= \frac{\partial F_1}{\partial t} + \frac{\partial F_1}{\partial y}\frac{d}{dt}y(t) + \frac{\partial F_1}{\partial \frac{d}{dt}y(t)}F_1 \\
&= \frac{(4t^3 + 28t^2 + 58t + 37)\left(\frac{d}{dt}y(t)\right) - 5\left(t + \frac{9}{5}\right)y(t)(t+1)}{(t+2)^6} \\
F_3 &= \frac{dF_2}{dt} \\
&= \frac{\partial F_2}{\partial t} + \frac{\partial F_2}{\partial y}\frac{d}{dt}y(t) + \frac{\partial F_2}{\partial \frac{d}{dt}y(t)}F_2 \\
&= \frac{(-17t^4 - 142t^3 - 411t^2 - 496t - 211)\left(\frac{d}{dt}y(t)\right) + 16\left(t^2 + 2t + \frac{1}{2}\right)y(t)\left(t + \frac{15}{8}\right)}{(t+2)^8} \\
F_4 &= \frac{dF_3}{dt} \\
&= \frac{\partial F_3}{\partial t} + \frac{\partial F_3}{\partial y}\frac{d}{dt}y(t) + \frac{\partial F_3}{\partial \frac{d}{dt}y(t)}F_3 \\
&= \frac{(84t^5 + 819t^4 + 3000t^3 + 5180t^2 + 4188t + 1241)\left(\frac{d}{dt}y(t)\right) - 63\left(t^4 + \frac{14}{3}t^3 + \frac{41}{7}t^2 - \frac{8}{9}t - \frac{27}{7}\right)y(t)}{(t+2)^{10}}
\end{aligned}
$$

And so on. Evaluating all the above at initial conditions $t = 0$ and $y(0) = y(0)$ and

$y'(0) = y'(0)$ gives

$$
\begin{aligned}
F_0 &= \frac{y'(0)}{4} - \frac{y(0)}{4} \\
F_1 &= \frac{3y(0)}{16} - \frac{7y'(0)}{16} \\
F_2 &= -\frac{9y(0)}{64} + \frac{37y'(0)}{64} \\
F_3 &= \frac{15y(0)}{256} - \frac{211y'(0)}{256} \\
F_4 &= \frac{243y(0)}{1024} + \frac{1241y'(0)}{1024}
\end{aligned}
$$

Substituting all the above in (7) and simplifying gives the solution as

$$
\begin{aligned}
y(t) = {} & \left(1 - \frac{1}{8}t^2 + \frac{1}{32}t^3 - \frac{3}{512}t^4 + \frac{1}{2048}t^5 + \frac{27}{81920}t^6\right) y(0) \\
& + \left(t + \frac{1}{8}t^2 - \frac{7}{96}t^3 + \frac{37}{1536}t^4 - \frac{211}{30720}t^5 + \frac{1241}{737280}t^6\right) y'(0) + O(t^6)
\end{aligned}
$$

Since the expansion point $t = 0$ is an ordinary, we can also solve this using standard power series The ode is normalized to be

$$
(t^2 + 4t + 4) \left(\frac{d^2}{dt^2} y(t)\right) - \frac{d}{dt} y(t) + y(t) = 0
$$

Let the solution be represented as power series of the form

$$
y(t) = \sum_{n=0}^{\infty} a_n t^n
$$

Then

$$
\begin{aligned}
\frac{d}{dt} y(t) &= \sum_{n=1}^{\infty} n a_n t^{n-1} \\
\frac{d^2}{dt^2} y(t) &= \sum_{n=2}^{\infty} n(n-1)\, a_n t^{n-2}
\end{aligned}
$$

Substituting the above back into the ode gives

$$
(t^2 + 4t + 4) \left(\sum_{n=2}^{\infty} n(n-1)\, a_n t^{n-2}\right) - \left(\sum_{n=1}^{\infty} n a_n t^{n-1}\right) + \left(\sum_{n=0}^{\infty} a_n t^n\right) = 0 \qquad (1)
$$

Which simplifies to

$$\left(\sum_{n=2}^{\infty} t^n a_n n(n-1)\right) + \left(\sum_{n=2}^{\infty} 4n\, t^{n-1} a_n (n-1)\right) \\ + \left(\sum_{n=2}^{\infty} 4n(n-1)\, a_n t^{n-2}\right) + \sum_{n=1}^{\infty} \left(-n a_n t^{n-1}\right) + \left(\sum_{n=0}^{\infty} a_n t^n\right) = 0 \tag{2}$$

The next step is to make all powers of t be n in each summation term. Going over each summation term above with power of t in it which is not already t^n and adjusting the power and the corresponding index gives

$$\sum_{n=2}^{\infty} 4n\, t^{n-1} a_n (n-1) = \sum_{n=1}^{\infty} 4(n+1)\, a_{n+1} n\, t^n$$

$$\sum_{n=2}^{\infty} 4n(n-1)\, a_n t^{n-2} = \sum_{n=0}^{\infty} 4(n+2)\, a_{n+2} (n+1)\, t^n$$

$$\sum_{n=1}^{\infty} \left(-n a_n t^{n-1}\right) = \sum_{n=0}^{\infty} \left(-(n+1)\, a_{n+1} t^n\right)$$

Substituting all the above in Eq (2) gives the following equation where now all powers of t are the same and equal to n.

$$\left(\sum_{n=2}^{\infty} t^n a_n n(n-1)\right) + \left(\sum_{n=1}^{\infty} 4(n+1)\, a_{n+1} n\, t^n\right) \\ + \left(\sum_{n=0}^{\infty} 4(n+2)\, a_{n+2}(n+1)\, t^n\right) + \sum_{n=0}^{\infty} \left(-(n+1)\, a_{n+1} t^n\right) + \left(\sum_{n=0}^{\infty} a_n t^n\right) = 0 \tag{3}$$

$n = 0$ gives

$$8a_2 - a_1 + a_0 = 0$$

$$a_2 = -\frac{a_0}{8} + \frac{a_1}{8}$$

$n = 1$ gives

$$6a_2 + 24a_3 + a_1 = 0$$

Which after substituting earlier equations, simplifies to

$$a_3 = \frac{a_0}{32} - \frac{7a_1}{96}$$

For $2 \leq n$, the recurrence equation is

$$n a_n (n-1) + 4(n+1) a_{n+1} n + 4(n+2) a_{n+2} (n+1) - (n+1) a_{n+1} + a_n = 0 \quad (4)$$

Solving for a_{n+2}, gives

$$\begin{aligned} a_{n+2} &= -\frac{n^2 a_n + 4n^2 a_{n+1} - n a_n + 3n a_{n+1} + a_n - a_{n+1}}{4(n+2)(n+1)} \\ &= -\frac{(n^2 - n + 1) a_n}{4(n+2)(n+1)} - \frac{(4n^2 + 3n - 1) a_{n+1}}{4(n+2)(n+1)} \end{aligned} \quad (5)$$

For $n = 2$ the recurrence equation gives

$$3a_2 + 21a_3 + 48a_4 = 0$$

Which after substituting the earlier terms found becomes

$$a_4 = -\frac{3a_0}{512} + \frac{37a_1}{1536}$$

For $n = 3$ the recurrence equation gives

$$7a_3 + 44a_4 + 80a_5 = 0$$

Which after substituting the earlier terms found becomes

$$a_5 = \frac{a_0}{2048} - \frac{211a_1}{30720}$$

For $n = 4$ the recurrence equation gives

$$13a_4 + 75a_5 + 120a_6 = 0$$

Which after substituting the earlier terms found becomes

$$a_6 = \frac{27a_0}{81920} + \frac{1241a_1}{737280}$$

For $n = 5$ the recurrence equation gives

$$21a_5 + 114a_6 + 168a_7 = 0$$

Which after substituting the earlier terms found becomes

$$a_7 = -\frac{653a_0}{2293760} - \frac{1171a_1}{4128768}$$

And so on. Therefore the solution is

$$\begin{aligned} y(t) &= \sum_{n=0}^{\infty} a_n t^n \\ &= a_3 t^3 + a_2 t^2 + a_1 t + a_0 + \dots \end{aligned}$$

Substituting the values for a_n found above, the solution becomes

$$\begin{aligned} y(t) = a_0 + a_1 t + \left(-\frac{a_0}{8} + \frac{a_1}{8}\right) t^2 + \left(\frac{a_0}{32} - \frac{7a_1}{96}\right) t^3 \\ + \left(-\frac{3a_0}{512} + \frac{37a_1}{1536}\right) t^4 + \left(\frac{a_0}{2048} - \frac{211a_1}{30720}\right) t^5 + \dots \end{aligned}$$

Collecting terms, the solution becomes

$$\begin{aligned} y(t) = &\left(1 - \frac{1}{8}t^2 + \frac{1}{32}t^3 - \frac{3}{512}t^4 + \frac{1}{2048}t^5\right) a_0 \\ &+ \left(t + \frac{1}{8}t^2 - \frac{7}{96}t^3 + \frac{37}{1536}t^4 - \frac{211}{30720}t^5\right) a_1 + O(t^6) \end{aligned} \tag{3}$$

At $t = 0$ the solution above becomes

$$\begin{aligned} y(t) = &\left(1 - \frac{1}{8}t^2 + \frac{1}{32}t^3 - \frac{3}{512}t^4 + \frac{1}{2048}t^5\right) c_1 \\ &+ \left(t + \frac{1}{8}t^2 - \frac{7}{96}t^3 + \frac{37}{1536}t^4 - \frac{211}{30720}t^5\right) c_2 + O(t^6) \end{aligned}$$

Replacing t in the above with the original independent variable $x$susing $t = -2 + x$

results in

$$y = \left(1-\frac{(-2+x)^2}{8}+\frac{(-2+x)^3}{32}-\frac{3(-2+x)^4}{512}+\frac{(-2+x)^5}{2048}+\frac{27(-2+x)^6}{81920}\right)y(2) + \left(-2+x+\frac{(-2+x)^2}{8}-\frac{7(-2+x)^3}{96}+\frac{37(-2+x)^4}{1536}-\frac{211(-2+x)^5}{30720} + \frac{1241(-2+x)^6}{737280}\right)y'(2)+O\left((-2+x)^6\right)$$

Summary

The solution(s) found are the following

$$y = \left(1-\frac{(-2+x)^2}{8}+\frac{(-2+x)^3}{32}-\frac{3(-2+x)^4}{512}+\frac{(-2+x)^5}{2048}+\frac{27(-2+x)^6}{81920}\right)y(2) + \left(-2+x+\frac{(-2+x)^2}{8}-\frac{7(-2+x)^3}{96}+\frac{37(-2+x)^4}{1536}-\frac{211(-2+x)^5}{30720} + \frac{1241(-2+x)^6}{737280}\right)y'(2)+O\left((-2+x)^6\right) \tag{1}$$

Verification of solutions

$$y = \left(1-\frac{(-2+x)^2}{8}+\frac{(-2+x)^3}{32}-\frac{3(-2+x)^4}{512}+\frac{(-2+x)^5}{2048}+\frac{27(-2+x)^6}{81920}\right)y(2) + \left(-2+x+\frac{(-2+x)^2}{8}-\frac{7(-2+x)^3}{96}+\frac{37(-2+x)^4}{1536}-\frac{211(-2+x)^5}{30720} + \frac{1241(-2+x)^6}{737280}\right)y'(2)+O\left((-2+x)^6\right)$$

Verified OK.

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
<- No Liouvillian solutions exists
-> Trying a solution in terms of special functions:
   -> Bessel
   -> elliptic
   -> Legendre
   <- Kummer successful
<- special function solution successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 54

```
Order:=6;
dsolve(x^2*diff(y(x),x$2)-diff(y(x),x)+y(x)=0,y(x),type='series',x=2);
```

$$y(x) = \left(1-\frac{(-2+x)^2}{8}+\frac{(-2+x)^3}{32}-\frac{3(-2+x)^4}{512}+\frac{(-2+x)^5}{2048}\right)y(2) + \left(-2+x+\frac{(-2+x)^2}{8}-\frac{7(-2+x)^3}{96}+\frac{37(-2+x)^4}{1536}-\frac{211(-2+x)^5}{30720}\right)D(y)(2) + O(x^6)$$

✓ Solution by Mathematica

Time used: 0.001 (sec). Leaf size: 87

```
AsymptoticDSolveValue[x^2*y''[x]-y'[x]+y[x]==0,y[x],{x,2,5}]
```

$$y(x) \to c_1\left(\frac{(x-2)^5}{2048} - \frac{3}{512}(x-2)^4 + \frac{1}{32}(x-2)^3 - \frac{1}{8}(x-2)^2 + 1\right) + c_2\left(-\frac{211(x-2)^5}{30720} + \frac{37(x-2)^4}{1536} - \frac{7}{96}(x-2)^3 + \frac{1}{8}(x-2)^2 + x - 2\right)$$

6.12 problem 12

Internal problem ID [5030]
Internal file name [`OUTPUT/4523_Sunday_June_05_2022_03_00_08_PM_32147205/index.tex`]

Book: Fundamentals of Differential Equations. By Nagle, Saff and Snider. 9th edition. Boston. Pearson 2018.
Section: Chapter 8, Series solutions of differential equations. Section 8.4. page 449
Problem number: 12.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second order series method. Ordinary point", "second order series method. Taylor series method"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

$$\boxed{y'' + (3x-1)\,y' - y = 0}$$

With the expansion point for the power series method at $x=-1$.

The ode does not have its expansion point at $x=0$, therefore to simplify the computation of power series expansion, change of variable is made on the independent variable to shift the initial conditions and the expasion point back to zero. The new ode is then solved more easily since the expansion point is now at zero. The solution converted back to the original independent variable. Let

$$t = x+1$$

The ode is converted to be in terms of the new independent variable t. This results in

$$\frac{d^2}{dt^2}y(t) + (3t-4)\left(\frac{d}{dt}y(t)\right) - y(t) = 0$$

With its expansion point and initial conditions now at $t=0$. The transformed ODE is now solved. Solving ode using Taylor series method. This gives review on how the Taylor series method works for solving second order ode.

Let

$$y'' = f(x, y, y')$$

Assuming expansion is at $x_0 = 0$ (we can always shift the actual expansion point to 0 by change of variables) and assuming $f(x, y, y')$ is analytic at x_0 which must be the case for an ordinary point. Let initial conditions be $y(x_0) = y_0$ and $y'(x_0) = y_0'$. Using Taylor series gives

$$\begin{aligned}
y(x) &= y(x_0) + (x - x_0)\, y'(x_0) + \frac{(x - x_0)^2}{2} y''(x_0) + \frac{(x - x_0)^3}{3!} y'''(x_0) + \cdots \\
&= y_0 + x y_0' + \frac{x^2}{2} f|_{x_0, y_0, y_0'} + \frac{x^3}{3!} f'|_{x_0, y_0, y_0'} + \cdots \\
&= y_0 + x y_0' + \sum_{n=0}^{\infty} \frac{x^{n+2}}{(n+2)!} \left. \frac{d^n f}{dx^n} \right|_{x_0, y_0, y_0'}
\end{aligned}$$

But

$$\begin{aligned}
\frac{df}{dx} &= \frac{\partial f}{\partial x}\frac{dx}{dx} + \frac{\partial f}{\partial y}\frac{dy}{dx} + \frac{\partial f}{\partial y'}\frac{dy'}{dx} && (1) \\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} y' + \frac{\partial f}{\partial y'} y'' && (287) \\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} y' + \frac{\partial f}{\partial y'} f && (288) \\
\frac{d^2 f}{dx^2} &= \frac{d}{dx}\left(\frac{df}{dx}\right) \\
&= \frac{\partial}{\partial x}\left(\frac{df}{dx}\right) + \frac{\partial}{\partial y}\left(\frac{df}{dx}\right) y' + \frac{\partial}{\partial y'}\left(\frac{df}{dx}\right) f && (2) \\
\frac{d^3 f}{dx^3} &= \frac{d}{dx}\left(\frac{d^2 f}{dx^2}\right) \\
&= \frac{\partial}{\partial x}\left(\frac{d^2 f}{dx^2}\right) + \left(\frac{\partial}{\partial y}\frac{d^2 f}{dx^2}\right) y' + \frac{\partial}{\partial y'}\left(\frac{d^2 f}{dx^2}\right) f && (3) \\
&\vdots
\end{aligned}$$

And so on. Hence if we name $F_0 = f(x, y, y')$ then the above can be written as

$$
\begin{aligned}
F_0 &= f(x, y, y') && (4)\\
F_1 &= \frac{df}{dx}\\
&= \frac{dF_0}{dx}\\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} y' + \frac{\partial f}{\partial y'} y''\\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} y' + \frac{\partial f}{\partial y'} f && (5)\\
&= \frac{\partial F_0}{\partial x} + \frac{\partial F_0}{\partial y} y' + \frac{\partial F_0}{\partial y'} F_0\\
F_2 &= \frac{d}{dx}\left(\frac{d}{dx} f\right)\\
&= \frac{d}{dx}(F_1)\\
&= \frac{\partial}{\partial x} F_1 + \left(\frac{\partial F_1}{\partial y}\right) y' + \left(\frac{\partial F_1}{\partial y'}\right) y''\\
&= \frac{\partial}{\partial x} F_1 + \left(\frac{\partial F_1}{\partial y}\right) y' + \left(\frac{\partial F_1}{\partial y'}\right) F_0\\
&\vdots\\
F_n &= \frac{d}{dx}(F_{n-1})\\
&= \frac{\partial}{\partial x} F_{n-1} + \left(\frac{\partial F_{n-1}}{\partial y}\right) y' + \left(\frac{\partial F_{n-1}}{\partial y'}\right) y''\\
&= \frac{\partial}{\partial x} F_{n-1} + \left(\frac{\partial F_{n-1}}{\partial y}\right) y' + \left(\frac{\partial F_{n-1}}{\partial y'}\right) F_0 && (6)
\end{aligned}
$$

Therefore (6) can be used from now on along with

$$
y(x) = y_0 + x y_0' + \sum_{n=0}^{\infty} \frac{x^{n+2}}{(n+2)!} F_n|_{x_0, y_0, y_0'} \tag{7}
$$

To find $y(x)$ series solution around $x = 0$. Hence

$$
\begin{aligned}
F_0 &= -3t\left(\frac{d}{dt}y(t)\right) + y(t) + 4\frac{d}{dt}y(t) \\
F_1 &= \frac{dF_0}{dt} \\
&= \frac{\partial F_0}{\partial t} + \frac{\partial F_0}{\partial y}\frac{d}{dt}y(t) + \frac{\partial F_0}{\partial \frac{d}{dt}y\left(t\right)}F_0 \\
&= \left(9t^2 - 24t + 14\right)\left(\frac{d}{dt}y(t)\right) + \left(-3t + 4\right)y(t) \\
F_2 &= \frac{dF_1}{dt} \\
&= \frac{\partial F_1}{\partial t} + \frac{\partial F_1}{\partial y}\frac{d}{dt}y(t) + \frac{\partial F_1}{\partial \frac{d}{dt}y\left(t\right)}F_1 \\
&= \left(-27t^3 + 108t^2 - 123t + 36\right)\left(\frac{d}{dt}y(t)\right) + 9y(t)\left(t^2 - \frac{8}{3}t + \frac{11}{9}\right) \\
F_3 &= \frac{dF_2}{dt} \\
&= \frac{\partial F_2}{\partial t} + \frac{\partial F_2}{\partial y}\frac{d}{dt}y(t) + \frac{\partial F_2}{\partial \frac{d}{dt}y\left(t\right)}F_2 \\
&= \left(81t^4 - 432t^3 + 729t^2 - 408t + 32\right)\left(\frac{d}{dt}y(t)\right) - 27\left(t^2 - \frac{8}{3}t + \frac{1}{3}\right)y(t)\left(-\frac{4}{3} + t\right) \\
F_4 &= \frac{dF_3}{dt} \\
&= \frac{\partial F_3}{\partial t} + \frac{\partial F_3}{\partial y}\frac{d}{dt}y(t) + \frac{\partial F_3}{\partial \frac{d}{dt}y\left(t\right)}F_3 \\
&= \left(-243t^5 + 1620t^4 - 3618t^3 + 2952t^2 - 375t - 268\right)\left(\frac{d}{dt}y(t)\right) + 81y(t)\left(t^4 - \frac{16}{3}t^3 + 8t^2 - \frac{64}{27}t - \frac{73}{81}\right.
\end{aligned}
$$

And so on. Evaluating all the above at initial conditions $t = 0$ and $y(0) = y(0)$ and $y'(0) = y'(0)$ gives

$$
\begin{aligned}
F_0 &= y(0) + 4y'(0) \\
F_1 &= 14y'(0) + 4y(0) \\
F_2 &= 36y'(0) + 11y(0) \\
F_3 &= 32y'(0) + 12y(0) \\
F_4 &= -268y'(0) - 73y(0)
\end{aligned}
$$

Substituting all the above in (7) and simplifying gives the solution as

$$
\begin{aligned}
y(t) = &\left(1+\frac{1}{2}t^2+\frac{2}{3}t^3+\frac{11}{24}t^4+\frac{1}{10}t^5-\frac{73}{720}t^6\right)y(0)\\
&+\left(t+2t^2+\frac{7}{3}t^3+\frac{3}{2}t^4+\frac{4}{15}t^5-\frac{67}{180}t^6\right)y'(0)+O(t^6)
\end{aligned}
$$

Since the expansion point $t = 0$ is an ordinary, we can also solve this using standard power series Let the solution be represented as power series of the form

$$
y(t) = \sum_{n=0}^{\infty} a_n t^n
$$

Then

$$
\begin{aligned}
\frac{d}{dt}y(t) &= \sum_{n=1}^{\infty} n a_n t^{n-1}\\
\frac{d^2}{dt^2}y(t) &= \sum_{n=2}^{\infty} n(n-1)\, a_n t^{n-2}
\end{aligned}
$$

Substituting the above back into the ode gives

$$
\sum_{n=2}^{\infty} n(n-1)\, a_n t^{n-2} = -3t\left(\sum_{n=1}^{\infty} n a_n t^{n-1}\right)+\left(\sum_{n=0}^{\infty} a_n t^n\right)+4\left(\sum_{n=1}^{\infty} n a_n t^{n-1}\right) \tag{1}
$$

Which simplifies to

$$
\left(\sum_{n=2}^{\infty} n(n-1)\, a_n t^{n-2}\right)+\left(\sum_{n=1}^{\infty} 3n a_n t^n\right)+\sum_{n=1}^{\infty}\left(-4n a_n t^{n-1}\right)+\sum_{n=0}^{\infty}\left(-a_n t^n\right)=0 \tag{2}
$$

The next step is to make all powers of t be n in each summation term. Going over each summation term above with power of t in it which is not already t^n and adjusting the power and the corresponding index gives

$$
\begin{aligned}
\sum_{n=2}^{\infty} n(n-1)\, a_n t^{n-2} &= \sum_{n=0}^{\infty} (n+2)\, a_{n+2}(n+1)\, t^n\\
\sum_{n=1}^{\infty} \left(-4n a_n t^{n-1}\right) &= \sum_{n=0}^{\infty} \left(-4(n+1)\, a_{n+1} t^n\right)
\end{aligned}
$$

Substituting all the above in Eq (2) gives the following equation where now all powers of t are the same and equal to n.

$$\left(\sum_{n=0}^{\infty} (n+2)\, a_{n+2}(n+1)\, t^{n}\right)+\left(\sum_{n=1}^{\infty} 3n a_{n} t^{n}\right)+\sum_{n\,=0}^{\infty} (-4(n+1)\, a_{n+1} t^{n})+\sum_{n\,=0}^{\infty} (-a_{n} t^{n})=0 \tag{3}$$

$n = 0$ gives

$$2a_2 - 4a_1 - a_0 = 0$$

$$a_2 = \frac{a_0}{2} + 2a_1$$

For $1 \leq n$, the recurrence equation is

$$(n+2)\, a_{n+2}(n+1) + 3n a_n - 4(n+1)\, a_{n+1} - a_n = 0 \tag{4}$$

Solving for a_{n+2}, gives

$$\begin{aligned} a_{n+2} &= -\frac{3n a_n - 4n a_{n+1} - a_n - 4a_{n+1}}{(n+2)\,(n+1)} \\ &= -\frac{(3n-1)\, a_n}{(n+2)\,(n+1)} - \frac{(-4n-4)\, a_{n+1}}{(n+2)\,(n+1)} \end{aligned} \tag{5}$$

For $n = 1$ the recurrence equation gives

$$6a_3 + 2a_1 - 8a_2 = 0$$

Which after substituting the earlier terms found becomes

$$a_3 = \frac{7a_1}{3} + \frac{2a_0}{3}$$

For $n = 2$ the recurrence equation gives

$$12a_4 + 5a_2 - 12a_3 = 0$$

Which after substituting the earlier terms found becomes

$$a_4 = \frac{11a_0}{24} + \frac{3a_1}{2}$$

For $n = 3$ the recurrence equation gives

$$20a_5 + 8a_3 - 16a_4 = 0$$

Which after substituting the earlier terms found becomes

$$a_5 = \frac{4a_1}{15} + \frac{a_0}{10}$$

For $n = 4$ the recurrence equation gives

$$30a_6 + 11a_4 - 20a_5 = 0$$

Which after substituting the earlier terms found becomes

$$a_6 = -\frac{73a_0}{720} - \frac{67a_1}{180}$$

For $n = 5$ the recurrence equation gives

$$42a_7 + 14a_5 - 24a_6 = 0$$

Which after substituting the earlier terms found becomes

$$a_7 = -\frac{19a_1}{63} - \frac{23a_0}{252}$$

And so on. Therefore the solution is

$$\begin{aligned} y(t) &= \sum_{n=0}^{\infty} a_n t^n \\ &= a_3 t^3 + a_2 t^2 + a_1 t + a_0 + \dots \end{aligned}$$

Substituting the values for a_n found above, the solution becomes

$$y(t) = a_0 + a_1 t + \left(\frac{a_0}{2} + 2a_1\right) t^2 + \left(\frac{7a_1}{3} + \frac{2a_0}{3}\right) t^3 + \left(\frac{11a_0}{24} + \frac{3a_1}{2}\right) t^4 + \left(\frac{4a_1}{15} + \frac{a_0}{10}\right) t^5 + \dots$$

Collecting terms, the solution becomes

$$y(t) = \left(1 + \frac{1}{2}t^2 + \frac{2}{3}t^3 + \frac{11}{24}t^4 + \frac{1}{10}t^5\right) a_0 + \left(t + 2t^2 + \frac{7}{3}t^3 + \frac{3}{2}t^4 + \frac{4}{15}t^5\right) a_1 + O(t^6) \quad (3)$$

At $t = 0$ the solution above becomes

$$y(t) = \left(1 + \frac{1}{2}t^2 + \frac{2}{3}t^3 + \frac{11}{24}t^4 + \frac{1}{10}t^5\right)c_1 + \left(t + 2t^2 + \frac{7}{3}t^3 + \frac{3}{2}t^4 + \frac{4}{15}t^5\right)c_2 + O(t^6)$$

Replacing t in the above with the original independent variable $x$susing $t = x+1$ results in

$$\begin{aligned} y = &\left(1 + \frac{(x+1)^2}{2} + \frac{2(x+1)^3}{3} + \frac{11(x+1)^4}{24} + \frac{(x+1)^5}{10} - \frac{73(x+1)^6}{720}\right)y(-1) \\ &+ \left(x + 1 + 2(x+1)^2 + \frac{7(x+1)^3}{3} + \frac{3(x+1)^4}{2} + \frac{4(x+1)^5}{15} - \frac{67(x+1)^6}{180}\right)y'(-1) \\ &+ O\left((x+1)^6\right) \end{aligned}$$

Summary

The solution(s) found are the following

$$\begin{aligned} y = &\left(1 + \frac{(x+1)^2}{2} + \frac{2(x+1)^3}{3} + \frac{11(x+1)^4}{24} + \frac{(x+1)^5}{10} - \frac{73(x+1)^6}{720}\right)y(-1) \\ &+ \left(x + 1 + 2(x+1)^2 + \frac{7(x+1)^3}{3} + \frac{3(x+1)^4}{2} + \frac{4(x+1)^5}{15} - \frac{67(x+1)^6}{180}\right)y'(-1) \\ &+ O\left((x+1)^6\right) \end{aligned} \tag{1}$$

Verification of solutions

$$\begin{aligned} y = &\left(1 + \frac{(x+1)^2}{2} + \frac{2(x+1)^3}{3} + \frac{11(x+1)^4}{24} + \frac{(x+1)^5}{10} - \frac{73(x+1)^6}{720}\right)y(-1) \\ &+ \left(x + 1 + 2(x+1)^2 + \frac{7(x+1)^3}{3} + \frac{3(x+1)^4}{2} + \frac{4(x+1)^5}{15} - \frac{67(x+1)^6}{180}\right)y'(-1) \\ &+ O\left((x+1)^6\right) \end{aligned}$$

Verified OK.

6.12.1 Maple step by step solution

Let's solve

$y'' + (3x-1)\,y' - y = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Assume series solution for y

  $y = \sum_{k=0}^{\infty} a_k x^k$

□ Rewrite DE with series expansions

  ◦ Convert $x^m \cdot y'$ to series expansion for $m = 0..1$

  $x^m \cdot y' = \sum_{k=\max(0,1-m)}^{\infty} a_k k\, x^{k-1+m}$

  ◦ Shift index using $k- >k+1-m$

  $x^m \cdot y' = \sum_{k=\max(0,1-m)+m-1}^{\infty} a_{k+1-m}(k+1-m)\, x^k$

  ◦ Convert y'' to series expansion

  $y'' = \sum_{k=2}^{\infty} a_k k(k-1)\, x^{k-2}$

  ◦ Shift index using $k- >k+2$

  $y'' = \sum_{k=0}^{\infty} a_{k+2}(k+2)\,(k+1)\, x^k$

  Rewrite DE with series expansions

  $\sum_{k=0}^{\infty} \left(a_{k+2}(k+2)\,(k+1) - a_{k+1}(k+1) + a_k(3k-1)\right) x^k = 0$

- Each term in the series must be 0, giving the recursion relation

  $k^2 a_{k+2} + (3a_k - a_{k+1} + 3a_{k+2})\,k - a_k - a_{k+1} + 2a_{k+2} = 0$

- Recursion relation that defines the series solution to the ODE

  $\left[y = \sum_{k=0}^{\infty} a_k x^k, a_{k+2} = -\frac{3a_k k - a_{k+1} k - a_k - a_{k+1}}{k^2+3k+2}\right]$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
<- No Liouvillian solutions exists
-> Trying a solution in terms of special functions:
   -> Bessel
   -> elliptic
   -> Legendre
   -> Kummer
      -> hyper3: Equivalence to 1F1 under a power @ Moebius
   -> hypergeometric
      -> heuristic approach
      <- heuristic approach successful
   <- hypergeometric successful
<- special function solution successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 54

```
Order:=6;
dsolve(diff(y(x),x$2)+(3*x-1)*diff(y(x),x)-y(x)=0,y(x),type='series',x=-1);
```

$$y(x) = \left(1+\frac{(x+1)^{2}}{2}+\frac{2(x+1)^{3}}{3}+\frac{11(x+1)^{4}}{24}+\frac{(x+1)^{5}}{10}\right)y(-1) + \left(x+1+2(x+1)^{2}+\frac{7(x+1)^{3}}{3}+\frac{3(x+1)^{4}}{2}+\frac{4(x+1)^{5}}{15}\right)D(y)(-1)+O(x^{6})$$

✓ Solution by Mathematica

Time used: 0.001 (sec). Leaf size: 85

```
AsymptoticDSolveValue[y''[x]+(3*x-1)*y'[x]-y[x]==0,y[x],{x,-1,5}]
```

$$y(x) \to c_1\left(\frac{1}{10}(x+1)^5+\frac{11}{24}(x+1)^4+\frac{2}{3}(x+1)^3+\frac{1}{2}(x+1)^2+1\right) + c_2\left(\frac{4}{15}(x+1)^5+\frac{3}{2}(x+1)^4+\frac{7}{3}(x+1)^3+2(x+1)^2+x+1\right)$$

6.13 problem 13

Internal problem ID [5031]
Internal file name [`OUTPUT/4524_Sunday_June_05_2022_03_00_10_PM_3396783/index.tex`]

Book: Fundamentals of Differential Equations. By Nagle, Saff and Snider. 9th edition. Boston. Pearson 2018.
Section: Chapter 8, Series solutions of differential equations. Section 8.4. page 449
Problem number: 13.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"exact", "linear", "separable", "homogeneousTypeD2", "first_order_ode_lie_symmetry_lookup", "first order ode series method. Ordinary point", "first order ode series method. Taylor series method"**

Maple gives the following as the ode type

```
[_separable]
```

$$\boxed{x' + \sin(t)\, x = 0}$$

With initial conditions

$$[x(0) = 1]$$

With the expansion point for the power series method at $t = 0$.

6.13.1 Existence and uniqueness analysis

This is a linear ODE. In canonical form it is written as

$$x' + p(t)x = q(t)$$

Where here

$$p(t) = \sin(t)$$
$$q(t) = 0$$

Hence the ode is

$$x' + \sin(t)\,x = 0$$

The domain of $p(t) = \sin(t)$ is

$$\{-\infty < t < \infty\}$$

And the point $t_0 = 0$ is inside this domain. Hence solution exists and is unique.

6.13.2 Solving as series ode

Solving ode using Taylor series method. This gives review on how the Taylor series method works for solving first order ode. Let

$$y' = f(x, y)$$

Where $f(x, y)$ is analytic at expansion point x_0. We can always shift to $x_0 = 0$ if x_0 is not zero. So from now we assume $x_0 = 0$. Assume also that $y(x_0) = y_0$. Using Taylor series

$$\begin{aligned}
y(x) &= y(x_0) + (x - x_0)\,y'(x_0) + \frac{(x - x_0)^2}{2} y''(x_0) + \frac{(x - x_0)^3}{3!} y''(x_0) + \cdots \\
&= y_0 + xf + \frac{x^2}{2} \frac{df}{dx}\bigg|_{x_0,y_0} + \frac{x^3}{3!} \frac{d^2 f}{dx^2}\bigg|_{x_0,y_0} + \cdots \\
&= y_0 + \sum_{n=0}^{\infty} \frac{x^{n+1}}{(n+1)!} \frac{d^n f}{dx^n}\bigg|_{x_0,y_0}
\end{aligned}$$

But

$$\begin{aligned}
\frac{df}{dx} &= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} f && (1) \\
\frac{d^2 f}{dx^2} &= \frac{d}{dx}\left(\frac{df}{dx}\right) \\
&= \frac{\partial}{\partial x}\left(\frac{df}{dx}\right) + \frac{\partial}{\partial y}\left(\frac{df}{dx}\right) f && (2) \\
\frac{d^3 f}{dx^3} &= \frac{d}{dx}\left(\frac{d^2 f}{dx^2}\right) \\
&= \frac{\partial}{\partial x}\left(\frac{d^2 f}{dx^2}\right) + \left(\frac{\partial}{\partial y}\frac{d^2 f}{dx^2}\right) f && (3) \\
&\vdots
\end{aligned}$$

And so on. Hence if we name $F_0 = f(x,y)$ then the above can be written as

$$
\begin{aligned}
F_0 &= f(x,y) &(4)\\
F_n &= \frac{d}{dx}(F_{n-1})\\
&= \frac{\partial}{\partial x}F_{n-1} + \left(\frac{\partial F_{n-1}}{\partial y}\right)F_0 &(5)
\end{aligned}
$$

For example, for $n=1$ we see that

$$
\begin{aligned}
F_1 &= \frac{d}{dx}(F_0)\\
&= \frac{\partial}{\partial x}F_0 + \left(\frac{\partial F_0}{\partial y}\right)F_0\\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}f
\end{aligned}
$$

Which is (1). And when $n=2$

$$
\begin{aligned}
F_2 &= \frac{d}{dx}(F_1)\\
&= \frac{\partial}{\partial x}F_1 + \left(\frac{\partial F_1}{\partial y}\right)F_0\\
&= \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}f\right) + \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}f\right)f\\
&= \frac{\partial}{\partial x}\left(\frac{df}{dx}\right) + \frac{\partial}{\partial y}\left(\frac{df}{dx}\right)f
\end{aligned}
$$

Which is (2) and so on. Therefore (4,5) can be used from now on along with

$$
y(x) = y_0 + \sum_{n=0}^{\infty} \frac{x^{n+1}}{(n+1)!} F_n\big|_{x_0,y_0} \tag{6}
$$

Hence

$$
\begin{aligned}
F_0 &= -\sin(t)\, x \\
F_1 &= \frac{dF_0}{dt} \\
&= \frac{\partial F_0}{\partial t} + \frac{\partial F_0}{\partial x} F_0 \\
&= x\left(-\cos(t) + \sin(t)^2\right) \\
F_2 &= \frac{dF_1}{dt} \\
&= \frac{\partial F_1}{\partial t} + \frac{\partial F_1}{\partial x} F_1 \\
&= \sin(t)\cos(t)\left(\cos(t) + 3\right) x \\
F_3 &= \frac{dF_2}{dt} \\
&= \frac{\partial F_2}{\partial t} + \frac{\partial F_2}{\partial x} F_2 \\
&= \left(\cos(t)^4 + 6\cos(t)^3 + 5\cos(t)^2 - 5\cos(t) - 3\right) x \\
F_4 &= \frac{dF_3}{dt} \\
&= \frac{\partial F_3}{\partial t} + \frac{\partial F_3}{\partial x} F_3 \\
&= \sin(t)\, x\left(-\cos(t)^4 - 10\cos(t)^3 - 23\cos(t)^2 - 5\cos(t) + 8\right)
\end{aligned}
$$

And so on. Evaluating all the above at initial conditions $t(0) = 0$ and $x(0) = 1$ gives

$$
\begin{aligned}
F_0 &= 0 \\
F_1 &= -1 \\
F_2 &= 0 \\
F_3 &= 4 \\
F_4 &= 0
\end{aligned}
$$

Substituting all the above in (6) and simplifying gives the solution as

$$
x = -\frac{t^2}{2} + 1 + \frac{t^4}{6} + O\left(t^6\right)
$$

Now we substitute the given initial conditions in the above to solve for $x(0)$. Solving for $x(0)$ from initial conditions gives

$$
x(0) = x(0)
$$

Therefore the solution becomes

$$x = -\frac{1}{2}t^2 + 1 + \frac{1}{6}t^4$$

Hence the solution can be written as

$$x = -\frac{t^2}{2} + 1 + \frac{t^4}{6} + O(t^6)$$

which simplifies to

$$x = -\frac{t^2}{2} + 1 + \frac{t^4}{6} + O(t^6)$$

Since $t = 0$ is also an ordinary point, then standard power series can also be used. Writing the ODE as

$$\begin{aligned} x' + q(t)x &= p(t) \\ x' + \sin(t)\, x &= 0 \end{aligned}$$

Where

$$\begin{aligned} q(t) &= \sin(t) \\ p(t) &= 0 \end{aligned}$$

Next, the type of the expansion point $t = 0$ is determined. This point can be an ordinary point, a regular singular point (also called removable singularity), or irregular singular point (also called non-removable singularity or essential singularity). When $t = 0$ is an ordinary point, then the standard power series is used. If the point is a regular singular point, Frobenius series is used instead. Irregular singular point requires more advanced methods (asymptotic methods) and is not supported now. Hopefully this will be added in the future. $t = 0$ is called an ordinary point $q(t)$ has a Taylor series expansion around the point $t = 0$. $t = 0$ is called a regular singular point if $q(t)$ is not not analytic at $t = 0$ but $tq(t)$ has Taylor series expansion. And finally, $t = 0$ is an irregular singular point if the point is not ordinary and not regular singular. This is the most complicated case. Now the expansion point $t = 0$ is checked to see if it is an ordinary point or not. Let the solution be represented as power series of the form

$$x = \sum_{n=0}^{\infty} a_n t^n$$

Then

$$x' = \sum_{n=1}^{\infty} n a_n t^{n-1}$$

Substituting the above back into the ode gives

$$\left(\sum_{n=1}^{\infty} n a_n t^{n-1}\right) + \sin(t) \left(\sum_{n=0}^{\infty} a_n t^n\right) = 0 \tag{1}$$

Expanding $\sin(t)$ as Taylor series around $t = 0$ and keeping only the first 6 terms gives

$$\begin{aligned}\sin(t) &= t - \frac{1}{6}t^3 + \frac{1}{120}t^5 - \frac{1}{5040}t^7 + \dots \\ &= t - \frac{1}{6}t^3 + \frac{1}{120}t^5 - \frac{1}{5040}t^7\end{aligned}$$

Hence the ODE in Eq (1) becomes

$$\left(\sum_{n=1}^{\infty} n a_n t^{n-1}\right) + \left(t - \frac{1}{6}t^3 + \frac{1}{120}t^5 - \frac{1}{5040}t^7\right) \left(\sum_{n=0}^{\infty} a_n t^n\right) = 0 \tag{1}$$

Expanding the second term in (1) gives

$$\left(\sum_{n=1}^{\infty} n a_n t^{n-1}\right) + t \cdot \left(\sum_{n=0}^{\infty} a_n t^n\right) - \frac{t^3}{6} \cdot \left(\sum_{n=0}^{\infty} a_n t^n\right) + \frac{t^5}{120} \cdot \left(\sum_{n=0}^{\infty} a_n t^n\right) - \frac{t^7}{5040} \cdot \left(\sum_{n=0}^{\infty} a_n t^n\right) = 0 \tag{1}$$

Which simplifies to

$$\begin{aligned}\left(\sum_{n=1}^{\infty} n a_n t^{n-1}\right) + \left(\sum_{n=0}^{\infty} t^{1+n} a_n\right) + \sum_{n=0}^{\infty} \left(-\frac{t^{n+3} a_n}{6}\right) \\ + \left(\sum_{n=0}^{\infty} \frac{t^{n+5} a_n}{120}\right) + \sum_{n=0}^{\infty} \left(-\frac{t^{n+7} a_n}{5040}\right) = 0\end{aligned} \tag{2}$$

The next step is to make all powers of t be n in each summation term. Going over each summation term above with power of t in it which is not already t^n and adjusting the power and the corresponding index gives

$$\sum_{n=1}^{\infty} n a_n t^{n-1} = \sum_{n=0}^{\infty} (1+n) a_{1+n} t^n$$

$$\sum_{n=0}^{\infty} t^{1+n} a_n = \sum_{n=1}^{\infty} a_{n-1} t^n$$

$$\sum_{n=0}^{\infty} \left(-\frac{t^{n+3} a_n}{6}\right) = \sum_{n=3}^{\infty} \left(-\frac{a_{n-3} t^n}{6}\right)$$

$$\sum_{n=0}^{\infty} \frac{t^{n+5} a_n}{120} = \sum_{n=5}^{\infty} \frac{a_{n-5} t^n}{120}$$

$$\sum_{n=0}^{\infty} \left(-\frac{t^{n+7} a_n}{5040}\right) = \sum_{n=7}^{\infty} \left(-\frac{a_{n-7} t^n}{5040}\right)$$

Substituting all the above in Eq (2) gives the following equation where now all powers of t are the same and equal to n.

$$\left(\sum_{n=0}^{\infty} (1+n)\, a_{1+n} t^n\right) + \left(\sum_{n=1}^{\infty} a_{n-1} t^n\right) + \sum_{n=3}^{\infty} \left(-\frac{a_{n-3} t^n}{6}\right) + \left(\sum_{n=5}^{\infty} \frac{a_{n-5} t^n}{120}\right) + \sum_{n=7}^{\infty} \left(-\frac{a_{n-7} t^n}{5040}\right) = 0 \tag{3}$$

$n = 1$ gives

$$2a_2 + a_0 = 0$$

Which after substituting earlier equations, simplifies to

$$a_2 = -\frac{a_0}{2}$$

$n = 2$ gives

$$3a_3 + a_1 = 0$$

Which after substituting earlier equations, simplifies to

$$3a_3 = 0$$

Or

$$a_3 = 0$$

$n = 3$ gives

$$4a_4 + a_2 - \frac{a_0}{6} = 0$$

Which after substituting earlier equations, simplifies to

$$a_4 = \frac{a_0}{6}$$

$n = 4$ gives

$$5a_5 + a_3 - \frac{a_1}{6} = 0$$

Which after substituting earlier equations, simplifies to

$$5a_5 = 0$$

Or

$$a_5 = 0$$

$n = 5$ gives

$$6a_6 + a_4 - \frac{a_2}{6} + \frac{a_0}{120} = 0$$

Which after substituting earlier equations, simplifies to

$$a_6 = -\frac{31a_0}{720}$$

For $7 \leq n$, the recurrence equation is

$$(1+n)\, a_{1+n} + a_{n-1} - \frac{a_{n-3}}{6} + \frac{a_{n-5}}{120} - \frac{a_{n-7}}{5040} = 0 \tag{4}$$

Solving for a_{1+n}, gives

$$a_{1+n} = \frac{-5040a_{n-1} + 840a_{n-3} - 42a_{n-5} + a_{n-7}}{5040 + 5040n} \tag{5}$$

And so on. Therefore the solution is

$$\begin{aligned} x &= \sum_{n=0}^{\infty} a_n t^n \\ &= a_3 t^3 + a_2 t^2 + a_1 t + a_0 + \dots \end{aligned}$$

Substituting the values for a_n found above, the solution becomes

$$x = a_0 - \frac{1}{2} a_0 t^2 + \frac{1}{6} a_0 t^4 + \dots$$

Collecting terms, the solution becomes

$$x = \left(-\frac{1}{2}t^2 + 1 + \frac{1}{6}t^4\right) a_0 + O(t^6) \tag{3}$$

At $t = 0$ the solution above becomes

$$x(0) = a_0$$

Therefore the solution in Eq(3) now can be written as

$$x = \left(-\frac{1}{2}t^2 + 1 + \frac{1}{6}t^4\right) x(0) + O(t^6)$$

Now we substitute the given initial conditions in the above to solve for $x(0)$. Solving for $x(0)$ from initial conditions gives

$$x(0) = 1$$

Therefore the solution becomes

$$x = -\frac{1}{2}t^2 + 1 + \frac{1}{6}t^4$$

Hence the solution can be written as

$$x = -\frac{t^2}{2} + 1 + \frac{t^4}{6} + O(t^6)$$

which simplifies to

$$x = -\frac{t^2}{2} + 1 + \frac{t^4}{6} + O(t^6)$$

Summary

The solution(s) found are the following

$$x = -\frac{t^2}{2} + 1 + \frac{t^4}{6} + O(t^6) \tag{1}$$

$$x = -\frac{t^2}{2} + 1 + \frac{t^4}{6} + O(t^6) \tag{2}$$

Verification of solutions

$$x = -\frac{t^2}{2} + 1 + \frac{t^4}{6} + O(t^6)$$

Verified OK.

$$x = -\frac{t^2}{2} + 1 + \frac{t^4}{6} + O(t^6)$$

Verified OK.

6.13.3 Maple step by step solution

Let's solve

$[x' + \sin(t)\, x = 0, x(0) = 1]$

- Highest derivative means the order of the ODE is 1

  x'

- Separate variables

  $\frac{x'}{x} = -\sin(t)$

- Integrate both sides with respect to t

  $\int \frac{x'}{x} dt = \int -\sin(t)\, dt + c_1$

- Evaluate integral

  $\ln(x) = \cos(t) + c_1$

- Solve for x

  $x = \mathrm{e}^{\cos(t)+c_1}$

- Use initial condition $x(0) = 1$

  $1 = \mathrm{e}^{c_1+1}$

- Solve for c_1

  $c_1 = -1$

- Substitute $c_1 = -1$ into general solution and simplify

  $x = \mathrm{e}^{\cos(t)-1}$

- Solution to the IVP

  $x = \mathrm{e}^{\cos(t)-1}$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
<- 1st order linear successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 14

```
Order:=6;
dsolve([diff(x(t),t)+sin(t)*x(t)=0,x(0) = 1],x(t),type='series',t=0);
```

$$x(t) = 1 - \frac{1}{2}t^2 + \frac{1}{6}t^4 + \mathrm{O}\left(t^6\right)$$

✓ Solution by Mathematica

Time used: 0.001 (sec). Leaf size: 19

```
AsymptoticDSolveValue[{x'[t]+Sin[t]*x[t]==0,{x[0]==1}},x[t],{t,0,5}]
```

$$x(t) \to \frac{t^4}{6} - \frac{t^2}{2} + 1$$

6.14 problem 14

Internal problem ID [5032]
Internal file name [`OUTPUT/4525_Sunday_June_05_2022_03_00_11_PM_77108776/index.tex`]

Book: Fundamentals of Differential Equations. By Nagle, Saff and Snider. 9th edition. Boston. Pearson 2018.
Section: Chapter 8, Series solutions of differential equations. Section 8.4. page 449
Problem number: 14.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"exact", "linear", "separable", "homogeneousTypeD2", "first_order_ode_lie_symmetry_lookup", "first order ode series method. Ordinary point", "first order ode series method. Taylor series method"**

Maple gives the following as the ode type

```
[_separable]
```

$$\boxed{y' - e^x y = 0}$$

With initial conditions

$$[y(0) = 1]$$

With the expansion point for the power series method at $x = 0$.

6.14.1 Existence and uniqueness analysis

This is a linear ODE. In canonical form it is written as

$$y' + p(x)y = q(x)$$

Where here

$$p(x) = -e^x$$
$$q(x) = 0$$

Hence the ode is

$$y' - e^x y = 0$$

The domain of $p(x) = -e^x$ is

$$\{-\infty < x < \infty\}$$

And the point $x_0 = 0$ is inside this domain. Hence solution exists and is unique.

6.14.2 Solving as series ode

Solving ode using Taylor series method. This gives review on how the Taylor series method works for solving first order ode. Let

$$y' = f(x, y)$$

Where $f(x, y)$ is analytic at expansion point x_0. We can always shift to $x_0 = 0$ if x_0 is not zero. So from now we assume $x_0 = 0$. Assume also that $y(x_0) = y_0$. Using Taylor series

$$\begin{aligned}
y(x) &= y(x_0) + (x - x_0)\, y'(x_0) + \frac{(x - x_0)^2}{2} y''(x_0) + \frac{(x - x_0)^3}{3!} y''(x_0) + \cdots \\
&= y_0 + xf + \frac{x^2}{2} \frac{df}{dx}\bigg|_{x_0,y_0} + \frac{x^3}{3!} \frac{d^2f}{dx^2}\bigg|_{x_0,y_0} + \cdots \\
&= y_0 + \sum_{n=0}^{\infty} \frac{x^{n+1}}{(n+1)!} \frac{d^n f}{dx^n}\bigg|_{x_0,y_0}
\end{aligned}$$

But

$$\begin{aligned}
\frac{df}{dx} &= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} f &(1)\\
\frac{d^2 f}{dx^2} &= \frac{d}{dx}\left(\frac{df}{dx}\right) \\
&= \frac{\partial}{\partial x}\left(\frac{df}{dx}\right) + \frac{\partial}{\partial y}\left(\frac{df}{dx}\right) f &(2)\\
\frac{d^3 f}{dx^3} &= \frac{d}{dx}\left(\frac{d^2 f}{dx^2}\right) \\
&= \frac{\partial}{\partial x}\left(\frac{d^2 f}{dx^2}\right) + \left(\frac{\partial}{\partial y}\frac{d^2 f}{dx^2}\right) f &(3)\\
&\vdots
\end{aligned}$$

And so on. Hence if we name $F_0 = f(x, y)$ then the above can be written as

$$
\begin{aligned}
F_0 &= f(x, y) \qquad (4) \\
F_n &= \frac{d}{dx}(F_{n-1}) \\
&= \frac{\partial}{\partial x}F_{n-1} + \left(\frac{\partial F_{n-1}}{\partial y}\right) F_0 \qquad (5)
\end{aligned}
$$

For example, for $n = 1$ we see that

$$
\begin{aligned}
F_1 &= \frac{d}{dx}(F_0) \\
&= \frac{\partial}{\partial x}F_0 + \left(\frac{\partial F_0}{\partial y}\right) F_0 \\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} f
\end{aligned}
$$

Which is (1). And when $n = 2$

$$
\begin{aligned}
F_2 &= \frac{d}{dx}(F_1) \\
&= \frac{\partial}{\partial x}F_1 + \left(\frac{\partial F_1}{\partial y}\right) F_0 \\
&= \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} f\right) + \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} f\right) f \\
&= \frac{\partial}{\partial x}\left(\frac{df}{dx}\right) + \frac{\partial}{\partial y}\left(\frac{df}{dx}\right) f
\end{aligned}
$$

Which is (2) and so on. Therefore (4,5) can be used from now on along with

$$
y(x) = y_0 + \sum_{n=0}^{\infty} \frac{x^{n+1}}{(n+1)!} F_n\Big|_{x_0, y_0} \qquad (6)
$$

Hence

$$
\begin{aligned}
F_0 &= e^{x} y \\
F_1 &= \frac{dF_0}{dx} \\
&= \frac{\partial F_0}{\partial x} + \frac{\partial F_0}{\partial y} F_0 \\
&= e^{x} y (1 + e^{x}) \\
F_2 &= \frac{dF_1}{dx} \\
&= \frac{\partial F_1}{\partial x} + \frac{\partial F_1}{\partial y} F_1 \\
&= y\, e^{x} \left(e^{2x} + 3\, e^{x} + 1\right) \\
F_3 &= \frac{dF_2}{dx} \\
&= \frac{\partial F_2}{\partial x} + \frac{\partial F_2}{\partial y} F_2 \\
&= y\, e^{x} \left(e^{3x} + 6\, e^{2x} + 7\, e^{x} + 1\right) \\
F_4 &= \frac{dF_3}{dx} \\
&= \frac{\partial F_3}{\partial x} + \frac{\partial F_3}{\partial y} F_3 \\
&= y\, e^{x} \left(e^{4x} + 10\, e^{3x} + 25\, e^{2x} + 15\, e^{x} + 1\right)
\end{aligned}
$$

And so on. Evaluating all the above at initial conditions $x(0) = 0$ and $y(0) = 1$ gives

$$
\begin{aligned}
F_0 &= 1 \\
F_1 &= 2 \\
F_2 &= 5 \\
F_3 &= 15 \\
F_4 &= 52
\end{aligned}
$$

Substituting all the above in (6) and simplifying gives the solution as

$$
y = x^2 + x + 1 + \frac{5x^3}{6} + \frac{5x^4}{8} + \frac{13x^5}{30} + O\left(x^6\right)
$$

Now we substitute the given initial conditions in the above to solve for $y(0)$. Solving for $y(0)$ from initial conditions gives

$$
y(0) = y(0)
$$

Therefore the solution becomes

$$y = x^2 + x + 1 + \frac{5}{6}x^3 + \frac{5}{8}x^4 + \frac{13}{30}x^5$$

Hence the solution can be written as

$$y = x^2 + x + 1 + \frac{5x^3}{6} + \frac{5x^4}{8} + \frac{13x^5}{30} + O(x^6)$$

which simplifies to

$$y = x^2 + x + 1 + \frac{5x^3}{6} + \frac{5x^4}{8} + \frac{13x^5}{30} + O(x^6)$$

Since $x = 0$ is also an ordinary point, then standard power series can also be used. Writing the ODE as

$$y' + q(x)y = p(x)$$
$$y' - e^x y = 0$$

Where

$$q(x) = -e^x$$
$$p(x) = 0$$

Next, the type of the expansion point $x = 0$ is determined. This point can be an ordinary point, a regular singular point (also called removable singularity), or irregular singular point (also called non-removable singularity or essential singularity). When $x = 0$ is an ordinary point, then the standard power series is used. If the point is a regular singular point, Frobenius series is used instead. Irregular singular point requires more advanced methods (asymptotic methods) and is not supported now. Hopefully this will be added in the future. $x = 0$ is called an ordinary point $q(x)$ has a Taylor series expansion around the point $x = 0$. $x = 0$ is called a regular singular point if $q(x)$ is not not analytic at $x = 0$ but $xq(x)$ has Taylor series expansion. And finally, $x = 0$ is an irregular singular point if the point is not ordinary and not regular singular. This is the most complicated case. Now the expansion point $x = 0$ is checked to see if it is an ordinary point or not. Let the solution be represented as power series of the form

$$y = \sum_{n=0}^{\infty} a_n x^n$$

Then

$$y' = \sum_{n=1}^{\infty} n a_n x^{n-1}$$

Substituting the above back into the ode gives

$$\left(\sum_{n=1}^{\infty} n a_n x^{n-1}\right) - \mathrm{e}^{x} \left(\sum_{n=0}^{\infty} a_n x^n\right) = 0 \tag{1}$$

Expanding $-\mathrm{e}^{x}$ as Taylor series around $x = 0$ and keeping only the first 6 terms gives

$$\begin{aligned}
-\mathrm{e}^{x} &= -1 - x - \frac{1}{2}x^2 - \frac{1}{6}x^3 - \frac{1}{24}x^4 - \frac{1}{120}x^5 - \frac{1}{720}x^6 + \dots \\
&= -1 - x - \frac{1}{2}x^2 - \frac{1}{6}x^3 - \frac{1}{24}x^4 - \frac{1}{120}x^5 - \frac{1}{720}x^6
\end{aligned}$$

Hence the ODE in Eq (1) becomes

$$\left(\sum_{n=1}^{\infty} n a_n x^{n-1}\right) + \left(-1 - x - \frac{1}{2}x^2 - \frac{1}{6}x^3 - \frac{1}{24}x^4 - \frac{1}{120}x^5 - \frac{1}{720}x^6\right) \left(\sum_{n=0}^{\infty} a_n x^n\right) = 0 \tag{1}$$

Expanding the second term in (1) gives

$$\left(\sum_{n=1}^{\infty} n a_n x^{n-1}\right) + -1 \cdot \left(\sum_{n=0}^{\infty} a_n x^n\right) - x \cdot \left(\sum_{n=0}^{\infty} a_n x^n\right) - \frac{x^2}{2} \cdot \left(\sum_{n=0}^{\infty} a_n x^n\right) - \frac{x^3}{6} \cdot \left(\sum_{n=0}^{\infty} a_n x^n\right) - \frac{x^4}{24} \cdot \left(\sum_{n=0}^{\infty} a_n x^n\right) - \tag{1}$$

Which simplifies to

$$\begin{aligned}
&\left(\sum_{n=1}^{\infty} n a_n x^{n-1}\right) + \sum_{n=0}^{\infty} (-a_n x^n) + \sum_{n=0}^{\infty} \left(-x^{1+n} a_n\right) \\
&+ \sum_{n=0}^{\infty} \left(-\frac{x^{n+2} a_n}{2}\right) + \sum_{n=0}^{\infty} \left(-\frac{x^{n+3} a_n}{6}\right) + \sum_{n=0}^{\infty} \left(-\frac{x^{n+4} a_n}{24}\right) \\
&+ \sum_{n=0}^{\infty} \left(-\frac{x^{n+5} a_n}{120}\right) + \sum_{n=0}^{\infty} \left(-\frac{x^{n+6} a_n}{720}\right) = 0
\end{aligned} \tag{2}$$

The next step is to make all powers of x be n in each summation term. Going over each summation term above with power of x in it which is not already x^n and adjusting the power and the corresponding index gives

$$\sum_{n=1}^{\infty} n a_n x^{n-1} = \sum_{n=0}^{\infty} (1+n) a_{1+n} x^n$$

$$\sum_{n=0}^{\infty} \left(-x^{1+n} a_n\right) = \sum_{n=1}^{\infty} (-a_{n-1} x^n)$$

$$\sum_{n=0}^{\infty}\left(-\frac{x^{n+2}a_n}{2}\right)=\sum_{n=2}^{\infty}\left(-\frac{a_{n-2}x^n}{2}\right)$$

$$\sum_{n=0}^{\infty}\left(-\frac{x^{n+3}a_n}{6}\right)=\sum_{n=3}^{\infty}\left(-\frac{a_{n-3}x^n}{6}\right)$$

$$\sum_{n=0}^{\infty}\left(-\frac{x^{n+4}a_n}{24}\right)=\sum_{n=4}^{\infty}\left(-\frac{a_{n-4}x^n}{24}\right)$$

$$\sum_{n=0}^{\infty}\left(-\frac{x^{n+5}a_n}{120}\right)=\sum_{n=5}^{\infty}\left(-\frac{a_{n-5}x^n}{120}\right)$$

$$\sum_{n=0}^{\infty}\left(-\frac{x^{n+6}a_n}{720}\right)=\sum_{n=6}^{\infty}\left(-\frac{a_{n-6}x^n}{720}\right)$$

Substituting all the above in Eq (2) gives the following equation where now all powers of x are the same and equal to n.

$$\begin{aligned}\left(\sum_{n=0}^{\infty}(1+n)\,a_{1+n}x^n\right)+\sum_{n=0}^{\infty}(-a_nx^n)+\sum_{n=1}^{\infty}(-a_{n-1}x^n)\\+\sum_{n=2}^{\infty}\left(-\frac{a_{n-2}x^n}{2}\right)+\sum_{n=3}^{\infty}\left(-\frac{a_{n-3}x^n}{6}\right)+\sum_{n=4}^{\infty}\left(-\frac{a_{n-4}x^n}{24}\right)\\+\sum_{n=5}^{\infty}\left(-\frac{a_{n-5}x^n}{120}\right)+\sum_{n=6}^{\infty}\left(-\frac{a_{n-6}x^n}{720}\right)=0\end{aligned}\tag{3}$$

$n=0$ gives

$$a_1-a_0=0$$
$$a_1=a_0$$

$n=1$ gives

$$2a_2-a_1-a_0=0$$

Which after substituting earlier equations, simplifies to

$$a_2=a_0$$

$n=2$ gives

$$3a_3-a_2-a_1-\frac{a_0}{2}=0$$

Which after substituting earlier equations, simplifies to

$$a_3 = \frac{5a_0}{6}$$

$n = 3$ gives

$$4a_4 - a_3 - a_2 - \frac{a_1}{2} - \frac{a_0}{6} = 0$$

Which after substituting earlier equations, simplifies to

$$a_4 = \frac{5a_0}{8}$$

$n = 4$ gives

$$5a_5 - a_4 - a_3 - \frac{a_2}{2} - \frac{a_1}{6} - \frac{a_0}{24} = 0$$

Which after substituting earlier equations, simplifies to

$$a_5 = \frac{13a_0}{30}$$

$n = 5$ gives

$$6a_6 - a_5 - a_4 - \frac{a_3}{2} - \frac{a_2}{6} - \frac{a_1}{24} - \frac{a_0}{120} = 0$$

Which after substituting earlier equations, simplifies to

$$a_6 = \frac{203a_0}{720}$$

For $6 \leq n$, the recurrence equation is

$$(1+n)\, a_{1+n} - a_n - a_{n-1} - \frac{a_{n-2}}{2} - \frac{a_{n-3}}{6} - \frac{a_{n-4}}{24} - \frac{a_{n-5}}{120} - \frac{a_{n-6}}{720} = 0 \tag{4}$$

Solving for a_{1+n}, gives

$$a_{1+n} = \frac{720a_n + 720a_{n-1} + 360a_{n-2} + 120a_{n-3} + 30a_{n-4} + 6a_{n-5} + a_{n-6}}{720 + 720n} \tag{5}$$

And so on. Therefore the solution is

$$\begin{aligned} y &= \sum_{n=0}^{\infty} a_n x^n \\ &= a_3 x^3 + a_2 x^2 + a_1 x + a_0 + \dots \end{aligned}$$

Substituting the values for a_n found above, the solution becomes

$$y = a_0 + a_0 x + a_0 x^2 + \frac{5}{6} a_0 x^3 + \frac{5}{8} a_0 x^4 + \frac{13}{30} a_0 x^5 + \dots$$

Collecting terms, the solution becomes

$$y = \left(x^2 + x + 1 + \frac{5}{6} x^3 + \frac{5}{8} x^4 + \frac{13}{30} x^5\right) a_0 + O(x^6) \tag{3}$$

At $x = 0$ the solution above becomes

$$y(0) = a_0$$

Therefore the solution in Eq(3) now can be written as

$$y = \left(x^2 + x + 1 + \frac{5}{6} x^3 + \frac{5}{8} x^4 + \frac{13}{30} x^5\right) y(0) + O(x^6)$$

Now we substitute the given initial conditions in the above to solve for $y(0)$. Solving for $y(0)$ from initial conditions gives

$$y(0) = 1$$

Therefore the solution becomes

$$y = x^2 + x + 1 + \frac{5}{6} x^3 + \frac{5}{8} x^4 + \frac{13}{30} x^5$$

Hence the solution can be written as

$$y = x^2 + x + 1 + \frac{5x^3}{6} + \frac{5x^4}{8} + \frac{13x^5}{30} + O(x^6)$$

which simplifies to

$$y = x^2 + x + 1 + \frac{5x^3}{6} + \frac{5x^4}{8} + \frac{13x^5}{30} + O(x^6)$$

Summary

The solution(s) found are the following

$$y = x^2 + x + 1 + \frac{5x^3}{6} + \frac{5x^4}{8} + \frac{13x^5}{30} + O(x^6) \tag{1}$$

$$y = x^2 + x + 1 + \frac{5x^3}{6} + \frac{5x^4}{8} + \frac{13x^5}{30} + O(x^6) \tag{2}$$

Verification of solutions

$$y = x^2 + x + 1 + \frac{5x^3}{6} + \frac{5x^4}{8} + \frac{13x^5}{30} + O(x^6)$$

Verified OK.

$$y = x^2 + x + 1 + \frac{5x^3}{6} + \frac{5x^4}{8} + \frac{13x^5}{30} + O(x^6)$$

Verified OK.

6.14.3 Maple step by step solution

Let's solve

$[y' - e^x y = 0, y(0) = 1]$

- Highest derivative means the order of the ODE is 1

  y'

- Separate variables

  $\frac{y'}{y} = e^x$

- Integrate both sides with respect to x

  $\int \frac{y'}{y} dx = \int e^x dx + c_1$

- Evaluate integral

  $\ln(y) = e^x + c_1$

- Solve for y

  $y = e^{e^x + c_1}$

- Use initial condition $y(0) = 1$

  $1 = e^{c_1 + 1}$

- Solve for c_1

  $c_1 = -1$

- Substitute $c_1 = -1$ into general solution and simplify

  $y = e^{e^x - 1}$

- Solution to the IVP

  $y = e^{e^x - 1}$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
<- 1st order linear successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 20

```
Order:=6;
dsolve([diff(y(x),x)-exp(x)*y(x)=0,y(0) = 1],y(x),type='series',x=0);
```

$$y(x) = 1 + x + x^2 + \frac{5}{6}x^3 + \frac{5}{8}x^4 + \frac{13}{30}x^5 + \mathrm{O}\left(x^6\right)$$

✓ Solution by Mathematica

Time used: 0.001 (sec). Leaf size: 30

```
AsymptoticDSolveValue[{y'[x]-Exp[x]*y[x]==0,{y[0]==1}},y[x],{x,0,5}]
```

$$y(x) \to \frac{13x^5}{30} + \frac{5x^4}{8} + \frac{5x^3}{6} + x^2 + x + 1$$

6.15 problem 15

Internal problem ID [5033]
Internal file name [`OUTPUT/4526_Sunday_June_05_2022_03_00_12_PM_43394477/index.tex`]

Book: Fundamentals of Differential Equations. By Nagle, Saff and Snider. 9th edition. Boston. Pearson 2018.
Section: Chapter 8, Series solutions of differential equations. Section 8.4. page 449
Problem number: 15.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second order series method. Ordinary point", "second order series method. Taylor series method"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

$$\boxed{(x^2+1)\, y''-y'\mathrm{e}^x+y=0}$$

With initial conditions

$$[y(0)=1, y'(0)=1]$$

With the expansion point for the power series method at $x=0$.

6.15.1 Existence and uniqueness analysis

This is a linear ODE. In canonical form it is written as

$$y''+p(x)y'+q(x)y=F$$

Where here

$$\begin{aligned}
p(x) &= -\frac{\mathrm{e}^x}{x^2+1} \\
q(x) &= \frac{1}{x^2+1} \\
F &= 0
\end{aligned}$$

Hence the ode is

$$y'' - \frac{e^x y'}{x^2+1} + \frac{y}{x^2+1} = 0$$

The domain of $p(x) = -\frac{e^x}{x^2+1}$ is

$$\{-\infty < x < \infty\}$$

And the point $x_0 = 0$ is inside this domain. The domain of $q(x) = \frac{1}{x^2+1}$ is

$$\{-\infty < x < \infty\}$$

And the point $x_0 = 0$ is also inside this domain. Hence solution exists and is unique.

Solving ode using Taylor series method. This gives review on how the Taylor series method works for solving second order ode.

Let

$$y'' = f(x, y, y')$$

Assuming expansion is at $x_0 = 0$ (we can always shift the actual expansion point to 0 by change of variables) and assuming $f(x, y, y')$ is analytic at x_0 which must be the case for an ordinary point. Let initial conditions be $y(x_0) = y_0$ and $y'(x_0) = y_0'$. Using Taylor series gives

$$\begin{aligned}
y(x) &= y(x_0) + (x - x_0)\, y'(x_0) + \frac{(x - x_0)^2}{2} y''(x_0) + \frac{(x - x_0)^3}{3!} y'''(x_0) + \cdots \\
&= y_0 + x y_0' + \frac{x^2}{2} f|_{x_0, y_0, y_0'} + \frac{x^3}{3!} f'|_{x_0, y_0, y_0'} + \cdots \\
&= y_0 + x y_0' + \sum_{n=0}^{\infty} \frac{x^{n+2}}{(n+2)\,!} \frac{d^n f}{dx^n}\bigg|_{x_0, y_0, y_0'}
\end{aligned}$$

But

$$\frac{df}{dx} = \frac{\partial f}{\partial x}\frac{dx}{dx} + \frac{\partial f}{\partial y}\frac{dy}{dx} + \frac{\partial f}{\partial y'}\frac{dy'}{dx} \tag{1}$$

$$= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}y' + \frac{\partial f}{\partial y'}y'' \tag{292}$$

$$= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}y' + \frac{\partial f}{\partial y'}f \tag{293}$$

$$\frac{d^2 f}{dx^2} = \frac{d}{dx}\left(\frac{df}{dx}\right)$$

$$= \frac{\partial}{\partial x}\left(\frac{df}{dx}\right) + \frac{\partial}{\partial y}\left(\frac{df}{dx}\right)y' + \frac{\partial}{\partial y'}\left(\frac{df}{dx}\right)f \tag{2}$$

$$\frac{d^3 f}{dx^3} = \frac{d}{dx}\left(\frac{d^2 f}{dx^2}\right)$$

$$= \frac{\partial}{\partial x}\left(\frac{d^2 f}{dx^2}\right) + \left(\frac{\partial}{\partial y}\frac{d^2 f}{dx^2}\right)y' + \frac{\partial}{\partial y'}\left(\frac{d^2 f}{dx^2}\right)f \tag{3}$$

$$\vdots$$

And so on. Hence if we name $F_0 = f(x, y, y')$ then the above can be written as

$$
\begin{aligned}
F_0 &= f(x, y, y') \qquad (4)\\
F_1 &= \frac{df}{dx}\\
&= \frac{dF_0}{dx}\\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} y' + \frac{\partial f}{\partial y'} y''\\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} y' + \frac{\partial f}{\partial y'} f \qquad (5)\\
&= \frac{\partial F_0}{\partial x} + \frac{\partial F_0}{\partial y} y' + \frac{\partial F_0}{\partial y'} F_0\\
F_2 &= \frac{d}{dx}\left(\frac{d}{dx} f\right)\\
&= \frac{d}{dx}(F_1)\\
&= \frac{\partial}{\partial x} F_1 + \left(\frac{\partial F_1}{\partial y}\right) y' + \left(\frac{\partial F_1}{\partial y'}\right) y''\\
&= \frac{\partial}{\partial x} F_1 + \left(\frac{\partial F_1}{\partial y}\right) y' + \left(\frac{\partial F_1}{\partial y'}\right) F_0\\
&\vdots\\
F_n &= \frac{d}{dx}(F_{n-1})\\
&= \frac{\partial}{\partial x} F_{n-1} + \left(\frac{\partial F_{n-1}}{\partial y}\right) y' + \left(\frac{\partial F_{n-1}}{\partial y'}\right) y''\\
&= \frac{\partial}{\partial x} F_{n-1} + \left(\frac{\partial F_{n-1}}{\partial y}\right) y' + \left(\frac{\partial F_{n-1}}{\partial y'}\right) F_0 \qquad (6)
\end{aligned}
$$

Therefore (6) can be used from now on along with

$$
y(x) = y_0 + x y_0' + \sum_{n=0}^{\infty} \frac{x^{n+2}}{(n+2)!} F_n|_{x_0, y_0, y_0'} \qquad (7)
$$

To find $y(x)$ series solution around $x=0$. Hence

$$
\begin{aligned}
F_0 &= -\frac{-y'\mathrm{e}^x + y}{x^2+1}\\
F_1 &= \frac{dF_0}{dx}\\
&= \frac{\partial F_0}{\partial x} + \frac{\partial F_0}{\partial y}y' + \frac{\partial F_0}{\partial y'}F_0\\
&= \frac{y'\mathrm{e}^{2x} + \left((x-1)^2\mathrm{e}^x - x^2 - 1\right)y' + 2\left(x - \frac{\mathrm{e}^x}{2}\right)y}{\left(x^2+1\right)^2}\\
F_2 &= \frac{dF_1}{dx}\\
&= \frac{\partial F_1}{\partial x} + \frac{\partial F_1}{\partial y}y' + \frac{\partial F_1}{\partial y'}F_1\\
&= \frac{\left(3(x-1)^2y' - y\right)\mathrm{e}^{2x} + y'\mathrm{e}^{3x} + \left(\left(x^4 - 4x^3 + 6x^2 - 4x - 3\right)\mathrm{e}^x + 4x^3 + 4x\right)y' - 2\Big(\left(x^2-3x+1\right)\mathrm{e}^x +}{\left(x^2+1\right)^3}\\
F_3 &= \frac{dF_2}{dx}\\
&= \frac{\partial F_2}{\partial x} + \frac{\partial F_2}{\partial y}y' + \frac{\partial F_2}{\partial y'}F_2\\
&= \frac{\left(\left(7x^4 - 28x^3 + 47x^2 - 28x - 4\right)y' - 5\left(x^2 - \frac{12}{5}x + 1\right)y\right)\mathrm{e}^{2x} + \left(6(x-1)^2y' - y\right)\mathrm{e}^{3x} + y'\mathrm{e}^{4x} + \left(\left(x^6\right.\right.}{}\\
F_4 &= \frac{dF_3}{dx}\\
&= \frac{\partial F_3}{\partial x} + \frac{\partial F_3}{\partial y}y' + \frac{\partial F_3}{\partial y'}F_3\\
&= \frac{\left(\left(15x^6 - 90x^5 + 260x^4 - 372x^3 + 195x^2 + 118x - 50\right)y' - 17y\left(x^4 - \frac{82}{17}x^3 + \frac{151}{17}x^2 - \frac{82}{17}x - \frac{6}{17}\right)\right)\mathrm{e}^{2x}}{}
\end{aligned}
$$

And so on. Evaluating all the above at initial conditions $x=0$ and $y(0)=1$ and $y'(0)=1$ gives

$$
\begin{aligned}
F_0 &= 0\\
F_1 &= 0\\
F_2 &= 1\\
F_3 &= 2\\
F_4 &= -6
\end{aligned}
$$

Substituting all the above in (7) and simplifying gives the solution as

$$y = x + 1 + \frac{x^4}{24} + \frac{x^5}{60} - \frac{x^6}{120} + O(x^6)$$

$$y = x + 1 + \frac{x^4}{24} + \frac{x^5}{60} - \frac{x^6}{120} + O(x^6)$$

Since the expansion point $x = 0$ is an ordinary, we can also solve this using standard power series The ode is normalized to be

$$(x^2 + 1)\, y'' - y' \mathrm{e}^x + y = 0$$

Let the solution be represented as power series of the form

$$y = \sum_{n=0}^{\infty} a_n x^n$$

Then

$$y' = \sum_{n=1}^{\infty} n a_n x^{n-1}$$
$$y'' = \sum_{n=2}^{\infty} n(n-1)\, a_n x^{n-2}$$

Substituting the above back into the ode gives

$$(x^2 + 1) \left(\sum_{n=2}^{\infty} n(n-1)\, a_n x^{n-2}\right) - \left(\sum_{n=1}^{\infty} n a_n x^{n-1}\right) \mathrm{e}^x + \left(\sum_{n=0}^{\infty} a_n x^n\right) = 0 \tag{1}$$

Expanding $-\mathrm{e}^x$ as Taylor series around $x = 0$ and keeping only the first 6 terms gives

$$\begin{aligned}
-\mathrm{e}^x &= -1 - x - \frac{1}{2}x^2 - \frac{1}{6}x^3 - \frac{1}{24}x^4 - \frac{1}{120}x^5 - \frac{1}{720}x^6 + \dots \\
&= -1 - x - \frac{1}{2}x^2 - \frac{1}{6}x^3 - \frac{1}{24}x^4 - \frac{1}{120}x^5 - \frac{1}{720}x^6
\end{aligned}$$

Hence the ODE in Eq (1) becomes

$$\begin{aligned}
&(x^2 + 1) \left(\sum_{n=2}^{\infty} n(n-1)\, a_n x^{n-2}\right) \\
&+ \left(-1 - x - \frac{1}{2}x^2 - \frac{1}{6}x^3 - \frac{1}{24}x^4 - \frac{1}{120}x^5 - \frac{1}{720}x^6\right) \left(\sum_{n=1}^{\infty} n a_n x^{n-1}\right) + \left(\sum_{n=0}^{\infty} a_n x^n\right) = 0
\end{aligned}$$

Expanding the second term in (1) gives

$$(x^2+1)\left(\sum_{n=2}^{\infty} n(n-1)\, a_n x^{n-2}\right) + -1\cdot\left(\sum_{n=1}^{\infty} n a_n x^{n-1}\right) - x\cdot\left(\sum_{n=1}^{\infty} n a_n x^{n-1}\right) - \frac{x^2}{2}\cdot\left(\sum_{n=1}^{\infty} n a_n x^{n-1}\right) - \frac{x^3}{6}\cdot\left(\sum_{n=1}^{\infty} n a_n x^{n-1}\right) - \frac{x^4}{24}\cdot\left(\sum_{n=1}^{\infty} n a_n x^{n-1}\right) - \frac{x^5}{120}\cdot\left(\sum_{n=1}^{\infty} n a_n x^{n-1}\right) - \frac{x^6}{720}\cdot\left(\sum_{n=1}^{\infty} n a_n x^{n-1}\right) + \left(\sum_{n=0}^{\infty} a_n x^n\right) = 0$$

Which simplifies to

$$\left(\sum_{n=2}^{\infty} x^n a_n n(n-1)\right) + \left(\sum_{n=2}^{\infty} n(n-1)\, a_n x^{n-2}\right) + \sum_{n=1}^{\infty}\left(-n a_n x^{n-1}\right) + \sum_{n=1}^{\infty}\left(-n a_n x^n\right) + \sum_{n=1}^{\infty}\left(-\frac{n\, x^{1+n} a_n}{2}\right) + \sum_{n=1}^{\infty}\left(-\frac{n\, x^{n+2} a_n}{6}\right) + \sum_{n=1}^{\infty}\left(-\frac{n\, x^{n+3} a_n}{24}\right) + \sum_{n=1}^{\infty}\left(-\frac{n\, x^{n+4} a_n}{120}\right) + \sum_{n=1}^{\infty}\left(-\frac{n\, x^{n+5} a_n}{720}\right) + \left(\sum_{n=0}^{\infty} a_n x^n\right) = 0 \tag{2}$$

The next step is to make all powers of x be n in each summation term. Going over each summation term above with power of x in it which is not already x^n and adjusting the power and the corresponding index gives

$$\sum_{n=2}^{\infty} n(n-1)\, a_n x^{n-2} = \sum_{n=0}^{\infty} (n+2)\, a_{n+2}(1+n)\, x^n$$

$$\sum_{n=1}^{\infty}\left(-n a_n x^{n-1}\right) = \sum_{n=0}^{\infty}\left(-(1+n)\, a_{1+n} x^n\right)$$

$$\sum_{n=1}^{\infty}\left(-\frac{n\, x^{1+n} a_n}{2}\right) = \sum_{n=2}^{\infty}\left(-\frac{(n-1)\, a_{n-1} x^n}{2}\right)$$

$$\sum_{n=1}^{\infty}\left(-\frac{n\, x^{n+2} a_n}{6}\right) = \sum_{n=3}^{\infty}\left(-\frac{(n-2)\, a_{n-2} x^n}{6}\right)$$

$$\sum_{n=1}^{\infty}\left(-\frac{n\,x^{n+3} a_{n}}{24}\right) = \sum_{n=4}^{\infty}\left(-\frac{(n-3)\,a_{n-3} x^{n}}{24}\right)$$

$$\sum_{n=1}^{\infty}\left(-\frac{n\,x^{n+4} a_{n}}{120}\right) = \sum_{n=5}^{\infty}\left(-\frac{(n-4)\,a_{n-4} x^{n}}{120}\right)$$

$$\sum_{n=1}^{\infty}\left(-\frac{n\,x^{n+5} a_{n}}{720}\right) = \sum_{n=6}^{\infty}\left(-\frac{(n-5)\,a_{n-5} x^{n}}{720}\right)$$

Substituting all the above in Eq (2) gives the following equation where now all powers of x are the same and equal to n.

$$\begin{aligned}
&\left(\sum_{n=2}^{\infty} x^{n} a_{n} n(n-1)\right) + \left(\sum_{n=0}^{\infty} (n+2)\,a_{n+2}(1+n)\,x^{n}\right) \\
&+ \sum_{n=0}^{\infty} \left(-(1+n)\,a_{1+n} x^{n}\right) + \sum_{n=1}^{\infty} \left(-n a_{n} x^{n}\right) + \sum_{n=2}^{\infty}\left(-\frac{(n-1)\,a_{n-1} x^{n}}{2}\right) \\
&+ \sum_{n=3}^{\infty}\left(-\frac{(n-2)\,a_{n-2} x^{n}}{6}\right) + \sum_{n=4}^{\infty}\left(-\frac{(n-3)\,a_{n-3} x^{n}}{24}\right) \\
&+ \sum_{n=5}^{\infty}\left(-\frac{(n-4)\,a_{n-4} x^{n}}{120}\right) + \sum_{n=6}^{\infty}\left(-\frac{(n-5)\,a_{n-5} x^{n}}{720}\right) + \left(\sum_{n=0}^{\infty} a_{n} x^{n}\right) = 0
\end{aligned} \tag{3}$$

$n = 0$ gives

$$2a_{2} - a_{1} + a_{0} = 0$$

$$a_{2} = -\frac{a_{0}}{2} + \frac{a_{1}}{2}$$

$n = 1$ gives

$$6a_{3} - 2a_{2} = 0$$

Which after substituting earlier equations, simplifies to

$$6a_{3} + a_{0} - a_{1} = 0$$

Or

$$a_{3} = -\frac{a_{0}}{6} + \frac{a_{1}}{6}$$

$n = 2$ gives

$$12a_4 - 3a_3 - \frac{a_1}{2} + a_2 = 0$$

Which after substituting earlier equations, simplifies to

$$a_4 = \frac{a_1}{24}$$

$n = 3$ gives

$$4a_3 + 20a_5 - 4a_4 - a_2 - \frac{a_1}{6} = 0$$

Which after substituting earlier equations, simplifies to

$$a_5 = \frac{a_0}{120} + \frac{a_1}{120}$$

$n = 4$ gives

$$9a_4 + 30a_6 - 5a_5 - \frac{3a_3}{2} - \frac{a_2}{3} - \frac{a_1}{24} = 0$$

Which after substituting earlier equations, simplifies to

$$a_6 = -\frac{a_0}{80} + \frac{a_1}{240}$$

$n = 5$ gives

$$16a_5 + 42a_7 - 6a_6 - 2a_4 - \frac{a_3}{2} - \frac{a_2}{12} - \frac{a_1}{120} = 0$$

Which after substituting earlier equations, simplifies to

$$a_7 = -\frac{a_0}{126} + \frac{13a_1}{5040}$$

For $6 \leq n$, the recurrence equation is

$$\begin{aligned} &na_n(n-1) + (n+2)\, a_{n+2}(1+n) - (1+n)\, a_{1+n} - na_n - \frac{(n-1)\, a_{n-1}}{2} \\ &- \frac{(n-2)\, a_{n-2}}{6} - \frac{(n-3)\, a_{n-3}}{24} - \frac{(n-4)\, a_{n-4}}{120} - \frac{(n-5)\, a_{n-5}}{720} + a_n = 0 \end{aligned} \tag{4}$$

Solving for a_{n+2}, gives

$$a_{n+2} = -\frac{720n^2 a_n - 1440na_n - 720na_{1+n} - na_{n-5} - 6na_{n-4} - 30na_{n-3} - 120na_{n-2} - 360na_{n-1} + 720a_n - 72}{720\,(n+2)\,(1+n)}$$

$$\begin{aligned}
&= -\frac{(720n^2 - 1440n + 720)\,a_n}{720\,(n+2)\,(1+n)} - \frac{(-720n - 720)\,a_{1+n}}{720\,(n+2)\,(1+n)} \\
(5)\quad &- \frac{(-n+5)\,a_{n-5}}{720\,(n+2)\,(1+n)} - \frac{(-6n+24)\,a_{n-4}}{720\,(n+2)\,(1+n)} - \frac{(-30n+90)\,a_{n-3}}{720\,(n+2)\,(1+n)} \\
&- \frac{(-120n+240)\,a_{n-2}}{720\,(n+2)\,(1+n)} - \frac{(-360n+360)\,a_{n-1}}{720\,(n+2)\,(1+n)}
\end{aligned}$$

And so on. Therefore the solution is

$$\begin{aligned}
y &= \sum_{n=0}^{\infty} a_n x^n \\
&= a_3x^3 + a_2x^2 + a_1x + a_0 + \dots
\end{aligned}$$

Substituting the values for a_n found above, the solution becomes

$$y = a_0 + a_1x + \left(-\frac{a_0}{2} + \frac{a_1}{2}\right)x^2 + \left(-\frac{a_0}{6} + \frac{a_1}{6}\right)x^3 + \frac{a_1x^4}{24} + \left(\frac{a_0}{120} + \frac{a_1}{120}\right)x^5 + \dots$$

Collecting terms, the solution becomes

$$y = \left(1 - \frac{1}{2}x^2 - \frac{1}{6}x^3 + \frac{1}{120}x^5\right)a_0 + \left(x + \frac{1}{2}x^2 + \frac{1}{6}x^3 + \frac{1}{24}x^4 + \frac{1}{120}x^5\right)a_1 + O(x^6) \quad (3)$$

At $x = 0$ the solution above becomes

$$y = \left(1 - \frac{1}{2}x^2 - \frac{1}{6}x^3 + \frac{1}{120}x^5\right)c_1 + \left(x + \frac{1}{2}x^2 + \frac{1}{6}x^3 + \frac{1}{24}x^4 + \frac{1}{120}x^5\right)c_2 + O(x^6)$$

$$y = 1 + \frac{x^5}{60} + x + \frac{x^4}{24} + O(x^6)$$

Summary

The solution(s) found are the following

$$y = x + 1 + \frac{x^4}{24} + \frac{x^5}{60} - \frac{x^6}{120} + O(x^6) \quad (1)$$

$$y = 1 + \frac{x^5}{60} + x + \frac{x^4}{24} + O(x^6) \quad (2)$$

Verification of solutions

$$y = x + 1 + \frac{x^4}{24} + \frac{x^5}{60} - \frac{x^6}{120} + O(x^6)$$

Verified OK.

$$y = 1 + \frac{x^5}{60} + x + \frac{x^4}{24} + O(x^6)$$

Verified OK.

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Heun: Equivalence to the GHE or one of its 4 confluent cases under a power @ Moebius
-> trying a solution of the form r0(x) * Y + r1(x) * Y where Y = exp(int(r(x), dx)) * 2F1([a
-> Trying changes of variables to rationalize or make the ODE simpler
   trying a symmetry of the form [xi=0, eta=F(x)]
   checking if the LODE is missing y
   -> Heun: Equivalence to the GHE or one of its 4 confluent cases under a power @ Moebius
   -> trying a solution of the form r0(x) * Y + r1(x) * Y where Y = exp(int(r(x), dx)) * 2F1
      trying a symmetry of the form [xi=0, eta=F(x)]
      trying 2nd order exact linear
      trying symmetries linear in x and y(x)
      trying to convert to a linear ODE with constant coefficients
<- unable to find a useful change of variables
   trying a symmetry of the form [xi=0, eta=F(x)]
   trying differential order: 2; exact nonlinear
   trying symmetries linear in x and y(x)
   trying to convert to a linear ODE with constant coefficients
   trying 2nd order, integrating factor of the form mu(x,y)
   trying a symmetry of the form [xi=0, eta=F(x)]
   checking if the LODE is missing y
   -> Heun: Equivalence to the GHE or one of its 4 confluent cases under a power @ Moebius
   -> trying a solution of the form r0(x) * Y + r1(x) * Y where Y = exp(int(r(x), dx)) * 2F1
   -> Trying changes of variables to rationalize or make the ODE simpler
      trying a symmetry of the form [xi=0, eta=F(x)]
      checking if the LODE is missing y
      -> Heun: Equivalence to the GHE or one of its 4 confluent cases under a power @ Moebiu
      -> trying a solution of the form r0(x) * Y + r1(x) * Y where Y = exp(int(r(x), dx)) *
         trying a symmetry of the form [xi=0, eta=F(x)]
         trying 2nd order exact linear
         trying symmetries linear in x and y(x)
         trying to convert to a linear ODE with constant coefficients
   <- unable to find a useful change of variables
      trying a symmetry of the form [xi=0, eta=F(x)]
   trying to convert to an ODE of Bessel type
   -> trying reduction of order to Riccati
      trying Riccati sub-methods:
         trying Riccati_symmetries
         -> trying a symmetry pattern of the form [F(x)*G(y), 0]
         -> trying a symmetry pattern of the form [0, F(x)*G(y)]
         -> trying a symmetry pattern of the form [F(x),G(x)*y+H(x)]
--- Trying Lie symmetry methods, 2nd order ---
`, `-> Computing symmetries using: way = 3`[0, y]
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 16

```
Order:=6;
dsolve([(x^2+1)*diff(y(x),x$2)-exp(x)*diff(y(x),x)+y(x)=0,y(0) = 1, D(y)(0) = 1],y(x),type='
```

$$y(x) = 1 + x + \frac{1}{24}x^4 + \frac{1}{60}x^5 + \mathrm{O}\left(x^6\right)$$

✓ Solution by Mathematica

Time used: 0.001 (sec). Leaf size: 20

```
AsymptoticDSolveValue[{(x^2+1)*y''[x]-Exp[x]*y'[x]+y[x]==0,{y[0]==1,y'[0]==1}},y[x],{x,0,5}]
```

$$y(x) \to \frac{x^5}{60} + \frac{x^4}{24} + x + 1$$

6.16 problem 16

Internal problem ID [5034]
Internal file name [`OUTPUT/4527_Sunday_June_05_2022_03_00_14_PM_69245814/index.tex`]

Book: Fundamentals of Differential Equations. By Nagle, Saff and Snider. 9th edition. Boston. Pearson 2018.
Section: Chapter 8, Series solutions of differential equations. Section 8.4. page 449
Problem number: 16.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second order series method. Ordinary point", "second order series method. Taylor series method"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

$$\boxed{y'' + ty' + e^{t}y = 0}$$

With initial conditions

$$[y(0) = 1, y'(0) = -1]$$

With the expansion point for the power series method at $t = 0$.

6.16.1 Existence and uniqueness analysis

This is a linear ODE. In canonical form it is written as

$$y'' + p(t)y' + q(t)y = F$$

Where here

$$\begin{aligned} p(t) &= t \\ q(t) &= e^{t} \\ F &= 0 \end{aligned}$$

Hence the ode is

$$y'' + ty' + e^{t}y = 0$$

The domain of $p(t) = t$ is

$$\{-\infty < t < \infty\}$$

And the point $t_0 = 0$ is inside this domain. The domain of $q(t) = e^{t}$ is

$$\{-\infty < t < \infty\}$$

And the point $t_0 = 0$ is also inside this domain. Hence solution exists and is unique.

Solving ode using Taylor series method. This gives review on how the Taylor series method works for solving second order ode.

Let

$$y'' = f(x, y, y')$$

Assuming expansion is at $x_0 = 0$ (we can always shift the actual expansion point to 0 by change of variables) and assuming $f(x, y, y')$ is analytic at x_0 which must be the case for an ordinary point. Let initial conditions be $y(x_0) = y_0$ and $y'(x_0) = y'_0$. Using Taylor series gives

$$\begin{aligned}
y(x) &= y(x_0) + (x - x_0)\, y'(x_0) + \frac{(x - x_0)^2}{2} y''(x_0) + \frac{(x - x_0)^3}{3!} y'''(x_0) + \cdots \\
&= y_0 + xy'_0 + \frac{x^2}{2} f|_{x_0, y_0, y'_0} + \frac{x^3}{3!} f'|_{x_0, y_0, y'_0} + \cdots \\
&= y_0 + xy'_0 + \sum_{n=0}^{\infty} \frac{x^{n+2}}{(n+2)\,!} \frac{d^n f}{dx^n}\bigg|_{x_0, y_0, y'_0}
\end{aligned}$$

But

$$\frac{df}{dx} = \frac{\partial f}{\partial x}\frac{dx}{dx} + \frac{\partial f}{\partial y}\frac{dy}{dx} + \frac{\partial f}{\partial y'}\frac{dy'}{dx} \tag{1}$$

$$= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}y' + \frac{\partial f}{\partial y'}y'' \tag{295}$$

$$= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}y' + \frac{\partial f}{\partial y'}f \tag{296}$$

$$\frac{d^2 f}{dx^2} = \frac{d}{dx}\left(\frac{df}{dx}\right)$$

$$= \frac{\partial}{\partial x}\left(\frac{df}{dx}\right) + \frac{\partial}{\partial y}\left(\frac{df}{dx}\right)y' + \frac{\partial}{\partial y'}\left(\frac{df}{dx}\right)f \tag{2}$$

$$\frac{d^3 f}{dx^3} = \frac{d}{dx}\left(\frac{d^2 f}{dx^2}\right)$$

$$= \frac{\partial}{\partial x}\left(\frac{d^2 f}{dx^2}\right) + \left(\frac{\partial}{\partial y}\frac{d^2 f}{dx^2}\right)y' + \frac{\partial}{\partial y'}\left(\frac{d^2 f}{dx^2}\right)f \tag{3}$$

$$\vdots$$

And so on. Hence if we name $F_0 = f(x,y,y')$ then the above can be written as

$$
\begin{aligned}
F_0 &= f(x,y,y') \qquad (4)\\
F_1 &= \frac{df}{dx}\\
&= \frac{dF_0}{dx}\\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}y' + \frac{\partial f}{\partial y'}y''\\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}y' + \frac{\partial f}{\partial y'}f \qquad (5)\\
&= \frac{\partial F_0}{\partial x} + \frac{\partial F_0}{\partial y}y' + \frac{\partial F_0}{\partial y'}F_0\\
F_2 &= \frac{d}{dx}\left(\frac{d}{dx}f\right)\\
&= \frac{d}{dx}(F_1)\\
&= \frac{\partial}{\partial x}F_1 + \left(\frac{\partial F_1}{\partial y}\right)y' + \left(\frac{\partial F_1}{\partial y'}\right)y''\\
&= \frac{\partial}{\partial x}F_1 + \left(\frac{\partial F_1}{\partial y}\right)y' + \left(\frac{\partial F_1}{\partial y'}\right)F_0\\
&\vdots\\
F_n &= \frac{d}{dx}(F_{n-1})\\
&= \frac{\partial}{\partial x}F_{n-1} + \left(\frac{\partial F_{n-1}}{\partial y}\right)y' + \left(\frac{\partial F_{n-1}}{\partial y'}\right)y''\\
&= \frac{\partial}{\partial x}F_{n-1} + \left(\frac{\partial F_{n-1}}{\partial y}\right)y' + \left(\frac{\partial F_{n-1}}{\partial y'}\right)F_0 \qquad (6)
\end{aligned}
$$

Therefore (6) can be used from now on along with

$$
y(x) = y_0 + xy_0' + \sum_{n=0}^{\infty}\frac{x^{n+2}}{(n+2)!}F_n|_{x_0,y_0,y_0'} \tag{7}
$$

To find $y(x)$ series solution around $x = 0$. Hence

$$
\begin{aligned}
F_0 &= -ty' - \mathrm{e}^t y \\
F_1 &= \frac{dF_0}{dt} \\
&= \frac{\partial F_0}{\partial t} + \frac{\partial F_0}{\partial y} y' + \frac{\partial F_0}{\partial y'} F_0 \\
&= \left(t^2 - \mathrm{e}^t - 1\right) y' + \mathrm{e}^t y(t-1) \\
F_2 &= \frac{dF_1}{dt} \\
&= \frac{\partial F_1}{\partial t} + \frac{\partial F_1}{\partial y} y' + \frac{\partial F_1}{\partial y'} F_1 \\
&= \mathrm{e}^{2t} y + \left((2t-2)\,\mathrm{e}^t - t^3 + 3t\right) y' - \mathrm{e}^t y\left(t^2 - t - 1\right) \\
F_3 &= \frac{dF_2}{dt} \\
&= \frac{\partial F_2}{\partial t} + \frac{\partial F_2}{\partial y} y' + \frac{\partial F_2}{\partial y'} F_2 \\
&= \left(y' + (-2t+4)\, y\right) \mathrm{e}^{2t} + \left(\left(-3t^2 + 5t + 1\right) \mathrm{e}^t + t^4 - 6t^2 + 3\right) y' + \mathrm{e}^t y\left(t^3 - t^2 - 4t + 2\right) \\
F_4 &= \frac{dF_3}{dt} \\
&= \frac{\partial F_3}{\partial t} + \frac{\partial F_3}{\partial y} y' + \frac{\partial F_3}{\partial y'} F_3 \\
&= \left((-3t+6)\, y' + 3y\left(t^2 - 3t + \frac{5}{3}\right)\right) \mathrm{e}^{2t} - y\,\mathrm{e}^{3t} + \left(\left(4t^3 - 9t^2 - 6t + 8\right) \mathrm{e}^t - t^5 + 10t^3 - 15t\right) y' - \mathrm{e}^t y(
\end{aligned}
$$

And so on. Evaluating all the above at initial conditions $t = 0$ and $y(0) = 1$ and $y'(0) = -1$ gives

$$
\begin{aligned}
F_0 &= -1 \\
F_1 &= 1 \\
F_2 &= 4 \\
F_3 &= 1 \\
F_4 &= -15
\end{aligned}
$$

Substituting all the above in (7) and simplifying gives the solution as

$$
y = -t + 1 - \frac{t^2}{2} + \frac{t^3}{6} + \frac{t^4}{6} + \frac{t^5}{120} - \frac{t^6}{48} + O\left(t^6\right)
$$

$$
y = -t + 1 - \frac{t^2}{2} + \frac{t^3}{6} + \frac{t^4}{6} + \frac{t^5}{120} - \frac{t^6}{48} + O\left(t^6\right)
$$

Since the expansion point $t = 0$ is an ordinary, we can also solve this using standard power series Let the solution be represented as power series of the form

$$y = \sum_{n=0}^{\infty} a_n t^n$$

Then

$$y' = \sum_{n=1}^{\infty} n a_n t^{n-1}$$
$$y'' = \sum_{n=2}^{\infty} n(n-1)\, a_n t^{n-2}$$

Substituting the above back into the ode gives

$$\sum_{n=2}^{\infty} n(n-1)\, a_n t^{n-2} = -t\left(\sum_{n=1}^{\infty} n a_n t^{n-1}\right) - \mathrm{e}^t\left(\sum_{n=0}^{\infty} a_n t^n\right) \tag{1}$$

Expanding e^t as Taylor series around $t = 0$ and keeping only the first 6 terms gives

$$\begin{aligned}\mathrm{e}^t &= 1 + t + \frac{1}{2}t^2 + \frac{1}{6}t^3 + \frac{1}{24}t^4 + \frac{1}{120}t^5 + \frac{1}{720}t^6 + \dots \\ &= 1 + t + \frac{1}{2}t^2 + \frac{1}{6}t^3 + \frac{1}{24}t^4 + \frac{1}{120}t^5 + \frac{1}{720}t^6\end{aligned}$$

Hence the ODE in Eq (1) becomes

$$\begin{aligned}&\left(\sum_{n=2}^{\infty} n(n-1)\, a_n t^{n-2}\right) + t\left(\sum_{n=1}^{\infty} n a_n t^{n-1}\right) \\ &+ \left(1 + t + \frac{1}{2}t^2 + \frac{1}{6}t^3 + \frac{1}{24}t^4 + \frac{1}{120}t^5 + \frac{1}{720}t^6\right)\left(\sum_{n=0}^{\infty} a_n t^n\right) = 0\end{aligned}$$

Expanding the third term in (1) gives

$$\begin{aligned}&\left(\sum_{n=2}^{\infty} n(n-1)\, a_n t^{n-2}\right) + t\left(\sum_{n=1}^{\infty} n a_n t^{n-1}\right) + 1 \cdot \left(\sum_{n=0}^{\infty} a_n t^n\right) \\ &+ t \cdot \left(\sum_{n=0}^{\infty} a_n t^n\right) + \frac{t^2}{2} \cdot \left(\sum_{n=0}^{\infty} a_n t^n\right) + \frac{t^3}{6} \cdot \left(\sum_{n=0}^{\infty} a_n t^n\right) + \frac{t^4}{24} \\ &\cdot \left(\sum_{n=0}^{\infty} a_n t^n\right) + \frac{t^5}{120} \cdot \left(\sum_{n=0}^{\infty} a_n t^n\right) + \frac{t^6}{720} \cdot \left(\sum_{n=0}^{\infty} a_n t^n\right) = 0\end{aligned}$$

Which simplifies to

$$
\begin{aligned}
&\left(\sum_{n=2}^{\infty} n(n-1)\, a_n t^{n-2}\right)+\left(\sum_{n=1}^{\infty} n\, t^n a_n\right)+\left(\sum_{n=0}^{\infty} a_n t^n\right)\\
&+\left(\sum_{n=0}^{\infty} t^{1+n} a_n\right)+\left(\sum_{n=0}^{\infty} \frac{t^{n+2} a_n}{2}\right)+\left(\sum_{n=0}^{\infty} \frac{t^{n+3} a_n}{6}\right)\\
&+\left(\sum_{n=0}^{\infty} \frac{t^{n+4} a_n}{24}\right)+\left(\sum_{n=0}^{\infty} \frac{t^{n+5} a_n}{120}\right)+\left(\sum_{n=0}^{\infty} \frac{t^{n+6} a_n}{720}\right)=0
\end{aligned} \tag{2}
$$

The next step is to make all powers of t be n in each summation term. Going over each summation term above with power of t in it which is not already t^n and adjusting the power and the corresponding index gives

$$
\begin{aligned}
\sum_{n=2}^{\infty} n(n-1)\, a_n t^{n-2} &= \sum_{n=0}^{\infty} (n+2)\, a_{n+2}(1+n)\, t^n \\
\sum_{n=0}^{\infty} t^{1+n} a_n &= \sum_{n=1}^{\infty} a_{n-1} t^n \\
\sum_{n=0}^{\infty} \frac{t^{n+2} a_n}{2} &= \sum_{n=2}^{\infty} \frac{a_{n-2} t^n}{2} \\
\sum_{n=0}^{\infty} \frac{t^{n+3} a_n}{6} &= \sum_{n=3}^{\infty} \frac{a_{n-3} t^n}{6} \\
\sum_{n=0}^{\infty} \frac{t^{n+4} a_n}{24} &= \sum_{n=4}^{\infty} \frac{a_{n-4} t^n}{24} \\
\sum_{n=0}^{\infty} \frac{t^{n+5} a_n}{120} &= \sum_{n=5}^{\infty} \frac{a_{n-5} t^n}{120} \\
\sum_{n=0}^{\infty} \frac{t^{n+6} a_n}{720} &= \sum_{n=6}^{\infty} \frac{a_{n-6} t^n}{720}
\end{aligned}
$$

Substituting all the above in Eq (2) gives the following equation where now all powers of t are the same and equal to n.

$$\left(\sum_{n=0}^{\infty}(n+2)\,a_{n+2}(1+n)\,t^{n}\right)+\left(\sum_{n=1}^{\infty}n\,t^{n}a_{n}\right)+\left(\sum_{n=0}^{\infty}a_{n}t^{n}\right)$$
$$+\left(\sum_{n=1}^{\infty}a_{n-1}t^{n}\right)+\left(\sum_{n=2}^{\infty}\frac{a_{n-2}t^{n}}{2}\right)+\left(\sum_{n=3}^{\infty}\frac{a_{n-3}t^{n}}{6}\right) \tag{3}$$
$$+\left(\sum_{n=4}^{\infty}\frac{a_{n-4}t^{n}}{24}\right)+\left(\sum_{n=5}^{\infty}\frac{a_{n-5}t^{n}}{120}\right)+\left(\sum_{n=6}^{\infty}\frac{a_{n-6}t^{n}}{720}\right)=0$$

$n = 0$ gives

$$2a_2 + a_0 = 0$$

$$a_2 = -\frac{a_0}{2}$$

$n = 1$ gives

$$6a_3 + 2a_1 + a_0 = 0$$

Which after substituting earlier equations, simplifies to

$$a_3 = -\frac{a_0}{6} - \frac{a_1}{3}$$

$n = 2$ gives

$$12a_4 + 3a_2 + a_1 + \frac{a_0}{2} = 0$$

Which after substituting earlier equations, simplifies to

$$a_4 = \frac{a_0}{12} - \frac{a_1}{12}$$

$n = 3$ gives

$$20a_5 + 4a_3 + a_2 + \frac{a_1}{2} + \frac{a_0}{6} = 0$$

Which after substituting earlier equations, simplifies to

$$a_5 = \frac{a_0}{20} + \frac{a_1}{24}$$

$n = 4$ gives

$$30a_6 + 5a_4 + a_3 + \frac{a_2}{2} + \frac{a_1}{6} + \frac{a_0}{24} = 0$$

Which after substituting earlier equations, simplifies to

$$a_6 = -\frac{a_0}{720} + \frac{7a_1}{360}$$

$n = 5$ gives

$$42a_7 + 6a_5 + a_4 + \frac{a_3}{2} + \frac{a_2}{6} + \frac{a_1}{24} + \frac{a_0}{120} = 0$$

Which after substituting earlier equations, simplifies to

$$a_7 = -\frac{3a_0}{560} - \frac{a_1}{1008}$$

For $6 \leq n$, the recurrence equation is

$$(n+2)\, a_{n+2}(1+n) + na_n + a_n + a_{n-1} + \frac{a_{n-2}}{2} + \frac{a_{n-3}}{6} + \frac{a_{n-4}}{24} + \frac{a_{n-5}}{120} + \frac{a_{n-6}}{720} = 0 \quad (4)$$

Solving for a_{n+2}, gives

$$a_{n+2} = -\frac{720na_n + 720a_n + a_{n-6} + 6a_{n-5} + 30a_{n-4} + 120a_{n-3} + 360a_{n-2} + 720a_{n-1}}{720\,(n+2)\,(1+n)}$$

$$(5) \quad = -\frac{(720n + 720)\, a_n}{720\,(n+2)\,(1+n)} - \frac{a_{n-6}}{720\,(n+2)\,(1+n)} - \frac{a_{n-5}}{120\,(n+2)\,(1+n)} - \frac{a_{n-4}}{24\,(n+2)\,(1+n)} - \frac{a_{n-3}}{6\,(n+2)\,(1+n)} - \frac{a_{n-2}}{2\,(n+2)\,(1+n)} - \frac{a_{n-1}}{(n+2)\,(1+n)}$$

And so on. Therefore the solution is

$$\begin{aligned} y &= \sum_{n=0}^{\infty} a_n t^n \\ &= a_3 t^3 + a_2 t^2 + a_1 t + a_0 + \dots \end{aligned}$$

Substituting the values for a_n found above, the solution becomes

$$y = a_0 + a_1 t - \frac{a_0 t^2}{2} + \left(-\frac{a_0}{6} - \frac{a_1}{3}\right) t^3 + \left(\frac{a_0}{12} - \frac{a_1}{12}\right) t^4 + \left(\frac{a_0}{20} + \frac{a_1}{24}\right) t^5 + \dots$$

Collecting terms, the solution becomes

$$y = \left(1 - \frac{1}{2}t^2 - \frac{1}{6}t^3 + \frac{1}{12}t^4 + \frac{1}{20}t^5\right) a_0 + \left(t - \frac{1}{3}t^3 - \frac{1}{12}t^4 + \frac{1}{24}t^5\right) a_1 + O(t^6) \quad (3)$$

At $t = 0$ the solution above becomes

$$y = \left(1 - \frac{1}{2}t^2 - \frac{1}{6}t^3 + \frac{1}{12}t^4 + \frac{1}{20}t^5\right) c_1 + \left(t - \frac{1}{3}t^3 - \frac{1}{12}t^4 + \frac{1}{24}t^5\right) c_2 + O(t^6)$$

$$y = 1 - \frac{t^2}{2} + \frac{t^3}{6} + \frac{t^4}{6} + \frac{t^5}{120} - t + O(t^6)$$

Summary

The solution(s) found are the following

$$y = -t + 1 - \frac{t^2}{2} + \frac{t^3}{6} + \frac{t^4}{6} + \frac{t^5}{120} - \frac{t^6}{48} + O(t^6) \quad (1)$$

$$y = 1 - \frac{t^2}{2} + \frac{t^3}{6} + \frac{t^4}{6} + \frac{t^5}{120} - t + O(t^6) \quad (2)$$

Verification of solutions

$$y = -t + 1 - \frac{t^2}{2} + \frac{t^3}{6} + \frac{t^4}{6} + \frac{t^5}{120} - \frac{t^6}{48} + O(t^6)$$

Verified OK.

$$y = 1 - \frac{t^2}{2} + \frac{t^3}{6} + \frac{t^4}{6} + \frac{t^5}{120} - t + O(t^6)$$

Verified OK.

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Heun: Equivalence to the GHE or one of its 4 confluent cases under a power @ Moebius
-> trying a solution of the form r0(x) * Y + r1(x) * Y where Y = exp(int(r(x), dx)) * 2F1([a
-> Trying changes of variables to rationalize or make the ODE simpler
   trying a symmetry of the form [xi=0, eta=F(x)]
   checking if the LODE is missing y
   -> Heun: Equivalence to the GHE or one of its 4 confluent cases under a power @ Moebius
   -> trying a solution of the form r0(x) * Y + r1(x) * Y where Y = exp(int(r(x), dx)) * 2F1
      trying a symmetry of the form [xi=0, eta=F(x)]
      trying 2nd order exact linear
      trying symmetries linear in x and y(x)
      trying to convert to a linear ODE with constant coefficients
<- unable to find a useful change of variables
   trying a symmetry of the form [xi=0, eta=F(x)]
   trying 2nd order exact linear
   trying symmetries linear in x and y(x)
   trying to convert to a linear ODE with constant coefficients
   trying 2nd order, integrating factor of the form mu(x,y)
   trying a symmetry of the form [xi=0, eta=F(x)]
   checking if the LODE is missing y
   -> Heun: Equivalence to the GHE or one of its 4 confluent cases under a power @ Moebius
   -> trying a solution of the form r0(x) * Y + r1(x) * Y where Y = exp(int(r(x), dx)) * 2F1
   -> Trying changes of variables to rationalize or make the ODE simpler
      trying a symmetry of the form [xi=0, eta=F(x)]
      checking if the LODE is missing y
      -> Heun: Equivalence to the GHE or one of its 4 confluent cases under a power @ Moebiu
      -> trying a solution of the form r0(x) * Y + r1(x) * Y where Y = exp(int(r(x), dx)) *
         trying a symmetry of the form [xi=0, eta=F(x)]
         trying 2nd order exact linear
         trying symmetries linear in x and y(x)
         trying to convert to a linear ODE with constant coefficients
   <- unable to find a useful change of variables
      trying a symmetry of the form [xi=0, eta=F(x)]
   trying to convert to an ODE of Bessel type
   -> trying reduction of order to Riccati
      trying Riccati sub-methods:
         trying Riccati_symmetries
         -> trying a symmetry pattern of the form [F(x)*G(y), 0]
         -> trying a symmetry pattern of the form [0, F(x)*G(y)]
         -> trying a symmetry pattern of the form [F(x),G(x)*y+H(x)]
--- Trying Lie symmetry methods, 2nd order ---
`, `-> Computing symmetries using: way = 3`[0, y]
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 20

```
Order:=6;
dsolve([diff(y(t),t$2)+t*diff(y(t),t)+exp(t)*y(t)=0,y(0) = 1, D(y)(0) = -1],y(t),type='serie
```

$$y(t) = 1 - t - \frac{1}{2}t^2 + \frac{1}{6}t^3 + \frac{1}{6}t^4 + \frac{1}{120}t^5 + \mathrm{O}\left(t^6\right)$$

✓ Solution by Mathematica

Time used: 0.001 (sec). Leaf size: 36

```
AsymptoticDSolveValue[{y''[t]+t*y'[t]+Exp[t]*y[t]==0,{y[0]==1,y'[0]==-1}},y[t],{t,0,5}]
```

$$y(t) \to \frac{t^5}{120} + \frac{t^4}{6} + \frac{t^3}{6} - \frac{t^2}{2} - t + 1$$

6.17 problem 19

Internal problem ID [5035]
Internal file name [`OUTPUT/4528_Sunday_June_05_2022_03_00_16_PM_12748421/index.tex`]

Book: Fundamentals of Differential Equations. By Nagle, Saff and Snider. 9th edition. Boston. Pearson 2018.
Section: Chapter 8, Series solutions of differential equations. Section 8.4. page 449
Problem number: 19.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second order series method. Ordinary point", "second order series method. Taylor series method"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

$$\boxed{y'' - y'\mathrm{e}^{2x} + y\cos(x) = 0}$$

With initial conditions

$$[y(0) = -1, y'(0) = 1]$$

With the expansion point for the power series method at $x = 0$.

6.17.1 Existence and uniqueness analysis

This is a linear ODE. In canonical form it is written as

$$y'' + p(x)y' + q(x)y = F$$

Where here

$$\begin{aligned} p(x) &= -\mathrm{e}^{2x} \\ q(x) &= \cos(x) \\ F &= 0 \end{aligned}$$

Hence the ode is

$$y'' - y'\mathrm{e}^{2x} + y\cos(x) = 0$$

The domain of $p(x) = -\mathrm{e}^{2x}$ is

$$\{-\infty < x < \infty\}$$

And the point $x_0 = 0$ is inside this domain. The domain of $q(x) = \cos(x)$ is

$$\{-\infty < x < \infty\}$$

And the point $x_0 = 0$ is also inside this domain. Hence solution exists and is unique.

Solving ode using Taylor series method. This gives review on how the Taylor series method works for solving second order ode.

Let

$$y'' = f(x, y, y')$$

Assuming expansion is at $x_0 = 0$ (we can always shift the actual expansion point to 0 by change of variables) and assuming $f(x, y, y')$ is analytic at x_0 which must be the case for an ordinary point. Let initial conditions be $y(x_0) = y_0$ and $y'(x_0) = y'_0$. Using Taylor series gives

$$\begin{aligned} y(x) &= y(x_0) + (x - x_0)\,y'(x_0) + \frac{(x - x_0)^2}{2}y''(x_0) + \frac{(x - x_0)^3}{3!}y'''(x_0) + \cdots \\ &= y_0 + xy'_0 + \frac{x^2}{2}\,f|_{x_0,y_0,y'_0} + \frac{x^3}{3!}\,f'|_{x_0,y_0,y'_0} + \cdots \\ &= y_0 + xy'_0 + \sum_{n=0}^{\infty} \frac{x^{n+2}}{(n+2)\,!}\,\frac{d^n f}{dx^n}\bigg|_{x_0,y_0,y'_0} \end{aligned}$$

But

$$\begin{aligned}
\frac{df}{dx} &= \frac{\partial f}{\partial x}\frac{dx}{dx} + \frac{\partial f}{\partial y}\frac{dy}{dx} + \frac{\partial f}{\partial y'}\frac{dy'}{dx} && (1) \\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}y' + \frac{\partial f}{\partial y'}y'' && (298) \\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}y' + \frac{\partial f}{\partial y'}f && (299) \\
\frac{d^2 f}{dx^2} &= \frac{d}{dx}\left(\frac{df}{dx}\right) \\
&= \frac{\partial}{\partial x}\left(\frac{df}{dx}\right) + \frac{\partial}{\partial y}\left(\frac{df}{dx}\right)y' + \frac{\partial}{\partial y'}\left(\frac{df}{dx}\right)f && (2) \\
\frac{d^3 f}{dx^3} &= \frac{d}{dx}\left(\frac{d^2 f}{dx^2}\right) \\
&= \frac{\partial}{\partial x}\left(\frac{d^2 f}{dx^2}\right) + \left(\frac{\partial}{\partial y}\frac{d^2 f}{dx^2}\right)y' + \frac{\partial}{\partial y'}\left(\frac{d^2 f}{dx^2}\right)f && (3) \\
&\vdots
\end{aligned}$$

And so on. Hence if we name $F_0 = f(x, y, y')$ then the above can be written as

$$\begin{aligned}
F_0 &= f(x,y,y') \qquad (4)\\
F_1 &= \frac{df}{dx}\\
&= \frac{dF_0}{dx}\\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}y' + \frac{\partial f}{\partial y'}y''\\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}y' + \frac{\partial f}{\partial y'}f \qquad (5)\\
&= \frac{\partial F_0}{\partial x} + \frac{\partial F_0}{\partial y}y' + \frac{\partial F_0}{\partial y'}F_0\\
F_2 &= \frac{d}{dx}\left(\frac{d}{dx}f\right)\\
&= \frac{d}{dx}(F_1)\\
&= \frac{\partial}{\partial x}F_1 + \left(\frac{\partial F_1}{\partial y}\right)y' + \left(\frac{\partial F_1}{\partial y'}\right)y''\\
&= \frac{\partial}{\partial x}F_1 + \left(\frac{\partial F_1}{\partial y}\right)y' + \left(\frac{\partial F_1}{\partial y'}\right)F_0\\
&\vdots\\
F_n &= \frac{d}{dx}(F_{n-1})\\
&= \frac{\partial}{\partial x}F_{n-1} + \left(\frac{\partial F_{n-1}}{\partial y}\right)y' + \left(\frac{\partial F_{n-1}}{\partial y'}\right)y''\\
&= \frac{\partial}{\partial x}F_{n-1} + \left(\frac{\partial F_{n-1}}{\partial y}\right)y' + \left(\frac{\partial F_{n-1}}{\partial y'}\right)F_0 \qquad (6)
\end{aligned}$$

Therefore (6) can be used from now on along with

$$y(x) = y_0 + xy_0' + \sum_{n=0}^{\infty}\frac{x^{n+2}}{(n+2)!}F_n|_{x_0,y_0,y_0'} \qquad (7)$$

To find $y(x)$ series solution around $x=0$. Hence

$$
\begin{aligned}
F_0 &= y'\mathrm{e}^{2x} - y\cos(x) \\
F_1 &= \frac{dF_0}{dx} \\
&= \frac{\partial F_0}{\partial x} + \frac{\partial F_0}{\partial y}y' + \frac{\partial F_0}{\partial y'}F_0 \\
&= \left(-y\cos(x) + 2y'\right)\mathrm{e}^{2x} + \sin(x)\,y - \cos(x)\,y' + y'\mathrm{e}^{4x} \\
F_2 &= \frac{dF_1}{dx} \\
&= \frac{\partial F_1}{\partial x} + \frac{\partial F_1}{\partial y}y' + \frac{\partial F_1}{\partial y'}F_1 \\
&= \left(y'(-2\cos(x)+4) - 4y\left(\cos(x) - \frac{\sin(x)}{4}\right)\right)\mathrm{e}^{2x} + \left(-y\cos(x)+6y'\right)\mathrm{e}^{4x} + y'\mathrm{e}^{6x} + 2\sin(x)\,y' + \mathrm{co} \\
F_3 &= \frac{dF_2}{dx} \\
&= \frac{\partial F_2}{\partial x} + \frac{\partial F_2}{\partial y}y' + \frac{\partial F_2}{\partial y'}F_2 \\
&= \left(y'(-8\cos(x)+5\sin(x)+8) + 2y\left(\cos(x)^2 - \frac{11\cos(x)}{2} + 3\sin(x)\right)\right)\mathrm{e}^{2x} + \left(y'(-3\cos(x)+28)\right. \\
F_4 &= \frac{dF_3}{dx} \\
&= \frac{\partial F_3}{\partial x} + \frac{\partial F_3}{\partial y}y' + \frac{\partial F_3}{\partial y'}F_3 \\
&= \left(y'\left(3\cos(x)^2 - 19\cos(x) + 24\sin(x) + 16\right) + 12\left(\cos(x)^2 + \left(-\frac{3\sin(x)}{4} - 2\right)\cos(x) + \frac{23\sin(x)}{12}\right.\right.
\end{aligned}
$$

And so on. Evaluating all the above at initial conditions $x=0$ and $y(0)=-1$ and $y'(0)=1$ gives

$$
\begin{aligned}
F_0 &= 2 \\
F_1 &= 3 \\
F_2 &= 12 \\
F_3 &= 62 \\
F_4 &= 311
\end{aligned}
$$

Substituting all the above in (7) and simplifying gives the solution as

$$
y = x^2 + x - 1 + \frac{x^3}{2} + \frac{x^4}{2} + \frac{31x^5}{60} + \frac{311x^6}{720} + O\left(x^6\right)
$$

$$y = x^2 + x - 1 + \frac{x^3}{2} + \frac{x^4}{2} + \frac{31x^5}{60} + \frac{311x^6}{720} + O(x^6)$$

Since the expansion point $x = 0$ is an ordinary, we can also solve this using standard power series Let the solution be represented as power series of the form

$$y = \sum_{n=0}^{\infty} a_n x^n$$

Then

$$y' = \sum_{n=1}^{\infty} n a_n x^{n-1}$$
$$y'' = \sum_{n=2}^{\infty} n(n-1)\, a_n x^{n-2}$$

Substituting the above back into the ode gives

$$\sum_{n=2}^{\infty} n(n-1)\, a_n x^{n-2} = \left(\sum_{n=1}^{\infty} n a_n x^{n-1}\right) e^{2x} - \left(\sum_{n=0}^{\infty} a_n x^n\right) \cos(x) \tag{1}$$

Expanding $-e^{2x}$ as Taylor series around $x = 0$ and keeping only the first 6 terms gives

$$\begin{aligned} -e^{2x} &= -1 - 2x - 2x^2 - \frac{4}{3}x^3 - \frac{2}{3}x^4 - \frac{4}{15}x^5 - \frac{4}{45}x^6 + \dots \\ &= -1 - 2x - 2x^2 - \frac{4}{3}x^3 - \frac{2}{3}x^4 - \frac{4}{15}x^5 - \frac{4}{45}x^6 \end{aligned}$$

Expanding $\cos(x)$ as Taylor series around $x = 0$ and keeping only the first 6 terms gives

$$\begin{aligned} \cos(x) &= -\frac{1}{720}x^6 + \frac{1}{24}x^4 - \frac{1}{2}x^2 + 1 + \dots \\ &= -\frac{1}{720}x^6 + \frac{1}{24}x^4 - \frac{1}{2}x^2 + 1 \end{aligned}$$

Hence the ODE in Eq (1) becomes

$$\begin{aligned} &\left(\sum_{n=2}^{\infty} n(n-1)\, a_n x^{n-2}\right) \\ &+ \left(-1 - 2x - 2x^2 - \frac{4}{3}x^3 - \frac{2}{3}x^4 - \frac{4}{15}x^5 - \frac{4}{45}x^6\right)\left(\sum_{n=1}^{\infty} n a_n x^{n-1}\right) \\ &+ \left(-\frac{1}{720}x^6 + \frac{1}{24}x^4 - \frac{1}{2}x^2 + 1\right)\left(\sum_{n=0}^{\infty} a_n x^n\right) = 0 \end{aligned}$$

Expanding the second term in (1) gives

$$\left(\sum_{n=2}^{\infty} n(n-1)\, a_n x^{n-2}\right) + -1 \cdot \left(\sum_{n=1}^{\infty} n a_n x^{n-1}\right) - 2x \cdot \left(\sum_{n=1}^{\infty} n a_n x^{n-1}\right) - 2x^2 \cdot \left(\sum_{n=1}^{\infty} n a_n x^{n-1}\right) - \frac{4x^3}{3} \cdot \left(\sum_{n=1}^{\infty} n a_n x^{n-1}\right) - \frac{2x^4}{3} \cdot \left(\sum_{n=1}^{\infty} n a_n x^{n-1}\right) - \frac{4x^5}{15} \cdot \left(\sum_{n=1}^{\infty} n a_n x^{n-1}\right) - \frac{4x^6}{45} \cdot \left(\sum_{n=1}^{\infty} n a_n x^{n-1}\right) + \left(-\frac{1}{720}x^6 + \frac{1}{24}x^4 - \frac{1}{2}x^2 + 1\right) \left(\sum_{n=0}^{\infty} a_n x^n\right) = 0$$

Expanding the third term in (1) gives

$$\left(\sum_{n=2}^{\infty} n(n-1)\, a_n x^{n-2}\right) + -1 \cdot \left(\sum_{n=1}^{\infty} n a_n x^{n-1}\right) - 2x \cdot \left(\sum_{n=1}^{\infty} n a_n x^{n-1}\right) - 2x^2 \cdot \left(\sum_{n=1}^{\infty} n a_n x^{n-1}\right) - \frac{4x^3}{3} \cdot \left(\sum_{n=1}^{\infty} n a_n x^{n-1}\right) - \frac{2x^4}{3} \cdot \left(\sum_{n=1}^{\infty} n a_n x^{n-1}\right) - \frac{4x^5}{15} \cdot \left(\sum_{n=1}^{\infty} n a_n x^{n-1}\right) - \frac{4x^6}{45} \cdot \left(\sum_{n=1}^{\infty} n a_n x^{n-1}\right) + -\frac{x^6}{720} \cdot \left(\sum_{n=0}^{\infty} a_n x^n\right) + \frac{x^4}{24} \cdot \left(\sum_{n=0}^{\infty} a_n x^n\right) - \frac{x^2}{2} \cdot \left(\sum_{n=0}^{\infty} a_n x^n\right) + 1 \cdot \left(\sum_{n=0}^{\infty} a_n x^n\right) = 0$$

Which simplifies to

$$\left(\sum_{n=2}^{\infty} n(n-1)\, a_n x^{n-2}\right) + \sum_{n=1}^{\infty} \left(-n a_n x^{n-1}\right) + \sum_{n=1}^{\infty} \left(-2n a_n x^n\right) + \sum_{n=1}^{\infty} \left(-2n\, x^{1+n} a_n\right) + \sum_{n=1}^{\infty} \left(-\frac{4n\, x^{n+2} a_n}{3}\right) + \sum_{n=1}^{\infty} \left(-\frac{2n\, x^{n+3} a_n}{3}\right) + \sum_{n=1}^{\infty} \left(-\frac{4n\, x^{n+4} a_n}{15}\right) + \sum_{n=1}^{\infty} \left(-\frac{4n\, x^{n+5} a_n}{45}\right) + \sum_{n=0}^{\infty} \left(-\frac{x^{n+6} a_n}{720}\right) + \left(\sum_{n=0}^{\infty} \frac{x^{n+4} a_n}{24}\right) + \sum_{n=0}^{\infty} \left(-\frac{x^{n+2} a_n}{2}\right) + \left(\sum_{n=0}^{\infty} a_n x^n\right) = 0 \tag{2}$$

The next step is to make all powers of x be n in each summation term. Going over each summation term above with power of x in it which is not already x^n and adjusting the power and the corresponding index gives

$$
\begin{aligned}
\sum_{n=2}^{\infty} n(n-1)\, a_n x^{n-2} &= \sum_{n=0}^{\infty} (n+2)\, a_{n+2}(1+n)\, x^n \\
\sum_{n=1}^{\infty} \left(-n a_n x^{n-1}\right) &= \sum_{n=0}^{\infty} \left(-(1+n)\, a_{1+n} x^n\right) \\
\sum_{n=1}^{\infty} \left(-2n\, x^{1+n} a_n\right) &= \sum_{n=2}^{\infty} \left(-2(n-1)\, a_{n-1} x^n\right) \\
\sum_{n=1}^{\infty} \left(-\frac{4n\, x^{n+2} a_n}{3}\right) &= \sum_{n=3}^{\infty} \left(-\frac{4(n-2)\, a_{n-2} x^n}{3}\right) \\
\sum_{n=1}^{\infty} \left(-\frac{2n\, x^{n+3} a_n}{3}\right) &= \sum_{n=4}^{\infty} \left(-\frac{2(n-3)\, a_{n-3} x^n}{3}\right) \\
\sum_{n=1}^{\infty} \left(-\frac{4n\, x^{n+4} a_n}{15}\right) &= \sum_{n=5}^{\infty} \left(-\frac{4(n-4)\, a_{n-4} x^n}{15}\right) \\
\sum_{n=1}^{\infty} \left(-\frac{4n\, x^{n+5} a_n}{45}\right) &= \sum_{n=6}^{\infty} \left(-\frac{4(n-5)\, a_{n-5} x^n}{45}\right) \\
\sum_{n=0}^{\infty} \left(-\frac{x^{n+6} a_n}{720}\right) &= \sum_{n=6}^{\infty} \left(-\frac{a_{n-6} x^n}{720}\right) \\
\sum_{n=0}^{\infty} \frac{x^{n+4} a_n}{24} &= \sum_{n=4}^{\infty} \frac{a_{n-4} x^n}{24} \\
\sum_{n=0}^{\infty} \left(-\frac{x^{n+2} a_n}{2}\right) &= \sum_{n=2}^{\infty} \left(-\frac{a_{n-2} x^n}{2}\right)
\end{aligned}
$$

Substituting all the above in Eq (2) gives the following equation where now all powers of x are the same and equal to n.

$$\left(\sum_{n=0}^{\infty} (n+2)\, a_{n+2}(1+n)\, x^{n}\right) + \sum_{n=0}^{\infty} \left(-(1+n)\, a_{1+n} x^{n}\right) + \sum_{n=1}^{\infty} \left(-2n a_{n} x^{n}\right) + \sum_{n=2}^{\infty} \left(-2(n-1)\, a_{n-1} x^{n}\right) + \sum_{n=3}^{\infty} \left(-\frac{4(n-2)\, a_{n-2} x^{n}}{3}\right) + \sum_{n=4}^{\infty} \left(-\frac{2(n-3)\, a_{n-3} x^{n}}{3}\right) + \sum_{n=5}^{\infty} \left(-\frac{4(n-4)\, a_{n-4} x^{n}}{15}\right) + \sum_{n=6}^{\infty} \left(-\frac{4(n-5)\, a_{n-5} x^{n}}{45}\right) + \sum_{n=6}^{\infty} \left(-\frac{a_{n-6} x^{n}}{720}\right) + \left(\sum_{n=4}^{\infty} \frac{a_{n-4} x^{n}}{24}\right) + \sum_{n=2}^{\infty} \left(-\frac{a_{n-2} x^{n}}{2}\right) + \left(\sum_{n=0}^{\infty} a_{n} x^{n}\right) = 0 \tag{3}$$

$n = 0$ gives

$$2a_2 - a_1 + a_0 = 0$$

$$a_2 = -\frac{a_0}{2} + \frac{a_1}{2}$$

$n = 1$ gives

$$6a_3 - 2a_2 - a_1 = 0$$

Which after substituting earlier equations, simplifies to

$$a_3 = -\frac{a_0}{6} + \frac{a_1}{3}$$

$n = 2$ gives

$$12a_4 - 3a_3 - 3a_2 - 2a_1 - \frac{a_0}{2} = 0$$

Which after substituting earlier equations, simplifies to

$$a_4 = -\frac{a_0}{8} + \frac{3a_1}{8}$$

$n = 3$ gives

$$20a_5 - 4a_4 - 5a_3 - 4a_2 - \frac{11a_1}{6} = 0$$

Which after substituting earlier equations, simplifies to

$$a_5 = -\frac{a_0}{6} + \frac{7a_1}{20}$$

$n = 4$ gives

$$30a_6 - 5a_5 - 7a_4 - 6a_3 - \frac{19a_2}{6} - \frac{2a_1}{3} + \frac{a_0}{24} = 0$$

Which after substituting earlier equations, simplifies to

$$a_6 = -\frac{13a_0}{90} + \frac{23a_1}{80}$$

$n = 5$ gives

$$42a_7 - 6a_6 - 9a_5 - 8a_4 - \frac{9a_3}{2} - \frac{4a_2}{3} - \frac{9a_1}{40} = 0$$

Which after substituting earlier equations, simplifies to

$$a_7 = -\frac{41a_0}{360} + \frac{11a_1}{45}$$

For $6 \leq n$, the recurrence equation is

$$\begin{aligned}(n+2)\, a_{n+2}(1+n) - (1+n)\, a_{1+n} - 2na_n - 2(n-1)\, a_{n-1} - \frac{4(n-2)\, a_{n-2}}{3} \\ - \frac{2(n-3)\, a_{n-3}}{3} - \frac{4(n-4)\, a_{n-4}}{15} - \frac{4(n-5)\, a_{n-5}}{45} - \frac{a_{n-6}}{720} + \frac{a_{n-4}}{24} - \frac{a_{n-2}}{2} + a_n = 0\end{aligned} \tag{4}$$

Solving for a_{n+2}, gives

$$a_{n+2} = \frac{1440na_n + 720na_{1+n} + 64na_{n-5} + 192na_{n-4} + 480na_{n-3} + 960na_{n-2} + 1440na_{n-1} - 720a_n + 720a_{1+n}}{720\,(n+2)\,(1+n)}$$

$$\begin{aligned}(5) \quad &= \frac{(1440n - 720)\, a_n}{720\,(n+2)\,(1+n)} + \frac{(720n + 720)\, a_{1+n}}{720\,(n+2)\,(1+n)} + \frac{a_{n-6}}{720\,(n+2)\,(1+n)} \\ &+ \frac{(64n - 320)\, a_{n-5}}{720\,(n+2)\,(1+n)} + \frac{(192n - 798)\, a_{n-4}}{720\,(n+2)\,(1+n)} + \frac{(480n - 1440)\, a_{n-3}}{720\,(n+2)\,(1+n)} \\ &+ \frac{(960n - 1560)\, a_{n-2}}{720\,(n+2)\,(1+n)} + \frac{(1440n - 1440)\, a_{n-1}}{720\,(n+2)\,(1+n)}\end{aligned}$$

And so on. Therefore the solution is

$$
\begin{aligned}
y &= \sum_{n=0}^{\infty} a_n x^n \\
&= a_3 x^3 + a_2 x^2 + a_1 x + a_0 + \dots
\end{aligned}
$$

Substituting the values for a_n found above, the solution becomes

$$y = a_0 + a_1 x + \left(-\frac{a_0}{2} + \frac{a_1}{2}\right) x^2 + \left(-\frac{a_0}{6} + \frac{a_1}{3}\right) x^3 + \left(-\frac{a_0}{8} + \frac{3a_1}{8}\right) x^4 + \left(-\frac{a_0}{6} + \frac{7a_1}{20}\right) x^5 + \dots$$

Collecting terms, the solution becomes

$$y = \left(1 - \frac{1}{2}x^2 - \frac{1}{6}x^3 - \frac{1}{8}x^4 - \frac{1}{6}x^5\right) a_0 + \left(x + \frac{1}{2}x^2 + \frac{1}{3}x^3 + \frac{3}{8}x^4 + \frac{7}{20}x^5\right) a_1 + O(x^6) \quad (3)$$

At $x = 0$ the solution above becomes

$$y = \left(1 - \frac{1}{2}x^2 - \frac{1}{6}x^3 - \frac{1}{8}x^4 - \frac{1}{6}x^5\right) c_1 + \left(x + \frac{1}{2}x^2 + \frac{1}{3}x^3 + \frac{3}{8}x^4 + \frac{7}{20}x^5\right) c_2 + O(x^6)$$

$$y = -1 + x^2 + \frac{x^3}{2} + \frac{x^4}{2} + \frac{31x^5}{60} + x + O(x^6)$$

Summary

The solution(s) found are the following

$$y = x^2 + x - 1 + \frac{x^3}{2} + \frac{x^4}{2} + \frac{31x^5}{60} + \frac{311x^6}{720} + O(x^6) \quad (1)$$

$$y = -1 + x^2 + \frac{x^3}{2} + \frac{x^4}{2} + \frac{31x^5}{60} + x + O(x^6) \quad (2)$$

Verification of solutions

$$y = x^2 + x - 1 + \frac{x^3}{2} + \frac{x^4}{2} + \frac{31x^5}{60} + \frac{311x^6}{720} + O(x^6)$$

Verified OK.

$$y = -1 + x^2 + \frac{x^3}{2} + \frac{x^4}{2} + \frac{31x^5}{60} + x + O(x^6)$$

Verified OK.

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Heun: Equivalence to the GHE or one of its 4 confluent cases under a power @ Moebius
-> trying a solution of the form r0(x) * Y + r1(x) * Y where Y = exp(int(r(x), dx)) * 2F1([a
-> Trying changes of variables to rationalize or make the ODE simpler
   trying a symmetry of the form [xi=0, eta=F(x)]
   checking if the LODE is missing y
   -> Heun: Equivalence to the GHE or one of its 4 confluent cases under a power @ Moebius
   -> trying a solution of the form r0(x) * Y + r1(x) * Y where Y = exp(int(r(x), dx)) * 2F1
      trying a symmetry of the form [xi=0, eta=F(x)]
      trying 2nd order exact linear
      trying symmetries linear in x and y(x)
      trying to convert to a linear ODE with constant coefficients
      -> trying with_periodic_functions in the coefficients
         --- Trying Lie symmetry methods, 2nd order ---
         `, `-> Computing symmetries using: way = 5`[0, u]
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 20

```
Order:=6;
dsolve([diff(y(x),x$2)-exp(2*x)*diff(y(x),x)+cos(x)*y(x)=0,y(0) = -1, D(y)(0) = 1],y(x),type
```

$$y(x) = -1 + x + x^2 + \frac{1}{2}x^3 + \frac{1}{2}x^4 + \frac{31}{60}x^5 + \mathrm{O}\left(x^6\right)$$

✓ Solution by Mathematica

Time used: 0.001 (sec). Leaf size: 30

```
AsymptoticDSolveValue[{y''[x]-Exp[2*x]*y'[x]+Cos[x]*y[x]==0,{y[0]==-1,y'[0]==1}},y[x],{x,0,5
```

$$y(x) \to \frac{31x^5}{60} + \frac{x^4}{2} + \frac{x^3}{2} + x^2 + x - 1$$

6.18 problem 21

Internal problem ID [5036]
Internal file name [`OUTPUT/4529_Sunday_June_05_2022_03_00_21_PM_6381161/index.tex`]

Book: Fundamentals of Differential Equations. By Nagle, Saff and Snider. 9th edition. Boston. Pearson 2018.
Section: Chapter 8, Series solutions of differential equations. Section 8.4. page 449
Problem number: 21.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"first order ode series method. Ordinary point", "first order ode series method. Taylor series method"**

Maple gives the following as the ode type

```
[_linear]
```

$$\boxed{y' - xy = \sin(x)}$$

With the expansion point for the power series method at $x = 0$.

6.18.1 Solving as series ode

Solving ode using Taylor series method. This gives review on how the Taylor series method works for solving first order ode. Let

$$y' = f(x, y)$$

Where $f(x, y)$ is analytic at expansion point x_0. We can always shift to $x_0 = 0$ if x_0 is not zero. So from now we assume $x_0 = 0$. Assume also that $y(x_0) = y_0$. Using Taylor series

$$\begin{aligned}
y(x) &= y(x_0) + (x - x_0)\, y'(x_0) + \frac{(x - x_0)^2}{2} y''(x_0) + \frac{(x - x_0)^3}{3!} y''(x_0) + \cdots \\
&= y_0 + xf + \frac{x^2}{2} \frac{df}{dx}\bigg|_{x_0,y_0} + \frac{x^3}{3!} \frac{d^2 f}{dx^2}\bigg|_{x_0,y_0} + \cdots \\
&= y_0 + \sum_{n=0}^{\infty} \frac{x^{n+1}}{(n+1)\,!} \frac{d^n f}{dx^n}\bigg|_{x_0,y_0}
\end{aligned}$$

But

$$
\begin{aligned}
\frac{df}{dx} &= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} f && (1)\\
\frac{d^2 f}{dx^2} &= \frac{d}{dx}\left(\frac{df}{dx}\right)\\
&= \frac{\partial}{\partial x}\left(\frac{df}{dx}\right) + \frac{\partial}{\partial y}\left(\frac{df}{dx}\right) f && (2)\\
\frac{d^3 f}{dx^3} &= \frac{d}{dx}\left(\frac{d^2 f}{dx^2}\right)\\
&= \frac{\partial}{\partial x}\left(\frac{d^2 f}{dx^2}\right) + \left(\frac{\partial}{\partial y}\frac{d^2 f}{dx^2}\right) f && (3)\\
&\vdots
\end{aligned}
$$

And so on. Hence if we name $F_0 = f(x,y)$ then the above can be written as

$$
\begin{aligned}
F_0 &= f(x,y) && (4)\\
F_n &= \frac{d}{dx}(F_{n-1})\\
&= \frac{\partial}{\partial x}F_{n-1} + \left(\frac{\partial F_{n-1}}{\partial y}\right) F_0 && (5)
\end{aligned}
$$

For example, for $n = 1$ we see that

$$
\begin{aligned}
F_1 &= \frac{d}{dx}(F_0)\\
&= \frac{\partial}{\partial x}F_0 + \left(\frac{\partial F_0}{\partial y}\right) F_0\\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} f
\end{aligned}
$$

Which is (1). And when $n = 2$

$$
\begin{aligned}
F_2 &= \frac{d}{dx}(F_1)\\
&= \frac{\partial}{\partial x}F_1 + \left(\frac{\partial F_1}{\partial y}\right) F_0\\
&= \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} f\right) + \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} f\right) f\\
&= \frac{\partial}{\partial x}\left(\frac{df}{dx}\right) + \frac{\partial}{\partial y}\left(\frac{df}{dx}\right) f
\end{aligned}
$$

Which is (2) and so on. Therefore (4,5) can be used from now on along with

$$y(x) = y_0 + \sum_{n=0}^{\infty} \frac{x^{n+1}}{(n+1)!} F_n|_{x_0,y_0} \tag{6}$$

Hence

$$\begin{aligned}
F_0 &= xy + \sin(x) \\
F_1 &= \frac{dF_0}{dx} \\
&= \frac{\partial F_0}{\partial x} + \frac{\partial F_0}{\partial y} F_0 \\
&= yx^2 + \sin(x)\,x + y + \cos(x) \\
F_2 &= \frac{dF_1}{dx} \\
&= \frac{\partial F_1}{\partial x} + \frac{\partial F_1}{\partial y} F_1 \\
&= (x^2+1)\sin(x) + x\left((x^2+3)\,y + \cos(x)\right) \\
F_3 &= \frac{dF_2}{dx} \\
&= \frac{\partial F_2}{\partial x} + \frac{\partial F_2}{\partial y} F_2 \\
&= (x^4+6x^2+3)\,y + \sin(x)\,x^3 + \cos(x)\,x^2 + 4\sin(x)\,x + 2\cos(x) \\
F_4 &= \frac{dF_3}{dx} \\
&= \frac{\partial F_3}{\partial x} + \frac{\partial F_3}{\partial y} F_3 \\
&= (x^4+8x^2+5)\sin(x) + \left((x^4+10x^2+15)\,y + \cos(x)\,(x^2+6)\right) x
\end{aligned}$$

And so on. Evaluating all the above at initial conditions $x(0) = 0$ and $y(0) = y(0)$ gives

$$\begin{aligned}
F_0 &= 0 \\
F_1 &= 1 + y(0) \\
F_2 &= 0 \\
F_3 &= 2 + 3y(0) \\
F_4 &= 0
\end{aligned}$$

Substituting all the above in (6) and simplifying gives the solution as

$$y = \left(1 + \frac{1}{2}x^2 + \frac{1}{8}x^4\right) y(0) + \frac{x^2}{2} + \frac{x^4}{12} + O(x^6)$$

Since $x = 0$ is also an ordinary point, then standard power series can also be used. Writing the ODE as

$$\begin{aligned} y' + q(x)y &= p(x) \\ y' - xy &= \sin(x) \end{aligned}$$

Where

$$\begin{aligned} q(x) &= -x \\ p(x) &= \sin(x) \end{aligned}$$

Next, the type of the expansion point $x = 0$ is determined. This point can be an ordinary point, a regular singular point (also called removable singularity), or irregular singular point (also called non-removable singularity or essential singularity). When $x = 0$ is an ordinary point, then the standard power series is used. If the point is a regular singular point, Frobenius series is used instead. Irregular singular point requires more advanced methods (asymptotic methods) and is not supported now. Hopefully this will be added in the future. $x = 0$ is called an ordinary point $q(x)$ has a Taylor series expansion around the point $x = 0$. $x = 0$ is called a regular singular point if $q(x)$ is not not analytic at $x = 0$ but $xq(x)$ has Taylor series expansion. And finally, $x = 0$ is an irregular singular point if the point is not ordinary and not regular singular. This is the most complicated case. Now the expansion point $x = 0$ is checked to see if it is an ordinary point or not. Let the solution be represented as power series of the form

$$y = \sum_{n=0}^{\infty} a_n x^n$$

Then

$$y' = \sum_{n=1}^{\infty} n a_n x^{n-1}$$

Substituting the above back into the ode gives

$$\left(\sum_{n=1}^{\infty} n a_n x^{n-1}\right) - x\left(\sum_{n=0}^{\infty} a_n x^n\right) = \sin(x) \tag{1}$$

Expanding $\sin(x)$ as Taylor series around $x = 0$ and keeping only the first 6 terms gives

$$\begin{aligned} \sin(x) &= x - \frac{1}{6}x^3 + \frac{1}{120}x^5 + \dots \\ &= x - \frac{1}{6}x^3 + \frac{1}{120}x^5 \end{aligned}$$

Hence the ODE in Eq (1) becomes

$$\left(\sum_{n=1}^{\infty} n a_n x^{n-1}\right) - x\left(\sum_{n=0}^{\infty} a_n x^n\right) = x - \frac{1}{6}x^3 + \frac{1}{120}x^5 \tag{1}$$

Which simplifies to

$$\left(\sum_{n=1}^{\infty} n a_n x^{n-1}\right) + \sum_{n=0}^{\infty} \left(-x^{1+n} a_n\right) = x - \frac{1}{6}x^3 + \frac{1}{120}x^5 \tag{2}$$

The next step is to make all powers of x be n in each summation term. Going over each summation term above with power of x in it which is not already x^n and adjusting the power and the corresponding index gives

$$\sum_{n=1}^{\infty} n a_n x^{n-1} = \sum_{n=0}^{\infty} (1+n) a_{1+n} x^n$$

$$\sum_{n=0}^{\infty} \left(-x^{1+n} a_n\right) = \sum_{n=1}^{\infty} \left(-a_{n-1} x^n\right)$$

Substituting all the above in Eq (2) gives the following equation where now all powers of x are the same and equal to n.

$$\left(\sum_{n=0}^{\infty} (1+n) a_{1+n} x^n\right) + \sum_{n=1}^{\infty} \left(-a_{n-1} x^n\right) = x - \frac{1}{6}x^3 + \frac{1}{120}x^5 \tag{3}$$

For $1 \leq n$, the recurrence equation is

$$\left((1+n) a_{1+n} - a_{n-1}\right) x^n = x - \frac{1}{6}x^3 + \frac{1}{120}x^5 \tag{4}$$

For $n = 1$ the recurrence equation gives

$$(2a_2 - a_0)\, x = x$$
$$2a_2 - a_0 = 1$$

Which after substituting the earlier terms found becomes

$$a_2 = \frac{1}{2} + \frac{a_0}{2}$$

For $n = 2$ the recurrence equation gives

$$(3a_3 - a_1)\, x^2 = 0$$
$$3a_3 - a_1 = 0$$

Which after substituting the earlier terms found becomes

$$a_3 = 0$$

For $n = 3$ the recurrence equation gives

$$(4a_4 - a_2)\, x^3 = -\frac{x^3}{6}$$
$$4a_4 - a_2 = -\frac{1}{6}$$

Which after substituting the earlier terms found becomes

$$a_4 = \frac{1}{12} + \frac{a_0}{8}$$

For $n = 4$ the recurrence equation gives

$$(5a_5 - a_3)\, x^4 = 0$$
$$5a_5 - a_3 = 0$$

Which after substituting the earlier terms found becomes

$$a_5 = 0$$

For $n = 5$ the recurrence equation gives

$$(6a_6 - a_4)\, x^5 = \frac{x^5}{120}$$
$$6a_6 - a_4 = \frac{1}{120}$$

Which after substituting the earlier terms found becomes

$$a_6 = \frac{11}{720} + \frac{a_0}{48}$$

And so on. Therefore the solution is

$$\begin{aligned} y &= \sum_{n=0}^{\infty} a_n x^n \\ &= a_3 x^3 + a_2 x^2 + a_1 x + a_0 + \dots \end{aligned}$$

Substituting the values for a_n found above, the solution becomes

$$y = a_0 + \left(\frac{1}{2} + \frac{a_0}{2}\right) x^2 + \left(\frac{1}{12} + \frac{a_0}{8}\right) x^4 + \dots$$

Collecting terms, the solution becomes

$$y = \left(1 + \frac{1}{2}x^2 + \frac{1}{8}x^4\right) a_0 + \frac{x^2}{2} + \frac{x^4}{12} + O(x^6) \tag{3}$$

<u>Summary</u>

The solution(s) found are the following

$$y = \left(1 + \frac{1}{2}x^2 + \frac{1}{8}x^4\right) y(0) + \frac{x^2}{2} + \frac{x^4}{12} + O(x^6) \tag{1}$$

$$y = \left(1 + \frac{1}{2}x^2 + \frac{1}{8}x^4\right) c_1 + \frac{x^2}{2} + \frac{x^4}{12} + O(x^6) \tag{2}$$

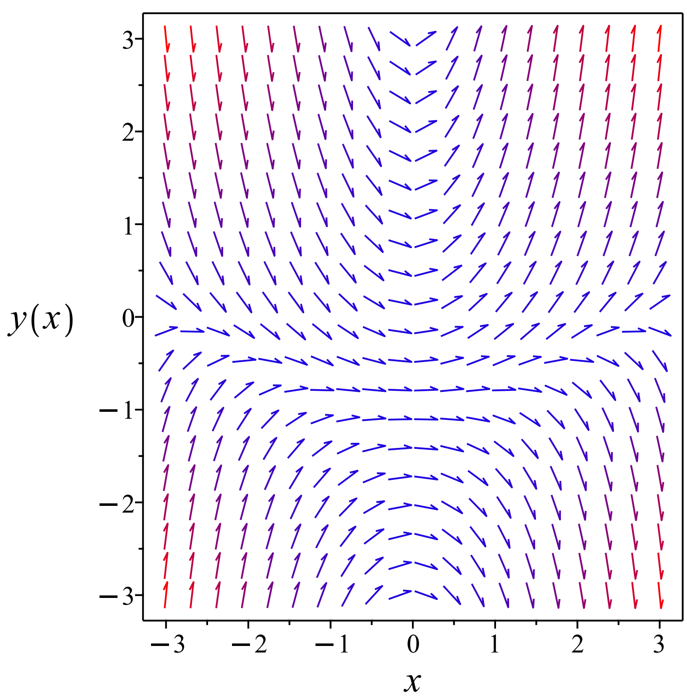


Figure 211: Slope field plot

Verification of solutions

$$y = \left(1+\frac{1}{2}x^2+\frac{1}{8}x^4\right) y(0)+\frac{x^2}{2}+\frac{x^4}{12}+O(x^6)$$

Verified OK.

$$y = \left(1+\frac{1}{2}x^2+\frac{1}{8}x^4\right) c_1+\frac{x^2}{2}+\frac{x^4}{12}+O(x^6)$$

Verified OK.

6.18.2 Maple step by step solution

Let's solve

$y' - xy = \sin(x)$

- Highest derivative means the order of the ODE is 1

  y'

- Isolate the derivative

  $y' = xy + \sin(x)$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE

  $y' - xy = \sin(x)$

- The ODE is linear; multiply by an integrating factor $\mu(x)$

  $\mu(x)(y' - xy) = \mu(x)\sin(x)$

- Assume the lhs of the ODE is the total derivative $\frac{d}{dx}(\mu(x)\, y)$

  $\mu(x)(y' - xy) = \mu'(x)\, y + \mu(x)\, y'$

- Isolate $\mu'(x)$

  $\mu'(x) = -\mu(x)\, x$

- Solve to find the integrating factor

  $\mu(x) = e^{-\frac{x^2}{2}}$

- Integrate both sides with respect to x

  $\int \left(\frac{d}{dx}(\mu(x)\, y)\right) dx = \int \mu(x)\sin(x)\, dx + c_1$

- Evaluate the integral on the lhs

  $\mu(x)\, y = \int \mu(x)\sin(x)\, dx + c_1$

- Solve for y

$$y = \frac{\int \mu(x) \sin(x) dx + c_1}{\mu(x)}$$

- Substitute $\mu(x) = e^{-\frac{x^2}{2}}$

$$y = \frac{\int e^{-\frac{x^2}{2}} \sin(x) dx + c_1}{e^{-\frac{x^2}{2}}}$$

- Evaluate the integrals on the rhs

$$y = \frac{-\frac{I\sqrt{\pi}\, e^{-\frac{1}{2}} \sqrt{2}\, \mathrm{erf}\left(\frac{x\sqrt{2}}{2} - \frac{I\sqrt{2}}{2}\right)}{4} + \frac{I\sqrt{\pi}\, e^{-\frac{1}{2}} \sqrt{2}\, \mathrm{erf}\left(\frac{x\sqrt{2}}{2} + \frac{I\sqrt{2}}{2}\right)}{4} + c_1}{e^{-\frac{x^2}{2}}}$$

- Simplify

$$y = \frac{\left(I\sqrt{\pi}\, e^{-\frac{1}{2}} \sqrt{2}\, \mathrm{erf}\left(\frac{\sqrt{2}\,(-x+I)}{2}\right) + I\sqrt{\pi}\, e^{-\frac{1}{2}} \sqrt{2}\, \mathrm{erf}\left(\frac{\sqrt{2}\,(I+x)}{2}\right) + 4c_1\right) e^{\frac{x^2}{2}}}{4}$$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
<- 1st order linear successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 25

```
Order:=6;
dsolve(diff(y(x),x)-x*y(x)=sin(x),y(x),type='series',x=0);
```

$$y(x) = \left(1 + \frac{1}{2}x^2 + \frac{1}{8}x^4\right) y(0) + \frac{x^2}{2} + \frac{x^4}{12} + O(x^6)$$

✓ Solution by Mathematica

Time used: 0.02 (sec). Leaf size: 37

```
AsymptoticDSolveValue[y'[x]-x*y[x]==Sin[x],y[x],{x,0,5}]
```

$$y(x) \to \frac{x^4}{12} + \frac{x^2}{2} + c_1\left(\frac{x^4}{8} + \frac{x^2}{2} + 1\right)$$

6.19 problem 22

Internal problem ID [5037]
Internal file name [`OUTPUT/4530_Sunday_June_05_2022_03_00_22_PM_58226421/index.tex`]

Book: Fundamentals of Differential Equations. By Nagle, Saff and Snider. 9th edition. Boston. Pearson 2018.
Section: Chapter 8, Series solutions of differential equations. Section 8.4. page 449
Problem number: 22.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"first order ode series method. Ordinary point", "first order ode series method. Taylor series method"**

Maple gives the following as the ode type

`[_linear]`

$$\boxed{w' + wx = e^x}$$

With the expansion point for the power series method at $x = 0$.

6.19.1 Solving as series ode

Solving ode using Taylor series method. This gives review on how the Taylor series method works for solving first order ode. Let

$$y' = f(x, y)$$

Where $f(x, y)$ is analytic at expansion point x_0. We can always shift to $x_0 = 0$ if x_0 is not zero. So from now we assume $x_0 = 0$. Assume also that $y(x_0) = y_0$. Using Taylor series

$$\begin{aligned}
y(x) &= y(x_0) + (x - x_0)\, y'(x_0) + \frac{(x - x_0)^2}{2} y''(x_0) + \frac{(x - x_0)^3}{3!} y''(x_0) + \cdots \\
&= y_0 + xf + \frac{x^2}{2} \frac{df}{dx}\bigg|_{x_0, y_0} + \frac{x^3}{3!} \frac{d^2 f}{dx^2}\bigg|_{x_0, y_0} + \cdots \\
&= y_0 + \sum_{n=0}^{\infty} \frac{x^{n+1}}{(n+1)!} \frac{d^n f}{dx^n}\bigg|_{x_0, y_0}
\end{aligned}$$

But

$$\frac{df}{dx} = \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} f \tag{1}$$

$$\begin{aligned}\frac{d^2 f}{dx^2} &= \frac{d}{dx}\left(\frac{df}{dx}\right)\\ &= \frac{\partial}{\partial x}\left(\frac{df}{dx}\right) + \frac{\partial}{\partial y}\left(\frac{df}{dx}\right) f \qquad (2)\\ \frac{d^3 f}{dx^3} &= \frac{d}{dx}\left(\frac{d^2 f}{dx^2}\right)\\ &= \frac{\partial}{\partial x}\left(\frac{d^2 f}{dx^2}\right) + \left(\frac{\partial}{\partial y}\frac{d^2 f}{dx^2}\right) f \qquad (3)\\ &\vdots\end{aligned}$$

And so on. Hence if we name $F_0 = f(x,y)$ then the above can be written as

$$F_0 = f(x,y) \tag{4}$$

$$\begin{aligned}F_n &= \frac{d}{dx}(F_{n-1})\\ &= \frac{\partial}{\partial x}F_{n-1} + \left(\frac{\partial F_{n-1}}{\partial y}\right) F_0 \qquad (5)\end{aligned}$$

For example, for $n=1$ we see that

$$\begin{aligned}F_1 &= \frac{d}{dx}(F_0)\\ &= \frac{\partial}{\partial x}F_0 + \left(\frac{\partial F_0}{\partial y}\right) F_0\\ &= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} f\end{aligned}$$

Which is (1). And when $n=2$

$$\begin{aligned}F_2 &= \frac{d}{dx}(F_1)\\ &= \frac{\partial}{\partial x}F_1 + \left(\frac{\partial F_1}{\partial y}\right) F_0\\ &= \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} f\right) + \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} f\right) f\\ &= \frac{\partial}{\partial x}\left(\frac{df}{dx}\right) + \frac{\partial}{\partial y}\left(\frac{df}{dx}\right) f\end{aligned}$$

Which is (2) and so on. Therefore (4,5) can be used from now on along with

$$y(x) = y_0 + \sum_{n=0}^{\infty} \frac{x^{n+1}}{(n+1)!} F_n|_{x_0,y_0} \tag{6}$$

Hence

$$\begin{aligned}
F_0 &= -wx + e^x \\
F_1 &= \frac{dF_0}{dx} \\
&= \frac{\partial F_0}{\partial x} + \frac{\partial F_0}{\partial w} F_0 \\
&= (-e^x + (x+1)w)(x-1) \\
F_2 &= \frac{dF_1}{dx} \\
&= \frac{\partial F_1}{\partial x} + \frac{\partial F_1}{\partial w} F_1 \\
&= \left(x^2 - x - 1\right) e^x - xw\left(x^2 - 3\right) \\
F_3 &= \frac{dF_2}{dx} \\
&= \frac{\partial F_2}{\partial x} + \frac{\partial F_2}{\partial w} F_2 \\
&= \left(-x^3 + x^2 + 4x - 2\right) e^x + w\left(x^4 - 6x^2 + 3\right) \\
F_4 &= \frac{dF_3}{dx} \\
&= \frac{\partial F_3}{\partial x} + \frac{\partial F_3}{\partial w} F_3 \\
&= \left(x^4 - x^3 - 8x^2 + 6x + 5\right) e^x - xw\left(x^4 - 10x^2 + 15\right)
\end{aligned}$$

And so on. Evaluating all the above at initial conditions $x(0) = 0$ and $w(0) = w(0)$ gives

$$\begin{aligned}
F_0 &= 1 \\
F_1 &= -w(0) + 1 \\
F_2 &= -1 \\
F_3 &= -2 + 3w(0) \\
F_4 &= 5
\end{aligned}$$

Substituting all the above in (6) and simplifying gives the solution as

$$w = \left(1 - \frac{1}{2}x^2 + \frac{1}{8}x^4\right) w(0) + x + \frac{x^2}{2} - \frac{x^3}{6} - \frac{x^4}{12} + \frac{x^5}{24} + O\left(x^6\right)$$

Since $x = 0$ is also an ordinary point, then standard power series can also be used. Writing the ODE as

$$\begin{aligned} w' + q(x)w &= p(x) \\ w' + wx &= e^x \end{aligned}$$

Where

$$\begin{aligned} q(x) &= x \\ p(x) &= e^x \end{aligned}$$

Next, the type of the expansion point $x = 0$ is determined. This point can be an ordinary point, a regular singular point (also called removable singularity), or irregular singular point (also called non-removable singularity or essential singularity). When $x = 0$ is an ordinary point, then the standard power series is used. If the point is a regular singular point, Frobenius series is used instead. Irregular singular point requires more advanced methods (asymptotic methods) and is not supported now. Hopefully this will be added in the future. $x = 0$ is called an ordinary point $q(x)$ has a Taylor series expansion around the point $x = 0$. $x = 0$ is called a regular singular point if $q(x)$ is not not analytic at $x = 0$ but $xq(x)$ has Taylor series expansion. And finally, $x = 0$ is an irregular singular point if the point is not ordinary and not regular singular. This is the most complicated case. Now the expansion point $x = 0$ is checked to see if it is an ordinary point or not. Let the solution be represented as power series of the form

$$w = \sum_{n=0}^{\infty} a_n x^n$$

Then

$$w' = \sum_{n=1}^{\infty} n a_n x^{n-1}$$

Substituting the above back into the ode gives

$$\left(\sum_{n=1}^{\infty} n a_n x^{n-1}\right) + \left(\sum_{n=0}^{\infty} a_n x^n\right) x = e^x \tag{1}$$

Expanding e^x as Taylor series around $x = 0$ and keeping only the first 6 terms gives

$$\begin{aligned} e^x &= 1 + x + \frac{1}{2}x^2 + \frac{1}{6}x^3 + \frac{1}{24}x^4 + \frac{1}{120}x^5 + \dots \\ &= 1 + x + \frac{1}{2}x^2 + \frac{1}{6}x^3 + \frac{1}{24}x^4 + \frac{1}{120}x^5 \end{aligned}$$

Hence the ODE in Eq (1) becomes

$$\left(\sum_{n=1}^{\infty} n a_n x^{n-1}\right) + \left(\sum_{n=0}^{\infty} a_n x^n\right) x = 1 + x + \frac{1}{2}x^2 + \frac{1}{6}x^3 + \frac{1}{24}x^4 + \frac{1}{120}x^5 \tag{1}$$

Which simplifies to

$$\left(\sum_{n=1}^{\infty} n a_n x^{n-1}\right) + \left(\sum_{n=0}^{\infty} x^{1+n} a_n\right) = 1 + x + \frac{1}{2}x^2 + \frac{1}{6}x^3 + \frac{1}{24}x^4 + \frac{1}{120}x^5 \tag{2}$$

The next step is to make all powers of x be n in each summation term. Going over each summation term above with power of x in it which is not already x^n and adjusting the power and the corresponding index gives

$$\sum_{n=1}^{\infty} n a_n x^{n-1} = \sum_{n=0}^{\infty} (1+n) a_{1+n} x^n$$
$$\sum_{n=0}^{\infty} x^{1+n} a_n = \sum_{n=1}^{\infty} a_{n-1} x^n$$

Substituting all the above in Eq (2) gives the following equation where now all powers of x are the same and equal to n.

$$\left(\sum_{n=0}^{\infty} (1+n) a_{1+n} x^n\right) + \left(\sum_{n=1}^{\infty} a_{n-1} x^n\right) = 1 + x + \frac{1}{2}x^2 + \frac{1}{6}x^3 + \frac{1}{24}x^4 + \frac{1}{120}x^5 \tag{3}$$

$n = 0$ gives

$$(a_1) 1 = 1$$
$$a_1 = 1$$

Or

$$a_1 = 1$$

For $1 \leq n$, the recurrence equation is

$$((1+n) a_{1+n} + a_{n-1}) x^n = 1 + x + \frac{1}{2}x^2 + \frac{1}{6}x^3 + \frac{1}{24}x^4 + \frac{1}{120}x^5 \tag{4}$$

For $n = 1$ the recurrence equation gives

$$(2a_2 + a_0)\, x = x$$
$$2a_2 + a_0 = 1$$

Which after substituting the earlier terms found becomes

$$a_2 = \frac{1}{2} - \frac{a_0}{2}$$

For $n = 2$ the recurrence equation gives

$$(3a_3 + a_1)\, x^2 = \frac{x^2}{2}$$
$$3a_3 + a_1 = \frac{1}{2}$$

Which after substituting the earlier terms found becomes

$$a_3 = -\frac{1}{6}$$

For $n = 3$ the recurrence equation gives

$$(4a_4 + a_2)\, x^3 = \frac{x^3}{6}$$
$$4a_4 + a_2 = \frac{1}{6}$$

Which after substituting the earlier terms found becomes

$$a_4 = -\frac{1}{12} + \frac{a_0}{8}$$

For $n = 4$ the recurrence equation gives

$$(5a_5 + a_3)\, x^4 = \frac{x^4}{24}$$
$$5a_5 + a_3 = \frac{1}{24}$$

Which after substituting the earlier terms found becomes

$$a_5 = \frac{1}{24}$$

For $n = 5$ the recurrence equation gives

$$\begin{aligned}(6a_6 + a_4)\, x^5 &= \frac{x^5}{120} \\ 6a_6 + a_4 &= \frac{1}{120}\end{aligned}$$

Which after substituting the earlier terms found becomes

$$a_6 = \frac{11}{720} - \frac{a_0}{48}$$

And so on. Therefore the solution is

$$\begin{aligned}w &= \sum_{n=0}^{\infty} a_n x^n \\ &= a_3 x^3 + a_2 x^2 + a_1 x + a_0 + \dots\end{aligned}$$

Substituting the values for a_n found above, the solution becomes

$$w = a_0 + x + \left(\frac{1}{2} - \frac{a_0}{2}\right) x^2 - \frac{x^3}{6} + \left(-\frac{1}{12} + \frac{a_0}{8}\right) x^4 + \frac{x^5}{24} + \dots$$

Collecting terms, the solution becomes

$$w = \left(1 - \frac{1}{2}x^2 + \frac{1}{8}x^4\right) a_0 + x + \frac{x^2}{2} - \frac{x^3}{6} - \frac{x^4}{12} + \frac{x^5}{24} + O(x^6) \tag{3}$$

Summary

The solution(s) found are the following

$$w = \left(1 - \frac{1}{2}x^2 + \frac{1}{8}x^4\right) w(0) + x + \frac{x^2}{2} - \frac{x^3}{6} - \frac{x^4}{12} + \frac{x^5}{24} + O(x^6) \tag{1}$$

$$w = \left(1 - \frac{1}{2}x^2 + \frac{1}{8}x^4\right) c_1 + x + \frac{x^2}{2} - \frac{x^3}{6} - \frac{x^4}{12} + \frac{x^5}{24} + O(x^6) \tag{2}$$

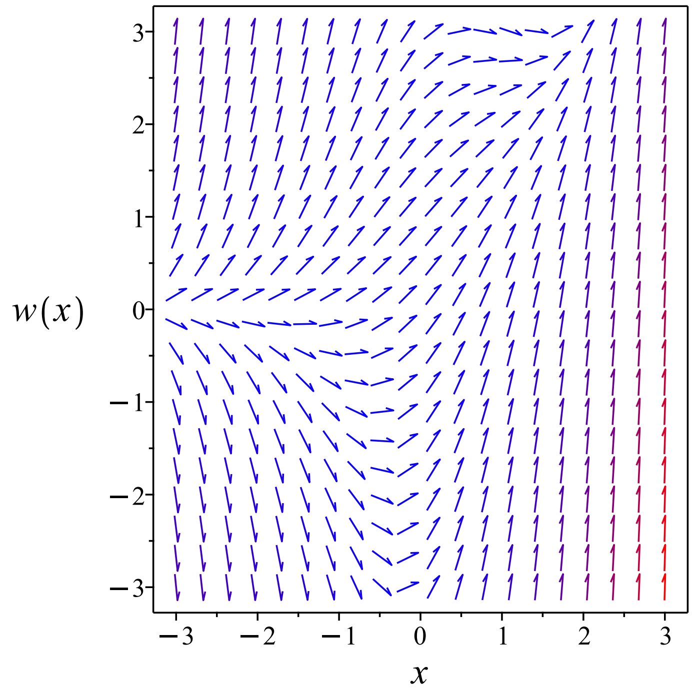


Figure 212: Slope field plot

<u>Verification of solutions</u>

$$w = \left(1 - \frac{1}{2}x^2 + \frac{1}{8}x^4\right) w(0) + x + \frac{x^2}{2} - \frac{x^3}{6} - \frac{x^4}{12} + \frac{x^5}{24} + O(x^6)$$

Verified OK.

$$w = \left(1 - \frac{1}{2}x^2 + \frac{1}{8}x^4\right) c_1 + x + \frac{x^2}{2} - \frac{x^3}{6} - \frac{x^4}{12} + \frac{x^5}{24} + O(x^6)$$

Verified OK.

6.19.2 Maple step by step solution

Let's solve

$w' + wx = e^x$

- Highest derivative means the order of the ODE is 1

  w'

- Isolate the derivative

  $w' = -wx + e^x$

- Group terms with w on the lhs of the ODE and the rest on the rhs of the ODE

  $$w' + wx = e^{x}$$

- The ODE is linear; multiply by an integrating factor $\mu(x)$

  $$\mu(x)\left(w' + wx\right) = \mu(x)\,e^{x}$$

- Assume the lhs of the ODE is the total derivative $\frac{d}{dx}(\mu(x)\,w)$

  $$\mu(x)\left(w' + wx\right) = \mu'(x)\,w + \mu(x)\,w'$$

- Isolate $\mu'(x)$

  $$\mu'(x) = \mu(x)\,x$$

- Solve to find the integrating factor

  $$\mu(x) = e^{\frac{x^2}{2}}$$

- Integrate both sides with respect to x

  $$\int \left(\frac{d}{dx}(\mu(x)\,w)\right)dx = \int \mu(x)\,e^{x}dx + c_1$$

- Evaluate the integral on the lhs

  $$\mu(x)\,w = \int \mu(x)\,e^{x}dx + c_1$$

- Solve for w

  $$w = \frac{\int \mu(x)e^{x}dx + c_1}{\mu(x)}$$

- Substitute $\mu(x) = e^{\frac{x^2}{2}}$

  $$w = \frac{\int e^{\frac{x^2}{2}}\,e^{x}dx + c_1}{e^{\frac{x^2}{2}}}$$

- Evaluate the integrals on the rhs

  $$w = \frac{-\frac{I\sqrt{\pi}\,e^{-\frac{1}{2}}\sqrt{2}\,\mathrm{erf}\left(\frac{I\sqrt{2}\,x}{2}+\frac{I\sqrt{2}}{2}\right)}{2} + c_1}{e^{\frac{x^2}{2}}}$$

- Simplify

  $$w = -\frac{\left(I\sqrt{\pi}\,e^{-\frac{1}{2}}\sqrt{2}\,\mathrm{erf}\left(\frac{I}{2}\sqrt{2}\,(x+1)\right) - 2c_1\right)e^{-\frac{x^2}{2}}}{2}$$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
<- 1st order linear successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 31

```
Order:=6;
dsolve(diff(w(x),x)+x*w(x)=exp(x),w(x),type='series',x=0);
```

$$w(x) = \left(1 - \frac{1}{2}x^2 + \frac{1}{8}x^4\right) w(0) + x + \frac{x^2}{2} - \frac{x^3}{6} - \frac{x^4}{12} + \frac{x^5}{24} + O(x^6)$$

✓ Solution by Mathematica

Time used: 0.011 (sec). Leaf size: 52

```
AsymptoticDSolveValue[w'[x]-x*w[x]==Exp[x],w[x],{x,0,5}]
```

$$w(x) \to \frac{13x^5}{120} + \frac{x^4}{6} + \frac{x^3}{2} + \frac{x^2}{2} + c_1\left(\frac{x^4}{8} + \frac{x^2}{2} + 1\right) + x$$

6.20 problem 23

Internal problem ID [5038]
Internal file name [`OUTPUT/4531_Sunday_June_05_2022_03_00_23_PM_26157909/index.tex`]

Book: Fundamentals of Differential Equations. By Nagle, Saff and Snider. 9th edition. Boston. Pearson 2018.
Section: Chapter 8, Series solutions of differential equations. Section 8.4. page 449
Problem number: 23.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"exact linear second order ode", "second_order_integrable_as_is", "second order series method. Ordinary point", "second order series method. Taylor series method"**

Maple gives the following as the ode type

```
[[_2nd_order, _exact, _linear, _nonhomogeneous]]
```

$$\boxed{z'' + xz' + z = x^2 + 2x + 1}$$

With the expansion point for the power series method at $x = 0$.

Solving ode using Taylor series method. This gives review on how the Taylor series method works for solving second order ode.

Let

$$y'' = f(x, y, y')$$

Assuming expansion is at $x_0 = 0$ (we can always shift the actual expansion point to 0 by change of variables) and assuming $f(x, y, y')$ is analytic at x_0 which must be the case for an ordinary point. Let initial conditions be $y(x_0) = y_0$ and $y'(x_0) = y'_0$. Using Taylor series gives

$$\begin{aligned}
y(x) &= y(x_0) + (x - x_0)\, y'(x_0) + \frac{(x - x_0)^2}{2} y''(x_0) + \frac{(x - x_0)^3}{3!} y'''(x_0) + \cdots \\
&= y_0 + xy'_0 + \frac{x^2}{2} f|_{x_0,y_0,y'_0} + \frac{x^3}{3!} f'|_{x_0,y_0,y'_0} + \cdots \\
&= y_0 + xy'_0 + \sum_{n=0}^{\infty} \frac{x^{n+2}}{(n+2)!} \frac{d^n f}{dx^n}\bigg|_{x_0,y_0,y'_0}
\end{aligned}$$

But

$$
\begin{aligned}
\frac{df}{dx} &= \frac{\partial f}{\partial x}\frac{dx}{dx} + \frac{\partial f}{\partial y}\frac{dy}{dx} + \frac{\partial f}{\partial y'}\frac{dy'}{dx} && (1)\\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}y' + \frac{\partial f}{\partial y'}y'' && (303)\\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}y' + \frac{\partial f}{\partial y'}f && (304)\\
\frac{d^2 f}{dx^2} &= \frac{d}{dx}\left(\frac{df}{dx}\right)\\
&= \frac{\partial}{\partial x}\left(\frac{df}{dx}\right) + \frac{\partial}{\partial y}\left(\frac{df}{dx}\right)y' + \frac{\partial}{\partial y'}\left(\frac{df}{dx}\right)f && (2)\\
\frac{d^3 f}{dx^3} &= \frac{d}{dx}\left(\frac{d^2 f}{dx^2}\right)\\
&= \frac{\partial}{\partial x}\left(\frac{d^2 f}{dx^2}\right) + \left(\frac{\partial}{\partial y}\frac{d^2 f}{dx^2}\right)y' + \frac{\partial}{\partial y'}\left(\frac{d^2 f}{dx^2}\right)f && (3)\\
&\vdots
\end{aligned}
$$

And so on. Hence if we name $F_0 = f(x,y,y')$ then the above can be written as

$$
\begin{aligned}
F_0 &= f(x,y,y') \qquad (4)\\
F_1 &= \frac{df}{dx}\\
&= \frac{dF_0}{dx}\\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}y' + \frac{\partial f}{\partial y'}y''\\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}y' + \frac{\partial f}{\partial y'}f \qquad (5)\\
&= \frac{\partial F_0}{\partial x} + \frac{\partial F_0}{\partial y}y' + \frac{\partial F_0}{\partial y'}F_0\\
F_2 &= \frac{d}{dx}\left(\frac{d}{dx}f\right)\\
&= \frac{d}{dx}(F_1)\\
&= \frac{\partial}{\partial x}F_1 + \left(\frac{\partial F_1}{\partial y}\right)y' + \left(\frac{\partial F_1}{\partial y'}\right)y''\\
&= \frac{\partial}{\partial x}F_1 + \left(\frac{\partial F_1}{\partial y}\right)y' + \left(\frac{\partial F_1}{\partial y'}\right)F_0\\
&\vdots\\
F_n &= \frac{d}{dx}(F_{n-1})\\
&= \frac{\partial}{\partial x}F_{n-1} + \left(\frac{\partial F_{n-1}}{\partial y}\right)y' + \left(\frac{\partial F_{n-1}}{\partial y'}\right)y''\\
&= \frac{\partial}{\partial x}F_{n-1} + \left(\frac{\partial F_{n-1}}{\partial y}\right)y' + \left(\frac{\partial F_{n-1}}{\partial y'}\right)F_0 \qquad (6)
\end{aligned}
$$

Therefore (6) can be used from now on along with

$$
y(x) = y_0 + xy_0' + \sum_{n=0}^{\infty}\frac{x^{n+2}}{(n+2)\,!}F_n|_{x_0,y_0,y_0'} \qquad (7)
$$

To find $y(x)$ series solution around $x=0$. Hence

$$
\begin{aligned}
F_0 &= -xz' - z + x^2 + 2x + 1 \\
F_1 &= \frac{dF_0}{dx} \\
&= \frac{\partial F_0}{\partial x} + \frac{\partial F_0}{\partial z} z' + \frac{\partial F_0}{\partial z'} F_0 \\
&= \left(x^2 - 2\right) z' - x^3 - 2x^2 + xz + x + 2 \\
F_2 &= \frac{dF_1}{dx} \\
&= \frac{\partial F_1}{\partial x} + \frac{\partial F_1}{\partial z} z' + \frac{\partial F_1}{\partial z'} F_1 \\
&= -z'x^3 + x^4 - x^2 z + 2x^3 + 5xz' - 4x^2 + 3z - 8x - 1 \\
F_3 &= \frac{dF_2}{dx} \\
&= \frac{\partial F_2}{\partial x} + \frac{\partial F_2}{\partial z} z' + \frac{\partial F_2}{\partial z'} F_2 \\
&= \left(x^4 - 9x^2 + 8\right) z' + \left(x^3 - 7x\right) z - x^5 - 2x^4 + 8x^3 + 16x^2 - 3x - 8 \\
F_4 &= \frac{dF_3}{dx} \\
&= \frac{\partial F_3}{\partial x} + \frac{\partial F_3}{\partial z} z' + \frac{\partial F_3}{\partial z'} F_3 \\
&= \left(-x^5 + 14x^3 - 33x\right) z' + \left(-x^4 + 12x^2 - 15\right) z + x^6 + 2x^5 - 13x^4 - 26x^3 + 23x^2 + 48x + 5
\end{aligned}
$$

And so on. Evaluating all the above at initial conditions $x=0$ and $z(0)=z(0)$ and $z'(0)=z'(0)$ gives

$$
\begin{aligned}
F_0 &= -z(0) + 1 \\
F_1 &= -2z'(0) + 2 \\
F_2 &= 3z(0) - 1 \\
F_3 &= 8z'(0) - 8 \\
F_4 &= 5 - 15z(0)
\end{aligned}
$$

Substituting all the above in (7) and simplifying gives the solution as

$$
\begin{aligned}
z = &\left(1 - \frac{1}{2}x^2 + \frac{1}{8}x^4 - \frac{1}{48}x^6\right) z(0) + \left(x - \frac{1}{3}x^3 + \frac{1}{15}x^5\right) z'(0) \\
&+ \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{24} - \frac{x^5}{15} + \frac{x^6}{144} + O\left(x^6\right)
\end{aligned}
$$

Since the expansion point $x = 0$ is an ordinary, we can also solve this using standard power series Let the solution be represented as power series of the form

$$z = \sum_{n=0}^{\infty} a_n x^n$$

Then

$$\begin{aligned} z' &= \sum_{n=1}^{\infty} n a_n x^{n-1} \\ z'' &= \sum_{n=2}^{\infty} n(n-1)\, a_n x^{n-2} \end{aligned}$$

Substituting the above back into the ode gives

$$\sum_{n=2}^{\infty} n(n-1)\, a_n x^{n-2} = -x\left(\sum_{n=1}^{\infty} n a_n x^{n-1}\right) - \left(\sum_{n=0}^{\infty} a_n x^n\right) + x^2 + 2x + 1 \tag{1}$$

Expanding $(x+1)^2$ as Taylor series around $x = 0$ and keeping only the first 6 terms gives

$$\begin{aligned} (x+1)^2 &= x^2 + 2x + 1 + \dots \\ &= x^2 + 2x + 1 \end{aligned}$$

Hence the ODE in Eq (1) becomes

$$\left(\sum_{n=2}^{\infty} n(n-1)\, a_n x^{n-2}\right) + x\left(\sum_{n=1}^{\infty} n a_n x^{n-1}\right) + \left(\sum_{n=0}^{\infty} a_n x^n\right) = x^2 + 2x + 1$$

Which simplifies to

$$\left(\sum_{n=2}^{\infty} n(n-1)\, a_n x^{n-2}\right) + \left(\sum_{n=1}^{\infty} n\, x^n a_n\right) + \left(\sum_{n=0}^{\infty} a_n x^n\right) = x^2 + 2x + 1 \tag{2}$$

The next step is to make all powers of x be n in each summation term. Going over each summation term above with power of x in it which is not already x^n and adjusting the power and the corresponding index gives

$$\sum_{n\,=2}^{\infty} n(n-1)\, a_n x^{n-2} = \sum_{n=0}^{\infty} (n+2)\, a_{n+2}(n+1)\, x^n$$

Substituting all the above in Eq (2) gives the following equation where now all powers of x are the same and equal to n.

$$\left(\sum_{n=0}^{\infty}(n+2)\, a_{n+2}(n+1)\, x^{n}\right)+\left(\sum_{n=1}^{\infty}n\, x^{n} a_{n}\right)+\left(\sum_{n=0}^{\infty}a_{n} x^{n}\right) = x^{2}+2x+1 \quad (3)$$

$n = 0$ gives

$$(2a_{2}+a_{0})\, x^{0} = 1$$
$$2a_{2}+a_{0} = 1$$

$$a_{2} = -\frac{a_{0}}{2}+\frac{1}{2}$$

For $1 \leq n$, the recurrence equation is

$$((n+2)\, a_{n+2}(n+1)+n a_{n}+a_{n})\, x^{n} = x^{2}+2x+1 \tag{4}$$

For $n = 1$ the recurrence equation gives

$$(6a_{3}+2a_{1})\, x = 2x$$
$$6a_{3}+2a_{1} = 2$$

Which after substituting the earlier terms found becomes

$$a_{3} = \frac{1}{3}-\frac{a_{1}}{3}$$

For $n = 2$ the recurrence equation gives

$$(12a_{4}+3a_{2})\, x^{2} = x^{2}$$
$$12a_{4}+3a_{2} = 1$$

Which after substituting the earlier terms found becomes

$$a_{4} = -\frac{1}{24}+\frac{a_{0}}{8}$$

For $n = 3$ the recurrence equation gives

$$\left(20a_5 + 4a_3\right)x^3 = 0$$
$$20a_5 + 4a_3 = 0$$

Which after substituting the earlier terms found becomes

$$a_5 = -\frac{1}{15} + \frac{a_1}{15}$$

For $n = 4$ the recurrence equation gives

$$\left(30a_6 + 5a_4\right)x^4 = 0$$
$$30a_6 + 5a_4 = 0$$

Which after substituting the earlier terms found becomes

$$a_6 = \frac{1}{144} - \frac{a_0}{48}$$

For $n = 5$ the recurrence equation gives

$$\left(42a_7 + 6a_5\right)x^5 = 0$$
$$42a_7 + 6a_5 = 0$$

Which after substituting the earlier terms found becomes

$$a_7 = \frac{1}{105} - \frac{a_1}{105}$$

And so on. Therefore the solution is

$$\begin{aligned} z &= \sum_{n=0}^{\infty} a_n x^n \\ &= a_3x^3 + a_2x^2 + a_1x + a_0 + \dots \end{aligned}$$

Substituting the values for a_n found above, the solution becomes

$$z = a_0 + a_1x + \left(-\frac{a_0}{2} + \frac{1}{2}\right)x^2 + \left(\frac{1}{3} - \frac{a_1}{3}\right)x^3 + \left(-\frac{1}{24} + \frac{a_0}{8}\right)x^4 + \left(-\frac{1}{15} + \frac{a_1}{15}\right)x^5 + \dots$$

Collecting terms, the solution becomes

$$z = \left(1-\frac{1}{2}x^{2}+\frac{1}{8}x^{4}\right) a_{0}+\left(x-\frac{1}{3}x^{3}+\frac{1}{15}x^{5}\right) a_{1}+\frac{x^{2}}{2}+\frac{x^{3}}{3}-\frac{x^{4}}{24}-\frac{x^{5}}{15}+O\left(x^{6}\right) \tag{3}$$

At $x = 0$ the solution above becomes

$$z = \left(1-\frac{1}{2}x^{2}+\frac{1}{8}x^{4}\right) c_{1}+\left(x-\frac{1}{3}x^{3}+\frac{1}{15}x^{5}\right) c_{2}+\frac{x^{2}}{2}+\frac{x^{3}}{3}-\frac{x^{4}}{24}-\frac{x^{5}}{15}+O\left(x^{6}\right)$$

Summary

The solution(s) found are the following

$$z = \left(1-\frac{1}{2}x^{2}+\frac{1}{8}x^{4}-\frac{1}{48}x^{6}\right) z(0)+\left(x-\frac{1}{3}x^{3}+\frac{1}{15}x^{5}\right) z'(0) + \frac{x^{2}}{2}+\frac{x^{3}}{3}-\frac{x^{4}}{24}-\frac{x^{5}}{15}+\frac{x^{6}}{144}+O\left(x^{6}\right) \tag{1}$$

$$z = \left(1-\frac{1}{2}x^{2}+\frac{1}{8}x^{4}\right) c_{1}+\left(x-\frac{1}{3}x^{3}+\frac{1}{15}x^{5}\right) c_{2}+\frac{x^{2}}{2}+\frac{x^{3}}{3}-\frac{x^{4}}{24}-\frac{x^{5}}{15}+O\left(x^{6}\right) \tag{2}$$

Verification of solutions

$$z = \left(1-\frac{1}{2}x^{2}+\frac{1}{8}x^{4}-\frac{1}{48}x^{6}\right) z(0)+\left(x-\frac{1}{3}x^{3}+\frac{1}{15}x^{5}\right) z'(0) + \frac{x^{2}}{2}+\frac{x^{3}}{3}-\frac{x^{4}}{24}-\frac{x^{5}}{15}+\frac{x^{6}}{144}+O\left(x^{6}\right)$$

Verified OK.

$$z = \left(1-\frac{1}{2}x^{2}+\frac{1}{8}x^{4}\right) c_{1}+\left(x-\frac{1}{3}x^{3}+\frac{1}{15}x^{5}\right) c_{2}+\frac{x^{2}}{2}+\frac{x^{3}}{3}-\frac{x^{4}}{24}-\frac{x^{5}}{15}+O\left(x^{6}\right)$$

Verified OK.

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
<- high order exact linear fully integrable successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 42

```
Order:=6;
dsolve(diff(z(x),x$2)+x*diff(z(x),x)+z(x)=x^2+2*x+1,z(x),type='series',x=0);
```

$$z(x) = \left(1-\frac{1}{2}x^2+\frac{1}{8}x^4\right) z(0)+\left(x-\frac{1}{3}x^3+\frac{1}{15}x^5\right) D(z)\,(0)+\frac{x^2}{2}+\frac{x^3}{3}-\frac{x^4}{24}-\frac{x^5}{15}+O\left(x^6\right)$$

✓ Solution by Mathematica

Time used: 0.014 (sec). Leaf size: 70

```
AsymptoticDSolveValue[z''[x]+x*z'[x]+z[x]==x^2+2*x+1,z[x],{x,0,5}]
```

$$z(x) \to -\frac{x^5}{15}-\frac{x^4}{24}+\frac{x^3}{3}+\frac{x^2}{2}+c_2\left(\frac{x^5}{15}-\frac{x^3}{3}+x\right)+c_1\left(\frac{x^4}{8}-\frac{x^2}{2}+1\right)$$

6.21 problem 24

Internal problem ID [5039]
Internal file name [`OUTPUT/4532_Sunday_June_05_2022_03_00_24_PM_51804296/index.tex`]

Book: Fundamentals of Differential Equations. By Nagle, Saff and Snider. 9th edition. Boston. Pearson 2018.
Section: Chapter 8, Series solutions of differential equations. Section 8.4. page 449
Problem number: 24.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second order series method. Ordinary point", "second order series method. Taylor series method"**

Maple gives the following as the ode type

```
[[_2nd_order, _linear, _nonhomogeneous]]
```

$$\boxed{y'' - 2xy' + 3y = x^2}$$

With the expansion point for the power series method at $x = 0$.

Solving ode using Taylor series method. This gives review on how the Taylor series method works for solving second order ode.

Let

$$y'' = f(x, y, y')$$

Assuming expansion is at $x_0 = 0$ (we can always shift the actual expansion point to 0 by change of variables) and assuming $f(x, y, y')$ is analytic at x_0 which must be the case for an ordinary point. Let initial conditions be $y(x_0) = y_0$ and $y'(x_0) = y'_0$. Using Taylor series gives

$$\begin{aligned} y(x) &= y(x_0) + (x - x_0)\, y'(x_0) + \frac{(x - x_0)^2}{2} y''(x_0) + \frac{(x - x_0)^3}{3!} y'''(x_0) + \cdots \\ &= y_0 + xy'_0 + \frac{x^2}{2} f|_{x_0,y_0,y'_0} + \frac{x^3}{3!} f'|_{x_0,y_0,y'_0} + \cdots \\ &= y_0 + xy'_0 + \sum_{n=0}^{\infty} \frac{x^{n+2}}{(n+2)!} \frac{d^n f}{dx^n}\bigg|_{x_0,y_0,y'_0} \end{aligned}$$

But

$$\begin{aligned}
\frac{df}{dx} &= \frac{\partial f}{\partial x}\frac{dx}{dx} + \frac{\partial f}{\partial y}\frac{dy}{dx} + \frac{\partial f}{\partial y'}\frac{dy'}{dx} && (1) \\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} y' + \frac{\partial f}{\partial y'} y'' && (306) \\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} y' + \frac{\partial f}{\partial y'} f && (307) \\
\frac{d^2 f}{dx^2} &= \frac{d}{dx}\left(\frac{df}{dx}\right) \\
&= \frac{\partial}{\partial x}\left(\frac{df}{dx}\right) + \frac{\partial}{\partial y}\left(\frac{df}{dx}\right) y' + \frac{\partial}{\partial y'}\left(\frac{df}{dx}\right) f && (2) \\
\frac{d^3 f}{dx^3} &= \frac{d}{dx}\left(\frac{d^2 f}{dx^2}\right) \\
&= \frac{\partial}{\partial x}\left(\frac{d^2 f}{dx^2}\right) + \left(\frac{\partial}{\partial y}\frac{d^2 f}{dx^2}\right) y' + \frac{\partial}{\partial y'}\left(\frac{d^2 f}{dx^2}\right) f && (3) \\
&\vdots
\end{aligned}$$

And so on. Hence if we name $F_0 = f(x,y,y')$ then the above can be written as

$$
\begin{aligned}
F_0 &= f(x,y,y') \qquad (4)\\
F_1 &= \frac{df}{dx}\\
&= \frac{dF_0}{dx}\\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}y' + \frac{\partial f}{\partial y'}y''\\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}y' + \frac{\partial f}{\partial y'}f \qquad (5)\\
&= \frac{\partial F_0}{\partial x} + \frac{\partial F_0}{\partial y}y' + \frac{\partial F_0}{\partial y'}F_0\\
F_2 &= \frac{d}{dx}\left(\frac{d}{dx}f\right)\\
&= \frac{d}{dx}(F_1)\\
&= \frac{\partial}{\partial x}F_1 + \left(\frac{\partial F_1}{\partial y}\right)y' + \left(\frac{\partial F_1}{\partial y'}\right)y''\\
&= \frac{\partial}{\partial x}F_1 + \left(\frac{\partial F_1}{\partial y}\right)y' + \left(\frac{\partial F_1}{\partial y'}\right)F_0\\
&\vdots\\
F_n &= \frac{d}{dx}(F_{n-1})\\
&= \frac{\partial}{\partial x}F_{n-1} + \left(\frac{\partial F_{n-1}}{\partial y}\right)y' + \left(\frac{\partial F_{n-1}}{\partial y'}\right)y''\\
&= \frac{\partial}{\partial x}F_{n-1} + \left(\frac{\partial F_{n-1}}{\partial y}\right)y' + \left(\frac{\partial F_{n-1}}{\partial y'}\right)F_0 \qquad (6)
\end{aligned}
$$

Therefore (6) can be used from now on along with

$$
y(x) = y_0 + xy_0' + \sum_{n=0}^{\infty} \frac{x^{n+2}}{(n+2)\,!} F_n|_{x_0,y_0,y_0'} \tag{7}
$$

To find $y(x)$ series solution around $x = 0$. Hence

$$
\begin{aligned}
F_0 &= 2xy' - 3y + x^2 \\
F_1 &= \frac{dF_0}{dx} \\
&= \frac{\partial F_0}{\partial x} + \frac{\partial F_0}{\partial y} y' + \frac{\partial F_0}{\partial y'} F_0 \\
&= 4x^2 y' + 2x^3 - 6xy - y' + 2x \\
F_2 &= \frac{dF_1}{dx} \\
&= \frac{\partial F_1}{\partial x} + \frac{\partial F_1}{\partial y} y' + \frac{\partial F_1}{\partial y'} F_1 \\
&= 8y'x^3 + 4x^4 - 12yx^2 + 5x^2 - 3y + 2 \\
F_3 &= \frac{dF_2}{dx} \\
&= \frac{\partial F_2}{\partial x} + \frac{\partial F_2}{\partial y} y' + \frac{\partial F_2}{\partial y'} F_2 \\
&= \left(16x^4 + 12x^2 - 3\right) y' + \left(-24x^3 - 24x\right) y + 8x^5 + 16x^3 + 10x \\
F_4 &= \frac{dF_3}{dx} \\
&= \frac{\partial F_3}{\partial x} + \frac{\partial F_3}{\partial y} y' + \frac{\partial F_3}{\partial y'} F_3 \\
&= \left(32x^5 + 64x^3 - 6x\right) y' + \left(-48x^4 - 108x^2 - 15\right) y + 16x^6 + 52x^4 + 45x^2 + 10
\end{aligned}
$$

And so on. Evaluating all the above at initial conditions $x = 0$ and $y(0) = y(0)$ and $y'(0) = y'(0)$ gives

$$
\begin{aligned}
F_0 &= -3y(0) \\
F_1 &= -y'(0) \\
F_2 &= -3y(0) + 2 \\
F_3 &= -3y'(0) \\
F_4 &= 10 - 15y(0)
\end{aligned}
$$

Substituting all the above in (7) and simplifying gives the solution as

$$
y = \left(1 - \frac{3}{2}x^2 - \frac{1}{8}x^4 - \frac{1}{48}x^6\right) y(0) + \left(x - \frac{1}{6}x^3 - \frac{1}{40}x^5\right) y'(0) + \frac{x^4}{12} + \frac{x^6}{72} + O\left(x^6\right)
$$

Since the expansion point $x = 0$ is an ordinary, we can also solve this using standard power series Let the solution be represented as power series of the form

$$y = \sum_{n=0}^{\infty} a_n x^n$$

Then

$$y' = \sum_{n=1}^{\infty} n a_n x^{n-1}$$
$$y'' = \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2}$$

Substituting the above back into the ode gives

$$\sum_{n=2}^{\infty} n(n-1) a_n x^{n-2} = 2x\left(\sum_{n=1}^{\infty} n a_n x^{n-1}\right) - 3\left(\sum_{n=0}^{\infty} a_n x^n\right) + x^2 \tag{1}$$

Expanding x^2 as Taylor series around $x = 0$ and keeping only the first 6 terms gives

$$\begin{aligned} x^2 &= x^2 + \dots \\ &= x^2 \end{aligned}$$

Hence the ODE in Eq (1) becomes

$$\left(\sum_{n=2}^{\infty} n(n-1) a_n x^{n-2}\right) - 2x\left(\sum_{n=1}^{\infty} n a_n x^{n-1}\right) + 3\left(\sum_{n=0}^{\infty} a_n x^n\right) = x^2$$

Which simplifies to

$$\left(\sum_{n=2}^{\infty} n(n-1) a_n x^{n-2}\right) + \sum_{n=1}^{\infty} \left(-2n\, x^n a_n\right) + \left(\sum_{n=0}^{\infty} 3a_n x^n\right) = x^2 \tag{2}$$

The next step is to make all powers of x be n in each summation term. Going over each summation term above with power of x in it which is not already x^n and adjusting the power and the corresponding index gives

$$\sum_{n=2}^{\infty} n(n-1) a_n x^{n-2} = \sum_{n=0}^{\infty} (n+2) a_{n+2} (n+1) x^n$$

Substituting all the above in Eq (2) gives the following equation where now all powers of x are the same and equal to n.

$$\left(\sum_{n=0}^{\infty} (n+2)\, a_{n+2}(n+1)\, x^n\right) + \sum_{n=1}^{\infty} (-2n\, x^n a_n) + \left(\sum_{n=0}^{\infty} 3a_n x^n\right) = x^2 \tag{3}$$

$n = 0$ gives

$$2a_2 + 3a_0 = 0$$

$$a_2 = -\frac{3a_0}{2}$$

For $1 \leq n$, the recurrence equation is

$$((n+2)\, a_{n+2}(n+1) - 2na_n + 3a_n)\, x^n = x^2 \tag{4}$$

For $n = 1$ the recurrence equation gives

$$(6a_3 + a_1)\, x = 0$$
$$6a_3 + a_1 = 0$$

Which after substituting the earlier terms found becomes

$$a_3 = -\frac{a_1}{6}$$

For $n = 2$ the recurrence equation gives

$$(12a_4 - a_2)\, x^2 = x^2$$
$$12a_4 - a_2 = 1$$

Which after substituting the earlier terms found becomes

$$a_4 = \frac{1}{12} - \frac{a_0}{8}$$

For $n = 3$ the recurrence equation gives

$$(20a_5 - 3a_3)\, x^3 = 0$$
$$20a_5 - 3a_3 = 0$$

Which after substituting the earlier terms found becomes

$$a_5 = -\frac{a_1}{40}$$

For $n = 4$ the recurrence equation gives

$$\begin{aligned}(30a_6 - 5a_4)\, x^4 &= 0\\ 30a_6 - 5a_4 &= 0\end{aligned}$$

Which after substituting the earlier terms found becomes

$$a_6 = \frac{1}{72} - \frac{a_0}{48}$$

For $n = 5$ the recurrence equation gives

$$\begin{aligned}(42a_7 - 7a_5)\, x^5 &= 0\\ 42a_7 - 7a_5 &= 0\end{aligned}$$

Which after substituting the earlier terms found becomes

$$a_7 = -\frac{a_1}{240}$$

And so on. Therefore the solution is

$$\begin{aligned}y &= \sum_{n=0}^{\infty} a_n x^n\\ &= a_3x^3 + a_2x^2 + a_1x + a_0 + \dots\end{aligned}$$

Substituting the values for a_n found above, the solution becomes

$$y = a_0 + a_1x - \frac{3a_0x^2}{2} - \frac{a_1x^3}{6} + \left(\frac{1}{12} - \frac{a_0}{8}\right)x^4 - \frac{a_1x^5}{40} + \dots$$

Collecting terms, the solution becomes

$$y = \left(1 - \frac{3}{2}x^2 - \frac{1}{8}x^4\right)a_0 + \left(x - \frac{1}{6}x^3 - \frac{1}{40}x^5\right)a_1 + \frac{x^4}{12} + O(x^6) \tag{3}$$

At $x = 0$ the solution above becomes

$$y = \left(1 - \frac{3}{2}x^2 - \frac{1}{8}x^4\right) c_1 + \left(x - \frac{1}{6}x^3 - \frac{1}{40}x^5\right) c_2 + \frac{x^4}{12} + O(x^6)$$

Summary

The solution(s) found are the following

$$y = \left(1 - \frac{3}{2}x^2 - \frac{1}{8}x^4 - \frac{1}{48}x^6\right) y(0) + \left(x - \frac{1}{6}x^3 - \frac{1}{40}x^5\right) y'(0) + \frac{x^4}{12} + \frac{x^6}{72} + O(x^6) \tag{1}$$

$$y = \left(1 - \frac{3}{2}x^2 - \frac{1}{8}x^4\right) c_1 + \left(x - \frac{1}{6}x^3 - \frac{1}{40}x^5\right) c_2 + \frac{x^4}{12} + O(x^6) \tag{2}$$

Verification of solutions

$$y = \left(1 - \frac{3}{2}x^2 - \frac{1}{8}x^4 - \frac{1}{48}x^6\right) y(0) + \left(x - \frac{1}{6}x^3 - \frac{1}{40}x^5\right) y'(0) + \frac{x^4}{12} + \frac{x^6}{72} + O(x^6)$$

Verified OK.

$$y = \left(1 - \frac{3}{2}x^2 - \frac{1}{8}x^4\right) c_1 + \left(x - \frac{1}{6}x^3 - \frac{1}{40}x^5\right) c_2 + \frac{x^4}{12} + O(x^6)$$

Verified OK.

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 2; linear nonhomogeneous with symmetry [0,1]
trying a double symmetry of the form [xi=0, eta=F(x)]
-> Try solving first the homogeneous part of the ODE
   checking if the LODE has constant coefficients
   checking if the LODE is of Euler type
   trying a symmetry of the form [xi=0, eta=F(x)]
   checking if the LODE is missing y
   -> Trying a Liouvillian solution using Kovacics algorithm
   <- No Liouvillian solutions exists
   -> Trying a solution in terms of special functions:
      -> Bessel
      -> elliptic
      -> Legendre
      <- Kummer successful
   <- special function solution successful
<- solving first the homogeneous part of the ODE successful`
```

✓ Solution by Maple

Time used: 0.015 (sec). Leaf size: 36

```
Order:=6;
dsolve(diff(y(x),x$2)-2*x*diff(y(x),x)+3*y(x)=x^2,y(x),type='series',x=0);
```

$$y(x) = \left(1 - \frac{3}{2}x^2 - \frac{1}{8}x^4\right) y(0) + \left(x - \frac{1}{6}x^3 - \frac{1}{40}x^5\right) D(y)(0) + \frac{x^4}{12} + O(x^6)$$

✓ Solution by Mathematica

Time used: 0.004 (sec). Leaf size: 49

```
AsymptoticDSolveValue[y''[x]-2*x*y'[x]+3*y[x]==x^2,y[x],{x,0,5}]
```

$$y(x) \to \frac{x^4}{12} + c_2\left(-\frac{x^5}{40} - \frac{x^3}{6} + x\right) + c_1\left(-\frac{x^4}{8} - \frac{3x^2}{2} + 1\right)$$

6.22 problem 25

Internal problem ID [5040]
Internal file name [`OUTPUT/4533_Sunday_June_05_2022_03_00_25_PM_85413879/index.tex`]

Book: Fundamentals of Differential Equations. By Nagle, Saff and Snider. 9th edition. Boston. Pearson 2018.
Section: Chapter 8, Series solutions of differential equations. Section 8.4. page 449
Problem number: 25.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second order series method. Ordinary point", "second_order_change_of_variable_on_y_method_2", "second order series method. Taylor series method", "second_order_ode_non_constant_coeff_transformation_on_B"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

$$\boxed{(x^2+1)\, y''-xy'+y=\cos(x)}$$

With the expansion point for the power series method at $x=0$.

Solving ode using Taylor series method. This gives review on how the Taylor series method works for solving second order ode.

Let

$$y''=f(x,y,y')$$

Assuming expansion is at $x_0=0$ (we can always shift the actual expansion point to 0 by change of variables) and assuming $f(x,y,y')$ is analytic at x_0 which must be the case for an ordinary point. Let initial conditions be $y(x_0)=y_0$ and $y'(x_0)=y'_0$. Using Taylor series gives

$$\begin{aligned}
y(x) &= y(x_0)+(x-x_0)\, y'(x_0)+\frac{(x-x_0)^2}{2}y''(x_0)+\frac{(x-x_0)^3}{3!}y'''(x_0)+\cdots \\
&= y_0+xy'_0+\frac{x^2}{2}\, f|_{x_0,y_0,y'_0}+\frac{x^3}{3!}\, f'|_{x_0,y_0,y'_0}+\cdots \\
&= y_0+xy'_0+\sum_{n=0}^{\infty}\frac{x^{n+2}}{(n+2)\,!}\, \frac{d^n f}{dx^n}\bigg|_{x_0,y_0,y'_0}
\end{aligned}$$

But

$$\frac{df}{dx} = \frac{\partial f}{\partial x}\frac{dx}{dx} + \frac{\partial f}{\partial y}\frac{dy}{dx} + \frac{\partial f}{\partial y'}\frac{dy'}{dx} \tag{1}$$

$$= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}y' + \frac{\partial f}{\partial y'}y'' \tag{309}$$

$$= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}y' + \frac{\partial f}{\partial y'}f \tag{310}$$

$$\frac{d^2 f}{dx^2} = \frac{d}{dx}\left(\frac{df}{dx}\right)$$

$$= \frac{\partial}{\partial x}\left(\frac{df}{dx}\right) + \frac{\partial}{\partial y}\left(\frac{df}{dx}\right)y' + \frac{\partial}{\partial y'}\left(\frac{df}{dx}\right)f \tag{2}$$

$$\frac{d^3 f}{dx^3} = \frac{d}{dx}\left(\frac{d^2 f}{dx^2}\right)$$

$$= \frac{\partial}{\partial x}\left(\frac{d^2 f}{dx^2}\right) + \left(\frac{\partial}{\partial y}\frac{d^2 f}{dx^2}\right)y' + \frac{\partial}{\partial y'}\left(\frac{d^2 f}{dx^2}\right)f \tag{3}$$

$$\vdots$$

And so on. Hence if we name $F_0 = f(x, y, y')$ then the above can be written as

$$\begin{aligned}
F_0 &= f(x, y, y') \qquad (4)\\
F_1 &= \frac{df}{dx}\\
&= \frac{dF_0}{dx}\\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}y' + \frac{\partial f}{\partial y'}y''\\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}y' + \frac{\partial f}{\partial y'}f \qquad (5)\\
&= \frac{\partial F_0}{\partial x} + \frac{\partial F_0}{\partial y}y' + \frac{\partial F_0}{\partial y'}F_0\\
F_2 &= \frac{d}{dx}\left(\frac{d}{dx}f\right)\\
&= \frac{d}{dx}(F_1)\\
&= \frac{\partial}{\partial x}F_1 + \left(\frac{\partial F_1}{\partial y}\right)y' + \left(\frac{\partial F_1}{\partial y'}\right)y''\\
&= \frac{\partial}{\partial x}F_1 + \left(\frac{\partial F_1}{\partial y}\right)y' + \left(\frac{\partial F_1}{\partial y'}\right)F_0\\
&\vdots\\
F_n &= \frac{d}{dx}(F_{n-1})\\
&= \frac{\partial}{\partial x}F_{n-1} + \left(\frac{\partial F_{n-1}}{\partial y}\right)y' + \left(\frac{\partial F_{n-1}}{\partial y'}\right)y''\\
&= \frac{\partial}{\partial x}F_{n-1} + \left(\frac{\partial F_{n-1}}{\partial y}\right)y' + \left(\frac{\partial F_{n-1}}{\partial y'}\right)F_0 \qquad (6)
\end{aligned}$$

Therefore (6) can be used from now on along with

$$y(x) = y_0 + xy'_0 + \sum_{n=0}^{\infty} \frac{x^{n+2}}{(n+2)!} F_n|_{x_0,y_0,y'_0} \tag{7}$$

To find $y(x)$ series solution around $x=0$. Hence

$$
\begin{aligned}
F_0 &= \frac{xy'-y+\cos(x)}{x^2+1} \\
F_1 &= \frac{dF_0}{dx} \\
&= \frac{\partial F_0}{\partial x}+\frac{\partial F_0}{\partial y}y'+\frac{\partial F_0}{\partial y'}F_0 \\
&= \frac{-x^2y'+(-x^2-1)\sin(x)+x(y-\cos(x))}{(x^2+1)^2} \\
F_2 &= \frac{dF_1}{dx} \\
&= \frac{\partial F_1}{\partial x}+\frac{\partial F_1}{\partial y}y'+\frac{\partial F_1}{\partial y'}F_1 \\
&= \frac{(2x^3-x)\,y'-x^4\cos(x)+3\sin(x)\,x^3-2yx^2+3\sin(x)\,x+y-2\cos(x)}{(x^2+1)^3} \\
F_3 &= \frac{dF_2}{dx} \\
&= \frac{\partial F_2}{\partial x}+\frac{\partial F_2}{\partial y}y'+\frac{\partial F_2}{\partial y'}F_2 \\
&= \frac{(-6x^4+9x^2)\,y'+(x^6-8x^4-4x^2+5)\sin(x)+(5x^5+4x^3+14x)\cos(x)+(6x^3-9x)\,y}{(x^2+1)^4} \\
F_4 &= \frac{dF_3}{dx} \\
&= \frac{\partial F_3}{\partial x}+\frac{\partial F_3}{\partial y}y'+\frac{\partial F_3}{\partial y'}F_3 \\
&= \frac{(24x^5-72x^3+9x)\,y'+(x^8-22x^6-13x^4-76x^2+19)\cos(x)+(-7x^7+29x^5-26x^3-62x)\sin(}{(x^2+1)^5}
\end{aligned}
$$

And so on. Evaluating all the above at initial conditions $x=0$ and $y(0)=y(0)$ and $y'(0)=y'(0)$ gives

$$
\begin{aligned}
F_0 &= 1-y(0) \\
F_1 &= 0 \\
F_2 &= y(0)-2 \\
F_3 &= 0 \\
F_4 &= 19-9y(0)
\end{aligned}
$$

Substituting all the above in (7) and simplifying gives the solution as

$$
y=\left(1-\frac{1}{2}x^2+\frac{1}{24}x^4-\frac{1}{80}x^6\right)y(0)+xy'(0)+\frac{x^2}{2}-\frac{x^4}{12}+\frac{19x^6}{720}+O(x^6)
$$

Since the expansion point $x = 0$ is an ordinary, we can also solve this using standard power series The ode is normalized to be

$$(x^2+1)\, y'' - xy' + y = \cos(x)$$

Let the solution be represented as power series of the form

$$y = \sum_{n=0}^{\infty} a_n x^n$$

Then

$$\begin{aligned} y' &= \sum_{n=1}^{\infty} n a_n x^{n-1} \\ y'' &= \sum_{n=2}^{\infty} n(n-1)\, a_n x^{n-2} \end{aligned}$$

Substituting the above back into the ode gives

$$(x^2+1)\left(\sum_{n=2}^{\infty} n(n-1)\, a_n x^{n-2}\right) - x\left(\sum_{n=1}^{\infty} n a_n x^{n-1}\right) + \left(\sum_{n=0}^{\infty} a_n x^n\right) = \cos(x) \quad (1)$$

Expanding $\cos(x)$ as Taylor series around $x = 0$ and keeping only the first 6 terms gives

$$\begin{aligned} \cos(x) &= \frac{1}{24}x^4 - \frac{1}{2}x^2 + 1 + \dots \\ &= \frac{1}{24}x^4 - \frac{1}{2}x^2 + 1 \end{aligned}$$

Hence the ODE in Eq (1) becomes

$$(x^2+1)\left(\sum_{n=2}^{\infty} n(n-1)\, a_n x^{n-2}\right) - x\left(\sum_{n=1}^{\infty} n a_n x^{n-1}\right) + \left(\sum_{n=0}^{\infty} a_n x^n\right) = \frac{1}{24}x^4 - \frac{1}{2}x^2 + 1$$

Which simplifies to

$$\left(\sum_{n=2}^{\infty} x^n a_n n(n-1)\right) + \left(\sum_{n=2}^{\infty} n(n-1)\, a_n x^{n-2}\right) + \sum_{n=1}^{\infty} (-n a_n x^n) + \left(\sum_{n=0}^{\infty} a_n x^n\right) \quad (2)$$
$$= \frac{1}{24}x^4 - \frac{1}{2}x^2 + 1$$

The next step is to make all powers of x be n in each summation term. Going over each summation term above with power of x in it which is not already x^n and adjusting the power and the corresponding index gives

$$\sum_{n=2}^{\infty} n(n-1)\, a_n x^{n-2} = \sum_{n=0}^{\infty} (n+2)\, a_{n+2}(n+1)\, x^n$$

Substituting all the above in Eq (2) gives the following equation where now all powers of x are the same and equal to n.

$$\left(\sum_{n=2}^{\infty} x^n a_n n(n-1)\right) + \left(\sum_{n=0}^{\infty} (n+2)\, a_{n+2}(n+1)\, x^n\right) + \sum_{n=1}^{\infty} (-n a_n x^n) + \left(\sum_{n=0}^{\infty} a_n x^n\right) \tag{3}$$
$$= \frac{1}{24}x^4 - \frac{1}{2}x^2 + 1$$

$n = 0$ gives

$$(2a_2 + a_0)\, x^0 = 1$$
$$2a_2 + a_0 = 1$$

$$a_2 = -\frac{a_0}{2} + \frac{1}{2}$$

For $2 \leq n$, the recurrence equation is

$$(n a_n(n-1) + (n+2)\, a_{n+2}(n+1) - n a_n + a_n)\, x^n = \frac{1}{24}x^4 - \frac{1}{2}x^2 + 1 \tag{4}$$

For $n = 2$ the recurrence equation gives

$$(a_2 + 12a_4)\, x^2 = -\frac{x^2}{2}$$
$$a_2 + 12a_4 = -\frac{1}{2}$$

Which after substituting the earlier terms found becomes

$$a_4 = -\frac{1}{12} + \frac{a_0}{24}$$

For $n = 3$ the recurrence equation gives

$$\begin{aligned}(4a_3 + 20a_5)\,x^3 &= 0\\ 4a_3 + 20a_5 &= 0\end{aligned}$$

Which after substituting the earlier terms found becomes

$$a_5 = 0$$

For $n = 4$ the recurrence equation gives

$$\begin{aligned}(9a_4 + 30a_6)\,x^4 &= \frac{x^4}{24}\\ 9a_4 + 30a_6 &= \frac{1}{24}\end{aligned}$$

Which after substituting the earlier terms found becomes

$$a_6 = \frac{19}{720} - \frac{a_0}{80}$$

For $n = 5$ the recurrence equation gives

$$\begin{aligned}(16a_5 + 42a_7)\,x^5 &= 0\\ 16a_5 + 42a_7 &= 0\end{aligned}$$

Which after substituting the earlier terms found becomes

$$a_7 = 0$$

And so on. Therefore the solution is

$$\begin{aligned}y &= \sum_{n=0}^{\infty} a_n x^n\\ &= a_3x^3 + a_2x^2 + a_1x + a_0 + \dots\end{aligned}$$

Substituting the values for a_n found above, the solution becomes

$$y = a_0 + a_1x + \left(-\frac{a_0}{2} + \frac{1}{2}\right)x^2 + \left(-\frac{1}{12} + \frac{a_0}{24}\right)x^4 + \dots$$

Collecting terms, the solution becomes

$$y = \left(\frac{1}{24}x^4 - \frac{1}{2}x^2 + 1\right) a_0 + a_1 x + \frac{x^2}{2} - \frac{x^4}{12} + O\left(x^6\right) \tag{3}$$

At $x = 0$ the solution above becomes

$$y = \left(\frac{1}{24}x^4 - \frac{1}{2}x^2 + 1\right) c_1 + c_2 x + \frac{x^2}{2} - \frac{x^4}{12} + O\left(x^6\right)$$

<u>Summary</u>
The solution(s) found are the following

$$y = \left(1 - \frac{1}{2}x^2 + \frac{1}{24}x^4 - \frac{1}{80}x^6\right) y(0) + xy'(0) + \frac{x^2}{2} - \frac{x^4}{12} + \frac{19x^6}{720} + O\left(x^6\right) \tag{1}$$

$$y = \left(\frac{1}{24}x^4 - \frac{1}{2}x^2 + 1\right) c_1 + c_2 x + \frac{x^2}{2} - \frac{x^4}{12} + O\left(x^6\right) \tag{2}$$

<u>Verification of solutions</u>

$$y = \left(1 - \frac{1}{2}x^2 + \frac{1}{24}x^4 - \frac{1}{80}x^6\right) y(0) + xy'(0) + \frac{x^2}{2} - \frac{x^4}{12} + \frac{19x^6}{720} + O\left(x^6\right)$$

Verified OK.

$$y = \left(\frac{1}{24}x^4 - \frac{1}{2}x^2 + 1\right) c_1 + c_2 x + \frac{x^2}{2} - \frac{x^4}{12} + O\left(x^6\right)$$

Verified OK.

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 2; linear nonhomogeneous with symmetry [0,1]
trying a double symmetry of the form [xi=0, eta=F(x)]
-> Try solving first the homogeneous part of the ODE
   checking if the LODE has constant coefficients
   checking if the LODE is of Euler type
   trying a symmetry of the form [xi=0, eta=F(x)]
   checking if the LODE is missing y
   -> Trying a Liouvillian solution using Kovacics algorithm
      A Liouvillian solution exists
      Reducible group (found an exponential solution)
      Group is reducible, not completely reducible
   <- Kovacics algorithm successful
<- solving first the homogeneous part of the ODE successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 28

```
Order:=6;
dsolve((1+x^2)*diff(y(x),x$2)-x*diff(y(x),x)+y(x)=cos(x),y(x),type='series',x=0);
```

$$y(x) = \left(1 - \frac{1}{2}x^2 + \frac{1}{24}x^4\right) y(0) + D(y)(0)\, x + \frac{x^2}{2} - \frac{x^4}{12} + O(x^6)$$

✓ Solution by Mathematica

Time used: 0.023 (sec). Leaf size: 41

```
AsymptoticDSolveValue[(1+x^2)*y''[x]-x*y'[x]+y[x]==Cos[x],y[x],{x,0,5}]
```

$$y(x) \to -\frac{x^4}{12} + \frac{x^2}{2} + c_1\left(\frac{x^4}{24} - \frac{x^2}{2} + 1\right) + c_2 x$$

6.23 problem 26

Internal problem ID [5041]
Internal file name [`OUTPUT/4534_Sunday_June_05_2022_03_00_26_PM_61570481/index.tex`]

Book: Fundamentals of Differential Equations. By Nagle, Saff and Snider. 9th edition. Boston. Pearson 2018.
Section: Chapter 8, Series solutions of differential equations. Section 8.4. page 449
Problem number: 26.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second order series method. Ordinary point", "second order series method. Taylor series method"**

Maple gives the following as the ode type

```
[[_2nd_order, _linear, _nonhomogeneous]]
```

$$\boxed{y'' - xy' + 2y = \cos(x)}$$

With the expansion point for the power series method at $x = 0$.

Solving ode using Taylor series method. This gives review on how the Taylor series method works for solving second order ode.

Let

$$y'' = f(x, y, y')$$

Assuming expansion is at $x_0 = 0$ (we can always shift the actual expansion point to 0 by change of variables) and assuming $f(x, y, y')$ is analytic at x_0 which must be the case for an ordinary point. Let initial conditions be $y(x_0) = y_0$ and $y'(x_0) = y'_0$. Using Taylor series gives

$$\begin{aligned}
y(x) &= y(x_0) + (x - x_0)\, y'(x_0) + \frac{(x - x_0)^2}{2} y''(x_0) + \frac{(x - x_0)^3}{3!} y'''(x_0) + \cdots \\
&= y_0 + xy'_0 + \frac{x^2}{2} f|_{x_0,y_0,y'_0} + \frac{x^3}{3!} f'|_{x_0,y_0,y'_0} + \cdots \\
&= y_0 + xy'_0 + \sum_{n=0}^{\infty} \frac{x^{n+2}}{(n+2)!} \frac{d^n f}{dx^n}\bigg|_{x_0,y_0,y'_0}
\end{aligned}$$

But

$$
\begin{aligned}
\frac{df}{dx} &= \frac{\partial f}{\partial x}\frac{dx}{dx} + \frac{\partial f}{\partial y}\frac{dy}{dx} + \frac{\partial f}{\partial y'}\frac{dy'}{dx} && (1)\\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}y' + \frac{\partial f}{\partial y'}y'' && (312)\\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}y' + \frac{\partial f}{\partial y'}f && (313)\\
\frac{d^2 f}{dx^2} &= \frac{d}{dx}\left(\frac{df}{dx}\right)\\
&= \frac{\partial}{\partial x}\left(\frac{df}{dx}\right) + \frac{\partial}{\partial y}\left(\frac{df}{dx}\right)y' + \frac{\partial}{\partial y'}\left(\frac{df}{dx}\right)f && (2)\\
\frac{d^3 f}{dx^3} &= \frac{d}{dx}\left(\frac{d^2 f}{dx^2}\right)\\
&= \frac{\partial}{\partial x}\left(\frac{d^2 f}{dx^2}\right) + \left(\frac{\partial}{\partial y}\frac{d^2 f}{dx^2}\right)y' + \frac{\partial}{\partial y'}\left(\frac{d^2 f}{dx^2}\right)f && (3)\\
&\vdots
\end{aligned}
$$

And so on. Hence if we name $F_0 = f(x,y,y')$ then the above can be written as

$$\begin{aligned}
F_0 &= f(x,y,y') \qquad (4)\\
F_1 &= \frac{df}{dx}\\
&= \frac{dF_0}{dx}\\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}y' + \frac{\partial f}{\partial y'}y''\\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}y' + \frac{\partial f}{\partial y'}f \qquad (5)\\
&= \frac{\partial F_0}{\partial x} + \frac{\partial F_0}{\partial y}y' + \frac{\partial F_0}{\partial y'}F_0\\
F_2 &= \frac{d}{dx}\left(\frac{d}{dx}f\right)\\
&= \frac{d}{dx}(F_1)\\
&= \frac{\partial}{\partial x}F_1 + \left(\frac{\partial F_1}{\partial y}\right)y' + \left(\frac{\partial F_1}{\partial y'}\right)y''\\
&= \frac{\partial}{\partial x}F_1 + \left(\frac{\partial F_1}{\partial y}\right)y' + \left(\frac{\partial F_1}{\partial y'}\right)F_0\\
&\vdots\\
F_n &= \frac{d}{dx}(F_{n-1})\\
&= \frac{\partial}{\partial x}F_{n-1} + \left(\frac{\partial F_{n-1}}{\partial y}\right)y' + \left(\frac{\partial F_{n-1}}{\partial y'}\right)y''\\
&= \frac{\partial}{\partial x}F_{n-1} + \left(\frac{\partial F_{n-1}}{\partial y}\right)y' + \left(\frac{\partial F_{n-1}}{\partial y'}\right)F_0 \qquad (6)
\end{aligned}$$

Therefore (6) can be used from now on along with

$$y(x) = y_0 + xy_0' + \sum_{n=0}^{\infty}\frac{x^{n+2}}{(n+2)!}F_n|_{x_0,y_0,y_0'} \qquad (7)$$

To find $y(x)$ series solution around $x = 0$. Hence

$$
\begin{aligned}
F_0 &= xy' - 2y + \cos(x) \\
F_1 &= \frac{dF_0}{dx} \\
&= \frac{\partial F_0}{\partial x} + \frac{\partial F_0}{\partial y} y' + \frac{\partial F_0}{\partial y'} F_0 \\
&= \left(x^2 - 1\right) y' - 2xy + \cos(x)\, x - \sin(x) \\
F_2 &= \frac{dF_1}{dx} \\
&= \frac{\partial F_1}{\partial x} + \frac{\partial F_1}{\partial y} y' + \frac{\partial F_1}{\partial y'} F_1 \\
&= \left(x^3 - x\right) y' - 2yx^2 + \cos(x)\, x^2 - \sin(x)\, x - \cos(x) \\
F_3 &= \frac{dF_2}{dx} \\
&= \frac{\partial F_2}{\partial x} + \frac{\partial F_2}{\partial y} y' + \frac{\partial F_2}{\partial y'} F_2 \\
&= y'x^4 + \cos(x)\, x^3 - 2yx^3 - x^2 \sin(x) - 2xy - y' \\
F_4 &= \frac{dF_3}{dx} \\
&= \frac{\partial F_3}{\partial x} + \frac{\partial F_3}{\partial y} y' + \frac{\partial F_3}{\partial y'} F_3 \\
&= \left(x^5 + 2x^3 - 3x\right) y' + \left(x^4 + 2x^2 - 1\right) \cos(x) + \left(-x^3 - 2x\right) \sin(x) - 2yx^2 \left(x^2 + 3\right)
\end{aligned}
$$

And so on. Evaluating all the above at initial conditions $x = 0$ and $y(0) = y(0)$ and $y'(0) = y'(0)$ gives

$$
\begin{aligned}
F_0 &= 1 - 2y(0) \\
F_1 &= -y'(0) \\
F_2 &= -1 \\
F_3 &= -y'(0) \\
F_4 &= -1
\end{aligned}
$$

Substituting all the above in (7) and simplifying gives the solution as

$$
y = \left(-x^2 + 1\right) y(0) + \left(x - \frac{1}{6}x^3 - \frac{1}{120}x^5\right) y'(0) + \frac{x^2}{2} - \frac{x^4}{24} - \frac{x^6}{720} + O\left(x^6\right)
$$

Since the expansion point $x = 0$ is an ordinary, we can also solve this using standard power series Let the solution be represented as power series of the form

$$y = \sum_{n=0}^{\infty} a_n x^n$$

Then

$$y' = \sum_{n=1}^{\infty} n a_n x^{n-1}$$
$$y'' = \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2}$$

Substituting the above back into the ode gives

$$\sum_{n=2}^{\infty} n(n-1) a_n x^{n-2} = x\left(\sum_{n=1}^{\infty} n a_n x^{n-1}\right) - 2\left(\sum_{n=0}^{\infty} a_n x^n\right) + \cos(x) \tag{1}$$

Expanding $\cos(x)$ as Taylor series around $x = 0$ and keeping only the first 6 terms gives

$$\begin{aligned}\cos(x) &= \frac{1}{24}x^4 - \frac{1}{2}x^2 + 1 + \dots \\ &= \frac{1}{24}x^4 - \frac{1}{2}x^2 + 1\end{aligned}$$

Hence the ODE in Eq (1) becomes

$$\left(\sum_{n=2}^{\infty} n(n-1) a_n x^{n-2}\right) - x\left(\sum_{n=1}^{\infty} n a_n x^{n-1}\right) + 2\left(\sum_{n=0}^{\infty} a_n x^n\right) = \frac{1}{24}x^4 - \frac{1}{2}x^2 + 1$$

Which simplifies to

$$\left(\sum_{n=2}^{\infty} n(n-1) a_n x^{n-2}\right) + \sum_{n=1}^{\infty} (-n\, x^n a_n) + \left(\sum_{n=0}^{\infty} 2 a_n x^n\right) = \frac{1}{24}x^4 - \frac{1}{2}x^2 + 1 \quad (2)$$

The next step is to make all powers of x be n in each summation term. Going over each summation term above with power of x in it which is not already x^n and adjusting the power and the corresponding index gives

$$\sum_{n=2}^{\infty} n(n-1) a_n x^{n-2} = \sum_{n=0}^{\infty} (n+2) a_{n+2}(n+1) x^n$$

Substituting all the above in Eq (2) gives the following equation where now all powers of x are the same and equal to n.

$$\left(\sum_{n=0}^{\infty} (n+2)\, a_{n+2}(n+1)\, x^{n}\right) + \sum_{n\,=1}^{\infty} \left(-n\, x^{n} a_{n}\right) + \left(\sum_{n=0}^{\infty} 2 a_{n} x^{n}\right) = \frac{1}{24} x^{4} - \frac{1}{2} x^{2} + 1 \quad (3)$$

$n = 0$ gives

$$\left(2 a_{2} + 2 a_{0}\right) x^{0} = 1$$
$$2 a_{2} + 2 a_{0} = 1$$

$$a_{2} = -a_{0} + \frac{1}{2}$$

For $1 \leq n$, the recurrence equation is

$$\left(\left(n+2\right) a_{n+2}(n+1) - n a_{n} + 2 a_{n}\right) x^{n} = \frac{1}{24} x^{4} - \frac{1}{2} x^{2} + 1 \tag{4}$$

For $n = 1$ the recurrence equation gives

$$\left(6 a_{3} + a_{1}\right) x = 0$$
$$6 a_{3} + a_{1} = 0$$

Which after substituting the earlier terms found becomes

$$a_{3} = -\frac{a_{1}}{6}$$

For $n = 2$ the recurrence equation gives

$$\left(12 a_{4}\right) x^{2} = -\frac{x^{2}}{2}$$
$$12 a_{4} = -\frac{1}{2}$$

Which after substituting the earlier terms found becomes

$$a_{4} = -\frac{1}{24}$$

For $n = 3$ the recurrence equation gives

$$
\begin{aligned}
(20a_5 - a_3)\, x^3 &= 0 \\
20a_5 - a_3 &= 0
\end{aligned}
$$

Which after substituting the earlier terms found becomes

$$
a_5 = -\frac{a_1}{120}
$$

For $n = 4$ the recurrence equation gives

$$
\begin{aligned}
(30a_6 - 2a_4)\, x^4 &= \frac{x^4}{24} \\
30a_6 - 2a_4 &= \frac{1}{24}
\end{aligned}
$$

Which after substituting the earlier terms found becomes

$$
a_6 = -\frac{1}{720}
$$

For $n = 5$ the recurrence equation gives

$$
\begin{aligned}
(42a_7 - 3a_5)\, x^5 &= 0 \\
42a_7 - 3a_5 &= 0
\end{aligned}
$$

Which after substituting the earlier terms found becomes

$$
a_7 = -\frac{a_1}{1680}
$$

And so on. Therefore the solution is

$$
\begin{aligned}
y &= \sum_{n=0}^{\infty} a_n x^n \\
&= a_3 x^3 + a_2 x^2 + a_1 x + a_0 + \dots
\end{aligned}
$$

Substituting the values for a_n found above, the solution becomes

$$
y = a_0 + a_1 x + \left(-a_0 + \frac{1}{2}\right) x^2 - \frac{a_1 x^3}{6} - \frac{x^4}{24} - \frac{a_1 x^5}{120} + \dots
$$

Collecting terms, the solution becomes

$$y = (-x^2+1)\,a_0 + \left(x - \frac{1}{6}x^3 - \frac{1}{120}x^5\right)a_1 + \frac{x^2}{2} - \frac{x^4}{24} + O(x^6) \tag{3}$$

At $x = 0$ the solution above becomes

$$y = (-x^2+1)\,c_1 + \left(x - \frac{1}{6}x^3 - \frac{1}{120}x^5\right)c_2 + \frac{x^2}{2} - \frac{x^4}{24} + O(x^6)$$

Summary

The solution(s) found are the following

$$y = (-x^2+1)\,y(0) + \left(x - \frac{1}{6}x^3 - \frac{1}{120}x^5\right)y'(0) + \frac{x^2}{2} - \frac{x^4}{24} - \frac{x^6}{720} + O(x^6) \tag{1}$$

$$y = (-x^2+1)\,c_1 + \left(x - \frac{1}{6}x^3 - \frac{1}{120}x^5\right)c_2 + \frac{x^2}{2} - \frac{x^4}{24} + O(x^6) \tag{2}$$

Verification of solutions

$$y = (-x^2+1)\,y(0) + \left(x - \frac{1}{6}x^3 - \frac{1}{120}x^5\right)y'(0) + \frac{x^2}{2} - \frac{x^4}{24} - \frac{x^6}{720} + O(x^6)$$

Verified OK.

$$y = (-x^2+1)\,c_1 + \left(x - \frac{1}{6}x^3 - \frac{1}{120}x^5\right)c_2 + \frac{x^2}{2} - \frac{x^4}{24} + O(x^6)$$

Verified OK.

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 2; linear nonhomogeneous with symmetry [0,1]
trying a double symmetry of the form [xi=0, eta=F(x)]
-> Try solving first the homogeneous part of the ODE
   checking if the LODE has constant coefficients
   checking if the LODE is of Euler type
   trying a symmetry of the form [xi=0, eta=F(x)]
   checking if the LODE is missing y
   -> Trying a Liouvillian solution using Kovacics algorithm
      A Liouvillian solution exists
      Reducible group (found an exponential solution)
      Group is reducible, not completely reducible
      Solution has integrals. Trying a special function solution free of integrals...
      -> Trying a solution in terms of special functions:
         -> Bessel
         -> elliptic
         -> Legendre
         <- Kummer successful
      <- special function solution successful
         -> Trying to convert hypergeometric functions to elementary form...
         <- elementary form is not straightforward to achieve - returning special function s
      <- Kovacics algorithm successful
<- solving first the homogeneous part of the ODE successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 33

```
Order:=6;
dsolve(diff(y(x),x$2)-x*diff(y(x),x)+2*y(x)=cos(x),y(x),type='series',x=0);
```

$$y(x) = (-x^2+1)\,y(0) + \left(x - \frac{1}{6}x^3 - \frac{1}{120}x^5\right) D(y)\,(0) + \frac{x^2}{2} - \frac{x^4}{24} + O(x^6)$$

✓ Solution by Mathematica

Time used: 0.021 (sec). Leaf size: 47

```
AsymptoticDSolveValue[y''[x]-x*y'[x]+2*y[x]==Cos[x],y[x],{x,0,5}]
```

$$y(x) \to -\frac{x^4}{24} + \frac{x^2}{2} + c_1(1 - x^2) + c_2\left(-\frac{x^5}{120} - \frac{x^3}{6} + x\right)$$

6.24 problem 27

Internal problem ID [5042]
Internal file name [`OUTPUT/4535_Sunday_June_05_2022_03_00_28_PM_23284395/index.tex`]

Book: Fundamentals of Differential Equations. By Nagle, Saff and Snider. 9th edition. Boston. Pearson 2018.
Section: Chapter 8, Series solutions of differential equations. Section 8.4. page 449
Problem number: 27.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second order series method. Ordinary point", "second order series method. Taylor series method"**

Maple gives the following as the ode type

```
[[_2nd_order, _linear, _nonhomogeneous]]
```

$$\boxed{(-x^2+1)\,y''-y'+y=\tan(x)}$$

With the expansion point for the power series method at $x=0$.

Solving ode using Taylor series method. This gives review on how the Taylor series method works for solving second order ode.

Let

$$y''=f(x,y,y')$$

Assuming expansion is at $x_0=0$ (we can always shift the actual expansion point to 0 by change of variables) and assuming $f(x,y,y')$ is analytic at x_0 which must be the case for an ordinary point. Let initial conditions be $y(x_0)=y_0$ and $y'(x_0)=y'_0$. Using Taylor series gives

$$\begin{aligned}
y(x) &= y(x_0)+(x-x_0)\,y'(x_0)+\frac{(x-x_0)^2}{2}y''(x_0)+\frac{(x-x_0)^3}{3!}y'''(x_0)+\cdots \\
&= y_0+xy'_0+\frac{x^2}{2}\,f|_{x_0,y_0,y'_0}+\frac{x^3}{3!}\,f'|_{x_0,y_0,y'_0}+\cdots \\
&= y_0+xy'_0+\sum_{n=0}^{\infty}\frac{x^{n+2}}{(n+2)\,!}\,\frac{d^n f}{dx^n}\bigg|_{x_0,y_0,y'_0}
\end{aligned}$$

But

$$\frac{df}{dx} = \frac{\partial f}{\partial x}\frac{dx}{dx} + \frac{\partial f}{\partial y}\frac{dy}{dx} + \frac{\partial f}{\partial y'}\frac{dy'}{dx} \tag{1}$$

$$= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}y' + \frac{\partial f}{\partial y'}y'' \tag{315}$$

$$= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}y' + \frac{\partial f}{\partial y'}f \tag{316}$$

$$\frac{d^2 f}{dx^2} = \frac{d}{dx}\left(\frac{df}{dx}\right)$$

$$= \frac{\partial}{\partial x}\left(\frac{df}{dx}\right) + \frac{\partial}{\partial y}\left(\frac{df}{dx}\right)y' + \frac{\partial}{\partial y'}\left(\frac{df}{dx}\right)f \tag{2}$$

$$\frac{d^3 f}{dx^3} = \frac{d}{dx}\left(\frac{d^2 f}{dx^2}\right)$$

$$= \frac{\partial}{\partial x}\left(\frac{d^2 f}{dx^2}\right) + \left(\frac{\partial}{\partial y}\frac{d^2 f}{dx^2}\right)y' + \frac{\partial}{\partial y'}\left(\frac{d^2 f}{dx^2}\right)f \tag{3}$$

$$\vdots$$

And so on. Hence if we name $F_0 = f(x, y, y')$ then the above can be written as

$$
\begin{aligned}
F_0 &= f(x, y, y') \qquad (4)\\
F_1 &= \frac{df}{dx}\\
&= \frac{dF_0}{dx}\\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} y' + \frac{\partial f}{\partial y'} y''\\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} y' + \frac{\partial f}{\partial y'} f \qquad (5)\\
&= \frac{\partial F_0}{\partial x} + \frac{\partial F_0}{\partial y} y' + \frac{\partial F_0}{\partial y'} F_0\\
F_2 &= \frac{d}{dx}\left(\frac{d}{dx} f\right)\\
&= \frac{d}{dx}(F_1)\\
&= \frac{\partial}{\partial x} F_1 + \left(\frac{\partial F_1}{\partial y}\right) y' + \left(\frac{\partial F_1}{\partial y'}\right) y''\\
&= \frac{\partial}{\partial x} F_1 + \left(\frac{\partial F_1}{\partial y}\right) y' + \left(\frac{\partial F_1}{\partial y'}\right) F_0\\
&\vdots\\
F_n &= \frac{d}{dx}(F_{n-1})\\
&= \frac{\partial}{\partial x} F_{n-1} + \left(\frac{\partial F_{n-1}}{\partial y}\right) y' + \left(\frac{\partial F_{n-1}}{\partial y'}\right) y''\\
&= \frac{\partial}{\partial x} F_{n-1} + \left(\frac{\partial F_{n-1}}{\partial y}\right) y' + \left(\frac{\partial F_{n-1}}{\partial y'}\right) F_0 \qquad (6)
\end{aligned}
$$

Therefore (6) can be used from now on along with

$$
y(x) = y_0 + x y_0' + \sum_{n=0}^{\infty} \frac{x^{n+2}}{(n+2)!} F_n|_{x_0, y_0, y_0'} \qquad (7)
$$

To find $y(x)$ series solution around $x = 0$. Hence

$$
\begin{aligned}
F_0 &= -\frac{y' - y + \tan(x)}{x^2 - 1} \\
F_1 &= \frac{dF_0}{dx} \\
&= \frac{\partial F_0}{\partial x} + \frac{\partial F_0}{\partial y} y' + \frac{\partial F_0}{\partial y'} F_0 \\
&= \frac{(x^2 + 2x)\, y' + (-x^2 + 1)\sec(x)^2 - 2\left(\frac{1}{2} + x\right)(y - \tan(x))}{(x^2 - 1)^2} \\
F_2 &= \frac{dF_1}{dx} \\
&= \frac{\partial F_1}{\partial x} + \frac{\partial F_1}{\partial y} y' + \frac{\partial F_1}{\partial y'} F_1 \\
&= \frac{(-4x^3 - 8x^2 - 2x - 1)\, y' + ((-2x^4 + 4x^2 - 2)\tan(x) + 4x^3 + x^2 - 4x - 1)\sec(x)^2 + 7(y - \tan(x)}{(x^2 - 1)^3} \\
F_3 &= \frac{dF_2}{dx} \\
&= \frac{\partial F_2}{\partial x} + \frac{\partial F_2}{\partial y} y' + \frac{\partial F_2}{\partial y'} F_2 \\
&= \frac{(19x^4 + 42x^3 + 25x^2 + 18x + 1)\, y' - 6(x-1)^3 (x+1)^3 \sec(x)^4 + 4(x-1)(x+1)\left(\left(3x^3 + \frac{1}{2}x^2 - 3\right.\right.}{(x^2 - 1)^4} \\
F_4 &= \frac{dF_3}{dx} \\
&= \frac{\partial F_3}{\partial x} + \frac{\partial F_3}{\partial y} y' + \frac{\partial F_3}{\partial y'} F_3 \\
&= \frac{(-108x^5 - 267x^4 - 264x^3 - 246x^2 - 48x - 12)\, y' + \left(\left(\left(8x^8 - 106x^6\right)\cos(x)^2 - 24(x-1)^4 (x+1)^4\right.\right.}{}
\end{aligned}
$$

And so on. Evaluating all the above at initial conditions $x = 0$ and $y(0) = y(0)$ and $y'(0) = y'(0)$ gives

$$
\begin{aligned}
F_0 &= -y(0) + y'(0) \\
F_1 &= 1 - y(0) \\
F_2 &= 1 - 2y(0) + y'(0) \\
F_3 &= 8 + y'(0) - 7y(0) \\
F_4 &= 19 + 12y'(0) - 29y(0)
\end{aligned}
$$

Substituting all the above in (7) and simplifying gives the solution as

$$
\begin{aligned}
y = & \left(1-\frac{1}{2}x^2-\frac{1}{6}x^3-\frac{1}{12}x^4-\frac{7}{120}x^5-\frac{29}{720}x^6\right) y(0) \\
& + \left(x+\frac{1}{2}x^2+\frac{1}{24}x^4+\frac{1}{120}x^5+\frac{1}{60}x^6\right) y'(0)+\frac{x^3}{6}+\frac{x^4}{24}+\frac{x^5}{15}+\frac{19x^6}{720}+O(x^6)
\end{aligned}
$$

Since the expansion point $x = 0$ is an ordinary, we can also solve this using standard power series The ode is normalized to be

$$
(-x^2+1)\, y''-y'+y = \tan(x)
$$

Let the solution be represented as power series of the form

$$
y = \sum_{n=0}^{\infty} a_n x^n
$$

Then

$$
\begin{aligned}
y' &= \sum_{n=1}^{\infty} n a_n x^{n-1} \\
y'' &= \sum_{n=2}^{\infty} n(n-1)\, a_n x^{n-2}
\end{aligned}
$$

Substituting the above back into the ode gives

$$
(-x^2+1)\left(\sum_{n=2}^{\infty} n(n-1)\, a_n x^{n-2}\right)-\left(\sum_{n=1}^{\infty} n a_n x^{n-1}\right)+\left(\sum_{n=0}^{\infty} a_n x^n\right) = \tan(x) \qquad (1)
$$

Expanding $\tan(x)$ as Taylor series around $x = 0$ and keeping only the first 6 terms gives

$$
\begin{aligned}
\tan(x) &= \frac{1}{3}x^3+x+\frac{2}{15}x^5+\dots \\
&= \frac{1}{3}x^3+x+\frac{2}{15}x^5
\end{aligned}
$$

Hence the ODE in Eq (1) becomes

$$
(-x^2+1)\left(\sum_{n=2}^{\infty} n(n-1)\, a_n x^{n-2}\right)-\left(\sum_{n=1}^{\infty} n a_n x^{n-1}\right)+\left(\sum_{n=0}^{\infty} a_n x^n\right) = \frac{1}{3}x^3+x+\frac{2}{15}x^5
$$

Which simplifies to

$$\sum_{n=2}^{\infty}\left(-x^n a_n n(n-1)\right)+\left(\sum_{n=2}^{\infty} n(n-1)\, a_n x^{n-2}\right)+\sum_{n=1}^{\infty}\left(-n a_n x^{n-1}\right)+\left(\sum_{n=0}^{\infty} a_n x^n\right) = \frac{1}{3}x^3+x+\frac{2}{15}x^5 \tag{2}$$

The next step is to make all powers of x be n in each summation term. Going over each summation term above with power of x in it which is not already x^n and adjusting the power and the corresponding index gives

$$\sum_{n=2}^{\infty} n(n-1)\, a_n x^{n-2}=\sum_{n=0}^{\infty}(n+2)\, a_{n+2}(n+1)\, x^n$$
$$\sum_{n=1}^{\infty}\left(-n a_n x^{n-1}\right)=\sum_{n=0}^{\infty}\left(-(n+1)\, a_{n+1} x^n\right)$$

Substituting all the above in Eq (2) gives the following equation where now all powers of x are the same and equal to n.

$$\sum_{n=2}^{\infty}\left(-x^n a_n n(n-1)\right)+\left(\sum_{n=0}^{\infty}(n+2)\, a_{n+2}(n+1)\, x^n\right) + \sum_{n=0}^{\infty}\left(-(n+1)\, a_{n+1} x^n\right)+\left(\sum_{n=0}^{\infty} a_n x^n\right)=\frac{1}{3}x^3+x+\frac{2}{15}x^5 \tag{3}$$

$n=0$ gives

$$2a_2-a_1+a_0=0$$

$$a_2=-\frac{a_0}{2}+\frac{a_1}{2}$$

$n=1$ gives

$$(6a_3-2a_2+a_1)\, x=x$$
$$6a_3-2a_2+a_1=1$$

Which after substituting earlier equations, simplifies to

$$a_3 = -\frac{a_0}{6} + \frac{1}{6}$$

For $2 \leq n$, the recurrence equation is

$$\left(-n a_n (n-1) + (n+2)\, a_{n+2}(n+1) - (n+1)\, a_{n+1} + a_n\right) x^n = \frac{1}{3}x^3 + x + \frac{2}{15}x^5 \tag{4}$$

For $n = 2$ the recurrence equation gives

$$\left(-a_2 + 12a_4 - 3a_3\right) x^2 = 0$$
$$-a_2 + 12a_4 - 3a_3 = 0$$

Which after substituting the earlier terms found becomes

$$a_4 = \frac{1}{24} - \frac{a_0}{12} + \frac{a_1}{24}$$

For $n = 3$ the recurrence equation gives

$$\left(-5a_3 + 20a_5 - 4a_4\right) x^3 = \frac{x^3}{3}$$
$$-5a_3 + 20a_5 - 4a_4 = \frac{1}{3}$$

Which after substituting the earlier terms found becomes

$$a_5 = \frac{1}{15} - \frac{7a_0}{120} + \frac{a_1}{120}$$

For $n = 4$ the recurrence equation gives

$$\left(-11a_4 + 30a_6 - 5a_5\right) x^4 = 0$$
$$-11a_4 + 30a_6 - 5a_5 = 0$$

Which after substituting the earlier terms found becomes

$$a_6 = \frac{19}{720} - \frac{29a_0}{720} + \frac{a_1}{60}$$

For $n = 5$ the recurrence equation gives

$$
\begin{aligned}
(-19a_5 + 42a_7 - 6a_6)\, x^5 &= \frac{2x^5}{15} \\
-19a_5 + 42a_7 - 6a_6 &= \frac{2}{15}
\end{aligned}
$$

Which after substituting the earlier terms found becomes

$$
a_7 = \frac{187}{5040} - \frac{9a_0}{280} + \frac{31a_1}{5040}
$$

And so on. Therefore the solution is

$$
\begin{aligned}
y &= \sum_{n=0}^{\infty} a_n x^n \\
&= a_3 x^3 + a_2 x^2 + a_1 x + a_0 + \dots
\end{aligned}
$$

Substituting the values for a_n found above, the solution becomes

$$
\begin{aligned}
y = a_0 + a_1 x + \left(-\frac{a_0}{2} + \frac{a_1}{2}\right) x^2 + \left(-\frac{a_0}{6} + \frac{1}{6}\right) x^3 \\
+ \left(\frac{1}{24} - \frac{a_0}{12} + \frac{a_1}{24}\right) x^4 + \left(\frac{1}{15} - \frac{7a_0}{120} + \frac{a_1}{120}\right) x^5 + \dots
\end{aligned}
$$

Collecting terms, the solution becomes

$$
\begin{aligned}
y = \left(1 - \frac{1}{2}x^2 - \frac{1}{6}x^3 - \frac{1}{12}x^4 - \frac{7}{120}x^5\right) a_0 \\
+ \left(x + \frac{1}{2}x^2 + \frac{1}{24}x^4 + \frac{1}{120}x^5\right) a_1 + \frac{x^3}{6} + \frac{x^4}{24} + \frac{x^5}{15} + O(x^6)
\end{aligned}
\tag{3}
$$

At $x = 0$ the solution above becomes

$$
\begin{aligned}
y = \left(1 - \frac{1}{2}x^2 - \frac{1}{6}x^3 - \frac{1}{12}x^4 - \frac{7}{120}x^5\right) c_1 \\
+ \left(x + \frac{1}{2}x^2 + \frac{1}{24}x^4 + \frac{1}{120}x^5\right) c_2 + \frac{x^3}{6} + \frac{x^4}{24} + \frac{x^5}{15} + O(x^6)
\end{aligned}
$$

Summary

The solution(s) found are the following

$$
\begin{aligned}
y = &\left(1-\frac{1}{2}x^{2}-\frac{1}{6}x^{3}-\frac{1}{12}x^{4}-\frac{7}{120}x^{5}-\frac{29}{720}x^{6}\right) y(0) \\
&+\left(x+\frac{1}{2}x^{2}+\frac{1}{24}x^{4}+\frac{1}{120}x^{5}+\frac{1}{60}x^{6}\right) y'(0)+\frac{x^{3}}{6}+\frac{x^{4}}{24}+\frac{x^{5}}{15}+\frac{19x^{6}}{720}+O\left(x^{6}\right)
\end{aligned} \tag{1}
$$

$$
\begin{aligned}
y = &\left(1-\frac{1}{2}x^{2}-\frac{1}{6}x^{3}-\frac{1}{12}x^{4}-\frac{7}{120}x^{5}\right) c_{1} \\
&+\left(x+\frac{1}{2}x^{2}+\frac{1}{24}x^{4}+\frac{1}{120}x^{5}\right) c_{2}+\frac{x^{3}}{6}+\frac{x^{4}}{24}+\frac{x^{5}}{15}+O\left(x^{6}\right)
\end{aligned} \tag{2}
$$

Verification of solutions

$$
\begin{aligned}
y = &\left(1-\frac{1}{2}x^{2}-\frac{1}{6}x^{3}-\frac{1}{12}x^{4}-\frac{7}{120}x^{5}-\frac{29}{720}x^{6}\right) y(0) \\
&+\left(x+\frac{1}{2}x^{2}+\frac{1}{24}x^{4}+\frac{1}{120}x^{5}+\frac{1}{60}x^{6}\right) y'(0)+\frac{x^{3}}{6}+\frac{x^{4}}{24}+\frac{x^{5}}{15}+\frac{19x^{6}}{720}+O\left(x^{6}\right)
\end{aligned}
$$

Verified OK.

$$
\begin{aligned}
y = &\left(1-\frac{1}{2}x^{2}-\frac{1}{6}x^{3}-\frac{1}{12}x^{4}-\frac{7}{120}x^{5}\right) c_{1} \\
&+\left(x+\frac{1}{2}x^{2}+\frac{1}{24}x^{4}+\frac{1}{120}x^{5}\right) c_{2}+\frac{x^{3}}{6}+\frac{x^{4}}{24}+\frac{x^{5}}{15}+O\left(x^{6}\right)
\end{aligned}
$$

Verified OK.

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 2; linear nonhomogeneous with symmetry [0,1]
trying a double symmetry of the form [xi=0, eta=F(x)]
-> Try solving first the homogeneous part of the ODE
   checking if the LODE has constant coefficients
   checking if the LODE is of Euler type
   trying a symmetry of the form [xi=0, eta=F(x)]
   checking if the LODE is missing y
   -> Trying a Liouvillian solution using Kovacics algorithm
      A Liouvillian solution exists
      Group is reducible or imprimitive
   <- Kovacics algorithm successful
<- solving first the homogeneous part of the ODE successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 53

```
Order:=6;
dsolve((1-x^2)*diff(y(x),x$2)-diff(y(x),x)+y(x)=tan(x),y(x),type='series',x=0);
```

$$y(x) = \left(1 - \frac{1}{2}x^2 - \frac{1}{6}x^3 - \frac{1}{12}x^4 - \frac{7}{120}x^5\right) y(0) + \left(x + \frac{1}{2}x^2 + \frac{1}{24}x^4 + \frac{1}{120}x^5\right) D(y)(0) + \frac{x^3}{6} + \frac{x^4}{24} + \frac{x^5}{15} + O(x^6)$$

✓ Solution by Mathematica

Time used: 0.028 (sec). Leaf size: 197

```
AsymptoticDSolveValue[(1-x^2)*y''[x]-y'[x]+y[x]==Tan[x],y[x],{x,0,5}]
```

$$
\begin{aligned}
y(x) \to\ & c_2\left(\frac{x^6}{60}+\frac{x^5}{120}+\frac{x^4}{24}+\frac{x^2}{2}+x\right)+c_1\left(-\frac{7x^5}{120}-\frac{x^4}{12}-\frac{x^3}{6}-\frac{x^2}{2}+1\right) \\
& +\left(-\frac{7x^5}{120}-\frac{x^4}{12}-\frac{x^3}{6}-\frac{x^2}{2}+1\right)\left(\frac{7x^6}{48}-\frac{4x^5}{15}+\frac{x^4}{8}-\frac{x^3}{3}\right) \\
& +\left(\frac{x^6}{60}+\frac{x^5}{120}+\frac{x^4}{24}+\frac{x^2}{2}+x\right)\left(\frac{67x^6}{240}-\frac{3x^5}{10}+\frac{x^4}{3}-\frac{x^3}{3}+\frac{x^2}{2}\right)
\end{aligned}
$$

6.25 problem 28

Internal problem ID [5043]
Internal file name [`OUTPUT/4536_Sunday_June_05_2022_03_00_30_PM_53218504/index.tex`]

Book: Fundamentals of Differential Equations. By Nagle, Saff and Snider. 9th edition. Boston. Pearson 2018.
Section: Chapter 8, Series solutions of differential equations. Section 8.4. page 449
Problem number: 28.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second order series method. Ordinary point", "second order series method. Taylor series method"**

Maple gives the following as the ode type

```
[[_2nd_order, _linear, _nonhomogeneous]]
```

$$\boxed{y'' - \sin(x)\, y = \cos(x)}$$

With the expansion point for the power series method at $x = 0$.

Solving ode using Taylor series method. This gives review on how the Taylor series method works for solving second order ode.

Let

$$y'' = f(x, y, y')$$

Assuming expansion is at $x_0 = 0$ (we can always shift the actual expansion point to 0 by change of variables) and assuming $f(x, y, y')$ is analytic at x_0 which must be the case for an ordinary point. Let initial conditions be $y(x_0) = y_0$ and $y'(x_0) = y'_0$. Using Taylor series gives

$$\begin{aligned}
y(x) &= y(x_0) + (x - x_0)\, y'(x_0) + \frac{(x - x_0)^2}{2} y''(x_0) + \frac{(x - x_0)^3}{3!} y'''(x_0) + \cdots \\
&= y_0 + x y'_0 + \frac{x^2}{2} f|_{x_0, y_0, y'_0} + \frac{x^3}{3!} f'|_{x_0, y_0, y'_0} + \cdots \\
&= y_0 + x y'_0 + \sum_{n=0}^{\infty} \frac{x^{n+2}}{(n+2)!} \frac{d^n f}{dx^n}\bigg|_{x_0, y_0, y'_0}
\end{aligned}$$

But

$$\begin{aligned}
\frac{df}{dx} &= \frac{\partial f}{\partial x}\frac{dx}{dx} + \frac{\partial f}{\partial y}\frac{dy}{dx} + \frac{\partial f}{\partial y'}\frac{dy'}{dx} && (1)\\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}y' + \frac{\partial f}{\partial y'}y'' && (318)\\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}y' + \frac{\partial f}{\partial y'}f && (319)\\
\frac{d^2 f}{dx^2} &= \frac{d}{dx}\left(\frac{df}{dx}\right)\\
&= \frac{\partial}{\partial x}\left(\frac{df}{dx}\right) + \frac{\partial}{\partial y}\left(\frac{df}{dx}\right)y' + \frac{\partial}{\partial y'}\left(\frac{df}{dx}\right)f && (2)\\
\frac{d^3 f}{dx^3} &= \frac{d}{dx}\left(\frac{d^2 f}{dx^2}\right)\\
&= \frac{\partial}{\partial x}\left(\frac{d^2 f}{dx^2}\right) + \left(\frac{\partial}{\partial y}\frac{d^2 f}{dx^2}\right)y' + \frac{\partial}{\partial y'}\left(\frac{d^2 f}{dx^2}\right)f && (3)\\
&\vdots
\end{aligned}$$

And so on. Hence if we name $F_0 = f(x, y, y')$ then the above can be written as

$$
\begin{aligned}
F_0 &= f(x, y, y') \qquad (4)\\
F_1 &= \frac{df}{dx}\\
&= \frac{dF_0}{dx}\\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} y' + \frac{\partial f}{\partial y'} y''\\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} y' + \frac{\partial f}{\partial y'} f \qquad (5)\\
&= \frac{\partial F_0}{\partial x} + \frac{\partial F_0}{\partial y} y' + \frac{\partial F_0}{\partial y'} F_0\\
F_2 &= \frac{d}{dx}\left(\frac{d}{dx} f\right)\\
&= \frac{d}{dx}(F_1)\\
&= \frac{\partial}{\partial x} F_1 + \left(\frac{\partial F_1}{\partial y}\right) y' + \left(\frac{\partial F_1}{\partial y'}\right) y''\\
&= \frac{\partial}{\partial x} F_1 + \left(\frac{\partial F_1}{\partial y}\right) y' + \left(\frac{\partial F_1}{\partial y'}\right) F_0\\
&\vdots\\
F_n &= \frac{d}{dx}(F_{n-1})\\
&= \frac{\partial}{\partial x} F_{n-1} + \left(\frac{\partial F_{n-1}}{\partial y}\right) y' + \left(\frac{\partial F_{n-1}}{\partial y'}\right) y''\\
&= \frac{\partial}{\partial x} F_{n-1} + \left(\frac{\partial F_{n-1}}{\partial y}\right) y' + \left(\frac{\partial F_{n-1}}{\partial y'}\right) F_0 \qquad (6)
\end{aligned}
$$

Therefore (6) can be used from now on along with

$$
y(x) = y_0 + x y_0' + \sum_{n=0}^{\infty} \frac{x^{n+2}}{(n+2)!} F_n|_{x_0, y_0, y_0'} \qquad (7)
$$

To find $y(x)$ series solution around $x = 0$. Hence

$$
\begin{aligned}
F_0 &= \sin(x)\,y + \cos(x) \\
F_1 &= \frac{dF_0}{dx} \\
&= \frac{\partial F_0}{\partial x} + \frac{\partial F_0}{\partial y} y' + \frac{\partial F_0}{\partial y'} F_0 \\
&= \sin(x)\,y' + y\cos(x) - \sin(x) \\
F_2 &= \frac{dF_1}{dx} \\
&= \frac{\partial F_1}{\partial x} + \frac{\partial F_1}{\partial y} y' + \frac{\partial F_1}{\partial y'} F_1 \\
&= 2\cos(x)\,y' + (-1 + \sin(x))\,(\sin(x)\,y + \cos(x)) \\
F_3 &= \frac{dF_2}{dx} \\
&= \frac{\partial F_2}{\partial x} + \frac{\partial F_2}{\partial y} y' + \frac{\partial F_2}{\partial y'} F_2 \\
&= \sin(x)\,(\sin(x) - 3)\,y' + y(4\sin(x) - 1)\cos(x) + 4\cos(x)^2 + \sin(x) - 1 \\
F_4 &= \frac{dF_3}{dx} \\
&= \frac{\partial F_3}{\partial x} + \frac{\partial F_3}{\partial y} y' + \frac{\partial F_3}{\partial y'} F_3 \\
&= (6\sin(x) - 4)\cos(x)\,y' - \cos(x)^3 - y(\sin(x) - 11)\cos(x)^2 + (-11\sin(x) + 2)\cos(x) + 2\sin(x)\,y
\end{aligned}
$$

And so on. Evaluating all the above at initial conditions $x = 0$ and $y(0) = y(0)$ and $y'(0) = y'(0)$ gives

$$
\begin{aligned}
F_0 &= 1 \\
F_1 &= y(0) \\
F_2 &= -1 + 2y'(0) \\
F_3 &= 3 - y(0) \\
F_4 &= 1 + 4y(0) - 4y'(0)
\end{aligned}
$$

Substituting all the above in (7) and simplifying gives the solution as

$$
\begin{aligned}
y = {} & \left(1 + \frac{1}{6}x^3 - \frac{1}{120}x^5 + \frac{1}{180}x^6\right) y(0) \\
& + \left(x + \frac{1}{12}x^4 - \frac{1}{180}x^6\right) y'(0) + \frac{x^2}{2} - \frac{x^4}{24} + \frac{x^5}{40} + \frac{x^6}{720} + O(x^6)
\end{aligned}
$$

Since the expansion point $x = 0$ is an ordinary, we can also solve this using standard power series Let the solution be represented as power series of the form

$$y = \sum_{n=0}^{\infty} a_n x^n$$

Then

$$y' = \sum_{n=1}^{\infty} n a_n x^{n-1}$$
$$y'' = \sum_{n=2}^{\infty} n(n-1)\, a_n x^{n-2}$$

Substituting the above back into the ode gives

$$\sum_{n=2}^{\infty} n(n-1)\, a_n x^{n-2} = \sin(x) \left(\sum_{n=0}^{\infty} a_n x^n\right) + \cos(x) \tag{1}$$

Expanding $-\sin(x)$ as Taylor series around $x = 0$ and keeping only the first 6 terms gives

$$\begin{aligned} -\sin(x) &= -x + \frac{1}{6}x^3 - \frac{1}{120}x^5 + \frac{1}{5040}x^7 + \dots \\ &= -x + \frac{1}{6}x^3 - \frac{1}{120}x^5 + \frac{1}{5040}x^7 \end{aligned}$$

Expanding $\cos(x)$ as Taylor series around $x = 0$ and keeping only the first 6 terms gives

$$\begin{aligned} \cos(x) &= \frac{1}{24}x^4 - \frac{1}{2}x^2 + 1 + \dots \\ &= \frac{1}{24}x^4 - \frac{1}{2}x^2 + 1 \end{aligned}$$

Hence the ODE in Eq (1) becomes

$$\left(\sum_{n=2}^{\infty} n(n-1)\, a_n x^{n-2}\right) + \left(-x + \frac{1}{6}x^3 - \frac{1}{120}x^5 + \frac{1}{5040}x^7\right) \left(\sum_{n=0}^{\infty} a_n x^n\right) = \frac{1}{24}x^4 - \frac{1}{2}x^2 + 1$$

Expanding the second term in (1) gives

$$\begin{aligned} &\left(\sum_{n=2}^{\infty} n(n-1)\, a_n x^{n-2}\right) + -x \cdot \left(\sum_{n=0}^{\infty} a_n x^n\right) + \frac{x^3}{6} \cdot \left(\sum_{n=0}^{\infty} a_n x^n\right) \\ &- \frac{x^5}{120} \cdot \left(\sum_{n=0}^{\infty} a_n x^n\right) + \frac{x^7}{5040} \cdot \left(\sum_{n=0}^{\infty} a_n x^n\right) = \frac{1}{24}x^4 - \frac{1}{2}x^2 + 1 \end{aligned}$$

Which simplifies to

$$\left(\sum_{n=2}^{\infty} n(n-1)\, a_n x^{n-2}\right) + \sum_{n=0}^{\infty} \left(-x^{1+n} a_n\right) + \left(\sum_{n=0}^{\infty} \frac{x^{n+3} a_n}{6}\right) + \sum_{n=0}^{\infty} \left(-\frac{x^{n+5} a_n}{120}\right) + \left(\sum_{n=0}^{\infty} \frac{x^{n+7} a_n}{5040}\right) = \frac{1}{24} x^4 - \frac{1}{2} x^2 + 1 \tag{2}$$

The next step is to make all powers of x be n in each summation term. Going over each summation term above with power of x in it which is not already x^n and adjusting the power and the corresponding index gives

$$\sum_{n=2}^{\infty} n(n-1)\, a_n x^{n-2} = \sum_{n=0}^{\infty} (n+2)\, a_{n+2}(1+n)\, x^n$$

$$\sum_{n=0}^{\infty} \left(-x^{1+n} a_n\right) = \sum_{n=1}^{\infty} \left(-a_{n-1} x^n\right)$$

$$\sum_{n=0}^{\infty} \frac{x^{n+3} a_n}{6} = \sum_{n=3}^{\infty} \frac{a_{n-3} x^n}{6}$$

$$\sum_{n=0}^{\infty} \left(-\frac{x^{n+5} a_n}{120}\right) = \sum_{n=5}^{\infty} \left(-\frac{a_{n-5} x^n}{120}\right)$$

$$\sum_{n=0}^{\infty} \frac{x^{n+7} a_n}{5040} = \sum_{n=7}^{\infty} \frac{a_{n-7} x^n}{5040}$$

Substituting all the above in Eq (2) gives the following equation where now all powers of x are the same and equal to n.

$$\left(\sum_{n=0}^{\infty} (n+2)\, a_{n+2}(1+n)\, x^n\right) + \sum_{n=1}^{\infty} \left(-a_{n-1} x^n\right) + \left(\sum_{n=3}^{\infty} \frac{a_{n-3} x^n}{6}\right) + \sum_{n=5}^{\infty} \left(-\frac{a_{n-5} x^n}{120}\right) + \left(\sum_{n=7}^{\infty} \frac{a_{n-7} x^n}{5040}\right) = \frac{1}{24} x^4 - \frac{1}{2} x^2 + 1 \tag{3}$$

$n = 0$ gives

$$(2a_2)\, 1 = 1$$
$$2a_2 = 1$$

Or

$$a_2 = \frac{1}{2}$$

$n = 1$ gives

$$6a_3 - a_0 = 0$$

Which after substituting earlier equations, simplifies to

$$a_3 = \frac{a_0}{6}$$

$n = 2$ gives

$$\begin{aligned}(12a_4 - a_1) x^2 &= -\frac{x^2}{2} \\ 12a_4 - a_1 &= -\frac{1}{2}\end{aligned}$$

Which after substituting earlier equations, simplifies to

$$a_4 = \frac{a_1}{12} - \frac{1}{24}$$

$n = 3$ gives

$$20a_5 - a_2 + \frac{a_0}{6} = 0$$

Which after substituting earlier equations, simplifies to

$$a_5 = -\frac{a_0}{120} + \frac{1}{40}$$

$n = 4$ gives

$$\begin{aligned}\left(30a_6 - a_3 + \frac{a_1}{6}\right) x^4 &= \frac{x^4}{24} \\ 30a_6 - a_3 + \frac{a_1}{6} &= \frac{1}{24}\end{aligned}$$

Which after substituting earlier equations, simplifies to

$$a_6 = \frac{a_0}{180} - \frac{a_1}{180} + \frac{1}{720}$$

$n = 5$ gives

$$42a_7 - a_4 + \frac{a_2}{6} - \frac{a_0}{120} = 0$$

Which after substituting earlier equations, simplifies to

$$a_7 = \frac{a_0}{5040} + \frac{a_1}{504} - \frac{1}{336}$$

For $7 \leq n$, the recurrence equation is

$$\left((n+2)\, a_{n+2}(1+n) - a_{n-1} + \frac{a_{n-3}}{6} - \frac{a_{n-5}}{120} + \frac{a_{n-7}}{5040}\right) x^n = \frac{1}{24}x^4 - \frac{1}{2}x^2 + 1 \tag{4}$$

And so on. Therefore the solution is

$$\begin{aligned} y &= \sum_{n=0}^{\infty} a_n x^n \\ &= a_3 x^3 + a_2 x^2 + a_1 x + a_0 + \dots \end{aligned}$$

Substituting the values for a_n found above, the solution becomes

$$y = a_0 + a_1 x + \frac{x^2}{2} + \frac{a_0 x^3}{6} + \left(\frac{a_1}{12} - \frac{1}{24}\right) x^4 + \left(-\frac{a_0}{120} + \frac{1}{40}\right) x^5 + \dots$$

Collecting terms, the solution becomes

$$y = \left(1 + \frac{1}{6}x^3 - \frac{1}{120}x^5\right) a_0 + \left(x + \frac{1}{12}x^4\right) a_1 + \frac{x^2}{2} - \frac{x^4}{24} + \frac{x^5}{40} + O(x^6) \tag{3}$$

At $x = 0$ the solution above becomes

$$y = \left(1 + \frac{1}{6}x^3 - \frac{1}{120}x^5\right) c_1 + \left(x + \frac{1}{12}x^4\right) c_2 + \frac{x^2}{2} - \frac{x^4}{24} + \frac{x^5}{40} + O(x^6)$$

Summary

The solution(s) found are the following

$$\begin{aligned} y = {} & \left(1 + \frac{1}{6}x^3 - \frac{1}{120}x^5 + \frac{1}{180}x^6\right) y(0) \\ & + \left(x + \frac{1}{12}x^4 - \frac{1}{180}x^6\right) y'(0) + \frac{x^2}{2} - \frac{x^4}{24} + \frac{x^5}{40} + \frac{x^6}{720} + O(x^6) \end{aligned} \tag{1}$$

$$y = \left(1 + \frac{1}{6}x^3 - \frac{1}{120}x^5\right) c_1 + \left(x + \frac{1}{12}x^4\right) c_2 + \frac{x^2}{2} - \frac{x^4}{24} + \frac{x^5}{40} + O(x^6) \tag{2}$$

Verification of solutions

$$
\begin{aligned}
y = &\left(1+\frac{1}{6}x^{3}-\frac{1}{120}x^{5}+\frac{1}{180}x^{6}\right) y(0) \\
&+ \left(x+\frac{1}{12}x^{4}-\frac{1}{180}x^{6}\right) y'(0)+\frac{x^{2}}{2}-\frac{x^{4}}{24}+\frac{x^{5}}{40}+\frac{x^{6}}{720}+O\left(x^{6}\right)
\end{aligned}
$$

Verified OK.

$$
y = \left(1+\frac{1}{6}x^{3}-\frac{1}{120}x^{5}\right) c_{1}+\left(x+\frac{1}{12}x^{4}\right) c_{2}+\frac{x^{2}}{2}-\frac{x^{4}}{24}+\frac{x^{5}}{40}+O\left(x^{6}\right)
$$

Verified OK.

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 2; linear nonhomogeneous with symmetry [0,1]
trying a double symmetry of the form [xi=0, eta=F(x)]
trying symmetries linear in x and y(x)
-> Try solving first the homogeneous part of the ODE
   trying a symmetry of the form [xi=0, eta=F(x)]
   checking if the LODE is missing y
   -> Heun: Equivalence to the GHE or one of its 4 confluent cases under a power @ Moebius
   -> trying a solution of the form r0(x) * Y + r1(x) * Y where Y = exp(int(r(x), dx)) * 2F1
   -> Trying changes of variables to rationalize or make the ODE simpler
      trying a symmetry of the form [xi=0, eta=F(x)]
      checking if the LODE is missing y
      -> Trying an equivalence, under non-integer power transformations,
         to LODEs admitting Liouvillian solutions.
         -> Trying a Liouvillian solution using Kovacics algorithm
         <- No Liouvillian solutions exists
      -> Trying a solution in terms of special functions:
         -> Bessel
         -> elliptic
         -> Legendre
         -> Whittaker
            -> hyper3: Equivalence to 1F1 under a power @ Moebius
         -> hypergeometric
            -> heuristic approach
            -> hyper3: Equivalence to 2F1, 1F1 or 0F1 under a power @ Moebius
         -> Mathieu
            -> Equivalence to the rational form of Mathieu ODE under a power @ Moebius
            Equivalence transformation and function parameters: {t = 1/2*t+1/2}, {kappa = 12
            <- Equivalence to the rational form of Mathieu ODE successful
         <- Mathieu successful
      <- special function solution successful
      Change of variables used:
         [x = arccos(t)]
      Linear ODE actually solved:
         -(-t^2+1)^(1/2)*u(t)-t*diff(u(t),t)+(-t^2+1)*diff(diff(u(t),t),t) = 0
   <- change of variables successful
<- solving first the homogeneous part of the ODE successful`
```

✓ Solution by Maple

Time used: 0.015 (sec). Leaf size: 35

```
Order:=6;
dsolve(diff(y(x),x$2)-sin(x)*y(x)=cos(x),y(x),type='series',x=0);
```

$$y(x) = \left(1+\frac{1}{6}x^3-\frac{1}{120}x^5\right)y(0)+\left(x+\frac{1}{12}x^4\right)D(y)(0)+\frac{x^2}{2}-\frac{x^4}{24}+\frac{x^5}{40}+O(x^6)$$

✓ Solution by Mathematica

Time used: 0.02 (sec). Leaf size: 56

```
AsymptoticDSolveValue[y''[x]-Sin[x]*y[x]==Cos[x],y[x],{x,0,5}]
```

$$y(x) \to \frac{x^5}{40}-\frac{x^4}{24}+c_2\left(\frac{x^4}{12}+x\right)+\frac{x^2}{2}+c_1\left(-\frac{x^5}{120}+\frac{x^3}{6}+1\right)$$

6.26 problem 29

Internal problem ID [5044]
Internal file name [`OUTPUT/4537_Sunday_June_05_2022_03_00_31_PM_51242300/index.tex`]

Book: Fundamentals of Differential Equations. By Nagle, Saff and Snider. 9th edition. Boston. Pearson 2018.
Section: Chapter 8, Series solutions of differential equations. Section 8.4. page 449
Problem number: 29.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second order series method. Ordinary point", "second order series method. Taylor series method"**

Maple gives the following as the ode type

`[_Gegenbauer]`

$$\boxed{\left(-x^{2}+1\right) y''-2 x y'+n (1+n) y=0}$$

With the expansion point for the power series method at $x = 0$.

Solving ode using Taylor series method. This gives review on how the Taylor series method works for solving second order ode.

Let

$$y'' = f(x, y, y')$$

Assuming expansion is at $x_0 = 0$ (we can always shift the actual expansion point to 0 by change of variables) and assuming $f(x, y, y')$ is analytic at x_0 which must be the case for an ordinary point. Let initial conditions be $y(x_0) = y_0$ and $y'(x_0) = y'_0$. Using Taylor series gives

$$\begin{aligned} y(x) &= y(x_0) + (x - x_0) y'(x_0) + \frac{(x - x_0)^2}{2} y''(x_0) + \frac{(x - x_0)^3}{3!} y'''(x_0) + \cdots \\ &= y_0 + x y'_0 + \frac{x^2}{2} f|_{x_0,y_0,y'_0} + \frac{x^3}{3!} f'|_{x_0,y_0,y'_0} + \cdots \\ &= y_0 + x y'_0 + \sum_{n=0}^{\infty} \frac{x^{n+2}}{(n+2)!} \left.\frac{d^n f}{dx^n}\right|_{x_0,y_0,y'_0} \end{aligned}$$

But

$$\frac{df}{dx} = \frac{\partial f}{\partial x}\frac{dx}{dx} + \frac{\partial f}{\partial y}\frac{dy}{dx} + \frac{\partial f}{\partial y'}\frac{dy'}{dx} \tag{1}$$

$$= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}y' + \frac{\partial f}{\partial y'}y'' \tag{321}$$

$$= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}y' + \frac{\partial f}{\partial y'}f \tag{322}$$

$$\frac{d^2 f}{dx^2} = \frac{d}{dx}\left(\frac{df}{dx}\right)$$

$$= \frac{\partial}{\partial x}\left(\frac{df}{dx}\right) + \frac{\partial}{\partial y}\left(\frac{df}{dx}\right)y' + \frac{\partial}{\partial y'}\left(\frac{df}{dx}\right)f \tag{2}$$

$$\frac{d^3 f}{dx^3} = \frac{d}{dx}\left(\frac{d^2 f}{dx^2}\right)$$

$$= \frac{\partial}{\partial x}\left(\frac{d^2 f}{dx^2}\right) + \left(\frac{\partial}{\partial y}\frac{d^2 f}{dx^2}\right)y' + \frac{\partial}{\partial y'}\left(\frac{d^2 f}{dx^2}\right)f \tag{3}$$

$$\vdots$$

And so on. Hence if we name $F_0 = f(x, y, y')$ then the above can be written as

$$
\begin{aligned}
F_0 &= f(x,y,y') \qquad (4)\\
F_1 &= \frac{df}{dx}\\
&= \frac{dF_0}{dx}\\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} y' + \frac{\partial f}{\partial y'} y''\\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} y' + \frac{\partial f}{\partial y'} f \qquad (5)\\
&= \frac{\partial F_0}{\partial x} + \frac{\partial F_0}{\partial y} y' + \frac{\partial F_0}{\partial y'} F_0\\
F_2 &= \frac{d}{dx}\left(\frac{d}{dx} f\right)\\
&= \frac{d}{dx}(F_1)\\
&= \frac{\partial}{\partial x} F_1 + \left(\frac{\partial F_1}{\partial y}\right) y' + \left(\frac{\partial F_1}{\partial y'}\right) y''\\
&= \frac{\partial}{\partial x} F_1 + \left(\frac{\partial F_1}{\partial y}\right) y' + \left(\frac{\partial F_1}{\partial y'}\right) F_0\\
&\vdots\\
F_n &= \frac{d}{dx}(F_{n-1})\\
&= \frac{\partial}{\partial x} F_{n-1} + \left(\frac{\partial F_{n-1}}{\partial y}\right) y' + \left(\frac{\partial F_{n-1}}{\partial y'}\right) y''\\
&= \frac{\partial}{\partial x} F_{n-1} + \left(\frac{\partial F_{n-1}}{\partial y}\right) y' + \left(\frac{\partial F_{n-1}}{\partial y'}\right) F_0 \qquad (6)
\end{aligned}
$$

Therefore (6) can be used from now on along with

$$
y(x) = y_0 + x y_0' + \sum_{n=0}^{\infty} \frac{x^{n+2}}{(n+2)!} F_n|_{x_0, y_0, y_0'} \qquad (7)
$$

To find $y(x)$ series solution around $x=0$. Hence

$$
\begin{aligned}
F_0 &= \frac{yn^2+ny-2xy'}{x^2-1} \\
F_1 &= \frac{dF_0}{dx} \\
&= \frac{\partial F_0}{\partial x}+\frac{\partial F_0}{\partial y}y'+\frac{\partial F_0}{\partial y'}F_0 \\
&= \frac{(n^2x^2+x^2n-n^2+6x^2-n+2)\,y'-4xny(1+n)}{(x^2-1)^2} \\
F_2 &= \frac{dF_1}{dx} \\
&= \frac{\partial F_1}{\partial x}+\frac{\partial F_1}{\partial y}y'+\frac{\partial F_1}{\partial y'}F_1 \\
&= \frac{(-8x((n^2+n+3)\,x^2-n^2-n+3)\,y'+y(1+n)\,((n^2+n+18)\,x^2-n^2-n+6)\,n)\,(x+1)\,(x-1)}{(x^2-1)^4} \\
F_3 &= \frac{dF_2}{dx} \\
&= \frac{\partial F_2}{\partial x}+\frac{\partial F_2}{\partial y}y'+\frac{\partial F_2}{\partial y'}F_2 \\
&= \frac{(x+1)\,(x-1)\,(((n^4+2n^3+59n^2+58n+120)\,x^4+(-2n^4-4n^3-46n^2-44n+240)\,x^2+n^4+2}{(x^2-1)^5} \\
F_4 &= \frac{dF_3}{dx} \\
&= \frac{\partial F_3}{\partial x}+\frac{\partial F_3}{\partial y}y'+\frac{\partial F_3}{\partial y'}F_3 \\
&= \frac{(x+1)^2\,(x-1)^2\,\big(-18x\big(\big(n^4+2n^3+\frac{77}{3}n^2+\frac{74}{3}n+40\big)\,x^4-2\big(n^2+n-\frac{20}{3}\big)\,(n^2+n+10)\,x^2+n^4+}{}
\end{aligned}
$$

And so on. Evaluating all the above at initial conditions $x=0$ and $y(0)=y(0)$ and $y'(0)=y'(0)$ gives

$$
\begin{aligned}
F_0 &= -y(0)\,(1+n)\,n \\
F_1 &= -y'(0)\,n^2-y'(0)\,n+2y'(0) \\
F_2 &= y(0)\,n^4+2y(0)\,n^3-5y(0)\,n^2-6y(0)\,n \\
F_3 &= y'(0)\,n^4+2y'(0)\,n^3-13y'(0)\,n^2-14y'(0)\,n+24y'(0) \\
F_4 &= -y(0)\,n^6-3y(0)\,n^5+23y(0)\,n^4+51y(0)\,n^3-94y(0)\,n^2-120y(0)\,n
\end{aligned}
$$

Substituting all the above in (7) and simplifying gives the solution as

$$y = \left(1-\frac{1}{2}x^2 n-\frac{1}{2}n^2 x^2+\frac{1}{24}n^4 x^4+\frac{1}{12}n^3 x^4-\frac{5}{24}n^2 x^4-\frac{1}{4}n\,x^4-\frac{1}{720}x^6 n^6-\frac{1}{240}x^6 n^5 + \frac{23}{720}x^6 n^4+\frac{17}{240}x^6 n^3-\frac{47}{360}x^6 n^2-\frac{1}{6}x^6 n\right) y(0) + \left(x-\frac{1}{6}n^2 x^3-\frac{1}{6}n\,x^3+\frac{1}{3}x^3+\frac{1}{120}x^5 n^4+\frac{1}{60}x^5 n^3-\frac{13}{120}x^5 n^2-\frac{7}{60}x^5 n+\frac{1}{5}x^5\right) y'(0) + O\left(x^6\right)$$

Since the expansion point $x = 0$ is an ordinary, we can also solve this using standard power series The ode is normalized to be

$$\left(-x^2+1\right) y'' - 2xy' + \left(n^2+n\right) y = 0$$

Let the solution be represented as power series of the form

$$y = \sum_{n=0}^{\infty} a_n x^n$$

Then

$$y' = \sum_{n=1}^{\infty} n a_n x^{n-1}$$
$$y'' = \sum_{n=2}^{\infty} n(n-1)\, a_n x^{n-2}$$

Substituting the above back into the ode gives

$$\left(-x^2+1\right)\left(\sum_{n=2}^{\infty} n(n-1)\, a_n x^{n-2}\right) - 2x\left(\sum_{n=1}^{\infty} n a_n x^{n-1}\right) + \left(n^2+n\right)\left(\sum_{n=0}^{\infty} a_n x^n\right) = 0 \tag{1}$$

Which simplifies to

$$\sum_{n=2}^{\infty}\left(-x^n a_n n(n-1)\right) + \left(\sum_{n=2}^{\infty} n(n-1)\, a_n x^{n-2}\right) + \sum_{n=1}^{\infty}\left(-2n a_n x^n\right) + \left(\sum_{n=0}^{\infty}\left(n^2+n\right) a_n x^n\right) = 0 \tag{2}$$

The next step is to make all powers of x be n in each summation term. Going over each summation term above with power of x in it which is not already x^n and adjusting the power and the corresponding index gives

$$\sum_{n=2}^{\infty} n(n-1)\, a_n x^{n-2} = \sum_{n=0}^{\infty} (n+2)\, a_{n+2}(n+1)\, x^n$$

Substituting all the above in Eq (2) gives the following equation where now all powers of x are the same and equal to n.

$$\begin{aligned}\sum_{n=2}^{\infty} (-x^n a_n n(n-1)) + \left(\sum_{n=0}^{\infty} (n+2)\, a_{n+2}(n+1)\, x^n\right) \\ + \sum_{n=1}^{\infty} (-2n a_n x^n) + \left(\sum_{n=0}^{\infty} (n^2+n)\, a_n x^n\right) = 0\end{aligned} \tag{3}$$

$n = 0$ gives

$$2a_2 + a_0 n(1+n) = 0$$

$$a_2 = -\frac{1}{2} a_0 n^2 - \frac{1}{2} a_0 n$$

$n = 1$ gives

$$6a_3 - 2a_1 + a_1 n(1+n) = 0$$

Which after substituting earlier equations, simplifies to

$$a_3 = -\frac{1}{6} a_1 n^2 - \frac{1}{6} a_1 n + \frac{1}{3} a_1$$

For $2 \leq n$, the recurrence equation is

$$-n a_n(n-1) + (n+2)\, a_{n+2}(n+1) - 2n a_n + a_n n(1+n) = 0 \tag{4}$$

Solving for a_{n+2}, gives

$$a_{n+2} = -\frac{a_n(n^2 - n^2 + n - n)}{(n+2)\,(n+1)} \tag{5}$$

For $n = 2$ the recurrence equation gives

$$-6a_2 + 12a_4 + a_2 n(1+n) = 0$$

Which after substituting the earlier terms found becomes

$$a_4 = -\frac{5}{24}a_0 n^2 - \frac{1}{4}a_0 n + \frac{1}{12}a_0 n^3 + \frac{1}{24}a_0 n^4$$

For $n = 3$ the recurrence equation gives

$$-12a_3 + 20a_5 + a_3 n(1+n) = 0$$

Which after substituting the earlier terms found becomes

$$a_5 = -\frac{13}{120}a_1 n^2 - \frac{7}{60}a_1 n + \frac{1}{5}a_1 + \frac{1}{60}a_1 n^3 + \frac{1}{120}a_1 n^4$$

For $n = 4$ the recurrence equation gives

$$-20a_4 + 30a_6 + a_4 n(1+n) = 0$$

Which after substituting the earlier terms found becomes

$$a_6 = -\frac{47}{360}a_0 n^2 - \frac{1}{6}a_0 n + \frac{17}{240}a_0 n^3 + \frac{23}{720}a_0 n^4 - \frac{1}{240}a_0 n^5 - \frac{1}{720}a_0 n^6$$

For $n = 5$ the recurrence equation gives

$$-30a_5 + 42a_7 + a_5 n(1+n) = 0$$

Which after substituting the earlier terms found becomes

$$a_7 = -\frac{5}{63}a_1 n^2 - \frac{37}{420}a_1 n + \frac{1}{7}a_1 + \frac{29}{1680}a_1 n^3 + \frac{41}{5040}a_1 n^4 - \frac{1}{1680}a_1 n^5 - \frac{1}{5040}a_1 n^6$$

And so on. Therefore the solution is

$$\begin{aligned} y &= \sum_{n=0}^{\infty} a_n x^n \\ &= a_3 x^3 + a_2 x^2 + a_1 x + a_0 + \dots \end{aligned}$$

Substituting the values for a_n found above, the solution becomes

$$
\begin{aligned}
y = a_0 + a_1 x &+ \left(-\frac{1}{2}a_0 n^2 - \frac{1}{2}a_0 n\right)x^2 + \left(-\frac{1}{6}a_1 n^2 - \frac{1}{6}a_1 n + \frac{1}{3}a_1\right)x^3 \\
&+ \left(-\frac{5}{24}a_0 n^2 - \frac{1}{4}a_0 n + \frac{1}{12}a_0 n^3 + \frac{1}{24}a_0 n^4\right)x^4 \\
&+ \left(-\frac{13}{120}a_1 n^2 - \frac{7}{60}a_1 n + \frac{1}{5}a_1 + \frac{1}{60}a_1 n^3 + \frac{1}{120}a_1 n^4\right)x^5 + \dots
\end{aligned}
$$

Collecting terms, the solution becomes

$$
\begin{aligned}
y = &\left(1 + \left(-\frac{1}{2}n^2 - \frac{1}{2}n\right)x^2 + \left(-\frac{5}{24}n^2 - \frac{1}{4}n + \frac{1}{12}n^3 + \frac{1}{24}n^4\right)x^4\right)a_0 \\
&+ \left(x + \left(-\frac{1}{6}n^2 - \frac{1}{6}n + \frac{1}{3}\right)x^3 + \left(-\frac{13}{120}n^2 - \frac{7}{60}n + \frac{1}{5} + \frac{1}{60}n^3 + \frac{1}{120}n^4\right)x^5\right)a_1 + O(x^6)
\end{aligned}
\tag{3}
$$

At $x = 0$ the solution above becomes

$$
\begin{aligned}
y = &\left(1 + \left(-\frac{1}{2}n^2 - \frac{1}{2}n\right)x^2 + \left(-\frac{5}{24}n^2 - \frac{1}{4}n + \frac{1}{12}n^3 + \frac{1}{24}n^4\right)x^4\right)c_1 \\
&+ \left(x + \left(-\frac{1}{6}n^2 - \frac{1}{6}n + \frac{1}{3}\right)x^3 + \left(-\frac{13}{120}n^2 - \frac{7}{60}n + \frac{1}{5} + \frac{1}{60}n^3 + \frac{1}{120}n^4\right)x^5\right)c_2 + O(x^6)
\end{aligned}
$$

<u>Summary</u>

The solution(s) found are the following

$$
\begin{aligned}
y = &\left(1 - \frac{1}{2}x^2 n - \frac{1}{2}n^2 x^2 + \frac{1}{24}n^4 x^4 + \frac{1}{12}n^3 x^4 - \frac{5}{24}n^2 x^4 - \frac{1}{4}n\,x^4 - \frac{1}{720}x^6 n^6 - \frac{1}{240}x^6 n^5 \right.\\
&\left. + \frac{23}{720}x^6 n^4 + \frac{17}{240}x^6 n^3 - \frac{47}{360}x^6 n^2 - \frac{1}{6}x^6 n\right) y(0) + \left(x - \frac{1}{6}n^2 x^3 - \frac{1}{6}n\,x^3 + \frac{1}{3}x^3 \right.\\
&\left. + \frac{1}{120}x^5 n^4 + \frac{1}{60}x^5 n^3 - \frac{13}{120}x^5 n^2 - \frac{7}{60}x^5 n + \frac{1}{5}x^5\right) y'(0) + O(x^6)
\end{aligned}
\tag{1}
$$

$$
\begin{aligned}
y = &\left(1 + \left(-\frac{1}{2}n^2 - \frac{1}{2}n\right)x^2 + \left(-\frac{5}{24}n^2 - \frac{1}{4}n + \frac{1}{12}n^3 + \frac{1}{24}n^4\right)x^4\right)c_1 \\
&+ \left(x + \left(-\frac{1}{6}n^2 - \frac{1}{6}n + \frac{1}{3}\right)x^3 + \left(-\frac{13}{120}n^2 - \frac{7}{60}n + \frac{1}{5} + \frac{1}{60}n^3 + \frac{1}{120}n^4\right)x^5\right)c_2 \\
&+ O(x^6)
\end{aligned}
\tag{2}
$$

Verification of solutions

$$y = \left(1 - \frac{1}{2}x^2 n - \frac{1}{2}n^2 x^2 + \frac{1}{24}n^4 x^4 + \frac{1}{12}n^3 x^4 - \frac{5}{24}n^2 x^4 - \frac{1}{4}n\,x^4 - \frac{1}{720}x^6 n^6 - \frac{1}{240}x^6 n^5 + \frac{23}{720}x^6 n^4 + \frac{17}{240}x^6 n^3 - \frac{47}{360}x^6 n^2 - \frac{1}{6}x^6 n\right) y(0) + \left(x - \frac{1}{6}n^2 x^3 - \frac{1}{6}n\,x^3 + \frac{1}{3}x^3 + \frac{1}{120}x^5 n^4 + \frac{1}{60}x^5 n^3 - \frac{13}{120}x^5 n^2 - \frac{7}{60}x^5 n + \frac{1}{5}x^5\right) y'(0) + O(x^6)$$

Verified OK.

$$y = \left(1 + \left(-\frac{1}{2}n^2 - \frac{1}{2}n\right) x^2 + \left(-\frac{5}{24}n^2 - \frac{1}{4}n + \frac{1}{12}n^3 + \frac{1}{24}n^4\right) x^4\right) c_1 + \left(x + \left(-\frac{1}{6}n^2 - \frac{1}{6}n + \frac{1}{3}\right) x^3 + \left(-\frac{13}{120}n^2 - \frac{7}{60}n + \frac{1}{5} + \frac{1}{60}n^3 + \frac{1}{120}n^4\right) x^5\right) c_2 + O(x^6)$$

Verified OK.

6.26.1 Maple step by step solution

Let's solve

$(-x^2 + 1)\, y'' - 2xy' + (n^2 + n)\, y = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y'' = \frac{n(1+n)y}{x^2-1} - \frac{2xy'}{x^2-1}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y'' + \frac{2xy'}{x^2-1} - \frac{n(1+n)y}{x^2-1} = 0$

□ Check to see if x_0 is a regular singular point

  ◦ Define functions

  $\left[P_2(x) = \frac{2x}{x^2-1}, P_3(x) = -\frac{n(1+n)}{x^2-1}\right]$

  ◦ $(x+1) \cdot P_2(x)$ is analytic at $x = -1$

  $\left((x+1) \cdot P_2(x)\right)\Big|_{x=-1} = 1$

  ◦ $(x+1)^2 \cdot P_3(x)$ is analytic at $x = -1$

$$\left((x+1)^2 \cdot P_3(x)\right)\Bigg|_{x=-1} = 0$$

- $x=-1$is a regular singular point

  Check to see if x_0 is a regular singular point

  $x_0 = -1$

- Multiply by denominators

  $(x^2-1)\, y''+2xy'-n(1+n)\, y=0$

- Change variables using $x = u-1$ so that the regular singular point is at $u = 0$

  $(u^2-2u)\left(\frac{d^2}{du^2}y(u)\right)+(2u-2)\left(\frac{d}{du}y(u)\right)+(-n^2-n)\, y(u)=0$

- Assume series solution for $y(u)$

  $y(u)=\sum_{k=0}^{\infty} a_k u^{k+r}$

☐ Rewrite ODE with series expansions

- Convert $u^m \cdot \left(\frac{d}{du}y(u)\right)$ to series expansion for $m = 0..1$

  $u^m \cdot \left(\frac{d}{du}y(u)\right)=\sum_{k=0}^{\infty} a_k(k+r)\, u^{k+r-1+m}$

- Shift index using $k->k+1-m$

  $u^m \cdot \left(\frac{d}{du}y(u)\right)=\sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\, u^{k+r}$

- Convert $u^m \cdot \left(\frac{d^2}{du^2}y(u)\right)$ to series expansion for $m = 1..2$

  $u^m \cdot \left(\frac{d^2}{du^2}y(u)\right)=\sum_{k=0}^{\infty} a_k(k+r)\,(k+r-1)\, u^{k+r-2+m}$

- Shift index using $k->k+2-m$

  $u^m \cdot \left(\frac{d^2}{du^2}y(u)\right)=\sum_{k=-2+m}^{\infty} a_{k+2-m}(k+2-m+r)\,(k+1-m+r)\, u^{k+r}$

  Rewrite ODE with series expansions

  $-2a_0r^2u^{-1+r}+\left(\sum_{k=0}^{\infty}\left(-2a_{k+1}(k+1+r)^2+a_k(r+1+n+k)\,(r-n+k)\right)u^{k+r}\right)=0$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $-2r^2=0$

- Values of r that satisfy the indicial equation

$r = 0$

- Each term in the series must be 0, giving the recursion relation

  $-2a_{k+1}(k+1)^2 + a_k(1+n+k)(-n+k) = 0$

- Recursion relation that defines series solution to ODE

  $a_{k+1} = \frac{a_k(1+n+k)(-n+k)}{2(k+1)^2}$

- Recursion relation for $r = 0$

  $a_{k+1} = \frac{a_k(1+n+k)(-n+k)}{2(k+1)^2}$

- Solution for $r = 0$

  $\left[y(u) = \sum_{k=0}^{\infty} a_k u^k, a_{k+1} = \frac{a_k(1+n+k)(-n+k)}{2(k+1)^2}\right]$

- Revert the change of variables $u = x+1$

  $\left[y = \sum_{k=0}^{\infty} a_k (x+1)^k, a_{k+1} = \frac{a_k(1+n+k)(-n+k)}{2(k+1)^2}\right]$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
<- No Liouvillian solutions exists
-> Trying a solution in terms of special functions:
   -> Bessel
   -> elliptic
   -> Legendre
   <- Legendre successful
<- special function solution successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 101

```
Order:=6;
dsolve((1-x^2)*diff(y(x),x$2)-2*x*diff(y(x),x)+n*(n+1)*y(x)=0,y(x),type='series',x=0);
```

$$y(x) = \left(1 - \frac{n(n+1)\,x^2}{2} + \frac{n(n^3+2n^2-5n-6)\,x^4}{24}\right) y(0) + \left(x - \frac{(n^2+n-2)\,x^3}{6} + \frac{(n^4+2n^3-13n^2-14n+24)\,x^5}{120}\right) D(y)\,(0) + O(x^6)$$

✓ Solution by Mathematica

Time used: 0.002 (sec). Leaf size: 120

```
AsymptoticDSolveValue[(1-x^2)*y''[x]-2*x*y'[x]+n*(n+1)*y[x]==0,y[x],{x,0,5}]
```

$$y(x) \to c_2\left(\frac{1}{120}(n^2+n)^2 x^5 + \frac{7}{60}(-n^2-n)\,x^5 + \frac{1}{6}(-n^2-n)\,x^3 + \frac{x^5}{5} + \frac{x^3}{3} + x\right) + c_1\left(\frac{1}{24}(n^2+n)^2 x^4 + \frac{1}{4}(-n^2-n)\,x^4 + \frac{1}{2}(-n^2-n)\,x^2 + 1\right)$$