A Solution Manual For

First order enumerated odes

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Contents

1 section 1

1.1 problem 1

Internal problem ID [7317]
Internal file name [`OUTPUT/6298_Sunday_June_05_2022_04_39_08_PM_28030563/index.tex`]

Book: First order enumerated odes
Section: section 1
Problem number: 1.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"quadrature"**

Maple gives the following as the ode type

`[_quadrature]`

$$\boxed{y' = 0}$$

1.1.1 Solving as quadrature ode

Integrating both sides gives

$$\begin{aligned} y &= \int 0 \, \mathrm{d}x \\ &= c_1 \end{aligned}$$

Summary
The solution(s) found are the following

$$y = c_1 \tag{1}$$

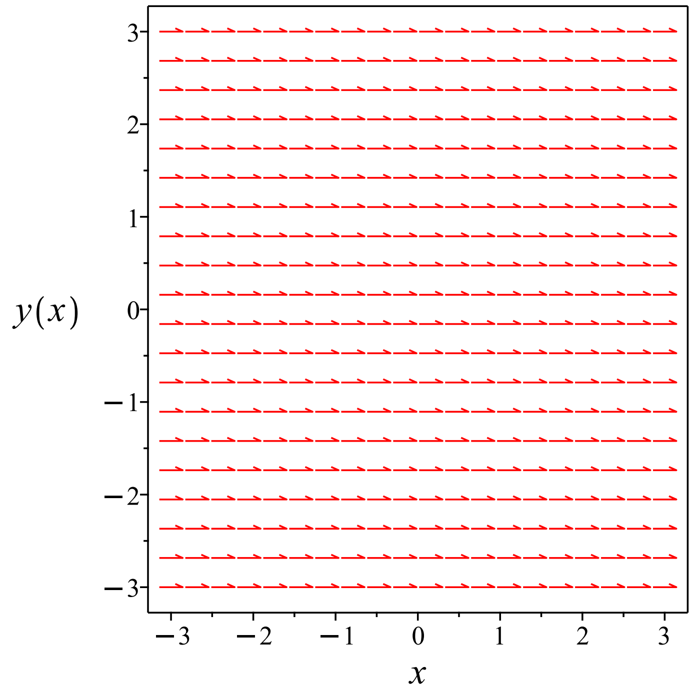


Figure 1: Slope field plot

Verification of solutions

$$y = c_1$$

Verified OK.

1.1.2 Maple step by step solution

Let's solve

$y' = 0$

- Highest derivative means the order of the ODE is 1

  y'

- Integrate both sides with respect to x

  $\int y' dx = \int 0 dx + c_1$

- Evaluate integral

  $y = c_1$

- Solve for y

  $y = c_1$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
<- 1st order linear successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 5

```
dsolve(diff(y(x),x)=0,y(x), singsol=all)
```

$$y(x) = c_1$$

✓ Solution by Mathematica

Time used: 0.002 (sec). Leaf size: 7

```
DSolve[y'[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to c_1$$

1.2 problem 2

Internal problem ID [7318]
Internal file name [`OUTPUT/6299_Sunday_June_05_2022_04_39_10_PM_29992107/index.tex`]

Book: First order enumerated odes
Section: section 1
Problem number: 2.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"quadrature"**

Maple gives the following as the ode type

`[_quadrature]`

$$\boxed{y' = a}$$

1.2.1 Solving as quadrature ode

Integrating both sides gives

$$\begin{aligned} y &= \int a \, \mathrm{d}x \\ &= xa + c_1 \end{aligned}$$

Summary
The solution(s) found are the following

$$y = xa + c_1 \tag{1}$$

Verification of solutions

$$y = xa + c_1$$

Verified OK.

1.2.2 Maple step by step solution

Let's solve

$y' = a$

- Highest derivative means the order of the ODE is 1

  y'

- Integrate both sides with respect to x

  $\int y' dx = \int a dx + c_1$

- Evaluate integral

  $y = xa + c_1$

- Solve for y

  $y = xa + c_1$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
<- 1st order linear successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 9

```
dsolve(diff(y(x),x)=a,y(x), singsol=all)
```

$$y(x) = ax + c_1$$

✓ Solution by Mathematica

Time used: 0.002 (sec). Leaf size: 11

```
DSolve[y'[x]==a,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to ax + c_1$$

1.3 problem 3

Internal problem ID [7319]
Internal file name [`OUTPUT/6300_Sunday_June_05_2022_04_39_11_PM_31027850/index.tex`]

Book: First order enumerated odes
Section: section 1
Problem number: 3.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"quadrature"**

Maple gives the following as the ode type

`[_quadrature]`

$$\boxed{y' = x}$$

1.3.1 Solving as quadrature ode

Integrating both sides gives

$$\begin{aligned} y &= \int x \, \mathrm{d}x \\ &= \frac{x^2}{2} + c_1 \end{aligned}$$

Summary
The solution(s) found are the following

$$y = \frac{x^2}{2} + c_1 \tag{1}$$

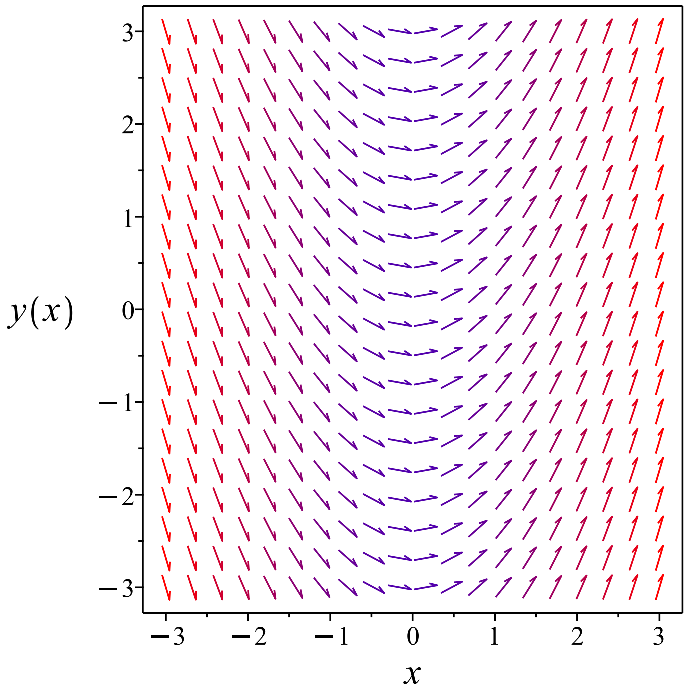


Figure 2: Slope field plot

Verification of solutions

$$y = \frac{x^2}{2} + c_1$$

Verified OK.

1.3.2 Maple step by step solution

Let's solve

$y' = x$

- Highest derivative means the order of the ODE is 1

  y'

- Integrate both sides with respect to x

  $\int y' dx = \int x dx + c_1$

- Evaluate integral

  $y = \frac{x^2}{2} + c_1$

- Solve for y

$y = \frac{x^2}{2} + c_1$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
<- quadrature successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 11

```
dsolve(diff(y(x),x)=x,y(x), singsol=all)
```

$$y(x) = \frac{x^2}{2} + c_1$$

✓ Solution by Mathematica

Time used: 0.002 (sec). Leaf size: 15

```
DSolve[y'[x]==x,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \frac{x^2}{2} + c_1$$

1.4 problem 4

Internal problem ID [7320]
Internal file name [`OUTPUT/6301_Sunday_June_05_2022_04_39_13_PM_53209903/index.tex`]

Book: First order enumerated odes
Section: section 1
Problem number: 4.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"quadrature"**

Maple gives the following as the ode type

`[_quadrature]`

$$\boxed{y' = 1}$$

1.4.1 Solving as quadrature ode

Integrating both sides gives

$$\begin{aligned} y &= \int 1 \, \mathrm{d}x \\ &= x + c_1 \end{aligned}$$

Summary
The solution(s) found are the following

$$y = x + c_1 \tag{1}$$

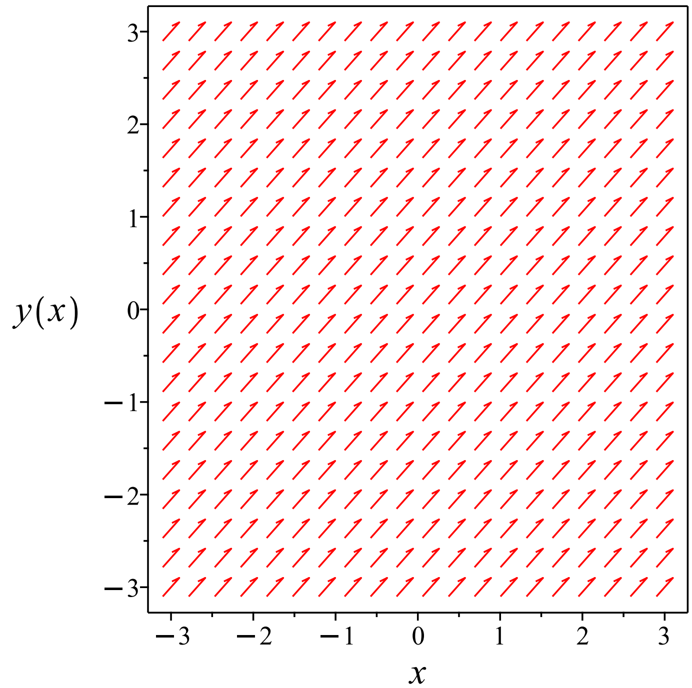


Figure 3: Slope field plot

Verification of solutions

$$y = x + c_1$$

Verified OK.

1.4.2 Maple step by step solution

Let's solve

$y' = 1$

- Highest derivative means the order of the ODE is 1

  y'

- Integrate both sides with respect to x

  $\int y' dx = \int 1 dx + c_1$

- Evaluate integral

  $y = x + c_1$

- Solve for y

  $y = x + c_1$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
<- 1st order linear successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 7

```
dsolve(diff(y(x),x)=1,y(x), singsol=all)
```

$$y(x) = x + c_1$$

✓ Solution by Mathematica

Time used: 0.002 (sec). Leaf size: 9

```
DSolve[y'[x]==1,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to x + c_1$$

1.5 problem 5

Internal problem ID [7321]
Internal file name [`OUTPUT/6302_Sunday_June_05_2022_04_39_14_PM_63413816/index.tex`]

Book: First order enumerated odes
Section: section 1
Problem number: 5.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"quadrature"**

Maple gives the following as the ode type

`[_quadrature]`

$$\boxed{y' = xa}$$

1.5.1 Solving as quadrature ode

Integrating both sides gives

$$\begin{aligned} y &= \int xa \,\mathrm{d}x \\ &= \frac{a\,x^2}{2} + c_1 \end{aligned}$$

Summary
The solution(s) found are the following

$$y = \frac{a\,x^2}{2} + c_1 \tag{1}$$

Verification of solutions

$$y = \frac{a\,x^2}{2} + c_1$$

Verified OK.

1.5.2 Maple step by step solution

Let's solve

$y' = xa$

- Highest derivative means the order of the ODE is 1

  y'

- Integrate both sides with respect to x

  $\int y' dx = \int xa dx + c_1$

- Evaluate integral

  $y = \frac{a\,x^2}{2} + c_1$

- Solve for y

  $y = \frac{a\,x^2}{2} + c_1$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
<- quadrature successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 12

```
dsolve(diff(y(x),x)=a*x,y(x), singsol=all)
```

$$y(x) = \frac{a\,x^2}{2} + c_1$$

✓ Solution by Mathematica

Time used: 0.002 (sec). Leaf size: 16

```
DSolve[y'[x]==a*x,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \frac{ax^2}{2} + c_1$$

1.6 problem 6

Internal problem ID [7322]
Internal file name [`OUTPUT/6303_Sunday_June_05_2022_04_39_16_PM_84203885/index.tex`]

Book: First order enumerated odes
Section: section 1
Problem number: 6.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"exact", "linear", "separable", "homogeneousTypeD2", "first_order_ode_lie_symmetry_lookup"**

Maple gives the following as the ode type

`[_separable]`

$$\boxed{y' - axy = 0}$$

1.6.1 Solving as separable ode

In canonical form the ODE is

$$\begin{aligned} y' &= F(x, y) \\ &= f(x)g(y) \\ &= axy \end{aligned}$$

Where $f(x) = xa$ and $g(y) = y$. Integrating both sides gives

$$\begin{aligned}
\frac{1}{y}\,dy &= xa\,dx \\
\int \frac{1}{y}\,dy &= \int xa\,dx \\
\ln\left(y\right) &= \frac{a\,x^{2}}{2}+c_{1} \\
y &= \mathrm{e}^{\frac{a\,x^{2}}{2}+c_{1}} \\
&= c_{1}\mathrm{e}^{\frac{a\,x^{2}}{2}}
\end{aligned}$$

Summary

The solution(s) found are the following

$$y = c_{1}\mathrm{e}^{\frac{a\,x^{2}}{2}} \tag{1}$$

Verification of solutions

$$y = c_{1}\mathrm{e}^{\frac{a\,x^{2}}{2}}$$

Verified OK.

1.6.2 Solving as linear ode

Entering Linear first order ODE solver. In canonical form a linear first order is

$$y' + p(x)y = q(x)$$

Where here

$$\begin{aligned}
p(x) &= -xa \\
q(x) &= 0
\end{aligned}$$

Hence the ode is

$$y' - axy = 0$$

The integrating factor μ is

$$\begin{aligned}
\mu &= \mathrm{e}^{\int -xa dx} \\
&= \mathrm{e}^{-\frac{a\,x^{2}}{2}}
\end{aligned}$$

The ode becomes

$$\frac{d}{dx}\mu y = 0$$
$$\frac{d}{dx}\left(e^{-\frac{a x^2}{2}} y\right) = 0$$

Integrating gives

$$e^{-\frac{a x^2}{2}} y = c_1$$

Dividing both sides by the integrating factor $\mu = e^{-\frac{a x^2}{2}}$ results in

$$y = c_1 e^{\frac{a x^2}{2}}$$

Summary
The solution(s) found are the following

$$y = c_1 e^{\frac{a x^2}{2}} \tag{1}$$

Verification of solutions

$$y = c_1 e^{\frac{a x^2}{2}}$$

Verified OK.

1.6.3 Solving as homogeneousTypeD2 ode

Using the change of variables $y = u(x) x$ on the above ode results in new ode in $u(x)$

$$u'(x) x + u(x) - a x^2 u(x) = 0$$

In canonical form the ODE is

$$\begin{aligned} u' &= F(x, u) \\ &= f(x) g(u) \\ &= \frac{u(a x^2 - 1)}{x} \end{aligned}$$

Where $f(x) = \frac{a\,x^2-1}{x}$ and $g(u) = u$. Integrating both sides gives

$$
\begin{aligned}
\frac{1}{u}\,du &= \frac{a\,x^2-1}{x}\,dx \\
\int \frac{1}{u}\,du &= \int \frac{a\,x^2-1}{x}\,dx \\
\ln(u) &= \frac{a\,x^2}{2} - \ln(x) + c_2 \\
u &= \mathrm{e}^{\frac{a\,x^2}{2}-\ln(x)+c_2} \\
&= c_2 \mathrm{e}^{\frac{a\,x^2}{2}-\ln(x)}
\end{aligned}
$$

Which simplifies to

$$
u(x) = \frac{c_2 \mathrm{e}^{\frac{a\,x^2}{2}}}{x}
$$

Therefore the solution y is

$$
\begin{aligned}
y &= xu \\
&= c_2 \mathrm{e}^{\frac{a\,x^2}{2}}
\end{aligned}
$$

Summary

The solution(s) found are the following

$$
y = c_2 \mathrm{e}^{\frac{a\,x^2}{2}} \tag{1}
$$

Verification of solutions

$$
y = c_2 \mathrm{e}^{\frac{a\,x^2}{2}}
$$

Verified OK.

1.6.4 Solving as first order ode lie symmetry lookup ode

Writing the ode as

$$
\begin{aligned}
y' &= axy \\
y' &= \omega(x,y)
\end{aligned}
$$

The condition of Lie symmetry is the linearized PDE given by

$$
\eta_x + \omega(\eta_y - \xi_x) - \omega^2 \xi_y - \omega_x \xi - \omega_y \eta = 0 \tag{A}
$$

The type of this ode is known. It is of type `linear`. Therefore we do not need to solve the PDE (A), and can just use the lookup table shown below to find ξ, η

Table 6: Lie symmetry infinitesimal lookup table for known first order ODE's

ODE class	Form	ξ	η
linear ode	$y' = f(x)y(x) + g(x)$	0	$e^{\int f dx}$
separable ode	$y' = f(x)\, g(y)$	$\frac{1}{f}$	0
quadrature ode	$y' = f(x)$	0	1
quadrature ode	$y' = g(y)$	1	0
homogeneous ODEs of Class A	$y' = f\left(\frac{y}{x}\right)$	x	y
homogeneous ODEs of Class C	$y' = (a + bx + cy)^{\frac{n}{m}}$	1	$-\frac{b}{c}$
homogeneous class D	$y' = \frac{y}{x} + g(x)\, F\left(\frac{y}{x}\right)$	x^2	xy
First order special form ID 1	$y' = g(x)\, e^{h(x)+by} + f(x)$	$\frac{e^{-\int bf(x)dx-h(x)}}{g(x)}$	$\frac{f(x)e^{-\int bf(x)dx-h(x)}}{g(x)}$
polynomial type ode	$y' = \frac{a_1x+b_1y+c_1}{a_2x+b_2y+c_2}$	$\frac{a_1b_2x-a_2b_1x-b_1c_2+b_2c_1}{a_1b_2-a_2b_1}$	$\frac{a_1b_2y-a_2b_1y-a_1c_2-a_2c_1}{a_1b_2-a_2b_1}$
Bernoulli ode	$y' = f(x)\, y + g(x)\, y^n$	0	$e^{-\int (n-1)f(x)dx} y^n$
Reduced Riccati	$y' = f_1(x)\, y + f_2(x)\, y^2$	0	$e^{-\int f_1 dx}$

The above table shows that

$$
\begin{aligned}
\xi(x, y) &= 0 \\
\eta(x, y) &= \mathrm{e}^{\frac{a\,x^2}{2}} \qquad \text{(A1)}
\end{aligned}
$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x, y) \to (R, S)$ where (R, S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$
\frac{dx}{\xi} = \frac{dy}{\eta} = dS \qquad (1)
$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y}\right) S(x, y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$R = x$$

S is found from

$$\begin{aligned} S &= \int \frac{1}{\eta} dy \\ &= \int \frac{1}{e^{\frac{a x^2}{2}}} dy \end{aligned}$$

Which results in

$$S = e^{-\frac{a x^2}{2}} y$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x, y) S_y}{R_x + \omega(x, y) R_y} \tag{2}$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x, y)$ is the right hand side of the original ode given by

$$\omega(x, y) = axy$$

Evaluating all the partial derivatives gives

$$\begin{aligned} R_x &= 1 \\ R_y &= 0 \\ S_x &= -xa\, e^{-\frac{a x^2}{2}} y \\ S_y &= e^{-\frac{a x^2}{2}} \end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = 0 \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = 0$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$S(R) = c_1 \tag{4}$$

To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$\mathrm{e}^{-\frac{a\,x^2}{2}} y = c_1$$

Which simplifies to

$$\mathrm{e}^{-\frac{a\,x^2}{2}} y = c_1$$

Which gives

$$y = c_1 \mathrm{e}^{\frac{a\,x^2}{2}}$$

Summary
The solution(s) found are the following

$$y = c_1 \mathrm{e}^{\frac{a\,x^2}{2}} \tag{1}$$

Verification of solutions

$$y = c_1 \mathrm{e}^{\frac{a\,x^2}{2}}$$

Verified OK.

1.6.5 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x, y) + N(x, y) \frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x, y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx} \phi(x, y) = 0$$

Hence

$$\frac{\partial \phi}{\partial x} + \frac{\partial \phi}{\partial y}\frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$\begin{aligned}\frac{\partial \phi}{\partial x} &= M\\ \frac{\partial \phi}{\partial y} &= N\end{aligned}$$

But since $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x, y)$ but at least we know now that we can do that since the condition $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x, y)\,\mathrm{d}x + N(x, y)\,\mathrm{d}y = 0 \tag{1A}$$

Therefore

$$\begin{aligned}\left(\frac{1}{ay}\right)\mathrm{d}y &= (x)\,\mathrm{d}x\\ (-x)\,\mathrm{d}x + \left(\frac{1}{ay}\right)\mathrm{d}y &= 0 \qquad (2A)\end{aligned}$$

Comparing (1A) and (2A) shows that

$$\begin{aligned}M(x, y) &= -x\\ N(x, y) &= \frac{1}{ay}\end{aligned}$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Using result found above gives

$$\begin{aligned}\frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}(-x)\\ &= 0\end{aligned}$$

And

$$\begin{aligned}\frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}\left(\frac{1}{ay}\right)\\ &= 0\end{aligned}$$

Since $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$, then the ODE is <u>exact</u> The following equations are now set up to solve for the function $\phi(x, y)$

$$\frac{\partial \phi}{\partial x} = M \tag{1}$$

$$\frac{\partial \phi}{\partial y} = N \tag{2}$$

Integrating (1) w.r.t. x gives

$$\begin{aligned}\int \frac{\partial \phi}{\partial x}\,\mathrm{d}x &= \int M\,\mathrm{d}x\\ \int \frac{\partial \phi}{\partial x}\,\mathrm{d}x &= \int -x\,\mathrm{d}x\end{aligned}$$

$$\phi = -\frac{x^2}{2} + f(y) \tag{3}$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$\frac{\partial \phi}{\partial y} = 0 + f'(y) \tag{4}$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = \frac{1}{ay}$. Therefore equation (4) becomes

$$\frac{1}{ay} = 0 + f'(y) \tag{5}$$

Solving equation (5) for $f'(y)$ gives

$$f'(y) = \frac{1}{ay}$$

Integrating the above w.r.t y gives

$$\int f'(y)\,\mathrm{d}y = \int \left(\frac{1}{ay}\right)\mathrm{d}y$$
$$f(y) = \frac{\ln (y)}{a} + c_1$$

Where c_1 is constant of integration. Substituting result found above for $f(y)$ into equation (3) gives ϕ

$$\phi = -\frac{x^2}{2} + \frac{\ln (y)}{a} + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = -\frac{x^2}{2} + \frac{\ln (y)}{a}$$

The solution becomes

$$y = \mathrm{e}^{\frac{1}{2}a\,x^2 + c_1 a}$$

Summary

The solution(s) found are the following

$$y = \mathrm{e}^{\frac{1}{2}a\,x^2 + c_1 a} \tag{1}$$

Verification of solutions

$$y = \mathrm{e}^{\frac{1}{2}a\,x^2 + c_1 a}$$

Verified OK.

1.6.6 Maple step by step solution

Let's solve

$y' - axy = 0$

- Highest derivative means the order of the ODE is 1

  y'

- Separate variables

$\frac{y'}{y} = xa$

- Integrate both sides with respect to x

  $\int \frac{y'}{y} dx = \int xa dx + c_1$

- Evaluate integral

  $\ln(y) = \frac{a x^2}{2} + c_1$

- Solve for y

  $y = e^{\frac{a x^2}{2} + c_1}$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
<- 1st order linear successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 13

```
dsolve(diff(y(x),x)=a*x*y(x),y(x), singsol=all)
```

$$y(x) = c_1 e^{\frac{a x^2}{2}}$$

✓ Solution by Mathematica

Time used: 0.032 (sec). Leaf size: 23

```
DSolve[y'[x]==a*x*y[x],y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to c_1 e^{\frac{a x^2}{2}}$$
$$y(x) \to 0$$

1.7 problem 7

Internal problem ID [7323]
Internal file name [`OUTPUT/6304_Sunday_June_05_2022_04_39_17_PM_62816172/index.tex`]

Book: First order enumerated odes
Section: section 1
Problem number: 7.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"linear", "exactWithIntegrationFactor", "first_order_ode_lie_symmetry_lookup"**

Maple gives the following as the ode type

```
[[_linear, `class A`]]
```

$$\boxed{y' - y = xa}$$

1.7.1 Solving as linear ode

Entering Linear first order ODE solver. In canonical form a linear first order is

$$y' + p(x)y = q(x)$$

Where here

$$\begin{aligned} p(x) &= -1 \\ q(x) &= xa \end{aligned}$$

Hence the ode is

$$y' - y = xa$$

The integrating factor μ is

$$\begin{aligned} \mu &= e^{\int (-1)dx} \\ &= e^{-x} \end{aligned}$$

The ode becomes

$$
\begin{aligned}
\frac{d}{dx}(\mu y) &= (\mu)(xa) \\
\frac{d}{dx}\left(e^{-x}y\right) &= \left(e^{-x}\right)(xa) \\
d\left(e^{-x}y\right) &= \left(xa\,e^{-x}\right)\,dx
\end{aligned}
$$

Integrating gives

$$
\begin{aligned}
e^{-x}y &= \int xa\,e^{-x}\,dx \\
e^{-x}y &= -(1+x)\,a\,e^{-x} + c_1
\end{aligned}
$$

Dividing both sides by the integrating factor $\mu = e^{-x}$ results in

$$
y = -e^{x}(1+x)\,a\,e^{-x} + c_1 e^{x}
$$

which simplifies to

$$
y = -a(1+x) + c_1 e^{x}
$$

Summary
The solution(s) found are the following

$$
y = -a(1+x) + c_1 e^{x} \tag{1}
$$

Verification of solutions

$$
y = -a(1+x) + c_1 e^{x}
$$

Verified OK.

1.7.2 Solving as first order ode lie symmetry lookup ode

Writing the ode as

$$
\begin{aligned}
y' &= xa + y \\
y' &= \omega(x,y)
\end{aligned}
$$

The condition of Lie symmetry is the linearized PDE given by

$$
\eta_x + \omega(\eta_y - \xi_x) - \omega^2 \xi_y - \omega_x \xi - \omega_y \eta = 0 \tag{A}
$$

The type of this ode is known. It is of type `linear`. Therefore we do not need to solve the PDE (A), and can just use the lookup table shown below to find ξ, η

Table 9: Lie symmetry infinitesimal lookup table for known first order ODE's

ODE class	Form	ξ	η
linear ode	$y' = f(x)y(x) + g(x)$	0	$e^{\int f dx}$
separable ode	$y' = f(x)\, g(y)$	$\frac{1}{f}$	0
quadrature ode	$y' = f(x)$	0	1
quadrature ode	$y' = g(y)$	1	0
homogeneous ODEs of Class A	$y' = f\left(\frac{y}{x}\right)$	x	y
homogeneous ODEs of Class C	$y' = (a + bx + cy)^{\frac{n}{m}}$	1	$-\frac{b}{c}$
homogeneous class D	$y' = \frac{y}{x} + g(x)\, F\left(\frac{y}{x}\right)$	x^2	xy
First order special form ID 1	$y' = g(x)\, e^{h(x)+by} + f(x)$	$\frac{e^{-\int bf(x)dx - h(x)}}{g(x)}$	$\frac{f(x)e^{-\int bf(x)dx - h(x)}}{g(x)}$
polynomial type ode	$y' = \frac{a_1x+b_1y+c_1}{a_2x+b_2y+c_2}$	$\frac{a_1b_2x-a_2b_1x-b_1c_2+b_2c_1}{a_1b_2-a_2b_1}$	$\frac{a_1b_2y-a_2b_1y-a_1c_2-a_2c_1}{a_1b_2-a_2b_1}$
Bernoulli ode	$y' = f(x)\, y + g(x)\, y^n$	0	$e^{-\int (n-1)f(x)dx} y^n$
Reduced Riccati	$y' = f_1(x)\, y + f_2(x)\, y^2$	0	$e^{-\int f_1 dx}$

The above table shows that

$$
\begin{aligned}
\xi(x, y) &= 0 \\
\eta(x, y) &= e^x
\end{aligned} \tag{A1}
$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x, y) \to (R, S)$ where (R, S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$
\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}
$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y}\right) S(x,y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$R = x$$

S is found from

$$\begin{aligned} S &= \int \frac{1}{\eta} dy \\ &= \int \frac{1}{e^{x}} dy \end{aligned}$$

Which results in

$$S = e^{-x} y$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x,y) S_y}{R_x + \omega(x,y) R_y} \tag{2}$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x,y)$ is the right hand side of the original ode given by

$$\omega(x,y) = xa + y$$

Evaluating all the partial derivatives gives

$$\begin{aligned} R_x &= 1 \\ R_y &= 0 \\ S_x &= -e^{-x} y \\ S_y &= e^{-x} \end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = xa\, e^{-x} \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = Ra\, e^{-R}$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$S(R) = -(R+1)\, a\, \mathrm{e}^{-R} + c_1 \tag{4}$$

To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$\mathrm{e}^{-x} y = -(1+x)\, a\, \mathrm{e}^{-x} + c_1$$

Which simplifies to

$$(a(1+x) + y)\, \mathrm{e}^{-x} - c_1 = 0$$

Which gives

$$y = -\left(x a\, \mathrm{e}^{-x} + a\, \mathrm{e}^{-x} - c_1\right) \mathrm{e}^{x}$$

Summary

The solution(s) found are the following

$$y = -\left(x a\, \mathrm{e}^{-x} + a\, \mathrm{e}^{-x} - c_1\right) \mathrm{e}^{x} \tag{1}$$

Verification of solutions

$$y = -\left(x a\, \mathrm{e}^{-x} + a\, \mathrm{e}^{-x} - c_1\right) \mathrm{e}^{x}$$

Verified OK.

1.7.3 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x,y) + N(x,y)\frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x,y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y) = 0$$

Hence

$$\frac{\partial \phi}{\partial x} + \frac{\partial \phi}{\partial y}\frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$\frac{\partial \phi}{\partial x} = M$$
$$\frac{\partial \phi}{\partial y} = N$$

But since $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x, y)$ but at least we know now that we can do that since the condition $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x, y)\,\mathrm{d}x + N(x, y)\,\mathrm{d}y = 0 \tag{1A}$$

Therefore

$$\mathrm{d}y = (xa + y)\,\mathrm{d}x$$
$$(-xa - y)\,\mathrm{d}x + \mathrm{d}y = 0 \tag{2A}$$

Comparing (1A) and (2A) shows that

$$M(x, y) = -xa - y$$
$$N(x, y) = 1$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Using result found above gives

$$\begin{aligned}\frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}(-xa - y) \\ &= -1\end{aligned}$$

And

$$\begin{aligned}\frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}(1) \\ &= 0\end{aligned}$$

Since $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$, then the ODE is not exact. Since the ODE is not exact, we will try to find an integrating factor to make it exact. Let

$$
\begin{aligned}
A &= \frac{1}{N}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right) \\
&= 1((-1) - (0)) \\
&= -1
\end{aligned}
$$

Since A does not depend on y, then it can be used to find an integrating factor. The integrating factor μ is

$$
\begin{aligned}
\mu &= e^{\int A \, \mathrm{d}x} \\
&= e^{\int -1 \, \mathrm{d}x}
\end{aligned}
$$

The result of integrating gives

$$
\begin{aligned}
\mu &= e^{-x} \\
&= \mathrm{e}^{-x}
\end{aligned}
$$

M and N are multiplied by this integrating factor, giving new M and new N which are called $\overline{M}$ and $\overline{N}$ for now so not to confuse them with the original M and N.

$$
\begin{aligned}
\overline{M} &= \mu M \\
&= \mathrm{e}^{-x}(-xa - y) \\
&= -\mathrm{e}^{-x}(xa + y)
\end{aligned}
$$

And

$$
\begin{aligned}
\overline{N} &= \mu N \\
&= \mathrm{e}^{-x}(1) \\
&= \mathrm{e}^{-x}
\end{aligned}
$$

Now a modified ODE is ontained from the original ODE, which is exact and can be solved. The modified ODE is

$$
\begin{aligned}
\overline{M} + \overline{N}\frac{\mathrm{d}y}{\mathrm{d}x} &= 0 \\
\left(-\mathrm{e}^{-x}(xa + y)\right) + \left(\mathrm{e}^{-x}\right)\frac{\mathrm{d}y}{\mathrm{d}x} &= 0
\end{aligned}
$$

The following equations are now set up to solve for the function $\phi(x, y)$

$$
\frac{\partial \phi}{\partial x} = \overline{M} \tag{1}
$$

$$
\frac{\partial \phi}{\partial y} = \overline{N} \tag{2}
$$

Integrating (1) w.r.t. x gives

$$\int \frac{\partial \phi}{\partial x} \,\mathrm{d}x = \int \overline{M} \,\mathrm{d}x$$
$$\int \frac{\partial \phi}{\partial x} \,\mathrm{d}x = \int -\mathrm{e}^{-x}(xa + y) \,\mathrm{d}x$$
$$\phi = (xa + a + y)\,\mathrm{e}^{-x} + f(y) \tag{3}$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$\frac{\partial \phi}{\partial y} = \mathrm{e}^{-x} + f'(y) \tag{4}$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = \mathrm{e}^{-x}$. Therefore equation (4) becomes

$$\mathrm{e}^{-x} = \mathrm{e}^{-x} + f'(y) \tag{5}$$

Solving equation (5) for $f'(y)$ gives

$$f'(y) = 0$$

Therefore

$$f(y) = c_1$$

Where c_1 is constant of integration. Substituting this result for $f(y)$ into equation (3) gives ϕ

$$\phi = (xa + a + y)\,\mathrm{e}^{-x} + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = (xa + a + y)\,\mathrm{e}^{-x}$$

The solution becomes

$$y = -\left(xa\,\mathrm{e}^{-x} + a\,\mathrm{e}^{-x} - c_1\right)\mathrm{e}^{x}$$

Summary

The solution(s) found are the following

$$y = -\left(xa\,\mathrm{e}^{-x} + a\,\mathrm{e}^{-x} - c_1\right)\mathrm{e}^{x} \tag{1}$$

Verification of solutions

$$y = -\left(xa\,\mathrm{e}^{-x} + a\,\mathrm{e}^{-x} - c_1\right)\mathrm{e}^{x}$$

Verified OK.

1.7.4 Maple step by step solution

Let's solve

$y' - y = xa$

- Highest derivative means the order of the ODE is 1

  y'

- Isolate the derivative

  $y' = xa + y$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE

  $y' - y = xa$

- The ODE is linear; multiply by an integrating factor $\mu(x)$

  $\mu(x)\left(y' - y\right) = \mu(x)\,xa$

- Assume the lhs of the ODE is the total derivative $\frac{d}{dx}(\mu(x)\,y)$

  $\mu(x)\left(y' - y\right) = \mu'(x)\,y + \mu(x)\,y'$

- Isolate $\mu'(x)$

  $\mu'(x) = -\mu(x)$

- Solve to find the integrating factor

  $\mu(x) = \mathrm{e}^{-x}$

- Integrate both sides with respect to x

  $\int \left(\frac{d}{dx}(\mu(x)\,y)\right) dx = \int \mu(x)\,xadx + c_1$

- Evaluate the integral on the lhs

  $\mu(x)\,y = \int \mu(x)\,xadx + c_1$

- Solve for y

  $y = \frac{\int \mu(x)xadx + c_1}{\mu(x)}$

- Substitute $\mu(x) = \mathrm{e}^{-x}$

  $y = \frac{\int xa\,\mathrm{e}^{-x}dx + c_1}{\mathrm{e}^{-x}}$

- Evaluate the integrals on the rhs

  $y = \frac{-(1+x)a\,e^{-x} + c_1}{e^{-x}}$

- Simplify

  $y = -a(1 + x) + c_1 e^x$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
<- 1st order linear successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 15

```
dsolve(diff(y(x),x)=a*x+y(x),y(x), singsol=all)
```

$$y(x) = c_1 e^x - a(x + 1)$$

✓ Solution by Mathematica

Time used: 0.03 (sec). Leaf size: 18

```
DSolve[y'[x]==a*x+y[x],y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to -a(x + 1) + c_1 e^x$$

1.8 problem 8

Internal problem ID [7324]
Internal file name [OUTPUT/6305_Sunday_June_05_2022_04_39_19_PM_55719783/index.tex]

Book: First order enumerated odes
Section: section 1
Problem number: 8.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"linear", "exactWithIntegrationFactor", "first_order_ode_lie_symmetry_lookup"**

Maple gives the following as the ode type

```
[[_linear, `class A`]]
```

$$\boxed{y' - by = xa}$$

1.8.1 Solving as linear ode

Entering Linear first order ODE solver. In canonical form a linear first order is

$$y' + p(x)y = q(x)$$

Where here

$$p(x) = -b$$
$$q(x) = xa$$

Hence the ode is

$$y' - by = xa$$

The integrating factor μ is

$$\mu = e^{\int -b dx}$$
$$= e^{-xb}$$

The ode becomes

$$\frac{\mathrm{d}}{\mathrm{d}x}(\mu y) = (\mu)\,(xa)$$
$$\frac{\mathrm{d}}{\mathrm{d}x}\left(\mathrm{e}^{-xb}y\right) = \left(\mathrm{e}^{-xb}\right)(xa)$$
$$\mathrm{d}\left(\mathrm{e}^{-xb}y\right) = \left(xa\,\mathrm{e}^{-xb}\right)\,\mathrm{d}x$$

Integrating gives

$$\mathrm{e}^{-xb}y = \int xa\,\mathrm{e}^{-xb}\,\mathrm{d}x$$
$$\mathrm{e}^{-xb}y = -\frac{(xb+1)\,a\,\mathrm{e}^{-xb}}{b^2} + c_1$$

Dividing both sides by the integrating factor $\mu = \mathrm{e}^{-xb}$ results in

$$y = -\frac{\mathrm{e}^{xb}(xb+1)\,a\,\mathrm{e}^{-xb}}{b^2} + c_1\mathrm{e}^{xb}$$

which simplifies to

$$y = \frac{c_1\mathrm{e}^{xb}b^2 - abx - a}{b^2}$$

Summary
The solution(s) found are the following

$$y = \frac{c_1\mathrm{e}^{xb}b^2 - abx - a}{b^2} \tag{1}$$

Verification of solutions

$$y = \frac{c_1\mathrm{e}^{xb}b^2 - abx - a}{b^2}$$

Verified OK.

1.8.2 Solving as first order ode lie symmetry lookup ode

Writing the ode as

$$y' = xa + by$$
$$y' = \omega(x, y)$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x + \omega(\eta_y - \xi_x) - \omega^2\xi_y - \omega_x\xi - \omega_y\eta = 0 \tag{A}$$

The type of this ode is known. It is of type `linear`. Therefore we do not need to solve the PDE (A), and can just use the lookup table shown below to find ξ, η

Table 12: Lie symmetry infinitesimal lookup table for known first order ODE's

ODE class	Form	ξ	η
linear ode	$y' = f(x)y(x) + g(x)$	0	$e^{\int f dx}$
separable ode	$y' = f(x)\, g(y)$	$\frac{1}{f}$	0
quadrature ode	$y' = f(x)$	0	1
quadrature ode	$y' = g(y)$	1	0
homogeneous ODEs of Class A	$y' = f\left(\frac{y}{x}\right)$	x	y
homogeneous ODEs of Class C	$y' = (a + bx + cy)^{\frac{n}{m}}$	1	$-\frac{b}{c}$
homogeneous class D	$y' = \frac{y}{x} + g(x)\, F\left(\frac{y}{x}\right)$	x^2	xy
First order special form ID 1	$y' = g(x)\, e^{h(x)+by} + f(x)$	$\frac{e^{-\int bf(x)dx - h(x)}}{g(x)}$	$\frac{f(x)e^{-\int bf(x)dx - h(x)}}{g(x)}$
polynomial type ode	$y' = \frac{a_1x+b_1y+c_1}{a_2x+b_2y+c_2}$	$\frac{a_1b_2x - a_2b_1x - b_1c_2 + b_2c_1}{a_1b_2 - a_2b_1}$	$\frac{a_1b_2y - a_2b_1y - a_1c_2 - a_2c_1}{a_1b_2 - a_2b_1}$
Bernoulli ode	$y' = f(x)\, y + g(x)\, y^n$	0	$e^{-\int (n-1)f(x)dx} y^n$
Reduced Riccati	$y' = f_1(x)\, y + f_2(x)\, y^2$	0	$e^{-\int f_1 dx}$

The above table shows that

$$\begin{aligned} \xi(x, y) &= 0 \\ \eta(x, y) &= e^{xb} \end{aligned} \tag{A1}$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x, y) \to (R, S)$ where (R, S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y}\right) S(x, y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$R = x$$

S is found from

$$\begin{aligned} S &= \int \frac{1}{\eta} dy \\ &= \int \frac{1}{e^{xb}} dy \end{aligned}$$

Which results in

$$S = e^{-xb} y$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x, y) S_y}{R_x + \omega(x, y) R_y} \tag{2}$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x, y)$ is the right hand side of the original ode given by

$$\omega(x, y) = xa + by$$

Evaluating all the partial derivatives gives

$$\begin{aligned} R_x &= 1 \\ R_y &= 0 \\ S_x &= -b\, e^{-xb} y \\ S_y &= e^{-xb} \end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = xa\, e^{-xb} \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = Ra\, e^{-Rb}$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$S(R) = -\frac{(Rb+1)\, a\, \mathrm{e}^{-Rb}}{b^2} + c_1 \tag{4}$$

To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$\mathrm{e}^{-xb} y = -\frac{(xb+1)\, a\, \mathrm{e}^{-xb}}{b^2} + c_1$$

Which simplifies to

$$\mathrm{e}^{-xb} y = -\frac{(xb+1)\, a\, \mathrm{e}^{-xb}}{b^2} + c_1$$

Which gives

$$y = -\frac{\left(xab\, \mathrm{e}^{-xb} - c_1 b^2 + a\, \mathrm{e}^{-xb}\right) \mathrm{e}^{xb}}{b^2}$$

Summary

The solution(s) found are the following

$$y = -\frac{\left(xab\, \mathrm{e}^{-xb} - c_1 b^2 + a\, \mathrm{e}^{-xb}\right) \mathrm{e}^{xb}}{b^2} \tag{1}$$

Verification of solutions

$$y = -\frac{\left(xab\, \mathrm{e}^{-xb} - c_1 b^2 + a\, \mathrm{e}^{-xb}\right) \mathrm{e}^{xb}}{b^2}$$

Verified OK.

1.8.3 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x, y) + N(x, y) \frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x, y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx} \phi(x, y) = 0$$

Hence

$$\frac{\partial \phi}{\partial x}+\frac{\partial \phi}{\partial y}\frac{dy}{dx}=0 \tag{B}$$

Comparing (A,B) shows that

$$\begin{aligned}\frac{\partial \phi}{\partial x}&=M\\ \frac{\partial \phi}{\partial y}&=N\end{aligned}$$

But since $\frac{\partial^2\phi}{\partial x\partial y}=\frac{\partial^2\phi}{\partial y\partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y}=\frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x,y)$ but at least we know now that we can do that since the condition $\frac{\partial^2\phi}{\partial x\partial y}=\frac{\partial^2\phi}{\partial y\partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x,y)\,\mathrm{d}x+N(x,y)\,\mathrm{d}y=0 \tag{1A}$$

Therefore

$$\begin{aligned}\mathrm{d}y&=(xa+by)\,\mathrm{d}x\\ (-xa-by)\,\mathrm{d}x+\mathrm{d}y&=0 \end{aligned}\tag{2A}$$

Comparing (1A) and (2A) shows that

$$\begin{aligned}M(x,y)&=-xa-by\\ N(x,y)&=1\end{aligned}$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y}=\frac{\partial N}{\partial x}$$

Using result found above gives

$$\begin{aligned}\frac{\partial M}{\partial y}&=\frac{\partial}{\partial y}(-xa-by)\\ &=-b\end{aligned}$$

And

$$\begin{aligned}\frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}(1)\\ &= 0\end{aligned}$$

Since $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$, then the ODE is not exact. Since the ODE is not exact, we will try to find an integrating factor to make it exact. Let

$$\begin{aligned}A &= \frac{1}{N}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right)\\ &= 1((-b) - (0))\\ &= -b\end{aligned}$$

Since A does not depend on y, then it can be used to find an integrating factor. The integrating factor μ is

$$\begin{aligned}\mu &= e^{\int A \,\mathrm{d}x}\\ &= e^{\int -b \,\mathrm{d}x}\end{aligned}$$

The result of integrating gives

$$\begin{aligned}\mu &= e^{-xb}\\ &= \mathrm{e}^{-xb}\end{aligned}$$

M and N are multiplied by this integrating factor, giving new M and new N which are called $\overline{M}$ and $\overline{N}$ for now so not to confuse them with the original M and N.

$$\begin{aligned}\overline{M} &= \mu M\\ &= \mathrm{e}^{-xb}(-xa - by)\\ &= -\mathrm{e}^{-xb}(xa + by)\end{aligned}$$

And

$$\begin{aligned}\overline{N} &= \mu N\\ &= \mathrm{e}^{-xb}(1)\\ &= \mathrm{e}^{-xb}\end{aligned}$$

Now a modified ODE is ontained from the original ODE, which is exact and can be solved. The modified ODE is

$$\begin{aligned}\overline{M} + \overline{N}\frac{\mathrm{d}y}{\mathrm{d}x} &= 0\\ \left(-\mathrm{e}^{-xb}(xa + by)\right) + \left(\mathrm{e}^{-xb}\right)\frac{\mathrm{d}y}{\mathrm{d}x} &= 0\end{aligned}$$

The following equations are now set up to solve for the function $\phi(x,y)$

$$\frac{\partial\phi}{\partial x} = \overline{M} \tag{1}$$
$$\frac{\partial\phi}{\partial y} = \overline{N} \tag{2}$$

Integrating (1) w.r.t. x gives

$$\int \frac{\partial\phi}{\partial x}\,\mathrm{d}x = \int \overline{M}\,\mathrm{d}x$$
$$\int \frac{\partial\phi}{\partial x}\,\mathrm{d}x = \int -\mathrm{e}^{-xb}(xa+by)\,\mathrm{d}x$$
$$\phi = \frac{(abx+b^2y+a)\,\mathrm{e}^{-xb}}{b^2} + f(y) \tag{3}$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$\frac{\partial\phi}{\partial y} = \mathrm{e}^{-xb} + f'(y) \tag{4}$$

But equation (2) says that $\frac{\partial\phi}{\partial y} = \mathrm{e}^{-xb}$. Therefore equation (4) becomes

$$\mathrm{e}^{-xb} = \mathrm{e}^{-xb} + f'(y) \tag{5}$$

Solving equation (5) for $f'(y)$ gives

$$f'(y) = 0$$

Therefore

$$f(y) = c_1$$

Where c_1 is constant of integration. Substituting this result for $f(y)$ into equation (3) gives ϕ

$$\phi = \frac{(abx+b^2y+a)\,\mathrm{e}^{-xb}}{b^2} + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = \frac{(abx+b^2y+a)\,\mathrm{e}^{-xb}}{b^2}$$

The solution becomes

$$y = -\frac{\left(xab\,\mathrm{e}^{-xb} - c_1 b^2 + a\,\mathrm{e}^{-xb}\right)\mathrm{e}^{xb}}{b^2}$$

Summary
The solution(s) found are the following

$$y = -\frac{\left(xab\,\mathrm{e}^{-xb} - c_1 b^2 + a\,\mathrm{e}^{-xb}\right)\mathrm{e}^{xb}}{b^2} \tag{1}$$

Verification of solutions

$$y = -\frac{\left(xab\,\mathrm{e}^{-xb} - c_1 b^2 + a\,\mathrm{e}^{-xb}\right)\mathrm{e}^{xb}}{b^2}$$

Verified OK.

1.8.4 Maple step by step solution

Let's solve

$y' - by = xa$

- Highest derivative means the order of the ODE is 1

  y'

- Isolate the derivative

  $y' = xa + by$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE

  $y' - by = xa$

- The ODE is linear; multiply by an integrating factor $\mu(x)$

  $\mu(x)\,(y' - by) = \mu(x)\,xa$

- Assume the lhs of the ODE is the total derivative $\frac{d}{dx}(\mu(x)\,y)$

  $\mu(x)\,(y' - by) = \mu'(x)\,y + \mu(x)\,y'$

- Isolate $\mu'(x)$

  $\mu'(x) = -\mu(x)\,b$

- Solve to find the integrating factor

  $\mu(x) = \mathrm{e}^{-xb}$

- Integrate both sides with respect to x

$$\int \left(\frac{d}{dx}(\mu(x)\, y)\right) dx = \int \mu(x)\, xadx + c_1$$

- Evaluate the integral on the lhs

$$\mu(x)\, y = \int \mu(x)\, xadx + c_1$$

- Solve for y

$$y = \frac{\int \mu(x)xadx + c_1}{\mu(x)}$$

- Substitute $\mu(x) = \mathrm{e}^{-xb}$

$$y = \frac{\int xa\,\mathrm{e}^{-xb}dx + c_1}{\mathrm{e}^{-xb}}$$

- Evaluate the integrals on the rhs

$$y = \frac{-\frac{(xb+1)a\,\mathrm{e}^{-xb}}{b^2} + c_1}{\mathrm{e}^{-xb}}$$

- Simplify

$$y = \frac{c_1\mathrm{e}^{xb}b^2 - abx - a}{b^2}$$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
<- 1st order linear successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 26

```
dsolve(diff(y(x),x)=a*x+b*y(x),y(x), singsol=all)
```

$$y(x) = \frac{\mathrm{e}^{bx}c_1b^2 - axb - a}{b^2}$$

✓ Solution by Mathematica

Time used: 0.054 (sec). Leaf size: 25

```
DSolve[y'[x]==a*x+b*y[x],y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to -\frac{abx + a}{b^2} + c_1 e^{bx}$$

1.9 problem 9

Internal problem ID [7325]
Internal file name [OUTPUT/6306_Sunday_June_05_2022_04_39_22_PM_77223765/index.tex]

Book: First order enumerated odes
Section: section 1
Problem number: 9.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"quadrature"**

Maple gives the following as the ode type

```
[_quadrature]
```

$$\boxed{y' - y = 0}$$

1.9.1 Solving as quadrature ode

Integrating both sides gives

$$\begin{aligned}
\int \frac{1}{y} dy &= x + c_1 \\
\ln(y) &= x + c_1 \\
y &= \mathrm{e}^{x+c_1} \\
y &= c_1 \mathrm{e}^{x}
\end{aligned}$$

Summary
The solution(s) found are the following

$$y = c_1 \mathrm{e}^{x} \tag{1}$$

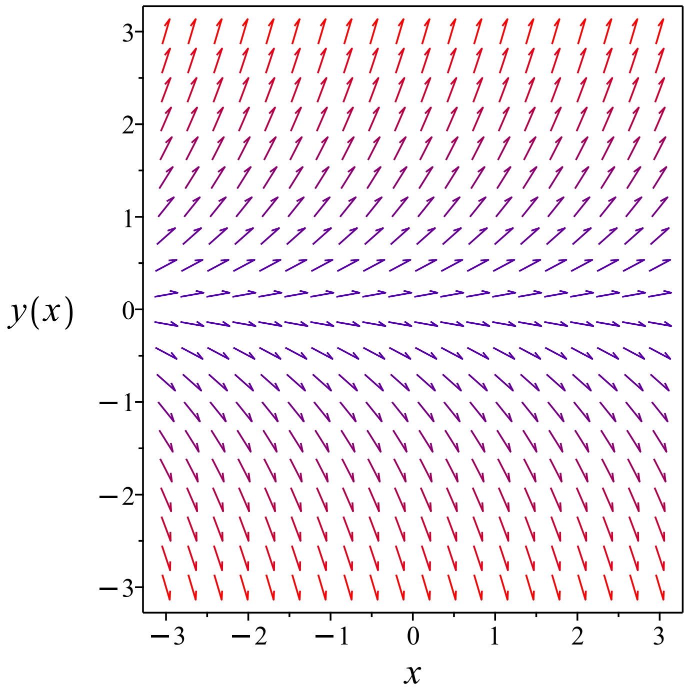


Figure 4: Slope field plot

Verification of solutions

$$y = c_1 e^x$$

Verified OK.

1.9.2 Maple step by step solution

Let's solve

$y' - y = 0$

- Highest derivative means the order of the ODE is 1

  y'

- Separate variables

  $\frac{y'}{y} = 1$

- Integrate both sides with respect to x

  $\int \frac{y'}{y} dx = \int 1 dx + c_1$

- Evaluate integral

$$\ln(y) = x + c_1$$

- Solve for y

$$y = e^{x+c_1}$$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
<- 1st order linear successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 8

```
dsolve(diff(y(x),x)=y(x),y(x), singsol=all)
```

$$y(x) = c_1 e^x$$

✓ Solution by Mathematica

Time used: 0.022 (sec). Leaf size: 16

```
DSolve[y'[x]==y[x],y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to c_1 e^x$$
$$y(x) \to 0$$

1.10 problem 10

Internal problem ID [7326]
Internal file name [OUTPUT/6307_Sunday_June_05_2022_04_39_24_PM_88182402/index.tex]

Book: First order enumerated odes
Section: section 1
Problem number: 10.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"quadrature"**

Maple gives the following as the ode type

`[_quadrature]`

$$\boxed{y' - by = 0}$$

1.10.1 Solving as quadrature ode

Integrating both sides gives

$$\int \frac{1}{by} dy = \int dx$$
$$\frac{\ln(y)}{b} = x + c_1$$

Raising both side to exponential gives

$$e^{\frac{\ln(y)}{b}} = e^{x+c_1}$$

Which simplifies to

$$y^{\frac{1}{b}} = c_2 e^x$$

Summary
The solution(s) found are the following

$$y = (c_2 e^x)^b \tag{1}$$

Verification of solutions

$$y = (c_2 e^x)^b$$

Verified OK.

1.10.2 Maple step by step solution

Let's solve

$y' - by = 0$

- Highest derivative means the order of the ODE is 1

  y'

- Separate variables

  $\frac{y'}{y} = b$

- Integrate both sides with respect to x

  $\int \frac{y'}{y} dx = \int b dx + c_1$

- Evaluate integral

  $\ln(y) = xb + c_1$

- Solve for y

  $y = e^{xb + c_1}$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
<- 1st order linear successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 10

```
dsolve(diff(y(x),x)=b*y(x),y(x), singsol=all)
```

$$y(x) = e^{bx}c_1$$

✓ Solution by Mathematica

Time used: 0.03 (sec). Leaf size: 18

```
DSolve[y'[x]==b*y[x],y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to c_1 e^{bx}$$
$$y(x) \to 0$$

1.11 problem 11

Internal problem ID [7327]
Internal file name [OUTPUT/6308_Sunday_June_05_2022_04_39_26_PM_47091222/index.tex]

Book: First order enumerated odes
Section: section 1
Problem number: 11.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"riccati"**

Maple gives the following as the ode type

```
[[_Riccati, _special]]
```

$$\boxed{y' - by^2 = xa}$$

1.11.1 Solving as riccati ode

In canonical form the ODE is

$$\begin{aligned} y' &= F(x, y) \\ &= b\,y^2 + xa \end{aligned}$$

This is a Riccati ODE. Comparing the ODE to solve

$$y' = b\,y^2 + xa$$

With Riccati ODE standard form

$$y' = f_0(x) + f_1(x)y + f_2(x)y^2$$

Shows that $f_0(x) = xa$, $f_1(x) = 0$ and $f_2(x) = b$. Let

$$\begin{aligned} y &= \frac{-u'}{f_2 u} \\ &= \frac{-u'}{bu} \end{aligned} \tag{1}$$

Using the above substitution in the given ODE results (after some simplification)in a second order ODE to solve for $u(x)$ which is

$$f_2 u''(x) - (f_2' + f_1 f_2)\, u'(x) + f_2^2 f_0 u(x) = 0 \tag{2}$$

But

$$\begin{aligned} f_2' &= 0 \\ f_1 f_2 &= 0 \\ f_2^2 f_0 &= b^2 x a \end{aligned}$$

Substituting the above terms back in equation (2) gives

$$b u''(x) + b^2 x a u(x) = 0$$

Solving the above ODE (this ode solved using Maple, not this program), gives

$$u(x) = c_1 \,\text{AiryAi}\left(-(ab)^{\frac{1}{3}}\, x\right) + c_2 \,\text{AiryBi}\left(-(ab)^{\frac{1}{3}}\, x\right)$$

The above shows that

$$u'(x) = \left(-\,\text{AiryAi}\left(1, -(ab)^{\frac{1}{3}}\, x\right) c_1 - \text{AiryBi}\left(1, -(ab)^{\frac{1}{3}}\, x\right) c_2\right) (ab)^{\frac{1}{3}}$$

Using the above in (1) gives the solution

$$y = -\frac{\left(-\,\text{AiryAi}\left(1, -(ab)^{\frac{1}{3}}\, x\right) c_1 - \text{AiryBi}\left(1, -(ab)^{\frac{1}{3}}\, x\right) c_2\right) (ab)^{\frac{1}{3}}}{b\left(c_1 \,\text{AiryAi}\left(-\,(ab)^{\frac{1}{3}}\, x\right) + c_2 \,\text{AiryBi}\left(-\,(ab)^{\frac{1}{3}}\, x\right)\right)}$$

Dividing both numerator and denominator by c_1 gives, after renaming the constant $\frac{c_2}{c_1} = c_3$ the following solution

$$y = \frac{\left(\text{AiryAi}\left(1, -(ab)^{\frac{1}{3}}\, x\right) c_3 + \text{AiryBi}\left(1, -(ab)^{\frac{1}{3}}\, x\right)\right) (ab)^{\frac{1}{3}}}{b\left(c_3 \,\text{AiryAi}\left(-\,(ab)^{\frac{1}{3}}\, x\right) + \text{AiryBi}\left(-\,(ab)^{\frac{1}{3}}\, x\right)\right)}$$

Summary

The solution(s) found are the following

$$y = \frac{\left(\text{AiryAi}\left(1, -(ab)^{\frac{1}{3}}\, x\right) c_3 + \text{AiryBi}\left(1, -(ab)^{\frac{1}{3}}\, x\right)\right) (ab)^{\frac{1}{3}}}{b\left(c_3 \,\text{AiryAi}\left(-\,(ab)^{\frac{1}{3}}\, x\right) + \text{AiryBi}\left(-\,(ab)^{\frac{1}{3}}\, x\right)\right)} \tag{1}$$

Verification of solutions

$$y = \frac{\left(\operatorname{AiryAi}\left(1, -(ab)^{\frac{1}{3}} x\right) c_3 + \operatorname{AiryBi}\left(1, -(ab)^{\frac{1}{3}} x\right)\right) (ab)^{\frac{1}{3}}}{b \left(c_3 \operatorname{AiryAi}\left(-(ab)^{\frac{1}{3}} x\right) + \operatorname{AiryBi}\left(-(ab)^{\frac{1}{3}} x\right)\right)}$$

Verified OK.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
trying homogeneous types:
trying Chini
differential order: 1; looking for linear symmetries
trying exact
Looking for potential symmetries
trying Riccati
trying Riccati Special
<- Riccati Special successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 59

```
dsolve(diff(y(x),x)=a*x+b*y(x)^2,y(x), singsol=all)
```

$$y(x) = \frac{(ab)^{\frac{1}{3}} \left(\operatorname{AiryAi}\left(1, -(ab)^{\frac{1}{3}} x\right) c_1 + \operatorname{AiryBi}\left(1, -(ab)^{\frac{1}{3}} x\right)\right)}{b \left(c_1 \operatorname{AiryAi}\left(-(ab)^{\frac{1}{3}} x\right) + \operatorname{AiryBi}\left(-(ab)^{\frac{1}{3}} x\right)\right)}$$

✓ Solution by Mathematica

Time used: 0.163 (sec). Leaf size: 331

```
DSolve[y'[x]==a*x+b*y[x]^2,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \frac{\sqrt{a}\sqrt{b}x^{3/2}\left(-2\,\text{BesselJ}\left(-\frac{2}{3},\frac{2}{3}\sqrt{a}\sqrt{b}x^{3/2}\right)+c_1\left(\text{BesselJ}\left(\frac{2}{3},\frac{2}{3}\sqrt{a}\sqrt{b}x^{3/2}\right)-\text{BesselJ}\left(-\frac{4}{3},\frac{2}{3}\sqrt{a}\sqrt{b}x^{3/2}\right)\right.\right.}{2bx\left(\text{BesselJ}\left(\frac{1}{3},\frac{2}{3}\sqrt{a}\sqrt{b}x^{3/2}\right)+c_1\,\text{BesselJ}\left(-\frac{1}{3},\frac{2}{3}\sqrt{a}\sqrt{b}x^{3/2}\right)\right)}$$

$$y(x) \to -\frac{\sqrt{a}\sqrt{b}x^{3/2}\,\text{BesselJ}\left(-\frac{4}{3},\frac{2}{3}\sqrt{a}\sqrt{b}x^{3/2}\right)-\sqrt{a}\sqrt{b}x^{3/2}\,\text{BesselJ}\left(\frac{2}{3},\frac{2}{3}\sqrt{a}\sqrt{b}x^{3/2}\right)+\text{BesselJ}\left(-\frac{1}{3},\frac{2}{3}\sqrt{a}\sqrt{b}x\right.}{2bx\,\text{BesselJ}\left(-\frac{1}{3},\frac{2}{3}\sqrt{a}\sqrt{b}x^{3/2}\right)}$$

1.12 problem 12

Internal problem ID [7328]
Internal file name [`OUTPUT/6309_Sunday_June_05_2022_04_39_29_PM_10393416/index.tex`]

Book: First order enumerated odes
Section: section 1
Problem number: 12.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"quadrature"**

Maple gives the following as the ode type

`[_quadrature]`

$$\boxed{y'c = 0}$$

1.12.1 Solving as quadrature ode

Integrating both sides gives

$$\begin{aligned} y &= \int 0 \, \mathrm{d}x \\ &= c_1 \end{aligned}$$

Summary
The solution(s) found are the following

$$y = c_1 \tag{1}$$

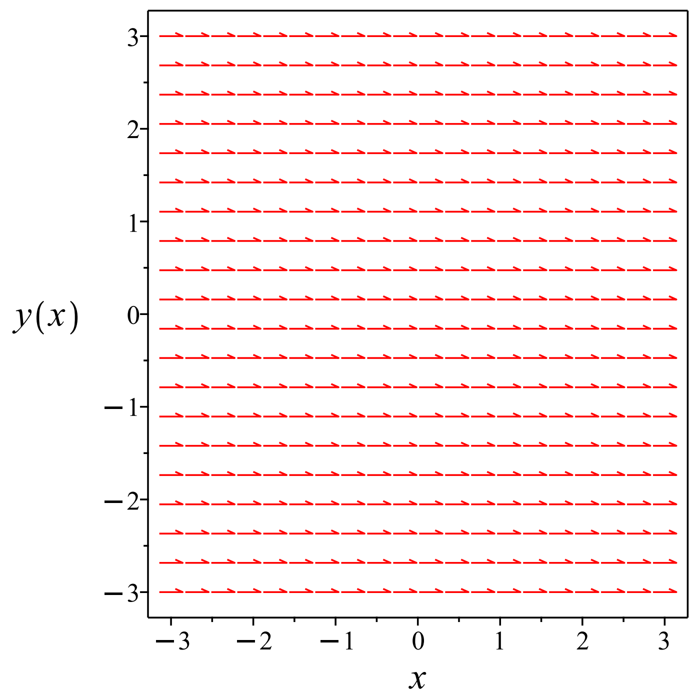


Figure 5: Slope field plot

Verification of solutions

$$y = c_1$$

Verified OK.

1.12.2 Maple step by step solution

Let's solve

$y'c = 0$

- Highest derivative means the order of the ODE is 1

  y'

- Integrate both sides with respect to x

  $\int y'c dx = \int 0 dx + c_1$

- Evaluate integral

  $cy = c_1$

- Solve for y

  $y = \frac{c_1}{c}$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
<- 1st order linear successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 5

```
dsolve(c*diff(y(x),x)=0,y(x), singsol=all)
```

$$y(x) = c_1$$

✓ Solution by Mathematica

Time used: 0.002 (sec). Leaf size: 7

```
DSolve[c*y'[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to c_1$$

1.13 problem 13

Internal problem ID [7329]
Internal file name [`OUTPUT/6310_Sunday_June_05_2022_04_39_31_PM_47734585/index.tex`]

Book: First order enumerated odes
Section: section 1
Problem number: 13.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"quadrature"**

Maple gives the following as the ode type

```
[_quadrature]
```

$$\boxed{y'c = a}$$

1.13.1 Solving as quadrature ode

Integrating both sides gives

$$\begin{aligned} y &= \int \frac{a}{c}\,\mathrm{d}x \\ &= \frac{xa}{c} + c_1 \end{aligned}$$

Summary
The solution(s) found are the following

$$y = \frac{xa}{c} + c_1 \tag{1}$$

Verification of solutions

$$y = \frac{xa}{c} + c_1$$

Verified OK.

1.13.2 Maple step by step solution

Let's solve

$y'c = a$

- Highest derivative means the order of the ODE is 1

  y'

- Integrate both sides with respect to x

  $\int y'cdx = \int adx + c_1$

- Evaluate integral

  $cy = xa + c_1$

- Solve for y

  $y = \frac{xa+c_1}{c}$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
<- 1st order linear successful`
```

✓ Solution by Maple

Time used: 0.015 (sec). Leaf size: 12

```
dsolve(c*diff(y(x),x)=a,y(x), singsol=all)
```

$$y(x) = \frac{ax}{c} + c_1$$

✓ Solution by Mathematica

Time used: 0.002 (sec). Leaf size: 14

```
DSolve[c*y'[x]==a,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \frac{ax}{c} + c_1$$

1.14 problem 14

Internal problem ID [7330]
Internal file name [`OUTPUT/6311_Sunday_June_05_2022_04_39_32_PM_74264084/index.tex`]

Book: First order enumerated odes
Section: section 1
Problem number: 14.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"quadrature"**

Maple gives the following as the ode type

`[_quadrature]`

$$\boxed{y'c = xa}$$

1.14.1 Solving as quadrature ode

Integrating both sides gives

$$\begin{aligned} y &= \int \frac{xa}{c}\, \mathrm{d}x \\ &= \frac{x^2 a}{2c} + c_1 \end{aligned}$$

Summary
The solution(s) found are the following

$$y = \frac{x^2 a}{2c} + c_1 \tag{1}$$

Verification of solutions

$$y = \frac{x^2 a}{2c} + c_1$$

Verified OK.

1.14.2 Maple step by step solution

Let's solve

$y'c = xa$

- Highest derivative means the order of the ODE is 1

  y'

- Integrate both sides with respect to x

  $\int y'cdx = \int xadx + c_1$

- Evaluate integral

  $cy = \frac{a\,x^2}{2} + c_1$

- Solve for y

  $y = \frac{a\,x^2+2c_1}{2c}$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
<- quadrature successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 15

```
dsolve(c*diff(y(x),x)=a*x,y(x), singsol=all)
```

$$y(x) = \frac{a\,x^2}{2c} + c_1$$

✓ Solution by Mathematica

Time used: 0.002 (sec). Leaf size: 19

```
DSolve[c*y'[x]==a*x,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \frac{ax^2}{2c} + c_1$$

1.15 problem 15

Internal problem ID [7331]
Internal file name [`OUTPUT/6312_Sunday_June_05_2022_04_39_34_PM_10323726/index.tex`]

Book: First order enumerated odes
Section: section 1
Problem number: 15.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"linear", "exactWithIntegrationFactor", "first_order_ode_lie_symmetry_lookup"**

Maple gives the following as the ode type

```
[[_linear, `class A`]]
```

$$\boxed{y'c - y = xa}$$

1.15.1 Solving as linear ode

Entering Linear first order ODE solver. In canonical form a linear first order is

$$y' + p(x)y = q(x)$$

Where here

$$p(x) = -\frac{1}{c}$$
$$q(x) = \frac{xa}{c}$$

Hence the ode is

$$y' - \frac{y}{c} = \frac{xa}{c}$$

The integrating factor μ is

$$\mu = e^{\int -\frac{1}{c}dx}$$
$$= e^{-\frac{x}{c}}$$

The ode becomes

$$\frac{d}{dx}(\mu y) = (\mu)\left(\frac{xa}{c}\right)$$
$$\frac{d}{dx}\left(e^{-\frac{x}{c}}y\right) = \left(e^{-\frac{x}{c}}\right)\left(\frac{xa}{c}\right)$$
$$d\left(e^{-\frac{x}{c}}y\right) = \left(\frac{xa\,e^{-\frac{x}{c}}}{c}\right) dx$$

Integrating gives

$$e^{-\frac{x}{c}}y = \int \frac{xa\,e^{-\frac{x}{c}}}{c}\, dx$$
$$e^{-\frac{x}{c}}y = -(c+x)\,a\,e^{-\frac{x}{c}} + c_1$$

Dividing both sides by the integrating factor $\mu = e^{-\frac{x}{c}}$ results in

$$y = -e^{\frac{x}{c}}(c+x)\,a\,e^{-\frac{x}{c}} + c_1 e^{\frac{x}{c}}$$

which simplifies to

$$y = -a(c+x) + c_1 e^{\frac{x}{c}}$$

Summary

The solution(s) found are the following

$$y = -a(c+x) + c_1 e^{\frac{x}{c}} \tag{1}$$

Verification of solutions

$$y = -a(c+x) + c_1 e^{\frac{x}{c}}$$

Verified OK.

1.15.2 Solving as first order ode lie symmetry lookup ode

Writing the ode as

$$y' = \frac{xa + y}{c}$$
$$y' = \omega(x, y)$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x + \omega(\eta_y - \xi_x) - \omega^2 \xi_y - \omega_x \xi - \omega_y \eta = 0 \tag{A}$$

The type of this ode is known. It is of type `linear`. Therefore we do not need to solve the PDE (A), and can just use the lookup table shown below to find ξ, η

Table 20: Lie symmetry infinitesimal lookup table for known first order ODE's

ODE class	Form	ξ	η
linear ode	$y' = f(x)y(x) + g(x)$	0	$e^{\int f dx}$
separable ode	$y' = f(x)\, g(y)$	$\frac{1}{f}$	0
quadrature ode	$y' = f(x)$	0	1
quadrature ode	$y' = g(y)$	1	0
homogeneous ODEs of Class A	$y' = f\left(\frac{y}{x}\right)$	x	y
homogeneous ODEs of Class C	$y' = (a + bx + cy)^{\frac{n}{m}}$	1	$-\frac{b}{c}$
homogeneous class D	$y' = \frac{y}{x} + g(x)\, F\left(\frac{y}{x}\right)$	x^2	xy
First order special form ID 1	$y' = g(x)\, e^{h(x)+by} + f(x)$	$\frac{e^{-\int bf(x)dx - h(x)}}{g(x)}$	$\frac{f(x)e^{-\int bf(x)dx - h(x)}}{g(x)}$
polynomial type ode	$y' = \frac{a_1x+b_1y+c_1}{a_2x+b_2y+c_2}$	$\frac{a_1b_2x - a_2b_1x - b_1c_2 + b_2c_1}{a_1b_2 - a_2b_1}$	$\frac{a_1b_2y - a_2b_1y - a_1c_2 - a_2c_1}{a_1b_2 - a_2b_1}$
Bernoulli ode	$y' = f(x)\, y + g(x)\, y^n$	0	$e^{-\int (n-1)f(x)dx} y^n$
Reduced Riccati	$y' = f_1(x)\, y + f_2(x)\, y^2$	0	$e^{-\int f_1 dx}$

The above table shows that

$$
\begin{aligned}
\xi(x, y) &= 0 \\
\eta(x, y) &= \mathrm{e}^{\frac{x}{c}}
\end{aligned} \tag{A1}
$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x, y) \to (R, S)$ where (R, S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$
\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}
$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y}\right) S(x, y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$
R = x
$$

S is found from

$$
\begin{aligned}
S &= \int \frac{1}{\eta} dy \\
&= \int \frac{1}{\mathrm{e}^{\frac{x}{c}}} dy
\end{aligned}
$$

Which results in

$$
S = \mathrm{e}^{-\frac{x}{c}} y
$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$
\frac{dS}{dR} = \frac{S_x + \omega(x, y) S_y}{R_x + \omega(x, y) R_y} \tag{2}
$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x, y)$ is the right hand side of the original ode given by

$$
\omega(x, y) = \frac{xa + y}{c}
$$

Evaluating all the partial derivatives gives

$$
\begin{aligned}
R_x &= 1 \\
R_y &= 0 \\
S_x &= -\frac{\mathrm{e}^{-\frac{x}{c}} y}{c} \\
S_y &= \mathrm{e}^{-\frac{x}{c}}
\end{aligned}
$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = \frac{x a \,\mathrm{e}^{-\frac{x}{c}}}{c} \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = \frac{R a \,\mathrm{e}^{-\frac{R}{c}}}{c}$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$S(R) = -(R + c)\, a \,\mathrm{e}^{-\frac{R}{c}} + c_1 \tag{4}$$

To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$\mathrm{e}^{-\frac{x}{c}} y = -(c + x)\, a \,\mathrm{e}^{-\frac{x}{c}} + c_1$$

Which simplifies to

$$(a(c + x) + y)\,\mathrm{e}^{-\frac{x}{c}} - c_1 = 0$$

Which gives

$$y = -\left(a \,\mathrm{e}^{-\frac{x}{c}} c + x a \,\mathrm{e}^{-\frac{x}{c}} - c_1\right) \mathrm{e}^{\frac{x}{c}}$$

Summary

The solution(s) found are the following

$$y = -\left(a \,\mathrm{e}^{-\frac{x}{c}} c + x a \,\mathrm{e}^{-\frac{x}{c}} - c_1\right) \mathrm{e}^{\frac{x}{c}} \tag{1}$$

Verification of solutions

$$y = -\left(a \,\mathrm{e}^{-\frac{x}{c}} c + x a \,\mathrm{e}^{-\frac{x}{c}} - c_1\right) \mathrm{e}^{\frac{x}{c}}$$

Verified OK.

1.15.3 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x,y)+N(x,y)\frac{dy}{dx}=0 \tag{A}$$

We assume there exists a function $\phi(x,y)=c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y)=0$$

Hence

$$\frac{\partial\phi}{\partial x}+\frac{\partial\phi}{\partial y}\frac{dy}{dx}=0 \tag{B}$$

Comparing (A,B) shows that

$$\begin{aligned}\frac{\partial\phi}{\partial x}&=M\\ \frac{\partial\phi}{\partial y}&=N\end{aligned}$$

But since $\frac{\partial^2\phi}{\partial x\partial y}=\frac{\partial^2\phi}{\partial y\partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y}=\frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x,y)$ but at least we know now that we can do that since the condition $\frac{\partial^2\phi}{\partial x\partial y}=\frac{\partial^2\phi}{\partial y\partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x,y)\,\mathrm{d}x+N(x,y)\,\mathrm{d}y=0 \tag{1A}$$

Therefore

$$\begin{aligned}(c)\,\mathrm{d}y&=(xa+y)\,\mathrm{d}x\\ (-xa-y)\,\mathrm{d}x+(c)\,\mathrm{d}y&=0 \end{aligned}\tag{2A}$$

Comparing (1A) and (2A) shows that

$$\begin{aligned}M(x,y)&=-xa-y\\ N(x,y)&=c\end{aligned}$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Using result found above gives

$$\begin{aligned}\frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}(-xa - y)\\ &= -1\end{aligned}$$

And

$$\begin{aligned}\frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}(c)\\ &= 0\end{aligned}$$

Since $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$, then the ODE is <u>not exact</u>. Since the ODE is not exact, we will try to find an integrating factor to make it exact. Let

$$\begin{aligned}A &= \frac{1}{N}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right)\\ &= \frac{1}{c}((-1) - (0))\\ &= -\frac{1}{c}\end{aligned}$$

Since A does not depend on y, then it can be used to find an integrating factor. The integrating factor μ is

$$\begin{aligned}\mu &= e^{\int A\,\mathrm{d}x}\\ &= e^{\int -\frac{1}{c}\,\mathrm{d}x}\end{aligned}$$

The result of integrating gives

$$\begin{aligned}\mu &= e^{-\frac{x}{c}}\\ &= \mathrm{e}^{-\frac{x}{c}}\end{aligned}$$

M and N are multiplied by this integrating factor, giving new M and new N which are called $\overline{M}$ and $\overline{N}$ for now so not to confuse them with the original M and N.

$$\begin{aligned}\overline{M} &= \mu M\\ &= \mathrm{e}^{-\frac{x}{c}}(-xa - y)\\ &= -\mathrm{e}^{-\frac{x}{c}}(xa + y)\end{aligned}$$

And

$$\begin{aligned}\overline{N} &= \mu N \\ &= \mathrm{e}^{-\frac{x}{c}}(c) \\ &= c\,\mathrm{e}^{-\frac{x}{c}}\end{aligned}$$

Now a modified ODE is ontained from the original ODE, which is exact and can be solved. The modified ODE is

$$\begin{aligned}\overline{M} + \overline{N}\frac{\mathrm{d}y}{\mathrm{d}x} &= 0 \\ \left(-\mathrm{e}^{-\frac{x}{c}}(xa+y)\right) + \left(c\,\mathrm{e}^{-\frac{x}{c}}\right)\frac{\mathrm{d}y}{\mathrm{d}x} &= 0\end{aligned}$$

The following equations are now set up to solve for the function $\phi(x,y)$

$$\frac{\partial\phi}{\partial x} = \overline{M} \tag{1}$$

$$\frac{\partial\phi}{\partial y} = \overline{N} \tag{2}$$

Integrating (1) w.r.t. x gives

$$\begin{aligned}\int \frac{\partial\phi}{\partial x}\,\mathrm{d}x &= \int \overline{M}\,\mathrm{d}x \\ \int \frac{\partial\phi}{\partial x}\,\mathrm{d}x &= \int -\mathrm{e}^{-\frac{x}{c}}(xa+y)\,\mathrm{d}x \\ \phi &= (a(c+x)+y)\,\mathrm{e}^{-\frac{x}{c}}c + f(y) \qquad (3)\end{aligned}$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$\frac{\partial\phi}{\partial y} = c\,\mathrm{e}^{-\frac{x}{c}} + f'(y) \tag{4}$$

But equation (2) says that $\frac{\partial\phi}{\partial y} = c\,\mathrm{e}^{-\frac{x}{c}}$. Therefore equation (4) becomes

$$c\,\mathrm{e}^{-\frac{x}{c}} = c\,\mathrm{e}^{-\frac{x}{c}} + f'(y) \tag{5}$$

Solving equation (5) for $f'(y)$ gives

$$f'(y) = 0$$

Therefore

$$f(y) = c_1$$

Where c_1 is constant of integration. Substituting this result for $f(y)$ into equation (3) gives ϕ

$$\phi = (a(c + x) + y)\,\mathrm{e}^{-\frac{x}{c}} c + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = (a(c + x) + y)\,\mathrm{e}^{-\frac{x}{c}} c$$

The solution becomes

$$y = -\frac{\left(\mathrm{e}^{-\frac{x}{c}} a\, c^2 + \mathrm{e}^{-\frac{x}{c}} acx - c_1\right)\mathrm{e}^{\frac{x}{c}}}{c}$$

Summary

The solution(s) found are the following

$$y = -\frac{\left(\mathrm{e}^{-\frac{x}{c}} a\, c^2 + \mathrm{e}^{-\frac{x}{c}} acx - c_1\right)\mathrm{e}^{\frac{x}{c}}}{c} \tag{1}$$

Verification of solutions

$$y = -\frac{\left(\mathrm{e}^{-\frac{x}{c}} a\, c^2 + \mathrm{e}^{-\frac{x}{c}} acx - c_1\right)\mathrm{e}^{\frac{x}{c}}}{c}$$

Verified OK.

1.15.4 Maple step by step solution

Let's solve

$y'c - y = xa$

- Highest derivative means the order of the ODE is 1

  y'

- Isolate the derivative

  $y' = \frac{y}{c} + \frac{xa}{c}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE

  $y' - \frac{y}{c} = \frac{xa}{c}$

- The ODE is linear; multiply by an integrating factor $\mu(x)$

  $\mu(x)\left(y' - \frac{y}{c}\right) = \frac{\mu(x)xa}{c}$
- Assume the lhs of the ODE is the total derivative $\frac{d}{dx}(\mu(x)\, y)$

  $\mu(x)\left(y' - \frac{y}{c}\right) = \mu'(x)\, y + \mu(x)\, y'$
- Isolate $\mu'(x)$

  $\mu'(x) = -\frac{\mu(x)}{c}$
- Solve to find the integrating factor

  $\mu(x) = \mathrm{e}^{-\frac{x}{c}}$
- Integrate both sides with respect to x

  $\int \left(\frac{d}{dx}(\mu(x)\, y)\right) dx = \int \frac{\mu(x)xa}{c} dx + c_1$
- Evaluate the integral on the lhs

  $\mu(x)\, y = \int \frac{\mu(x)xa}{c} dx + c_1$
- Solve for y

  $y = \frac{\int \frac{\mu(x)xa}{c} dx + c_1}{\mu(x)}$
- Substitute $\mu(x) = \mathrm{e}^{-\frac{x}{c}}$

  $y = \frac{\int \frac{xa\,\mathrm{e}^{-\frac{x}{c}}}{c} dx + c_1}{\mathrm{e}^{-\frac{x}{c}}}$
- Evaluate the integrals on the rhs

  $y = \frac{-(c+x)a\,\mathrm{e}^{-\frac{x}{c}} + c_1}{\mathrm{e}^{-\frac{x}{c}}}$
- Simplify

  $y = -a(c+x) + c_1\mathrm{e}^{\frac{x}{c}}$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
<- 1st order linear successful`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 19

```
dsolve(c*diff(y(x),x)=a*x+y(x),y(x), singsol=all)
```

$$y(x) = \mathrm{e}^{\frac{x}{c}} c_1 - a(c + x)$$

✓ Solution by Mathematica

Time used: 0.05 (sec). Leaf size: 22

```
DSolve[c*y'[x]==a*x+y[x],y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to -a(c + x) + c_1 e^{\frac{x}{c}}$$

1.16 problem 16

Internal problem ID [7332]
Internal file name [`OUTPUT/6313_Sunday_June_05_2022_04_39_36_PM_41636995/index.tex`]

Book: First order enumerated odes
Section: section 1
Problem number: 16.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"linear", "exactWithIntegrationFactor", "first_order_ode_lie_symmetry_lookup"**

Maple gives the following as the ode type

```
[[_linear, `class A`]]
```

$$\boxed{y'c - by = xa}$$

1.16.1 Solving as linear ode

Entering Linear first order ODE solver. In canonical form a linear first order is

$$y' + p(x)y = q(x)$$

Where here

$$p(x) = -\frac{b}{c}$$
$$q(x) = \frac{xa}{c}$$

Hence the ode is

$$y' - \frac{by}{c} = \frac{xa}{c}$$

The integrating factor μ is

$$\mu = e^{\int -\frac{b}{c}dx} \\ = e^{-\frac{bx}{c}}$$

The ode becomes

$$\frac{d}{dx}(\mu y) = (\mu)\left(\frac{xa}{c}\right)$$
$$\frac{d}{dx}\left(e^{-\frac{bx}{c}} y\right) = \left(e^{-\frac{bx}{c}}\right)\left(\frac{xa}{c}\right)$$
$$d\left(e^{-\frac{bx}{c}} y\right) = \left(\frac{xa\, e^{-\frac{bx}{c}}}{c}\right) dx$$

Integrating gives

$$e^{-\frac{bx}{c}} y = \int \frac{xa\, e^{-\frac{bx}{c}}}{c}\, dx$$
$$e^{-\frac{bx}{c}} y = -\frac{(xb+c)\, a\, e^{-\frac{bx}{c}}}{b^2} + c_1$$

Dividing both sides by the integrating factor $\mu = e^{-\frac{bx}{c}}$ results in

$$y = -\frac{e^{\frac{bx}{c}} (xb+c)\, a\, e^{-\frac{bx}{c}}}{b^2} + c_1 e^{\frac{bx}{c}}$$

which simplifies to

$$y = \frac{c_1 e^{\frac{bx}{c}} b^2 - a(xb+c)}{b^2}$$

Summary

The solution(s) found are the following

$$y = \frac{c_1 e^{\frac{bx}{c}} b^2 - a(xb+c)}{b^2} \tag{1}$$

Verification of solutions

$$y = \frac{c_1 e^{\frac{bx}{c}} b^2 - a(xb+c)}{b^2}$$

Verified OK.

1.16.2 Solving as first order ode lie symmetry lookup ode

Writing the ode as

$$y' = \frac{xa + by}{c}$$
$$y' = \omega(x, y)$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x + \omega(\eta_y - \xi_x) - \omega^2\xi_y - \omega_x\xi - \omega_y\eta = 0 \tag{A}$$

The type of this ode is known. It is of type `linear`. Therefore we do not need to solve the PDE (A), and can just use the lookup table shown below to find ξ, η

Table 23: Lie symmetry infinitesimal lookup table for known first order ODE's

ODE class	Form	ξ	η
linear ode	$y' = f(x)y(x) + g(x)$	0	$e^{\int f dx}$
separable ode	$y' = f(x)\, g(y)$	$\frac{1}{f}$	0
quadrature ode	$y' = f(x)$	0	1
quadrature ode	$y' = g(y)$	1	0
homogeneous ODEs of Class A	$y' = f\left(\frac{y}{x}\right)$	x	y
homogeneous ODEs of Class C	$y' = (a + bx + cy)^{\frac{n}{m}}$	1	$-\frac{b}{c}$
homogeneous class D	$y' = \frac{y}{x} + g(x)\, F\left(\frac{y}{x}\right)$	x^2	xy
First order special form ID 1	$y' = g(x)\, e^{h(x)+by} + f(x)$	$\frac{e^{-\int bf(x)dx - h(x)}}{g(x)}$	$\frac{f(x)e^{-\int bf(x)dx - h(x)}}{g(x)}$
polynomial type ode	$y' = \frac{a_1x+b_1y+c_1}{a_2x+b_2y+c_2}$	$\frac{a_1b_2x - a_2b_1x - b_1c_2 + b_2c_1}{a_1b_2 - a_2b_1}$	$\frac{a_1b_2y - a_2b_1y - a_1c_2 - a_2c_1}{a_1b_2 - a_2b_1}$
Bernoulli ode	$y' = f(x)\, y + g(x)\, y^n$	0	$e^{-\int (n-1)f(x)dx} y^n$
Reduced Riccati	$y' = f_1(x)\, y + f_2(x)\, y^2$	0	$e^{-\int f_1 dx}$

The above table shows that

$$
\begin{aligned}
\xi(x,y) &= 0 \\
\eta(x,y) &= \mathrm{e}^{\frac{bx}{c}}
\end{aligned}
\tag{A1}
$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x,y) \to (R,S)$ where (R,S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$
\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}
$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y}\right) S(x,y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$
R = x
$$

S is found from

$$
\begin{aligned}
S &= \int \frac{1}{\eta} dy \\
&= \int \frac{1}{\mathrm{e}^{\frac{bx}{c}}} dy
\end{aligned}
$$

Which results in

$$
S = \mathrm{e}^{-\frac{bx}{c}} y
$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$
\frac{dS}{dR} = \frac{S_x + \omega(x,y) S_y}{R_x + \omega(x,y) R_y} \tag{2}
$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x,y)$ is the right hand side of the original ode given by

$$
\omega(x,y) = \frac{xa + by}{c}
$$

Evaluating all the partial derivatives gives

$$
\begin{aligned}
R_x &= 1 \\
R_y &= 0 \\
S_x &= -\frac{b\,\mathrm{e}^{-\frac{bx}{c}} y}{c} \\
S_y &= \mathrm{e}^{-\frac{bx}{c}}
\end{aligned}
$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$
\frac{dS}{dR} = \frac{xa\,\mathrm{e}^{-\frac{bx}{c}}}{c} \tag{2A}
$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$
\frac{dS}{dR} = \frac{Ra\,\mathrm{e}^{-\frac{bR}{c}}}{c}
$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$
S(R) = -\frac{(bR + c)\,a\,\mathrm{e}^{-\frac{bR}{c}}}{b^2} + c_1 \tag{4}
$$

To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$
\mathrm{e}^{-\frac{bx}{c}} y = -\frac{(xb + c)\,a\,\mathrm{e}^{-\frac{bx}{c}}}{b^2} + c_1
$$

Which simplifies to

$$
\mathrm{e}^{-\frac{bx}{c}} y = -\frac{(xb + c)\,a\,\mathrm{e}^{-\frac{bx}{c}}}{b^2} + c_1
$$

Which gives

$$
y = -\frac{\left(xab\,\mathrm{e}^{-\frac{bx}{c}} + a\,\mathrm{e}^{-\frac{bx}{c}} c - c_1 b^2\right) \mathrm{e}^{\frac{bx}{c}}}{b^2}
$$

<u>Summary</u>
The solution(s) found are the following

$$
y = -\frac{\left(xab\,\mathrm{e}^{-\frac{bx}{c}} + a\,\mathrm{e}^{-\frac{bx}{c}} c - c_1 b^2\right) \mathrm{e}^{\frac{bx}{c}}}{b^2} \tag{1}
$$

Verification of solutions

$$y = -\frac{\left(xab\,\mathrm{e}^{-\frac{bx}{c}} + a\,\mathrm{e}^{-\frac{bx}{c}}c - c_1 b^2\right)\mathrm{e}^{\frac{bx}{c}}}{b^2}$$

Verified OK.

1.16.3 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x,y) + N(x,y)\frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x,y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y) = 0$$

Hence

$$\frac{\partial \phi}{\partial x} + \frac{\partial \phi}{\partial y}\frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$\begin{aligned}\frac{\partial \phi}{\partial x} &= M\\ \frac{\partial \phi}{\partial y} &= N\end{aligned}$$

But since $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x,y)$ but at least we know now that we can do that since the condition $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x,y)\,\mathrm{d}x + N(x,y)\,\mathrm{d}y = 0 \tag{1A}$$

Therefore

$$
\begin{aligned}
(c)\,\mathrm{d}y &= (xa+by)\,\mathrm{d}x \\
(-xa-by)\,\mathrm{d}x+(c)\,\mathrm{d}y &= 0 \qquad \text{(2A)}
\end{aligned}
$$

Comparing (1A) and (2A) shows that

$$
\begin{aligned}
M(x,y) &= -xa-by \\
N(x,y) &= c
\end{aligned}
$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$
\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}
$$

Using result found above gives

$$
\begin{aligned}
\frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}(-xa-by) \\
&= -b
\end{aligned}
$$

And

$$
\begin{aligned}
\frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}(c) \\
&= 0
\end{aligned}
$$

Since $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$, then the ODE is <u>not exact</u>. Since the ODE is not exact, we will try to find an integrating factor to make it exact. Let

$$
\begin{aligned}
A &= \frac{1}{N}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right) \\
&= \frac{1}{c}((-b)-(0)) \\
&= -\frac{b}{c}
\end{aligned}
$$

Since A does not depend on y, then it can be used to find an integrating factor. The integrating factor μ is

$$
\begin{aligned}
\mu &= e^{\int A\,\mathrm{d}x} \\
&= e^{\int -\frac{b}{c}\,\mathrm{d}x}
\end{aligned}
$$

The result of integrating gives

$$
\begin{aligned}
\mu &= e^{-\frac{bx}{c}} \\
&= \mathrm{e}^{-\frac{bx}{c}}
\end{aligned}
$$

M and N are multiplied by this integrating factor, giving new M and new N which are called $\overline{M}$ and $\overline{N}$ for now so not to confuse them with the original M and N.

$$
\begin{aligned}
\overline{M} &= \mu M \\
&= \mathrm{e}^{-\frac{bx}{c}}(-xa - by) \\
&= -\mathrm{e}^{-\frac{bx}{c}}(xa + by)
\end{aligned}
$$

And

$$
\begin{aligned}
\overline{N} &= \mu N \\
&= \mathrm{e}^{-\frac{bx}{c}}(c) \\
&= c\,\mathrm{e}^{-\frac{bx}{c}}
\end{aligned}
$$

Now a modified ODE is ontained from the original ODE, which is exact and can be solved. The modified ODE is

$$
\begin{aligned}
\overline{M} + \overline{N}\frac{\mathrm{d}y}{\mathrm{d}x} &= 0 \\
\left(-\mathrm{e}^{-\frac{bx}{c}}(xa + by)\right) + \left(c\,\mathrm{e}^{-\frac{bx}{c}}\right)\frac{\mathrm{d}y}{\mathrm{d}x} &= 0
\end{aligned}
$$

The following equations are now set up to solve for the function $\phi(x, y)$

$$
\frac{\partial \phi}{\partial x} = \overline{M} \tag{1}
$$

$$
\frac{\partial \phi}{\partial y} = \overline{N} \tag{2}
$$

Integrating (1) w.r.t. x gives

$$
\begin{aligned}
\int \frac{\partial \phi}{\partial x}\,\mathrm{d}x &= \int \overline{M}\,\mathrm{d}x \\
\int \frac{\partial \phi}{\partial x}\,\mathrm{d}x &= \int -\mathrm{e}^{-\frac{bx}{c}}(xa + by)\,\mathrm{d}x
\end{aligned}
$$

$$
\phi = \frac{c(abx + b^2 y + ac)\,\mathrm{e}^{-\frac{bx}{c}}}{b^2} + f(y) \tag{3}
$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$\frac{\partial \phi}{\partial y} = c\,\mathrm{e}^{-\frac{bx}{c}} + f'(y) \tag{4}$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = c\,\mathrm{e}^{-\frac{bx}{c}}$. Therefore equation (4) becomes

$$c\,\mathrm{e}^{-\frac{bx}{c}} = c\,\mathrm{e}^{-\frac{bx}{c}} + f'(y) \tag{5}$$

Solving equation (5) for $f'(y)$ gives

$$f'(y) = 0$$

Therefore

$$f(y) = c_1$$

Where c_1 is constant of integration. Substituting this result for $f(y)$ into equation (3) gives ϕ

$$\phi = \frac{c(abx + b^2 y + ac)\,\mathrm{e}^{-\frac{bx}{c}}}{b^2} + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = \frac{c(abx + b^2 y + ac)\,\mathrm{e}^{-\frac{bx}{c}}}{b^2}$$

The solution becomes

$$y = -\frac{\left(\mathrm{e}^{-\frac{bx}{c}} abcx + \mathrm{e}^{-\frac{bx}{c}} a\,c^2 - c_1 b^2\right) \mathrm{e}^{\frac{bx}{c}}}{b^2 c}$$

Summary

The solution(s) found are the following

$$y = -\frac{\left(\mathrm{e}^{-\frac{bx}{c}} abcx + \mathrm{e}^{-\frac{bx}{c}} a\,c^2 - c_1 b^2\right) \mathrm{e}^{\frac{bx}{c}}}{b^2 c} \tag{1}$$

Verification of solutions

$$y = -\frac{\left(\mathrm{e}^{-\frac{bx}{c}} abcx + \mathrm{e}^{-\frac{bx}{c}} a\,c^2 - c_1 b^2\right) \mathrm{e}^{\frac{bx}{c}}}{b^2 c}$$

Verified OK.

1.16.4 Maple step by step solution

Let's solve

$y'c - by = xa$

- Highest derivative means the order of the ODE is 1

  y'

- Isolate the derivative

  $y' = \frac{by}{c} + \frac{xa}{c}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE

  $y' - \frac{by}{c} = \frac{xa}{c}$

- The ODE is linear; multiply by an integrating factor $\mu(x)$

  $\mu(x)\left(y' - \frac{by}{c}\right) = \frac{\mu(x)xa}{c}$

- Assume the lhs of the ODE is the total derivative $\frac{d}{dx}(\mu(x)\, y)$

  $\mu(x)\left(y' - \frac{by}{c}\right) = \mu'(x)\, y + \mu(x)\, y'$

- Isolate $\mu'(x)$

  $\mu'(x) = -\frac{\mu(x)b}{c}$

- Solve to find the integrating factor

  $\mu(x) = \mathrm{e}^{-\frac{bx}{c}}$

- Integrate both sides with respect to x

  $\int \left(\frac{d}{dx}(\mu(x)\, y)\right) dx = \int \frac{\mu(x)xa}{c} dx + c_1$

- Evaluate the integral on the lhs

  $\mu(x)\, y = \int \frac{\mu(x)xa}{c} dx + c_1$

- Solve for y

  $y = \frac{\int \frac{\mu(x)xa}{c} dx + c_1}{\mu(x)}$

- Substitute $\mu(x) = \mathrm{e}^{-\frac{bx}{c}}$

  $y = \frac{\int \frac{xa\,\mathrm{e}^{-\frac{bx}{c}}}{c} dx + c_1}{\mathrm{e}^{-\frac{bx}{c}}}$

- Evaluate the integrals on the rhs

  $y = \frac{-\frac{(xb+c)a\,\mathrm{e}^{-\frac{bx}{c}}}{b^2} + c_1}{\mathrm{e}^{-\frac{bx}{c}}}$

- Simplify

  $y = \frac{c_1 e^{\frac{bx}{c}} b^2 - a(xb+c)}{b^2}$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
<- 1st order linear successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 29

```
dsolve(c*diff(y(x),x)=a*x+b*y(x),y(x), singsol=all)
```

$$y(x) = \frac{e^{\frac{bx}{c}} c_1 b^2 - a(bx+c)}{b^2}$$

✓ Solution by Mathematica

Time used: 0.055 (sec). Leaf size: 28

```
DSolve[c*y'[x]==a*x+b*y[x],y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to -\frac{a(bx+c)}{b^2} + c_1 e^{\frac{bx}{c}}$$

1.17 problem 17

Internal problem ID [7333]
Internal file name [`OUTPUT/6314_Sunday_June_05_2022_04_39_37_PM_13778214/index.tex`]

Book: First order enumerated odes
Section: section 1
Problem number: 17.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"quadrature"**

Maple gives the following as the ode type

`[_quadrature]`

$$\boxed{y'c - y = 0}$$

1.17.1 Solving as quadrature ode

Integrating both sides gives

$$\int \frac{c}{y}dy = \int dx$$
$$c\ln(y) = x + c_1$$

Raising both side to exponential gives

$$e^{c\ln(y)} = e^{x+c_1}$$

Which simplifies to

$$y^c = c_2 e^x$$

Summary
The solution(s) found are the following

$$y = (c_2 e^x)^{\frac{1}{c}} \tag{1}$$

Verification of solutions

$$y = (c_2 e^x)^{\frac{1}{c}}$$

Verified OK.

1.17.2 Maple step by step solution

Let's solve

$y'c - y = 0$

- Highest derivative means the order of the ODE is 1

  y'

- Separate variables

  $\frac{y'}{y} = \frac{1}{c}$

- Integrate both sides with respect to x

  $\int \frac{y'}{y} dx = \int \frac{1}{c} dx + c_1$

- Evaluate integral

  $\ln(y) = \frac{x}{c} + c_1$

- Solve for y

  $y = \mathrm{e}^{\frac{c_1 c + x}{c}}$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
<- 1st order linear successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 12

```
dsolve(c*diff(y(x),x)=y(x),y(x), singsol=all)
```

$$y(x) = e^{\frac{x}{c}} c_1$$

✓ Solution by Mathematica

Time used: 0.023 (sec). Leaf size: 20

```
DSolve[c*y'[x]==y[x],y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to c_1 e^{\frac{x}{c}}$$
$$y(x) \to 0$$

1.18 problem 18

Internal problem ID [7334]
Internal file name [`OUTPUT/6315_Sunday_June_05_2022_04_39_39_PM_83415849/index.tex`]

Book: First order enumerated odes
Section: section 1
Problem number: 18.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"quadrature"**

Maple gives the following as the ode type

`[_quadrature]`

$$\boxed{y'c - by = 0}$$

1.18.1 Solving as quadrature ode

Integrating both sides gives

$$\int \frac{c}{by} dy = \int dx$$
$$\frac{c \ln(y)}{b} = x + c_1$$

Raising both side to exponential gives

$$e^{\frac{c \ln(y)}{b}} = e^{x+c_1}$$

Which simplifies to

$$y^{\frac{c}{b}} = c_2 e^x$$

Summary
The solution(s) found are the following

$$y = (c_2 e^x)^{\frac{b}{c}} \tag{1}$$

Verification of solutions

$$y = (c_2 e^x)^{\frac{b}{c}}$$

Verified OK.

1.18.2 Maple step by step solution

Let's solve

$y'c - by = 0$

- Highest derivative means the order of the ODE is 1

  y'

- Separate variables

  $\frac{y'}{y} = \frac{b}{c}$

- Integrate both sides with respect to x

  $\int \frac{y'}{y} dx = \int \frac{b}{c} dx + c_1$

- Evaluate integral

  $\ln(y) = \frac{bx}{c} + c_1$

- Solve for y

  $y = e^{\frac{c_1 c + xb}{c}}$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
<- 1st order linear successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 13

```
dsolve(c*diff(y(x),x)=b*y(x),y(x), singsol=all)
```

$$y(x) = e^{\frac{bx}{c}} c_1$$

✓ Solution by Mathematica

Time used: 0.025 (sec). Leaf size: 21

```
DSolve[c*y'[x]==b*y[x],y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to c_1 e^{\frac{bx}{c}}$$
$$y(x) \to 0$$

1.19 problem 19

Internal problem ID [7335]
Internal file name [OUTPUT/6316_Sunday_June_05_2022_04_39_41_PM_32606779/index.tex]

Book: First order enumerated odes
Section: section 1
Problem number: 19.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"riccati"**

Maple gives the following as the ode type

```
[[_Riccati, _special]]
```

$$\boxed{y'c - by^2 = xa}$$

1.19.1 Solving as riccati ode

In canonical form the ODE is

$$\begin{aligned} y' &= F(x,y) \\ &= \frac{b\,y^2 + xa}{c} \end{aligned}$$

This is a Riccati ODE. Comparing the ODE to solve

$$y' = \frac{b\,y^2}{c} + \frac{xa}{c}$$

With Riccati ODE standard form

$$y' = f_0(x) + f_1(x)y + f_2(x)y^2$$

Shows that $f_0(x) = \frac{xa}{c}$, $f_1(x) = 0$ and $f_2(x) = \frac{b}{c}$. Let

$$\begin{aligned} y &= \frac{-u'}{f_2 u} \\ &= \frac{-u'}{\frac{bu}{c}} \end{aligned} \tag{1}$$

Using the above substitution in the given ODE results (after some simplification)in a second order ODE to solve for $u(x)$ which is

$$f_2 u''(x) - (f_2' + f_1 f_2) u'(x) + f_2^2 f_0 u(x) = 0 \tag{2}$$

But

$$\begin{aligned} f_2' &= 0 \\ f_1 f_2 &= 0 \\ f_2^2 f_0 &= \frac{b^2 x a}{c^3} \end{aligned}$$

Substituting the above terms back in equation (2) gives

$$\frac{b u''(x)}{c} + \frac{b^2 x a u(x)}{c^3} = 0$$

Solving the above ODE (this ode solved using Maple, not this program), gives

$$u(x) = c_1 \operatorname{AiryAi}\left(-\left(\frac{ab}{c^2}\right)^{\frac{1}{3}} x\right) + c_2 \operatorname{AiryBi}\left(-\left(\frac{ab}{c^2}\right)^{\frac{1}{3}} x\right)$$

The above shows that

$$u'(x) = \left(-\operatorname{AiryBi}\left(1, -\left(\frac{ab}{c^2}\right)^{\frac{1}{3}} x\right) c_2 - \operatorname{AiryAi}\left(1, -\left(\frac{ab}{c^2}\right)^{\frac{1}{3}} x\right) c_1\right) \left(\frac{ab}{c^2}\right)^{\frac{1}{3}}$$

Using the above in (1) gives the solution

$$y = -\frac{\left(-\operatorname{AiryBi}\left(1, -\left(\frac{ab}{c^2}\right)^{\frac{1}{3}} x\right) c_2 - \operatorname{AiryAi}\left(1, -\left(\frac{ab}{c^2}\right)^{\frac{1}{3}} x\right) c_1\right) \left(\frac{ab}{c^2}\right)^{\frac{1}{3}} c}{b \left(c_1 \operatorname{AiryAi}\left(-\left(\frac{ab}{c^2}\right)^{\frac{1}{3}} x\right) + c_2 \operatorname{AiryBi}\left(-\left(\frac{ab}{c^2}\right)^{\frac{1}{3}} x\right)\right)}$$

Dividing both numerator and denominator by c_1 gives, after renaming the constant $\frac{c_2}{c_1} = c_3$ the following solution

$$y = \frac{\left(\operatorname{AiryAi}\left(1, -\left(\frac{ab}{c^2}\right)^{\frac{1}{3}} x\right) c_3 + \operatorname{AiryBi}\left(1, -\left(\frac{ab}{c^2}\right)^{\frac{1}{3}} x\right)\right) \left(\frac{ab}{c^2}\right)^{\frac{1}{3}} c}{b \left(c_3 \operatorname{AiryAi}\left(-\left(\frac{ab}{c^2}\right)^{\frac{1}{3}} x\right) + \operatorname{AiryBi}\left(-\left(\frac{ab}{c^2}\right)^{\frac{1}{3}} x\right)\right)}$$

Summary
The solution(s) found are the following

$$y = \frac{\left(\operatorname{AiryAi}\left(1, -\left(\frac{ab}{c^2}\right)^{\frac{1}{3}} x\right) c_3 + \operatorname{AiryBi}\left(1, -\left(\frac{ab}{c^2}\right)^{\frac{1}{3}} x\right)\right) \left(\frac{ab}{c^2}\right)^{\frac{1}{3}} c}{b \left(c_3 \operatorname{AiryAi}\left(-\left(\frac{ab}{c^2}\right)^{\frac{1}{3}} x\right) + \operatorname{AiryBi}\left(-\left(\frac{ab}{c^2}\right)^{\frac{1}{3}} x\right)\right)} \tag{1}$$

Verification of solutions

$$y = \frac{\left(\operatorname{AiryAi}\left(1, -\left(\frac{ab}{c^2}\right)^{\frac{1}{3}} x\right) c_3 + \operatorname{AiryBi}\left(1, -\left(\frac{ab}{c^2}\right)^{\frac{1}{3}} x\right)\right) \left(\frac{ab}{c^2}\right)^{\frac{1}{3}} c}{b \left(c_3 \operatorname{AiryAi}\left(-\left(\frac{ab}{c^2}\right)^{\frac{1}{3}} x\right) + \operatorname{AiryBi}\left(-\left(\frac{ab}{c^2}\right)^{\frac{1}{3}} x\right)\right)}$$

Verified OK.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
trying homogeneous types:
trying Chini
differential order: 1; looking for linear symmetries
trying exact
Looking for potential symmetries
trying Riccati
trying Riccati Special
<- Riccati Special successful`
```

✓ Solution by Maple
Time used: 0.0 (sec). Leaf size: 75

```
dsolve(c*diff(y(x),x)=a*x+b*y(x)^2,y(x), singsol=all)
```

$$y(x) = \frac{\left(\frac{ba}{c^2}\right)^{\frac{1}{3}} \left(\operatorname{AiryAi}\left(1, -\left(\frac{ba}{c^2}\right)^{\frac{1}{3}} x\right) c_1 + \operatorname{AiryBi}\left(1, -\left(\frac{ba}{c^2}\right)^{\frac{1}{3}} x\right)\right) c}{b \left(c_1 \operatorname{AiryAi}\left(-\left(\frac{ba}{c^2}\right)^{\frac{1}{3}} x\right) + \operatorname{AiryBi}\left(-\left(\frac{ba}{c^2}\right)^{\frac{1}{3}} x\right)\right)}$$

✓ Solution by Mathematica

Time used: 0.21 (sec). Leaf size: 628

```
DSolve[c*y'[x]==a*x+b*y[x]^2,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \frac{c\left(x^{3/2}\sqrt{\frac{a}{c}}\sqrt{\frac{b}{c}}\left(-2\,\text{BesselJ}\left(-\frac{2}{3},\frac{2}{3}\sqrt{\frac{a}{c}}\sqrt{\frac{b}{c}}x^{3/2}\right)+c_1\left(\text{BesselJ}\left(\frac{2}{3},\frac{2}{3}\sqrt{\frac{a}{c}}\sqrt{\frac{b}{c}}x^{3/2}\right)-\text{BesselJ}\left(-\frac{4}{3},\frac{2}{3}\sqrt{\frac{a}{c}}\sqrt{\cdots}\right.\right.\right.\right.}{2bx\left(\text{BesselJ}\left(\frac{1}{3},\frac{2}{3}\sqrt{\frac{a}{c}}\sqrt{\frac{b}{c}}x^{3/2}\right)+c_1\,\text{BesselJ}\left(-\frac{1}{3},\frac{2}{3}\sqrt{\frac{a}{c}}\sqrt{\frac{b}{c}}x^{3/2}\cdots\right.\right.}$$

$$y(x) \to -\frac{c\left(x^{3/2}\sqrt{\frac{a}{c}}\sqrt{\frac{b}{c}}\,\text{BesselJ}\left(-\frac{4}{3},\frac{2}{3}\sqrt{\frac{a}{c}}\sqrt{\frac{b}{c}}x^{3/2}\right)-x^{3/2}\sqrt{\frac{a}{c}}\sqrt{\frac{b}{c}}\,\text{BesselJ}\left(\frac{2}{3},\frac{2}{3}\sqrt{\frac{a}{c}}\sqrt{\frac{b}{c}}x^{3/2}\right)+\text{BesselJ}\left(-\frac{1}{3},\frac{2}{3}\cdots\right.\right.}{2bx\,\text{BesselJ}\left(-\frac{1}{3},\frac{2}{3}\sqrt{\frac{a}{c}}\sqrt{\frac{b}{c}}x^{3/2}\right)}$$

$$y(x) \to -\frac{c\left(x^{3/2}\sqrt{\frac{a}{c}}\sqrt{\frac{b}{c}}\,\text{BesselJ}\left(-\frac{4}{3},\frac{2}{3}\sqrt{\frac{a}{c}}\sqrt{\frac{b}{c}}x^{3/2}\right)-x^{3/2}\sqrt{\frac{a}{c}}\sqrt{\frac{b}{c}}\,\text{BesselJ}\left(\frac{2}{3},\frac{2}{3}\sqrt{\frac{a}{c}}\sqrt{\frac{b}{c}}x^{3/2}\right)+\text{BesselJ}\left(-\frac{1}{3},\frac{2}{3}\cdots\right.\right.}{2bx\,\text{BesselJ}\left(-\frac{1}{3},\frac{2}{3}\sqrt{\frac{a}{c}}\sqrt{\frac{b}{c}}x^{3/2}\right)}$$

1.20 problem 20

Internal problem ID [7336]
Internal file name [OUTPUT/6317_Sunday_June_05_2022_04_39_43_PM_56669206/index.tex]

Book: First order enumerated odes
Section: section 1
Problem number: 20.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"riccati"**

Maple gives the following as the ode type

```
[[_Riccati, _special]]
```

$$\boxed{y'c - \frac{xa + by^2}{r} = 0}$$

1.20.1 Solving as riccati ode

In canonical form the ODE is

$$\begin{aligned} y' &= F(x,y) \\ &= \frac{b\,y^2 + xa}{rc} \end{aligned}$$

This is a Riccati ODE. Comparing the ODE to solve

$$y' = \frac{b\,y^2}{rc} + \frac{xa}{rc}$$

With Riccati ODE standard form

$$y' = f_0(x) + f_1(x)y + f_2(x)y^2$$

Shows that $f_0(x) = \frac{xa}{rc}$, $f_1(x) = 0$ and $f_2(x) = \frac{b}{cr}$. Let

$$\begin{aligned} y &= \frac{-u'}{f_2 u} \\ &= \frac{-u'}{\frac{bu}{cr}} \end{aligned} \tag{1}$$

Using the above substitution in the given ODE results (after some simplification)in a second order ODE to solve for $u(x)$ which is

$$f_2 u''(x) - (f_2' + f_1 f_2) u'(x) + f_2^2 f_0 u(x) = 0 \tag{2}$$

But

$$\begin{aligned} f_2' &= 0 \\ f_1 f_2 &= 0 \\ f_2^2 f_0 &= \frac{b^2 x a}{c^3 r^3} \end{aligned}$$

Substituting the above terms back in equation (2) gives

$$\frac{b u''(x)}{c r} + \frac{b^2 x a u(x)}{c^3 r^3} = 0$$

Solving the above ODE (this ode solved using Maple, not this program), gives

$$u(x) = c_1 \operatorname{AiryAi}\left(-\left(\frac{ab}{r^2 c^2}\right)^{\frac{1}{3}} x\right) + c_2 \operatorname{AiryBi}\left(-\left(\frac{ab}{r^2 c^2}\right)^{\frac{1}{3}} x\right)$$

The above shows that

$$u'(x) = \left(-\operatorname{AiryAi}\left(1, -\left(\frac{ab}{r^2 c^2}\right)^{\frac{1}{3}} x\right) c_1 - \operatorname{AiryBi}\left(1, -\left(\frac{ab}{r^2 c^2}\right)^{\frac{1}{3}} x\right) c_2\right) \left(\frac{ab}{r^2 c^2}\right)^{\frac{1}{3}}$$

Using the above in (1) gives the solution

$$y = -\frac{\left(-\operatorname{AiryAi}\left(1, -\left(\frac{ab}{r^2 c^2}\right)^{\frac{1}{3}} x\right) c_1 - \operatorname{AiryBi}\left(1, -\left(\frac{ab}{r^2 c^2}\right)^{\frac{1}{3}} x\right) c_2\right) \left(\frac{ab}{r^2 c^2}\right)^{\frac{1}{3}} c r}{b \left(c_1 \operatorname{AiryAi}\left(-\left(\frac{ab}{r^2 c^2}\right)^{\frac{1}{3}} x\right) + c_2 \operatorname{AiryBi}\left(-\left(\frac{ab}{r^2 c^2}\right)^{\frac{1}{3}} x\right)\right)}$$

Dividing both numerator and denominator by c_1 gives, after renaming the constant $\frac{c_2}{c_1} = c_3$ the following solution

$$y = \frac{\left(\operatorname{AiryAi}\left(1, -\left(\frac{ab}{r^2 c^2}\right)^{\frac{1}{3}} x\right) c_3 + \operatorname{AiryBi}\left(1, -\left(\frac{ab}{r^2 c^2}\right)^{\frac{1}{3}} x\right)\right) \left(\frac{ab}{r^2 c^2}\right)^{\frac{1}{3}} c r}{b \left(c_3 \operatorname{AiryAi}\left(-\left(\frac{ab}{r^2 c^2}\right)^{\frac{1}{3}} x\right) + \operatorname{AiryBi}\left(-\left(\frac{ab}{r^2 c^2}\right)^{\frac{1}{3}} x\right)\right)}$$

Summary

The solution(s) found are the following

$$y = \frac{\left(\operatorname{AiryAi}\left(1, -\left(\frac{ab}{r^2c^2}\right)^{\frac{1}{3}} x\right) c_3 + \operatorname{AiryBi}\left(1, -\left(\frac{ab}{r^2c^2}\right)^{\frac{1}{3}} x\right)\right) \left(\frac{ab}{r^2c^2}\right)^{\frac{1}{3}} cr}{b \left(c_3 \operatorname{AiryAi}\left(-\left(\frac{ab}{r^2c^2}\right)^{\frac{1}{3}} x\right) + \operatorname{AiryBi}\left(-\left(\frac{ab}{r^2c^2}\right)^{\frac{1}{3}} x\right)\right)} \tag{1}$$

Verification of solutions

$$y = \frac{\left(\operatorname{AiryAi}\left(1, -\left(\frac{ab}{r^2c^2}\right)^{\frac{1}{3}} x\right) c_3 + \operatorname{AiryBi}\left(1, -\left(\frac{ab}{r^2c^2}\right)^{\frac{1}{3}} x\right)\right) \left(\frac{ab}{r^2c^2}\right)^{\frac{1}{3}} cr}{b \left(c_3 \operatorname{AiryAi}\left(-\left(\frac{ab}{r^2c^2}\right)^{\frac{1}{3}} x\right) + \operatorname{AiryBi}\left(-\left(\frac{ab}{r^2c^2}\right)^{\frac{1}{3}} x\right)\right)}$$

Verified OK.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
trying homogeneous types:
trying Chini
differential order: 1; looking for linear symmetries
trying exact
Looking for potential symmetries
trying Riccati
trying Riccati Special
<- Riccati Special successful`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 91

```
dsolve(c*diff(y(x),x)=(a*x+b*y(x)^2)/r,y(x), singsol=all)
```

$$y(x) = \frac{\left(\frac{ba}{r^2c^2}\right)^{\frac{1}{3}} \left(\operatorname{AiryAi}\left(1, -\left(\frac{ba}{r^2c^2}\right)^{\frac{1}{3}} x\right) c_1 + \operatorname{AiryBi}\left(1, -\left(\frac{ba}{r^2c^2}\right)^{\frac{1}{3}} x\right)\right) rc}{b \left(c_1 \operatorname{AiryAi}\left(-\left(\frac{ba}{r^2c^2}\right)^{\frac{1}{3}} x\right) + \operatorname{AiryBi}\left(-\left(\frac{ba}{r^2c^2}\right)^{\frac{1}{3}} x\right)\right)}$$

✓ Solution by Mathematica

Time used: 0.222 (sec). Leaf size: 517

```
DSolve[c*y'[x]==(a*x+b*y[x]^2)/r,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \rightarrow \frac{cr\left(x^{3/2}\sqrt{\frac{a}{cr}}\sqrt{\frac{b}{cr}}\left(-2\,\text{BesselJ}\left(-\frac{2}{3},\frac{2}{3}\sqrt{\frac{a}{cr}}\sqrt{\frac{b}{cr}}x^{3/2}\right)+c_1\left(\text{BesselJ}\left(\frac{2}{3},\frac{2}{3}\sqrt{\frac{a}{cr}}\sqrt{\frac{b}{cr}}x^{3/2}\right)-\text{BesselJ}\left(-\frac{4}{3},\frac{2}{3}\right.\right.\right.\right.}{2bx\left(\text{BesselJ}\left(\frac{1}{3},\frac{2}{3}\sqrt{\frac{a}{cr}}\sqrt{\frac{b}{cr}}x^{3/2}\right)+c_1\,\text{BesselJ}\left(-\frac{1}{3},\frac{2}{3}\sqrt{\frac{a}{cr}}\right.\right.}$$

$$y(x) \rightarrow -\frac{cr\left(x^{3/2}\sqrt{\frac{a}{cr}}\sqrt{\frac{b}{cr}}\,\text{BesselJ}\left(-\frac{4}{3},\frac{2}{3}\sqrt{\frac{a}{cr}}\sqrt{\frac{b}{cr}}x^{3/2}\right)-x^{3/2}\sqrt{\frac{a}{cr}}\sqrt{\frac{b}{cr}}\,\text{BesselJ}\left(\frac{2}{3},\frac{2}{3}\sqrt{\frac{a}{cr}}\sqrt{\frac{b}{cr}}x^{3/2}\right)+\text{BesselJ}\right.}{2bx\,\text{BesselJ}\left(-\frac{1}{3},\frac{2}{3}\sqrt{\frac{a}{cr}}\sqrt{\frac{b}{cr}}x^{3/2}\right)}$$

1.21 problem 21

Internal problem ID [7337]
Internal file name [OUTPUT/6318_Sunday_June_05_2022_04_39_45_PM_69908985/index.tex]

Book: First order enumerated odes
Section: section 1
Problem number: 21.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"riccati"**

Maple gives the following as the ode type

```
[_rational, _Riccati]
```

$$\boxed{y'c - \frac{xa + by^2}{rx} = 0}$$

1.21.1 Solving as riccati ode

In canonical form the ODE is

$$\begin{aligned} y' &= F(x,y) \\ &= \frac{b\,y^2 + xa}{rxc} \end{aligned}$$

This is a Riccati ODE. Comparing the ODE to solve

$$y' = \frac{b\,y^2}{rxc} + \frac{a}{rc}$$

With Riccati ODE standard form

$$y' = f_0(x) + f_1(x)y + f_2(x)y^2$$

Shows that $f_0(x) = \frac{a}{rc}$, $f_1(x) = 0$ and $f_2(x) = \frac{b}{crx}$. Let

$$\begin{aligned} y &= \frac{-u'}{f_2 u} \\ &= \frac{-u'}{\frac{bu}{crx}} \end{aligned} \tag{1}$$

Using the above substitution in the given ODE results (after some simplification)in a second order ODE to solve for $u(x)$ which is

$$f_2 u''(x) - (f_2' + f_1 f_2) u'(x) + f_2^2 f_0 u(x) = 0 \tag{2}$$

But

$$\begin{aligned} f_2' &= -\frac{b}{cr\,x^2} \\ f_1 f_2 &= 0 \\ f_2^2 f_0 &= \frac{b^2 a}{c^3 r^3 x^2} \end{aligned}$$

Substituting the above terms back in equation (2) gives

$$\frac{bu''(x)}{crx} + \frac{bu'(x)}{cr\,x^2} + \frac{b^2 a u(x)}{c^3 r^3 x^2} = 0$$

Solving the above ODE (this ode solved using Maple, not this program), gives

$$u(x) = c_1 \operatorname{BesselJ}\left(0, \frac{2\sqrt{ab}\,\sqrt{x}}{rc}\right) + c_2 \operatorname{BesselY}\left(0, \frac{2\sqrt{ab}\,\sqrt{x}}{rc}\right)$$

The above shows that

$$u'(x) = \frac{\left(-\operatorname{BesselY}\left(1, \frac{2\sqrt{ab}\,\sqrt{x}}{rc}\right) c_2 - \operatorname{BesselJ}\left(1, \frac{2\sqrt{ab}\,\sqrt{x}}{rc}\right) c_1\right)\sqrt{ab}}{rc\sqrt{x}}$$

Using the above in (1) gives the solution

$$y = -\frac{\left(-\operatorname{BesselY}\left(1, \frac{2\sqrt{ab}\,\sqrt{x}}{rc}\right) c_2 - \operatorname{BesselJ}\left(1, \frac{2\sqrt{ab}\,\sqrt{x}}{rc}\right) c_1\right)\sqrt{ab}\,\sqrt{x}}{b\left(c_1 \operatorname{BesselJ}\left(0, \frac{2\sqrt{ab}\,\sqrt{x}}{rc}\right) + c_2 \operatorname{BesselY}\left(0, \frac{2\sqrt{ab}\,\sqrt{x}}{rc}\right)\right)}$$

Dividing both numerator and denominator by c_1 gives, after renaming the constant $\frac{c_2}{c_1} = c_3$ the following solution

$$y = \frac{\left(\operatorname{BesselJ}\left(1, \frac{2\sqrt{ab}\,\sqrt{x}}{rc}\right) c_3 + \operatorname{BesselY}\left(1, \frac{2\sqrt{ab}\,\sqrt{x}}{rc}\right)\right)\sqrt{ab}\,\sqrt{x}}{b\left(c_3 \operatorname{BesselJ}\left(0, \frac{2\sqrt{ab}\,\sqrt{x}}{rc}\right) + \operatorname{BesselY}\left(0, \frac{2\sqrt{ab}\,\sqrt{x}}{rc}\right)\right)}$$

Summary

The solution(s) found are the following

$$y = \frac{\left(\text{BesselJ}\left(1, \frac{2\sqrt{ab}\,\sqrt{x}}{rc}\right) c_3 + \text{BesselY}\left(1, \frac{2\sqrt{ab}\,\sqrt{x}}{rc}\right)\right)\sqrt{ab}\,\sqrt{x}}{b\left(c_3 \,\text{BesselJ}\left(0, \frac{2\sqrt{ab}\,\sqrt{x}}{rc}\right) + \text{BesselY}\left(0, \frac{2\sqrt{ab}\,\sqrt{x}}{rc}\right)\right)} \tag{1}$$

Verification of solutions

$$y = \frac{\left(\text{BesselJ}\left(1, \frac{2\sqrt{ab}\,\sqrt{x}}{rc}\right) c_3 + \text{BesselY}\left(1, \frac{2\sqrt{ab}\,\sqrt{x}}{rc}\right)\right)\sqrt{ab}\,\sqrt{x}}{b\left(c_3 \,\text{BesselJ}\left(0, \frac{2\sqrt{ab}\,\sqrt{x}}{rc}\right) + \text{BesselY}\left(0, \frac{2\sqrt{ab}\,\sqrt{x}}{rc}\right)\right)}$$

Verified OK.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
trying homogeneous types:
trying Chini
differential order: 1; looking for linear symmetries
trying exact
Looking for potential symmetries
trying Riccati
trying Riccati sub-methods:
      -> Trying a Liouvillian solution using Kovacics algorithm
      <- No Liouvillian solutions exists
   <- Abel AIR successful: ODE belongs to the 0F1 1-parameter (Bessel type) class`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 94

```
dsolve(c*diff(y(x),x)=(a*x+b*y(x)^2)/(r*x),y(x), singsol=all)
```

$$y(x) = \frac{\sqrt{\frac{xba}{r^2c^2}}\, cr \left(\text{BesselY}\left(1, 2\sqrt{\frac{xba}{r^2c^2}}\right) c_1 + \text{BesselJ}\left(1, 2\sqrt{\frac{xba}{r^2c^2}}\right)\right)}{b\left(c_1 \,\text{BesselY}\left(0, 2\sqrt{\frac{xba}{r^2c^2}}\right) + \text{BesselJ}\left(0, 2\sqrt{\frac{xba}{r^2c^2}}\right)\right)}$$

✓ Solution by Mathematica

Time used: 0.295 (sec). Leaf size: 207

```
DSolve[c*y'[x]==(a*x+b*y[x]^2)/(r*x),y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \frac{\sqrt{a}\sqrt{x}\left(2\,\text{BesselY}\left(1, \frac{2\sqrt{a}\sqrt{b}\sqrt{x}}{cr}\right) + c_1 \,\text{BesselJ}\left(1, \frac{2\sqrt{a}\sqrt{b}\sqrt{x}}{cr}\right)\right)}{\sqrt{b}\left(2\,\text{BesselY}\left(0, \frac{2\sqrt{a}\sqrt{b}\sqrt{x}}{cr}\right) + c_1 \,\text{BesselJ}\left(0, \frac{2\sqrt{a}\sqrt{b}\sqrt{x}}{cr}\right)\right)}$$

$$y(x) \to \frac{\sqrt{a}\sqrt{x}\,\text{BesselJ}\left(1, \frac{2\sqrt{a}\sqrt{b}\sqrt{x}}{cr}\right)}{\sqrt{b}\,\text{BesselJ}\left(0, \frac{2\sqrt{a}\sqrt{b}\sqrt{x}}{cr}\right)}$$

1.22 problem 22

Internal problem ID [7338]
Internal file name [`OUTPUT/6319_Sunday_June_05_2022_04_39_48_PM_63185761/index.tex`]

Book: First order enumerated odes
Section: section 1
Problem number: 22.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"riccati"**

Maple gives the following as the ode type

```
[_rational, _Riccati]
```

$$\boxed{y'c - \frac{xa + by^2}{r\,x^2} = 0}$$

1.22.1 Solving as riccati ode

In canonical form the ODE is

$$\begin{aligned} y' &= F(x,y) \\ &= \frac{b\,y^2 + xa}{r\,x^2 c} \end{aligned}$$

This is a Riccati ODE. Comparing the ODE to solve

$$y' = \frac{b\,y^2}{r\,x^2 c} + \frac{a}{rxc}$$

With Riccati ODE standard form

$$y' = f_0(x) + f_1(x)y + f_2(x)y^2$$

Shows that $f_0(x) = \frac{a}{rxc}$, $f_1(x) = 0$ and $f_2(x) = \frac{b}{cr\,x^2}$. Let

$$\begin{aligned} y &= \frac{-u'}{f_2 u} \\ &= \frac{-u'}{\frac{bu}{cr\,x^2}} \end{aligned} \tag{1}$$

Using the above substitution in the given ODE results (after some simplification)in a second order ODE to solve for $u(x)$ which is

$$f_2u''(x) - (f_2' + f_1f_2)\,u'(x) + f_2^2 f_0 u(x) = 0 \tag{2}$$

But

$$\begin{aligned} f_2' &= -\frac{2b}{cr\,x^3} \\ f_1f_2 &= 0 \\ f_2^2 f_0 &= \frac{b^2 a}{c^3 r^3 x^5} \end{aligned}$$

Substituting the above terms back in equation (2) gives

$$\frac{bu''(x)}{cr\,x^2} + \frac{2bu'(x)}{cr\,x^3} + \frac{b^2 a u(x)}{c^3 r^3 x^5} = 0$$

Solving the above ODE (this ode solved using Maple, not this program), gives

$$u(x) = \frac{c_1 \operatorname{BesselJ}\left(1, \frac{2\sqrt{ab}}{rc\sqrt{x}}\right) + c_2 \operatorname{BesselY}\left(1, \frac{2\sqrt{ab}}{rc\sqrt{x}}\right)}{\sqrt{x}}$$

The above shows that

$$u'(x) = \frac{\left(-\operatorname{BesselY}\left(0, \frac{2\sqrt{ab}}{rc\sqrt{x}}\right) c_2 - \operatorname{BesselJ}\left(0, \frac{2\sqrt{ab}}{rc\sqrt{x}}\right) c_1\right)\sqrt{ab}}{x^2 rc}$$

Using the above in (1) gives the solution

$$y = -\frac{\left(-\operatorname{BesselY}\left(0, \frac{2\sqrt{ab}}{rc\sqrt{x}}\right) c_2 - \operatorname{BesselJ}\left(0, \frac{2\sqrt{ab}}{rc\sqrt{x}}\right) c_1\right)\sqrt{ab}\,\sqrt{x}}{b\left(c_1 \operatorname{BesselJ}\left(1, \frac{2\sqrt{ab}}{rc\sqrt{x}}\right) + c_2 \operatorname{BesselY}\left(1, \frac{2\sqrt{ab}}{rc\sqrt{x}}\right)\right)}$$

Dividing both numerator and denominator by c_1 gives, after renaming the constant $\frac{c_2}{c_1} = c_3$ the following solution

$$y = \frac{\left(\operatorname{BesselJ}\left(0, \frac{2\sqrt{ab}}{rc\sqrt{x}}\right) c_3 + \operatorname{BesselY}\left(0, \frac{2\sqrt{ab}}{rc\sqrt{x}}\right)\right)\sqrt{ab}\,\sqrt{x}}{b\left(c_3 \operatorname{BesselJ}\left(1, \frac{2\sqrt{ab}}{rc\sqrt{x}}\right) + \operatorname{BesselY}\left(1, \frac{2\sqrt{ab}}{rc\sqrt{x}}\right)\right)}$$

Summary

The solution(s) found are the following

$$
y = \frac{\left(\mathrm{BesselJ}\left(0, \frac{2\sqrt{ab}}{rc\sqrt{x}}\right) c_3 + \mathrm{BesselY}\left(0, \frac{2\sqrt{ab}}{rc\sqrt{x}}\right)\right) \sqrt{ab}\,\sqrt{x}}{b\left(c_3 \,\mathrm{BesselJ}\left(1, \frac{2\sqrt{ab}}{rc\sqrt{x}}\right) + \mathrm{BesselY}\left(1, \frac{2\sqrt{ab}}{rc\sqrt{x}}\right)\right)} \tag{1}
$$

Verification of solutions

$$
y = \frac{\left(\mathrm{BesselJ}\left(0, \frac{2\sqrt{ab}}{rc\sqrt{x}}\right) c_3 + \mathrm{BesselY}\left(0, \frac{2\sqrt{ab}}{rc\sqrt{x}}\right)\right) \sqrt{ab}\,\sqrt{x}}{b\left(c_3 \,\mathrm{BesselJ}\left(1, \frac{2\sqrt{ab}}{rc\sqrt{x}}\right) + \mathrm{BesselY}\left(1, \frac{2\sqrt{ab}}{rc\sqrt{x}}\right)\right)}
$$

Verified OK.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
trying homogeneous types:
trying Chini
differential order: 1; looking for linear symmetries
trying exact
Looking for potential symmetries
trying Riccati
trying Riccati sub-methods:
      -> Trying a Liouvillian solution using Kovacics algorithm
      <- No Liouvillian solutions exists
   <- Abel AIR successful: ODE belongs to the 0F1 1-parameter (Bessel type) class`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 106

```
dsolve(c*diff(y(x),x)=(a*x+b*y(x)^2)/(r*x^2),y(x), singsol=all)
```

$$y(x) = \frac{a\left(\text{BesselY}\left(0, 2\sqrt{\frac{ba}{c^2r^2x}}\right)c_1 + \text{BesselJ}\left(0, 2\sqrt{\frac{ba}{c^2r^2x}}\right)\right)}{cr\sqrt{\frac{ba}{c^2r^2x}}\left(c_1\,\text{BesselY}\left(1, 2\sqrt{\frac{ba}{c^2r^2x}}\right) + \text{BesselJ}\left(1, 2\sqrt{\frac{ba}{c^2r^2x}}\right)\right)}$$

✓ Solution by Mathematica

Time used: 0.358 (sec). Leaf size: 492

```
DSolve[c*y'[x]==(a*x+b*y[x]^2)/(r*x^2),y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \frac{2\sqrt{a}\sqrt{b}\,\text{BesselY}\left(0, \frac{2\sqrt{a}\sqrt{b}\sqrt{\frac{1}{x}}}{cr}\right) + \frac{2cr\,\text{BesselY}\left(1, \frac{2\sqrt{a}\sqrt{b}\sqrt{\frac{1}{x}}}{cr}\right)}{\sqrt{\frac{1}{x}}} - 2\sqrt{a}\sqrt{b}\,\text{BesselY}\left(2, \frac{2\sqrt{a}\sqrt{b}\sqrt{\frac{1}{x}}}{cr}\right) - i\sqrt{a}\sqrt{b}c_1\,\text{Be}}{2b\sqrt{\frac{1}{x}}\left(2\,\text{BesselY}\left(1, \frac{2\sqrt{a}\sqrt{b}\sqrt{\frac{1}{x}}}{cr}\right) - ic_1\,\text{Bess}\right.}$$

$$y(x) \to \frac{x\left(\sqrt{a}\sqrt{b}\sqrt{\frac{1}{x}}\,\text{BesselJ}\left(0, \frac{2\sqrt{a}\sqrt{b}\sqrt{\frac{1}{x}}}{cr}\right) + cr\,\text{BesselJ}\left(1, \frac{2\sqrt{a}\sqrt{b}\sqrt{\frac{1}{x}}}{cr}\right) - \sqrt{a}\sqrt{b}\sqrt{\frac{1}{x}}\,\text{BesselJ}\left(2, \frac{2\sqrt{a}\sqrt{b}\sqrt{\frac{1}{x}}}{cr}\right)\right)}{2b\,\text{BesselJ}\left(1, \frac{2\sqrt{a}\sqrt{b}\sqrt{\frac{1}{x}}}{cr}\right)}$$

1.23 problem 23

Internal problem ID [7339]
Internal file name [`OUTPUT/6320_Sunday_June_05_2022_04_39_50_PM_19538571/index.tex`]

Book: First order enumerated odes
Section: section 1
Problem number: 23.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"bernoulli", "exactWithIntegrationFactor", "first_order_ode_lie_symmetry_lookup"**

Maple gives the following as the ode type

```
[_rational, _Bernoulli]
```

$$\boxed{y'c - \frac{xa + by^2}{y} = 0}$$

1.23.1 Solving as first order ode lie symmetry lookup ode

Writing the ode as

$$y' = \frac{b\,y^2 + xa}{yc}$$
$$y' = \omega(x, y)$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x + \omega(\eta_y - \xi_x) - \omega^2 \xi_y - \omega_x \xi - \omega_y \eta = 0 \tag{A}$$

The type of this ode is known. It is of type `Bernoulli`. Therefore we do not need to solve the PDE (A), and can just use the lookup table shown below to find ξ, η

Table 28: Lie symmetry infinitesimal lookup table for known first order ODE's

ODE class	Form	ξ	η
linear ode	$y' = f(x)y(x) + g(x)$	0	$e^{\int f dx}$
separable ode	$y' = f(x)\, g(y)$	$\frac{1}{f}$	0
quadrature ode	$y' = f(x)$	0	1
quadrature ode	$y' = g(y)$	1	0
homogeneous ODEs of Class A	$y' = f\left(\frac{y}{x}\right)$	x	y
homogeneous ODEs of Class C	$y' = (a + bx + cy)^{\frac{n}{m}}$	1	$-\frac{b}{c}$
homogeneous class D	$y' = \frac{y}{x} + g(x)\, F\left(\frac{y}{x}\right)$	x^2	xy
First order special form ID 1	$y' = g(x)\, e^{h(x)+by} + f(x)$	$\frac{e^{-\int bf(x)dx - h(x)}}{g(x)}$	$\frac{f(x)e^{-\int bf(x)dx - h(x)}}{g(x)}$
polynomial type ode	$y' = \frac{a_1x+b_1y+c_1}{a_2x+b_2y+c_2}$	$\frac{a_1b_2x - a_2b_1x - b_1c_2 + b_2c_1}{a_1b_2 - a_2b_1}$	$\frac{a_1b_2y - a_2b_1y - a_1c_2 - a_2c_1}{a_1b_2 - a_2b_1}$
Bernoulli ode	$y' = f(x)\, y + g(x)\, y^n$	0	$e^{-\int (n-1)f(x)dx} y^n$
Reduced Riccati	$y' = f_1(x)\, y + f_2(x)\, y^2$	0	$e^{-\int f_1 dx}$

The above table shows that

$$
\begin{aligned}
\xi(x, y) &= 0 \\
\eta(x, y) &= \frac{\mathrm{e}^{\frac{2bx}{c}}}{y}
\end{aligned} \tag{A1}
$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x, y) \to (R, S)$ where (R, S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$
\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}
$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y}\right) S(x, y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the

canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$R = x$$

S is found from

$$\begin{aligned} S &= \int \frac{1}{\eta} dy \\ &= \int \frac{1}{\frac{e^{\frac{2bx}{c}}}{y}} dy \end{aligned}$$

Which results in

$$S = \frac{y^2 e^{-\frac{2bx}{c}}}{2}$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x,y) S_y}{R_x + \omega(x,y) R_y} \tag{2}$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x,y)$ is the right hand side of the original ode given by

$$\omega(x,y) = \frac{b\,y^2 + xa}{yc}$$

Evaluating all the partial derivatives gives

$$\begin{aligned} R_x &= 1 \\ R_y &= 0 \\ S_x &= -\frac{y^2 b\, e^{-\frac{2bx}{c}}}{c} \\ S_y &= y\, e^{-\frac{2bx}{c}} \end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = \frac{e^{-\frac{2bx}{c}} xa}{c} \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = \frac{e^{-\frac{2bR}{c}} Ra}{c}$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$S(R) = -\frac{(2bR+c)\,\mathrm{e}^{-\frac{2bR}{c}} a}{4b^2} + c_1 \tag{4}$$

To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$\frac{y^2 \mathrm{e}^{-\frac{2bx}{c}}}{2} = -\frac{(2xb+c)\,\mathrm{e}^{-\frac{2bx}{c}} a}{4b^2} + c_1$$

Which simplifies to

$$\frac{y^2 \mathrm{e}^{-\frac{2bx}{c}}}{2} = -\frac{(2xb+c)\,\mathrm{e}^{-\frac{2bx}{c}} a}{4b^2} + c_1$$

Summary

The solution(s) found are the following

$$\frac{y^2 \mathrm{e}^{-\frac{2bx}{c}}}{2} = -\frac{(2xb+c)\,\mathrm{e}^{-\frac{2bx}{c}} a}{4b^2} + c_1 \tag{1}$$

Verification of solutions

$$\frac{y^2 \mathrm{e}^{-\frac{2bx}{c}}}{2} = -\frac{(2xb+c)\,\mathrm{e}^{-\frac{2bx}{c}} a}{4b^2} + c_1$$

Verified OK.

1.23.2 Solving as bernoulli ode

In canonical form, the ODE is

$$\begin{aligned} y' &= F(x, y) \\ &= \frac{b\,y^2 + xa}{yc} \end{aligned}$$

This is a Bernoulli ODE.

$$y' = \frac{b}{c} y + \frac{xa}{c} \frac{1}{y} \tag{1}$$

The standard Bernoulli ODE has the form

$$y' = f_0(x) y + f_1(x) y^n \tag{2}$$

The first step is to divide the above equation by y^n which gives

$$\frac{y'}{y^n} = f_0(x)y^{1-n} + f_1(x) \tag{3}$$

The next step is use the substitution $w = y^{1-n}$ in equation (3) which generates a new ODE in $w(x)$ which will be linear and can be easily solved using an integrating factor. Backsubstitution then gives the solution $y(x)$ which is what we want.

This method is now applied to the ODE at hand. Comparing the ODE (1) With (2) Shows that

$$\begin{aligned} f_0(x) &= \frac{b}{c} \\ f_1(x) &= \frac{xa}{c} \\ n &= -1 \end{aligned}$$

Dividing both sides of ODE (1) by $y^n = \frac{1}{y}$ gives

$$y'y = \frac{b\,y^2}{c} + \frac{xa}{c} \tag{4}$$

Let

$$\begin{aligned} w &= y^{1-n} \\ &= y^2 \end{aligned} \tag{5}$$

Taking derivative of equation (5) w.r.t x gives

$$w' = 2yy' \tag{6}$$

Substituting equations (5) and (6) into equation (4) gives

$$\begin{aligned} \frac{w'(x)}{2} &= \frac{bw(x)}{c} + \frac{xa}{c} \\ w' &= \frac{2bw}{c} + \frac{2xa}{c} \end{aligned} \tag{7}$$

The above now is a linear ODE in $w(x)$ which is now solved.

Entering Linear first order ODE solver. In canonical form a linear first order is

$$w'(x) + p(x)w(x) = q(x)$$

Where here

$$
\begin{aligned}
p(x) &= -\frac{2b}{c} \\
q(x) &= \frac{2xa}{c}
\end{aligned}
$$

Hence the ode is

$$
w'(x) - \frac{2bw(x)}{c} = \frac{2xa}{c}
$$

The integrating factor μ is

$$
\begin{aligned}
\mu &= \mathrm{e}^{\int -\frac{2b}{c} dx} \\
&= \mathrm{e}^{-\frac{2bx}{c}}
\end{aligned}
$$

The ode becomes

$$
\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}x}(\mu w) &= (\mu)\left(\frac{2xa}{c}\right) \\
\frac{\mathrm{d}}{\mathrm{d}x}\left(\mathrm{e}^{-\frac{2bx}{c}} w\right) &= \left(\mathrm{e}^{-\frac{2bx}{c}}\right)\left(\frac{2xa}{c}\right) \\
\mathrm{d}\left(\mathrm{e}^{-\frac{2bx}{c}} w\right) &= \left(\frac{2\,\mathrm{e}^{-\frac{2bx}{c}} xa}{c}\right) \mathrm{d}x
\end{aligned}
$$

Integrating gives

$$
\begin{aligned}
\mathrm{e}^{-\frac{2bx}{c}} w &= \int \frac{2\,\mathrm{e}^{-\frac{2bx}{c}} xa}{c} \,\mathrm{d}x \\
\mathrm{e}^{-\frac{2bx}{c}} w &= -\frac{(2xb + c)\,\mathrm{e}^{-\frac{2bx}{c}} a}{2b^2} + c_1
\end{aligned}
$$

Dividing both sides by the integrating factor $\mu = \mathrm{e}^{-\frac{2bx}{c}}$ results in

$$
w(x) = -\frac{\mathrm{e}^{\frac{2bx}{c}} (2xb + c)\,\mathrm{e}^{-\frac{2bx}{c}} a}{2b^2} + c_1 \mathrm{e}^{\frac{2bx}{c}}
$$

which simplifies to

$$
w(x) = \frac{c_1 \mathrm{e}^{\frac{2bx}{c}} b^2 - \left(xb + \frac{c}{2}\right) a}{b^2}
$$

Replacing w in the above by y^2 using equation (5) gives the final solution.

$$
y^2 = \frac{c_1 \mathrm{e}^{\frac{2bx}{c}} b^2 - \left(xb + \frac{c}{2}\right) a}{b^2}
$$

Solving for y gives

$$y(x) = \frac{\sqrt{4c_1 \mathrm{e}^{\frac{2bx}{c}} b^2 - 4abx - 2ac}}{2b}$$
$$y(x) = -\frac{\sqrt{4c_1 \mathrm{e}^{\frac{2bx}{c}} b^2 - 4abx - 2ac}}{2b}$$

Summary

The solution(s) found are the following

$$y = \frac{\sqrt{4c_1 \mathrm{e}^{\frac{2bx}{c}} b^2 - 4abx - 2ac}}{2b} \tag{1}$$
$$y = -\frac{\sqrt{4c_1 \mathrm{e}^{\frac{2bx}{c}} b^2 - 4abx - 2ac}}{2b} \tag{2}$$

Verification of solutions

$$y = \frac{\sqrt{4c_1 \mathrm{e}^{\frac{2bx}{c}} b^2 - 4abx - 2ac}}{2b}$$

Verified OK.

$$y = -\frac{\sqrt{4c_1 \mathrm{e}^{\frac{2bx}{c}} b^2 - 4abx - 2ac}}{2b}$$

Verified OK.

1.23.3 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x,y) + N(x,y)\frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x,y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y) = 0$$

Hence

$$\frac{\partial \phi}{\partial x} + \frac{\partial \phi}{\partial y}\frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$\begin{aligned}\frac{\partial \phi}{\partial x} &= M\\ \frac{\partial \phi}{\partial y} &= N\end{aligned}$$

But since $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x, y)$ but at least we know now that we can do that since the condition $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x, y)\,\mathrm{d}x + N(x, y)\,\mathrm{d}y = 0 \tag{1A}$$

Therefore

$$\begin{aligned}(yc)\,\mathrm{d}y &= \left(b\,y^2 + xa\right)\mathrm{d}x\\ \left(-b\,y^2 - xa\right)\mathrm{d}x + (yc)\,\mathrm{d}y &= 0 \qquad (2A)\end{aligned}$$

Comparing (1A) and (2A) shows that

$$\begin{aligned}M(x, y) &= -b\,y^2 - xa\\ N(x, y) &= yc\end{aligned}$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Using result found above gives

$$\begin{aligned}\frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}\left(-b\,y^2 - xa\right)\\ &= -2by\end{aligned}$$

And

$$\begin{aligned}\frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}(yc)\\ &= 0\end{aligned}$$

Since $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$, then the ODE is <u>not exact</u>. Since the ODE is not exact, we will try to find an integrating factor to make it exact. Let

$$
\begin{aligned}
A &= \frac{1}{N}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right) \\
&= \frac{1}{yc}((-2by) - (0)) \\
&= -\frac{2b}{c}
\end{aligned}
$$

Since A does not depend on y, then it can be used to find an integrating factor. The integrating factor μ is

$$
\begin{aligned}
\mu &= e^{\int A \, \mathrm{d}x} \\
&= e^{\int -\frac{2b}{c} \, \mathrm{d}x}
\end{aligned}
$$

The result of integrating gives

$$
\begin{aligned}
\mu &= e^{-\frac{2bx}{c}} \\
&= \mathrm{e}^{-\frac{2bx}{c}}
\end{aligned}
$$

M and N are multiplied by this integrating factor, giving new M and new N which are called $\overline{M}$ and $\overline{N}$ for now so not to confuse them with the original M and N.

$$
\begin{aligned}
\overline{M} &= \mu M \\
&= \mathrm{e}^{-\frac{2bx}{c}}\left(-b\,y^2 - xa\right) \\
&= -\mathrm{e}^{-\frac{2bx}{c}}\left(b\,y^2 + xa\right)
\end{aligned}
$$

And

$$
\begin{aligned}
\overline{N} &= \mu N \\
&= \mathrm{e}^{-\frac{2bx}{c}}(yc) \\
&= yc\,\mathrm{e}^{-\frac{2bx}{c}}
\end{aligned}
$$

Now a modified ODE is ontained from the original ODE, which is exact and can be solved. The modified ODE is

$$
\begin{aligned}
\overline{M} + \overline{N}\frac{\mathrm{d}y}{\mathrm{d}x} &= 0 \\
\left(-\mathrm{e}^{-\frac{2bx}{c}}\left(b\,y^2 + xa\right)\right) + \left(yc\,\mathrm{e}^{-\frac{2bx}{c}}\right)\frac{\mathrm{d}y}{\mathrm{d}x} &= 0
\end{aligned}
$$

The following equations are now set up to solve for the function $\phi(x, y)$

$$\frac{\partial \phi}{\partial x} = \overline{M} \tag{1}$$
$$\frac{\partial \phi}{\partial y} = \overline{N} \tag{2}$$

Integrating (1) w.r.t. x gives

$$\int \frac{\partial \phi}{\partial x} \, \mathrm{d}x = \int \overline{M} \, \mathrm{d}x$$
$$\int \frac{\partial \phi}{\partial x} \, \mathrm{d}x = \int -\mathrm{e}^{-\frac{2bx}{c}} \left(b\, y^2 + xa\right) \mathrm{d}x$$
$$\phi = \frac{c(2y^2b^2 + 2abx + ac)\, \mathrm{e}^{-\frac{2bx}{c}}}{4b^2} + f(y) \tag{3}$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$\frac{\partial \phi}{\partial y} = yc\, \mathrm{e}^{-\frac{2bx}{c}} + f'(y) \tag{4}$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = yc\, \mathrm{e}^{-\frac{2bx}{c}}$. Therefore equation (4) becomes

$$yc\, \mathrm{e}^{-\frac{2bx}{c}} = yc\, \mathrm{e}^{-\frac{2bx}{c}} + f'(y) \tag{5}$$

Solving equation (5) for $f'(y)$ gives

$$f'(y) = 0$$

Therefore

$$f(y) = c_1$$

Where c_1 is constant of integration. Substituting this result for $f(y)$ into equation (3) gives ϕ

$$\phi = \frac{c(2y^2b^2 + 2abx + ac)\, \mathrm{e}^{-\frac{2bx}{c}}}{4b^2} + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = \frac{c(2y^2b^2 + 2abx + ac)\, \mathrm{e}^{-\frac{2bx}{c}}}{4b^2}$$

Summary

The solution(s) found are the following

$$\frac{c(2b^2y^2+2abx+ac)\,e^{-\frac{2bx}{c}}}{4b^2}=c_1 \tag{1}$$

Verification of solutions

$$\frac{c(2b^2y^2+2abx+ac)\,e^{-\frac{2bx}{c}}}{4b^2}=c_1$$

Verified OK.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
<- Bernoulli successful`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 69

```
dsolve(c*diff(y(x),x)=(a*x+b*y(x)^2)/y(x),y(x), singsol=all)
```

$$y(x)=-\frac{\sqrt{4\,e^{\frac{2bx}{c}}c_1b^2-4axb-2ac}}{2b}$$
$$y(x)=\frac{\sqrt{4\,e^{\frac{2bx}{c}}c_1b^2-4axb-2ac}}{2b}$$

✓ Solution by Mathematica

Time used: 5.371 (sec). Leaf size: 85

```
DSolve[c*y'[x]==(a*x+b*y[x]^2)/y[x],y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to -\frac{i\sqrt{abx + \frac{ac}{2} + b^2 c_1 \left(-e^{\frac{2bx}{c}}\right)}}{b}$$

$$y(x) \to \frac{i\sqrt{abx + \frac{ac}{2} + b^2 c_1 \left(-e^{\frac{2bx}{c}}\right)}}{b}$$

1.24 problem 24

Internal problem ID [7340]
Internal file name [`OUTPUT/6321_Sunday_June_05_2022_04_39_53_PM_21514280/index.tex`]

Book: First order enumerated odes
Section: section 1
Problem number: 24.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"quadrature"**

Maple gives the following as the ode type

`[_quadrature]`

$$\boxed{a \sin(x) y x y' = 0}$$

1.24.1 Solving as quadrature ode

Integrating both sides gives

$$\begin{aligned} y &= \int 0 \, \mathrm{d}x \\ &= c_1 \end{aligned}$$

Summary
The solution(s) found are the following

$$y = c_1 \tag{1}$$

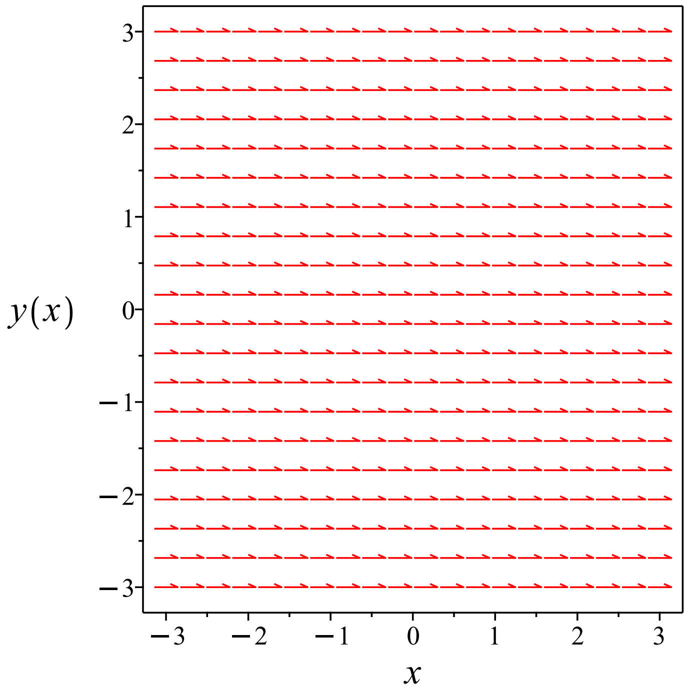


Figure 6: Slope field plot

Verification of solutions

$$y = c_1$$

Verified OK.

1.24.2 Maple step by step solution

Let's solve

$a \sin(x) y x y' = 0$

- Highest derivative means the order of the ODE is 1

  y'

- Integrate both sides with respect to x

  $\int a \sin(x) y x y' dx = \int 0 dx + c_1$

- Cannot compute integral

  $\int a \sin(x) y x y' dx = c_1$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
<- 1st order linear successful`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 9

```
dsolve(a*sin(x)*y(x)*x*diff(y(x),x)=0,y(x), singsol=all)
```

$$y(x) = 0$$
$$y(x) = c_1$$

✓ Solution by Mathematica

Time used: 0.003 (sec). Leaf size: 12

```
DSolve[a*Sin[x]*y[x]*x*y'[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to 0$$
$$y(x) \to c_1$$

1.25 problem 25

Internal problem ID [7341]
Internal file name [`OUTPUT/6322_Sunday_June_05_2022_04_39_55_PM_63703193/index.tex`]

Book: First order enumerated odes
Section: section 1
Problem number: 25.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"quadrature"**

Maple gives the following as the ode type

`[_quadrature]`

$$\boxed{f(x)\sin(x)\,yxy'\pi = 0}$$

1.25.1 Solving as quadrature ode

Integrating both sides gives

$$\begin{aligned} y &= \int 0 \,\mathrm{d}x \\ &= c_1 \end{aligned}$$

Summary
The solution(s) found are the following

$$y = c_1 \tag{1}$$

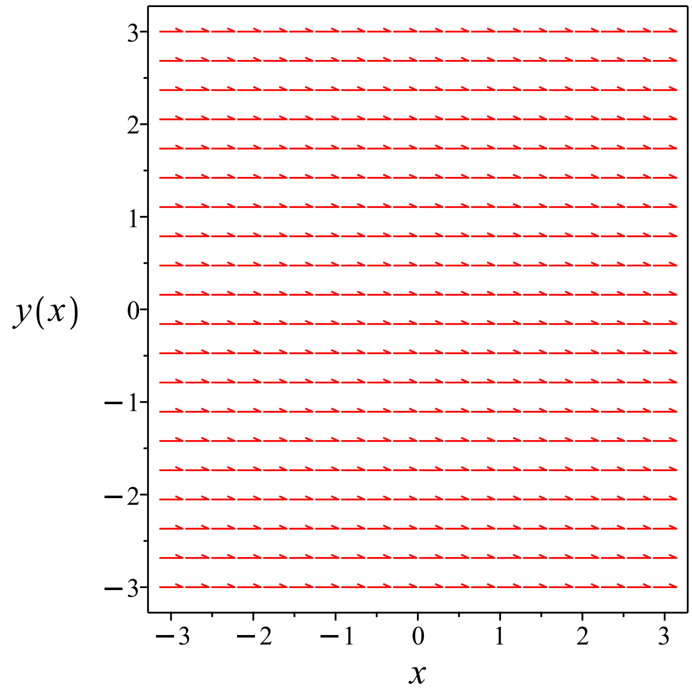


Figure 7: Slope field plot

Verification of solutions

$$y = c_1$$

Verified OK.

1.25.2 Maple step by step solution

Let's solve

$f(x)\sin(x)\, yxy'\pi = 0$

- Highest derivative means the order of the ODE is 1

  y'

- Integrate both sides with respect to x

  $\int f(x)\sin(x)\, yxy'\pi dx = \int 0dx + c_1$

- Cannot compute integral

  $\int f(x)\sin(x)\, yxy'\pi dx = c_1$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
<- 1st order linear successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 9

```
dsolve(f(x)*sin(x)*y(x)*x*diff(y(x),x)*Pi=0,y(x), singsol=all)
```

$$y(x) = 0$$
$$y(x) = c_1$$

✓ Solution by Mathematica

Time used: 0.004 (sec). Leaf size: 12

```
DSolve[f(x)*Sin[x]*y[x]*x*y'[x]*Pi==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to 0$$
$$y(x) \to c_1$$

1.26 problem 26

Internal problem ID [7342]
Internal file name [`OUTPUT/6323_Sunday_June_05_2022_04_39_57_PM_86742712/index.tex`]

Book: First order enumerated odes
Section: section 1
Problem number: 26.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"linear", "exactWithIntegrationFactor", "first_order_ode_lie_symmetry_lookup"**

Maple gives the following as the ode type

```
[[_linear, `class A`]]
```

$$\boxed{y' - y = \sin(x)}$$

1.26.1 Solving as linear ode

Entering Linear first order ODE solver. In canonical form a linear first order is

$$y' + p(x)y = q(x)$$

Where here

$$\begin{aligned} p(x) &= -1 \\ q(x) &= \sin(x) \end{aligned}$$

Hence the ode is

$$y' - y = \sin(x)$$

The integrating factor μ is

$$\begin{aligned} \mu &= e^{\int (-1)dx} \\ &= e^{-x} \end{aligned}$$

The ode becomes

$$\begin{aligned}
\frac{d}{dx}(\mu y) &= (\mu)(\sin(x)) \\
\frac{d}{dx}\left(e^{-x} y\right) &= \left(e^{-x}\right)(\sin(x)) \\
d\left(e^{-x} y\right) &= \left(\sin(x)\, e^{-x}\right)\, dx
\end{aligned}$$

Integrating gives

$$\begin{aligned}
e^{-x} y &= \int \sin(x)\, e^{-x}\, dx \\
e^{-x} y &= -\frac{\cos(x)\, e^{-x}}{2} - \frac{\sin(x)\, e^{-x}}{2} + c_1
\end{aligned}$$

Dividing both sides by the integrating factor $\mu = e^{-x}$ results in

$$y = e^{x}\left(-\frac{\cos(x)\, e^{-x}}{2} - \frac{\sin(x)\, e^{-x}}{2}\right) + c_1 e^{x}$$

which simplifies to

$$y = c_1 e^{x} - \frac{\sin(x)}{2} - \frac{\cos(x)}{2}$$

Summary

The solution(s) found are the following

$$y = c_1 e^{x} - \frac{\sin(x)}{2} - \frac{\cos(x)}{2} \tag{1}$$

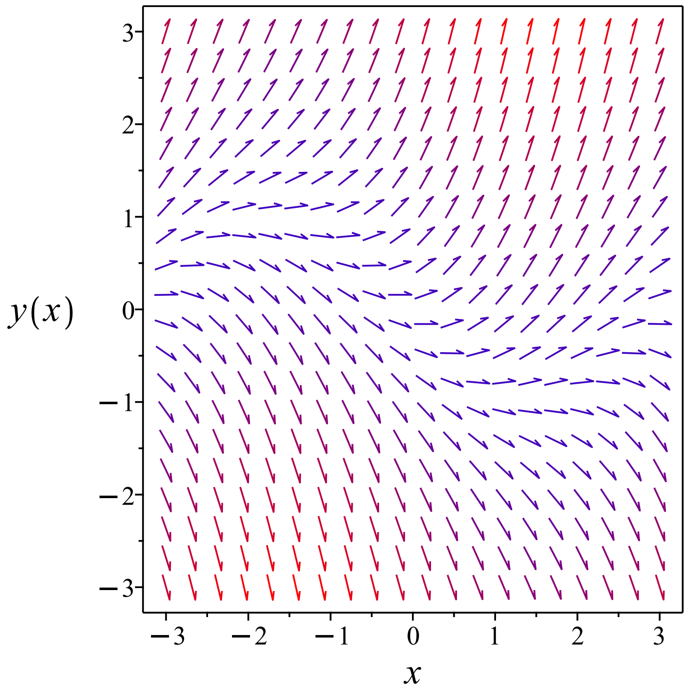


Figure 8: Slope field plot

Verification of solutions

$$y = c_1 e^{x} - \frac{\sin(x)}{2} - \frac{\cos(x)}{2}$$

Verified OK.

1.26.2 Solving as first order ode lie symmetry lookup ode

Writing the ode as

$$\begin{aligned} y' &= \sin(x) + y \\ y' &= \omega(x, y) \end{aligned}$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x + \omega(\eta_y - \xi_x) - \omega^2 \xi_y - \omega_x \xi - \omega_y \eta = 0 \tag{A}$$

The type of this ode is known. It is of type `linear`. Therefore we do not need to solve the PDE (A), and can just use the lookup table shown below to find ξ, η

Table 32: Lie symmetry infinitesimal lookup table for known first order ODE's

ODE class	Form	ξ	η
linear ode	$y' = f(x)y(x) + g(x)$	0	$e^{\int f dx}$
separable ode	$y' = f(x)\, g(y)$	$\frac{1}{f}$	0
quadrature ode	$y' = f(x)$	0	1
quadrature ode	$y' = g(y)$	1	0
homogeneous ODEs of Class A	$y' = f\left(\frac{y}{x}\right)$	x	y
homogeneous ODEs of Class C	$y' = (a + bx + cy)^{\frac{n}{m}}$	1	$-\frac{b}{c}$
homogeneous class D	$y' = \frac{y}{x} + g(x)\, F\left(\frac{y}{x}\right)$	x^2	xy
First order special form ID 1	$y' = g(x)\, e^{h(x)+by} + f(x)$	$\frac{e^{-\int bf(x)dx-h(x)}}{g(x)}$	$\frac{f(x)e^{-\int bf(x)dx-h(x)}}{g(x)}$
polynomial type ode	$y' = \frac{a_1x+b_1y+c_1}{a_2x+b_2y+c_2}$	$\frac{a_1b_2x-a_2b_1x-b_1c_2+b_2c_1}{a_1b_2-a_2b_1}$	$\frac{a_1b_2y-a_2b_1y-a_1c_2-a_2c_1}{a_1b_2-a_2b_1}$
Bernoulli ode	$y' = f(x)\, y + g(x)\, y^n$	0	$e^{-\int (n-1)f(x)dx} y^n$
Reduced Riccati	$y' = f_1(x)\, y + f_2(x)\, y^2$	0	$e^{-\int f_1 dx}$

The above table shows that

$$
\begin{aligned}
\xi(x, y) &= 0 \\
\eta(x, y) &= e^x
\end{aligned} \tag{A1}
$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x, y) \to (R, S)$ where (R, S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$
\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}
$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y}\right) S(x, y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the

canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$R = x$$

S is found from

$$\begin{aligned} S &= \int \frac{1}{\eta} dy \\ &= \int \frac{1}{e^{x}} dy \end{aligned}$$

Which results in

$$S = e^{-x} y$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x,y) S_y}{R_x + \omega(x,y) R_y} \tag{2}$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x,y)$ is the right hand side of the original ode given by

$$\omega(x,y) = \sin(x) + y$$

Evaluating all the partial derivatives gives

$$\begin{aligned} R_x &= 1 \\ R_y &= 0 \\ S_x &= -e^{-x} y \\ S_y &= e^{-x} \end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = \sin(x) e^{-x} \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = \sin(R) e^{-R}$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$S(R) = c_1 - \frac{e^{-R}(\cos(R) + \sin(R))}{2} \tag{4}$$

To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$e^{-x}y = -\frac{(\cos(x) + \sin(x))\, e^{-x}}{2} + c_1$$

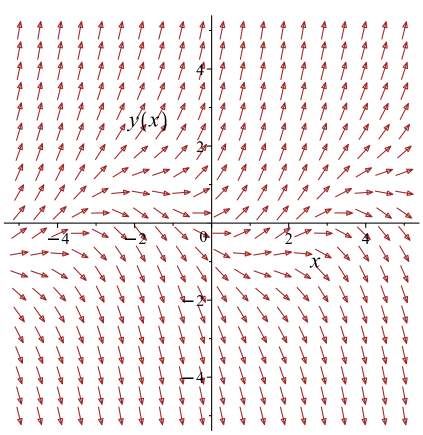
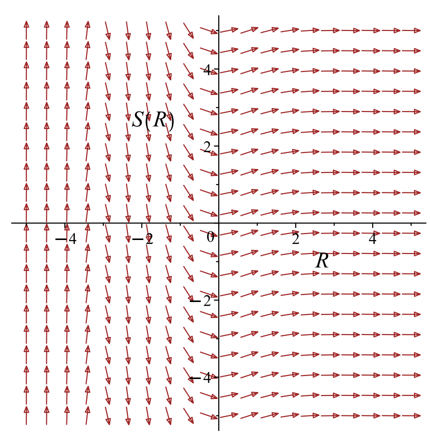
Which simplifies to

$$e^{-x}y = -\frac{(\cos(x) + \sin(x))\, e^{-x}}{2} + c_1$$

Which gives

$$y = -\frac{e^{x}(\sin(x)\, e^{-x} + \cos(x)\, e^{-x} - 2c_1)}{2}$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in x, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dx} = \sin(x) + y$ 	$R = x$ $S = e^{-x}y$	$\frac{dS}{dR} = \sin(R)\, e^{-R}$ 

Summary
The solution(s) found are the following

$$y = -\frac{e^{x}(\sin(x)\, e^{-x} + \cos(x)\, e^{-x} - 2c_1)}{2} \tag{1}$$

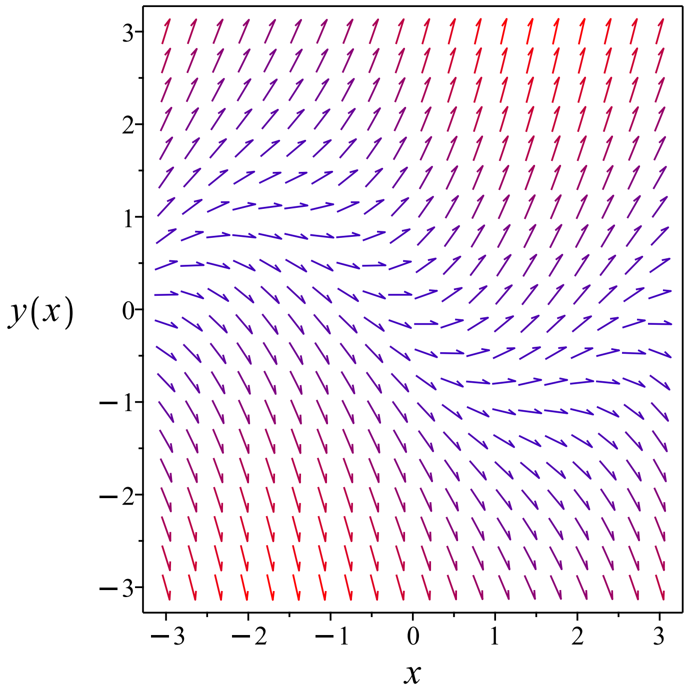


Figure 9: Slope field plot

Verification of solutions

$$y = -\frac{e^{x}(\sin(x)\,e^{-x} + \cos(x)\,e^{-x} - 2c_1)}{2}$$

Verified OK.

1.26.3 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x,y) + N(x,y)\frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x,y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y) = 0$$

Hence

$$\frac{\partial\phi}{\partial x} + \frac{\partial\phi}{\partial y}\frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$\begin{aligned}\frac{\partial \phi}{\partial x} &= M \\ \frac{\partial \phi}{\partial y} &= N\end{aligned}$$

But since $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x, y)$ but at least we know now that we can do that since the condition $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x, y)\,\mathrm{d}x + N(x, y)\,\mathrm{d}y = 0 \tag{1A}$$

Therefore

$$\begin{aligned}\mathrm{d}y &= (\sin(x) + y)\,\mathrm{d}x \\ (-\sin(x) - y)\,\mathrm{d}x + \mathrm{d}y &= 0 \qquad (2A)\end{aligned}$$

Comparing (1A) and (2A) shows that

$$\begin{aligned}M(x, y) &= -\sin(x) - y \\ N(x, y) &= 1\end{aligned}$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Using result found above gives

$$\begin{aligned}\frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}(-\sin(x) - y) \\ &= -1\end{aligned}$$

And

$$\begin{aligned}\frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}(1) \\ &= 0\end{aligned}$$

Since $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$, then the ODE is not exact. Since the ODE is not exact, we will try to find an integrating factor to make it exact. Let

$$\begin{aligned}A &= \frac{1}{N}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right)\\ &= 1((-1) - (0))\\ &= -1\end{aligned}$$

Since A does not depend on y, then it can be used to find an integrating factor. The integrating factor μ is

$$\begin{aligned}\mu &= e^{\int A\,\mathrm{d}x}\\ &= e^{\int -1\,\mathrm{d}x}\end{aligned}$$

The result of integrating gives

$$\begin{aligned}\mu &= e^{-x}\\ &= \mathrm{e}^{-x}\end{aligned}$$

M and N are multiplied by this integrating factor, giving new M and new N which are called $\overline{M}$ and $\overline{N}$ for now so not to confuse them with the original M and N.

$$\begin{aligned}\overline{M} &= \mu M\\ &= \mathrm{e}^{-x}(-\sin(x) - y)\\ &= -\mathrm{e}^{-x}(\sin(x) + y)\end{aligned}$$

And

$$\begin{aligned}\overline{N} &= \mu N\\ &= \mathrm{e}^{-x}(1)\\ &= \mathrm{e}^{-x}\end{aligned}$$

Now a modified ODE is ontained from the original ODE, which is exact and can be solved. The modified ODE is

$$\begin{aligned}\overline{M} + \overline{N}\frac{\mathrm{d}y}{\mathrm{d}x} &= 0\\ \left(-\mathrm{e}^{-x}(\sin(x) + y)\right) + \left(\mathrm{e}^{-x}\right)\frac{\mathrm{d}y}{\mathrm{d}x} &= 0\end{aligned}$$

The following equations are now set up to solve for the function $\phi(x, y)$

$$\frac{\partial \phi}{\partial x} = \overline{M} \tag{1}$$

$$\frac{\partial \phi}{\partial y} = \overline{N} \tag{2}$$

Integrating (1) w.r.t. x gives

$$\int \frac{\partial \phi}{\partial x}\, \mathrm{d}x = \int \overline{M}\, \mathrm{d}x$$

$$\int \frac{\partial \phi}{\partial x}\, \mathrm{d}x = \int -\mathrm{e}^{-x}(\sin(x) + y)\, \mathrm{d}x$$

$$\phi = \frac{(2y + \cos(x) + \sin(x))\,\mathrm{e}^{-x}}{2} + f(y) \tag{3}$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$\frac{\partial \phi}{\partial y} = \mathrm{e}^{-x} + f'(y) \tag{4}$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = \mathrm{e}^{-x}$. Therefore equation (4) becomes

$$\mathrm{e}^{-x} = \mathrm{e}^{-x} + f'(y) \tag{5}$$

Solving equation (5) for $f'(y)$ gives

$$f'(y) = 0$$

Therefore

$$f(y) = c_1$$

Where c_1 is constant of integration. Substituting this result for $f(y)$ into equation (3) gives ϕ

$$\phi = \frac{(2y + \cos(x) + \sin(x))\,\mathrm{e}^{-x}}{2} + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = \frac{(2y + \cos(x) + \sin(x))\,\mathrm{e}^{-x}}{2}$$

The solution becomes

$$y = -\frac{\mathrm{e}^{x}(\sin(x)\,\mathrm{e}^{-x} + \cos(x)\,\mathrm{e}^{-x} - 2c_1)}{2}$$

Summary
The solution(s) found are the following

$$y = -\frac{e^{x}(\sin(x) e^{-x} + \cos(x) e^{-x} - 2c_1)}{2} \tag{1}$$

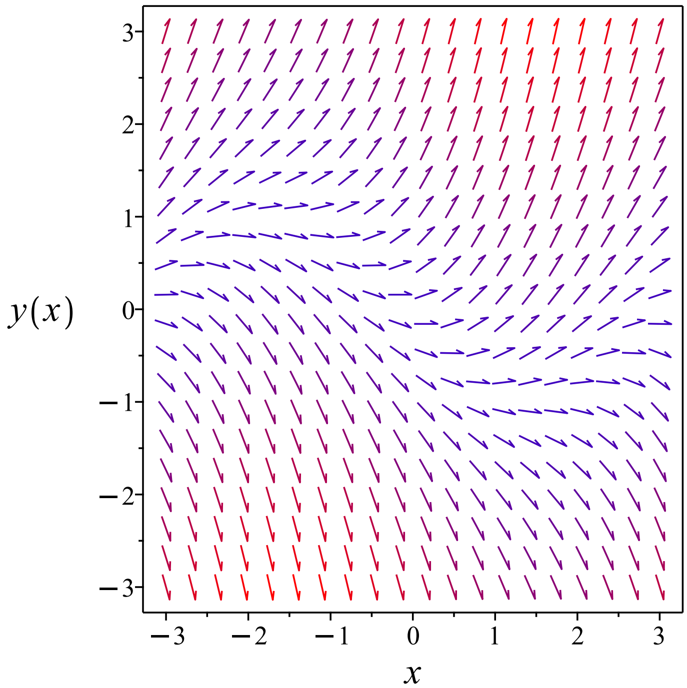


Figure 10: Slope field plot

Verification of solutions

$$y = -\frac{e^{x}(\sin(x) e^{-x} + \cos(x) e^{-x} - 2c_1)}{2}$$

Verified OK.

1.26.4 Maple step by step solution

Let's solve

$y' - y = \sin(x)$

- Highest derivative means the order of the ODE is 1

y'

- Isolate the derivative

$y' = \sin(x) + y$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE

  $y' - y = \sin(x)$

- The ODE is linear; multiply by an integrating factor $\mu(x)$

  $\mu(x)(y' - y) = \mu(x)\sin(x)$

- Assume the lhs of the ODE is the total derivative $\frac{d}{dx}(\mu(x)\,y)$

  $\mu(x)(y' - y) = \mu'(x)\,y + \mu(x)\,y'$

- Isolate $\mu'(x)$

  $\mu'(x) = -\mu(x)$

- Solve to find the integrating factor

  $\mu(x) = e^{-x}$

- Integrate both sides with respect to x

  $\int \left(\frac{d}{dx}(\mu(x)\,y)\right) dx = \int \mu(x)\sin(x)\,dx + c_1$

- Evaluate the integral on the lhs

  $\mu(x)\,y = \int \mu(x)\sin(x)\,dx + c_1$

- Solve for y

  $y = \frac{\int \mu(x)\sin(x)dx + c_1}{\mu(x)}$

- Substitute $\mu(x) = e^{-x}$

  $y = \frac{\int \sin(x)e^{-x}dx + c_1}{e^{-x}}$

- Evaluate the integrals on the rhs

  $y = \frac{-\frac{\sin(x)e^{-x}}{2} - \frac{\cos(x)e^{-x}}{2} + c_1}{e^{-x}}$

- Simplify

  $y = c_1 e^{x} - \frac{\sin(x)}{2} - \frac{\cos(x)}{2}$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
<- 1st order linear successful`
```

✓ Solution by Maple

Time used: 0.015 (sec). Leaf size: 17

```
dsolve(diff(y(x),x)=sin(x)+y(x),y(x), singsol=all)
```

$$y(x) = -\frac{\cos(x)}{2} - \frac{\sin(x)}{2} + c_1 e^x$$

✓ Solution by Mathematica

Time used: 0.04 (sec). Leaf size: 24

```
DSolve[y'[x]==Sin[x]+y[x],y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to -\frac{\sin(x)}{2} - \frac{\cos(x)}{2} + c_1 e^x$$

1.27 problem 27

Internal problem ID [7343]
Internal file name [`OUTPUT/6324_Sunday_June_05_2022_04_39_59_PM_66980461/index.tex`]

Book: First order enumerated odes
Section: section 1
Problem number: 27.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"riccati"**

Maple gives the following as the ode type

`[_Riccati]`

$$\boxed{y' - y^2 = \sin(x)}$$

1.27.1 Solving as riccati ode

In canonical form the ODE is

$$\begin{aligned} y' &= F(x, y) \\ &= \sin(x) + y^2 \end{aligned}$$

This is a Riccati ODE. Comparing the ODE to solve

$$y' = \sin(x) + y^2$$

With Riccati ODE standard form

$$y' = f_0(x) + f_1(x)y + f_2(x)y^2$$

Shows that $f_0(x) = \sin(x)$, $f_1(x) = 0$ and $f_2(x) = 1$. Let

$$\begin{aligned} y &= \frac{-u'}{f_2 u} \\ &= \frac{-u'}{u} \end{aligned} \tag{1}$$

Using the above substitution in the given ODE results (after some simplification)in a second order ODE to solve for $u(x)$ which is

$$f_2u''(x) - (f_2' + f_1f_2)\,u'(x) + f_2^2f_0u(x) = 0 \tag{2}$$

But

$$\begin{aligned} f_2' &= 0 \\ f_1f_2 &= 0 \\ f_2^2f_0 &= \sin(x) \end{aligned}$$

Substituting the above terms back in equation (2) gives

$$u''(x) + \sin(x)\,u(x) = 0$$

Solving the above ODE (this ode solved using Maple, not this program), gives

$$u(x) = c_1\,\text{MathieuC}\left(0, -2, -\frac{\pi}{4} + \frac{x}{2}\right) + c_2\,\text{MathieuS}\left(0, -2, -\frac{\pi}{4} + \frac{x}{2}\right)$$

The above shows that

$$u'(x) = \frac{c_1\,\text{MathieuCPrime}\left(0, -2, -\frac{\pi}{4} + \frac{x}{2}\right)}{2} + \frac{c_2\,\text{MathieuSPrime}\left(0, -2, -\frac{\pi}{4} + \frac{x}{2}\right)}{2}$$

Using the above in (1) gives the solution

$$y = -\frac{\frac{c_1\,\text{MathieuCPrime}\left(0,-2,-\frac{\pi}{4}+\frac{x}{2}\right)}{2} + \frac{c_2\,\text{MathieuSPrime}\left(0,-2,-\frac{\pi}{4}+\frac{x}{2}\right)}{2}}{c_1\,\text{MathieuC}\left(0, -2, -\frac{\pi}{4} + \frac{x}{2}\right) + c_2\,\text{MathieuS}\left(0, -2, -\frac{\pi}{4} + \frac{x}{2}\right)}$$

Dividing both numerator and denominator by c_1 gives, after renaming the constant $\frac{c_2}{c_1} = c_3$ the following solution

$$y = \frac{-c_3\,\text{MathieuCPrime}\left(0, -2, -\frac{\pi}{4} + \frac{x}{2}\right) - \text{MathieuSPrime}\left(0, -2, -\frac{\pi}{4} + \frac{x}{2}\right)}{2c_3\,\text{MathieuC}\left(0, -2, -\frac{\pi}{4} + \frac{x}{2}\right) + 2\,\text{MathieuS}\left(0, -2, -\frac{\pi}{4} + \frac{x}{2}\right)}$$

Summary

The solution(s) found are the following

$$y = \frac{-c_3\,\text{MathieuCPrime}\left(0, -2, -\frac{\pi}{4} + \frac{x}{2}\right) - \text{MathieuSPrime}\left(0, -2, -\frac{\pi}{4} + \frac{x}{2}\right)}{2c_3\,\text{MathieuC}\left(0, -2, -\frac{\pi}{4} + \frac{x}{2}\right) + 2\,\text{MathieuS}\left(0, -2, -\frac{\pi}{4} + \frac{x}{2}\right)} \tag{1}$$

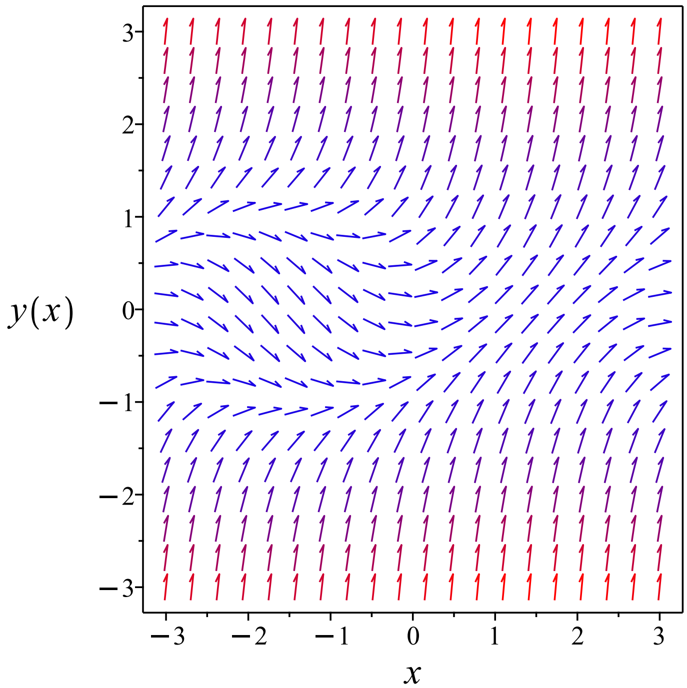


Figure 11: Slope field plot

Verification of solutions

$$y = \frac{-c_3 \operatorname{MathieuCPrime}\left(0, -2, -\frac{\pi}{4} + \frac{x}{2}\right) - \operatorname{MathieuSPrime}\left(0, -2, -\frac{\pi}{4} + \frac{x}{2}\right)}{2c_3 \operatorname{MathieuC}\left(0, -2, -\frac{\pi}{4} + \frac{x}{2}\right) + 2 \operatorname{MathieuS}\left(0, -2, -\frac{\pi}{4} + \frac{x}{2}\right)}$$

Verified OK.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
trying homogeneous types:
trying Chini
differential order: 1; looking for linear symmetries
trying exact
Looking for potential symmetries
trying Riccati
trying Riccati Special
trying Riccati sub-methods:
   trying Riccati_symmetries
   trying Riccati to 2nd Order
   -> Calling odsolve with the ODE`, diff(diff(y(x), x), x) = -y(x)*sin(x), y(x)`      *** S
      Methods for second order ODEs:
      --- Trying classification methods ---
      trying a symmetry of the form [xi=0, eta=F(x)]
      checking if the LODE is missing y
      -> Heun: Equivalence to the GHE or one of its 4 confluent cases under a power @ Moebiu
      -> trying a solution of the form r0(x) * Y + r1(x) * Y where Y = exp(int(r(x), dx)) *
      -> Trying changes of variables to rationalize or make the ODE simpler
         trying a symmetry of the form [xi=0, eta=F(x)]
         checking if the LODE is missing y
         -> Trying an equivalence, under non-integer power transformations,
            to LODEs admitting Liouvillian solutions.
            -> Trying a Liouvillian solution using Kovacics algorithm
            <- No Liouvillian solutions exists
         -> Trying a solution in terms of special functions:
            -> Bessel
            -> elliptic
            -> Legendre
            -> Whittaker
               -> hyper3: Equivalence to 1F1 under a power @ Moebius
            -> hypergeometric
               -> heuristic approach
               -> hyper3: Equivalence to 2F1, 1F1 or 0F1 under a power @ Moebius
            -> Mathieu
               -> Equivalence to the rational form of Mathieu ODE under a power @ Moebius
               Equivalence transformation and function parameters: {t = 1/2*t+1/2}, {kappa =
               <- Equivalence to the rational form of Mathieu ODE successful
            <- Mathieu successful
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 59

```
dsolve(diff(y(x),x)=sin(x)+y(x)^2,y(x), singsol=all)
```

$$y(x) = \frac{-c_1 \,\mathrm{MathieuSPrime}\left(0, -2, -\frac{\pi}{4} + \frac{x}{2}\right) - \mathrm{MathieuCPrime}\left(0, -2, -\frac{\pi}{4} + \frac{x}{2}\right)}{2c_1 \,\mathrm{MathieuS}\left(0, -2, -\frac{\pi}{4} + \frac{x}{2}\right) + 2\,\mathrm{MathieuC}\left(0, -2, -\frac{\pi}{4} + \frac{x}{2}\right)}$$

✓ Solution by Mathematica

Time used: 0.208 (sec). Leaf size: 105

```
DSolve[y'[x]==Sin[x]+y[x]^2,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \frac{-\mathrm{MathieuSPrime}\left[0, -2, \frac{1}{4}(\pi - 2x)\right] + c_1 \mathrm{MathieuCPrime}\left[0, -2, \frac{1}{4}(\pi - 2x)\right]}{2\left(\mathrm{MathieuS}\left[0, -2, \frac{1}{4}(2x - \pi)\right] + c_1 \mathrm{MathieuC}\left[0, -2, \frac{1}{4}(\pi - 2x)\right]\right)}$$

$$y(x) \to \frac{\mathrm{MathieuCPrime}\left[0, -2, \frac{1}{4}(\pi - 2x)\right]}{2\mathrm{MathieuC}\left[0, -2, \frac{1}{4}(\pi - 2x)\right]}$$

1.28 problem 28

Internal problem ID [7344]
Internal file name [`OUTPUT/6325_Sunday_June_05_2022_04_40_08_PM_2578982/index.tex`]

Book: First order enumerated odes
Section: section 1
Problem number: 28.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"linear", "homogeneousTypeD2", "exactWithIntegrationFactor", "first_order_ode_lie_symmetry_lookup"**

Maple gives the following as the ode type

`[_linear]`

$$\boxed{y' - \frac{y}{x} = \cos(x)}$$

1.28.1 Solving as linear ode

Entering Linear first order ODE solver. In canonical form a linear first order is

$$y' + p(x)y = q(x)$$

Where here

$$\begin{aligned} p(x) &= -\frac{1}{x} \\ q(x) &= \cos(x) \end{aligned}$$

Hence the ode is

$$y' - \frac{y}{x} = \cos(x)$$

The integrating factor μ is

$$\begin{aligned}\mu &= e^{\int -\frac{1}{x}dx} \\ &= \frac{1}{x}\end{aligned}$$

The ode becomes

$$\begin{aligned}\frac{d}{dx}(\mu y) &= (\mu)(\cos(x)) \\ \frac{d}{dx}\left(\frac{y}{x}\right) &= \left(\frac{1}{x}\right)(\cos(x)) \\ d\left(\frac{y}{x}\right) &= \left(\frac{\cos(x)}{x}\right) dx\end{aligned}$$

Integrating gives

$$\begin{aligned}\frac{y}{x} &= \int \frac{\cos(x)}{x} dx \\ \frac{y}{x} &= \operatorname{Ci}(x) + c_1\end{aligned}$$

Dividing both sides by the integrating factor $\mu = \frac{1}{x}$ results in

$$y = x \operatorname{Ci}(x) + c_1 x$$

which simplifies to

$$y = x(\operatorname{Ci}(x) + c_1)$$

Summary

The solution(s) found are the following

$$y = x(\operatorname{Ci}(x) + c_1) \tag{1}$$

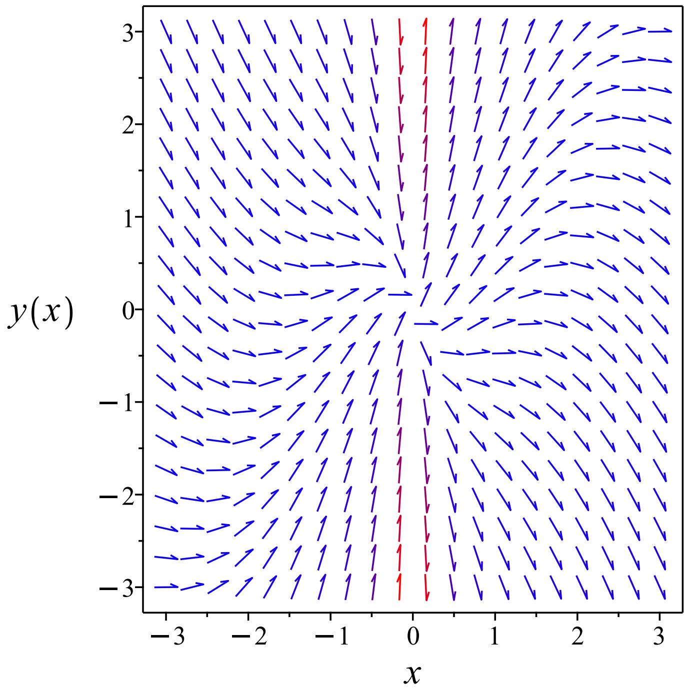


Figure 12: Slope field plot

Verification of solutions

$$y = x(\mathrm{Ci}(x) + c_1)$$

Verified OK.

1.28.2 Solving as homogeneousTypeD2 ode

Using the change of variables $y = u(x)\,x$ on the above ode results in new ode in $u(x)$

$$u'(x)\,x = \cos(x)$$

Integrating both sides gives

$$\begin{aligned} u(x) &= \int \frac{\cos(x)}{x}\, dx \\ &= \mathrm{Ci}(x) + c_2 \end{aligned}$$

Therefore the solution y is

$$\begin{aligned} y &= xu \\ &= x(\mathrm{Ci}(x) + c_2) \end{aligned}$$

Summary

The solution(s) found are the following

$$y = x(\mathrm{Ci}(x) + c_2) \tag{1}$$

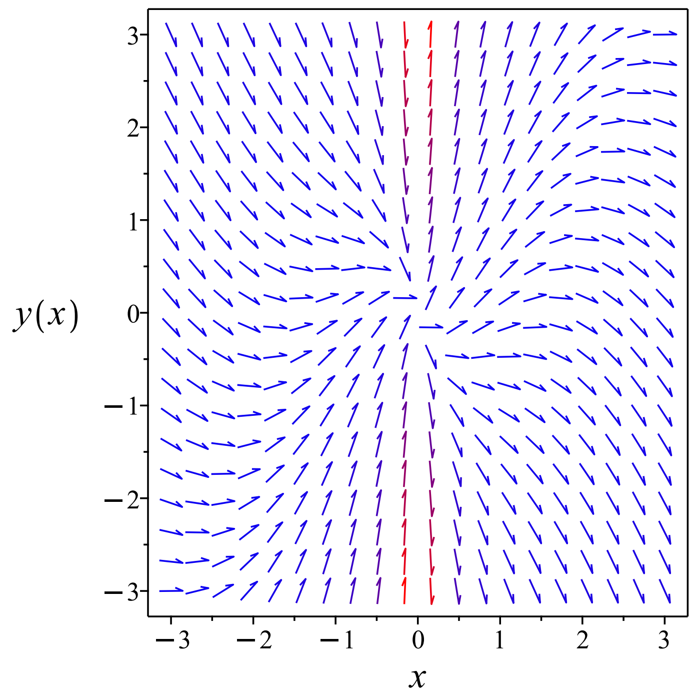


Figure 13: Slope field plot

Verification of solutions

$$y = x(\mathrm{Ci}(x) + c_2)$$

Verified OK.

1.28.3 Solving as first order ode lie symmetry lookup ode

Writing the ode as

$$y' = \frac{x\cos(x) + y}{x}$$
$$y' = \omega(x, y)$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x + \omega(\eta_y - \xi_x) - \omega^2\xi_y - \omega_x\xi - \omega_y\eta = 0 \tag{A}$$

The type of this ode is known. It is of type `linear`. Therefore we do not need to solve the PDE (A), and can just use the lookup table shown below to find ξ, η

Table 35: Lie symmetry infinitesimal lookup table for known first order ODE's

ODE class	Form	ξ	η
linear ode	$y' = f(x)y(x) + g(x)$	0	$e^{\int f dx}$
separable ode	$y' = f(x)\, g(y)$	$\frac{1}{f}$	0
quadrature ode	$y' = f(x)$	0	1
quadrature ode	$y' = g(y)$	1	0
homogeneous ODEs of Class A	$y' = f\left(\frac{y}{x}\right)$	x	y
homogeneous ODEs of Class C	$y' = (a + bx + cy)^{\frac{n}{m}}$	1	$-\frac{b}{c}$
homogeneous class D	$y' = \frac{y}{x} + g(x)\, F\left(\frac{y}{x}\right)$	x^2	xy
First order special form ID 1	$y' = g(x)\, e^{h(x)+by} + f(x)$	$\frac{e^{-\int bf(x)dx - h(x)}}{g(x)}$	$\frac{f(x)e^{-\int bf(x)dx - h(x)}}{g(x)}$
polynomial type ode	$y' = \frac{a_1x+b_1y+c_1}{a_2x+b_2y+c_2}$	$\frac{a_1b_2x-a_2b_1x-b_1c_2+b_2c_1}{a_1b_2-a_2b_1}$	$\frac{a_1b_2y-a_2b_1y-a_1c_2-a_2c_1}{a_1b_2-a_2b_1}$
Bernoulli ode	$y' = f(x)\, y + g(x)\, y^n$	0	$e^{-\int (n-1)f(x)dx} y^n$
Reduced Riccati	$y' = f_1(x)\, y + f_2(x)\, y^2$	0	$e^{-\int f_1 dx}$

The above table shows that

$$
\begin{aligned}
\xi(x, y) &= 0 \\
\eta(x, y) &= x
\end{aligned} \tag{A1}
$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x, y) \to (R, S)$ where (R, S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$
\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}
$$

The above comes from the requirements that $\left(\xi\frac{\partial}{\partial x}+\eta\frac{\partial}{\partial y}\right)S(x,y)=1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the canonical coordinates, where $S(R)$. Since $\xi=0$ then in this special case

$$R = x$$

S is found from

$$\begin{aligned} S &= \int \frac{1}{\eta} dy \\ &= \int \frac{1}{x} dy \end{aligned}$$

Which results in

$$S = \frac{y}{x}$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x,y)S_y}{R_x + \omega(x,y)R_y} \tag{2}$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x,y)$ is the right hand side of the original ode given by

$$\omega(x,y) = \frac{x\cos(x) + y}{x}$$

Evaluating all the partial derivatives gives

$$\begin{aligned} R_x &= 1 \\ R_y &= 0 \\ S_x &= -\frac{y}{x^2} \\ S_y &= \frac{1}{x} \end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = \frac{\cos(x)}{x} \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = \frac{\cos(R)}{R}$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordinates R, S. Integrating the above gives

$$S(R) = \operatorname{Ci}(R) + c_1 \tag{4}$$

To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$\frac{y}{x} = \operatorname{Ci}(x) + c_1$$

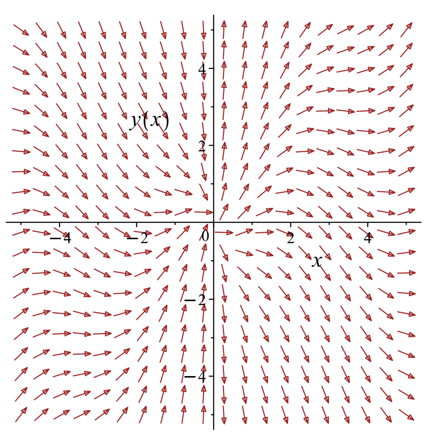
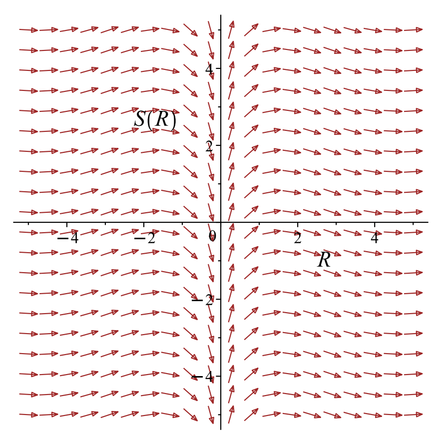
Which simplifies to

$$\frac{y}{x} = \operatorname{Ci}(x) + c_1$$

Which gives

$$y = x(\operatorname{Ci}(x) + c_1)$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in x, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dx} = \frac{x\cos(x)+y}{x}$ 	$R = x$ $S = \frac{y}{x}$	$\frac{dS}{dR} = \frac{\cos(R)}{R}$ 

Summary
The solution(s) found are the following

$$y = x(\operatorname{Ci}(x) + c_1) \tag{1}$$

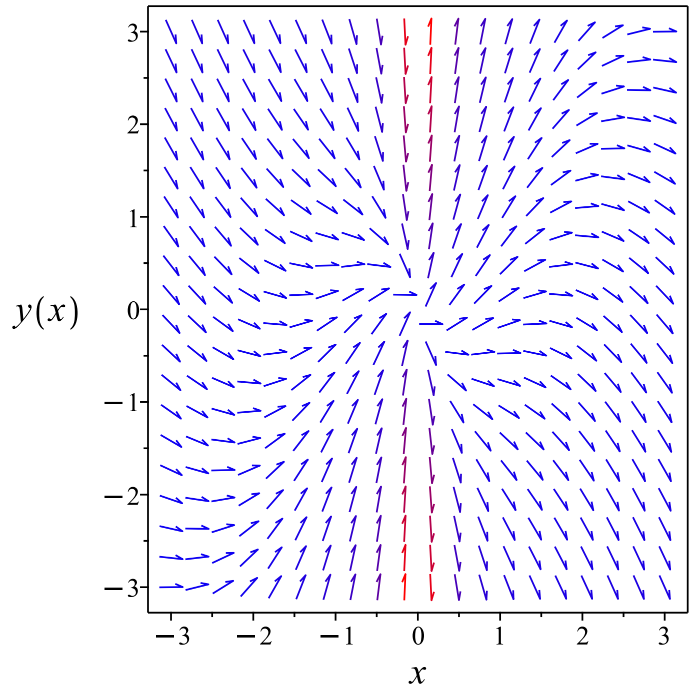


Figure 14: Slope field plot

Verification of solutions

$$y = x(\mathrm{Ci}\,(x) + c_1)$$

Verified OK.

1.28.4 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x, y) + N(x, y)\frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x, y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x, y) = 0$$

Hence

$$\frac{\partial \phi}{\partial x} + \frac{\partial \phi}{\partial y}\frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$\frac{\partial \phi}{\partial x} = M$$
$$\frac{\partial \phi}{\partial y} = N$$

But since $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x,y)$ but at least we know now that we can do that since the condition $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x,y)\,\mathrm{d}x + N(x,y)\,\mathrm{d}y = 0 \tag{1A}$$

Therefore

$$\mathrm{d}y = \left(\cos(x) + \frac{y}{x}\right)\mathrm{d}x$$
$$\left(-\cos(x) - \frac{y}{x}\right)\mathrm{d}x + \mathrm{d}y = 0 \tag{2A}$$

Comparing (1A) and (2A) shows that

$$M(x,y) = -\cos(x) - \frac{y}{x}$$
$$N(x,y) = 1$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Using result found above gives

$$\begin{aligned}\frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}\left(-\cos(x) - \frac{y}{x}\right)\\ &= -\frac{1}{x}\end{aligned}$$

And

$$\begin{aligned}
\frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}(1) \\
&= 0
\end{aligned}$$

Since $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$, then the ODE is not exact. Since the ODE is not exact, we will try to find an integrating factor to make it exact. Let

$$\begin{aligned}
A &= \frac{1}{N}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right) \\
&= 1\left(\left(-\frac{1}{x}\right) - (0)\right) \\
&= -\frac{1}{x}
\end{aligned}$$

Since A does not depend on y, then it can be used to find an integrating factor. The integrating factor μ is

$$\begin{aligned}
\mu &= e^{\int A \, \mathrm{d}x} \\
&= e^{\int -\frac{1}{x} \, \mathrm{d}x}
\end{aligned}$$

The result of integrating gives

$$\begin{aligned}
\mu &= e^{-\ln(x)} \\
&= \frac{1}{x}
\end{aligned}$$

M and N are multiplied by this integrating factor, giving new M and new N which are called $\overline{M}$ and $\overline{N}$ for now so not to confuse them with the original M and N.

$$\begin{aligned}
\overline{M} &= \mu M \\
&= \frac{1}{x}\left(-\cos(x) - \frac{y}{x}\right) \\
&= \frac{-x\cos(x) - y}{x^2}
\end{aligned}$$

And

$$\begin{aligned}
\overline{N} &= \mu N \\
&= \frac{1}{x}(1) \\
&= \frac{1}{x}
\end{aligned}$$

Now a modified ODE is ontained from the original ODE, which is exact and can be solved. The modified ODE is

$$\overline{M} + \overline{N}\frac{\mathrm{d}y}{\mathrm{d}x} = 0$$
$$\left(\frac{-x\cos(x) - y}{x^2}\right) + \left(\frac{1}{x}\right)\frac{\mathrm{d}y}{\mathrm{d}x} = 0$$

The following equations are now set up to solve for the function $\phi(x, y)$

$$\frac{\partial\phi}{\partial x} = \overline{M} \tag{1}$$
$$\frac{\partial\phi}{\partial y} = \overline{N} \tag{2}$$

Integrating (1) w.r.t. x gives

$$\int \frac{\partial\phi}{\partial x}\,\mathrm{d}x = \int \overline{M}\,\mathrm{d}x$$
$$\int \frac{\partial\phi}{\partial x}\,\mathrm{d}x = \int \frac{-x\cos(x) - y}{x^2}\,\mathrm{d}x$$
$$\phi = -\operatorname{Ci}(x) + \frac{y}{x} + f(y) \tag{3}$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$\frac{\partial\phi}{\partial y} = \frac{1}{x} + f'(y) \tag{4}$$

But equation (2) says that $\frac{\partial\phi}{\partial y} = \frac{1}{x}$. Therefore equation (4) becomes

$$\frac{1}{x} = \frac{1}{x} + f'(y) \tag{5}$$

Solving equation (5) for $f'(y)$ gives

$$f'(y) = 0$$

Therefore

$$f(y) = c_1$$

Where c_1 is constant of integration. Substituting this result for $f(y)$ into equation (3) gives ϕ

$$\phi = -\operatorname{Ci}(x) + \frac{y}{x} + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = -\operatorname{Ci}(x) + \frac{y}{x}$$

The solution becomes

$$y = x(\operatorname{Ci}(x) + c_1)$$

Summary

The solution(s) found are the following

$$y = x(\operatorname{Ci}(x) + c_1) \tag{1}$$

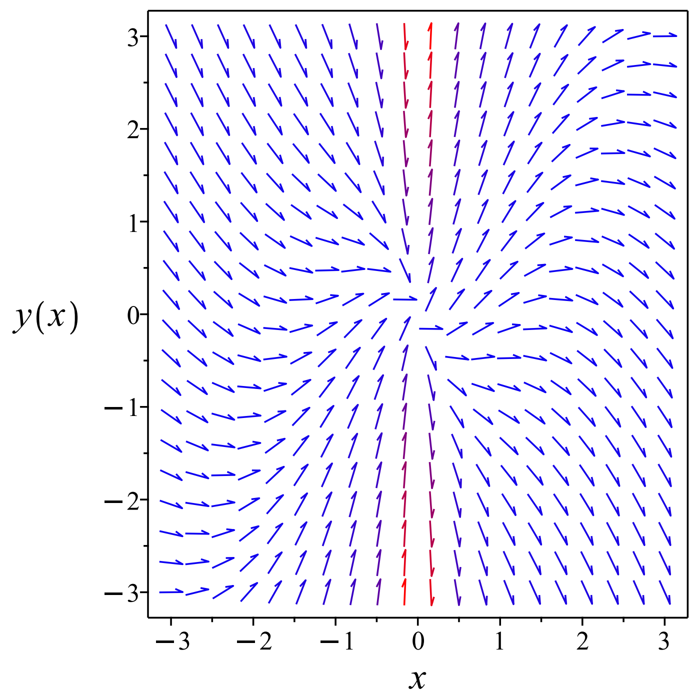


Figure 15: Slope field plot

Verification of solutions

$$y = x(\operatorname{Ci}(x) + c_1)$$

Verified OK.

1.28.5 Maple step by step solution

Let's solve

$y' - \frac{y}{x} = \cos(x)$

- Highest derivative means the order of the ODE is 1

  y'

- Isolate the derivative

  $y' = \cos(x) + \frac{y}{x}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE

  $y' - \frac{y}{x} = \cos(x)$

- The ODE is linear; multiply by an integrating factor $\mu(x)$

  $\mu(x)\left(y' - \frac{y}{x}\right) = \mu(x)\cos(x)$

- Assume the lhs of the ODE is the total derivative $\frac{d}{dx}(\mu(x)\,y)$

  $\mu(x)\left(y' - \frac{y}{x}\right) = \mu'(x)\,y + \mu(x)\,y'$

- Isolate $\mu'(x)$

  $\mu'(x) = -\frac{\mu(x)}{x}$

- Solve to find the integrating factor

  $\mu(x) = \frac{1}{x}$

- Integrate both sides with respect to x

  $\int \left(\frac{d}{dx}(\mu(x)\,y)\right) dx = \int \mu(x)\cos(x)\,dx + c_1$

- Evaluate the integral on the lhs

  $\mu(x)\,y = \int \mu(x)\cos(x)\,dx + c_1$

- Solve for y

  $y = \frac{\int \mu(x)\cos(x)dx + c_1}{\mu(x)}$

- Substitute $\mu(x) = \frac{1}{x}$

  $y = x\left(\int \frac{\cos(x)}{x}dx + c_1\right)$

- Evaluate the integrals on the rhs

  $y = x(\mathrm{Ci}(x) + c_1)$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
<- 1st order linear successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 10

```
dsolve(diff(y(x),x)=cos(x)+y(x)/x,y(x), singsol=all)
```

$$y(x) = (\mathrm{Ci}\,(x) + c_1)\,x$$

✓ Solution by Mathematica

Time used: 0.038 (sec). Leaf size: 12

```
DSolve[y'[x]==Cos[x]+y[x]/x,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to x(\mathrm{CosIntegral}(x) + c_1)$$

1.29 problem 29

Internal problem ID [7345]
Internal file name [`OUTPUT/6326_Sunday_June_05_2022_04_40_10_PM_8112400/index.tex`]

Book: First order enumerated odes
Section: section 1
Problem number: 29.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"riccati"**

Maple gives the following as the ode type

`[_Riccati]`

$$\boxed{y' - \frac{y^2}{x} = \cos(x)}$$

1.29.1 Solving as riccati ode

In canonical form the ODE is

$$\begin{aligned} y' &= F(x,y) \\ &= \frac{x\cos(x) + y^2}{x} \end{aligned}$$

This is a Riccati ODE. Comparing the ODE to solve

$$y' = \cos(x) + \frac{y^2}{x}$$

With Riccati ODE standard form

$$y' = f_0(x) + f_1(x)y + f_2(x)y^2$$

Shows that $f_0(x) = \cos(x)$, $f_1(x) = 0$ and $f_2(x) = \frac{1}{x}$. Let

$$\begin{aligned} y &= \frac{-u'}{f_2 u} \\ &= \frac{-u'}{\frac{u}{x}} \end{aligned} \tag{1}$$

Using the above substitution in the given ODE results (after some simplification)in a second order ODE to solve for $u(x)$ which is

$$f_2 u''(x) - (f_2' + f_1 f_2)\, u'(x) + f_2^2 f_0 u(x) = 0 \tag{2}$$

But

$$\begin{aligned} f_2' &= -\frac{1}{x^2} \\ f_1 f_2 &= 0 \\ f_2^2 f_0 &= \frac{\cos(x)}{x^2} \end{aligned}$$

Substituting the above terms back in equation (2) gives

$$\frac{u''(x)}{x} + \frac{u'(x)}{x^2} + \frac{\cos(x)\, u(x)}{x^2} = 0$$

Solving the above ODE (this ode solved using Maple, not this program), gives

$$u(x) = \mathrm{DESol}\left(\left\{\_Y''(x) + \frac{\_Y'(x)}{x} + \frac{\cos(x)\_Y(x)}{x}\right\}, \{\_Y(x)\}\right)$$

The above shows that

$$u'(x) = \frac{d}{dx}\mathrm{DESol}\left(\left\{\_Y''(x) + \frac{\_Y'(x)}{x} + \frac{\cos(x)\_Y(x)}{x}\right\}, \{\_Y(x)\}\right)$$

Using the above in (1) gives the solution

$$y = -\frac{\left(\frac{d}{dx}\mathrm{DESol}\left(\left\{\_Y''(x) + \frac{\_Y'(x)}{x} + \frac{\cos(x)\_Y(x)}{x}\right\}, \{\_Y(x)\}\right)\right) x}{\mathrm{DESol}\left(\left\{\_Y''(x) + \frac{\_Y'(x)}{x} + \frac{\cos(x)\_Y(x)}{x}\right\}, \{\_Y(x)\}\right)}$$

Dividing both numerator and denominator by c_1 gives, after renaming the constant $\frac{c_2}{c_1} = c_3$ the following solution

$$y = -\frac{\left(\frac{d}{dx}\mathrm{DESol}\left(\left\{\_Y''(x) + \frac{\_Y'(x)}{x} + \frac{\cos(x)\_Y(x)}{x}\right\}, \{\_Y(x)\}\right)\right) x}{\mathrm{DESol}\left(\left\{\_Y''(x) + \frac{\_Y'(x)}{x} + \frac{\cos(x)\_Y(x)}{x}\right\}, \{\_Y(x)\}\right)}$$

Summary

The solution(s) found are the following

$$y = -\frac{\left(\frac{d}{dx}\,\mathrm{DESol}\left(\left\{\_Y''(x) + \frac{\_Y'(x)}{x} + \frac{\cos(x)\,\_Y(x)}{x}\right\}, \{\_Y(x)\}\right)\right) x}{\mathrm{DESol}\left(\left\{\_Y''(x) + \frac{\_Y'(x)}{x} + \frac{\cos(x)\,\_Y(x)}{x}\right\}, \{\_Y(x)\}\right)} \tag{1}$$

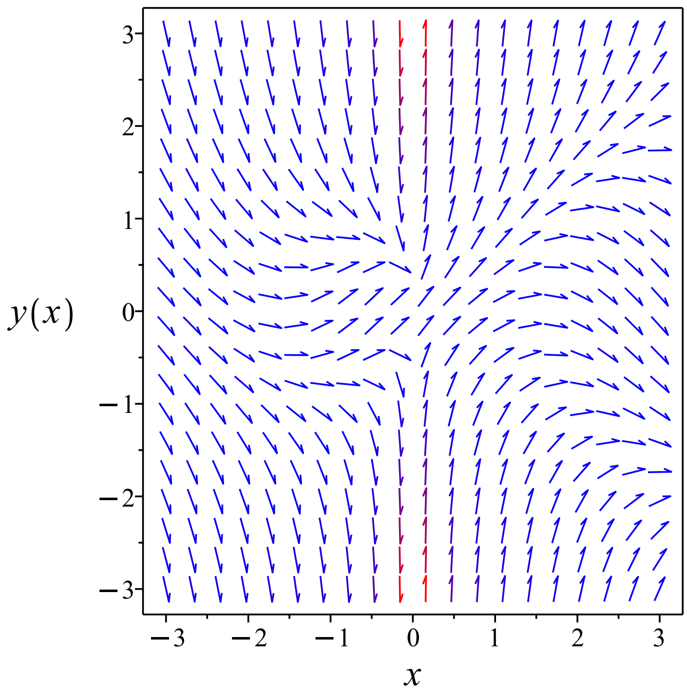


Figure 16: Slope field plot

Verification of solutions

$$y = -\frac{\left(\frac{d}{dx}\,\mathrm{DESol}\left(\left\{\_Y''(x) + \frac{\_Y'(x)}{x} + \frac{\cos(x)\,\_Y(x)}{x}\right\}, \{\_Y(x)\}\right)\right) x}{\mathrm{DESol}\left(\left\{\_Y''(x) + \frac{\_Y'(x)}{x} + \frac{\cos(x)\,\_Y(x)}{x}\right\}, \{\_Y(x)\}\right)}$$

Verified OK.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
trying homogeneous types:
trying Chini
differential order: 1; looking for linear symmetries
trying exact
Looking for potential symmetries
trying Riccati
trying Riccati sub-methods:
   trying Riccati_symmetries
   trying Riccati to 2nd Order
   -> Calling odsolve with the ODE`, diff(diff(y(x), x), x) = -(diff(y(x), x))/x-cos(x)*y(x)
      Methods for second order ODEs:
      --- Trying classification methods ---
      trying a symmetry of the form [xi=0, eta=F(x)]
      checking if the LODE is missing y
      -> Heun: Equivalence to the GHE or one of its 4 confluent cases under a power @ Moebiu
      -> trying a solution of the form r0(x) * Y + r1(x) * Y where Y = exp(int(r(x), dx)) *
      -> Trying changes of variables to rationalize or make the ODE simpler
         trying a symmetry of the form [xi=0, eta=F(x)]
         checking if the LODE is missing y
         -> Heun: Equivalence to the GHE or one of its 4 confluent cases under a power @ Moe
         -> trying a solution of the form r0(x) * Y + r1(x) * Y where Y = exp(int(r(x), dx))
            trying a symmetry of the form [xi=0, eta=F(x)]
            trying 2nd order exact linear
            trying symmetries linear in x and y(x)
            trying to convert to a linear ODE with constant coefficients
            -> trying with_periodic_functions in the coefficients
         trying a symmetry of the form [xi=0, eta=F(x)]
         checking if the LODE is missing y
         -> Heun: Equivalence to the GHE or one of its 4 confluent cases under a power @ Moe
         -> trying a solution of the form r0(x) * Y + r1(x) * Y where Y = exp(int(r(x), dx))
            trying a symmetry of the form [xi=0, eta=F(x)]
            trying 2nd order exact linear
            trying symmetries linear in x and y(x)
            trying to convert to a linear ODE with constant coefficients
      <- unable to find a useful change of variables
         trying a symmetry of the form [xi=0, eta=F(x)]
         trying 2nd order exact linear
```

✗ Solution by Maple

```
dsolve(diff(y(x),x)=cos(x)+y(x)^2/x,y(x), singsol=all)
```

No solution found

✗ Solution by Mathematica

Time used: 0.0 (sec). Leaf size: 0

```
DSolve[y'[x]==Cos[x]+y[x]^2/x,y[x],x,IncludeSingularSolutions -> True]
```

Not solved

1.30 problem 30

Internal problem ID [7346]
Internal file name [`OUTPUT/6327_Sunday_June_05_2022_04_40_15_PM_24901680/index.tex`]

Book: First order enumerated odes
Section: section 1
Problem number: 30.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"riccati"**

Maple gives the following as the ode type

`[_Riccati]`

$$\boxed{y' - y - by^2 = x}$$

1.30.1 Solving as riccati ode

In canonical form the ODE is

$$\begin{aligned} y' &= F(x, y) \\ &= b\,y^2 + x + y \end{aligned}$$

This is a Riccati ODE. Comparing the ODE to solve

$$y' = b\,y^2 + x + y$$

With Riccati ODE standard form

$$y' = f_0(x) + f_1(x)y + f_2(x)y^2$$

Shows that $f_0(x) = x$, $f_1(x) = 1$ and $f_2(x) = b$. Let

$$\begin{aligned} y &= \frac{-u'}{f_2 u} \\ &= \frac{-u'}{bu} \end{aligned} \tag{1}$$

Using the above substitution in the given ODE results (after some simplification)in a second order ODE to solve for $u(x)$ which is

$$f_2u''(x) - (f_2' + f_1f_2)\,u'(x) + f_2^2f_0u(x) = 0 \tag{2}$$

But

$$\begin{aligned} f_2' &= 0 \\ f_1f_2 &= b \\ f_2^2f_0 &= x\,b^2 \end{aligned}$$

Substituting the above terms back in equation (2) gives

$$bu''(x) - bu'(x) + x\,b^2u(x) = 0$$

Solving the above ODE (this ode solved using Maple, not this program), gives

$$u(x) = \mathrm{e}^{\frac{x}{2}}\left(\mathrm{AiryAi}\left(-\frac{4xb-1}{4b^{\frac{2}{3}}}\right)c_1 + \mathrm{AiryBi}\left(-\frac{4xb-1}{4b^{\frac{2}{3}}}\right)c_2\right)$$

The above shows that

$$u'(x) = -\mathrm{e}^{\frac{x}{2}}\left(b^{\frac{1}{3}}\,\mathrm{AiryAi}\left(1, -\frac{4xb-1}{4b^{\frac{2}{3}}}\right)c_1 + b^{\frac{1}{3}}\,\mathrm{AiryBi}\left(1, -\frac{4xb-1}{4b^{\frac{2}{3}}}\right)c_2 - \frac{\mathrm{AiryAi}\left(-\frac{4xb-1}{4b^{\frac{2}{3}}}\right)c_1}{2} - \frac{\mathrm{AiryBi}\left(-\frac{4xb-1}{4b^{\frac{2}{3}}}\right)c_2}{2}\right)$$

Using the above in (1) gives the solution

$$y = \frac{b^{\frac{1}{3}}\,\mathrm{AiryAi}\left(1, -\frac{4xb-1}{4b^{\frac{2}{3}}}\right)c_1 + b^{\frac{1}{3}}\,\mathrm{AiryBi}\left(1, -\frac{4xb-1}{4b^{\frac{2}{3}}}\right)c_2 - \frac{\mathrm{AiryAi}\left(-\frac{4xb-1}{4b^{\frac{2}{3}}}\right)c_1}{2} - \frac{\mathrm{AiryBi}\left(-\frac{4xb-1}{4b^{\frac{2}{3}}}\right)c_2}{2}}{b\left(\mathrm{AiryAi}\left(-\frac{4xb-1}{4b^{\frac{2}{3}}}\right)c_1 + \mathrm{AiryBi}\left(-\frac{4xb-1}{4b^{\frac{2}{3}}}\right)c_2\right)}$$

Dividing both numerator and denominator by c_1 gives, after renaming the constant $\frac{c_2}{c_1} = c_3$ the following solution

$$y = \frac{2b^{\frac{1}{3}} \operatorname{AiryAi}\left(1, -\frac{4xb-1}{4b^{\frac{2}{3}}}\right) c_3 + 2b^{\frac{1}{3}} \operatorname{AiryBi}\left(1, -\frac{4xb-1}{4b^{\frac{2}{3}}}\right) - \operatorname{AiryAi}\left(-\frac{4xb-1}{4b^{\frac{2}{3}}}\right) c_3 - \operatorname{AiryBi}\left(-\frac{4xb-1}{4b^{\frac{2}{3}}}\right)}{2b\left(\operatorname{AiryAi}\left(-\frac{4xb-1}{4b^{\frac{2}{3}}}\right) c_3 + \operatorname{AiryBi}\left(-\frac{4xb-1}{4b^{\frac{2}{3}}}\right)\right)}$$

Summary

The solution(s) found are the following

$$y = \frac{2b^{\frac{1}{3}} \operatorname{AiryAi}\left(1, -\frac{4xb-1}{4b^{\frac{2}{3}}}\right) c_3 + 2b^{\frac{1}{3}} \operatorname{AiryBi}\left(1, -\frac{4xb-1}{4b^{\frac{2}{3}}}\right) - \operatorname{AiryAi}\left(-\frac{4xb-1}{4b^{\frac{2}{3}}}\right) c_3 - \operatorname{AiryBi}\left(-\frac{4xb-1}{4b^{\frac{2}{3}}}\right)}{2b\left(\operatorname{AiryAi}\left(-\frac{4xb-1}{4b^{\frac{2}{3}}}\right) c_3 + \operatorname{AiryBi}\left(-\frac{4xb-1}{4b^{\frac{2}{3}}}\right)\right)} \tag{1}$$

Verification of solutions

$$y = \frac{2b^{\frac{1}{3}} \operatorname{AiryAi}\left(1, -\frac{4xb-1}{4b^{\frac{2}{3}}}\right) c_3 + 2b^{\frac{1}{3}} \operatorname{AiryBi}\left(1, -\frac{4xb-1}{4b^{\frac{2}{3}}}\right) - \operatorname{AiryAi}\left(-\frac{4xb-1}{4b^{\frac{2}{3}}}\right) c_3 - \operatorname{AiryBi}\left(-\frac{4xb-1}{4b^{\frac{2}{3}}}\right)}{2b\left(\operatorname{AiryAi}\left(-\frac{4xb-1}{4b^{\frac{2}{3}}}\right) c_3 + \operatorname{AiryBi}\left(-\frac{4xb-1}{4b^{\frac{2}{3}}}\right)\right)}$$

Verified OK.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
trying homogeneous types:
trying Chini
differential order: 1; looking for linear symmetries
trying exact
Looking for potential symmetries
trying Riccati
trying Riccati sub-methods:
   <- Abel AIR successful: ODE belongs to the 0F1 0-parameter (Airy type) class`
```

✓ Solution by Maple

Time used: 0.015 (sec). Leaf size: 105

```
dsolve(diff(y(x),x)=x+y(x)+b*y(x)^2,y(x), singsol=all)
```

$$y(x) = \frac{2b^{\frac{1}{3}} \operatorname{AiryAi}\left(1, -\frac{4bx-1}{4b^{\frac{2}{3}}}\right) c_1 + 2 \operatorname{AiryBi}\left(1, -\frac{4bx-1}{4b^{\frac{2}{3}}}\right) b^{\frac{1}{3}} - \operatorname{AiryAi}\left(-\frac{4bx-1}{4b^{\frac{2}{3}}}\right) c_1 - \operatorname{AiryBi}\left(-\frac{4bx-1}{4b^{\frac{2}{3}}}\right)}{2b\left(\operatorname{AiryAi}\left(-\frac{4bx-1}{4b^{\frac{2}{3}}}\right) c_1 + \operatorname{AiryBi}\left(-\frac{4bx-1}{4b^{\frac{2}{3}}}\right)\right)}$$

✓ Solution by Mathematica

Time used: 0.222 (sec). Leaf size: 211

```
DSolve[y'[x]==x+y[x]+b*y[x]^2,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to -\frac{-(-b)^{2/3} \operatorname{AiryBi}\left(\frac{\frac{1}{4}-bx}{(-b)^{2/3}}\right) + 2b \operatorname{AiryBiPrime}\left(\frac{\frac{1}{4}-bx}{(-b)^{2/3}}\right) + c_1\left(2b \operatorname{AiryAiPrime}\left(\frac{\frac{1}{4}-bx}{(-b)^{2/3}}\right) - (-b)^{2/3} \operatorname{Airy}\right.}{2(-b)^{5/3}\left(\operatorname{AiryBi}\left(\frac{\frac{1}{4}-bx}{(-b)^{2/3}}\right) + c_1 \operatorname{AiryAi}\left(\frac{\frac{1}{4}-bx}{(-b)^{2/3}}\right)\right)}$$

$$y(x) \to -\frac{\frac{2\sqrt[3]{-b}\operatorname{AiryAiPrime}\left(\frac{\frac{1}{4}-bx}{(-b)^{2/3}}\right)}{\operatorname{AiryAi}\left(\frac{\frac{1}{4}-bx}{(-b)^{2/3}}\right)} + 1}{2b}$$

1.31 problem 31

Internal problem ID [7347]
Internal file name [`OUTPUT/6328_Sunday_June_05_2022_04_40_18_PM_23675789/index.tex`]

Book: First order enumerated odes
Section: section 1
Problem number: 31.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"quadrature"**

Maple gives the following as the ode type

`[_quadrature]`

$$\boxed{xy' = 0}$$

1.31.1 Solving as quadrature ode

Integrating both sides gives

$$\begin{aligned} y &= \int 0 \, \mathrm{d}x \\ &= c_1 \end{aligned}$$

Summary
The solution(s) found are the following

$$y = c_1 \tag{1}$$

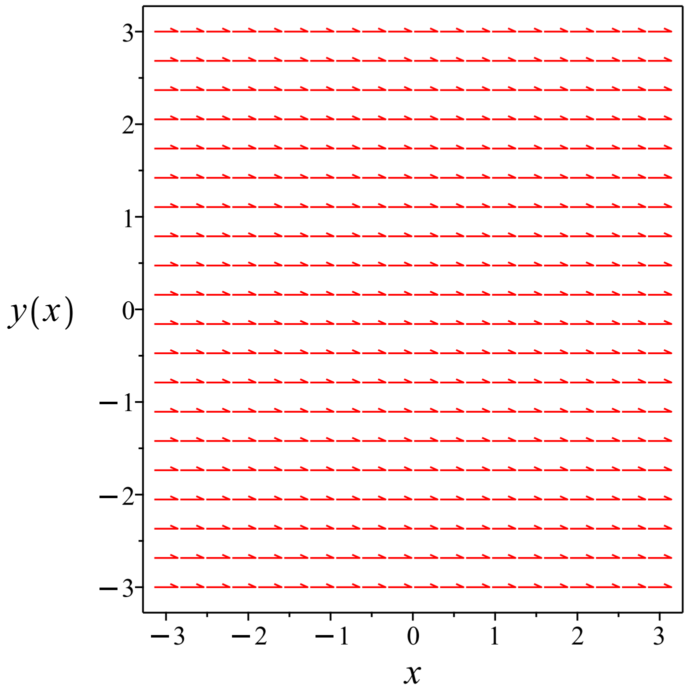


Figure 17: Slope field plot

Verification of solutions

$$y = c_1$$

Verified OK.

1.31.2 Maple step by step solution

Let's solve

$xy' = 0$

- Highest derivative means the order of the ODE is 1

  y'

- Integrate both sides with respect to x

  $\int xy'dx = \int 0dx + c_1$

- Cannot compute integral

  $\int xy'dx = c_1$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
<- 1st order linear successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 5

```
dsolve(x*diff(y(x),x)=0,y(x), singsol=all)
```

$$y(x) = c_1$$

✓ Solution by Mathematica

Time used: 0.002 (sec). Leaf size: 7

```
DSolve[x*y'[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to c_1$$

1.32 problem 32

Internal problem ID [7348]
Internal file name [`OUTPUT/6329_Sunday_June_05_2022_04_40_19_PM_96928607/index.tex`]

Book: First order enumerated odes
Section: section 1
Problem number: 32.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"quadrature"**

Maple gives the following as the ode type

`[_quadrature]`

$$\boxed{5y' = 0}$$

1.32.1 Solving as quadrature ode

Integrating both sides gives

$$\begin{aligned} y &= \int 0 \, \mathrm{d}x \\ &= c_1 \end{aligned}$$

Summary
The solution(s) found are the following

$$y = c_1 \tag{1}$$

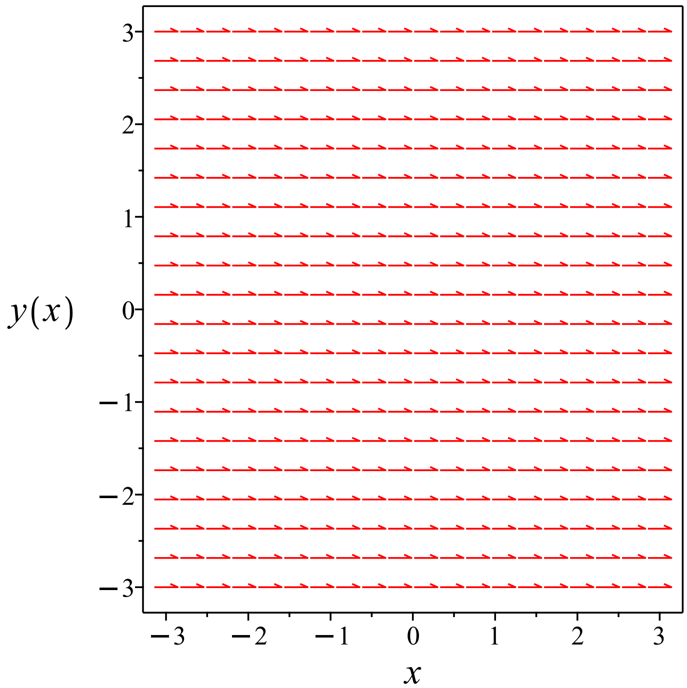


Figure 18: Slope field plot

Verification of solutions

$$y = c_1$$

Verified OK.

1.32.2 Maple step by step solution

Let's solve

$5y' = 0$

- Highest derivative means the order of the ODE is 1

  y'

- Integrate both sides with respect to x

  $\int 5y'dx = \int 0dx + c_1$

- Evaluate integral

  $5y = c_1$

- Solve for y

  $y = \frac{c_1}{5}$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
<- 1st order linear successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 5

```
dsolve(5*diff(y(x),x)=0,y(x), singsol=all)
```

$$y(x) = c_1$$

✓ Solution by Mathematica

Time used: 0.002 (sec). Leaf size: 7

```
DSolve[5*y'[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to c_1$$

1.33 problem 33

Internal problem ID [7349]
Internal file name [`OUTPUT/6330_Sunday_June_05_2022_04_40_21_PM_37418029/index.tex`]

Book: First order enumerated odes
Section: section 1
Problem number: 33.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"quadrature"**

Maple gives the following as the ode type

```
[_quadrature]
```

$$\boxed{e y' = 0}$$

1.33.1 Solving as quadrature ode

Integrating both sides gives

$$\begin{aligned} y &= \int 0 \,\mathrm{d}x \\ &= c_1 \end{aligned}$$

Summary
The solution(s) found are the following

$$y = c_1 \tag{1}$$

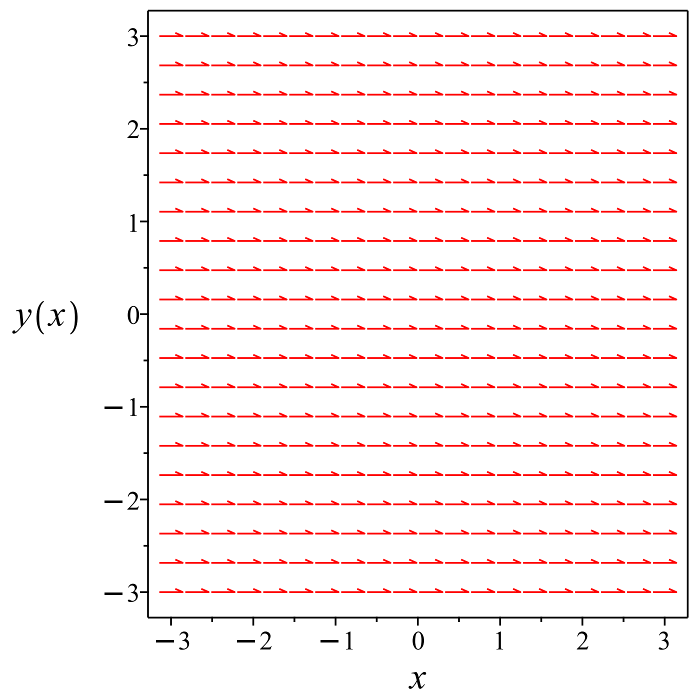


Figure 19: Slope field plot

<u>Verification of solutions</u>

$$y = c_1$$

Verified OK.

1.33.2 Maple step by step solution

Let's solve

$\mathrm{e}y' = 0$

- Highest derivative means the order of the ODE is 1

  y'

- Integrate both sides with respect to x

  $\int \mathrm{e}y' dx = \int 0 dx + c_1$

- Evaluate integral

  $\mathrm{e}y = c_1$

- Solve for y

  $y = \frac{c_1}{\mathrm{e}}$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
<- 1st order linear successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 5

```
dsolve(exp(1)*diff(y(x),x)=0,y(x), singsol=all)
```

$$y(x) = c_1$$

✓ Solution by Mathematica

Time used: 0.002 (sec). Leaf size: 7

```
DSolve[Exp[1]*y'[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to c_1$$

1.34 problem 34

Internal problem ID [7350]
Internal file name [`OUTPUT/6331_Sunday_June_05_2022_04_40_22_PM_77794828/index.tex`]

Book: First order enumerated odes
Section: section 1
Problem number: 34.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"quadrature"**

Maple gives the following as the ode type

`[_quadrature]`

$$\boxed{\pi y' = 0}$$

1.34.1 Solving as quadrature ode

Integrating both sides gives

$$\begin{aligned} y &= \int 0 \, \mathrm{d}x \\ &= c_1 \end{aligned}$$

Summary
The solution(s) found are the following

$$y = c_1 \tag{1}$$

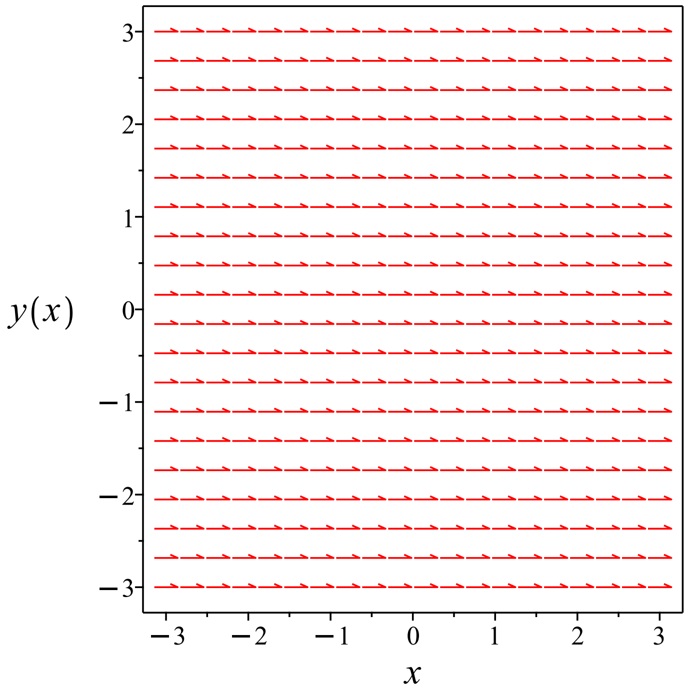


Figure 20: Slope field plot

Verification of solutions

$$y = c_1$$

Verified OK.

1.34.2 Maple step by step solution

Let's solve

$\pi y' = 0$

- Highest derivative means the order of the ODE is 1

  y'

- Integrate both sides with respect to x

  $\int \pi y' dx = \int 0 dx + c_1$

- Evaluate integral

  $\pi y = c_1$

- Solve for y

  $y = \frac{c_1}{\pi}$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
<- 1st order linear successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 5

```
dsolve(Pi*diff(y(x),x)=0,y(x), singsol=all)
```

$$y(x) = c_1$$

✓ Solution by Mathematica

Time used: 0.002 (sec). Leaf size: 7

```
DSolve[Pi*y'[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to c_1$$

1.35 problem 35

Internal problem ID [7351]
Internal file name [`OUTPUT/6332_Sunday_June_05_2022_04_40_24_PM_66895747/index.tex`]

Book: First order enumerated odes
Section: section 1
Problem number: 35.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"quadrature"**

Maple gives the following as the ode type

`[_quadrature]`

$$\boxed{y' \sin(x) = 0}$$

1.35.1 Solving as quadrature ode

Integrating both sides gives

$$\begin{aligned} y &= \int 0 \, \mathrm{d}x \\ &= c_1 \end{aligned}$$

Summary
The solution(s) found are the following

$$y = c_1 \tag{1}$$

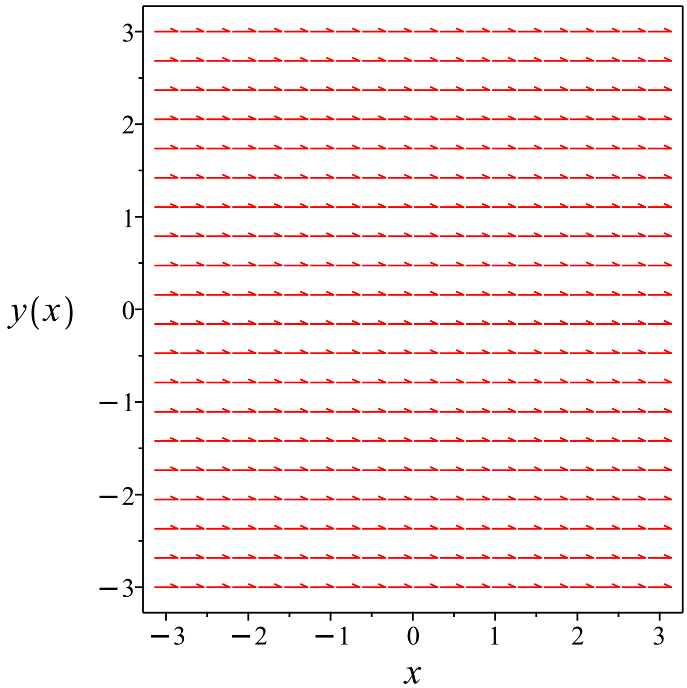


Figure 21: Slope field plot

Verification of solutions

$$y = c_1$$

Verified OK.

1.35.2 Maple step by step solution

Let's solve

$y' \sin(x) = 0$

- Highest derivative means the order of the ODE is 1

  y'

- Integrate both sides with respect to x

  $\int y' \sin(x)\, dx = \int 0 dx + c_1$

- Cannot compute integral

  $\int y' \sin(x)\, dx = c_1$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
<- 1st order linear successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 5

```
dsolve(sin(x)*diff(y(x),x)=0,y(x), singsol=all)
```

$$y(x) = c_1$$

✓ Solution by Mathematica

Time used: 0.002 (sec). Leaf size: 7

```
DSolve[Sin[x]*y'[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to c_1$$

1.36 problem 36

Internal problem ID [7352]
Internal file name [`OUTPUT/6333_Sunday_June_05_2022_04_40_26_PM_13619755/index.tex`]

Book: First order enumerated odes
Section: section 1
Problem number: 36.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"quadrature"**

Maple gives the following as the ode type

`[_quadrature]`

$$\boxed{f(x)\, y' = 0}$$

1.36.1 Solving as quadrature ode

Integrating both sides gives

$$\begin{aligned} y &= \int 0 \, \mathrm{d}x \\ &= c_1 \end{aligned}$$

Summary
The solution(s) found are the following

$$y = c_1 \tag{1}$$

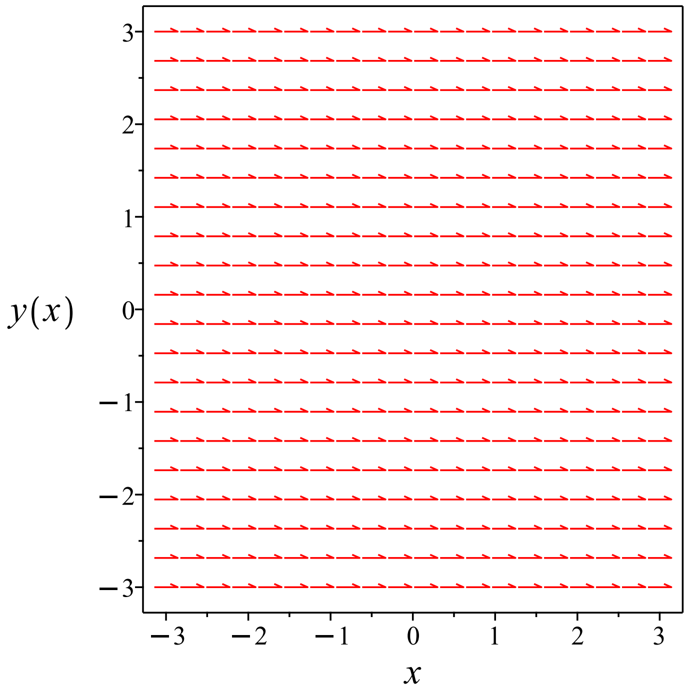


Figure 22: Slope field plot

Verification of solutions

$$y = c_1$$

Verified OK.

1.36.2 Maple step by step solution

Let's solve

$f(x)\, y' = 0$

- Highest derivative means the order of the ODE is 1

  y'

- Integrate both sides with respect to x

  $\int f(x)\, y' dx = \int 0 dx + c_1$

- Cannot compute integral

  $\int f(x)\, y' dx = c_1$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
<- 1st order linear successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 5

```
dsolve(f(x)*diff(y(x),x)=0,y(x), singsol=all)
```

$$y(x) = c_1$$

✓ Solution by Mathematica

Time used: 0.002 (sec). Leaf size: 7

```
DSolve[f[x]*y'[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to c_1$$

1.37 problem 37

Internal problem ID [7353]
Internal file name [`OUTPUT/6334_Sunday_June_05_2022_04_40_27_PM_93838805/index.tex`]

Book: First order enumerated odes
Section: section 1
Problem number: 37.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"quadrature"**

Maple gives the following as the ode type

```
[_quadrature]
```

$$\boxed{xy' = 1}$$

1.37.1 Solving as quadrature ode

Integrating both sides gives

$$\begin{aligned} y &= \int \frac{1}{x}\, \mathrm{d}x \\ &= \ln(x) + c_1 \end{aligned}$$

Summary
The solution(s) found are the following

$$y = \ln(x) + c_1 \tag{1}$$

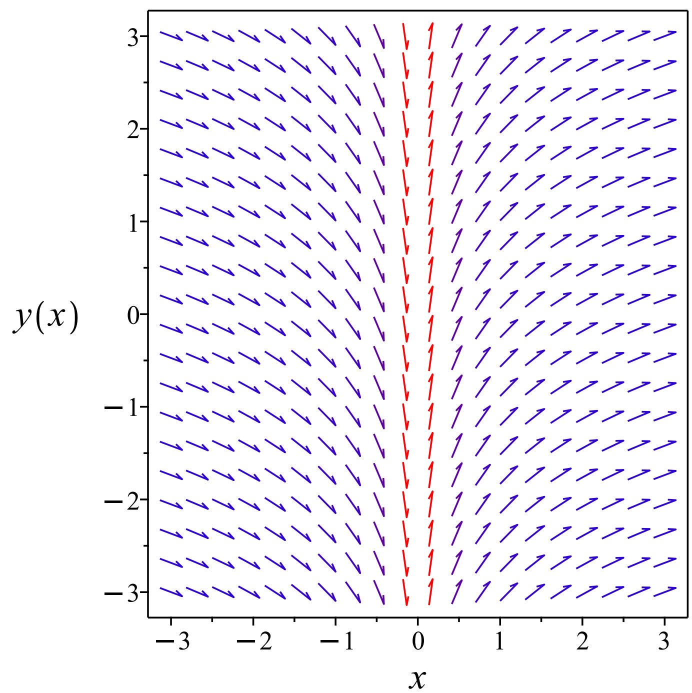


Figure 23: Slope field plot

Verification of solutions

$$y = \ln(x) + c_1$$

Verified OK.

1.37.2 Maple step by step solution

Let's solve

$xy' = 1$

- Highest derivative means the order of the ODE is 1

  y'

- Separate variables

  $y' = \frac{1}{x}$

- Integrate both sides with respect to x

  $\int y' dx = \int \frac{1}{x} dx + c_1$

- Evaluate integral

$$y = \ln(x) + c_1$$

- Solve for y

$$y = \ln(x) + c_1$$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
<- quadrature successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 8

```
dsolve(x*diff(y(x),x)=1,y(x), singsol=all)
```

$$y(x) = \ln(x) + c_1$$

✓ Solution by Mathematica

Time used: 0.002 (sec). Leaf size: 10

```
DSolve[x*y'[x]==1,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \log(x) + c_1$$

1.38 problem 38

Internal problem ID [7354]
Internal file name [`OUTPUT/6335_Sunday_June_05_2022_04_40_29_PM_52270518/index.tex`]

Book: First order enumerated odes
Section: section 1
Problem number: 38.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"quadrature"**

Maple gives the following as the ode type

`[_quadrature]`

$$\boxed{xy' = \sin(x)}$$

1.38.1 Solving as quadrature ode

Integrating both sides gives

$$\begin{aligned} y &= \int \frac{\sin(x)}{x}\, \mathrm{d}x \\ &= \mathrm{Si}(x) + c_1 \end{aligned}$$

Summary
The solution(s) found are the following

$$y = \mathrm{Si}(x) + c_1 \tag{1}$$

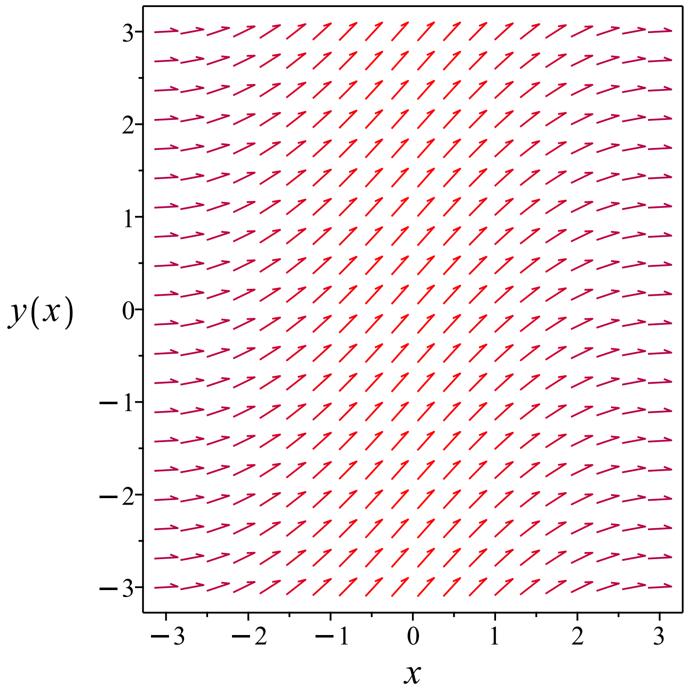


Figure 24: Slope field plot

Verification of solutions

$$y = \mathrm{Si}(x) + c_1$$

Verified OK.

1.38.2 Maple step by step solution

Let's solve

$xy' = \sin(x)$

- Highest derivative means the order of the ODE is 1

  y'

- Separate variables

  $y' = \frac{\sin(x)}{x}$

- Integrate both sides with respect to x

  $\int y' dx = \int \frac{\sin(x)}{x} dx + c_1$

- Evaluate integral

$$y = \mathrm{Si}(x) + c_1$$

- Solve for y

$$y = \mathrm{Si}(x) + c_1$$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
<- quadrature successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 8

```
dsolve(x*diff(y(x),x)=sin(x),y(x), singsol=all)
```

$$y(x) = \mathrm{Si}\,(x) + c_1$$

✓ Solution by Mathematica

Time used: 0.015 (sec). Leaf size: 10

```
DSolve[x*y'[x]==Sin[x],y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \mathrm{Si}(x) + c_1$$

1.39 problem 39

Internal problem ID [7355]
Internal file name [`OUTPUT/6336_Sunday_June_05_2022_04_40_31_PM_79609722/index.tex`]

Book: First order enumerated odes
Section: section 1
Problem number: 39.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"quadrature"**

Maple gives the following as the ode type

```
[_quadrature]
```

$$\boxed{(x-1)\,y'=0}$$

1.39.1 Solving as quadrature ode

Integrating both sides gives

$$\begin{aligned} y &= \int 0 \,\mathrm{d}x \\ &= c_1 \end{aligned}$$

Summary
The solution(s) found are the following

$$y = c_1 \tag{1}$$

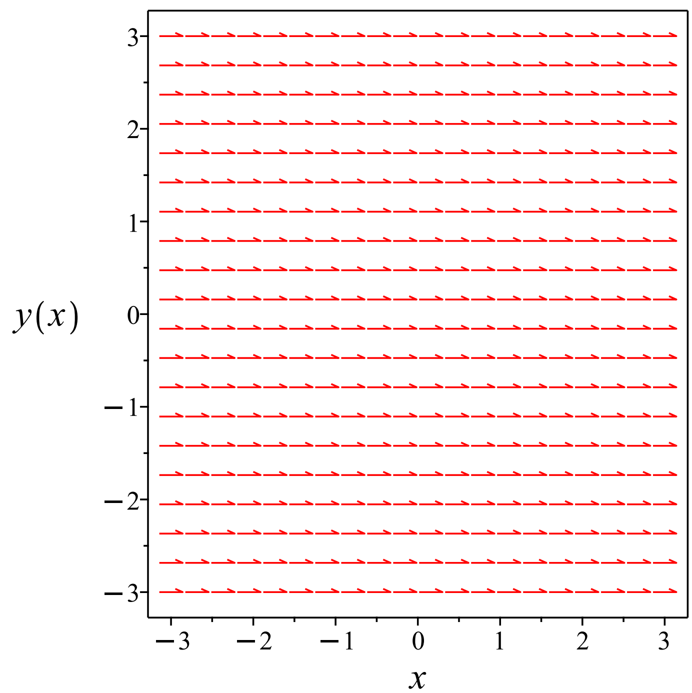


Figure 25: Slope field plot

<u>Verification of solutions</u>

$$y = c_1$$

Verified OK.

1.39.2 Maple step by step solution

Let's solve

$(x-1)\,y' = 0$

- Highest derivative means the order of the ODE is 1

  y'

- Integrate both sides with respect to x

  $\int (x-1)\,y' dx = \int 0 dx + c_1$

- Cannot compute integral

  $\int (x-1)\,y' dx = c_1$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
<- 1st order linear successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 5

```
dsolve((x-1)*diff(y(x),x)=0,y(x), singsol=all)
```

$$y(x) = c_1$$

✓ Solution by Mathematica

Time used: 0.002 (sec). Leaf size: 7

```
DSolve[(x-1)*y'[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to c_1$$

1.40 problem 40

Internal problem ID [7356]
Internal file name [`OUTPUT/6337_Sunday_June_05_2022_04_40_32_PM_82675100/index.tex`]

Book: First order enumerated odes
Section: section 1
Problem number: 40.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"quadrature"**

Maple gives the following as the ode type

```
[_quadrature]
```

$$\boxed{yy' = 0}$$

1.40.1 Solving as quadrature ode

Integrating both sides gives

$$\begin{aligned} y &= \int 0 \, \mathrm{d}x \\ &= c_1 \end{aligned}$$

Summary
The solution(s) found are the following

$$y = c_1 \tag{1}$$

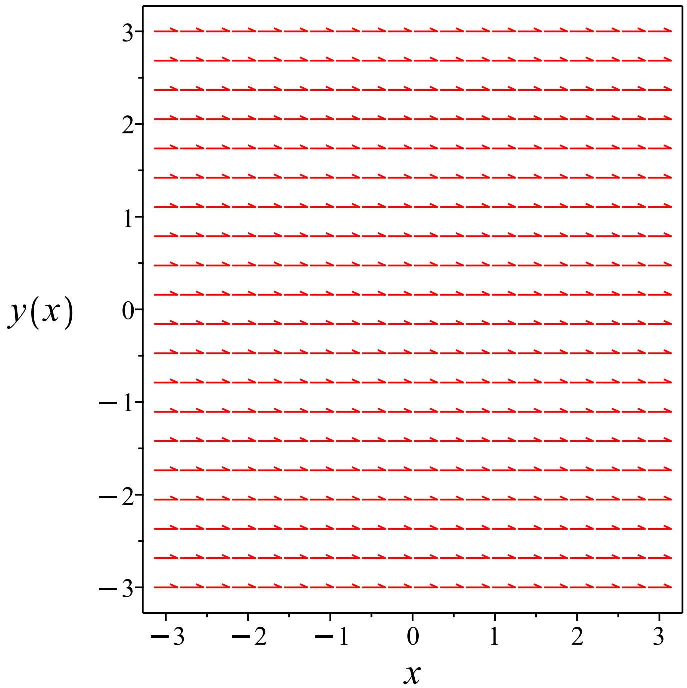


Figure 26: Slope field plot

Verification of solutions

$$y = c_1$$

Verified OK.

1.40.2 Maple step by step solution

Let's solve

$yy' = 0$

- Highest derivative means the order of the ODE is 1

  y'

- Integrate both sides with respect to x

  $\int yy'dx = \int 0dx + c_1$

- Evaluate integral

  $\frac{y^2}{2} = c_1$

- Solve for y

  $\{y = \sqrt{c_1}\,\sqrt{2}, y = -\sqrt{c_1}\,\sqrt{2}\}$

Maple trace

```
`Classification methods on request
Methods to be used are: [exact]
----------------------------
* Tackling ODE using method: exact
--- Trying classification methods ---
trying exact
<- exact successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 11

```
dsolve(y(x)*diff(y(x),x)=0,y(x), singsol=all)
```

$$y(x) = 0$$
$$y(x) = -c_1$$

✓ Solution by Mathematica

Time used: 0.002 (sec). Leaf size: 12

```
DSolve[y[x]*y'[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to 0$$
$$y(x) \to c_1$$

1.41 problem 41

Internal problem ID [7357]
Internal file name [`OUTPUT/6338_Sunday_June_05_2022_04_40_34_PM_33765645/index.tex`]

Book: First order enumerated odes
Section: section 1
Problem number: 41.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"quadrature"**

Maple gives the following as the ode type

```
[_quadrature]
```

$$\boxed{xy'y = 0}$$

1.41.1 Solving as quadrature ode

Integrating both sides gives

$$\begin{aligned} y &= \int 0 \, \mathrm{d}x \\ &= c_1 \end{aligned}$$

Summary
The solution(s) found are the following

$$y = c_1 \tag{1}$$

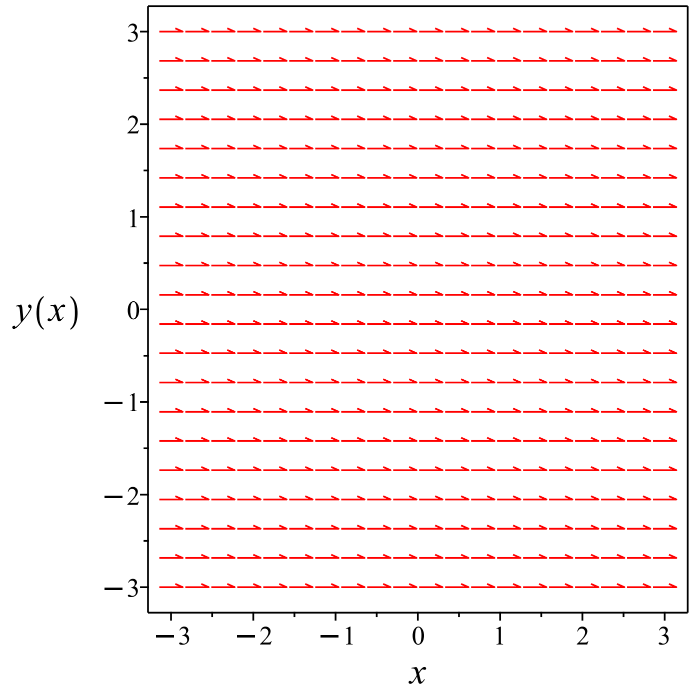


Figure 27: Slope field plot

Verification of solutions

$$y = c_1$$

Verified OK.

1.41.2 Maple step by step solution

Let's solve

$xy'y = 0$

- Highest derivative means the order of the ODE is 1

  y'

- Integrate both sides with respect to x

  $\int xy'ydx = \int 0dx + c_1$

- Cannot compute integral

  $\int xy'ydx = c_1$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
<- 1st order linear successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 9

```
dsolve(x*y(x)*diff(y(x),x)=0,y(x), singsol=all)
```

$$y(x) = 0$$
$$y(x) = c_1$$

✓ Solution by Mathematica

Time used: 0.002 (sec). Leaf size: 12

```
DSolve[x*y[x]*y'[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to 0$$
$$y(x) \to c_1$$

1.42 problem 42

Internal problem ID [7358]
Internal file name [`OUTPUT/6339_Sunday_June_05_2022_04_40_36_PM_8332014/index.tex`]

Book: First order enumerated odes
Section: section 1
Problem number: 42.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"quadrature"**

Maple gives the following as the ode type

`[_quadrature]`

$$\boxed{xy\sin(x)\,y' = 0}$$

1.42.1 Solving as quadrature ode

Integrating both sides gives

$$\begin{aligned} y &= \int 0 \,\mathrm{d}x \\ &= c_1 \end{aligned}$$

Summary
The solution(s) found are the following

$$y = c_1 \tag{1}$$

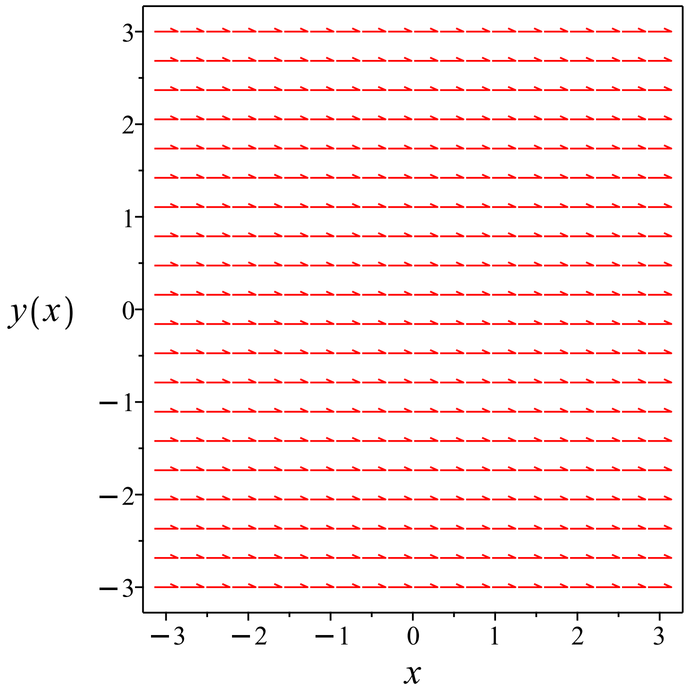


Figure 28: Slope field plot

<u>Verification of solutions</u>

$$y = c_1$$

Verified OK.

1.42.2 Maple step by step solution

Let's solve

$xy \sin(x) y' = 0$

- Highest derivative means the order of the ODE is 1

  y'

- Integrate both sides with respect to x

  $\int xy \sin(x) y' dx = \int 0 dx + c_1$

- Cannot compute integral

  $\int xy \sin(x) y' dx = c_1$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
<- 1st order linear successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 9

```
dsolve(x*y(x)*sin(x)*diff(y(x),x)=0,y(x), singsol=all)
```

$$y(x) = 0$$
$$y(x) = c_1$$

✓ Solution by Mathematica

Time used: 0.002 (sec). Leaf size: 12

```
DSolve[x*y[x]*Sin[x]*y'[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to 0$$
$$y(x) \to c_1$$

1.43 problem 43

Internal problem ID [7359]
Internal file name [`OUTPUT/6340_Sunday_June_05_2022_04_40_38_PM_29944544/index.tex`]

Book: First order enumerated odes
Section: section 1
Problem number: 43.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"quadrature"**

Maple gives the following as the ode type

```
[_quadrature]
```

$$\boxed{\pi y \sin(x)\, y' = 0}$$

1.43.1 Solving as quadrature ode

Integrating both sides gives

$$\begin{aligned} y &= \int 0 \, \mathrm{d}x \\ &= c_1 \end{aligned}$$

Summary
The solution(s) found are the following

$$y = c_1 \tag{1}$$

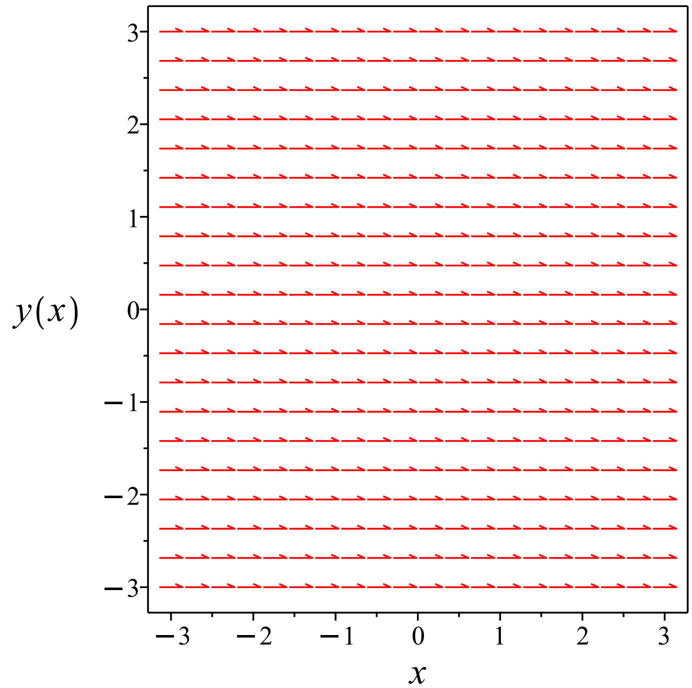


Figure 29: Slope field plot

Verification of solutions

$$y = c_1$$

Verified OK.

1.43.2 Maple step by step solution

Let's solve

$\pi y \sin(x) y' = 0$

- Highest derivative means the order of the ODE is 1

  y'

- Integrate both sides with respect to x

  $\int \pi y \sin(x) y' dx = \int 0 dx + c_1$

- Cannot compute integral

  $\int \pi y \sin(x) y' dx = c_1$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
<- 1st order linear successful`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 9

```
dsolve(Pi*y(x)*sin(x)*diff(y(x),x)=0,y(x), singsol=all)
```

$$y(x) = 0$$
$$y(x) = c_1$$

✓ Solution by Mathematica

Time used: 0.002 (sec). Leaf size: 12

```
DSolve[Pi*y[x]*Sin[x]*y'[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to 0$$
$$y(x) \to c_1$$

1.44 problem 44

Internal problem ID [7360]
Internal file name [`OUTPUT/6341_Sunday_June_05_2022_04_40_40_PM_90922818/index.tex`]

Book: First order enumerated odes
Section: section 1
Problem number: 44.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"quadrature"**

Maple gives the following as the ode type

`[_quadrature]`

$$\boxed{\sin(x)\, y' x = 0}$$

1.44.1 Solving as quadrature ode

Integrating both sides gives

$$\begin{aligned} y &= \int 0 \, \mathrm{d}x \\ &= c_1 \end{aligned}$$

Summary
The solution(s) found are the following

$$y = c_1 \tag{1}$$

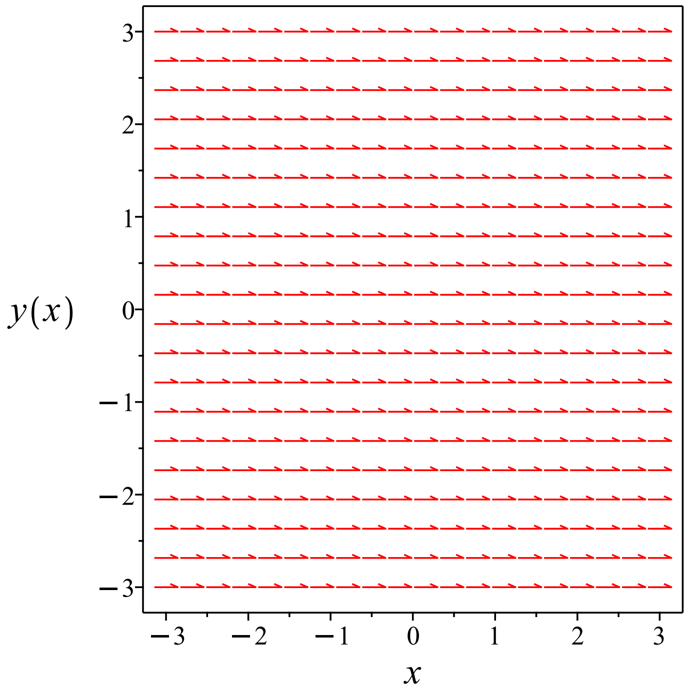


Figure 30: Slope field plot

Verification of solutions

$$y = c_1$$

Verified OK.

1.44.2 Maple step by step solution

Let's solve

$\sin(x) y' x = 0$

- Highest derivative means the order of the ODE is 1

  y'

- Integrate both sides with respect to x

  $\int \sin(x) y' x dx = \int 0 dx + c_1$

- Cannot compute integral

  $\int \sin(x) y' x dx = c_1$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
<- 1st order linear successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 5

```
dsolve(x*sin(x)*diff(y(x),x)=0,y(x), singsol=all)
```

$$y(x) = c_1$$

✓ Solution by Mathematica

Time used: 0.003 (sec). Leaf size: 7

```
DSolve[x*Sin[x]*y'[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to c_1$$

1.45 problem 45

Internal problem ID [7361]
Internal file name [`OUTPUT/6342_Sunday_June_05_2022_04_40_41_PM_12780467/index.tex`]

Book: First order enumerated odes
Section: section 1
Problem number: 45.
ODE order: 1.
ODE degree: 2.

The type(s) of ODE detected by this program : **"quadrature"**

Maple gives the following as the ode type

`[_quadrature]`

$$\boxed{x \sin(x) y'^2 = 0}$$

Solving the given ode for y' results in 2 differential equations to solve. Each one of these will generate a solution. The equations generated are

$$y' = 0 \tag{1}$$
$$y' = 0 \tag{2}$$

Now each one of the above ODE is solved.

Solving equation (1)

Integrating both sides gives

$$\begin{aligned} y &= \int 0 \, \mathrm{d}x \\ &= c_1 \end{aligned}$$

Summary
The solution(s) found are the following

$$y = c_1 \tag{1}$$

Verification of solutions

$$y = c_1$$

Verified OK.

Solving equation (2)

Integrating both sides gives

$$
\begin{aligned}
y &= \int 0 \,\mathrm{d}x \\
&= c_2
\end{aligned}
$$

Summary

The solution(s) found are the following

$$
y = c_2 \tag{1}
$$

Verification of solutions

$$
y = c_2
$$

Verified OK.

1.45.1 Maple step by step solution

Let's solve

$x \sin(x) y'^2 = 0$

- Highest derivative means the order of the ODE is 1

  y'

- Integrate both sides with respect to x

  $\int x \sin(x) y'^2 dx = \int 0 dx + c_1$

- Cannot compute integral

  $\int x \sin(x) y'^2 dx = c_1$

Maple trace

```
`Methods for first order ODEs:
-> Solving 1st order ODE of high degree, 1st attempt
trying 1st order WeierstrassP solution for high degree ODE
trying 1st order WeierstrassPPrime solution for high degree ODE
trying 1st order JacobiSN solution for high degree ODE
trying 1st order ODE linearizable_by_differentiation
trying differential order: 1; missing variables
<- differential order: 1; missing  y(x)  successful`
```

✓ Solution by Maple

Time used: 0.015 (sec). Leaf size: 5

```
dsolve(x*sin(x)*diff(y(x),x)^2=0,y(x), singsol=all)
```

$$y(x) = c_1$$

✓ Solution by Mathematica

Time used: 0.002 (sec). Leaf size: 7

```
DSolve[x*Sin[x]*y'[x]^2==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to c_1$$

1.46 problem 46

Internal problem ID [7362]
Internal file name [OUTPUT/6343_Sunday_June_05_2022_04_40_43_PM_41266990/index.tex]

Book: First order enumerated odes
Section: section 1
Problem number: 46.
ODE order: 1.
ODE degree: 2.

The type(s) of ODE detected by this program : **"quadrature"**

Maple gives the following as the ode type

```
[_quadrature]
```

$$\boxed{y{y'}^2 = 0}$$

Solving the given ode for y' results in 2 differential equations to solve. Each one of these will generate a solution. The equations generated are

$$y' = 0 \tag{1}$$
$$y' = 0 \tag{2}$$

Now each one of the above ODE is solved.

Solving equation (1)

Integrating both sides gives

$$\begin{aligned} y &= \int 0 \, \mathrm{d}x \\ &= c_1 \end{aligned}$$

Summary
The solution(s) found are the following

$$y = c_1 \tag{1}$$

Verification of solutions

$$y = c_1$$

Verified OK.

Solving equation (2)

Integrating both sides gives

$$y = \int 0 \, \mathrm{d}x$$
$$= c_2$$

Summary
The solution(s) found are the following

$$y = c_2 \tag{1}$$

Verification of solutions

$$y = c_2$$

Verified OK.

1.46.1 Maple step by step solution

Let's solve

$yy'^2 = 0$

- Highest derivative means the order of the ODE is 1

  y'

- Integrate both sides with respect to x

  $\int yy'^2 dx = \int 0 dx + c_1$

- Cannot compute integral

  $\int yy'^2 dx = c_1$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
<- 1st order linear successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 9

```
dsolve(y(x)*diff(y(x),x)^2=0,y(x), singsol=all)
```

$$y(x) = 0$$
$$y(x) = c_1$$

✓ Solution by Mathematica

Time used: 0.002 (sec). Leaf size: 12

```
DSolve[y[x]*(y'[x])^2==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to 0$$
$$y(x) \to c_1$$

1.47 problem 47

Internal problem ID [7363]
Internal file name [OUTPUT/6344_Sunday_June_05_2022_04_40_44_PM_95841037/index.tex]

Book: First order enumerated odes
Section: section 1
Problem number: 47.
ODE order: 1.
ODE degree: 0.

The type(s) of ODE detected by this program : **"quadrature"**

Maple gives the following as the ode type

```
[_quadrature]
```

$$\boxed{y'^n = 0}$$

1.47.1 Solving as quadrature ode

Integrating both sides gives

$$\begin{aligned} y &= \int 0 \, \mathrm{d}x \\ &= c_1 \end{aligned}$$

Summary
The solution(s) found are the following

$$y = c_1 \tag{1}$$

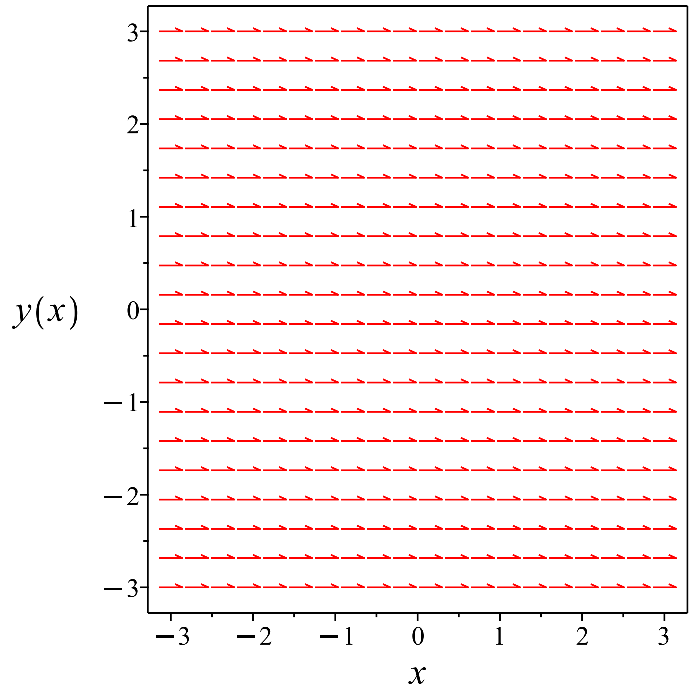


Figure 31: Slope field plot

<u>Verification of solutions</u>

$$y = c_1$$

Verified OK.

1.47.2 Maple step by step solution

Let's solve

$y'^n = 0$

- Highest derivative means the order of the ODE is 1

  y'

- Integrate both sides with respect to x

  $\int y'^n dx = \int 0 dx + c_1$

- Cannot compute integral

  $\int y'^n dx = c_1$

Maple trace

```
`Methods for first order ODEs:
-> Solving 1st order ODE of high degree, 1st attempt
trying 1st order WeierstrassP solution for high degree ODE
trying 1st order WeierstrassPPrime solution for high degree ODE
trying 1st order JacobiSN solution for high degree ODE
trying 1st order ODE linearizable_by_differentiation
trying differential order: 1; missing variables
<- differential order: 1; missing  y(x)  successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 5

```
dsolve(diff(y(x),x)^n=0,y(x), singsol=all)
```

$$y(x) = c_1$$

✓ Solution by Mathematica

Time used: 0.003 (sec). Leaf size: 15

```
DSolve[(y'[x])^n==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to 0^{\frac{1}{n}} x + c_1$$

1.48 problem 48

Internal problem ID [7364]
Internal file name [OUTPUT/6345_Sunday_June_05_2022_04_40_46_PM_4803324/index.tex]

Book: First order enumerated odes
Section: section 1
Problem number: 48.
ODE order: 1.
ODE degree: 0.

The type(s) of ODE detected by this program : **"quadrature"**

Maple gives the following as the ode type

`[_quadrature]`

$$\boxed{x{y'}^n = 0}$$

1.48.1 Solving as quadrature ode

Integrating both sides gives

$$\begin{aligned} y &= \int 0 \, \mathrm{d}x \\ &= c_1 \end{aligned}$$

Summary
The solution(s) found are the following

$$y = c_1 \tag{1}$$

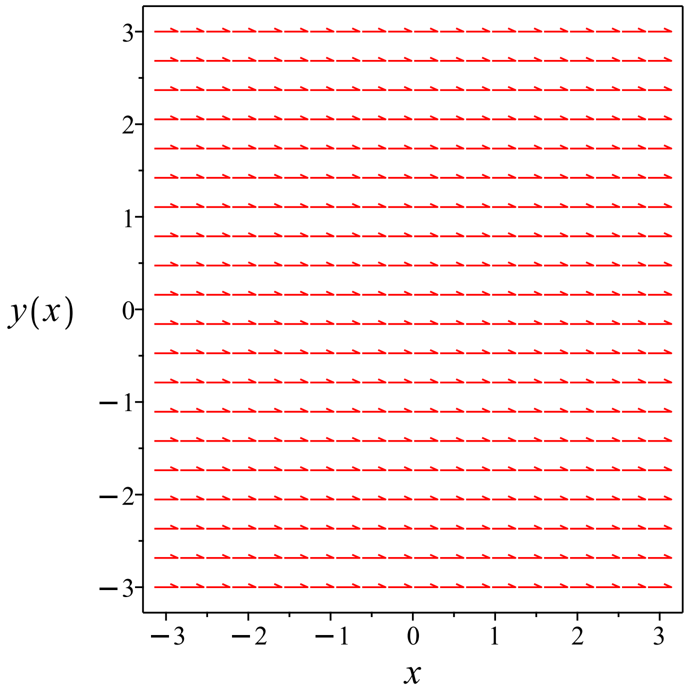


Figure 32: Slope field plot

Verification of solutions

$$y = c_1$$

Verified OK.

1.48.2 Maple step by step solution

Let's solve

$xy'^n = 0$

- Highest derivative means the order of the ODE is 1

  y'

- Integrate both sides with respect to x

  $\int xy'^n dx = \int 0 dx + c_1$

- Cannot compute integral

  $\int xy'^n dx = c_1$

Maple trace

```
`Methods for first order ODEs:
-> Solving 1st order ODE of high degree, 1st attempt
trying 1st order WeierstrassP solution for high degree ODE
trying 1st order WeierstrassPPrime solution for high degree ODE
trying 1st order JacobiSN solution for high degree ODE
trying 1st order ODE linearizable_by_differentiation
trying differential order: 1; missing variables
<- differential order: 1; missing  y(x)  successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 5

```
dsolve(x*diff(y(x),x)^n=0,y(x), singsol=all)
```

$$y(x) = c_1$$

✓ Solution by Mathematica

Time used: 0.003 (sec). Leaf size: 15

```
DSolve[x*(y'[x])^n==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to 0^{\frac{1}{n}}x + c_1$$

1.49 problem 49

Internal problem ID [7365]
Internal file name [`OUTPUT/6346_Sunday_June_05_2022_04_40_48_PM_7391507/index.tex`]

Book: First order enumerated odes
Section: section 1
Problem number: 49.
ODE order: 1.
ODE degree: 2.

The type(s) of ODE detected by this program : **"quadrature"**

Maple gives the following as the ode type

`[_quadrature]`

$$\boxed{y'^2 = x}$$

Solving the given ode for y' results in 2 differential equations to solve. Each one of these will generate a solution. The equations generated are

$$y' = \sqrt{x} \tag{1}$$
$$y' = -\sqrt{x} \tag{2}$$

Now each one of the above ODE is solved.

Solving equation (1)

Integrating both sides gives

$$\begin{aligned} y &= \int \sqrt{x}\, \mathrm{d}x \\ &= \frac{2x^{\frac{3}{2}}}{3} + c_1 \end{aligned}$$

Summary
The solution(s) found are the following

$$y = \frac{2x^{\frac{3}{2}}}{3} + c_1 \tag{1}$$

Verification of solutions

$$y = \frac{2x^{\frac{3}{2}}}{3} + c_1$$

Verified OK.

Solving equation (2)

Integrating both sides gives

$$\begin{aligned} y &= \int -\sqrt{x}\,\mathrm{d}x \\ &= -\frac{2x^{\frac{3}{2}}}{3} + c_2 \end{aligned}$$

Summary

The solution(s) found are the following

$$y = -\frac{2x^{\frac{3}{2}}}{3} + c_2 \tag{1}$$

Verification of solutions

$$y = -\frac{2x^{\frac{3}{2}}}{3} + c_2$$

Verified OK.

1.49.1 Maple step by step solution

Let's solve

$y'^2 = x$

- Highest derivative means the order of the ODE is 1

  y'

- Integrate both sides with respect to x

  $\int y'^2 dx = \int x dx + c_1$

- Cannot compute integral

  $\int y'^2 dx = \frac{x^2}{2} + c_1$

Maple trace

```
`Methods for first order ODEs:
-> Solving 1st order ODE of high degree, 1st attempt
trying 1st order WeierstrassP solution for high degree ODE
trying 1st order WeierstrassPPrime solution for high degree ODE
trying 1st order JacobiSN solution for high degree ODE
trying 1st order ODE linearizable_by_differentiation
trying differential order: 1; missing variables
<- differential order: 1; missing  y(x)  successful`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 21

```
dsolve(diff(y(x),x)^2=x,y(x), singsol=all)
```

$$y(x) = \frac{2x^{\frac{3}{2}}}{3} + c_1$$

$$y(x) = -\frac{2x^{\frac{3}{2}}}{3} + c_1$$

✓ Solution by Mathematica

Time used: 0.003 (sec). Leaf size: 33

```
DSolve[(y'[x])^2==x,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to -\frac{2x^{3/2}}{3} + c_1$$

$$y(x) \to \frac{2x^{3/2}}{3} + c_1$$

1.50 problem 50

Internal problem ID [7366]
Internal file name [OUTPUT/6347_Sunday_June_05_2022_04_40_50_PM_66979101/index.tex]

Book: First order enumerated odes
Section: section 1
Problem number: 50.
ODE order: 1.
ODE degree: 2.

The type(s) of ODE detected by this program : **"dAlembert", "first_order_nonlinear_p_but_linear_in_x_y"**

Maple gives the following as the ode type

```
[[_homogeneous, `class C`], _dAlembert]
```

$$\boxed{y'^2 - y = x}$$

1.50.1 Solving as first order nonlinear p but linear in x y ode

The ode has the form

$$(y')^{\frac{n}{m}} = ax + by + c \tag{1}$$

Where $n = 2, m = 1, a = 1, b = 1, c = 0$. Hence the ode is

$$(y')^2 = x + y$$

Let

$$u = ax + by + c$$

Hence

$$u' = a + by'$$
$$y' = \frac{u' - a}{b}$$

Substituting the above in (1) gives

$$\left(\frac{u' - a}{b}\right)^{\frac{n}{m}} = u$$
$$\left(\frac{u' - a}{b}\right)^{n} = u^{m}$$

Plugging in the above the values for n, m, a, b, c gives

$$(u'(x) - 1)^2 = u$$

Therefore the solutions are

$$u'(x) - 1 = \sqrt{u}$$
$$u'(x) - 1 = -\sqrt{u}$$

Rewriting the above gives

$$u'(x) = \sqrt{u} + 1$$
$$u'(x) = -\sqrt{u} + 1$$

Each of the above is a separable ODE in $u(x)$. This results in

$$\frac{du}{\sqrt{u} + 1} = dx$$
$$\frac{du}{-\sqrt{u} + 1} = dx$$

Integrating each of the above solutions gives

$$\int \frac{du}{\sqrt{u} + 1} = x + c_1$$
$$\int \frac{du}{-\sqrt{u} + 1} = x + c_1$$

But since

$$u = ax + by + c$$
$$= x + y$$

Then the solutions can be written as

$$\int^{x+y} \frac{1}{\sqrt{\tau} + 1} d\tau = x + c_1$$
$$\int^{x+y} \frac{1}{-\sqrt{\tau} + 1} d\tau = x + c_1$$

Summary

The solution(s) found are the following

$$\int^{x+y} \frac{1}{\sqrt{\tau}+1} d\tau = x + c_1 \tag{1}$$

$$\int^{x+y} \frac{1}{-\sqrt{\tau}+1} d\tau = x + c_1 \tag{2}$$

Verification of solutions

$$\int^{x+y} \frac{1}{\sqrt{\tau}+1} d\tau = x + c_1$$

Verified OK.

$$\int^{x+y} \frac{1}{-\sqrt{\tau}+1} d\tau = x + c_1$$

Verified OK.

1.50.2 Solving as dAlembert ode

Let $p = y'$ the ode becomes

$$p^2 - y = x$$

Solving for y from the above results in

$$y = p^2 - x \tag{1A}$$

This has the form

$$y = xf(p) + g(p) \tag{*}$$

Where f, g are functions of $p = y'(x)$. The above ode is dAlembert ode which is now solved. Taking derivative of (*) w.r.t. x gives

$$\begin{aligned} p &= f + (xf' + g') \frac{dp}{dx} \\ p - f &= (xf' + g') \frac{dp}{dx} \end{aligned} \tag{2}$$

Comparing the form $y = xf + g$ to (1A) shows that

$$\begin{aligned} f &= -1 \\ g &= p^2 \end{aligned}$$

Hence (2) becomes

$$p + 1 = 2pp'(x) \tag{2A}$$

The singular solution is found by setting $\frac{dp}{dx} = 0$ in the above which gives

$$p + 1 = 0$$

Solving for p from the above gives

$$p = -1$$

Substituting these in (1A) gives

$$y = 1 - x$$

The general solution is found when $\frac{\mathrm{d}p}{\mathrm{d}x} \neq 0$. From eq. (2A). This results in

$$p'(x) = \frac{p(x) + 1}{2p(x)} \tag{3}$$

This ODE is now solved for $p(x)$. Integrating both sides gives

$$\int \frac{2p}{p+1} dp = x + c_1$$
$$2p - 2\ln(p+1) = x + c_1$$

Solving for p gives these solutions

$$\begin{aligned} p_1 &= -\operatorname{LambertW}\left(-\mathrm{e}^{-1-\frac{x}{2}-\frac{c_1}{2}}\right) - 1 \\ &= -\operatorname{LambertW}\left(-c_1 \mathrm{e}^{-1-\frac{x}{2}}\right) - 1 \end{aligned}$$

Substituing the above solution for p in (2A) gives

$$y = \left(-\operatorname{LambertW}\left(-c_1 \mathrm{e}^{-1-\frac{x}{2}}\right) - 1\right)^2 - x$$

<u>Summary</u>

The solution(s) found are the following

$$y = 1 - x \tag{1}$$
$$y = \left(-\operatorname{LambertW}\left(-c_1 \mathrm{e}^{-1-\frac{x}{2}}\right) - 1\right)^2 - x \tag{2}$$

Verification of solutions

$$y = 1 - x$$

Verified OK.

$$y = \left(-\operatorname{LambertW}\left(-c_1 \mathrm{e}^{-1-\frac{x}{2}}\right) - 1\right)^2 - x$$

Verified OK.

Maple trace

```
`Methods for first order ODEs:
-> Solving 1st order ODE of high degree, 1st attempt
trying 1st order WeierstrassP solution for high degree ODE
trying 1st order WeierstrassPPrime solution for high degree ODE
trying 1st order JacobiSN solution for high degree ODE
trying 1st order ODE linearizable_by_differentiation
trying differential order: 1; missing variables
trying dAlembert
<- dAlembert successful`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 33

```
dsolve(diff(y(x),x)^2=x+y(x),y(x), singsol=all)
```

$$y(x) = \operatorname{LambertW}\left(-c_1 \mathrm{e}^{-\frac{x}{2}-1}\right)^2 + 2 \operatorname{LambertW}\left(-c_1 \mathrm{e}^{-\frac{x}{2}-1}\right) - x + 1$$

✓ Solution by Mathematica

Time used: 18.817 (sec). Leaf size: 100

```
DSolve[(y'[x])^2==x+y[x],y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to W\left(-e^{-\frac{x}{2}-1-\frac{c_1}{2}}\right)^2 + 2W\left(-e^{-\frac{x}{2}-1-\frac{c_1}{2}}\right) - x + 1$$
$$y(x) \to W\left(e^{\frac{1}{2}(-x-2+c_1)}\right)^2 + 2W\left(e^{\frac{1}{2}(-x-2+c_1)}\right) - x + 1$$
$$y(x) \to 1 - x$$

1.51 problem 51

Internal problem ID [7367]
Internal file name [`OUTPUT/6348_Sunday_June_05_2022_04_40_55_PM_62329261/index.tex`]

Book: First order enumerated odes
Section: section 1
Problem number: 51.
ODE order: 1.
ODE degree: 2.

The type(s) of ODE detected by this program : **"dAlembert", "first_order_nonlinear_p_but_separable"**

Maple gives the following as the ode type

```
[[_homogeneous, `class A`], _rational, _dAlembert]
```

$$\boxed{y'^2 - \frac{y}{x} = 0}$$

1.51.1 Solving as first order nonlinear p but separable ode

The ode has the form

$$(y')^{\frac{n}{m}} = f(x)g(y) \tag{1}$$

Where $n = 2, m = 1, f = \frac{1}{x}, g = y$. Hence the ode is

$$(y')^2 = \frac{y}{x}$$

Solving for y' from (1) gives

$$y' = \sqrt{fg}$$
$$y' = -\sqrt{fg}$$

To be able to solve as separable ode, we have to now assume that $f > 0, g > 0$.

$$\frac{1}{x} > 0$$
$$y > 0$$

Under the above assumption the differential equations become separable and can be written as

$$
\begin{aligned}
y' &= \sqrt{f}\,\sqrt{g} \\
y' &= -\sqrt{f}\,\sqrt{g}
\end{aligned}
$$

Therefore

$$
\begin{aligned}
\frac{1}{\sqrt{g}}\,dy &= \left(\sqrt{f}\right)\,dx \\
-\frac{1}{\sqrt{g}}\,dy &= \left(\sqrt{f}\right)\,dx
\end{aligned}
$$

Replacing $f(x), g(y)$ by their values gives

$$
\begin{aligned}
\frac{1}{\sqrt{y}}\,dy &= \left(\sqrt{\frac{1}{x}}\right)\,dx \\
-\frac{1}{\sqrt{y}}\,dy &= \left(\sqrt{\frac{1}{x}}\right)\,dx
\end{aligned}
$$

Integrating now gives the solutions.

$$
\begin{aligned}
\int \frac{1}{\sqrt{y}}dy &= \int \sqrt{\frac{1}{x}}dx + c_1 \\
\int -\frac{1}{\sqrt{y}}dy &= \int \sqrt{\frac{1}{x}}dx + c_1
\end{aligned}
$$

Integrating gives

$$
\begin{aligned}
2\sqrt{y} &= 2x\sqrt{\frac{1}{x}} + c_1 \\
-2\sqrt{y} &= 2x\sqrt{\frac{1}{x}} + c_1
\end{aligned}
$$

Therefore

$$
\begin{aligned}
y &= x\sqrt{\frac{1}{x}}\,c_1 + \frac{c_1^2}{4} + x \\
y &= x\sqrt{\frac{1}{x}}\,c_1 + \frac{c_1^2}{4} + x
\end{aligned}
$$

Summary
The solution(s) found are the following

$$y = x\sqrt{\frac{1}{x}c_1 + \frac{c_1^2}{4}} + x \tag{1}$$

$$y = x\sqrt{\frac{1}{x}c_1 + \frac{c_1^2}{4}} + x \tag{2}$$

Verification of solutions

$$y = x\sqrt{\frac{1}{x}c_1 + \frac{c_1^2}{4}} + x$$

Verified OK. `{0 < y, 0 < 1/x}`

$$y = x\sqrt{\frac{1}{x}c_1 + \frac{c_1^2}{4}} + x$$

Verified OK. `{0 < y, 0 < 1/x}`

1.51.2 Solving as dAlembert ode

Let $p = y'$ the ode becomes

$$p^2 - \frac{y}{x} = 0$$

Solving for y from the above results in

$$y = p^2 x \tag{1A}$$

This has the form

$$y = xf(p) + g(p) \tag{*}$$

Where f, g are functions of $p = y'(x)$. The above ode is dAlembert ode which is now solved. Taking derivative of (*) w.r.t. x gives

$$p = f + (xf' + g')\frac{dp}{dx}$$

$$p - f = (xf' + g')\frac{dp}{dx} \tag{2}$$

Comparing the form $y = xf + g$ to (1A) shows that

$$f = p^2$$
$$g = 0$$

Hence (2) becomes

$$-p^2 + p = 2xpp'(x) \tag{2A}$$

The singular solution is found by setting $\frac{dp}{dx} = 0$ in the above which gives

$$-p^2 + p = 0$$

Solving for p from the above gives

$$\begin{aligned} p &= 0 \\ p &= 1 \end{aligned}$$

Substituting these in (1A) gives

$$\begin{aligned} y &= 0 \\ y &= x \end{aligned}$$

The general solution is found when $\frac{\mathrm{d}p}{\mathrm{d}x} \neq 0$. From eq. (2A). This results in

$$p'(x) = \frac{-p(x)^2 + p(x)}{2xp(x)} \tag{3}$$

This ODE is now solved for $p(x)$.

Entering Linear first order ODE solver. In canonical form a linear first order is

$$p'(x) + p(x)p(x) = q(x)$$

Where here

$$\begin{aligned} p(x) &= \frac{1}{2x} \\ q(x) &= \frac{1}{2x} \end{aligned}$$

Hence the ode is

$$p'(x) + \frac{p(x)}{2x} = \frac{1}{2x}$$

The integrating factor μ is

$$\begin{aligned} \mu &= \mathrm{e}^{\int \frac{1}{2x} dx} \\ &= \sqrt{x} \end{aligned}$$

The ode becomes

$$
\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}x}(\mu p) &= (\mu)\left(\frac{1}{2x}\right) \\
\frac{\mathrm{d}}{\mathrm{d}x}\left(\sqrt{x}\, p\right) &= \left(\sqrt{x}\right)\left(\frac{1}{2x}\right) \\
\mathrm{d}\left(\sqrt{x}\, p\right) &= \left(\frac{1}{2\sqrt{x}}\right) \mathrm{d}x
\end{aligned}
$$

Integrating gives

$$
\begin{aligned}
\sqrt{x}\, p &= \int \frac{1}{2\sqrt{x}}\, \mathrm{d}x \\
\sqrt{x}\, p &= \sqrt{x} + c_1
\end{aligned}
$$

Dividing both sides by the integrating factor $\mu = \sqrt{x}$ results in

$$
p(x) = 1 + \frac{c_1}{\sqrt{x}}
$$

Substituing the above solution for p in (2A) gives

$$
y = \left(1 + \frac{c_1}{\sqrt{x}}\right)^2 x
$$

Summary

The solution(s) found are the following

$$
y = 0 \tag{1}
$$

$$
y = x \tag{2}
$$

$$
y = \left(1 + \frac{c_1}{\sqrt{x}}\right)^2 x \tag{3}
$$

Verification of solutions

$$
y = 0
$$

Verified OK. `{0 < y, 0 < 1/x}`

$$
y = x
$$

Verified OK. `{0 < y, 0 < 1/x}`

$$
y = \left(1 + \frac{c_1}{\sqrt{x}}\right)^2 x
$$

Verified OK. `{0 < y, 0 < 1/x}`

Maple trace

```
`Methods for first order ODEs:
   *** Sublevel 2 ***
   Methods for first order ODEs:
   -> Solving 1st order ODE of high degree, 1st attempt
   trying 1st order WeierstrassP solution for high degree ODE
   trying 1st order WeierstrassPPrime solution for high degree ODE
   trying 1st order JacobiSN solution for high degree ODE
   trying 1st order ODE linearizable_by_differentiation
   trying differential order: 1; missing variables
   trying dAlembert
   <- dAlembert successful`
```

✓ Solution by Maple

Time used: 0.015 (sec). Leaf size: 39

```
dsolve(diff(y(x),x)^2=y(x)/x,y(x), singsol=all)
```

$$y(x) = 0$$
$$y(x) = \frac{\left(x + \sqrt{c_1 x}\right)^2}{x}$$
$$y(x) = \frac{\left(-x + \sqrt{c_1 x}\right)^2}{x}$$

✓ Solution by Mathematica

Time used: 0.048 (sec). Leaf size: 46

```
DSolve[(y'[x])^2==y[x]/x,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \frac{1}{4}\left(-2\sqrt{x} + c_1\right)^2$$
$$y(x) \to \frac{1}{4}\left(2\sqrt{x} + c_1\right)^2$$
$$y(x) \to 0$$

1.52 problem 52

Internal problem ID [7368]
Internal file name [`OUTPUT/6349_Sunday_June_05_2022_04_40_59_PM_62601205/index.tex`]

Book: First order enumerated odes
Section: section 1
Problem number: 52.
ODE order: 1.
ODE degree: 2.

The type(s) of ODE detected by this program : **"first_order_nonlinear_p_but_separable"**

Maple gives the following as the ode type

`[_separable]`

$$\boxed{y'^2 - \frac{y^2}{x} = 0}$$

1.52.1 Solving as first order nonlinear p but separable ode

The ode has the form

$$(y')^{\frac{n}{m}} = f(x)g(y) \tag{1}$$

Where $n = 2, m = 1, f = \frac{1}{x}, g = y^2$. Hence the ode is

$$(y')^2 = \frac{y^2}{x}$$

Solving for y' from (1) gives

$$y' = \sqrt{fg}$$
$$y' = -\sqrt{fg}$$

To be able to solve as separable ode, we have to now assume that $f > 0, g > 0$.

$$\frac{1}{x} > 0$$
$$y^2 > 0$$

Under the above assumption the differential equations become separable and can be written as

$$
\begin{aligned}
y' &= \sqrt{f}\,\sqrt{g} \\
y' &= -\sqrt{f}\,\sqrt{g}
\end{aligned}
$$

Therefore

$$
\begin{aligned}
\frac{1}{\sqrt{g}}\,dy &= \left(\sqrt{f}\right)\,dx \\
-\frac{1}{\sqrt{g}}\,dy &= \left(\sqrt{f}\right)\,dx
\end{aligned}
$$

Replacing $f(x), g(y)$ by their values gives

$$
\begin{aligned}
\frac{1}{\sqrt{y^2}}\,dy &= \left(\sqrt{\frac{1}{x}}\right)\,dx \\
-\frac{1}{\sqrt{y^2}}\,dy &= \left(\sqrt{\frac{1}{x}}\right)\,dx
\end{aligned}
$$

Integrating now gives the solutions.

$$
\begin{aligned}
\int \frac{1}{\sqrt{y^2}}dy &= \int \sqrt{\frac{1}{x}}dx + c_1 \\
\int -\frac{1}{\sqrt{y^2}}dy &= \int \sqrt{\frac{1}{x}}dx + c_1
\end{aligned}
$$

Integrating gives

$$
\begin{aligned}
\frac{y \ln(y)}{\sqrt{y^2}} &= 2x\sqrt{\frac{1}{x}} + c_1 \\
-\frac{y \ln(y)}{\sqrt{y^2}} &= 2x\sqrt{\frac{1}{x}} + c_1
\end{aligned}
$$

Therefore

$$
\frac{y \ln(y)}{\sqrt{y^2}} = 2x\sqrt{\frac{1}{x}} + c_1
$$

$$
-\frac{y \ln(y)}{\sqrt{y^2}} = 2x\sqrt{\frac{1}{x}} + c_1
$$

Summary

The solution(s) found are the following

$$\frac{y \ln (y)}{\sqrt{y^2}} = 2x\sqrt{\frac{1}{x}} + c_1 \tag{1}$$

$$-\frac{y \ln (y)}{\sqrt{y^2}} = 2x\sqrt{\frac{1}{x}} + c_1 \tag{2}$$

Verification of solutions

$$\frac{y \ln (y)}{\sqrt{y^2}} = 2x\sqrt{\frac{1}{x}} + c_1$$

Verified OK. `{0 < 1/x, 0 < y^2}`

$$-\frac{y \ln (y)}{\sqrt{y^2}} = 2x\sqrt{\frac{1}{x}} + c_1$$

Verified OK. `{0 < 1/x, 0 < y^2}`

1.52.2 Maple step by step solution

Let's solve

$y'^2 - \frac{y^2}{x} = 0$

- Highest derivative means the order of the ODE is 1

  y'

- Separate variables

  $\frac{y'}{y} = \frac{1}{\sqrt{x}}$

- Integrate both sides with respect to x

  $\int \frac{y'}{y} dx = \int \frac{1}{\sqrt{x}} dx + c_1$

- Evaluate integral

  $\ln (y) = 2\sqrt{x} + c_1$

- Solve for y

  $y = e^{2\sqrt{x}+c_1}$

Maple trace

```
`Methods for first order ODEs:
-> Solving 1st order ODE of high degree, 1st attempt
trying 1st order WeierstrassP solution for high degree ODE
trying 1st order WeierstrassPPrime solution for high degree ODE
trying 1st order JacobiSN solution for high degree ODE
trying 1st order ODE linearizable_by_differentiation
trying differential order: 1; missing variables
trying simple symmetries for implicit equations
<- symmetries for implicit equations successful`
```

✓ Solution by Maple

Time used: 0.031 (sec). Leaf size: 27

```
dsolve(diff(y(x),x)^2=y(x)^2/x,y(x), singsol=all)
```

$$y(x) = 0$$
$$y(x) = c_1 \mathrm{e}^{-2\sqrt{x}}$$
$$y(x) = c_1 \mathrm{e}^{2\sqrt{x}}$$

✓ Solution by Mathematica

Time used: 0.068 (sec). Leaf size: 38

```
DSolve[(y'[x])^2==y[x]^2/x,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to c_1 e^{-2\sqrt{x}}$$
$$y(x) \to c_1 e^{2\sqrt{x}}$$
$$y(x) \to 0$$

1.53 problem 53

Internal problem ID [7369]
Internal file name [`OUTPUT/6350_Sunday_June_05_2022_04_41_01_PM_39470954/index.tex`]

Book: First order enumerated odes
Section: section 1
Problem number: 53.
ODE order: 1.
ODE degree: 2.

The type(s) of ODE detected by this program : **"first_order_nonlinear_p_but_separable"**

Maple gives the following as the ode type

```
[[_homogeneous, `class G`]]
```

$$\boxed{y'^2 - \frac{y^3}{x} = 0}$$

1.53.1 Solving as first order nonlinear p but separable ode

The ode has the form

$$(y')^{\frac{n}{m}} = f(x)g(y) \tag{1}$$

Where $n = 2, m = 1, f = \frac{1}{x}, g = y^3$. Hence the ode is

$$(y')^2 = \frac{y^3}{x}$$

Solving for y' from (1) gives

$$y' = \sqrt{fg}$$
$$y' = -\sqrt{fg}$$

To be able to solve as separable ode, we have to now assume that $f > 0, g > 0$.

$$\frac{1}{x} > 0$$
$$y^3 > 0$$

Under the above assumption the differential equations become separable and can be written as

$$
\begin{aligned}
y' &= \sqrt{f}\,\sqrt{g} \\
y' &= -\sqrt{f}\,\sqrt{g}
\end{aligned}
$$

Therefore

$$
\begin{aligned}
\frac{1}{\sqrt{g}}\,dy &= \left(\sqrt{f}\right)\,dx \\
-\frac{1}{\sqrt{g}}\,dy &= \left(\sqrt{f}\right)\,dx
\end{aligned}
$$

Replacing $f(x), g(y)$ by their values gives

$$
\begin{aligned}
\frac{1}{\sqrt{y^3}}\,dy &= \left(\sqrt{\frac{1}{x}}\right)\,dx \\
-\frac{1}{\sqrt{y^3}}\,dy &= \left(\sqrt{\frac{1}{x}}\right)\,dx
\end{aligned}
$$

Integrating now gives the solutions.

$$
\begin{aligned}
\int \frac{1}{\sqrt{y^3}}dy &= \int \sqrt{\frac{1}{x}}dx + c_1 \\
\int -\frac{1}{\sqrt{y^3}}dy &= \int \sqrt{\frac{1}{x}}dx + c_1
\end{aligned}
$$

Integrating gives

$$
\begin{aligned}
-\frac{2y}{\sqrt{y^3}} &= 2x\sqrt{\frac{1}{x}} + c_1 \\
\frac{2y}{\sqrt{y^3}} &= 2x\sqrt{\frac{1}{x}} + c_1
\end{aligned}
$$

Therefore

$$
y = \frac{8x\sqrt{\frac{1}{x}} + 4c_1}{8x^3\left(\frac{1}{x}\right)^{\frac{3}{2}} + 6x\sqrt{\frac{1}{x}}\,c_1^2 + c_1^3 + 12c_1x}
$$

$$
y = \frac{8x\sqrt{\frac{1}{x}} + 4c_1}{8x^3\left(\frac{1}{x}\right)^{\frac{3}{2}} + 6x\sqrt{\frac{1}{x}}\,c_1^2 + c_1^3 + 12c_1x}
$$

Summary

The solution(s) found are the following

$$y = \frac{8x\sqrt{\frac{1}{x}} + 4c_1}{8x^3 \left(\frac{1}{x}\right)^{\frac{3}{2}} + 6x\sqrt{\frac{1}{x}}\, c_1^2 + c_1^3 + 12c_1 x} \tag{1}$$

$$y = \frac{8x\sqrt{\frac{1}{x}} + 4c_1}{8x^3 \left(\frac{1}{x}\right)^{\frac{3}{2}} + 6x\sqrt{\frac{1}{x}}\, c_1^2 + c_1^3 + 12c_1 x} \tag{2}$$

Verification of solutions

$$y = \frac{8x\sqrt{\frac{1}{x}} + 4c_1}{8x^3 \left(\frac{1}{x}\right)^{\frac{3}{2}} + 6x\sqrt{\frac{1}{x}}\, c_1^2 + c_1^3 + 12c_1 x}$$

Verified OK. `{0 < 1/x, 0 < y^3}`

$$y = \frac{8x\sqrt{\frac{1}{x}} + 4c_1}{8x^3 \left(\frac{1}{x}\right)^{\frac{3}{2}} + 6x\sqrt{\frac{1}{x}}\, c_1^2 + c_1^3 + 12c_1 x}$$

Verified OK. `{0 < 1/x, 0 < y^3}`

Maple trace

```
`Methods for first order ODEs:
-> Solving 1st order ODE of high degree, 1st attempt
trying 1st order WeierstrassP solution for high degree ODE
<- 1st_order WeierstrassP successful`
```

✓ Solution by Maple

Time used: 0.015 (sec). Leaf size: 27

```
dsolve(diff(y(x),x)^2=y(x)^3/x,y(x), singsol=all)
```

$$y(x) = 0$$

$$y(x) = \frac{\text{WeierstrassP}\left(1, 0, 0\right) 2^{\frac{2}{3}}}{\left(\sqrt{x}\, 2^{\frac{1}{3}} + c_1\right)^2}$$

✓ Solution by Mathematica

Time used: 0.073 (sec). Leaf size: 42

```
DSolve[(y'[x])^2==y[x]^3/x,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \frac{4}{(-2\sqrt{x}+c_1)^2}$$
$$y(x) \to \frac{4}{(2\sqrt{x}+c_1)^2}$$
$$y(x) \to 0$$

1.54 problem 54

Internal problem ID [7370]
Internal file name [`OUTPUT/6351_Sunday_June_05_2022_04_41_05_PM_32553168/index.tex`]

Book: First order enumerated odes
Section: section 1
Problem number: 54.
ODE order: 1.
ODE degree: 3.

The type(s) of ODE detected by this program : **"first_order_nonlinear_p_but_separable"**

Maple gives the following as the ode type

```
[[_homogeneous, `class G`], _rational]
```

$$\boxed{y'^3 - \frac{y^2}{x} = 0}$$

1.54.1 Solving as first order nonlinear p but separable ode

The ode has the form

$$(y')^{\frac{n}{m}} = f(x)g(y) \tag{1}$$

Where $n = 3, m = 1, f = \frac{1}{x}, g = y^2$. Hence the ode is

$$(y')^3 = \frac{y^2}{x}$$

Solving for y' from (1) gives

$$\begin{aligned}
y' &= (fg)^{\frac{1}{3}} \\
y' &= -\frac{(fg)^{\frac{1}{3}}}{2} + \frac{i\sqrt{3}\,(fg)^{\frac{1}{3}}}{2} \\
y' &= -\frac{(fg)^{\frac{1}{3}}}{2} - \frac{i\sqrt{3}\,(fg)^{\frac{1}{3}}}{2}
\end{aligned}$$

To be able to solve as separable ode, we have to now assume that $f > 0, g > 0$.

$$\frac{1}{x} > 0$$
$$y^2 > 0$$

Under the above assumption the differential equations become separable and can be written as

$$y' = f^{\frac{1}{3}} g^{\frac{1}{3}}$$
$$y' = \frac{f^{\frac{1}{3}} g^{\frac{1}{3}} (i\sqrt{3} - 1)}{2}$$
$$y' = -\frac{f^{\frac{1}{3}} g^{\frac{1}{3}} (1 + i\sqrt{3})}{2}$$

Therefore

$$\frac{1}{g^{\frac{1}{3}}}\, dy = \left(f^{\frac{1}{3}}\right) dx$$
$$\frac{2}{g^{\frac{1}{3}} (i\sqrt{3} - 1)}\, dy = \left(f^{\frac{1}{3}}\right) dx$$
$$-\frac{2}{g^{\frac{1}{3}} (1 + i\sqrt{3})}\, dy = \left(f^{\frac{1}{3}}\right) dx$$

Replacing $f(x), g(y)$ by their values gives

$$\frac{1}{(y^2)^{\frac{1}{3}}}\, dy = \left(\left(\frac{1}{x}\right)^{\frac{1}{3}}\right) dx$$
$$\frac{2}{(y^2)^{\frac{1}{3}} (i\sqrt{3} - 1)}\, dy = \left(\left(\frac{1}{x}\right)^{\frac{1}{3}}\right) dx$$
$$-\frac{2}{(y^2)^{\frac{1}{3}} (1 + i\sqrt{3})}\, dy = \left(\left(\frac{1}{x}\right)^{\frac{1}{3}}\right) dx$$

Integrating now gives the solutions.

$$\int \frac{1}{(y^2)^{\frac{1}{3}}} dy = \int \left(\frac{1}{x}\right)^{\frac{1}{3}} dx + c_1$$
$$\int \frac{2}{(y^2)^{\frac{1}{3}} (i\sqrt{3} - 1)} dy = \int \left(\frac{1}{x}\right)^{\frac{1}{3}} dx + c_1$$
$$\int -\frac{2}{(y^2)^{\frac{1}{3}} (1 + i\sqrt{3})} dy = \int \left(\frac{1}{x}\right)^{\frac{1}{3}} dx + c_1$$

Integrating gives

$$\frac{3y}{(y^2)^{\frac{1}{3}}} = \frac{3x\left(\frac{1}{x}\right)^{\frac{1}{3}}}{2} + c_1$$

$$\frac{6y}{(y^2)^{\frac{1}{3}}\left(i\sqrt{3}-1\right)} = \frac{3x\left(\frac{1}{x}\right)^{\frac{1}{3}}}{2} + c_1$$

$$-\frac{6y}{(y^2)^{\frac{1}{3}}\left(1+i\sqrt{3}\right)} = \frac{3x\left(\frac{1}{x}\right)^{\frac{1}{3}}}{2} + c_1$$

Therefore

$$y = \frac{x^2}{8} + \frac{x^2\left(\frac{1}{x}\right)^{\frac{2}{3}} c_1}{4} + \frac{x\left(\frac{1}{x}\right)^{\frac{1}{3}} c_1^2}{6} + \frac{c_1^3}{27}$$

$$y = \frac{x^2}{8} + \frac{x^2\left(\frac{1}{x}\right)^{\frac{2}{3}} c_1}{4} + \frac{x\left(\frac{1}{x}\right)^{\frac{1}{3}} c_1^2}{6} + \frac{c_1^3}{27}$$

$$y = \frac{x^2}{8} + \frac{x^2\left(\frac{1}{x}\right)^{\frac{2}{3}} c_1}{4} + \frac{x\left(\frac{1}{x}\right)^{\frac{1}{3}} c_1^2}{6} + \frac{c_1^3}{27}$$

<u>Summary</u>

The solution(s) found are the following

$$y = \frac{x^2}{8} + \frac{x^2\left(\frac{1}{x}\right)^{\frac{2}{3}} c_1}{4} + \frac{x\left(\frac{1}{x}\right)^{\frac{1}{3}} c_1^2}{6} + \frac{c_1^3}{27} \tag{1}$$

$$y = \frac{x^2}{8} + \frac{x^2\left(\frac{1}{x}\right)^{\frac{2}{3}} c_1}{4} + \frac{x\left(\frac{1}{x}\right)^{\frac{1}{3}} c_1^2}{6} + \frac{c_1^3}{27} \tag{2}$$

$$y = \frac{x^2}{8} + \frac{x^2\left(\frac{1}{x}\right)^{\frac{2}{3}} c_1}{4} + \frac{x\left(\frac{1}{x}\right)^{\frac{1}{3}} c_1^2}{6} + \frac{c_1^3}{27} \tag{3}$$

Verification of solutions

$$y = \frac{x^2}{8} + \frac{x^2 \left(\frac{1}{x}\right)^{\frac{2}{3}} c_1}{4} + \frac{x \left(\frac{1}{x}\right)^{\frac{1}{3}} c_1^2}{6} + \frac{c_1^3}{27}$$

Verified OK. `{0 < 1/x, 0 < y^2}`

$$y = \frac{x^2}{8} + \frac{x^2 \left(\frac{1}{x}\right)^{\frac{2}{3}} c_1}{4} + \frac{x \left(\frac{1}{x}\right)^{\frac{1}{3}} c_1^2}{6} + \frac{c_1^3}{27}$$

Verified OK. `{0 < 1/x, 0 < y^2}`

$$y = \frac{x^2}{8} + \frac{x^2 \left(\frac{1}{x}\right)^{\frac{2}{3}} c_1}{4} + \frac{x \left(\frac{1}{x}\right)^{\frac{1}{3}} c_1^2}{6} + \frac{c_1^3}{27}$$

Verified OK. `{0 < 1/x, 0 < y^2}`

Maple trace

```
`Methods for first order ODEs:
   *** Sublevel 2 ***
   Methods for first order ODEs:
   -> Solving 1st order ODE of high degree, 1st attempt
   trying 1st order WeierstrassP solution for high degree ODE
   trying 1st order WeierstrassPPrime solution for high degree ODE
   trying 1st order JacobiSN solution for high degree ODE
   trying 1st order ODE linearizable_by_differentiation
   trying differential order: 1; missing variables
   trying simple symmetries for implicit equations
   Successful isolation of dy/dx: 3 solutions were found. Trying to solve each resulting ODE
      *** Sublevel 3 ***
      Methods for first order ODEs:
      --- Trying classification methods ---
      trying homogeneous types:
      trying homogeneous G
      trying an integrating factor from the invariance group
      <- integrating factor successful
      <- homogeneous successful
   -------------------
   * Tackling next ODE.
      *** Sublevel 3 ***
      Methods for first order ODEs:
      --- Trying classification methods ---
      trying homogeneous types:
      trying homogeneous G
      trying an integrating factor from the invariance group
      <- integrating factor successful
      <- homogeneous successful
   -------------------
   * Tackling next ODE.
      *** Sublevel 3 ***
      Methods for first order ODEs:
      --- Trying classification methods ---
      trying homogeneous types:
      trying homogeneous G
      trying an integrating factor from the invariance group
      <- integrating factor successful
      <- homogeneous successful`
```

✓ Solution by Maple

Time used: 0.047 (sec). Leaf size: 353

```
dsolve(diff(y(x),x)^3=y(x)^2/x,y(x), singsol=all)
```

$$y(x) = 0$$
$$y(x) = -\frac{3x^{\frac{4}{3}}c_1}{8} + \frac{3x^{\frac{2}{3}}c_1^2}{8} - \frac{c_1^3}{8} + \frac{x^2}{8}$$
$$y(x) = \frac{3(-i\sqrt{3}-1)\,c_1^2x^{\frac{2}{3}}}{16} + \frac{3c_1\,(1-i\sqrt{3})\,x^{\frac{4}{3}}}{16} - \frac{c_1^3}{8} + \frac{x^2}{8}$$
$$y(x) = \frac{3(i\sqrt{3}-1)\,c_1^2x^{\frac{2}{3}}}{16} + \frac{3(1+i\sqrt{3})\,c_1x^{\frac{4}{3}}}{16} - \frac{c_1^3}{8} + \frac{x^2}{8}$$
$$y(x) = \frac{3x^{\frac{4}{3}}c_1}{16} + \frac{3x^{\frac{2}{3}}c_1^2}{32} + \frac{c_1^3}{64} + \frac{x^2}{8}$$
$$y(x) = \frac{3(-i\sqrt{3}-1)\,c_1^2x^{\frac{2}{3}}}{64} + \frac{3(i\sqrt{3}-1)\,c_1x^{\frac{4}{3}}}{32} + \frac{c_1^3}{64} + \frac{x^2}{8}$$
$$y(x) = \frac{3(i\sqrt{3}-1)\,c_1^2x^{\frac{2}{3}}}{64} + \frac{3c_1\,(-i\sqrt{3}-1)\,x^{\frac{4}{3}}}{32} + \frac{c_1^3}{64} + \frac{x^2}{8}$$
$$y(x) = -\frac{3x^{\frac{4}{3}}c_1}{16} + \frac{3x^{\frac{2}{3}}c_1^2}{32} - \frac{c_1^3}{64} + \frac{x^2}{8}$$
$$y(x) = \frac{3(-i\sqrt{3}-1)\,c_1^2x^{\frac{2}{3}}}{64} + \frac{3c_1\,(1-i\sqrt{3})\,x^{\frac{4}{3}}}{32} - \frac{c_1^3}{64} + \frac{x^2}{8}$$
$$y(x) = \frac{3(i\sqrt{3}-1)\,c_1^2x^{\frac{2}{3}}}{64} + \frac{3(1+i\sqrt{3})\,c_1x^{\frac{4}{3}}}{32} - \frac{c_1^3}{64} + \frac{x^2}{8}$$

✓ Solution by Mathematica

Time used: 0.083 (sec). Leaf size: 152

```
DSolve[(y'[x])^3==y[x]^2/x,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \frac{1}{216}(3x^{2/3} + 2c_1)^3$$
$$y(x) \to \frac{1}{216}\left(18i\left(\sqrt{3}+i\right)c_1{}^2x^{2/3} - 27i\left(\sqrt{3}-i\right)c_1x^{4/3} + 27x^2 + 8c_1{}^3\right)$$
$$y(x) \to \frac{1}{216}\left(-18i\left(\sqrt{3}-i\right)c_1{}^2x^{2/3} + 27i\left(\sqrt{3}+i\right)c_1x^{4/3} + 27x^2 + 8c_1{}^3\right)$$
$$y(x) \to 0$$

1.55 problem 55

Internal problem ID [7371]
Internal file name [`OUTPUT/6352_Sunday_June_05_2022_04_41_12_PM_61719047/index.tex`]

Book: First order enumerated odes
Section: section 1
Problem number: 55.
ODE order: 1.
ODE degree: 2.

The type(s) of ODE detected by this program : **"first_order_nonlinear_p_but_separable"**

Maple gives the following as the ode type

```
[[_homogeneous, `class G`]]
```

$$\boxed{y'^2 - \frac{1}{yx} = 0}$$

1.55.1 Solving as first order nonlinear p but separable ode

The ode has the form

$$(y')^{\frac{n}{m}} = f(x)g(y) \tag{1}$$

Where $n = 2, m = 1, f = \frac{1}{x}, g = \frac{1}{y}$. Hence the ode is

$$(y')^2 = \frac{1}{xy}$$

Solving for y' from (1) gives

$$y' = \sqrt{fg}$$
$$y' = -\sqrt{fg}$$

To be able to solve as separable ode, we have to now assume that $f > 0, g > 0$.

$$\frac{1}{x} > 0$$
$$\frac{1}{y} > 0$$

Under the above assumption the differential equations become separable and can be written as

$$
\begin{aligned}
y' &= \sqrt{f}\,\sqrt{g} \\
y' &= -\sqrt{f}\,\sqrt{g}
\end{aligned}
$$

Therefore

$$
\begin{aligned}
\frac{1}{\sqrt{g}}\,dy &= \left(\sqrt{f}\right)\,dx \\
-\frac{1}{\sqrt{g}}\,dy &= \left(\sqrt{f}\right)\,dx
\end{aligned}
$$

Replacing $f(x), g(y)$ by their values gives

$$
\begin{aligned}
\frac{1}{\sqrt{\frac{1}{y}}}\,dy &= \left(\sqrt{\frac{1}{x}}\right)\,dx \\
-\frac{1}{\sqrt{\frac{1}{y}}}\,dy &= \left(\sqrt{\frac{1}{x}}\right)\,dx
\end{aligned}
$$

Integrating now gives the solutions.

$$
\begin{aligned}
\int \frac{1}{\sqrt{\frac{1}{y}}}dy &= \int \sqrt{\frac{1}{x}}dx + c_1 \\
\int -\frac{1}{\sqrt{\frac{1}{y}}}dy &= \int \sqrt{\frac{1}{x}}dx + c_1
\end{aligned}
$$

Integrating gives

$$
\begin{aligned}
\frac{2y}{3\sqrt{\frac{1}{y}}} &= 2x\sqrt{\frac{1}{x}} + c_1 \\
-\frac{2y}{3\sqrt{\frac{1}{y}}} &= 2x\sqrt{\frac{1}{x}} + c_1
\end{aligned}
$$

Therefore

$$
\frac{2y}{3\sqrt{\frac{1}{y}}} = 2x\sqrt{\frac{1}{x}} + c_1
$$

$$
-\frac{2y}{3\sqrt{\frac{1}{y}}} = 2x\sqrt{\frac{1}{x}} + c_1
$$

Summary

The solution(s) found are the following

$$\frac{2y}{3\sqrt{\frac{1}{y}}} = 2x\sqrt{\frac{1}{x}} + c_1 \tag{1}$$

$$-\frac{2y}{3\sqrt{\frac{1}{y}}} = 2x\sqrt{\frac{1}{x}} + c_1 \tag{2}$$

Verification of solutions

$$\frac{2y}{3\sqrt{\frac{1}{y}}} = 2x\sqrt{\frac{1}{x}} + c_1$$

Verified OK. `{0 < 1/x, 0 < 1/y}`

$$-\frac{2y}{3\sqrt{\frac{1}{y}}} = 2x\sqrt{\frac{1}{x}} + c_1$$

Verified OK. `{0 < 1/x, 0 < 1/y}`

Maple trace

```
`Methods for first order ODEs:
-> Solving 1st order ODE of high degree, 1st attempt
trying 1st order WeierstrassP solution for high degree ODE
trying 1st order WeierstrassPPrime solution for high degree ODE
trying 1st order JacobiSN solution for high degree ODE
trying 1st order ODE linearizable_by_differentiation
trying differential order: 1; missing variables
trying simple symmetries for implicit equations
Successful isolation of dy/dx: 2 solutions were found. Trying to solve each resulting ODE.
   *** Sublevel 2 ***
   Methods for first order ODEs:
   --- Trying classification methods ---
   trying homogeneous types:
   trying homogeneous G
   1st order, trying the canonical coordinates of the invariance group
   <- 1st order, canonical coordinates successful
   <- homogeneous successful
-------------------
* Tackling next ODE.
   *** Sublevel 2 ***
   Methods for first order ODEs:
   --- Trying classification methods ---
   trying homogeneous types:
   trying homogeneous G
   1st order, trying the canonical coordinates of the invariance group
   <- 1st order, canonical coordinates successful
   <- homogeneous successful`
```

✓ Solution by Maple

Time used: 0.031 (sec). Leaf size: 51

```
dsolve(diff(y(x),x)^2=1/(y(x)*x),y(x), singsol=all)
```

$$\frac{y(x)\sqrt{xy(x)} - c_1\sqrt{x} - 3x}{\sqrt{x}} = 0$$

$$\frac{y(x)\sqrt{xy(x)} - c_1\sqrt{x} + 3x}{\sqrt{x}} = 0$$

✓ Solution by Mathematica

Time used: 3.748 (sec). Leaf size: 53

```
DSolve[(y'[x])^2==1/(y[x]*x),y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \left(\frac{3}{2}\right)^{2/3} \left(-2\sqrt{x}+c_1\right)^{2/3}$$
$$y(x) \to \left(\frac{3}{2}\right)^{2/3} \left(2\sqrt{x}+c_1\right)^{2/3}$$

1.56 problem 56

Internal problem ID [7372]
Internal file name [`OUTPUT/6353_Sunday_June_05_2022_04_41_18_PM_57592104/index.tex`]

Book: First order enumerated odes
Section: section 1
Problem number: 56.
ODE order: 1.
ODE degree: 2.

The type(s) of ODE detected by this program : **"first_order_nonlinear_p_but_separable"**

Maple gives the following as the ode type

```
[[_homogeneous, `class G`]]
```

$$\boxed{y'^2 - \frac{1}{y^3 x} = 0}$$

1.56.1 Solving as first order nonlinear p but separable ode

The ode has the form

$$(y')^{\frac{n}{m}} = f(x)g(y) \tag{1}$$

Where $n = 2, m = 1, f = \frac{1}{x}, g = \frac{1}{y^3}$. Hence the ode is

$$(y')^2 = \frac{1}{y^3 x}$$

Solving for y' from (1) gives

$$y' = \sqrt{fg}$$
$$y' = -\sqrt{fg}$$

To be able to solve as separable ode, we have to now assume that $f > 0, g > 0$.

$$\frac{1}{x} > 0$$
$$\frac{1}{y^3} > 0$$

Under the above assumption the differential equations become separable and can be written as

$$
\begin{aligned}
y' &= \sqrt{f}\,\sqrt{g} \\
y' &= -\sqrt{f}\,\sqrt{g}
\end{aligned}
$$

Therefore

$$
\begin{aligned}
\frac{1}{\sqrt{g}}\,dy &= \left(\sqrt{f}\right)\,dx \\
-\frac{1}{\sqrt{g}}\,dy &= \left(\sqrt{f}\right)\,dx
\end{aligned}
$$

Replacing $f(x), g(y)$ by their values gives

$$
\begin{aligned}
\frac{1}{\sqrt{\frac{1}{y^3}}}\,dy &= \left(\sqrt{\frac{1}{x}}\right)\,dx \\
-\frac{1}{\sqrt{\frac{1}{y^3}}}\,dy &= \left(\sqrt{\frac{1}{x}}\right)\,dx
\end{aligned}
$$

Integrating now gives the solutions.

$$
\begin{aligned}
\int \frac{1}{\sqrt{\frac{1}{y^3}}}dy &= \int \sqrt{\frac{1}{x}}dx + c_1 \\
\int -\frac{1}{\sqrt{\frac{1}{y^3}}}dy &= \int \sqrt{\frac{1}{x}}dx + c_1
\end{aligned}
$$

Integrating gives

$$
\begin{aligned}
\frac{2y}{5\sqrt{\frac{1}{y^3}}} &= 2x\sqrt{\frac{1}{x}} + c_1 \\
-\frac{2y}{5\sqrt{\frac{1}{y^3}}} &= 2x\sqrt{\frac{1}{x}} + c_1
\end{aligned}
$$

Therefore

$$
\frac{2y}{5\sqrt{\frac{1}{y^3}}} = 2x\sqrt{\frac{1}{x}} + c_1
$$

$$
-\frac{2y}{5\sqrt{\frac{1}{y^3}}} = 2x\sqrt{\frac{1}{x}} + c_1
$$

Summary

The solution(s) found are the following

$$\frac{2y}{5\sqrt{\frac{1}{y^3}}} = 2x\sqrt{\frac{1}{x}} + c_1 \tag{1}$$

$$-\frac{2y}{5\sqrt{\frac{1}{y^3}}} = 2x\sqrt{\frac{1}{x}} + c_1 \tag{2}$$

Verification of solutions

$$\frac{2y}{5\sqrt{\frac{1}{y^3}}} = 2x\sqrt{\frac{1}{x}} + c_1$$

Verified OK. `{0 < 1/x, 0 < 1/y^3}`

$$-\frac{2y}{5\sqrt{\frac{1}{y^3}}} = 2x\sqrt{\frac{1}{x}} + c_1$$

Verified OK. `{0 < 1/x, 0 < 1/y^3}`

Maple trace

```
`Methods for first order ODEs:
-> Solving 1st order ODE of high degree, 1st attempt
trying 1st order WeierstrassP solution for high degree ODE
trying 1st order WeierstrassPPrime solution for high degree ODE
trying 1st order JacobiSN solution for high degree ODE
trying 1st order ODE linearizable_by_differentiation
trying differential order: 1; missing variables
trying simple symmetries for implicit equations
Successful isolation of dy/dx: 2 solutions were found. Trying to solve each resulting ODE.
   *** Sublevel 2 ***
   Methods for first order ODEs:
   --- Trying classification methods ---
   trying homogeneous types:
   trying homogeneous G
   1st order, trying the canonical coordinates of the invariance group
   <- 1st order, canonical coordinates successful
   <- homogeneous successful
-------------------
* Tackling next ODE.
   *** Sublevel 2 ***
   Methods for first order ODEs:
   --- Trying classification methods ---
   trying homogeneous types:
   trying homogeneous G
   1st order, trying the canonical coordinates of the invariance group
   <- 1st order, canonical coordinates successful
   <- homogeneous successful`
```

✓ Solution by Maple

Time used: 0.046 (sec). Leaf size: 55

```
dsolve(diff(y(x),x)^2=1/(x*y(x)^3),y(x), singsol=all)
```

$$\frac{\sqrt{xy(x)}\,y(x)^2 - c_1\sqrt{x} - 5x}{\sqrt{x}} = 0$$
$$\frac{\sqrt{xy(x)}\,y(x)^2 - c_1\sqrt{x} + 5x}{\sqrt{x}} = 0$$

✓ Solution by Mathematica

Time used: 0.112 (sec). Leaf size: 53

```
DSolve[(y'[x])^2==1/(x*y[x]^3),y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \left(\frac{5}{2}\right)^{2/5} \left(-2\sqrt{x}+c_1\right)^{2/5}$$
$$y(x) \to \left(\frac{5}{2}\right)^{2/5} \left(2\sqrt{x}+c_1\right)^{2/5}$$

1.57 problem 57

Internal problem ID [7373]
Internal file name [`OUTPUT/6354_Sunday_June_05_2022_04_41_22_PM_83742797/index.tex`]

Book: First order enumerated odes
Section: section 1
Problem number: 57.
ODE order: 1.
ODE degree: 2.

The type(s) of ODE detected by this program : **"first_order_nonlinear_p_but_separable"**

Maple gives the following as the ode type

`[_separable]`

$$\boxed{y'^2 - \frac{1}{y^3 x^2} = 0}$$

1.57.1 Solving as first order nonlinear p but separable ode

The ode has the form

$$(y')^{\frac{n}{m}} = f(x)g(y) \tag{1}$$

Where $n = 2, m = 1, f = \frac{1}{x^2}, g = \frac{1}{y^3}$. Hence the ode is

$$(y')^2 = \frac{1}{y^3 x^2}$$

Solving for y' from (1) gives

$$y' = \sqrt{fg}$$
$$y' = -\sqrt{fg}$$

To be able to solve as separable ode, we have to now assume that $f > 0, g > 0$.

$$\frac{1}{x^2} > 0$$
$$\frac{1}{y^3} > 0$$

Under the above assumption the differential equations become separable and can be written as

$$y' = \sqrt{f}\,\sqrt{g}$$
$$y' = -\sqrt{f}\,\sqrt{g}$$

Therefore

$$\frac{1}{\sqrt{g}}\,dy = \left(\sqrt{f}\right)\,dx$$
$$-\frac{1}{\sqrt{g}}\,dy = \left(\sqrt{f}\right)\,dx$$

Replacing $f(x), g(y)$ by their values gives

$$\frac{1}{\sqrt{\frac{1}{y^3}}}\,dy = \left(\sqrt{\frac{1}{x^2}}\right)\,dx$$
$$-\frac{1}{\sqrt{\frac{1}{y^3}}}\,dy = \left(\sqrt{\frac{1}{x^2}}\right)\,dx$$

Integrating now gives the solutions.

$$\int \frac{1}{\sqrt{\frac{1}{y^3}}}dy = \int \sqrt{\frac{1}{x^2}}dx + c_1$$
$$\int -\frac{1}{\sqrt{\frac{1}{y^3}}}dy = \int \sqrt{\frac{1}{x^2}}dx + c_1$$

Integrating gives

$$\frac{2y}{5\sqrt{\frac{1}{y^3}}} = \sqrt{\frac{1}{x^2}}\,x\ln(x) + c_1$$
$$-\frac{2y}{5\sqrt{\frac{1}{y^3}}} = \sqrt{\frac{1}{x^2}}\,x\ln(x) + c_1$$

Therefore

$$\frac{2y}{5\sqrt{\frac{1}{y^3}}} = \sqrt{\frac{1}{x^2}}\,x\ln(x) + c_1$$

$$-\frac{2y}{5\sqrt{\frac{1}{y^3}}} = \sqrt{\frac{1}{x^2}}\,x\ln(x) + c_1$$

Summary
The solution(s) found are the following

$$\frac{2y}{5\sqrt{\frac{1}{y^3}}} = \sqrt{\frac{1}{x^2}}\, x \ln(x) + c_1 \tag{1}$$

$$-\frac{2y}{5\sqrt{\frac{1}{y^3}}} = \sqrt{\frac{1}{x^2}}\, x \ln(x) + c_1 \tag{2}$$

Verification of solutions

$$\frac{2y}{5\sqrt{\frac{1}{y^3}}} = \sqrt{\frac{1}{x^2}}\, x \ln(x) + c_1$$

Verified OK. `{0 < 1/x^2, 0 < 1/y^3}`

$$-\frac{2y}{5\sqrt{\frac{1}{y^3}}} = \sqrt{\frac{1}{x^2}}\, x \ln(x) + c_1$$

Verified OK. `{0 < 1/x^2, 0 < 1/y^3}`

1.57.2 Maple step by step solution

Let's solve

$y'^2 - \frac{1}{y^3 x^2} = 0$

- Highest derivative means the order of the ODE is 1

  y'

- Separate variables

  $y' y^{\frac{3}{2}} = \frac{1}{x}$

- Integrate both sides with respect to x

  $\int y' y^{\frac{3}{2}} dx = \int \frac{1}{x} dx + c_1$

- Evaluate integral

  $\frac{2y^{\frac{5}{2}}}{5} = \ln(x) + c_1$

- Solve for y

  $y = \frac{(80\ln(x) + 80c_1)^{\frac{2}{5}}}{4}$

Maple trace

```
`Methods for first order ODEs:
-> Solving 1st order ODE of high degree, 1st attempt
trying 1st order WeierstrassP solution for high degree ODE
trying 1st order WeierstrassPPrime solution for high degree ODE
trying 1st order JacobiSN solution for high degree ODE
trying 1st order ODE linearizable_by_differentiation
trying differential order: 1; missing variables
trying simple symmetries for implicit equations
<- symmetries for implicit equations successful`
```

✓ Solution by Maple

Time used: 0.047 (sec). Leaf size: 29

```
dsolve(diff(y(x),x)^2=1/(x^2*y(x)^3),y(x), singsol=all)
```

$$\ln(x) - \frac{2y(x)^{\frac{5}{2}}}{5} - c_1 = 0$$
$$\ln(x) + \frac{2y(x)^{\frac{5}{2}}}{5} - c_1 = 0$$

✓ Solution by Mathematica

Time used: 0.139 (sec). Leaf size: 45

```
DSolve[(y'[x])^2==1/(x^2*y[x]^3),y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \left(\frac{5}{2}\right)^{2/5} (-\log(x) + c_1)^{2/5}$$
$$y(x) \to \left(\frac{5}{2}\right)^{2/5} (\log(x) + c_1)^{2/5}$$

1.58 problem 58

Internal problem ID [7374]
Internal file name [`OUTPUT/6355_Sunday_June_05_2022_04_41_26_PM_32107100/index.tex`]

Book: First order enumerated odes
Section: section 1
Problem number: 58.
ODE order: 1.
ODE degree: 4.

The type(s) of ODE detected by this program : **"first_order_nonlinear_p_but_separable"**

Maple gives the following as the ode type

```
[[_homogeneous, `class G`], _rational]
```

$$\boxed{y'^4 - \frac{1}{y^3 x} = 0}$$

1.58.1 Solving as first order nonlinear p but separable ode

The ode has the form

$$(y')^{\frac{n}{m}} = f(x)g(y) \tag{1}$$

Where $n = 4, m = 1, f = \frac{1}{x}, g = \frac{1}{y^3}$. Hence the ode is

$$(y')^4 = \frac{1}{y^3 x}$$

Solving for y' from (1) gives

$$y' = (fg)^{\frac{1}{4}}$$
$$y' = i(fg)^{\frac{1}{4}}$$
$$y' = -(fg)^{\frac{1}{4}}$$
$$y' = -i(fg)^{\frac{1}{4}}$$

To be able to solve as separable ode, we have to now assume that $f > 0, g > 0$.

$$\frac{1}{x} > 0$$
$$\frac{1}{y^3} > 0$$

Under the above assumption the differential equations become separable and can be written as

$$y' = f^{\frac{1}{4}} g^{\frac{1}{4}}$$
$$y' = i f^{\frac{1}{4}} g^{\frac{1}{4}}$$
$$y' = -f^{\frac{1}{4}} g^{\frac{1}{4}}$$
$$y' = -i f^{\frac{1}{4}} g^{\frac{1}{4}}$$

Therefore

$$\frac{1}{g^{\frac{1}{4}}} \, dy = \left(f^{\frac{1}{4}}\right) \, dx$$
$$-\frac{i}{g^{\frac{1}{4}}} \, dy = \left(f^{\frac{1}{4}}\right) \, dx$$
$$-\frac{1}{g^{\frac{1}{4}}} \, dy = \left(f^{\frac{1}{4}}\right) \, dx$$
$$\frac{i}{g^{\frac{1}{4}}} \, dy = \left(f^{\frac{1}{4}}\right) \, dx$$

Replacing $f(x), g(y)$ by their values gives

$$\frac{1}{\left(\frac{1}{y^3}\right)^{\frac{1}{4}}} \, dy = \left(\left(\frac{1}{x}\right)^{\frac{1}{4}}\right) \, dx$$
$$-\frac{i}{\left(\frac{1}{y^3}\right)^{\frac{1}{4}}} \, dy = \left(\left(\frac{1}{x}\right)^{\frac{1}{4}}\right) \, dx$$
$$-\frac{1}{\left(\frac{1}{y^3}\right)^{\frac{1}{4}}} \, dy = \left(\left(\frac{1}{x}\right)^{\frac{1}{4}}\right) \, dx$$
$$\frac{i}{\left(\frac{1}{y^3}\right)^{\frac{1}{4}}} \, dy = \left(\left(\frac{1}{x}\right)^{\frac{1}{4}}\right) \, dx$$

Integrating now gives the solutions.

$$\int \frac{1}{\left(\frac{1}{y^3}\right)^{\frac{1}{4}}} dy = \int \left(\frac{1}{x}\right)^{\frac{1}{4}} dx + c_1$$

$$\int -\frac{i}{\left(\frac{1}{y^3}\right)^{\frac{1}{4}}} dy = \int \left(\frac{1}{x}\right)^{\frac{1}{4}} dx + c_1$$

$$\int -\frac{1}{\left(\frac{1}{y^3}\right)^{\frac{1}{4}}} dy = \int \left(\frac{1}{x}\right)^{\frac{1}{4}} dx + c_1$$

$$\int \frac{i}{\left(\frac{1}{y^3}\right)^{\frac{1}{4}}} dy = \int \left(\frac{1}{x}\right)^{\frac{1}{4}} dx + c_1$$

Integrating gives

$$\frac{4y}{7\left(\frac{1}{y^3}\right)^{\frac{1}{4}}} = \frac{4x\left(\frac{1}{x}\right)^{\frac{1}{4}}}{3} + c_1$$

$$-\frac{4iy}{7\left(\frac{1}{y^3}\right)^{\frac{1}{4}}} = \frac{4x\left(\frac{1}{x}\right)^{\frac{1}{4}}}{3} + c_1$$

$$-\frac{4y}{7\left(\frac{1}{y^3}\right)^{\frac{1}{4}}} = \frac{4x\left(\frac{1}{x}\right)^{\frac{1}{4}}}{3} + c_1$$

$$\frac{4iy}{7\left(\frac{1}{y^3}\right)^{\frac{1}{4}}} = \frac{4x\left(\frac{1}{x}\right)^{\frac{1}{4}}}{3} + c_1$$

Therefore

$$\frac{4y}{7\left(\frac{1}{y^3}\right)^{\frac{1}{4}}} = \frac{4x\left(\frac{1}{x}\right)^{\frac{1}{4}}}{3} + c_1$$

$$-\frac{4iy}{7\left(\frac{1}{y^3}\right)^{\frac{1}{4}}} = \frac{4x\left(\frac{1}{x}\right)^{\frac{1}{4}}}{3} + c_1$$

$$-\frac{4y}{7\left(\frac{1}{y^3}\right)^{\frac{1}{4}}} = \frac{4x\left(\frac{1}{x}\right)^{\frac{1}{4}}}{3} + c_1$$

$$\frac{4iy}{7\left(\frac{1}{y^3}\right)^{\frac{1}{4}}} = \frac{4x\left(\frac{1}{x}\right)^{\frac{1}{4}}}{3} + c_1$$

Summary

The solution(s) found are the following

$$\frac{4y}{7\left(\frac{1}{y^3}\right)^{\frac{1}{4}}} = \frac{4x\left(\frac{1}{x}\right)^{\frac{1}{4}}}{3} + c_1 \tag{1}$$

$$-\frac{4iy}{7\left(\frac{1}{y^3}\right)^{\frac{1}{4}}} = \frac{4x\left(\frac{1}{x}\right)^{\frac{1}{4}}}{3} + c_1 \tag{2}$$

$$-\frac{4y}{7\left(\frac{1}{y^3}\right)^{\frac{1}{4}}} = \frac{4x\left(\frac{1}{x}\right)^{\frac{1}{4}}}{3} + c_1 \tag{3}$$

$$\frac{4iy}{7\left(\frac{1}{y^3}\right)^{\frac{1}{4}}} = \frac{4x\left(\frac{1}{x}\right)^{\frac{1}{4}}}{3} + c_1 \tag{4}$$

Verification of solutions

$$\frac{4y}{7\left(\frac{1}{y^3}\right)^{\frac{1}{4}}} = \frac{4x\left(\frac{1}{x}\right)^{\frac{1}{4}}}{3} + c_1$$

Verified OK. `{0 < 1/x, 0 < 1/y^3}`

$$-\frac{4iy}{7\left(\frac{1}{y^3}\right)^{\frac{1}{4}}} = \frac{4x\left(\frac{1}{x}\right)^{\frac{1}{4}}}{3} + c_1$$

Verified OK. `{0 < 1/x, 0 < 1/y^3}`

$$-\frac{4y}{7\left(\frac{1}{y^3}\right)^{\frac{1}{4}}} = \frac{4x\left(\frac{1}{x}\right)^{\frac{1}{4}}}{3} + c_1$$

Verified OK. `{0 < 1/x, 0 < 1/y^3}`

$$\frac{4iy}{7\left(\frac{1}{y^3}\right)^{\frac{1}{4}}} = \frac{4x\left(\frac{1}{x}\right)^{\frac{1}{4}}}{3} + c_1$$

Verified OK. `{0 < 1/x, 0 < 1/y^3}`

Maple trace

```
`Methods for first order ODEs:
-> Solving 1st order ODE of high degree, 1st attempt
trying 1st order WeierstrassP solution for high degree ODE
trying 1st order WeierstrassPPrime solution for high degree ODE
trying 1st order JacobiSN solution for high degree ODE
trying 1st order ODE linearizable_by_differentiation
trying differential order: 1; missing variables
trying simple symmetries for implicit equations
Successful isolation of dy/dx: 4 solutions were found. Trying to solve each resulting ODE.
   *** Sublevel 2 ***
   Methods for first order ODEs:
   --- Trying classification methods ---
   trying homogeneous types:
   trying homogeneous G
   1st order, trying the canonical coordinates of the invariance group
   <- 1st order, canonical coordinates successful
   <- homogeneous successful
-------------------
* Tackling next ODE.
   *** Sublevel 2 ***
   Methods for first order ODEs:
   --- Trying classification methods ---
   trying homogeneous types:
   trying homogeneous G
   1st order, trying the canonical coordinates of the invariance group
   <- 1st order, canonical coordinates successful
   <- homogeneous successful
-------------------
* Tackling next ODE.
   *** Sublevel 2 ***
   Methods for first order ODEs:
   --- Trying classification methods ---
   trying homogeneous types:
   trying homogeneous G
   1st order, trying the canonical coordinates of the invariance group
   <- 1st order, canonical coordinates successful
   <- homogeneous successful
-------------------
* Tackling next ODE.
   *** Sublevel 2 ***
   Methods for first order ODEs:
   --- Trying classification methods ---
   trying homogeneous types:
   trying homogeneous G
   1st order, trying the canonical coordinates of the invariance group
```

✓ Solution by Maple

Time used: 0.063 (sec). Leaf size: 123

```
dsolve(diff(y(x),x)^4=1/(x*y(x)^3),y(x), singsol=all)
```

$$-\frac{7x^3 - 3y(x)\left(x^3 y(x)\right)^{\frac{3}{4}} + c_1 x^{\frac{9}{4}}}{x^{\frac{9}{4}}} = 0$$

$$\frac{-7x^3 + 3iy(x)\left(x^3 y(x)\right)^{\frac{3}{4}} - c_1 x^{\frac{9}{4}}}{x^{\frac{9}{4}}} = 0$$

$$\frac{7x^3 + 3iy(x)\left(x^3 y(x)\right)^{\frac{3}{4}} - c_1 x^{\frac{9}{4}}}{x^{\frac{9}{4}}} = 0$$

$$\frac{7x^3 + 3y(x)\left(x^3 y(x)\right)^{\frac{3}{4}} - c_1 x^{\frac{9}{4}}}{x^{\frac{9}{4}}} = 0$$

✓ Solution by Mathematica

Time used: 7.225 (sec). Leaf size: 129

```
DSolve[(y'[x])^4==1/(x*y[x]^3),y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \frac{\left(-\frac{28x^{3/4}}{3} + 7c_1\right)^{4/7}}{2\sqrt[7]{2}}$$

$$y(x) \to \frac{\left(7c_1 - \frac{28}{3}ix^{3/4}\right)^{4/7}}{2\sqrt[7]{2}}$$

$$y(x) \to \frac{\left(\frac{28}{3}ix^{3/4} + 7c_1\right)^{4/7}}{2\sqrt[7]{2}}$$

$$y(x) \to \frac{\left(\frac{28x^{3/4}}{3} + 7c_1\right)^{4/7}}{2\sqrt[7]{2}}$$

1.59 problem 59

Internal problem ID [7375]
Internal file name [`OUTPUT/6356_Sunday_June_05_2022_04_41_31_PM_62105651/index.tex`]

Book: First order enumerated odes
Section: section 1
Problem number: 59.
ODE order: 1.
ODE degree: 2.

The type(s) of ODE detected by this program : **"first_order_nonlinear_p_but_separable"**

Maple gives the following as the ode type

`[_separable]`

$$\boxed{y'^2 - \frac{1}{x^3 y^4} = 0}$$

1.59.1 Solving as first order nonlinear p but separable ode

The ode has the form

$$(y')^{\frac{n}{m}} = f(x)g(y) \tag{1}$$

Where $n = 2, m = 1, f = \frac{1}{x^3}, g = \frac{1}{y^4}$. Hence the ode is

$$(y')^2 = \frac{1}{x^3 y^4}$$

Solving for y' from (1) gives

$$y' = \sqrt{fg}$$
$$y' = -\sqrt{fg}$$

To be able to solve as separable ode, we have to now assume that $f > 0, g > 0$.

$$\frac{1}{x^3} > 0$$
$$\frac{1}{y^4} > 0$$

Under the above assumption the differential equations become separable and can be written as

$$\begin{aligned} y' &= \sqrt{f}\,\sqrt{g} \\ y' &= -\sqrt{f}\,\sqrt{g} \end{aligned}$$

Therefore

$$\begin{aligned} \frac{1}{\sqrt{g}}\,dy &= \left(\sqrt{f}\right)\,dx \\ -\frac{1}{\sqrt{g}}\,dy &= \left(\sqrt{f}\right)\,dx \end{aligned}$$

Replacing $f(x), g(y)$ by their values gives

$$\begin{aligned} \frac{1}{\sqrt{\frac{1}{y^4}}}\,dy &= \left(\sqrt{\frac{1}{x^3}}\right)\,dx \\ -\frac{1}{\sqrt{\frac{1}{y^4}}}\,dy &= \left(\sqrt{\frac{1}{x^3}}\right)\,dx \end{aligned}$$

Integrating now gives the solutions.

$$\begin{aligned} \int \frac{1}{\sqrt{\frac{1}{y^4}}}dy &= \int \sqrt{\frac{1}{x^3}}dx + c_1 \\ \int -\frac{1}{\sqrt{\frac{1}{y^4}}}dy &= \int \sqrt{\frac{1}{x^3}}dx + c_1 \end{aligned}$$

Integrating gives

$$\begin{aligned} \frac{y}{3\sqrt{\frac{1}{y^4}}} &= -2x\sqrt{\frac{1}{x^3}} + c_1 \\ -\frac{y}{3\sqrt{\frac{1}{y^4}}} &= -2x\sqrt{\frac{1}{x^3}} + c_1 \end{aligned}$$

Therefore

$$\frac{y}{3\sqrt{\frac{1}{y^4}}} = -2x\sqrt{\frac{1}{x^3}} + c_1$$

$$-\frac{y}{3\sqrt{\frac{1}{y^4}}} = -2x\sqrt{\frac{1}{x^3}} + c_1$$

Summary
The solution(s) found are the following

$$\frac{y}{3\sqrt{\frac{1}{y^4}}} = -2x\sqrt{\frac{1}{x^3}} + c_1 \tag{1}$$

$$-\frac{y}{3\sqrt{\frac{1}{y^4}}} = -2x\sqrt{\frac{1}{x^3}} + c_1 \tag{2}$$

Verification of solutions

$$\frac{y}{3\sqrt{\frac{1}{y^4}}} = -2x\sqrt{\frac{1}{x^3}} + c_1$$

Verified OK. `{0 < 1/x^3, 0 < 1/y^4}`

$$-\frac{y}{3\sqrt{\frac{1}{y^4}}} = -2x\sqrt{\frac{1}{x^3}} + c_1$$

Verified OK. `{0 < 1/x^3, 0 < 1/y^4}`

1.59.2 Maple step by step solution

Let's solve

$y'^2 - \frac{1}{x^3 y^4} = 0$

- Highest derivative means the order of the ODE is 1

  y'

- Separate variables

  $y' y^2 = \frac{1}{x^{\frac{3}{2}}}$

- Integrate both sides with respect to x

  $\int y' y^2 dx = \int \frac{1}{x^{\frac{3}{2}}} dx + c_1$

- Evaluate integral

  $\frac{y^3}{3} = -\frac{2}{\sqrt{x}} + c_1$

- Solve for y

  $y = \left(\frac{3c_1\sqrt{x}-6}{\sqrt{x}}\right)^{\frac{1}{3}}$

Maple trace

```
`Methods for first order ODEs:
-> Solving 1st order ODE of high degree, 1st attempt
trying 1st order WeierstrassP solution for high degree ODE
trying 1st order WeierstrassPPrime solution for high degree ODE
trying 1st order JacobiSN solution for high degree ODE
trying 1st order ODE linearizable_by_differentiation
trying differential order: 1; missing variables
trying simple symmetries for implicit equations
Successful isolation of dy/dx: 2 solutions were found. Trying to solve each resulting ODE.
   *** Sublevel 2 ***
   Methods for first order ODEs:
   --- Trying classification methods ---
   trying a quadrature
   trying 1st order linear
   trying Bernoulli
   <- Bernoulli successful
-------------------
* Tackling next ODE.
   *** Sublevel 2 ***
   Methods for first order ODEs:
   --- Trying classification methods ---
   trying a quadrature
   trying 1st order linear
   trying Bernoulli
   <- Bernoulli successful`
```

✓ Solution by Maple

Time used: 0.047 (sec). Leaf size: 137

```
dsolve(diff(y(x),x)^2=1/(x^3*y(x)^4),y(x), singsol=all)
```

$$y(x) = \left(\frac{c_1\sqrt{x}-6}{\sqrt{x}}\right)^{\frac{1}{3}}$$

$$y(x) = -\frac{\left(\frac{c_1\sqrt{x}-6}{\sqrt{x}}\right)^{\frac{1}{3}}\left(1+i\sqrt{3}\right)}{2}$$

$$y(x) = \frac{\left(\frac{c_1\sqrt{x}-6}{\sqrt{x}}\right)^{\frac{1}{3}}\left(i\sqrt{3}-1\right)}{2}$$

$$y(x) = \left(\frac{c_1\sqrt{x}+6}{\sqrt{x}}\right)^{\frac{1}{3}}$$

$$y(x) = -\frac{\left(\frac{c_1\sqrt{x}+6}{\sqrt{x}}\right)^{\frac{1}{3}}\left(1+i\sqrt{3}\right)}{2}$$

$$y(x) = \frac{\left(\frac{c_1\sqrt{x}+6}{\sqrt{x}}\right)^{\frac{1}{3}}\left(i\sqrt{3}-1\right)}{2}$$

✓ Solution by Mathematica

Time used: 3.775 (sec). Leaf size: 157

```
DSolve[(y'[x])^2==1/(x^3*y[x]^4),y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to -\sqrt[3]{-3}\sqrt[3]{-\frac{2}{\sqrt{x}}+c_1}$$

$$y(x) \to \sqrt[3]{3}\sqrt[3]{-\frac{2}{\sqrt{x}}+c_1}$$

$$y(x) \to (-1)^{2/3}\sqrt[3]{3}\sqrt[3]{-\frac{2}{\sqrt{x}}+c_1}$$

$$y(x) \to -\sqrt[3]{-3}\sqrt[3]{\frac{2}{\sqrt{x}}+c_1}$$

$$y(x) \to \sqrt[3]{3}\sqrt[3]{\frac{2}{\sqrt{x}}+c_1}$$

$$y(x) \to (-1)^{2/3}\sqrt[3]{3}\sqrt[3]{\frac{2}{\sqrt{x}}+c_1}$$

1.60 problem 60

Internal problem ID [7376]
Internal file name [`OUTPUT/6356_Wednesday_July_13_2022_06_14_16_PM_9550685/index.tex`]

Book: First order enumerated odes
Section: section 1
Problem number: 60.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"homogeneousTypeC", "first_order_ode_lie_symmetry_lookup"**

Maple gives the following as the ode type

```
[[_homogeneous, `class C`], _dAlembert]
```

$$\boxed{y' - \sqrt{1 + 6x + y} = 0}$$

1.60.1 Solving as homogeneousTypeC ode

Let

$$z = 1 + 6x + y \tag{1}$$

Then

$$z'(x) = 6 + y'$$

Therefore

$$y' = z'(x) - 6$$

Hence the given ode can now be written as

$$z'(x) - 6 = \sqrt{z}$$

This is separable first order ode. Integrating

$$
\begin{aligned}
\int dx &= \int \frac{1}{\sqrt{z}+6} dz \\
x + c_1 &= 2\sqrt{z} - 6\ln\left(\sqrt{z}+6\right) + 6\ln\left(-6+\sqrt{z}\right) - 6\ln\left(-36+z\right)
\end{aligned}
$$

Replacing z back by its value from (1) then the above gives the solution as

$$
2\sqrt{1+6x+y} - 6\ln\left(\sqrt{1+6x+y}+6\right) \\
+ 6\ln\left(-6+\sqrt{1+6x+y}\right) - 6\ln\left(-35+6x+y\right) = x + c_1
$$

Summary

The solution(s) found are the following

$$
2\sqrt{1+6x+y} - 6\ln\left(\sqrt{1+6x+y}+6\right) \\
+ 6\ln\left(-6+\sqrt{1+6x+y}\right) - 6\ln\left(-35+6x+y\right) = x + c_1 \tag{1}
$$

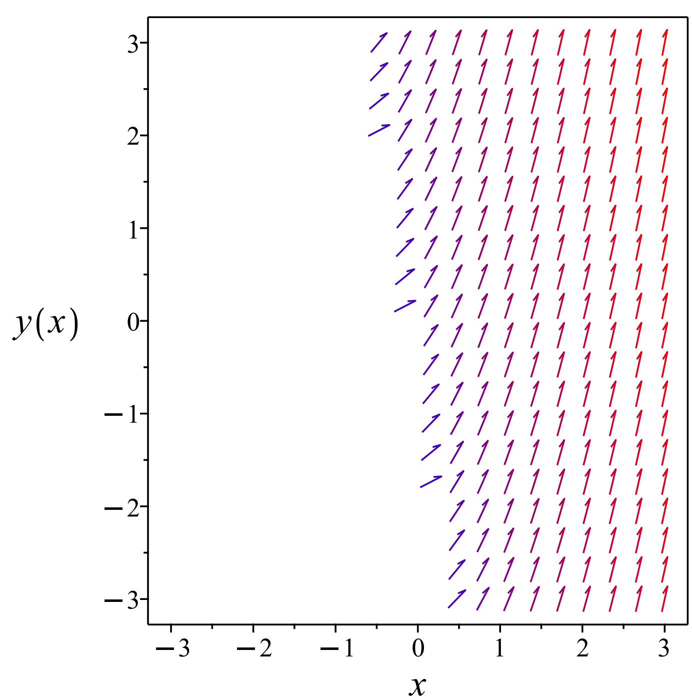


Figure 33: Slope field plot

Verification of solutions

$$
\begin{aligned}
&2\sqrt{1+6x+y}-6\ln\left(\sqrt{1+6x+y}+6\right)\\
&+6\ln\left(-6+\sqrt{1+6x+y}\right)-6\ln\left(-35+6x+y\right)=x+c_1
\end{aligned}
$$

Verified OK.

1.60.2 Solving as first order ode lie symmetry lookup ode

Writing the ode as

$$
\begin{aligned}
y' &= \sqrt{1+6x+y}\\
y' &= \omega(x,y)
\end{aligned}
$$

The condition of Lie symmetry is the linearized PDE given by

$$
\eta_x+\omega(\eta_y-\xi_x)-\omega^2\xi_y-\omega_x\xi-\omega_y\eta=0 \tag{A}
$$

The type of this ode is known. It is of type `homogeneous Type C`. Therefore we do not need to solve the PDE (A), and can just use the lookup table shown below to find ξ,η

Table 60: Lie symmetry infinitesimal lookup table for known first order ODE's

ODE class	Form	ξ	η
linear ode	$y' = f(x)y(x) + g(x)$	0	$e^{\int f dx}$
separable ode	$y' = f(x)\, g(y)$	$\frac{1}{f}$	0
quadrature ode	$y' = f(x)$	0	1
quadrature ode	$y' = g(y)$	1	0
homogeneous ODEs of Class A	$y' = f\left(\frac{y}{x}\right)$	x	y
homogeneous ODEs of Class C	$y' = (a + bx + cy)^{\frac{n}{m}}$	1	$-\frac{b}{c}$
homogeneous class D	$y' = \frac{y}{x} + g(x)\, F\left(\frac{y}{x}\right)$	x^2	xy
First order special form ID 1	$y' = g(x)\, e^{h(x)+by} + f(x)$	$\frac{e^{-\int bf(x)dx - h(x)}}{g(x)}$	$\frac{f(x)e^{-\int bf(x)dx - h(x)}}{g(x)}$
polynomial type ode	$y' = \frac{a_1x+b_1y+c_1}{a_2x+b_2y+c_2}$	$\frac{a_1b_2x - a_2b_1x - b_1c_2 + b_2c_1}{a_1b_2 - a_2b_1}$	$\frac{a_1b_2y - a_2b_1y - a_1c_2 - a_2c_1}{a_1b_2 - a_2b_1}$
Bernoulli ode	$y' = f(x)\, y + g(x)\, y^n$	0	$e^{-\int (n-1)f(x)dx} y^n$
Reduced Riccati	$y' = f_1(x)\, y + f_2(x)\, y^2$	0	$e^{-\int f_1 dx}$

The above table shows that

$$\begin{aligned} \xi(x,y) &= 1 \\ \eta(x,y) &= -6 \end{aligned} \tag{A1}$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x,y) \to (R,S)$ where (R,S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y}\right) S(x,y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the

canonical coordinates, where $S(R)$. Therefore

$$\begin{aligned}\frac{dy}{dx} &= \frac{\eta}{\xi} \\ &= \frac{-6}{1} \\ &= -6\end{aligned}$$

This is easily solved to give

$$y = -6x + c_1$$

Where now the coordinate R is taken as the constant of integration. Hence

$$R = 6x + y$$

And S is found from

$$\begin{aligned}dS &= \frac{dx}{\xi} \\ &= \frac{dx}{1}\end{aligned}$$

Integrating gives

$$\begin{aligned}S &= \int \frac{dx}{T} \\ &= x\end{aligned}$$

Where the constant of integration is set to zero as we just need one solution. Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x, y)S_y}{R_x + \omega(x, y)R_y} \tag{2}$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x, y)$ is the right hand side of the original ode given by

$$\omega(x, y) = \sqrt{1 + 6x + y}$$

Evaluating all the partial derivatives gives

$$\begin{aligned}R_x &= 6 \\ R_y &= 1 \\ S_x &= 1 \\ S_y &= 0\end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = \frac{1}{\sqrt{1+6x+y}+6} \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = \frac{1}{\sqrt{1+R}+6}$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$S(R) = 2\sqrt{1+R} - 6\ln\left(\sqrt{1+R}+6\right) + 6\ln\left(-6+\sqrt{1+R}\right) - 6\ln\left(-35+R\right) + c_1 \tag{4}$$

To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$x = 2\sqrt{1+6x+y} - 6\ln\left(\sqrt{1+6x+y}+6\right) + 6\ln\left(-6+\sqrt{1+6x+y}\right) - 6\ln\left(-35+6x+y\right) + c_1$$

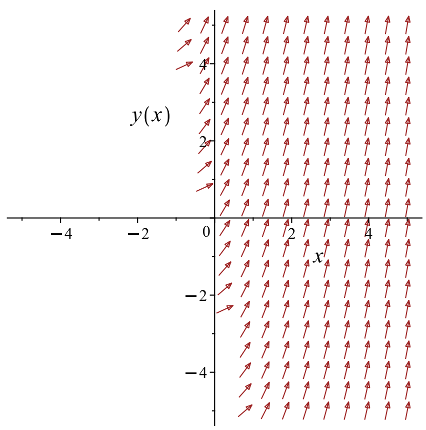
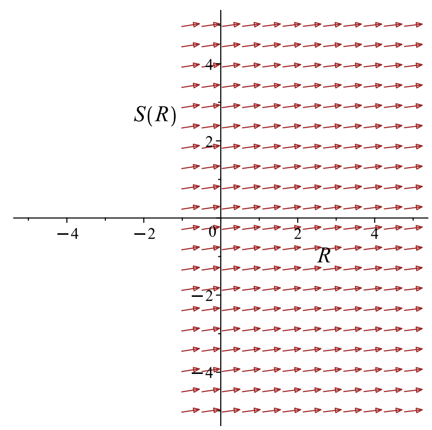
Which simplifies to

$$x = 2\sqrt{1+6x+y} - 6\ln\left(\sqrt{1+6x+y}+6\right) + 6\ln\left(-6+\sqrt{1+6x+y}\right) - 6\ln\left(-35+6x+y\right) + c_1$$

Which gives

$$y = \mathrm{e}^{-2\,\mathrm{LambertW}\left(-\frac{\mathrm{e}^{\frac{c_1}{12}-\frac{x}{12}-1}}{6}\right)+\frac{c_1}{6}-\frac{x}{6}-2} - 12\,\mathrm{e}^{-\mathrm{LambertW}\left(-\frac{\mathrm{e}^{\frac{c_1}{12}-\frac{x}{12}-1}}{6}\right)+\frac{c_1}{12}-\frac{x}{12}-1} - 6x + 35$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in x, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dx} = \sqrt{1+6x+y}$ 	$R = 6x + y$ $S = x$	$\frac{dS}{dR} = \frac{1}{\sqrt{1+R}+6}$ 

Summary
The solution(s) found are the following

$$y = e^{-2\,\mathrm{LambertW}\left(-\frac{e^{\frac{c_1}{12}-\frac{x}{12}-1}}{6}\right)+\frac{c_1}{6}-\frac{x}{6}-2} - 12\,e^{-\mathrm{LambertW}\left(-\frac{e^{\frac{c_1}{12}-\frac{x}{12}-1}}{6}\right)+\frac{c_1}{12}-\frac{x}{12}-1} - 6x + 35 \tag{15}$$

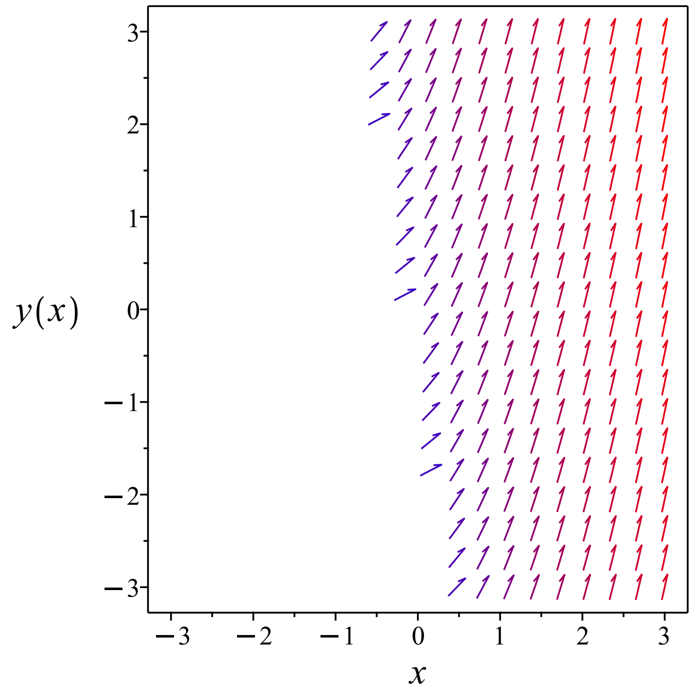


Figure 34: Slope field plot

Verification of solutions

$$y = \mathrm{e}^{-2\,\mathrm{LambertW}\left(-\frac{\mathrm{e}^{\frac{c_1}{12}-\frac{x}{12}-1}}{6}\right)+\frac{c_1}{6}-\frac{x}{6}-2} - 12\,\mathrm{e}^{-\mathrm{LambertW}\left(-\frac{\mathrm{e}^{\frac{c_1}{12}-\frac{x}{12}-1}}{6}\right)+\frac{c_1}{12}-\frac{x}{12}-1} - 6x + 35$$

Verified OK.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying homogeneous types:
trying homogeneous C
1st order, trying the canonical coordinates of the invariance group
   -> Calling odsolve with the ODE`, diff(y(x), x) = -6, y(x)`      *** Sublevel 2 ***
      Methods for first order ODEs:
      --- Trying classification methods ---
      trying a quadrature
      trying 1st order linear
      <- 1st order linear successful
<- 1st order, canonical coordinates successful
<- homogeneous successful`
```

✓ Solution by Maple

Time used: 0.015 (sec). Leaf size: 57

```
dsolve(diff(y(x),x)=(1+6*x+y(x))^(1/2),y(x), singsol=all)
```

$$x-2\sqrt{1+6x+y(x)}+6\ln\left(6+\sqrt{1+6x+y(x)}\right)-6\ln\left(-6+\sqrt{1+6x+y(x)}\right)+6\ln(-35+y(x)+6x)-c_1=0$$

✓ Solution by Mathematica

Time used: 13.35 (sec). Leaf size: 65

```
DSolve[y'[x]==(1+6*x+y[x])^(1/2),y[x],x,IncludeSingularSolutions -> True]
```

$$y(x)\to 36W\left(-\frac{1}{6}e^{\frac{1}{72}(-6x-73+6c_1)}\right)^2+72W\left(-\frac{1}{6}e^{\frac{1}{72}(-6x-73+6c_1)}\right)-6x+35$$
$$y(x)\to 35-6x$$

1.61 problem 61

Internal problem ID [7377]
Internal file name [`OUTPUT/6357_Wednesday_July_13_2022_06_14_18_PM_75656588/index.tex`]

Book: First order enumerated odes
Section: section 1
Problem number: 61.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"homogeneousTypeC", "first_order_ode_lie_symmetry_lookup"**

Maple gives the following as the ode type

```
[[_homogeneous, `class C`], _dAlembert]
```

$$\boxed{y' - (1 + 6x + y)^{\frac{1}{3}} = 0}$$

1.61.1 Solving as homogeneousTypeC ode

Let

$$z = 1 + 6x + y \tag{1}$$

Then

$$z'(x) = 6 + y'$$

Therefore

$$y' = z'(x) - 6$$

Hence the given ode can now be written as

$$z'(x) - 6 = z^{\frac{1}{3}}$$

This is separable first order ode. Integrating

$$\int dx = \int \frac{1}{z^{\frac{1}{3}} + 6} dz$$

$$x + c_1 = \frac{3z^{\frac{2}{3}}}{2} - 36\ln\left(z^{\frac{2}{3}} - 6z^{\frac{1}{3}} + 36\right) + 72\ln\left(z^{\frac{1}{3}} + 6\right) + 36\ln(216 + z) - 18z^{\frac{1}{3}}$$

Replacing z back by its value from (1) then the above gives the solution as

$$\frac{3(1+6x+y)^{\frac{2}{3}}}{2} - 36\ln\left((1+6x+y)^{\frac{2}{3}} - 6(1+6x+y)^{\frac{1}{3}} + 36\right) + 72\ln\left((1+6x+y)^{\frac{1}{3}} + 6\right) + 36\ln(217 + 6x + y) - 18(1+6x+y)^{\frac{1}{3}} = x + c_1$$

Summary

The solution(s) found are the following

$$\frac{3(1+6x+y)^{\frac{2}{3}}}{2} - 36\ln\left((1+6x+y)^{\frac{2}{3}} - 6(1+6x+y)^{\frac{1}{3}} + 36\right) + 72\ln\left((1+6x+y)^{\frac{1}{3}} + 6\right) + 36\ln(217 + 6x + y) - 18(1+6x+y)^{\frac{1}{3}} = x + c_1 \tag{1}$$

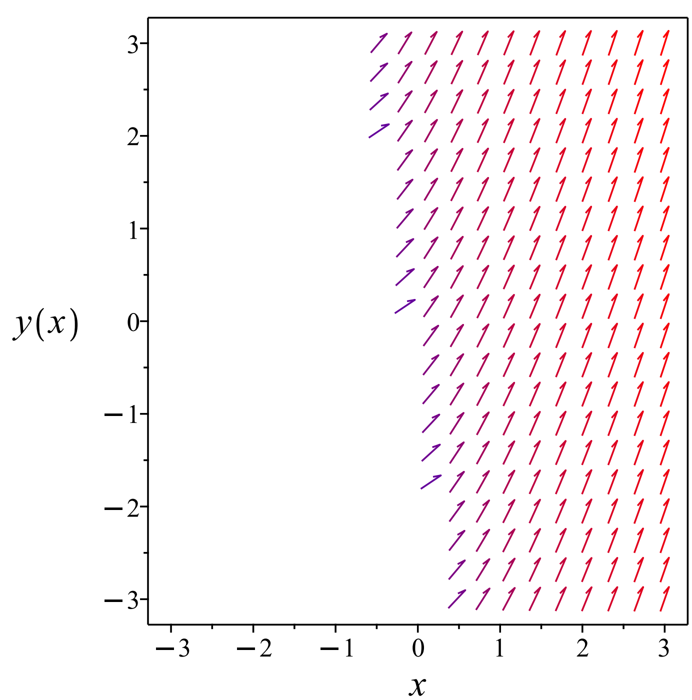


Figure 35: Slope field plot

Verification of solutions

$$
\frac{3(1+6x+y)^{\frac{2}{3}}}{2} - 36\ln\left((1+6x+y)^{\frac{2}{3}} - 6(1+6x+y)^{\frac{1}{3}} + 36\right) \\
+ 72\ln\left((1+6x+y)^{\frac{1}{3}} + 6\right) + 36\ln(217+6x+y) - 18(1+6x+y)^{\frac{1}{3}} = x + c_1
$$

Verified OK.

1.61.2 Solving as first order ode lie symmetry lookup ode

Writing the ode as

$$
\begin{aligned}
y' &= (1+6x+y)^{\frac{1}{3}} \\
y' &= \omega(x,y)
\end{aligned}
$$

The condition of Lie symmetry is the linearized PDE given by

$$
\eta_x + \omega(\eta_y - \xi_x) - \omega^2 \xi_y - \omega_x \xi - \omega_y \eta = 0 \tag{A}
$$

The type of this ode is known. It is of type `homogeneous Type C`. Therefore we do not need to solve the PDE (A), and can just use the lookup table shown below to find ξ, η

Table 62: Lie symmetry infinitesimal lookup table for known first order ODE's

ODE class	Form	ξ	η
linear ode	$y' = f(x)y(x) + g(x)$	0	$e^{\int f dx}$
separable ode	$y' = f(x)\, g(y)$	$\frac{1}{f}$	0
quadrature ode	$y' = f(x)$	0	1
quadrature ode	$y' = g(y)$	1	0
homogeneous ODEs of Class A	$y' = f\left(\frac{y}{x}\right)$	x	y
homogeneous ODEs of Class C	$y' = (a + bx + cy)^{\frac{n}{m}}$	1	$-\frac{b}{c}$
homogeneous class D	$y' = \frac{y}{x} + g(x)\, F\left(\frac{y}{x}\right)$	x^2	xy
First order special form ID 1	$y' = g(x)\, e^{h(x)+by} + f(x)$	$\frac{e^{-\int bf(x)dx - h(x)}}{g(x)}$	$\frac{f(x)e^{-\int bf(x)dx - h(x)}}{g(x)}$
polynomial type ode	$y' = \frac{a_1x+b_1y+c_1}{a_2x+b_2y+c_2}$	$\frac{a_1b_2x - a_2b_1x - b_1c_2 + b_2c_1}{a_1b_2 - a_2b_1}$	$\frac{a_1b_2y - a_2b_1y - a_1c_2 - a_2c_1}{a_1b_2 - a_2b_1}$
Bernoulli ode	$y' = f(x)\, y + g(x)\, y^n$	0	$e^{-\int (n-1)f(x)dx} y^n$
Reduced Riccati	$y' = f_1(x)\, y + f_2(x)\, y^2$	0	$e^{-\int f_1 dx}$

The above table shows that

$$
\begin{aligned}
\xi(x, y) &= 1 \\
\eta(x, y) &= -6
\end{aligned}
\tag{A1}
$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x, y) \to (R, S)$ where (R, S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$
\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}
$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y}\right) S(x, y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the

canonical coordinates, where $S(R)$. Therefore

$$\begin{aligned}\frac{dy}{dx} &= \frac{\eta}{\xi} \\ &= \frac{-6}{1} \\ &= -6\end{aligned}$$

This is easily solved to give

$$y = -6x + c_1$$

Where now the coordinate R is taken as the constant of integration. Hence

$$R = 6x + y$$

And S is found from

$$\begin{aligned}dS &= \frac{dx}{\xi} \\ &= \frac{dx}{1}\end{aligned}$$

Integrating gives

$$\begin{aligned}S &= \int \frac{dx}{T} \\ &= x\end{aligned}$$

Where the constant of integration is set to zero as we just need one solution. Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x,y)S_y}{R_x + \omega(x,y)R_y} \tag{2}$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x,y)$ is the right hand side of the original ode given by

$$\omega(x,y) = (1 + 6x + y)^{\frac{1}{3}}$$

Evaluating all the partial derivatives gives

$$\begin{aligned}R_x &= 6 \\ R_y &= 1 \\ S_x &= 1 \\ S_y &= 0\end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = \frac{1}{(1+6x+y)^{\frac{1}{3}}+6} \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = \frac{1}{(1+R)^{\frac{1}{3}}+6}$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$S(R) = \frac{3(1+R)^{\frac{2}{3}}}{2} - 36\ln\left((1+R)^{\frac{2}{3}} - 6(1+R)^{\frac{1}{3}} + 36\right) + 72\ln\left((1+R)^{\frac{1}{3}} + 6\right) + 36\ln(217+R) - 18(\tag{4}$$

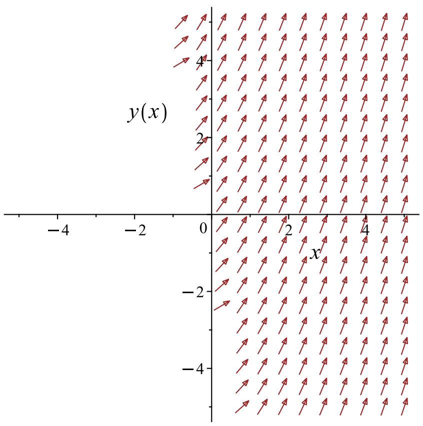
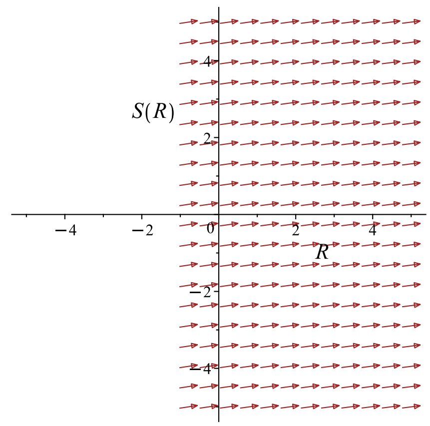
To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$x = \frac{3(1+6x+y)^{\frac{2}{3}}}{2} - 36\ln\left((1+6x+y)^{\frac{2}{3}} - 6(1+6x+y)^{\frac{1}{3}} + 36\right) + 72\ln\left((1+6x+y)^{\frac{1}{3}} + 6\right) + 36\ln($$

Which simplifies to

$$x = \frac{3(1+6x+y)^{\frac{2}{3}}}{2} - 36\ln\left((1+6x+y)^{\frac{2}{3}} - 6(1+6x+y)^{\frac{1}{3}} + 36\right) + 72\ln\left((1+6x+y)^{\frac{1}{3}} + 6\right) + 36\ln($$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in x, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dx} = (1+6x+y)^{\frac{1}{3}}$ 	$R = 6x + y$ $S = x$	$\frac{dS}{dR} = \frac{1}{(1+R)^{\frac{1}{3}}+6}$ 

Summary

The solution(s) found are the following

$$
\begin{aligned}
x = {} & \frac{3(1+6x+y)^{\frac{2}{3}}}{2} - 36\ln\left((1+6x+y)^{\frac{2}{3}} - 6(1+6x+y)^{\frac{1}{3}} + 36\right) \\
& + 72\ln\left((1+6x+y)^{\frac{1}{3}} + 6\right) + 36\ln(217+6x+y) - 18(1+6x+y)^{\frac{1}{3}} + c_1
\end{aligned} \tag{1}
$$

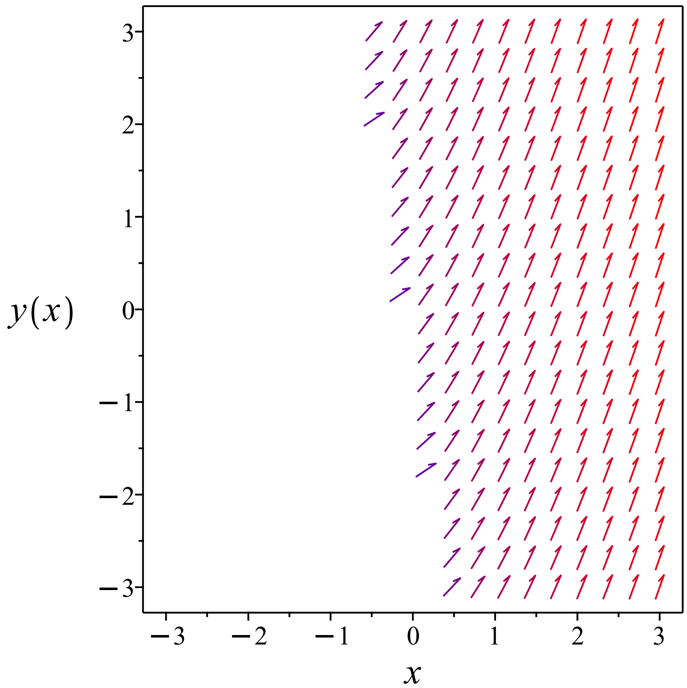


Figure 36: Slope field plot

Verification of solutions

$$x = \frac{3(1+6x+y)^{\frac{2}{3}}}{2} - 36 \ln\left((1+6x+y)^{\frac{2}{3}} - 6(1+6x+y)^{\frac{1}{3}} + 36\right) + 72 \ln\left((1+6x+y)^{\frac{1}{3}} + 6\right) + 36 \ln(217+6x+y) - 18(1+6x+y)^{\frac{1}{3}} + c_1$$

Verified OK.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying homogeneous types:
trying homogeneous C
1st order, trying the canonical coordinates of the invariance group
<- 1st order, canonical coordinates successful
<- homogeneous successful`
```

✓ Solution by Maple

Time used: 0.015 (sec). Leaf size: 79

```
dsolve(diff(y(x),x)=(1+6*x+y(x))^(1/3),y(x), singsol=all)
```

$$
\begin{aligned}
&x-\frac{3(1+6x+y(x))^{\frac{2}{3}}}{2}-72\ln\left(6+(1+6x+y(x))^{\frac{1}{3}}\right)\\
&+36\ln\left((1+6x+y(x))^{\frac{2}{3}}-6(1+6x+y(x))^{\frac{1}{3}}+36\right)\\
&-36\ln\left(217+y(x)+6x\right)+18(1+6x+y(x))^{\frac{1}{3}}-c_1=0
\end{aligned}
$$

✓ Solution by Mathematica

Time used: 0.246 (sec). Leaf size: 66

```
DSolve[y'[x]==(1+6*x+y[x])^(1/3),y[x],x,IncludeSingularSolutions -> True]
```

$$
\begin{aligned}
&\text{Solve}\left[\frac{1}{6}\left(y(x)-9(y(x)+6x+1)^{2/3}+108\sqrt[3]{y(x)+6x+1}\right.\right.\\
&\left.\left.-648\log\left(\sqrt[3]{y(x)+6x+1}+6\right)+6x+1\right)-\frac{y(x)}{6}=c_1,y(x)\right]
\end{aligned}
$$

1.62 problem 62

Internal problem ID [7378]
Internal file name [`OUTPUT/6358_Wednesday_July_13_2022_06_14_18_PM_49721640/index.tex`]

Book: First order enumerated odes
Section: section 1
Problem number: 62.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"homogeneousTypeC", "first_order_ode_lie_symmetry_lookup"**

Maple gives the following as the ode type

```
[[_homogeneous, `class C`], _dAlembert]
```

$$\boxed{y' - (1 + 6x + y)^{\frac{1}{4}} = 0}$$

1.62.1 Solving as homogeneousTypeC ode

Let

$$z = 1 + 6x + y \tag{1}$$

Then

$$z'(x) = 6 + y'$$

Therefore

$$y' = z'(x) - 6$$

Hence the given ode can now be written as

$$z'(x) - 6 = z^{\frac{1}{4}}$$

This is separable first order ode. Integrating

$$
\begin{aligned}
\int dx &= \int \frac{1}{z^{\frac{1}{4}}+6} dz \\
x+c_1 &= \frac{4z^{\frac{3}{4}}}{3} - 432\ln\left(z^{\frac{1}{4}}+6\right) + 432\ln\left(z^{\frac{1}{4}}-6\right) - 216\ln\left(z-1296\right) \\
&- 12\sqrt{z} - 216\ln\left(-36+\sqrt{z}\right) + 216\ln\left(\sqrt{z}+36\right) + 144z^{\frac{1}{4}}
\end{aligned}
$$

Replacing z back by its value from (1) then the above gives the solution as

$$
\begin{aligned}
&\frac{4(1+6x+y)^{\frac{3}{4}}}{3} - 432\ln\left((1+6x+y)^{\frac{1}{4}}+6\right) + 432\ln\left((1+6x+y)^{\frac{1}{4}}-6\right) \\
&- 216\ln\left(-1295+6x+y\right) - 12\sqrt{1+6x+y} - 216\ln\left(-36+\sqrt{1+6x+y}\right) \\
&+ 216\ln\left(\sqrt{1+6x+y}+36\right) + 144(1+6x+y)^{\frac{1}{4}} = x+c_1
\end{aligned}
$$

Summary

The solution(s) found are the following

$$
\begin{aligned}
&\frac{4(1+6x+y)^{\frac{3}{4}}}{3} - 432\ln\left((1+6x+y)^{\frac{1}{4}}+6\right) + 432\ln\left((1+6x+y)^{\frac{1}{4}}-6\right) \\
&- 216\ln\left(-1295+6x+y\right) - 12\sqrt{1+6x+y} - 216\ln\left(-36+\sqrt{1+6x+y}\right) \\
&+ 216\ln\left(\sqrt{1+6x+y}+36\right) + 144(1+6x+y)^{\frac{1}{4}} = x+c_1
\end{aligned} \tag{1}
$$

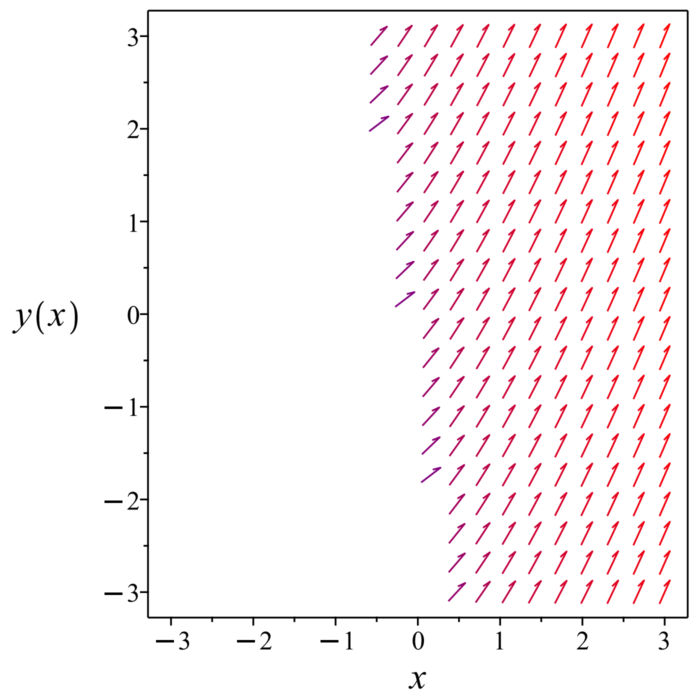


Figure 37: Slope field plot

Verification of solutions

$$\frac{4(1+6x+y)^{\frac{3}{4}}}{3} - 432\ln\left((1+6x+y)^{\frac{1}{4}}+6\right) + 432\ln\left((1+6x+y)^{\frac{1}{4}}-6\right) \\ - 216\ln(-1295+6x+y) - 12\sqrt{1+6x+y} - 216\ln\left(-36+\sqrt{1+6x+y}\right) \\ + 216\ln\left(\sqrt{1+6x+y}+36\right) + 144(1+6x+y)^{\frac{1}{4}} = x + c_1$$

Verified OK.

1.62.2 Solving as first order ode lie symmetry lookup ode

Writing the ode as

$$y' = (1+6x+y)^{\frac{1}{4}}$$
$$y' = \omega(x,y)$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x + \omega(\eta_y - \xi_x) - \omega^2\xi_y - \omega_x\xi - \omega_y\eta = 0 \tag{A}$$

The type of this ode is known. It is of type `homogeneous Type C`. Therefore we do not need to solve the PDE (A), and can just use the lookup table shown below to find ξ, η

Table 64: Lie symmetry infinitesimal lookup table for known first order ODE's

ODE class	Form	ξ	η
linear ode	$y' = f(x)y(x) + g(x)$	0	$e^{\int f dx}$
separable ode	$y' = f(x)\, g(y)$	$\frac{1}{f}$	0
quadrature ode	$y' = f(x)$	0	1
quadrature ode	$y' = g(y)$	1	0
homogeneous ODEs of Class A	$y' = f\left(\frac{y}{x}\right)$	x	y
homogeneous ODEs of Class C	$y' = (a + bx + cy)^{\frac{n}{m}}$	1	$-\frac{b}{c}$
homogeneous class D	$y' = \frac{y}{x} + g(x)\, F\left(\frac{y}{x}\right)$	x^2	xy
First order special form ID 1	$y' = g(x)\, e^{h(x)+by} + f(x)$	$\frac{e^{-\int bf(x)dx - h(x)}}{g(x)}$	$\frac{f(x)e^{-\int bf(x)dx - h(x)}}{g(x)}$
polynomial type ode	$y' = \frac{a_1x+b_1y+c_1}{a_2x+b_2y+c_2}$	$\frac{a_1b_2x - a_2b_1x - b_1c_2 + b_2c_1}{a_1b_2 - a_2b_1}$	$\frac{a_1b_2y - a_2b_1y - a_1c_2 - a_2c_1}{a_1b_2 - a_2b_1}$
Bernoulli ode	$y' = f(x)\, y + g(x)\, y^n$	0	$e^{-\int (n-1)f(x)dx} y^n$
Reduced Riccati	$y' = f_1(x)\, y + f_2(x)\, y^2$	0	$e^{-\int f_1 dx}$

The above table shows that

$$
\begin{aligned}
\xi(x, y) &= 1 \\
\eta(x, y) &= -6
\end{aligned}
\tag{A1}
$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x, y) \to (R, S)$ where (R, S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$
\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}
$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y}\right) S(x, y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the

canonical coordinates, where $S(R)$. Therefore

$$\begin{aligned}\frac{dy}{dx} &= \frac{\eta}{\xi}\\ &= \frac{-6}{1}\\ &= -6\end{aligned}$$

This is easily solved to give

$$y = -6x + c_1$$

Where now the coordinate R is taken as the constant of integration. Hence

$$R = 6x + y$$

And S is found from

$$\begin{aligned}dS &= \frac{dx}{\xi}\\ &= \frac{dx}{1}\end{aligned}$$

Integrating gives

$$\begin{aligned}S &= \int \frac{dx}{T}\\ &= x\end{aligned}$$

Where the constant of integration is set to zero as we just need one solution. Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x,y)S_y}{R_x + \omega(x,y)R_y} \tag{2}$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x,y)$ is the right hand side of the original ode given by

$$\omega(x,y) = (1 + 6x + y)^{\frac{1}{4}}$$

Evaluating all the partial derivatives gives

$$\begin{aligned}R_x &= 6\\ R_y &= 1\\ S_x &= 1\\ S_y &= 0\end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = \frac{1}{(1+6x+y)^{\frac{1}{4}} + 6} \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = \frac{1}{(1+R)^{\frac{1}{4}} + 6}$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$S(R) = -216\ln(-R+1295) - 12\sqrt{1+R} - 216\ln\left(-36+\sqrt{1+R}\right) + 216\ln\left(\sqrt{1+R}+36\right) + 144(1+ \tag{4}$$

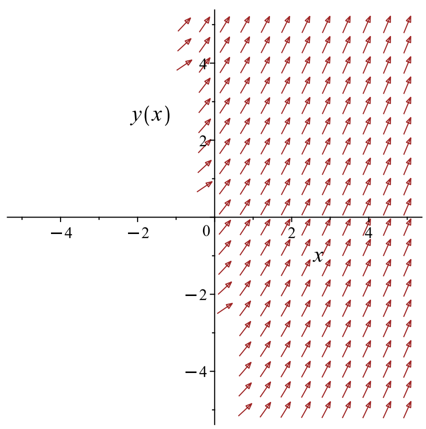
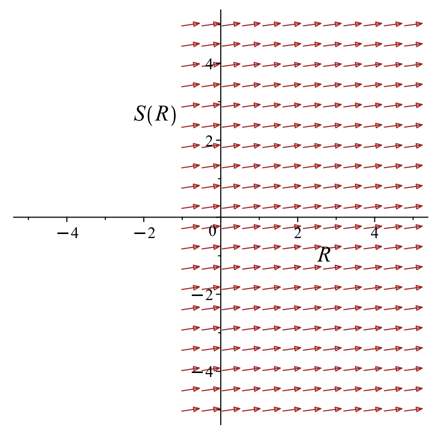
To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$x = -216\ln(-y-6x+1295) - 12\sqrt{1+6x+y} - 216\ln\left(-36+\sqrt{1+6x+y}\right) + 216\ln\left(\sqrt{1+6x+y}\right.$$

Which simplifies to

$$x = -216\ln(-y-6x+1295) - 12\sqrt{1+6x+y} - 216\ln\left(-36+\sqrt{1+6x+y}\right) + 216\ln\left(\sqrt{1+6x+y}\right.$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in x, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dx} = (1 + 6x + y)^{\frac{1}{4}}$ 	$R = 6x + y$ $S = x$	$\frac{dS}{dR} = \frac{1}{(1+R)^{\frac{1}{4}}+6}$ 

Summary

The solution(s) found are the following

$$
\begin{aligned}
x = {} & -216 \ln \left(-y - 6x + 1295\right) - 12\sqrt{1 + 6x + y} - 216 \ln \left(-36 + \sqrt{1 + 6x + y}\right) \\
& + 216 \ln \left(\sqrt{1 + 6x + y} + 36\right) + 144(1 + 6x + y)^{\frac{1}{4}} - 432 \ln \left((1 + 6x + y)^{\frac{1}{4}} + 6\right) \\
& + 432 \ln \left((1 + 6x + y)^{\frac{1}{4}} - 6\right) + \frac{4(1 + 6x + y)^{\frac{3}{4}}}{3} + c_1
\end{aligned} \tag{1}
$$

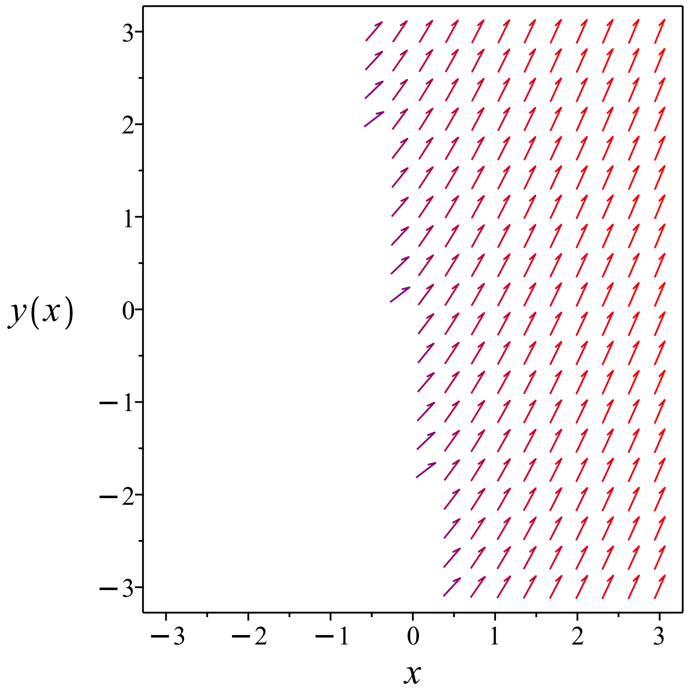


Figure 38: Slope field plot

Verification of solutions

$$x = -216\ln\left(-y-6x+1295\right)-12\sqrt{1+6x+y}-216\ln\left(-36+\sqrt{1+6x+y}\right) + 216\ln\left(\sqrt{1+6x+y}+36\right)+144(1+6x+y)^{\frac{1}{4}}-432\ln\left((1+6x+y)^{\frac{1}{4}}+6\right) + 432\ln\left((1+6x+y)^{\frac{1}{4}}-6\right)+\frac{4(1+6x+y)^{\frac{3}{4}}}{3}+c_1$$

Verified OK.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying homogeneous types:
trying homogeneous C
1st order, trying the canonical coordinates of the invariance group
<- 1st order, canonical coordinates successful
<- homogeneous successful`
```

✓ Solution by Maple

Time used: 0.015 (sec). Leaf size: 109

```
dsolve(diff(y(x),x)=(1+6*x+y(x))^(1/4),y(x), singsol=all)
```

$$x+216\ln\left(-y(x)-6x+1295\right)+12\sqrt{1+6x+y\left(x\right)}+216\ln\left(\sqrt{1+6x+y\left(x\right)}-36\right)-216\ln\left(\sqrt{1+6x+y\left(x\right)}+36\right)-144(1+6x+y(x))^{\frac{1}{4}}+432\ln\left(6+(1+6x+y(x))^{\frac{1}{4}}\right)-432\ln\left((1+6x+y(x))^{\frac{1}{4}}-6\right)-\frac{4(1+6x+y(x))^{\frac{3}{4}}}{3}-c_1=0$$

✓ Solution by Mathematica

Time used: 2.535 (sec). Leaf size: 79

```
DSolve[y'[x]==(1+6*x+y[x])^(1/4),y[x],x,IncludeSingularSolutions -> True]
```

$$\text{Solve}\left[\frac{1}{6}\left(y(x)-8(y(x)+6x+1)^{3/4}+72\sqrt{y(x)+6x+1}-864\sqrt[4]{y(x)+6x+1}+5184\log\left(\sqrt[4]{y(x)+6x+1}+6\right)+6x+1\right)-\frac{y(x)}{6}=c_1,y(x)\right]$$

1.63 problem 63

Internal problem ID [7379]
Internal file name [`OUTPUT/6359_Wednesday_July_13_2022_06_14_19_PM_21072012/index.tex`]

Book: First order enumerated odes
Section: section 1
Problem number: 63.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"homogeneousTypeC", "first_order_ode_lie_symmetry_lookup"**

Maple gives the following as the ode type

```
[[_homogeneous, `class C`], _dAlembert]
```

$$\boxed{y' - (a + xb + y)^4 = 0}$$

1.63.1 Solving as homogeneousTypeC ode

Let

$$z = a + xb + y \tag{1}$$

Then

$$z'(x) = b + y'$$

Therefore

$$y' = z'(x) - b$$

Hence the given ode can now be written as

$$z'(x) - b = z^4$$

This is separable first order ode. Integrating

$$\int dx = \int \frac{1}{z^4 + b} dz$$

$$x + c_1 = \frac{\sqrt{2}\left(\ln\left(\frac{z^2+b^{\frac{1}{4}}z\sqrt{2}+\sqrt{b}}{z^2-b^{\frac{1}{4}}z\sqrt{2}+\sqrt{b}}\right) + 2\arctan\left(\frac{\sqrt{2}\,z}{b^{\frac{1}{4}}}+1\right) + 2\arctan\left(\frac{\sqrt{2}\,z}{b^{\frac{1}{4}}}-1\right)\right)}{8b^{\frac{3}{4}}}$$

Replacing z back by its value from (1) then the above gives the solution as

$$\frac{\sqrt{2}\left(\ln\left(\frac{(a+xb+y)^2+b^{\frac{1}{4}}(a+xb+y)\sqrt{2}+\sqrt{b}}{(a+xb+y)^2-b^{\frac{1}{4}}(a+xb+y)\sqrt{2}+\sqrt{b}}\right) + 2\arctan\left(\frac{\sqrt{2}\,(a+xb+y)}{b^{\frac{1}{4}}}+1\right) + 2\arctan\left(\frac{\sqrt{2}\,(a+xb+y)}{b^{\frac{1}{4}}}-1\right)\right)}{8b^{\frac{3}{4}}}
= x + c_1$$

Summary

The solution(s) found are the following

$$\frac{\sqrt{2}\left(\ln\left(\frac{(a+xb+y)^2+b^{\frac{1}{4}}(a+xb+y)\sqrt{2}+\sqrt{b}}{(a+xb+y)^2-b^{\frac{1}{4}}(a+xb+y)\sqrt{2}+\sqrt{b}}\right) + 2\arctan\left(\frac{\sqrt{2}\,(a+xb+y)}{b^{\frac{1}{4}}}+1\right) + 2\arctan\left(\frac{\sqrt{2}\,(a+xb+y)}{b^{\frac{1}{4}}}-1\right)\right)}{8b^{\frac{3}{4}}}
= x + c_1 \tag{1}$$

Verification of solutions

$$\frac{\sqrt{2}\left(\ln\left(\frac{(a+xb+y)^2+b^{\frac{1}{4}}(a+xb+y)\sqrt{2}+\sqrt{b}}{(a+xb+y)^2-b^{\frac{1}{4}}(a+xb+y)\sqrt{2}+\sqrt{b}}\right) + 2\arctan\left(\frac{\sqrt{2}\,(a+xb+y)}{b^{\frac{1}{4}}}+1\right) + 2\arctan\left(\frac{\sqrt{2}\,(a+xb+y)}{b^{\frac{1}{4}}}-1\right)\right)}{8b^{\frac{3}{4}}}
= x + c_1$$

Verified OK.

1.63.2 Solving as first order ode lie symmetry lookup ode

Writing the ode as

$$y' = (xb + a + y)^4$$
$$y' = \omega(x, y)$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x + \omega(\eta_y - \xi_x) - \omega^2\xi_y - \omega_x\xi - \omega_y\eta = 0 \tag{A}$$

The type of this ode is known. It is of type `homogeneous Type C`. Therefore we do not need to solve the PDE (A), and can just use the lookup table shown below to find ξ, η

Table 66: Lie symmetry infinitesimal lookup table for known first order ODE's

ODE class	Form	ξ	η
linear ode	$y' = f(x)y(x) + g(x)$	0	$e^{\int f dx}$
separable ode	$y' = f(x)\, g(y)$	$\frac{1}{f}$	0
quadrature ode	$y' = f(x)$	0	1
quadrature ode	$y' = g(y)$	1	0
homogeneous ODEs of Class A	$y' = f\left(\frac{y}{x}\right)$	x	y
homogeneous ODEs of Class C	$y' = (a + bx + cy)^{\frac{n}{m}}$	1	$-\frac{b}{c}$
homogeneous class D	$y' = \frac{y}{x} + g(x)\, F\left(\frac{y}{x}\right)$	x^2	xy
First order special form ID 1	$y' = g(x)\, e^{h(x)+by} + f(x)$	$\frac{e^{-\int bf(x)dx - h(x)}}{g(x)}$	$\frac{f(x)e^{-\int bf(x)dx - h(x)}}{g(x)}$
polynomial type ode	$y' = \frac{a_1x+b_1y+c_1}{a_2x+b_2y+c_2}$	$\frac{a_1b_2x - a_2b_1x - b_1c_2 + b_2c_1}{a_1b_2 - a_2b_1}$	$\frac{a_1b_2y - a_2b_1y - a_1c_2 - a_2c_1}{a_1b_2 - a_2b_1}$
Bernoulli ode	$y' = f(x)\, y + g(x)\, y^n$	0	$e^{-\int (n-1)f(x)dx} y^n$
Reduced Riccati	$y' = f_1(x)\, y + f_2(x)\, y^2$	0	$e^{-\int f_1 dx}$

The above table shows that

$$
\begin{aligned}
\xi(x, y) &= 1 \\
\eta(x, y) &= -b
\end{aligned} \tag{A1}
$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x, y) \to (R, S)$ where (R, S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$
\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}
$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y}\right) S(x,y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the canonical coordinates, where $S(R)$. Therefore

$$\begin{aligned}\frac{dy}{dx} &= \frac{\eta}{\xi} \\ &= \frac{-b}{1} \\ &= -b\end{aligned}$$

This is easily solved to give

$$y = -xb + c_1$$

Where now the coordinate R is taken as the constant of integration. Hence

$$R = xb + y$$

And S is found from

$$\begin{aligned}dS &= \frac{dx}{\xi} \\ &= \frac{dx}{1}\end{aligned}$$

Integrating gives

$$\begin{aligned}S &= \int \frac{dx}{T} \\ &= x\end{aligned}$$

Where the constant of integration is set to zero as we just need one solution. Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x,y)S_y}{R_x + \omega(x,y)R_y} \tag{2}$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x,y)$ is the right hand side of the original ode given by

$$\omega(x,y) = (xb + a + y)^4$$

Evaluating all the partial derivatives gives

$$
\begin{aligned}
R_x &= b \\
R_y &= 1 \\
S_x &= 1 \\
S_y &= 0
\end{aligned}
$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$
\frac{dS}{dR} = \frac{1}{b + (xb + a + y)^4} \tag{2A}
$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$
\frac{dS}{dR} = \frac{1}{b + (R + a)^4}
$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$
S(R) = \int \frac{1}{R^4 + 4R^3a + 6R^2a^2 + 4R\,a^3 + a^4 + b} dR + c_1 \tag{4}
$$

To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$
x = \int^y \frac{1}{(xb + \_a)^4 + 4\,(xb + \_a)^3\,a + 6\,(xb + \_a)^2\,a^2 + 4\,(xb + \_a)\,a^3 + a^4 + b} d\_a + c_1
$$

Which simplifies to

$$
x = \int^y \frac{1}{(xb + \_a)^4 + 4\,(xb + \_a)^3\,a + 6\,(xb + \_a)^2\,a^2 + 4\,(xb + \_a)\,a^3 + a^4 + b} d\_a + c_1
$$

This results in

$$
x = \int^y \frac{1}{(xb + \_a)^4 + 4\,(xb + \_a)^3\,a + 6\,(xb + \_a)^2\,a^2 + 4\,(xb + \_a)\,a^3 + a^4 + b} d\_a + c_1
$$

Summary

The solution(s) found are the following

$$
\begin{aligned}
x = {} & \int^y \frac{1}{(xb + \_a)^4 + 4\,(xb + \_a)^3\,a + 6\,(xb + \_a)^2\,a^2 + 4\,(xb + \_a)\,a^3 + a^4 + b} d\_a \\
& + c_1
\end{aligned} \tag{1}
$$

Verification of solutions

$$x = \int^{y} \frac{1}{(xb+\_a)^4 + 4(xb+\_a)^3 a + 6(xb+\_a)^2 a^2 + 4(xb+\_a) a^3 + a^4 + b} d\_a + c_1$$

Verified OK.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
trying homogeneous types:
trying homogeneous C
1st order, trying the canonical coordinates of the invariance group
   -> Calling odsolve with the ODE`, diff(y(x), x) = -b, y(x)`      *** Sublevel 2 ***
      Methods for first order ODEs:
      --- Trying classification methods ---
      trying a quadrature
      trying 1st order linear
      <- 1st order linear successful
<- 1st order, canonical coordinates successful
<- homogeneous successful`
```

✓ Solution by Maple

Time used: 0.015 (sec). Leaf size: 49

```
dsolve(diff(y(x),x)=(a+b*x+y(x))^(4),y(x), singsol=all)
```

$$y(x) = -bx + \mathrm{RootOf}\left(-x + \int^{\_Z} \frac{1}{\_a^4 + 4\_a^3 a + 6\_a^2 a^2 + 4\_a\, a^3 + a^4 + b} d\_a + c_1\right)$$

✓ Solution by Mathematica

Time used: 0.429 (sec). Leaf size: 163

```
DSolve[y'[x]==(a+b*x+y[x])^(4),y[x],x,IncludeSingularSolutions -> True]
```

$$\text{Solve}\left[\frac{2\sqrt{2}\arctan\left(1-\frac{\sqrt{2}(a+bx+y(x))}{\sqrt[4]{b}}\right)-2\sqrt{2}\arctan\left(\frac{\sqrt{2}(a+bx+y(x))}{\sqrt[4]{b}}+1\right)+\sqrt{2}\log\left((a+bx+y(x))^2-\sqrt{}\right.}{8b^3}\right.$$

1.64 problem 64

Internal problem ID [7380]
Internal file name [`OUTPUT/6360_Wednesday_July_13_2022_06_14_20_PM_26296455/index.tex`]

Book: First order enumerated odes
Section: section 1
Problem number: 64.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"homogeneousTypeC", "first_order_ode_lie_symmetry_lookup"**

Maple gives the following as the ode type

```
[[_homogeneous, `class C`], _dAlembert]
```

$$\boxed{y' - (\pi + x + 7y)^{\frac{7}{2}} = 0}$$

1.64.1 Solving as homogeneousTypeC ode

Let

$$z = \pi + x + 7y \tag{1}$$

Then

$$z'(x) = 1 + 7y'$$

Therefore

$$y' = \frac{z'(x)}{7} - \frac{1}{7}$$

Hence the given ode can now be written as

$$\frac{z'(x)}{7} - \frac{1}{7} = z^{\frac{7}{2}}$$

This is separable first order ode. Integrating

$$\int dx = \int \frac{1}{7z^{\frac{7}{2}} + 1} dz$$

$$x + c_1 = -\frac{\left(\sum_{\_R=\mathrm{RootOf}(49\_Z^7-1)} \frac{\ln(z-\_R)}{\_R^6}\right)}{343} + \frac{\left(\sum_{\_R=\mathrm{RootOf}(7\_Z^7+1)} \frac{\ln(\sqrt{z}-\_R)}{\_R^5}\right)}{49} + \frac{\left(\sum_{\_R=\mathrm{RootOf}(7\_Z^7-1)} \frac{\ln(\sqrt{z}-\_R)}{\_R^5}\right)}{49}$$

Replacing z back by its value from (1) then the above gives the solution as

$$-\frac{\left(\sum_{\_R=\mathrm{RootOf}(49\_Z^7-1)} \frac{\ln(\pi+x+7y-\_R)}{\_R^6}\right)}{343} + \frac{\left(\sum_{\_R=\mathrm{RootOf}(7\_Z^7+1)} \frac{\ln(\sqrt{\pi+x+7y}-\_R)}{\_R^5}\right)}{49} + \frac{\left(\sum_{\_R=\mathrm{RootOf}(7\_Z^7-1)} \frac{\ln(\sqrt{\pi+x+7y}-\_R)}{\_R^5}\right)}{49} = x + c_1$$

Summary

The solution(s) found are the following

$$-\frac{\left(\sum_{\_R=\mathrm{RootOf}(49\_Z^7-1)} \frac{\ln(\pi+x+7y-\_R)}{\_R^6}\right)}{343} + \frac{\left(\sum_{\_R=\mathrm{RootOf}(7\_Z^7+1)} \frac{\ln(\sqrt{\pi+x+7y}-\_R)}{\_R^5}\right)}{49} + \frac{\left(\sum_{\_R=\mathrm{RootOf}(7\_Z^7-1)} \frac{\ln(\sqrt{\pi+x+7y}-\_R)}{\_R^5}\right)}{49} = x + c_1 \tag{1}$$

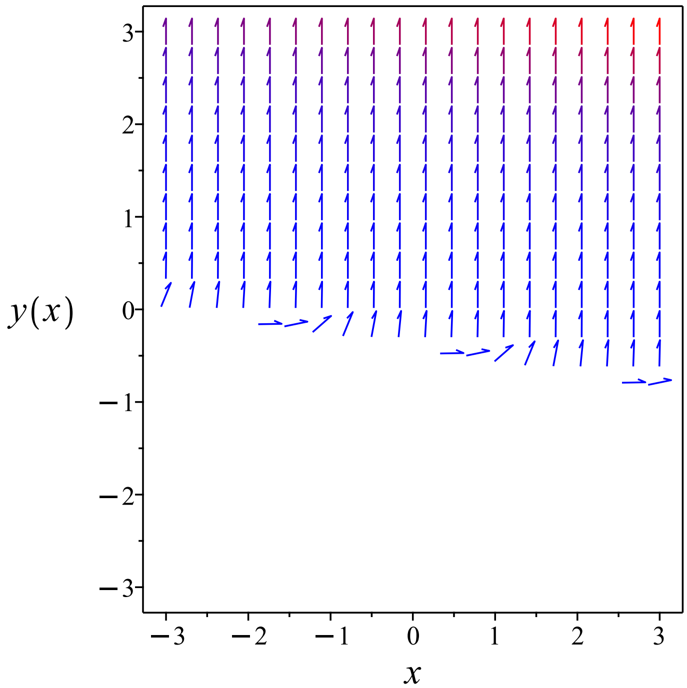


Figure 39: Slope field plot

Verification of solutions

$$-\frac{\left(\sum_{\_R=\operatorname{RootOf}(49\_Z^7-1)}\frac{\ln(\pi+x+7y-\_R)}{\_R^6}\right)}{343}+\frac{\left(\sum_{\_R=\operatorname{RootOf}(7\_Z^7+1)}\frac{\ln\left(\sqrt{\pi+x+7y}-\_R\right)}{\_R^5}\right)}{49}$$
$$+\frac{\left(\sum_{\_R=\operatorname{RootOf}(7\_Z^7-1)}\frac{\ln\left(\sqrt{\pi+x+7y}-\_R\right)}{\_R^5}\right)}{49}=x+c_1$$

Verified OK.

1.64.2 Solving as first order ode lie symmetry lookup ode

Writing the ode as

$$y'=(\pi+x+7y)^{\frac{7}{2}}$$
$$y'=\omega(x,y)$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x+\omega(\eta_y-\xi_x)-\omega^2\xi_y-\omega_x\xi-\omega_y\eta=0 \tag{A}$$

The type of this ode is known. It is of type `homogeneous Type C`. Therefore we do not need to solve the PDE (A), and can just use the lookup table shown below to find ξ, η

Table 68: Lie symmetry infinitesimal lookup table for known first order ODE's

ODE class	Form	ξ	η
linear ode	$y' = f(x)y(x) + g(x)$	0	$e^{\int f dx}$
separable ode	$y' = f(x)\, g(y)$	$\frac{1}{f}$	0
quadrature ode	$y' = f(x)$	0	1
quadrature ode	$y' = g(y)$	1	0
homogeneous ODEs of Class A	$y' = f\left(\frac{y}{x}\right)$	x	y
homogeneous ODEs of Class C	$y' = (a + bx + cy)^{\frac{n}{m}}$	1	$-\frac{b}{c}$
homogeneous class D	$y' = \frac{y}{x} + g(x)\, F\left(\frac{y}{x}\right)$	x^2	xy
First order special form ID 1	$y' = g(x)\, e^{h(x)+by} + f(x)$	$\frac{e^{-\int bf(x)dx - h(x)}}{g(x)}$	$\frac{f(x)e^{-\int bf(x)dx - h(x)}}{g(x)}$
polynomial type ode	$y' = \frac{a_1x+b_1y+c_1}{a_2x+b_2y+c_2}$	$\frac{a_1b_2x - a_2b_1x - b_1c_2 + b_2c_1}{a_1b_2 - a_2b_1}$	$\frac{a_1b_2y - a_2b_1y - a_1c_2 - a_2c_1}{a_1b_2 - a_2b_1}$
Bernoulli ode	$y' = f(x)\, y + g(x)\, y^n$	0	$e^{-\int (n-1)f(x)dx} y^n$
Reduced Riccati	$y' = f_1(x)\, y + f_2(x)\, y^2$	0	$e^{-\int f_1 dx}$

The above table shows that

$$
\begin{aligned}
\xi(x, y) &= 1 \\
\eta(x, y) &= -\frac{1}{7}
\end{aligned} \tag{A1}
$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x, y) \to (R, S)$ where (R, S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$
\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}
$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y}\right) S(x,y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the canonical coordinates, where $S(R)$. Therefore

$$\begin{aligned}\frac{dy}{dx} &= \frac{\eta}{\xi} \\ &= \frac{-\frac{1}{7}}{1} \\ &= -\frac{1}{7}\end{aligned}$$

This is easily solved to give

$$y = -\frac{x}{7} + c_1$$

Where now the coordinate R is taken as the constant of integration. Hence

$$R = y + \frac{x}{7}$$

And S is found from

$$\begin{aligned}dS &= \frac{dx}{\xi} \\ &= \frac{dx}{1}\end{aligned}$$

Integrating gives

$$\begin{aligned}S &= \int \frac{dx}{T} \\ &= x\end{aligned}$$

Where the constant of integration is set to zero as we just need one solution. Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x,y) S_y}{R_x + \omega(x,y) R_y} \tag{2}$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x,y)$ is the right hand side of the original ode given by

$$\omega(x,y) = (\pi + x + 7y)^{\frac{7}{2}}$$

Evaluating all the partial derivatives gives

$$
\begin{aligned}
R_x &= \frac{1}{7} \\
R_y &= 1 \\
S_x &= 1 \\
S_y &= 0
\end{aligned}
$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = \frac{7}{7\left(\pi + x + 7y\right)^{\frac{7}{2}} + 1} \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = \frac{7}{7\left(\pi + 7R\right)^{\frac{7}{2}} + 1}$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$S(R) = \int \frac{7}{7\pi^3\sqrt{\pi+7R} + 147\pi^2 R\sqrt{\pi+7R} + 1029\pi\, R^2\sqrt{\pi+7R} + 2401R^3\sqrt{\pi+7R} + 1} dR + c_1 \tag{4}$$

To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$x = \int^y \frac{7}{7\pi^3\sqrt{\pi+x+7\_a} + 147\pi^2\left(\_a+\frac{x}{7}\right)\sqrt{\pi+x+7\_a} + 1029\pi\left(\_a+\frac{x}{7}\right)^2\sqrt{\pi+x+7\_a} + 2401\left(\_\right.}$$

Which simplifies to

$$x - 7\left(\int^y \frac{1}{7\left(\pi+x+7\_a\right)^{\frac{7}{2}}+1} d\_a\right) - c_1 = 0$$

This results in

$$x - 7\left(\int^y \frac{1}{7\left(\pi+x+7\_a\right)^{\frac{7}{2}}+1} d\_a\right) - c_1 = 0$$

Summary

The solution(s) found are the following

$$x - 7\left(\int^{y} \frac{1}{7\left(\pi + x + 7\_a\right)^{\frac{7}{2}} + 1}d\_a\right) - c_1 = 0 \tag{1}$$

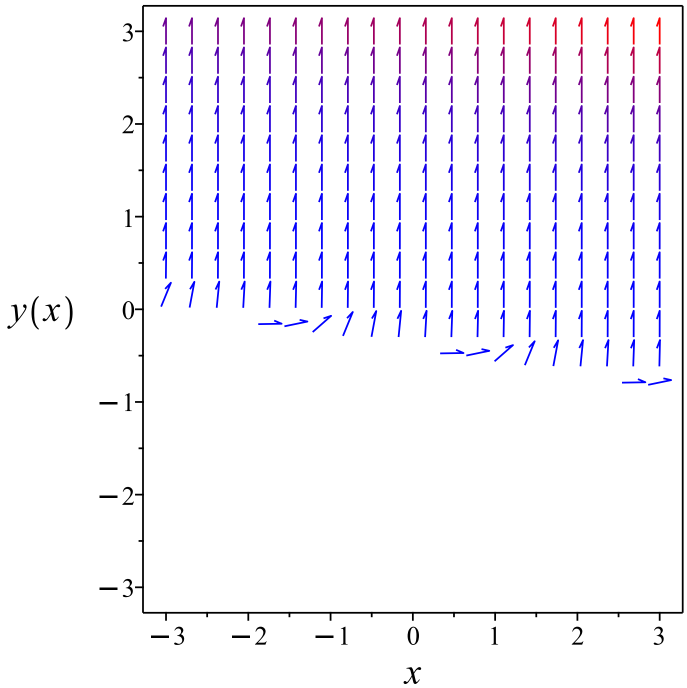


Figure 40: Slope field plot

Verification of solutions

$$x - 7\left(\int^{y} \frac{1}{7\left(\pi + x + 7\_a\right)^{\frac{7}{2}} + 1}d\_a\right) - c_1 = 0$$

Verified OK.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying homogeneous types:
trying homogeneous C
1st order, trying the canonical coordinates of the invariance group
   -> Calling odsolve with the ODE`, diff(y(x), x) = -1/7, y(x)`      *** Sublevel 2 ***
      Methods for first order ODEs:
      --- Trying classification methods ---
      trying a quadrature
      trying 1st order linear
      <- 1st order linear successful
<- 1st order, canonical coordinates successful
<- homogeneous successful`
```

✓ Solution by Maple

Time used: 0.031 (sec). Leaf size: 33

```
dsolve(diff(y(x),x)=(Pi+x+7*y(x))^(7/2),y(x), singsol=all)
```

$$y(x) = -\frac{x}{7} + \mathrm{RootOf}\left(-x + 7\left(\int^{\_Z} \frac{1}{1 + 7\left(\pi + 7\_a\right)^{\frac{7}{2}}} d\_a\right) + c_1\right)$$

✓ Solution by Mathematica

Time used: 30.556 (sec). Leaf size: 43

```
DSolve[y'[x]==(Pi+x+7*y[x])^(7/2),y[x],x,IncludeSingularSolutions -> True]
```

$$\mathrm{Solve}\left[-(7y(x)+x+\pi)\left(\mathrm{Hypergeometric2F1}\left(\frac{2}{7},1,\frac{9}{7},-7(x+7y(x)+\pi)^{7/2}\right)-1\right) - 7y(x) = c_1, y(x)\right]$$

1.65 problem 65

Internal problem ID [7381]
Internal file name [`OUTPUT/6361_Wednesday_July_13_2022_06_14_22_PM_50797711/index.tex`]

Book: First order enumerated odes
Section: section 1
Problem number: 65.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"homogeneousTypeC", "first_order_ode_lie_symmetry_lookup"**

Maple gives the following as the ode type

```
[[_homogeneous, `class C`], _dAlembert]
```

$$\boxed{y' - (a + xb + cy)^6 = 0}$$

1.65.1 Solving as homogeneousTypeC ode

Let

$$z = a + xb + cy \tag{1}$$

Then

$$z'(x) = b + y'c$$

Therefore

$$y' = \frac{z'(x) - b}{c}$$

Hence the given ode can now be written as

$$\frac{z'(x) - b}{c} = z^6$$

This is separable first order ode. Integrating

$$\int dx = \int \frac{1}{c\,z^6 + b} dz$$

$$x + c_1 = \frac{\sqrt{3}\left(\frac{b}{c}\right)^{\frac{1}{6}} \ln\left(z^2 + \sqrt{3}\left(\frac{b}{c}\right)^{\frac{1}{6}} z + \left(\frac{b}{c}\right)^{\frac{1}{3}}\right)}{12b} + \frac{\left(\frac{b}{c}\right)^{\frac{1}{6}} \arctan\left(\frac{2z}{\left(\frac{b}{c}\right)^{\frac{1}{6}}} + \sqrt{3}\right)}{6b} - \frac{\sqrt{3}\left(\frac{b}{c}\right)^{\frac{1}{6}} \ln\left(z^2 - \sqrt{3}\left(\frac{b}{c}\right)^{\frac{1}{6}} z + \left(\frac{b}{c}\right)^{\frac{1}{3}}\right)}{12b} + \frac{\left(\frac{b}{c}\right)^{\frac{1}{6}} \arctan\left(\frac{2z}{\left(\frac{b}{c}\right)^{\frac{1}{6}}} - \sqrt{3}\right)}{6b} + \frac{\left(\frac{b}{c}\right)^{\frac{1}{6}} \arctan\left(\frac{z}{\left(\frac{b}{c}\right)^{\frac{1}{6}}}\right)}{3b}$$

Replacing z back by its value from (1) then the above gives the solution as

$$\frac{\sqrt{3}\left(\frac{b}{c}\right)^{\frac{1}{6}} \ln\left((a + xb + cy)^2 + \sqrt{3}\left(\frac{b}{c}\right)^{\frac{1}{6}} (a + xb + cy) + \left(\frac{b}{c}\right)^{\frac{1}{3}}\right)}{12b} + \frac{\left(\frac{b}{c}\right)^{\frac{1}{6}} \arctan\left(\frac{2a+2xb+2cy}{\left(\frac{b}{c}\right)^{\frac{1}{6}}} + \sqrt{3}\right)}{6b} - \frac{\sqrt{3}\left(\frac{b}{c}\right)^{\frac{1}{6}} \ln\left((a + xb + cy)^2 - \sqrt{3}\left(\frac{b}{c}\right)^{\frac{1}{6}} (a + xb + cy) + \left(\frac{b}{c}\right)^{\frac{1}{3}}\right)}{12b} + \frac{\left(\frac{b}{c}\right)^{\frac{1}{6}} \arctan\left(\frac{2a+2xb+2cy}{\left(\frac{b}{c}\right)^{\frac{1}{6}}} - \sqrt{3}\right)}{6b} + \frac{\left(\frac{b}{c}\right)^{\frac{1}{6}} \arctan\left(\frac{a+xb+cy}{\left(\frac{b}{c}\right)^{\frac{1}{6}}}\right)}{3b} = x + c_1$$

Summary
The solution(s) found are the following

$$
\begin{aligned}
&\frac{\sqrt{3}\left(\frac{b}{c}\right)^{\frac{1}{6}} \ln\left(\left(a+xb+cy\right)^{2}+\sqrt{3}\left(\frac{b}{c}\right)^{\frac{1}{6}}\left(a+xb+cy\right)+\left(\frac{b}{c}\right)^{\frac{1}{3}}\right)}{12b} \\
&+\frac{\left(\frac{b}{c}\right)^{\frac{1}{6}} \arctan\left(\frac{2a+2xb+2cy}{\left(\frac{b}{c}\right)^{\frac{1}{6}}}+\sqrt{3}\right)}{6b} \\
&-\frac{\sqrt{3}\left(\frac{b}{c}\right)^{\frac{1}{6}} \ln\left(\left(a+xb+cy\right)^{2}-\sqrt{3}\left(\frac{b}{c}\right)^{\frac{1}{6}}\left(a+xb+cy\right)+\left(\frac{b}{c}\right)^{\frac{1}{3}}\right)}{12b} \\
&+\frac{\left(\frac{b}{c}\right)^{\frac{1}{6}} \arctan\left(\frac{2a+2xb+2cy}{\left(\frac{b}{c}\right)^{\frac{1}{6}}}-\sqrt{3}\right)}{6b}+\frac{\left(\frac{b}{c}\right)^{\frac{1}{6}} \arctan\left(\frac{a+xb+cy}{\left(\frac{b}{c}\right)^{\frac{1}{6}}}\right)}{3b}=x+c_{1}
\end{aligned}
\tag{1}
$$

Verification of solutions

$$
\begin{aligned}
&\frac{\sqrt{3}\left(\frac{b}{c}\right)^{\frac{1}{6}} \ln\left(\left(a+xb+cy\right)^{2}+\sqrt{3}\left(\frac{b}{c}\right)^{\frac{1}{6}}\left(a+xb+cy\right)+\left(\frac{b}{c}\right)^{\frac{1}{3}}\right)}{12b} \\
&+\frac{\left(\frac{b}{c}\right)^{\frac{1}{6}} \arctan\left(\frac{2a+2xb+2cy}{\left(\frac{b}{c}\right)^{\frac{1}{6}}}+\sqrt{3}\right)}{6b} \\
&-\frac{\sqrt{3}\left(\frac{b}{c}\right)^{\frac{1}{6}} \ln\left(\left(a+xb+cy\right)^{2}-\sqrt{3}\left(\frac{b}{c}\right)^{\frac{1}{6}}\left(a+xb+cy\right)+\left(\frac{b}{c}\right)^{\frac{1}{3}}\right)}{12b} \\
&+\frac{\left(\frac{b}{c}\right)^{\frac{1}{6}} \arctan\left(\frac{2a+2xb+2cy}{\left(\frac{b}{c}\right)^{\frac{1}{6}}}-\sqrt{3}\right)}{6b}+\frac{\left(\frac{b}{c}\right)^{\frac{1}{6}} \arctan\left(\frac{a+xb+cy}{\left(\frac{b}{c}\right)^{\frac{1}{6}}}\right)}{3b}=x+c_{1}
\end{aligned}
$$

Verified OK.

1.65.2 Solving as first order ode lie symmetry lookup ode

Writing the ode as

$$
\begin{aligned}
y' &= (xb+cy+a)^{6} \\
y' &= \omega(x,y)
\end{aligned}
$$

The condition of Lie symmetry is the linearized PDE given by

$$
\eta_{x}+\omega(\eta_{y}-\xi_{x})-\omega^{2}\xi_{y}-\omega_{x}\xi-\omega_{y}\eta=0 \tag{A}
$$

The type of this ode is known. It is of type `homogeneous Type C`. Therefore we do not need to solve the PDE (A), and can just use the lookup table shown below to find ξ, η

Table 70: Lie symmetry infinitesimal lookup table for known first order ODE's

ODE class	Form	ξ	η
linear ode	$y' = f(x)y(x) + g(x)$	0	$e^{\int f dx}$
separable ode	$y' = f(x)\, g(y)$	$\frac{1}{f}$	0
quadrature ode	$y' = f(x)$	0	1
quadrature ode	$y' = g(y)$	1	0
homogeneous ODEs of Class A	$y' = f\left(\frac{y}{x}\right)$	x	y
homogeneous ODEs of Class C	$y' = (a + bx + cy)^{\frac{n}{m}}$	1	$-\frac{b}{c}$
homogeneous class D	$y' = \frac{y}{x} + g(x)\, F\left(\frac{y}{x}\right)$	x^2	xy
First order special form ID 1	$y' = g(x)\, e^{h(x)+by} + f(x)$	$\frac{e^{-\int bf(x)dx-h(x)}}{g(x)}$	$\frac{f(x)e^{-\int bf(x)dx-h(x)}}{g(x)}$
polynomial type ode	$y' = \frac{a_1x+b_1y+c_1}{a_2x+b_2y+c_2}$	$\frac{a_1b_2x-a_2b_1x-b_1c_2+b_2c_1}{a_1b_2-a_2b_1}$	$\frac{a_1b_2y-a_2b_1y-a_1c_2-a_2c_1}{a_1b_2-a_2b_1}$
Bernoulli ode	$y' = f(x)\, y + g(x)\, y^n$	0	$e^{-\int (n-1)f(x)dx} y^n$
Reduced Riccati	$y' = f_1(x)\, y + f_2(x)\, y^2$	0	$e^{-\int f_1 dx}$

The above table shows that

$$
\begin{aligned}
\xi(x,y) &= 1 \\
\eta(x,y) &= -\frac{b}{c}
\end{aligned}
\tag{A1}
$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x,y) \to (R,S)$ where (R,S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$
\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}
$$

The above comes from the requirements that $\left(\xi\frac{\partial}{\partial x}+\eta\frac{\partial}{\partial y}\right)S(x,y)=1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the canonical coordinates, where $S(R)$. Therefore

$$\begin{aligned}\frac{dy}{dx} &= \frac{\eta}{\xi}\\ &= \frac{-\frac{b}{c}}{1}\\ &= -\frac{b}{c}\end{aligned}$$

This is easily solved to give

$$y = -\frac{bx}{c}+c_1$$

Where now the coordinate R is taken as the constant of integration. Hence

$$R = \frac{xb+cy}{c}$$

And S is found from

$$\begin{aligned}dS &= \frac{dx}{\xi}\\ &= \frac{dx}{1}\end{aligned}$$

Integrating gives

$$\begin{aligned}S &= \int \frac{dx}{T}\\ &= x\end{aligned}$$

Where the constant of integration is set to zero as we just need one solution. Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x+\omega(x,y)S_y}{R_x+\omega(x,y)R_y} \tag{2}$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x,y)$ is the right hand side of the original ode given by

$$\omega(x,y) = (xb+cy+a)^6$$

Evaluating all the partial derivatives gives

$$
\begin{aligned}
R_x &= \frac{b}{c} \\
R_y &= 1 \\
S_x &= 1 \\
S_y &= 0
\end{aligned}
$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$
\frac{dS}{dR} = \frac{1}{\frac{b}{c} + (xb + cy + a)^6} \tag{2A}
$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$
\frac{dS}{dR} = \frac{1}{\frac{b}{c} + (Rc + a)^6}
$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$
S(R) = \int \frac{c}{R^6c^7 + 6R^5a\,c^6 + 15R^4a^2c^5 + 20R^3a^3c^4 + 15R^2a^4c^3 + 6R\,a^5c^2 + a^6c + b} dR + c_1 \tag{4}
$$

To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$
x = \int^{\frac{cy+xb}{c}} \frac{c}{\_a^6c^7 + 6\_a^5a\,c^6 + 15\_a^4a^2c^5 + 20\_a^3a^3c^4 + 15\_a^2a^4c^3 + 6\_a\,a^5c^2 + a^6c + b} d\_a + c_1
$$

Which simplifies to

$$
x = \int^{\frac{cy+xb}{c}} \frac{c}{\_a^6c^7 + 6\_a^5a\,c^6 + 15\_a^4a^2c^5 + 20\_a^3a^3c^4 + 15\_a^2a^4c^3 + 6\_a\,a^5c^2 + a^6c + b} d\_a + c_1
$$

Summary

The solution(s) found are the following

$$
x = \int^{\frac{cy+xb}{c}} \frac{c}{\_a^6c^7 + 6\_a^5a\,c^6 + 15\_a^4a^2c^5 + 20\_a^3a^3c^4 + 15\_a^2a^4c^3 + 6\_a\,a^5c^2 + a^6c + b} d\_a + c_1 \tag{1}
$$

Verification of solutions

$$x = \int^{\frac{cy+xb}{c}} \frac{c}{\_a^6c^7 + 6\_a^5a\,c^6 + 15\_a^4a^2c^5 + 20\_a^3a^3c^4 + 15\_a^2a^4c^3 + 6\_a\,a^5c^2 + a^6c + b} d\_a + c_1$$

Verified OK.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
trying homogeneous types:
trying homogeneous C
1st order, trying the canonical coordinates of the invariance group
   -> Calling odsolve with the ODE`, diff(y(x), x) = -b/c, y(x)`      *** Sublevel 2 ***
      Methods for first order ODEs:
      --- Trying classification methods ---
      trying a quadrature
      trying 1st order linear
      <- 1st order linear successful
<- 1st order, canonical coordinates successful
<- homogeneous successful`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 94

```
dsolve(diff(y(x),x)=(a+b*x+c*y(x))^6,y(x), singsol=all)
```

$$y(x) = \frac{\operatorname{RootOf}\left(\left(\int^{\_Z} \frac{1}{c^7\_a^6+6\_a^5a\,c^6+15\_a^4a^2c^5+20\_a^3a^3c^4+15\_a^2a^4c^3+6\_a\,a^5c^2+a^6c+b} d\_a\right) c - x + c_1\right) c - bx}{c}$$

✓ Solution by Mathematica

Time used: 1.941 (sec). Leaf size: 274

```
DSolve[y'[x]==(a+b*x+c*y[x])^6,y[x],x,IncludeSingularSolutions -> True]
```

$$\text{Solve}\left[\frac{-4\sqrt[6]{b}\arctan\left(\frac{\sqrt[6]{c}(a+bx+cy(x))}{\sqrt[6]{b}}\right)+2\sqrt[6]{b}\arctan\left(\sqrt{3}-\frac{2\sqrt[6]{c}(a+bx+cy(x))}{\sqrt[6]{b}}\right)-2\sqrt[6]{b}\arctan\left(\frac{2\sqrt[6]{c}(a+bx+cy}{\sqrt[6]{b}}\right.}{}\right.$$

$$\left.-\frac{cy(x)}{b}=c_1,y(x)\right]$$

1.66 problem 66

Internal problem ID [7382]
Internal file name [`OUTPUT/6483_Saturday_August_06_2022_05_19_34_AM_9550685/index.tex`]

Book: First order enumerated odes
Section: section 1
Problem number: 66.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"exact", "separable", "first order special form ID 1", "first_order_ode_lie_symmetry_lookup"**

Maple gives the following as the ode type

```
[_separable]
```

$$\boxed{y' - \mathrm{e}^{x+y} = 0}$$

1.66.1 Solving as separable ode

In canonical form the ODE is

$$\begin{aligned} y' &= F(x, y) \\ &= f(x)g(y) \\ &= \mathrm{e}^{x}\mathrm{e}^{y} \end{aligned}$$

Where $f(x) = \mathrm{e}^{x}$ and $g(y) = \mathrm{e}^{y}$. Integrating both sides gives

$$\begin{aligned} \frac{1}{\mathrm{e}^{y}}\,dy &= \mathrm{e}^{x}\,dx \\ \int \frac{1}{\mathrm{e}^{y}}\,dy &= \int \mathrm{e}^{x}\,dx \\ -\mathrm{e}^{-y} &= \mathrm{e}^{x} + c_1 \end{aligned}$$

Which results in

$$y = \ln\left(-\frac{1}{e^{x} + c_1}\right)$$

Summary

The solution(s) found are the following

$$y = \ln\left(-\frac{1}{e^{x} + c_1}\right) \tag{1}$$

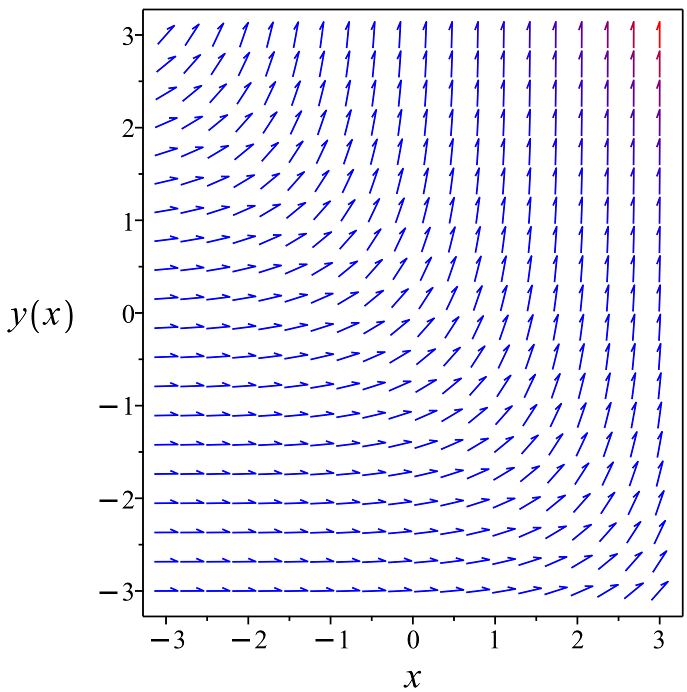


Figure 41: Slope field plot

Verification of solutions

$$y = \ln\left(-\frac{1}{e^{x} + c_1}\right)$$

Verified OK.

1.66.2 Solving as first order special form ID 1 ode

Writing the ode as

$$y' = e^{x+y} \tag{1}$$

And using the substitution $u = e^{-y}$ then

$$u' = -y' e^{-y}$$

The above shows that

$$\begin{aligned} y' &= -u'(x)\, e^{y} \\ &= -\frac{u'(x)}{u} \end{aligned}$$

Substituting this in (1) gives

$$-\frac{u'(x)}{u} = \frac{e^{x}}{u}$$

The above simplifies to

$$u'(x) = -e^{x} \tag{2}$$

Now ode (2) is solved for $u(x)$ Integrating both sides gives

$$\begin{aligned} u(x) &= \int -e^{x}\, dx \\ &= -e^{x} + c_1 \end{aligned}$$

Substituting the solution found for $u(x)$ in $u = e^{-y}$ gives

$$\begin{aligned} y &= -\ln(u(x)) \\ &= -\ln(-e^{x} + c_1) \\ &= -\ln(-e^{x} + c_1) \end{aligned}$$

Summary

The solution(s) found are the following

$$y = -\ln(-e^{x} + c_1) \tag{1}$$

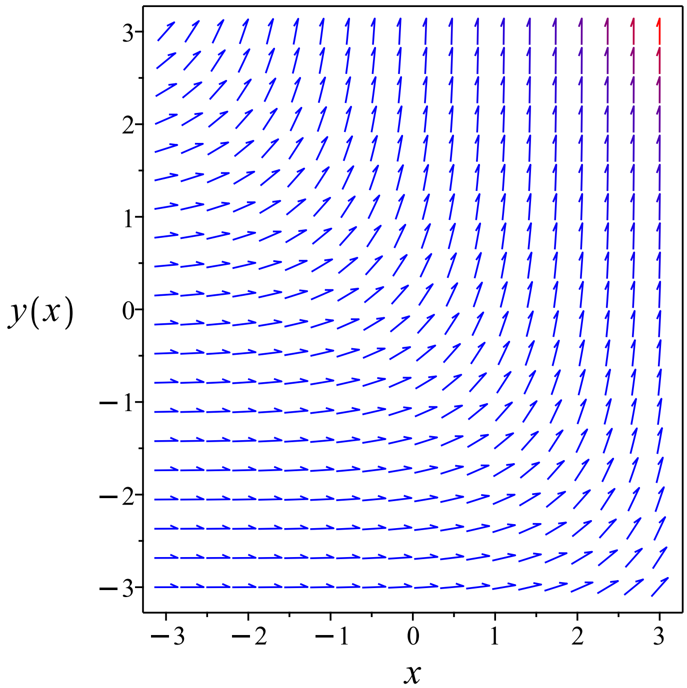


Figure 42: Slope field plot

Verification of solutions

$$y = -\ln\left(-\mathrm{e}^{x} + c_1\right)$$

Verified OK.

1.66.3 Solving as first order ode lie symmetry lookup ode

Writing the ode as

$$y' = \mathrm{e}^{x+y}$$
$$y' = \omega(x, y)$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x + \omega(\eta_y - \xi_x) - \omega^2 \xi_y - \omega_x \xi - \omega_y \eta = 0 \tag{A}$$

The type of this ode is known. It is of type `separable`. Therefore we do not need to solve the PDE (A), and can just use the lookup table shown below to find ξ, η

Table 72: Lie symmetry infinitesimal lookup table for known first order ODE's

ODE class	Form	ξ	η
linear ode	$y' = f(x)y(x) + g(x)$	0	$e^{\int f dx}$
separable ode	$y' = f(x)\, g(y)$	$\frac{1}{f}$	0
quadrature ode	$y' = f(x)$	0	1
quadrature ode	$y' = g(y)$	1	0
homogeneous ODEs of Class A	$y' = f\left(\frac{y}{x}\right)$	x	y
homogeneous ODEs of Class C	$y' = (a + bx + cy)^{\frac{n}{m}}$	1	$-\frac{b}{c}$
homogeneous class D	$y' = \frac{y}{x} + g(x)\, F\left(\frac{y}{x}\right)$	x^2	xy
First order special form ID 1	$y' = g(x)\, e^{h(x)+by} + f(x)$	$\frac{e^{-\int bf(x)dx - h(x)}}{g(x)}$	$\frac{f(x)e^{-\int bf(x)dx - h(x)}}{g(x)}$
polynomial type ode	$y' = \frac{a_1x+b_1y+c_1}{a_2x+b_2y+c_2}$	$\frac{a_1b_2x - a_2b_1x - b_1c_2 + b_2c_1}{a_1b_2 - a_2b_1}$	$\frac{a_1b_2y - a_2b_1y - a_1c_2 - a_2c_1}{a_1b_2 - a_2b_1}$
Bernoulli ode	$y' = f(x)\, y + g(x)\, y^n$	0	$e^{-\int (n-1)f(x)dx} y^n$
Reduced Riccati	$y' = f_1(x)\, y + f_2(x)\, y^2$	0	$e^{-\int f_1 dx}$

The above table shows that

$$\begin{aligned} \xi(x, y) &= \mathrm{e}^{-x} \\ \eta(x, y) &= 0 \end{aligned} \tag{A1}$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x, y) \to (R, S)$ where (R, S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y}\right) S(x, y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the

canonical coordinates, where $S(R)$. Since $\eta = 0$ then in this special case

$$R = y$$

S is found from

$$\begin{aligned} S &= \int \frac{1}{\xi} dx \\ &= \int \frac{1}{\mathrm{e}^{-x}} dx \end{aligned}$$

Which results in

$$S = \mathrm{e}^{x}$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x,y) S_y}{R_x + \omega(x,y) R_y} \tag{2}$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x, y)$ is the right hand side of the original ode given by

$$\omega(x,y) = \mathrm{e}^{x+y}$$

Evaluating all the partial derivatives gives

$$\begin{aligned} R_x &= 0 \\ R_y &= 1 \\ S_x &= \mathrm{e}^{x} \\ S_y &= 0 \end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = \mathrm{e}^{-y} \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = \mathrm{e}^{-R}$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$S(R) = -e^{-R} + c_1 \tag{4}$$

To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$e^x = -e^{-y} + c_1$$

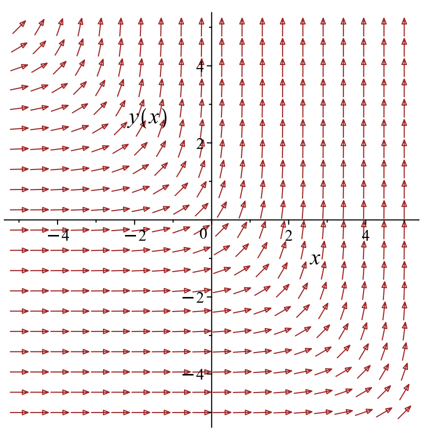
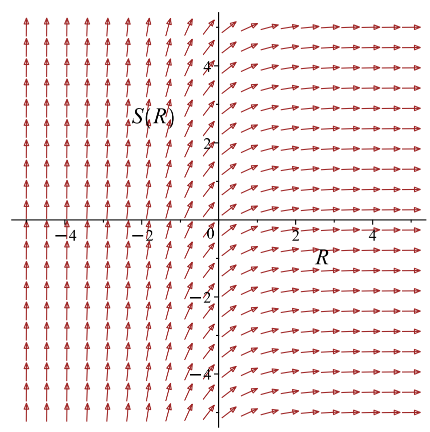
Which simplifies to

$$e^x = -e^{-y} + c_1$$

Which gives

$$y = -\ln(-e^x + c_1)$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

<table>
<tr><th>Original ode in x, y coordinates</th><th>Canonical coordinates transformation</th><th>ODE in canonical coordinates (R, S)</th></tr>
<tr><td>$\frac{dy}{dx} = e^{x+y}$

</td><td>$R = y$
$S = e^x$</td><td>$\frac{dS}{dR} = e^{-R}$

</td></tr>
</table>

Summary

The solution(s) found are the following

$$y = -\ln(-e^x + c_1) \tag{1}$$

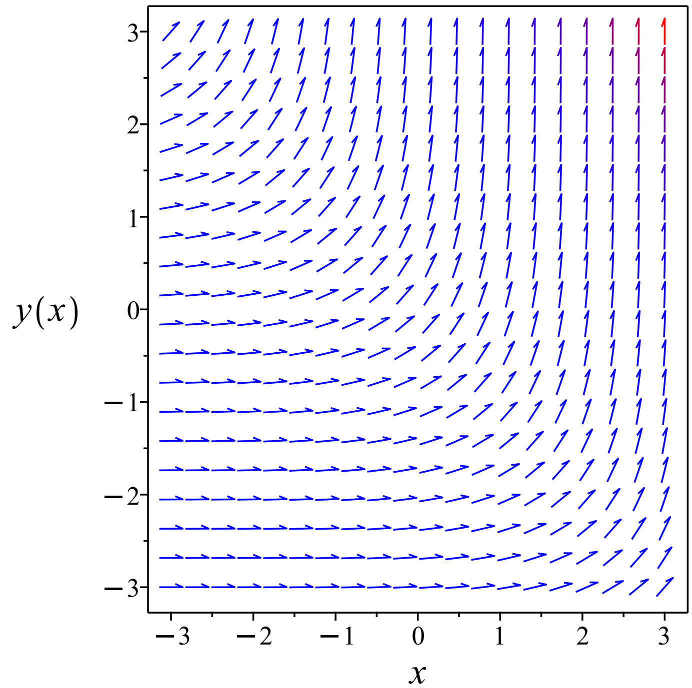


Figure 43: Slope field plot

Verification of solutions

$$y = -\ln\left(-e^{x} + c_1\right)$$

Verified OK.

1.66.4 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x, y) + N(x, y)\frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x, y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x, y) = 0$$

Hence

$$\frac{\partial \phi}{\partial x} + \frac{\partial \phi}{\partial y}\frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$\frac{\partial \phi}{\partial x} = M$$
$$\frac{\partial \phi}{\partial y} = N$$

But since $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x, y)$ but at least we know now that we can do that since the condition $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x, y)\,\mathrm{d}x + N(x, y)\,\mathrm{d}y = 0 \tag{1A}$$

Therefore

$$\left(\mathrm{e}^{-y}\right)\mathrm{d}y = \left(\mathrm{e}^{x}\right)\mathrm{d}x$$
$$\left(-\mathrm{e}^{x}\right)\mathrm{d}x + \left(\mathrm{e}^{-y}\right)\mathrm{d}y = 0 \tag{2A}$$

Comparing (1A) and (2A) shows that

$$M(x, y) = -\mathrm{e}^{x}$$
$$N(x, y) = \mathrm{e}^{-y}$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Using result found above gives

$$\begin{aligned}\frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}\left(-\mathrm{e}^{x}\right)\\ &= 0\end{aligned}$$

And

$$\begin{aligned}\frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}\left(\mathrm{e}^{-y}\right)\\ &= 0\end{aligned}$$

Since $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$, then the ODE is <u>exact</u> The following equations are now set up to solve for the function $\phi(x, y)$

$$\frac{\partial \phi}{\partial x} = M \tag{1}$$

$$\frac{\partial \phi}{\partial y} = N \tag{2}$$

Integrating (1) w.r.t. x gives

$$\int \frac{\partial \phi}{\partial x} \, \mathrm{d}x = \int M \, \mathrm{d}x$$

$$\int \frac{\partial \phi}{\partial x} \, \mathrm{d}x = \int -\mathrm{e}^{x} \, \mathrm{d}x$$

$$\phi = -\mathrm{e}^{x} + f(y) \tag{3}$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$\frac{\partial \phi}{\partial y} = 0 + f'(y) \tag{4}$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = \mathrm{e}^{-y}$. Therefore equation (4) becomes

$$\mathrm{e}^{-y} = 0 + f'(y) \tag{5}$$

Solving equation (5) for $f'(y)$ gives

$$f'(y) = \mathrm{e}^{-y}$$

Integrating the above w.r.t y gives

$$\int f'(y) \, \mathrm{d}y = \int \left(\mathrm{e}^{-y}\right) \mathrm{d}y$$

$$f(y) = -\mathrm{e}^{-y} + c_1$$

Where c_1 is constant of integration. Substituting result found above for $f(y)$ into equation (3) gives ϕ

$$\phi = -\mathrm{e}^{x} - \mathrm{e}^{-y} + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = -e^x - e^{-y}$$

The solution becomes

$$y = -\ln\left(-e^x - c_1\right)$$

Summary

The solution(s) found are the following

$$y = -\ln\left(-e^x - c_1\right) \tag{1}$$

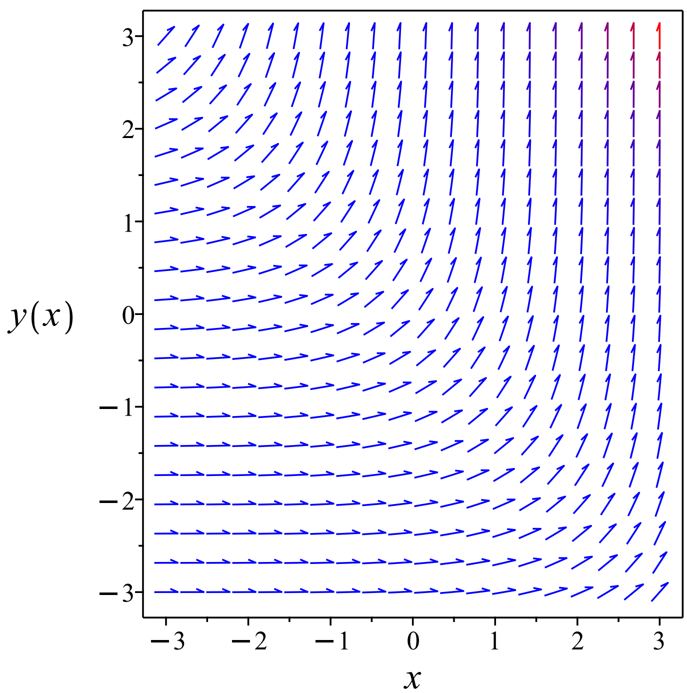


Figure 44: Slope field plot

Verification of solutions

$$y = -\ln\left(-e^x - c_1\right)$$

Verified OK.

1.66.5 Maple step by step solution

Let's solve

$y' - e^{x+y} = 0$

- Highest derivative means the order of the ODE is 1

  y'

- Separate variables

  $\frac{y'}{e^{y}} = e^{x}$

- Integrate both sides with respect to x

  $\int \frac{y'}{e^{y}} dx = \int e^{x} dx + c_1$

- Evaluate integral

  $-\frac{1}{e^{y}} = e^{x} + c_1$

- Solve for y

  $y = \ln\left(-\frac{1}{e^{x}+c_1}\right)$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
<- separable successful`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 13

```
dsolve(diff(y(x),x)=exp(x+y(x)),y(x), singsol=all)
```

$$y(x) = \ln\left(-\frac{1}{e^{x}+c_1}\right)$$

✓ Solution by Mathematica

Time used: 0.876 (sec). Leaf size: 18

```
DSolve[y'[x]==Exp[x+y[x]],y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to -\log \left(-e^x - c_1\right)$$

1.67 problem 67

Internal problem ID [7383]
Internal file name [`OUTPUT/6484_Saturday_August_06_2022_05_19_35_AM_71634107/index.tex`]

Book: First order enumerated odes
Section: section 1
Problem number: 67.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"first order special form ID 1", "first_order_ode_lie_symmetry_lookup"**

Maple gives the following as the ode type

```
[[_homogeneous, `class C`], _dAlembert]
```

$$\boxed{y' - e^{x+y} = 10}$$

1.67.1 Solving as first order special form ID 1 ode

Writing the ode as

$$y' = 10 + e^{x+y} \tag{1}$$

And using the substitution $u = e^{-y}$ then

$$u' = -y' e^{-y}$$

The above shows that

$$\begin{aligned} y' &= -u'(x)\, e^{y} \\ &= -\frac{u'(x)}{u} \end{aligned}$$

Substituting this in (1) gives

$$-\frac{u'(x)}{u} = \frac{e^{x}}{u} + 10$$

The above simplifies to

$$
\begin{aligned}
-u'(x) &= \mathrm{e}^{x} + 10u(x) \\
u'(x) + 10u(x) &= -\mathrm{e}^{x} \qquad (2)
\end{aligned}
$$

Now ode (2) is solved for $u(x)$

Entering Linear first order ODE solver. In canonical form a linear first order is

$$
u'(x) + p(x)u(x) = q(x)
$$

Where here

$$
\begin{aligned}
p(x) &= 10 \\
q(x) &= -\mathrm{e}^{x}
\end{aligned}
$$

Hence the ode is

$$
u'(x) + 10u(x) = -\mathrm{e}^{x}
$$

The integrating factor μ is

$$
\begin{aligned}
\mu &= \mathrm{e}^{\int 10dx} \\
&= \mathrm{e}^{10x}
\end{aligned}
$$

The ode becomes

$$
\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}x}(\mu u) &= (\mu)\left(-\mathrm{e}^{x}\right) \\
\frac{\mathrm{d}}{\mathrm{d}x}\left(\mathrm{e}^{10x}u\right) &= \left(\mathrm{e}^{10x}\right)\left(-\mathrm{e}^{x}\right) \\
\mathrm{d}\left(\mathrm{e}^{10x}u\right) &= \left(-\mathrm{e}^{11x}\right)\,\mathrm{d}x
\end{aligned}
$$

Integrating gives

$$
\begin{aligned}
\mathrm{e}^{10x}u &= \int -\mathrm{e}^{11x}\,\mathrm{d}x \\
\mathrm{e}^{10x}u &= -\frac{\mathrm{e}^{11x}}{11} + c_1
\end{aligned}
$$

Dividing both sides by the integrating factor $\mu = \mathrm{e}^{10x}$ results in

$$
u(x) = -\frac{\mathrm{e}^{-10x}\mathrm{e}^{11x}}{11} + c_1\mathrm{e}^{-10x}
$$

which simplifies to

$$u(x) = -\frac{(e^{11x} - 11c_1)\, e^{-10x}}{11}$$

Substituting the solution found for $u(x)$ in $u = e^{-y}$ gives

$$\begin{aligned} y &= -\ln(u(x)) \\ &= -\ln\left(-\frac{(e^{11x} - 11c_1)\, e^{-10x}}{11}\right) \\ &= \ln(11) - \ln\left((-e^{11x} + 11c_1)\, e^{-10x}\right) \end{aligned}$$

Summary

The solution(s) found are the following

$$y = \ln(11) - \ln\left((-e^{11x} + 11c_1)\, e^{-10x}\right) \tag{1}$$

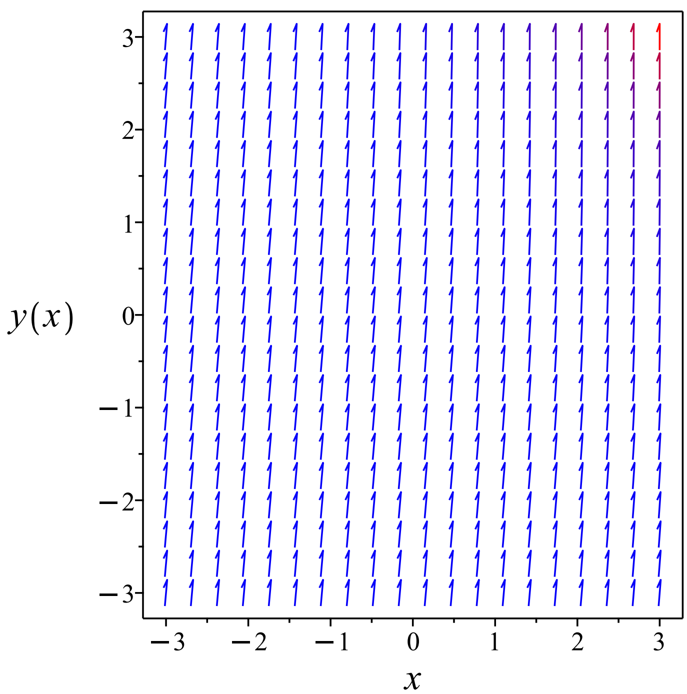


Figure 45: Slope field plot

Verification of solutions

$$y = \ln(11) - \ln\left((-e^{11x} + 11c_1)\, e^{-10x}\right)$$

Verified OK.

1.67.2 Solving as first order ode lie symmetry lookup ode

Writing the ode as

$$y' = 10 + \mathrm{e}^{x+y}$$
$$y' = \omega(x, y)$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x + \omega(\eta_y - \xi_x) - \omega^2 \xi_y - \omega_x \xi - \omega_y \eta = 0 \tag{A}$$

The type of this ode is known. It is of type `first order special form ID 1`. Therefore we do not need to solve the PDE (A), and can just use the lookup table shown below to find ξ, η

Table 75: Lie symmetry infinitesimal lookup table for known first order ODE's

ODE class	Form	ξ	η
linear ode	$y' = f(x)y(x) + g(x)$	0	$e^{\int f dx}$
separable ode	$y' = f(x)\, g(y)$	$\frac{1}{f}$	0
quadrature ode	$y' = f(x)$	0	1
quadrature ode	$y' = g(y)$	1	0
homogeneous ODEs of Class A	$y' = f\left(\frac{y}{x}\right)$	x	y
homogeneous ODEs of Class C	$y' = (a + bx + cy)^{\frac{n}{m}}$	1	$-\frac{b}{c}$
homogeneous class D	$y' = \frac{y}{x} + g(x)\, F\left(\frac{y}{x}\right)$	x^2	xy
First order special form ID 1	$y' = g(x)\, e^{h(x)+by} + f(x)$	$\frac{e^{-\int b f(x) dx - h(x)}}{g(x)}$	$\frac{f(x) e^{-\int b f(x) dx - h(x)}}{g(x)}$
polynomial type ode	$y' = \frac{a_1 x + b_1 y + c_1}{a_2 x + b_2 y + c_2}$	$\frac{a_1 b_2 x - a_2 b_1 x - b_1 c_2 + b_2 c_1}{a_1 b_2 - a_2 b_1}$	$\frac{a_1 b_2 y - a_2 b_1 y - a_1 c_2 - a_2 c_1}{a_1 b_2 - a_2 b_1}$
Bernoulli ode	$y' = f(x)\, y + g(x)\, y^n$	0	$e^{-\int (n-1) f(x) dx} y^n$
Reduced Riccati	$y' = f_1(x)\, y + f_2(x)\, y^2$	0	$e^{-\int f_1 dx}$

The above table shows that

$$
\begin{aligned}
\xi(x, y) &= \mathrm{e}^{-11x} \\
\eta(x, y) &= 10 + \mathrm{e}^{-11x}
\end{aligned} \tag{A1}
$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x, y) \to (R, S)$ where (R, S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$
\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}
$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y}\right) S(x, y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the canonical coordinates, where $S(R)$. Therefore

$$
\begin{aligned}
\frac{dy}{dx} &= \frac{\eta}{\xi} \\
&= \frac{10 + \mathrm{e}^{-11x}}{\mathrm{e}^{-11x}} \\
&= 1 + 10\,\mathrm{e}^{11x}
\end{aligned}
$$

This is easily solved to give

$$
y = x + \frac{10\,\mathrm{e}^{11x}}{11} + c_1
$$

Where now the coordinate R is taken as the constant of integration. Hence

$$
R = -x - \frac{10\,\mathrm{e}^{11x}}{11} + y
$$

And S is found from

$$
\begin{aligned}
dS &= \frac{dx}{\xi} \\
&= \frac{dx}{\mathrm{e}^{-11x}}
\end{aligned}
$$

Integrating gives

$$
\begin{aligned}
S &= \int \frac{dx}{T} \\
&= \frac{\mathrm{e}^{11x}}{11}
\end{aligned}
$$

Where the constant of integration is set to zero as we just need one solution. Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x,y)S_y}{R_x + \omega(x,y)R_y} \tag{2}$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x,y)$ is the right hand side of the original ode given by

$$\omega(x,y) = 10 + e^{x+y}$$

Evaluating all the partial derivatives gives

$$\begin{aligned} R_x &= -1 - 10\,e^{11x} \\ R_y &= 1 \\ S_x &= e^{11x} \\ S_y &= 0 \end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = \frac{e^{11x}}{9 - 10\,e^{11x} + e^{x+y}} \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = -\frac{11S(R)\,11^{\frac{9}{11}}}{-11S(R)^{\frac{2}{11}}\,e^{R+10S(R)} + (110S(R) - 9)\,11^{\frac{9}{11}}}$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$-\frac{2\ln(S(R))}{11} + \frac{2\ln\left(S(R)^{\frac{2}{11}}\,e^{R+10S(R)}\right)}{11} - \frac{2\ln\left(11^{\frac{9}{11}} + S(R)^{\frac{2}{11}}\,e^{R+10S(R)}\right)}{11} - c_1 = 0 \tag{4}$$

To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$-\frac{2\ln\left(\frac{e^{11x}}{11}\right)}{11} + \frac{2\ln\left(\frac{11^{\frac{9}{11}}\left(e^{11x}\right)^{\frac{2}{11}}\,e^{y-x}}{11}\right)}{11} - \frac{2\ln\left(11^{\frac{9}{11}} + \frac{11^{\frac{9}{11}}\left(e^{11x}\right)^{\frac{2}{11}}\,e^{y-x}}{11}\right)}{11} - c_1 = 0$$

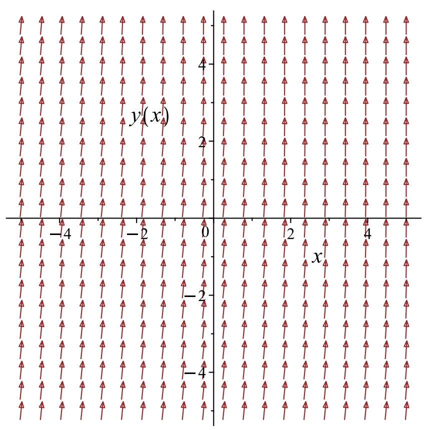
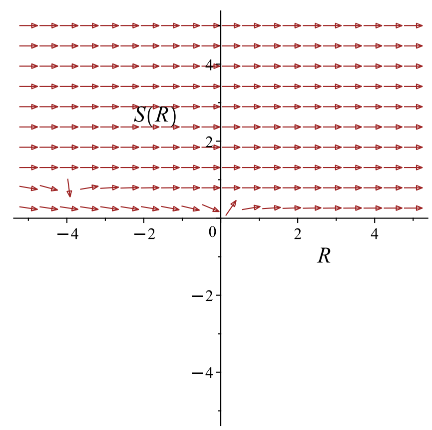
Which simplifies to

$$\frac{2\ln(11)}{11}-\frac{20x}{11}+\frac{2y}{11}-\frac{2\ln(11+e^{x+y})}{11}-c_1=0$$

Which gives

$$y=10x+\ln\left(-\frac{11}{-11+e^{11x+\frac{11c_1}{2}}}\right)+\frac{11c_1}{2}$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in x, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dx}=10+e^{x+y}$ 	$R=-x-\frac{10\,e^{11x}}{11}+y$ $S=\frac{e^{11x}}{11}$	$\frac{dS}{dR}=-\frac{11S(R)11^{\frac{9}{11}}}{-11S(R)^{\frac{2}{11}}e^{R+10S(R)}+(110S(R)-9)11^{\frac{9}{11}}}$ 

Summary

The solution(s) found are the following

$$y=10x+\ln\left(-\frac{11}{-11+e^{11x+\frac{11c_1}{2}}}\right)+\frac{11c_1}{2} \tag{1}$$

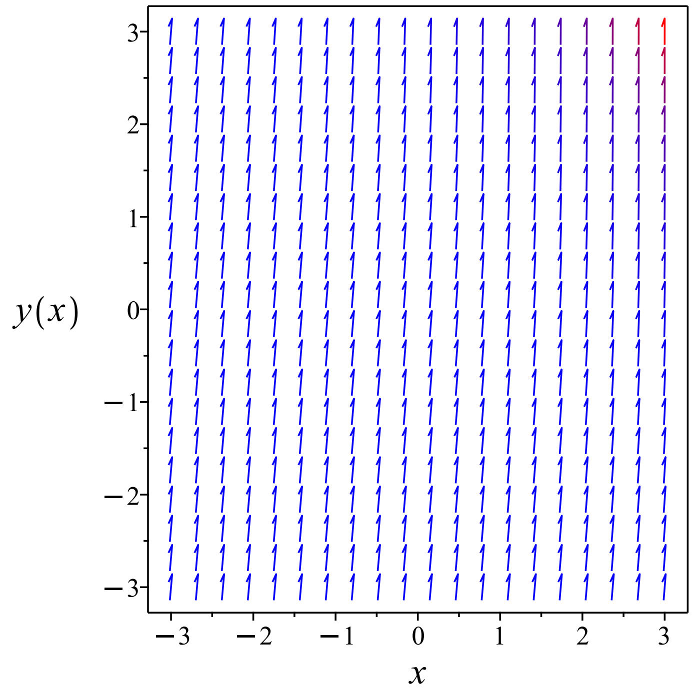


Figure 46: Slope field plot

Verification of solutions

$$y = 10x + \ln\left(-\frac{11}{-11 + e^{11x + \frac{11c_1}{2}}}\right) + \frac{11c_1}{2}$$

Verified OK.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
trying homogeneous types:
trying homogeneous C
1st order, trying the canonical coordinates of the invariance group
   -> Calling odsolve with the ODE`, diff(y(x), x) = -1, y(x)`      *** Sublevel 2 ***
      Methods for first order ODEs:
      --- Trying classification methods ---
      trying a quadrature
      trying 1st order linear
      <- 1st order linear successful
<- 1st order, canonical coordinates successful
<- homogeneous successful`
```

✓ Solution by Maple

Time used: 0.031 (sec). Leaf size: 26

```
dsolve(diff(y(x),x)=10+exp(x+y(x)),y(x), singsol=all)
```

$$y(x) = -x + \ln(11) + \ln\left(\frac{e^{11x}}{-e^{11x} + c_1}\right)$$

✓ Solution by Mathematica

Time used: 3.4 (sec). Leaf size: 42

```
DSolve[y'[x]==10+Exp[x+y[x]],y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \log\left(-\frac{11e^{10x+11c_1}}{-1+e^{11(x+c_1)}}\right)$$
$$y(x) \to \log\left(-11e^{-x}\right)$$

1.68 problem 68

Internal problem ID [7384]
Internal file name [`OUTPUT/6485_Saturday_August_06_2022_05_19_37_AM_49721640/index.tex`]

Book: First order enumerated odes
Section: section 1
Problem number: 68.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"first order special form ID 1", "first_order_ode_lie_symmetry_lookup"**

Maple gives the following as the ode type

```
[[_1st_order, `_with_symmetry_[F(x),G(x)]`]]
```

$$\boxed{y' - 10\,\mathrm{e}^{x+y} = x^2}$$

1.68.1 Solving as first order special form ID 1 ode

Writing the ode as

$$y' = 10\,\mathrm{e}^{x+y} + x^2 \tag{1}$$

And using the substitution $u = \mathrm{e}^{-y}$ then

$$u' = -y'\mathrm{e}^{-y}$$

The above shows that

$$\begin{aligned} y' &= -u'(x)\,\mathrm{e}^{y} \\ &= -\frac{u'(x)}{u} \end{aligned}$$

Substituting this in (1) gives

$$-\frac{u'(x)}{u} = \frac{10\,\mathrm{e}^{x}}{u} + x^2$$

The above simplifies to

$$
\begin{aligned}
-u'(x) &= 10\,\mathrm{e}^{x}+x^{2}u(x) \\
u'(x)+x^{2}u(x) &= -10\,\mathrm{e}^{x}
\end{aligned} \tag{2}
$$

Now ode (2) is solved for $u(x)$

Entering Linear first order ODE solver. In canonical form a linear first order is

$$
u'(x)+p(x)u(x) = q(x)
$$

Where here

$$
\begin{aligned}
p(x) &= x^{2} \\
q(x) &= -10\,\mathrm{e}^{x}
\end{aligned}
$$

Hence the ode is

$$
u'(x)+x^{2}u(x) = -10\,\mathrm{e}^{x}
$$

The integrating factor μ is

$$
\begin{aligned}
\mu &= \mathrm{e}^{\int x^{2}dx} \\
&= \mathrm{e}^{\frac{x^{3}}{3}}
\end{aligned}
$$

The ode becomes

$$
\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}x}(\mu u) &= (\mu)\left(-10\,\mathrm{e}^{x}\right) \\
\frac{\mathrm{d}}{\mathrm{d}x}\left(\mathrm{e}^{\frac{x^{3}}{3}}u\right) &= \left(\mathrm{e}^{\frac{x^{3}}{3}}\right)\left(-10\,\mathrm{e}^{x}\right) \\
\mathrm{d}\left(\mathrm{e}^{\frac{x^{3}}{3}}u\right) &= \left(-10\,\mathrm{e}^{\frac{x(x^{2}+3)}{3}}\right)\,\mathrm{d}x
\end{aligned}
$$

Integrating gives

$$
\begin{aligned}
\mathrm{e}^{\frac{x^{3}}{3}}u &= \int -10\,\mathrm{e}^{\frac{x(x^{2}+3)}{3}}\,\mathrm{d}x \\
\mathrm{e}^{\frac{x^{3}}{3}}u &= \int -10\,\mathrm{e}^{\frac{x(x^{2}+3)}{3}}dx + c_{1}
\end{aligned}
$$

Dividing both sides by the integrating factor $\mu = \mathrm{e}^{\frac{x^{3}}{3}}$ results in

$$
u(x) = \mathrm{e}^{-\frac{x^{3}}{3}}\left(\int -10\,\mathrm{e}^{\frac{x(x^{2}+3)}{3}}dx\right) + c_{1}\mathrm{e}^{-\frac{x^{3}}{3}}
$$

which simplifies to

$$u(x) = e^{-\frac{x^3}{3}} \left(-10\left(\int e^{\frac{x\left(x^2+3\right)}{3}}dx\right) + c_1\right)$$

Substituting the solution found for $u(x)$ in $u = e^{-y}$ gives

$$\begin{aligned}
y &= -\ln\left(u(x)\right) \\
&= -\ln\left(e^{-\frac{x^3}{3}} \left(-10\left(\int e^{\frac{x\left(x^2+3\right)}{3}}dx\right) + c_1\right)\right) \\
&= -\ln\left(e^{-\frac{x^3}{3}} \left(-10\left(\int e^{\frac{x\left(x^2+3\right)}{3}}dx\right) + c_1\right)\right)
\end{aligned}$$

Summary

The solution(s) found are the following

$$y = -\ln\left(e^{-\frac{x^3}{3}} \left(-10\left(\int e^{\frac{x\left(x^2+3\right)}{3}}dx\right) + c_1\right)\right) \tag{1}$$

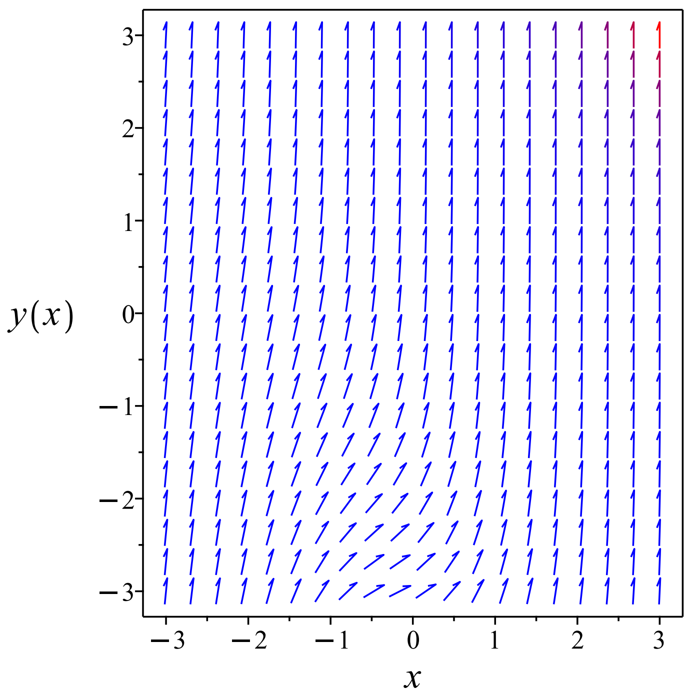


Figure 47: Slope field plot

Verification of solutions

$$y = -\ln\left(e^{-\frac{x^3}{3}} \left(-10\left(\int e^{\frac{x\left(x^2+3\right)}{3}}dx\right) + c_1\right)\right)$$

Verified OK.

1.68.2 Solving as first order ode lie symmetry lookup ode

Writing the ode as

$$y' = 10\,e^{x+y} + x^2$$
$$y' = \omega(x, y)$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x + \omega(\eta_y - \xi_x) - \omega^2\xi_y - \omega_x\xi - \omega_y\eta = 0 \tag{A}$$

The type of this ode is known. It is of type `first order special form ID 1`. Therefore we do not need to solve the PDE (A), and can just use the lookup table shown below to find ξ, η

Table 77: Lie symmetry infinitesimal lookup table for known first order ODE's

ODE class	Form	ξ	η
linear ode	$y' = f(x)y(x) + g(x)$	0	$e^{\int f dx}$
separable ode	$y' = f(x)\, g(y)$	$\frac{1}{f}$	0
quadrature ode	$y' = f(x)$	0	1
quadrature ode	$y' = g(y)$	1	0
homogeneous ODEs of Class A	$y' = f\left(\frac{y}{x}\right)$	x	y
homogeneous ODEs of Class C	$y' = (a + bx + cy)^{\frac{n}{m}}$	1	$-\frac{b}{c}$
homogeneous class D	$y' = \frac{y}{x} + g(x)\, F\left(\frac{y}{x}\right)$	x^2	xy
First order special form ID 1	$y' = g(x)\, e^{h(x)+by} + f(x)$	$\frac{e^{-\int b f(x)dx - h(x)}}{g(x)}$	$\frac{f(x)e^{-\int b f(x)dx - h(x)}}{g(x)}$
polynomial type ode	$y' = \frac{a_1x+b_1y+c_1}{a_2x+b_2y+c_2}$	$\frac{a_1b_2x-a_2b_1x-b_1c_2+b_2c_1}{a_1b_2-a_2b_1}$	$\frac{a_1b_2y-a_2b_1y-a_1c_2-a_2c_1}{a_1b_2-a_2b_1}$
Bernoulli ode	$y' = f(x)\, y + g(x)\, y^n$	0	$e^{-\int (n-1)f(x)dx} y^n$
Reduced Riccati	$y' = f_1(x)\, y + f_2(x)\, y^2$	0	$e^{-\int f_1 dx}$

The above table shows that

$$\xi(x,y) = \frac{e^{-\frac{1}{3}x^3-x}}{10}$$
$$\eta(x,y) = x^2 + \frac{e^{-\frac{1}{3}x^3-x}}{10} \tag{A1}$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x,y) \to (R,S)$ where (R,S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}$$

The above comes from the requirements that $\left(\xi\frac{\partial}{\partial x} + \eta\frac{\partial}{\partial y}\right) S(x,y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the canonical coordinates, where $S(R)$. Therefore

$$\begin{aligned}\frac{dy}{dx} &= \frac{\eta}{\xi}\\ &= \frac{x^2 + \frac{e^{-\frac{1}{3}x^3-x}}{10}}{\frac{e^{-\frac{1}{3}x^3-x}}{10}}\\ &= 10\,e^{\frac{x(x^2+3)}{3}}x^2 + 1\end{aligned}$$

This is easily solved to give

$$y = \int \left(10\,e^{\frac{x(x^2+3)}{3}}x^2 + 1\right) dx + c_1$$

Where now the coordinate R is taken as the constant of integration. Hence

$$R = -\left(\int \left(10\,e^{\frac{x(x^2+3)}{3}}x^2 + 1\right) dx\right) + y$$

And S is found from

$$\begin{aligned}dS &= \frac{dx}{\xi}\\ &= \frac{dx}{\frac{e^{-\frac{1}{3}x^3-x}}{10}}\end{aligned}$$

Integrating gives

$$
\begin{aligned}
S &= \int \frac{dx}{T} \\
&= \int 10\,\mathrm{e}^{x+\frac{1}{3}x^3} dx
\end{aligned}
$$

Where the constant of integration is set to zero as we just need one solution.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
trying homogeneous types:
trying Chini
differential order: 1; looking for linear symmetries
trying exact
Looking for potential symmetries
trying inverse_Riccati
trying an equivalence to an Abel ODE
differential order: 1; trying a linearization to 2nd order
--- trying a change of variables {x -> y(x), y(x) -> x}
differential order: 1; trying a linearization to 2nd order
trying 1st order ODE linearizable_by_differentiation
--- Trying Lie symmetry methods, 1st order ---
`, `-> Computing symmetries using: way = 3
`, `-> Computing symmetries using: way = 4
`, `-> Computing symmetries using: way = 5
trying symmetry patterns for 1st order ODEs
-> trying a symmetry pattern of the form [F(x)*G(y), 0]
-> trying a symmetry pattern of the form [0, F(x)*G(y)]
<- symmetry pattern of the form [0, F(x)*G(y)] successful`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 30

```
dsolve(diff(y(x),x)=10*exp(x+y(x))+x^2,y(x), singsol=all)
```

$$y(x) = \frac{x^3}{3} - \ln\left(-c_1 - 10\left(\int e^{\frac{x(x^2+3)}{3}} dx\right)\right)$$

✓ Solution by Mathematica

Time used: 0.431 (sec). Leaf size: 115

```
DSolve[y'[x]==10*Exp[x+y[x]]+x^2,y[x],x,IncludeSingularSolutions -> True]
```

$$\text{Solve}\left[\int_1^{y(x)} -\frac{1}{10}e^{-K[2]}\left(10e^{K[2]}\int_1^x -\frac{1}{10}e^{\frac{K[1]^3}{3}-K[2]}K[1]^2 dK[1] + e^{\frac{x^3}{3}}\right) dK[2] + \int_1^x \left(\frac{1}{10}e^{\frac{K[1]^3}{3}-y(x)}K[1]^2 + e^{\frac{K[1]^3}{3}+K[1]}\right) dK[1] = c_1, y(x)\right]$$

1.69 problem 69

Internal problem ID [7385]
Internal file name [`OUTPUT/6486_Saturday_August_06_2022_05_19_40_AM_11615872/index.tex`]

Book: First order enumerated odes
Section: section 1
Problem number: 69.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"first order special form ID 1", "first_order_ode_lie_symmetry_lookup"**

Maple gives the following as the ode type

```
[[_1st_order, `_with_symmetry_[F(x),G(x)]`]]
```

$$\boxed{y' - \mathrm{e}^{x+y} x = \sin(x)}$$

1.69.1 Solving as first order special form ID 1 ode

Writing the ode as

$$y' = \mathrm{e}^{x+y} x + \sin(x) \tag{1}$$

And using the substitution $u = \mathrm{e}^{-y}$ then

$$u' = -y' \mathrm{e}^{-y}$$

The above shows that

$$\begin{aligned} y' &= -u'(x)\,\mathrm{e}^{y} \\ &= -\frac{u'(x)}{u} \end{aligned}$$

Substituting this in (1) gives

$$-\frac{u'(x)}{u} = \frac{x\,\mathrm{e}^{x}}{u} + \sin(x)$$

The above simplifies to

$$
\begin{aligned}
-u'(x) &= x\,\mathrm{e}^{x} + \sin(x)\,u(x) \\
u'(x) + \sin(x)\,u(x) &= -x\,\mathrm{e}^{x}
\end{aligned} \tag{2}
$$

Now ode (2) is solved for $u(x)$

Entering Linear first order ODE solver. In canonical form a linear first order is

$$
u'(x) + p(x)u(x) = q(x)
$$

Where here

$$
\begin{aligned}
p(x) &= \sin(x) \\
q(x) &= -x\,\mathrm{e}^{x}
\end{aligned}
$$

Hence the ode is

$$
u'(x) + \sin(x)\,u(x) = -x\,\mathrm{e}^{x}
$$

The integrating factor μ is

$$
\begin{aligned}
\mu &= \mathrm{e}^{\int \sin(x)dx} \\
&= \mathrm{e}^{-\cos(x)}
\end{aligned}
$$

The ode becomes

$$
\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}x}(\mu u) &= (\mu)\,(-x\,\mathrm{e}^{x}) \\
\frac{\mathrm{d}}{\mathrm{d}x}\left(\mathrm{e}^{-\cos(x)}u\right) &= \left(\mathrm{e}^{-\cos(x)}\right)(-x\,\mathrm{e}^{x}) \\
\mathrm{d}\left(\mathrm{e}^{-\cos(x)}u\right) &= \left(-x\,\mathrm{e}^{x-\cos(x)}\right)\,\mathrm{d}x
\end{aligned}
$$

Integrating gives

$$
\begin{aligned}
\mathrm{e}^{-\cos(x)}u &= \int -x\,\mathrm{e}^{x-\cos(x)}\,\mathrm{d}x \\
\mathrm{e}^{-\cos(x)}u &= \int -x\,\mathrm{e}^{x-\cos(x)}dx + c_1
\end{aligned}
$$

Dividing both sides by the integrating factor $\mu = \mathrm{e}^{-\cos(x)}$ results in

$$
u(x) = \mathrm{e}^{\cos(x)}\left(\int -x\,\mathrm{e}^{x-\cos(x)}dx\right) + c_1\mathrm{e}^{\cos(x)}
$$

which simplifies to

$$u(x) = \mathrm{e}^{\cos(x)} \left(-\left(\int x\,\mathrm{e}^{x-\cos(x)} dx\right) + c_1\right)$$

Substituting the solution found for $u(x)$ in $u = \mathrm{e}^{-y}$ gives

$$\begin{aligned} y &= -\ln\left(u(x)\right) \\ &= -\ln\left(\mathrm{e}^{\cos(x)} \left(-\left(\int x\,\mathrm{e}^{x-\cos(x)} dx\right) + c_1\right)\right) \\ &= -\ln\left(\mathrm{e}^{\cos(x)} \left(-\left(\int x\,\mathrm{e}^{x-\cos(x)} dx\right) + c_1\right)\right) \end{aligned}$$

Summary

The solution(s) found are the following

$$y = -\ln\left(\mathrm{e}^{\cos(x)} \left(-\left(\int x\,\mathrm{e}^{x-\cos(x)} dx\right) + c_1\right)\right) \tag{1}$$

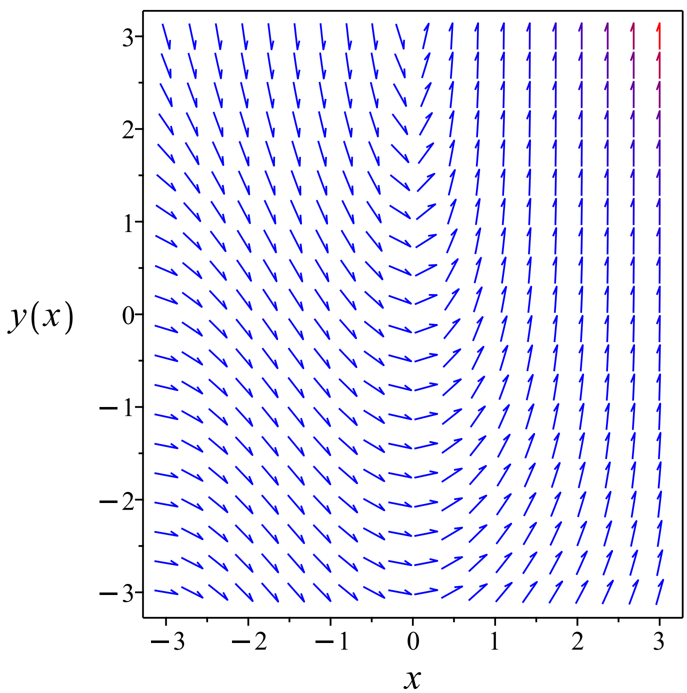


Figure 48: Slope field plot

Verification of solutions

$$y = -\ln\left(\mathrm{e}^{\cos(x)} \left(-\left(\int x\,\mathrm{e}^{x-\cos(x)} dx\right) + c_1\right)\right)$$

Verified OK.

1.69.2 Solving as first order ode lie symmetry lookup ode

Writing the ode as

$$\begin{aligned} y' &= x\,\mathrm{e}^{x+y} + \sin(x) \\ y' &= \omega(x,y) \end{aligned}$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x + \omega(\eta_y - \xi_x) - \omega^2 \xi_y - \omega_x \xi - \omega_y \eta = 0 \tag{A}$$

The type of this ode is known. It is of type `first order special form ID 1`. Therefore we do not need to solve the PDE (A), and can just use the lookup table shown below to find ξ, η

Table 79: Lie symmetry infinitesimal lookup table for known first order ODE's

ODE class	Form	ξ	η
linear ode	$y' = f(x)y(x) + g(x)$	0	$e^{\int f dx}$
separable ode	$y' = f(x)\,g(y)$	$\frac{1}{f}$	0
quadrature ode	$y' = f(x)$	0	1
quadrature ode	$y' = g(y)$	1	0
homogeneous ODEs of Class A	$y' = f\left(\frac{y}{x}\right)$	x	y
homogeneous ODEs of Class C	$y' = (a + bx + cy)^{\frac{n}{m}}$	1	$-\frac{b}{c}$
homogeneous class D	$y' = \frac{y}{x} + g(x)\,F\left(\frac{y}{x}\right)$	x^2	xy
First order special form ID 1	$y' = g(x)\,e^{h(x)+by} + f(x)$	$\frac{e^{-\int b f(x) dx - h(x)}}{g(x)}$	$\frac{f(x) e^{-\int b f(x) dx - h(x)}}{g(x)}$
polynomial type ode	$y' = \frac{a_1 x + b_1 y + c_1}{a_2 x + b_2 y + c_2}$	$\frac{a_1 b_2 x - a_2 b_1 x - b_1 c_2 + b_2 c_1}{a_1 b_2 - a_2 b_1}$	$\frac{a_1 b_2 y - a_2 b_1 y - a_1 c_2 - a_2 c_1}{a_1 b_2 - a_2 b_1}$
Bernoulli ode	$y' = f(x)\,y + g(x)\,y^n$	0	$e^{-\int (n-1) f(x) dx} y^n$
Reduced Riccati	$y' = f_1(x)\,y + f_2(x)\,y^2$	0	$e^{-\int f_1 dx}$

The above table shows that

$$\xi(x,y) = \frac{e^{\cos(x)-x}}{x}$$
$$\eta(x,y) = \sin(x) + \frac{e^{\cos(x)-x}}{x} \tag{A1}$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x,y) \to (R,S)$ where (R,S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y}\right) S(x,y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the canonical coordinates, where $S(R)$. Therefore

$$\begin{aligned}\frac{dy}{dx} &= \frac{\eta}{\xi} \\ &= \frac{\sin(x) + \frac{e^{\cos(x)-x}}{x}}{\frac{e^{\cos(x)-x}}{x}} \\ &= e^{x-\cos(x)}\left(x\sin(x) + e^{\cos(x)-x}\right)\end{aligned}$$

This is easily solved to give

$$y = \int e^{x-\cos(x)}\left(x\sin(x) + e^{\cos(x)-x}\right) dx + c_1$$

Where now the coordinate R is taken as the constant of integration. Hence

$$R = -\left(\int e^{x-\cos(x)}\left(x\sin(x) + e^{\cos(x)-x}\right) dx\right) + y$$

And S is found from

$$\begin{aligned}dS &= \frac{dx}{\xi} \\ &= \frac{dx}{\frac{e^{\cos(x)-x}}{x}}\end{aligned}$$

Integrating gives

$$\begin{aligned}S &= \int \frac{dx}{T} \\ &= \int x\, e^{x-\cos(x)} dx\end{aligned}$$

Where the constant of integration is set to zero as we just need one solution.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
trying homogeneous types:
trying Chini
differential order: 1; looking for linear symmetries
trying exact
Looking for potential symmetries
trying inverse_Riccati
trying an equivalence to an Abel ODE
differential order: 1; trying a linearization to 2nd order
--- trying a change of variables {x -> y(x), y(x) -> x}
differential order: 1; trying a linearization to 2nd order
trying 1st order ODE linearizable_by_differentiation
--- Trying Lie symmetry methods, 1st order ---
`, `-> Computing symmetries using: way = 3
`, `-> Computing symmetries using: way = 4
`, `-> Computing symmetries using: way = 5
trying symmetry patterns for 1st order ODEs
-> trying a symmetry pattern of the form [F(x)*G(y), 0]
-> trying a symmetry pattern of the form [0, F(x)*G(y)]
<- symmetry pattern of the form [0, F(x)*G(y)] successful`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 29

```
dsolve(diff(y(x),x)=x*exp(x+y(x))+sin(x),y(x), singsol=all)
```

$$y(x) = -\cos(x) - \ln\left(-c_1 - \left(\int x\,\mathrm{e}^{x-\cos(x)}dx\right)\right)$$

✓ Solution by Mathematica

Time used: 3.93 (sec). Leaf size: 100

```
DSolve[y'[x]==x*Exp[x+y[x]]+Sin[x],y[x],x,IncludeSingularSolutions -> True]
```

$$\text{Solve}\left[\int_1^x \left(-e^{K[1]-\cos(K[1])}K[1]-e^{-\cos(K[1])-y(x)}\sin(K[1])\right)dK[1]+\int_1^{y(x)} -e^{-\cos(x)-K[2]}\left(e^{\cos(x)+K[2]}\int_1^x e^{-\cos(K[1])-K[2]}\sin(K[1])dK[1]-1\right)dK[2]=c_1,y(x)\right]$$

1.70 problem 70

Internal problem ID [7386]
Internal file name [OUTPUT/6487_Saturday_August_06_2022_05_19_42_AM_71234662/index.tex]

Book: First order enumerated odes
Section: section 1
Problem number: 70.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"first order special form ID 1", "first_order_ode_lie_symmetry_lookup"**

Maple gives the following as the ode type

```
[[_1st_order, `_with_symmetry_[F(x),G(x)]`]]
```

$$\boxed{y' - 5\,\mathrm{e}^{x^2+20y} = \sin(x)}$$

1.70.1 Solving as first order special form ID 1 ode

Writing the ode as

$$y' = 5\,\mathrm{e}^{x^2+20y} + \sin(x) \tag{1}$$

And using the substitution $u = \mathrm{e}^{-20y}$ then

$$u' = -20y'\mathrm{e}^{-20y}$$

The above shows that

$$\begin{aligned} y' &= -\frac{u'(x)\,\mathrm{e}^{20y}}{20} \\ &= -\frac{u'(x)}{20u} \end{aligned}$$

Substituting this in (1) gives

$$-\frac{u'(x)}{20u} = \frac{5\,\mathrm{e}^{x^2}}{u} + \sin(x)$$

The above simplifies to

$$
\begin{aligned}
-\frac{u'(x)}{20} &= 5\,\mathrm{e}^{x^2} + \sin(x)\,u(x) \\
u'(x) + 20\sin(x)\,u(x) &= -100\,\mathrm{e}^{x^2} \qquad (2)
\end{aligned}
$$

Now ode (2) is solved for $u(x)$

Entering Linear first order ODE solver. In canonical form a linear first order is

$$
u'(x) + p(x)u(x) = q(x)
$$

Where here

$$
\begin{aligned}
p(x) &= 20\sin(x) \\
q(x) &= -100\,\mathrm{e}^{x^2}
\end{aligned}
$$

Hence the ode is

$$
u'(x) + 20\sin(x)\,u(x) = -100\,\mathrm{e}^{x^2}
$$

The integrating factor μ is

$$
\begin{aligned}
\mu &= \mathrm{e}^{\int 20\sin(x)dx} \\
&= \mathrm{e}^{-20\cos(x)}
\end{aligned}
$$

The ode becomes

$$
\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}x}(\mu u) &= (\mu)\left(-100\,\mathrm{e}^{x^2}\right) \\
\frac{\mathrm{d}}{\mathrm{d}x}\left(\mathrm{e}^{-20\cos(x)}u\right) &= \left(\mathrm{e}^{-20\cos(x)}\right)\left(-100\,\mathrm{e}^{x^2}\right) \\
\mathrm{d}\left(\mathrm{e}^{-20\cos(x)}u\right) &= \left(-100\,\mathrm{e}^{x^2-20\cos(x)}\right)\,\mathrm{d}x
\end{aligned}
$$

Integrating gives

$$
\begin{aligned}
\mathrm{e}^{-20\cos(x)}u &= \int -100\,\mathrm{e}^{x^2-20\cos(x)}\,\mathrm{d}x \\
\mathrm{e}^{-20\cos(x)}u &= \int -100\,\mathrm{e}^{x^2-20\cos(x)}dx + c_1
\end{aligned}
$$

Dividing both sides by the integrating factor $\mu = \mathrm{e}^{-20\cos(x)}$ results in

$$
u(x) = \mathrm{e}^{20\cos(x)}\left(\int -100\,\mathrm{e}^{x^2-20\cos(x)}dx\right) + c_1\mathrm{e}^{20\cos(x)}
$$

which simplifies to

$$u(x) = e^{20\cos(x)} \left(-100\left(\int e^{x^2-20\cos(x)}dx\right) + c_1\right)$$

Substituting the solution found for $u(x)$ in $u = e^{-20y}$ gives

$$\begin{aligned} y &= -\frac{\ln(u(x))}{20} \\ &= -\frac{\ln\left(e^{20\cos(x)}\left(-100\left(\int e^{x^2-20\cos(x)}dx\right) + c_1\right)\right)}{20} \\ &= -\frac{\ln\left(e^{20\cos(x)}\left(-100\left(\int e^{x^2-20\cos(x)}dx\right) + c_1\right)\right)}{20} \end{aligned}$$

<u>Summary</u>
The solution(s) found are the following

$$y = -\frac{\ln\left(e^{20\cos(x)}\left(-100\left(\int e^{x^2-20\cos(x)}dx\right) + c_1\right)\right)}{20} \tag{1}$$

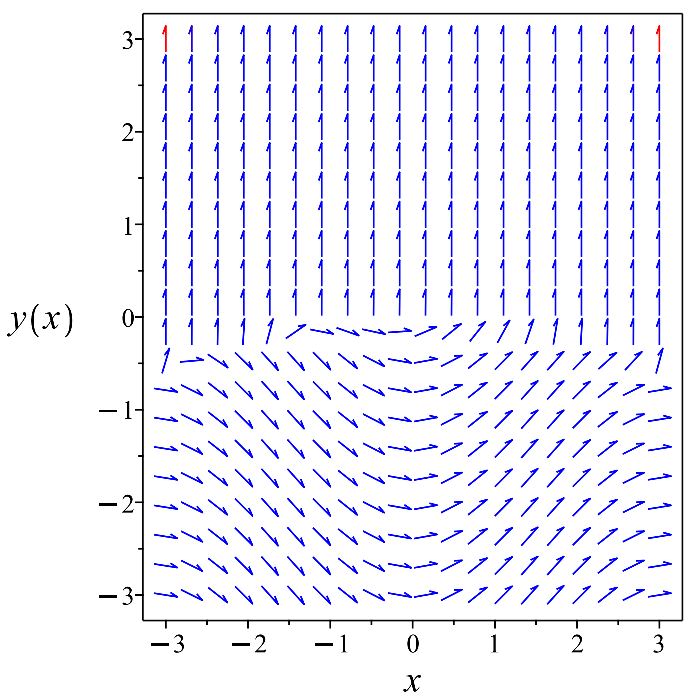


Figure 49: Slope field plot

Verification of solutions

$$y = -\frac{\ln\left(\mathrm{e}^{20\cos(x)}\left(-100\left(\int \mathrm{e}^{x^2-20\cos(x)}dx\right)+c_1\right)\right)}{20}$$

Verified OK.

1.70.2 Solving as first order ode lie symmetry lookup ode

Writing the ode as

$$y' = 5\,\mathrm{e}^{x^2+20y} + \sin(x)$$
$$y' = \omega(x,y)$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x + \omega(\eta_y - \xi_x) - \omega^2\xi_y - \omega_x\xi - \omega_y\eta = 0 \tag{A}$$

The type of this ode is known. It is of type `first order special form ID 1`. Therefore we do not need to solve the PDE (A), and can just use the lookup table shown below to find ξ, η

Table 81: Lie symmetry infinitesimal lookup table for known first order ODE's

ODE class	Form	ξ	η
linear ode	$y' = f(x)y(x) + g(x)$	0	$e^{\int f dx}$
separable ode	$y' = f(x)\,g(y)$	$\frac{1}{f}$	0
quadrature ode	$y' = f(x)$	0	1
quadrature ode	$y' = g(y)$	1	0
homogeneous ODEs of Class A	$y' = f\left(\frac{y}{x}\right)$	x	y
homogeneous ODEs of Class C	$y' = (a + bx + cy)^{\frac{n}{m}}$	1	$-\frac{b}{c}$
homogeneous class D	$y' = \frac{y}{x} + g(x)\,F\left(\frac{y}{x}\right)$	x^2	xy
First order special form ID 1	$y' = g(x)\,e^{h(x)+by} + f(x)$	$\frac{e^{-\int bf(x)dx - h(x)}}{g(x)}$	$\frac{f(x)e^{-\int bf(x)dx - h(x)}}{g(x)}$
polynomial type ode	$y' = \frac{a_1x+b_1y+c_1}{a_2x+b_2y+c_2}$	$\frac{a_1b_2x - a_2b_1x - b_1c_2 + b_2c_1}{a_1b_2 - a_2b_1}$	$\frac{a_1b_2y - a_2b_1y - a_1c_2 - a_2c_1}{a_1b_2 - a_2b_1}$
Bernoulli ode	$y' = f(x)\,y + g(x)\,y^n$	0	$e^{-\int (n-1)f(x)dx}y^n$
Reduced Riccati	$y' = f_1(x)\,y + f_2(x)\,y^2$	0	$e^{-\int f_1 dx}$

The above table shows that

$$\xi(x, y) = \frac{e^{20\cos(x)-x^2}}{5}$$
$$\eta(x, y) = \sin(x) + \frac{e^{20\cos(x)-x^2}}{5} \tag{A1}$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x, y) \to (R, S)$ where (R, S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}$$

The above comes from the requirements that $\left(\xi\frac{\partial}{\partial x} + \eta\frac{\partial}{\partial y}\right) S(x, y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the

canonical coordinates, where $S(R)$. Therefore

$$\begin{aligned}\frac{dy}{dx} &= \frac{\eta}{\xi} \\ &= \frac{\sin(x) + \frac{e^{20\cos(x)-x^2}}{5}}{\frac{e^{20\cos(x)-x^2}}{5}} \\ &= \left(5\sin(x) + e^{20\cos(x)-x^2}\right) e^{x^2-20\cos(x)}\end{aligned}$$

This is easily solved to give

$$y = \int \left(5\sin(x) + e^{20\cos(x)-x^2}\right) e^{x^2-20\cos(x)} dx + c_1$$

Where now the coordinate R is taken as the constant of integration. Hence

$$R = -\left(\int \left(5\sin(x) + e^{20\cos(x)-x^2}\right) e^{x^2-20\cos(x)} dx\right) + y$$

And S is found from

$$\begin{aligned}dS &= \frac{dx}{\xi} \\ &= \frac{dx}{\frac{e^{20\cos(x)-x^2}}{5}}\end{aligned}$$

Integrating gives

$$\begin{aligned}S &= \int \frac{dx}{T} \\ &= \int 5\, e^{x^2-20\cos(x)} dx\end{aligned}$$

Where the constant of integration is set to zero as we just need one solution.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
trying homogeneous types:
trying Chini
differential order: 1; looking for linear symmetries
trying exact
Looking for potential symmetries
trying inverse_Riccati
trying an equivalence to an Abel ODE
differential order: 1; trying a linearization to 2nd order
--- trying a change of variables {x -> y(x), y(x) -> x}
differential order: 1; trying a linearization to 2nd order
trying 1st order ODE linearizable_by_differentiation
--- Trying Lie symmetry methods, 1st order ---
`, `-> Computing symmetries using: way = 3
`, `-> Computing symmetries using: way = 4
`, `-> Computing symmetries using: way = 5
trying symmetry patterns for 1st order ODEs
-> trying a symmetry pattern of the form [F(x)*G(y), 0]
-> trying a symmetry pattern of the form [0, F(x)*G(y)]
<- symmetry pattern of the form [0, F(x)*G(y)] successful`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 33

```
dsolve(diff(y(x),x)=5*exp(x^2+20*y(x))+sin(x),y(x), singsol=all)
```

$$y(x) = -\cos(x) - \frac{\ln(20)}{20} - \frac{\ln\left(-c_1 - 5\left(\int e^{x^2-20\cos(x)}dx\right)\right)}{20}$$

✓ Solution by Mathematica

Time used: 10.354 (sec). Leaf size: 140

```
DSolve[y'[x]==5*Exp[x^2+20*y[x]]+Sin[x],y[x],x,IncludeSingularSolutions -> True]
```

$$\text{Solve}\left[\int_1^x -\frac{1}{100}e^{-20\cos(K[1])-20y(x)}\left(\sin(K[1])+5e^{K[1]^2+20y(x)}\right)dK[1]+\int_1^{y(x)} -\frac{1}{100}e^{-20\cos(x)-20K[2]}\left(100e^{20\cos(x)+20K[2]}\int_1^x\left(\frac{1}{5}e^{-20\cos(K[1])-20K[2]}\left(\sin(K[1])+5e^{K[1]^2+20K[2]}\right)-e^{K[1]^2-20\cos\right.\right.\right.$$
$$\left.\left.-1\right)dK[2]=c_1,y(x)\right]$$

2 section 2 (system of first order ode's)

2.1 problem 1

Internal problem ID [7387]
Internal file name [`OUTPUT/6633_Monday_November_27_2023_11_02_13_PM_16279652/index.tex`]

Book: First order enumerated odes
Section: section 2 (system of first order ode's)
Problem number: 1.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"system of linear ODEs"**

Solve

$$x'(t) + y'(t) = x(t) + y(t) + t$$
$$x'(t) + y'(t) = 2x(t) + 3y(t) + e^t$$

The system is

$$x'(t) + y'(t) = x(t) + y(t) + t \tag{1}$$
$$x'(t) + y'(t) = 2x(t) + 3y(t) + e^t \tag{2}$$

Since the left side is the same, this implies

$$x(t) + y(t) + t = 2x(t) + 3y(t) + e^t$$
$$y(t) = -\frac{x(t)}{2} - \frac{e^t}{2} + \frac{t}{2} \tag{3}$$

Taking derivative of the above w.r.t. t gives

$$y'(t) = -\frac{x'(t)}{2} - \frac{e^t}{2} + \frac{1}{2} \tag{4}$$

Substituting (3,4) in (1) to eliminate $y(t)\,, y'(t)$ gives

$$\frac{x'(t)}{2} - \frac{e^t}{2} + \frac{1}{2} = \frac{x(t)}{2} - \frac{e^t}{2} + \frac{3t}{2}$$
$$x'(t) = x(t) + 3t - 1 \tag{5}$$

Which is now solved for $x(t)$.

Entering Linear first order ODE solver. In canonical form a linear first order is

$$x'(t) + p(t)x(t) = q(t)$$

Where here

$$
\begin{aligned}
p(t) &= -1 \\
q(t) &= 3t - 1
\end{aligned}
$$

Hence the ode is

$$
x'(t) - x(t) = 3t - 1
$$

The integrating factor μ is

$$
\begin{aligned}
\mu &= e^{\int (-1)dt} \\
&= e^{-t}
\end{aligned}
$$

The ode becomes

$$
\begin{aligned}
\frac{d}{dt}(\mu x) &= (\mu)(3t - 1) \\
\frac{d}{dt}(e^{-t}x) &= (e^{-t})(3t - 1) \\
d(e^{-t}x) &= ((3t - 1)e^{-t})\ dt
\end{aligned}
$$

Integrating gives

$$
\begin{aligned}
e^{-t}x &= \int (3t - 1)e^{-t}\,dt \\
e^{-t}x &= -(3t + 2)e^{-t} + c_1
\end{aligned}
$$

Dividing both sides by the integrating factor $\mu = e^{-t}$ results in

$$
x(t) = -e^{t}(3t + 2)e^{-t} + c_1 e^{t}
$$

which simplifies to

$$
x(t) = -3t - 2 + c_1 e^{t}
$$

Given now that we have the solution

$$
x(t) = -3t - 2 + c_1 e^{t} \tag{6}
$$

Then substituting (6) into (3) gives

$$
y(t) = 2t + 1 - \frac{c_1 e^{t}}{2} - \frac{e^{t}}{2} \tag{7}
$$

✓ Solution by Maple

Time used: 0.015 (sec). Leaf size: 31

```
dsolve([diff(x(t),t)+diff(y(t),t)-x(t)=y(t)+t,diff(x(t),t)+diff(y(t),t)=2*x(t)+3*y(t)+exp(t)
```

$$x(t) = -3t - 2 + c_1 e^t$$
$$y(t) = 2t + 1 - \frac{c_1 e^t}{2} - \frac{e^t}{2}$$

✓ Solution by Mathematica

Time used: 0.042 (sec). Leaf size: 37

```
DSolve[{x'[t]+y'[t]-x[t]==y[t]+t,x'[t]+y'[t]==2*x[t]+3*y[t]+Exp[t]},{x[t],y[t]},t,IncludeSin
```

$$x(t) \to -3t + (1 + 2c_1)e^t - 2$$
$$y(t) \to 2t - (1 + c_1)e^t + 1$$

2.2 problem 2

Internal problem ID [7388]
Internal file name [`OUTPUT/6634_Monday_November_27_2023_11_02_14_PM_68593190/index.tex`]

Book: First order enumerated odes
Section: section 2 (system of first order ode's)
Problem number: 2.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"system of linear ODEs"**

Solve

$$x'(t) = -x(t) - 2y(t) + t - e^{t}$$
$$y'(t) = 3x(t) + 5y(t) - t + 2\,e^{t}$$

2.2.1 Solution using Matrix exponential method

In this method, we will assume we have found the matrix exponential e^{At} allready. There are different methods to determine this but will not be shown here. This is a system of linear ODE's given as

$$\vec{x}'(t) = A\,\vec{x}(t) + \vec{G}(t)$$

Or

$$\begin{bmatrix} x'(t) \\ y'(t) \end{bmatrix} = \begin{bmatrix} -1 & -2 \\ 3 & 5 \end{bmatrix} \begin{bmatrix} x(t) \\ y(t) \end{bmatrix} + \begin{bmatrix} t - e^{t} \\ -t + 2\,e^{t} \end{bmatrix}$$

Since the system is nonhomogeneous, then the solution is given by

$$\vec{x}(t) = \vec{x}_h(t) + \vec{x}_p(t)$$

Where $\vec{x}_h(t)$ is the homogeneous solution to $\vec{x}'(t) = A\,\vec{x}(t)$ and $\vec{x}_p(t)$ is a particular solution to $\vec{x}'(t) = A\,\vec{x}(t) + \vec{G}(t)$. The particular solution will be found using variation of parameters method applied to the fundamental matrix. For the above matrix A, the matrix exponential can be found to be

$$e^{At} = \begin{bmatrix} \frac{(1+\sqrt{3})e^{-(\sqrt{3}-2)t}}{2} - \frac{e^{(2+\sqrt{3})t}(\sqrt{3}-1)}{2} & \frac{\left(-e^{(2+\sqrt{3})t}+e^{-(\sqrt{3}-2)t}\right)\sqrt{3}}{3} \\ -\frac{\left(-e^{(2+\sqrt{3})t}+e^{-(\sqrt{3}-2)t}\right)\sqrt{3}}{2} & \frac{(-\sqrt{3}+1)e^{-(\sqrt{3}-2)t}}{2} + \frac{e^{(2+\sqrt{3})t}(1+\sqrt{3})}{2} \end{bmatrix}$$

Therefore the homogeneous solution is

$$\begin{aligned}
\vec{x}_h(t) &= e^{At}\vec{c} \\
&= \begin{bmatrix} \frac{(1+\sqrt{3})e^{-(\sqrt{3}-2)t}}{2} - \frac{e^{(2+\sqrt{3})t}(\sqrt{3}-1)}{2} & \frac{\left(-e^{(2+\sqrt{3})t}+e^{-(\sqrt{3}-2)t}\right)\sqrt{3}}{3} \\ -\frac{\left(-e^{(2+\sqrt{3})t}+e^{-(\sqrt{3}-2)t}\right)\sqrt{3}}{2} & \frac{(-\sqrt{3}+1)e^{-(\sqrt{3}-2)t}}{2} + \frac{e^{(2+\sqrt{3})t}(1+\sqrt{3})}{2} \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} \\
&= \begin{bmatrix} \left(\frac{(1+\sqrt{3})e^{-(\sqrt{3}-2)t}}{2} - \frac{e^{(2+\sqrt{3})t}(\sqrt{3}-1)}{2}\right)c_1 + \frac{\left(-e^{(2+\sqrt{3})t}+e^{-(\sqrt{3}-2)t}\right)\sqrt{3}\,c_2}{3} \\ -\frac{\left(-e^{(2+\sqrt{3})t}+e^{-(\sqrt{3}-2)t}\right)\sqrt{3}\,c_1}{2} + \left(\frac{(-\sqrt{3}+1)e^{-(\sqrt{3}-2)t}}{2} + \frac{e^{(2+\sqrt{3})t}(1+\sqrt{3})}{2}\right)c_2 \end{bmatrix} \\
&= \begin{bmatrix} \frac{\left((3c_1+2c_2)\sqrt{3}+3c_1\right)e^{-(\sqrt{3}-2)t}}{6} - \frac{\left(\left(c_1+\frac{2c_2}{3}\right)\sqrt{3}-c_1\right)e^{(2+\sqrt{3})t}}{2} \\ \frac{\left((-c_1-c_2)\sqrt{3}+c_2\right)e^{-(\sqrt{3}-2)t}}{2} + \frac{\left((c_1+c_2)\sqrt{3}+c_2\right)e^{(2+\sqrt{3})t}}{2} \end{bmatrix}
\end{aligned}$$

The particular solution given by

$$\vec{x}_p(t) = e^{At} \int e^{-At}\vec{G}(t)\, dt$$

But

$$\begin{aligned}
e^{-At} &= (e^{At})^{-1} \\
&= \begin{bmatrix} \frac{\left((-\sqrt{3}+1)e^{-(\sqrt{3}-2)t}+e^{(2+\sqrt{3})t}(1+\sqrt{3})\right)e^{-4t}}{2} & -\frac{\sqrt{3}\,e^{-4t}\left(-e^{(2+\sqrt{3})t}+e^{-(\sqrt{3}-2)t}\right)}{3} \\ \frac{\sqrt{3}\,e^{-4t}\left(-e^{(2+\sqrt{3})t}+e^{-(\sqrt{3}-2)t}\right)}{2} & \frac{e^{-4t}\left(\sqrt{3}\,e^{-(\sqrt{3}-2)t}-\sqrt{3}\,e^{(2+\sqrt{3})t}+e^{-(\sqrt{3}-2)t}+e^{(2+\sqrt{3})t}\right)}{2} \end{bmatrix}
\end{aligned}$$

Hence

$$\vec{x}_p(t) = \left[\begin{array}{cc} \frac{\left(1+\sqrt{3}\right)\mathrm{e}^{-\left(\sqrt{3}-2\right)t}}{2}-\frac{\mathrm{e}^{\left(2+\sqrt{3}\right)t}\left(\sqrt{3}-1\right)}{2} & \frac{\left(-\mathrm{e}^{\left(2+\sqrt{3}\right)t}+\mathrm{e}^{-\left(\sqrt{3}-2\right)t}\right)\sqrt{3}}{3} \\ -\frac{\left(-\mathrm{e}^{\left(2+\sqrt{3}\right)t}+\mathrm{e}^{-\left(\sqrt{3}-2\right)t}\right)\sqrt{3}}{2} & \frac{\left(-\sqrt{3}+1\right)\mathrm{e}^{-\left(\sqrt{3}-2\right)t}}{2}+\frac{\mathrm{e}^{\left(2+\sqrt{3}\right)t}\left(1+\sqrt{3}\right)}{2} \end{array}\right] \int \left[\begin{array}{c} \frac{\left(\left(-\sqrt{3}+1\right)\mathrm{e}^{-\left(\sqrt{3}-2\right)t}+\mathrm{e}\cdots}{2} \\ \frac{\sqrt{3}\,\mathrm{e}^{-4t}\left(-\mathrm{e}^{\left(2+\sqrt{3}\right.}\cdots}{2} \end{array}\right.$$

$$= \left[\begin{array}{cc} \frac{\left(1+\sqrt{3}\right)\mathrm{e}^{-\left(\sqrt{3}-2\right)t}}{2}-\frac{\mathrm{e}^{\left(2+\sqrt{3}\right)t}\left(\sqrt{3}-1\right)}{2} & \frac{\left(-\mathrm{e}^{\left(2+\sqrt{3}\right)t}+\mathrm{e}^{-\left(\sqrt{3}-2\right)t}\right)\sqrt{3}}{3} \\ -\frac{\left(-\mathrm{e}^{\left(2+\sqrt{3}\right)t}+\mathrm{e}^{-\left(\sqrt{3}-2\right)t}\right)\sqrt{3}}{2} & \frac{\left(-\sqrt{3}+1\right)\mathrm{e}^{-\left(\sqrt{3}-2\right)t}}{2}+\frac{\mathrm{e}^{\left(2+\sqrt{3}\right)t}\left(1+\sqrt{3}\right)}{2} \end{array}\right] \left[\begin{array}{c} \frac{\left(\left(5t+19\right)\sqrt{3}-9t-33\right)\mathrm{e}^{-\left(2+\right.}\cdots}{6} \\ \frac{\left(7+\left(-4-t\right)\sqrt{3}+2t\right)\mathrm{e}^{-\left(2+\sqrt{3}\right)}\cdots}{2} \end{array}\right.$$

$$= \left[\begin{array}{c} -3t-11 \\ 2t+7-\frac{\mathrm{e}^{t}}{2} \end{array}\right]$$

Hence the complete solution is

$$\begin{aligned} \vec{x}(t) &= \vec{x}_h(t)+\vec{x}_p(t) \\ &= \left[\begin{array}{c} \frac{\left(\left(3c_1+2c_2\right)\sqrt{3}+3c_1\right)\mathrm{e}^{-\left(\sqrt{3}-2\right)t}}{6}+\frac{\left(\left(-3c_1-2c_2\right)\sqrt{3}+3c_1\right)\mathrm{e}^{\left(2+\sqrt{3}\right)t}}{6}-3t-11 \\ \frac{\left(\left(-c_1-c_2\right)\sqrt{3}+c_2\right)\mathrm{e}^{-\left(\sqrt{3}-2\right)t}}{2}+\frac{\left(\left(c_1+c_2\right)\sqrt{3}+c_2\right)\mathrm{e}^{\left(2+\sqrt{3}\right)t}}{2}+2t+7-\frac{\mathrm{e}^{t}}{2} \end{array}\right] \end{aligned}$$

2.2.2 Solution using explicit Eigenvalue and Eigenvector method

This is a system of linear ODE's given as

$$\vec{x}'(t) = A\,\vec{x}(t)+\vec{G}(t)$$

Or

$$\left[\begin{array}{c} x'(t) \\ y'(t) \end{array}\right] = \left[\begin{array}{cc} -1 & -2 \\ 3 & 5 \end{array}\right]\left[\begin{array}{c} x(t) \\ y(t) \end{array}\right]+\left[\begin{array}{c} t-\mathrm{e}^{t} \\ -t+2\,\mathrm{e}^{t} \end{array}\right]$$

Since the system is nonhomogeneous, then the solution is given by

$$\vec{x}(t) = \vec{x}_h(t)+\vec{x}_p(t)$$

Where $\vec{x}_h(t)$ is the homogeneous solution to $\vec{x}'(t) = A\,\vec{x}(t)$ and $\vec{x}_p(t)$ is a particular solution to $\vec{x}'(t) = A\,\vec{x}(t)+\vec{G}(t)$. The particular solution will be found using variation of parameters method applied to the fundamental matrix.

The first step is find the homogeneous solution. We start by finding the eigenvalues of A. This is done by solving the following equation for the eigenvalues λ

$$\det(A - \lambda I) = 0$$

Expanding gives

$$\det\left(\begin{bmatrix} -1 & -2 \\ 3 & 5 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right) = 0$$

Therefore

$$\det\left(\begin{bmatrix} -1-\lambda & -2 \\ 3 & 5-\lambda \end{bmatrix}\right) = 0$$

Which gives the characteristic equation

$$\lambda^2 - 4\lambda + 1 = 0$$

The roots of the above are the eigenvalues.

$$\lambda_1 = 2 + \sqrt{3}$$
$$\lambda_2 = 2 - \sqrt{3}$$

This table summarises the above result

eigenvalue	algebraic multiplicity	type of eigenvalue
$2-\sqrt{3}$	1	real eigenvalue
$2+\sqrt{3}$	1	real eigenvalue

Now the eigenvector for each eigenvalue are found.

Considering the eigenvalue $\lambda_1 = 2 - \sqrt{3}$

We need to solve $A\vec{v} = \lambda\vec{v}$ or $(A - \lambda I)\vec{v} = \vec{0}$ which becomes

$$\left(\begin{bmatrix} -1 & -2 \\ 3 & 5 \end{bmatrix} - \left(2-\sqrt{3}\right)\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right)\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$
$$\begin{bmatrix} -3+\sqrt{3} & -2 \\ 3 & 3+\sqrt{3} \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Now forward elimination is applied to solve for the eigenvector $\vec{v}$. The augmented matrix is

$$\left[\begin{array}{cc|c} -3+\sqrt{3} & -2 & 0 \\ 3 & 3+\sqrt{3} & 0 \end{array}\right]$$

$$R_2 = R_2 - \frac{3R_1}{-3+\sqrt{3}} \implies \left[\begin{array}{cc|c} -3+\sqrt{3} & -2 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

Therefore the system in Echelon form is

$$\begin{bmatrix} -3+\sqrt{3} & -2 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

The free variables are $\{v_2\}$ and the leading variables are $\{v_1\}$. Let $v_2 = t$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\left\{v_1 = \frac{2t}{-3+\sqrt{3}}\right\}$

Hence the solution is

$$\begin{bmatrix} \frac{2t}{-3+\sqrt{3}} \\ t \end{bmatrix} = \begin{bmatrix} \frac{2t}{-3+\sqrt{3}} \\ t \end{bmatrix}$$

Since there is one free Variable, we have found one eigenvector associated with this eigenvalue. The above can be written as

$$\begin{bmatrix} \frac{2t}{-3+\sqrt{3}} \\ t \end{bmatrix} = t \begin{bmatrix} \frac{2}{-3+\sqrt{3}} \\ 1 \end{bmatrix}$$

Let $t = 1$ the eigenvector becomes

$$\begin{bmatrix} \frac{2t}{-3+\sqrt{3}} \\ t \end{bmatrix} = \begin{bmatrix} \frac{2}{-3+\sqrt{3}} \\ 1 \end{bmatrix}$$

Which is normalized to

$$\begin{bmatrix} \frac{2t}{-3+\sqrt{3}} \\ t \end{bmatrix} = \begin{bmatrix} \frac{2}{-3+\sqrt{3}} \\ 1 \end{bmatrix}$$

Considering the eigenvalue $\lambda_2 = 2+\sqrt{3}$

We need to solve $A\vec{v} = \lambda\vec{v}$ or $(A - \lambda I)\vec{v} = \vec{0}$ which becomes

$$\left(\begin{bmatrix} -1 & -2 \\ 3 & 5 \end{bmatrix} - \left(2+\sqrt{3}\right)\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right)\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} -3-\sqrt{3} & -2 \\ 3 & 3-\sqrt{3} \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Now forward elimination is applied to solve for the eigenvector $\vec{v}$. The augmented matrix is

$$\left[\begin{array}{cc|c} -3-\sqrt{3} & -2 & 0 \\ 3 & 3-\sqrt{3} & 0 \end{array}\right]$$

$$R_2 = R_2 - \frac{3R_1}{-3-\sqrt{3}} \implies \left[\begin{array}{cc|c} -3-\sqrt{3} & -2 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

Therefore the system in Echelon form is

$$\begin{bmatrix} -3-\sqrt{3} & -2 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

The free variables are $\{v_2\}$ and the leading variables are $\{v_1\}$. Let $v_2 = t$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\left\{v_1 = -\frac{2t}{3+\sqrt{3}}\right\}$

Hence the solution is

$$\begin{bmatrix} -\frac{2t}{3+\sqrt{3}} \\ t \end{bmatrix} = \begin{bmatrix} -\frac{2t}{3+\sqrt{3}} \\ t \end{bmatrix}$$

Since there is one free Variable, we have found one eigenvector associated with this eigenvalue. The above can be written as

$$\begin{bmatrix} -\frac{2t}{3+\sqrt{3}} \\ t \end{bmatrix} = t\begin{bmatrix} -\frac{2}{3+\sqrt{3}} \\ 1 \end{bmatrix}$$

Let $t = 1$ the eigenvector becomes

$$\begin{bmatrix} -\frac{2t}{3+\sqrt{3}} \\ t \end{bmatrix} = \begin{bmatrix} -\frac{2}{3+\sqrt{3}} \\ 1 \end{bmatrix}$$

Which is normalized to

$$\begin{bmatrix} -\frac{2t}{3+\sqrt{3}} \\ t \end{bmatrix} = \begin{bmatrix} -\frac{2}{3+\sqrt{3}} \\ 1 \end{bmatrix}$$

The following table gives a summary of this result. It shows for each eigenvalue the algebraic multiplicity m, and its geometric multiplicity k and the eigenvectors associated with the eigenvalue. If $m > k$ then the eigenvalue is defective which means the number of normal linearly independent eigenvectors associated with this eigenvalue (called the geometric multiplicity k) does not equal the algebraic multiplicity m, and we need to determine an additional $m - k$ generalized eigenvectors for this eigenvalue.

	multiplicity			
eigenvalue	algebraic m	geometric k	defective?	eigenvectors
$2+\sqrt{3}$	1	1	No	$\begin{bmatrix} -\frac{2}{3+\sqrt{3}} \\ 1 \end{bmatrix}$
$2-\sqrt{3}$	1	1	No	$\begin{bmatrix} -\frac{2}{3-\sqrt{3}} \\ 1 \end{bmatrix}$

Now that we found the eigenvalues and associated eigenvectors, we will go over each eigenvalue and generate the solution basis. The only problem we need to take care of is if the eigenvalue is defective. Since eigenvalue $2+\sqrt{3}$ is real and distinct then the corresponding eigenvector solution is

$$\begin{aligned} \vec{x}_1(t) &= \vec{v}_1 e^{\left(2+\sqrt{3}\right)t} \\ &= \begin{bmatrix} -\frac{2}{3+\sqrt{3}} \\ 1 \end{bmatrix} e^{\left(2+\sqrt{3}\right)t} \end{aligned}$$

Since eigenvalue $2-\sqrt{3}$ is real and distinct then the corresponding eigenvector solution is

$$\begin{aligned} \vec{x}_2(t) &= \vec{v}_2 e^{\left(2-\sqrt{3}\right)t} \\ &= \begin{bmatrix} -\frac{2}{3-\sqrt{3}} \\ 1 \end{bmatrix} e^{\left(2-\sqrt{3}\right)t} \end{aligned}$$

Therefore the homogeneous solution is

$$\vec{x}_h(t) = c_1 \vec{x}_1(t) + c_2 \vec{x}_2(t)$$

Which is written as

$$\begin{bmatrix} x(t) \\ y(t) \end{bmatrix} = c_1 \begin{bmatrix} -\frac{2\,\mathrm{e}^{\left(2+\sqrt{3}\right)t}}{3+\sqrt{3}} \\ \mathrm{e}^{\left(2+\sqrt{3}\right)t} \end{bmatrix} + c_2 \begin{bmatrix} -\frac{2\,\mathrm{e}^{\left(2-\sqrt{3}\right)t}}{3-\sqrt{3}} \\ \mathrm{e}^{\left(2-\sqrt{3}\right)t} \end{bmatrix}$$

Now that we found homogeneous solution above, we need to find a particular solution $\vec{x}_p(t)$. We will use Variation of parameters. The fundamental matrix is

$$\Phi = \begin{bmatrix} \vec{x}_1 & \vec{x}_2 & \cdots \end{bmatrix}$$

Where $\vec{x}_i$ are the solution basis found above. Therefore the fundamental matrix is

$$\Phi(t) = \begin{bmatrix} -\frac{2\,\mathrm{e}^{\left(2+\sqrt{3}\right)t}}{3+\sqrt{3}} & -\frac{2\,\mathrm{e}^{\left(2-\sqrt{3}\right)t}}{3-\sqrt{3}} \\ \mathrm{e}^{\left(2+\sqrt{3}\right)t} & \mathrm{e}^{\left(2-\sqrt{3}\right)t} \end{bmatrix}$$

The particular solution is then given by

$$\vec{x}_p(t) = \Phi \int \Phi^{-1} \vec{G}(t)\, dt$$

But

$$\Phi^{-1} = \begin{bmatrix} \frac{\sqrt{3}\,\mathrm{e}^{-\left(2+\sqrt{3}\right)t}}{2} & \frac{\sqrt{3}\left(3+\sqrt{3}\right)\mathrm{e}^{-\left(2+\sqrt{3}\right)t}}{6} \\ -\frac{\sqrt{3}\,\mathrm{e}^{\left(\sqrt{3}-2\right)t}}{2} & \frac{\mathrm{e}^{\left(\sqrt{3}-2\right)t}\sqrt{3}\left(-3+\sqrt{3}\right)}{6} \end{bmatrix}$$

Hence

$$\vec{x}_p(t) = \begin{bmatrix} -\frac{2\,\mathrm{e}^{\left(2+\sqrt{3}\right)t}}{3+\sqrt{3}} & -\frac{2\,\mathrm{e}^{\left(2-\sqrt{3}\right)t}}{3-\sqrt{3}} \\ \mathrm{e}^{\left(2+\sqrt{3}\right)t} & \mathrm{e}^{\left(2-\sqrt{3}\right)t} \end{bmatrix} \int \begin{bmatrix} \frac{\sqrt{3}\,\mathrm{e}^{-\left(2+\sqrt{3}\right)t}}{2} & \frac{\sqrt{3}\left(3+\sqrt{3}\right)\mathrm{e}^{-\left(2+\sqrt{3}\right)t}}{6} \\ -\frac{\sqrt{3}\,\mathrm{e}^{\left(\sqrt{3}-2\right)t}}{2} & \frac{\mathrm{e}^{\left(\sqrt{3}-2\right)t}\sqrt{3}\left(-3+\sqrt{3}\right)}{6} \end{bmatrix} \begin{bmatrix} t-\mathrm{e}^{t} \\ -t+2\,\mathrm{e}^{t} \end{bmatrix} dt$$

$$= \begin{bmatrix} -\frac{2\,\mathrm{e}^{\left(2+\sqrt{3}\right)t}}{3+\sqrt{3}} & -\frac{2\,\mathrm{e}^{\left(2-\sqrt{3}\right)t}}{3-\sqrt{3}} \\ \mathrm{e}^{\left(2+\sqrt{3}\right)t} & \mathrm{e}^{\left(2-\sqrt{3}\right)t} \end{bmatrix} \int \begin{bmatrix} \frac{\sqrt{3}\,\mathrm{e}^{-t\left(1+\sqrt{3}\right)}}{2} + \mathrm{e}^{-t\left(1+\sqrt{3}\right)} - \frac{\mathrm{e}^{-\left(2+\sqrt{3}\right)t}t}{2} \\ -\frac{\sqrt{3}\,\mathrm{e}^{t\left(\sqrt{3}-1\right)}}{2} + \mathrm{e}^{t\left(\sqrt{3}-1\right)} - \frac{\mathrm{e}^{\left(\sqrt{3}-2\right)t}t}{2} \end{bmatrix} dt$$

$$= \begin{bmatrix} -\frac{2\,\mathrm{e}^{\left(2+\sqrt{3}\right)t}}{3+\sqrt{3}} & -\frac{2\,\mathrm{e}^{\left(2-\sqrt{3}\right)t}}{3-\sqrt{3}} \\ \mathrm{e}^{\left(2+\sqrt{3}\right)t} & \mathrm{e}^{\left(2-\sqrt{3}\right)t} \end{bmatrix} \begin{bmatrix} \frac{5\sqrt{3}\left(\left(\left(t+\frac{1}{5}\right)\sqrt{3}+\frac{9t}{5}+\frac{3}{5}\right)\mathrm{e}^{-\left(2+\sqrt{3}\right)t}+\mathrm{e}^{-t\left(1+\sqrt{3}\right)}\left(-\frac{26\sqrt{3}}{5}-9\right)\right)}{6\left(1+\sqrt{3}\right)\left(2+\sqrt{3}\right)^{2}} \\ -\frac{5\left(\left(\left(t+\frac{1}{5}\right)\sqrt{3}-\frac{9t}{5}-\frac{3}{5}\right)\mathrm{e}^{\left(\sqrt{3}-2\right)t}+\mathrm{e}^{t\left(\sqrt{3}-1\right)}\left(-\frac{26\sqrt{3}}{5}+9\right)\right)\sqrt{3}}{6\left(\sqrt{3}-1\right)\left(\sqrt{3}-2\right)^{2}} \end{bmatrix}$$

$$= \begin{bmatrix} -3t-11 \\ 2t+7-\frac{\mathrm{e}^{t}}{2} \end{bmatrix}$$

Now that we found particular solution, the final solution is

$$\vec{x}(t) = \vec{x}_h(t) + \vec{x}_p(t)$$

$$\begin{bmatrix} x(t) \\ y(t) \end{bmatrix} = \begin{bmatrix} -\frac{2c_1\mathrm{e}^{\left(2+\sqrt{3}\right)t}}{3+\sqrt{3}} \\ c_1\mathrm{e}^{\left(2+\sqrt{3}\right)t} \end{bmatrix} + \begin{bmatrix} -\frac{2c_2\mathrm{e}^{\left(2-\sqrt{3}\right)t}}{3-\sqrt{3}} \\ c_2\mathrm{e}^{\left(2-\sqrt{3}\right)t} \end{bmatrix} + \begin{bmatrix} -3t-11 \\ 2t+7-\frac{\mathrm{e}^{t}}{2} \end{bmatrix}$$

Which becomes

$$\begin{bmatrix} x(t) \\ y(t) \end{bmatrix} = \begin{bmatrix} -\frac{c_2\left(3+\sqrt{3}\right)\mathrm{e}^{-\left(\sqrt{3}-2\right)t}}{3} + \frac{c_1\left(-3+\sqrt{3}\right)\mathrm{e}^{\left(2+\sqrt{3}\right)t}}{3} - 3t - 11 \\ c_1\mathrm{e}^{\left(2+\sqrt{3}\right)t} + c_2\mathrm{e}^{-\left(\sqrt{3}-2\right)t} + 2t + 7 - \frac{\mathrm{e}^{t}}{2} \end{bmatrix}$$

2.2.3 Maple step by step solution

Let's solve

$[x'(t) = -x(t) - 2y(t) + t - \mathrm{e}^{t}, y'(t) = 3x(t) + 5y(t) - t + 2\,\mathrm{e}^{t}]$

- Define vector

$$\vec{x}(t) = \begin{bmatrix} x(t) \\ y(t) \end{bmatrix}$$

- Convert system into a vector equation

$$\vec{x}'(t) = \begin{bmatrix} -1 & -2 \\ 3 & 5 \end{bmatrix} \cdot \vec{x}(t) + \begin{bmatrix} t - e^{t} \\ -t + 2\,e^{t} \end{bmatrix}$$

- System to solve

$$\vec{x}'(t) = \begin{bmatrix} -1 & -2 \\ 3 & 5 \end{bmatrix} \cdot \vec{x}(t) + \begin{bmatrix} t - e^{t} \\ -t + 2\,e^{t} \end{bmatrix}$$

- Define the forcing function

$$\vec{f}(t) = \begin{bmatrix} t - e^{t} \\ -t + 2\,e^{t} \end{bmatrix}$$

- Define the coefficient matrix

$$A = \begin{bmatrix} -1 & -2 \\ 3 & 5 \end{bmatrix}$$

- Rewrite the system as

$$\vec{x}'(t) = A \cdot \vec{x}(t) + \vec{f}$$

- To solve the system, find the eigenvalues and eigenvectors of A
- Eigenpairs of A

$$\left[\left[2 - \sqrt{3}, \begin{bmatrix} -\frac{2}{3-\sqrt{3}} \\ 1 \end{bmatrix}\right], \left[2 + \sqrt{3}, \begin{bmatrix} -\frac{2}{3+\sqrt{3}} \\ 1 \end{bmatrix}\right]\right]$$

- Consider eigenpair

$$\left[2 - \sqrt{3}, \begin{bmatrix} -\frac{2}{3-\sqrt{3}} \\ 1 \end{bmatrix}\right]$$

- Solution to homogeneous system from eigenpair

$$\vec{x}_1 = e^{\left(2-\sqrt{3}\right)t} \cdot \begin{bmatrix} -\frac{2}{3-\sqrt{3}} \\ 1 \end{bmatrix}$$

- Consider eigenpair

$$\left[2+\sqrt{3}, \left[\begin{array}{c} -\frac{2}{3+\sqrt{3}} \\ 1 \end{array}\right]\right]$$

- Solution to homogeneous system from eigenpair

$$\vec{x}_2 = \mathrm{e}^{\left(2+\sqrt{3}\right)t} \cdot \left[\begin{array}{c} -\frac{2}{3+\sqrt{3}} \\ 1 \end{array}\right]$$

- General solution of the system of ODEs can be written in terms of the particular solution $\vec{x}_p($

$$\vec{x}(t) = c_1\vec{x}_1 + c_2\vec{x}_2 + \vec{x}_p(t)$$

☐ Fundamental matrix

- ○ Let $\phi(t)$ be the matrix whose columns are the independent solutions of the homogeneous syste

$$\phi(t) = \left[\begin{array}{cc} -\frac{2\,\mathrm{e}^{\left(2-\sqrt{3}\right)t}}{3-\sqrt{3}} & -\frac{2\,\mathrm{e}^{\left(2+\sqrt{3}\right)t}}{3+\sqrt{3}} \\ \mathrm{e}^{\left(2-\sqrt{3}\right)t} & \mathrm{e}^{\left(2+\sqrt{3}\right)t} \end{array}\right]$$

- ○ The fundamental matrix, $\Phi(t)$ is a normalized version of $\phi(t)$ satisfying $\Phi(0) = I$ where I is th

$$\Phi(t) = \phi(t) \cdot \frac{1}{\phi(0)}$$

- ○ Substitute the value of $\phi(t)$ and $\phi(0)$

$$\Phi(t) = \left[\begin{array}{cc} -\frac{2\,\mathrm{e}^{\left(2-\sqrt{3}\right)t}}{3-\sqrt{3}} & -\frac{2\,\mathrm{e}^{\left(2+\sqrt{3}\right)t}}{3+\sqrt{3}} \\ \mathrm{e}^{\left(2-\sqrt{3}\right)t} & \mathrm{e}^{\left(2+\sqrt{3}\right)t} \end{array}\right] \cdot \frac{1}{\left[\begin{array}{cc} -\frac{2}{3-\sqrt{3}} & -\frac{2}{3+\sqrt{3}} \\ 1 & 1 \end{array}\right]}$$

- ○ Evaluate and simplify to get the fundamental matrix

$$\Phi(t) = \left[\begin{array}{cc} \frac{\left(\left(3+\sqrt{3}\right)\mathrm{e}^{-\left(\sqrt{3}-2\right)t}+\mathrm{e}^{\left(2+\sqrt{3}\right)t}\left(-3+\sqrt{3}\right)\right)\sqrt{3}}{6} & \frac{\left(-\mathrm{e}^{\left(2+\sqrt{3}\right)t}+\mathrm{e}^{-\left(\sqrt{3}-2\right)t}\right)\sqrt{3}}{3} \\ -\frac{\left(-\mathrm{e}^{\left(2+\sqrt{3}\right)t}+\mathrm{e}^{-\left(\sqrt{3}-2\right)t}\right)\sqrt{3}}{2} & \frac{\left(-\sqrt{3}+1\right)\mathrm{e}^{-\left(\sqrt{3}-2\right)t}}{2} + \frac{\mathrm{e}^{\left(2+\sqrt{3}\right)t}\left(1+\sqrt{3}\right)}{2} \end{array}\right]$$

☐ Find a particular solution of the system of ODEs using variation of parameters

- ○ Let the particular solution be the fundamental matrix multiplied by $\vec{v}(t)$ and solve for $\vec{v}(t)$

$$\vec{x}_p(t) = \Phi(t) \cdot \vec{v}(t)$$

- ○ Take the derivative of the particular solution

$\overrightarrow{x}_p'(t) = \Phi'(t) \cdot \overrightarrow{v}(t) + \Phi(t) \cdot \overrightarrow{v}'(t)$

- Substitute particular solution and its derivative into the system of ODEs

  $\Phi'(t) \cdot \overrightarrow{v}(t) + \Phi(t) \cdot \overrightarrow{v}'(t) = A \cdot \Phi(t) \cdot \overrightarrow{v}(t) + \overrightarrow{f}(t)$

- The fundamental matrix has columns that are solutions to the homogeneous system so its der

  $A \cdot \Phi(t) \cdot \overrightarrow{v}(t) + \Phi(t) \cdot \overrightarrow{v}'(t) = A \cdot \Phi(t) \cdot \overrightarrow{v}(t) + \overrightarrow{f}(t)$

- Cancel like terms

  $\Phi(t) \cdot \overrightarrow{v}'(t) = \overrightarrow{f}(t)$

- Multiply by the inverse of the fundamental matrix

  $\overrightarrow{v}'(t) = \frac{1}{\Phi(t)} \cdot \overrightarrow{f}(t)$

- Integrate to solve for $\overrightarrow{v}(t)$

  $\overrightarrow{v}(t) = \int_0^t \frac{1}{\Phi(s)} \cdot \overrightarrow{f}(s)\, ds$

- Plug $\overrightarrow{v}(t)$ into the equation for the particular solution

  $\overrightarrow{x}_p(t) = \Phi(t) \cdot \left(\int_0^t \frac{1}{\Phi(s)} \cdot \overrightarrow{f}(s)\, ds\right)$

- Plug in the fundamental matrix and the forcing function and compute

$$\overrightarrow{x}_p(t) = \left[\begin{array}{c} \frac{\left(33+20\sqrt{3}\right)\mathrm{e}^{-\left(\sqrt{3}-2\right)t}+\left(33-20\sqrt{3}\right)\mathrm{e}^{\left(2+\sqrt{3}\right)t}-18t-66}{6\left(\sqrt{3}-2\right)^{2}\left(2+\sqrt{3}\right)^{2}} \\ \frac{\left(-9\sqrt{3}-13\right)\mathrm{e}^{-\left(\sqrt{3}-2\right)t}}{4}+\frac{\left(9\sqrt{3}-13\right)\mathrm{e}^{\left(2+\sqrt{3}\right)t}}{4}+2t-\frac{\mathrm{e}^{t}}{2}+7 \end{array}\right]$$

- Plug particular solution back into general solution

$$\overrightarrow{x}(t) = c_1\overrightarrow{x}_1 + c_2\overrightarrow{x}_2 + \left[\begin{array}{c} \frac{\left(33+20\sqrt{3}\right)\mathrm{e}^{-\left(\sqrt{3}-2\right)t}+\left(33-20\sqrt{3}\right)\mathrm{e}^{\left(2+\sqrt{3}\right)t}-18t-66}{6\left(\sqrt{3}-2\right)^{2}\left(2+\sqrt{3}\right)^{2}} \\ \frac{\left(-9\sqrt{3}-13\right)\mathrm{e}^{-\left(\sqrt{3}-2\right)t}}{4}+\frac{\left(9\sqrt{3}-13\right)\mathrm{e}^{\left(2+\sqrt{3}\right)t}}{4}+2t-\frac{\mathrm{e}^{t}}{2}+7 \end{array}\right]$$

- Substitute in vector of dependent variables

$$\left[\begin{array}{c} x(t) \\ y(t) \end{array}\right] = \left[\begin{array}{c} \frac{\left((-2c_1+20)\sqrt{3}-6c_1+33\right)\mathrm{e}^{-\left(\sqrt{3}-2\right)t}}{6}+\frac{\left((2c_2-20)\sqrt{3}-6c_2+33\right)\mathrm{e}^{\left(2+\sqrt{3}\right)t}}{6}-3t-11 \\ \frac{\left(4c_1-9\sqrt{3}-13\right)\mathrm{e}^{-\left(\sqrt{3}-2\right)t}}{4}+\frac{\left(4c_2+9\sqrt{3}-13\right)\mathrm{e}^{\left(2+\sqrt{3}\right)t}}{4}+2t-\frac{\mathrm{e}^{t}}{2}+7 \end{array}\right]$$

- Solution to the system of ODEs

$$\left\{x(t)=\frac{\left((-2c_1+20)\sqrt{3}-6c_1+33\right)\mathrm{e}^{-\left(\sqrt{3}-2\right)t}}{6}+\frac{\left((2c_2-20)\sqrt{3}-6c_2+33\right)\mathrm{e}^{\left(2+\sqrt{3}\right)t}}{6}-3t-11,y(t)=\frac{\left(4c_1-9\sqrt{3}-13\right)\mathrm{e}}{4}\right.$$

✓ Solution by Maple

Time used: 0.015 (sec). Leaf size: 95

```
dsolve([2*diff(x(t),t)+diff(y(t),t)-x(t)=y(t)+t,diff(x(t),t)+diff(y(t),t)=2*x(t)+3*y(t)+exp(
```

$$x(t)=\mathrm{e}^{\left(2+\sqrt{3}\right)t}c_2+\mathrm{e}^{-\left(-2+\sqrt{3}\right)t}c_1-3t-11$$

$$y(t)=-\frac{\mathrm{e}^{\left(2+\sqrt{3}\right)t}c_2\sqrt{3}}{2}+\frac{\mathrm{e}^{-\left(-2+\sqrt{3}\right)t}c_1\sqrt{3}}{2}-\frac{3\,\mathrm{e}^{\left(2+\sqrt{3}\right)t}c_2}{2}-\frac{3\,\mathrm{e}^{-\left(-2+\sqrt{3}\right)t}c_1}{2}-\frac{\mathrm{e}^{t}}{2}+2t+7$$

✓ Solution by Mathematica

Time used: 10.209 (sec). Leaf size: 174

```
DSolve[{2*x'[t]+y'[t]-x[t]==y[t]+t,x'[t]+y'[t]==2*x[t]+3*y[t]+Exp[t]},{x[t],y[t]},t,IncludeS
```

$$x(t)\rightarrow\frac{1}{6}e^{-\left(\left(\sqrt{3}-2\right)t\right)}\left(-6e^{\left(\sqrt{3}-2\right)t}(3t+11)+\left(-3\left(\sqrt{3}-1\right)c_1-2\sqrt{3}c_2\right)e^{2\sqrt{3}t}+3\left(1+\sqrt{3}\right)c_1+2\sqrt{3}c_2\right)$$

$$y(t)\rightarrow\frac{1}{2}\left(4t-e^{t}+\left(-\sqrt{3}c_1-\sqrt{3}c_2+c_2\right)e^{-\left(\left(\sqrt{3}-2\right)t\right)}+\left(\sqrt{3}c_1+\left(1+\sqrt{3}\right)c_2\right)e^{\left(2+\sqrt{3}\right)t}+14\right)$$

2.3 problem 3

Internal problem ID [7389]
Internal file name [`OUTPUT/6635_Monday_November_27_2023_11_02_14_PM_64086885/index.tex`]

Book: First order enumerated odes
Section: section 2 (system of first order ode's)
Problem number: 3.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"system of linear ODEs"**

Solve

$$
\begin{aligned}
x'(t) + y'(t) &= x(t) + y(t) + t + \sin(t) + \cos(t) \\
x'(t) + y'(t) &= 2x(t) + 3y(t) + \mathrm{e}^{t}
\end{aligned}
$$

The system is

$$
x'(t) + y'(t) = x(t) + y(t) + t + \sin(t) + \cos(t) \tag{1}
$$
$$
x'(t) + y'(t) = 2x(t) + 3y(t) + \mathrm{e}^{t} \tag{2}
$$

Since the left side is the same, this implies

$$
x(t) + y(t) + t + \sin(t) + \cos(t) = 2x(t) + 3y(t) + \mathrm{e}^{t}
$$
$$
y(t) = -\frac{x(t)}{2} - \frac{\mathrm{e}^{t}}{2} + \frac{t}{2} + \frac{\sin(t)}{2} + \frac{\cos(t)}{2} \tag{3}
$$

Taking derivative of the above w.r.t. t gives

$$
y'(t) = -\frac{x'(t)}{2} - \frac{\mathrm{e}^{t}}{2} + \frac{1}{2} + \frac{\cos(t)}{2} - \frac{\sin(t)}{2} \tag{4}
$$

Substituting (3,4) in (1) to eliminate $y(t), y'(t)$ gives

$$
\frac{x'(t)}{2} - \frac{\mathrm{e}^{t}}{2} + \frac{1}{2} + \frac{\cos(t)}{2} - \frac{\sin(t)}{2} = \frac{x(t)}{2} - \frac{\mathrm{e}^{t}}{2} + \frac{3t}{2} + \frac{3\sin(t)}{2} + \frac{3\cos(t)}{2}
$$
$$
x'(t) = x(t) + 3t + 4\sin(t) + 2\cos(t) - 1 \tag{5}
$$

Which is now solved for $x(t)$.

Entering Linear first order ODE solver. In canonical form a linear first order is

$$
x'(t) + p(t)x(t) = q(t)
$$

Where here

$$\begin{aligned} p(t) &= -1 \\ q(t) &= 3t + 4\sin(t) + 2\cos(t) - 1 \end{aligned}$$

Hence the ode is

$$x'(t) - x(t) = 3t + 4\sin(t) + 2\cos(t) - 1$$

The integrating factor μ is

$$\begin{aligned} \mu &= \mathrm{e}^{\int (-1)dt} \\ &= \mathrm{e}^{-t} \end{aligned}$$

The ode becomes

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t}(\mu x) &= (\mu)\,(3t + 4\sin(t) + 2\cos(t) - 1) \\ \frac{\mathrm{d}}{\mathrm{d}t}\left(\mathrm{e}^{-t}x\right) &= \left(\mathrm{e}^{-t}\right)(3t + 4\sin(t) + 2\cos(t) - 1) \\ \mathrm{d}\left(\mathrm{e}^{-t}x\right) &= \left((3t + 4\sin(t) + 2\cos(t) - 1)\,\mathrm{e}^{-t}\right)\,\mathrm{d}t \end{aligned}$$

Integrating gives

$$\begin{aligned} \mathrm{e}^{-t}x &= \int (3t + 4\sin(t) + 2\cos(t) - 1)\,\mathrm{e}^{-t}\,\mathrm{d}t \\ \mathrm{e}^{-t}x &= -3\,\mathrm{e}^{-t}t - 2\,\mathrm{e}^{-t} - 3\,\mathrm{e}^{-t}\cos(t) - \sin(t)\,\mathrm{e}^{-t} + c_1 \end{aligned}$$

Dividing both sides by the integrating factor $\mu = \mathrm{e}^{-t}$ results in

$$x(t) = \mathrm{e}^{t}\left(-3\,\mathrm{e}^{-t}t - 2\,\mathrm{e}^{-t} - 3\,\mathrm{e}^{-t}\cos(t) - \sin(t)\,\mathrm{e}^{-t}\right) + c_1\mathrm{e}^{t}$$

which simplifies to

$$x(t) = -2 + c_1\mathrm{e}^{t} - 3t - \sin(t) - 3\cos(t)$$

Given now that we have the solution

$$x(t) = -2 + c_1\mathrm{e}^{t} - 3t - \sin(t) - 3\cos(t) \tag{6}$$

Then substituting (6) into (3) gives

$$y(t) = 1 - \frac{c_1\mathrm{e}^{t}}{2} + 2t + \sin(t) + 2\cos(t) - \frac{\mathrm{e}^{t}}{2} \tag{7}$$

✓ Solution by Maple

Time used: 0.078 (sec). Leaf size: 45

```
dsolve([diff(x(t),t)+diff(y(t),t)-x(t)=y(t)+t+sin(t)+cos(t),diff(x(t),t)+diff(y(t),t)=2*x(t)
```

$$x(t) = -\sin(t) - 3\cos(t) + c_1 e^t - 3t - 2$$
$$y(t) = \sin(t) + 2\cos(t) - \frac{c_1 e^t}{2} + 2t + 1 - \frac{e^t}{2}$$

✓ Solution by Mathematica

Time used: 0.055 (sec). Leaf size: 54

```
DSolve[{x'[t]+y'[t]-x[t]==y[t]+t+Sin[t]+Cos[t],x'[t]+y'[t]==2*x[t]+3*y[t]+Exp[t]},{x[t],y[t]
```

$$x(t) \to -3t + e^t - \sin(t) - 3\cos(t) + 2c_1 e^t - 2$$
$$y(t) \to 2t - e^t + \sin(t) + 2\cos(t) - c_1 e^t + 1$$