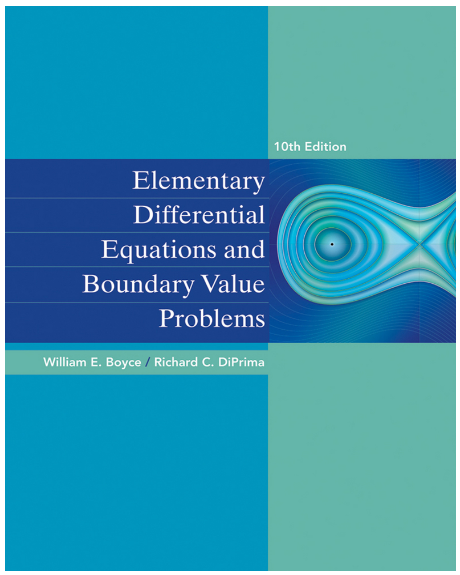
A Solution Manual For

Elementary differential equations and boundary value problems, 10th ed., Boyce and DiPrima



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May 15, 2024

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1.1 problem 1

Internal problem ID [448]
Internal file name [`OUTPUT/448_Sunday_June_05_2022_01_41_39_AM_4868104/index.tex`]

Book: Elementary differential equations and boundary value problems, 10th ed., Boyce and DiPrima
Section: Section 2.1. Page 40
Problem number: 1.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"linear", "exactWithIntegrationFactor", "first_order_ode_lie_symmetry_lookup"**

Maple gives the following as the ode type

```
[[_linear, `class A`]]
```

$$\boxed{3y + y' = e^{-2t} + t}$$

1.1.1 Solving as linear ode

Entering Linear first order ODE solver. In canonical form a linear first order is

$$y' + p(t)y = q(t)$$

Where here

$$p(t) = 3$$
$$q(t) = e^{-2t} + t$$

Hence the ode is

$$3y + y' = e^{-2t} + t$$

The integrating factor μ is

$$\begin{aligned}\mu &= e^{\int 3dt} \\ &= e^{3t}\end{aligned}$$

The ode becomes

$$\begin{aligned}\frac{d}{dt}(\mu y) &= (\mu)\left(e^{-2t}+t\right) \\ \frac{d}{dt}\left(e^{3t}y\right) &= \left(e^{3t}\right)\left(e^{-2t}+t\right) \\ d\left(e^{3t}y\right) &= \left(\left(e^{2t}t+1\right)e^{t}\right)\, dt\end{aligned}$$

Integrating gives

$$\begin{aligned}e^{3t}y &= \int \left(e^{2t}t+1\right)e^{t}\, dt \\ e^{3t}y &= \frac{e^{3t}t}{3}-\frac{e^{3t}}{9}+e^{t}+c_1\end{aligned}$$

Dividing both sides by the integrating factor $\mu = e^{3t}$ results in

$$y = e^{-3t}\left(\frac{e^{3t}t}{3}-\frac{e^{3t}}{9}+e^{t}\right)+c_1 e^{-3t}$$

which simplifies to

$$y = \frac{t}{3}-\frac{1}{9}+e^{-2t}+c_1 e^{-3t}$$

Summary

The solution(s) found are the following

$$y = \frac{t}{3}-\frac{1}{9}+e^{-2t}+c_1 e^{-3t} \tag{1}$$

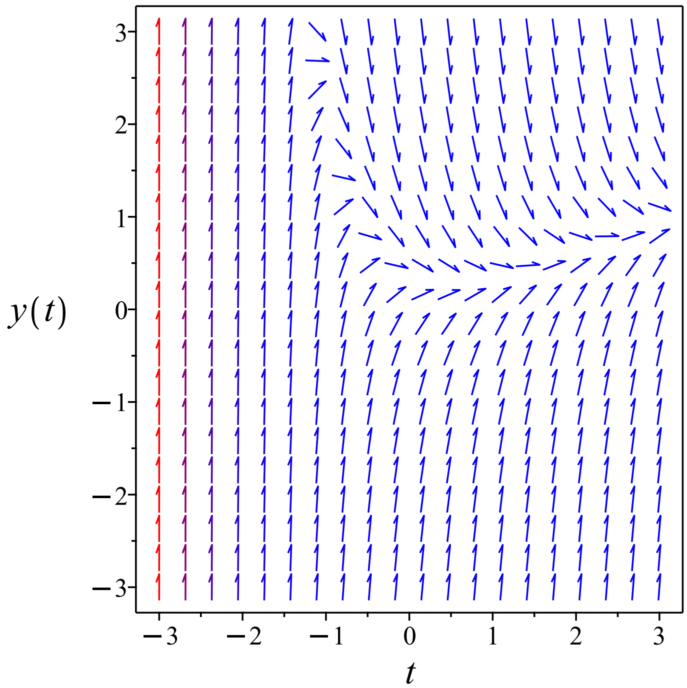


Figure 1: Slope field plot

Verification of solutions

$$y = \frac{t}{3} - \frac{1}{9} + e^{-2t} + c_1 e^{-3t}$$

Verified OK.

1.1.2 Solving as first order ode lie symmetry lookup ode

Writing the ode as

$$y' = -3y + e^{-2t} + t$$
$$y' = \omega(t, y)$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_t + \omega(\eta_y - \xi_t) - \omega^2 \xi_y - \omega_t \xi - \omega_y \eta = 0 \tag{A}$$

The type of this ode is known. It is of type `linear`. Therefore we do not need to solve the PDE (A), and can just use the lookup table shown below to find ξ, η

Table 1: Lie symmetry infinitesimal lookup table for known first order ODE's

ODE class	Form	ξ	η
linear ode	$y' = f(x)y(x) + g(x)$	0	$e^{\int f dx}$
separable ode	$y' = f(x)\, g(y)$	$\frac{1}{f}$	0
quadrature ode	$y' = f(x)$	0	1
quadrature ode	$y' = g(y)$	1	0
homogeneous ODEs of Class A	$y' = f\left(\frac{y}{x}\right)$	x	y
homogeneous ODEs of Class C	$y' = (a + bx + cy)^{\frac{n}{m}}$	1	$-\frac{b}{c}$
homogeneous class D	$y' = \frac{y}{x} + g(x)\, F\left(\frac{y}{x}\right)$	x^2	xy
First order special form ID 1	$y' = g(x)\, e^{h(x)+by} + f(x)$	$\frac{e^{-\int bf(x)dx - h(x)}}{g(x)}$	$\frac{f(x)e^{-\int bf(x)dx - h(x)}}{g(x)}$
polynomial type ode	$y' = \frac{a_1x+b_1y+c_1}{a_2x+b_2y+c_2}$	$\frac{a_1b_2x-a_2b_1x-b_1c_2+b_2c_1}{a_1b_2-a_2b_1}$	$\frac{a_1b_2y-a_2b_1y-a_1c_2-a_2c_1}{a_1b_2-a_2b_1}$
Bernoulli ode	$y' = f(x)\, y + g(x)\, y^n$	0	$e^{-\int (n-1)f(x)dx} y^n$
Reduced Riccati	$y' = f_1(x)\, y + f_2(x)\, y^2$	0	$e^{-\int f_1 dx}$

The above table shows that

$$
\begin{aligned}
\xi(t, y) &= 0 \\
\eta(t, y) &= \mathrm{e}^{-3t}
\end{aligned}
\tag{A1}
$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(t, y) \to (R, S)$ where (R, S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$
\frac{dt}{\xi} = \frac{dy}{\eta} = dS \tag{1}
$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial t} + \eta \frac{\partial}{\partial y}\right) S(t, y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the

canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$R = t$$

S is found from

$$\begin{aligned} S &= \int \frac{1}{\eta} dy \\ &= \int \frac{1}{e^{-3t}} dy \end{aligned}$$

Which results in

$$S = e^{3t} y$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_t + \omega(t,y) S_y}{R_t + \omega(t,y) R_y} \tag{2}$$

Where in the above R_t, R_y, S_t, S_y are all partial derivatives and $\omega(t,y)$ is the right hand side of the original ode given by

$$\omega(t,y) = -3y + e^{-2t} + t$$

Evaluating all the partial derivatives gives

$$\begin{aligned} R_t &= 1 \\ R_y &= 0 \\ S_t &= 3\,e^{3t} y \\ S_y &= e^{3t} \end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = e^{t} + e^{3t} t \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for t, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = e^{R} + e^{3R} R$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by

integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$S(R) = \frac{e^{3R}R}{3} - \frac{e^{3R}}{9} + e^{R} + c_1 \tag{4}$$

To complete the solution, we just need to transform (4) back to t, y coordinates. This results in

$$e^{3t}y = \frac{e^{3t}t}{3} - \frac{e^{3t}}{9} + e^{t} + c_1$$

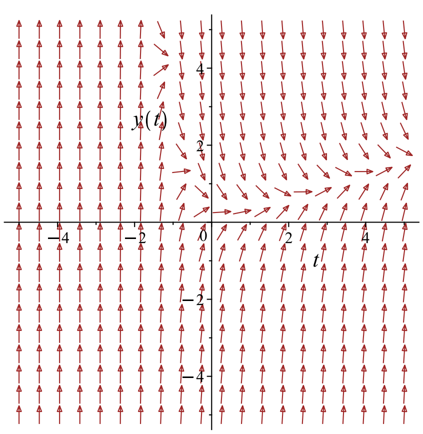
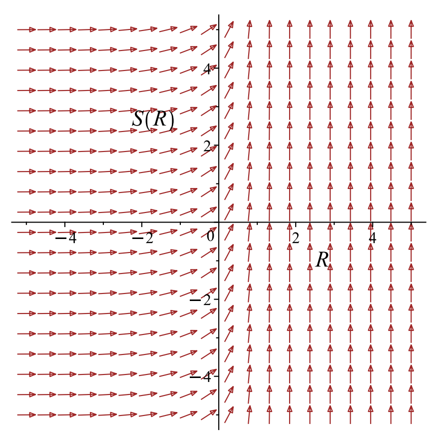
Which simplifies to

$$e^{3t}y = \frac{e^{3t}t}{3} - \frac{e^{3t}}{9} + e^{t} + c_1$$

Which gives

$$y = \frac{(3\,e^{3t}t - e^{3t} + 9\,e^{t} + 9c_1)\,e^{-3t}}{9}$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in t, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dt} = -3y + e^{-2t} + t$ 	$R = t$ $S = e^{3t}y$	$\frac{dS}{dR} = e^{R} + e^{3R}R$ 

Summary

The solution(s) found are the following

$$y = \frac{(3\,e^{3t}t - e^{3t} + 9\,e^{t} + 9c_1)\,e^{-3t}}{9} \tag{1}$$

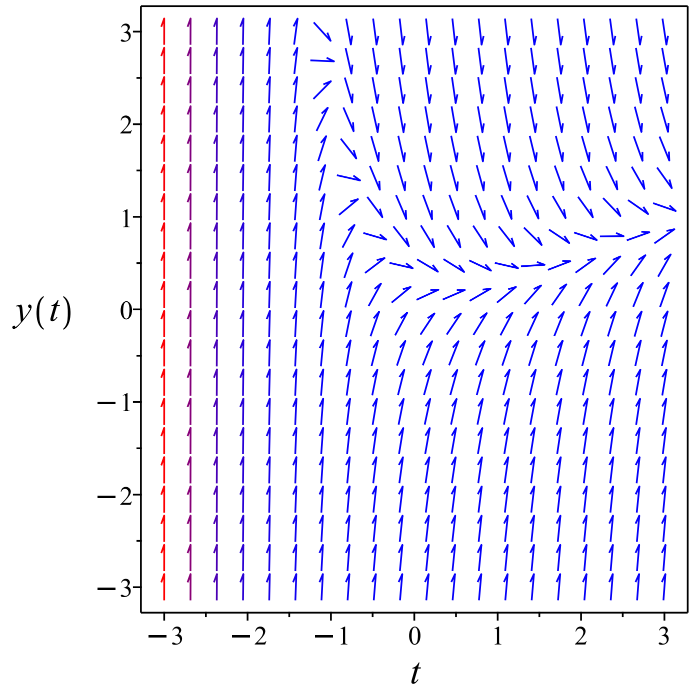


Figure 2: Slope field plot

Verification of solutions

$$y = \frac{(3\,e^{3t}t - e^{3t} + 9\,e^{t} + 9c_1)\,e^{-3t}}{9}$$

Verified OK.

1.1.3 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x,y) + N(x,y)\frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x,y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y) = 0$$

Hence

$$\frac{\partial\phi}{\partial x} + \frac{\partial\phi}{\partial y}\frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$\frac{\partial \phi}{\partial x} = M$$
$$\frac{\partial \phi}{\partial y} = N$$

But since $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x, y)$ but at least we know now that we can do that since the condition $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(t, y)\,\mathrm{d}t + N(t, y)\,\mathrm{d}y = 0 \tag{1A}$$

Therefore

$$\mathrm{d}y = \left(-3y + \mathrm{e}^{-2t} + t\right) \mathrm{d}t$$
$$\left(3y - \mathrm{e}^{-2t} - t\right) \mathrm{d}t + \mathrm{d}y = 0 \tag{2A}$$

Comparing (1A) and (2A) shows that

$$M(t, y) = 3y - \mathrm{e}^{-2t} - t$$
$$N(t, y) = 1$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial t}$$

Using result found above gives

$$\begin{aligned} \frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}\left(3y - \mathrm{e}^{-2t} - t\right) \\ &= 3 \end{aligned}$$

And

$$\begin{aligned} \frac{\partial N}{\partial t} &= \frac{\partial}{\partial t}(1) \\ &= 0 \end{aligned}$$

Since $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial t}$, then the ODE is not exact. Since the ODE is not exact, we will try to find an integrating factor to make it exact. Let

$$\begin{aligned} A &= \frac{1}{N}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial t}\right) \\ &= 1((3) - (0)) \\ &= 3 \end{aligned}$$

Since A does not depend on y, then it can be used to find an integrating factor. The integrating factor μ is

$$\begin{aligned} \mu &= e^{\int A \, \mathrm{d}t} \\ &= e^{\int 3 \, \mathrm{d}t} \end{aligned}$$

The result of integrating gives

$$\begin{aligned} \mu &= e^{3t} \\ &= \mathrm{e}^{3t} \end{aligned}$$

M and N are multiplied by this integrating factor, giving new M and new N which are called $\overline{M}$ and $\overline{N}$ for now so not to confuse them with the original M and N.

$$\begin{aligned} \overline{M} &= \mu M \\ &= \mathrm{e}^{3t}\left(3y - \mathrm{e}^{-2t} - t\right) \\ &= -\left(1 + (-3y + t)\,\mathrm{e}^{2t}\right)\mathrm{e}^{t} \end{aligned}$$

And

$$\begin{aligned} \overline{N} &= \mu N \\ &= \mathrm{e}^{3t}(1) \\ &= \mathrm{e}^{3t} \end{aligned}$$

Now a modified ODE is ontained from the original ODE, which is exact and can be solved. The modified ODE is

$$\begin{aligned} \overline{M} + \overline{N}\frac{\mathrm{d}y}{\mathrm{d}t} &= 0 \\ \left(-\left(1 + (-3y + t)\,\mathrm{e}^{2t}\right)\mathrm{e}^{t}\right) + \left(\mathrm{e}^{3t}\right)\frac{\mathrm{d}y}{\mathrm{d}t} &= 0 \end{aligned}$$

The following equations are now set up to solve for the function $\phi(t, y)$

$$\frac{\partial \phi}{\partial t} = \overline{M} \tag{1}$$

$$\frac{\partial \phi}{\partial y} = \overline{N} \tag{2}$$

Integrating (1) w.r.t. t gives

$$
\begin{aligned}
\int \frac{\partial \phi}{\partial t} \, \mathrm{d}t &= \int \overline{M} \, \mathrm{d}t \\
\int \frac{\partial \phi}{\partial t} \, \mathrm{d}t &= \int -\left(1 + (-3y + t)\,\mathrm{e}^{2t}\right) \mathrm{e}^{t} \, \mathrm{d}t \\
\phi &= \frac{(-3t + 9y + 1)\,\mathrm{e}^{3t}}{9} - \mathrm{e}^{t} + f(y) \qquad (3)
\end{aligned}
$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both t and y. Taking derivative of equation (3) w.r.t y gives

$$
\frac{\partial \phi}{\partial y} = \mathrm{e}^{3t} + f'(y) \qquad (4)
$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = \mathrm{e}^{3t}$. Therefore equation (4) becomes

$$
\mathrm{e}^{3t} = \mathrm{e}^{3t} + f'(y) \qquad (5)
$$

Solving equation (5) for $f'(y)$ gives

$$
f'(y) = 0
$$

Therefore

$$
f(y) = c_1
$$

Where c_1 is constant of integration. Substituting this result for $f(y)$ into equation (3) gives ϕ

$$
\phi = \frac{(-3t + 9y + 1)\,\mathrm{e}^{3t}}{9} - \mathrm{e}^{t} + c_1
$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$
c_1 = \frac{(-3t + 9y + 1)\,\mathrm{e}^{3t}}{9} - \mathrm{e}^{t}
$$

The solution becomes

$$
y = \frac{(3\,\mathrm{e}^{3t} t - \mathrm{e}^{3t} + 9\,\mathrm{e}^{t} + 9c_1)\,\mathrm{e}^{-3t}}{9}
$$

Summary
The solution(s) found are the following

$$y = \frac{(3\,e^{3t}t - e^{3t} + 9\,e^{t} + 9c_1)\,e^{-3t}}{9} \tag{1}$$

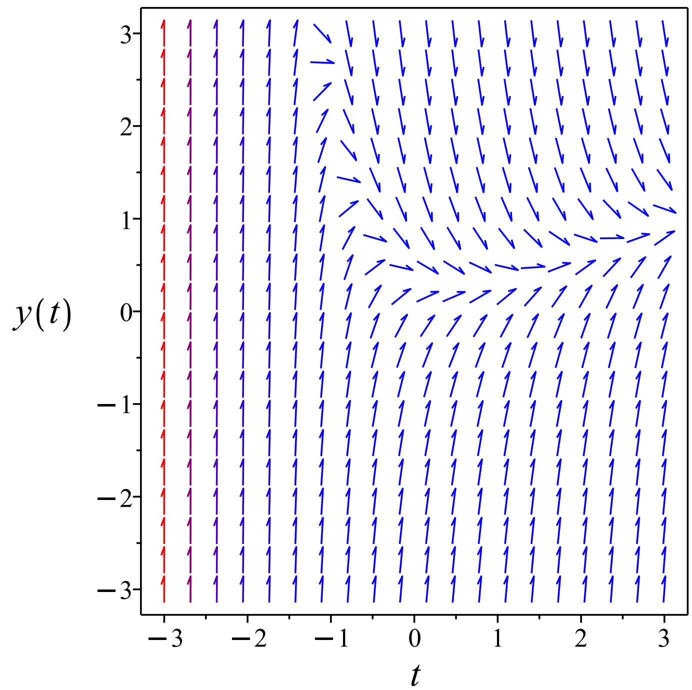


Figure 3: Slope field plot

Verification of solutions

$$y = \frac{(3\,e^{3t}t - e^{3t} + 9\,e^{t} + 9c_1)\,e^{-3t}}{9}$$

Verified OK.

1.1.4 Maple step by step solution

Let's solve

$3y + y' = e^{-2t} + t$

- Highest derivative means the order of the ODE is 1

  y'

- Isolate the derivative

$y' = -3y + e^{-2t} + t$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE

  $3y + y' = e^{-2t} + t$

- The ODE is linear; multiply by an integrating factor $\mu(t)$

  $\mu(t)(3y + y') = \mu(t)(e^{-2t} + t)$

- Assume the lhs of the ODE is the total derivative $\frac{d}{dt}(\mu(t)\, y)$

  $\mu(t)(3y + y') = \mu'(t)\, y + \mu(t)\, y'$

- Isolate $\mu'(t)$

  $\mu'(t) = 3\mu(t)$

- Solve to find the integrating factor

  $\mu(t) = e^{3t}$

- Integrate both sides with respect to t

  $\int \left(\frac{d}{dt}(\mu(t)\, y)\right) dt = \int \mu(t)(e^{-2t} + t)\, dt + c_1$

- Evaluate the integral on the lhs

  $\mu(t)\, y = \int \mu(t)(e^{-2t} + t)\, dt + c_1$

- Solve for y

  $y = \frac{\int \mu(t)(e^{-2t}+t)\, dt + c_1}{\mu(t)}$

- Substitute $\mu(t) = e^{3t}$

  $y = \frac{\int (e^{-2t}+t)e^{3t}\, dt + c_1}{e^{3t}}$

- Evaluate the integrals on the rhs

  $y = \frac{\frac{(e^t)^3 t}{3} - \frac{(e^t)^3}{9} + e^t + c_1}{e^{3t}}$

- Simplify

  $y = \frac{(3\, e^{3t} t - e^{3t} + 9\, e^t + 9c_1)\, e^{-3t}}{9}$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
<- 1st order linear successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 19

```
dsolve(3*y(t)+diff(y(t),t) = exp(-2*t)+t,y(t), singsol=all)
```

$$y(t) = \frac{t}{3} - \frac{1}{9} + \mathrm{e}^{-2t} + c_1 \mathrm{e}^{-3t}$$

✓ Solution by Mathematica

Time used: 0.086 (sec). Leaf size: 27

```
DSolve[3*y[t]+y'[t] == Exp[-2*t]+t,y[t],t,IncludeSingularSolutions -> True]
```

$$y(t) \to \frac{t}{3} + e^{-2t} + c_1 e^{-3t} - \frac{1}{9}$$

1.2 problem 2

Internal problem ID [449]
Internal file name [`OUTPUT/449_Sunday_June_05_2022_01_41_40_AM_16129924/index.tex`]

Book: Elementary differential equations and boundary value problems, 10th ed., Boyce and DiPrima
Section: Section 2.1. Page 40
Problem number: 2.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"linear", "exactWithIntegrationFactor", "first_order_ode_lie_symmetry_lookup"**

Maple gives the following as the ode type

```
[[_linear, `class A`]]
```

$$\boxed{-2y + y' = e^{2t}t^2}$$

1.2.1 Solving as linear ode

Entering Linear first order ODE solver. In canonical form a linear first order is

$$y' + p(t)y = q(t)$$

Where here

$$p(t) = -2$$
$$q(t) = e^{2t}t^2$$

Hence the ode is

$$-2y + y' = e^{2t}t^2$$

The integrating factor μ is

$$\begin{aligned}\mu &= e^{\int (-2)dt} \\ &= e^{-2t}\end{aligned}$$

The ode becomes

$$\begin{aligned}\frac{d}{dt}(\mu y) &= (\mu)\left(e^{2t}t^2\right) \\ \frac{d}{dt}\left(e^{-2t}y\right) &= \left(e^{-2t}\right)\left(e^{2t}t^2\right) \\ d\left(e^{-2t}y\right) &= t^2\, dt\end{aligned}$$

Integrating gives

$$\begin{aligned}e^{-2t}y &= \int t^2\, dt \\ e^{-2t}y &= \frac{t^3}{3} + c_1\end{aligned}$$

Dividing both sides by the integrating factor $\mu = e^{-2t}$ results in

$$y = \frac{e^{2t}t^3}{3} + c_1 e^{2t}$$

which simplifies to

$$y = e^{2t}\left(\frac{t^3}{3} + c_1\right)$$

Summary
The solution(s) found are the following

$$y = e^{2t}\left(\frac{t^3}{3} + c_1\right) \tag{1}$$

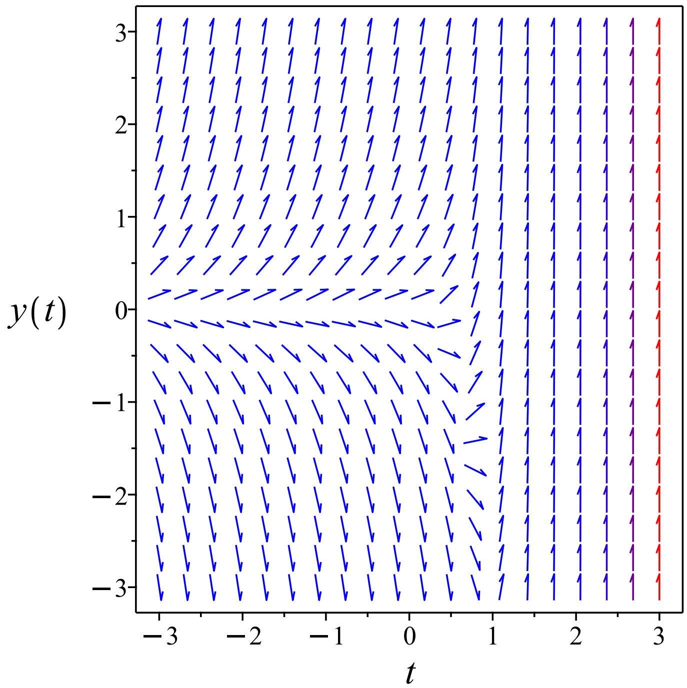


Figure 4: Slope field plot

Verification of solutions

$$y = e^{2t}\left(\frac{t^3}{3} + c_1\right)$$

Verified OK.

1.2.2 Solving as first order ode lie symmetry lookup ode

Writing the ode as

$$y' = 2y + e^{2t}t^2$$
$$y' = \omega(t, y)$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_t + \omega(\eta_y - \xi_t) - \omega^2\xi_y - \omega_t\xi - \omega_y\eta = 0 \tag{A}$$

The type of this ode is known. It is of type `linear`. Therefore we do not need to solve the PDE (A), and can just use the lookup table shown below to find ξ, η

Table 4: Lie symmetry infinitesimal lookup table for known first order ODE's

ODE class	Form	ξ	η
linear ode	$y' = f(x)y(x) + g(x)$	0	$e^{\int f dx}$
separable ode	$y' = f(x)\, g(y)$	$\frac{1}{f}$	0
quadrature ode	$y' = f(x)$	0	1
quadrature ode	$y' = g(y)$	1	0
homogeneous ODEs of Class A	$y' = f\left(\frac{y}{x}\right)$	x	y
homogeneous ODEs of Class C	$y' = (a + bx + cy)^{\frac{n}{m}}$	1	$-\frac{b}{c}$
homogeneous class D	$y' = \frac{y}{x} + g(x)\, F\left(\frac{y}{x}\right)$	x^2	xy
First order special form ID 1	$y' = g(x)\, e^{h(x)+by} + f(x)$	$\frac{e^{-\int bf(x)dx - h(x)}}{g(x)}$	$\frac{f(x)e^{-\int bf(x)dx - h(x)}}{g(x)}$
polynomial type ode	$y' = \frac{a_1x+b_1y+c_1}{a_2x+b_2y+c_2}$	$\frac{a_1b_2x-a_2b_1x-b_1c_2+b_2c_1}{a_1b_2-a_2b_1}$	$\frac{a_1b_2y-a_2b_1y-a_1c_2-a_2c_1}{a_1b_2-a_2b_1}$
Bernoulli ode	$y' = f(x)\, y + g(x)\, y^n$	0	$e^{-\int (n-1)f(x)dx} y^n$
Reduced Riccati	$y' = f_1(x)\, y + f_2(x)\, y^2$	0	$e^{-\int f_1 dx}$

The above table shows that

$$
\begin{aligned}
\xi(t, y) &= 0 \\
\eta(t, y) &= e^{2t}
\end{aligned} \tag{A1}
$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(t, y) \to (R, S)$ where (R, S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$
\frac{dt}{\xi} = \frac{dy}{\eta} = dS \tag{1}
$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial t} + \eta \frac{\partial}{\partial y}\right) S(t, y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the

canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$R = t$$

S is found from

$$\begin{aligned} S &= \int \frac{1}{\eta} dy \\ &= \int \frac{1}{e^{2t}} dy \end{aligned}$$

Which results in

$$S = e^{-2t} y$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_t + \omega(t,y) S_y}{R_t + \omega(t,y) R_y} \tag{2}$$

Where in the above R_t, R_y, S_t, S_y are all partial derivatives and $\omega(t,y)$ is the right hand side of the original ode given by

$$\omega(t,y) = 2y + e^{2t} t^2$$

Evaluating all the partial derivatives gives

$$\begin{aligned} R_t &= 1 \\ R_y &= 0 \\ S_t &= -2\, e^{-2t} y \\ S_y &= e^{-2t} \end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = t^2 \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for t, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = R^2$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by

integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$S(R) = \frac{R^3}{3} + c_1 \tag{4}$$

To complete the solution, we just need to transform (4) back to t, y coordinates. This results in

$$e^{-2t}y = \frac{t^3}{3} + c_1$$

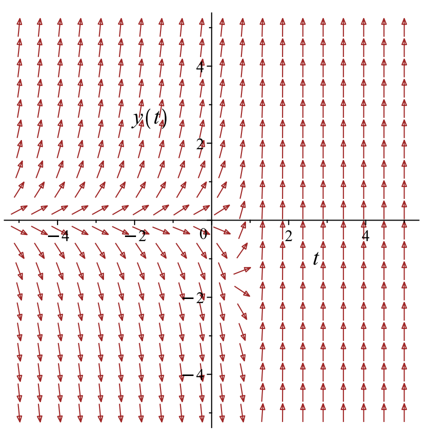
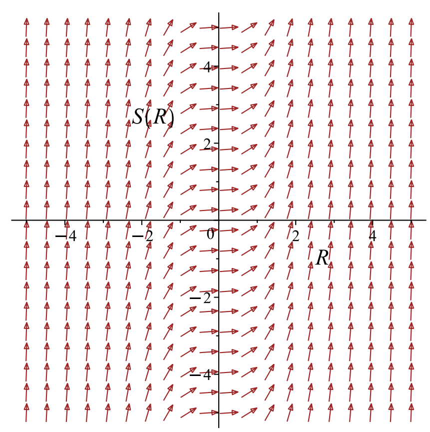
Which simplifies to

$$e^{-2t}y = \frac{t^3}{3} + c_1$$

Which gives

$$y = \frac{e^{2t}(t^3 + 3c_1)}{3}$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in t, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dt} = 2y + e^{2t}t^2$ 	$R = t$ $S = e^{-2t}y$	$\frac{dS}{dR} = R^2$ 

Summary

The solution(s) found are the following

$$y = \frac{e^{2t}(t^3 + 3c_1)}{3} \tag{1}$$

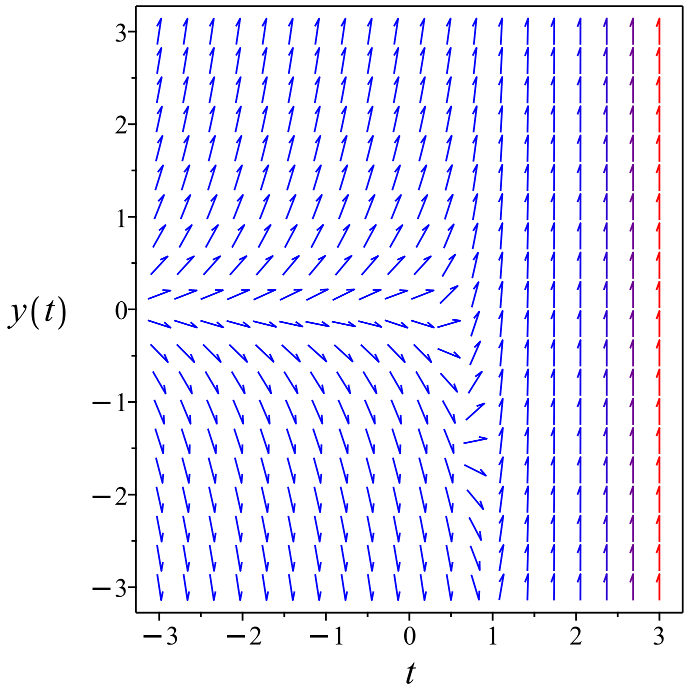


Figure 5: Slope field plot

Verification of solutions

$$y = \frac{e^{2t}(t^3 + 3c_1)}{3}$$

Verified OK.

1.2.3 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x,y) + N(x,y)\frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x,y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y) = 0$$

Hence

$$\frac{\partial \phi}{\partial x} + \frac{\partial \phi}{\partial y}\frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$\frac{\partial \phi}{\partial x} = M$$
$$\frac{\partial \phi}{\partial y} = N$$

But since $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x, y)$ but at least we know now that we can do that since the condition $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(t, y) \, \mathrm{d}t + N(t, y) \, \mathrm{d}y = 0 \tag{1A}$$

Therefore

$$\begin{aligned} \mathrm{d}y &= \left(2y + \mathrm{e}^{2t} t^2\right) \mathrm{d}t \\ \left(-2y - \mathrm{e}^{2t} t^2\right) \mathrm{d}t + \mathrm{d}y &= 0 \end{aligned} \tag{2A}$$

Comparing (1A) and (2A) shows that

$$M(t, y) = -2y - \mathrm{e}^{2t} t^2$$
$$N(t, y) = 1$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial t}$$

Using result found above gives

$$\begin{aligned} \frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}\left(-2y - \mathrm{e}^{2t} t^2\right) \\ &= -2 \end{aligned}$$

And

$$\begin{aligned} \frac{\partial N}{\partial t} &= \frac{\partial}{\partial t}(1) \\ &= 0 \end{aligned}$$

Since $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial t}$, then the ODE is not exact. Since the ODE is not exact, we will try to find an integrating factor to make it exact. Let

$$\begin{aligned}A &= \frac{1}{N}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial t}\right)\\ &= 1((-2) - (0))\\ &= -2\end{aligned}$$

Since A does not depend on y, then it can be used to find an integrating factor. The integrating factor μ is

$$\begin{aligned}\mu &= e^{\int A\,\mathrm{d}t}\\ &= e^{\int -2\,\mathrm{d}t}\end{aligned}$$

The result of integrating gives

$$\begin{aligned}\mu &= e^{-2t}\\ &= \mathrm{e}^{-2t}\end{aligned}$$

M and N are multiplied by this integrating factor, giving new M and new N which are called $\overline{M}$ and $\overline{N}$ for now so not to confuse them with the original M and N.

$$\begin{aligned}\overline{M} &= \mu M\\ &= \mathrm{e}^{-2t}\left(-2y - \mathrm{e}^{2t}t^2\right)\\ &= -2\,\mathrm{e}^{-2t}y - t^2\end{aligned}$$

And

$$\begin{aligned}\overline{N} &= \mu N\\ &= \mathrm{e}^{-2t}(1)\\ &= \mathrm{e}^{-2t}\end{aligned}$$

Now a modified ODE is ontained from the original ODE, which is exact and can be solved. The modified ODE is

$$\begin{aligned}\overline{M} + \overline{N}\frac{\mathrm{d}y}{\mathrm{d}t} &= 0\\ \left(-2\,\mathrm{e}^{-2t}y - t^2\right) + \left(\mathrm{e}^{-2t}\right)\frac{\mathrm{d}y}{\mathrm{d}t} &= 0\end{aligned}$$

The following equations are now set up to solve for the function $\phi(t, y)$

$$\frac{\partial \phi}{\partial t} = \overline{M} \tag{1}$$

$$\frac{\partial \phi}{\partial y} = \overline{N} \tag{2}$$

Integrating (1) w.r.t. t gives

$$\int \frac{\partial \phi}{\partial t} \, \mathrm{d}t = \int \overline{M} \, \mathrm{d}t$$

$$\int \frac{\partial \phi}{\partial t} \, \mathrm{d}t = \int -2\,\mathrm{e}^{-2t} y - t^2 \, \mathrm{d}t$$

$$\phi = -\frac{t^3}{3} + \mathrm{e}^{-2t} y + f(y) \tag{3}$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both t and y. Taking derivative of equation (3) w.r.t y gives

$$\frac{\partial \phi}{\partial y} = \mathrm{e}^{-2t} + f'(y) \tag{4}$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = \mathrm{e}^{-2t}$. Therefore equation (4) becomes

$$\mathrm{e}^{-2t} = \mathrm{e}^{-2t} + f'(y) \tag{5}$$

Solving equation (5) for $f'(y)$ gives

$$f'(y) = 0$$

Therefore

$$f(y) = c_1$$

Where c_1 is constant of integration. Substituting this result for $f(y)$ into equation (3) gives ϕ

$$\phi = -\frac{t^3}{3} + \mathrm{e}^{-2t} y + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = -\frac{t^3}{3} + \mathrm{e}^{-2t} y$$

The solution becomes

$$y = \frac{\mathrm{e}^{2t}(t^3 + 3c_1)}{3}$$

Summary

The solution(s) found are the following

$$y = \frac{e^{2t}(t^3 + 3c_1)}{3} \tag{1}$$

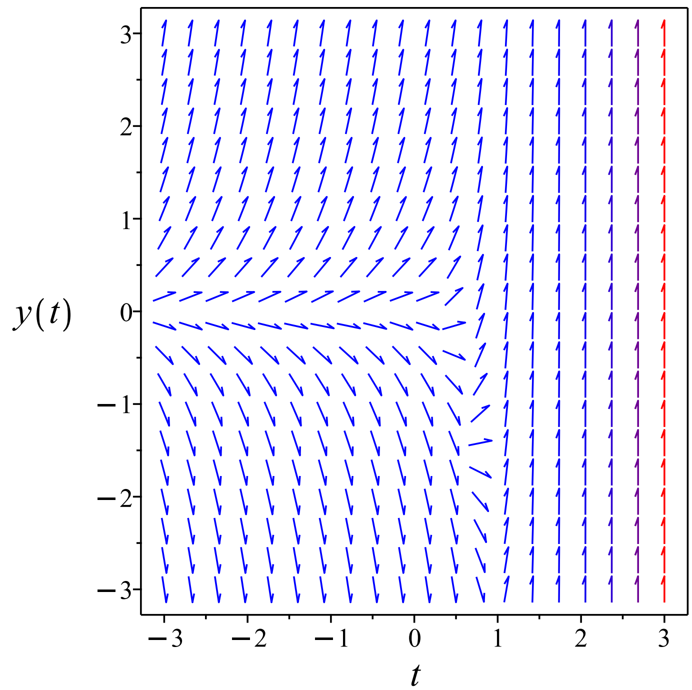


Figure 6: Slope field plot

Verification of solutions

$$y = \frac{e^{2t}(t^3 + 3c_1)}{3}$$

Verified OK.

1.2.4 Maple step by step solution

Let's solve

$-2y + y' = e^{2t}t^2$

- Highest derivative means the order of the ODE is 1

  y'

- Isolate the derivative

$y' = 2y + e^{2t}t^2$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE

  $-2y + y' = e^{2t}t^2$
- The ODE is linear; multiply by an integrating factor $\mu(t)$

  $\mu(t)\left(-2y + y'\right) = \mu(t)\, e^{2t}t^2$
- Assume the lhs of the ODE is the total derivative $\frac{d}{dt}(\mu(t)\, y)$

  $\mu(t)\left(-2y + y'\right) = \mu'(t)\, y + \mu(t)\, y'$
- Isolate $\mu'(t)$

  $\mu'(t) = -2\mu(t)$
- Solve to find the integrating factor

  $\mu(t) = e^{-2t}$
- Integrate both sides with respect to t

  $\int \left(\frac{d}{dt}(\mu(t)\, y)\right) dt = \int \mu(t)\, e^{2t}t^2 dt + c_1$
- Evaluate the integral on the lhs

  $\mu(t)\, y = \int \mu(t)\, e^{2t}t^2 dt + c_1$
- Solve for y

  $y = \frac{\int \mu(t) e^{2t}t^2 dt + c_1}{\mu(t)}$
- Substitute $\mu(t) = e^{-2t}$

  $y = \frac{\int e^{2t}t^2 e^{-2t} dt + c_1}{e^{-2t}}$
- Evaluate the integrals on the rhs

  $y = \frac{\frac{t^3}{3} + c_1}{e^{-2t}}$
- Simplify

  $y = \frac{e^{2t}(t^3 + 3c_1)}{3}$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
<- 1st order linear successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 17

```
dsolve(-2*y(t)+diff(y(t),t) = exp(2*t)*t^2,y(t), singsol=all)
```

$$y(t) = \frac{(t^3 + 3c_1)\,\mathrm{e}^{2t}}{3}$$

✓ Solution by Mathematica

Time used: 0.043 (sec). Leaf size: 22

```
DSolve[-2*y[t]+y'[t]== Exp[2*t]*t^2,y[t],t,IncludeSingularSolutions -> True]
```

$$y(t) \to \frac{1}{3}e^{2t}(t^3 + 3c_1)$$

1.3 problem 3

Internal problem ID [450]
Internal file name [`OUTPUT/450_Sunday_June_05_2022_01_41_41_AM_58171829/index.tex`]

Book: Elementary differential equations and boundary value problems, 10th ed., Boyce and DiPrima
Section: Section 2.1. Page 40
Problem number: 3.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"exact", "linear", "first_order_ode_lie_symmetry_lookup"**

Maple gives the following as the ode type

```
[[_linear, `class A`]]
```

$$\boxed{y + y' = 1 + t\,\mathrm{e}^{-t}}$$

1.3.1 Solving as linear ode

Entering Linear first order ODE solver. In canonical form a linear first order is

$$y' + p(t)y = q(t)$$

Where here

$$p(t) = 1$$
$$q(t) = 1 + t\,\mathrm{e}^{-t}$$

Hence the ode is

$$y + y' = 1 + t\,\mathrm{e}^{-t}$$

The integrating factor μ is

$$\mu = e^{\int 1dt} \\ = e^{t}$$

The ode becomes

$$\frac{d}{dt}(\mu y) = (\mu)\left(1 + t\,e^{-t}\right)$$
$$\frac{d}{dt}\left(y\,e^{t}\right) = \left(e^{t}\right)\left(1 + t\,e^{-t}\right)$$
$$d\left(y\,e^{t}\right) = \left(e^{t} + t\right)\,dt$$

Integrating gives

$$y\,e^{t} = \int e^{t} + t\,dt$$
$$y\,e^{t} = \frac{t^2}{2} + e^{t} + c_1$$

Dividing both sides by the integrating factor $\mu = e^{t}$ results in

$$y = e^{-t}\left(\frac{t^2}{2} + e^{t}\right) + c_1 e^{-t}$$

which simplifies to

$$y = 1 + \frac{(t^2 + 2c_1)\,e^{-t}}{2}$$

Summary

The solution(s) found are the following

$$y = 1 + \frac{(t^2 + 2c_1)\,e^{-t}}{2} \tag{1}$$

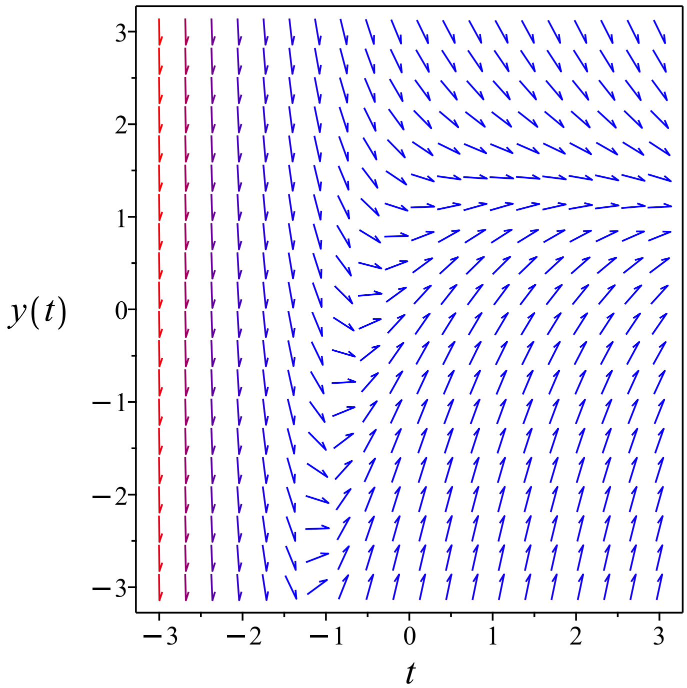


Figure 7: Slope field plot

Verification of solutions

$$y = 1 + \frac{(t^2 + 2c_1)\,\mathrm{e}^{-t}}{2}$$

Verified OK.

1.3.2 Solving as first order ode lie symmetry lookup ode

Writing the ode as

$$\begin{aligned} y' &= -\left(y\,\mathrm{e}^{t} - \mathrm{e}^{t} - t\right)\mathrm{e}^{-t} \\ y' &= \omega(t, y) \end{aligned}$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_t + \omega(\eta_y - \xi_t) - \omega^2 \xi_y - \omega_t \xi - \omega_y \eta = 0 \tag{A}$$

The type of this ode is known. It is of type `linear`. Therefore we do not need to solve the PDE (A), and can just use the lookup table shown below to find ξ, η

Table 7: Lie symmetry infinitesimal lookup table for known first order ODE's

ODE class	Form	ξ	η
linear ode	$y' = f(x)y(x) + g(x)$	0	$e^{\int f dx}$
separable ode	$y' = f(x)\, g(y)$	$\frac{1}{f}$	0
quadrature ode	$y' = f(x)$	0	1
quadrature ode	$y' = g(y)$	1	0
homogeneous ODEs of Class A	$y' = f\left(\frac{y}{x}\right)$	x	y
homogeneous ODEs of Class C	$y' = (a + bx + cy)^{\frac{n}{m}}$	1	$-\frac{b}{c}$
homogeneous class D	$y' = \frac{y}{x} + g(x)\, F\left(\frac{y}{x}\right)$	x^2	xy
First order special form ID 1	$y' = g(x)\, e^{h(x)+by} + f(x)$	$\frac{e^{-\int bf(x)dx - h(x)}}{g(x)}$	$\frac{f(x)e^{-\int bf(x)dx - h(x)}}{g(x)}$
polynomial type ode	$y' = \frac{a_1 x + b_1 y + c_1}{a_2 x + b_2 y + c_2}$	$\frac{a_1 b_2 x - a_2 b_1 x - b_1 c_2 + b_2 c_1}{a_1 b_2 - a_2 b_1}$	$\frac{a_1 b_2 y - a_2 b_1 y - a_1 c_2 - a_2 c_1}{a_1 b_2 - a_2 b_1}$
Bernoulli ode	$y' = f(x)\, y + g(x)\, y^n$	0	$e^{-\int (n-1)f(x)dx} y^n$
Reduced Riccati	$y' = f_1(x)\, y + f_2(x)\, y^2$	0	$e^{-\int f_1 dx}$

The above table shows that

$$
\begin{aligned}
\xi(t, y) &= 0 \\
\eta(t, y) &= \mathrm{e}^{-t}
\end{aligned} \tag{A1}
$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(t, y) \to (R, S)$ where (R, S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$
\frac{dt}{\xi} = \frac{dy}{\eta} = dS \tag{1}
$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial t} + \eta \frac{\partial}{\partial y}\right) S(t, y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the

canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$R = t$$

S is found from

$$\begin{aligned} S &= \int \frac{1}{\eta} dy \\ &= \int \frac{1}{\mathrm{e}^{-t}} dy \end{aligned}$$

Which results in

$$S = y\,\mathrm{e}^{t}$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_t + \omega(t, y) S_y}{R_t + \omega(t, y) R_y} \tag{2}$$

Where in the above R_t, R_y, S_t, S_y are all partial derivatives and $\omega(t, y)$ is the right hand side of the original ode given by

$$\omega(t, y) = -\left(y\,\mathrm{e}^{t} - \mathrm{e}^{t} - t\right) \mathrm{e}^{-t}$$

Evaluating all the partial derivatives gives

$$\begin{aligned} R_t &= 1 \\ R_y &= 0 \\ S_t &= y\,\mathrm{e}^{t} \\ S_y &= \mathrm{e}^{t} \end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = \mathrm{e}^{t} + t \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for t, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = \mathrm{e}^{R} + R$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by

integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$S(R) = \frac{R^2}{2} + e^R + c_1 \tag{4}$$

To complete the solution, we just need to transform (4) back to t, y coordinates. This results in

$$y\,e^t = \frac{t^2}{2} + e^t + c_1$$

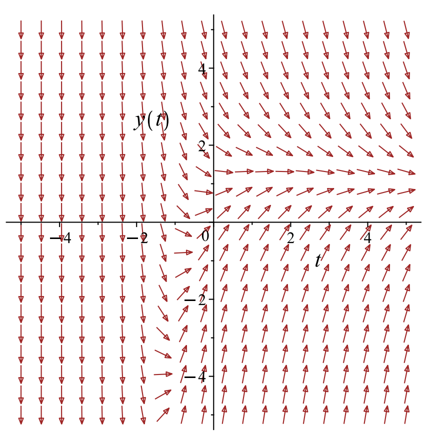
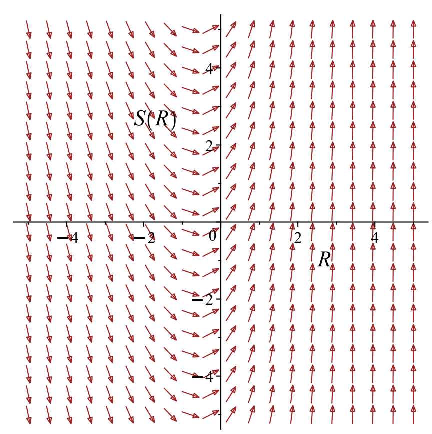
Which simplifies to

$$y\,e^t = \frac{t^2}{2} + e^t + c_1$$

Which gives

$$y = \frac{(t^2 + 2\,e^t + 2c_1)\,e^{-t}}{2}$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

<table>
<tr><th>Original ode in t, y coordinates</th><th>Canonical coordinates transformation</th><th>ODE in canonical coordinates (R, S)</th></tr>
<tr><td>$\frac{dy}{dt} = -(y\,e^t - e^t - t)\,e^{-t}$
</td><td>$R = t$
$S = y\,e^t$</td><td>$\frac{dS}{dR} = e^R + R$
</td></tr>
</table>

Summary
The solution(s) found are the following

$$y = \frac{(t^2 + 2\,e^t + 2c_1)\,e^{-t}}{2} \tag{1}$$

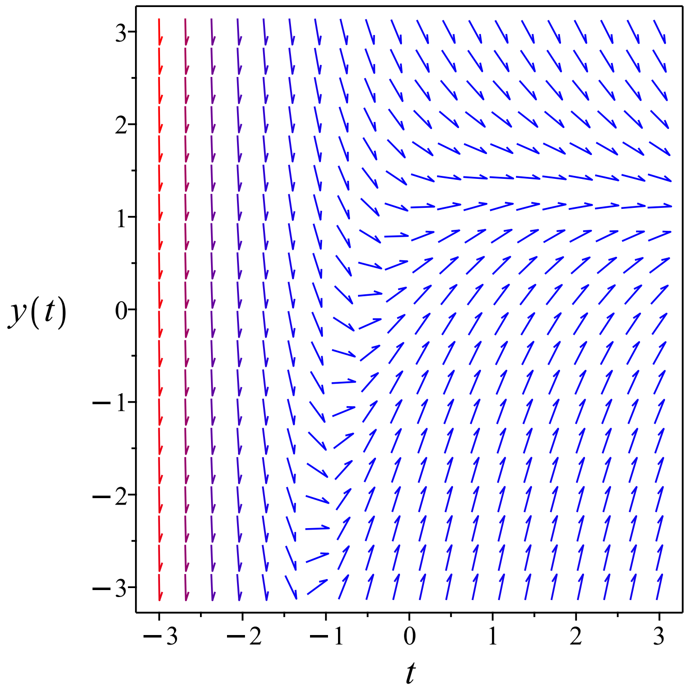


Figure 8: Slope field plot

<u>Verification of solutions</u>

$$y = \frac{(t^2 + 2\,e^t + 2c_1)\,e^{-t}}{2}$$

Verified OK.

1.3.3 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x,y) + N(x,y)\frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x,y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y) = 0$$

Hence

$$\frac{\partial\phi}{\partial x} + \frac{\partial\phi}{\partial y}\frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$\begin{aligned}\frac{\partial\phi}{\partial x} &= M\\ \frac{\partial\phi}{\partial y} &= N\end{aligned}$$

But since $\frac{\partial^2\phi}{\partial x\partial y} = \frac{\partial^2\phi}{\partial y\partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x,y)$ but at least we know now that we can do that since the condition $\frac{\partial^2\phi}{\partial x\partial y} = \frac{\partial^2\phi}{\partial y\partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(t,y)\,\mathrm{d}t + N(t,y)\,\mathrm{d}y = 0 \tag{1A}$$

Therefore

$$\begin{aligned}(\mathrm{e}^t)\,\mathrm{d}y &= \left(-y\,\mathrm{e}^t + \mathrm{e}^t + t\right)\mathrm{d}t\\ \left(y\,\mathrm{e}^t - \mathrm{e}^t - t\right)\mathrm{d}t + (\mathrm{e}^t)\,\mathrm{d}y &= 0 \qquad (2A)\end{aligned}$$

Comparing (1A) and (2A) shows that

$$\begin{aligned}M(t,y) &= y\,\mathrm{e}^t - \mathrm{e}^t - t\\ N(t,y) &= \mathrm{e}^t\end{aligned}$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial t}$$

Using result found above gives

$$\begin{aligned}\frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}\left(y\,\mathrm{e}^t - \mathrm{e}^t - t\right)\\ &= \mathrm{e}^t\end{aligned}$$

And

$$\begin{aligned}\frac{\partial N}{\partial t} &= \frac{\partial}{\partial t}(\mathrm{e}^t)\\ &= \mathrm{e}^t\end{aligned}$$

Since $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial t}$, then the ODE is exact The following equations are now set up to solve for the function $\phi(t, y)$

$$
\begin{aligned}
\frac{\partial \phi}{\partial t} &= M \qquad (1) \\
\frac{\partial \phi}{\partial y} &= N \qquad (2)
\end{aligned}
$$

Integrating (1) w.r.t. t gives

$$
\begin{aligned}
\int \frac{\partial \phi}{\partial t} \, \mathrm{d}t &= \int M \, \mathrm{d}t \\
\int \frac{\partial \phi}{\partial t} \, \mathrm{d}t &= \int y \, \mathrm{e}^t - \mathrm{e}^t - t \, \mathrm{d}t \\
\phi &= \mathrm{e}^t (y-1) - \frac{t^2}{2} + f(y) \qquad (3)
\end{aligned}
$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both t and y. Taking derivative of equation (3) w.r.t y gives

$$
\frac{\partial \phi}{\partial y} = \mathrm{e}^t + f'(y) \qquad (4)
$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = \mathrm{e}^t$. Therefore equation (4) becomes

$$
\mathrm{e}^t = \mathrm{e}^t + f'(y) \qquad (5)
$$

Solving equation (5) for $f'(y)$ gives

$$
f'(y) = 0
$$

Therefore

$$
f(y) = c_1
$$

Where c_1 is constant of integration. Substituting this result for $f(y)$ into equation (3) gives ϕ

$$
\phi = \mathrm{e}^t (y-1) - \frac{t^2}{2} + c_1
$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$
c_1 = \mathrm{e}^t (y-1) - \frac{t^2}{2}
$$

The solution becomes

$$y = \frac{(t^2 + 2\,e^t + 2c_1)\,e^{-t}}{2}$$

Summary

The solution(s) found are the following

$$y = \frac{(t^2 + 2\,e^t + 2c_1)\,e^{-t}}{2} \tag{1}$$

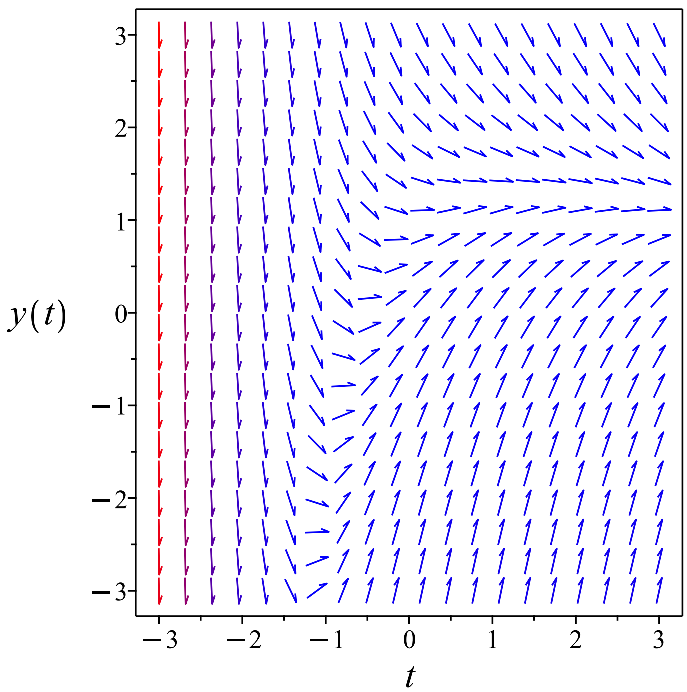


Figure 9: Slope field plot

Verification of solutions

$$y = \frac{(t^2 + 2\,e^t + 2c_1)\,e^{-t}}{2}$$

Verified OK.

1.3.4 Maple step by step solution

Let's solve

$y + y' = 1 + \frac{t}{e^t}$

- Highest derivative means the order of the ODE is 1

  y'

- Isolate the derivative

  $y' = -y + \frac{e^t + t}{e^t}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE

  $y + y' = \frac{e^t + t}{e^t}$

- The ODE is linear; multiply by an integrating factor $\mu(t)$

  $\mu(t)\,(y + y') = \frac{\mu(t)(e^t + t)}{e^t}$

- Assume the lhs of the ODE is the total derivative $\frac{d}{dt}(\mu(t)\,y)$

  $\mu(t)\,(y + y') = \mu'(t)\,y + \mu(t)\,y'$

- Isolate $\mu'(t)$

  $\mu'(t) = \mu(t)$

- Solve to find the integrating factor

  $\mu(t) = (e^t)^2\,e^{-t}$

- Integrate both sides with respect to t

  $\int \left(\frac{d}{dt}(\mu(t)\,y)\right) dt = \int \frac{\mu(t)(e^t + t)}{e^t} dt + c_1$

- Evaluate the integral on the lhs

  $\mu(t)\,y = \int \frac{\mu(t)(e^t + t)}{e^t} dt + c_1$

- Solve for y

  $y = \frac{\int \frac{\mu(t)\left(e^t + t\right)}{e^t} dt + c_1}{\mu(t)}$

- Substitute $\mu(t) = (e^t)^2\,e^{-t}$

  $y = \frac{\int e^t e^{-t}(e^t + t)dt + c_1}{(e^t)^2 e^{-t}}$

- Evaluate the integrals on the rhs

  $y = \frac{\frac{t^2}{2} + e^t + c_1}{(e^t)^2 e^{-t}}$

- Simplify

  $y = 1 + \frac{(t^2+2c_1)e^{-t}}{2}$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
<- 1st order linear successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 19

```
dsolve(y(t)+diff(y(t),t) = 1+t/exp(t),y(t), singsol=all)
```

$$y(t) = 1 + \frac{(t^2 + 2c_1)\,e^{-t}}{2}$$

✓ Solution by Mathematica

Time used: 0.052 (sec). Leaf size: 27

```
DSolve[y[t]+y'[t] == 1+t/Exp[t],y[t],t,IncludeSingularSolutions -> True]
```

$$y(t) \to \frac{1}{2}e^{-t}(t^2 + 2e^t + 2c_1)$$

1.4 problem 4

Internal problem ID [451]
Internal file name [`OUTPUT/451_Sunday_June_05_2022_01_41_42_AM_48741648/index.tex`]

Book: Elementary differential equations and boundary value problems, 10th ed., Boyce and DiPrima
Section: Section 2.1. Page 40
Problem number: 4.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"exact", "linear", "first_order_ode_lie_symmetry_lookup"**

Maple gives the following as the ode type

`[_linear]`

$$\boxed{\frac{y}{t} + y' = 3\cos(2t)}$$

1.4.1 Solving as linear ode

Entering Linear first order ODE solver. In canonical form a linear first order is

$$y' + p(t)y = q(t)$$

Where here

$$p(t) = \frac{1}{t}$$
$$q(t) = 3\cos(2t)$$

Hence the ode is

$$\frac{y}{t} + y' = 3\cos(2t)$$

The integrating factor μ is

$$
\begin{aligned}
\mu &= e^{\int \frac{1}{t}dt} \\
&= t
\end{aligned}
$$

The ode becomes

$$
\begin{aligned}
\frac{d}{dt}(\mu y) &= (\mu)\left(3\cos(2t)\right) \\
\frac{d}{dt}(ty) &= (t)\left(3\cos(2t)\right) \\
d(ty) &= (3\cos(2t)\,t)\; dt
\end{aligned}
$$

Integrating gives

$$
\begin{aligned}
ty &= \int 3\cos(2t)\,t\, dt \\
ty &= \frac{3\cos(2t)}{4} + \frac{3\sin(2t)\,t}{2} + c_1
\end{aligned}
$$

Dividing both sides by the integrating factor $\mu = t$ results in

$$
y = \frac{\frac{3\cos(2t)}{4} + \frac{3\sin(2t)t}{2}}{t} + \frac{c_1}{t}
$$

which simplifies to

$$
y = \frac{6\sin(2t)\,t + 3\cos(2t) + 4c_1}{4t}
$$

Summary

The solution(s) found are the following

$$
y = \frac{6\sin(2t)\,t + 3\cos(2t) + 4c_1}{4t} \tag{1}
$$

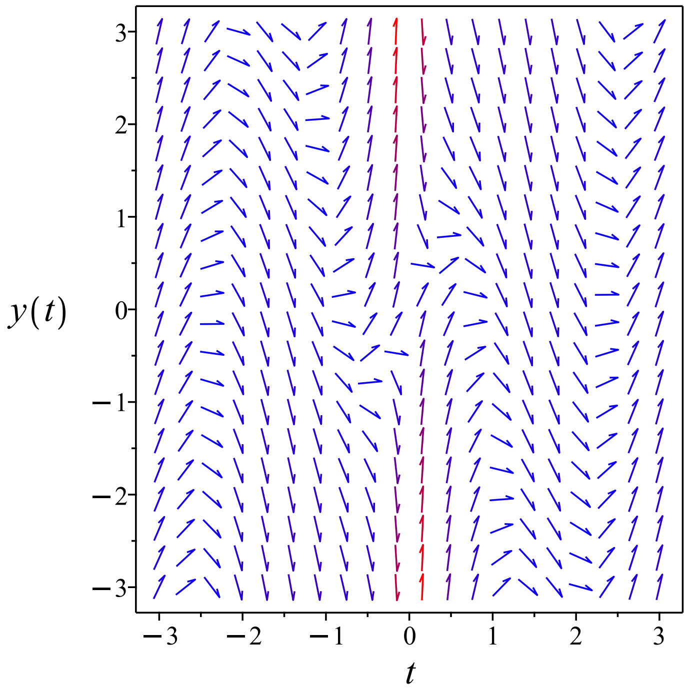


Figure 10: Slope field plot

Verification of solutions

$$y = \frac{6\sin(2t)\,t + 3\cos(2t) + 4c_1}{4t}$$

Verified OK.

1.4.2 Solving as first order ode lie symmetry lookup ode

Writing the ode as

$$y' = \frac{-y + 3\cos(2t)\,t}{t}$$
$$y' = \omega(t, y)$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_t + \omega(\eta_y - \xi_t) - \omega^2\xi_y - \omega_t\xi - \omega_y\eta = 0 \tag{A}$$

The type of this ode is known. It is of type `linear`. Therefore we do not need to solve the PDE (A), and can just use the lookup table shown below to find ξ, η

Table 10: Lie symmetry infinitesimal lookup table for known first order ODE's

ODE class	Form	ξ	η
linear ode	$y' = f(x)y(x) + g(x)$	0	$e^{\int f dx}$
separable ode	$y' = f(x)\, g(y)$	$\frac{1}{f}$	0
quadrature ode	$y' = f(x)$	0	1
quadrature ode	$y' = g(y)$	1	0
homogeneous ODEs of Class A	$y' = f\left(\frac{y}{x}\right)$	x	y
homogeneous ODEs of Class C	$y' = (a + bx + cy)^{\frac{n}{m}}$	1	$-\frac{b}{c}$
homogeneous class D	$y' = \frac{y}{x} + g(x)\, F\left(\frac{y}{x}\right)$	x^2	xy
First order special form ID 1	$y' = g(x)\, e^{h(x)+by} + f(x)$	$\frac{e^{-\int bf(x)dx - h(x)}}{g(x)}$	$\frac{f(x)e^{-\int bf(x)dx - h(x)}}{g(x)}$
polynomial type ode	$y' = \frac{a_1x+b_1y+c_1}{a_2x+b_2y+c_2}$	$\frac{a_1b_2x - a_2b_1x - b_1c_2 + b_2c_1}{a_1b_2 - a_2b_1}$	$\frac{a_1b_2y - a_2b_1y - a_1c_2 - a_2c_1}{a_1b_2 - a_2b_1}$
Bernoulli ode	$y' = f(x)\, y + g(x)\, y^n$	0	$e^{-\int (n-1)f(x)dx} y^n$
Reduced Riccati	$y' = f_1(x)\, y + f_2(x)\, y^2$	0	$e^{-\int f_1 dx}$

The above table shows that

$$
\begin{aligned}
\xi(t, y) &= 0 \\
\eta(t, y) &= \frac{1}{t}
\end{aligned} \tag{A1}
$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(t, y) \to (R, S)$ where (R, S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$
\frac{dt}{\xi} = \frac{dy}{\eta} = dS \tag{1}
$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial t} + \eta \frac{\partial}{\partial y}\right) S(t, y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the

canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$R = t$$

S is found from

$$\begin{aligned} S &= \int \frac{1}{\eta} dy \\ &= \int \frac{1}{\frac{1}{t}} dy \end{aligned}$$

Which results in

$$S = ty$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_t + \omega(t, y) S_y}{R_t + \omega(t, y) R_y} \tag{2}$$

Where in the above R_t, R_y, S_t, S_y are all partial derivatives and $\omega(t, y)$ is the right hand side of the original ode given by

$$\omega(t, y) = \frac{-y + 3\cos(2t)\, t}{t}$$

Evaluating all the partial derivatives gives

$$\begin{aligned} R_t &= 1 \\ R_y &= 0 \\ S_t &= y \\ S_y &= t \end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = 3\cos(2t)\, t \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for t, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = 3\cos(2R)\, R$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by

integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$S(R) = \frac{3\cos(2R)}{4} + \frac{3\sin(2R)\,R}{2} + c_1 \tag{4}$$

To complete the solution, we just need to transform (4) back to t, y coordinates. This results in

$$yt = \frac{3\cos(2t)}{4} + \frac{3\sin(2t)\,t}{2} + c_1$$

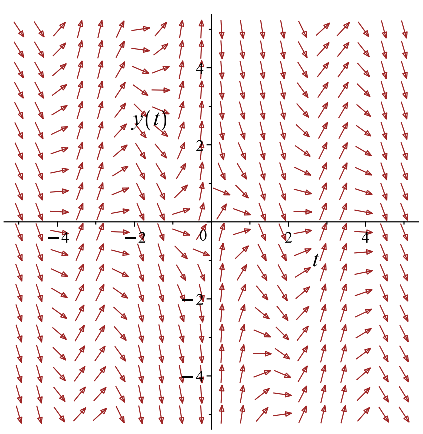
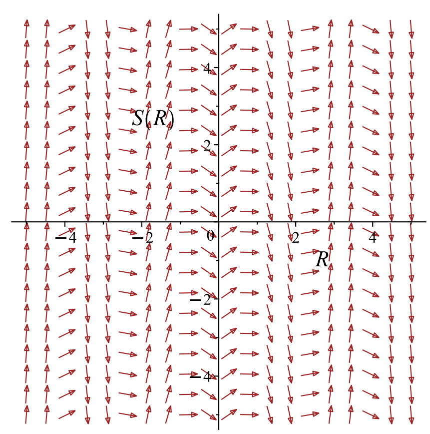
Which simplifies to

$$yt = \frac{3\cos(2t)}{4} + \frac{3\sin(2t)\,t}{2} + c_1$$

Which gives

$$y = \frac{6\sin(2t)\,t + 3\cos(2t) + 4c_1}{4t}$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in t, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dt} = \frac{-y+3\cos(2t)t}{t}$ 	$R = t$ $S = ty$	$\frac{dS}{dR} = 3\cos(2R)\,R$ 

Summary

The solution(s) found are the following

$$y = \frac{6\sin(2t)\,t + 3\cos(2t) + 4c_1}{4t} \tag{1}$$

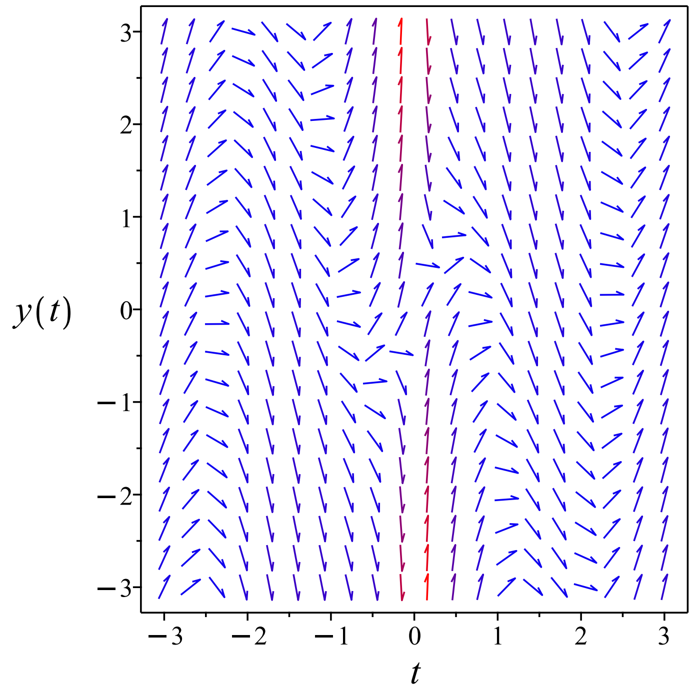


Figure 11: Slope field plot

Verification of solutions

$$y = \frac{6 \sin (2t) t + 3 \cos (2t) + 4c_1}{4t}$$

Verified OK.

1.4.3 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x,y) + N(x,y) \frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x,y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y) = 0$$

Hence

$$\frac{\partial \phi}{\partial x} + \frac{\partial \phi}{\partial y}\frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$\begin{aligned}\frac{\partial\phi}{\partial x} &= M\\ \frac{\partial\phi}{\partial y} &= N\end{aligned}$$

But since $\frac{\partial^2\phi}{\partial x\partial y} = \frac{\partial^2\phi}{\partial y\partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x,y)$ but at least we know now that we can do that since the condition $\frac{\partial^2\phi}{\partial x\partial y} = \frac{\partial^2\phi}{\partial y\partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(t,y)\,\mathrm{d}t + N(t,y)\,\mathrm{d}y = 0 \tag{1A}$$

Therefore

$$\begin{aligned}(t)\,\mathrm{d}y &= (-y + 3\cos(2t)\,t)\,\mathrm{d}t\\ (y - 3\cos(2t)\,t)\,\mathrm{d}t + (t)\,\mathrm{d}y &= 0 \qquad (2A)\end{aligned}$$

Comparing (1A) and (2A) shows that

$$\begin{aligned}M(t,y) &= y - 3\cos(2t)\,t\\ N(t,y) &= t\end{aligned}$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial t}$$

Using result found above gives

$$\begin{aligned}\frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}(y - 3\cos(2t)\,t)\\ &= 1\end{aligned}$$

And

$$\begin{aligned}\frac{\partial N}{\partial t} &= \frac{\partial}{\partial t}(t)\\ &= 1\end{aligned}$$

Since $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial t}$, then the ODE is exact The following equations are now set up to solve for the function $\phi(t, y)$

$$\frac{\partial \phi}{\partial t} = M \tag{1}$$
$$\frac{\partial \phi}{\partial y} = N \tag{2}$$

Integrating (1) w.r.t. t gives

$$\int \frac{\partial \phi}{\partial t} \, \mathrm{d}t = \int M \, \mathrm{d}t$$
$$\int \frac{\partial \phi}{\partial t} \, \mathrm{d}t = \int y - 3\cos(2t) \, t \, \mathrm{d}t$$
$$\phi = ty - \frac{3\cos(2t)}{4} - \frac{3\sin(2t)\, t}{2} + f(y) \tag{3}$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both t and y. Taking derivative of equation (3) w.r.t y gives

$$\frac{\partial \phi}{\partial y} = t + f'(y) \tag{4}$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = t$. Therefore equation (4) becomes

$$t = t + f'(y) \tag{5}$$

Solving equation (5) for $f'(y)$ gives

$$f'(y) = 0$$

Therefore

$$f(y) = c_1$$

Where c_1 is constant of integration. Substituting this result for $f(y)$ into equation (3) gives ϕ

$$\phi = ty - \frac{3\cos(2t)}{4} - \frac{3\sin(2t)\, t}{2} + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = ty - \frac{3\cos(2t)}{4} - \frac{3\sin(2t)\, t}{2}$$

The solution becomes

$$y = \frac{6 \sin(2t) t + 3 \cos(2t) + 4c_1}{4t}$$

Summary

The solution(s) found are the following

$$y = \frac{6 \sin(2t) t + 3 \cos(2t) + 4c_1}{4t} \tag{1}$$

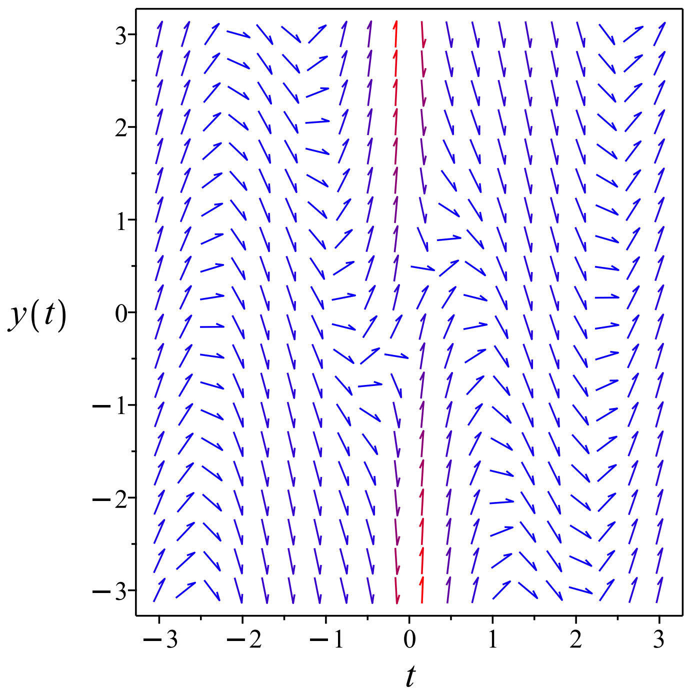


Figure 12: Slope field plot

Verification of solutions

$$y = \frac{6 \sin(2t) t + 3 \cos(2t) + 4c_1}{4t}$$

Verified OK.

1.4.4 Maple step by step solution

Let's solve

$\frac{y}{t}+y'=3\cos(2t)$

- Highest derivative means the order of the ODE is 1

y'

- Isolate the derivative

$y'=-\frac{y}{t}+3\cos(2t)$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE

$\frac{y}{t}+y'=3\cos(2t)$

- The ODE is linear; multiply by an integrating factor $\mu(t)$

$\mu(t)\left(\frac{y}{t}+y'\right)=3\mu(t)\cos(2t)$

- Assume the lhs of the ODE is the total derivative $\frac{d}{dt}(\mu(t)\,y)$

$\mu(t)\left(\frac{y}{t}+y'\right)=\mu'(t)\,y+\mu(t)\,y'$

- Isolate $\mu'(t)$

$\mu'(t)=\frac{\mu(t)}{t}$

- Solve to find the integrating factor

$\mu(t)=t$

- Integrate both sides with respect to t

$\int\left(\frac{d}{dt}(\mu(t)\,y)\right)dt=\int 3\mu(t)\cos(2t)\,dt+c_1$

- Evaluate the integral on the lhs

$\mu(t)\,y=\int 3\mu(t)\cos(2t)\,dt+c_1$

- Solve for y

$y=\frac{\int 3\mu(t)\cos(2t)dt+c_1}{\mu(t)}$

- Substitute $\mu(t)=t$

$y=\frac{\int 3\cos(2t)t dt+c_1}{t}$

- Evaluate the integrals on the rhs

$y=\frac{\frac{3\cos(2t)}{4}+\frac{3\sin(2t)t}{2}+c_1}{t}$

- Simplify

$$y = \frac{6\sin(2t)t+3\cos(2t)+4c_1}{4t}$$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
<- 1st order linear successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 26

```
dsolve(y(t)/t+diff(y(t),t) = 3*cos(2*t),y(t), singsol=all)
```

$$y(t) = \frac{4c_1 + 6\sin(2t)\, t + 3\cos(2t)}{4t}$$

✓ Solution by Mathematica

Time used: 0.073 (sec). Leaf size: 30

```
DSolve[y[t]/t+y'[t] == 3*Cos[2*t],y[t],t,IncludeSingularSolutions -> True]
```

$$y(t) \to \frac{6t\sin(2t) + 3\cos(2t) + 4c_1}{4t}$$

1.5 problem 5

Internal problem ID [452]
Internal file name [`OUTPUT/452_Sunday_June_05_2022_01_41_43_AM_90446891/index.tex`]

Book: Elementary differential equations and boundary value problems, 10th ed., Boyce and DiPrima
Section: Section 2.1. Page 40
Problem number: 5.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"linear", "exactWithIntegrationFactor", "first_order_ode_lie_symmetry_lookup"**

Maple gives the following as the ode type

```
[[_linear, `class A`]]
```

$$\boxed{-2y + y' = 3\,\mathrm{e}^{t}}$$

1.5.1 Solving as linear ode

Entering Linear first order ODE solver. In canonical form a linear first order is

$$y' + p(t)y = q(t)$$

Where here

$$p(t) = -2$$
$$q(t) = 3\,\mathrm{e}^{t}$$

Hence the ode is

$$-2y + y' = 3\,\mathrm{e}^{t}$$

The integrating factor μ is

$$\begin{aligned}\mu &= \mathrm{e}^{\int (-2)dt} \\ &= \mathrm{e}^{-2t}\end{aligned}$$

The ode becomes

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}(\mu y) &= (\mu)\left(3\,\mathrm{e}^{t}\right) \\ \frac{\mathrm{d}}{\mathrm{d}t}\left(\mathrm{e}^{-2t} y\right) &= \left(\mathrm{e}^{-2t}\right)\left(3\,\mathrm{e}^{t}\right) \\ \mathrm{d}\left(\mathrm{e}^{-2t} y\right) &= \left(3\,\mathrm{e}^{-t}\right)\,\mathrm{d}t\end{aligned}$$

Integrating gives

$$\begin{aligned}\mathrm{e}^{-2t} y &= \int 3\,\mathrm{e}^{-t}\,\mathrm{d}t \\ \mathrm{e}^{-2t} y &= -3\,\mathrm{e}^{-t} + c_1\end{aligned}$$

Dividing both sides by the integrating factor $\mu = \mathrm{e}^{-2t}$ results in

$$y = -3\,\mathrm{e}^{2t}\mathrm{e}^{-t} + c_1 \mathrm{e}^{2t}$$

which simplifies to

$$y = -3\,\mathrm{e}^{t} + c_1 \mathrm{e}^{2t}$$

Summary

The solution(s) found are the following

$$y = -3\,\mathrm{e}^{t} + c_1 \mathrm{e}^{2t} \tag{1}$$

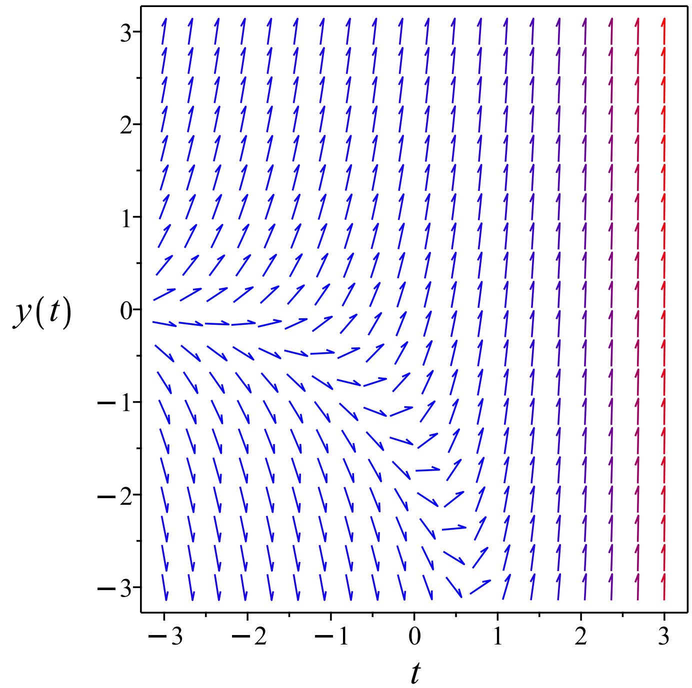


Figure 13: Slope field plot

Verification of solutions

$$y = -3\,e^{t} + c_1 e^{2t}$$

Verified OK.

1.5.2 Solving as first order ode lie symmetry lookup ode

Writing the ode as

$$\begin{aligned} y' &= 2y + 3\,e^{t} \\ y' &= \omega(t, y) \end{aligned}$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_t + \omega(\eta_y - \xi_t) - \omega^2 \xi_y - \omega_t \xi - \omega_y \eta = 0 \tag{A}$$

The type of this ode is known. It is of type `linear`. Therefore we do not need to solve the PDE (A), and can just use the lookup table shown below to find ξ, η

Table 13: Lie symmetry infinitesimal lookup table for known first order ODE's

ODE class	Form	ξ	η
linear ode	$y' = f(x)y(x) + g(x)$	0	$e^{\int f dx}$
separable ode	$y' = f(x)\, g(y)$	$\frac{1}{f}$	0
quadrature ode	$y' = f(x)$	0	1
quadrature ode	$y' = g(y)$	1	0
homogeneous ODEs of Class A	$y' = f\left(\frac{y}{x}\right)$	x	y
homogeneous ODEs of Class C	$y' = (a + bx + cy)^{\frac{n}{m}}$	1	$-\frac{b}{c}$
homogeneous class D	$y' = \frac{y}{x} + g(x)\, F\left(\frac{y}{x}\right)$	x^2	xy
First order special form ID 1	$y' = g(x)\, e^{h(x)+by} + f(x)$	$\frac{e^{-\int bf(x)dx-h(x)}}{g(x)}$	$\frac{f(x)e^{-\int bf(x)dx-h(x)}}{g(x)}$
polynomial type ode	$y' = \frac{a_1x+b_1y+c_1}{a_2x+b_2y+c_2}$	$\frac{a_1b_2x-a_2b_1x-b_1c_2+b_2c_1}{a_1b_2-a_2b_1}$	$\frac{a_1b_2y-a_2b_1y-a_1c_2-a_2c_1}{a_1b_2-a_2b_1}$
Bernoulli ode	$y' = f(x)\, y + g(x)\, y^n$	0	$e^{-\int (n-1)f(x)dx} y^n$
Reduced Riccati	$y' = f_1(x)\, y + f_2(x)\, y^2$	0	$e^{-\int f_1 dx}$

The above table shows that

$$
\begin{aligned}
\xi(t,y) &= 0 \\
\eta(t,y) &= \mathrm{e}^{2t}
\end{aligned} \tag{A1}
$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(t,y) \to (R,S)$ where (R,S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$
\frac{dt}{\xi} = \frac{dy}{\eta} = dS \tag{1}
$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial t} + \eta \frac{\partial}{\partial y}\right) S(t,y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the

canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$R = t$$

S is found from

$$\begin{aligned} S &= \int \frac{1}{\eta} dy \\ &= \int \frac{1}{e^{2t}} dy \end{aligned}$$

Which results in

$$S = e^{-2t} y$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_t + \omega(t,y) S_y}{R_t + \omega(t,y) R_y} \tag{2}$$

Where in the above R_t, R_y, S_t, S_y are all partial derivatives and $\omega(t,y)$ is the right hand side of the original ode given by

$$\omega(t,y) = 2y + 3\,e^t$$

Evaluating all the partial derivatives gives

$$\begin{aligned} R_t &= 1 \\ R_y &= 0 \\ S_t &= -2\,e^{-2t} y \\ S_y &= e^{-2t} \end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = 3\,e^{-t} \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for t, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = 3\,e^{-R}$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by

integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$S(R) = -3\,\mathrm{e}^{-R} + c_1 \tag{4}$$

To complete the solution, we just need to transform (4) back to t, y coordinates. This results in

$$\mathrm{e}^{-2t} y = -3\,\mathrm{e}^{-t} + c_1$$

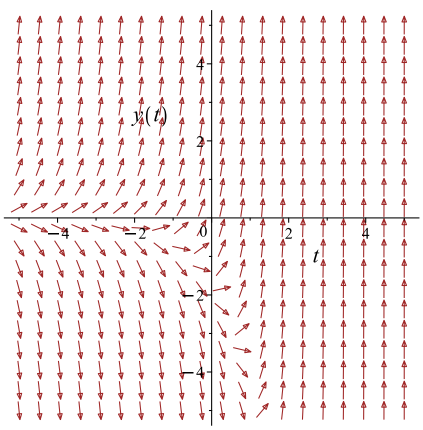
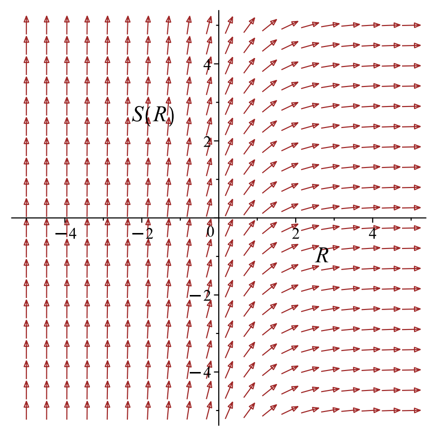
Which simplifies to

$$\mathrm{e}^{-2t} y = -3\,\mathrm{e}^{-t} + c_1$$

Which gives

$$y = -\left(3\,\mathrm{e}^{-t} - c_1\right) \mathrm{e}^{2t}$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

<table>
<tr><th>Original ode in t, y coordinates</th><th>Canonical coordinates transformation</th><th>ODE in canonical coordinates (R, S)</th></tr>
<tr><td>$\frac{dy}{dt} = 2y + 3\,\mathrm{e}^{t}$
</td><td>$R = t$
$S = \mathrm{e}^{-2t} y$</td><td>$\frac{dS}{dR} = 3\,\mathrm{e}^{-R}$
</td></tr>
</table>

Summary

The solution(s) found are the following

$$y = -\left(3\,\mathrm{e}^{-t} - c_1\right) \mathrm{e}^{2t} \tag{1}$$

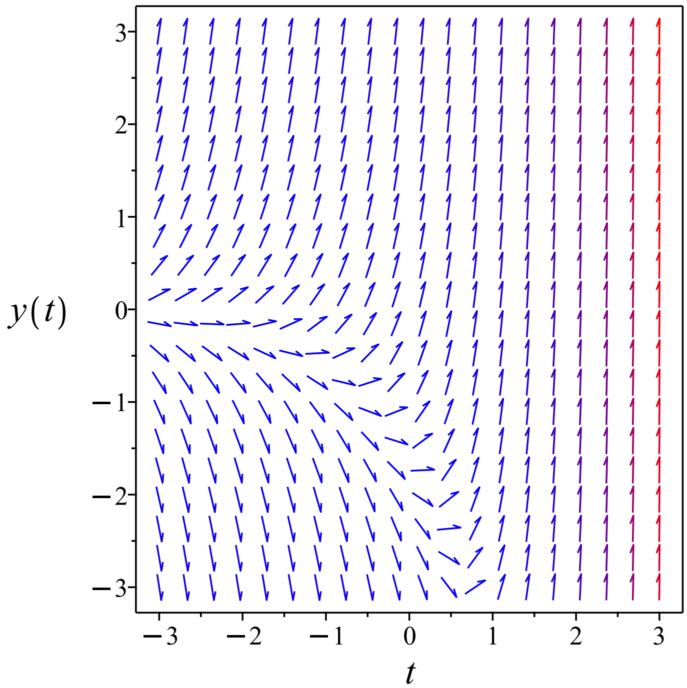


Figure 14: Slope field plot

Verification of solutions

$$y = -\left(3\,\mathrm{e}^{-t} - c_1\right)\mathrm{e}^{2t}$$

Verified OK.

1.5.3 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x,y) + N(x,y)\,\frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x,y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y) = 0$$

Hence

$$\frac{\partial \phi}{\partial x} + \frac{\partial \phi}{\partial y}\frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$
\begin{aligned}
\frac{\partial \phi}{\partial x} &= M \\
\frac{\partial \phi}{\partial y} &= N
\end{aligned}
$$

But since $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ then for the above to be valid, we require that

$$
\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}
$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x, y)$ but at least we know now that we can do that since the condition $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$
M(t, y)\, \mathrm{d}t + N(t, y)\, \mathrm{d}y = 0 \tag{1A}
$$

Therefore

$$
\begin{aligned}
\mathrm{d}y &= \left(2y + 3\,\mathrm{e}^{t}\right) \mathrm{d}t \\
\left(-2y - 3\,\mathrm{e}^{t}\right) \mathrm{d}t + \mathrm{d}y &= 0 \qquad (2A)
\end{aligned}
$$

Comparing (1A) and (2A) shows that

$$
\begin{aligned}
M(t, y) &= -2y - 3\,\mathrm{e}^{t} \\
N(t, y) &= 1
\end{aligned}
$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$
\frac{\partial M}{\partial y} = \frac{\partial N}{\partial t}
$$

Using result found above gives

$$
\begin{aligned}
\frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}\left(-2y - 3\,\mathrm{e}^{t}\right) \\
&= -2
\end{aligned}
$$

And

$$
\begin{aligned}
\frac{\partial N}{\partial t} &= \frac{\partial}{\partial t}(1) \\
&= 0
\end{aligned}
$$

Since $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial t}$, then the ODE is not exact. Since the ODE is not exact, we will try to find an integrating factor to make it exact. Let

$$\begin{aligned} A &= \frac{1}{N}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial t}\right) \\ &= 1((-2) - (0)) \\ &= -2 \end{aligned}$$

Since A does not depend on y, then it can be used to find an integrating factor. The integrating factor μ is

$$\begin{aligned} \mu &= e^{\int A \, \mathrm{d}t} \\ &= e^{\int -2 \, \mathrm{d}t} \end{aligned}$$

The result of integrating gives

$$\begin{aligned} \mu &= e^{-2t} \\ &= \mathrm{e}^{-2t} \end{aligned}$$

M and N are multiplied by this integrating factor, giving new M and new N which are called $\overline{M}$ and $\overline{N}$ for now so not to confuse them with the original M and N.

$$\begin{aligned} \overline{M} &= \mu M \\ &= \mathrm{e}^{-2t}\left(-2y - 3\,\mathrm{e}^{t}\right) \\ &= \left(-2y - 3\,\mathrm{e}^{t}\right)\mathrm{e}^{-2t} \end{aligned}$$

And

$$\begin{aligned} \overline{N} &= \mu N \\ &= \mathrm{e}^{-2t}(1) \\ &= \mathrm{e}^{-2t} \end{aligned}$$

Now a modified ODE is ontained from the original ODE, which is exact and can be solved. The modified ODE is

$$\begin{aligned} \overline{M} + \overline{N}\frac{\mathrm{d}y}{\mathrm{d}t} &= 0 \\ \left(\left(-2y - 3\,\mathrm{e}^{t}\right)\mathrm{e}^{-2t}\right) + \left(\mathrm{e}^{-2t}\right)\frac{\mathrm{d}y}{\mathrm{d}t} &= 0 \end{aligned}$$

The following equations are now set up to solve for the function $\phi(t, y)$

$$\frac{\partial \phi}{\partial t} = \overline{M} \tag{1}$$

$$\frac{\partial \phi}{\partial y} = \overline{N} \tag{2}$$

Integrating (1) w.r.t. t gives

$$\int \frac{\partial \phi}{\partial t} \,\mathrm{d}t = \int \overline{M} \,\mathrm{d}t$$

$$\int \frac{\partial \phi}{\partial t} \,\mathrm{d}t = \int \left(-2y - 3\,\mathrm{e}^{t}\right) \mathrm{e}^{-2t} \,\mathrm{d}t$$

$$\phi = 3\,\mathrm{e}^{-t} + \mathrm{e}^{-2t} y + f(y) \tag{3}$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both t and y. Taking derivative of equation (3) w.r.t y gives

$$\frac{\partial \phi}{\partial y} = \mathrm{e}^{-2t} + f'(y) \tag{4}$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = \mathrm{e}^{-2t}$. Therefore equation (4) becomes

$$\mathrm{e}^{-2t} = \mathrm{e}^{-2t} + f'(y) \tag{5}$$

Solving equation (5) for $f'(y)$ gives

$$f'(y) = 0$$

Therefore

$$f(y) = c_1$$

Where c_1 is constant of integration. Substituting this result for $f(y)$ into equation (3) gives ϕ

$$\phi = 3\,\mathrm{e}^{-t} + \mathrm{e}^{-2t} y + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = 3\,\mathrm{e}^{-t} + \mathrm{e}^{-2t} y$$

The solution becomes

$$y = -\left(3\,\mathrm{e}^{-t} - c_1\right) \mathrm{e}^{2t}$$

Summary

The solution(s) found are the following

$$y = -\left(3\,\mathrm{e}^{-t} - c_1\right) \mathrm{e}^{2t} \tag{1}$$

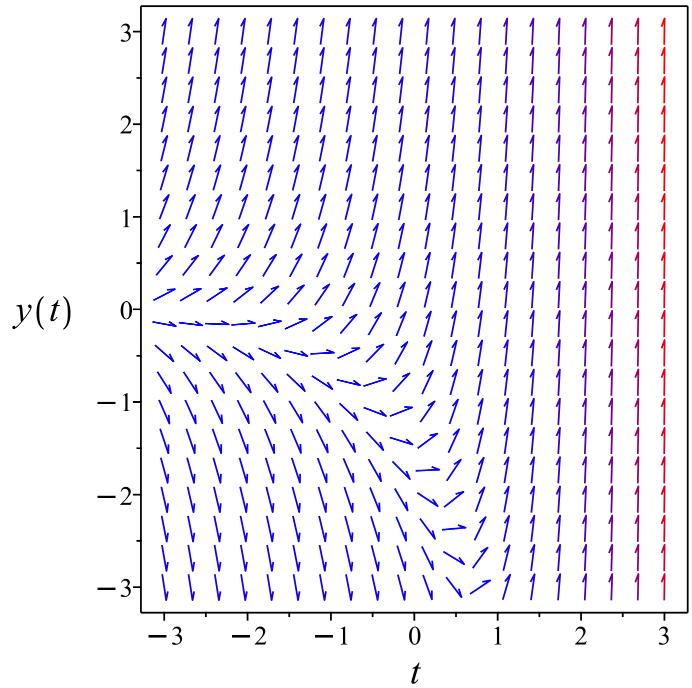


Figure 15: Slope field plot

Verification of solutions

$$y = -\left(3\,\mathrm{e}^{-t} - c_1\right)\mathrm{e}^{2t}$$

Verified OK.

1.5.4 Maple step by step solution

Let's solve

$-2y + y' = 3\,\mathrm{e}^{t}$

- Highest derivative means the order of the ODE is 1

  y'

- Isolate the derivative

  $y' = 2y + 3\,\mathrm{e}^{t}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE

  $-2y + y' = 3\,\mathrm{e}^{t}$

- The ODE is linear; multiply by an integrating factor $\mu(t)$

  $\mu(t)\left(-2y + y'\right) = 3\mu(t)\,\mathrm{e}^{t}$

- Assume the lhs of the ODE is the total derivative $\frac{d}{dt}(\mu(t)\, y)$

  $\mu(t)\,(-2y+y') = \mu'(t)\, y + \mu(t)\, y'$
- Isolate $\mu'(t)$

  $\mu'(t) = -2\mu(t)$
- Solve to find the integrating factor

  $\mu(t) = \mathrm{e}^{-2t}$
- Integrate both sides with respect to t

  $\int \left(\frac{d}{dt}(\mu(t)\, y)\right) dt = \int 3\mu(t)\, \mathrm{e}^{t} dt + c_1$
- Evaluate the integral on the lhs

  $\mu(t)\, y = \int 3\mu(t)\, \mathrm{e}^{t} dt + c_1$
- Solve for y

  $y = \frac{\int 3\mu(t)\mathrm{e}^{t} dt + c_1}{\mu(t)}$
- Substitute $\mu(t) = \mathrm{e}^{-2t}$

  $y = \frac{\int 3\,\mathrm{e}^{t}\mathrm{e}^{-2t} dt + c_1}{\mathrm{e}^{-2t}}$
- Evaluate the integrals on the rhs

  $y = \frac{-3\,\mathrm{e}^{-t} + c_1}{\mathrm{e}^{-2t}}$
- Simplify

  $y = -3\,\mathrm{e}^{t} + c_1 \mathrm{e}^{2t}$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
<- 1st order linear successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 15

```
dsolve(-2*y(t)+diff(y(t),t) = 3*exp(t),y(t), singsol=all)
```

$$y(t) = -3\,\mathrm{e}^{t} + c_1\mathrm{e}^{2t}$$

✓ Solution by Mathematica

Time used: 0.046 (sec). Leaf size: 17

```
DSolve[-2*y[t]+y'[t] == 3*Exp[t],y[t],t,IncludeSingularSolutions -> True]
```

$$y(t) \to e^t\left(-3 + c_1e^t\right)$$

1.6 problem 6

Internal problem ID [453]
Internal file name [`OUTPUT/453_Sunday_June_05_2022_01_41_43_AM_13381902/index.tex`]

Book: Elementary differential equations and boundary value problems, 10th ed., Boyce and DiPrima
Section: Section 2.1. Page 40
Problem number: 6.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"linear", "exactWithIntegrationFactor", "first_order_ode_lie_symmetry_lookup"**

Maple gives the following as the ode type

`[_linear]`

$$\boxed{2y + ty' = \sin(t)}$$

1.6.1 Solving as linear ode

Entering Linear first order ODE solver. In canonical form a linear first order is

$$y' + p(t)y = q(t)$$

Where here

$$p(t) = \frac{2}{t}$$
$$q(t) = \frac{\sin(t)}{t}$$

Hence the ode is

$$y' + \frac{2y}{t} = \frac{\sin(t)}{t}$$

The integrating factor μ is

$$\mu = e^{\int \frac{2}{t}dt}$$
$$= t^2$$

The ode becomes

$$\frac{d}{dt}(\mu y) = (\mu)\left(\frac{\sin(t)}{t}\right)$$
$$\frac{d}{dt}(t^2 y) = (t^2)\left(\frac{\sin(t)}{t}\right)$$
$$d(t^2 y) = (t \sin(t))\ dt$$

Integrating gives

$$t^2 y = \int t \sin(t)\ dt$$
$$t^2 y = -t\cos(t) + \sin(t) + c_1$$

Dividing both sides by the integrating factor $\mu = t^2$ results in

$$y = \frac{-t\cos(t) + \sin(t)}{t^2} + \frac{c_1}{t^2}$$

which simplifies to

$$y = \frac{-t\cos(t) + \sin(t) + c_1}{t^2}$$

Summary

The solution(s) found are the following

$$y = \frac{-t\cos(t) + \sin(t) + c_1}{t^2} \tag{1}$$

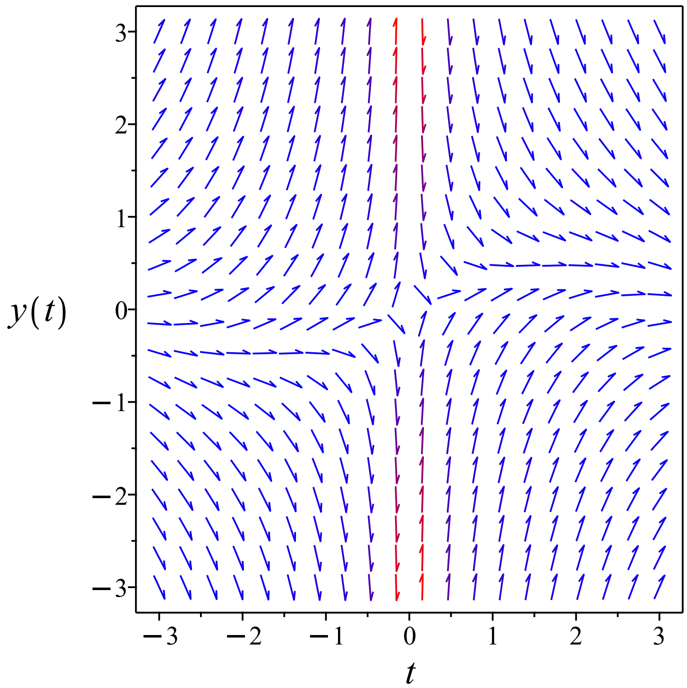


Figure 16: Slope field plot

Verification of solutions

$$y = \frac{-t\cos(t) + \sin(t) + c_1}{t^2}$$

Verified OK.

1.6.2 Solving as first order ode lie symmetry lookup ode

Writing the ode as

$$y' = \frac{-2y + \sin(t)}{t}$$
$$y' = \omega(t, y)$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_t + \omega(\eta_y - \xi_t) - \omega^2 \xi_y - \omega_t \xi - \omega_y \eta = 0 \tag{A}$$

The type of this ode is known. It is of type `linear`. Therefore we do not need to solve the PDE (A), and can just use the lookup table shown below to find ξ, η

Table 16: Lie symmetry infinitesimal lookup table for known first order ODE's

ODE class	Form	ξ	η
linear ode	$y' = f(x)y(x) + g(x)$	0	$e^{\int f dx}$
separable ode	$y' = f(x)\,g(y)$	$\frac{1}{f}$	0
quadrature ode	$y' = f(x)$	0	1
quadrature ode	$y' = g(y)$	1	0
homogeneous ODEs of Class A	$y' = f\left(\frac{y}{x}\right)$	x	y
homogeneous ODEs of Class C	$y' = (a + bx + cy)^{\frac{n}{m}}$	1	$-\frac{b}{c}$
homogeneous class D	$y' = \frac{y}{x} + g(x)\,F\left(\frac{y}{x}\right)$	x^2	xy
First order special form ID 1	$y' = g(x)\,e^{h(x)+by} + f(x)$	$\frac{e^{-\int bf(x)dx - h(x)}}{g(x)}$	$\frac{f(x)e^{-\int bf(x)dx - h(x)}}{g(x)}$
polynomial type ode	$y' = \frac{a_1x+b_1y+c_1}{a_2x+b_2y+c_2}$	$\frac{a_1b_2x - a_2b_1x - b_1c_2 + b_2c_1}{a_1b_2 - a_2b_1}$	$\frac{a_1b_2y - a_2b_1y - a_1c_2 - a_2c_1}{a_1b_2 - a_2b_1}$
Bernoulli ode	$y' = f(x)\,y + g(x)\,y^n$	0	$e^{-\int (n-1)f(x)dx}y^n$
Reduced Riccati	$y' = f_1(x)\,y + f_2(x)\,y^2$	0	$e^{-\int f_1 dx}$

The above table shows that

$$\begin{aligned} \xi(t,y) &= 0 \\ \eta(t,y) &= \frac{1}{t^2} \end{aligned} \tag{A1}$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(t,y) \to (R,S)$ where (R,S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$\frac{dt}{\xi} = \frac{dy}{\eta} = dS \tag{1}$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial t} + \eta \frac{\partial}{\partial y}\right) S(t,y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the

canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$R = t$$

S is found from

$$\begin{aligned} S &= \int \frac{1}{\eta} dy \\ &= \int \frac{1}{\frac{1}{t^2}} dy \end{aligned}$$

Which results in

$$S = t^2 y$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_t + \omega(t,y) S_y}{R_t + \omega(t,y) R_y} \tag{2}$$

Where in the above R_t, R_y, S_t, S_y are all partial derivatives and $\omega(t,y)$ is the right hand side of the original ode given by

$$\omega(t,y) = \frac{-2y + \sin(t)}{t}$$

Evaluating all the partial derivatives gives

$$\begin{aligned} R_t &= 1 \\ R_y &= 0 \\ S_t &= 2ty \\ S_y &= t^2 \end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = t \sin(t) \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for t, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = R \sin(R)$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by

integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$S(R) = -R\cos(R) + \sin(R) + c_1 \tag{4}$$

To complete the solution, we just need to transform (4) back to t, y coordinates. This results in

$$yt^2 = -t\cos(t) + \sin(t) + c_1$$

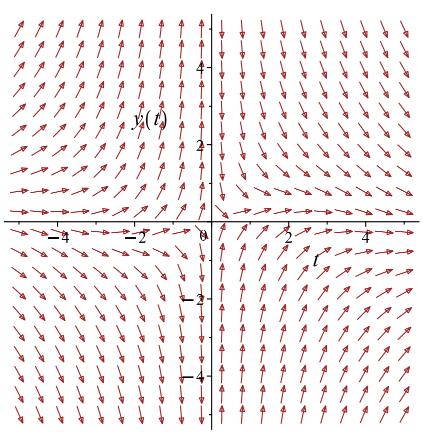
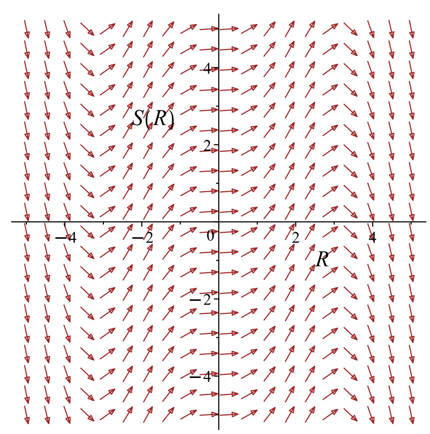
Which simplifies to

$$yt^2 = -t\cos(t) + \sin(t) + c_1$$

Which gives

$$y = \frac{-t\cos(t) + \sin(t) + c_1}{t^2}$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in t, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dt} = \frac{-2y+\sin(t)}{t}$ 	$R = t$ $S = t^2 y$	$\frac{dS}{dR} = R\sin(R)$ 

Summary

The solution(s) found are the following

$$y = \frac{-t\cos(t) + \sin(t) + c_1}{t^2} \tag{1}$$

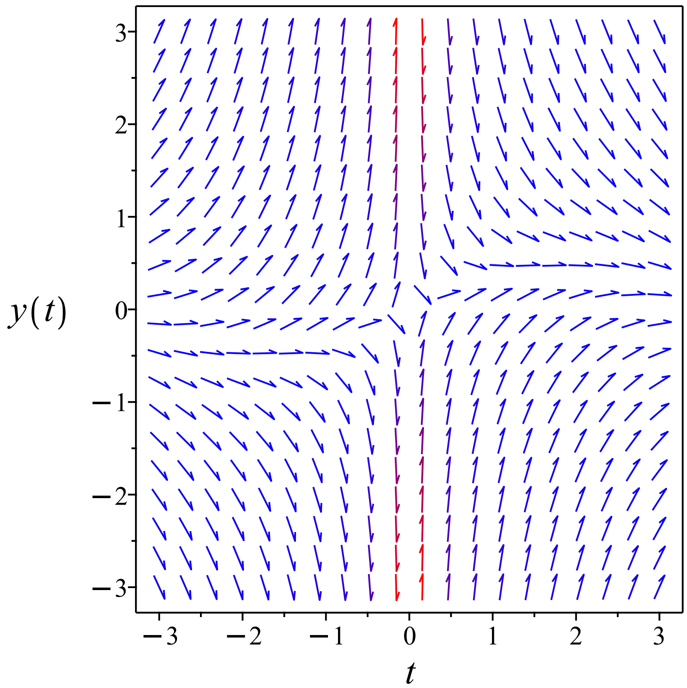


Figure 17: Slope field plot

Verification of solutions

$$y = \frac{-t\cos(t) + \sin(t) + c_1}{t^2}$$

Verified OK.

1.6.3 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x,y) + N(x,y)\frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x,y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y) = 0$$

Hence

$$\frac{\partial\phi}{\partial x} + \frac{\partial\phi}{\partial y}\frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$\begin{aligned}
\frac{\partial \phi}{\partial x} &= M \\
\frac{\partial \phi}{\partial y} &= N
\end{aligned}$$

But since $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x, y)$ but at least we know now that we can do that since the condition $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(t, y)\,\mathrm{d}t + N(t, y)\,\mathrm{d}y = 0 \tag{1A}$$

Therefore

$$\begin{aligned}
(t)\,\mathrm{d}y &= (-2y + \sin(t))\,\mathrm{d}t \\
(2y - \sin(t))\,\mathrm{d}t + (t)\,\mathrm{d}y &= 0 \qquad (2A)
\end{aligned}$$

Comparing (1A) and (2A) shows that

$$\begin{aligned}
M(t, y) &= 2y - \sin(t) \\
N(t, y) &= t
\end{aligned}$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial t}$$

Using result found above gives

$$\begin{aligned}
\frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}(2y - \sin(t)) \\
&= 2
\end{aligned}$$

And

$$\begin{aligned}
\frac{\partial N}{\partial t} &= \frac{\partial}{\partial t}(t) \\
&= 1
\end{aligned}$$

Since $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial t}$, then the ODE is <u>not exact</u>. Since the ODE is not exact, we will try to find an integrating factor to make it exact. Let

$$\begin{aligned} A &= \frac{1}{N}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial t}\right) \\ &= \frac{1}{t}((2) - (1)) \\ &= \frac{1}{t} \end{aligned}$$

Since A does not depend on y, then it can be used to find an integrating factor. The integrating factor μ is

$$\begin{aligned} \mu &= e^{\int A \, \mathrm{d}t} \\ &= e^{\int \frac{1}{t} \, \mathrm{d}t} \end{aligned}$$

The result of integrating gives

$$\begin{aligned} \mu &= e^{\ln(t)} \\ &= t \end{aligned}$$

M and N are multiplied by this integrating factor, giving new M and new N which are called $\overline{M}$ and $\overline{N}$ for now so not to confuse them with the original M and N.

$$\begin{aligned} \overline{M} &= \mu M \\ &= t(2y - \sin(t)) \\ &= (2y - \sin(t))\, t \end{aligned}$$

And

$$\begin{aligned} \overline{N} &= \mu N \\ &= t(t) \\ &= t^2 \end{aligned}$$

Now a modified ODE is ontained from the original ODE, which is exact and can be solved. The modified ODE is

$$\begin{aligned} \overline{M} + \overline{N} \frac{\mathrm{d}y}{\mathrm{d}t} &= 0 \\ ((2y - \sin(t))\, t) + (t^2) \frac{\mathrm{d}y}{\mathrm{d}t} &= 0 \end{aligned}$$

The following equations are now set up to solve for the function $\phi(t, y)$

$$\frac{\partial \phi}{\partial t} = \overline{M} \tag{1}$$
$$\frac{\partial \phi}{\partial y} = \overline{N} \tag{2}$$

Integrating (1) w.r.t. t gives

$$\int \frac{\partial \phi}{\partial t}\, \mathrm{d}t = \int \overline{M}\, \mathrm{d}t$$
$$\int \frac{\partial \phi}{\partial t}\, \mathrm{d}t = \int (2y - \sin(t))\, t\, \mathrm{d}t$$
$$\phi = t^2 y + t\cos(t) - \sin(t) + f(y) \tag{3}$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both t and y. Taking derivative of equation (3) w.r.t y gives

$$\frac{\partial \phi}{\partial y} = t^2 + f'(y) \tag{4}$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = t^2$. Therefore equation (4) becomes

$$t^2 = t^2 + f'(y) \tag{5}$$

Solving equation (5) for $f'(y)$ gives

$$f'(y) = 0$$

Therefore

$$f(y) = c_1$$

Where c_1 is constant of integration. Substituting this result for $f(y)$ into equation (3) gives ϕ

$$\phi = t^2 y + t\cos(t) - \sin(t) + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = t^2 y + t\cos(t) - \sin(t)$$

The solution becomes

$$y = \frac{-t\cos(t) + \sin(t) + c_1}{t^2}$$

Summary

The solution(s) found are the following

$$y = \frac{-t\cos(t) + \sin(t) + c_1}{t^2} \tag{1}$$

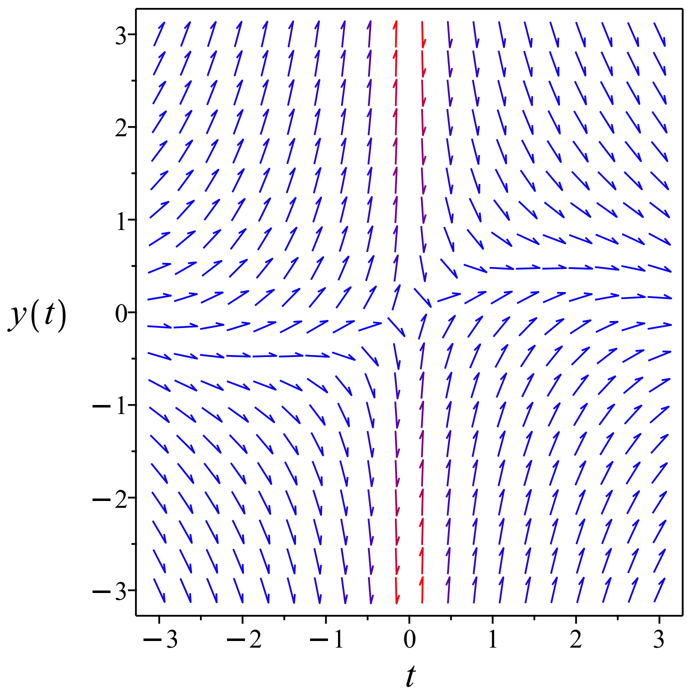


Figure 18: Slope field plot

Verification of solutions

$$y = \frac{-t\cos(t) + \sin(t) + c_1}{t^2}$$

Verified OK.

1.6.4 Maple step by step solution

Let's solve

$2y + ty' = \sin(t)$

- Highest derivative means the order of the ODE is 1

  y'

- Isolate the derivative

  $y' = -\frac{2y}{t} + \frac{\sin(t)}{t}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE

  $y' + \frac{2y}{t} = \frac{\sin(t)}{t}$

- The ODE is linear; multiply by an integrating factor $\mu(t)$

  $\mu(t)\left(y' + \frac{2y}{t}\right) = \frac{\mu(t)\sin(t)}{t}$

- Assume the lhs of the ODE is the total derivative $\frac{d}{dt}(\mu(t)\, y)$

  $\mu(t)\left(y' + \frac{2y}{t}\right) = \mu'(t)\, y + \mu(t)\, y'$

- Isolate $\mu'(t)$

  $\mu'(t) = \frac{2\mu(t)}{t}$

- Solve to find the integrating factor

  $\mu(t) = t^2$

- Integrate both sides with respect to t

  $\int \left(\frac{d}{dt}(\mu(t)\, y)\right) dt = \int \frac{\mu(t)\sin(t)}{t} dt + c_1$

- Evaluate the integral on the lhs

  $\mu(t)\, y = \int \frac{\mu(t)\sin(t)}{t} dt + c_1$

- Solve for y

  $y = \frac{\int \frac{\mu(t)\sin(t)}{t} dt + c_1}{\mu(t)}$

- Substitute $\mu(t) = t^2$

  $y = \frac{\int t\sin(t) dt + c_1}{t^2}$

- Evaluate the integrals on the rhs

  $y = \frac{-t\cos(t) + \sin(t) + c_1}{t^2}$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
<- 1st order linear successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 17

```
dsolve(2*y(t)+t*diff(y(t),t) = sin(t),y(t), singsol=all)
```

$$y(t) = \frac{\sin(t) - \cos(t) t + c_1}{t^2}$$

✓ Solution by Mathematica

Time used: 0.039 (sec). Leaf size: 19

```
DSolve[2*y[t]+t*y'[t]== Sin[t],y[t],t,IncludeSingularSolutions -> True]
```

$$y(t) \to \frac{\sin(t) - t\cos(t) + c_1}{t^2}$$

1.7 problem 7

Internal problem ID [454]
Internal file name [`OUTPUT/454_Sunday_June_05_2022_01_41_44_AM_75941329/index.tex`]

Book: Elementary differential equations and boundary value problems, 10th ed., Boyce and DiPrima
Section: Section 2.1. Page 40
Problem number: 7.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"exact", "linear", "first_order_ode_lie_symmetry_lookup"**

Maple gives the following as the ode type

```
[_linear]
```

$$\boxed{2yt + y' = 2t\,\mathrm{e}^{-t^2}}$$

1.7.1 Solving as linear ode

Entering Linear first order ODE solver. In canonical form a linear first order is

$$y' + p(t)y = q(t)$$

Where here

$$p(t) = 2t$$
$$q(t) = 2t\,\mathrm{e}^{-t^2}$$

Hence the ode is

$$2yt + y' = 2t\,\mathrm{e}^{-t^2}$$

The integrating factor μ is

$$
\begin{aligned}
\mu &= e^{\int 2t dt} \\
&= e^{t^2}
\end{aligned}
$$

The ode becomes

$$
\begin{aligned}
\frac{d}{dt}(\mu y) &= (\mu)\left(2t\,e^{-t^2}\right) \\
\frac{d}{dt}\left(y\,e^{t^2}\right) &= \left(e^{t^2}\right)\left(2t\,e^{-t^2}\right) \\
d\left(y\,e^{t^2}\right) &= (2t)\ dt
\end{aligned}
$$

Integrating gives

$$
\begin{aligned}
y\,e^{t^2} &= \int 2t\,dt \\
y\,e^{t^2} &= t^2 + c_1
\end{aligned}
$$

Dividing both sides by the integrating factor $\mu = e^{t^2}$ results in

$$
y = t^2 e^{-t^2} + c_1 e^{-t^2}
$$

which simplifies to

$$
y = e^{-t^2}\left(t^2 + c_1\right)
$$

Summary

The solution(s) found are the following

$$
y = e^{-t^2}\left(t^2 + c_1\right) \tag{1}
$$

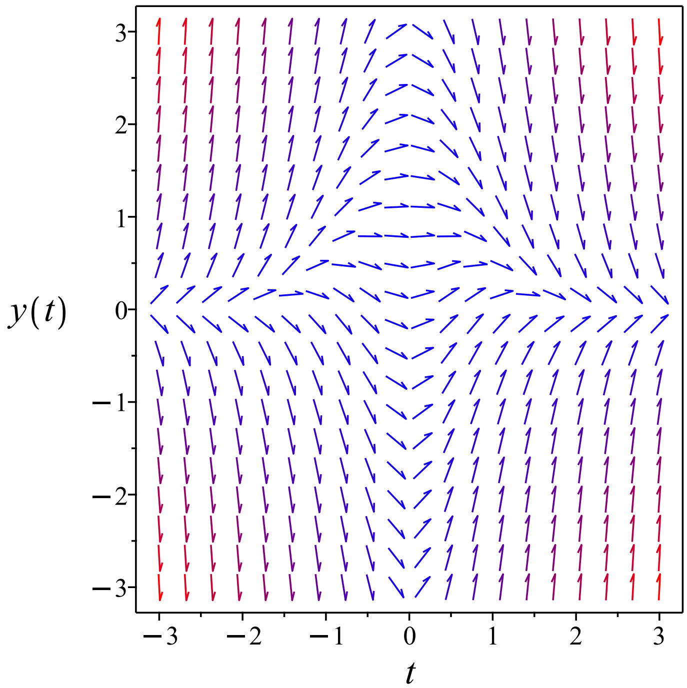


Figure 19: Slope field plot

Verification of solutions

$$y = e^{-t^2}\left(t^2 + c_1\right)$$

Verified OK.

1.7.2 Solving as first order ode lie symmetry lookup ode

Writing the ode as

$$y' = -2t\left(y\, e^{t^2} - 1\right) e^{-t^2}$$
$$y' = \omega(t, y)$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_t + \omega(\eta_y - \xi_t) - \omega^2 \xi_y - \omega_t \xi - \omega_y \eta = 0 \tag{A}$$

The type of this ode is known. It is of type `linear`. Therefore we do not need to solve the PDE (A), and can just use the lookup table shown below to find ξ, η

Table 19: Lie symmetry infinitesimal lookup table for known first order ODE's

ODE class	Form	ξ	η
linear ode	$y' = f(x)y(x) + g(x)$	0	$e^{\int f dx}$
separable ode	$y' = f(x)\, g(y)$	$\frac{1}{f}$	0
quadrature ode	$y' = f(x)$	0	1
quadrature ode	$y' = g(y)$	1	0
homogeneous ODEs of Class A	$y' = f\left(\frac{y}{x}\right)$	x	y
homogeneous ODEs of Class C	$y' = (a + bx + cy)^{\frac{n}{m}}$	1	$-\frac{b}{c}$
homogeneous class D	$y' = \frac{y}{x} + g(x)\, F\left(\frac{y}{x}\right)$	x^2	xy
First order special form ID 1	$y' = g(x)\, e^{h(x)+by} + f(x)$	$\frac{e^{-\int bf(x)dx - h(x)}}{g(x)}$	$\frac{f(x)e^{-\int bf(x)dx - h(x)}}{g(x)}$
polynomial type ode	$y' = \frac{a_1x+b_1y+c_1}{a_2x+b_2y+c_2}$	$\frac{a_1b_2x - a_2b_1x - b_1c_2 + b_2c_1}{a_1b_2 - a_2b_1}$	$\frac{a_1b_2y - a_2b_1y - a_1c_2 - a_2c_1}{a_1b_2 - a_2b_1}$
Bernoulli ode	$y' = f(x)\, y + g(x)\, y^n$	0	$e^{-\int (n-1)f(x)dx} y^n$
Reduced Riccati	$y' = f_1(x)\, y + f_2(x)\, y^2$	0	$e^{-\int f_1 dx}$

The above table shows that

$$\begin{aligned} \xi(t, y) &= 0 \\ \eta(t, y) &= \mathrm{e}^{-t^2} \end{aligned} \tag{A1}$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(t, y) \to (R, S)$ where (R, S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$\frac{dt}{\xi} = \frac{dy}{\eta} = dS \tag{1}$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial t} + \eta \frac{\partial}{\partial y}\right) S(t, y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the

canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$R = t$$

S is found from

$$\begin{aligned} S &= \int \frac{1}{\eta} dy \\ &= \int \frac{1}{\mathrm{e}^{-t^2}} dy \end{aligned}$$

Which results in

$$S = y\,\mathrm{e}^{t^2}$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_t + \omega(t,y)S_y}{R_t + \omega(t,y)R_y} \tag{2}$$

Where in the above R_t, R_y, S_t, S_y are all partial derivatives and $\omega(t,y)$ is the right hand side of the original ode given by

$$\omega(t,y) = -2t\left(y\,\mathrm{e}^{t^2} - 1\right)\mathrm{e}^{-t^2}$$

Evaluating all the partial derivatives gives

$$\begin{aligned} R_t &= 1 \\ R_y &= 0 \\ S_t &= 2ty\,\mathrm{e}^{t^2} \\ S_y &= \mathrm{e}^{t^2} \end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = 2t \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for t, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = 2R$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by

integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$S(R) = R^2 + c_1 \tag{4}$$

To complete the solution, we just need to transform (4) back to t, y coordinates. This results in

$$y\,\mathrm{e}^{t^2} = t^2 + c_1$$

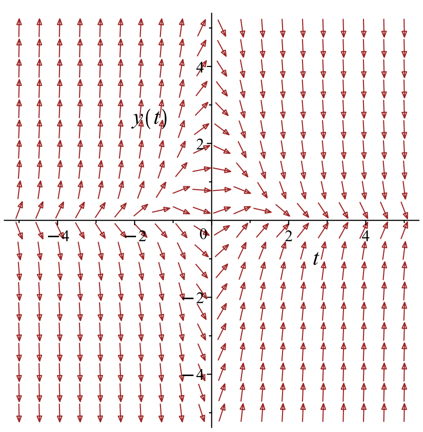
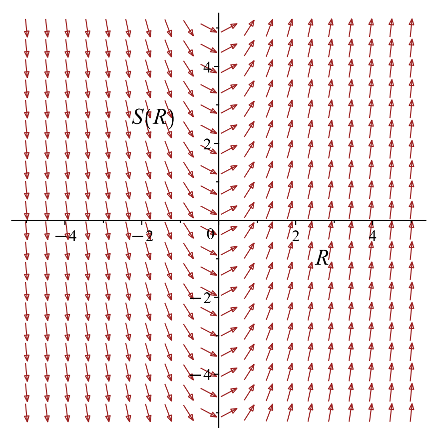
Which simplifies to

$$y\,\mathrm{e}^{t^2} = t^2 + c_1$$

Which gives

$$y = \mathrm{e}^{-t^2}\left(t^2 + c_1\right)$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in t, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dt} = -2t\left(y\,\mathrm{e}^{t^2} - 1\right)\mathrm{e}^{-t^2}$ 	$R = t$ $S = y\,\mathrm{e}^{t^2}$	$\frac{dS}{dR} = 2R$ 

Summary
The solution(s) found are the following

$$y = \mathrm{e}^{-t^2}\left(t^2 + c_1\right) \tag{1}$$

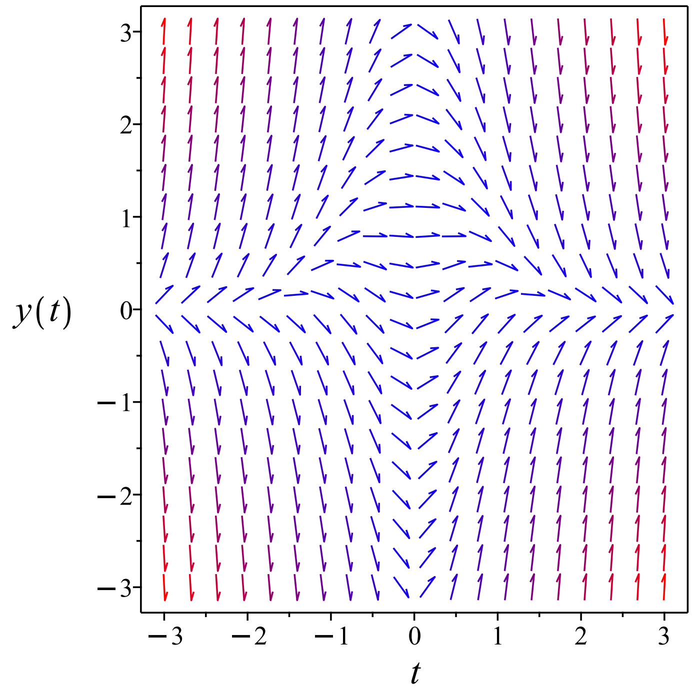


Figure 20: Slope field plot

Verification of solutions

$$y = e^{-t^2}\left(t^2 + c_1\right)$$

Verified OK.

1.7.3 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x,y) + N(x,y)\frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x,y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y) = 0$$

Hence

$$\frac{\partial \phi}{\partial x} + \frac{\partial \phi}{\partial y}\frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$\begin{aligned}\frac{\partial \phi}{\partial x} &= M\\ \frac{\partial \phi}{\partial y} &= N\end{aligned}$$

But since $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x, y)$ but at least we know now that we can do that since the condition $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(t, y)\,\mathrm{d}t + N(t, y)\,\mathrm{d}y = 0 \tag{1A}$$

Therefore

$$\begin{aligned}\left(\mathrm{e}^{t^2}\right)\mathrm{d}y &= \left(-2t\left(y\,\mathrm{e}^{t^2} - 1\right)\right)\mathrm{d}t\\ \left(2t\left(y\,\mathrm{e}^{t^2} - 1\right)\right)\mathrm{d}t + \left(\mathrm{e}^{t^2}\right)\mathrm{d}y &= 0 \qquad (2A)\end{aligned}$$

Comparing (1A) and (2A) shows that

$$\begin{aligned}M(t, y) &= 2t\left(y\,\mathrm{e}^{t^2} - 1\right)\\ N(t, y) &= \mathrm{e}^{t^2}\end{aligned}$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial t}$$

Using result found above gives

$$\begin{aligned}\frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}\left(2t\left(y\,\mathrm{e}^{t^2} - 1\right)\right)\\ &= 2t\,\mathrm{e}^{t^2}\end{aligned}$$

And

$$\begin{aligned}\frac{\partial N}{\partial t} &= \frac{\partial}{\partial t}\left(\mathrm{e}^{t^2}\right)\\ &= 2t\,\mathrm{e}^{t^2}\end{aligned}$$

Since $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial t}$, then the ODE is <u>exact</u> The following equations are now set up to solve for the function $\phi(t, y)$

$$\frac{\partial \phi}{\partial t} = M \tag{1}$$
$$\frac{\partial \phi}{\partial y} = N \tag{2}$$

Integrating (1) w.r.t. t gives

$$\begin{aligned}\int \frac{\partial \phi}{\partial t}\,\mathrm{d}t &= \int M\,\mathrm{d}t\\ \int \frac{\partial \phi}{\partial t}\,\mathrm{d}t &= \int 2t\left(y\,\mathrm{e}^{t^2} - 1\right)\mathrm{d}t\end{aligned}$$
$$\phi = -t^2 + y\,\mathrm{e}^{t^2} + f(y) \tag{3}$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both t and y. Taking derivative of equation (3) w.r.t y gives

$$\frac{\partial \phi}{\partial y} = \mathrm{e}^{t^2} + f'(y) \tag{4}$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = \mathrm{e}^{t^2}$. Therefore equation (4) becomes

$$\mathrm{e}^{t^2} = \mathrm{e}^{t^2} + f'(y) \tag{5}$$

Solving equation (5) for $f'(y)$ gives

$$f'(y) = 0$$

Therefore

$$f(y) = c_1$$

Where c_1 is constant of integration. Substituting this result for $f(y)$ into equation (3) gives ϕ

$$\phi = -t^2 + y\,\mathrm{e}^{t^2} + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = -t^2 + y\,\mathrm{e}^{t^2}$$

The solution becomes

$$y = \mathrm{e}^{-t^2}\left(t^2 + c_1\right)$$

Summary

The solution(s) found are the following

$$y = \mathrm{e}^{-t^2}\left(t^2 + c_1\right) \tag{1}$$

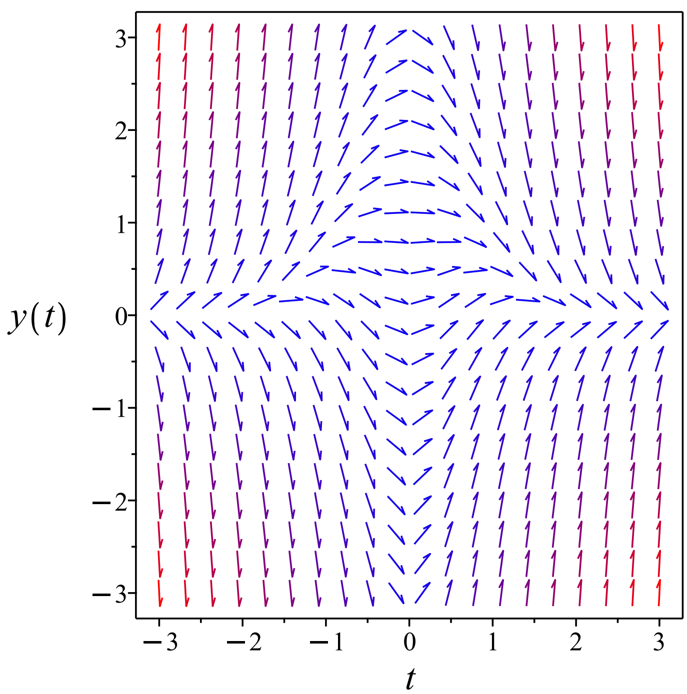


Figure 21: Slope field plot

Verification of solutions

$$y = \mathrm{e}^{-t^2}\left(t^2 + c_1\right)$$

Verified OK.

1.7.4 Maple step by step solution

Let's solve

$2yt + y' = \frac{2t}{e^{t^2}}$

- Highest derivative means the order of the ODE is 1

  y'

- Isolate the derivative

  $y' = -2yt + \frac{2t}{e^{t^2}}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE

  $2yt + y' = \frac{2t}{e^{t^2}}$

- The ODE is linear; multiply by an integrating factor $\mu(t)$

  $\mu(t)\,(2yt + y') = \frac{2\mu(t)t}{e^{t^2}}$

- Assume the lhs of the ODE is the total derivative $\frac{d}{dt}(\mu(t)\,y)$

  $\mu(t)\,(2yt + y') = \mu'(t)\,y + \mu(t)\,y'$

- Isolate $\mu'(t)$

  $\mu'(t) = 2\mu(t)\,t$

- Solve to find the integrating factor

  $\mu(t) = \left(e^{t^2}\right)^2 e^{-t^2}$

- Integrate both sides with respect to t

  $\int \left(\frac{d}{dt}(\mu(t)\,y)\right) dt = \int \frac{2\mu(t)t}{e^{t^2}} dt + c_1$

- Evaluate the integral on the lhs

  $\mu(t)\,y = \int \frac{2\mu(t)t}{e^{t^2}} dt + c_1$

- Solve for y

  $y = \frac{\int \frac{2\mu(t)t}{e^{t^2}} dt + c_1}{\mu(t)}$

- Substitute $\mu(t) = \left(e^{t^2}\right)^2 e^{-t^2}$

  $y = \frac{\int 2t\,e^{-t^2} e^{t^2} dt + c_1}{\left(e^{t^2}\right)^2 e^{-t^2}}$

- Evaluate the integrals on the rhs

$$y = \frac{t^2+c_1}{\left(e^{t^2}\right)^2 e^{-t^2}}$$

- Simplify

$$y = e^{-t^2}(t^2 + c_1)$$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
<- 1st order linear successful`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 16

```
dsolve(2*t*y(t)+diff(y(t),t) = 2*t/exp(t^2),y(t), singsol=all)
```

$$y(t) = \left(t^2 + c_1\right) e^{-t^2}$$

✓ Solution by Mathematica

Time used: 0.055 (sec). Leaf size: 19

```
DSolve[2*t*y[t]+y'[t] == 2*t/Exp[t^2],y[t],t,IncludeSingularSolutions -> True]
```

$$y(t) \to e^{-t^2}\left(t^2 + c_1\right)$$

1.8 problem 8

Internal problem ID [455]
Internal file name [`OUTPUT/455_Sunday_June_05_2022_01_41_45_AM_39090059/index.tex`]

Book: Elementary differential equations and boundary value problems, 10th ed., Boyce and DiPrima
Section: Section 2.1. Page 40
Problem number: 8.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"linear", "exactWithIntegrationFactor", "first_order_ode_lie_symmetry_lookup"**

Maple gives the following as the ode type

`[_linear]`

$$\boxed{4yt + \left(t^2+1\right)y' = \frac{1}{\left(t^2+1\right)^2}}$$

1.8.1 Solving as linear ode

Entering Linear first order ODE solver. In canonical form a linear first order is

$$y' + p(t)y = q(t)$$

Where here

$$p(t) = \frac{4t}{t^2+1}$$
$$q(t) = \frac{1}{\left(t^2+1\right)^3}$$

Hence the ode is

$$y' + \frac{4yt}{t^2+1} = \frac{1}{\left(t^2+1\right)^3}$$

The integrating factor μ is

$$\begin{aligned}\mu &= \mathrm{e}^{\int \frac{4t}{t^2+1}dt} \\ &= \left(t^2+1\right)^2\end{aligned}$$

The ode becomes

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}(\mu y) &= (\mu)\left(\frac{1}{\left(t^2+1\right)^3}\right) \\ \frac{\mathrm{d}}{\mathrm{d}t}\left(\left(t^2+1\right)^2 y\right) &= \left(\left(t^2+1\right)^2\right)\left(\frac{1}{\left(t^2+1\right)^3}\right) \\ \mathrm{d}\left(\left(t^2+1\right)^2 y\right) &= \frac{1}{t^2+1}\,\mathrm{d}t\end{aligned}$$

Integrating gives

$$\begin{aligned}\left(t^2+1\right)^2 y &= \int \frac{1}{t^2+1}\,\mathrm{d}t \\ \left(t^2+1\right)^2 y &= \arctan(t) + c_1\end{aligned}$$

Dividing both sides by the integrating factor $\mu = \left(t^2+1\right)^2$ results in

$$y = \frac{\arctan(t)}{\left(t^2+1\right)^2} + \frac{c_1}{\left(t^2+1\right)^2}$$

which simplifies to

$$y = \frac{\arctan(t) + c_1}{\left(t^2+1\right)^2}$$

Summary

The solution(s) found are the following

$$y = \frac{\arctan(t) + c_1}{\left(t^2+1\right)^2} \tag{1}$$

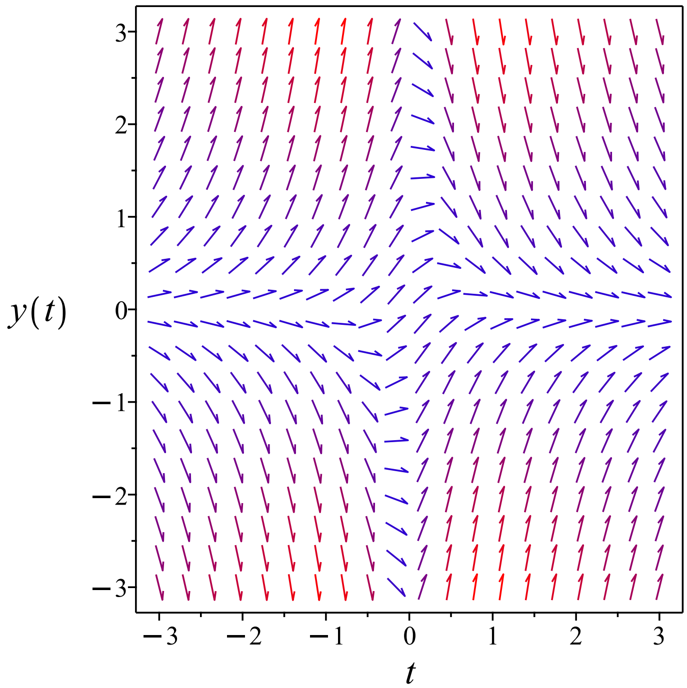


Figure 22: Slope field plot

Verification of solutions

$$y = \frac{\arctan(t) + c_1}{(t^2+1)^2}$$

Verified OK.

1.8.2 Solving as first order ode lie symmetry lookup ode

Writing the ode as

$$y' = -\frac{4t^5 y + 8t^3 y + 4ty - 1}{(t^2+1)^3}$$
$$y' = \omega(t, y)$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_t + \omega(\eta_y - \xi_t) - \omega^2 \xi_y - \omega_t \xi - \omega_y \eta = 0 \tag{A}$$

The type of this ode is known. It is of type `linear`. Therefore we do not need to solve the PDE (A), and can just use the lookup table shown below to find ξ, η

Table 22: Lie symmetry infinitesimal lookup table for known first order ODE's

ODE class	Form	ξ	η
linear ode	$y' = f(x)y(x) + g(x)$	0	$e^{\int f dx}$
separable ode	$y' = f(x)\, g(y)$	$\frac{1}{f}$	0
quadrature ode	$y' = f(x)$	0	1
quadrature ode	$y' = g(y)$	1	0
homogeneous ODEs of Class A	$y' = f\left(\frac{y}{x}\right)$	x	y
homogeneous ODEs of Class C	$y' = (a + bx + cy)^{\frac{n}{m}}$	1	$-\frac{b}{c}$
homogeneous class D	$y' = \frac{y}{x} + g(x)\, F\left(\frac{y}{x}\right)$	x^2	xy
First order special form ID 1	$y' = g(x)\, e^{h(x)+by} + f(x)$	$\frac{e^{-\int bf(x)dx - h(x)}}{g(x)}$	$\frac{f(x)e^{-\int bf(x)dx - h(x)}}{g(x)}$
polynomial type ode	$y' = \frac{a_1 x + b_1 y + c_1}{a_2 x + b_2 y + c_2}$	$\frac{a_1 b_2 x - a_2 b_1 x - b_1 c_2 + b_2 c_1}{a_1 b_2 - a_2 b_1}$	$\frac{a_1 b_2 y - a_2 b_1 y - a_1 c_2 - a_2 c_1}{a_1 b_2 - a_2 b_1}$
Bernoulli ode	$y' = f(x)\, y + g(x)\, y^n$	0	$e^{-\int (n-1) f(x) dx} y^n$
Reduced Riccati	$y' = f_1(x)\, y + f_2(x)\, y^2$	0	$e^{-\int f_1 dx}$

The above table shows that

$$
\begin{aligned}
\xi(t, y) &= 0 \\
\eta(t, y) &= \frac{1}{(t^2 + 1)^2}
\end{aligned} \tag{A1}
$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(t, y) \to (R, S)$ where (R, S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$
\frac{dt}{\xi} = \frac{dy}{\eta} = dS \tag{1}
$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial t} + \eta \frac{\partial}{\partial y}\right) S(t, y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the

canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$R = t$$

S is found from

$$\begin{aligned} S &= \int \frac{1}{\eta} dy \\ &= \int \frac{1}{\frac{1}{(t^2+1)^2}} dy \end{aligned}$$

Which results in

$$S = (t^2 + 1)^2 y$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_t + \omega(t, y) S_y}{R_t + \omega(t, y) R_y} \tag{2}$$

Where in the above R_t, R_y, S_t, S_y are all partial derivatives and $\omega(t, y)$ is the right hand side of the original ode given by

$$\omega(t, y) = -\frac{4t^5 y + 8t^3 y + 4ty - 1}{(t^2 + 1)^3}$$

Evaluating all the partial derivatives gives

$$\begin{aligned} R_t &= 1 \\ R_y &= 0 \\ S_t &= 4(t^2 + 1) yt \\ S_y &= (t^2 + 1)^2 \end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = \frac{1}{t^2 + 1} \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for t, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = \frac{1}{R^2 + 1}$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordinates R, S. Integrating the above gives

$$S(R) = \arctan(R) + c_1 \tag{4}$$

To complete the solution, we just need to transform (4) back to t, y coordinates. This results in

$$\left(t^2+1\right)^2 y = \arctan(t) + c_1$$

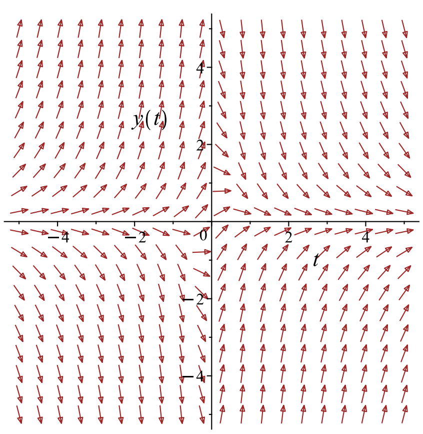
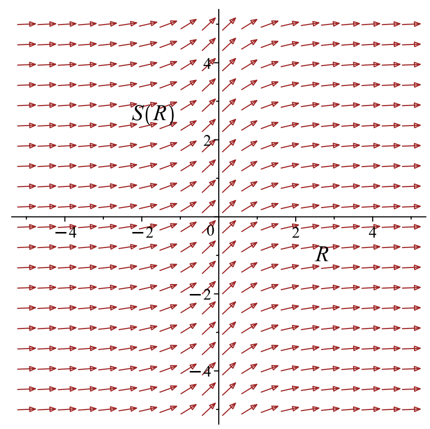
Which simplifies to

$$\left(t^2+1\right)^2 y = \arctan(t) + c_1$$

Which gives

$$y = \frac{\arctan(t) + c_1}{\left(t^2+1\right)^2}$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in t, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dt} = -\frac{4t^5y+8t^3y+4ty-1}{(t^2+1)^3}$ 	$R = t$ $S = \left(t^2+1\right)^2 y$	$\frac{dS}{dR} = \frac{1}{R^2+1}$ 

Summary
The solution(s) found are the following

$$y = \frac{\arctan(t) + c_1}{(t^2 + 1)^2} \tag{1}$$

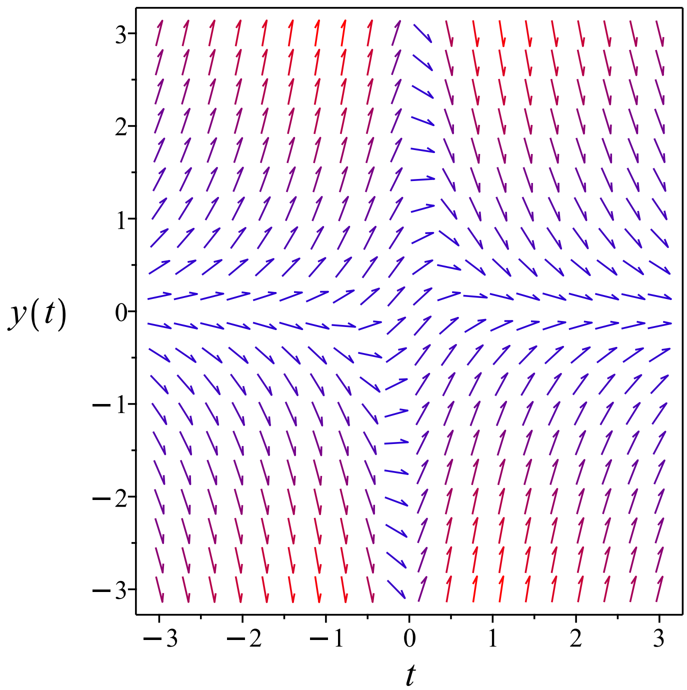


Figure 23: Slope field plot

Verification of solutions

$$y = \frac{\arctan(t) + c_1}{(t^2 + 1)^2}$$

Verified OK.

1.8.3 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x, y) + N(x, y) \frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x, y) = c$ where c is constant, that satisfies the

ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y)=0$$

Hence

$$\frac{\partial \phi}{\partial x}+\frac{\partial \phi}{\partial y}\frac{dy}{dx}=0 \tag{B}$$

Comparing (A,B) shows that

$$\begin{aligned}\frac{\partial \phi}{\partial x}&=M\\ \frac{\partial \phi}{\partial y}&=N\end{aligned}$$

But since $\frac{\partial^2\phi}{\partial x\partial y}=\frac{\partial^2\phi}{\partial y\partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y}=\frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x,y)$ but at least we know now that we can do that since the condition $\frac{\partial^2\phi}{\partial x\partial y}=\frac{\partial^2\phi}{\partial y\partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(t,y)\,\mathrm{d}t+N(t,y)\,\mathrm{d}y=0 \tag{1A}$$

Therefore

$$\begin{aligned}(t^2+1)\,\mathrm{d}y&=\left(-4ty+\frac{1}{(t^2+1)^2}\right)\mathrm{d}t\\ \left(4ty-\frac{1}{(t^2+1)^2}\right)\mathrm{d}t+(t^2+1)\,\mathrm{d}y&=0 \qquad \text{(2A)}\end{aligned}$$

Comparing (1A) and (2A) shows that

$$\begin{aligned}M(t,y)&=4ty-\frac{1}{(t^2+1)^2}\\ N(t,y)&=t^2+1\end{aligned}$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y}=\frac{\partial N}{\partial t}$$

Using result found above gives

$$
\begin{aligned}
\frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}\left(4ty - \frac{1}{\left(t^2+1\right)^2}\right) \\
&= 4t
\end{aligned}
$$

And

$$
\begin{aligned}
\frac{\partial N}{\partial t} &= \frac{\partial}{\partial t}\left(t^2+1\right) \\
&= 2t
\end{aligned}
$$

Since $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial t}$, then the ODE is not exact. Since the ODE is not exact, we will try to find an integrating factor to make it exact. Let

$$
\begin{aligned}
A &= \frac{1}{N}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial t}\right) \\
&= \frac{1}{t^2+1}((4t) - (2t)) \\
&= \frac{2t}{t^2+1}
\end{aligned}
$$

Since A does not depend on y, then it can be used to find an integrating factor. The integrating factor μ is

$$
\begin{aligned}
\mu &= e^{\int A \, \mathrm{d}t} \\
&= e^{\int \frac{2t}{t^2+1} \, \mathrm{d}t}
\end{aligned}
$$

The result of integrating gives

$$
\begin{aligned}
\mu &= e^{\ln(t^2+1)} \\
&= t^2+1
\end{aligned}
$$

M and N are multiplied by this integrating factor, giving new M and new N which are called $\overline{M}$ and $\overline{N}$ for now so not to confuse them with the original M and N.

$$
\begin{aligned}
\overline{M} &= \mu M \\
&= t^2+1\left(4ty - \frac{1}{\left(t^2+1\right)^2}\right) \\
&= \left(4ty - \frac{1}{\left(t^2+1\right)^2}\right)\left(t^2+1\right)
\end{aligned}
$$

And

$$
\begin{aligned}
\overline{N} &= \mu N \\
&= t^2 + 1\left(t^2 + 1\right) \\
&= \left(t^2 + 1\right)^2
\end{aligned}
$$

Now a modified ODE is ontained from the original ODE, which is exact and can be solved. The modified ODE is

$$
\begin{aligned}
\overline{M} + \overline{N}\frac{\mathrm{d}y}{\mathrm{d}t} &= 0 \\
\left(\left(4ty - \frac{1}{\left(t^2+1\right)^2}\right)\left(t^2+1\right)\right) + \left(\left(t^2+1\right)^2\right)\frac{\mathrm{d}y}{\mathrm{d}t} &= 0
\end{aligned}
$$

The following equations are now set up to solve for the function $\phi(t, y)$

$$
\frac{\partial \phi}{\partial t} = \overline{M} \tag{1}
$$

$$
\frac{\partial \phi}{\partial y} = \overline{N} \tag{2}
$$

Integrating (1) w.r.t. t gives

$$
\begin{aligned}
\int \frac{\partial \phi}{\partial t}\,\mathrm{d}t &= \int \overline{M}\,\mathrm{d}t \\
\int \frac{\partial \phi}{\partial t}\,\mathrm{d}t &= \int \left(4ty - \frac{1}{\left(t^2+1\right)^2}\right)\left(t^2+1\right)\mathrm{d}t
\end{aligned}
$$

$$
\phi = t^4 y + 2t^2 y - \arctan(t) + f(y) \tag{3}
$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both t and y. Taking derivative of equation (3) w.r.t y gives

$$
\frac{\partial \phi}{\partial y} = t^4 + 2t^2 + f'(y) \tag{4}
$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = \left(t^2+1\right)^2$. Therefore equation (4) becomes

$$
\left(t^2+1\right)^2 = t^4 + 2t^2 + f'(y) \tag{5}
$$

Solving equation (5) for $f'(y)$ gives

$$
f'(y) = 1
$$

Integrating the above w.r.t y gives

$$\int f'(y)\,\mathrm{d}y = \int (1)\,\mathrm{d}y$$
$$f(y) = y + c_1$$

Where c_1 is constant of integration. Substituting result found above for $f(y)$ into equation (3) gives ϕ

$$\phi = t^4 y + 2t^2 y - \arctan(t) + y + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = t^4 y + 2t^2 y - \arctan(t) + y$$

Summary
The solution(s) found are the following

$$y t^4 + 2y t^2 - \arctan(t) + y = c_1 \tag{1}$$

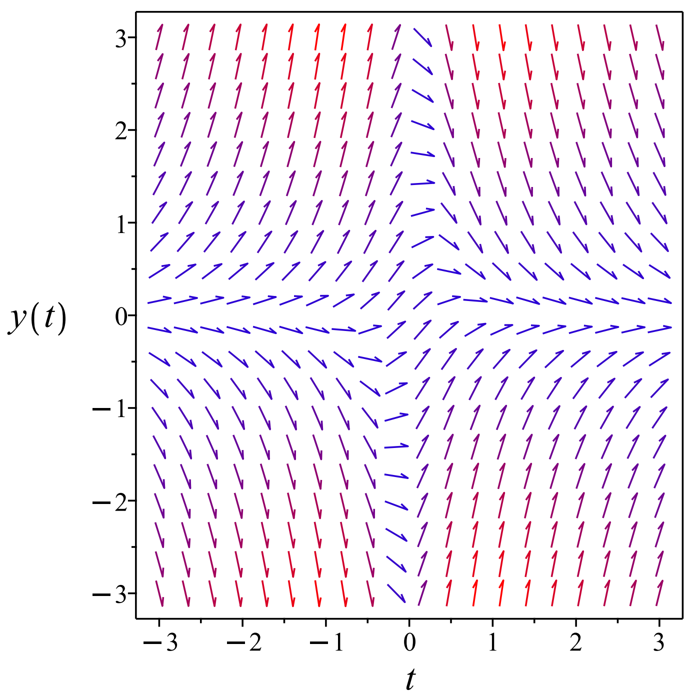


Figure 24: Slope field plot

Verification of solutions

$$yt^4 + 2yt^2 - \arctan(t) + y = c_1$$

Verified OK.

1.8.4 Maple step by step solution

Let's solve

$4yt + (t^2+1)\,y' = \frac{1}{(t^2+1)^2}$

- Highest derivative means the order of the ODE is 1

  y'

- Isolate the derivative

  $y' = -\frac{4yt}{t^2+1} + \frac{1}{(t^2+1)^3}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE

  $y' + \frac{4yt}{t^2+1} = \frac{1}{(t^2+1)^3}$

- The ODE is linear; multiply by an integrating factor $\mu(t)$

  $\mu(t)\left(y' + \frac{4yt}{t^2+1}\right) = \frac{\mu(t)}{(t^2+1)^3}$

- Assume the lhs of the ODE is the total derivative $\frac{d}{dt}(\mu(t)\,y)$

  $\mu(t)\left(y' + \frac{4yt}{t^2+1}\right) = \mu'(t)\,y + \mu(t)\,y'$

- Isolate $\mu'(t)$

  $\mu'(t) = \frac{4\mu(t)t}{t^2+1}$

- Solve to find the integrating factor

  $\mu(t) = (t^2+1)^2$

- Integrate both sides with respect to t

  $\int \left(\frac{d}{dt}(\mu(t)\,y)\right) dt = \int \frac{\mu(t)}{(t^2+1)^3} dt + c_1$

- Evaluate the integral on the lhs

  $\mu(t)\,y = \int \frac{\mu(t)}{(t^2+1)^3} dt + c_1$

- Solve for y

  $y = \frac{\int \frac{\mu(t)}{(t^2+1)^3} dt + c_1}{\mu(t)}$

- Substitute $\mu(t) = (t^2+1)^2$

  $y = \frac{\int \frac{1}{t^2+1}dt + c_1}{(t^2+1)^2}$

- Evaluate the integrals on the rhs

  $y = \frac{\arctan(t)+c_1}{(t^2+1)^2}$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
<- 1st order linear successful`
```

✓ Solution by Maple

Time used: 0.015 (sec). Leaf size: 16

```
dsolve(4*t*y(t)+(t^2+1)*diff(y(t),t) = 1/(t^2+1)^2,y(t), singsol=all)
```

$$y(t) = \frac{\arctan(t) + c_1}{(t^2+1)^2}$$

✓ Solution by Mathematica

Time used: 0.041 (sec). Leaf size: 18

```
DSolve[4*t*y[t]+(t^2+1)*y'[t] == 1/(t^2+1)^2,y[t],t,IncludeSingularSolutions -> True]
```

$$y(t) \to \frac{\arctan(t) + c_1}{(t^2+1)^2}$$

1.9 problem 9

Internal problem ID [456]
Internal file name [`OUTPUT/456_Sunday_June_05_2022_01_41_46_AM_4296471/index.tex`]

Book: Elementary differential equations and boundary value problems, 10th ed., Boyce and DiPrima
Section: Section 2.1. Page 40
Problem number: 9.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"linear", "exactWithIntegrationFactor", "first_order_ode_lie_symmetry_lookup"**

Maple gives the following as the ode type

```
[[_linear, `class A`]]
```

$$\boxed{y + 2y' = 3t}$$

1.9.1 Solving as linear ode

Entering Linear first order ODE solver. In canonical form a linear first order is

$$y' + p(t)y = q(t)$$

Where here

$$p(t) = \frac{1}{2}$$
$$q(t) = \frac{3t}{2}$$

Hence the ode is

$$y' + \frac{y}{2} = \frac{3t}{2}$$

The integrating factor μ is

$$
\begin{aligned}
\mu &= e^{\int \frac{1}{2}dt} \\
&= e^{\frac{t}{2}}
\end{aligned}
$$

The ode becomes

$$
\begin{aligned}
\frac{d}{dt}(\mu y) &= (\mu)\left(\frac{3t}{2}\right) \\
\frac{d}{dt}\left(e^{\frac{t}{2}} y\right) &= \left(e^{\frac{t}{2}}\right)\left(\frac{3t}{2}\right) \\
d\left(e^{\frac{t}{2}} y\right) &= \left(\frac{3t\, e^{\frac{t}{2}}}{2}\right) dt
\end{aligned}
$$

Integrating gives

$$
\begin{aligned}
e^{\frac{t}{2}} y &= \int \frac{3t\, e^{\frac{t}{2}}}{2}\, dt \\
e^{\frac{t}{2}} y &= 3(t-2)\, e^{\frac{t}{2}} + c_1
\end{aligned}
$$

Dividing both sides by the integrating factor $\mu = e^{\frac{t}{2}}$ results in

$$
y = 3\, e^{-\frac{t}{2}}(t-2)\, e^{\frac{t}{2}} + c_1 e^{-\frac{t}{2}}
$$

which simplifies to

$$
y = 3t - 6 + c_1 e^{-\frac{t}{2}}
$$

Summary

The solution(s) found are the following

$$
y = 3t - 6 + c_1 e^{-\frac{t}{2}} \tag{1}
$$

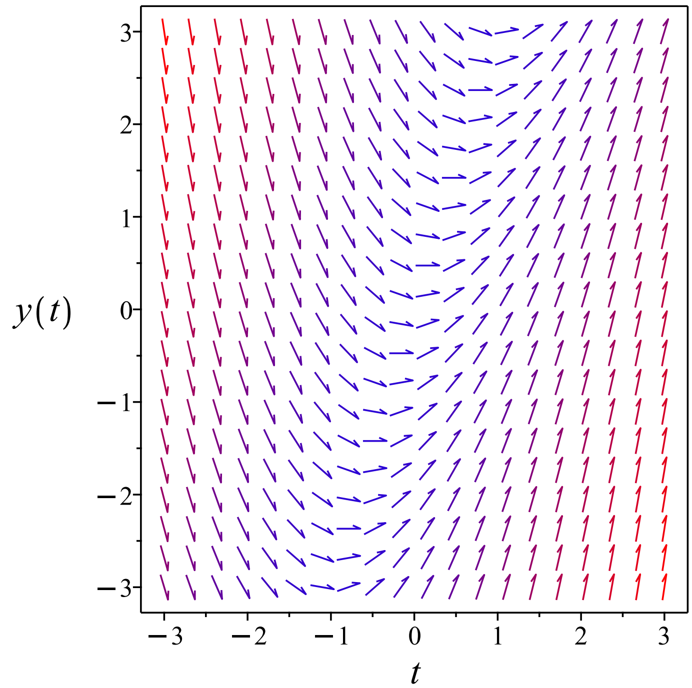


Figure 25: Slope field plot

Verification of solutions

$$y = 3t - 6 + c_1 e^{-\frac{t}{2}}$$

Verified OK.

1.9.2 Solving as first order ode lie symmetry lookup ode

Writing the ode as

$$y' = -\frac{y}{2} + \frac{3t}{2}$$
$$y' = \omega(t, y)$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_t + \omega(\eta_y - \xi_t) - \omega^2 \xi_y - \omega_t \xi - \omega_y \eta = 0 \tag{A}$$

The type of this ode is known. It is of type `linear`. Therefore we do not need to solve the PDE (A), and can just use the lookup table shown below to find ξ, η

Table 25: Lie symmetry infinitesimal lookup table for known first order ODE's

ODE class	Form	ξ	η
linear ode	$y' = f(x)y(x) + g(x)$	0	$e^{\int f dx}$
separable ode	$y' = f(x)\, g(y)$	$\frac{1}{f}$	0
quadrature ode	$y' = f(x)$	0	1
quadrature ode	$y' = g(y)$	1	0
homogeneous ODEs of Class A	$y' = f\left(\frac{y}{x}\right)$	x	y
homogeneous ODEs of Class C	$y' = (a + bx + cy)^{\frac{n}{m}}$	1	$-\frac{b}{c}$
homogeneous class D	$y' = \frac{y}{x} + g(x)\, F\left(\frac{y}{x}\right)$	x^2	xy
First order special form ID 1	$y' = g(x)\, e^{h(x)+by} + f(x)$	$\frac{e^{-\int bf(x)dx - h(x)}}{g(x)}$	$\frac{f(x)e^{-\int bf(x)dx - h(x)}}{g(x)}$
polynomial type ode	$y' = \frac{a_1x+b_1y+c_1}{a_2x+b_2y+c_2}$	$\frac{a_1b_2x-a_2b_1x-b_1c_2+b_2c_1}{a_1b_2-a_2b_1}$	$\frac{a_1b_2y-a_2b_1y-a_1c_2-a_2c_1}{a_1b_2-a_2b_1}$
Bernoulli ode	$y' = f(x)\, y + g(x)\, y^n$	0	$e^{-\int (n-1)f(x)dx} y^n$
Reduced Riccati	$y' = f_1(x)\, y + f_2(x)\, y^2$	0	$e^{-\int f_1 dx}$

The above table shows that

$$
\begin{aligned}
\xi(t, y) &= 0 \\
\eta(t, y) &= \mathrm{e}^{-\frac{t}{2}}
\end{aligned}
\tag{A1}
$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(t, y) \to (R, S)$ where (R, S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$
\frac{dt}{\xi} = \frac{dy}{\eta} = dS \tag{1}
$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial t} + \eta \frac{\partial}{\partial y}\right) S(t, y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the

canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$R = t$$

S is found from

$$\begin{aligned} S &= \int \frac{1}{\eta} dy \\ &= \int \frac{1}{e^{-\frac{t}{2}}} dy \end{aligned}$$

Which results in

$$S = e^{\frac{t}{2}} y$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_t + \omega(t, y) S_y}{R_t + \omega(t, y) R_y} \tag{2}$$

Where in the above R_t, R_y, S_t, S_y are all partial derivatives and $\omega(t, y)$ is the right hand side of the original ode given by

$$\omega(t, y) = -\frac{y}{2} + \frac{3t}{2}$$

Evaluating all the partial derivatives gives

$$\begin{aligned} R_t &= 1 \\ R_y &= 0 \\ S_t &= \frac{e^{\frac{t}{2}} y}{2} \\ S_y &= e^{\frac{t}{2}} \end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = \frac{3t\, e^{\frac{t}{2}}}{2} \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for t, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = \frac{3R\, e^{\frac{R}{2}}}{2}$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordinates R, S. Integrating the above gives

$$S(R) = 3(R-2)\,\mathrm{e}^{\frac{R}{2}} + c_1 \tag{4}$$

To complete the solution, we just need to transform (4) back to t, y coordinates. This results in

$$\mathrm{e}^{\frac{t}{2}} y = 3(t-2)\,\mathrm{e}^{\frac{t}{2}} + c_1$$

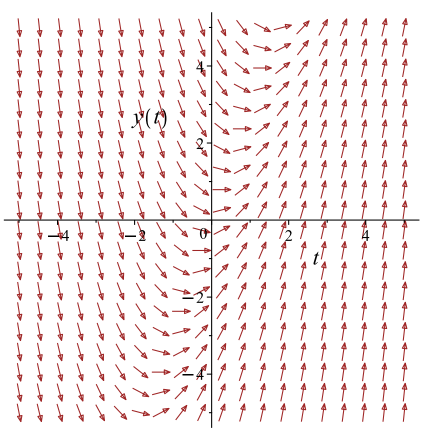
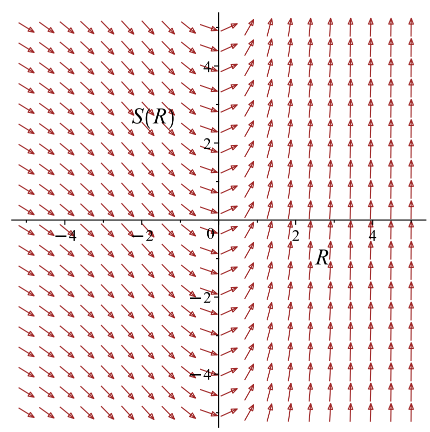
Which simplifies to

$$(-3t + y + 6)\,\mathrm{e}^{\frac{t}{2}} - c_1 = 0$$

Which gives

$$y = \left(3t\,\mathrm{e}^{\frac{t}{2}} - 6\,\mathrm{e}^{\frac{t}{2}} + c_1\right)\mathrm{e}^{-\frac{t}{2}}$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in t, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dt} = -\frac{y}{2} + \frac{3t}{2}$ 	$R = t$ $S = \mathrm{e}^{\frac{t}{2}} y$	$\frac{dS}{dR} = \frac{3R\,\mathrm{e}^{\frac{R}{2}}}{2}$ 

Summary

The solution(s) found are the following

$$y = \left(3t\,\mathrm{e}^{\frac{t}{2}} - 6\,\mathrm{e}^{\frac{t}{2}} + c_1\right)\mathrm{e}^{-\frac{t}{2}} \tag{1}$$

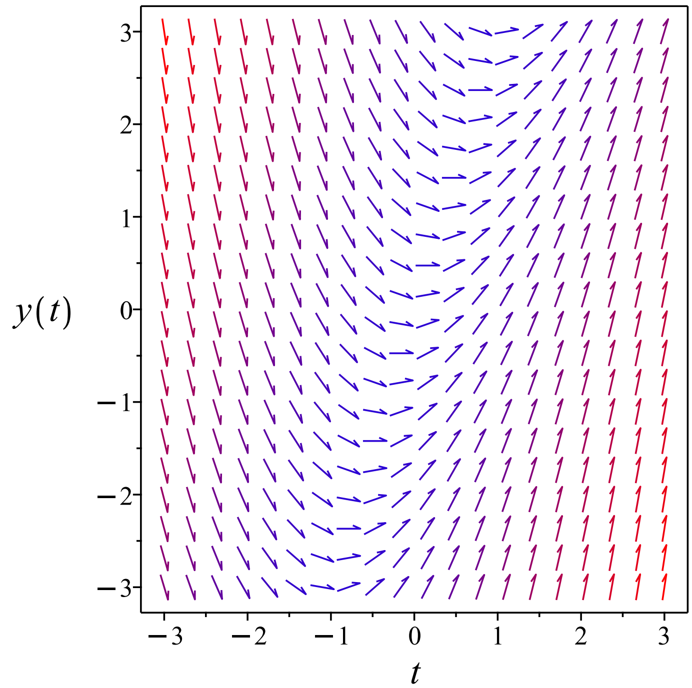


Figure 26: Slope field plot

<u>Verification of solutions</u>

$$y = \left(3t\,e^{\frac{t}{2}} - 6\,e^{\frac{t}{2}} + c_1\right) e^{-\frac{t}{2}}$$

Verified OK.

1.9.3 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x, y) + N(x, y)\frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x, y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x, y) = 0$$

Hence

$$\frac{\partial \phi}{\partial x} + \frac{\partial \phi}{\partial y}\frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$
\begin{aligned}
\frac{\partial \phi}{\partial x} &= M \\
\frac{\partial \phi}{\partial y} &= N
\end{aligned}
$$

But since $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ then for the above to be valid, we require that

$$
\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}
$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x, y)$ but at least we know now that we can do that since the condition $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$
M(t, y) \, \mathrm{d}t + N(t, y) \, \mathrm{d}y = 0 \tag{1A}
$$

Therefore

$$
\begin{aligned}
(2) \, \mathrm{d}y &= (-y + 3t) \, \mathrm{d}t \\
(y - 3t) \, \mathrm{d}t + (2) \, \mathrm{d}y &= 0 \qquad \text{(2A)}
\end{aligned}
$$

Comparing (1A) and (2A) shows that

$$
\begin{aligned}
M(t, y) &= y - 3t \\
N(t, y) &= 2
\end{aligned}
$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$
\frac{\partial M}{\partial y} = \frac{\partial N}{\partial t}
$$

Using result found above gives

$$
\begin{aligned}
\frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}(y - 3t) \\
&= 1
\end{aligned}
$$

And

$$
\begin{aligned}
\frac{\partial N}{\partial t} &= \frac{\partial}{\partial t}(2) \\
&= 0
\end{aligned}
$$

Since $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial t}$, then the ODE is not exact. Since the ODE is not exact, we will try to find an integrating factor to make it exact. Let

$$\begin{aligned} A &= \frac{1}{N}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial t}\right) \\ &= \frac{1}{2}((1) - (0)) \\ &= \frac{1}{2} \end{aligned}$$

Since A does not depend on y, then it can be used to find an integrating factor. The integrating factor μ is

$$\begin{aligned} \mu &= e^{\int A \, \mathrm{d}t} \\ &= e^{\int \frac{1}{2} \, \mathrm{d}t} \end{aligned}$$

The result of integrating gives

$$\begin{aligned} \mu &= e^{\frac{t}{2}} \\ &= \mathrm{e}^{\frac{t}{2}} \end{aligned}$$

M and N are multiplied by this integrating factor, giving new M and new N which are called $\overline{M}$ and $\overline{N}$ for now so not to confuse them with the original M and N.

$$\begin{aligned} \overline{M} &= \mu M \\ &= \mathrm{e}^{\frac{t}{2}}(y - 3t) \\ &= (y - 3t)\,\mathrm{e}^{\frac{t}{2}} \end{aligned}$$

And

$$\begin{aligned} \overline{N} &= \mu N \\ &= \mathrm{e}^{\frac{t}{2}}(2) \\ &= 2\,\mathrm{e}^{\frac{t}{2}} \end{aligned}$$

Now a modified ODE is ontained from the original ODE, which is exact and can be solved. The modified ODE is

$$\begin{aligned} \overline{M} + \overline{N}\frac{\mathrm{d}y}{\mathrm{d}t} &= 0 \\ \left((y - 3t)\,\mathrm{e}^{\frac{t}{2}}\right) + \left(2\,\mathrm{e}^{\frac{t}{2}}\right)\frac{\mathrm{d}y}{\mathrm{d}t} &= 0 \end{aligned}$$

The following equations are now set up to solve for the function $\phi(t,y)$

$$\frac{\partial\phi}{\partial t} = \overline{M} \tag{1}$$

$$\frac{\partial\phi}{\partial y} = \overline{N} \tag{2}$$

Integrating (1) w.r.t. t gives

$$\begin{aligned}
\int \frac{\partial\phi}{\partial t}\,\mathrm{d}t &= \int \overline{M}\,\mathrm{d}t \\
\int \frac{\partial\phi}{\partial t}\,\mathrm{d}t &= \int (y-3t)\,\mathrm{e}^{\frac{t}{2}}\,\mathrm{d}t
\end{aligned}$$

$$\phi = (-6t + 2y + 12)\,\mathrm{e}^{\frac{t}{2}} + f(y) \tag{3}$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both t and y. Taking derivative of equation (3) w.r.t y gives

$$\frac{\partial\phi}{\partial y} = 2\,\mathrm{e}^{\frac{t}{2}} + f'(y) \tag{4}$$

But equation (2) says that $\frac{\partial\phi}{\partial y} = 2\,\mathrm{e}^{\frac{t}{2}}$. Therefore equation (4) becomes

$$2\,\mathrm{e}^{\frac{t}{2}} = 2\,\mathrm{e}^{\frac{t}{2}} + f'(y) \tag{5}$$

Solving equation (5) for $f'(y)$ gives

$$f'(y) = 0$$

Therefore

$$f(y) = c_1$$

Where c_1 is constant of integration. Substituting this result for $f(y)$ into equation (3) gives ϕ

$$\phi = (-6t + 2y + 12)\,\mathrm{e}^{\frac{t}{2}} + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = (-6t + 2y + 12)\,\mathrm{e}^{\frac{t}{2}}$$

The solution becomes

$$y = \frac{\left(6t\,e^{\frac{t}{2}} - 12\,e^{\frac{t}{2}} + c_1\right) e^{-\frac{t}{2}}}{2}$$

Summary

The solution(s) found are the following

$$y = \frac{\left(6t\,e^{\frac{t}{2}} - 12\,e^{\frac{t}{2}} + c_1\right) e^{-\frac{t}{2}}}{2} \tag{1}$$

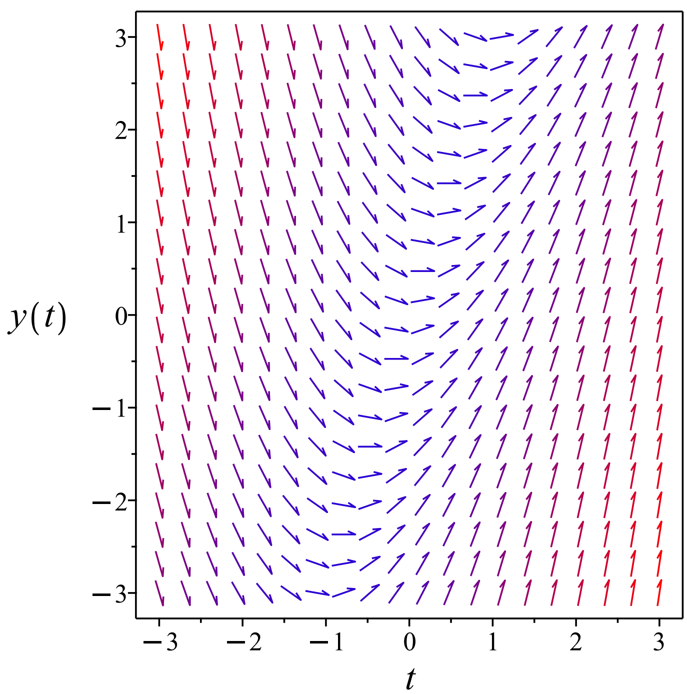


Figure 27: Slope field plot

Verification of solutions

$$y = \frac{\left(6t\,e^{\frac{t}{2}} - 12\,e^{\frac{t}{2}} + c_1\right) e^{-\frac{t}{2}}}{2}$$

Verified OK.

1.9.4 Maple step by step solution

Let's solve

$y + 2y' = 3t$

- Highest derivative means the order of the ODE is 1

  y'

- Isolate the derivative

  $y' = -\frac{y}{2} + \frac{3t}{2}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE

  $y' + \frac{y}{2} = \frac{3t}{2}$

- The ODE is linear; multiply by an integrating factor $\mu(t)$

  $\mu(t)\left(y' + \frac{y}{2}\right) = \frac{3\mu(t)t}{2}$

- Assume the lhs of the ODE is the total derivative $\frac{d}{dt}(\mu(t)\,y)$

  $\mu(t)\left(y' + \frac{y}{2}\right) = \mu'(t)\,y + \mu(t)\,y'$

- Isolate $\mu'(t)$

  $\mu'(t) = \frac{\mu(t)}{2}$

- Solve to find the integrating factor

  $\mu(t) = \mathrm{e}^{\frac{t}{2}}$

- Integrate both sides with respect to t

  $\int \left(\frac{d}{dt}(\mu(t)\,y)\right) dt = \int \frac{3\mu(t)t}{2} dt + c_1$

- Evaluate the integral on the lhs

  $\mu(t)\,y = \int \frac{3\mu(t)t}{2} dt + c_1$

- Solve for y

  $y = \frac{\int \frac{3\mu(t)t}{2} dt + c_1}{\mu(t)}$

- Substitute $\mu(t) = \mathrm{e}^{\frac{t}{2}}$

  $y = \frac{\int \frac{3t\,\mathrm{e}^{\frac{t}{2}}}{2} dt + c_1}{\mathrm{e}^{\frac{t}{2}}}$

- Evaluate the integrals on the rhs

  $y = \frac{3(t-2)\mathrm{e}^{\frac{t}{2}} + c_1}{\mathrm{e}^{\frac{t}{2}}}$

- Simplify

$$y = 3t - 6 + c_1 e^{-\frac{t}{2}}$$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
<- 1st order linear successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 15

```
dsolve(y(t)+2*diff(y(t),t) = 3*t,y(t), singsol=all)
```

$$y(t) = 3t - 6 + e^{-\frac{t}{2}} c_1$$

✓ Solution by Mathematica

Time used: 0.045 (sec). Leaf size: 20

```
DSolve[y[t]+2*y'[t] == 3*t,y[t],t,IncludeSingularSolutions -> True]
```

$$y(t) \to 3t + c_1 e^{-t/2} - 6$$

1.10 problem 10

Internal problem ID [457]
Internal file name [`OUTPUT/457_Sunday_June_05_2022_01_41_47_AM_99710545/index.tex`]

Book: Elementary differential equations and boundary value problems, 10th ed., Boyce and DiPrima
Section: Section 2.1. Page 40
Problem number: 10.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"linear", "homogeneousTypeD2", "exactWithIntegrationFactor", "first_order_ode_lie_symmetry_lookup"**

Maple gives the following as the ode type

`[_linear]`

$$\boxed{ty' - y = t^2 e^{-t}}$$

1.10.1 Solving as linear ode

Entering Linear first order ODE solver. In canonical form a linear first order is

$$y' + p(t)y = q(t)$$

Where here

$$p(t) = -\frac{1}{t}$$
$$q(t) = t\,e^{-t}$$

Hence the ode is

$$y' - \frac{y}{t} = t\,e^{-t}$$

The integrating factor μ is

$$\begin{aligned}\mu &= e^{\int -\frac{1}{t}dt} \\ &= \frac{1}{t}\end{aligned}$$

The ode becomes

$$\begin{aligned}\frac{d}{dt}(\mu y) &= (\mu)\left(t\,e^{-t}\right) \\ \frac{d}{dt}\left(\frac{y}{t}\right) &= \left(\frac{1}{t}\right)\left(t\,e^{-t}\right) \\ d\left(\frac{y}{t}\right) &= e^{-t}\,dt\end{aligned}$$

Integrating gives

$$\begin{aligned}\frac{y}{t} &= \int e^{-t}\,dt \\ \frac{y}{t} &= -e^{-t} + c_1\end{aligned}$$

Dividing both sides by the integrating factor $\mu = \frac{1}{t}$ results in

$$y = -t\,e^{-t} + c_1 t$$

which simplifies to

$$y = t\left(-e^{-t} + c_1\right)$$

Summary

The solution(s) found are the following

$$y = t\left(-e^{-t} + c_1\right) \tag{1}$$

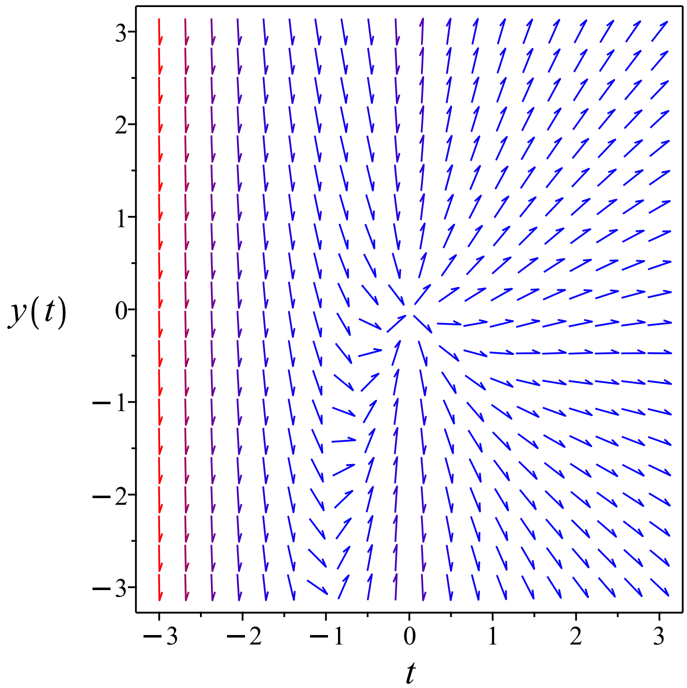


Figure 28: Slope field plot

Verification of solutions

$$y = t\left(-e^{-t} + c_1\right)$$

Verified OK.

1.10.2 Solving as homogeneousTypeD2 ode

Using the change of variables $y = u(t)\, t$ on the above ode results in new ode in $u(t)$

$$t(u'(t)\, t + u(t)) - u(t)\, t = t^2 e^{-t}$$

Integrating both sides gives

$$\begin{aligned} u(t) &= \int e^{-t}\, dt \\ &= -e^{-t} + c_2 \end{aligned}$$

Therefore the solution y is

$$\begin{aligned} y &= ut \\ &= t\left(-e^{-t} + c_2\right) \end{aligned}$$

Summary
The solution(s) found are the following

$$y = t\left(-\mathrm{e}^{-t} + c_2\right) \tag{1}$$

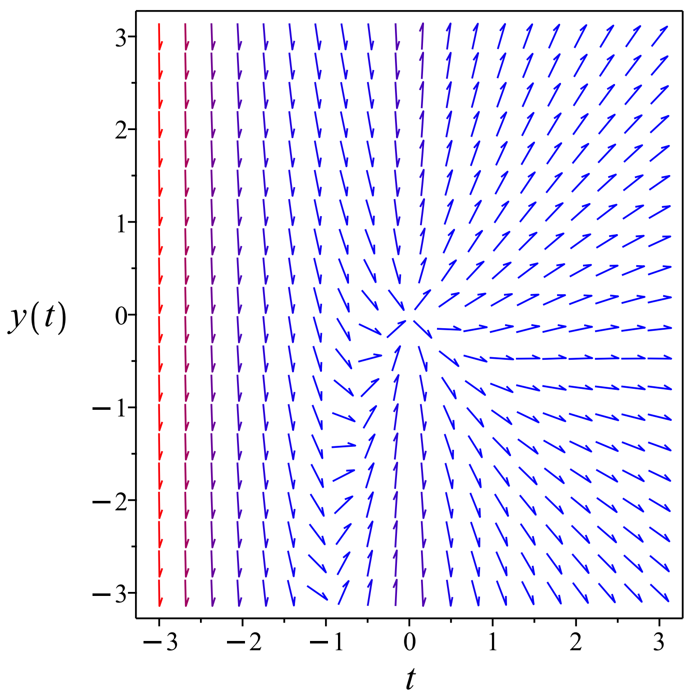


Figure 29: Slope field plot

Verification of solutions

$$y = t\left(-\mathrm{e}^{-t} + c_2\right)$$

Verified OK.

1.10.3 Solving as first order ode lie symmetry lookup ode

Writing the ode as

$$y' = \frac{(y\,\mathrm{e}^{t} + t^2)\,\mathrm{e}^{-t}}{t}$$
$$y' = \omega(t, y)$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_t + \omega(\eta_y - \xi_t) - \omega^2 \xi_y - \omega_t \xi - \omega_y \eta = 0 \tag{A}$$

The type of this ode is known. It is of type `linear`. Therefore we do not need to solve the PDE (A), and can just use the lookup table shown below to find ξ, η

Table 28: Lie symmetry infinitesimal lookup table for known first order ODE's

ODE class	Form	ξ	η
linear ode	$y' = f(x)y(x) + g(x)$	0	$e^{\int f dx}$
separable ode	$y' = f(x)\, g(y)$	$\frac{1}{f}$	0
quadrature ode	$y' = f(x)$	0	1
quadrature ode	$y' = g(y)$	1	0
homogeneous ODEs of Class A	$y' = f\left(\frac{y}{x}\right)$	x	y
homogeneous ODEs of Class C	$y' = (a + bx + cy)^{\frac{n}{m}}$	1	$-\frac{b}{c}$
homogeneous class D	$y' = \frac{y}{x} + g(x)\, F\left(\frac{y}{x}\right)$	x^2	xy
First order special form ID 1	$y' = g(x)\, e^{h(x)+by} + f(x)$	$\frac{e^{-\int b f(x) dx - h(x)}}{g(x)}$	$\frac{f(x) e^{-\int b f(x) dx - h(x)}}{g(x)}$
polynomial type ode	$y' = \frac{a_1 x + b_1 y + c_1}{a_2 x + b_2 y + c_2}$	$\frac{a_1 b_2 x - a_2 b_1 x - b_1 c_2 + b_2 c_1}{a_1 b_2 - a_2 b_1}$	$\frac{a_1 b_2 y - a_2 b_1 y - a_1 c_2 - a_2 c_1}{a_1 b_2 - a_2 b_1}$
Bernoulli ode	$y' = f(x)\, y + g(x)\, y^n$	0	$e^{-\int (n-1) f(x) dx} y^n$
Reduced Riccati	$y' = f_1(x)\, y + f_2(x)\, y^2$	0	$e^{-\int f_1 dx}$

The above table shows that

$$
\begin{aligned}
\xi(t, y) &= 0 \\
\eta(t, y) &= t
\end{aligned} \tag{A1}
$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(t, y) \to (R, S)$ where (R, S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$
\frac{dt}{\xi} = \frac{dy}{\eta} = dS \tag{1}
$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial t} + \eta \frac{\partial}{\partial y}\right) S(t, y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$R = t$$

S is found from

$$\begin{aligned} S &= \int \frac{1}{\eta} dy \\ &= \int \frac{1}{t} dy \end{aligned}$$

Which results in

$$S = \frac{y}{t}$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_t + \omega(t, y) S_y}{R_t + \omega(t, y) R_y} \tag{2}$$

Where in the above R_t, R_y, S_t, S_y are all partial derivatives and $\omega(t, y)$ is the right hand side of the original ode given by

$$\omega(t, y) = \frac{(y\,\mathrm{e}^t + t^2)\,\mathrm{e}^{-t}}{t}$$

Evaluating all the partial derivatives gives

$$\begin{aligned} R_t &= 1 \\ R_y &= 0 \\ S_t &= -\frac{y}{t^2} \\ S_y &= \frac{1}{t} \end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = \mathrm{e}^{-t} \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for t, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = \mathrm{e}^{-R}$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordinates R, S. Integrating the above gives

$$S(R) = -\mathrm{e}^{-R} + c_1 \tag{4}$$

To complete the solution, we just need to transform (4) back to t, y coordinates. This results in

$$\frac{y}{t} = -\mathrm{e}^{-t} + c_1$$

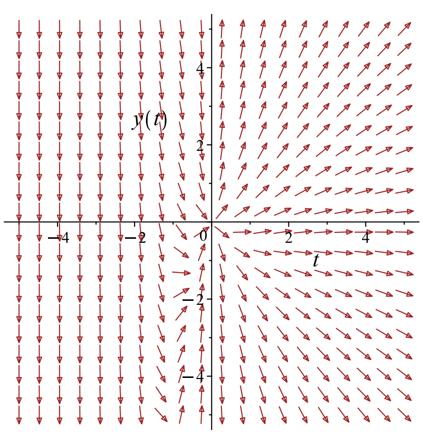
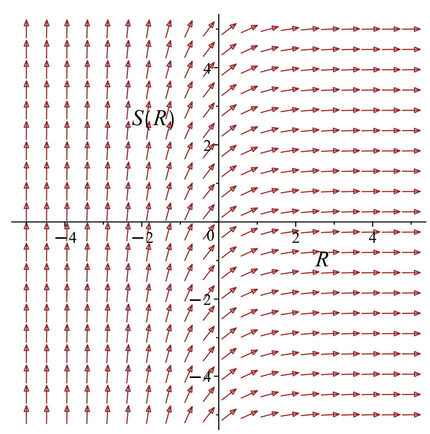
Which simplifies to

$$\frac{y}{t} = -\mathrm{e}^{-t} + c_1$$

Which gives

$$y = -t(\mathrm{e}^{-t} - c_1)$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in t, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dt} = \frac{(y\,\mathrm{e}^{t}+t^2)\mathrm{e}^{-t}}{t}$ 	$R = t$ $S = \frac{y}{t}$	$\frac{dS}{dR} = \mathrm{e}^{-R}$ 

Summary
The solution(s) found are the following

$$y = -t(\mathrm{e}^{-t} - c_1) \tag{1}$$

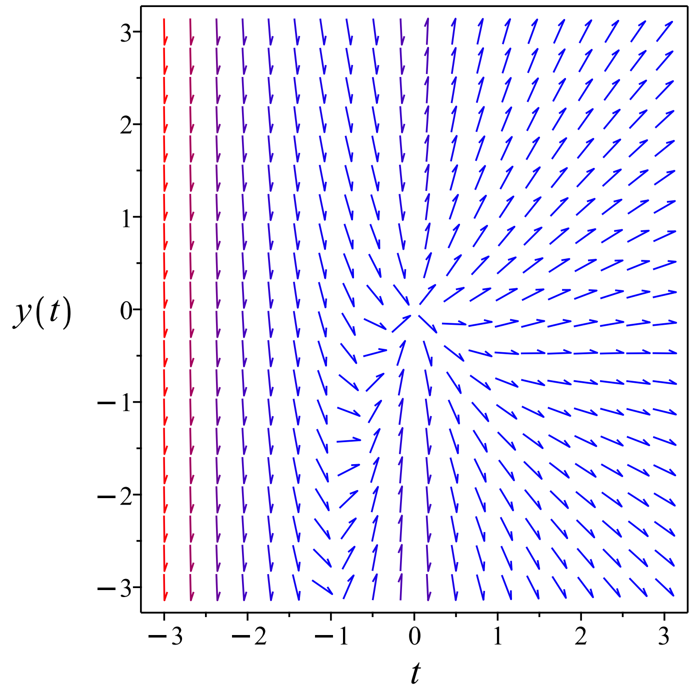


Figure 30: Slope field plot

Verification of solutions

$$
y = -t\left(e^{-t} - c_1\right)
$$

Verified OK.

1.10.4 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$
M(x, y) + N(x, y)\frac{dy}{dx} = 0 \tag{A}
$$

We assume there exists a function $\phi(x, y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$
\frac{d}{dx}\phi(x, y) = 0
$$

Hence

$$
\frac{\partial \phi}{\partial x} + \frac{\partial \phi}{\partial y}\frac{dy}{dx} = 0 \tag{B}
$$

Comparing (A,B) shows that

$$\begin{aligned}\frac{\partial \phi}{\partial x} &= M\\ \frac{\partial \phi}{\partial y} &= N\end{aligned}$$

But since $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x, y)$ but at least we know now that we can do that since the condition $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(t, y)\,\mathrm{d}t + N(t, y)\,\mathrm{d}y = 0 \tag{1A}$$

Therefore

$$\begin{aligned}(t)\,\mathrm{d}y &= \left(t^2 \mathrm{e}^{-t} + y\right) \mathrm{d}t\\ \left(-t^2 \mathrm{e}^{-t} - y\right) \mathrm{d}t + (t)\,\mathrm{d}y &= 0 \qquad (2A)\end{aligned}$$

Comparing (1A) and (2A) shows that

$$\begin{aligned}M(t, y) &= -t^2 \mathrm{e}^{-t} - y\\ N(t, y) &= t\end{aligned}$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial t}$$

Using result found above gives

$$\begin{aligned}\frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}\left(-t^2 \mathrm{e}^{-t} - y\right)\\ &= -1\end{aligned}$$

And

$$\begin{aligned}\frac{\partial N}{\partial t} &= \frac{\partial}{\partial t}(t)\\ &= 1\end{aligned}$$

Since $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial t}$, then the ODE is not exact. Since the ODE is not exact, we will try to find an integrating factor to make it exact. Let

$$
\begin{aligned}
A &= \frac{1}{N}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial t}\right) \\
&= \frac{1}{t}((-1) - (1)) \\
&= -\frac{2}{t}
\end{aligned}
$$

Since A does not depend on y, then it can be used to find an integrating factor. The integrating factor μ is

$$
\begin{aligned}
\mu &= e^{\int A \, \mathrm{d}t} \\
&= e^{\int -\frac{2}{t} \, \mathrm{d}t}
\end{aligned}
$$

The result of integrating gives

$$
\begin{aligned}
\mu &= e^{-2\ln(t)} \\
&= \frac{1}{t^2}
\end{aligned}
$$

M and N are multiplied by this integrating factor, giving new M and new N which are called $\overline{M}$ and $\overline{N}$ for now so not to confuse them with the original M and N.

$$
\begin{aligned}
\overline{M} &= \mu M \\
&= \frac{1}{t^2}\left(-t^2 \mathrm{e}^{-t} - y\right) \\
&= \frac{-t^2 \mathrm{e}^{-t} - y}{t^2}
\end{aligned}
$$

And

$$
\begin{aligned}
\overline{N} &= \mu N \\
&= \frac{1}{t^2}(t) \\
&= \frac{1}{t}
\end{aligned}
$$

Now a modified ODE is ontained from the original ODE, which is exact and can be solved. The modified ODE is

$$
\begin{aligned}
\overline{M} + \overline{N}\frac{\mathrm{d}y}{\mathrm{d}t} &= 0 \\
\left(\frac{-t^2 \mathrm{e}^{-t} - y}{t^2}\right) + \left(\frac{1}{t}\right)\frac{\mathrm{d}y}{\mathrm{d}t} &= 0
\end{aligned}
$$

The following equations are now set up to solve for the function $\phi(t,y)$

$$\frac{\partial\phi}{\partial t} = \overline{M} \tag{1}$$

$$\frac{\partial\phi}{\partial y} = \overline{N} \tag{2}$$

Integrating (1) w.r.t. t gives

$$\int \frac{\partial\phi}{\partial t}\, \mathrm{d}t = \int \overline{M}\, \mathrm{d}t$$

$$\int \frac{\partial\phi}{\partial t}\, \mathrm{d}t = \int \frac{-t^2 \mathrm{e}^{-t} - y}{t^2}\, \mathrm{d}t$$

$$\phi = \frac{t\,\mathrm{e}^{-t} + y}{t} + f(y) \tag{3}$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both t and y. Taking derivative of equation (3) w.r.t y gives

$$\frac{\partial\phi}{\partial y} = \frac{1}{t} + f'(y) \tag{4}$$

But equation (2) says that $\frac{\partial\phi}{\partial y} = \frac{1}{t}$. Therefore equation (4) becomes

$$\frac{1}{t} = \frac{1}{t} + f'(y) \tag{5}$$

Solving equation (5) for $f'(y)$ gives

$$f'(y) = 0$$

Therefore

$$f(y) = c_1$$

Where c_1 is constant of integration. Substituting this result for $f(y)$ into equation (3) gives ϕ

$$\phi = \frac{t\,\mathrm{e}^{-t} + y}{t} + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = \frac{t\,\mathrm{e}^{-t} + y}{t}$$

The solution becomes

$$y = -t\left(\mathrm{e}^{-t} - c_1\right)$$

Summary

The solution(s) found are the following

$$y = -t\left(\mathrm{e}^{-t} - c_1\right) \tag{1}$$

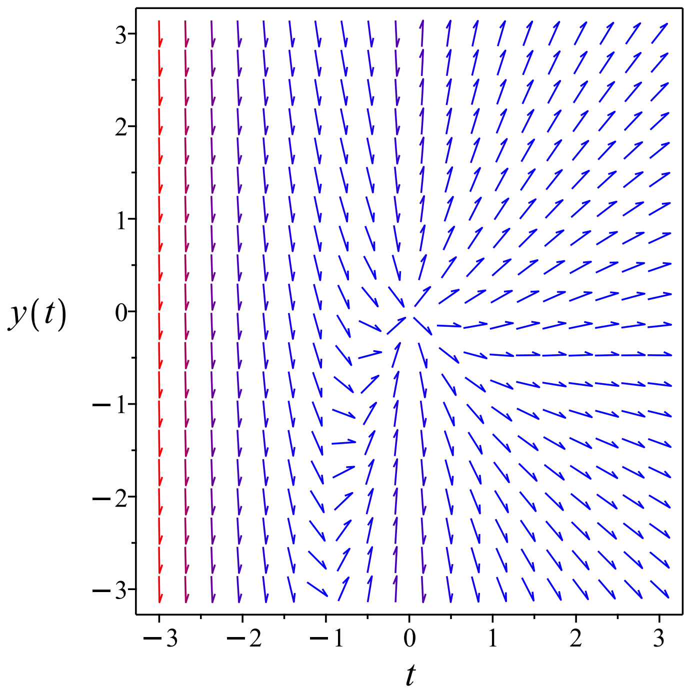


Figure 31: Slope field plot

Verification of solutions

$$y = -t\left(\mathrm{e}^{-t} - c_1\right)$$

Verified OK.

1.10.5 Maple step by step solution

Let's solve

$ty' - y = \frac{t^2}{e^t}$

- Highest derivative means the order of the ODE is 1

  y'

- Isolate the derivative

  $y' = \frac{y}{t} + \frac{t}{e^t}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE

  $y' - \frac{y}{t} = \frac{t}{e^t}$

- The ODE is linear; multiply by an integrating factor $\mu(t)$

  $\mu(t)\left(y' - \frac{y}{t}\right) = \frac{\mu(t)t}{e^t}$

- Assume the lhs of the ODE is the total derivative $\frac{d}{dt}(\mu(t)\,y)$

  $\mu(t)\left(y' - \frac{y}{t}\right) = \mu'(t)\,y + \mu(t)\,y'$

- Isolate $\mu'(t)$

  $\mu'(t) = -\frac{\mu(t)}{t}$

- Solve to find the integrating factor

  $\mu(t) = \frac{e^{-t}e^t}{t}$

- Integrate both sides with respect to t

  $\int \left(\frac{d}{dt}(\mu(t)\,y)\right) dt = \int \frac{\mu(t)t}{e^t} dt + c_1$

- Evaluate the integral on the lhs

  $\mu(t)\,y = \int \frac{\mu(t)t}{e^t} dt + c_1$

- Solve for y

  $y = \frac{\int \frac{\mu(t)t}{e^t} dt + c_1}{\mu(t)}$

- Substitute $\mu(t) = \frac{e^{-t}e^t}{t}$

  $y = \frac{t\left(\int e^{-t} dt + c_1\right)}{e^{-t}e^t}$

- Evaluate the integrals on the rhs

  $y = \frac{t\left(-e^{-t} + c_1\right)}{e^{-t}e^t}$

- Simplify

$$y = t(-e^{-t} + c_1)$$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
<- 1st order linear successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 14

```
dsolve(-y(t)+t*diff(y(t),t) = t^2/exp(t),y(t), singsol=all)
```

$$y(t) = \left(-e^{-t} + c_1\right) t$$

✓ Solution by Mathematica

Time used: 0.053 (sec). Leaf size: 17

```
DSolve[-y[t]+t*y'[t] == t^2/Exp[t],y[t],t,IncludeSingularSolutions -> True]
```

$$y(t) \to t\left(-e^{-t} + c_1\right)$$

1.11 problem 11

Internal problem ID [458]
Internal file name [`OUTPUT/458_Sunday_June_05_2022_01_41_48_AM_82720145/index.tex`]

Book: Elementary differential equations and boundary value problems, 10th ed., Boyce and DiPrima
Section: Section 2.1. Page 40
Problem number: 11.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"linear", "exactWithIntegrationFactor", "first_order_ode_lie_symmetry_lookup"**

Maple gives the following as the ode type

```
[[_linear, `class A`]]
```

$$\boxed{y + y' = 5\sin(2t)}$$

1.11.1 Solving as linear ode

Entering Linear first order ODE solver. In canonical form a linear first order is

$$y' + p(t)y = q(t)$$

Where here

$$p(t) = 1$$
$$q(t) = 5\sin(2t)$$

Hence the ode is

$$y + y' = 5\sin(2t)$$

The integrating factor μ is

$$\begin{aligned}\mu &= \mathrm{e}^{\int 1dt} \\ &= \mathrm{e}^{t}\end{aligned}$$

The ode becomes

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}(\mu y) &= (\mu)\,(5\sin(2t)) \\ \frac{\mathrm{d}}{\mathrm{d}t}\left(y\,\mathrm{e}^{t}\right) &= \left(\mathrm{e}^{t}\right)(5\sin(2t)) \\ \mathrm{d}\left(y\,\mathrm{e}^{t}\right) &= \left(5\,\mathrm{e}^{t}\sin(2t)\right)\,\mathrm{d}t\end{aligned}$$

Integrating gives

$$\begin{aligned}y\,\mathrm{e}^{t} &= \int 5\,\mathrm{e}^{t}\sin(2t)\,\mathrm{d}t \\ y\,\mathrm{e}^{t} &= -2\,\mathrm{e}^{t}\cos(2t) + \mathrm{e}^{t}\sin(2t) + c_1\end{aligned}$$

Dividing both sides by the integrating factor $\mu = \mathrm{e}^{t}$ results in

$$y = \mathrm{e}^{-t}\left(-2\,\mathrm{e}^{t}\cos(2t) + \mathrm{e}^{t}\sin(2t)\right) + c_1\mathrm{e}^{-t}$$

which simplifies to

$$y = \sin(2t) - 2\cos(2t) + c_1\mathrm{e}^{-t}$$

Summary

The solution(s) found are the following

$$y = \sin(2t) - 2\cos(2t) + c_1\mathrm{e}^{-t} \tag{1}$$

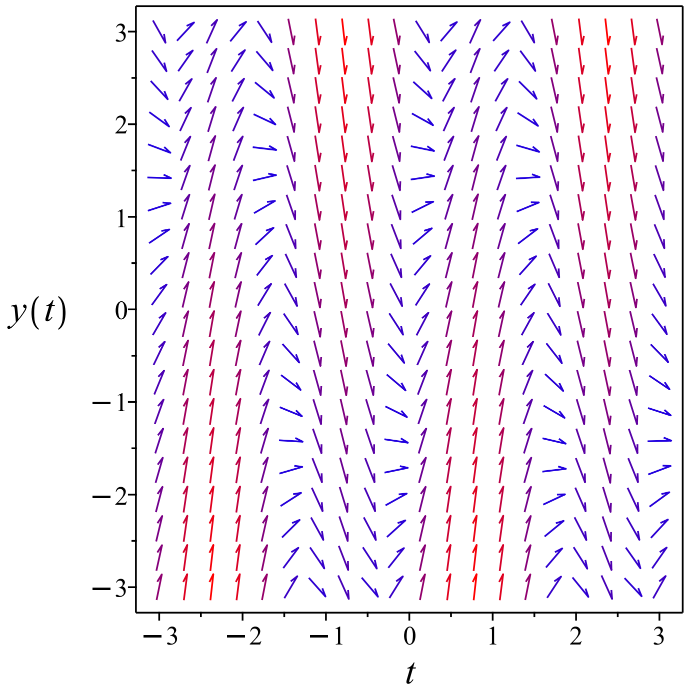


Figure 32: Slope field plot

Verification of solutions

$$y = \sin(2t) - 2\cos(2t) + c_1 e^{-t}$$

Verified OK.

1.11.2 Solving as first order ode lie symmetry lookup ode

Writing the ode as

$$y' = -y + 5\sin(2t)$$
$$y' = \omega(t, y)$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_t + \omega(\eta_y - \xi_t) - \omega^2 \xi_y - \omega_t \xi - \omega_y \eta = 0 \tag{A}$$

The type of this ode is known. It is of type `linear`. Therefore we do not need to solve the PDE (A), and can just use the lookup table shown below to find ξ, η

Table 31: Lie symmetry infinitesimal lookup table for known first order ODE's

ODE class	Form	ξ	η
linear ode	$y' = f(x)y(x) + g(x)$	0	$e^{\int f dx}$
separable ode	$y' = f(x)\, g(y)$	$\frac{1}{f}$	0
quadrature ode	$y' = f(x)$	0	1
quadrature ode	$y' = g(y)$	1	0
homogeneous ODEs of Class A	$y' = f\left(\frac{y}{x}\right)$	x	y
homogeneous ODEs of Class C	$y' = (a + bx + cy)^{\frac{n}{m}}$	1	$-\frac{b}{c}$
homogeneous class D	$y' = \frac{y}{x} + g(x)\, F\left(\frac{y}{x}\right)$	x^2	xy
First order special form ID 1	$y' = g(x)\, e^{h(x)+by} + f(x)$	$\frac{e^{-\int bf(x)dx - h(x)}}{g(x)}$	$\frac{f(x)e^{-\int bf(x)dx - h(x)}}{g(x)}$
polynomial type ode	$y' = \frac{a_1x+b_1y+c_1}{a_2x+b_2y+c_2}$	$\frac{a_1b_2x - a_2b_1x - b_1c_2 + b_2c_1}{a_1b_2 - a_2b_1}$	$\frac{a_1b_2y - a_2b_1y - a_1c_2 - a_2c_1}{a_1b_2 - a_2b_1}$
Bernoulli ode	$y' = f(x)\, y + g(x)\, y^n$	0	$e^{-\int (n-1)f(x)dx} y^n$
Reduced Riccati	$y' = f_1(x)\, y + f_2(x)\, y^2$	0	$e^{-\int f_1 dx}$

The above table shows that

$$\begin{aligned} \xi(t, y) &= 0 \\ \eta(t, y) &= e^{-t} \end{aligned} \tag{A1}$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(t, y) \to (R, S)$ where (R, S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$\frac{dt}{\xi} = \frac{dy}{\eta} = dS \tag{1}$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial t} + \eta \frac{\partial}{\partial y}\right) S(t, y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the

canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$R = t$$

S is found from

$$\begin{aligned} S &= \int \frac{1}{\eta} dy \\ &= \int \frac{1}{e^{-t}} dy \end{aligned}$$

Which results in

$$S = y\,e^{t}$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_t + \omega(t, y) S_y}{R_t + \omega(t, y) R_y} \tag{2}$$

Where in the above R_t, R_y, S_t, S_y are all partial derivatives and $\omega(t, y)$ is the right hand side of the original ode given by

$$\omega(t, y) = -y + 5\sin(2t)$$

Evaluating all the partial derivatives gives

$$\begin{aligned} R_t &= 1 \\ R_y &= 0 \\ S_t &= y\,e^{t} \\ S_y &= e^{t} \end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = 5\,e^{t}\sin(2t) \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for t, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = 5\,e^{R}\sin(2R)$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by

integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$S(R) = c_1 - e^R(2\cos(2R) - \sin(2R)) \tag{4}$$

To complete the solution, we just need to transform (4) back to t, y coordinates. This results in

$$y\,e^t = c_1 - e^t(2\cos(2t) - \sin(2t))$$

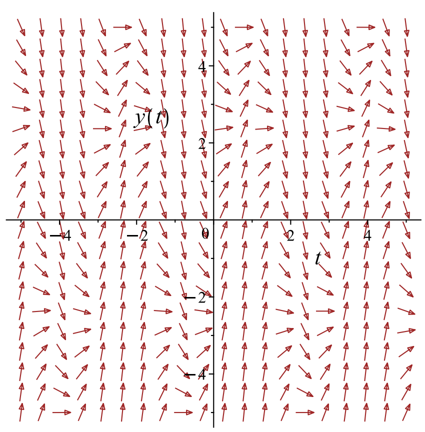
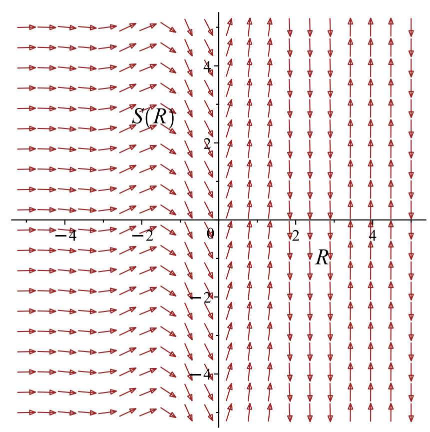
Which simplifies to

$$y\,e^t = c_1 - e^t(2\cos(2t) - \sin(2t))$$

Which gives

$$y = e^{-t}\left(e^t \sin(2t) - 2\,e^t \cos(2t) + c_1\right)$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

<table>
<tr><th>Original ode in t, y coordinates</th><th>Canonical coordinates transformation</th><th>ODE in canonical coordinates (R, S)</th></tr>
<tr><td>$\frac{dy}{dt} = -y + 5\sin(2t)$

</td><td>$R = t$
$S = y\,e^t$</td><td>$\frac{dS}{dR} = 5\,e^R \sin(2R)$

</td></tr>
</table>

Summary
The solution(s) found are the following

$$y = e^{-t}\left(e^t \sin(2t) - 2\,e^t \cos(2t) + c_1\right) \tag{1}$$

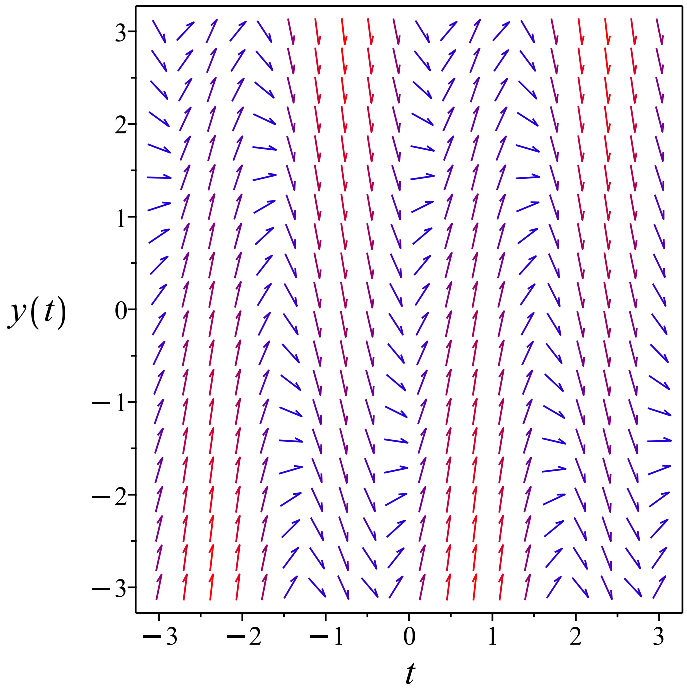


Figure 33: Slope field plot

Verification of solutions

$$y = e^{-t}\left(e^t \sin(2t) - 2\,e^t \cos(2t) + c_1\right)$$

Verified OK.

1.11.3 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x,y) + N(x,y)\frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x,y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y) = 0$$

Hence

$$\frac{\partial \phi}{\partial x} + \frac{\partial \phi}{\partial y}\frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$\begin{aligned}\frac{\partial \phi}{\partial x} &= M \\ \frac{\partial \phi}{\partial y} &= N\end{aligned}$$

But since $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x, y)$ but at least we know now that we can do that since the condition $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(t, y)\,\mathrm{d}t + N(t, y)\,\mathrm{d}y = 0 \tag{1A}$$

Therefore

$$\begin{aligned}\mathrm{d}y &= (-y + 5\sin(2t))\,\mathrm{d}t \\ (y - 5\sin(2t))\,\mathrm{d}t + \mathrm{d}y &= 0 \qquad (2A)\end{aligned}$$

Comparing (1A) and (2A) shows that

$$\begin{aligned}M(t, y) &= y - 5\sin(2t) \\ N(t, y) &= 1\end{aligned}$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial t}$$

Using result found above gives

$$\begin{aligned}\frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}(y - 5\sin(2t)) \\ &= 1\end{aligned}$$

And

$$\begin{aligned}\frac{\partial N}{\partial t} &= \frac{\partial}{\partial t}(1) \\ &= 0\end{aligned}$$

Since $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial t}$, then the ODE is <u>not exact</u>. Since the ODE is not exact, we will try to find an integrating factor to make it exact. Let

$$\begin{aligned} A &= \frac{1}{N}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial t}\right) \\ &= 1((1) - (0)) \\ &= 1 \end{aligned}$$

Since A does not depend on y, then it can be used to find an integrating factor. The integrating factor μ is

$$\begin{aligned} \mu &= e^{\int A \, \mathrm{d}t} \\ &= e^{\int 1 \, \mathrm{d}t} \end{aligned}$$

The result of integrating gives

$$\begin{aligned} \mu &= e^{t} \\ &= \mathrm{e}^{t} \end{aligned}$$

M and N are multiplied by this integrating factor, giving new M and new N which are called $\overline{M}$ and $\overline{N}$ for now so not to confuse them with the original M and N.

$$\begin{aligned} \overline{M} &= \mu M \\ &= \mathrm{e}^{t}(y - 5\sin(2t)) \\ &= (y - 5\sin(2t))\,\mathrm{e}^{t} \end{aligned}$$

And

$$\begin{aligned} \overline{N} &= \mu N \\ &= \mathrm{e}^{t}(1) \\ &= \mathrm{e}^{t} \end{aligned}$$

Now a modified ODE is ontained from the original ODE, which is exact and can be solved. The modified ODE is

$$\begin{aligned} \overline{M} + \overline{N}\frac{\mathrm{d}y}{\mathrm{d}t} &= 0 \\ \left((y - 5\sin(2t))\,\mathrm{e}^{t}\right) + \left(\mathrm{e}^{t}\right)\frac{\mathrm{d}y}{\mathrm{d}t} &= 0 \end{aligned}$$

The following equations are now set up to solve for the function $\phi(t, y)$

$$\frac{\partial \phi}{\partial t} = \overline{M} \tag{1}$$

$$\frac{\partial \phi}{\partial y} = \overline{N} \tag{2}$$

Integrating (1) w.r.t. t gives

$$\int \frac{\partial \phi}{\partial t} \, \mathrm{d}t = \int \overline{M} \, \mathrm{d}t$$

$$\int \frac{\partial \phi}{\partial t} \, \mathrm{d}t = \int (y - 5\sin(2t)) \, \mathrm{e}^{t} \, \mathrm{d}t$$

$$\phi = \mathrm{e}^{t}(y + 2\cos(2t) - \sin(2t)) + f(y) \tag{3}$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both t and y. Taking derivative of equation (3) w.r.t y gives

$$\frac{\partial \phi}{\partial y} = \mathrm{e}^{t} + f'(y) \tag{4}$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = \mathrm{e}^{t}$. Therefore equation (4) becomes

$$\mathrm{e}^{t} = \mathrm{e}^{t} + f'(y) \tag{5}$$

Solving equation (5) for $f'(y)$ gives

$$f'(y) = 0$$

Therefore

$$f(y) = c_1$$

Where c_1 is constant of integration. Substituting this result for $f(y)$ into equation (3) gives ϕ

$$\phi = \mathrm{e}^{t}(y + 2\cos(2t) - \sin(2t)) + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = \mathrm{e}^{t}(y + 2\cos(2t) - \sin(2t))$$

The solution becomes

$$y = \mathrm{e}^{-t}\left(\mathrm{e}^{t}\sin(2t) - 2\,\mathrm{e}^{t}\cos(2t) + c_1\right)$$

Summary
The solution(s) found are the following

$$y = e^{-t}\left(e^{t} \sin(2t) - 2 e^{t} \cos(2t) + c_1\right) \tag{1}$$

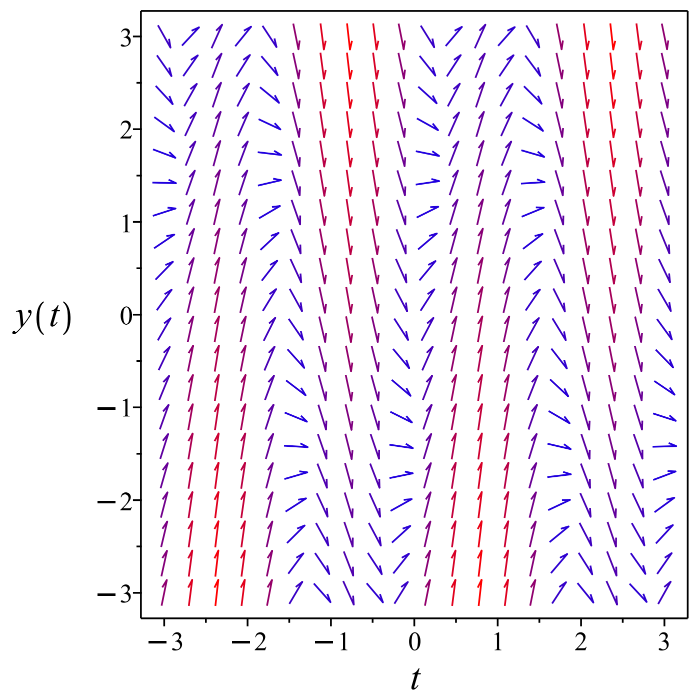


Figure 34: Slope field plot

Verification of solutions

$$y = e^{-t}\left(e^{t} \sin(2t) - 2 e^{t} \cos(2t) + c_1\right)$$

Verified OK.

1.11.4 Maple step by step solution

Let's solve

$y + y' = 5\sin(2t)$

- Highest derivative means the order of the ODE is 1

  y'

- Isolate the derivative

  $y' = -y + 5\sin(2t)$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE

  $y + y' = 5 \sin(2t)$
- The ODE is linear; multiply by an integrating factor $\mu(t)$

  $\mu(t)(y + y') = 5\mu(t) \sin(2t)$
- Assume the lhs of the ODE is the total derivative $\frac{d}{dt}(\mu(t) y)$

  $\mu(t)(y + y') = \mu'(t) y + \mu(t) y'$
- Isolate $\mu'(t)$

  $\mu'(t) = \mu(t)$
- Solve to find the integrating factor

  $\mu(t) = \mathrm{e}^{t}$
- Integrate both sides with respect to t

  $\int \left(\frac{d}{dt}(\mu(t) y)\right) dt = \int 5\mu(t) \sin(2t) dt + c_1$
- Evaluate the integral on the lhs

  $\mu(t) y = \int 5\mu(t) \sin(2t) dt + c_1$
- Solve for y

  $y = \frac{\int 5\mu(t) \sin(2t)dt + c_1}{\mu(t)}$
- Substitute $\mu(t) = \mathrm{e}^{t}$

  $y = \frac{\int 5\,\mathrm{e}^{t} \sin(2t)dt + c_1}{\mathrm{e}^{t}}$
- Evaluate the integrals on the rhs

  $y = \frac{\mathrm{e}^{t} \sin(2t) - 2\,\mathrm{e}^{t} \cos(2t) + c_1}{\mathrm{e}^{t}}$
- Simplify

  $y = \sin(2t) - 2\cos(2t) + c_1 \mathrm{e}^{-t}$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
<- 1st order linear successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 21

```
dsolve(y(t)+diff(y(t),t) = 5*sin(2*t),y(t), singsol=all)
```

$$y(t) = \sin(2t) - 2\cos(2t) + e^{-t} c_1$$

✓ Solution by Mathematica

Time used: 0.087 (sec). Leaf size: 24

```
DSolve[y[t]+y'[t] == 5*Sin[2*t],y[t],t,IncludeSingularSolutions -> True]
```

$$y(t) \to \sin(2t) - 2\cos(2t) + c_1 e^{-t}$$

1.12 problem 12

Internal problem ID [459]
Internal file name [`OUTPUT/459_Sunday_June_05_2022_01_41_49_AM_20042804/index.tex`]

Book: Elementary differential equations and boundary value problems, 10th ed., Boyce and DiPrima
Section: Section 2.1. Page 40
Problem number: 12.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"linear", "exactWithIntegrationFactor", "first_order_ode_lie_symmetry_lookup"**

Maple gives the following as the ode type

```
[[_linear, `class A`]]
```

$$\boxed{y + 2y' = 3t^2}$$

1.12.1 Solving as linear ode

Entering Linear first order ODE solver. In canonical form a linear first order is

$$y' + p(t)y = q(t)$$

Where here

$$p(t) = \frac{1}{2}$$
$$q(t) = \frac{3t^2}{2}$$

Hence the ode is

$$y' + \frac{y}{2} = \frac{3t^2}{2}$$

The integrating factor μ is

$$\mu = e^{\int \frac{1}{2}dt}$$
$$= e^{\frac{t}{2}}$$

The ode becomes

$$\frac{d}{dt}(\mu y) = (\mu)\left(\frac{3t^2}{2}\right)$$
$$\frac{d}{dt}\left(e^{\frac{t}{2}}y\right) = \left(e^{\frac{t}{2}}\right)\left(\frac{3t^2}{2}\right)$$
$$d\left(e^{\frac{t}{2}}y\right) = \left(\frac{3t^2e^{\frac{t}{2}}}{2}\right) dt$$

Integrating gives

$$e^{\frac{t}{2}}y = \int \frac{3t^2e^{\frac{t}{2}}}{2} dt$$
$$e^{\frac{t}{2}}y = 3(t^2 - 4t + 8) e^{\frac{t}{2}} + c_1$$

Dividing both sides by the integrating factor $\mu = e^{\frac{t}{2}}$ results in

$$y = 3 e^{-\frac{t}{2}}(t^2 - 4t + 8) e^{\frac{t}{2}} + c_1e^{-\frac{t}{2}}$$

which simplifies to

$$y = 3t^2 - 12t + 24 + c_1e^{-\frac{t}{2}}$$

Summary

The solution(s) found are the following

$$y = 3t^2 - 12t + 24 + c_1e^{-\frac{t}{2}} \tag{1}$$

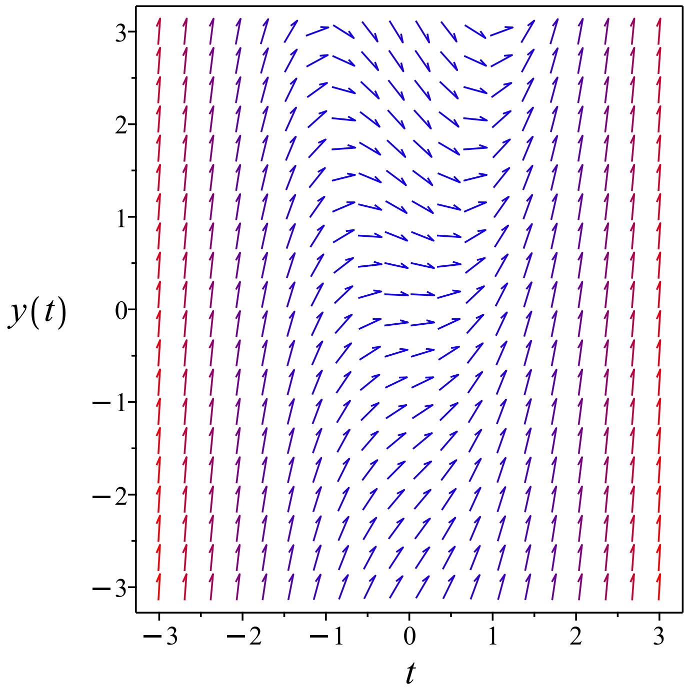


Figure 35: Slope field plot

Verification of solutions

$$y = 3t^2 - 12t + 24 + c_1 e^{-\frac{t}{2}}$$

Verified OK.

1.12.2 Solving as first order ode lie symmetry lookup ode

Writing the ode as

$$y' = -\frac{y}{2} + \frac{3t^2}{2}$$
$$y' = \omega(t, y)$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_t + \omega(\eta_y - \xi_t) - \omega^2 \xi_y - \omega_t \xi - \omega_y \eta = 0 \tag{A}$$

The type of this ode is known. It is of type `linear`. Therefore we do not need to solve the PDE (A), and can just use the lookup table shown below to find ξ, η

Table 34: Lie symmetry infinitesimal lookup table for known first order ODE's

ODE class	Form	ξ	η
linear ode	$y' = f(x)y(x) + g(x)$	0	$e^{\int f dx}$
separable ode	$y' = f(x)\, g(y)$	$\frac{1}{f}$	0
quadrature ode	$y' = f(x)$	0	1
quadrature ode	$y' = g(y)$	1	0
homogeneous ODEs of Class A	$y' = f\left(\frac{y}{x}\right)$	x	y
homogeneous ODEs of Class C	$y' = (a + bx + cy)^{\frac{n}{m}}$	1	$-\frac{b}{c}$
homogeneous class D	$y' = \frac{y}{x} + g(x)\, F\left(\frac{y}{x}\right)$	x^2	xy
First order special form ID 1	$y' = g(x)\, e^{h(x)+by} + f(x)$	$\frac{e^{-\int bf(x)dx - h(x)}}{g(x)}$	$\frac{f(x)e^{-\int bf(x)dx - h(x)}}{g(x)}$
polynomial type ode	$y' = \frac{a_1x+b_1y+c_1}{a_2x+b_2y+c_2}$	$\frac{a_1b_2x - a_2b_1x - b_1c_2 + b_2c_1}{a_1b_2 - a_2b_1}$	$\frac{a_1b_2y - a_2b_1y - a_1c_2 - a_2c_1}{a_1b_2 - a_2b_1}$
Bernoulli ode	$y' = f(x)\, y + g(x)\, y^n$	0	$e^{-\int (n-1)f(x)dx} y^n$
Reduced Riccati	$y' = f_1(x)\, y + f_2(x)\, y^2$	0	$e^{-\int f_1 dx}$

The above table shows that

$$
\begin{aligned}
\xi(t, y) &= 0 \\
\eta(t, y) &= \mathrm{e}^{-\frac{t}{2}}
\end{aligned}
\tag{A1}
$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(t, y) \to (R, S)$ where (R, S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$
\frac{dt}{\xi} = \frac{dy}{\eta} = dS \tag{1}
$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial t} + \eta \frac{\partial}{\partial y}\right) S(t, y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the

canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$R = t$$

S is found from

$$\begin{aligned} S &= \int \frac{1}{\eta} dy \\ &= \int \frac{1}{e^{-\frac{t}{2}}} dy \end{aligned}$$

Which results in

$$S = e^{\frac{t}{2}} y$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_t + \omega(t, y) S_y}{R_t + \omega(t, y) R_y} \tag{2}$$

Where in the above R_t, R_y, S_t, S_y are all partial derivatives and $\omega(t, y)$ is the right hand side of the original ode given by

$$\omega(t, y) = -\frac{y}{2} + \frac{3t^2}{2}$$

Evaluating all the partial derivatives gives

$$\begin{aligned} R_t &= 1 \\ R_y &= 0 \\ S_t &= \frac{e^{\frac{t}{2}} y}{2} \\ S_y &= e^{\frac{t}{2}} \end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = \frac{3t^2 e^{\frac{t}{2}}}{2} \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for t, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = \frac{3R^2 e^{\frac{R}{2}}}{2}$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordinates R, S. Integrating the above gives

$$S(R) = 3\left(R^2 - 4R + 8\right) e^{\frac{R}{2}} + c_1 \tag{4}$$

To complete the solution, we just need to transform (4) back to t, y coordinates. This results in

$$e^{\frac{t}{2}} y = 3\left(t^2 - 4t + 8\right) e^{\frac{t}{2}} + c_1$$

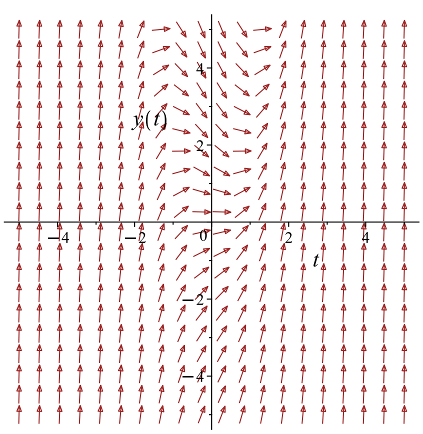
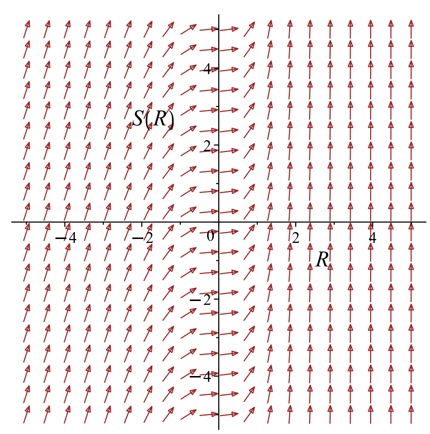
Which simplifies to

$$\left(-3t^2 + 12t + y - 24\right) e^{\frac{t}{2}} - c_1 = 0$$

Which gives

$$y = \left(3t^2 e^{\frac{t}{2}} - 12t\, e^{\frac{t}{2}} + 24\, e^{\frac{t}{2}} + c_1\right) e^{-\frac{t}{2}}$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in t, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dt} = -\frac{y}{2} + \frac{3t^2}{2}$ 	$R = t$ $S = e^{\frac{t}{2}} y$	$\frac{dS}{dR} = \frac{3R^2 e^{\frac{R}{2}}}{2}$ 

Summary

The solution(s) found are the following

$$y = \left(3t^2 e^{\frac{t}{2}} - 12t\, e^{\frac{t}{2}} + 24\, e^{\frac{t}{2}} + c_1\right) e^{-\frac{t}{2}} \tag{1}$$

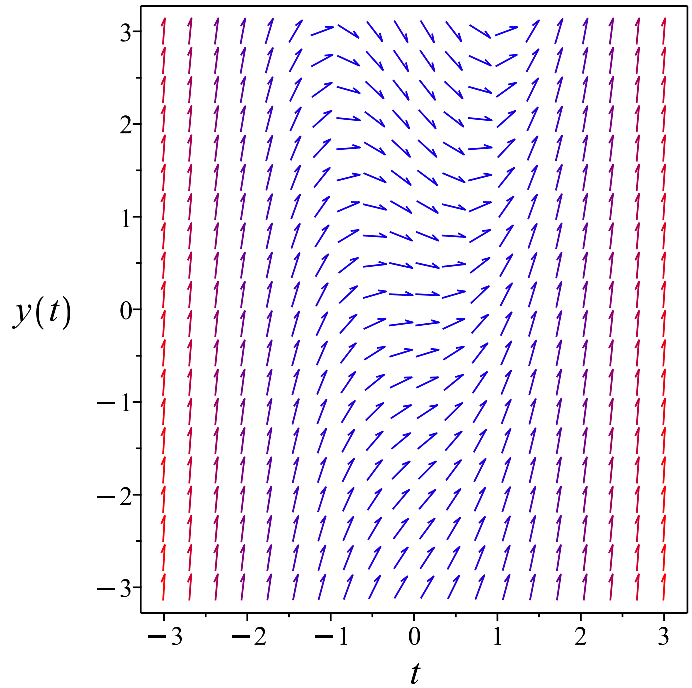


Figure 36: Slope field plot

Verification of solutions

$$y = \left(3t^2 e^{\frac{t}{2}} - 12t\, e^{\frac{t}{2}} + 24\, e^{\frac{t}{2}} + c_1\right) e^{-\frac{t}{2}}$$

Verified OK.

1.12.3 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x,y) + N(x,y)\,\frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x,y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y) = 0$$

Hence

$$\frac{\partial \phi}{\partial x} + \frac{\partial \phi}{\partial y}\frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$
\begin{aligned}
\frac{\partial \phi}{\partial x} &= M \\
\frac{\partial \phi}{\partial y} &= N
\end{aligned}
$$

But since $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ then for the above to be valid, we require that

$$
\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}
$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x, y)$ but at least we know now that we can do that since the condition $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$
M(t, y) \,\mathrm{d}t + N(t, y) \,\mathrm{d}y = 0 \tag{1A}
$$

Therefore

$$
\begin{aligned}
(2) \,\mathrm{d}y &= \left(3t^2 - y\right) \mathrm{d}t \\
\left(-3t^2 + y\right) \mathrm{d}t + (2) \,\mathrm{d}y &= 0
\end{aligned} \tag{2A}
$$

Comparing (1A) and (2A) shows that

$$
\begin{aligned}
M(t, y) &= -3t^2 + y \\
N(t, y) &= 2
\end{aligned}
$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$
\frac{\partial M}{\partial y} = \frac{\partial N}{\partial t}
$$

Using result found above gives

$$
\begin{aligned}
\frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}\left(-3t^2 + y\right) \\
&= 1
\end{aligned}
$$

And

$$
\begin{aligned}
\frac{\partial N}{\partial t} &= \frac{\partial}{\partial t}(2) \\
&= 0
\end{aligned}
$$

Since $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial t}$, then the ODE is <u>not exact</u>. Since the ODE is not exact, we will try to find an integrating factor to make it exact. Let

$$\begin{aligned} A &= \frac{1}{N}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial t}\right) \\ &= \frac{1}{2}((1) - (0)) \\ &= \frac{1}{2} \end{aligned}$$

Since A does not depend on y, then it can be used to find an integrating factor. The integrating factor μ is

$$\begin{aligned} \mu &= e^{\int A \, \mathrm{d}t} \\ &= e^{\int \frac{1}{2} \, \mathrm{d}t} \end{aligned}$$

The result of integrating gives

$$\begin{aligned} \mu &= e^{\frac{t}{2}} \\ &= \mathrm{e}^{\frac{t}{2}} \end{aligned}$$

M and N are multiplied by this integrating factor, giving new M and new N which are called $\overline{M}$ and $\overline{N}$ for now so not to confuse them with the original M and N.

$$\begin{aligned} \overline{M} &= \mu M \\ &= \mathrm{e}^{\frac{t}{2}}\left(-3t^2 + y\right) \\ &= \left(-3t^2 + y\right)\mathrm{e}^{\frac{t}{2}} \end{aligned}$$

And

$$\begin{aligned} \overline{N} &= \mu N \\ &= \mathrm{e}^{\frac{t}{2}}(2) \\ &= 2\,\mathrm{e}^{\frac{t}{2}} \end{aligned}$$

Now a modified ODE is ontained from the original ODE, which is exact and can be solved. The modified ODE is

$$\begin{aligned} \overline{M} + \overline{N}\frac{\mathrm{d}y}{\mathrm{d}t} &= 0 \\ \left(\left(-3t^2 + y\right)\mathrm{e}^{\frac{t}{2}}\right) + \left(2\,\mathrm{e}^{\frac{t}{2}}\right)\frac{\mathrm{d}y}{\mathrm{d}t} &= 0 \end{aligned}$$

The following equations are now set up to solve for the function $\phi(t,y)$

$$\frac{\partial \phi}{\partial t} = \overline{M} \tag{1}$$

$$\frac{\partial \phi}{\partial y} = \overline{N} \tag{2}$$

Integrating (1) w.r.t. t gives

$$\int \frac{\partial \phi}{\partial t} \mathrm{d}t = \int \overline{M} \mathrm{d}t$$

$$\int \frac{\partial \phi}{\partial t} \mathrm{d}t = \int \left(-3t^2 + y\right) \mathrm{e}^{\frac{t}{2}} \mathrm{d}t$$

$$\phi = -6\left(t^2 - 4t - \frac{1}{3}y + 8\right) \mathrm{e}^{\frac{t}{2}} + f(y) \tag{3}$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both t and y. Taking derivative of equation (3) w.r.t y gives

$$\frac{\partial \phi}{\partial y} = 2\,\mathrm{e}^{\frac{t}{2}} + f'(y) \tag{4}$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = 2\,\mathrm{e}^{\frac{t}{2}}$. Therefore equation (4) becomes

$$2\,\mathrm{e}^{\frac{t}{2}} = 2\,\mathrm{e}^{\frac{t}{2}} + f'(y) \tag{5}$$

Solving equation (5) for $f'(y)$ gives

$$f'(y) = 0$$

Therefore

$$f(y) = c_1$$

Where c_1 is constant of integration. Substituting this result for $f(y)$ into equation (3) gives ϕ

$$\phi = -6\left(t^2 - 4t - \frac{1}{3}y + 8\right) \mathrm{e}^{\frac{t}{2}} + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = -6\left(t^2 - 4t - \frac{1}{3}y + 8\right) \mathrm{e}^{\frac{t}{2}}$$

The solution becomes

$$y = \frac{\left(6t^2 e^{\frac{t}{2}} - 24t\, e^{\frac{t}{2}} + 48\, e^{\frac{t}{2}} + c_1\right) e^{-\frac{t}{2}}}{2}$$

Summary
The solution(s) found are the following

$$y = \frac{\left(6t^2 e^{\frac{t}{2}} - 24t\, e^{\frac{t}{2}} + 48\, e^{\frac{t}{2}} + c_1\right) e^{-\frac{t}{2}}}{2} \tag{1}$$

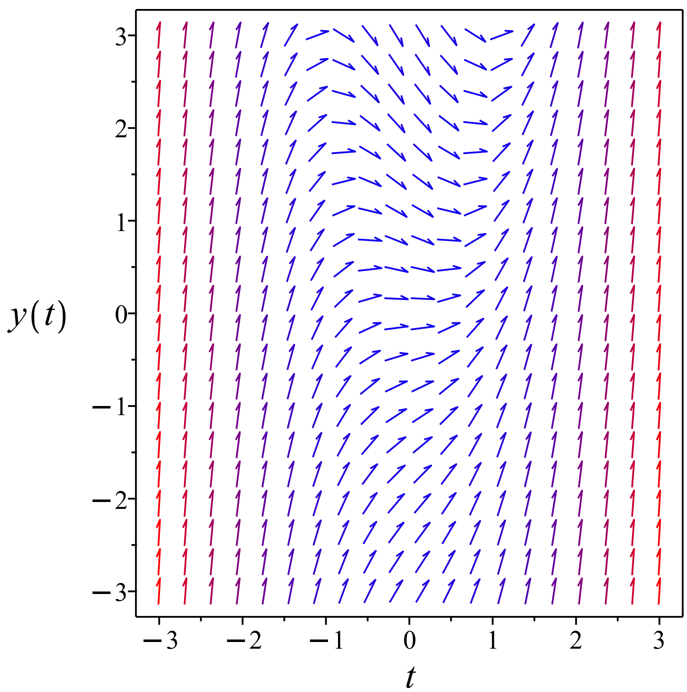


Figure 37: Slope field plot

Verification of solutions

$$y = \frac{\left(6t^2 e^{\frac{t}{2}} - 24t\, e^{\frac{t}{2}} + 48\, e^{\frac{t}{2}} + c_1\right) e^{-\frac{t}{2}}}{2}$$

Verified OK.

1.12.4 Maple step by step solution

Let's solve

$y + 2y' = 3t^2$

- Highest derivative means the order of the ODE is 1

  y'

- Isolate the derivative

  $y' = -\frac{y}{2} + \frac{3t^2}{2}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE

  $y' + \frac{y}{2} = \frac{3t^2}{2}$

- The ODE is linear; multiply by an integrating factor $\mu(t)$

  $\mu(t)\left(y' + \frac{y}{2}\right) = \frac{3\mu(t)t^2}{2}$

- Assume the lhs of the ODE is the total derivative $\frac{d}{dt}(\mu(t)\,y)$

  $\mu(t)\left(y' + \frac{y}{2}\right) = \mu'(t)\,y + \mu(t)\,y'$

- Isolate $\mu'(t)$

  $\mu'(t) = \frac{\mu(t)}{2}$

- Solve to find the integrating factor

  $\mu(t) = e^{\frac{t}{2}}$

- Integrate both sides with respect to t

  $\int \left(\frac{d}{dt}(\mu(t)\,y)\right) dt = \int \frac{3\mu(t)t^2}{2} dt + c_1$

- Evaluate the integral on the lhs

  $\mu(t)\,y = \int \frac{3\mu(t)t^2}{2} dt + c_1$

- Solve for y

  $y = \frac{\int \frac{3\mu(t)t^2}{2} dt + c_1}{\mu(t)}$

- Substitute $\mu(t) = e^{\frac{t}{2}}$

  $y = \frac{\int \frac{3t^2 e^{\frac{t}{2}}}{2} dt + c_1}{e^{\frac{t}{2}}}$

- Evaluate the integrals on the rhs

  $y = \frac{3(t^2-4t+8)e^{\frac{t}{2}} + c_1}{e^{\frac{t}{2}}}$

- Simplify

  $y = 3t^2 - 12t + 24 + c_1 e^{-\frac{t}{2}}$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
<- 1st order linear successful`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 20

```
dsolve(y(t)+2*diff(y(t),t) = 3*t^2,y(t), singsol=all)
```

$$y(t) = 3t^2 - 12t + 24 + e^{-\frac{t}{2}} c_1$$

✓ Solution by Mathematica

Time used: 0.047 (sec). Leaf size: 25

```
DSolve[y[t]+2*y'[t] == 3*t^2,y[t],t,IncludeSingularSolutions -> True]
```

$$y(t) \to 3t^2 - 12t + c_1 e^{-t/2} + 24$$

1.13 problem 13

Internal problem ID [460]
Internal file name [`OUTPUT/460_Sunday_June_05_2022_01_41_49_AM_58508263/index.tex`]

Book: Elementary differential equations and boundary value problems, 10th ed., Boyce and DiPrima
Section: Section 2.1. Page 40
Problem number: 13.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"linear", "exactWithIntegrationFactor", "first_order_ode_lie_symmetry_lookup"**

Maple gives the following as the ode type

```
[[_linear, `class A`]]
```

$$\boxed{-y + y' = 2\,\mathrm{e}^{2t}t}$$

With initial conditions

$$[y(0) = 1]$$

1.13.1 Existence and uniqueness analysis

This is a linear ODE. In canonical form it is written as

$$y' + p(t)y = q(t)$$

Where here

$$p(t) = -1$$
$$q(t) = 2\,\mathrm{e}^{2t}t$$

Hence the ode is

$$-y + y' = 2\,\mathrm{e}^{2t} t$$

The domain of $p(t) = -1$ is

$$\{-\infty < t < \infty\}$$

And the point $t_0 = 0$ is inside this domain. The domain of $q(t) = 2\,\mathrm{e}^{2t} t$ is

$$\{-\infty < t < \infty\}$$

And the point $t_0 = 0$ is also inside this domain. Hence solution exists and is unique.

1.13.2 Solving as linear ode

Entering Linear first order ODE solver. The integrating factor μ is

$$\begin{aligned}\mu &= \mathrm{e}^{\int (-1) dt} \\ &= \mathrm{e}^{-t}\end{aligned}$$

The ode becomes

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}(\mu y) &= (\mu)\left(2\,\mathrm{e}^{2t} t\right) \\ \frac{\mathrm{d}}{\mathrm{d}t}\left(\mathrm{e}^{-t} y\right) &= \left(\mathrm{e}^{-t}\right)\left(2\,\mathrm{e}^{2t} t\right) \\ \mathrm{d}\left(\mathrm{e}^{-t} y\right) &= \left(2t\,\mathrm{e}^{t}\right)\,\mathrm{d}t\end{aligned}$$

Integrating gives

$$\begin{aligned}\mathrm{e}^{-t} y &= \int 2t\,\mathrm{e}^{t}\,\mathrm{d}t \\ \mathrm{e}^{-t} y &= 2\,\mathrm{e}^{t}(-1 + t) + c_1\end{aligned}$$

Dividing both sides by the integrating factor $\mu = \mathrm{e}^{-t}$ results in

$$y = 2\,\mathrm{e}^{2t}(-1 + t) + c_1 \mathrm{e}^{t}$$

Initial conditions are used to solve for c_1. Substituting $t = 0$ and $y = 1$ in the above solution gives an equation to solve for the constant of integration.

$$1 = c_1 - 2$$

$$c_1 = 3$$

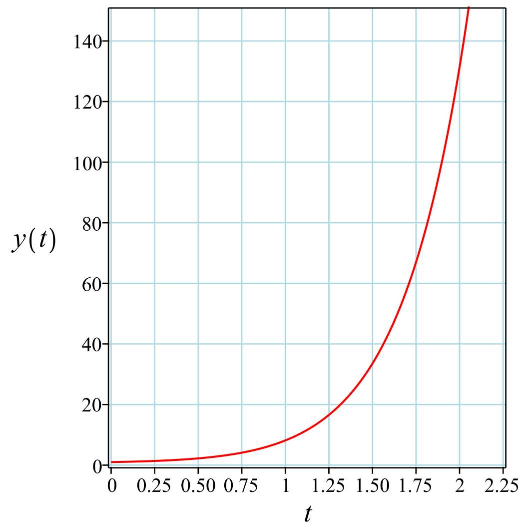
Substituting c_1 found above in the general solution gives

$$y = 2\,\mathrm{e}^{2t}t - 2\,\mathrm{e}^{2t} + 3\,\mathrm{e}^{t}$$

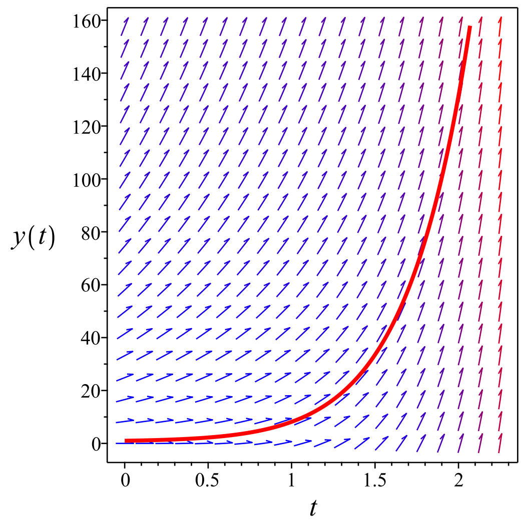
Summary

The solution(s) found are the following

$$y = 2\,\mathrm{e}^{2t}t - 2\,\mathrm{e}^{2t} + 3\,\mathrm{e}^{t} \tag{1}$$



(a) Solution plot



(b) Slope field plot

Verification of solutions

$$y = 2\,\mathrm{e}^{2t}t - 2\,\mathrm{e}^{2t} + 3\,\mathrm{e}^{t}$$

Verified OK.

1.13.3 Solving as first order ode lie symmetry lookup ode

Writing the ode as

$$y' = y + 2\,\mathrm{e}^{2t}t$$
$$y' = \omega(t, y)$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_t + \omega(\eta_y - \xi_t) - \omega^2 \xi_y - \omega_t \xi - \omega_y \eta = 0 \tag{A}$$

The type of this ode is known. It is of type `linear`. Therefore we do not need to solve the PDE (A), and can just use the lookup table shown below to find ξ, η

Table 37: Lie symmetry infinitesimal lookup table for known first order ODE's

ODE class	Form	ξ	η
linear ode	$y' = f(x)y(x) + g(x)$	0	$e^{\int f dx}$
separable ode	$y' = f(x)\, g(y)$	$\frac{1}{f}$	0
quadrature ode	$y' = f(x)$	0	1
quadrature ode	$y' = g(y)$	1	0
homogeneous ODEs of Class A	$y' = f\left(\frac{y}{x}\right)$	x	y
homogeneous ODEs of Class C	$y' = (a + bx + cy)^{\frac{n}{m}}$	1	$-\frac{b}{c}$
homogeneous class D	$y' = \frac{y}{x} + g(x)\, F\left(\frac{y}{x}\right)$	x^2	xy
First order special form ID 1	$y' = g(x)\, e^{h(x)+by} + f(x)$	$\frac{e^{-\int bf(x)dx - h(x)}}{g(x)}$	$\frac{f(x)e^{-\int bf(x)dx - h(x)}}{g(x)}$
polynomial type ode	$y' = \frac{a_1x+b_1y+c_1}{a_2x+b_2y+c_2}$	$\frac{a_1b_2x - a_2b_1x - b_1c_2 + b_2c_1}{a_1b_2 - a_2b_1}$	$\frac{a_1b_2y - a_2b_1y - a_1c_2 - a_2c_1}{a_1b_2 - a_2b_1}$
Bernoulli ode	$y' = f(x)\, y + g(x)\, y^n$	0	$e^{-\int (n-1)f(x)dx} y^n$
Reduced Riccati	$y' = f_1(x)\, y + f_2(x)\, y^2$	0	$e^{-\int f_1 dx}$

The above table shows that

$$
\begin{aligned}
\xi(t, y) &= 0 \\
\eta(t, y) &= e^t
\end{aligned} \tag{A1}
$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(t, y) \to (R, S)$ where (R, S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$
\frac{dt}{\xi} = \frac{dy}{\eta} = dS \tag{1}
$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial t} + \eta \frac{\partial}{\partial y}\right) S(t,y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$R = t$$

S is found from

$$\begin{aligned} S &= \int \frac{1}{\eta} dy \\ &= \int \frac{1}{e^{t}} dy \end{aligned}$$

Which results in

$$S = e^{-t} y$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_t + \omega(t,y) S_y}{R_t + \omega(t,y) R_y} \tag{2}$$

Where in the above R_t, R_y, S_t, S_y are all partial derivatives and $\omega(t,y)$ is the right hand side of the original ode given by

$$\omega(t,y) = y + 2\,e^{2t} t$$

Evaluating all the partial derivatives gives

$$\begin{aligned} R_t &= 1 \\ R_y &= 0 \\ S_t &= -e^{-t} y \\ S_y &= e^{-t} \end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = 2t\,e^{t} \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for t, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = 2R\,e^{R}$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$S(R) = 2(R-1)\,\mathrm{e}^{R} + c_1 \tag{4}$$

To complete the solution, we just need to transform (4) back to t, y coordinates. This results in

$$\mathrm{e}^{-t}y = 2\,\mathrm{e}^{t}(-1+t) + c_1$$

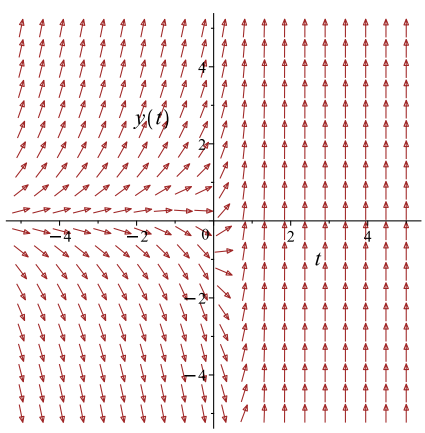
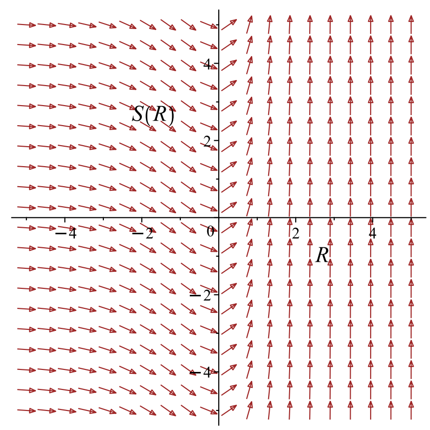
Which simplifies to

$$\mathrm{e}^{-t}y = 2\,\mathrm{e}^{t}(-1+t) + c_1$$

Which gives

$$y = \mathrm{e}^{t}\left(2t\,\mathrm{e}^{t} - 2\,\mathrm{e}^{t} + c_1\right)$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in t, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dt} = y + 2\,\mathrm{e}^{2t}t$ 	$R = t$ $S = \mathrm{e}^{-t}y$	$\frac{dS}{dR} = 2R\,\mathrm{e}^{R}$ 

Initial conditions are used to solve for c_1. Substituting $t = 0$ and $y = 1$ in the above solution gives an equation to solve for the constant of integration.

$$1 = c_1 - 2$$

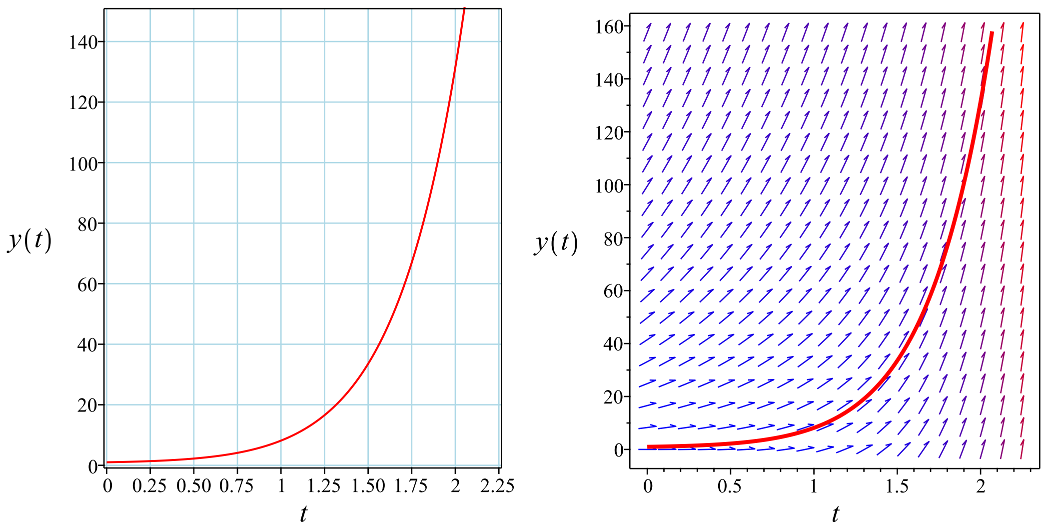
$$c_1 = 3$$

Substituting c_1 found above in the general solution gives

$$y = 2\,\mathrm{e}^{2t}t - 2\,\mathrm{e}^{2t} + 3\,\mathrm{e}^{t}$$

Summary
The solution(s) found are the following

$$y = 2\,\mathrm{e}^{2t}t - 2\,\mathrm{e}^{2t} + 3\,\mathrm{e}^{t} \tag{1}$$



(a) Solution plot

(b) Slope field plot

Verification of solutions

$$y = 2\,\mathrm{e}^{2t}t - 2\,\mathrm{e}^{2t} + 3\,\mathrm{e}^{t}$$

Verified OK.

1.13.4 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x, y) + N(x, y)\,\frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x, y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x, y) = 0$$

Hence

$$\frac{\partial \phi}{\partial x}+\frac{\partial \phi}{\partial y}\frac{dy}{dx}=0 \tag{B}$$

Comparing (A,B) shows that

$$\begin{aligned}\frac{\partial \phi}{\partial x}&=M\\ \frac{\partial \phi}{\partial y}&=N\end{aligned}$$

But since $\frac{\partial^2\phi}{\partial x\partial y}=\frac{\partial^2\phi}{\partial y\partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y}=\frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x,y)$ but at least we know now that we can do that since the condition $\frac{\partial^2\phi}{\partial x\partial y}=\frac{\partial^2\phi}{\partial y\partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(t,y)\,\mathrm{d}t+N(t,y)\,\mathrm{d}y=0 \tag{1A}$$

Therefore

$$\begin{aligned}\mathrm{d}y&=\left(y+2\,\mathrm{e}^{2t}t\right)\mathrm{d}t\\ \left(-y-2\,\mathrm{e}^{2t}t\right)\mathrm{d}t+\mathrm{d}y&=0\end{aligned} \tag{2A}$$

Comparing (1A) and (2A) shows that

$$\begin{aligned}M(t,y)&=-y-2\,\mathrm{e}^{2t}t\\ N(t,y)&=1\end{aligned}$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y}=\frac{\partial N}{\partial t}$$

Using result found above gives

$$\begin{aligned}\frac{\partial M}{\partial y}&=\frac{\partial}{\partial y}\left(-y-2\,\mathrm{e}^{2t}t\right)\\ &=-1\end{aligned}$$

And

$$\begin{aligned}\frac{\partial N}{\partial t} &= \frac{\partial}{\partial t}(1) \\ &= 0\end{aligned}$$

Since $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial t}$, then the ODE is <u>not exact</u>. Since the ODE is not exact, we will try to find an integrating factor to make it exact. Let

$$\begin{aligned}A &= \frac{1}{N}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial t}\right) \\ &= 1((-1) - (0)) \\ &= -1\end{aligned}$$

Since A does not depend on y, then it can be used to find an integrating factor. The integrating factor μ is

$$\begin{aligned}\mu &= e^{\int A \, \mathrm{d}t} \\ &= e^{\int -1 \, \mathrm{d}t}\end{aligned}$$

The result of integrating gives

$$\begin{aligned}\mu &= e^{-t} \\ &= \mathrm{e}^{-t}\end{aligned}$$

M and N are multiplied by this integrating factor, giving new M and new N which are called $\overline{M}$ and $\overline{N}$ for now so not to confuse them with the original M and N.

$$\begin{aligned}\overline{M} &= \mu M \\ &= \mathrm{e}^{-t}\left(-y - 2\,\mathrm{e}^{2t} t\right) \\ &= -\mathrm{e}^{-t} y - 2t\,\mathrm{e}^{t}\end{aligned}$$

And

$$\begin{aligned}\overline{N} &= \mu N \\ &= \mathrm{e}^{-t}(1) \\ &= \mathrm{e}^{-t}\end{aligned}$$

Now a modified ODE is ontained from the original ODE, which is exact and can be solved. The modified ODE is

$$\begin{aligned}\overline{M} + \overline{N}\frac{\mathrm{d}y}{\mathrm{d}t} &= 0 \\ \left(-\mathrm{e}^{-t} y - 2t\,\mathrm{e}^{t}\right) + \left(\mathrm{e}^{-t}\right)\frac{\mathrm{d}y}{\mathrm{d}t} &= 0\end{aligned}$$

The following equations are now set up to solve for the function $\phi(t, y)$

$$\frac{\partial \phi}{\partial t} = \overline{M} \tag{1}$$
$$\frac{\partial \phi}{\partial y} = \overline{N} \tag{2}$$

Integrating (1) w.r.t. t gives

$$\int \frac{\partial \phi}{\partial t} \, \mathrm{d}t = \int \overline{M} \, \mathrm{d}t$$
$$\int \frac{\partial \phi}{\partial t} \, \mathrm{d}t = \int -\mathrm{e}^{-t} y - 2t \, \mathrm{e}^{t} \, \mathrm{d}t$$
$$\phi = \mathrm{e}^{-t} y - 2 \, \mathrm{e}^{t}(-1 + t) + f(y) \tag{3}$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both t and y. Taking derivative of equation (3) w.r.t y gives

$$\frac{\partial \phi}{\partial y} = \mathrm{e}^{-t} + f'(y) \tag{4}$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = \mathrm{e}^{-t}$. Therefore equation (4) becomes

$$\mathrm{e}^{-t} = \mathrm{e}^{-t} + f'(y) \tag{5}$$

Solving equation (5) for $f'(y)$ gives

$$f'(y) = 0$$

Therefore

$$f(y) = c_1$$

Where c_1 is constant of integration. Substituting this result for $f(y)$ into equation (3) gives ϕ

$$\phi = \mathrm{e}^{-t} y - 2 \, \mathrm{e}^{t}(-1 + t) + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = \mathrm{e}^{-t} y - 2 \, \mathrm{e}^{t}(-1 + t)$$

The solution becomes

$$y = e^t \left(2t\,e^t - 2\,e^t + c_1\right)$$

Initial conditions are used to solve for c_1. Substituting $t = 0$ and $y = 1$ in the above solution gives an equation to solve for the constant of integration.

$$1 = c_1 - 2$$

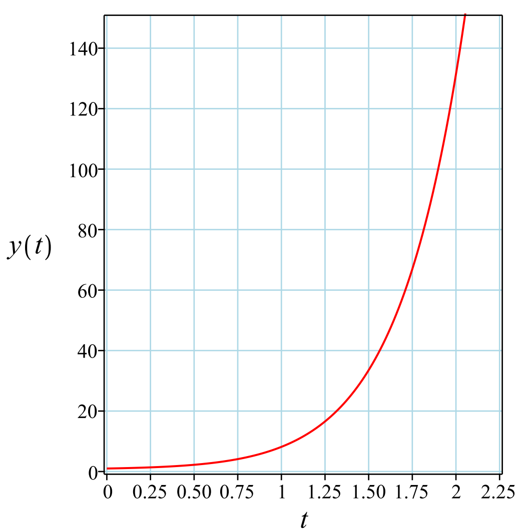
$$c_1 = 3$$

Substituting c_1 found above in the general solution gives

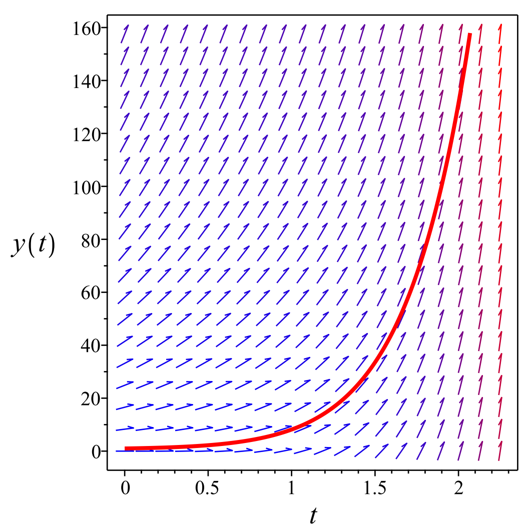
$$y = 2\,e^{2t}t - 2\,e^{2t} + 3\,e^t$$

Summary
The solution(s) found are the following

$$y = 2\,e^{2t}t - 2\,e^{2t} + 3\,e^t \tag{1}$$



(a) Solution plot



(b) Slope field plot

Verification of solutions

$$y = 2\,e^{2t}t - 2\,e^{2t} + 3\,e^t$$

Verified OK.

1.13.5 Maple step by step solution

Let's solve

$[-y + y' = 2\,\mathrm{e}^{2t} t, y(0) = 1]$

- Highest derivative means the order of the ODE is 1

  y'

- Isolate the derivative

  $y' = y + 2\,\mathrm{e}^{2t} t$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE

  $-y + y' = 2\,\mathrm{e}^{2t} t$

- The ODE is linear; multiply by an integrating factor $\mu(t)$

  $\mu(t)\,(-y + y') = 2\mu(t)\,\mathrm{e}^{2t} t$

- Assume the lhs of the ODE is the total derivative $\frac{d}{dt}(\mu(t)\,y)$

  $\mu(t)\,(-y + y') = \mu'(t)\,y + \mu(t)\,y'$

- Isolate $\mu'(t)$

  $\mu'(t) = -\mu(t)$

- Solve to find the integrating factor

  $\mu(t) = \mathrm{e}^{-t}$

- Integrate both sides with respect to t

  $\int \left(\frac{d}{dt}(\mu(t)\,y)\right) dt = \int 2\mu(t)\,\mathrm{e}^{2t} t dt + c_1$

- Evaluate the integral on the lhs

  $\mu(t)\,y = \int 2\mu(t)\,\mathrm{e}^{2t} t dt + c_1$

- Solve for y

  $y = \frac{\int 2\mu(t)\mathrm{e}^{2t} t dt + c_1}{\mu(t)}$

- Substitute $\mu(t) = \mathrm{e}^{-t}$

  $y = \frac{\int 2\,\mathrm{e}^{2t} t\,\mathrm{e}^{-t} dt + c_1}{\mathrm{e}^{-t}}$

- Evaluate the integrals on the rhs

  $y = \frac{2\,\mathrm{e}^{t}(-1+t) + c_1}{\mathrm{e}^{-t}}$

- Simplify

  $y = (2t - 2)\,(\mathrm{e}^{t})^2 + c_1 \mathrm{e}^{t}$

- Use initial condition $y(0) = 1$

  $1 = c_1 - 2$
- Solve for c_1

  $c_1 = 3$
- Substitute $c_1 = 3$ into general solution and simplify

  $y = (2t - 2)\,e^{2t} + 3\,e^{t}$
- Solution to the IVP

  $y = (2t - 2)\,e^{2t} + 3\,e^{t}$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
<- 1st order linear successful`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 16

```
dsolve([-y(t)+diff(y(t),t) = 2*exp(2*t)*t,y(0) = 1],y(t), singsol=all)
```

$$y(t) = (2t - 2)\,e^{2t} + 3\,e^{t}$$

✓ Solution by Mathematica

Time used: 0.053 (sec). Leaf size: 19

```
DSolve[{-y[t]+y'[t] == 2*Exp[2*t]*t,y[0]==1},y[t],t,IncludeSingularSolutions -> True]
```

$$y(t) \to e^t\left(2e^t(t-1)+3\right)$$

1.14 problem 14

Internal problem ID [461]
Internal file name [`OUTPUT/461_Sunday_June_05_2022_01_41_50_AM_10145323/index.tex`]

Book: Elementary differential equations and boundary value problems, 10th ed., Boyce and DiPrima
Section: Section 2.1. Page 40
Problem number: 14.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"exact", "linear", "first_order_ode_lie_symmetry_lookup"**

Maple gives the following as the ode type

```
[[_linear, `class A`]]
```

$$\boxed{2y + y' = t\,\mathrm{e}^{-2t}}$$

With initial conditions

$$[y(1) = 0]$$

1.14.1 Existence and uniqueness analysis

This is a linear ODE. In canonical form it is written as

$$y' + p(t)y = q(t)$$

Where here

$$\begin{aligned} p(t) &= 2 \\ q(t) &= t\,\mathrm{e}^{-2t} \end{aligned}$$

Hence the ode is

$$2y + y' = t\,\mathrm{e}^{-2t}$$

The domain of $p(t) = 2$ is

$$\{-\infty < t < \infty\}$$

And the point $t_0 = 1$ is inside this domain. The domain of $q(t) = t\,\mathrm{e}^{-2t}$ is

$$\{-\infty < t < \infty\}$$

And the point $t_0 = 1$ is also inside this domain. Hence solution exists and is unique.

1.14.2 Solving as linear ode

Entering Linear first order ODE solver. The integrating factor μ is

$$\begin{aligned}\mu &= \mathrm{e}^{\int 2dt} \\ &= \mathrm{e}^{2t}\end{aligned}$$

The ode becomes

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}(\mu y) &= (\mu)\left(t\,\mathrm{e}^{-2t}\right) \\ \frac{\mathrm{d}}{\mathrm{d}t}\left(y\,\mathrm{e}^{2t}\right) &= \left(\mathrm{e}^{2t}\right)\left(t\,\mathrm{e}^{-2t}\right) \\ \mathrm{d}\left(y\,\mathrm{e}^{2t}\right) &= t\,\mathrm{d}t\end{aligned}$$

Integrating gives

$$\begin{aligned}y\,\mathrm{e}^{2t} &= \int t\,\mathrm{d}t \\ y\,\mathrm{e}^{2t} &= \frac{t^2}{2} + c_1\end{aligned}$$

Dividing both sides by the integrating factor $\mu = \mathrm{e}^{2t}$ results in

$$y = \frac{\mathrm{e}^{-2t}t^2}{2} + c_1\mathrm{e}^{-2t}$$

which simplifies to

$$y = \mathrm{e}^{-2t}\left(\frac{t^2}{2} + c_1\right)$$

Initial conditions are used to solve for c_1. Substituting $t = 1$ and $y = 0$ in the above solution gives an equation to solve for the constant of integration.

$$0 = \frac{e^{-2}(2c_1 + 1)}{2}$$

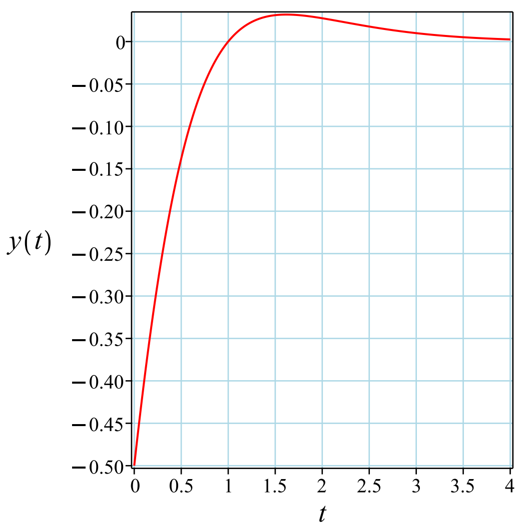
$$c_1 = -\frac{1}{2}$$

Substituting c_1 found above in the general solution gives

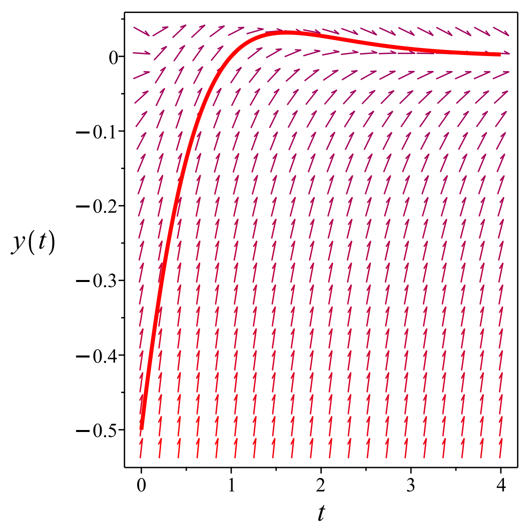
$$y = \frac{e^{-2t}(t^2 - 1)}{2}$$

Summary
The solution(s) found are the following

$$y = \frac{e^{-2t}(t^2 - 1)}{2} \tag{1}$$



(a) Solution plot



(b) Slope field plot

Verification of solutions

$$y = \frac{e^{-2t}(t^2 - 1)}{2}$$

Verified OK.

1.14.3 Solving as first order ode lie symmetry lookup ode

Writing the ode as

$$
\begin{aligned}
y' &= -\left(2y\,\mathrm{e}^{2t} - t\right)\mathrm{e}^{-2t} \\
y' &= \omega(t, y)
\end{aligned}
$$

The condition of Lie symmetry is the linearized PDE given by

$$
\eta_t + \omega(\eta_y - \xi_t) - \omega^2 \xi_y - \omega_t \xi - \omega_y \eta = 0 \tag{A}
$$

The type of this ode is known. It is of type `linear`. Therefore we do not need to solve the PDE (A), and can just use the lookup table shown below to find ξ, η

Table 40: Lie symmetry infinitesimal lookup table for known first order ODE's

ODE class	Form	ξ	η
linear ode	$y' = f(x)y(x) + g(x)$	0	$e^{\int f dx}$
separable ode	$y' = f(x)\,g(y)$	$\frac{1}{f}$	0
quadrature ode	$y' = f(x)$	0	1
quadrature ode	$y' = g(y)$	1	0
homogeneous ODEs of Class A	$y' = f\left(\frac{y}{x}\right)$	x	y
homogeneous ODEs of Class C	$y' = (a + bx + cy)^{\frac{n}{m}}$	1	$-\frac{b}{c}$
homogeneous class D	$y' = \frac{y}{x} + g(x)\,F\left(\frac{y}{x}\right)$	x^2	xy
First order special form ID 1	$y' = g(x)\,e^{h(x)+by} + f(x)$	$\frac{e^{-\int bf(x)dx - h(x)}}{g(x)}$	$\frac{f(x)e^{-\int bf(x)dx - h(x)}}{g(x)}$
polynomial type ode	$y' = \frac{a_1x+b_1y+c_1}{a_2x+b_2y+c_2}$	$\frac{a_1b_2x-a_2b_1x-b_1c_2+b_2c_1}{a_1b_2-a_2b_1}$	$\frac{a_1b_2y-a_2b_1y-a_1c_2-a_2c_1}{a_1b_2-a_2b_1}$
Bernoulli ode	$y' = f(x)\,y + g(x)\,y^n$	0	$e^{-\int (n-1)f(x)dx}y^n$
Reduced Riccati	$y' = f_1(x)\,y + f_2(x)\,y^2$	0	$e^{-\int f_1 dx}$

The above table shows that

$$
\begin{aligned}
\xi(t, y) &= 0 \\
\eta(t, y) &= e^{-2t}
\end{aligned} \tag{A1}
$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(t, y) \to (R, S)$ where (R, S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$
\frac{dt}{\xi} = \frac{dy}{\eta} = dS \tag{1}
$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial t} + \eta \frac{\partial}{\partial y}\right) S(t, y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$
R = t
$$

S is found from

$$
\begin{aligned}
S &= \int \frac{1}{\eta} dy \\
&= \int \frac{1}{e^{-2t}} dy
\end{aligned}
$$

Which results in

$$
S = y\, e^{2t}
$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$
\frac{dS}{dR} = \frac{S_t + \omega(t, y) S_y}{R_t + \omega(t, y) R_y} \tag{2}
$$

Where in the above R_t, R_y, S_t, S_y are all partial derivatives and $\omega(t, y)$ is the right hand side of the original ode given by

$$
\omega(t, y) = -\left(2y\, e^{2t} - t\right) e^{-2t}
$$

Evaluating all the partial derivatives gives

$$
\begin{aligned}
R_t &= 1 \\
R_y &= 0 \\
S_t &= 2y\, e^{2t} \\
S_y &= e^{2t}
\end{aligned}
$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = t \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for t, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = R$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$S(R) = \frac{R^2}{2} + c_1 \tag{4}$$

To complete the solution, we just need to transform (4) back to t, y coordinates. This results in

$$y\,\mathrm{e}^{2t} = \frac{t^2}{2} + c_1$$

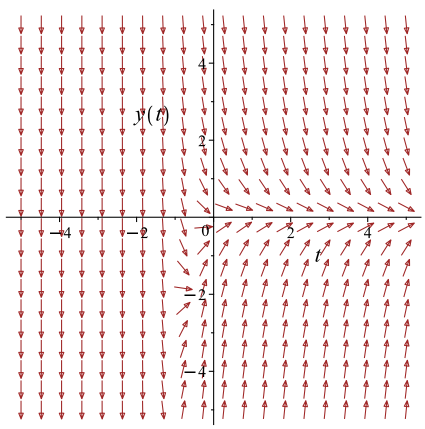
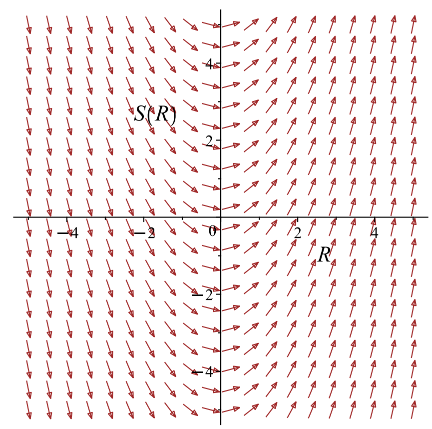
Which simplifies to

$$y\,\mathrm{e}^{2t} = \frac{t^2}{2} + c_1$$

Which gives

$$y = \frac{\mathrm{e}^{-2t}(t^2 + 2c_1)}{2}$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in t, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dt} = -(2y\,\mathrm{e}^{2t} - t)\,\mathrm{e}^{-2t}$  	$R = t$ $S = y\,\mathrm{e}^{2t}$	$\frac{dS}{dR} = R$ 

Initial conditions are used to solve for c_1. Substituting $t = 1$ and $y = 0$ in the above solution gives an equation to solve for the constant of integration.

$$0 = \mathrm{e}^{-2}c_1 + \frac{\mathrm{e}^{-2}}{2}$$

$$c_1 = -\frac{1}{2}$$

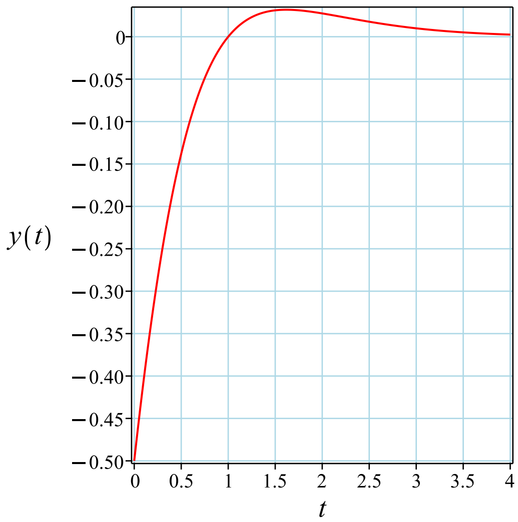
Substituting c_1 found above in the general solution gives

$$y = \frac{\mathrm{e}^{-2t}t^2}{2} - \frac{\mathrm{e}^{-2t}}{2}$$

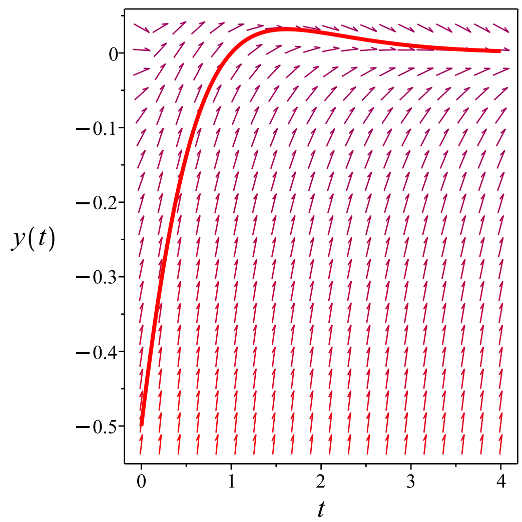
Summary

The solution(s) found are the following

$$y = \frac{\mathrm{e}^{-2t}t^2}{2} - \frac{\mathrm{e}^{-2t}}{2} \tag{1}$$



(a) Solution plot



(b) Slope field plot

Verification of solutions

$$y = \frac{e^{-2t}t^2}{2} - \frac{e^{-2t}}{2}$$

Verified OK.

1.14.4 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x, y) + N(x, y) \frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x, y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x, y) = 0$$

Hence

$$\frac{\partial \phi}{\partial x} + \frac{\partial \phi}{\partial y}\frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$\frac{\partial \phi}{\partial x} = M$$
$$\frac{\partial \phi}{\partial y} = N$$

But since $\frac{\partial^2\phi}{\partial x\partial y} = \frac{\partial^2\phi}{\partial y\partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x, y)$ but at least we know now that we can do that since the condition $\frac{\partial^2\phi}{\partial x\partial y} = \frac{\partial^2\phi}{\partial y\partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(t, y)\,\mathrm{d}t + N(t, y)\,\mathrm{d}y = 0 \tag{1A}$$

Therefore

$$\begin{aligned}(\mathrm{e}^{2t})\,\mathrm{d}y &= (-2y\,\mathrm{e}^{2t} + t)\,\mathrm{d}t\\ (2y\,\mathrm{e}^{2t} - t)\,\mathrm{d}t + (\mathrm{e}^{2t})\,\mathrm{d}y &= 0 \end{aligned} \tag{2A}$$

Comparing (1A) and (2A) shows that

$$\begin{aligned}M(t, y) &= 2y\,\mathrm{e}^{2t} - t\\ N(t, y) &= \mathrm{e}^{2t}\end{aligned}$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial t}$$

Using result found above gives

$$\begin{aligned}\frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}(2y\,\mathrm{e}^{2t} - t)\\ &= 2\,\mathrm{e}^{2t}\end{aligned}$$

And

$$\begin{aligned}\frac{\partial N}{\partial t} &= \frac{\partial}{\partial t}(\mathrm{e}^{2t})\\ &= 2\,\mathrm{e}^{2t}\end{aligned}$$

Since $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial t}$, then the ODE is exact The following equations are now set up to solve for the function $\phi(t, y)$

$$\frac{\partial \phi}{\partial t} = M \tag{1}$$

$$\frac{\partial \phi}{\partial y} = N \tag{2}$$

Integrating (1) w.r.t. t gives

$$\int \frac{\partial \phi}{\partial t}\, \mathrm{d}t = \int M \, \mathrm{d}t$$
$$\int \frac{\partial \phi}{\partial t}\, \mathrm{d}t = \int 2y\, \mathrm{e}^{2t} - t \, \mathrm{d}t$$
$$\phi = -\frac{t^2}{2} + y\, \mathrm{e}^{2t} + f(y) \tag{3}$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both t and y. Taking derivative of equation (3) w.r.t y gives

$$\frac{\partial \phi}{\partial y} = \mathrm{e}^{2t} + f'(y) \tag{4}$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = \mathrm{e}^{2t}$. Therefore equation (4) becomes

$$\mathrm{e}^{2t} = \mathrm{e}^{2t} + f'(y) \tag{5}$$

Solving equation (5) for $f'(y)$ gives

$$f'(y) = 0$$

Therefore

$$f(y) = c_1$$

Where c_1 is constant of integration. Substituting this result for $f(y)$ into equation (3) gives ϕ

$$\phi = -\frac{t^2}{2} + y\, \mathrm{e}^{2t} + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = -\frac{t^2}{2} + y\, \mathrm{e}^{2t}$$

The solution becomes

$$y = \frac{\mathrm{e}^{-2t}(t^2 + 2c_1)}{2}$$

Initial conditions are used to solve for c_1. Substituting $t = 1$ and $y = 0$ in the above solution gives an equation to solve for the constant of integration.

$$0 = e^{-2}c_1 + \frac{e^{-2}}{2}$$

$$c_1 = -\frac{1}{2}$$

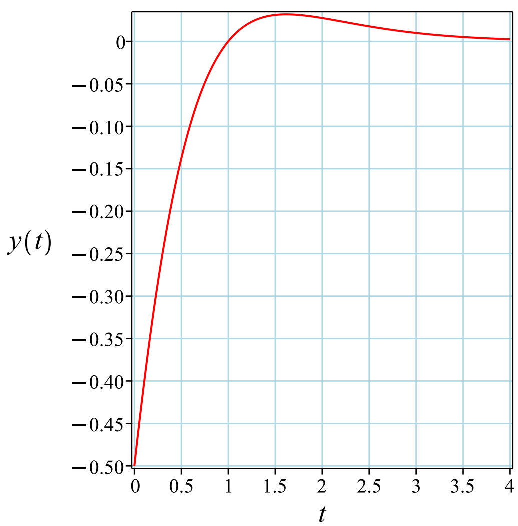
Substituting c_1 found above in the general solution gives

$$y = \frac{e^{-2t}t^2}{2} - \frac{e^{-2t}}{2}$$

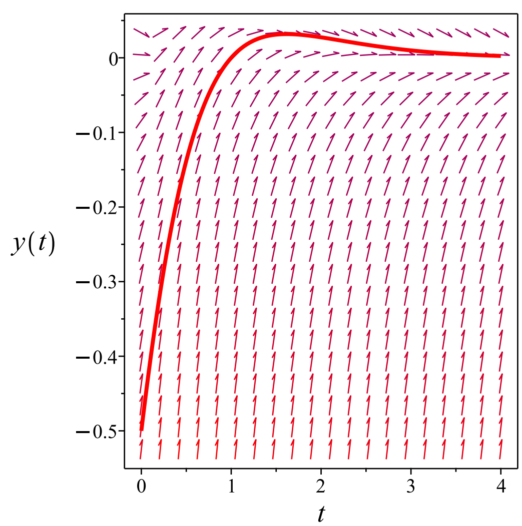
Summary

The solution(s) found are the following

$$y = \frac{e^{-2t}t^2}{2} - \frac{e^{-2t}}{2} \tag{1}$$



(a) Solution plot



(b) Slope field plot

Verification of solutions

$$y = \frac{e^{-2t}t^2}{2} - \frac{e^{-2t}}{2}$$

Verified OK.

1.14.5 Maple step by step solution

Let's solve

$\left[2y+y'=\frac{t}{e^{2t}}, y(1)=0\right]$

- Highest derivative means the order of the ODE is 1

  y'

- Isolate the derivative

  $y'=-2y+\frac{t}{e^{2t}}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE

  $2y+y'=\frac{t}{e^{2t}}$

- The ODE is linear; multiply by an integrating factor $\mu(t)$

  $\mu(t)\left(2y+y'\right)=\frac{\mu(t)t}{e^{2t}}$

- Assume the lhs of the ODE is the total derivative $\frac{d}{dt}(\mu(t)\,y)$

  $\mu(t)\left(2y+y'\right)=\mu'(t)\,y+\mu(t)\,y'$

- Isolate $\mu'(t)$

  $\mu'(t)=2\mu(t)$

- Solve to find the integrating factor

  $\mu(t)=\left(e^{2t}\right)^2 e^{-2t}$

- Integrate both sides with respect to t

  $\int\left(\frac{d}{dt}(\mu(t)\,y)\right)dt=\int\frac{\mu(t)t}{e^{2t}}dt+c_1$

- Evaluate the integral on the lhs

  $\mu(t)\,y=\int\frac{\mu(t)t}{e^{2t}}dt+c_1$

- Solve for y

  $y=\frac{\int\frac{\mu(t)t}{e^{2t}}dt+c_1}{\mu(t)}$

- Substitute $\mu(t)=\left(e^{2t}\right)^2 e^{-2t}$

  $y=\frac{\int t\,e^{-2t}e^{2t}dt+c_1}{\left(e^{2t}\right)^2 e^{-2t}}$

- Evaluate the integrals on the rhs

  $y=\frac{\frac{t^2}{2}+c_1}{\left(e^{2t}\right)^2 e^{-2t}}$

- Simplify

$$y = \frac{e^{-2t}(t^2+2c_1)}{2}$$

- Use initial condition $y(1) = 0$

$$0 = \frac{e^{-2}(2c_1+1)}{2}$$

- Solve for c_1

$$c_1 = -\frac{1}{2}$$

- Substitute $c_1 = -\frac{1}{2}$ into general solution and simplify

$$y = \frac{e^{-2t}(t^2-1)}{2}$$

- Solution to the IVP

$$y = \frac{e^{-2t}(t^2-1)}{2}$$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
<- 1st order linear successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 15

```
dsolve([2*y(t)+diff(y(t),t) = t/exp(2*t),y(1) = 0],y(t), singsol=all)
```

$$y(t) = \frac{(t^2-1)\,e^{-2t}}{2}$$

✓ Solution by Mathematica

Time used: 0.054 (sec). Leaf size: 19

```
DSolve[{2*y[t]+y'[t] == t/Exp[2*t],y[1]==0},y[t],t,IncludeSingularSolutions -> True]
```

$$y(t) \to \frac{1}{2}e^{-2t}(t^2-1)$$

1.15 problem 15

Internal problem ID [462]
Internal file name [`OUTPUT/462_Sunday_June_05_2022_01_41_51_AM_24139145/index.tex`]

Book: Elementary differential equations and boundary value problems, 10th ed., Boyce and DiPrima
Section: Section 2.1. Page 40
Problem number: 15.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"linear", "exactWithIntegrationFactor", "first_order_ode_lie_symmetry_lookup"**

Maple gives the following as the ode type

`[_linear]`

$$\boxed{2y + ty' = t^2 - t + 1}$$

With initial conditions

$$\left[y(1) = \frac{1}{2}\right]$$

1.15.1 Existence and uniqueness analysis

This is a linear ODE. In canonical form it is written as

$$y' + p(t)y = q(t)$$

Where here

$$p(t) = \frac{2}{t}$$
$$q(t) = \frac{t^2 - t + 1}{t}$$

Hence the ode is

$$y' + \frac{2y}{t} = \frac{t^2 - t + 1}{t}$$

The domain of $p(t) = \frac{2}{t}$ is

$$\{t < 0 \lor 0 < t\}$$

And the point $t_0 = 1$ is inside this domain. The domain of $q(t) = \frac{t^2-t+1}{t}$ is

$$\{t < 0 \lor 0 < t\}$$

And the point $t_0 = 1$ is also inside this domain. Hence solution exists and is unique.

1.15.2 Solving as linear ode

Entering Linear first order ODE solver. The integrating factor μ is

$$\begin{aligned}\mu &= e^{\int \frac{2}{t} dt} \\ &= t^2\end{aligned}$$

The ode becomes

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}(\mu y) &= (\mu)\left(\frac{t^2 - t + 1}{t}\right) \\ \frac{\mathrm{d}}{\mathrm{d}t}(t^2 y) &= (t^2)\left(\frac{t^2 - t + 1}{t}\right) \\ \mathrm{d}(t^2 y) &= \left((t^2 - t + 1)\, t\right)\, \mathrm{d}t\end{aligned}$$

Integrating gives

$$\begin{aligned}t^2 y &= \int (t^2 - t + 1)\, t \,\mathrm{d}t \\ t^2 y &= \frac{1}{4}t^4 - \frac{1}{3}t^3 + \frac{1}{2}t^2 + c_1\end{aligned}$$

Dividing both sides by the integrating factor $\mu = t^2$ results in

$$y = \frac{\frac{1}{4}t^4 - \frac{1}{3}t^3 + \frac{1}{2}t^2}{t^2} + \frac{c_1}{t^2}$$

which simplifies to

$$y = \frac{3t^4 - 4t^3 + 6t^2 + 12c_1}{12t^2}$$

Initial conditions are used to solve for c_1. Substituting $t = 1$ and $y = \frac{1}{2}$ in the above solution gives an equation to solve for the constant of integration.

$$\frac{1}{2} = c_1 + \frac{5}{12}$$

$$c_1 = \frac{1}{12}$$

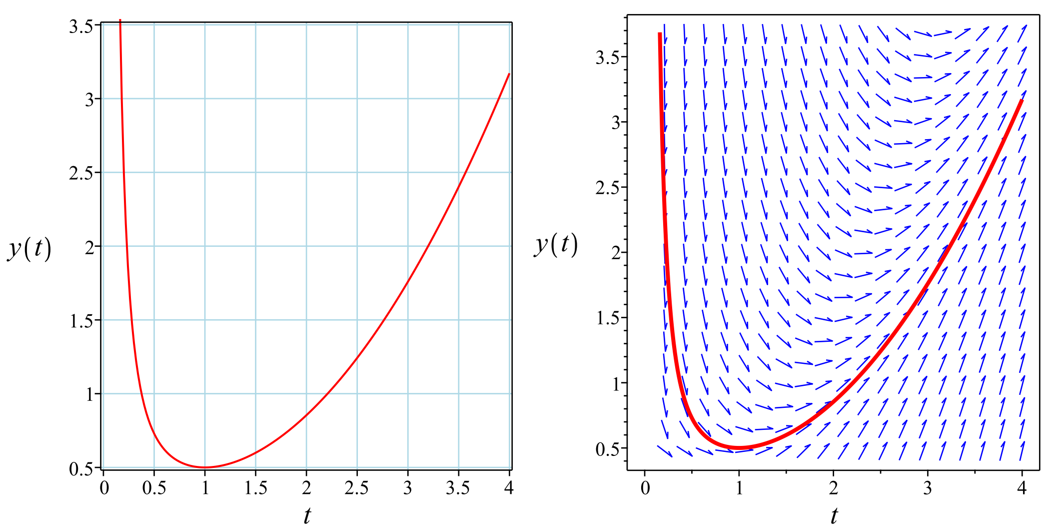
Substituting c_1 found above in the general solution gives

$$y = \frac{3t^4 - 4t^3 + 6t^2 + 1}{12t^2}$$

Summary

The solution(s) found are the following

$$y = \frac{3t^4 - 4t^3 + 6t^2 + 1}{12t^2} \tag{1}$$



(a) Solution plot

(b) Slope field plot

Verification of solutions

$$y = \frac{3t^4 - 4t^3 + 6t^2 + 1}{12t^2}$$

Verified OK.

1.15.3 Solving as first order ode lie symmetry lookup ode

Writing the ode as

$$
\begin{aligned}
y' &= -\frac{-t^2+t+2y-1}{t} \\
y' &= \omega(t,y)
\end{aligned}
$$

The condition of Lie symmetry is the linearized PDE given by

$$
\eta_t+\omega(\eta_y-\xi_t)-\omega^2\xi_y-\omega_t\xi-\omega_y\eta=0 \tag{A}
$$

The type of this ode is known. It is of type `linear`. Therefore we do not need to solve the PDE (A), and can just use the lookup table shown below to find ξ,η

Table 43: Lie symmetry infinitesimal lookup table for known first order ODE's

ODE class	Form	ξ	η
linear ode	$y'=f(x)y(x)+g(x)$	0	$e^{\int f dx}$
separable ode	$y'=f(x)\,g(y)$	$\frac{1}{f}$	0
quadrature ode	$y'=f(x)$	0	1
quadrature ode	$y'=g(y)$	1	0
homogeneous ODEs of Class A	$y'=f\left(\frac{y}{x}\right)$	x	y
homogeneous ODEs of Class C	$y'=(a+bx+cy)^{\frac{n}{m}}$	1	$-\frac{b}{c}$
homogeneous class D	$y'=\frac{y}{x}+g(x)\,F\left(\frac{y}{x}\right)$	x^2	xy
First order special form ID 1	$y'=g(x)\,e^{h(x)+by}+f(x)$	$\frac{e^{-\int bf(x)dx-h(x)}}{g(x)}$	$\frac{f(x)e^{-\int bf(x)dx-h(x)}}{g(x)}$
polynomial type ode	$y'=\frac{a_1x+b_1y+c_1}{a_2x+b_2y+c_2}$	$\frac{a_1b_2x-a_2b_1x-b_1c_2+b_2c_1}{a_1b_2-a_2b_1}$	$\frac{a_1b_2y-a_2b_1y-a_1c_2-a_2c_1}{a_1b_2-a_2b_1}$
Bernoulli ode	$y'=f(x)\,y+g(x)\,y^n$	0	$e^{-\int(n-1)f(x)dx}y^n$
Reduced Riccati	$y'=f_1(x)\,y+f_2(x)\,y^2$	0	$e^{-\int f_1 dx}$

The above table shows that

$$\begin{aligned}\xi(t,y) &= 0\\ \eta(t,y) &= \frac{1}{t^2}\end{aligned} \tag{A1}$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(t,y) \to (R,S)$ where (R,S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$\frac{dt}{\xi} = \frac{dy}{\eta} = dS \tag{1}$$

The above comes from the requirements that $\left(\xi\frac{\partial}{\partial t} + \eta\frac{\partial}{\partial y}\right) S(t,y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$R = t$$

S is found from

$$\begin{aligned}S &= \int \frac{1}{\eta} dy\\ &= \int \frac{1}{\frac{1}{t^2}} dy\end{aligned}$$

Which results in

$$S = t^2 y$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_t + \omega(t,y)S_y}{R_t + \omega(t,y)R_y} \tag{2}$$

Where in the above R_t, R_y, S_t, S_y are all partial derivatives and $\omega(t,y)$ is the right hand side of the original ode given by

$$\omega(t,y) = -\frac{-t^2 + t + 2y - 1}{t}$$

Evaluating all the partial derivatives gives

$$
\begin{aligned}
R_t &= 1 \\
R_y &= 0 \\
S_t &= 2ty \\
S_y &= t^2
\end{aligned}
$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = t^3 - t^2 + t \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for t, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = R^3 - R^2 + R$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$S(R) = \frac{1}{4}R^4 - \frac{1}{3}R^3 + \frac{1}{2}R^2 + c_1 \tag{4}$$

To complete the solution, we just need to transform (4) back to t, y coordinates. This results in

$$yt^2 = \frac{1}{4}t^4 - \frac{1}{3}t^3 + \frac{1}{2}t^2 + c_1$$

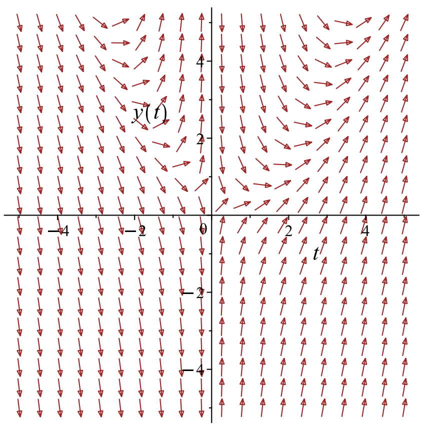
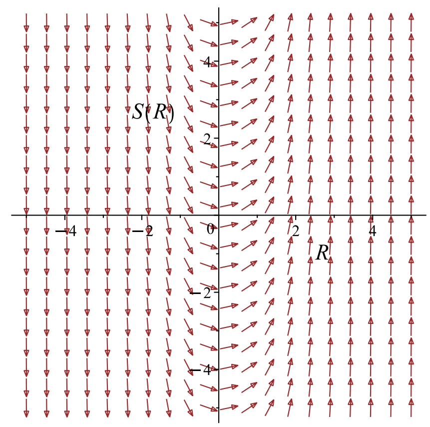
Which simplifies to

$$yt^2 = \frac{1}{4}t^4 - \frac{1}{3}t^3 + \frac{1}{2}t^2 + c_1$$

Which gives

$$y = \frac{3t^4 - 4t^3 + 6t^2 + 12c_1}{12t^2}$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in t, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dt} = -\frac{-t^2+t+2y-1}{t}$ 	$R = t$ $S = t^2 y$	$\frac{dS}{dR} = R^3 - R^2 + R$ 

Initial conditions are used to solve for c_1. Substituting $t = 1$ and $y = \frac{1}{2}$ in the above solution gives an equation to solve for the constant of integration.

$$\frac{1}{2} = c_1 + \frac{5}{12}$$

$$c_1 = \frac{1}{12}$$

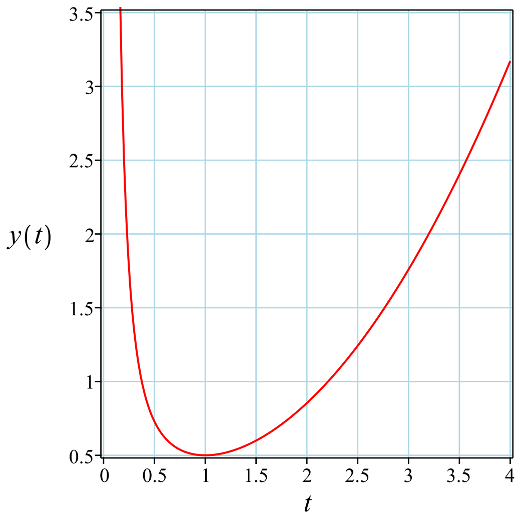
Substituting c_1 found above in the general solution gives

$$y = \frac{3t^4 - 4t^3 + 6t^2 + 1}{12t^2}$$

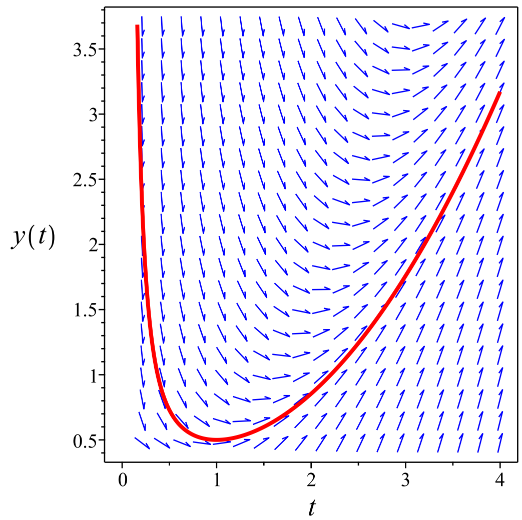
Summary

The solution(s) found are the following

$$y = \frac{3t^4 - 4t^3 + 6t^2 + 1}{12t^2} \tag{1}$$



(a) Solution plot



(b) Slope field plot

Verification of solutions

$$y = \frac{3t^4 - 4t^3 + 6t^2 + 1}{12t^2}$$

Verified OK.

1.15.4 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x,y) + N(x,y)\frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x,y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y) = 0$$

Hence

$$\frac{\partial\phi}{\partial x} + \frac{\partial\phi}{\partial y}\frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$\frac{\partial\phi}{\partial x} = M$$
$$\frac{\partial\phi}{\partial y} = N$$

But since $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x, y)$ but at least we know now that we can do that since the condition $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(t, y)\, \mathrm{d}t + N(t, y)\, \mathrm{d}y = 0 \tag{1A}$$

Therefore

$$\begin{aligned} (t)\, \mathrm{d}y &= \left(t^2 - t - 2y + 1\right) \mathrm{d}t \\ \left(-t^2 + t + 2y - 1\right) \mathrm{d}t + (t)\, \mathrm{d}y &= 0 \end{aligned} \tag{2A}$$

Comparing (1A) and (2A) shows that

$$\begin{aligned} M(t, y) &= -t^2 + t + 2y - 1 \\ N(t, y) &= t \end{aligned}$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial t}$$

Using result found above gives

$$\begin{aligned} \frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}\left(-t^2 + t + 2y - 1\right) \\ &= 2 \end{aligned}$$

And

$$\begin{aligned} \frac{\partial N}{\partial t} &= \frac{\partial}{\partial t}(t) \\ &= 1 \end{aligned}$$

Since $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial t}$, then the ODE is not exact. Since the ODE is not exact, we will try to find an integrating factor to make it exact. Let

$$\begin{aligned} A &= \frac{1}{N}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial t}\right) \\ &= \frac{1}{t}((2) - (1)) \\ &= \frac{1}{t} \end{aligned}$$

Since A does not depend on y, then it can be used to find an integrating factor. The integrating factor μ is

$$\begin{aligned}\mu &= e^{\int A \,\mathrm{d}t} \\ &= e^{\int \frac{1}{t} \,\mathrm{d}t}\end{aligned}$$

The result of integrating gives

$$\begin{aligned}\mu &= e^{\ln(t)} \\ &= t\end{aligned}$$

M and N are multiplied by this integrating factor, giving new M and new N which are called $\overline{M}$ and $\overline{N}$ for now so not to confuse them with the original M and N.

$$\begin{aligned}\overline{M} &= \mu M \\ &= t\left(-t^2 + t + 2y - 1\right) \\ &= -t\left(t^2 - t - 2y + 1\right)\end{aligned}$$

And

$$\begin{aligned}\overline{N} &= \mu N \\ &= t(t) \\ &= t^2\end{aligned}$$

Now a modified ODE is ontained from the original ODE, which is exact and can be solved. The modified ODE is

$$\begin{aligned}\overline{M} + \overline{N}\frac{\mathrm{d}y}{\mathrm{d}t} &= 0 \\ \left(-t\left(t^2 - t - 2y + 1\right)\right) + \left(t^2\right)\frac{\mathrm{d}y}{\mathrm{d}t} &= 0\end{aligned}$$

The following equations are now set up to solve for the function $\phi(t, y)$

$$\frac{\partial \phi}{\partial t} = \overline{M} \tag{1}$$

$$\frac{\partial \phi}{\partial y} = \overline{N} \tag{2}$$

Integrating (1) w.r.t. t gives

$$\begin{aligned}\int \frac{\partial \phi}{\partial t} \,\mathrm{d}t &= \int \overline{M} \,\mathrm{d}t \\ \int \frac{\partial \phi}{\partial t} \,\mathrm{d}t &= \int -t\left(t^2 - t - 2y + 1\right) \mathrm{d}t\end{aligned}$$

$$\phi = -\frac{t^2\left(t^2 - \frac{4}{3}t - 4y + 2\right)}{4} + f(y) \tag{3}$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both t and y. Taking derivative of equation (3) w.r.t y gives

$$\frac{\partial \phi}{\partial y} = t^2 + f'(y) \tag{4}$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = t^2$. Therefore equation (4) becomes

$$t^2 = t^2 + f'(y) \tag{5}$$

Solving equation (5) for $f'(y)$ gives

$$f'(y) = 0$$

Therefore

$$f(y) = c_1$$

Where c_1 is constant of integration. Substituting this result for $f(y)$ into equation (3) gives ϕ

$$\phi = -\frac{t^2\left(t^2 - \frac{4}{3}t - 4y + 2\right)}{4} + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = -\frac{t^2\left(t^2 - \frac{4}{3}t - 4y + 2\right)}{4}$$

The solution becomes

$$y = \frac{3t^4 - 4t^3 + 6t^2 + 12c_1}{12t^2}$$

Initial conditions are used to solve for c_1. Substituting $t = 1$ and $y = \frac{1}{2}$ in the above solution gives an equation to solve for the constant of integration.

$$\frac{1}{2} = c_1 + \frac{5}{12}$$

$$c_1 = \frac{1}{12}$$

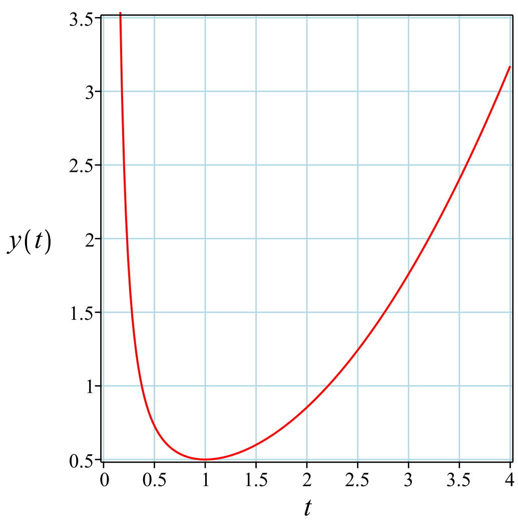
Substituting c_1 found above in the general solution gives

$$y = \frac{3t^4 - 4t^3 + 6t^2 + 1}{12t^2}$$

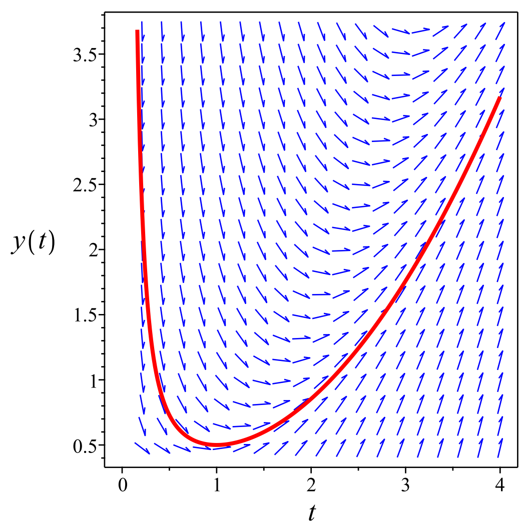
Summary

The solution(s) found are the following

$$y = \frac{3t^4 - 4t^3 + 6t^2 + 1}{12t^2} \tag{1}$$



(a) Solution plot



(b) Slope field plot

Verification of solutions

$$y = \frac{3t^4 - 4t^3 + 6t^2 + 1}{12t^2}$$

Verified OK.

1.15.5 Maple step by step solution

Let's solve

$\left[2y + ty' = t^2 - t + 1, y(1) = \frac{1}{2}\right]$

- Highest derivative means the order of the ODE is 1

  y'

- Isolate the derivative

$y' = -\frac{2y}{t} + \frac{t^2-t+1}{t}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE

  $y' + \frac{2y}{t} = \frac{t^2-t+1}{t}$

- The ODE is linear; multiply by an integrating factor $\mu(t)$

  $\mu(t) \left(y' + \frac{2y}{t}\right) = \frac{\mu(t)(t^2-t+1)}{t}$

- Assume the lhs of the ODE is the total derivative $\frac{d}{dt}(\mu(t)\, y)$

  $\mu(t) \left(y' + \frac{2y}{t}\right) = \mu'(t)\, y + \mu(t)\, y'$

- Isolate $\mu'(t)$

  $\mu'(t) = \frac{2\mu(t)}{t}$

- Solve to find the integrating factor

  $\mu(t) = t^2$

- Integrate both sides with respect to t

  $\int \left(\frac{d}{dt}(\mu(t)\, y)\right) dt = \int \frac{\mu(t)(t^2-t+1)}{t} dt + c_1$

- Evaluate the integral on the lhs

  $\mu(t)\, y = \int \frac{\mu(t)(t^2-t+1)}{t} dt + c_1$

- Solve for y

  $y = \frac{\int \frac{\mu(t)\left(t^2-t+1\right)}{t} dt + c_1}{\mu(t)}$

- Substitute $\mu(t) = t^2$

  $y = \frac{\int (t^2-t+1) t dt + c_1}{t^2}$

- Evaluate the integrals on the rhs

  $y = \frac{\frac{1}{4}t^4 - \frac{1}{3}t^3 + \frac{1}{2}t^2 + c_1}{t^2}$

- Simplify

  $y = \frac{3t^4 - 4t^3 + 6t^2 + 12c_1}{12t^2}$

- Use initial condition $y(1) = \frac{1}{2}$

  $\frac{1}{2} = c_1 + \frac{5}{12}$

- Solve for c_1

  $c_1 = \frac{1}{12}$

- Substitute $c_1 = \frac{1}{12}$ into general solution and simplify

$y = \frac{3t^4 - 4t^3 + 6t^2 + 1}{12t^2}$

- Solution to the IVP

  $y = \frac{3t^4 - 4t^3 + 6t^2 + 1}{12t^2}$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
<- 1st order linear successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 19

```
dsolve([2*y(t)+t*diff(y(t),t) = t^2-t+1,y(1) = 1/2],y(t), singsol=all)
```

$$y(t) = \frac{t^2}{4} - \frac{t}{3} + \frac{1}{2} + \frac{1}{12t^2}$$

✓ Solution by Mathematica

Time used: 0.028 (sec). Leaf size: 22

```
DSolve[{2*y[t]+t*y'[t] == t^2-t+1,y[1]==1/2},y[t],t,IncludeSingularSolutions -> True]
```

$$y(t) \to \frac{1}{12}\left(3t^2 + \frac{1}{t^2} - 4t + 6\right)$$

1.16 problem 16

Internal problem ID [463]
Internal file name [`OUTPUT/463_Sunday_June_05_2022_01_41_52_AM_63516619/index.tex`]

Book: Elementary differential equations and boundary value problems, 10th ed., Boyce and DiPrima
Section: Section 2.1. Page 40
Problem number: 16.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"exact", "linear", "first_order_ode_lie_symmetry_lookup"**

Maple gives the following as the ode type

`[_linear]`

$$\boxed{y' + \frac{2y}{t} = \frac{\cos(t)}{t^2}}$$

With initial conditions

$$[y(\pi) = 0]$$

1.16.1 Existence and uniqueness analysis

This is a linear ODE. In canonical form it is written as

$$y' + p(t)y = q(t)$$

Where here

$$p(t) = \frac{2}{t}$$
$$q(t) = \frac{\cos(t)}{t^2}$$

Hence the ode is

$$y' + \frac{2y}{t} = \frac{\cos(t)}{t^2}$$

The domain of $p(t) = \frac{2}{t}$ is

$$\{t < 0 \lor 0 < t\}$$

And the point $t_0 = \pi$ is inside this domain. The domain of $q(t) = \frac{\cos(t)}{t^2}$ is

$$\{t < 0 \lor 0 < t\}$$

And the point $t_0 = \pi$ is also inside this domain. Hence solution exists and is unique.

1.16.2 Solving as linear ode

Entering Linear first order ODE solver. The integrating factor μ is

$$\begin{aligned}\mu &= e^{\int \frac{2}{t} dt} \\ &= t^2\end{aligned}$$

The ode becomes

$$\begin{aligned}\frac{d}{dt}(\mu y) &= (\mu)\left(\frac{\cos(t)}{t^2}\right) \\ \frac{d}{dt}(t^2 y) &= (t^2)\left(\frac{\cos(t)}{t^2}\right) \\ d(t^2 y) &= \cos(t)\, dt\end{aligned}$$

Integrating gives

$$\begin{aligned}t^2 y &= \int \cos(t)\, dt \\ t^2 y &= \sin(t) + c_1\end{aligned}$$

Dividing both sides by the integrating factor $\mu = t^2$ results in

$$y = \frac{\sin(t)}{t^2} + \frac{c_1}{t^2}$$

which simplifies to

$$y = \frac{\sin(t) + c_1}{t^2}$$

Initial conditions are used to solve for c_1. Substituting $t = \pi$ and $y = 0$ in the above solution gives an equation to solve for the constant of integration.

$$0 = \frac{c_1}{\pi^2}$$

$$c_1 = 0$$

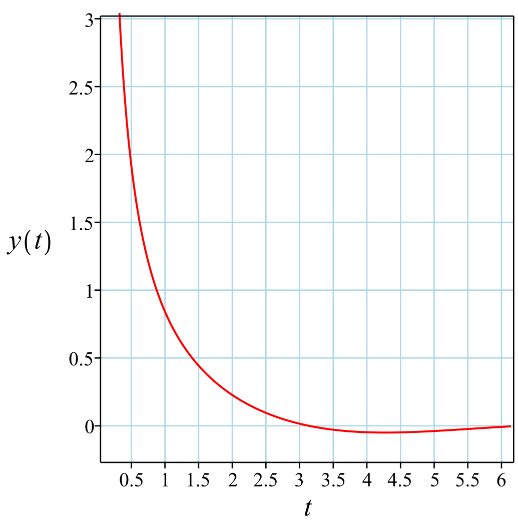
Substituting c_1 found above in the general solution gives

$$y = \frac{\sin(t)}{t^2}$$

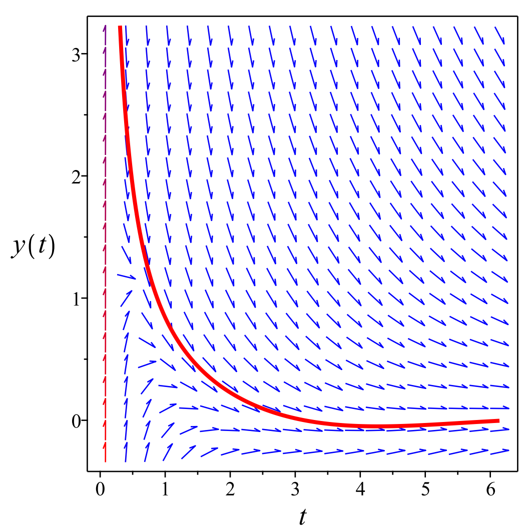
Summary

The solution(s) found are the following

$$y = \frac{\sin(t)}{t^2} \tag{1}$$



(a) Solution plot



(b) Slope field plot

Verification of solutions

$$y = \frac{\sin(t)}{t^2}$$

Verified OK.

1.16.3 Solving as first order ode lie symmetry lookup ode

Writing the ode as

$$
\begin{aligned}
y' &= \frac{-2ty + \cos(t)}{t^2} \\
y' &= \omega(t, y)
\end{aligned}
$$

The condition of Lie symmetry is the linearized PDE given by

$$
\eta_t + \omega(\eta_y - \xi_t) - \omega^2 \xi_y - \omega_t \xi - \omega_y \eta = 0 \tag{A}
$$

The type of this ode is known. It is of type `linear`. Therefore we do not need to solve the PDE (A), and can just use the lookup table shown below to find ξ, η

Table 46: Lie symmetry infinitesimal lookup table for known first order ODE's

ODE class	Form	ξ	η
linear ode	$y' = f(x)y(x) + g(x)$	0	$e^{\int f dx}$
separable ode	$y' = f(x)\, g(y)$	$\frac{1}{f}$	0
quadrature ode	$y' = f(x)$	0	1
quadrature ode	$y' = g(y)$	1	0
homogeneous ODEs of Class A	$y' = f\left(\frac{y}{x}\right)$	x	y
homogeneous ODEs of Class C	$y' = (a + bx + cy)^{\frac{n}{m}}$	1	$-\frac{b}{c}$
homogeneous class D	$y' = \frac{y}{x} + g(x)\, F\left(\frac{y}{x}\right)$	x^2	xy
First order special form ID 1	$y' = g(x)\, e^{h(x)+by} + f(x)$	$\frac{e^{-\int bf(x)dx - h(x)}}{g(x)}$	$\frac{f(x)e^{-\int bf(x)dx - h(x)}}{g(x)}$
polynomial type ode	$y' = \frac{a_1x+b_1y+c_1}{a_2x+b_2y+c_2}$	$\frac{a_1b_2x - a_2b_1x - b_1c_2 + b_2c_1}{a_1b_2 - a_2b_1}$	$\frac{a_1b_2y - a_2b_1y - a_1c_2 - a_2c_1}{a_1b_2 - a_2b_1}$
Bernoulli ode	$y' = f(x)\, y + g(x)\, y^n$	0	$e^{-\int (n-1)f(x)dx} y^n$
Reduced Riccati	$y' = f_1(x)\, y + f_2(x)\, y^2$	0	$e^{-\int f_1 dx}$

The above table shows that

$$\xi(t,y) = 0$$
$$\eta(t,y) = \frac{1}{t^2} \tag{A1}$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(t,y) \to (R,S)$ where (R,S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$\frac{dt}{\xi} = \frac{dy}{\eta} = dS \tag{1}$$

The above comes from the requirements that $\left(\xi\frac{\partial}{\partial t} + \eta\frac{\partial}{\partial y}\right) S(t,y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$R = t$$

S is found from

$$\begin{aligned} S &= \int \frac{1}{\eta} dy \\ &= \int \frac{1}{\frac{1}{t^2}} dy \end{aligned}$$

Which results in

$$S = t^2 y$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_t + \omega(t,y)S_y}{R_t + \omega(t,y)R_y} \tag{2}$$

Where in the above R_t, R_y, S_t, S_y are all partial derivatives and $\omega(t,y)$ is the right hand side of the original ode given by

$$\omega(t,y) = \frac{-2ty + \cos(t)}{t^2}$$

Evaluating all the partial derivatives gives

$$
\begin{aligned}
R_t &= 1 \\
R_y &= 0 \\
S_t &= 2ty \\
S_y &= t^2
\end{aligned}
$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = \cos(t) \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for t, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = \cos(R)$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$S(R) = \sin(R) + c_1 \tag{4}$$

To complete the solution, we just need to transform (4) back to t, y coordinates. This results in

$$yt^2 = \sin(t) + c_1$$

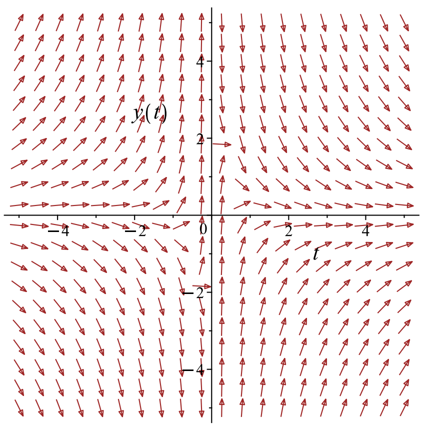
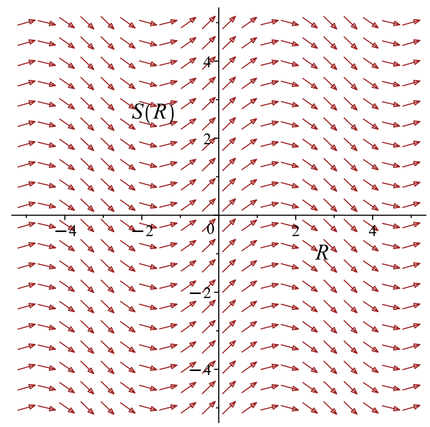
Which simplifies to

$$yt^2 = \sin(t) + c_1$$

Which gives

$$y = \frac{\sin(t) + c_1}{t^2}$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in t, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dt} = \frac{-2ty+\cos(t)}{t^2}$  	$R = t$ $S = t^2 y$	$\frac{dS}{dR} = \cos(R)$ 

Initial conditions are used to solve for c_1. Substituting $t = \pi$ and $y = 0$ in the above solution gives an equation to solve for the constant of integration.

$$0 = \frac{c_1}{\pi^2}$$

$$c_1 = 0$$

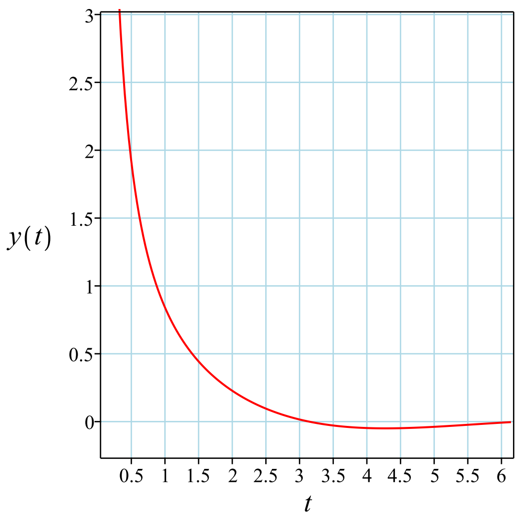
Substituting c_1 found above in the general solution gives

$$y = \frac{\sin(t)}{t^2}$$

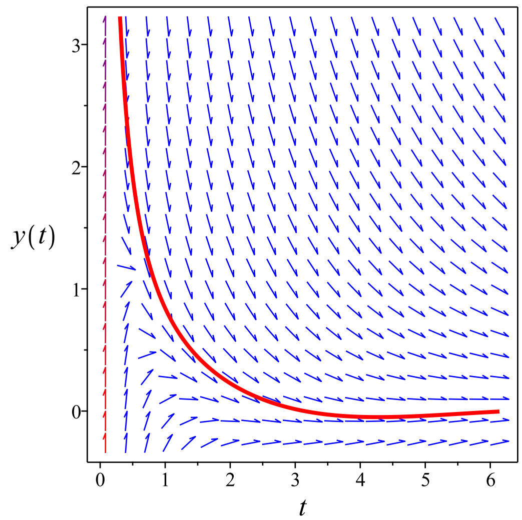
Summary

The solution(s) found are the following

$$y = \frac{\sin(t)}{t^2} \tag{1}$$



(a) Solution plot



(b) Slope field plot

Verification of solutions

$$y = \frac{\sin(t)}{t^2}$$

Verified OK.

1.16.4 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x, y) + N(x, y)\frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x, y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x, y) = 0$$

Hence

$$\frac{\partial \phi}{\partial x} + \frac{\partial \phi}{\partial y}\frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$\frac{\partial \phi}{\partial x} = M$$
$$\frac{\partial \phi}{\partial y} = N$$

But since $\frac{\partial^2\phi}{\partial x\partial y} = \frac{\partial^2\phi}{\partial y\partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x,y)$ but at least we know now that we can do that since the condition $\frac{\partial^2\phi}{\partial x\partial y} = \frac{\partial^2\phi}{\partial y\partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(t,y)\,\mathrm{d}t + N(t,y)\,\mathrm{d}y = 0 \tag{1A}$$

Therefore

$$\begin{aligned}(t^2)\,\mathrm{d}y &= (-2ty + \cos(t))\,\mathrm{d}t\\ (2ty - \cos(t))\,\mathrm{d}t + (t^2)\,\mathrm{d}y &= 0 \qquad (2A)\end{aligned}$$

Comparing (1A) and (2A) shows that

$$\begin{aligned}M(t,y) &= 2ty - \cos(t)\\ N(t,y) &= t^2\end{aligned}$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial t}$$

Using result found above gives

$$\begin{aligned}\frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}(2ty - \cos(t))\\ &= 2t\end{aligned}$$

And

$$\begin{aligned}\frac{\partial N}{\partial t} &= \frac{\partial}{\partial t}(t^2)\\ &= 2t\end{aligned}$$

Since $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial t}$, then the ODE is exact The following equations are now set up to solve for the function $\phi(t,y)$

$$\frac{\partial \phi}{\partial t} = M \tag{1}$$

$$\frac{\partial \phi}{\partial y} = N \tag{2}$$

Integrating (1) w.r.t. t gives

$$\begin{aligned}
\int \frac{\partial \phi}{\partial t} \,\mathrm{d}t &= \int M \,\mathrm{d}t \\
\int \frac{\partial \phi}{\partial t} \,\mathrm{d}t &= \int 2ty - \cos(t) \,\mathrm{d}t \\
\phi &= t^2 y - \sin(t) + f(y) \qquad (3)
\end{aligned}$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both t and y. Taking derivative of equation (3) w.r.t y gives

$$\frac{\partial \phi}{\partial y} = t^2 + f'(y) \qquad (4)$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = t^2$. Therefore equation (4) becomes

$$t^2 = t^2 + f'(y) \qquad (5)$$

Solving equation (5) for $f'(y)$ gives

$$f'(y) = 0$$

Therefore

$$f(y) = c_1$$

Where c_1 is constant of integration. Substituting this result for $f(y)$ into equation (3) gives ϕ

$$\phi = t^2 y - \sin(t) + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = t^2 y - \sin(t)$$

The solution becomes

$$y = \frac{\sin(t) + c_1}{t^2}$$

Initial conditions are used to solve for c_1. Substituting $t = \pi$ and $y = 0$ in the above solution gives an equation to solve for the constant of integration.

$$0 = \frac{c_1}{\pi^2}$$

$$c_1 = 0$$

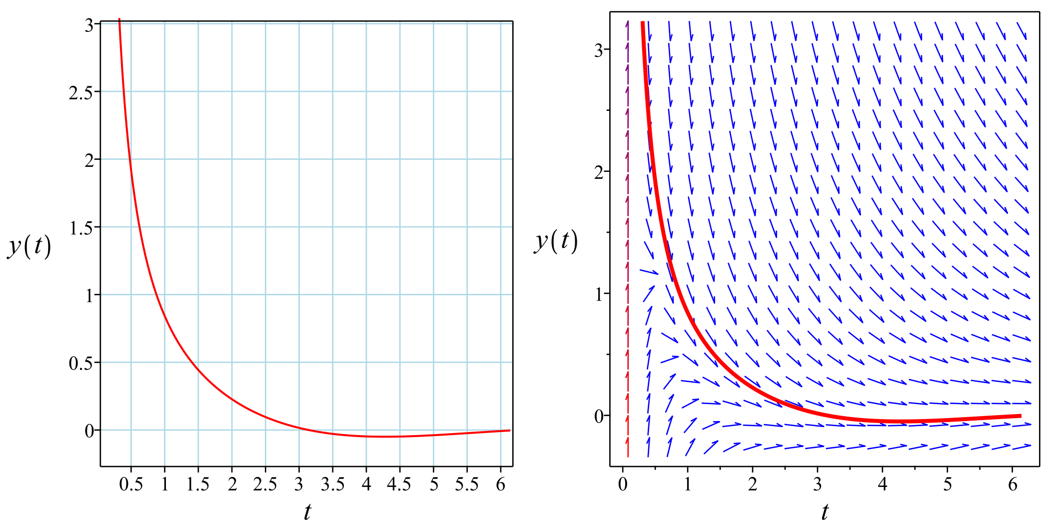
Substituting c_1 found above in the general solution gives

$$y = \frac{\sin(t)}{t^2}$$

Summary

The solution(s) found are the following

$$y = \frac{\sin(t)}{t^2} \tag{1}$$



(a) Solution plot (b) Slope field plot

Verification of solutions

$$y = \frac{\sin(t)}{t^2}$$

Verified OK.

1.16.5 Maple step by step solution

Let's solve

$\left[y' + \frac{2y}{t} = \frac{\cos(t)}{t^2}, y(\pi) = 0\right]$

- Highest derivative means the order of the ODE is 1

  y'

- Isolate the derivative

  $y' = -\frac{2y}{t} + \frac{\cos(t)}{t^2}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE

  $y' + \frac{2y}{t} = \frac{\cos(t)}{t^2}$

- The ODE is linear; multiply by an integrating factor $\mu(t)$

  $\mu(t)\left(y' + \frac{2y}{t}\right) = \frac{\mu(t)\cos(t)}{t^2}$

- Assume the lhs of the ODE is the total derivative $\frac{d}{dt}(\mu(t)\, y)$

  $\mu(t)\left(y' + \frac{2y}{t}\right) = \mu'(t)\, y + \mu(t)\, y'$

- Isolate $\mu'(t)$

  $\mu'(t) = \frac{2\mu(t)}{t}$

- Solve to find the integrating factor

  $\mu(t) = t^2$

- Integrate both sides with respect to t

  $\int \left(\frac{d}{dt}(\mu(t)\, y)\right) dt = \int \frac{\mu(t)\cos(t)}{t^2} dt + c_1$

- Evaluate the integral on the lhs

  $\mu(t)\, y = \int \frac{\mu(t)\cos(t)}{t^2} dt + c_1$

- Solve for y

  $y = \frac{\int \frac{\mu(t)\cos(t)}{t^2} dt + c_1}{\mu(t)}$

- Substitute $\mu(t) = t^2$

  $y = \frac{\int \cos(t) dt + c_1}{t^2}$

- Evaluate the integrals on the rhs

  $y = \frac{\sin(t) + c_1}{t^2}$

- Use initial condition $y(\pi) = 0$

$0 = \frac{c_1}{\pi^2}$

- Solve for c_1

  $c_1 = 0$

- Substitute $c_1 = 0$ into general solution and simplify

  $y = \frac{\sin(t)}{t^2}$

- Solution to the IVP

  $y = \frac{\sin(t)}{t^2}$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
<- 1st order linear successful`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 10

```
dsolve([2*y(t)/t+diff(y(t),t) = cos(t)/t^2,y(Pi) = 0],y(t), singsol=all)
```

$$y(t) = \frac{\sin(t)}{t^2}$$

✓ Solution by Mathematica

Time used: 0.041 (sec). Leaf size: 11

```
DSolve[{2*y[t]/t+y'[t] == Cos[t]/t^2,y[Pi]==0},y[t],t,IncludeSingularSolutions -> True]
```

$$y(t) \to \frac{\sin(t)}{t^2}$$

1.17 problem 17

Internal problem ID [464]
Internal file name [`OUTPUT/464_Sunday_June_05_2022_01_41_53_AM_28340272/index.tex`]

Book: Elementary differential equations and boundary value problems, 10th ed., Boyce and DiPrima
Section: Section 2.1. Page 40
Problem number: 17.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"linear", "exactWithIntegrationFactor", "first_order_ode_lie_symmetry_lookup"**

Maple gives the following as the ode type

```
[[_linear, `class A`]]
```

$$\boxed{-2y + y' = e^{2t}}$$

With initial conditions

$$[y(0) = 2]$$

1.17.1 Existence and uniqueness analysis

This is a linear ODE. In canonical form it is written as

$$y' + p(t)y = q(t)$$

Where here

$$p(t) = -2$$
$$q(t) = e^{2t}$$

Hence the ode is

$$-2y + y' = e^{2t}$$

The domain of $p(t) = -2$ is

$$\{-\infty < t < \infty\}$$

And the point $t_0 = 0$ is inside this domain. The domain of $q(t) = e^{2t}$ is

$$\{-\infty < t < \infty\}$$

And the point $t_0 = 0$ is also inside this domain. Hence solution exists and is unique.

1.17.2 Solving as linear ode

Entering Linear first order ODE solver. The integrating factor μ is

$$\begin{aligned}\mu &= e^{\int (-2)dt} \\ &= e^{-2t}\end{aligned}$$

The ode becomes

$$\begin{aligned}\frac{d}{dt}(\mu y) &= (\mu)\left(e^{2t}\right) \\ \frac{d}{dt}\left(e^{-2t}y\right) &= \left(e^{-2t}\right)\left(e^{2t}\right) \\ d\left(e^{-2t}y\right) &= dt\end{aligned}$$

Integrating gives

$$\begin{aligned}e^{-2t}y &= \int dt \\ e^{-2t}y &= t + c_1\end{aligned}$$

Dividing both sides by the integrating factor $\mu = e^{-2t}$ results in

$$y = e^{2t}t + c_1 e^{2t}$$

which simplifies to

$$y = e^{2t}(t + c_1)$$

Initial conditions are used to solve for c_1. Substituting $t = 0$ and $y = 2$ in the above solution gives an equation to solve for the constant of integration.

$$2 = c_1$$

$$c_1 = 2$$

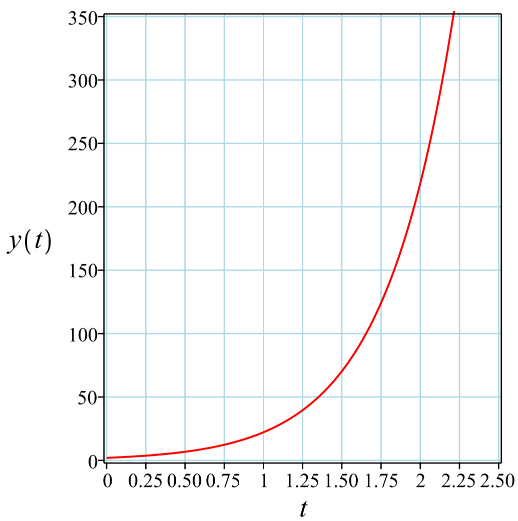
Substituting c_1 found above in the general solution gives

$$y = e^{2t}(2 + t)$$

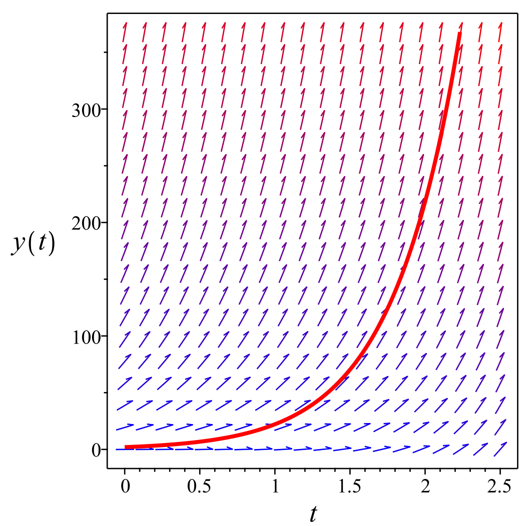
Summary

The solution(s) found are the following

$$y = e^{2t}(2 + t) \tag{1}$$



(a) Solution plot



(b) Slope field plot

Verification of solutions

$$y = e^{2t}(2 + t)$$

Verified OK.

1.17.3 Solving as first order ode lie symmetry lookup ode

Writing the ode as

$$
\begin{aligned}
y' &= 2y + e^{2t} \\
y' &= \omega(t, y)
\end{aligned}
$$

The condition of Lie symmetry is the linearized PDE given by

$$
\eta_t + \omega(\eta_y - \xi_t) - \omega^2 \xi_y - \omega_t \xi - \omega_y \eta = 0 \tag{A}
$$

The type of this ode is known. It is of type `linear`. Therefore we do not need to solve the PDE (A), and can just use the lookup table shown below to find ξ, η

Table 49: Lie symmetry infinitesimal lookup table for known first order ODE's

ODE class	Form	ξ	η
linear ode	$y' = f(x)y(x) + g(x)$	0	$e^{\int f dx}$
separable ode	$y' = f(x)\, g(y)$	$\frac{1}{f}$	0
quadrature ode	$y' = f(x)$	0	1
quadrature ode	$y' = g(y)$	1	0
homogeneous ODEs of Class A	$y' = f\left(\frac{y}{x}\right)$	x	y
homogeneous ODEs of Class C	$y' = (a + bx + cy)^{\frac{n}{m}}$	1	$-\frac{b}{c}$
homogeneous class D	$y' = \frac{y}{x} + g(x)\, F\left(\frac{y}{x}\right)$	x^2	xy
First order special form ID 1	$y' = g(x)\, e^{h(x)+by} + f(x)$	$\frac{e^{-\int bf(x)dx-h(x)}}{g(x)}$	$\frac{f(x)e^{-\int bf(x)dx-h(x)}}{g(x)}$
polynomial type ode	$y' = \frac{a_1x+b_1y+c_1}{a_2x+b_2y+c_2}$	$\frac{a_1b_2x-a_2b_1x-b_1c_2+b_2c_1}{a_1b_2-a_2b_1}$	$\frac{a_1b_2y-a_2b_1y-a_1c_2-a_2c_1}{a_1b_2-a_2b_1}$
Bernoulli ode	$y' = f(x)\, y + g(x)\, y^n$	0	$e^{-\int (n-1)f(x)dx} y^n$
Reduced Riccati	$y' = f_1(x)\, y + f_2(x)\, y^2$	0	$e^{-\int f_1 dx}$

The above table shows that

$$\begin{aligned} \xi(t,y) &= 0 \\ \eta(t,y) &= e^{2t} \end{aligned} \tag{A1}$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(t,y) \to (R,S)$ where (R,S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$\frac{dt}{\xi} = \frac{dy}{\eta} = dS \tag{1}$$

The above comes from the requirements that $\left(\xi\frac{\partial}{\partial t} + \eta\frac{\partial}{\partial y}\right) S(t,y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$R = t$$

S is found from

$$\begin{aligned} S &= \int \frac{1}{\eta} dy \\ &= \int \frac{1}{e^{2t}} dy \end{aligned}$$

Which results in

$$S = e^{-2t} y$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_t + \omega(t,y)S_y}{R_t + \omega(t,y)R_y} \tag{2}$$

Where in the above R_t, R_y, S_t, S_y are all partial derivatives and $\omega(t,y)$ is the right hand side of the original ode given by

$$\omega(t,y) = 2y + e^{2t}$$

Evaluating all the partial derivatives gives

$$\begin{aligned} R_t &= 1 \\ R_y &= 0 \\ S_t &= -2\,e^{-2t} y \\ S_y &= e^{-2t} \end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = 1 \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for t, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = 1$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$S(R) = R + c_1 \tag{4}$$

To complete the solution, we just need to transform (4) back to t, y coordinates. This results in

$$e^{-2t}y = t + c_1$$

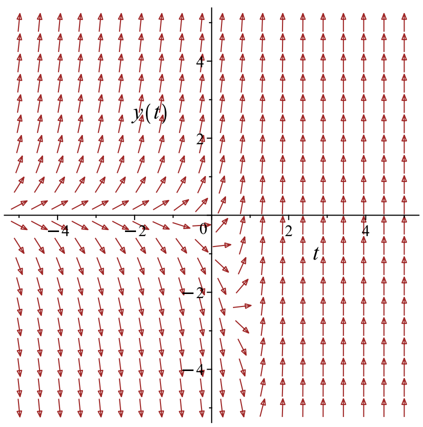
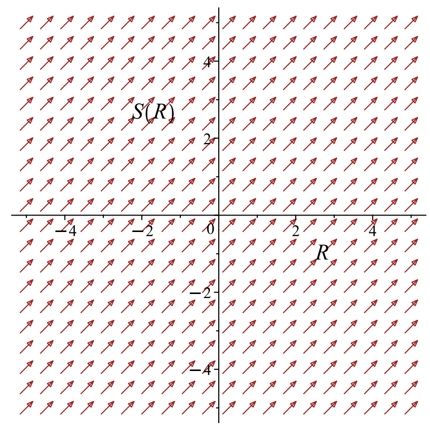
Which simplifies to

$$e^{-2t}y = t + c_1$$

Which gives

$$y = e^{2t}(t + c_1)$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in t, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dt} = 2y + e^{2t}$ 	$R = t$ $S = e^{-2t}y$	$\frac{dS}{dR} = 1$ 

Initial conditions are used to solve for c_1. Substituting $t = 0$ and $y = 2$ in the above solution gives an equation to solve for the constant of integration.

$$2 = c_1$$

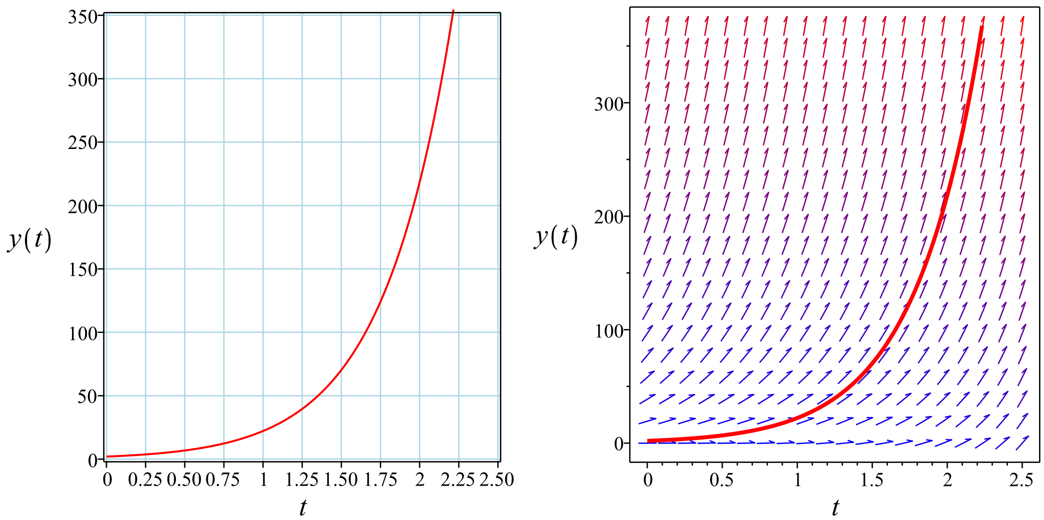
$$c_1 = 2$$

Substituting c_1 found above in the general solution gives

$$y = e^{2t}t + 2e^{2t}$$

<u>Summary</u>
The solution(s) found are the following

$$y = e^{2t}t + 2e^{2t} \tag{1}$$



(a) Solution plot (b) Slope field plot

Verification of solutions

$$y = e^{2t}t + 2\,e^{2t}$$

Verified OK.

1.17.4 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x,y) + N(x,y)\frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x,y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y) = 0$$

Hence

$$\frac{\partial \phi}{\partial x} + \frac{\partial \phi}{\partial y}\frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$\frac{\partial \phi}{\partial x} = M$$
$$\frac{\partial \phi}{\partial y} = N$$

But since $\frac{\partial^2\phi}{\partial x\partial y} = \frac{\partial^2\phi}{\partial y\partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x, y)$ but at least we know now that we can do that since the condition $\frac{\partial^2\phi}{\partial x\partial y} = \frac{\partial^2\phi}{\partial y\partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(t, y)\,\mathrm{d}t + N(t, y)\,\mathrm{d}y = 0 \tag{1A}$$

Therefore

$$\begin{aligned} \mathrm{d}y &= \left(2y + \mathrm{e}^{2t}\right)\mathrm{d}t \\ \left(-2y - \mathrm{e}^{2t}\right)\mathrm{d}t + \mathrm{d}y &= 0 \end{aligned} \tag{2A}$$

Comparing (1A) and (2A) shows that

$$\begin{aligned} M(t, y) &= -2y - \mathrm{e}^{2t} \\ N(t, y) &= 1 \end{aligned}$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial t}$$

Using result found above gives

$$\begin{aligned} \frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}\left(-2y - \mathrm{e}^{2t}\right) \\ &= -2 \end{aligned}$$

And

$$\begin{aligned} \frac{\partial N}{\partial t} &= \frac{\partial}{\partial t}(1) \\ &= 0 \end{aligned}$$

Since $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial t}$, then the ODE is <u>not exact</u>. Since the ODE is not exact, we will try to find an integrating factor to make it exact. Let

$$\begin{aligned} A &= \frac{1}{N}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial t}\right) \\ &= 1((-2) - (0)) \\ &= -2 \end{aligned}$$

Since A does not depend on y, then it can be used to find an integrating factor. The integrating factor μ is

$$\begin{aligned}\mu &= e^{\int A \,\mathrm{d}t} \\ &= e^{\int -2 \,\mathrm{d}t}\end{aligned}$$

The result of integrating gives

$$\begin{aligned}\mu &= e^{-2t} \\ &= \mathrm{e}^{-2t}\end{aligned}$$

M and N are multiplied by this integrating factor, giving new M and new N which are called $\overline{M}$ and $\overline{N}$ for now so not to confuse them with the original M and N.

$$\begin{aligned}\overline{M} &= \mu M \\ &= \mathrm{e}^{-2t}\left(-2y - \mathrm{e}^{2t}\right) \\ &= -2\,\mathrm{e}^{-2t} y - 1\end{aligned}$$

And

$$\begin{aligned}\overline{N} &= \mu N \\ &= \mathrm{e}^{-2t}(1) \\ &= \mathrm{e}^{-2t}\end{aligned}$$

Now a modified ODE is ontained from the original ODE, which is exact and can be solved. The modified ODE is

$$\begin{aligned}\overline{M} + \overline{N}\frac{\mathrm{d}y}{\mathrm{d}t} &= 0 \\ \left(-2\,\mathrm{e}^{-2t} y - 1\right) + \left(\mathrm{e}^{-2t}\right)\frac{\mathrm{d}y}{\mathrm{d}t} &= 0\end{aligned}$$

The following equations are now set up to solve for the function $\phi(t, y)$

$$\frac{\partial \phi}{\partial t} = \overline{M} \tag{1}$$

$$\frac{\partial \phi}{\partial y} = \overline{N} \tag{2}$$

Integrating (1) w.r.t. t gives

$$\begin{aligned}\int \frac{\partial \phi}{\partial t} \,\mathrm{d}t &= \int \overline{M} \,\mathrm{d}t \\ \int \frac{\partial \phi}{\partial t} \,\mathrm{d}t &= \int -2\,\mathrm{e}^{-2t} y - 1 \,\mathrm{d}t\end{aligned}$$

$$\phi = -t + \mathrm{e}^{-2t} y + f(y) \tag{3}$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both t and y. Taking derivative of equation (3) w.r.t y gives

$$\frac{\partial \phi}{\partial y} = e^{-2t} + f'(y) \tag{4}$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = e^{-2t}$. Therefore equation (4) becomes

$$e^{-2t} = e^{-2t} + f'(y) \tag{5}$$

Solving equation (5) for $f'(y)$ gives

$$f'(y) = 0$$

Therefore

$$f(y) = c_1$$

Where c_1 is constant of integration. Substituting this result for $f(y)$ into equation (3) gives ϕ

$$\phi = -t + e^{-2t}y + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = -t + e^{-2t}y$$

The solution becomes

$$y = e^{2t}(t + c_1)$$

Initial conditions are used to solve for c_1. Substituting $t = 0$ and $y = 2$ in the above solution gives an equation to solve for the constant of integration.

$$2 = c_1$$

$$c_1 = 2$$

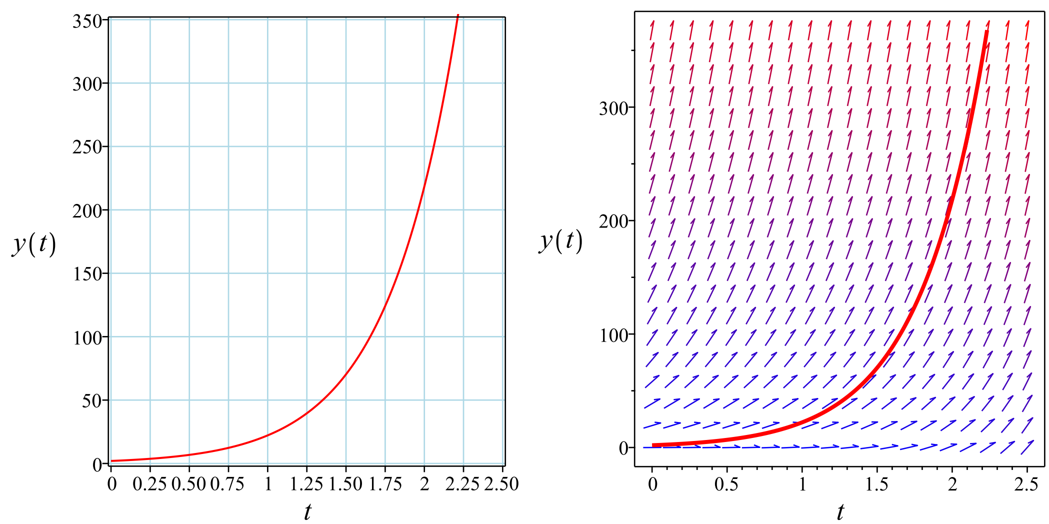
Substituting c_1 found above in the general solution gives

$$y = e^{2t}t + 2\,e^{2t}$$

Summary

The solution(s) found are the following

$$y = e^{2t}t + 2\,e^{2t} \tag{1}$$



(a) Solution plot

(b) Slope field plot

Verification of solutions

$$y = e^{2t}t + 2\,e^{2t}$$

Verified OK.

1.17.5 Maple step by step solution

Let's solve

$[-2y + y' = e^{2t}, y(0) = 2]$

- Highest derivative means the order of the ODE is 1

  y'

- Isolate the derivative

  $y' = 2y + e^{2t}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE

  $-2y + y' = e^{2t}$

- The ODE is linear; multiply by an integrating factor $\mu(t)$

$\mu(t)\left(-2y+y'\right)=\mu(t)\,\mathrm{e}^{2t}$

- Assume the lhs of the ODE is the total derivative $\frac{d}{dt}\left(\mu(t)\,y\right)$

$\mu(t)\left(-2y+y'\right)=\mu'(t)\,y+\mu(t)\,y'$

- Isolate $\mu'(t)$

$\mu'(t)=-2\mu(t)$

- Solve to find the integrating factor

$\mu(t)=\mathrm{e}^{-2t}$

- Integrate both sides with respect to t

$\int\left(\frac{d}{dt}\left(\mu(t)\,y\right)\right)dt=\int\mu(t)\,\mathrm{e}^{2t}dt+c_1$

- Evaluate the integral on the lhs

$\mu(t)\,y=\int\mu(t)\,\mathrm{e}^{2t}dt+c_1$

- Solve for y

$y=\frac{\int\mu(t)\mathrm{e}^{2t}dt+c_1}{\mu(t)}$

- Substitute $\mu(t)=\mathrm{e}^{-2t}$

$y=\frac{\int\mathrm{e}^{-2t}\mathrm{e}^{2t}dt+c_1}{\mathrm{e}^{-2t}}$

- Evaluate the integrals on the rhs

$y=\frac{t+c_1}{\mathrm{e}^{-2t}}$

- Simplify

$y=\mathrm{e}^{2t}(t+c_1)$

- Use initial condition $y(0)=2$

$2=c_1$

- Solve for c_1

$c_1=2$

- Substitute $c_1=2$ into general solution and simplify

$y=\mathrm{e}^{2t}(2+t)$

- Solution to the IVP

$y=\mathrm{e}^{2t}(2+t)$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
<- 1st order linear successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 12

```
dsolve([-2*y(t)+diff(y(t),t) = exp(2*t),y(0) = 2],y(t), singsol=all)
```

$$y(t) = (2+t)\,e^{2t}$$

✓ Solution by Mathematica

Time used: 0.039 (sec). Leaf size: 14

```
DSolve[{-2*y[t]+y'[t] == Exp[2*t],y[0]==2},y[t],t,IncludeSingularSolutions -> True]
```

$$y(t) \to e^{2t}(t+2)$$

1.18 problem 18

Internal problem ID [465]
Internal file name [`OUTPUT/465_Sunday_June_05_2022_01_41_54_AM_98318111/index.tex`]

Book: Elementary differential equations and boundary value problems, 10th ed., Boyce and DiPrima
Section: Section 2.1. Page 40
Problem number: 18.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"linear", "exactWithIntegrationFactor", "first_order_ode_lie_symmetry_lookup"**

Maple gives the following as the ode type

`[_linear]`

$$\boxed{2y + ty' = \sin(t)}$$

With initial conditions

$$\left[y\left(\frac{\pi}{2}\right) = 1\right]$$

1.18.1 Existence and uniqueness analysis

This is a linear ODE. In canonical form it is written as

$$y' + p(t)y = q(t)$$

Where here

$$p(t) = \frac{2}{t}$$
$$q(t) = \frac{\sin(t)}{t}$$

Hence the ode is

$$y' + \frac{2y}{t} = \frac{\sin(t)}{t}$$

The domain of $p(t) = \frac{2}{t}$ is

$$\{t < 0 \lor 0 < t\}$$

And the point $t_0 = \frac{\pi}{2}$ is inside this domain. The domain of $q(t) = \frac{\sin(t)}{t}$ is

$$\{t < 0 \lor 0 < t\}$$

And the point $t_0 = \frac{\pi}{2}$ is also inside this domain. Hence solution exists and is unique.

1.18.2 Solving as linear ode

Entering Linear first order ODE solver. The integrating factor μ is

$$\begin{aligned}\mu &= e^{\int \frac{2}{t} dt} \\ &= t^2\end{aligned}$$

The ode becomes

$$\begin{aligned}\frac{d}{dt}(\mu y) &= (\mu)\left(\frac{\sin(t)}{t}\right) \\ \frac{d}{dt}(t^2 y) &= (t^2)\left(\frac{\sin(t)}{t}\right) \\ d(t^2 y) &= (t\sin(t))\ dt\end{aligned}$$

Integrating gives

$$\begin{aligned}t^2 y &= \int t\sin(t)\ dt \\ t^2 y &= -t\cos(t) + \sin(t) + c_1\end{aligned}$$

Dividing both sides by the integrating factor $\mu = t^2$ results in

$$y = \frac{-t\cos(t) + \sin(t)}{t^2} + \frac{c_1}{t^2}$$

which simplifies to

$$y = \frac{-t\cos(t) + \sin(t) + c_1}{t^2}$$

Initial conditions are used to solve for c_1. Substituting $t = \frac{\pi}{2}$ and $y = 1$ in the above solution gives an equation to solve for the constant of integration.

$$1 = \frac{4 + 4c_1}{\pi^2}$$

$$c_1 = \frac{\pi^2}{4} - 1$$

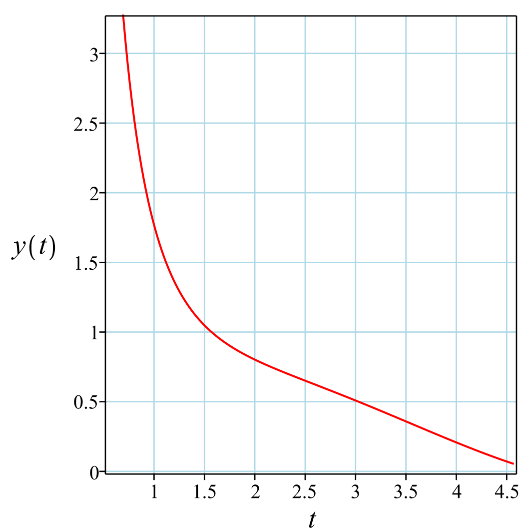
Substituting c_1 found above in the general solution gives

$$y = \frac{-4t\cos(t) + \pi^2 + 4\sin(t) - 4}{4t^2}$$

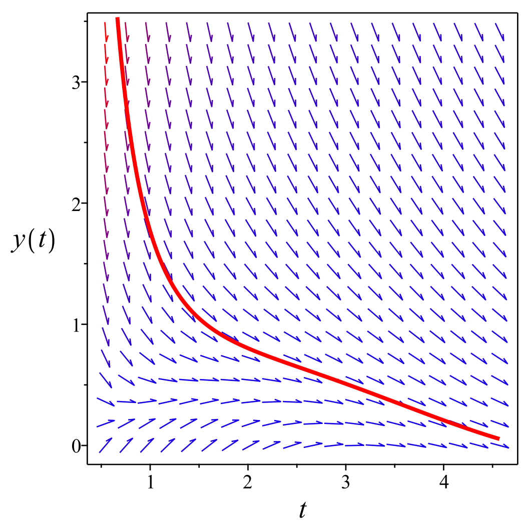
Summary

The solution(s) found are the following

$$y = \frac{-4t\cos(t) + \pi^2 + 4\sin(t) - 4}{4t^2} \tag{1}$$



(a) Solution plot



(b) Slope field plot

Verification of solutions

$$y = \frac{-4t\cos(t) + \pi^2 + 4\sin(t) - 4}{4t^2}$$

Verified OK.

1.18.3 Solving as first order ode lie symmetry lookup ode

Writing the ode as

$$y' = -\frac{2y - \sin(t)}{t}$$
$$y' = \omega(t, y)$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_t + \omega(\eta_y - \xi_t) - \omega^2 \xi_y - \omega_t \xi - \omega_y \eta = 0 \tag{A}$$

The type of this ode is known. It is of type `linear`. Therefore we do not need to solve the PDE (A), and can just use the lookup table shown below to find ξ, η

Table 52: Lie symmetry infinitesimal lookup table for known first order ODE's

ODE class	Form	ξ	η
linear ode	$y' = f(x)y(x) + g(x)$	0	$e^{\int f dx}$
separable ode	$y' = f(x)\, g(y)$	$\frac{1}{f}$	0
quadrature ode	$y' = f(x)$	0	1
quadrature ode	$y' = g(y)$	1	0
homogeneous ODEs of Class A	$y' = f\left(\frac{y}{x}\right)$	x	y
homogeneous ODEs of Class C	$y' = (a + bx + cy)^{\frac{n}{m}}$	1	$-\frac{b}{c}$
homogeneous class D	$y' = \frac{y}{x} + g(x)\, F\left(\frac{y}{x}\right)$	x^2	xy
First order special form ID 1	$y' = g(x)\, e^{h(x)+by} + f(x)$	$\frac{e^{-\int bf(x)dx - h(x)}}{g(x)}$	$\frac{f(x)e^{-\int bf(x)dx - h(x)}}{g(x)}$
polynomial type ode	$y' = \frac{a_1x+b_1y+c_1}{a_2x+b_2y+c_2}$	$\frac{a_1b_2x - a_2b_1x - b_1c_2 + b_2c_1}{a_1b_2 - a_2b_1}$	$\frac{a_1b_2y - a_2b_1y - a_1c_2 - a_2c_1}{a_1b_2 - a_2b_1}$
Bernoulli ode	$y' = f(x)\, y + g(x)\, y^n$	0	$e^{-\int (n-1)f(x)dx} y^n$
Reduced Riccati	$y' = f_1(x)\, y + f_2(x)\, y^2$	0	$e^{-\int f_1 dx}$

The above table shows that

$$\begin{aligned}\xi(t,y) &= 0\\ \eta(t,y) &= \frac{1}{t^2}\end{aligned} \tag{A1}$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(t,y) \to (R,S)$ where (R,S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$\frac{dt}{\xi} = \frac{dy}{\eta} = dS \tag{1}$$

The above comes from the requirements that $\left(\xi\frac{\partial}{\partial t} + \eta\frac{\partial}{\partial y}\right) S(t,y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$R = t$$

S is found from

$$\begin{aligned}S &= \int \frac{1}{\eta} dy\\ &= \int \frac{1}{\frac{1}{t^2}} dy\end{aligned}$$

Which results in

$$S = t^2 y$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_t + \omega(t,y)S_y}{R_t + \omega(t,y)R_y} \tag{2}$$

Where in the above R_t, R_y, S_t, S_y are all partial derivatives and $\omega(t,y)$ is the right hand side of the original ode given by

$$\omega(t,y) = -\frac{2y - \sin(t)}{t}$$

Evaluating all the partial derivatives gives

$$
\begin{aligned}
R_t &= 1 \\
R_y &= 0 \\
S_t &= 2ty \\
S_y &= t^2
\end{aligned}
$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = t \sin(t) \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for t, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = R \sin(R)$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$S(R) = -R \cos(R) + \sin(R) + c_1 \tag{4}$$

To complete the solution, we just need to transform (4) back to t, y coordinates. This results in

$$yt^2 = -t \cos(t) + \sin(t) + c_1$$

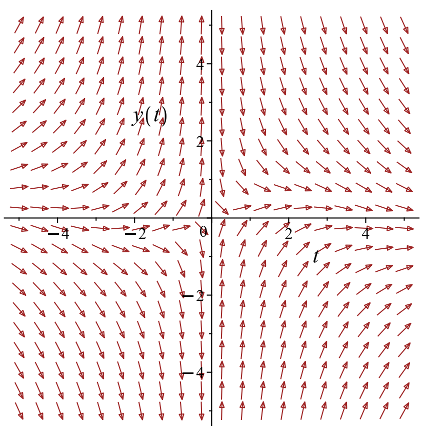
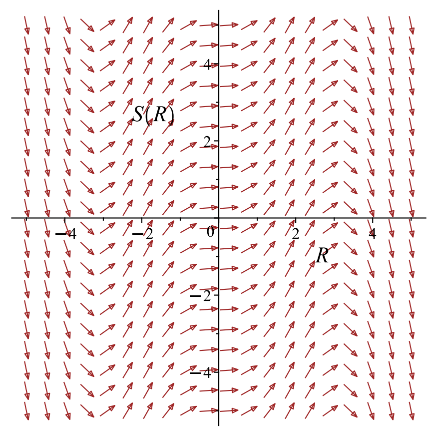
Which simplifies to

$$yt^2 = -t \cos(t) + \sin(t) + c_1$$

Which gives

$$y = -\frac{t \cos(t) - \sin(t) - c_1}{t^2}$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in t, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dt} = -\frac{2y-\sin(t)}{t}$ 	$R = t$ $S = t^2 y$	$\frac{dS}{dR} = R\sin(R)$ 

Initial conditions are used to solve for c_1. Substituting $t = \frac{\pi}{2}$ and $y = 1$ in the above solution gives an equation to solve for the constant of integration.

$$1 = \frac{4 + 4c_1}{\pi^2}$$

$$c_1 = \frac{\pi^2}{4} - 1$$

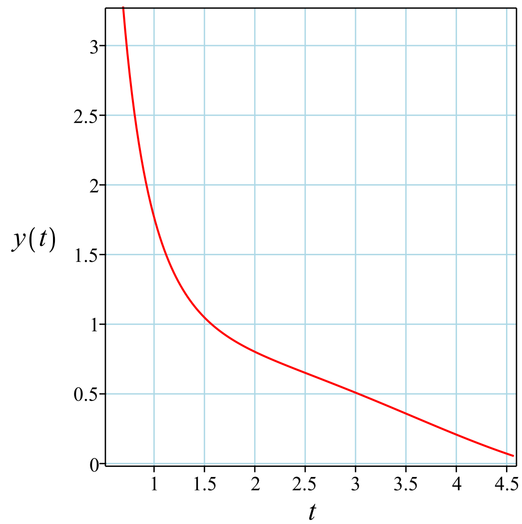
Substituting c_1 found above in the general solution gives

$$y = \frac{-4t\cos(t) + \pi^2 + 4\sin(t) - 4}{4t^2}$$

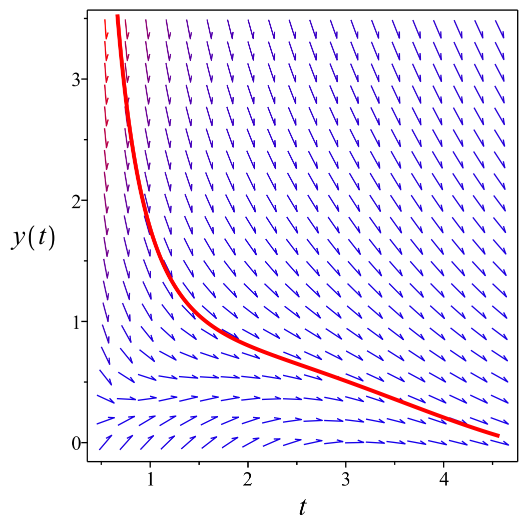
Summary

The solution(s) found are the following

$$y = \frac{-4t\cos(t) + \pi^2 + 4\sin(t) - 4}{4t^2} \tag{1}$$



(a) Solution plot



(b) Slope field plot

Verification of solutions

$$y = \frac{-4t\cos(t) + \pi^2 + 4\sin(t) - 4}{4t^2}$$

Verified OK.

1.18.4 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x,y) + N(x,y)\frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x,y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y) = 0$$

Hence

$$\frac{\partial\phi}{\partial x} + \frac{\partial\phi}{\partial y}\frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$\frac{\partial\phi}{\partial x} = M$$
$$\frac{\partial\phi}{\partial y} = N$$

But since $\frac{\partial^2\phi}{\partial x\partial y} = \frac{\partial^2\phi}{\partial y\partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x,y)$ but at least we know now that we can do that since the condition $\frac{\partial^2\phi}{\partial x\partial y} = \frac{\partial^2\phi}{\partial y\partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(t,y)\,\mathrm{d}t + N(t,y)\,\mathrm{d}y = 0 \tag{1A}$$

Therefore

$$\begin{aligned}(t)\,\mathrm{d}y &= (-2y + \sin(t))\,\mathrm{d}t\\ (2y - \sin(t))\,\mathrm{d}t + (t)\,\mathrm{d}y &= 0 \qquad (2A)\end{aligned}$$

Comparing (1A) and (2A) shows that

$$\begin{aligned}M(t,y) &= 2y - \sin(t)\\ N(t,y) &= t\end{aligned}$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial t}$$

Using result found above gives

$$\begin{aligned}\frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}(2y - \sin(t))\\ &= 2\end{aligned}$$

And

$$\begin{aligned}\frac{\partial N}{\partial t} &= \frac{\partial}{\partial t}(t)\\ &= 1\end{aligned}$$

Since $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial t}$, then the ODE is <u>not exact</u>. Since the ODE is not exact, we will try to find an integrating factor to make it exact. Let

$$\begin{aligned}A &= \frac{1}{N}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial t}\right)\\ &= \frac{1}{t}((2) - (1))\\ &= \frac{1}{t}\end{aligned}$$

Since A does not depend on y, then it can be used to find an integrating factor. The integrating factor μ is

$$\begin{aligned}\mu &= e^{\int A\,\mathrm{d}t} \\ &= e^{\int \frac{1}{t}\,\mathrm{d}t}\end{aligned}$$

The result of integrating gives

$$\begin{aligned}\mu &= e^{\ln(t)} \\ &= t\end{aligned}$$

M and N are multiplied by this integrating factor, giving new M and new N which are called $\overline{M}$ and $\overline{N}$ for now so not to confuse them with the original M and N.

$$\begin{aligned}\overline{M} &= \mu M \\ &= t(2y - \sin(t)) \\ &= (2y - \sin(t))\, t\end{aligned}$$

And

$$\begin{aligned}\overline{N} &= \mu N \\ &= t(t) \\ &= t^2\end{aligned}$$

Now a modified ODE is ontained from the original ODE, which is exact and can be solved. The modified ODE is

$$\begin{aligned}\overline{M} + \overline{N}\frac{\mathrm{d}y}{\mathrm{d}t} &= 0 \\ ((2y - \sin(t))\, t) + \left(t^2\right)\frac{\mathrm{d}y}{\mathrm{d}t} &= 0\end{aligned}$$

The following equations are now set up to solve for the function $\phi(t, y)$

$$\frac{\partial \phi}{\partial t} = \overline{M} \tag{1}$$

$$\frac{\partial \phi}{\partial y} = \overline{N} \tag{2}$$

Integrating (1) w.r.t. t gives

$$\begin{aligned}\int \frac{\partial \phi}{\partial t}\,\mathrm{d}t &= \int \overline{M}\,\mathrm{d}t \\ \int \frac{\partial \phi}{\partial t}\,\mathrm{d}t &= \int (2y - \sin(t))\, t\,\mathrm{d}t\end{aligned}$$

$$\phi = t^2 y + t\cos(t) - \sin(t) + f(y) \tag{3}$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both t and y. Taking derivative of equation (3) w.r.t y gives

$$\frac{\partial \phi}{\partial y} = t^2 + f'(y) \tag{4}$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = t^2$. Therefore equation (4) becomes

$$t^2 = t^2 + f'(y) \tag{5}$$

Solving equation (5) for $f'(y)$ gives

$$f'(y) = 0$$

Therefore

$$f(y) = c_1$$

Where c_1 is constant of integration. Substituting this result for $f(y)$ into equation (3) gives ϕ

$$\phi = t^2 y + t\cos(t) - \sin(t) + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = t^2 y + t\cos(t) - \sin(t)$$

The solution becomes

$$y = -\frac{t\cos(t) - \sin(t) - c_1}{t^2}$$

Initial conditions are used to solve for c_1. Substituting $t = \frac{\pi}{2}$ and $y = 1$ in the above solution gives an equation to solve for the constant of integration.

$$1 = \frac{4 + 4c_1}{\pi^2}$$

$$c_1 = \frac{\pi^2}{4} - 1$$

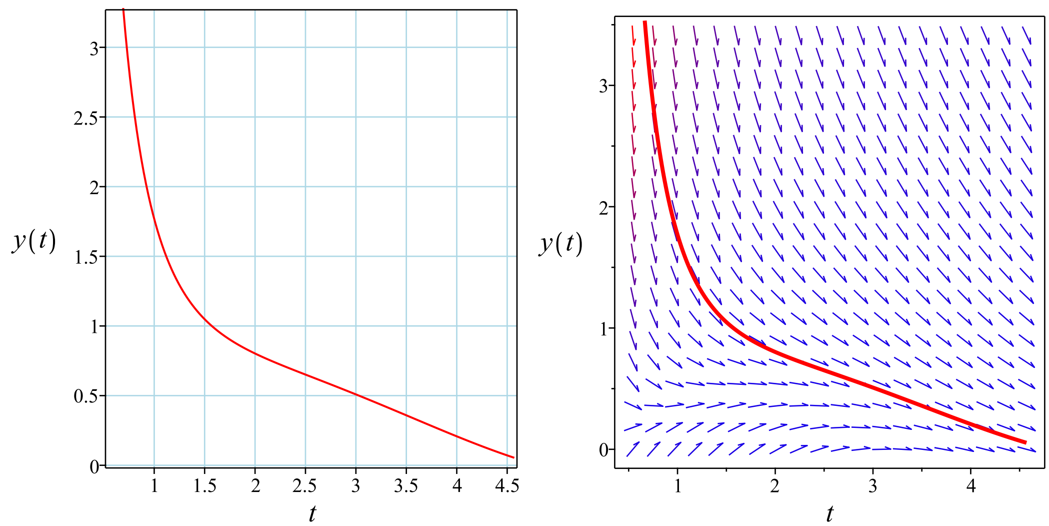
Substituting c_1 found above in the general solution gives

$$y = \frac{-4t\cos(t) + \pi^2 + 4\sin(t) - 4}{4t^2}$$

Summary

The solution(s) found are the following

$$y = \frac{-4t\cos(t) + \pi^2 + 4\sin(t) - 4}{4t^2} \tag{1}$$



(a) Solution plot

(b) Slope field plot

Verification of solutions

$$y = \frac{-4t\cos(t) + \pi^2 + 4\sin(t) - 4}{4t^2}$$

Verified OK.

1.18.5 Maple step by step solution

Let's solve

$\left[2y + ty' = \sin(t), y\left(\frac{\pi}{2}\right) = 1\right]$

- Highest derivative means the order of the ODE is 1

  y'

- Isolate the derivative

  $y' = -\frac{2y}{t} + \frac{\sin(t)}{t}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE

  $y' + \frac{2y}{t} = \frac{\sin(t)}{t}$

- The ODE is linear; multiply by an integrating factor $\mu(t)$

  $\mu(t)\left(y'+\frac{2y}{t}\right)=\frac{\mu(t)\sin(t)}{t}$

- Assume the lhs of the ODE is the total derivative $\frac{d}{dt}(\mu(t)\,y)$

  $\mu(t)\left(y'+\frac{2y}{t}\right)=\mu'(t)\,y+\mu(t)\,y'$

- Isolate $\mu'(t)$

  $\mu'(t)=\frac{2\mu(t)}{t}$

- Solve to find the integrating factor

  $\mu(t)=t^2$

- Integrate both sides with respect to t

  $\int\left(\frac{d}{dt}(\mu(t)\,y)\right)dt=\int\frac{\mu(t)\sin(t)}{t}dt+c_1$

- Evaluate the integral on the lhs

  $\mu(t)\,y=\int\frac{\mu(t)\sin(t)}{t}dt+c_1$

- Solve for y

  $y=\frac{\int\frac{\mu(t)\sin(t)}{t}dt+c_1}{\mu(t)}$

- Substitute $\mu(t)=t^2$

  $y=\frac{\int t\sin(t)dt+c_1}{t^2}$

- Evaluate the integrals on the rhs

  $y=\frac{-t\cos(t)+\sin(t)+c_1}{t^2}$

- Use initial condition $y\left(\frac{\pi}{2}\right)=1$

  $1=\frac{4(1+c_1)}{\pi^2}$

- Solve for c_1

  $c_1=\frac{\pi^2}{4}-1$

- Substitute $c_1=\frac{\pi^2}{4}-1$ into general solution and simplify

  $y=\frac{-t\cos(t)+\sin(t)+\frac{\pi^2}{4}-1}{t^2}$

- Solution to the IVP

  $y=\frac{-t\cos(t)+\sin(t)+\frac{\pi^2}{4}-1}{t^2}$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
<- 1st order linear successful`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 22

```
dsolve([2*y(t)+t*diff(y(t),t) = sin(t),y(1/2*Pi) = 1],y(t), singsol=all)
```

$$y(t) = \frac{\sin(t) - \cos(t)\, t + \frac{\pi^2}{4} - 1}{t^2}$$

✓ Solution by Mathematica

Time used: 0.034 (sec). Leaf size: 26

```
DSolve[{2*y[t]+t*y'[t] == Sin[t],y[Pi/2]==1},y[t],t,IncludeSingularSolutions -> True]
```

$$y(t) \to \frac{4\sin(t) - 4t\cos(t) + \pi^2 - 4}{4t^2}$$

1.19 problem 19

Internal problem ID [466]
Internal file name [`OUTPUT/466_Sunday_June_05_2022_01_41_55_AM_9788455/index.tex`]

Book: Elementary differential equations and boundary value problems, 10th ed., Boyce and DiPrima
Section: Section 2.1. Page 40
Problem number: 19.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"linear", "exactWithIntegrationFactor", "first_order_ode_lie_symmetry_lookup"**

Maple gives the following as the ode type

`[_linear]`

$$\boxed{4yt^2 + y't^3 = e^{-t}}$$

With initial conditions

$$[y(-1) = 0]$$

1.19.1 Existence and uniqueness analysis

This is a linear ODE. In canonical form it is written as

$$y' + p(t)y = q(t)$$

Where here

$$p(t) = \frac{4}{t}$$
$$q(t) = \frac{e^{-t}}{t^3}$$

Hence the ode is

$$y' + \frac{4y}{t} = \frac{e^{-t}}{t^3}$$

The domain of $p(t) = \frac{4}{t}$ is

$$\{t < 0 \lor 0 < t\}$$

And the point $t_0 = -1$ is inside this domain. The domain of $q(t) = \frac{e^{-t}}{t^3}$ is

$$\{t < 0 \lor 0 < t\}$$

And the point $t_0 = -1$ is also inside this domain. Hence solution exists and is unique.

1.19.2 Solving as linear ode

Entering Linear first order ODE solver. The integrating factor μ is

$$\begin{aligned}\mu &= e^{\int \frac{4}{t} dt} \\ &= t^4\end{aligned}$$

The ode becomes

$$\begin{aligned}\frac{d}{dt}(\mu y) &= (\mu)\left(\frac{e^{-t}}{t^3}\right) \\ \frac{d}{dt}(t^4 y) &= (t^4)\left(\frac{e^{-t}}{t^3}\right) \\ d(t^4 y) &= (t\,e^{-t})\,dt\end{aligned}$$

Integrating gives

$$\begin{aligned}t^4 y &= \int t\,e^{-t}\,dt \\ t^4 y &= -e^{-t}(t+1) + c_1\end{aligned}$$

Dividing both sides by the integrating factor $\mu = t^4$ results in

$$y = -\frac{e^{-t}(t+1)}{t^4} + \frac{c_1}{t^4}$$

which simplifies to

$$y = \frac{(-t-1)\,e^{-t} + c_1}{t^4}$$

Initial conditions are used to solve for c_1. Substituting $t = -1$ and $y = 0$ in the above solution gives an equation to solve for the constant of integration.

$$0 = c_1$$

$$c_1 = 0$$

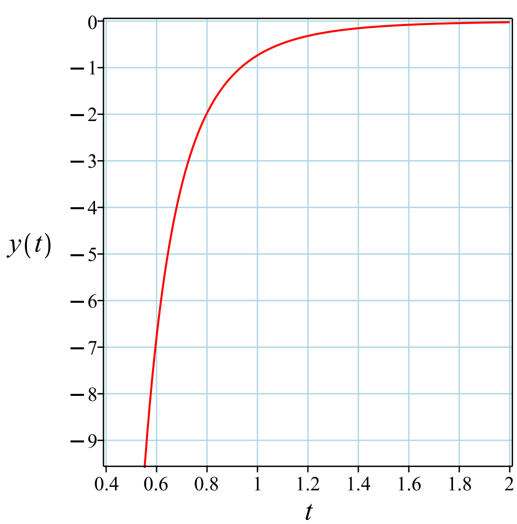
Substituting c_1 found above in the general solution gives

$$y = -\frac{e^{-t}(t+1)}{t^4}$$

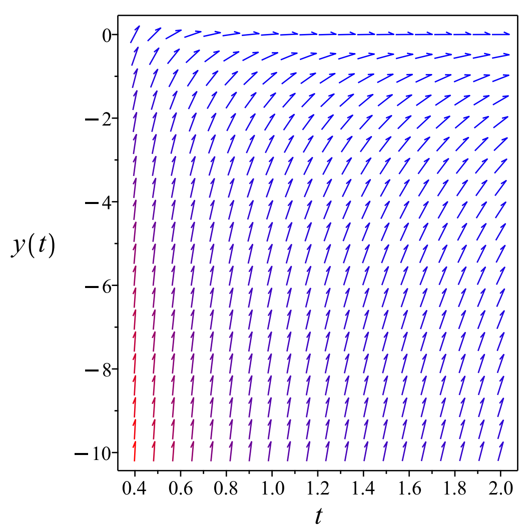
Summary

The solution(s) found are the following

$$y = -\frac{e^{-t}(t+1)}{t^4} \tag{1}$$



(a) Solution plot



(b) Slope field plot

Verification of solutions

$$y = -\frac{e^{-t}(t+1)}{t^4}$$

Verified OK.

1.19.3 Solving as first order ode lie symmetry lookup ode

Writing the ode as

$$y' = \frac{-4t^2 y + e^{-t}}{t^3}$$
$$y' = \omega(t, y)$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_t + \omega(\eta_y - \xi_t) - \omega^2 \xi_y - \omega_t \xi - \omega_y \eta = 0 \tag{A}$$

The type of this ode is known. It is of type `linear`. Therefore we do not need to solve the PDE (A), and can just use the lookup table shown below to find ξ, η

Table 55: Lie symmetry infinitesimal lookup table for known first order ODE's

ODE class	Form	ξ	η
linear ode	$y' = f(x)y(x) + g(x)$	0	$e^{\int f dx}$
separable ode	$y' = f(x)\, g(y)$	$\frac{1}{f}$	0
quadrature ode	$y' = f(x)$	0	1
quadrature ode	$y' = g(y)$	1	0
homogeneous ODEs of Class A	$y' = f\left(\frac{y}{x}\right)$	x	y
homogeneous ODEs of Class C	$y' = (a + bx + cy)^{\frac{n}{m}}$	1	$-\frac{b}{c}$
homogeneous class D	$y' = \frac{y}{x} + g(x)\, F\left(\frac{y}{x}\right)$	x^2	xy
First order special form ID 1	$y' = g(x)\, e^{h(x)+by} + f(x)$	$\frac{e^{-\int bf(x)dx - h(x)}}{g(x)}$	$\frac{f(x)e^{-\int bf(x)dx - h(x)}}{g(x)}$
polynomial type ode	$y' = \frac{a_1x+b_1y+c_1}{a_2x+b_2y+c_2}$	$\frac{a_1b_2x - a_2b_1x - b_1c_2 + b_2c_1}{a_1b_2 - a_2b_1}$	$\frac{a_1b_2y - a_2b_1y - a_1c_2 - a_2c_1}{a_1b_2 - a_2b_1}$
Bernoulli ode	$y' = f(x)\, y + g(x)\, y^n$	0	$e^{-\int (n-1)f(x)dx} y^n$
Reduced Riccati	$y' = f_1(x)\, y + f_2(x)\, y^2$	0	$e^{-\int f_1 dx}$

The above table shows that

$$
\begin{aligned}
\xi(t,y) &= 0 \\
\eta(t,y) &= \frac{1}{t^4}
\end{aligned}
\tag{A1}
$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(t,y) \to (R,S)$ where (R,S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$
\frac{dt}{\xi} = \frac{dy}{\eta} = dS \tag{1}
$$

The above comes from the requirements that $\left(\xi\frac{\partial}{\partial t} + \eta\frac{\partial}{\partial y}\right) S(t,y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$
R = t
$$

S is found from

$$
\begin{aligned}
S &= \int \frac{1}{\eta} dy \\
&= \int \frac{1}{\frac{1}{t^4}} dy
\end{aligned}
$$

Which results in

$$
S = t^4 y
$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$
\frac{dS}{dR} = \frac{S_t + \omega(t,y) S_y}{R_t + \omega(t,y) R_y} \tag{2}
$$

Where in the above R_t, R_y, S_t, S_y are all partial derivatives and $\omega(t,y)$ is the right hand side of the original ode given by

$$
\omega(t,y) = \frac{-4t^2 y + e^{-t}}{t^3}
$$

Evaluating all the partial derivatives gives

$$
\begin{aligned}
R_t &= 1 \\
R_y &= 0 \\
S_t &= 4t^3 y \\
S_y &= t^4
\end{aligned}
$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$
\frac{dS}{dR} = t\,\mathrm{e}^{-t} \tag{2A}
$$

We now need to express the RHS as function of R only. This is done by solving for t, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$
\frac{dS}{dR} = R\,\mathrm{e}^{-R}
$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$
S(R) = -(R+1)\,\mathrm{e}^{-R} + c_1 \tag{4}
$$

To complete the solution, we just need to transform (4) back to t, y coordinates. This results in

$$
y t^4 = -\mathrm{e}^{-t}(t+1) + c_1
$$

Which simplifies to

$$
y t^4 = -\mathrm{e}^{-t}(t+1) + c_1
$$

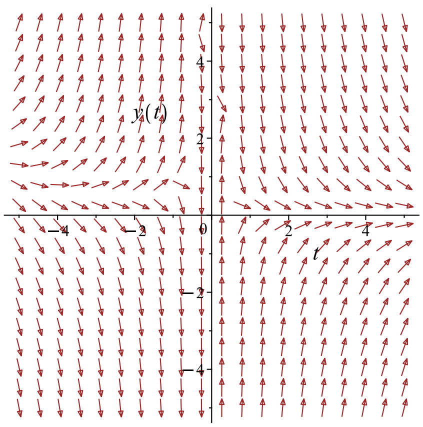
Which gives

$$
y = -\frac{t\,\mathrm{e}^{-t} + \mathrm{e}^{-t} - c_1}{t^4}
$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

<table>
<tr><th>Original ode in t, y coordinates</th><th>Canonical coordinates transformation</th><th>ODE in canonical coordinates (R, S)</th></tr>
<tr><td>

$\frac{dy}{dt} = \frac{-4t^2y+e^{-t}}{t^3}$



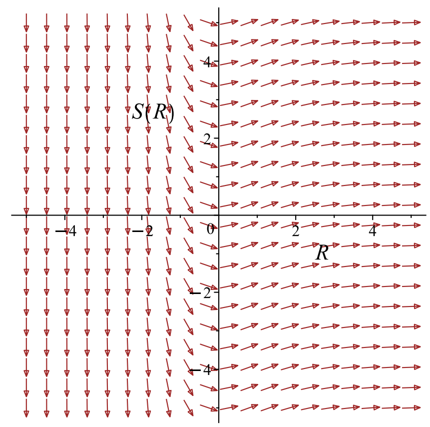
</td><td>

$R = t$

$S = t^4y$

</td><td>

$\frac{dS}{dR} = R\,e^{-R}$



</td></tr>
</table>

Initial conditions are used to solve for c_1. Substituting $t = -1$ and $y = 0$ in the above solution gives an equation to solve for the constant of integration.

$$0 = c_1$$

$$c_1 = 0$$

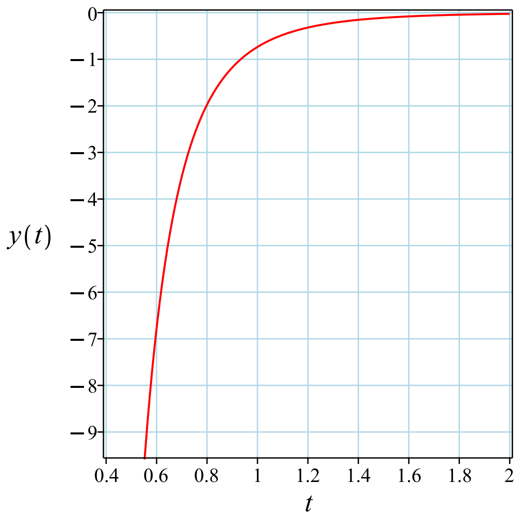
Substituting c_1 found above in the general solution gives

$$y = -\frac{e^{-t}(t+1)}{t^4}$$

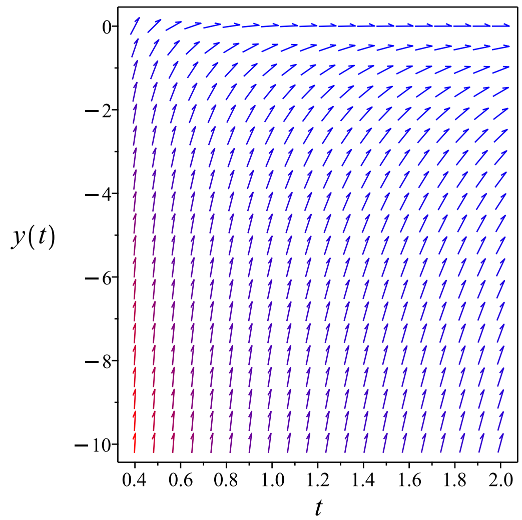
Summary

The solution(s) found are the following

$$y = -\frac{e^{-t}(t+1)}{t^4} \tag{1}$$



(a) Solution plot



(b) Slope field plot

Verification of solutions

$$y = -\frac{e^{-t}(t+1)}{t^4}$$

Verified OK.

1.19.4 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x,y) + N(x,y)\frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x,y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y) = 0$$

Hence

$$\frac{\partial \phi}{\partial x} + \frac{\partial \phi}{\partial y}\frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$\begin{aligned}\frac{\partial \phi}{\partial x} &= M \\ \frac{\partial \phi}{\partial y} &= N\end{aligned}$$

But since $\frac{\partial^2\phi}{\partial x\partial y} = \frac{\partial^2\phi}{\partial y\partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x, y)$ but at least we know now that we can do that since the condition $\frac{\partial^2\phi}{\partial x\partial y} = \frac{\partial^2\phi}{\partial y\partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(t, y)\,\mathrm{d}t + N(t, y)\,\mathrm{d}y = 0 \tag{1A}$$

Therefore

$$\begin{aligned}(t^3)\,\mathrm{d}y &= (-4t^2y + \mathrm{e}^{-t})\,\mathrm{d}t\\ (4t^2y - \mathrm{e}^{-t})\,\mathrm{d}t + (t^3)\,\mathrm{d}y &= 0 \end{aligned} \tag{2A}$$

Comparing (1A) and (2A) shows that

$$\begin{aligned}M(t, y) &= 4t^2y - \mathrm{e}^{-t}\\ N(t, y) &= t^3\end{aligned}$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial t}$$

Using result found above gives

$$\begin{aligned}\frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}(4t^2y - \mathrm{e}^{-t})\\ &= 4t^2\end{aligned}$$

And

$$\begin{aligned}\frac{\partial N}{\partial t} &= \frac{\partial}{\partial t}(t^3)\\ &= 3t^2\end{aligned}$$

Since $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial t}$, then the ODE is <u>not exact</u>. Since the ODE is not exact, we will try to find an integrating factor to make it exact. Let

$$\begin{aligned}A &= \frac{1}{N}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial t}\right)\\ &= \frac{1}{t^3}\left((4t^2) - (3t^2)\right)\\ &= \frac{1}{t}\end{aligned}$$

Since A does not depend on y, then it can be used to find an integrating factor. The integrating factor μ is

$$\begin{aligned}\mu &= e^{\int A\,\mathrm{d}t}\\ &= e^{\int \frac{1}{t}\,\mathrm{d}t}\end{aligned}$$

The result of integrating gives

$$\begin{aligned}\mu &= e^{\ln(t)}\\ &= t\end{aligned}$$

M and N are multiplied by this integrating factor, giving new M and new N which are called $\overline{M}$ and $\overline{N}$ for now so not to confuse them with the original M and N.

$$\begin{aligned}\overline{M} &= \mu M\\ &= t\left(4t^2y - \mathrm{e}^{-t}\right)\\ &= 4t^3y - t\,\mathrm{e}^{-t}\end{aligned}$$

And

$$\begin{aligned}\overline{N} &= \mu N\\ &= t\left(t^3\right)\\ &= t^4\end{aligned}$$

Now a modified ODE is ontained from the original ODE, which is exact and can be solved. The modified ODE is

$$\begin{aligned}\overline{M} + \overline{N}\frac{\mathrm{d}y}{\mathrm{d}t} &= 0\\ \left(4t^3y - t\,\mathrm{e}^{-t}\right) + \left(t^4\right)\frac{\mathrm{d}y}{\mathrm{d}t} &= 0\end{aligned}$$

The following equations are now set up to solve for the function $\phi(t, y)$

$$\frac{\partial\phi}{\partial t} = \overline{M} \tag{1}$$

$$\frac{\partial\phi}{\partial y} = \overline{N} \tag{2}$$

Integrating (1) w.r.t. t gives

$$\begin{aligned}\int \frac{\partial\phi}{\partial t}\,\mathrm{d}t &= \int \overline{M}\,\mathrm{d}t\\ \int \frac{\partial\phi}{\partial t}\,\mathrm{d}t &= \int 4t^3y - t\,\mathrm{e}^{-t}\,\mathrm{d}t\end{aligned}$$

$$\phi = t\,\mathrm{e}^{-t} + \mathrm{e}^{-t} + t^4y + f(y) \tag{3}$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both t and y. Taking derivative of equation (3) w.r.t y gives

$$\frac{\partial \phi}{\partial y} = t^4 + f'(y) \tag{4}$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = t^4$. Therefore equation (4) becomes

$$t^4 = t^4 + f'(y) \tag{5}$$

Solving equation (5) for $f'(y)$ gives

$$f'(y) = 0$$

Therefore

$$f(y) = c_1$$

Where c_1 is constant of integration. Substituting this result for $f(y)$ into equation (3) gives ϕ

$$\phi = t\,\mathrm{e}^{-t} + \mathrm{e}^{-t} + t^4 y + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = t\,\mathrm{e}^{-t} + \mathrm{e}^{-t} + t^4 y$$

The solution becomes

$$y = -\frac{t\,\mathrm{e}^{-t} + \mathrm{e}^{-t} - c_1}{t^4}$$

Initial conditions are used to solve for c_1. Substituting $t = -1$ and $y = 0$ in the above solution gives an equation to solve for the constant of integration.

$$0 = c_1$$

$$c_1 = 0$$

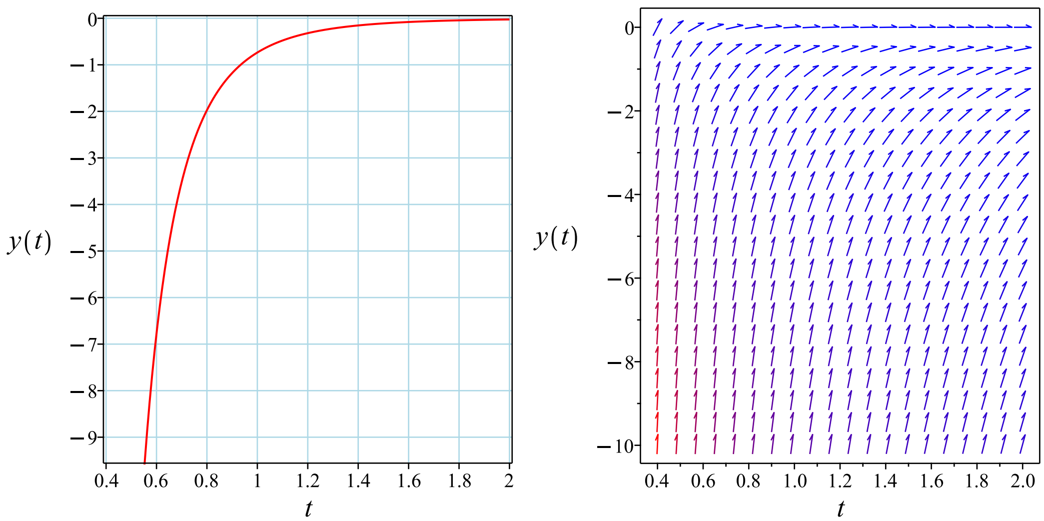
Substituting c_1 found above in the general solution gives

$$y = -\frac{\mathrm{e}^{-t}(t+1)}{t^4}$$

Summary

The solution(s) found are the following

$$y = -\frac{e^{-t}(t+1)}{t^4} \tag{1}$$



(a) Solution plot (b) Slope field plot

Verification of solutions

$$y = -\frac{e^{-t}(t+1)}{t^4}$$

Verified OK.

1.19.5 Maple step by step solution

Let's solve

$[4yt^2 + y't^3 = e^{-t}, y(-1) = 0]$

- Highest derivative means the order of the ODE is 1

  y'

- Isolate the derivative

  $y' = -\frac{4y}{t} + \frac{e^{-t}}{t^3}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE

  $y' + \frac{4y}{t} = \frac{e^{-t}}{t^3}$

- The ODE is linear; multiply by an integrating factor $\mu(t)$

  $\mu(t)\left(y'+\frac{4y}{t}\right)=\frac{\mu(t)\mathrm{e}^{-t}}{t^3}$
- Assume the lhs of the ODE is the total derivative $\frac{d}{dt}(\mu(t)\,y)$

  $\mu(t)\left(y'+\frac{4y}{t}\right)=\mu'(t)\,y+\mu(t)\,y'$
- Isolate $\mu'(t)$

  $\mu'(t)=\frac{4\mu(t)}{t}$
- Solve to find the integrating factor

  $\mu(t)=t^4$
- Integrate both sides with respect to t

  $\int\left(\frac{d}{dt}(\mu(t)\,y)\right)dt=\int\frac{\mu(t)\mathrm{e}^{-t}}{t^3}dt+c_1$
- Evaluate the integral on the lhs

  $\mu(t)\,y=\int\frac{\mu(t)\mathrm{e}^{-t}}{t^3}dt+c_1$
- Solve for y

  $y=\frac{\int\frac{\mu(t)\mathrm{e}^{-t}}{t^3}dt+c_1}{\mu(t)}$
- Substitute $\mu(t)=t^4$

  $y=\frac{\int t\,\mathrm{e}^{-t}dt+c_1}{t^4}$
- Evaluate the integrals on the rhs

  $y=\frac{-\mathrm{e}^{-t}(t+1)+c_1}{t^4}$
- Simplify

  $y=\frac{(-t-1)\mathrm{e}^{-t}+c_1}{t^4}$
- Use initial condition $y(-1)=0$

  $0=c_1$
- Solve for c_1

  $c_1=0$
- Substitute $c_1=0$ into general solution and simplify

  $y=-\frac{\mathrm{e}^{-t}(t+1)}{t^4}$
- Solution to the IVP

  $y=-\frac{\mathrm{e}^{-t}(t+1)}{t^4}$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
<- 1st order linear successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 16

```
dsolve([4*t^2*y(t)+t^3*diff(y(t),t) = exp(-t),y(-1) = 0],y(t), singsol=all)
```

$$y(t) = -\frac{(t+1)\,\mathrm{e}^{-t}}{t^4}$$

✓ Solution by Mathematica

Time used: 0.067 (sec). Leaf size: 18

```
DSolve[{4*t^2*y[t]+t^3*y'[t] == Exp[-t],y[-1]==0},y[t],t,IncludeSingularSolutions -> True]
```

$$y(t) \to -\frac{e^{-t}(t+1)}{t^4}$$

1.20 problem 20

Internal problem ID [467]
Internal file name [`OUTPUT/467_Sunday_June_05_2022_01_41_56_AM_59241620/index.tex`]

Book: Elementary differential equations and boundary value problems, 10th ed., Boyce and DiPrima
Section: Section 2.1. Page 40
Problem number: 20.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"linear", "exactWithIntegrationFactor", "first_order_ode_lie_symmetry_lookup"**

Maple gives the following as the ode type

`[_linear]`

$$\boxed{(t+1)\,y+ty'=t}$$

With initial conditions

$$[y(\ln(2))=1]$$

1.20.1 Existence and uniqueness analysis

This is a linear ODE. In canonical form it is written as

$$y'+p(t)y=q(t)$$

Where here

$$\begin{aligned}p(t)&=-\frac{-t-1}{t}\\ q(t)&=1\end{aligned}$$

Hence the ode is

$$y' - \frac{(-t-1)\, y}{t} = 1$$

The domain of $p(t) = -\frac{-t-1}{t}$ is

$$\{t < 0 \lor 0 < t\}$$

And the point $t_0 = \ln(2)$ is inside this domain. The domain of $q(t) = 1$ is

$$\{-\infty < t < \infty\}$$

And the point $t_0 = \ln(2)$ is also inside this domain. Hence solution exists and is unique.

1.20.2 Solving as linear ode

Entering Linear first order ODE solver. The integrating factor μ is

$$\begin{aligned}\mu &= \mathrm{e}^{\int -\frac{-t-1}{t}dt} \\ &= \mathrm{e}^{t+\ln(t)}\end{aligned}$$

Which simplifies to

$$\mu = t\,\mathrm{e}^{t}$$

The ode becomes

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}(\mu y) &= \mu \\ \frac{\mathrm{d}}{\mathrm{d}t}\left(t\,\mathrm{e}^{t} y\right) &= t\,\mathrm{e}^{t} \\ \mathrm{d}\left(t\,\mathrm{e}^{t} y\right) &= t\,\mathrm{e}^{t} \mathrm{d}t\end{aligned}$$

Integrating gives

$$\begin{aligned}t\,\mathrm{e}^{t} y &= \int t\,\mathrm{e}^{t}\,\mathrm{d}t \\ t\,\mathrm{e}^{t} y &= \mathrm{e}^{t}(-1+t) + c_1\end{aligned}$$

Dividing both sides by the integrating factor $\mu = t\,\mathrm{e}^{t}$ results in

$$y = \frac{\mathrm{e}^{-t}\mathrm{e}^{t}(-1+t)}{t} + \frac{c_1 \mathrm{e}^{-t}}{t}$$

which simplifies to

$$y = \frac{c_1 e^{-t} + t - 1}{t}$$

Initial conditions are used to solve for c_1. Substituting $t = \ln(2)$ and $y = 1$ in the above solution gives an equation to solve for the constant of integration.

$$1 = \frac{c_1 + 2\ln(2) - 2}{2\ln(2)}$$

$$c_1 = 2$$

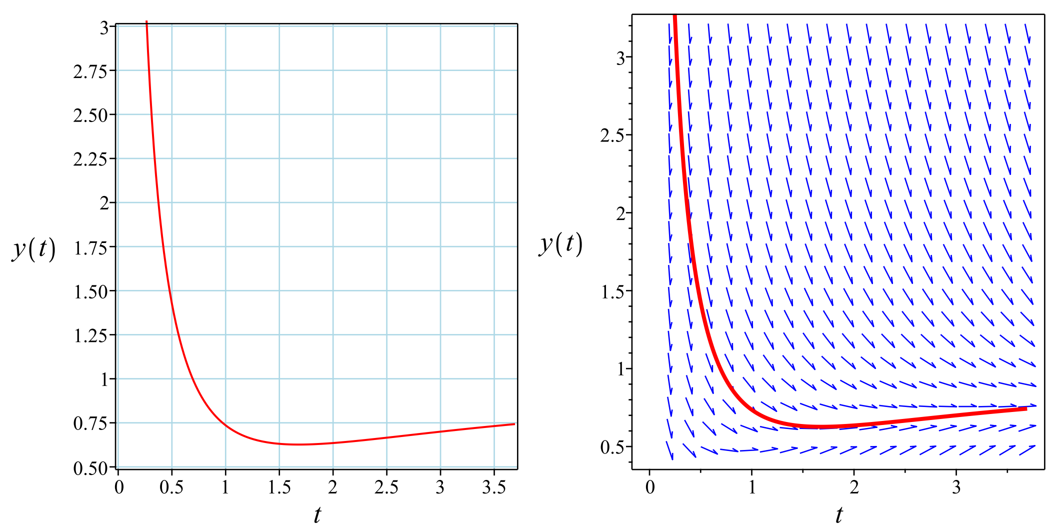
Substituting c_1 found above in the general solution gives

$$y = \frac{-1 + 2\,e^{-t} + t}{t}$$

Summary

The solution(s) found are the following

$$y = \frac{-1 + 2\,e^{-t} + t}{t} \tag{1}$$



(a) Solution plot

(b) Slope field plot

Verification of solutions

$$y = \frac{-1 + 2\,e^{-t} + t}{t}$$

Verified OK.

1.20.3 Solving as first order ode lie symmetry lookup ode

Writing the ode as

$$
\begin{aligned}
y' &= -\frac{ty-t+y}{t} \\
y' &= \omega(t,y)
\end{aligned}
$$

The condition of Lie symmetry is the linearized PDE given by

$$
\eta_t + \omega(\eta_y - \xi_t) - \omega^2 \xi_y - \omega_t \xi - \omega_y \eta = 0 \tag{A}
$$

The type of this ode is known. It is of type `linear`. Therefore we do not need to solve the PDE (A), and can just use the lookup table shown below to find ξ, η

Table 58: Lie symmetry infinitesimal lookup table for known first order ODE's

ODE class	Form	ξ	η
linear ode	$y' = f(x)y(x) + g(x)$	0	$e^{\int f dx}$
separable ode	$y' = f(x)\, g(y)$	$\frac{1}{f}$	0
quadrature ode	$y' = f(x)$	0	1
quadrature ode	$y' = g(y)$	1	0
homogeneous ODEs of Class A	$y' = f\left(\frac{y}{x}\right)$	x	y
homogeneous ODEs of Class C	$y' = (a + bx + cy)^{\frac{n}{m}}$	1	$-\frac{b}{c}$
homogeneous class D	$y' = \frac{y}{x} + g(x)\, F\left(\frac{y}{x}\right)$	x^2	xy
First order special form ID 1	$y' = g(x)\, e^{h(x)+by} + f(x)$	$\frac{e^{-\int bf(x)dx-h(x)}}{g(x)}$	$\frac{f(x)e^{-\int bf(x)dx-h(x)}}{g(x)}$
polynomial type ode	$y' = \frac{a_1x+b_1y+c_1}{a_2x+b_2y+c_2}$	$\frac{a_1b_2x-a_2b_1x-b_1c_2+b_2c_1}{a_1b_2-a_2b_1}$	$\frac{a_1b_2y-a_2b_1y-a_1c_2-a_2c_1}{a_1b_2-a_2b_1}$
Bernoulli ode	$y' = f(x)\, y + g(x)\, y^n$	0	$e^{-\int (n-1)f(x)dx} y^n$
Reduced Riccati	$y' = f_1(x)\, y + f_2(x)\, y^2$	0	$e^{-\int f_1 dx}$

The above table shows that

$$
\begin{aligned}
\xi(t,y) &= 0 \\
\eta(t,y) &= \mathrm{e}^{-t-\ln(t)}
\end{aligned} \tag{A1}
$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(t,y) \to (R,S)$ where (R,S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$
\frac{dt}{\xi} = \frac{dy}{\eta} = dS \tag{1}
$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial t} + \eta \frac{\partial}{\partial y}\right) S(t,y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$
R = t
$$

S is found from

$$
\begin{aligned}
S &= \int \frac{1}{\eta} dy \\
&= \int \frac{1}{\mathrm{e}^{-t-\ln(t)}} dy
\end{aligned}
$$

Which results in

$$
S = t\,\mathrm{e}^{t} y
$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$
\frac{dS}{dR} = \frac{S_t + \omega(t,y) S_y}{R_t + \omega(t,y) R_y} \tag{2}
$$

Where in the above R_t, R_y, S_t, S_y are all partial derivatives and $\omega(t,y)$ is the right hand side of the original ode given by

$$
\omega(t,y) = -\frac{ty - t + y}{t}
$$

Evaluating all the partial derivatives gives

$$
\begin{aligned}
R_t &= 1 \\
R_y &= 0 \\
S_t &= y\,\mathrm{e}^{t}(t+1) \\
S_y &= t\,\mathrm{e}^{t}
\end{aligned}
$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = t\,\mathrm{e}^{t} \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for t, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = R\,\mathrm{e}^{R}$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$S(R) = (R-1)\,\mathrm{e}^{R} + c_1 \tag{4}$$

To complete the solution, we just need to transform (4) back to t, y coordinates. This results in

$$t\,\mathrm{e}^{t} y = \mathrm{e}^{t}(-1+t) + c_1$$

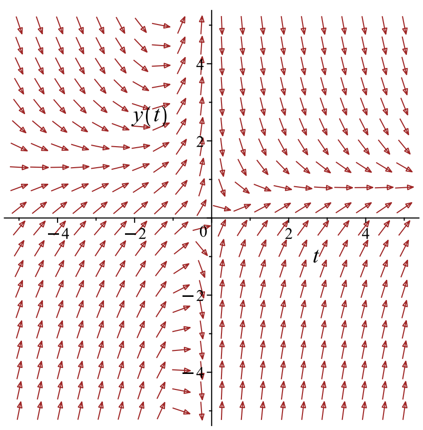
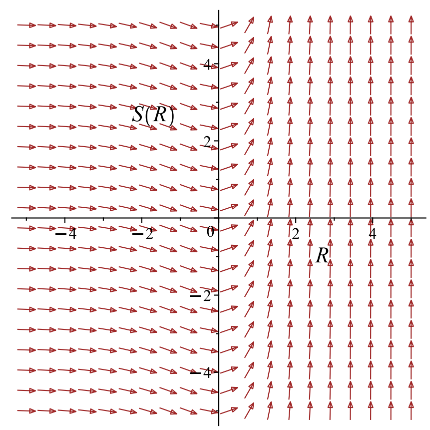
Which simplifies to

$$t\,\mathrm{e}^{t} y = \mathrm{e}^{t}(-1+t) + c_1$$

Which gives

$$y = \frac{(t\,\mathrm{e}^{t} - \mathrm{e}^{t} + c_1)\,\mathrm{e}^{-t}}{t}$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in t, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dt} = -\frac{ty-t+y}{t}$ 	$R = t$ $S = t\,\mathrm{e}^{t} y$	$\frac{dS}{dR} = R\,\mathrm{e}^{R}$ 

Initial conditions are used to solve for c_1. Substituting $t = \ln(2)$ and $y = 1$ in the above solution gives an equation to solve for the constant of integration.

$$1 = \frac{c_1 + 2\ln(2) - 2}{2\ln(2)}$$

$$c_1 = 2$$

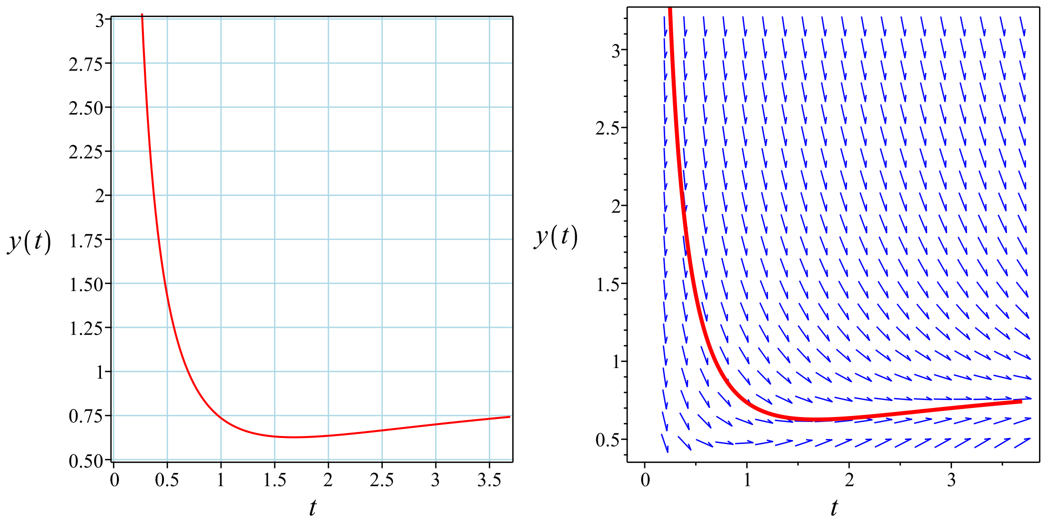
Substituting c_1 found above in the general solution gives

$$y = \frac{\mathrm{e}^{-t}\mathrm{e}^{t}t - \mathrm{e}^{-t}\mathrm{e}^{t} + 2\,\mathrm{e}^{-t}}{t}$$

Summary

The solution(s) found are the following

$$y = \frac{\mathrm{e}^{-t}\mathrm{e}^{t}t - \mathrm{e}^{-t}\mathrm{e}^{t} + 2\,\mathrm{e}^{-t}}{t} \tag{1}$$



(a) Solution plot (b) Slope field plot

Verification of solutions

$$y = \frac{e^{-t}e^{t}t - e^{-t}e^{t} + 2\,e^{-t}}{t}$$

Verified OK.

1.20.4 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x,y) + N(x,y)\frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x,y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y) = 0$$

Hence

$$\frac{\partial\phi}{\partial x} + \frac{\partial\phi}{\partial y}\frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$\frac{\partial\phi}{\partial x} = M$$
$$\frac{\partial\phi}{\partial y} = N$$

But since $\frac{\partial^2\phi}{\partial x\partial y} = \frac{\partial^2\phi}{\partial y\partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x,y)$ but at least we know now that we can do that since the condition $\frac{\partial^2\phi}{\partial x\partial y} = \frac{\partial^2\phi}{\partial y\partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(t,y)\,\mathrm{d}t + N(t,y)\,\mathrm{d}y = 0 \tag{1A}$$

Therefore

$$\begin{aligned}(t)\,\mathrm{d}y &= (-(t+1)\,y + t)\,\mathrm{d}t\\ ((t+1)\,y - t)\,\mathrm{d}t + (t)\,\mathrm{d}y &= 0 \qquad (2A)\end{aligned}$$

Comparing (1A) and (2A) shows that

$$\begin{aligned}M(t,y) &= (t+1)\,y - t\\ N(t,y) &= t\end{aligned}$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial t}$$

Using result found above gives

$$\begin{aligned}\frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}((t+1)\,y - t)\\ &= t+1\end{aligned}$$

And

$$\begin{aligned}\frac{\partial N}{\partial t} &= \frac{\partial}{\partial t}(t)\\ &= 1\end{aligned}$$

Since $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial t}$, then the ODE is <u>not exact</u>. Since the ODE is not exact, we will try to find an integrating factor to make it exact. Let

$$\begin{aligned}A &= \frac{1}{N}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial t}\right)\\ &= \frac{1}{t}((t+1) - (1))\\ &= 1\end{aligned}$$

Since A does not depend on y, then it can be used to find an integrating factor. The integrating factor μ is

$$\begin{aligned}\mu &= e^{\int A \, \mathrm{d}t} \\ &= e^{\int 1 \, \mathrm{d}t}\end{aligned}$$

The result of integrating gives

$$\begin{aligned}\mu &= e^{t} \\ &= \mathrm{e}^{t}\end{aligned}$$

M and N are multiplied by this integrating factor, giving new M and new N which are called $\overline{M}$ and $\overline{N}$ for now so not to confuse them with the original M and N.

$$\begin{aligned}\overline{M} &= \mu M \\ &= \mathrm{e}^{t}((t+1)\,y - t) \\ &= ((y-1)\,t + y)\,\mathrm{e}^{t}\end{aligned}$$

And

$$\begin{aligned}\overline{N} &= \mu N \\ &= \mathrm{e}^{t}(t) \\ &= t\,\mathrm{e}^{t}\end{aligned}$$

Now a modified ODE is ontained from the original ODE, which is exact and can be solved. The modified ODE is

$$\begin{aligned}\overline{M} + \overline{N}\frac{\mathrm{d}y}{\mathrm{d}t} &= 0 \\ \left(((y-1)\,t + y)\,\mathrm{e}^{t}\right) + \left(t\,\mathrm{e}^{t}\right)\frac{\mathrm{d}y}{\mathrm{d}t} &= 0\end{aligned}$$

The following equations are now set up to solve for the function $\phi(t, y)$

$$\frac{\partial \phi}{\partial t} = \overline{M} \tag{1}$$

$$\frac{\partial \phi}{\partial y} = \overline{N} \tag{2}$$

Integrating (1) w.r.t. t gives

$$\begin{aligned}\int \frac{\partial \phi}{\partial t} \, \mathrm{d}t &= \int \overline{M} \, \mathrm{d}t \\ \int \frac{\partial \phi}{\partial t} \, \mathrm{d}t &= \int ((y-1)\,t + y)\,\mathrm{e}^{t} \, \mathrm{d}t\end{aligned}$$

$$\phi = (1 + (y-1)\,t)\,\mathrm{e}^{t} + f(y) \tag{3}$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both t and y. Taking derivative of equation (3) w.r.t y gives

$$\frac{\partial \phi}{\partial y} = t\,\mathrm{e}^{t} + f'(y) \tag{4}$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = t\,\mathrm{e}^{t}$. Therefore equation (4) becomes

$$t\,\mathrm{e}^{t} = t\,\mathrm{e}^{t} + f'(y) \tag{5}$$

Solving equation (5) for $f'(y)$ gives

$$f'(y) = 0$$

Therefore

$$f(y) = c_1$$

Where c_1 is constant of integration. Substituting this result for $f(y)$ into equation (3) gives ϕ

$$\phi = (1 + (y-1)\,t)\,\mathrm{e}^{t} + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = (1 + (y-1)\,t)\,\mathrm{e}^{t}$$

The solution becomes

$$y = \frac{(t\,\mathrm{e}^{t} - \mathrm{e}^{t} + c_1)\,\mathrm{e}^{-t}}{t}$$

Initial conditions are used to solve for c_1. Substituting $t = \ln(2)$ and $y = 1$ in the above solution gives an equation to solve for the constant of integration.

$$1 = \frac{c_1 + 2\ln(2) - 2}{2\ln(2)}$$

$$c_1 = 2$$

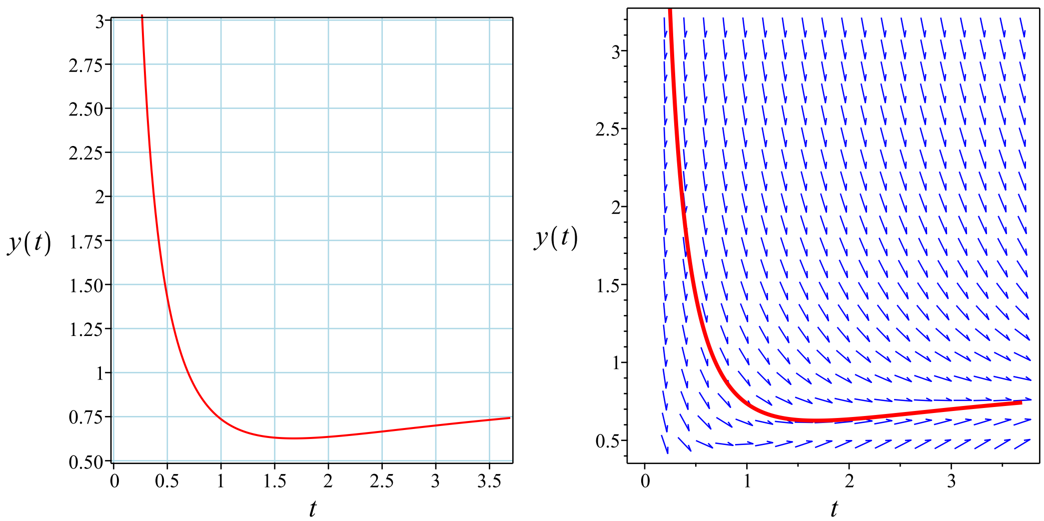
Substituting c_1 found above in the general solution gives

$$y = \frac{\mathrm{e}^{-t}\mathrm{e}^{t}t - \mathrm{e}^{-t}\mathrm{e}^{t} + 2\,\mathrm{e}^{-t}}{t}$$

Summary

The solution(s) found are the following

$$y = \frac{e^{-t}e^{t}t - e^{-t}e^{t} + 2\,e^{-t}}{t} \tag{1}$$



(a) Solution plot

(b) Slope field plot

Verification of solutions

$$y = \frac{e^{-t}e^{t}t - e^{-t}e^{t} + 2\,e^{-t}}{t}$$

Verified OK.

1.20.5 Maple step by step solution

Let's solve

$[(t+1)\,y + ty' = t, y(\ln(2)) = 1]$

- Highest derivative means the order of the ODE is 1

  y'

- Isolate the derivative

  $y' = 1 - \frac{(t+1)y}{t}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE

  $y' + \frac{(t+1)y}{t} = 1$

- The ODE is linear; multiply by an integrating factor $\mu(t)$

  $\mu(t)\left(y' + \frac{(t+1)y}{t}\right) = \mu(t)$
- Assume the lhs of the ODE is the total derivative $\frac{d}{dt}(\mu(t)\, y)$

  $\mu(t)\left(y' + \frac{(t+1)y}{t}\right) = \mu'(t)\, y + \mu(t)\, y'$
- Isolate $\mu'(t)$

  $\mu'(t) = \frac{\mu(t)(t+1)}{t}$
- Solve to find the integrating factor

  $\mu(t) = t\, \mathrm{e}^{t}$
- Integrate both sides with respect to t

  $\int \left(\frac{d}{dt}(\mu(t)\, y)\right) dt = \int \mu(t)\, dt + c_1$
- Evaluate the integral on the lhs

  $\mu(t)\, y = \int \mu(t)\, dt + c_1$
- Solve for y

  $y = \frac{\int \mu(t) dt + c_1}{\mu(t)}$
- Substitute $\mu(t) = t\mathrm{e}^{t}$

  $y = \frac{\int t\, \mathrm{e}^{t} dt + c_1}{t\, \mathrm{e}^{t}}$
- Evaluate the integrals on the rhs

  $y = \frac{\mathrm{e}^{t}(-1+t) + c_1}{t\, \mathrm{e}^{t}}$
- Simplify

  $y = \frac{c_1 \mathrm{e}^{-t} + t - 1}{t}$
- Use initial condition $y(\ln(2)) = 1$

  $1 = \frac{\frac{c_1}{2} + \ln(2) - 1}{\ln(2)}$
- Solve for c_1

  $c_1 = 2$
- Substitute $c_1 = 2$ into general solution and simplify

  $y = \frac{-1 + 2\, \mathrm{e}^{-t} + t}{t}$
- Solution to the IVP

  $y = \frac{-1 + 2\, \mathrm{e}^{-t} + t}{t}$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
<- 1st order linear successful`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 17

```
dsolve([(1+t)*y(t)+t*diff(y(t),t) = t,y(ln(2)) = 1],y(t), singsol=all)
```

$$y(t) = \frac{t - 1 + 2\,\mathrm{e}^{-t}}{t}$$

✓ Solution by Mathematica

Time used: 0.034 (sec). Leaf size: 23

```
DSolve[{(1+t)*y[t]+t*y'[t]== t,y[Log[2]]==1},y[t],t,IncludeSingularSolutions -> True]
```

$$y(t) \to \frac{e^{-t}(e^t(t-1)+2)}{t}$$

1.21 problem 21

Internal problem ID [468]
Internal file name [`OUTPUT/468_Sunday_June_05_2022_01_41_57_AM_10339982/index.tex`]

Book: Elementary differential equations and boundary value problems, 10th ed., Boyce and DiPrima
Section: Section 2.1. Page 40
Problem number: 21.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"linear", "exactWithIntegrationFactor", "first_order_ode_lie_symmetry_lookup"**

Maple gives the following as the ode type

```
[[_linear, `class A`]]
```

$$\boxed{-\frac{y}{2}+y'=2\cos\left(t\right)}$$

With initial conditions

$$[y(0)=a]$$

1.21.1 Existence and uniqueness analysis

This is a linear ODE. In canonical form it is written as

$$y'+p(t)y=q(t)$$

Where here

$$p(t)=-\frac{1}{2}$$
$$q(t)=2\cos\left(t\right)$$

Hence the ode is

$$-\frac{y}{2}+y'=2\cos\left(t\right)$$

The domain of $p(t)=-\frac{1}{2}$ is

$$\{-\infty<t<\infty\}$$

And the point $t_0=0$ is inside this domain. The domain of $q(t)=2\cos\left(t\right)$ is

$$\{-\infty<t<\infty\}$$

And the point $t_0=0$ is also inside this domain. Hence solution exists and is unique.

1.21.2 Solving as linear ode

Entering Linear first order ODE solver. The integrating factor μ is

$$\begin{aligned}\mu &= \mathrm{e}^{\int -\frac{1}{2}dt}\\ &= \mathrm{e}^{-\frac{t}{2}}\end{aligned}$$

The ode becomes

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}(\mu y) &= (\mu)\left(2\cos\left(t\right)\right)\\ \frac{\mathrm{d}}{\mathrm{d}t}\left(\mathrm{e}^{-\frac{t}{2}}y\right) &= \left(\mathrm{e}^{-\frac{t}{2}}\right)\left(2\cos\left(t\right)\right)\\ \mathrm{d}\left(\mathrm{e}^{-\frac{t}{2}}y\right) &= \left(2\cos\left(t\right)\mathrm{e}^{-\frac{t}{2}}\right)\,\mathrm{d}t\end{aligned}$$

Integrating gives

$$\begin{aligned}\mathrm{e}^{-\frac{t}{2}}y &= \int 2\cos\left(t\right)\mathrm{e}^{-\frac{t}{2}}\,\mathrm{d}t\\ \mathrm{e}^{-\frac{t}{2}}y &= -\frac{4\cos\left(t\right)\mathrm{e}^{-\frac{t}{2}}}{5}+\frac{8\sin\left(t\right)\mathrm{e}^{-\frac{t}{2}}}{5}+c_1\end{aligned}$$

Dividing both sides by the integrating factor $\mu=\mathrm{e}^{-\frac{t}{2}}$ results in

$$y=\mathrm{e}^{\frac{t}{2}}\left(-\frac{4\cos\left(t\right)\mathrm{e}^{-\frac{t}{2}}}{5}+\frac{8\sin\left(t\right)\mathrm{e}^{-\frac{t}{2}}}{5}\right)+c_1\mathrm{e}^{\frac{t}{2}}$$

which simplifies to

$$y = c_1 e^{\frac{t}{2}} + \frac{8 \sin(t)}{5} - \frac{4 \cos(t)}{5}$$

Initial conditions are used to solve for c_1. Substituting $t = 0$ and $y = a$ in the above solution gives an equation to solve for the constant of integration.

$$a = c_1 - \frac{4}{5}$$

$$c_1 = \frac{4}{5} + a$$

Substituting c_1 found above in the general solution gives

$$y = \left(\frac{4}{5} + a\right) e^{\frac{t}{2}} + \frac{8 \sin(t)}{5} - \frac{4 \cos(t)}{5}$$

Summary
The solution(s) found are the following

$$y = \left(\frac{4}{5} + a\right) e^{\frac{t}{2}} + \frac{8 \sin(t)}{5} - \frac{4 \cos(t)}{5} \tag{1}$$

Verification of solutions

$$y = \left(\frac{4}{5} + a\right) e^{\frac{t}{2}} + \frac{8 \sin(t)}{5} - \frac{4 \cos(t)}{5}$$

Verified OK.

1.21.3 Solving as first order ode lie symmetry lookup ode

Writing the ode as

$$y' = \frac{y}{2} + 2 \cos(t)$$
$$y' = \omega(t, y)$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_t + \omega(\eta_y - \xi_t) - \omega^2 \xi_y - \omega_t \xi - \omega_y \eta = 0 \tag{A}$$

The type of this ode is known. It is of type `linear`. Therefore we do not need to solve the PDE (A), and can just use the lookup table shown below to find ξ, η

Table 61: Lie symmetry infinitesimal lookup table for known first order ODE's

ODE class	Form	ξ	η
linear ode	$y' = f(x)y(x) + g(x)$	0	$e^{\int f dx}$
separable ode	$y' = f(x)\, g(y)$	$\frac{1}{f}$	0
quadrature ode	$y' = f(x)$	0	1
quadrature ode	$y' = g(y)$	1	0
homogeneous ODEs of Class A	$y' = f\left(\frac{y}{x}\right)$	x	y
homogeneous ODEs of Class C	$y' = (a + bx + cy)^{\frac{n}{m}}$	1	$-\frac{b}{c}$
homogeneous class D	$y' = \frac{y}{x} + g(x)\, F\left(\frac{y}{x}\right)$	x^2	xy
First order special form ID 1	$y' = g(x)\, e^{h(x)+by} + f(x)$	$\frac{e^{-\int bf(x)dx - h(x)}}{g(x)}$	$\frac{f(x)e^{-\int bf(x)dx - h(x)}}{g(x)}$
polynomial type ode	$y' = \frac{a_1 x + b_1 y + c_1}{a_2 x + b_2 y + c_2}$	$\frac{a_1 b_2 x - a_2 b_1 x - b_1 c_2 + b_2 c_1}{a_1 b_2 - a_2 b_1}$	$\frac{a_1 b_2 y - a_2 b_1 y - a_1 c_2 - a_2 c_1}{a_1 b_2 - a_2 b_1}$
Bernoulli ode	$y' = f(x)\, y + g(x)\, y^n$	0	$e^{-\int (n-1) f(x) dx} y^n$
Reduced Riccati	$y' = f_1(x)\, y + f_2(x)\, y^2$	0	$e^{-\int f_1 dx}$

The above table shows that

$$
\begin{aligned}
\xi(t, y) &= 0 \\
\eta(t, y) &= \mathrm{e}^{\frac{t}{2}}
\end{aligned} \tag{A1}
$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(t, y) \to (R, S)$ where (R, S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$
\frac{dt}{\xi} = \frac{dy}{\eta} = dS \tag{1}
$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial t} + \eta \frac{\partial}{\partial y}\right) S(t, y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the

canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$R = t$$

S is found from

$$\begin{aligned} S &= \int \frac{1}{\eta} dy \\ &= \int \frac{1}{e^{\frac{t}{2}}} dy \end{aligned}$$

Which results in

$$S = e^{-\frac{t}{2}} y$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_t + \omega(t, y) S_y}{R_t + \omega(t, y) R_y} \tag{2}$$

Where in the above R_t, R_y, S_t, S_y are all partial derivatives and $\omega(t, y)$ is the right hand side of the original ode given by

$$\omega(t, y) = \frac{y}{2} + 2\cos(t)$$

Evaluating all the partial derivatives gives

$$\begin{aligned} R_t &= 1 \\ R_y &= 0 \\ S_t &= -\frac{e^{-\frac{t}{2}} y}{2} \\ S_y &= e^{-\frac{t}{2}} \end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = 2\cos(t)\, e^{-\frac{t}{2}} \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for t, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = 2\cos(R)\, e^{-\frac{R}{2}}$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordinates R, S. Integrating the above gives

$$S(R) = c_1 - \frac{4\,\mathrm{e}^{-\frac{R}{2}}(\cos(R) - 2\sin(R))}{5} \tag{4}$$

To complete the solution, we just need to transform (4) back to t, y coordinates. This results in

$$\mathrm{e}^{-\frac{t}{2}} y = c_1 - \frac{4\,\mathrm{e}^{-\frac{t}{2}}(\cos(t) - 2\sin(t))}{5}$$

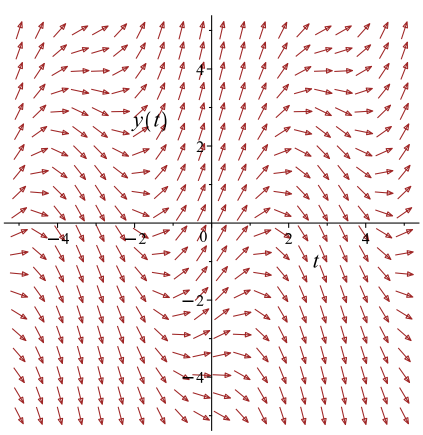
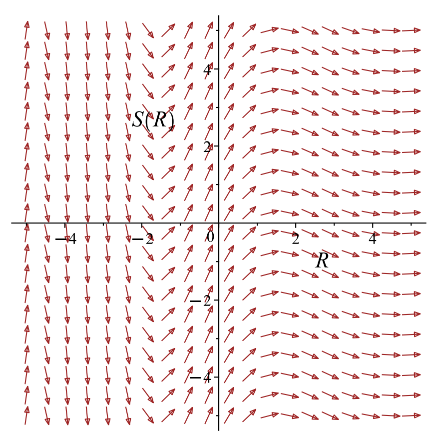
Which simplifies to

$$\mathrm{e}^{-\frac{t}{2}} y = c_1 - \frac{4\,\mathrm{e}^{-\frac{t}{2}}(\cos(t) - 2\sin(t))}{5}$$

Which gives

$$y = -\frac{\mathrm{e}^{\frac{t}{2}}\left(4\cos(t)\,\mathrm{e}^{-\frac{t}{2}} - 8\sin(t)\,\mathrm{e}^{-\frac{t}{2}} - 5c_1\right)}{5}$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in t, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dt} = \frac{y}{2} + 2\cos(t)$ 	$R = t$ $S = \mathrm{e}^{-\frac{t}{2}} y$	$\frac{dS}{dR} = 2\cos(R)\,\mathrm{e}^{-\frac{R}{2}}$ 

Initial conditions are used to solve for c_1. Substituting $t = 0$ and $y = a$ in the above solution gives an equation to solve for the constant of integration.

$$a = c_1 - \frac{4}{5}$$

$$c_1 = \frac{4}{5} + a$$

Substituting c_1 found above in the general solution gives

$$y = -\frac{\left(4\cos(t)\,\mathrm{e}^{-\frac{t}{2}} - 8\sin(t)\,\mathrm{e}^{-\frac{t}{2}} - 4 - 5a\right)\mathrm{e}^{\frac{t}{2}}}{5}$$

Summary

The solution(s) found are the following

$$y = -\frac{\left(4\cos(t)\,\mathrm{e}^{-\frac{t}{2}} - 8\sin(t)\,\mathrm{e}^{-\frac{t}{2}} - 4 - 5a\right)\mathrm{e}^{\frac{t}{2}}}{5} \tag{1}$$

Verification of solutions

$$y = -\frac{\left(4\cos(t)\,\mathrm{e}^{-\frac{t}{2}} - 8\sin(t)\,\mathrm{e}^{-\frac{t}{2}} - 4 - 5a\right)\mathrm{e}^{\frac{t}{2}}}{5}$$

Verified OK.

1.21.4 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x,y) + N(x,y)\frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x,y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y) = 0$$

Hence

$$\frac{\partial \phi}{\partial x} + \frac{\partial \phi}{\partial y}\frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$\begin{aligned}\frac{\partial\phi}{\partial x} &= M\\ \frac{\partial\phi}{\partial y} &= N\end{aligned}$$

But since $\frac{\partial^2\phi}{\partial x\partial y} = \frac{\partial^2\phi}{\partial y\partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x,y)$ but at least we know now that we can do that since the condition $\frac{\partial^2\phi}{\partial x\partial y} = \frac{\partial^2\phi}{\partial y\partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(t,y)\,\mathrm{d}t + N(t,y)\,\mathrm{d}y = 0 \tag{1A}$$

Therefore

$$\begin{aligned}\mathrm{d}y &= \left(\frac{y}{2} + 2\cos(t)\right)\mathrm{d}t\\ \left(-\frac{y}{2} - 2\cos(t)\right)\mathrm{d}t + \mathrm{d}y &= 0 \qquad (2A)\end{aligned}$$

Comparing (1A) and (2A) shows that

$$\begin{aligned}M(t,y) &= -\frac{y}{2} - 2\cos(t)\\ N(t,y) &= 1\end{aligned}$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial t}$$

Using result found above gives

$$\begin{aligned}\frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}\left(-\frac{y}{2} - 2\cos(t)\right)\\ &= -\frac{1}{2}\end{aligned}$$

And

$$
\begin{aligned}
\frac{\partial N}{\partial t} &= \frac{\partial}{\partial t}(1) \\
&= 0
\end{aligned}
$$

Since $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial t}$, then the ODE is <u>not exact</u>. Since the ODE is not exact, we will try to find an integrating factor to make it exact. Let

$$
\begin{aligned}
A &= \frac{1}{N}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial t}\right) \\
&= 1\left(\left(-\frac{1}{2}\right) - (0)\right) \\
&= -\frac{1}{2}
\end{aligned}
$$

Since A does not depend on y, then it can be used to find an integrating factor. The integrating factor μ is

$$
\begin{aligned}
\mu &= e^{\int A \, \mathrm{d}t} \\
&= e^{\int -\frac{1}{2} \, \mathrm{d}t}
\end{aligned}
$$

The result of integrating gives

$$
\begin{aligned}
\mu &= e^{-\frac{t}{2}} \\
&= \mathrm{e}^{-\frac{t}{2}}
\end{aligned}
$$

M and N are multiplied by this integrating factor, giving new M and new N which are called $\overline{M}$ and $\overline{N}$ for now so not to confuse them with the original M and N.

$$
\begin{aligned}
\overline{M} &= \mu M \\
&= \mathrm{e}^{-\frac{t}{2}}\left(-\frac{y}{2} - 2\cos(t)\right) \\
&= -\frac{(y + 4\cos(t))\,\mathrm{e}^{-\frac{t}{2}}}{2}
\end{aligned}
$$

And

$$
\begin{aligned}
\overline{N} &= \mu N \\
&= \mathrm{e}^{-\frac{t}{2}}(1) \\
&= \mathrm{e}^{-\frac{t}{2}}
\end{aligned}
$$

Now a modified ODE is ontained from the original ODE, which is exact and can be solved. The modified ODE is

$$\overline{M} + \overline{N}\frac{\mathrm{d}y}{\mathrm{d}t} = 0$$
$$\left(-\frac{(y + 4\cos(t))\,\mathrm{e}^{-\frac{t}{2}}}{2}\right) + \left(\mathrm{e}^{-\frac{t}{2}}\right)\frac{\mathrm{d}y}{\mathrm{d}t} = 0$$

The following equations are now set up to solve for the function $\phi(t, y)$

$$\frac{\partial\phi}{\partial t} = \overline{M} \tag{1}$$
$$\frac{\partial\phi}{\partial y} = \overline{N} \tag{2}$$

Integrating (1) w.r.t. t gives

$$\int \frac{\partial\phi}{\partial t}\,\mathrm{d}t = \int \overline{M}\,\mathrm{d}t$$
$$\int \frac{\partial\phi}{\partial t}\,\mathrm{d}t = \int -\frac{(y + 4\cos(t))\,\mathrm{e}^{-\frac{t}{2}}}{2}\,\mathrm{d}t$$
$$\phi = \frac{(5y + 4\cos(t) - 8\sin(t))\,\mathrm{e}^{-\frac{t}{2}}}{5} + f(y) \tag{3}$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both t and y. Taking derivative of equation (3) w.r.t y gives

$$\frac{\partial\phi}{\partial y} = \mathrm{e}^{-\frac{t}{2}} + f'(y) \tag{4}$$

But equation (2) says that $\frac{\partial\phi}{\partial y} = \mathrm{e}^{-\frac{t}{2}}$. Therefore equation (4) becomes

$$\mathrm{e}^{-\frac{t}{2}} = \mathrm{e}^{-\frac{t}{2}} + f'(y) \tag{5}$$

Solving equation (5) for $f'(y)$ gives

$$f'(y) = 0$$

Therefore

$$f(y) = c_1$$

Where c_1 is constant of integration. Substituting this result for $f(y)$ into equation (3) gives ϕ

$$\phi = \frac{\left(5y + 4\cos\left(t\right) - 8\sin\left(t\right)\right) e^{-\frac{t}{2}}}{5} + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = \frac{\left(5y + 4\cos\left(t\right) - 8\sin\left(t\right)\right) e^{-\frac{t}{2}}}{5}$$

The solution becomes

$$y = -\frac{e^{\frac{t}{2}} \left(4\cos\left(t\right) e^{-\frac{t}{2}} - 8\sin\left(t\right) e^{-\frac{t}{2}} - 5c_1\right)}{5}$$

Initial conditions are used to solve for c_1. Substituting $t = 0$ and $y = a$ in the above solution gives an equation to solve for the constant of integration.

$$a = c_1 - \frac{4}{5}$$

$$c_1 = \frac{4}{5} + a$$

Substituting c_1 found above in the general solution gives

$$y = -\frac{\left(4\cos\left(t\right) e^{-\frac{t}{2}} - 8\sin\left(t\right) e^{-\frac{t}{2}} - 4 - 5a\right) e^{\frac{t}{2}}}{5}$$

Summary

The solution(s) found are the following

$$y = -\frac{\left(4\cos\left(t\right) e^{-\frac{t}{2}} - 8\sin\left(t\right) e^{-\frac{t}{2}} - 4 - 5a\right) e^{\frac{t}{2}}}{5} \tag{1}$$

Verification of solutions

$$y = -\frac{\left(4\cos\left(t\right) e^{-\frac{t}{2}} - 8\sin\left(t\right) e^{-\frac{t}{2}} - 4 - 5a\right) e^{\frac{t}{2}}}{5}$$

Verified OK.

1.21.5 Maple step by step solution

Let's solve

$\left[-\frac{y}{2}+y'=2\cos\left(t\right), y(0)=a\right]$

- Highest derivative means the order of the ODE is 1

  y'

- Isolate the derivative

  $y'=\frac{y}{2}+2\cos\left(t\right)$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE

  $-\frac{y}{2}+y'=2\cos\left(t\right)$

- The ODE is linear; multiply by an integrating factor $\mu(t)$

  $\mu(t)\left(-\frac{y}{2}+y'\right)=2\mu(t)\cos\left(t\right)$

- Assume the lhs of the ODE is the total derivative $\frac{d}{dt}(\mu(t)\,y)$

  $\mu(t)\left(-\frac{y}{2}+y'\right)=\mu'(t)\,y+\mu(t)\,y'$

- Isolate $\mu'(t)$

  $\mu'(t)=-\frac{\mu(t)}{2}$

- Solve to find the integrating factor

  $\mu(t)=\mathrm{e}^{-\frac{t}{2}}$

- Integrate both sides with respect to t

  $\int\left(\frac{d}{dt}(\mu(t)\,y)\right)dt=\int 2\mu(t)\cos\left(t\right)dt+c_1$

- Evaluate the integral on the lhs

  $\mu(t)\,y=\int 2\mu(t)\cos\left(t\right)dt+c_1$

- Solve for y

  $y=\frac{\int 2\mu(t)\cos(t)dt+c_1}{\mu(t)}$

- Substitute $\mu(t)=\mathrm{e}^{-\frac{t}{2}}$

  $y=\frac{\int 2\cos(t)\mathrm{e}^{-\frac{t}{2}}dt+c_1}{\mathrm{e}^{-\frac{t}{2}}}$

- Evaluate the integrals on the rhs

  $y=\frac{-\frac{4\cos(t)\mathrm{e}^{-\frac{t}{2}}}{5}+\frac{8\sin(t)\mathrm{e}^{-\frac{t}{2}}}{5}+c_1}{\mathrm{e}^{-\frac{t}{2}}}$

- Simplify

$$y = c_1 e^{\frac{t}{2}} + \frac{8\sin(t)}{5} - \frac{4\cos(t)}{5}$$

- Use initial condition $y(0) = a$

$$a = c_1 - \frac{4}{5}$$

- Solve for c_1

$$c_1 = \frac{4}{5} + a$$

- Substitute $c_1 = \frac{4}{5} + a$ into general solution and simplify

$$y = e^{\frac{t}{2}} a - \frac{4\cos(t)}{5} + \frac{8\sin(t)}{5} + \frac{4 e^{\frac{t}{2}}}{5}$$

- Solution to the IVP

$$y = e^{\frac{t}{2}} a - \frac{4\cos(t)}{5} + \frac{8\sin(t)}{5} + \frac{4 e^{\frac{t}{2}}}{5}$$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
<- 1st order linear successful`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 21

```
dsolve([-1/2*y(t)+diff(y(t),t) = 2*cos(t),y(0) = a],y(t), singsol=all)
```

$$y(t) = -\frac{4\cos(t)}{5} + \frac{8\sin(t)}{5} + e^{\frac{t}{2}} a + \frac{4 e^{\frac{t}{2}}}{5}$$

✓ Solution by Mathematica

Time used: 0.054 (sec). Leaf size: 31

```
DSolve[{-1/2*y[t]+y'[t] == 2*Cos[t],y[0]==a},y[t],t,IncludeSingularSolutions -> True]
```

$$y(t) \to \frac{1}{5}\left((5a+4)e^{t/2} + 8\sin(t) - 4\cos(t)\right)$$

1.22 problem 22

Internal problem ID [469]
Internal file name [`OUTPUT/469_Sunday_June_05_2022_01_41_59_AM_35727842/index.tex`]

Book: Elementary differential equations and boundary value problems, 10th ed., Boyce and DiPrima
Section: Section 2.1. Page 40
Problem number: 22.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"linear", "exactWithIntegrationFactor", "first_order_ode_lie_symmetry_lookup"**

Maple gives the following as the ode type

```
[[_linear, `class A`]]
```

$$\boxed{-y + 2y' = \mathrm{e}^{\frac{t}{3}}}$$

With initial conditions

$$[y(0) = a]$$

1.22.1 Existence and uniqueness analysis

This is a linear ODE. In canonical form it is written as

$$y' + p(t)y = q(t)$$

Where here

$$p(t) = -\frac{1}{2}$$
$$q(t) = \frac{\mathrm{e}^{\frac{t}{3}}}{2}$$

Hence the ode is

$$-\frac{y}{2}+y'=\frac{\mathrm{e}^{\frac{t}{3}}}{2}$$

The domain of $p(t)=-\frac{1}{2}$ is

$$\{-\infty<t<\infty\}$$

And the point $t_0=0$ is inside this domain. The domain of $q(t)=\frac{\mathrm{e}^{\frac{t}{3}}}{2}$ is

$$\{-\infty<t<\infty\}$$

And the point $t_0=0$ is also inside this domain. Hence solution exists and is unique.

1.22.2 Solving as linear ode

Entering Linear first order ODE solver. The integrating factor μ is

$$\begin{aligned}\mu &= \mathrm{e}^{\int -\frac{1}{2}dt}\\ &= \mathrm{e}^{-\frac{t}{2}}\end{aligned}$$

The ode becomes

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}(\mu y) &= (\mu)\left(\frac{\mathrm{e}^{\frac{t}{3}}}{2}\right)\\ \frac{\mathrm{d}}{\mathrm{d}t}\left(\mathrm{e}^{-\frac{t}{2}}y\right) &= \left(\mathrm{e}^{-\frac{t}{2}}\right)\left(\frac{\mathrm{e}^{\frac{t}{3}}}{2}\right)\\ \mathrm{d}\left(\mathrm{e}^{-\frac{t}{2}}y\right) &= \left(\frac{\mathrm{e}^{-\frac{t}{6}}}{2}\right)\mathrm{d}t\end{aligned}$$

Integrating gives

$$\begin{aligned}\mathrm{e}^{-\frac{t}{2}}y &= \int \frac{\mathrm{e}^{-\frac{t}{6}}}{2}\,\mathrm{d}t\\ \mathrm{e}^{-\frac{t}{2}}y &= -3\,\mathrm{e}^{-\frac{t}{6}}+c_1\end{aligned}$$

Dividing both sides by the integrating factor $\mu=\mathrm{e}^{-\frac{t}{2}}$ results in

$$y=-3\,\mathrm{e}^{\frac{t}{2}}\mathrm{e}^{-\frac{t}{6}}+c_1\mathrm{e}^{\frac{t}{2}}$$

which simplifies to

$$y = -3\,\mathrm{e}^{\frac{t}{3}} + c_1 \mathrm{e}^{\frac{t}{2}}$$

Initial conditions are used to solve for c_1. Substituting $t = 0$ and $y = a$ in the above solution gives an equation to solve for the constant of integration.

$$a = -3 + c_1$$

$$c_1 = 3 + a$$

Substituting c_1 found above in the general solution gives

$$y = -3\,\mathrm{e}^{\frac{t}{3}} + (3 + a)\,\mathrm{e}^{\frac{t}{2}}$$

Summary
The solution(s) found are the following

$$y = -3\,\mathrm{e}^{\frac{t}{3}} + (3 + a)\,\mathrm{e}^{\frac{t}{2}} \tag{1}$$

Verification of solutions

$$y = -3\,\mathrm{e}^{\frac{t}{3}} + (3 + a)\,\mathrm{e}^{\frac{t}{2}}$$

Verified OK.

1.22.3 Solving as first order ode lie symmetry lookup ode

Writing the ode as

$$y' = \frac{y}{2} + \frac{\mathrm{e}^{\frac{t}{3}}}{2}$$
$$y' = \omega(t, y)$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_t + \omega(\eta_y - \xi_t) - \omega^2 \xi_y - \omega_t \xi - \omega_y \eta = 0 \tag{A}$$

The type of this ode is known. It is of type `linear`. Therefore we do not need to solve the PDE (A), and can just use the lookup table shown below to find ξ, η

Table 64: Lie symmetry infinitesimal lookup table for known first order ODE's

ODE class	Form	ξ	η
linear ode	$y' = f(x)y(x) + g(x)$	0	$e^{\int f dx}$
separable ode	$y' = f(x)\, g(y)$	$\frac{1}{f}$	0
quadrature ode	$y' = f(x)$	0	1
quadrature ode	$y' = g(y)$	1	0
homogeneous ODEs of Class A	$y' = f\left(\frac{y}{x}\right)$	x	y
homogeneous ODEs of Class C	$y' = (a + bx + cy)^{\frac{n}{m}}$	1	$-\frac{b}{c}$
homogeneous class D	$y' = \frac{y}{x} + g(x)\, F\left(\frac{y}{x}\right)$	x^2	xy
First order special form ID 1	$y' = g(x)\, e^{h(x)+by} + f(x)$	$\frac{e^{-\int bf(x)dx-h(x)}}{g(x)}$	$\frac{f(x)e^{-\int bf(x)dx-h(x)}}{g(x)}$
polynomial type ode	$y' = \frac{a_1x+b_1y+c_1}{a_2x+b_2y+c_2}$	$\frac{a_1b_2x-a_2b_1x-b_1c_2+b_2c_1}{a_1b_2-a_2b_1}$	$\frac{a_1b_2y-a_2b_1y-a_1c_2-a_2c_1}{a_1b_2-a_2b_1}$
Bernoulli ode	$y' = f(x)\, y + g(x)\, y^n$	0	$e^{-\int(n-1)f(x)dx} y^n$
Reduced Riccati	$y' = f_1(x)\, y + f_2(x)\, y^2$	0	$e^{-\int f_1 dx}$

The above table shows that

$$\begin{aligned} \xi(t, y) &= 0 \\ \eta(t, y) &= e^{\frac{t}{2}} \end{aligned} \tag{A1}$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(t, y) \to (R, S)$ where (R, S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$\frac{dt}{\xi} = \frac{dy}{\eta} = dS \tag{1}$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial t} + \eta \frac{\partial}{\partial y}\right) S(t, y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the

canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$R = t$$

S is found from

$$\begin{aligned} S &= \int \frac{1}{\eta} dy \\ &= \int \frac{1}{e^{\frac{t}{2}}} dy \end{aligned}$$

Which results in

$$S = e^{-\frac{t}{2}} y$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_t + \omega(t, y) S_y}{R_t + \omega(t, y) R_y} \tag{2}$$

Where in the above R_t, R_y, S_t, S_y are all partial derivatives and $\omega(t, y)$ is the right hand side of the original ode given by

$$\omega(t, y) = \frac{y}{2} + \frac{e^{\frac{t}{3}}}{2}$$

Evaluating all the partial derivatives gives

$$\begin{aligned} R_t &= 1 \\ R_y &= 0 \\ S_t &= -\frac{e^{-\frac{t}{2}} y}{2} \\ S_y &= e^{-\frac{t}{2}} \end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = \frac{e^{-\frac{t}{6}}}{2} \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for t, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = \frac{e^{-\frac{R}{6}}}{2}$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordinates R, S. Integrating the above gives

$$S(R) = -3\,\mathrm{e}^{-\frac{R}{6}} + c_1 \tag{4}$$

To complete the solution, we just need to transform (4) back to t, y coordinates. This results in

$$\mathrm{e}^{-\frac{t}{2}} y = -3\,\mathrm{e}^{-\frac{t}{6}} + c_1$$

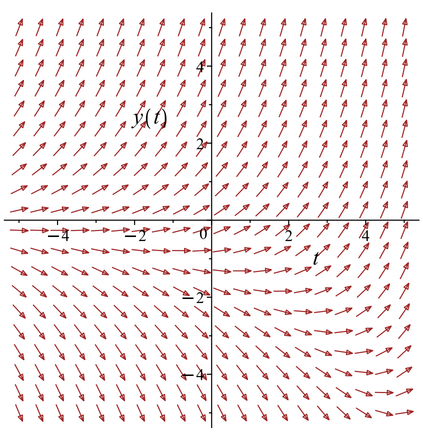
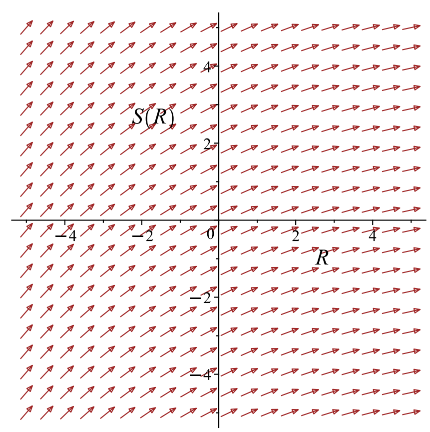
Which simplifies to

$$\mathrm{e}^{-\frac{t}{2}} y = -3\,\mathrm{e}^{-\frac{t}{6}} + c_1$$

Which gives

$$y = -\left(3\,\mathrm{e}^{-\frac{t}{6}} - c_1\right) \mathrm{e}^{\frac{t}{2}}$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in t, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dt} = \frac{y}{2} + \frac{\mathrm{e}^{\frac{t}{3}}}{2}$  	$R = t$ $S = \mathrm{e}^{-\frac{t}{2}} y$	$\frac{dS}{dR} = \frac{\mathrm{e}^{-\frac{R}{6}}}{2}$ 

Initial conditions are used to solve for c_1. Substituting $t = 0$ and $y = a$ in the above solution gives an equation to solve for the constant of integration.

$$a = -3 + c_1$$

$$c_1 = 3 + a$$

Substituting c_1 found above in the general solution gives

$$y = -\left(3\,\mathrm{e}^{-\frac{t}{6}} - 3 - a\right)\mathrm{e}^{\frac{t}{2}}$$

Summary

The solution(s) found are the following

$$y = -\left(3\,\mathrm{e}^{-\frac{t}{6}} - 3 - a\right)\mathrm{e}^{\frac{t}{2}} \tag{1}$$

Verification of solutions

$$y = -\left(3\,\mathrm{e}^{-\frac{t}{6}} - 3 - a\right)\mathrm{e}^{\frac{t}{2}}$$

Verified OK.

1.22.4 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x,y) + N(x,y)\frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x,y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y) = 0$$

Hence

$$\frac{\partial \phi}{\partial x} + \frac{\partial \phi}{\partial y}\frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$\frac{\partial \phi}{\partial x} = M$$
$$\frac{\partial \phi}{\partial y} = N$$

But since $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x,y)$ but at least we know now that we can do that since the condition $\frac{\partial^2\phi}{\partial x\partial y} = \frac{\partial^2\phi}{\partial y\partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(t,y)\,\mathrm{d}t + N(t,y)\,\mathrm{d}y = 0 \tag{1A}$$

Therefore

$$\begin{aligned}(2)\,\mathrm{d}y &= \left(y + \mathrm{e}^{\frac{t}{3}}\right)\mathrm{d}t\\ \left(-y - \mathrm{e}^{\frac{t}{3}}\right)\mathrm{d}t + (2)\,\mathrm{d}y &= 0 \end{aligned} \tag{2A}$$

Comparing (1A) and (2A) shows that

$$\begin{aligned}M(t,y) &= -y - \mathrm{e}^{\frac{t}{3}}\\ N(t,y) &= 2\end{aligned}$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial t}$$

Using result found above gives

$$\begin{aligned}\frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}\left(-y - \mathrm{e}^{\frac{t}{3}}\right)\\ &= -1\end{aligned}$$

And

$$\begin{aligned}\frac{\partial N}{\partial t} &= \frac{\partial}{\partial t}(2)\\ &= 0\end{aligned}$$

Since $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial t}$, then the ODE is <u>not exact</u>. Since the ODE is not exact, we will try to find an integrating factor to make it exact. Let

$$\begin{aligned}A &= \frac{1}{N}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial t}\right)\\ &= \frac{1}{2}((-1) - (0))\\ &= -\frac{1}{2}\end{aligned}$$

Since A does not depend on y, then it can be used to find an integrating factor. The integrating factor μ is

$$
\begin{aligned}
\mu &= e^{\int A \, \mathrm{d}t} \\
&= e^{\int -\frac{1}{2} \, \mathrm{d}t}
\end{aligned}
$$

The result of integrating gives

$$
\begin{aligned}
\mu &= e^{-\frac{t}{2}} \\
&= \mathrm{e}^{-\frac{t}{2}}
\end{aligned}
$$

M and N are multiplied by this integrating factor, giving new M and new N which are called $\overline{M}$ and $\overline{N}$ for now so not to confuse them with the original M and N.

$$
\begin{aligned}
\overline{M} &= \mu M \\
&= \mathrm{e}^{-\frac{t}{2}} \left(-y - \mathrm{e}^{\frac{t}{3}}\right) \\
&= -\left(y + \mathrm{e}^{\frac{t}{3}}\right) \mathrm{e}^{-\frac{t}{2}}
\end{aligned}
$$

And

$$
\begin{aligned}
\overline{N} &= \mu N \\
&= \mathrm{e}^{-\frac{t}{2}} (2) \\
&= 2\,\mathrm{e}^{-\frac{t}{2}}
\end{aligned}
$$

Now a modified ODE is ontained from the original ODE, which is exact and can be solved. The modified ODE is

$$
\begin{aligned}
\overline{M} + \overline{N} \frac{\mathrm{d}y}{\mathrm{d}t} &= 0 \\
\left(-\left(y + \mathrm{e}^{\frac{t}{3}}\right) \mathrm{e}^{-\frac{t}{2}}\right) + \left(2\,\mathrm{e}^{-\frac{t}{2}}\right) \frac{\mathrm{d}y}{\mathrm{d}t} &= 0
\end{aligned}
$$

The following equations are now set up to solve for the function $\phi(t, y)$

$$
\frac{\partial \phi}{\partial t} = \overline{M} \tag{1}
$$

$$
\frac{\partial \phi}{\partial y} = \overline{N} \tag{2}
$$

Integrating (1) w.r.t. t gives

$$
\begin{aligned}
\int \frac{\partial \phi}{\partial t} \, \mathrm{d}t &= \int \overline{M} \, \mathrm{d}t \\
\int \frac{\partial \phi}{\partial t} \, \mathrm{d}t &= \int -\left(y + \mathrm{e}^{\frac{t}{3}}\right) \mathrm{e}^{-\frac{t}{2}} \, \mathrm{d}t
\end{aligned}
$$

$$
\phi = 2\,\mathrm{e}^{-\frac{t}{2}} y + 6\,\mathrm{e}^{-\frac{t}{6}} + f(y) \tag{3}
$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both t and y. Taking derivative of equation (3) w.r.t y gives

$$\frac{\partial \phi}{\partial y} = 2\,\mathrm{e}^{-\frac{t}{2}} + f'(y) \tag{4}$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = 2\,\mathrm{e}^{-\frac{t}{2}}$. Therefore equation (4) becomes

$$2\,\mathrm{e}^{-\frac{t}{2}} = 2\,\mathrm{e}^{-\frac{t}{2}} + f'(y) \tag{5}$$

Solving equation (5) for $f'(y)$ gives

$$f'(y) = 0$$

Therefore

$$f(y) = c_1$$

Where c_1 is constant of integration. Substituting this result for $f(y)$ into equation (3) gives ϕ

$$\phi = 2\,\mathrm{e}^{-\frac{t}{2}} y + 6\,\mathrm{e}^{-\frac{t}{6}} + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = 2\,\mathrm{e}^{-\frac{t}{2}} y + 6\,\mathrm{e}^{-\frac{t}{6}}$$

The solution becomes

$$y = -\frac{\left(6\,\mathrm{e}^{-\frac{t}{6}} - c_1\right) \mathrm{e}^{\frac{t}{2}}}{2}$$

Initial conditions are used to solve for c_1. Substituting $t = 0$ and $y = a$ in the above solution gives an equation to solve for the constant of integration.

$$a = -3 + \frac{c_1}{2}$$

$$c_1 = 6 + 2a$$

Substituting c_1 found above in the general solution gives

$$y = -\frac{\left(6\,\mathrm{e}^{-\frac{t}{6}}-6-2a\right)\mathrm{e}^{\frac{t}{2}}}{2}$$

Summary

The solution(s) found are the following

$$y = -\frac{\left(6\,\mathrm{e}^{-\frac{t}{6}}-6-2a\right)\mathrm{e}^{\frac{t}{2}}}{2} \tag{1}$$

Verification of solutions

$$y = -\frac{\left(6\,\mathrm{e}^{-\frac{t}{6}}-6-2a\right)\mathrm{e}^{\frac{t}{2}}}{2}$$

Verified OK.

1.22.5 Maple step by step solution

Let's solve

$\left[-y+2y'=\mathrm{e}^{\frac{t}{3}}, y(0)=a\right]$

- Highest derivative means the order of the ODE is 1

  y'
- Isolate the derivative

  $y'=\frac{y}{2}+\frac{\mathrm{e}^{\frac{t}{3}}}{2}$
- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE

  $-\frac{y}{2}+y'=\frac{\mathrm{e}^{\frac{t}{3}}}{2}$
- The ODE is linear; multiply by an integrating factor $\mu(t)$

  $\mu(t)\left(-\frac{y}{2}+y'\right)=\frac{\mu(t)\mathrm{e}^{\frac{t}{3}}}{2}$
- Assume the lhs of the ODE is the total derivative $\frac{d}{dt}(\mu(t)\,y)$

  $\mu(t)\left(-\frac{y}{2}+y'\right)=\mu'(t)\,y+\mu(t)\,y'$
- Isolate $\mu'(t)$

  $\mu'(t)=-\frac{\mu(t)}{2}$
- Solve to find the integrating factor

$\mu(t) = \mathrm{e}^{-\frac{t}{2}}$

- Integrate both sides with respect to t

$$\int \left(\frac{d}{dt}(\mu(t)\,y)\right) dt = \int \frac{\mu(t)\mathrm{e}^{\frac{t}{3}}}{2} dt + c_1$$

- Evaluate the integral on the lhs

$$\mu(t)\,y = \int \frac{\mu(t)\mathrm{e}^{\frac{t}{3}}}{2} dt + c_1$$

- Solve for y

$$y = \frac{\int \frac{\mu(t)\mathrm{e}^{\frac{t}{3}}}{2} dt + c_1}{\mu(t)}$$

- Substitute $\mu(t) = \mathrm{e}^{-\frac{t}{2}}$

$$y = \frac{\int \frac{\mathrm{e}^{\frac{t}{3}}\mathrm{e}^{-\frac{t}{2}}}{2} dt + c_1}{\mathrm{e}^{-\frac{t}{2}}}$$

- Evaluate the integrals on the rhs

$$y = \frac{-3\,\mathrm{e}^{-\frac{t}{6}} + c_1}{\mathrm{e}^{-\frac{t}{2}}}$$

- Simplify

$$y = \mathrm{e}^{\frac{t}{3}}\left(\mathrm{e}^{\frac{t}{6}} c_1 - 3\right)$$

- Use initial condition $y(0) = a$

$$a = -3 + c_1$$

- Solve for c_1

$$c_1 = 3 + a$$

- Substitute $c_1 = 3 + a$ into general solution and simplify

$$y = \mathrm{e}^{\frac{t}{3}}\left(\mathrm{e}^{\frac{t}{6}}(3 + a) - 3\right)$$

- Solution to the IVP

$$y = \mathrm{e}^{\frac{t}{3}}\left(\mathrm{e}^{\frac{t}{6}}(3 + a) - 3\right)$$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
<- 1st order linear successful`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 18

```
dsolve([-y(t)+2*diff(y(t),t) = exp(1/3*t),y(0) = a],y(t), singsol=all)
```

$$y(t) = e^{\frac{t}{3}}\left(-3 + (a+3)\, e^{\frac{t}{6}}\right)$$

✓ Solution by Mathematica

Time used: 0.052 (sec). Leaf size: 26

```
DSolve[{-y[t]+2*y'[t] == Exp[1/3*t],y[0]==a},y[t],t,IncludeSingularSolutions -> True]
```

$$y(t) \rightarrow e^{t/3}\left((a+3)e^{t/6} - 3\right)$$

1.23 problem 23

Internal problem ID [470]
Internal file name [`OUTPUT/470_Sunday_June_05_2022_01_41_59_AM_31829908/index.tex`]

Book: Elementary differential equations and boundary value problems, 10th ed., Boyce and DiPrima
Section: Section 2.1. Page 40
Problem number: 23.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"linear", "exactWithIntegrationFactor", "first_order_ode_lie_symmetry_lookup"**

Maple gives the following as the ode type

```
[[_linear, `class A`]]
```

$$\boxed{-2y + 3y' = e^{-\frac{\pi t}{2}}}$$

With initial conditions

$$[y(0) = a]$$

1.23.1 Existence and uniqueness analysis

This is a linear ODE. In canonical form it is written as

$$y' + p(t)y = q(t)$$

Where here

$$p(t) = -\frac{2}{3}$$
$$q(t) = \frac{e^{-\frac{\pi t}{2}}}{3}$$

Hence the ode is

$$y' - \frac{2y}{3} = \frac{e^{-\frac{\pi t}{2}}}{3}$$

The domain of $p(t) = -\frac{2}{3}$ is

$$\{-\infty < t < \infty\}$$

And the point $t_0 = 0$ is inside this domain. The domain of $q(t) = \frac{e^{-\frac{\pi t}{2}}}{3}$ is

$$\{-\infty < t < \infty\}$$

And the point $t_0 = 0$ is also inside this domain. Hence solution exists and is unique.

1.23.2 Solving as linear ode

Entering Linear first order ODE solver. The integrating factor μ is

$$\begin{aligned}
\mu &= e^{\int -\frac{2}{3}dt} \\
&= e^{-\frac{2t}{3}}
\end{aligned}$$

The ode becomes

$$\begin{aligned}
\frac{d}{dt}(\mu y) &= (\mu)\left(\frac{e^{-\frac{\pi t}{2}}}{3}\right) \\
\frac{d}{dt}\left(e^{-\frac{2t}{3}} y\right) &= \left(e^{-\frac{2t}{3}}\right)\left(\frac{e^{-\frac{\pi t}{2}}}{3}\right) \\
d\left(e^{-\frac{2t}{3}} y\right) &= \left(\frac{e^{t\left(-\frac{\pi}{2}-\frac{2}{3}\right)}}{3}\right) dt
\end{aligned}$$

Integrating gives

$$\begin{aligned}
e^{-\frac{2t}{3}} y &= \int \frac{e^{t\left(-\frac{\pi}{2}-\frac{2}{3}\right)}}{3} dt \\
e^{-\frac{2t}{3}} y &= -\frac{2\,e^{-\frac{t(3\pi+4)}{6}}}{3\pi+4} + c_1
\end{aligned}$$

Dividing both sides by the integrating factor $\mu = e^{-\frac{2t}{3}}$ results in

$$y = -\frac{2\,e^{\frac{2t}{3}}\,e^{-\frac{t(3\pi+4)}{6}}}{3\pi+4} + c_1 e^{\frac{2t}{3}}$$

which simplifies to

$$y = \frac{-2\,e^{-\frac{\pi t}{2}} + (3\pi + 4)\,c_1 e^{\frac{2t}{3}}}{3\pi + 4}$$

Initial conditions are used to solve for c_1. Substituting $t = 0$ and $y = a$ in the above solution gives an equation to solve for the constant of integration.

$$a = \frac{3\pi c_1 + 4c_1 - 2}{3\pi + 4}$$

$$c_1 = \frac{3\pi a + 4a + 2}{3\pi + 4}$$

Substituting c_1 found above in the general solution gives

$$y = \frac{-2\,e^{-\frac{\pi t}{2}} + (3\pi a + 4a + 2)\,e^{\frac{2t}{3}}}{3\pi + 4}$$

Summary

The solution(s) found are the following

$$y = \frac{-2\,e^{-\frac{\pi t}{2}} + (3\pi a + 4a + 2)\,e^{\frac{2t}{3}}}{3\pi + 4} \tag{1}$$

Verification of solutions

$$y = \frac{-2\,e^{-\frac{\pi t}{2}} + (3\pi a + 4a + 2)\,e^{\frac{2t}{3}}}{3\pi + 4}$$

Verified OK.

1.23.3 Solving as first order ode lie symmetry lookup ode

Writing the ode as

$$y' = \frac{2y}{3} + \frac{e^{-\frac{\pi t}{2}}}{3}$$
$$y' = \omega(t, y)$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_t + \omega(\eta_y - \xi_t) - \omega^2 \xi_y - \omega_t \xi - \omega_y \eta = 0 \tag{A}$$

The type of this ode is known. It is of type `linear`. Therefore we do not need to solve the PDE (A), and can just use the lookup table shown below to find ξ, η

Table 67: Lie symmetry infinitesimal lookup table for known first order ODE's

ODE class	Form	ξ	η
linear ode	$y' = f(x)y(x) + g(x)$	0	$e^{\int f dx}$
separable ode	$y' = f(x)\, g(y)$	$\frac{1}{f}$	0
quadrature ode	$y' = f(x)$	0	1
quadrature ode	$y' = g(y)$	1	0
homogeneous ODEs of Class A	$y' = f\left(\frac{y}{x}\right)$	x	y
homogeneous ODEs of Class C	$y' = (a + bx + cy)^{\frac{n}{m}}$	1	$-\frac{b}{c}$
homogeneous class D	$y' = \frac{y}{x} + g(x)\, F\left(\frac{y}{x}\right)$	x^2	xy
First order special form ID 1	$y' = g(x)\, e^{h(x)+by} + f(x)$	$\frac{e^{-\int bf(x)dx-h(x)}}{g(x)}$	$\frac{f(x)e^{-\int bf(x)dx-h(x)}}{g(x)}$
polynomial type ode	$y' = \frac{a_1x+b_1y+c_1}{a_2x+b_2y+c_2}$	$\frac{a_1b_2x-a_2b_1x-b_1c_2+b_2c_1}{a_1b_2-a_2b_1}$	$\frac{a_1b_2y-a_2b_1y-a_1c_2-a_2c_1}{a_1b_2-a_2b_1}$
Bernoulli ode	$y' = f(x)\, y + g(x)\, y^n$	0	$e^{-\int (n-1)f(x)dx} y^n$
Reduced Riccati	$y' = f_1(x)\, y + f_2(x)\, y^2$	0	$e^{-\int f_1 dx}$

The above table shows that

$$
\begin{aligned}
\xi(t, y) &= 0 \\
\eta(t, y) &= \mathrm{e}^{\frac{2t}{3}} \qquad \text{(A1)}
\end{aligned}
$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(t, y) \to (R, S)$ where (R, S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$
\frac{dt}{\xi} = \frac{dy}{\eta} = dS \qquad (1)
$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial t} + \eta \frac{\partial}{\partial y}\right) S(t, y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$R = t$$

S is found from

$$\begin{aligned} S &= \int \frac{1}{\eta} dy \\ &= \int \frac{1}{e^{\frac{2t}{3}}} dy \end{aligned}$$

Which results in

$$S = e^{-\frac{2t}{3}} y$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_t + \omega(t, y) S_y}{R_t + \omega(t, y) R_y} \tag{2}$$

Where in the above R_t, R_y, S_t, S_y are all partial derivatives and $\omega(t, y)$ is the right hand side of the original ode given by

$$\omega(t, y) = \frac{2y}{3} + \frac{e^{-\frac{\pi t}{2}}}{3}$$

Evaluating all the partial derivatives gives

$$\begin{aligned} R_t &= 1 \\ R_y &= 0 \\ S_t &= -\frac{2 e^{-\frac{2t}{3}} y}{3} \\ S_y &= e^{-\frac{2t}{3}} \end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = \frac{e^{t\left(-\frac{\pi}{2} - \frac{2}{3}\right)}}{3} \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for t, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = \frac{e^{R\left(-\frac{\pi}{2} - \frac{2}{3}\right)}}{3}$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordinates R, S. Integrating the above gives

$$S(R) = -\frac{2\,\mathrm{e}^{-\frac{R(3\pi+4)}{6}}}{3\pi+4} + c_1 \tag{4}$$

To complete the solution, we just need to transform (4) back to t, y coordinates. This results in

$$\mathrm{e}^{-\frac{2t}{3}} y = -\frac{2\,\mathrm{e}^{-\frac{t(3\pi+4)}{6}}}{3\pi+4} + c_1$$

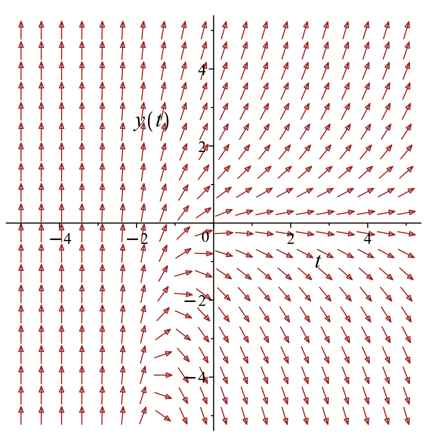
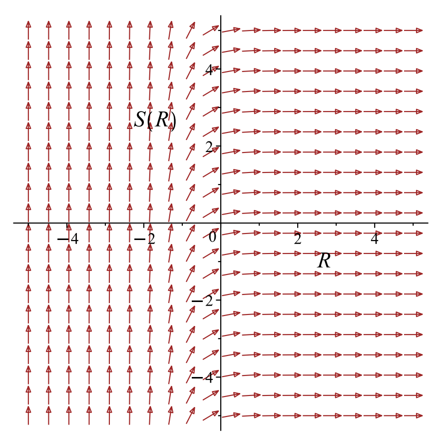
Which simplifies to

$$\mathrm{e}^{-\frac{2t}{3}} y = -\frac{2\,\mathrm{e}^{-\frac{t(3\pi+4)}{6}}}{3\pi+4} + c_1$$

Which gives

$$y = \frac{\left(3\pi c_1 - 2\,\mathrm{e}^{-\frac{t(3\pi+4)}{6}} + 4c_1\right)\mathrm{e}^{\frac{2t}{3}}}{3\pi+4}$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in t, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dt} = \frac{2y}{3} + \frac{\mathrm{e}^{-\frac{\pi t}{2}}}{3}$ 	$R = t$ $S = \mathrm{e}^{-\frac{2t}{3}} y$	$\frac{dS}{dR} = \frac{\mathrm{e}^{R\left(-\frac{\pi}{2}-\frac{2}{3}\right)}}{3}$ 

Initial conditions are used to solve for c_1. Substituting $t = 0$ and $y = a$ in the above solution gives an equation to solve for the constant of integration.

$$a = \frac{3\pi c_1 + 4c_1 - 2}{3\pi + 4}$$

$$c_1 = \frac{3\pi a + 4a + 2}{3\pi + 4}$$

Substituting c_1 found above in the general solution gives

$$y = \frac{\left(\frac{3\pi(3\pi a+4a+2)}{3\pi+4} - 2\,\mathrm{e}^{-\frac{t(3\pi+4)}{6}} + \frac{12\pi a+16a+8}{3\pi+4}\right)\mathrm{e}^{\frac{2t}{3}}}{3\pi + 4}$$

Summary

The solution(s) found are the following

$$y = \frac{\left(\frac{3\pi(3\pi a+4a+2)}{3\pi+4} - 2\,\mathrm{e}^{-\frac{t(3\pi+4)}{6}} + \frac{12\pi a+16a+8}{3\pi+4}\right)\mathrm{e}^{\frac{2t}{3}}}{3\pi + 4} \tag{1}$$

Verification of solutions

$$y = \frac{\left(\frac{3\pi(3\pi a+4a+2)}{3\pi+4} - 2\,\mathrm{e}^{-\frac{t(3\pi+4)}{6}} + \frac{12\pi a+16a+8}{3\pi+4}\right)\mathrm{e}^{\frac{2t}{3}}}{3\pi + 4}$$

Verified OK.

1.23.4 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x,y) + N(x,y)\frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x,y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y) = 0$$

Hence

$$\frac{\partial\phi}{\partial x} + \frac{\partial\phi}{\partial y}\frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$\frac{\partial \phi}{\partial x} = M$$
$$\frac{\partial \phi}{\partial y} = N$$

But since $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x, y)$ but at least we know now that we can do that since the condition $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(t, y)\, \mathrm{d}t + N(t, y)\, \mathrm{d}y = 0 \tag{1A}$$

Therefore

$$(3)\, \mathrm{d}y = \left(2y + \mathrm{e}^{-\frac{\pi t}{2}}\right) \mathrm{d}t$$
$$\left(-2y - \mathrm{e}^{-\frac{\pi t}{2}}\right) \mathrm{d}t + (3)\, \mathrm{d}y = 0 \tag{2A}$$

Comparing (1A) and (2A) shows that

$$M(t, y) = -2y - \mathrm{e}^{-\frac{\pi t}{2}}$$
$$N(t, y) = 3$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial t}$$

Using result found above gives

$$\frac{\partial M}{\partial y} = \frac{\partial}{\partial y}\left(-2y - \mathrm{e}^{-\frac{\pi t}{2}}\right)$$
$$= -2$$

And

$$\frac{\partial N}{\partial t} = \frac{\partial}{\partial t}(3)$$
$$= 0$$

Since $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial t}$, then the ODE is <u>not exact</u>. Since the ODE is not exact, we will try to find an integrating factor to make it exact. Let

$$
\begin{aligned}
A &= \frac{1}{N}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial t}\right) \\
&= \frac{1}{3}((-2) - (0)) \\
&= -\frac{2}{3}
\end{aligned}
$$

Since A does not depend on y, then it can be used to find an integrating factor. The integrating factor μ is

$$
\begin{aligned}
\mu &= e^{\int A \, \mathrm{d}t} \\
&= e^{\int -\frac{2}{3} \, \mathrm{d}t}
\end{aligned}
$$

The result of integrating gives

$$
\begin{aligned}
\mu &= e^{-\frac{2t}{3}} \\
&= \mathrm{e}^{-\frac{2t}{3}}
\end{aligned}
$$

M and N are multiplied by this integrating factor, giving new M and new N which are called $\overline{M}$ and $\overline{N}$ for now so not to confuse them with the original M and N.

$$
\begin{aligned}
\overline{M} &= \mu M \\
&= \mathrm{e}^{-\frac{2t}{3}}\left(-2y - \mathrm{e}^{-\frac{\pi t}{2}}\right) \\
&= -\left(2y + \mathrm{e}^{-\frac{\pi t}{2}}\right)\mathrm{e}^{-\frac{2t}{3}}
\end{aligned}
$$

And

$$
\begin{aligned}
\overline{N} &= \mu N \\
&= \mathrm{e}^{-\frac{2t}{3}}(3) \\
&= 3\,\mathrm{e}^{-\frac{2t}{3}}
\end{aligned}
$$

Now a modified ODE is ontained from the original ODE, which is exact and can be solved. The modified ODE is

$$
\begin{aligned}
\overline{M} + \overline{N}\frac{\mathrm{d}y}{\mathrm{d}t} &= 0 \\
\left(-\left(2y + \mathrm{e}^{-\frac{\pi t}{2}}\right)\mathrm{e}^{-\frac{2t}{3}}\right) + \left(3\,\mathrm{e}^{-\frac{2t}{3}}\right)\frac{\mathrm{d}y}{\mathrm{d}t} &= 0
\end{aligned}
$$

The following equations are now set up to solve for the function $\phi(t, y)$

$$\frac{\partial \phi}{\partial t} = \overline{M} \tag{1}$$

$$\frac{\partial \phi}{\partial y} = \overline{N} \tag{2}$$

Integrating (1) w.r.t. t gives

$$\int \frac{\partial \phi}{\partial t}\,\mathrm{d}t = \int \overline{M}\,\mathrm{d}t$$

$$\int \frac{\partial \phi}{\partial t}\,\mathrm{d}t = \int -\left(2y + \mathrm{e}^{-\frac{\pi t}{2}}\right)\mathrm{e}^{-\frac{2t}{3}}\,\mathrm{d}t$$

$$\phi = \frac{6\,\mathrm{e}^{t\left(-\frac{\pi}{2}-\frac{2}{3}\right)} + 3y(3\pi+4)\,\mathrm{e}^{-\frac{2t}{3}}}{3\pi+4} + f(y) \tag{3}$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both t and y. Taking derivative of equation (3) w.r.t y gives

$$\frac{\partial \phi}{\partial y} = 3\,\mathrm{e}^{-\frac{2t}{3}} + f'(y) \tag{4}$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = 3\,\mathrm{e}^{-\frac{2t}{3}}$. Therefore equation (4) becomes

$$3\,\mathrm{e}^{-\frac{2t}{3}} = 3\,\mathrm{e}^{-\frac{2t}{3}} + f'(y) \tag{5}$$

Solving equation (5) for $f'(y)$ gives

$$f'(y) = 0$$

Therefore

$$f(y) = c_1$$

Where c_1 is constant of integration. Substituting this result for $f(y)$ into equation (3) gives ϕ

$$\phi = \frac{6\,\mathrm{e}^{t\left(-\frac{\pi}{2}-\frac{2}{3}\right)} + 3y(3\pi+4)\,\mathrm{e}^{-\frac{2t}{3}}}{3\pi+4} + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = \frac{6\,\mathrm{e}^{t\left(-\frac{\pi}{2}-\frac{2}{3}\right)} + 3y(3\pi+4)\,\mathrm{e}^{-\frac{2t}{3}}}{3\pi+4}$$

The solution becomes

$$y = \frac{\left(3\pi c_1 - 6\,\mathrm{e}^{-\frac{t(3\pi+4)}{6}} + 4c_1\right)\mathrm{e}^{\frac{2t}{3}}}{9\pi + 12}$$

Initial conditions are used to solve for c_1. Substituting $t = 0$ and $y = a$ in the above solution gives an equation to solve for the constant of integration.

$$a = \frac{3\pi c_1 + 4c_1 - 6}{9\pi + 12}$$

$$c_1 = \frac{9\pi a + 12a + 6}{3\pi + 4}$$

Substituting c_1 found above in the general solution gives

$$y = \frac{\left(\frac{9\pi(3\pi a+4a+2)}{3\pi+4} - 6\,\mathrm{e}^{-\frac{t(3\pi+4)}{6}} + \frac{36\pi a+48a+24}{3\pi+4}\right)\mathrm{e}^{\frac{2t}{3}}}{9\pi + 12}$$

Summary

The solution(s) found are the following

$$y = \frac{\left(\frac{9\pi(3\pi a+4a+2)}{3\pi+4} - 6\,\mathrm{e}^{-\frac{t(3\pi+4)}{6}} + \frac{36\pi a+48a+24}{3\pi+4}\right)\mathrm{e}^{\frac{2t}{3}}}{9\pi + 12} \tag{1}$$

Verification of solutions

$$y = \frac{\left(\frac{9\pi(3\pi a+4a+2)}{3\pi+4} - 6\,\mathrm{e}^{-\frac{t(3\pi+4)}{6}} + \frac{36\pi a+48a+24}{3\pi+4}\right)\mathrm{e}^{\frac{2t}{3}}}{9\pi + 12}$$

Verified OK.

1.23.5 Maple step by step solution

Let's solve

$\left[-2y + 3y' = \mathrm{e}^{-\frac{\pi t}{2}}, y(0) = a\right]$

- Highest derivative means the order of the ODE is 1

  y'

- Isolate the derivative

  $y' = \frac{2y}{3} + \frac{\mathrm{e}^{-\frac{\pi t}{2}}}{3}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE

  $y' - \frac{2y}{3} = \frac{e^{-\frac{\pi t}{2}}}{3}$
- The ODE is linear; multiply by an integrating factor $\mu(t)$

  $\mu(t) \left(y' - \frac{2y}{3}\right) = \frac{\mu(t) e^{-\frac{\pi t}{2}}}{3}$
- Assume the lhs of the ODE is the total derivative $\frac{d}{dt}(\mu(t) \, y)$

  $\mu(t) \left(y' - \frac{2y}{3}\right) = \mu'(t) \, y + \mu(t) \, y'$
- Isolate $\mu'(t)$

  $\mu'(t) = -\frac{2\mu(t)}{3}$
- Solve to find the integrating factor

  $\mu(t) = e^{-\frac{2t}{3}}$
- Integrate both sides with respect to t

  $\int \left(\frac{d}{dt}(\mu(t) \, y)\right) dt = \int \frac{\mu(t) e^{-\frac{\pi t}{2}}}{3} dt + c_1$
- Evaluate the integral on the lhs

  $\mu(t) \, y = \int \frac{\mu(t) e^{-\frac{\pi t}{2}}}{3} dt + c_1$
- Solve for y

  $y = \frac{\int \frac{\mu(t) e^{-\frac{\pi t}{2}}}{3} dt + c_1}{\mu(t)}$
- Substitute $\mu(t) = e^{-\frac{2t}{3}}$

  $y = \frac{\int \frac{e^{-\frac{\pi t}{2}} e^{-\frac{2t}{3}}}{3} dt + c_1}{e^{-\frac{2t}{3}}}$
- Evaluate the integrals on the rhs

  $y = \frac{-\frac{2 e^{-\frac{1}{2}\pi t - \frac{2}{3}t}}{3\pi + 4} + c_1}{e^{-\frac{2t}{3}}}$
- Simplify

  $y = \frac{\left(3\pi c_1 - 2 e^{t\left(-\frac{\pi}{2} - \frac{2}{3}\right)} + 4c_1\right) e^{\frac{2t}{3}}}{3\pi + 4}$
- Use initial condition $y(0) = a$

  $a = \frac{3\pi c_1 + 4c_1 - 2}{3\pi + 4}$
- Solve for c_1

  $c_1 = \frac{3\pi a + 4a + 2}{3\pi + 4}$

- Substitute $c_1 = \frac{3\pi a+4a+2}{3\pi+4}$ into general solution and simplify

$$y = \frac{\left(3\pi a-2\,\mathrm{e}^{t\left(-\frac{\pi}{2}-\frac{2}{3}\right)}+4a+2\right)\mathrm{e}^{\frac{2t}{3}}}{3\pi+4}$$

- Solution to the IVP

$$y = \frac{\left(3\pi a-2\,\mathrm{e}^{t\left(-\frac{\pi}{2}-\frac{2}{3}\right)}+4a+2\right)\mathrm{e}^{\frac{2t}{3}}}{3\pi+4}$$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
<- 1st order linear successful`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 35

```
dsolve([-2*y(t)+3*diff(y(t),t) = exp(-1/2*Pi*t),y(0) = a],y(t), singsol=all)
```

$$y(t) = \frac{\left(3\pi a-2\,\mathrm{e}^{t\left(-\frac{\pi}{2}-\frac{2}{3}\right)}+4a+2\right)\mathrm{e}^{\frac{2t}{3}}}{3\pi+4}$$

✓ Solution by Mathematica

Time used: 0.083 (sec). Leaf size: 43

```
DSolve[{-2*y[t]+3*y'[t] == Exp[-1/2*Pi*t],y[0]==a},y[t],t,IncludeSingularSolutions -> True]
```

$$y(t) \to \frac{e^{2t/3}\left((4+3\pi)a-2e^{-\frac{1}{6}(4+3\pi)t}+2\right)}{4+3\pi}$$

1.24 problem 24

Internal problem ID [471]
Internal file name [`OUTPUT/471_Sunday_June_05_2022_01_42_00_AM_12055452/index.tex`]

Book: Elementary differential equations and boundary value problems, 10th ed., Boyce and DiPrima
Section: Section 2.1. Page 40
Problem number: 24.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"exact", "linear", "first_order_ode_lie_symmetry_lookup"**

Maple gives the following as the ode type

`[_linear]`

$$\boxed{(t+1)\,y+ty'=2t\,\mathrm{e}^{-t}}$$

With initial conditions

$$[y(1)=a]$$

1.24.1 Existence and uniqueness analysis

This is a linear ODE. In canonical form it is written as

$$y'+p(t)y=q(t)$$

Where here

$$p(t)=-\frac{-t-1}{t}$$
$$q(t)=2\,\mathrm{e}^{-t}$$

Hence the ode is

$$y' - \frac{(-t-1)\,y}{t} = 2\,\mathrm{e}^{-t}$$

The domain of $p(t) = -\frac{-t-1}{t}$ is

$$\{t < 0 \lor 0 < t\}$$

And the point $t_0 = 1$ is inside this domain. The domain of $q(t) = 2\,\mathrm{e}^{-t}$ is

$$\{-\infty < t < \infty\}$$

And the point $t_0 = 1$ is also inside this domain. Hence solution exists and is unique.

1.24.2 Solving as linear ode

Entering Linear first order ODE solver. The integrating factor μ is

$$\begin{aligned}\mu &= \mathrm{e}^{\int -\frac{-t-1}{t}dt} \\ &= \mathrm{e}^{t+\ln(t)}\end{aligned}$$

Which simplifies to

$$\mu = t\,\mathrm{e}^{t}$$

The ode becomes

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}(\mu y) &= (\mu)\left(2\,\mathrm{e}^{-t}\right) \\ \frac{\mathrm{d}}{\mathrm{d}t}\left(t\,\mathrm{e}^{t} y\right) &= \left(t\,\mathrm{e}^{t}\right)\left(2\,\mathrm{e}^{-t}\right) \\ \mathrm{d}\left(t\,\mathrm{e}^{t} y\right) &= (2t)\;\mathrm{d}t\end{aligned}$$

Integrating gives

$$\begin{aligned}t\,\mathrm{e}^{t} y &= \int 2t\,\mathrm{d}t \\ t\,\mathrm{e}^{t} y &= t^{2} + c_{1}\end{aligned}$$

Dividing both sides by the integrating factor $\mu = t\,\mathrm{e}^{t}$ results in

$$y = t\,\mathrm{e}^{-t} + \frac{c_{1}\mathrm{e}^{-t}}{t}$$

which simplifies to

$$y = \frac{e^{-t}(t^2 + c_1)}{t}$$

Initial conditions are used to solve for c_1. Substituting $t = 1$ and $y = a$ in the above solution gives an equation to solve for the constant of integration.

$$a = e^{-1}(1 + c_1)$$

$$c_1 = -(e^{-1} - a)\, e$$

Substituting c_1 found above in the general solution gives

$$y = \frac{e^{-t}(t^2 - (e^{-1} - a)\, e)}{t}$$

Summary

The solution(s) found are the following

$$y = \frac{e^{-t}(t^2 - (e^{-1} - a)\, e)}{t} \tag{1}$$

Verification of solutions

$$y = \frac{e^{-t}(t^2 - (e^{-1} - a)\, e)}{t}$$

Verified OK.

1.24.3 Solving as first order ode lie symmetry lookup ode

Writing the ode as

$$y' = -\frac{(t\, e^t y + y\, e^t - 2t)\, e^{-t}}{t}$$
$$y' = \omega(t, y)$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_t + \omega(\eta_y - \xi_t) - \omega^2 \xi_y - \omega_t \xi - \omega_y \eta = 0 \tag{A}$$

The type of this ode is known. It is of type `linear`. Therefore we do not need to solve the PDE (A), and can just use the lookup table shown below to find ξ, η

Table 70: Lie symmetry infinitesimal lookup table for known first order ODE's

ODE class	Form	ξ	η
linear ode	$y' = f(x)y(x) + g(x)$	0	$e^{\int f dx}$
separable ode	$y' = f(x)\, g(y)$	$\frac{1}{f}$	0
quadrature ode	$y' = f(x)$	0	1
quadrature ode	$y' = g(y)$	1	0
homogeneous ODEs of Class A	$y' = f\left(\frac{y}{x}\right)$	x	y
homogeneous ODEs of Class C	$y' = (a + bx + cy)^{\frac{n}{m}}$	1	$-\frac{b}{c}$
homogeneous class D	$y' = \frac{y}{x} + g(x)\, F\left(\frac{y}{x}\right)$	x^2	xy
First order special form ID 1	$y' = g(x)\, e^{h(x)+by} + f(x)$	$\frac{e^{-\int bf(x)dx - h(x)}}{g(x)}$	$\frac{f(x)e^{-\int bf(x)dx - h(x)}}{g(x)}$
polynomial type ode	$y' = \frac{a_1x+b_1y+c_1}{a_2x+b_2y+c_2}$	$\frac{a_1b_2x-a_2b_1x-b_1c_2+b_2c_1}{a_1b_2-a_2b_1}$	$\frac{a_1b_2y-a_2b_1y-a_1c_2-a_2c_1}{a_1b_2-a_2b_1}$
Bernoulli ode	$y' = f(x)\, y + g(x)\, y^n$	0	$e^{-\int (n-1)f(x)dx} y^n$
Reduced Riccati	$y' = f_1(x)\, y + f_2(x)\, y^2$	0	$e^{-\int f_1 dx}$

The above table shows that

$$
\begin{aligned}
\xi(t, y) &= 0 \\
\eta(t, y) &= \mathrm{e}^{-t-\ln(t)}
\end{aligned} \tag{A1}
$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(t, y) \to (R, S)$ where (R, S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$
\frac{dt}{\xi} = \frac{dy}{\eta} = dS \tag{1}
$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial t} + \eta \frac{\partial}{\partial y}\right) S(t, y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the

canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$R = t$$

S is found from

$$\begin{aligned} S &= \int \frac{1}{\eta} dy \\ &= \int \frac{1}{\mathrm{e}^{-t-\ln(t)}} dy \end{aligned}$$

Which results in

$$S = t\,\mathrm{e}^{t} y$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_t + \omega(t, y) S_y}{R_t + \omega(t, y) R_y} \tag{2}$$

Where in the above R_t, R_y, S_t, S_y are all partial derivatives and $\omega(t, y)$ is the right hand side of the original ode given by

$$\omega(t, y) = -\frac{(t\,\mathrm{e}^{t} y + y\,\mathrm{e}^{t} - 2t)\,\mathrm{e}^{-t}}{t}$$

Evaluating all the partial derivatives gives

$$\begin{aligned} R_t &= 1 \\ R_y &= 0 \\ S_t &= y\,\mathrm{e}^{t}(t + 1) \\ S_y &= t\,\mathrm{e}^{t} \end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = 2t \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for t, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = 2R$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by

integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$S(R) = R^2 + c_1 \tag{4}$$

To complete the solution, we just need to transform (4) back to t, y coordinates. This results in

$$t\,\mathrm{e}^t y = t^2 + c_1$$

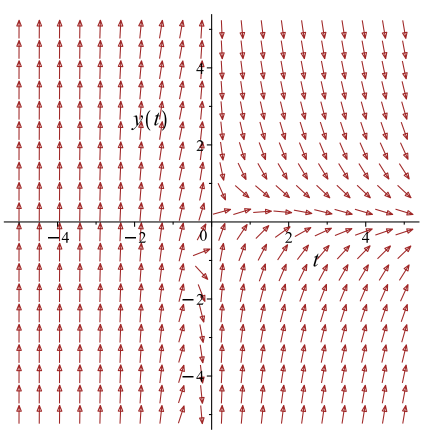
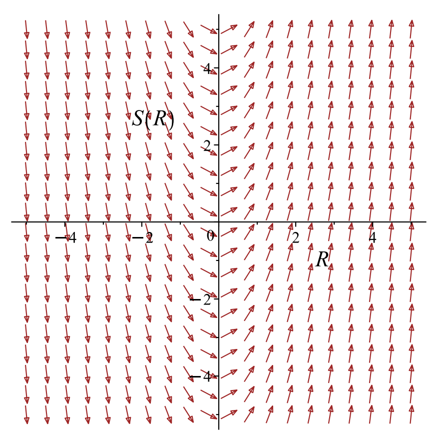
Which simplifies to

$$t\,\mathrm{e}^t y = t^2 + c_1$$

Which gives

$$y = \frac{\mathrm{e}^{-t}(t^2 + c_1)}{t}$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in t, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dt} = -\frac{(t\,\mathrm{e}^t y + y\,\mathrm{e}^t - 2t)\,\mathrm{e}^{-t}}{t}$ 	$R = t$ $S = t\,\mathrm{e}^t y$	$\frac{dS}{dR} = 2R$ 

Initial conditions are used to solve for c_1. Substituting $t = 1$ and $y = a$ in the above solution gives an equation to solve for the constant of integration.

$$a = \mathrm{e}^{-1}c_1 + \mathrm{e}^{-1}$$

$$c_1 = -\left(e^{-1} - a\right) e$$

Substituting c_1 found above in the general solution gives

$$y = \frac{e^{-t}(t^2 - (e^{-1} - a) e)}{t}$$

Summary

The solution(s) found are the following

$$y = \frac{e^{-t}(t^2 - (e^{-1} - a) e)}{t} \tag{1}$$

Verification of solutions

$$y = \frac{e^{-t}(t^2 - (e^{-1} - a) e)}{t}$$

Verified OK.

1.24.4 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x, y) + N(x, y)\frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x, y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x, y) = 0$$

Hence

$$\frac{\partial \phi}{\partial x} + \frac{\partial \phi}{\partial y}\frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$\begin{aligned}\frac{\partial \phi}{\partial x} &= M \\ \frac{\partial \phi}{\partial y} &= N\end{aligned}$$

But since $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x, y)$ but at least we know now that we can do that since the condition $\frac{\partial^2\phi}{\partial x \partial y} = \frac{\partial^2\phi}{\partial y \partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(t, y)\,\mathrm{d}t + N(t, y)\,\mathrm{d}y = 0 \tag{1A}$$

Therefore

$$\begin{aligned}
\left(t\,\mathrm{e}^{t}\right)\mathrm{d}y &= \left(-t\,\mathrm{e}^{t}y - y\,\mathrm{e}^{t} + 2t\right)\mathrm{d}t\\
\left(t\,\mathrm{e}^{t}y + y\,\mathrm{e}^{t} - 2t\right)\mathrm{d}t + \left(t\,\mathrm{e}^{t}\right)\mathrm{d}y &= 0
\end{aligned} \tag{2A}$$

Comparing (1A) and (2A) shows that

$$\begin{aligned}
M(t, y) &= t\,\mathrm{e}^{t}y + y\,\mathrm{e}^{t} - 2t\\
N(t, y) &= t\,\mathrm{e}^{t}
\end{aligned}$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial t}$$

Using result found above gives

$$\begin{aligned}
\frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}\left(t\,\mathrm{e}^{t}y + y\,\mathrm{e}^{t} - 2t\right)\\
&= \mathrm{e}^{t}(t + 1)
\end{aligned}$$

And

$$\begin{aligned}
\frac{\partial N}{\partial t} &= \frac{\partial}{\partial t}\left(t\,\mathrm{e}^{t}\right)\\
&= \mathrm{e}^{t}(t + 1)
\end{aligned}$$

Since $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial t}$, then the ODE is exact The following equations are now set up to solve for the function $\phi(t, y)$

$$\frac{\partial \phi}{\partial t} = M \tag{1}$$

$$\frac{\partial \phi}{\partial y} = N \tag{2}$$

Integrating (1) w.r.t. t gives

$$\int \frac{\partial \phi}{\partial t} \,\mathrm{d}t = \int M \,\mathrm{d}t$$
$$\int \frac{\partial \phi}{\partial t} \,\mathrm{d}t = \int t\,\mathrm{e}^t y + y\,\mathrm{e}^t - 2t \,\mathrm{d}t$$
$$\phi = -t\left(-y\,\mathrm{e}^t + t\right) + f(y) \tag{3}$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both t and y. Taking derivative of equation (3) w.r.t y gives

$$\frac{\partial \phi}{\partial y} = t\,\mathrm{e}^t + f'(y) \tag{4}$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = t\,\mathrm{e}^t$. Therefore equation (4) becomes

$$t\,\mathrm{e}^t = t\,\mathrm{e}^t + f'(y) \tag{5}$$

Solving equation (5) for $f'(y)$ gives

$$f'(y) = 0$$

Therefore

$$f(y) = c_1$$

Where c_1 is constant of integration. Substituting this result for $f(y)$ into equation (3) gives ϕ

$$\phi = -t\left(-y\,\mathrm{e}^t + t\right) + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = -t\left(-y\,\mathrm{e}^t + t\right)$$

The solution becomes

$$y = \frac{\mathrm{e}^{-t}(t^2 + c_1)}{t}$$

Initial conditions are used to solve for c_1. Substituting $t = 1$ and $y = a$ in the above solution gives an equation to solve for the constant of integration.

$$a = e^{-1}c_1 + e^{-1}$$

$$c_1 = -\left(e^{-1} - a\right) e$$

Substituting c_1 found above in the general solution gives

$$y = \frac{e^{-t}(t^2 - (e^{-1} - a) e)}{t}$$

Summary

The solution(s) found are the following

$$y = \frac{e^{-t}(t^2 - (e^{-1} - a) e)}{t} \tag{1}$$

Verification of solutions

$$y = \frac{e^{-t}(t^2 - (e^{-1} - a) e)}{t}$$

Verified OK.

1.24.5 Maple step by step solution

Let's solve

$\left[(t+1)\, y + ty' = \frac{2t}{e^t}, y(1) = a\right]$

- Highest derivative means the order of the ODE is 1

  y'

- Isolate the derivative

  $y' = -\frac{(t+1)y}{t} + \frac{2}{e^t}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE

  $y' + \frac{(t+1)y}{t} = \frac{2}{e^t}$

- The ODE is linear; multiply by an integrating factor $\mu(t)$

  $\mu(t) \left(y' + \frac{(t+1)y}{t}\right) = \frac{2\mu(t)}{e^t}$

- Assume the lhs of the ODE is the total derivative $\frac{d}{dt}(\mu(t)\, y)$

$\mu(t)\left(y'+\frac{(t+1)y}{t}\right)=\mu'(t)\,y+\mu(t)\,y'$

- Isolate $\mu'(t)$

  $\mu'(t)=\frac{\mu(t)(t+1)}{t}$

- Solve to find the integrating factor

  $\mu(t)=t(\mathrm{e}^{t})^{2}\,\mathrm{e}^{-t}$

- Integrate both sides with respect to t

  $\int\left(\frac{d}{dt}(\mu(t)\,y)\right)dt=\int\frac{2\mu(t)}{\mathrm{e}^{t}}dt+c_1$

- Evaluate the integral on the lhs

  $\mu(t)\,y=\int\frac{2\mu(t)}{\mathrm{e}^{t}}dt+c_1$

- Solve for y

  $y=\frac{\int\frac{2\mu(t)}{\mathrm{e}^{t}}dt+c_1}{\mu(t)}$

- Substitute $\mu(t)=t(\mathrm{e}^{t})^{2}\,\mathrm{e}^{-t}$

  $y=\frac{\int 2\,\mathrm{e}^{-t}\mathrm{e}^{t}t dt+c_1}{t(\mathrm{e}^{t})^{2}\mathrm{e}^{-t}}$

- Evaluate the integrals on the rhs

  $y=\frac{t^{2}+c_1}{t(\mathrm{e}^{t})^{2}\mathrm{e}^{-t}}$

- Simplify

  $y=\frac{\mathrm{e}^{-t}(t^{2}+c_1)}{t}$

- Use initial condition $y(1)=a$

  $a=\mathrm{e}^{-1}(1+c_1)$

- Solve for c_1

  $c_1=-\frac{\mathrm{e}^{-1}-a}{\mathrm{e}^{-1}}$

- Substitute $c_1=-\frac{\mathrm{e}^{-1}-a}{\mathrm{e}^{-1}}$ into general solution and simplify

  $y=\frac{\mathrm{e}^{-t}(-1+\mathrm{e}a+t^{2})}{t}$

- Solution to the IVP

  $y=\frac{\mathrm{e}^{-t}(-1+\mathrm{e}a+t^{2})}{t}$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
<- 1st order linear successful`
```

✓ Solution by Maple

Time used: 0.031 (sec). Leaf size: 21

```
dsolve([(1+t)*y(t)+t*diff(y(t),t) = 2*t/exp(t),y(1) = a],y(t), singsol=all)
```

$$y(t) = \frac{(t^2 + a\,\mathrm{e} - 1)\,\mathrm{e}^{-t}}{t}$$

✓ Solution by Mathematica

Time used: 0.065 (sec). Leaf size: 22

```
DSolve[{(1+t)*y[t]+t*y'[t] == 2*t/Exp[t],y[1]==a},y[t],t,IncludeSingularSolutions -> True]
```

$$y(t) \to \frac{e^{-t}(ea + t^2 - 1)}{t}$$

1.25 problem 25

Internal problem ID [472]
Internal file name [`OUTPUT/472_Sunday_June_05_2022_01_42_01_AM_79347504/index.tex`]

Book: Elementary differential equations and boundary value problems, 10th ed., Boyce and DiPrima
Section: Section 2.1. Page 40
Problem number: 25.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"exact", "linear", "first_order_ode_lie_symmetry_lookup"**

Maple gives the following as the ode type

`[_linear]`

$$\boxed{2y + ty' = \frac{\sin(t)}{t}}$$

With initial conditions

$$\left[y\left(-\frac{\pi}{2}\right) = a\right]$$

1.25.1 Existence and uniqueness analysis

This is a linear ODE. In canonical form it is written as

$$y' + p(t)y = q(t)$$

Where here

$$p(t) = \frac{2}{t}$$
$$q(t) = \frac{\sin(t)}{t^2}$$

Hence the ode is

$$y' + \frac{2y}{t} = \frac{\sin(t)}{t^2}$$

The domain of $p(t) = \frac{2}{t}$ is

$$\{t < 0 \lor 0 < t\}$$

And the point $t_0 = -\frac{\pi}{2}$ is inside this domain. The domain of $q(t) = \frac{\sin(t)}{t^2}$ is

$$\{t < 0 \lor 0 < t\}$$

And the point $t_0 = -\frac{\pi}{2}$ is also inside this domain. Hence solution exists and is unique.

1.25.2 Solving as linear ode

Entering Linear first order ODE solver. The integrating factor μ is

$$\begin{aligned}\mu &= e^{\int \frac{2}{t} dt} \\ &= t^2\end{aligned}$$

The ode becomes

$$\begin{aligned}\frac{d}{dt}(\mu y) &= (\mu)\left(\frac{\sin(t)}{t^2}\right) \\ \frac{d}{dt}(t^2 y) &= (t^2)\left(\frac{\sin(t)}{t^2}\right) \\ d(t^2 y) &= \sin(t)\ dt\end{aligned}$$

Integrating gives

$$\begin{aligned}t^2 y &= \int \sin(t)\ dt \\ t^2 y &= -\cos(t) + c_1\end{aligned}$$

Dividing both sides by the integrating factor $\mu = t^2$ results in

$$y = -\frac{\cos(t)}{t^2} + \frac{c_1}{t^2}$$

which simplifies to

$$y = \frac{-\cos(t) + c_1}{t^2}$$

Initial conditions are used to solve for c_1. Substituting $t = -\frac{\pi}{2}$ and $y = a$ in the above solution gives an equation to solve for the constant of integration.

$$a = \frac{4c_1}{\pi^2}$$

$$c_1 = \frac{\pi^2 a}{4}$$

Substituting c_1 found above in the general solution gives

$$y = \frac{-\cos(t) + \frac{\pi^2 a}{4}}{t^2}$$

Summary

The solution(s) found are the following

$$y = \frac{-\cos(t) + \frac{\pi^2 a}{4}}{t^2} \tag{1}$$

Verification of solutions

$$y = \frac{-\cos(t) + \frac{\pi^2 a}{4}}{t^2}$$

Verified OK.

1.25.3 Solving as first order ode lie symmetry lookup ode

Writing the ode as

$$y' = \frac{-2ty + \sin(t)}{t^2}$$
$$y' = \omega(t, y)$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_t + \omega(\eta_y - \xi_t) - \omega^2 \xi_y - \omega_t \xi - \omega_y \eta = 0 \tag{A}$$

The type of this ode is known. It is of type `linear`. Therefore we do not need to solve the PDE (A), and can just use the lookup table shown below to find ξ, η

Table 73: Lie symmetry infinitesimal lookup table for known first order ODE's

ODE class	Form	ξ	η
linear ode	$y' = f(x)y(x) + g(x)$	0	$e^{\int f dx}$
separable ode	$y' = f(x)\, g(y)$	$\frac{1}{f}$	0
quadrature ode	$y' = f(x)$	0	1
quadrature ode	$y' = g(y)$	1	0
homogeneous ODEs of Class A	$y' = f\left(\frac{y}{x}\right)$	x	y
homogeneous ODEs of Class C	$y' = (a + bx + cy)^{\frac{n}{m}}$	1	$-\frac{b}{c}$
homogeneous class D	$y' = \frac{y}{x} + g(x)\, F\left(\frac{y}{x}\right)$	x^2	xy
First order special form ID 1	$y' = g(x)\, e^{h(x)+by} + f(x)$	$\frac{e^{-\int bf(x)dx - h(x)}}{g(x)}$	$\frac{f(x)e^{-\int bf(x)dx - h(x)}}{g(x)}$
polynomial type ode	$y' = \frac{a_1x+b_1y+c_1}{a_2x+b_2y+c_2}$	$\frac{a_1b_2x-a_2b_1x-b_1c_2+b_2c_1}{a_1b_2-a_2b_1}$	$\frac{a_1b_2y-a_2b_1y-a_1c_2-a_2c_1}{a_1b_2-a_2b_1}$
Bernoulli ode	$y' = f(x)\, y + g(x)\, y^n$	0	$e^{-\int (n-1)f(x)dx} y^n$
Reduced Riccati	$y' = f_1(x)\, y + f_2(x)\, y^2$	0	$e^{-\int f_1 dx}$

The above table shows that

$$
\begin{aligned}
\xi(t, y) &= 0 \\
\eta(t, y) &= \frac{1}{t^2}
\end{aligned} \tag{A1}
$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(t, y) \to (R, S)$ where (R, S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$
\frac{dt}{\xi} = \frac{dy}{\eta} = dS \tag{1}
$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial t} + \eta \frac{\partial}{\partial y}\right) S(t, y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the

canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$R = t$$

S is found from

$$\begin{aligned} S &= \int \frac{1}{\eta} dy \\ &= \int \frac{1}{\frac{1}{t^2}} dy \end{aligned}$$

Which results in

$$S = t^2 y$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_t + \omega(t, y) S_y}{R_t + \omega(t, y) R_y} \tag{2}$$

Where in the above R_t, R_y, S_t, S_y are all partial derivatives and $\omega(t, y)$ is the right hand side of the original ode given by

$$\omega(t, y) = \frac{-2ty + \sin(t)}{t^2}$$

Evaluating all the partial derivatives gives

$$\begin{aligned} R_t &= 1 \\ R_y &= 0 \\ S_t &= 2ty \\ S_y &= t^2 \end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = \sin(t) \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for t, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = \sin(R)$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by

integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$S(R) = -\cos(R) + c_1 \tag{4}$$

To complete the solution, we just need to transform (4) back to t, y coordinates. This results in

$$yt^2 = -\cos(t) + c_1$$

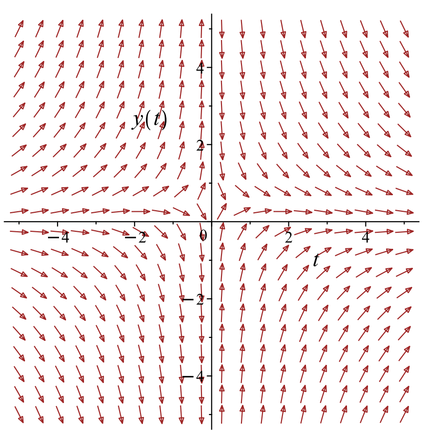
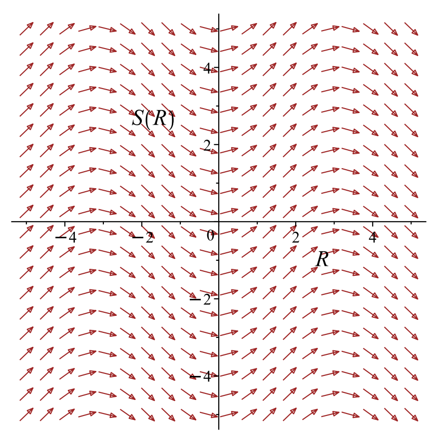
Which simplifies to

$$yt^2 = -\cos(t) + c_1$$

Which gives

$$y = -\frac{\cos(t) - c_1}{t^2}$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

<table>
<tr><th>Original ode in t, y coordinates</th><th>Canonical coordinates transformation</th><th>ODE in canonical coordinates (R, S)</th></tr>
<tr><td>$\frac{dy}{dt} = \frac{-2ty+\sin(t)}{t^2}$

</td><td>$R = t$
$S = t^2 y$</td><td>$\frac{dS}{dR} = \sin(R)$

</td></tr>
</table>

Initial conditions are used to solve for c_1. Substituting $t = -\frac{\pi}{2}$ and $y = a$ in the above solution gives an equation to solve for the constant of integration.

$$a = \frac{4c_1}{\pi^2}$$

$$c_1 = \frac{\pi^2 a}{4}$$

Substituting c_1 found above in the general solution gives

$$y = -\frac{\cos(t) - \frac{\pi^2 a}{4}}{t^2}$$

Summary

The solution(s) found are the following

$$y = -\frac{\cos(t) - \frac{\pi^2 a}{4}}{t^2} \tag{1}$$

Verification of solutions

$$y = -\frac{\cos(t) - \frac{\pi^2 a}{4}}{t^2}$$

Verified OK.

1.25.4 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x,y) + N(x,y)\frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x,y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y) = 0$$

Hence

$$\frac{\partial \phi}{\partial x} + \frac{\partial \phi}{\partial y}\frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$\frac{\partial \phi}{\partial x} = M$$
$$\frac{\partial \phi}{\partial y} = N$$

But since $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x,y)$ but at least we know now that we can do that since the condition $\frac{\partial^2\phi}{\partial x\partial y} = \frac{\partial^2\phi}{\partial y\partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(t,y)\,\mathrm{d}t + N(t,y)\,\mathrm{d}y = 0 \tag{1A}$$

Therefore

$$\begin{aligned}(t^2)\,\mathrm{d}y &= (-2ty + \sin(t))\,\mathrm{d}t\\ (2ty - \sin(t))\,\mathrm{d}t + (t^2)\,\mathrm{d}y &= 0 \end{aligned} \tag{2A}$$

Comparing (1A) and (2A) shows that

$$\begin{aligned} M(t,y) &= 2ty - \sin(t)\\ N(t,y) &= t^2 \end{aligned}$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial t}$$

Using result found above gives

$$\begin{aligned} \frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}(2ty - \sin(t))\\ &= 2t \end{aligned}$$

And

$$\begin{aligned} \frac{\partial N}{\partial t} &= \frac{\partial}{\partial t}(t^2)\\ &= 2t \end{aligned}$$

Since $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial t}$, then the ODE is exact The following equations are now set up to solve for the function $\phi(t,y)$

$$\frac{\partial \phi}{\partial t} = M \tag{1}$$

$$\frac{\partial \phi}{\partial y} = N \tag{2}$$

Integrating (1) w.r.t. t gives

$$\int \frac{\partial \phi}{\partial t} \, \mathrm{d}t = \int M \, \mathrm{d}t$$
$$\int \frac{\partial \phi}{\partial t} \, \mathrm{d}t = \int 2ty - \sin(t) \, \mathrm{d}t$$
$$\phi = t^2 y + \cos(t) + f(y) \tag{3}$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both t and y. Taking derivative of equation (3) w.r.t y gives

$$\frac{\partial \phi}{\partial y} = t^2 + f'(y) \tag{4}$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = t^2$. Therefore equation (4) becomes

$$t^2 = t^2 + f'(y) \tag{5}$$

Solving equation (5) for $f'(y)$ gives

$$f'(y) = 0$$

Therefore

$$f(y) = c_1$$

Where c_1 is constant of integration. Substituting this result for $f(y)$ into equation (3) gives ϕ

$$\phi = t^2 y + \cos(t) + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = t^2 y + \cos(t)$$

The solution becomes

$$y = -\frac{\cos(t) - c_1}{t^2}$$

Initial conditions are used to solve for c_1. Substituting $t = -\frac{\pi}{2}$ and $y = a$ in the above solution gives an equation to solve for the constant of integration.

$$a = \frac{4c_1}{\pi^2}$$

$$c_1 = \frac{\pi^2 a}{4}$$

Substituting c_1 found above in the general solution gives

$$y = -\frac{\cos(t) - \frac{\pi^2 a}{4}}{t^2}$$

Summary
The solution(s) found are the following

$$y = -\frac{\cos(t) - \frac{\pi^2 a}{4}}{t^2} \tag{1}$$

Verification of solutions

$$y = -\frac{\cos(t) - \frac{\pi^2 a}{4}}{t^2}$$

Verified OK.

1.25.5 Maple step by step solution

Let's solve

$\left[2y + ty' = \frac{\sin(t)}{t}, y\left(-\frac{\pi}{2}\right) = a\right]$

- Highest derivative means the order of the ODE is 1

  y'
- Isolate the derivative

  $y' = -\frac{2y}{t} + \frac{\sin(t)}{t^2}$
- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE

  $y' + \frac{2y}{t} = \frac{\sin(t)}{t^2}$
- The ODE is linear; multiply by an integrating factor $\mu(t)$

  $\mu(t)\left(y' + \frac{2y}{t}\right) = \frac{\mu(t)\sin(t)}{t^2}$

- Assume the lhs of the ODE is the total derivative $\frac{d}{dt}(\mu(t)\, y)$

  $\mu(t)\left(y'+\frac{2y}{t}\right)=\mu'(t)\, y+\mu(t)\, y'$
- Isolate $\mu'(t)$

  $\mu'(t)=\frac{2\mu(t)}{t}$
- Solve to find the integrating factor

  $\mu(t)=t^2$
- Integrate both sides with respect to t

  $\int\left(\frac{d}{dt}(\mu(t)\, y)\right)dt=\int\frac{\mu(t)\sin(t)}{t^2}dt+c_1$
- Evaluate the integral on the lhs

  $\mu(t)\, y=\int\frac{\mu(t)\sin(t)}{t^2}dt+c_1$
- Solve for y

  $y=\frac{\int\frac{\mu(t)\sin(t)}{t^2}dt+c_1}{\mu(t)}$
- Substitute $\mu(t)=t^2$

  $y=\frac{\int\sin(t)dt+c_1}{t^2}$
- Evaluate the integrals on the rhs

  $y=\frac{-\cos(t)+c_1}{t^2}$
- Use initial condition $y\left(-\frac{\pi}{2}\right)=a$

  $a=\frac{4c_1}{\pi^2}$
- Solve for c_1

  $c_1=\frac{\pi^2 a}{4}$
- Substitute $c_1=\frac{\pi^2 a}{4}$ into general solution and simplify

  $y=\frac{-\cos(t)+\frac{\pi^2 a}{4}}{t^2}$
- Solution to the IVP

  $y=\frac{-\cos(t)+\frac{\pi^2 a}{4}}{t^2}$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
<- 1st order linear successful`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 19

```
dsolve([2*y(t)+t*diff(y(t),t) = sin(t)/t,y(-1/2*Pi) = a],y(t), singsol=all)
```

$$y(t) = \frac{-\cos(t) + \frac{a\,\pi^2}{4}}{t^2}$$

✓ Solution by Mathematica

Time used: 0.042 (sec). Leaf size: 22

```
DSolve[{2*y[t]+t*y'[t] == Sin[t]/t,y[-Pi/2]==a},y[t],t,IncludeSingularSolutions -> True]
```

$$y(t) \to \frac{\pi^2 a - 4\cos(t)}{4t^2}$$

1.26 problem 26

Internal problem ID [473]
Internal file name [`OUTPUT/473_Sunday_June_05_2022_01_42_02_AM_28951275/index.tex`]

Book: Elementary differential equations and boundary value problems, 10th ed., Boyce and DiPrima
Section: Section 2.1. Page 40
Problem number: 26.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"exact", "linear", "first_order_ode_lie_symmetry_lookup"**

Maple gives the following as the ode type

`[_linear]`

$$\boxed{\cos(t) y + \sin(t) y' = e^t}$$

With initial conditions

$$[y(1) = a]$$

1.26.1 Solving as linear ode

Entering Linear first order ODE solver. The integrating factor μ is

$$\begin{aligned} \mu &= e^{\int \cot(t)dt} \\ &= \sin(t) \end{aligned}$$

The ode becomes

$$\begin{aligned} \frac{d}{dt}(\mu y) &= (\mu)\left(\csc(t) e^t\right) \\ \frac{d}{dt}(\sin(t) y) &= (\sin(t))\left(\csc(t) e^t\right) \\ d(\sin(t) y) &= e^t \, dt \end{aligned}$$

Integrating gives

$$\sin(t)\, y = \int \mathrm{e}^{t}\, \mathrm{d}t$$
$$\sin(t)\, y = \mathrm{e}^{t} + c_1$$

Dividing both sides by the integrating factor $\mu = \sin(t)$ results in

$$y = \csc(t)\, \mathrm{e}^{t} + c_1 \csc(t)$$

which simplifies to

$$y = \csc(t) \left(\mathrm{e}^{t} + c_1\right)$$

Initial conditions are used to solve for c_1. Substituting $t = 1$ and $y = a$ in the above solution gives an equation to solve for the constant of integration.

$$a = \frac{\mathrm{e} + c_1}{\sin(1)}$$

$$c_1 = -\frac{\csc(1)\, \mathrm{e} - a}{\csc(1)}$$

Substituting c_1 found above in the general solution gives

$$y = \csc(t) \left(\mathrm{e}^{t} - \frac{\csc(1)\, \mathrm{e} - a}{\csc(1)}\right)$$

Summary

The solution(s) found are the following

$$y = \csc(t) \left(\mathrm{e}^{t} - \frac{\csc(1)\, \mathrm{e} - a}{\csc(1)}\right) \tag{1}$$

Verification of solutions

$$y = \csc(t) \left(\mathrm{e}^{t} - \frac{\csc(1)\, \mathrm{e} - a}{\csc(1)}\right)$$

Verified OK.

1.26.2 Solving as first order ode lie symmetry lookup ode

Writing the ode as

$$y' = -\frac{\cos(t)\, y - e^t}{\sin(t)}$$
$$y' = \omega(t, y)$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_t + \omega(\eta_y - \xi_t) - \omega^2 \xi_y - \omega_t \xi - \omega_y \eta = 0 \tag{A}$$

The type of this ode is known. It is of type `linear`. Therefore we do not need to solve the PDE (A), and can just use the lookup table shown below to find ξ, η

Table 76: Lie symmetry infinitesimal lookup table for known first order ODE's

ODE class	Form	ξ	η
linear ode	$y' = f(x)y(x) + g(x)$	0	$e^{\int f dx}$
separable ode	$y' = f(x)\, g(y)$	$\frac{1}{f}$	0
quadrature ode	$y' = f(x)$	0	1
quadrature ode	$y' = g(y)$	1	0
homogeneous ODEs of Class A	$y' = f\left(\frac{y}{x}\right)$	x	y
homogeneous ODEs of Class C	$y' = (a + bx + cy)^{\frac{n}{m}}$	1	$-\frac{b}{c}$
homogeneous class D	$y' = \frac{y}{x} + g(x)\, F\left(\frac{y}{x}\right)$	x^2	xy
First order special form ID 1	$y' = g(x)\, e^{h(x)+by} + f(x)$	$\frac{e^{-\int bf(x)dx - h(x)}}{g(x)}$	$\frac{f(x)e^{-\int bf(x)dx - h(x)}}{g(x)}$
polynomial type ode	$y' = \frac{a_1x+b_1y+c_1}{a_2x+b_2y+c_2}$	$\frac{a_1b_2x-a_2b_1x-b_1c_2+b_2c_1}{a_1b_2-a_2b_1}$	$\frac{a_1b_2y-a_2b_1y-a_1c_2-a_2c_1}{a_1b_2-a_2b_1}$
Bernoulli ode	$y' = f(x)\, y + g(x)\, y^n$	0	$e^{-\int (n-1)f(x)dx} y^n$
Reduced Riccati	$y' = f_1(x)\, y + f_2(x)\, y^2$	0	$e^{-\int f_1 dx}$

The above table shows that

$$
\begin{aligned}
\xi(t, y) &= 0 \\
\eta(t, y) &= \frac{1}{\sin(t)}
\end{aligned} \tag{A1}
$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(t, y) \to (R, S)$ where (R, S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$
\frac{dt}{\xi} = \frac{dy}{\eta} = dS \tag{1}
$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial t} + \eta \frac{\partial}{\partial y}\right) S(t, y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$
R = t
$$

S is found from

$$
\begin{aligned}
S &= \int \frac{1}{\eta} dy \\
&= \int \frac{1}{\frac{1}{\sin(t)}} dy
\end{aligned}
$$

Which results in

$$
S = \sin(t) y
$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$
\frac{dS}{dR} = \frac{S_t + \omega(t, y) S_y}{R_t + \omega(t, y) R_y} \tag{2}
$$

Where in the above R_t, R_y, S_t, S_y are all partial derivatives and $\omega(t, y)$ is the right hand side of the original ode given by

$$
\omega(t, y) = -\frac{\cos(t) y - e^t}{\sin(t)}
$$

Evaluating all the partial derivatives gives

$$
\begin{aligned}
R_t &= 1 \\
R_y &= 0 \\
S_t &= \cos(t)\, y \\
S_y &= \sin(t)
\end{aligned}
$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = e^t \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for t, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = e^R$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$S(R) = e^R + c_1 \tag{4}$$

To complete the solution, we just need to transform (4) back to t, y coordinates. This results in

$$\sin(t)\, y = e^t + c_1$$

Which simplifies to

$$\sin(t)\, y = e^t + c_1$$

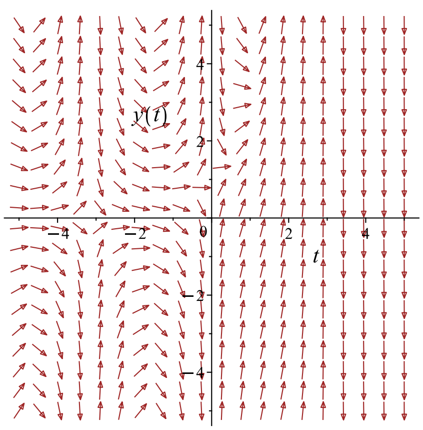
Which gives

$$y = \frac{e^t + c_1}{\sin(t)}$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

<table>
<tr><th>Original ode in t, y coordinates</th><th>Canonical coordinates transformation</th><th>ODE in canonical coordinates (R, S)</th></tr>
<tr>
<td>

$\frac{dy}{dt} = -\frac{\cos(t) y - e^{t}}{\sin(t)}$



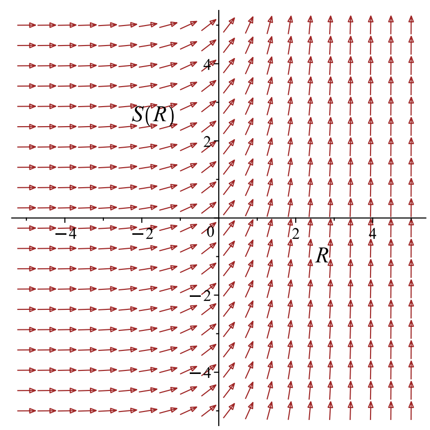
</td>
<td>

$R = t$

$S = \sin(t) y$

</td>
<td>

$\frac{dS}{dR} = e^{R}$



</td>
</tr>
</table>

Initial conditions are used to solve for c_1. Substituting $t = 1$ and $y = a$ in the above solution gives an equation to solve for the constant of integration.

$$a = \frac{e + c_1}{\sin(1)}$$

$$c_1 = -\frac{\csc(1) e - a}{\csc(1)}$$

Substituting c_1 found above in the general solution gives

$$y = \frac{e^{t} - \frac{\csc(1) e - a}{\csc(1)}}{\sin(t)}$$

<u>Summary</u>

The solution(s) found are the following

$$y = \frac{e^{t} - \frac{\csc(1) e - a}{\csc(1)}}{\sin(t)} \tag{1}$$

Verification of solutions

$$y = \frac{\mathrm{e}^{t} - \frac{\csc(1)\mathrm{e}-a}{\csc(1)}}{\sin(t)}$$

Verified OK.

1.26.3 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x,y) + N(x,y)\frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x,y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y) = 0$$

Hence

$$\frac{\partial\phi}{\partial x} + \frac{\partial\phi}{\partial y}\frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$\begin{aligned}\frac{\partial\phi}{\partial x} &= M\\ \frac{\partial\phi}{\partial y} &= N\end{aligned}$$

But since $\frac{\partial^2\phi}{\partial x\partial y} = \frac{\partial^2\phi}{\partial y\partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x,y)$ but at least we know now that we can do that since the condition $\frac{\partial^2\phi}{\partial x\partial y} = \frac{\partial^2\phi}{\partial y\partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(t,y)\,\mathrm{d}t + N(t,y)\,\mathrm{d}y = 0 \tag{1A}$$

Therefore

$$
\begin{aligned}
(\sin (t)) \mathrm{d}y &= \left(-\cos (t) y + \mathrm{e}^{t}\right) \mathrm{d}t \\
\left(\cos (t) y - \mathrm{e}^{t}\right) \mathrm{d}t + (\sin (t)) \mathrm{d}y &= 0 \qquad (2A)
\end{aligned}
$$

Comparing (1A) and (2A) shows that

$$
\begin{aligned}
M(t, y) &= \cos (t) y - \mathrm{e}^{t} \\
N(t, y) &= \sin (t)
\end{aligned}
$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$
\frac{\partial M}{\partial y} = \frac{\partial N}{\partial t}
$$

Using result found above gives

$$
\begin{aligned}
\frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}\left(\cos (t) y - \mathrm{e}^{t}\right) \\
&= \cos (t)
\end{aligned}
$$

And

$$
\begin{aligned}
\frac{\partial N}{\partial t} &= \frac{\partial}{\partial t}(\sin (t)) \\
&= \cos (t)
\end{aligned}
$$

Since $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial t}$, then the ODE is exact The following equations are now set up to solve for the function $\phi(t, y)$

$$
\frac{\partial \phi}{\partial t} = M \qquad (1)
$$

$$
\frac{\partial \phi}{\partial y} = N \qquad (2)
$$

Integrating (1) w.r.t. t gives

$$
\begin{aligned}
\int \frac{\partial \phi}{\partial t} \mathrm{d}t &= \int M \mathrm{d}t \\
\int \frac{\partial \phi}{\partial t} \mathrm{d}t &= \int \cos (t) y - \mathrm{e}^{t} \mathrm{d}t \\
\phi &= \sin (t) y - \mathrm{e}^{t} + f(y) \qquad (3)
\end{aligned}
$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both t and y. Taking derivative of equation (3) w.r.t y gives

$$\frac{\partial \phi}{\partial y} = \sin(t) + f'(y) \tag{4}$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = \sin(t)$. Therefore equation (4) becomes

$$\sin(t) = \sin(t) + f'(y) \tag{5}$$

Solving equation (5) for $f'(y)$ gives

$$f'(y) = 0$$

Therefore

$$f(y) = c_1$$

Where c_1 is constant of integration. Substituting this result for $f(y)$ into equation (3) gives ϕ

$$\phi = \sin(t)\, y - \mathrm{e}^t + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = \sin(t)\, y - \mathrm{e}^t$$

The solution becomes

$$y = \frac{\mathrm{e}^t + c_1}{\sin(t)}$$

Initial conditions are used to solve for c_1. Substituting $t = 1$ and $y = a$ in the above solution gives an equation to solve for the constant of integration.

$$a = \frac{\mathrm{e} + c_1}{\sin(1)}$$

$$c_1 = -\frac{\csc(1)\,\mathrm{e} - a}{\csc(1)}$$

Substituting c_1 found above in the general solution gives

$$y = \frac{e^{t} - \frac{\csc(1)e - a}{\csc(1)}}{\sin(t)}$$

Summary
The solution(s) found are the following

$$y = \frac{e^{t} - \frac{\csc(1)e - a}{\csc(1)}}{\sin(t)} \tag{1}$$

Verification of solutions

$$y = \frac{e^{t} - \frac{\csc(1)e - a}{\csc(1)}}{\sin(t)}$$

Verified OK.

1.26.4 Maple step by step solution

Let's solve

$[\cos(t)\, y + \sin(t)\, y' = e^{t}, y(1) = a]$

- Highest derivative means the order of the ODE is 1

  y'

- Isolate the derivative

  $y' = -\frac{\cos(t)y}{\sin(t)} + \frac{e^{t}}{\sin(t)}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE

  $y' + \frac{\cos(t)y}{\sin(t)} = \frac{e^{t}}{\sin(t)}$

- The ODE is linear; multiply by an integrating factor $\mu(t)$

  $\mu(t)\left(y' + \frac{\cos(t)y}{\sin(t)}\right) = \frac{\mu(t)e^{t}}{\sin(t)}$

- Assume the lhs of the ODE is the total derivative $\frac{d}{dt}(\mu(t)\, y)$

  $\mu(t)\left(y' + \frac{\cos(t)y}{\sin(t)}\right) = \mu'(t)\, y + \mu(t)\, y'$

- Isolate $\mu'(t)$

  $\mu'(t) = \frac{\mu(t)\cos(t)}{\sin(t)}$

- Solve to find the integrating factor

$\mu(t) = \sin(t)$

- Integrate both sides with respect to t

$\int \left(\frac{d}{dt}(\mu(t)\, y)\right) dt = \int \frac{\mu(t)\mathrm{e}^t}{\sin(t)} dt + c_1$

- Evaluate the integral on the lhs

$\mu(t)\, y = \int \frac{\mu(t)\mathrm{e}^t}{\sin(t)} dt + c_1$

- Solve for y

$y = \frac{\int \frac{\mu(t)\mathrm{e}^t}{\sin(t)} dt + c_1}{\mu(t)}$

- Substitute $\mu(t) = \sin(t)$

$y = \frac{\int \mathrm{e}^t dt + c_1}{\sin(t)}$

- Evaluate the integrals on the rhs

$y = \frac{\mathrm{e}^t + c_1}{\sin(t)}$

- Simplify

$y = \csc(t)\,(\mathrm{e}^t + c_1)$

- Use initial condition $y(1) = a$

$a = \csc(1)\,(\mathrm{e} + c_1)$

- Solve for c_1

$c_1 = -\frac{\csc(1)\mathrm{e} - a}{\csc(1)}$

- Substitute $c_1 = -\frac{\csc(1)\mathrm{e} - a}{\csc(1)}$ into general solution and simplify

$y = \csc(t)\,(a \sin(1) + \mathrm{e}^t - \mathrm{e})$

- Solution to the IVP

$y = \csc(t)\,(a \sin(1) + \mathrm{e}^t - \mathrm{e})$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
<- 1st order linear successful`
```

✓ Solution by Maple

Time used: 0.031 (sec). Leaf size: 18

```
dsolve([cos(t)*y(t)+sin(t)*diff(y(t),t) = exp(t),y(1) = a],y(t), singsol=all)
```

$$y(t) = \csc(t)\left(\mathrm{e}^{t} + a\sin(1) - \mathrm{e}\right)$$

✓ Solution by Mathematica

Time used: 0.062 (sec). Leaf size: 19

```
DSolve[{Cos[t]*y[t]+Sin[t]*y'[t] == Exp[t],y[1]==a},y[t],t,IncludeSingularSolutions -> True]
```

$$y(t) \to \csc(t)\left(a\sin(1) + e^{t} - e\right)$$

1.27 problem 27

Internal problem ID [474]
Internal file name [`OUTPUT/474_Sunday_June_05_2022_01_42_03_AM_40812399/index.tex`]

Book: Elementary differential equations and boundary value problems, 10th ed., Boyce and DiPrima
Section: Section 2.1. Page 40
Problem number: 27.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"linear", "exactWithIntegrationFactor", "first_order_ode_lie_symmetry_lookup"**

Maple gives the following as the ode type

```
[[_linear, `class A`]]
```

$$\boxed{y' + \frac{y}{2} = 2\cos(t)}$$

With initial conditions

$$[y(0) = -1]$$

1.27.1 Existence and uniqueness analysis

This is a linear ODE. In canonical form it is written as

$$y' + p(t)y = q(t)$$

Where here

$$p(t) = \frac{1}{2}$$
$$q(t) = 2\cos(t)$$

Hence the ode is

$$y' + \frac{y}{2} = 2\cos(t)$$

The domain of $p(t) = \frac{1}{2}$ is

$$\{-\infty < t < \infty\}$$

And the point $t_0 = 0$ is inside this domain. The domain of $q(t) = 2\cos(t)$ is

$$\{-\infty < t < \infty\}$$

And the point $t_0 = 0$ is also inside this domain. Hence solution exists and is unique.

1.27.2 Solving as linear ode

Entering Linear first order ODE solver. The integrating factor μ is

$$\begin{aligned}
\mu &= e^{\int \frac{1}{2} dt} \\
&= e^{\frac{t}{2}}
\end{aligned}$$

The ode becomes

$$\begin{aligned}
\frac{d}{dt}(\mu y) &= (\mu)(2\cos(t)) \\
\frac{d}{dt}\left(e^{\frac{t}{2}} y\right) &= \left(e^{\frac{t}{2}}\right)(2\cos(t)) \\
d\left(e^{\frac{t}{2}} y\right) &= \left(2\cos(t)\, e^{\frac{t}{2}}\right) dt
\end{aligned}$$

Integrating gives

$$\begin{aligned}
e^{\frac{t}{2}} y &= \int 2\cos(t)\, e^{\frac{t}{2}}\, dt \\
e^{\frac{t}{2}} y &= \frac{4\cos(t)\, e^{\frac{t}{2}}}{5} + \frac{8\sin(t)\, e^{\frac{t}{2}}}{5} + c_1
\end{aligned}$$

Dividing both sides by the integrating factor $\mu = e^{\frac{t}{2}}$ results in

$$y = e^{-\frac{t}{2}} \left(\frac{4\cos(t)\, e^{\frac{t}{2}}}{5} + \frac{8\sin(t)\, e^{\frac{t}{2}}}{5} \right) + c_1 e^{-\frac{t}{2}}$$

which simplifies to

$$y = \frac{8\sin(t)}{5} + \frac{4\cos(t)}{5} + c_1 e^{-\frac{t}{2}}$$

Initial conditions are used to solve for c_1. Substituting $t = 0$ and $y = -1$ in the above solution gives an equation to solve for the constant of integration.

$$-1 = \frac{4}{5} + c_1$$

$$c_1 = -\frac{9}{5}$$

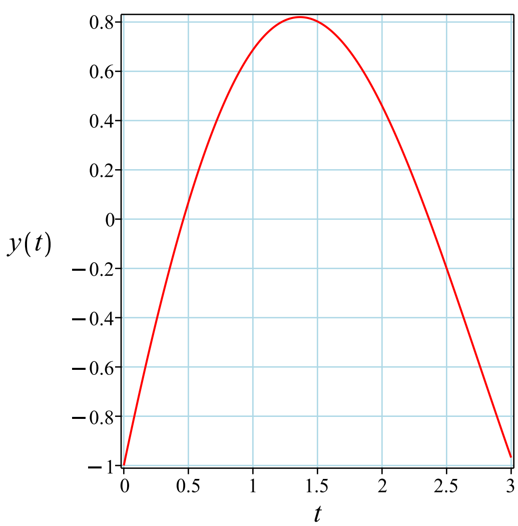
Substituting c_1 found above in the general solution gives

$$y = \frac{8\sin(t)}{5} + \frac{4\cos(t)}{5} - \frac{9\,e^{-\frac{t}{2}}}{5}$$

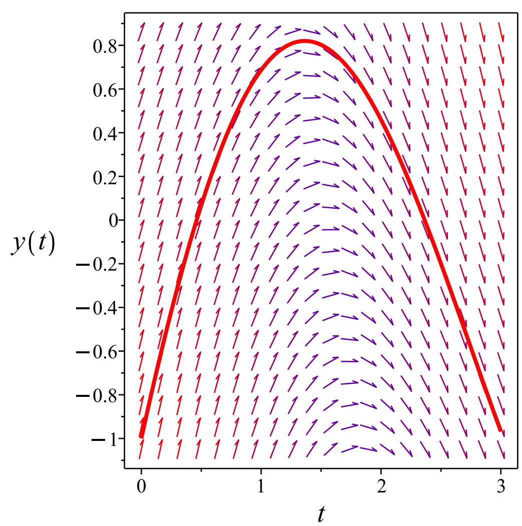
Summary

The solution(s) found are the following

$$y = \frac{8\sin(t)}{5} + \frac{4\cos(t)}{5} - \frac{9\,e^{-\frac{t}{2}}}{5} \tag{1}$$



(a) Solution plot



(b) Slope field plot

Verification of solutions

$$y = \frac{8\sin(t)}{5} + \frac{4\cos(t)}{5} - \frac{9\,e^{-\frac{t}{2}}}{5}$$

Verified OK.

1.27.3 Solving as first order ode lie symmetry lookup ode

Writing the ode as

$$
\begin{aligned}
y' &= -\frac{y}{2} + 2\cos(t) \\
y' &= \omega(t, y)
\end{aligned}
$$

The condition of Lie symmetry is the linearized PDE given by

$$
\eta_t + \omega(\eta_y - \xi_t) - \omega^2 \xi_y - \omega_t \xi - \omega_y \eta = 0 \tag{A}
$$

The type of this ode is known. It is of type `linear`. Therefore we do not need to solve the PDE (A), and can just use the lookup table shown below to find ξ, η

Table 79: Lie symmetry infinitesimal lookup table for known first order ODE's

ODE class	Form	ξ	η
linear ode	$y' = f(x)y(x) + g(x)$	0	$e^{\int f dx}$
separable ode	$y' = f(x)\, g(y)$	$\frac{1}{f}$	0
quadrature ode	$y' = f(x)$	0	1
quadrature ode	$y' = g(y)$	1	0
homogeneous ODEs of Class A	$y' = f\left(\frac{y}{x}\right)$	x	y
homogeneous ODEs of Class C	$y' = (a + bx + cy)^{\frac{n}{m}}$	1	$-\frac{b}{c}$
homogeneous class D	$y' = \frac{y}{x} + g(x)\, F\left(\frac{y}{x}\right)$	x^2	xy
First order special form ID 1	$y' = g(x)\, e^{h(x)+by} + f(x)$	$\frac{e^{-\int bf(x)dx - h(x)}}{g(x)}$	$\frac{f(x)e^{-\int bf(x)dx - h(x)}}{g(x)}$
polynomial type ode	$y' = \frac{a_1x+b_1y+c_1}{a_2x+b_2y+c_2}$	$\frac{a_1b_2x-a_2b_1x-b_1c_2+b_2c_1}{a_1b_2-a_2b_1}$	$\frac{a_1b_2y-a_2b_1y-a_1c_2-a_2c_1}{a_1b_2-a_2b_1}$
Bernoulli ode	$y' = f(x)\, y + g(x)\, y^n$	0	$e^{-\int (n-1)f(x)dx} y^n$
Reduced Riccati	$y' = f_1(x)\, y + f_2(x)\, y^2$	0	$e^{-\int f_1 dx}$

The above table shows that

$$\xi(t,y) = 0$$
$$\eta(t,y) = \mathrm{e}^{-\frac{t}{2}} \tag{A1}$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(t,y) \to (R,S)$ where (R,S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$\frac{dt}{\xi} = \frac{dy}{\eta} = dS \tag{1}$$

The above comes from the requirements that $\left(\xi\frac{\partial}{\partial t} + \eta\frac{\partial}{\partial y}\right) S(t,y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$R = t$$

S is found from

$$\begin{aligned} S &= \int \frac{1}{\eta} dy \\ &= \int \frac{1}{\mathrm{e}^{-\frac{t}{2}}} dy \end{aligned}$$

Which results in

$$S = \mathrm{e}^{\frac{t}{2}} y$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_t + \omega(t,y)S_y}{R_t + \omega(t,y)R_y} \tag{2}$$

Where in the above R_t, R_y, S_t, S_y are all partial derivatives and $\omega(t,y)$ is the right hand side of the original ode given by

$$\omega(t,y) = -\frac{y}{2} + 2\cos(t)$$

Evaluating all the partial derivatives gives

$$\begin{aligned} R_t &= 1 \\ R_y &= 0 \\ S_t &= \frac{\mathrm{e}^{\frac{t}{2}} y}{2} \\ S_y &= \mathrm{e}^{\frac{t}{2}} \end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = 2\cos(t)\,\mathrm{e}^{\frac{t}{2}} \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for t, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = 2\cos(R)\,\mathrm{e}^{\frac{R}{2}}$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$S(R) = c_1 + \frac{4\,\mathrm{e}^{\frac{R}{2}}(\cos(R) + 2\sin(R))}{5} \tag{4}$$

To complete the solution, we just need to transform (4) back to t, y coordinates. This results in

$$\mathrm{e}^{\frac{t}{2}} y = \frac{4\,\mathrm{e}^{\frac{t}{2}}(\cos(t) + 2\sin(t))}{5} + c_1$$

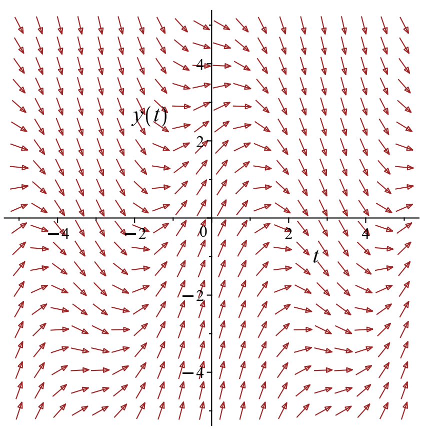
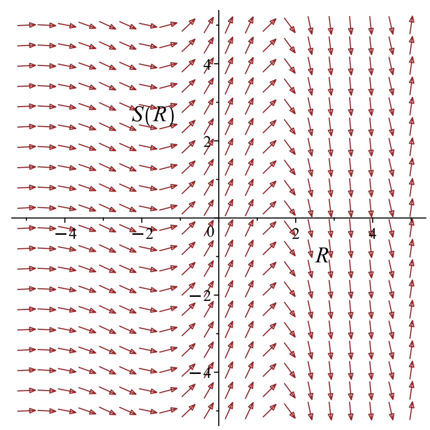
Which simplifies to

$$\mathrm{e}^{\frac{t}{2}} y = \frac{4\,\mathrm{e}^{\frac{t}{2}}(\cos(t) + 2\sin(t))}{5} + c_1$$

Which gives

$$y = \frac{\mathrm{e}^{-\frac{t}{2}}\left(4\cos(t)\,\mathrm{e}^{\frac{t}{2}} + 8\sin(t)\,\mathrm{e}^{\frac{t}{2}} + 5c_1\right)}{5}$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in t, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dt} = -\frac{y}{2} + 2\cos(t)$ 	$R = t$ $S = e^{\frac{t}{2}} y$	$\frac{dS}{dR} = 2\cos(R)\,e^{\frac{R}{2}}$ 

Initial conditions are used to solve for c_1. Substituting $t = 0$ and $y = -1$ in the above solution gives an equation to solve for the constant of integration.

$$-1 = \frac{4}{5} + c_1$$

$$c_1 = -\frac{9}{5}$$

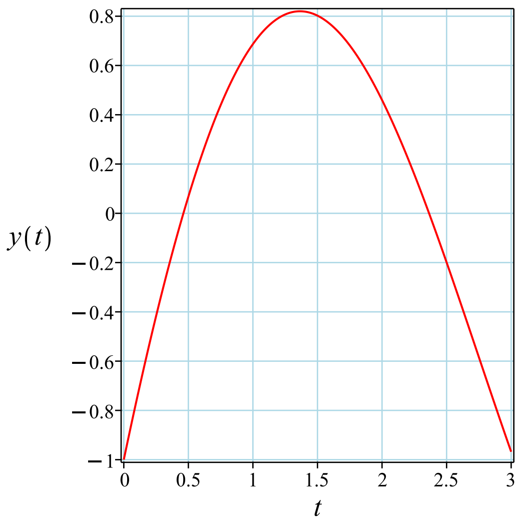
Substituting c_1 found above in the general solution gives

$$y = \frac{8\sin(t)}{5} + \frac{4\cos(t)}{5} - \frac{9\,e^{-\frac{t}{2}}}{5}$$

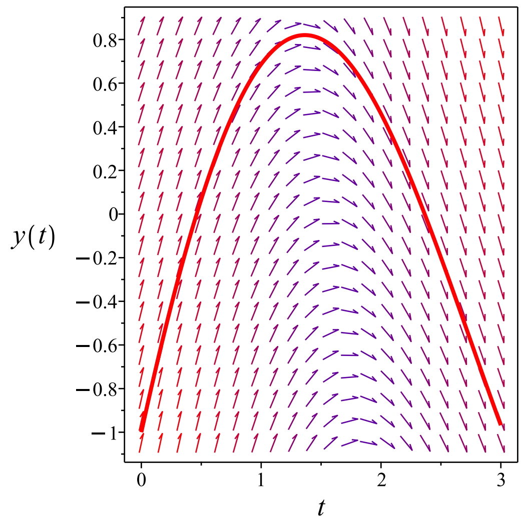
Summary

The solution(s) found are the following

$$y = \frac{8\sin(t)}{5} + \frac{4\cos(t)}{5} - \frac{9\,e^{-\frac{t}{2}}}{5} \tag{1}$$



(a) Solution plot



(b) Slope field plot

<u>Verification of solutions</u>

$$y = \frac{8\sin(t)}{5} + \frac{4\cos(t)}{5} - \frac{9\,\mathrm{e}^{-\frac{t}{2}}}{5}$$

Verified OK.

1.27.4 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x,y) + N(x,y)\frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x,y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y) = 0$$

Hence

$$\frac{\partial\phi}{\partial x} + \frac{\partial\phi}{\partial y}\frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$\frac{\partial\phi}{\partial x} = M$$
$$\frac{\partial\phi}{\partial y} = N$$

But since $\frac{\partial^2\phi}{\partial x\partial y} = \frac{\partial^2\phi}{\partial y\partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x, y)$ but at least we know now that we can do that since the condition $\frac{\partial^2\phi}{\partial x\partial y} = \frac{\partial^2\phi}{\partial y\partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(t, y)\,\mathrm{d}t + N(t, y)\,\mathrm{d}y = 0 \tag{1A}$$

Therefore

$$\mathrm{d}y = \left(-\frac{y}{2} + 2\cos(t)\right)\mathrm{d}t$$

$$\left(\frac{y}{2} - 2\cos(t)\right)\mathrm{d}t + \mathrm{d}y = 0 \tag{2A}$$

Comparing (1A) and (2A) shows that

$$\begin{aligned} M(t, y) &= \frac{y}{2} - 2\cos(t) \\ N(t, y) &= 1 \end{aligned}$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial t}$$

Using result found above gives

$$\begin{aligned} \frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}\left(\frac{y}{2} - 2\cos(t)\right) \\ &= \frac{1}{2} \end{aligned}$$

And

$$\begin{aligned} \frac{\partial N}{\partial t} &= \frac{\partial}{\partial t}(1) \\ &= 0 \end{aligned}$$

Since $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial t}$, then the ODE is not exact. Since the ODE is not exact, we will try to find an integrating factor to make it exact. Let

$$\begin{aligned} A &= \frac{1}{N}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial t}\right) \\ &= 1\left(\left(\frac{1}{2}\right) - (0)\right) \\ &= \frac{1}{2} \end{aligned}$$

Since A does not depend on y, then it can be used to find an integrating factor. The integrating factor μ is

$$\begin{aligned} \mu &= e^{\int A \, \mathrm{d}t} \\ &= e^{\int \frac{1}{2} \, \mathrm{d}t} \end{aligned}$$

The result of integrating gives

$$\begin{aligned} \mu &= e^{\frac{t}{2}} \\ &= \mathrm{e}^{\frac{t}{2}} \end{aligned}$$

M and N are multiplied by this integrating factor, giving new M and new N which are called $\overline{M}$ and $\overline{N}$ for now so not to confuse them with the original M and N.

$$\begin{aligned} \overline{M} &= \mu M \\ &= \mathrm{e}^{\frac{t}{2}}\left(\frac{y}{2} - 2\cos(t)\right) \\ &= \frac{(y - 4\cos(t))\,\mathrm{e}^{\frac{t}{2}}}{2} \end{aligned}$$

And

$$\begin{aligned} \overline{N} &= \mu N \\ &= \mathrm{e}^{\frac{t}{2}}(1) \\ &= \mathrm{e}^{\frac{t}{2}} \end{aligned}$$

Now a modified ODE is ontained from the original ODE, which is exact and can be solved. The modified ODE is

$$\begin{aligned} \overline{M} + \overline{N}\frac{\mathrm{d}y}{\mathrm{d}t} &= 0 \\ \left(\frac{(y - 4\cos(t))\,\mathrm{e}^{\frac{t}{2}}}{2}\right) + \left(\mathrm{e}^{\frac{t}{2}}\right)\frac{\mathrm{d}y}{\mathrm{d}t} &= 0 \end{aligned}$$

The following equations are now set up to solve for the function $\phi(t,y)$

$$\frac{\partial \phi}{\partial t} = \overline{M} \tag{1}$$

$$\frac{\partial \phi}{\partial y} = \overline{N} \tag{2}$$

Integrating (1) w.r.t. t gives

$$\int \frac{\partial \phi}{\partial t} \,\mathrm{d}t = \int \overline{M} \,\mathrm{d}t$$

$$\int \frac{\partial \phi}{\partial t} \,\mathrm{d}t = \int \frac{(y - 4\cos(t))\,\mathrm{e}^{\frac{t}{2}}}{2} \,\mathrm{d}t$$

$$\phi = -\frac{\mathrm{e}^{\frac{t}{2}}(-5y + 4\cos(t) + 8\sin(t))}{5} + f(y) \tag{3}$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both t and y. Taking derivative of equation (3) w.r.t y gives

$$\frac{\partial \phi}{\partial y} = \mathrm{e}^{\frac{t}{2}} + f'(y) \tag{4}$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = \mathrm{e}^{\frac{t}{2}}$. Therefore equation (4) becomes

$$\mathrm{e}^{\frac{t}{2}} = \mathrm{e}^{\frac{t}{2}} + f'(y) \tag{5}$$

Solving equation (5) for $f'(y)$ gives

$$f'(y) = 0$$

Therefore

$$f(y) = c_1$$

Where c_1 is constant of integration. Substituting this result for $f(y)$ into equation (3) gives ϕ

$$\phi = -\frac{\mathrm{e}^{\frac{t}{2}}(-5y + 4\cos(t) + 8\sin(t))}{5} + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = -\frac{\mathrm{e}^{\frac{t}{2}}(-5y + 4\cos(t) + 8\sin(t))}{5}$$

The solution becomes

$$y = \frac{\mathrm{e}^{-\frac{t}{2}} \left(4 \cos \left(t\right) \mathrm{e}^{\frac{t}{2}} + 8 \sin \left(t\right) \mathrm{e}^{\frac{t}{2}} + 5 c_1\right)}{5}$$

Initial conditions are used to solve for c_1. Substituting $t = 0$ and $y = -1$ in the above solution gives an equation to solve for the constant of integration.

$$-1 = \frac{4}{5} + c_1$$

$$c_1 = -\frac{9}{5}$$

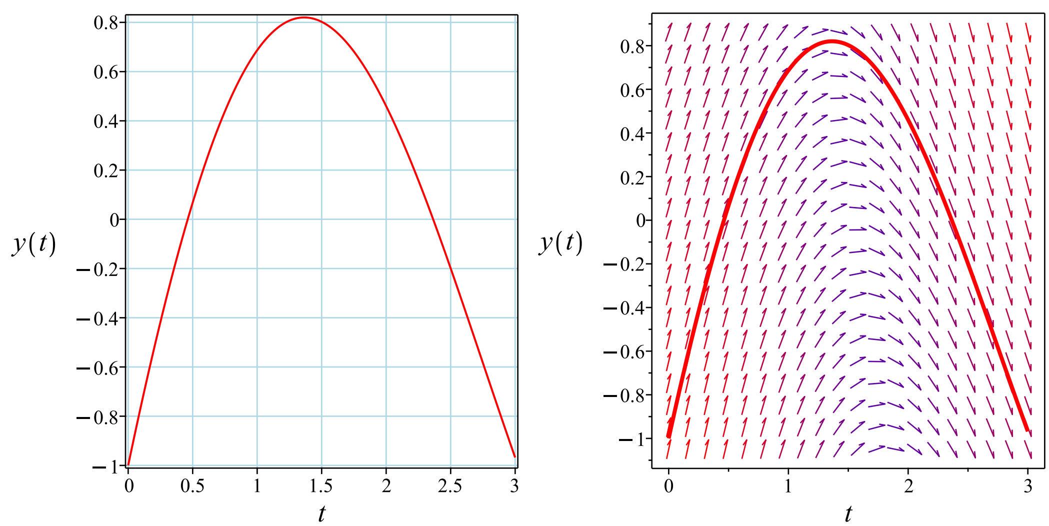
Substituting c_1 found above in the general solution gives

$$y = \frac{8 \sin \left(t\right)}{5} + \frac{4 \cos \left(t\right)}{5} - \frac{9 \,\mathrm{e}^{-\frac{t}{2}}}{5}$$

Summary

The solution(s) found are the following

$$y = \frac{8 \sin \left(t\right)}{5} + \frac{4 \cos \left(t\right)}{5} - \frac{9 \,\mathrm{e}^{-\frac{t}{2}}}{5} \tag{1}$$



(a) Solution plot (b) Slope field plot

Verification of solutions

$$y = \frac{8 \sin \left(t\right)}{5} + \frac{4 \cos \left(t\right)}{5} - \frac{9 \,\mathrm{e}^{-\frac{t}{2}}}{5}$$

Verified OK.

1.27.5 Maple step by step solution

Let's solve

$\left[y' + \frac{y}{2} = 2\cos(t), y(0) = -1\right]$

- Highest derivative means the order of the ODE is 1

  y'

- Isolate the derivative

  $y' = -\frac{y}{2} + 2\cos(t)$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE

  $y' + \frac{y}{2} = 2\cos(t)$

- The ODE is linear; multiply by an integrating factor $\mu(t)$

  $\mu(t)\left(y' + \frac{y}{2}\right) = 2\mu(t)\cos(t)$

- Assume the lhs of the ODE is the total derivative $\frac{d}{dt}(\mu(t)\, y)$

  $\mu(t)\left(y' + \frac{y}{2}\right) = \mu'(t)\, y + \mu(t)\, y'$

- Isolate $\mu'(t)$

  $\mu'(t) = \frac{\mu(t)}{2}$

- Solve to find the integrating factor

  $\mu(t) = e^{\frac{t}{2}}$

- Integrate both sides with respect to t

  $\int \left(\frac{d}{dt}(\mu(t)\, y)\right) dt = \int 2\mu(t)\cos(t)\, dt + c_1$

- Evaluate the integral on the lhs

  $\mu(t)\, y = \int 2\mu(t)\cos(t)\, dt + c_1$

- Solve for y

  $y = \frac{\int 2\mu(t)\cos(t)dt + c_1}{\mu(t)}$

- Substitute $\mu(t) = e^{\frac{t}{2}}$

  $y = \frac{\int 2\cos(t)e^{\frac{t}{2}}dt + c_1}{e^{\frac{t}{2}}}$

- Evaluate the integrals on the rhs

  $y = \frac{\frac{4\cos(t)e^{\frac{t}{2}}}{5} + \frac{8\sin(t)e^{\frac{t}{2}}}{5} + c_1}{e^{\frac{t}{2}}}$

- Simplify

$$y = \frac{8\sin(t)}{5} + \frac{4\cos(t)}{5} + c_1 e^{-\frac{t}{2}}$$

- Use initial condition $y(0) = -1$

  $$-1 = \frac{4}{5} + c_1$$

- Solve for c_1

  $$c_1 = -\frac{9}{5}$$

- Substitute $c_1 = -\frac{9}{5}$ into general solution and simplify

  $$y = \frac{8\sin(t)}{5} + \frac{4\cos(t)}{5} - \frac{9\,e^{-\frac{t}{2}}}{5}$$

- Solution to the IVP

  $$y = \frac{8\sin(t)}{5} + \frac{4\cos(t)}{5} - \frac{9\,e^{-\frac{t}{2}}}{5}$$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
<- 1st order linear successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 19

```
dsolve([1/2*y(t)+diff(y(t),t) = 2*cos(t),y(0) = -1],y(t), singsol=all)
```

$$y(t) = \frac{4\cos(t)}{5} + \frac{8\sin(t)}{5} - \frac{9\,e^{-\frac{t}{2}}}{5}$$

✓ Solution by Mathematica

Time used: 0.049 (sec). Leaf size: 27

```
DSolve[{1/2*y[t]+y'[t] == 2*Cos[t],y[0]==-1},y[t],t,IncludeSingularSolutions -> True]
```

$$y(t) \to \frac{1}{5}\left(-9e^{-t/2} + 8\sin(t) + 4\cos(t)\right)$$

1.28 problem 28

Internal problem ID [475]
Internal file name [`OUTPUT/475_Sunday_June_05_2022_01_42_04_AM_90970000/index.tex`]

Book: Elementary differential equations and boundary value problems, 10th ed., Boyce and DiPrima
Section: Section 2.1. Page 40
Problem number: 28.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"linear", "exactWithIntegrationFactor", "first_order_ode_lie_symmetry_lookup"**

Maple gives the following as the ode type

```
[[_linear, `class A`]]
```

$$\boxed{\frac{2y}{3}+y'=-\frac{t}{2}+1}$$

1.28.1 Solving as linear ode

Entering Linear first order ODE solver. In canonical form a linear first order is

$$y'+p(t)y=q(t)$$

Where here

$$p(t)=\frac{2}{3}$$
$$q(t)=-\frac{t}{2}+1$$

Hence the ode is

$$\frac{2y}{3}+y'=-\frac{t}{2}+1$$

The integrating factor μ is

$$\mu = e^{\int \frac{2}{3}dt}$$
$$= e^{\frac{2t}{3}}$$

The ode becomes

$$\frac{d}{dt}(\mu y) = (\mu)\left(-\frac{t}{2}+1\right)$$
$$\frac{d}{dt}\left(e^{\frac{2t}{3}} y\right) = \left(e^{\frac{2t}{3}}\right)\left(-\frac{t}{2}+1\right)$$
$$d\left(e^{\frac{2t}{3}} y\right) = \left(-\frac{(t-2)\,e^{\frac{2t}{3}}}{2}\right) dt$$

Integrating gives

$$e^{\frac{2t}{3}} y = \int -\frac{(t-2)\,e^{\frac{2t}{3}}}{2}\, dt$$
$$e^{\frac{2t}{3}} y = -\frac{3(2t-7)\,e^{\frac{2t}{3}}}{8} + c_1$$

Dividing both sides by the integrating factor $\mu = e^{\frac{2t}{3}}$ results in

$$y = -\frac{3\,e^{-\frac{2t}{3}}(2t-7)\,e^{\frac{2t}{3}}}{8} + c_1 e^{-\frac{2t}{3}}$$

which simplifies to

$$y = -\frac{3t}{4} + \frac{21}{8} + c_1 e^{-\frac{2t}{3}}$$

Summary
The solution(s) found are the following

$$y = -\frac{3t}{4} + \frac{21}{8} + c_1 e^{-\frac{2t}{3}} \tag{1}$$

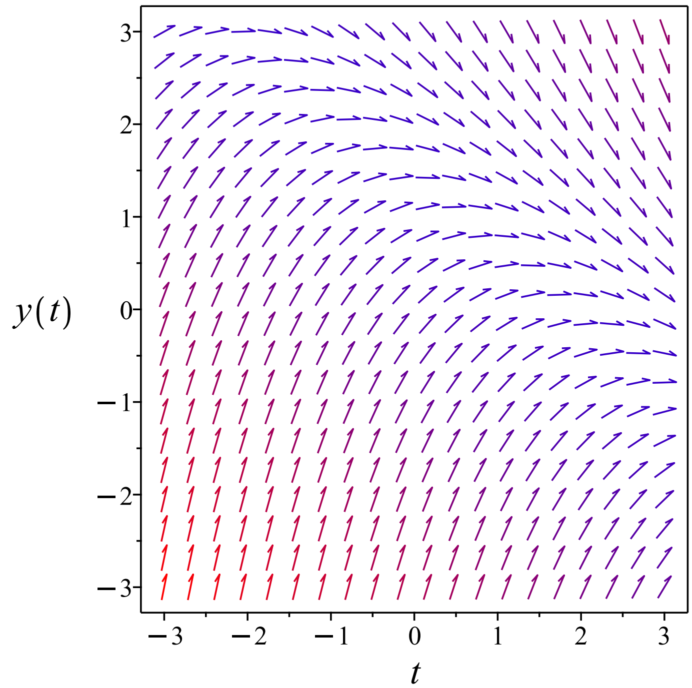


Figure 65: Slope field plot

Verification of solutions

$$y = -\frac{3t}{4} + \frac{21}{8} + c_1 e^{-\frac{2t}{3}}$$

Verified OK.

1.28.2 Solving as first order ode lie symmetry lookup ode

Writing the ode as

$$y' = -\frac{2y}{3} - \frac{t}{2} + 1$$
$$y' = \omega(t, y)$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_t + \omega(\eta_y - \xi_t) - \omega^2 \xi_y - \omega_t \xi - \omega_y \eta = 0 \tag{A}$$

The type of this ode is known. It is of type `linear`. Therefore we do not need to solve the PDE (A), and can just use the lookup table shown below to find ξ, η

Table 82: Lie symmetry infinitesimal lookup table for known first order ODE's

ODE class	Form	ξ	η
linear ode	$y' = f(x)y(x) + g(x)$	0	$e^{\int f dx}$
separable ode	$y' = f(x)\, g(y)$	$\frac{1}{f}$	0
quadrature ode	$y' = f(x)$	0	1
quadrature ode	$y' = g(y)$	1	0
homogeneous ODEs of Class A	$y' = f\left(\frac{y}{x}\right)$	x	y
homogeneous ODEs of Class C	$y' = (a + bx + cy)^{\frac{n}{m}}$	1	$-\frac{b}{c}$
homogeneous class D	$y' = \frac{y}{x} + g(x)\, F\left(\frac{y}{x}\right)$	x^2	xy
First order special form ID 1	$y' = g(x)\, e^{h(x)+by} + f(x)$	$\frac{e^{-\int bf(x)dx - h(x)}}{g(x)}$	$\frac{f(x)e^{-\int bf(x)dx - h(x)}}{g(x)}$
polynomial type ode	$y' = \frac{a_1x+b_1y+c_1}{a_2x+b_2y+c_2}$	$\frac{a_1b_2x - a_2b_1x - b_1c_2 + b_2c_1}{a_1b_2 - a_2b_1}$	$\frac{a_1b_2y - a_2b_1y - a_1c_2 - a_2c_1}{a_1b_2 - a_2b_1}$
Bernoulli ode	$y' = f(x)\, y + g(x)\, y^n$	0	$e^{-\int (n-1)f(x)dx} y^n$
Reduced Riccati	$y' = f_1(x)\, y + f_2(x)\, y^2$	0	$e^{-\int f_1 dx}$

The above table shows that

$$
\begin{aligned}
\xi(t, y) &= 0 \\
\eta(t, y) &= \mathrm{e}^{-\frac{2t}{3}}
\end{aligned} \tag{A1}
$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(t, y) \to (R, S)$ where (R, S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$
\frac{dt}{\xi} = \frac{dy}{\eta} = dS \tag{1}
$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial t} + \eta \frac{\partial}{\partial y}\right) S(t, y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the

canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$R = t$$

S is found from

$$\begin{aligned} S &= \int \frac{1}{\eta} dy \\ &= \int \frac{1}{e^{-\frac{2t}{3}}} dy \end{aligned}$$

Which results in

$$S = e^{\frac{2t}{3}} y$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_t + \omega(t, y) S_y}{R_t + \omega(t, y) R_y} \tag{2}$$

Where in the above R_t, R_y, S_t, S_y are all partial derivatives and $\omega(t, y)$ is the right hand side of the original ode given by

$$\omega(t, y) = -\frac{2y}{3} - \frac{t}{2} + 1$$

Evaluating all the partial derivatives gives

$$\begin{aligned} R_t &= 1 \\ R_y &= 0 \\ S_t &= \frac{2 e^{\frac{2t}{3}} y}{3} \\ S_y &= e^{\frac{2t}{3}} \end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = \left(-\frac{t}{2} + 1\right) e^{\frac{2t}{3}} \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for t, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = \left(-\frac{R}{2} + 1\right) e^{\frac{2R}{3}}$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordinates R, S. Integrating the above gives

$$S(R) = -\frac{3(2R-7)\,\mathrm{e}^{\frac{2R}{3}}}{8} + c_1 \tag{4}$$

To complete the solution, we just need to transform (4) back to t, y coordinates. This results in

$$\mathrm{e}^{\frac{2t}{3}} y = -\frac{3(2t-7)\,\mathrm{e}^{\frac{2t}{3}}}{8} + c_1$$

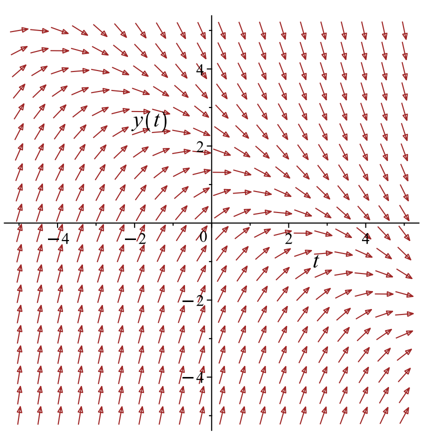
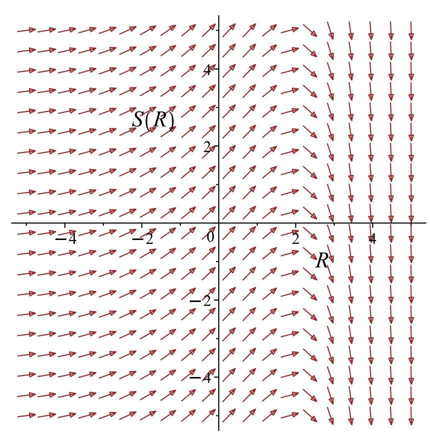
Which simplifies to

$$\mathrm{e}^{\frac{2t}{3}} y = -\frac{3(2t-7)\,\mathrm{e}^{\frac{2t}{3}}}{8} + c_1$$

Which gives

$$y = -\frac{\left(6\,\mathrm{e}^{\frac{2t}{3}} t - 21\,\mathrm{e}^{\frac{2t}{3}} - 8c_1\right) \mathrm{e}^{-\frac{2t}{3}}}{8}$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in t, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dt} = -\frac{2y}{3} - \frac{t}{2} + 1$ 	$R = t$ $S = \mathrm{e}^{\frac{2t}{3}} y$	$\frac{dS}{dR} = \left(-\frac{R}{2} + 1\right) \mathrm{e}^{\frac{2R}{3}}$ 

Summary
The solution(s) found are the following

$$y = -\frac{\left(6\,\mathrm{e}^{\frac{2t}{3}} t - 21\,\mathrm{e}^{\frac{2t}{3}} - 8c_1\right)\mathrm{e}^{-\frac{2t}{3}}}{8} \tag{1}$$

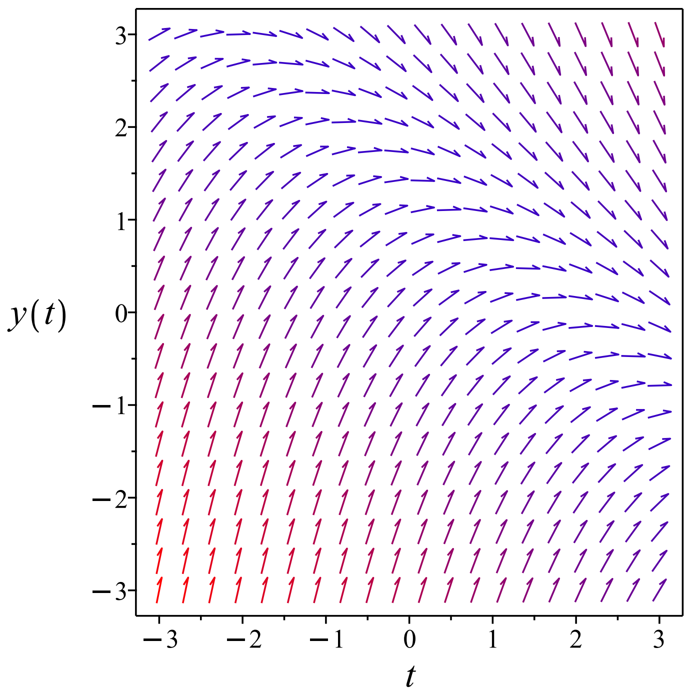


Figure 66: Slope field plot

Verification of solutions

$$y = -\frac{\left(6\,\mathrm{e}^{\frac{2t}{3}} t - 21\,\mathrm{e}^{\frac{2t}{3}} - 8c_1\right)\mathrm{e}^{-\frac{2t}{3}}}{8}$$

Verified OK.

1.28.3 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x,y) + N(x,y)\,\frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x,y) = c$ where c is constant, that satisfies the

ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y) = 0$$

Hence

$$\frac{\partial\phi}{\partial x} + \frac{\partial\phi}{\partial y}\frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$\begin{aligned}\frac{\partial\phi}{\partial x} &= M\\ \frac{\partial\phi}{\partial y} &= N\end{aligned}$$

But since $\frac{\partial^2\phi}{\partial x\partial y} = \frac{\partial^2\phi}{\partial y\partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x,y)$ but at least we know now that we can do that since the condition $\frac{\partial^2\phi}{\partial x\partial y} = \frac{\partial^2\phi}{\partial y\partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(t,y)\,\mathrm{d}t + N(t,y)\,\mathrm{d}y = 0 \tag{1A}$$

Therefore

$$\mathrm{d}y = \left(-\frac{2y}{3} - \frac{t}{2} + 1\right)\mathrm{d}t$$

$$\left(\frac{2y}{3} + \frac{t}{2} - 1\right)\mathrm{d}t + \mathrm{d}y = 0 \tag{2A}$$

Comparing (1A) and (2A) shows that

$$\begin{aligned}M(t,y) &= \frac{2y}{3} + \frac{t}{2} - 1\\ N(t,y) &= 1\end{aligned}$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial t}$$

Using result found above gives

$$\begin{aligned}\frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}\left(\frac{2y}{3}+\frac{t}{2}-1\right)\\ &= \frac{2}{3}\end{aligned}$$

And

$$\begin{aligned}\frac{\partial N}{\partial t} &= \frac{\partial}{\partial t}(1)\\ &= 0\end{aligned}$$

Since $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial t}$, then the ODE is not exact. Since the ODE is not exact, we will try to find an integrating factor to make it exact. Let

$$\begin{aligned}A &= \frac{1}{N}\left(\frac{\partial M}{\partial y}-\frac{\partial N}{\partial t}\right)\\ &= 1\left(\left(\frac{2}{3}\right)-(0)\right)\\ &= \frac{2}{3}\end{aligned}$$

Since A does not depend on y, then it can be used to find an integrating factor. The integrating factor μ is

$$\begin{aligned}\mu &= e^{\int A\,\mathrm{d}t}\\ &= e^{\int \frac{2}{3}\,\mathrm{d}t}\end{aligned}$$

The result of integrating gives

$$\begin{aligned}\mu &= e^{\frac{2t}{3}}\\ &= \mathrm{e}^{\frac{2t}{3}}\end{aligned}$$

M and N are multiplied by this integrating factor, giving new M and new N which are called $\overline{M}$ and $\overline{N}$ for now so not to confuse them with the original M and N.

$$\begin{aligned}\overline{M} &= \mu M\\ &= \mathrm{e}^{\frac{2t}{3}}\left(\frac{2y}{3}+\frac{t}{2}-1\right)\\ &= \frac{(4y+3t-6)\,\mathrm{e}^{\frac{2t}{3}}}{6}\end{aligned}$$

And

$$
\begin{aligned}
\overline{N} &= \mu N \\
&= \mathrm{e}^{\frac{2t}{3}}(1) \\
&= \mathrm{e}^{\frac{2t}{3}}
\end{aligned}
$$

Now a modified ODE is ontained from the original ODE, which is exact and can be solved. The modified ODE is

$$
\begin{aligned}
\overline{M} + \overline{N}\frac{\mathrm{d}y}{\mathrm{d}t} &= 0 \\
\left(\frac{(4y + 3t - 6)\,\mathrm{e}^{\frac{2t}{3}}}{6}\right) + \left(\mathrm{e}^{\frac{2t}{3}}\right)\frac{\mathrm{d}y}{\mathrm{d}t} &= 0
\end{aligned}
$$

The following equations are now set up to solve for the function $\phi(t, y)$

$$
\frac{\partial \phi}{\partial t} = \overline{M} \tag{1}
$$

$$
\frac{\partial \phi}{\partial y} = \overline{N} \tag{2}
$$

Integrating (1) w.r.t. t gives

$$
\begin{aligned}
\int \frac{\partial \phi}{\partial t}\,\mathrm{d}t &= \int \overline{M}\,\mathrm{d}t \\
\int \frac{\partial \phi}{\partial t}\,\mathrm{d}t &= \int \frac{(4y + 3t - 6)\,\mathrm{e}^{\frac{2t}{3}}}{6}\,\mathrm{d}t
\end{aligned}
$$

$$
\phi = \frac{(6t + 8y - 21)\,\mathrm{e}^{\frac{2t}{3}}}{8} + f(y) \tag{3}
$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both t and y. Taking derivative of equation (3) w.r.t y gives

$$
\frac{\partial \phi}{\partial y} = \mathrm{e}^{\frac{2t}{3}} + f'(y) \tag{4}
$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = \mathrm{e}^{\frac{2t}{3}}$. Therefore equation (4) becomes

$$
\mathrm{e}^{\frac{2t}{3}} = \mathrm{e}^{\frac{2t}{3}} + f'(y) \tag{5}
$$

Solving equation (5) for $f'(y)$ gives

$$f'(y) = 0$$

Therefore

$$f(y) = c_1$$

Where c_1 is constant of integration. Substituting this result for $f(y)$ into equation (3) gives ϕ

$$\phi = \frac{(6t + 8y - 21)\,\mathrm{e}^{\frac{2t}{3}}}{8} + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = \frac{(6t + 8y - 21)\,\mathrm{e}^{\frac{2t}{3}}}{8}$$

The solution becomes

$$y = -\frac{\left(6\,\mathrm{e}^{\frac{2t}{3}} t - 21\,\mathrm{e}^{\frac{2t}{3}} - 8c_1\right)\mathrm{e}^{-\frac{2t}{3}}}{8}$$

Summary

The solution(s) found are the following

$$y = -\frac{\left(6\,\mathrm{e}^{\frac{2t}{3}} t - 21\,\mathrm{e}^{\frac{2t}{3}} - 8c_1\right)\mathrm{e}^{-\frac{2t}{3}}}{8} \tag{1}$$

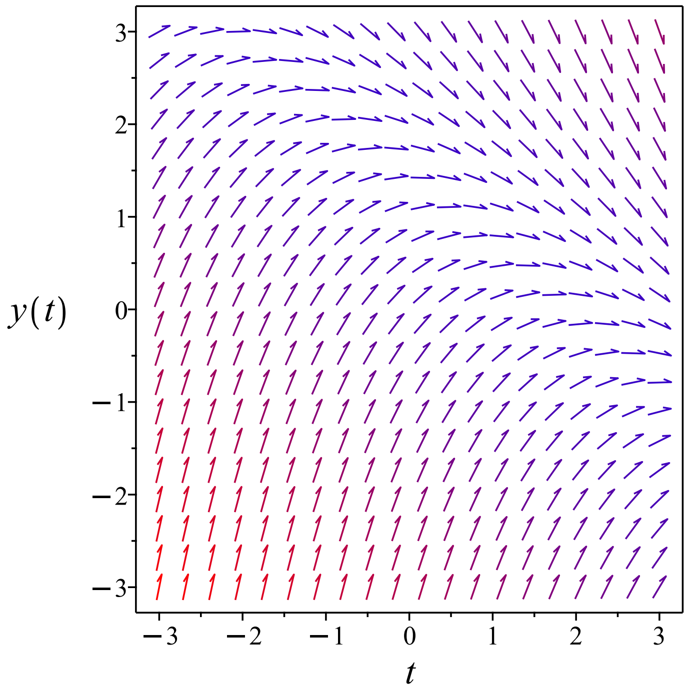


Figure 67: Slope field plot

Verification of solutions

$$y = -\frac{\left(6\,\mathrm{e}^{\frac{2t}{3}} t - 21\,\mathrm{e}^{\frac{2t}{3}} - 8c_1\right)\mathrm{e}^{-\frac{2t}{3}}}{8}$$

Verified OK.

1.28.4 Maple step by step solution

Let's solve

$\frac{2y}{3} + y' = -\frac{t}{2} + 1$

- Highest derivative means the order of the ODE is 1

  y'

- Isolate the derivative

  $y' = -\frac{2y}{3} - \frac{t}{2} + 1$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE

  $\frac{2y}{3} + y' = -\frac{t}{2} + 1$

- The ODE is linear; multiply by an integrating factor $\mu(t)$

$$\mu(t)\left(\frac{2y}{3}+y'\right)=\mu(t)\left(-\frac{t}{2}+1\right)$$

- Assume the lhs of the ODE is the total derivative $\frac{d}{dt}(\mu(t)\,y)$

$$\mu(t)\left(\frac{2y}{3}+y'\right)=\mu'(t)\,y+\mu(t)\,y'$$

- Isolate $\mu'(t)$

$$\mu'(t)=\frac{2\mu(t)}{3}$$

- Solve to find the integrating factor

$$\mu(t)=\mathrm{e}^{\frac{2t}{3}}$$

- Integrate both sides with respect to t

$$\int\left(\frac{d}{dt}(\mu(t)\,y)\right)dt=\int\mu(t)\left(-\frac{t}{2}+1\right)dt+c_1$$

- Evaluate the integral on the lhs

$$\mu(t)\,y=\int\mu(t)\left(-\frac{t}{2}+1\right)dt+c_1$$

- Solve for y

$$y=\frac{\int\mu(t)\left(-\frac{t}{2}+1\right)dt+c_1}{\mu(t)}$$

- Substitute $\mu(t)=\mathrm{e}^{\frac{2t}{3}}$

$$y=\frac{\int\left(-\frac{t}{2}+1\right)\mathrm{e}^{\frac{2t}{3}}dt+c_1}{\mathrm{e}^{\frac{2t}{3}}}$$

- Evaluate the integrals on the rhs

$$y=\frac{-\frac{3(2t-7)\,\mathrm{e}^{\frac{2t}{3}}}{8}+c_1}{\mathrm{e}^{\frac{2t}{3}}}$$

- Simplify

$$y=-\frac{3t}{4}+\frac{21}{8}+c_1\mathrm{e}^{-\frac{2t}{3}}$$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
<- 1st order linear successful`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 15

```
dsolve(2/3*y(t)+diff(y(t),t) = 1-1/2*t,y(t), singsol=all)
```

$$y(t) = -\frac{3t}{4} + \frac{21}{8} + \mathrm{e}^{-\frac{2t}{3}} c_1$$

✓ Solution by Mathematica

Time used: 0.061 (sec). Leaf size: 24

```
DSolve[2/3*y[t]+y'[t] == 1-1/2*t,y[t],t,IncludeSingularSolutions -> True]
```

$$y(t) \to -\frac{3t}{4} + c_1 e^{-2t/3} + \frac{21}{8}$$

1.29 problem 29

Internal problem ID [476]
Internal file name [`OUTPUT/476_Sunday_June_05_2022_01_42_05_AM_51193873/index.tex`]

Book: Elementary differential equations and boundary value problems, 10th ed., Boyce and DiPrima
Section: Section 2.1. Page 40
Problem number: 29.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"linear", "exactWithIntegrationFactor", "first_order_ode_lie_symmetry_lookup"**

Maple gives the following as the ode type

```
[[_linear, `class A`]]
```

$$\boxed{\frac{y}{4} + y' = 3 + 2\cos(2t)}$$

With initial conditions

$$[y(0) = 0]$$

1.29.1 Existence and uniqueness analysis

This is a linear ODE. In canonical form it is written as

$$y' + p(t)y = q(t)$$

Where here

$$p(t) = \frac{1}{4}$$
$$q(t) = 3 + 2\cos(2t)$$

Hence the ode is

$$\frac{y}{4}+y' = 3+2\cos(2t)$$

The domain of $p(t)=\frac{1}{4}$ is

$$\{-\infty < t < \infty\}$$

And the point $t_0=0$ is inside this domain. The domain of $q(t)=3+2\cos(2t)$ is

$$\{-\infty < t < \infty\}$$

And the point $t_0=0$ is also inside this domain. Hence solution exists and is unique.

1.29.2 Solving as linear ode

Entering Linear first order ODE solver. The integrating factor μ is

$$\begin{aligned}\mu &= \mathrm{e}^{\int \frac{1}{4}dt}\\ &= \mathrm{e}^{\frac{t}{4}}\end{aligned}$$

The ode becomes

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}(\mu y) &= (\mu)(3+2\cos(2t))\\ \frac{\mathrm{d}}{\mathrm{d}t}\left(\mathrm{e}^{\frac{t}{4}}y\right) &= \left(\mathrm{e}^{\frac{t}{4}}\right)(3+2\cos(2t))\\ \mathrm{d}\left(\mathrm{e}^{\frac{t}{4}}y\right) &= \left((3+2\cos(2t))\,\mathrm{e}^{\frac{t}{4}}\right)\,\mathrm{d}t\end{aligned}$$

Integrating gives

$$\begin{aligned}\mathrm{e}^{\frac{t}{4}}y &= \int (3+2\cos(2t))\,\mathrm{e}^{\frac{t}{4}}\,\mathrm{d}t\\ \mathrm{e}^{\frac{t}{4}}y &= 12\,\mathrm{e}^{\frac{t}{4}}+\frac{8\cos(2t)\,\mathrm{e}^{\frac{t}{4}}}{65}+\frac{64\sin(2t)\,\mathrm{e}^{\frac{t}{4}}}{65}+c_1\end{aligned}$$

Dividing both sides by the integrating factor $\mu=\mathrm{e}^{\frac{t}{4}}$ results in

$$y = \mathrm{e}^{-\frac{t}{4}}\left(12\,\mathrm{e}^{\frac{t}{4}}+\frac{8\cos(2t)\,\mathrm{e}^{\frac{t}{4}}}{65}+\frac{64\sin(2t)\,\mathrm{e}^{\frac{t}{4}}}{65}\right)+c_1\mathrm{e}^{-\frac{t}{4}}$$

which simplifies to

$$y = \frac{64\sin(2t)}{65} + \frac{8\cos(2t)}{65} + 12 + c_1 e^{-\frac{t}{4}}$$

Initial conditions are used to solve for c_1. Substituting $t = 0$ and $y = 0$ in the above solution gives an equation to solve for the constant of integration.

$$0 = \frac{788}{65} + c_1$$

$$c_1 = -\frac{788}{65}$$

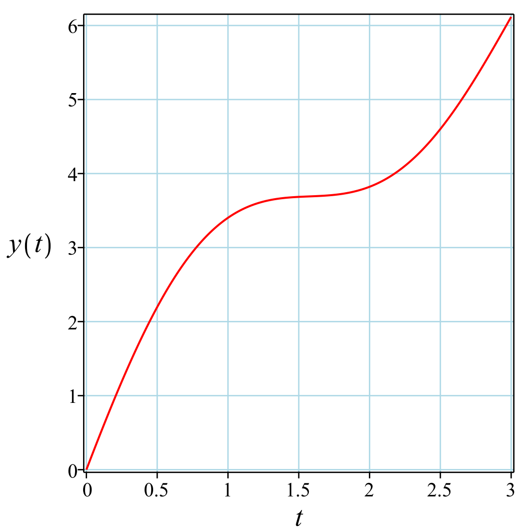
Substituting c_1 found above in the general solution gives

$$y = \frac{64\sin(2t)}{65} + \frac{8\cos(2t)}{65} + 12 - \frac{788\,e^{-\frac{t}{4}}}{65}$$

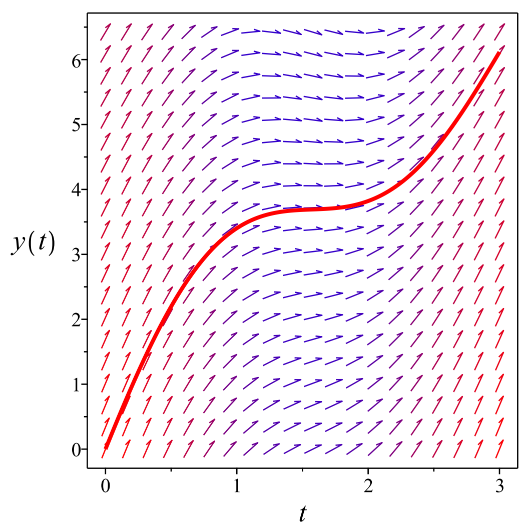
Summary

The solution(s) found are the following

$$y = \frac{64\sin(2t)}{65} + \frac{8\cos(2t)}{65} + 12 - \frac{788\,e^{-\frac{t}{4}}}{65} \tag{1}$$



(a) Solution plot



(b) Slope field plot

Verification of solutions

$$y = \frac{64\sin(2t)}{65} + \frac{8\cos(2t)}{65} + 12 - \frac{788\,e^{-\frac{t}{4}}}{65}$$

Verified OK.

1.29.3 Solving as first order ode lie symmetry lookup ode

Writing the ode as

$$
\begin{aligned}
y' &= -\frac{y}{4} + 3 + 2\cos(2t) \\
y' &= \omega(t, y)
\end{aligned}
$$

The condition of Lie symmetry is the linearized PDE given by

$$
\eta_t + \omega(\eta_y - \xi_t) - \omega^2 \xi_y - \omega_t \xi - \omega_y \eta = 0 \tag{A}
$$

The type of this ode is known. It is of type `linear`. Therefore we do not need to solve the PDE (A), and can just use the lookup table shown below to find ξ, η

Table 85: Lie symmetry infinitesimal lookup table for known first order ODE's

ODE class	Form	ξ	η
linear ode	$y' = f(x)y(x) + g(x)$	0	$e^{\int f dx}$
separable ode	$y' = f(x)\, g(y)$	$\frac{1}{f}$	0
quadrature ode	$y' = f(x)$	0	1
quadrature ode	$y' = g(y)$	1	0
homogeneous ODEs of Class A	$y' = f\left(\frac{y}{x}\right)$	x	y
homogeneous ODEs of Class C	$y' = (a + bx + cy)^{\frac{n}{m}}$	1	$-\frac{b}{c}$
homogeneous class D	$y' = \frac{y}{x} + g(x)\, F\left(\frac{y}{x}\right)$	x^2	xy
First order special form ID 1	$y' = g(x)\, e^{h(x)+by} + f(x)$	$\frac{e^{-\int bf(x)dx - h(x)}}{g(x)}$	$\frac{f(x)e^{-\int bf(x)dx - h(x)}}{g(x)}$
polynomial type ode	$y' = \frac{a_1x+b_1y+c_1}{a_2x+b_2y+c_2}$	$\frac{a_1b_2x - a_2b_1x - b_1c_2 + b_2c_1}{a_1b_2 - a_2b_1}$	$\frac{a_1b_2y - a_2b_1y - a_1c_2 - a_2c_1}{a_1b_2 - a_2b_1}$
Bernoulli ode	$y' = f(x)\, y + g(x)\, y^n$	0	$e^{-\int (n-1)f(x)dx} y^n$
Reduced Riccati	$y' = f_1(x)\, y + f_2(x)\, y^2$	0	$e^{-\int f_1 dx}$

The above table shows that

$$
\begin{aligned}
\xi(t,y) &= 0 \\
\eta(t,y) &= \mathrm{e}^{-\frac{t}{4}}
\end{aligned} \tag{A1}
$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(t,y) \to (R,S)$ where (R,S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$
\frac{dt}{\xi} = \frac{dy}{\eta} = dS \tag{1}
$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial t} + \eta \frac{\partial}{\partial y}\right) S(t,y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$
R = t
$$

S is found from

$$
\begin{aligned}
S &= \int \frac{1}{\eta} dy \\
&= \int \frac{1}{\mathrm{e}^{-\frac{t}{4}}} dy
\end{aligned}
$$

Which results in

$$
S = \mathrm{e}^{\frac{t}{4}} y
$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$
\frac{dS}{dR} = \frac{S_t + \omega(t,y) S_y}{R_t + \omega(t,y) R_y} \tag{2}
$$

Where in the above R_t, R_y, S_t, S_y are all partial derivatives and $\omega(t,y)$ is the right hand side of the original ode given by

$$
\omega(t,y) = -\frac{y}{4} + 3 + 2\cos(2t)
$$

Evaluating all the partial derivatives gives

$$
\begin{aligned}
R_t &= 1 \\
R_y &= 0 \\
S_t &= \frac{\mathrm{e}^{\frac{t}{4}} y}{4} \\
S_y &= \mathrm{e}^{\frac{t}{4}}
\end{aligned}
$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = (3 + 2\cos(2t))\,\mathrm{e}^{\frac{t}{4}} \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for t, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = (3 + 2\cos(2R))\,\mathrm{e}^{\frac{R}{4}}$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$S(R) = 12\,\mathrm{e}^{\frac{R}{4}} + c_1 + \frac{8\,\mathrm{e}^{\frac{R}{4}}(\cos(2R) + 8\sin(2R))}{65} \tag{4}$$

To complete the solution, we just need to transform (4) back to t, y coordinates. This results in

$$\mathrm{e}^{\frac{t}{4}} y = 12\,\mathrm{e}^{\frac{t}{4}} + c_1 + \frac{8\,\mathrm{e}^{\frac{t}{4}}(\cos(2t) + 8\sin(2t))}{65}$$

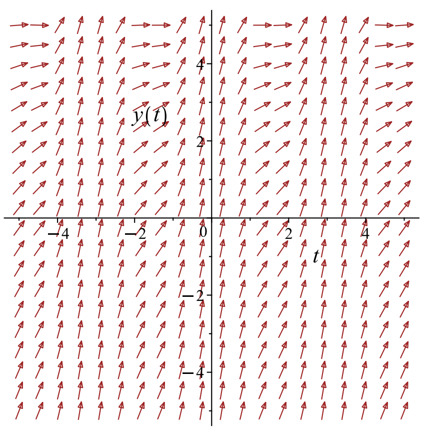
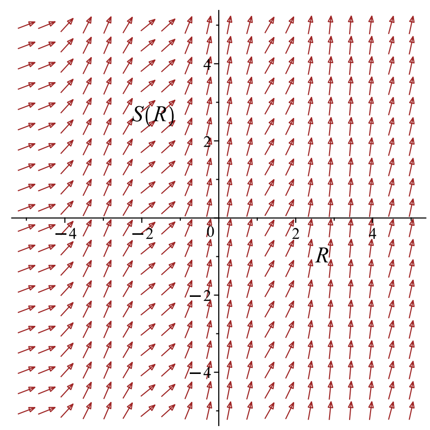
Which simplifies to

$$\frac{(65y - 8\cos(2t) - 64\sin(2t) - 780)\,\mathrm{e}^{\frac{t}{4}}}{65} - c_1 = 0$$

Which gives

$$y = \frac{\mathrm{e}^{-\frac{t}{4}}\left(8\cos(2t)\,\mathrm{e}^{\frac{t}{4}} + 64\sin(2t)\,\mathrm{e}^{\frac{t}{4}} + 780\,\mathrm{e}^{\frac{t}{4}} + 65c_1\right)}{65}$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in t, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dt} = -\frac{y}{4} + 3 + 2\cos(2t)$ 	$R = t$ $S = \mathrm{e}^{\frac{t}{4}} y$	$\frac{dS}{dR} = (3 + 2\cos(2R))\,\mathrm{e}^{\frac{R}{4}}$ 

Initial conditions are used to solve for c_1. Substituting $t = 0$ and $y = 0$ in the above solution gives an equation to solve for the constant of integration.

$$0 = \frac{788}{65} + c_1$$

$$c_1 = -\frac{788}{65}$$

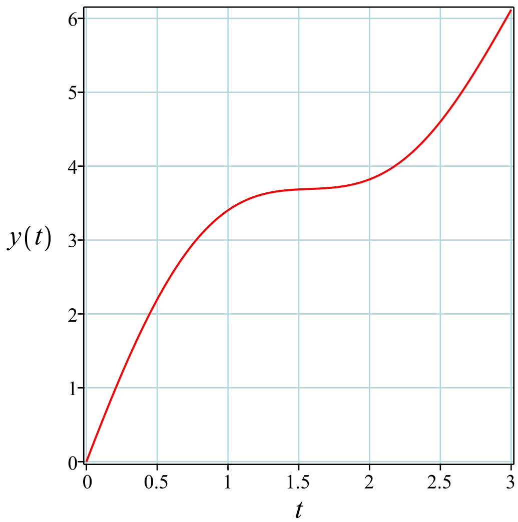
Substituting c_1 found above in the general solution gives

$$y = \frac{64\sin(2t)}{65} + \frac{8\cos(2t)}{65} + 12 - \frac{788\,\mathrm{e}^{-\frac{t}{4}}}{65}$$

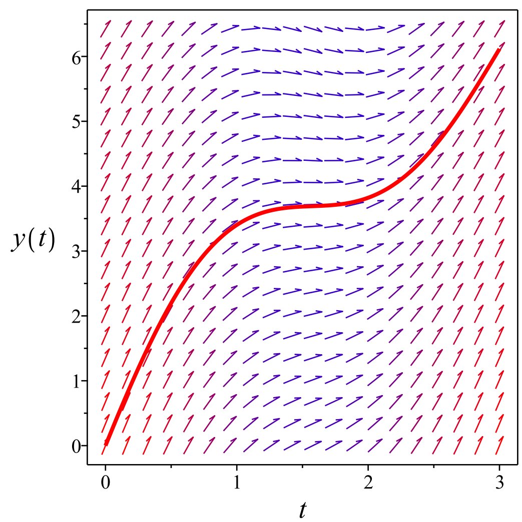
Summary

The solution(s) found are the following

$$y = \frac{64\sin(2t)}{65} + \frac{8\cos(2t)}{65} + 12 - \frac{788\,\mathrm{e}^{-\frac{t}{4}}}{65} \tag{1}$$



(a) Solution plot



(b) Slope field plot

<u>Verification of solutions</u>

$$y = \frac{64 \sin(2t)}{65} + \frac{8 \cos(2t)}{65} + 12 - \frac{788 \, e^{-\frac{t}{4}}}{65}$$

Verified OK.

1.29.4 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x,y) + N(x,y)\frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x,y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y) = 0$$

Hence

$$\frac{\partial \phi}{\partial x} + \frac{\partial \phi}{\partial y}\frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$\frac{\partial \phi}{\partial x} = M$$
$$\frac{\partial \phi}{\partial y} = N$$

But since $\frac{\partial^2\phi}{\partial x\partial y} = \frac{\partial^2\phi}{\partial y\partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x, y)$ but at least we know now that we can do that since the condition $\frac{\partial^2\phi}{\partial x\partial y} = \frac{\partial^2\phi}{\partial y\partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(t, y)\,\mathrm{d}t + N(t, y)\,\mathrm{d}y = 0 \tag{1A}$$

Therefore

$$\begin{aligned}
\mathrm{d}y &= \left(-\frac{y}{4} + 3 + 2\cos(2t)\right)\mathrm{d}t \\
\left(\frac{y}{4} - 3 - 2\cos(2t)\right)\mathrm{d}t + \mathrm{d}y &= 0 \qquad \text{(2A)}
\end{aligned}$$

Comparing (1A) and (2A) shows that

$$\begin{aligned}
M(t, y) &= \frac{y}{4} - 3 - 2\cos(2t) \\
N(t, y) &= 1
\end{aligned}$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial t}$$

Using result found above gives

$$\begin{aligned}
\frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}\left(\frac{y}{4} - 3 - 2\cos(2t)\right) \\
&= \frac{1}{4}
\end{aligned}$$

And

$$\begin{aligned}
\frac{\partial N}{\partial t} &= \frac{\partial}{\partial t}(1) \\
&= 0
\end{aligned}$$

Since $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial t}$, then the ODE is not exact. Since the ODE is not exact, we will try to find an integrating factor to make it exact. Let

$$\begin{aligned} A &= \frac{1}{N}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial t}\right) \\ &= 1\left(\left(\frac{1}{4}\right) - (0)\right) \\ &= \frac{1}{4} \end{aligned}$$

Since A does not depend on y, then it can be used to find an integrating factor. The integrating factor μ is

$$\begin{aligned} \mu &= e^{\int A \, \mathrm{d}t} \\ &= e^{\int \frac{1}{4} \, \mathrm{d}t} \end{aligned}$$

The result of integrating gives

$$\begin{aligned} \mu &= e^{\frac{t}{4}} \\ &= \mathrm{e}^{\frac{t}{4}} \end{aligned}$$

M and N are multiplied by this integrating factor, giving new M and new N which are called $\overline{M}$ and $\overline{N}$ for now so not to confuse them with the original M and N.

$$\begin{aligned} \overline{M} &= \mu M \\ &= \mathrm{e}^{\frac{t}{4}}\left(\frac{y}{4} - 3 - 2\cos(2t)\right) \\ &= \frac{(y - 12 - 8\cos(2t))\,\mathrm{e}^{\frac{t}{4}}}{4} \end{aligned}$$

And

$$\begin{aligned} \overline{N} &= \mu N \\ &= \mathrm{e}^{\frac{t}{4}}(1) \\ &= \mathrm{e}^{\frac{t}{4}} \end{aligned}$$

Now a modified ODE is ontained from the original ODE, which is exact and can be solved. The modified ODE is

$$\begin{aligned} \overline{M} + \overline{N}\frac{\mathrm{d}y}{\mathrm{d}t} &= 0 \\ \left(\frac{(y - 12 - 8\cos(2t))\,\mathrm{e}^{\frac{t}{4}}}{4}\right) + \left(\mathrm{e}^{\frac{t}{4}}\right)\frac{\mathrm{d}y}{\mathrm{d}t} &= 0 \end{aligned}$$

The following equations are now set up to solve for the function $\phi(t, y)$

$$\frac{\partial \phi}{\partial t} = \overline{M} \tag{1}$$

$$\frac{\partial \phi}{\partial y} = \overline{N} \tag{2}$$

Integrating (1) w.r.t. t gives

$$\int \frac{\partial \phi}{\partial t} \, \mathrm{d}t = \int \overline{M} \, \mathrm{d}t$$

$$\int \frac{\partial \phi}{\partial t} \, \mathrm{d}t = \int \frac{(y - 12 - 8\cos(2t))\, \mathrm{e}^{\frac{t}{4}}}{4} \, \mathrm{d}t$$

$$\phi = -\frac{\mathrm{e}^{\frac{t}{4}} (780 - 65y + 8\cos(2t) + 64\sin(2t))}{65} + f(y) \tag{3}$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both t and y. Taking derivative of equation (3) w.r.t y gives

$$\frac{\partial \phi}{\partial y} = \mathrm{e}^{\frac{t}{4}} + f'(y) \tag{4}$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = \mathrm{e}^{\frac{t}{4}}$. Therefore equation (4) becomes

$$\mathrm{e}^{\frac{t}{4}} = \mathrm{e}^{\frac{t}{4}} + f'(y) \tag{5}$$

Solving equation (5) for $f'(y)$ gives

$$f'(y) = 0$$

Therefore

$$f(y) = c_1$$

Where c_1 is constant of integration. Substituting this result for $f(y)$ into equation (3) gives ϕ

$$\phi = -\frac{\mathrm{e}^{\frac{t}{4}} (780 - 65y + 8\cos(2t) + 64\sin(2t))}{65} + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = -\frac{\mathrm{e}^{\frac{t}{4}} (780 - 65y + 8\cos(2t) + 64\sin(2t))}{65}$$

The solution becomes

$$y = \frac{e^{-\frac{t}{4}} \left(8 \cos \left(2t\right) e^{\frac{t}{4}} + 64 \sin \left(2t\right) e^{\frac{t}{4}} + 780 e^{\frac{t}{4}} + 65 c_1\right)}{65}$$

Initial conditions are used to solve for c_1. Substituting $t = 0$ and $y = 0$ in the above solution gives an equation to solve for the constant of integration.

$$0 = \frac{788}{65} + c_1$$

$$c_1 = -\frac{788}{65}$$

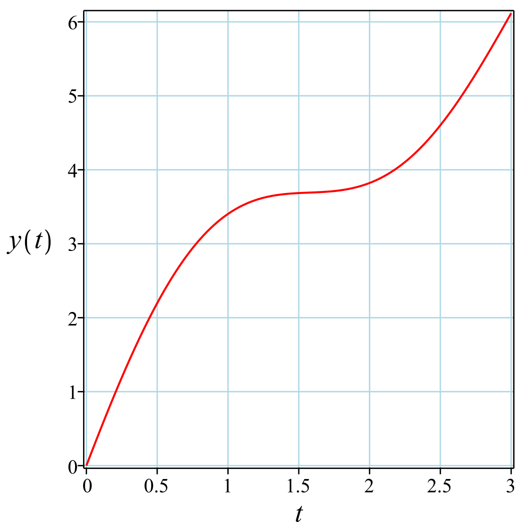
Substituting c_1 found above in the general solution gives

$$y = \frac{64 \sin \left(2t\right)}{65} + \frac{8 \cos \left(2t\right)}{65} + 12 - \frac{788 e^{-\frac{t}{4}}}{65}$$

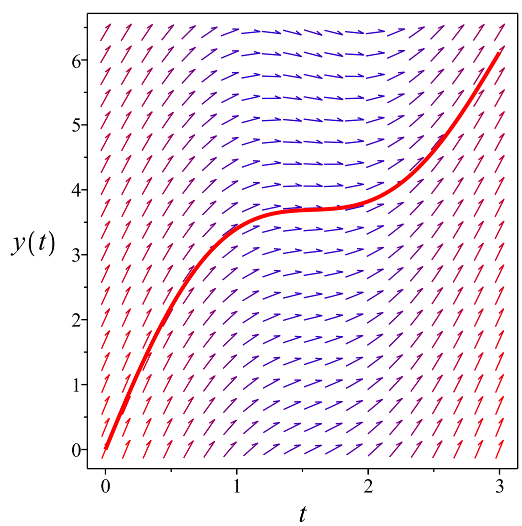
Summary

The solution(s) found are the following

$$y = \frac{64 \sin \left(2t\right)}{65} + \frac{8 \cos \left(2t\right)}{65} + 12 - \frac{788 e^{-\frac{t}{4}}}{65} \tag{1}$$



(a) Solution plot



(b) Slope field plot

Verification of solutions

$$y = \frac{64 \sin \left(2t\right)}{65} + \frac{8 \cos \left(2t\right)}{65} + 12 - \frac{788 e^{-\frac{t}{4}}}{65}$$

Verified OK.

1.29.5 Maple step by step solution

Let's solve

$\left[\frac{y}{4}+y'=3+2\cos(2t), y(0)=0\right]$

- Highest derivative means the order of the ODE is 1

  y'

- Isolate the derivative

  $y'=-\frac{y}{4}+3+2\cos(2t)$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE

  $\frac{y}{4}+y'=3+2\cos(2t)$

- The ODE is linear; multiply by an integrating factor $\mu(t)$

  $\mu(t)\left(\frac{y}{4}+y'\right)=\mu(t)\left(3+2\cos(2t)\right)$

- Assume the lhs of the ODE is the total derivative $\frac{d}{dt}(\mu(t)\, y)$

  $\mu(t)\left(\frac{y}{4}+y'\right)=\mu'(t)\, y+\mu(t)\, y'$

- Isolate $\mu'(t)$

  $\mu'(t)=\frac{\mu(t)}{4}$

- Solve to find the integrating factor

  $\mu(t)=\mathrm{e}^{\frac{t}{4}}$

- Integrate both sides with respect to t

  $\int\left(\frac{d}{dt}(\mu(t)\, y)\right)dt=\int\mu(t)\left(3+2\cos(2t)\right)dt+c_1$

- Evaluate the integral on the lhs

  $\mu(t)\, y=\int\mu(t)\left(3+2\cos(2t)\right)dt+c_1$

- Solve for y

  $y=\frac{\int\mu(t)(3+2\cos(2t))dt+c_1}{\mu(t)}$

- Substitute $\mu(t)=\mathrm{e}^{\frac{t}{4}}$

  $y=\frac{\int(3+2\cos(2t))\mathrm{e}^{\frac{t}{4}}dt+c_1}{\mathrm{e}^{\frac{t}{4}}}$

- Evaluate the integrals on the rhs

  $y=\frac{\frac{8\cos(2t)\mathrm{e}^{\frac{t}{4}}}{65}+\frac{64\sin(2t)\mathrm{e}^{\frac{t}{4}}}{65}+12\,\mathrm{e}^{\frac{t}{4}}+c_1}{\mathrm{e}^{\frac{t}{4}}}$

- Simplify

$$y = \frac{64\sin(2t)}{65} + \frac{8\cos(2t)}{65} + 12 + c_1 e^{-\frac{t}{4}}$$

- Use initial condition $y(0) = 0$

$$0 = \frac{788}{65} + c_1$$

- Solve for c_1

$$c_1 = -\frac{788}{65}$$

- Substitute $c_1 = -\frac{788}{65}$ into general solution and simplify

$$y = \frac{64\sin(2t)}{65} + \frac{8\cos(2t)}{65} + 12 - \frac{788\, e^{-\frac{t}{4}}}{65}$$

- Solution to the IVP

$$y = \frac{64\sin(2t)}{65} + \frac{8\cos(2t)}{65} + 12 - \frac{788\, e^{-\frac{t}{4}}}{65}$$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
<- 1st order linear successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 24

```
dsolve([1/4*y(t)+diff(y(t),t) = 3+2*cos(2*t),y(0) = 0],y(t), singsol=all)
```

$$y(t) = 12 + \frac{8\cos(2t)}{65} + \frac{64\sin(2t)}{65} - \frac{788\, e^{-\frac{t}{4}}}{65}$$

✓ Solution by Mathematica

Time used: 0.158 (sec). Leaf size: 32

```
DSolve[{1/4*y[t]+y'[t] == 3+2*Cos[2*t],y[0]==0},y[t],t,IncludeSingularSolutions -> True]
```

$$y(t) \to \frac{4}{65}\left(-197e^{-t/4} + 16\sin(2t) + 2\cos(2t) + 195\right)$$

1.30 problem 30

Internal problem ID [477]
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Book: Elementary differential equations and boundary value problems, 10th ed., Boyce and DiPrima
Section: Section 2.1. Page 40
Problem number: 30.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"linear", "exactWithIntegrationFactor", "first_order_ode_lie_symmetry_lookup"**

Maple gives the following as the ode type

```
[[_linear, `class A`]]
```

$$\boxed{-y + y' = 1 + 3\sin(t)}$$

1.30.1 Solving as linear ode

Entering Linear first order ODE solver. In canonical form a linear first order is

$$y' + p(t)y = q(t)$$

Where here

$$p(t) = -1$$
$$q(t) = 1 + 3\sin(t)$$

Hence the ode is

$$-y + y' = 1 + 3\sin(t)$$

The integrating factor μ is

$$\mu = e^{\int (-1)dt}$$
$$= e^{-t}$$

The ode becomes

$$\frac{d}{dt}(\mu y) = (\mu)(1 + 3\sin(t))$$
$$\frac{d}{dt}(e^{-t}y) = (e^{-t})(1 + 3\sin(t))$$
$$d(e^{-t}y) = ((1 + 3\sin(t))e^{-t})\ dt$$

Integrating gives

$$e^{-t}y = \int (1 + 3\sin(t))e^{-t}\,dt$$
$$e^{-t}y = -e^{-t} - \frac{3e^{-t}\cos(t)}{2} - \frac{3e^{-t}\sin(t)}{2} + c_1$$

Dividing both sides by the integrating factor $\mu = e^{-t}$ results in

$$y = e^{t}\left(-e^{-t} - \frac{3e^{-t}\cos(t)}{2} - \frac{3e^{-t}\sin(t)}{2}\right) + c_1 e^{t}$$

which simplifies to

$$y = -1 + c_1 e^{t} - \frac{3\cos(t)}{2} - \frac{3\sin(t)}{2}$$

Summary

The solution(s) found are the following

$$y = -1 + c_1 e^{t} - \frac{3\cos(t)}{2} - \frac{3\sin(t)}{2} \tag{1}$$

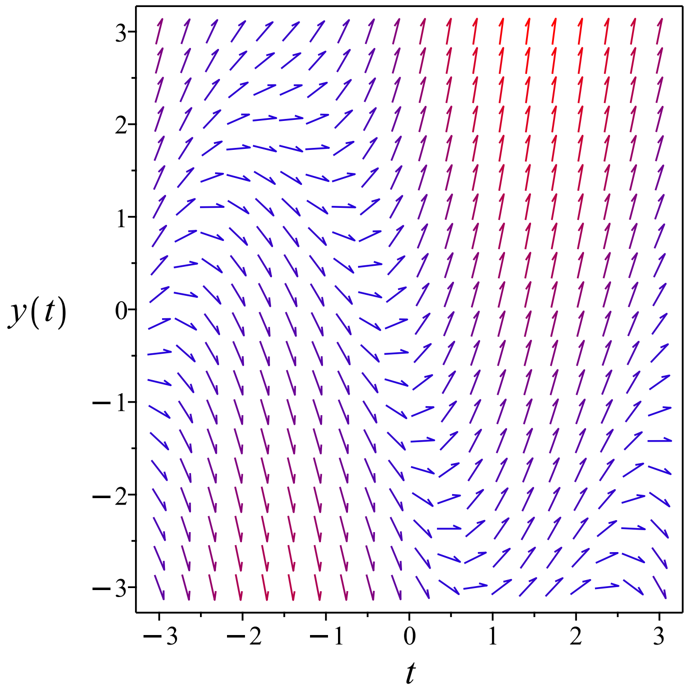


Figure 71: Slope field plot

Verification of solutions

$$y = -1 + c_1 e^t - \frac{3\cos(t)}{2} - \frac{3\sin(t)}{2}$$

Verified OK.

1.30.2 Solving as first order ode lie symmetry lookup ode

Writing the ode as

$$y' = y + 1 + 3\sin(t)$$
$$y' = \omega(t, y)$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_t + \omega(\eta_y - \xi_t) - \omega^2\xi_y - \omega_t\xi - \omega_y\eta = 0 \tag{A}$$

The type of this ode is known. It is of type `linear`. Therefore we do not need to solve the PDE (A), and can just use the lookup table shown below to find ξ, η

Table 88: Lie symmetry infinitesimal lookup table for known first order ODE's

ODE class	Form	ξ	η
linear ode	$y' = f(x)y(x) + g(x)$	0	$e^{\int f dx}$
separable ode	$y' = f(x)\, g(y)$	$\frac{1}{f}$	0
quadrature ode	$y' = f(x)$	0	1
quadrature ode	$y' = g(y)$	1	0
homogeneous ODEs of Class A	$y' = f\left(\frac{y}{x}\right)$	x	y
homogeneous ODEs of Class C	$y' = (a + bx + cy)^{\frac{n}{m}}$	1	$-\frac{b}{c}$
homogeneous class D	$y' = \frac{y}{x} + g(x)\, F\left(\frac{y}{x}\right)$	x^2	xy
First order special form ID 1	$y' = g(x)\, e^{h(x)+by} + f(x)$	$\frac{e^{-\int bf(x)dx - h(x)}}{g(x)}$	$\frac{f(x)e^{-\int bf(x)dx - h(x)}}{g(x)}$
polynomial type ode	$y' = \frac{a_1x+b_1y+c_1}{a_2x+b_2y+c_2}$	$\frac{a_1b_2x - a_2b_1x - b_1c_2 + b_2c_1}{a_1b_2 - a_2b_1}$	$\frac{a_1b_2y - a_2b_1y - a_1c_2 - a_2c_1}{a_1b_2 - a_2b_1}$
Bernoulli ode	$y' = f(x)\, y + g(x)\, y^n$	0	$e^{-\int (n-1)f(x)dx} y^n$
Reduced Riccati	$y' = f_1(x)\, y + f_2(x)\, y^2$	0	$e^{-\int f_1 dx}$

The above table shows that

$$
\begin{aligned}
\xi(t, y) &= 0 \\
\eta(t, y) &= \mathrm{e}^t
\end{aligned} \tag{A1}
$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(t, y) \to (R, S)$ where (R, S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$
\frac{dt}{\xi} = \frac{dy}{\eta} = dS \tag{1}
$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial t} + \eta \frac{\partial}{\partial y}\right) S(t, y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the

canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$R = t$$

S is found from

$$\begin{aligned} S &= \int \frac{1}{\eta} dy \\ &= \int \frac{1}{e^t} dy \end{aligned}$$

Which results in

$$S = e^{-t} y$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_t + \omega(t,y) S_y}{R_t + \omega(t,y) R_y} \tag{2}$$

Where in the above R_t, R_y, S_t, S_y are all partial derivatives and $\omega(t,y)$ is the right hand side of the original ode given by

$$\omega(t,y) = y + 1 + 3\sin(t)$$

Evaluating all the partial derivatives gives

$$\begin{aligned} R_t &= 1 \\ R_y &= 0 \\ S_t &= -e^{-t} y \\ S_y &= e^{-t} \end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = (1 + 3\sin(t)) e^{-t} \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for t, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = (1 + 3\sin(R)) e^{-R}$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by

integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$S(R) = -\mathrm{e}^{-R} + c_1 - \frac{3\,\mathrm{e}^{-R}(\cos(R) + \sin(R))}{2} \tag{4}$$

To complete the solution, we just need to transform (4) back to t, y coordinates. This results in

$$\mathrm{e}^{-t} y = -\mathrm{e}^{-t} + c_1 - \frac{3\,\mathrm{e}^{-t}(\cos(t) + \sin(t))}{2}$$

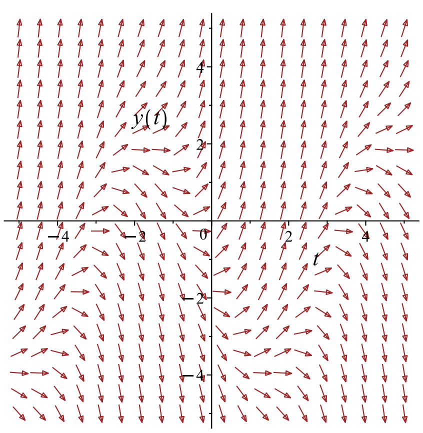
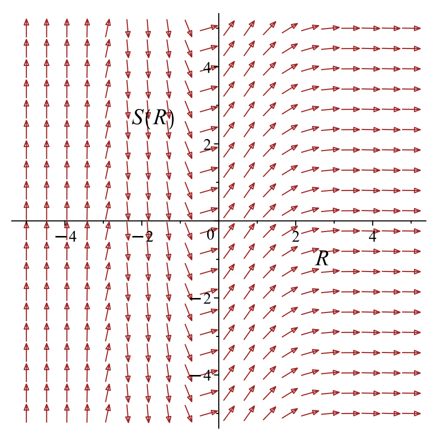
Which simplifies to

$$\frac{(2y + 3\cos(t) + 3\sin(t) + 2)\,\mathrm{e}^{-t}}{2} - c_1 = 0$$

Which gives

$$y = -\frac{(3\,\mathrm{e}^{-t}\sin(t) + 3\,\mathrm{e}^{-t}\cos(t) + 2\,\mathrm{e}^{-t} - 2c_1)\,\mathrm{e}^{t}}{2}$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in t, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dt} = y + 1 + 3\sin(t)$ 	$R = t$ $S = \mathrm{e}^{-t} y$	$\frac{dS}{dR} = (1 + 3\sin(R))\,\mathrm{e}^{-R}$ 

Summary

The solution(s) found are the following

$$y = -\frac{(3\,\mathrm{e}^{-t}\sin(t) + 3\,\mathrm{e}^{-t}\cos(t) + 2\,\mathrm{e}^{-t} - 2c_1)\,\mathrm{e}^{t}}{2} \tag{1}$$

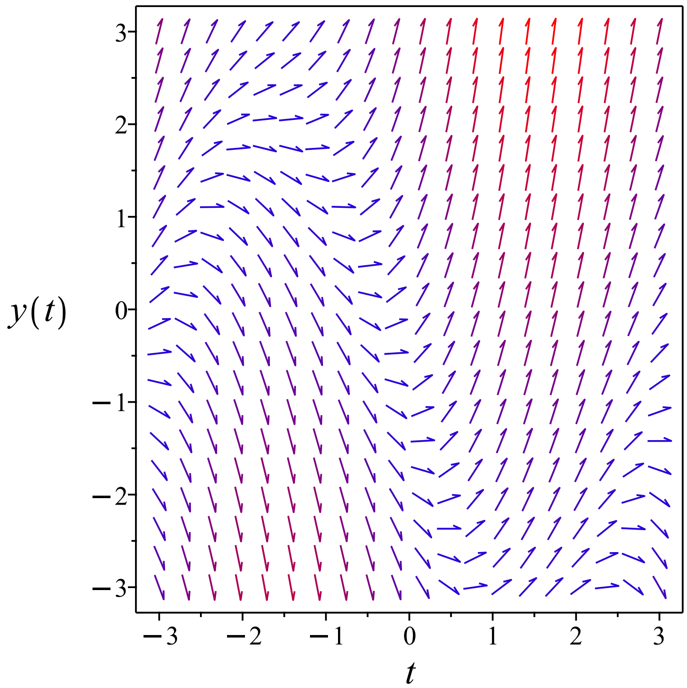


Figure 72: Slope field plot

Verification of solutions

$$y = -\frac{\left(3\,\mathrm{e}^{-t} \sin\left(t\right)+3\,\mathrm{e}^{-t} \cos\left(t\right)+2\,\mathrm{e}^{-t}-2 c_{1}\right) \mathrm{e}^{t}}{2}$$

Verified OK.

1.30.3 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x,y)+N(x,y)\frac{dy}{dx}=0 \tag{A}$$

We assume there exists a function $\phi(x,y)=c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y)=0$$

Hence

$$\frac{\partial\phi}{\partial x}+\frac{\partial\phi}{\partial y}\frac{dy}{dx}=0 \tag{B}$$

Comparing (A,B) shows that

$$\frac{\partial \phi}{\partial x} = M$$
$$\frac{\partial \phi}{\partial y} = N$$

But since $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x, y)$ but at least we know now that we can do that since the condition $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(t, y)\,\mathrm{d}t + N(t, y)\,\mathrm{d}y = 0 \tag{1A}$$

Therefore

$$\begin{aligned}\mathrm{d}y &= (y + 1 + 3\sin(t))\,\mathrm{d}t\\ (-y - 1 - 3\sin(t))\,\mathrm{d}t + \mathrm{d}y &= 0\end{aligned} \tag{2A}$$

Comparing (1A) and (2A) shows that

$$\begin{aligned}M(t, y) &= -y - 1 - 3\sin(t)\\ N(t, y) &= 1\end{aligned}$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial t}$$

Using result found above gives

$$\begin{aligned}\frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}(-y - 1 - 3\sin(t))\\ &= -1\end{aligned}$$

And

$$\begin{aligned}\frac{\partial N}{\partial t} &= \frac{\partial}{\partial t}(1)\\ &= 0\end{aligned}$$

Since $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial t}$, then the ODE is not exact. Since the ODE is not exact, we will try to find an integrating factor to make it exact. Let

$$
\begin{aligned}
A &= \frac{1}{N}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial t}\right) \\
&= 1((-1) - (0)) \\
&= -1
\end{aligned}
$$

Since A does not depend on y, then it can be used to find an integrating factor. The integrating factor μ is

$$
\begin{aligned}
\mu &= e^{\int A \, \mathrm{d}t} \\
&= e^{\int -1 \, \mathrm{d}t}
\end{aligned}
$$

The result of integrating gives

$$
\begin{aligned}
\mu &= e^{-t} \\
&= \mathrm{e}^{-t}
\end{aligned}
$$

M and N are multiplied by this integrating factor, giving new M and new N which are called $\overline{M}$ and $\overline{N}$ for now so not to confuse them with the original M and N.

$$
\begin{aligned}
\overline{M} &= \mu M \\
&= \mathrm{e}^{-t}(-y - 1 - 3\sin(t)) \\
&= -\mathrm{e}^{-t}(y + 1 + 3\sin(t))
\end{aligned}
$$

And

$$
\begin{aligned}
\overline{N} &= \mu N \\
&= \mathrm{e}^{-t}(1) \\
&= \mathrm{e}^{-t}
\end{aligned}
$$

Now a modified ODE is ontained from the original ODE, which is exact and can be solved. The modified ODE is

$$
\begin{aligned}
\overline{M} + \overline{N}\frac{\mathrm{d}y}{\mathrm{d}t} &= 0 \\
\left(-\mathrm{e}^{-t}(y + 1 + 3\sin(t))\right) + \left(\mathrm{e}^{-t}\right)\frac{\mathrm{d}y}{\mathrm{d}t} &= 0
\end{aligned}
$$

The following equations are now set up to solve for the function $\phi(t, y)$

$$
\frac{\partial \phi}{\partial t} = \overline{M} \tag{1}
$$

$$
\frac{\partial \phi}{\partial y} = \overline{N} \tag{2}
$$

Integrating (1) w.r.t. t gives

$$\int \frac{\partial \phi}{\partial t} \mathrm{d}t = \int \overline{M} \mathrm{d}t$$

$$\int \frac{\partial \phi}{\partial t} \mathrm{d}t = \int -\mathrm{e}^{-t}(y + 1 + 3\sin(t)) \mathrm{d}t$$

$$\phi = \frac{(2y + 3\cos(t) + 3\sin(t) + 2)\,\mathrm{e}^{-t}}{2} + f(y) \tag{3}$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both t and y. Taking derivative of equation (3) w.r.t y gives

$$\frac{\partial \phi}{\partial y} = \mathrm{e}^{-t} + f'(y) \tag{4}$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = \mathrm{e}^{-t}$. Therefore equation (4) becomes

$$\mathrm{e}^{-t} = \mathrm{e}^{-t} + f'(y) \tag{5}$$

Solving equation (5) for $f'(y)$ gives

$$f'(y) = 0$$

Therefore

$$f(y) = c_1$$

Where c_1 is constant of integration. Substituting this result for $f(y)$ into equation (3) gives ϕ

$$\phi = \frac{(2y + 3\cos(t) + 3\sin(t) + 2)\,\mathrm{e}^{-t}}{2} + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = \frac{(2y + 3\cos(t) + 3\sin(t) + 2)\,\mathrm{e}^{-t}}{2}$$

The solution becomes

$$y = -\frac{(3\,\mathrm{e}^{-t}\sin(t) + 3\,\mathrm{e}^{-t}\cos(t) + 2\,\mathrm{e}^{-t} - 2c_1)\,\mathrm{e}^{t}}{2}$$

Summary

The solution(s) found are the following

$$y = -\frac{\left(3\,\mathrm{e}^{-t}\sin\left(t\right)+3\,\mathrm{e}^{-t}\cos\left(t\right)+2\,\mathrm{e}^{-t}-2c_1\right)\mathrm{e}^{t}}{2} \tag{1}$$

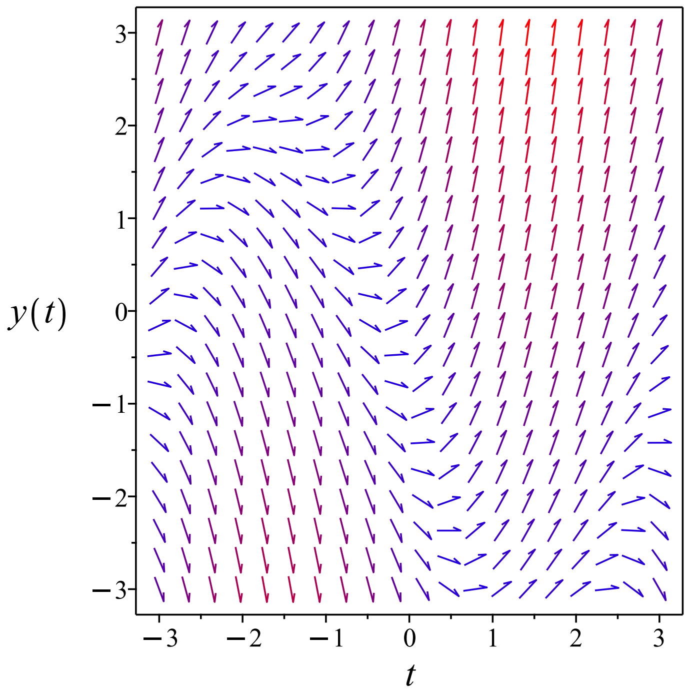


Figure 73: Slope field plot

Verification of solutions

$$y = -\frac{\left(3\,\mathrm{e}^{-t}\sin\left(t\right)+3\,\mathrm{e}^{-t}\cos\left(t\right)+2\,\mathrm{e}^{-t}-2c_1\right)\mathrm{e}^{t}}{2}$$

Verified OK.

1.30.4 Maple step by step solution

Let's solve

$-y+y' = 1+3\sin\left(t\right)$

- Highest derivative means the order of the ODE is 1

  y'

- Isolate the derivative

$y' = y + 1 + 3\sin(t)$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE

$-y + y' = 1 + 3\sin(t)$

- The ODE is linear; multiply by an integrating factor $\mu(t)$

$\mu(t)(-y + y') = \mu(t)(1 + 3\sin(t))$

- Assume the lhs of the ODE is the total derivative $\frac{d}{dt}(\mu(t)\,y)$

$\mu(t)(-y + y') = \mu'(t)\,y + \mu(t)\,y'$

- Isolate $\mu'(t)$

$\mu'(t) = -\mu(t)$

- Solve to find the integrating factor

$\mu(t) = \mathrm{e}^{-t}$

- Integrate both sides with respect to t

$\int \left(\frac{d}{dt}(\mu(t)\,y)\right) dt = \int \mu(t)(1 + 3\sin(t))\,dt + c_1$

- Evaluate the integral on the lhs

$\mu(t)\,y = \int \mu(t)(1 + 3\sin(t))\,dt + c_1$

- Solve for y

$y = \frac{\int \mu(t)(1+3\sin(t))dt + c_1}{\mu(t)}$

- Substitute $\mu(t) = \mathrm{e}^{-t}$

$y = \frac{\int (1+3\sin(t))\mathrm{e}^{-t}dt + c_1}{\mathrm{e}^{-t}}$

- Evaluate the integrals on the rhs

$y = \frac{-\frac{1}{\mathrm{e}^{t}} - \frac{3\,\mathrm{e}^{-t}\cos(t)}{2} - \frac{3\,\mathrm{e}^{-t}\sin(t)}{2} + c_1}{\mathrm{e}^{-t}}$

- Simplify

$y = -1 + c_1\mathrm{e}^{t} - \frac{3\cos(t)}{2} - \frac{3\sin(t)}{2}$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
<- 1st order linear successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 18

```
dsolve(-y(t)+diff(y(t),t) = 1+3*sin(t),y(t), singsol=all)
```

$$y(t) = -1 - \frac{3\cos(t)}{2} - \frac{3\sin(t)}{2} + e^{t} c_1$$

✓ Solution by Mathematica

Time used: 0.072 (sec). Leaf size: 25

```
DSolve[-y[t]+y'[t] == 1+3*Sin[t],y[t],t,IncludeSingularSolutions -> True]
```

$$y(t) \to -\frac{3\sin(t)}{2} - \frac{3\cos(t)}{2} + c_1 e^t - 1$$

1.31 problem 31

Internal problem ID [478]
Internal file name [`OUTPUT/478_Sunday_June_05_2022_01_42_07_AM_9283464/index.tex`]

Book: Elementary differential equations and boundary value problems, 10th ed., Boyce and DiPrima
Section: Section 2.1. Page 40
Problem number: 31.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"linear", "exactWithIntegrationFactor", "first_order_ode_lie_symmetry_lookup"**

Maple gives the following as the ode type

```
[[_linear, `class A`]]
```

$$\boxed{-\frac{3y}{2}+y'=2\,\mathrm{e}^{t}+3t}$$

1.31.1 Solving as linear ode

Entering Linear first order ODE solver. In canonical form a linear first order is

$$y'+p(t)y=q(t)$$

Where here

$$p(t)=-\frac{3}{2}$$
$$q(t)=2\,\mathrm{e}^{t}+3t$$

Hence the ode is

$$-\frac{3y}{2}+y'=2\,\mathrm{e}^{t}+3t$$

The integrating factor μ is

$$\mu = e^{\int -\frac{3}{2}dt}$$
$$= e^{-\frac{3t}{2}}$$

The ode becomes

$$\frac{d}{dt}(\mu y) = (\mu)\left(2\,e^{t} + 3t\right)$$
$$\frac{d}{dt}\left(e^{-\frac{3t}{2}} y\right) = \left(e^{-\frac{3t}{2}}\right)\left(2\,e^{t} + 3t\right)$$
$$d\left(e^{-\frac{3t}{2}} y\right) = \left(\left(2\,e^{t} + 3t\right) e^{-\frac{3t}{2}}\right) dt$$

Integrating gives

$$e^{-\frac{3t}{2}} y = \int \left(2\,e^{t} + 3t\right) e^{-\frac{3t}{2}}\,dt$$
$$e^{-\frac{3t}{2}} y = -2\,e^{-\frac{3t}{2}} t - \frac{4\,e^{-\frac{3t}{2}}}{3} - 4\,e^{-\frac{t}{2}} + c_1$$

Dividing both sides by the integrating factor $\mu = e^{-\frac{3t}{2}}$ results in

$$y = e^{\frac{3t}{2}}\left(-2\,e^{-\frac{3t}{2}} t - \frac{4\,e^{-\frac{3t}{2}}}{3} - 4\,e^{-\frac{t}{2}}\right) + c_1 e^{\frac{3t}{2}}$$

which simplifies to

$$y = -2t - \frac{4}{3} - 4\,e^{t} + c_1 e^{\frac{3t}{2}}$$

Summary
The solution(s) found are the following

$$y = -2t - \frac{4}{3} - 4\,e^{t} + c_1 e^{\frac{3t}{2}} \tag{1}$$

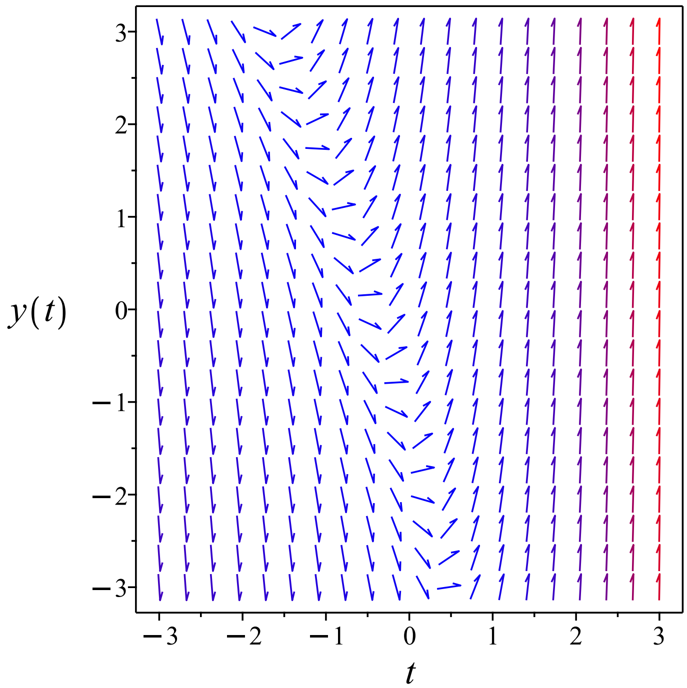


Figure 74: Slope field plot

Verification of solutions

$$
y = -2t - \frac{4}{3} - 4\,\mathrm{e}^{t} + c_1 \mathrm{e}^{\frac{3t}{2}}
$$

Verified OK.

1.31.2 Solving as first order ode lie symmetry lookup ode

Writing the ode as

$$
\begin{aligned}
y' &= \frac{3y}{2} + 2\,\mathrm{e}^{t} + 3t \\
y' &= \omega(t, y)
\end{aligned}
$$

The condition of Lie symmetry is the linearized PDE given by

$$
\eta_t + \omega(\eta_y - \xi_t) - \omega^2 \xi_y - \omega_t \xi - \omega_y \eta = 0 \tag{A}
$$

The type of this ode is known. It is of type `linear`. Therefore we do not need to solve the PDE (A), and can just use the lookup table shown below to find ξ, η

Table 91: Lie symmetry infinitesimal lookup table for known first order ODE's

ODE class	Form	ξ	η
linear ode	$y' = f(x)y(x) + g(x)$	0	$e^{\int f dx}$
separable ode	$y' = f(x)\, g(y)$	$\frac{1}{f}$	0
quadrature ode	$y' = f(x)$	0	1
quadrature ode	$y' = g(y)$	1	0
homogeneous ODEs of Class A	$y' = f\left(\frac{y}{x}\right)$	x	y
homogeneous ODEs of Class C	$y' = (a + bx + cy)^{\frac{n}{m}}$	1	$-\frac{b}{c}$
homogeneous class D	$y' = \frac{y}{x} + g(x)\, F\left(\frac{y}{x}\right)$	x^2	xy
First order special form ID 1	$y' = g(x)\, e^{h(x)+by} + f(x)$	$\frac{e^{-\int bf(x)dx-h(x)}}{g(x)}$	$\frac{f(x)e^{-\int bf(x)dx-h(x)}}{g(x)}$
polynomial type ode	$y' = \frac{a_1x+b_1y+c_1}{a_2x+b_2y+c_2}$	$\frac{a_1b_2x-a_2b_1x-b_1c_2+b_2c_1}{a_1b_2-a_2b_1}$	$\frac{a_1b_2y-a_2b_1y-a_1c_2-a_2c_1}{a_1b_2-a_2b_1}$
Bernoulli ode	$y' = f(x)\, y + g(x)\, y^n$	0	$e^{-\int (n-1)f(x)dx} y^n$
Reduced Riccati	$y' = f_1(x)\, y + f_2(x)\, y^2$	0	$e^{-\int f_1 dx}$

The above table shows that

$$
\begin{aligned}
\xi(t, y) &= 0 \\
\eta(t, y) &= \mathrm{e}^{\frac{3t}{2}} \qquad \text{(A1)}
\end{aligned}
$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(t, y) \to (R, S)$ where (R, S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$
\frac{dt}{\xi} = \frac{dy}{\eta} = dS \qquad (1)
$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial t} + \eta \frac{\partial}{\partial y}\right) S(t, y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the

canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$R = t$$

S is found from

$$\begin{aligned} S &= \int \frac{1}{\eta} dy \\ &= \int \frac{1}{e^{\frac{3t}{2}}} dy \end{aligned}$$

Which results in

$$S = e^{-\frac{3t}{2}} y$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_t + \omega(t, y) S_y}{R_t + \omega(t, y) R_y} \tag{2}$$

Where in the above R_t, R_y, S_t, S_y are all partial derivatives and $\omega(t, y)$ is the right hand side of the original ode given by

$$\omega(t, y) = \frac{3y}{2} + 2\,e^{t} + 3t$$

Evaluating all the partial derivatives gives

$$\begin{aligned} R_t &= 1 \\ R_y &= 0 \\ S_t &= -\frac{3\,e^{-\frac{3t}{2}} y}{2} \\ S_y &= e^{-\frac{3t}{2}} \end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = 2\,e^{-\frac{t}{2}} + 3\,e^{-\frac{3t}{2}} t \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for t, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = 2\,e^{-\frac{R}{2}} + 3\,e^{-\frac{3R}{2}} R$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$S(R) = -2\,\mathrm{e}^{-\frac{3R}{2}} R - \frac{4\,\mathrm{e}^{-\frac{3R}{2}}}{3} - 4\,\mathrm{e}^{-\frac{R}{2}} + c_1 \tag{4}$$

To complete the solution, we just need to transform (4) back to t, y coordinates. This results in

$$\mathrm{e}^{-\frac{3t}{2}} y = -2\,\mathrm{e}^{-\frac{3t}{2}} t - \frac{4\,\mathrm{e}^{-\frac{3t}{2}}}{3} - 4\,\mathrm{e}^{-\frac{t}{2}} + c_1$$

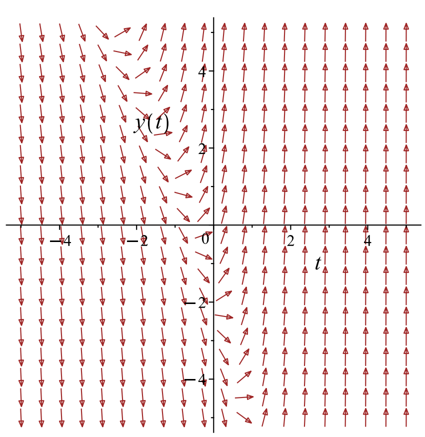
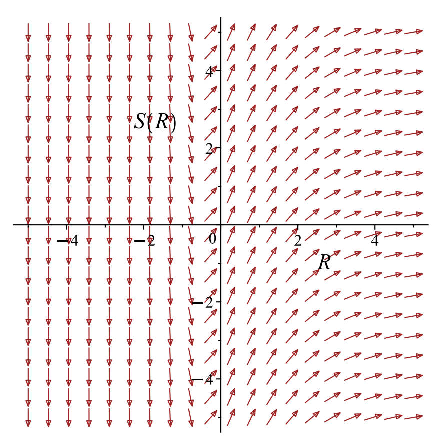
Which simplifies to

$$\frac{(6t + 3y + 4)\,\mathrm{e}^{-\frac{3t}{2}}}{3} - c_1 + 4\,\mathrm{e}^{-\frac{t}{2}} = 0$$

Which gives

$$y = -\frac{\left(6\,\mathrm{e}^{-\frac{3t}{2}} t + 4\,\mathrm{e}^{-\frac{3t}{2}} + 12\,\mathrm{e}^{-\frac{t}{2}} - 3c_1\right) \mathrm{e}^{\frac{3t}{2}}}{3}$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in t, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dt} = \frac{3y}{2} + 2\,\mathrm{e}^{t} + 3t$ 	$R = t$ $S = \mathrm{e}^{-\frac{3t}{2}} y$	$\frac{dS}{dR} = 2\,\mathrm{e}^{-\frac{R}{2}} + 3\,\mathrm{e}^{-\frac{3R}{2}} R$ 

Summary
The solution(s) found are the following

$$y = -\frac{\left(6\,\mathrm{e}^{-\frac{3t}{2}} t + 4\,\mathrm{e}^{-\frac{3t}{2}} + 12\,\mathrm{e}^{-\frac{t}{2}} - 3c_1\right) \mathrm{e}^{\frac{3t}{2}}}{3} \tag{1}$$

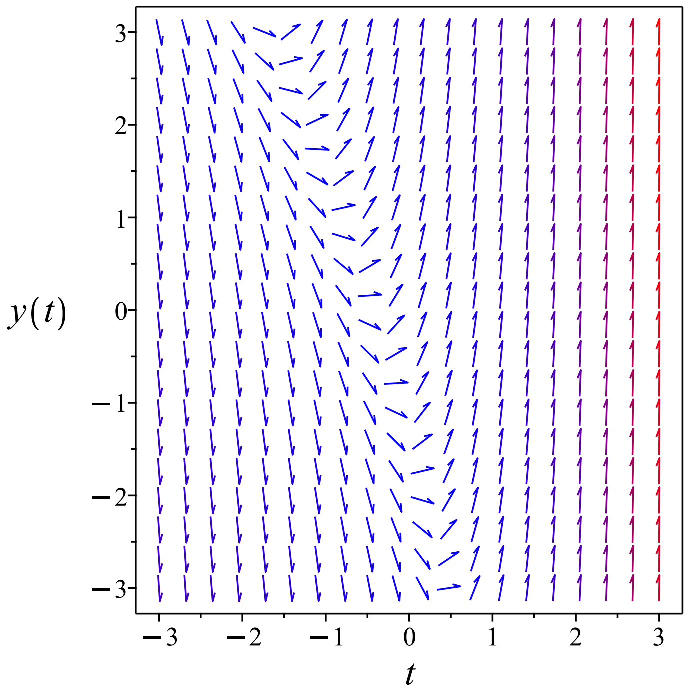


Figure 75: Slope field plot

Verification of solutions

$$y = -\frac{\left(6\,\mathrm{e}^{-\frac{3t}{2}} t + 4\,\mathrm{e}^{-\frac{3t}{2}} + 12\,\mathrm{e}^{-\frac{t}{2}} - 3c_1\right) \mathrm{e}^{\frac{3t}{2}}}{3}$$

Verified OK.

1.31.3 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x,y) + N(x,y)\frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x,y) = c$ where c is constant, that satisfies the

ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y)=0$$

Hence

$$\frac{\partial\phi}{\partial x}+\frac{\partial\phi}{\partial y}\frac{dy}{dx}=0 \tag{B}$$

Comparing (A,B) shows that

$$\begin{aligned}\frac{\partial\phi}{\partial x}&=M\\ \frac{\partial\phi}{\partial y}&=N\end{aligned}$$

But since $\frac{\partial^2\phi}{\partial x\partial y}=\frac{\partial^2\phi}{\partial y\partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y}=\frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x,y)$ but at least we know now that we can do that since the condition $\frac{\partial^2\phi}{\partial x\partial y}=\frac{\partial^2\phi}{\partial y\partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(t,y)\,\mathrm{d}t+N(t,y)\,\mathrm{d}y=0 \tag{1A}$$

Therefore

$$\mathrm{d}y=\left(\frac{3y}{2}+2\,\mathrm{e}^{t}+3t\right)\mathrm{d}t$$

$$\left(-\frac{3y}{2}-2\,\mathrm{e}^{t}-3t\right)\mathrm{d}t+\mathrm{d}y=0 \tag{2A}$$

Comparing (1A) and (2A) shows that

$$\begin{aligned}M(t,y)&=-\frac{3y}{2}-2\,\mathrm{e}^{t}-3t\\ N(t,y)&=1\end{aligned}$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y}=\frac{\partial N}{\partial t}$$

Using result found above gives

$$
\begin{aligned}
\frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}\left(-\frac{3y}{2} - 2\,\mathrm{e}^{t} - 3t\right) \\
&= -\frac{3}{2}
\end{aligned}
$$

And

$$
\begin{aligned}
\frac{\partial N}{\partial t} &= \frac{\partial}{\partial t}(1) \\
&= 0
\end{aligned}
$$

Since $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial t}$, then the ODE is not exact. Since the ODE is not exact, we will try to find an integrating factor to make it exact. Let

$$
\begin{aligned}
A &= \frac{1}{N}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial t}\right) \\
&= 1\left(\left(-\frac{3}{2}\right) - (0)\right) \\
&= -\frac{3}{2}
\end{aligned}
$$

Since A does not depend on y, then it can be used to find an integrating factor. The integrating factor μ is

$$
\begin{aligned}
\mu &= e^{\int A \,\mathrm{d}t} \\
&= e^{\int -\frac{3}{2} \,\mathrm{d}t}
\end{aligned}
$$

The result of integrating gives

$$
\begin{aligned}
\mu &= e^{-\frac{3t}{2}} \\
&= \mathrm{e}^{-\frac{3t}{2}}
\end{aligned}
$$

M and N are multiplied by this integrating factor, giving new M and new N which are called $\overline{M}$ and $\overline{N}$ for now so not to confuse them with the original M and N.

$$
\begin{aligned}
\overline{M} &= \mu M \\
&= \mathrm{e}^{-\frac{3t}{2}}\left(-\frac{3y}{2} - 2\,\mathrm{e}^{t} - 3t\right) \\
&= -\frac{(3y + 4\,\mathrm{e}^{t} + 6t)\,\mathrm{e}^{-\frac{3t}{2}}}{2}
\end{aligned}
$$

And

$$
\begin{aligned}
\overline{N} &= \mu N \\
&= \mathrm{e}^{-\frac{3t}{2}}(1) \\
&= \mathrm{e}^{-\frac{3t}{2}}
\end{aligned}
$$

Now a modified ODE is ontained from the original ODE, which is exact and can be solved. The modified ODE is

$$
\begin{aligned}
\overline{M} + \overline{N}\frac{\mathrm{d}y}{\mathrm{d}t} &= 0 \\
\left(-\frac{(3y + 4\,\mathrm{e}^{t} + 6t)\,\mathrm{e}^{-\frac{3t}{2}}}{2}\right) + \left(\mathrm{e}^{-\frac{3t}{2}}\right)\frac{\mathrm{d}y}{\mathrm{d}t} &= 0
\end{aligned}
$$

The following equations are now set up to solve for the function $\phi(t, y)$

$$
\frac{\partial \phi}{\partial t} = \overline{M} \tag{1}
$$

$$
\frac{\partial \phi}{\partial y} = \overline{N} \tag{2}
$$

Integrating (1) w.r.t. t gives

$$
\begin{aligned}
\int \frac{\partial \phi}{\partial t}\,\mathrm{d}t &= \int \overline{M}\,\mathrm{d}t \\
\int \frac{\partial \phi}{\partial t}\,\mathrm{d}t &= \int -\frac{(3y + 4\,\mathrm{e}^{t} + 6t)\,\mathrm{e}^{-\frac{3t}{2}}}{2}\,\mathrm{d}t
\end{aligned}
$$

$$
\phi = \frac{(6t + 3y + 4)\,\mathrm{e}^{-\frac{3t}{2}}}{3} + 4\,\mathrm{e}^{-\frac{t}{2}} + f(y) \tag{3}
$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both t and y. Taking derivative of equation (3) w.r.t y gives

$$
\frac{\partial \phi}{\partial y} = \mathrm{e}^{-\frac{3t}{2}} + f'(y) \tag{4}
$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = \mathrm{e}^{-\frac{3t}{2}}$. Therefore equation (4) becomes

$$
\mathrm{e}^{-\frac{3t}{2}} = \mathrm{e}^{-\frac{3t}{2}} + f'(y) \tag{5}
$$

Solving equation (5) for $f'(y)$ gives

$$f'(y) = 0$$

Therefore

$$f(y) = c_1$$

Where c_1 is constant of integration. Substituting this result for $f(y)$ into equation (3) gives ϕ

$$\phi = \frac{(6t + 3y + 4)\, e^{-\frac{3t}{2}}}{3} + 4\, e^{-\frac{t}{2}} + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = \frac{(6t + 3y + 4)\, e^{-\frac{3t}{2}}}{3} + 4\, e^{-\frac{t}{2}}$$

The solution becomes

$$y = -\frac{\left(6\, e^{-\frac{3t}{2}} t + 4\, e^{-\frac{3t}{2}} + 12\, e^{-\frac{t}{2}} - 3c_1\right) e^{\frac{3t}{2}}}{3}$$

Summary

The solution(s) found are the following

$$y = -\frac{\left(6\, e^{-\frac{3t}{2}} t + 4\, e^{-\frac{3t}{2}} + 12\, e^{-\frac{t}{2}} - 3c_1\right) e^{\frac{3t}{2}}}{3} \tag{1}$$

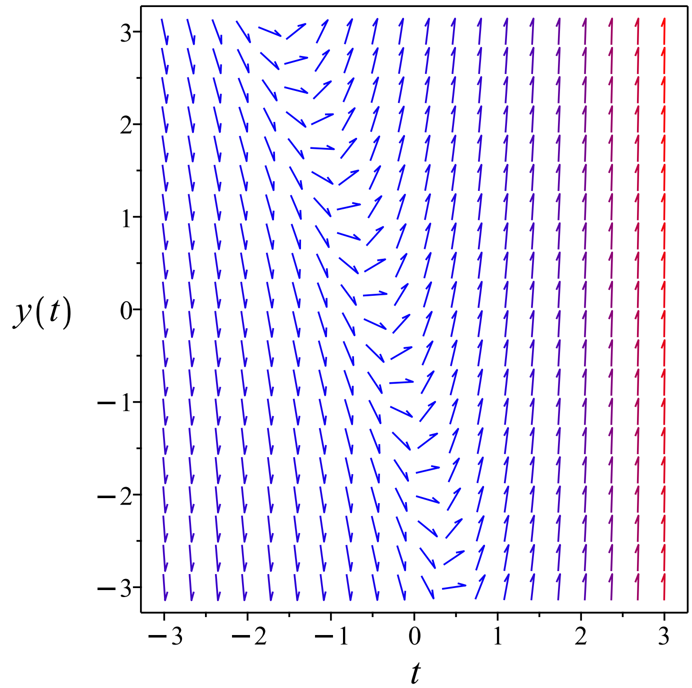


Figure 76: Slope field plot

Verification of solutions

$$y = -\frac{\left(6\,\mathrm{e}^{-\frac{3t}{2}} t + 4\,\mathrm{e}^{-\frac{3t}{2}} + 12\,\mathrm{e}^{-\frac{t}{2}} - 3c_1\right)\mathrm{e}^{\frac{3t}{2}}}{3}$$

Verified OK.

1.31.4 Maple step by step solution

Let's solve

$-\frac{3y}{2} + y' = 2\,\mathrm{e}^{t} + 3t$

- Highest derivative means the order of the ODE is 1

  y'

- Isolate the derivative

  $y' = \frac{3y}{2} + 2\,\mathrm{e}^{t} + 3t$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE

  $-\frac{3y}{2} + y' = 2\,\mathrm{e}^{t} + 3t$

- The ODE is linear; multiply by an integrating factor $\mu(t)$

$$\mu(t)\left(-\frac{3y}{2}+y'\right)=\mu(t)\left(2\,e^{t}+3t\right)$$

- Assume the lhs of the ODE is the total derivative $\frac{d}{dt}(\mu(t)\,y)$

$$\mu(t)\left(-\frac{3y}{2}+y'\right)=\mu'(t)\,y+\mu(t)\,y'$$

- Isolate $\mu'(t)$

$$\mu'(t)=-\frac{3\mu(t)}{2}$$

- Solve to find the integrating factor

$$\mu(t)=e^{-\frac{3t}{2}}$$

- Integrate both sides with respect to t

$$\int\left(\frac{d}{dt}(\mu(t)\,y)\right)dt=\int\mu(t)\left(2\,e^{t}+3t\right)dt+c_1$$

- Evaluate the integral on the lhs

$$\mu(t)\,y=\int\mu(t)\left(2\,e^{t}+3t\right)dt+c_1$$

- Solve for y

$$y=\frac{\int\mu(t)\left(2\,e^{t}+3t\right)dt+c_1}{\mu(t)}$$

- Substitute $\mu(t)=e^{-\frac{3t}{2}}$

$$y=\frac{\int\left(2\,e^{t}+3t\right)e^{-\frac{3t}{2}}dt+c_1}{e^{-\frac{3t}{2}}}$$

- Evaluate the integrals on the rhs

$$y=\frac{-2\,e^{-\frac{3t}{2}}t-\frac{4\,e^{-\frac{3t}{2}}}{3}-4\,e^{-\frac{t}{2}}+c_1}{e^{-\frac{3t}{2}}}$$

- Simplify

$$y=-2t-\frac{4}{3}-4\,e^{t}+c_1e^{\frac{3t}{2}}$$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
<- 1st order linear successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 19

```
dsolve(-3/2*y(t)+diff(y(t),t) = 2*exp(t)+3*t,y(t), singsol=all)
```

$$y(t) = -2t - \frac{4}{3} - 4\,\mathrm{e}^{t} + \mathrm{e}^{\frac{3t}{2}} c_1$$

✓ Solution by Mathematica

Time used: 0.1 (sec). Leaf size: 27

```
DSolve[-3/2*y[t]+y'[t] == 2*Exp[t]+3*t,y[t],t,IncludeSingularSolutions -> True]
```

$$y(t) \to -2t - 4e^t + c_1 e^{3t/2} - \frac{4}{3}$$

2 Section 2.2. Page 48

2.1 problem 1

Internal problem ID [479]
Internal file name [`OUTPUT/479_Sunday_June_05_2022_01_42_08_AM_65104678/index.tex`]

Book: Elementary differential equations and boundary value problems, 10th ed., Boyce and DiPrima
Section: Section 2.2. Page 48
Problem number: 1.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"exact", "separable", "differentialType", "first_order_ode_lie_symmetry_lookup"**

Maple gives the following as the ode type

`[_separable]`

$$\boxed{y' - \frac{x^2}{y} = 0}$$

2.1.1 Solving as separable ode

In canonical form the ODE is

$$\begin{aligned} y' &= F(x,y) \\ &= f(x)g(y) \\ &= \frac{x^2}{y} \end{aligned}$$

Where $f(x) = x^2$ and $g(y) = \frac{1}{y}$. Integrating both sides gives

$$\frac{1}{\frac{1}{y}}\, dy = x^2\, dx$$

$$\int \frac{1}{\frac{1}{y}}\, dy = \int x^2\, dx$$

$$\frac{y^2}{2} = \frac{x^3}{3} + c_1$$

Which results in

$$y = \frac{\sqrt{6x^3 + 18c_1}}{3}$$

$$y = -\frac{\sqrt{6x^3 + 18c_1}}{3}$$

<u>Summary</u>

The solution(s) found are the following

$$y = \frac{\sqrt{6x^3 + 18c_1}}{3} \tag{1}$$

$$y = -\frac{\sqrt{6x^3 + 18c_1}}{3} \tag{2}$$

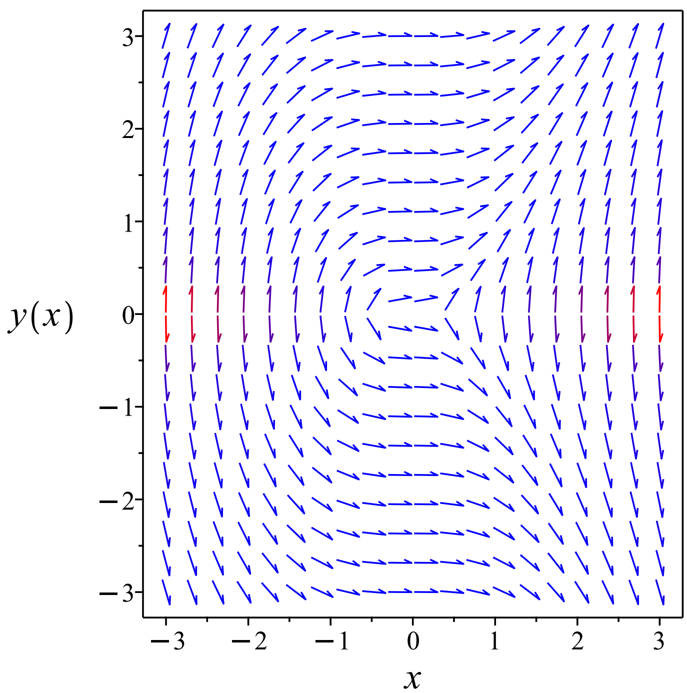


Figure 77: Slope field plot

Verification of solutions

$$y = \frac{\sqrt{6x^3 + 18c_1}}{3}$$

Verified OK.

$$y = -\frac{\sqrt{6x^3 + 18c_1}}{3}$$

Verified OK.

2.1.2 Solving as differentialType ode

Writing the ode as

$$y' = \frac{x^2}{y} \tag{1}$$

Which becomes

$$(y)\, dy = \left(x^2\right) dx \tag{2}$$

But the RHS is complete differential because

$$\left(x^2\right) dx = d\left(\frac{x^3}{3}\right)$$

Hence (2) becomes

$$(y)\, dy = d\left(\frac{x^3}{3}\right)$$

Integrating both sides gives gives these solutions

$$y = \frac{\sqrt{6x^3 + 18c_1}}{3} + c_1$$
$$y = -\frac{\sqrt{6x^3 + 18c_1}}{3} + c_1$$

Summary
The solution(s) found are the following

$$y = \frac{\sqrt{6x^3 + 18c_1}}{3} + c_1 \tag{1}$$
$$y = -\frac{\sqrt{6x^3 + 18c_1}}{3} + c_1 \tag{2}$$

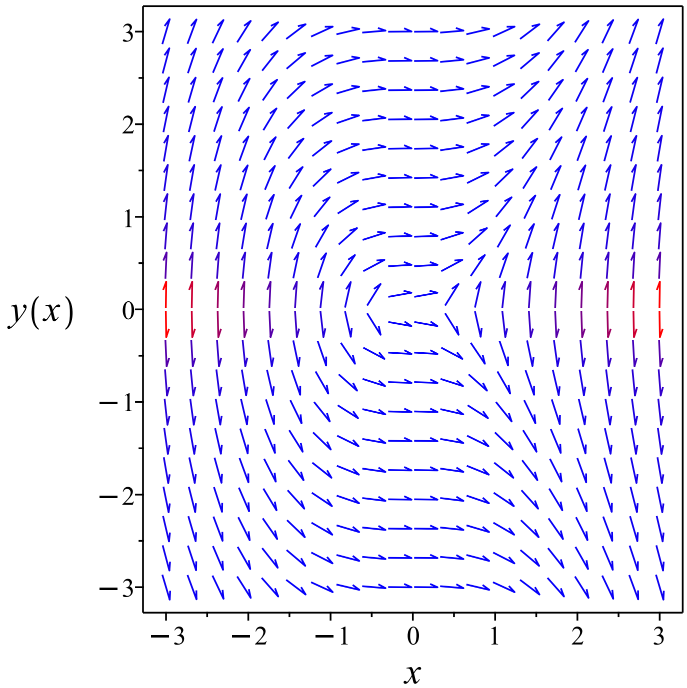


Figure 78: Slope field plot

Verification of solutions

$$y = \frac{\sqrt{6x^3 + 18c_1}}{3} + c_1$$

Verified OK.

$$y = -\frac{\sqrt{6x^3 + 18c_1}}{3} + c_1$$

Verified OK.

2.1.3 Solving as first order ode lie symmetry lookup ode

Writing the ode as

$$y' = \frac{x^2}{y}$$
$$y' = \omega(x, y)$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x + \omega(\eta_y - \xi_x) - \omega^2 \xi_y - \omega_x \xi - \omega_y \eta = 0 \tag{A}$$

The type of this ode is known. It is of type `separable`. Therefore we do not need to solve the PDE (A), and can just use the lookup table shown below to find ξ, η

Table 94: Lie symmetry infinitesimal lookup table for known first order ODE's

ODE class	Form	ξ	η
linear ode	$y' = f(x)y(x) + g(x)$	0	$e^{\int f dx}$
separable ode	$y' = f(x)\, g(y)$	$\frac{1}{f}$	0
quadrature ode	$y' = f(x)$	0	1
quadrature ode	$y' = g(y)$	1	0
homogeneous ODEs of Class A	$y' = f\left(\frac{y}{x}\right)$	x	y
homogeneous ODEs of Class C	$y' = (a + bx + cy)^{\frac{n}{m}}$	1	$-\frac{b}{c}$
homogeneous class D	$y' = \frac{y}{x} + g(x)\, F\left(\frac{y}{x}\right)$	x^2	xy
First order special form ID 1	$y' = g(x)\, e^{h(x)+by} + f(x)$	$\frac{e^{-\int bf(x)dx - h(x)}}{g(x)}$	$\frac{f(x)e^{-\int bf(x)dx - h(x)}}{g(x)}$
polynomial type ode	$y' = \frac{a_1x+b_1y+c_1}{a_2x+b_2y+c_2}$	$\frac{a_1b_2x - a_2b_1x - b_1c_2 + b_2c_1}{a_1b_2 - a_2b_1}$	$\frac{a_1b_2y - a_2b_1y - a_1c_2 - a_2c_1}{a_1b_2 - a_2b_1}$
Bernoulli ode	$y' = f(x)\, y + g(x)\, y^n$	0	$e^{-\int (n-1)f(x)dx} y^n$
Reduced Riccati	$y' = f_1(x)\, y + f_2(x)\, y^2$	0	$e^{-\int f_1 dx}$

The above table shows that

$$
\begin{aligned}
\xi(x, y) &= \frac{1}{x^2} \\
\eta(x, y) &= 0
\end{aligned} \tag{A1}
$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x, y) \to (R, S)$ where (R, S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$
\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}
$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y}\right) S(x,y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the canonical coordinates, where $S(R)$. Since $\eta = 0$ then in this special case

$$R = y$$

S is found from

$$\begin{aligned} S &= \int \frac{1}{\xi} dx \\ &= \int \frac{1}{\frac{1}{x^2}} dx \end{aligned}$$

Which results in

$$S = \frac{x^3}{3}$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x,y) S_y}{R_x + \omega(x,y) R_y} \tag{2}$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x,y)$ is the right hand side of the original ode given by

$$\omega(x,y) = \frac{x^2}{y}$$

Evaluating all the partial derivatives gives

$$\begin{aligned} R_x &= 0 \\ R_y &= 1 \\ S_x &= x^2 \\ S_y &= 0 \end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = y \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = R$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordinates R, S. Integrating the above gives

$$S(R) = \frac{R^2}{2} + c_1 \tag{4}$$

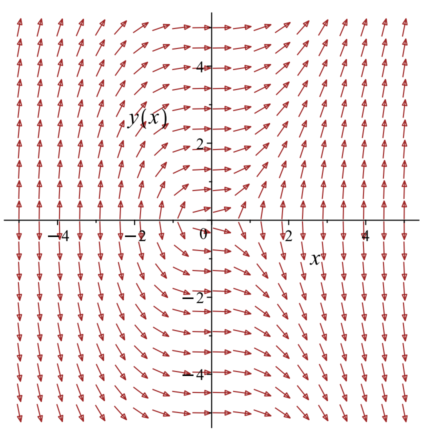
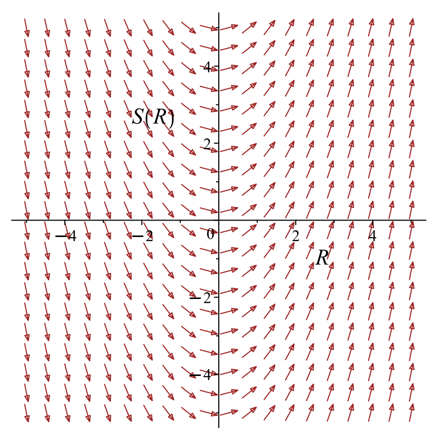
To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$\frac{x^3}{3} = \frac{y^2}{2} + c_1$$

Which simplifies to

$$\frac{x^3}{3} = \frac{y^2}{2} + c_1$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in x, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dx} = \frac{x^2}{y}$ 	$R = y$ $S = \frac{x^3}{3}$	$\frac{dS}{dR} = R$ 

Summary

The solution(s) found are the following

$$\frac{x^3}{3} = \frac{y^2}{2} + c_1 \tag{1}$$

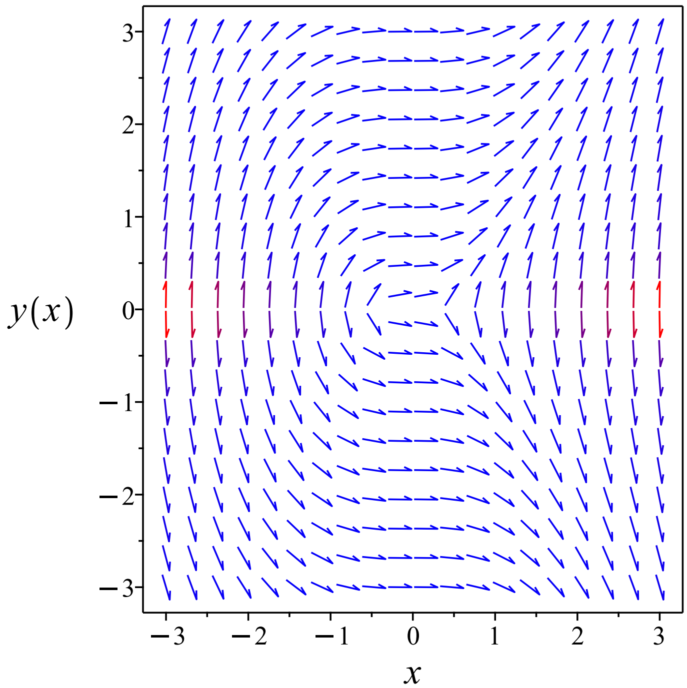


Figure 79: Slope field plot

Verification of solutions

$$\frac{x^3}{3} = \frac{y^2}{2} + c_1$$

Verified OK.

2.1.4 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x,y) + N(x,y)\frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x,y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y) = 0$$

Hence

$$\frac{\partial \phi}{\partial x} + \frac{\partial \phi}{\partial y}\frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$\frac{\partial \phi}{\partial x} = M$$
$$\frac{\partial \phi}{\partial y} = N$$

But since $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x, y)$ but at least we know now that we can do that since the condition $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x, y)\,\mathrm{d}x + N(x, y)\,\mathrm{d}y = 0 \tag{1A}$$

Therefore

$$(y)\,\mathrm{d}y = (x^2)\,\mathrm{d}x$$
$$(-x^2)\,\mathrm{d}x + (y)\,\mathrm{d}y = 0 \tag{2A}$$

Comparing (1A) and (2A) shows that

$$M(x, y) = -x^2$$
$$N(x, y) = y$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Using result found above gives

$$\begin{aligned}\frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}(-x^2)\\ &= 0\end{aligned}$$

And

$$\begin{aligned}\frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}(y)\\ &= 0\end{aligned}$$

Since $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$, then the ODE is exact The following equations are now set up to solve for the function $\phi(x,y)$

$$\frac{\partial \phi}{\partial x} = M \tag{1}$$
$$\frac{\partial \phi}{\partial y} = N \tag{2}$$

Integrating (1) w.r.t. x gives

$$\int \frac{\partial \phi}{\partial x}\, \mathrm{d}x = \int M\, \mathrm{d}x$$
$$\int \frac{\partial \phi}{\partial x}\, \mathrm{d}x = \int -x^2\, \mathrm{d}x$$
$$\phi = -\frac{x^3}{3} + f(y) \tag{3}$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$\frac{\partial \phi}{\partial y} = 0 + f'(y) \tag{4}$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = y$. Therefore equation (4) becomes

$$y = 0 + f'(y) \tag{5}$$

Solving equation (5) for $f'(y)$ gives

$$f'(y) = y$$

Integrating the above w.r.t y gives

$$\int f'(y)\, \mathrm{d}y = \int (y)\, \mathrm{d}y$$
$$f(y) = \frac{y^2}{2} + c_1$$

Where c_1 is constant of integration. Substituting result found above for $f(y)$ into equation (3) gives ϕ

$$\phi = -\frac{x^3}{3} + \frac{y^2}{2} + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = -\frac{x^3}{3} + \frac{y^2}{2}$$

Summary

The solution(s) found are the following

$$-\frac{x^3}{3} + \frac{y^2}{2} = c_1 \tag{1}$$

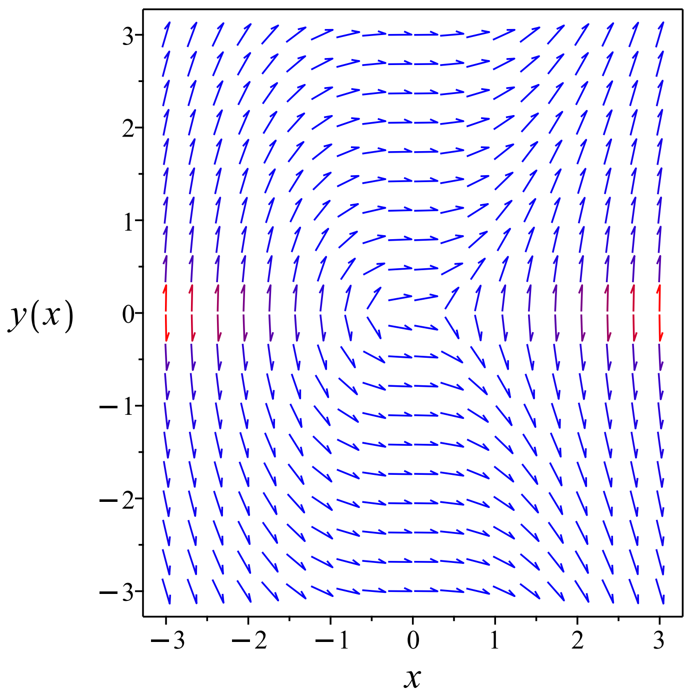


Figure 80: Slope field plot

Verification of solutions

$$-\frac{x^3}{3} + \frac{y^2}{2} = c_1$$

Verified OK.

2.1.5 Maple step by step solution

Let's solve

$y' - \frac{x^2}{y} = 0$

- Highest derivative means the order of the ODE is 1

  y'

- Separate variables

  $yy' = x^2$

- Integrate both sides with respect to x

  $\int yy'dx = \int x^2 dx + c_1$

- Evaluate integral

  $\frac{y^2}{2} = \frac{x^3}{3} + c_1$

- Solve for y

  $\left\{y = -\frac{\sqrt{6x^3+18c_1}}{3}, y = \frac{\sqrt{6x^3+18c_1}}{3}\right\}$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
<- Bernoulli successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 33

```
dsolve(diff(y(x),x) = x^2/y(x),y(x), singsol=all)
```

$$y(x) = -\frac{\sqrt{6x^3 + 9c_1}}{3}$$
$$y(x) = \frac{\sqrt{6x^3 + 9c_1}}{3}$$

✓ Solution by Mathematica

Time used: 0.093 (sec). Leaf size: 50

```
DSolve[y'[x] == x^2/y[x],y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to -\sqrt{\frac{2}{3}}\sqrt{x^3 + 3c_1}$$
$$y(x) \to \sqrt{\frac{2}{3}}\sqrt{x^3 + 3c_1}$$

2.2 problem 2

Internal problem ID [480]
Internal file name [`OUTPUT/480_Sunday_June_05_2022_01_42_09_AM_72882121/index.tex`]

Book: Elementary differential equations and boundary value problems, 10th ed., Boyce and DiPrima
Section: Section 2.2. Page 48
Problem number: 2.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"exact", "separable", "first_order_ode_lie_symmetry_lookup"**

Maple gives the following as the ode type

```
[_separable]
```

$$\boxed{y' - \frac{x^2}{(x^3+1)\,y} = 0}$$

2.2.1 Solving as separable ode

In canonical form the ODE is

$$\begin{aligned} y' &= F(x,y) \\ &= f(x)g(y) \\ &= \frac{x^2}{(x^3+1)\,y} \end{aligned}$$

Where $f(x) = \frac{x^2}{x^3+1}$ and $g(y) = \frac{1}{y}$. Integrating both sides gives

$$\frac{1}{\frac{1}{y}}\,dy = \frac{x^2}{x^3+1}\,dx$$

$$\int \frac{1}{\frac{1}{y}}dy = \int \frac{x^2}{x^3+1}dx$$
$$\frac{y^2}{2} = \frac{\ln(x^3+1)}{3} + c_1$$

Which results in

$$y = \frac{\sqrt{6\ln(x^3+1)+18c_1}}{3}$$
$$y = -\frac{\sqrt{6\ln(x^3+1)+18c_1}}{3}$$

Summary

The solution(s) found are the following

$$y = \frac{\sqrt{6\ln(x^3+1)+18c_1}}{3} \tag{1}$$
$$y = -\frac{\sqrt{6\ln(x^3+1)+18c_1}}{3} \tag{2}$$

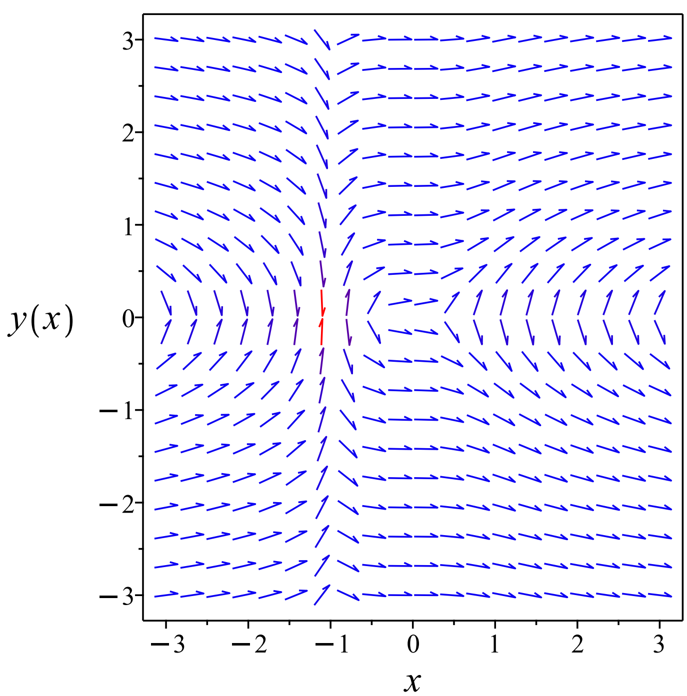


Figure 81: Slope field plot

Verification of solutions

$$y = \frac{\sqrt{6 \ln (x^3 + 1) + 18c_1}}{3}$$

Verified OK.

$$y = -\frac{\sqrt{6 \ln (x^3 + 1) + 18c_1}}{3}$$

Verified OK.

2.2.2 Solving as first order ode lie symmetry lookup ode

Writing the ode as

$$y' = \frac{x^2}{(x^3 + 1) y}$$
$$y' = \omega(x, y)$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x + \omega(\eta_y - \xi_x) - \omega^2 \xi_y - \omega_x \xi - \omega_y \eta = 0 \tag{A}$$

The type of this ode is known. It is of type `separable`. Therefore we do not need to solve the PDE (A), and can just use the lookup table shown below to find ξ, η

Table 97: Lie symmetry infinitesimal lookup table for known first order ODE's

ODE class	Form	ξ	η
linear ode	$y' = f(x)y(x) + g(x)$	0	$e^{\int f dx}$
separable ode	$y' = f(x)\, g(y)$	$\frac{1}{f}$	0
quadrature ode	$y' = f(x)$	0	1
quadrature ode	$y' = g(y)$	1	0
homogeneous ODEs of Class A	$y' = f\left(\frac{y}{x}\right)$	x	y
homogeneous ODEs of Class C	$y' = (a + bx + cy)^{\frac{n}{m}}$	1	$-\frac{b}{c}$
homogeneous class D	$y' = \frac{y}{x} + g(x)\, F\left(\frac{y}{x}\right)$	x^2	xy
First order special form ID 1	$y' = g(x)\, e^{h(x)+by} + f(x)$	$\frac{e^{-\int bf(x)dx-h(x)}}{g(x)}$	$\frac{f(x)e^{-\int bf(x)dx-h(x)}}{g(x)}$
polynomial type ode	$y' = \frac{a_1x+b_1y+c_1}{a_2x+b_2y+c_2}$	$\frac{a_1b_2x-a_2b_1x-b_1c_2+b_2c_1}{a_1b_2-a_2b_1}$	$\frac{a_1b_2y-a_2b_1y-a_1c_2-a_2c_1}{a_1b_2-a_2b_1}$
Bernoulli ode	$y' = f(x)\, y + g(x)\, y^n$	0	$e^{-\int (n-1)f(x)dx} y^n$
Reduced Riccati	$y' = f_1(x)\, y + f_2(x)\, y^2$	0	$e^{-\int f_1 dx}$

The above table shows that

$$
\begin{aligned}
\xi(x, y) &= \frac{x^3 + 1}{x^2} \\
\eta(x, y) &= 0
\end{aligned} \tag{A1}
$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x, y) \to (R, S)$ where (R, S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$
\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}
$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y}\right) S(x, y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the

canonical coordinates, where $S(R)$. Since $\eta = 0$ then in this special case

$$R = y$$

S is found from

$$\begin{aligned} S &= \int \frac{1}{\xi} dx \\ &= \int \frac{1}{\frac{x^3+1}{x^2}} dx \end{aligned}$$

Which results in

$$S = \frac{\ln(x^3 + 1)}{3}$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x, y)S_y}{R_x + \omega(x, y)R_y} \tag{2}$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x, y)$ is the right hand side of the original ode given by

$$\omega(x, y) = \frac{x^2}{(x^3 + 1)\, y}$$

Evaluating all the partial derivatives gives

$$\begin{aligned} R_x &= 0 \\ R_y &= 1 \\ S_x &= \frac{x^2}{(x^2 - x + 1)(x + 1)} \\ S_y &= 0 \end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = y \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = R$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordinates R, S. Integrating the above gives

$$S(R) = \frac{R^2}{2} + c_1 \tag{4}$$

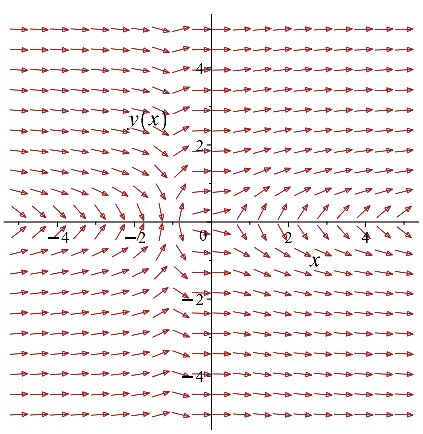
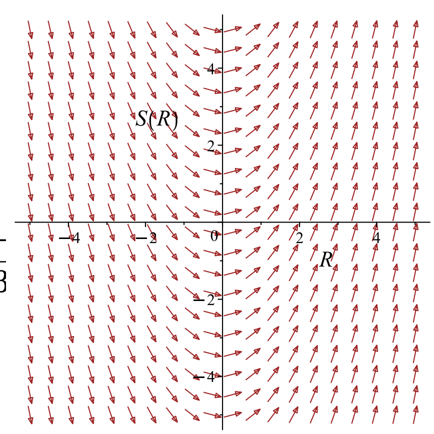
To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$\frac{\ln(x+1)}{3} + \frac{\ln(x^2-x+1)}{3} = \frac{y^2}{2} + c_1$$

Which simplifies to

$$\frac{\ln(x+1)}{3} + \frac{\ln(x^2-x+1)}{3} = \frac{y^2}{2} + c_1$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in x, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dx} = \frac{x^2}{(x^3+1)y}$ 	$R = y$ $S = \frac{\ln(x+1)}{3} + \frac{\ln(x^2 -}{3}$	$\frac{dS}{dR} = R$ 

Summary

The solution(s) found are the following

$$\frac{\ln(x+1)}{3} + \frac{\ln(x^2-x+1)}{3} = \frac{y^2}{2} + c_1 \tag{1}$$

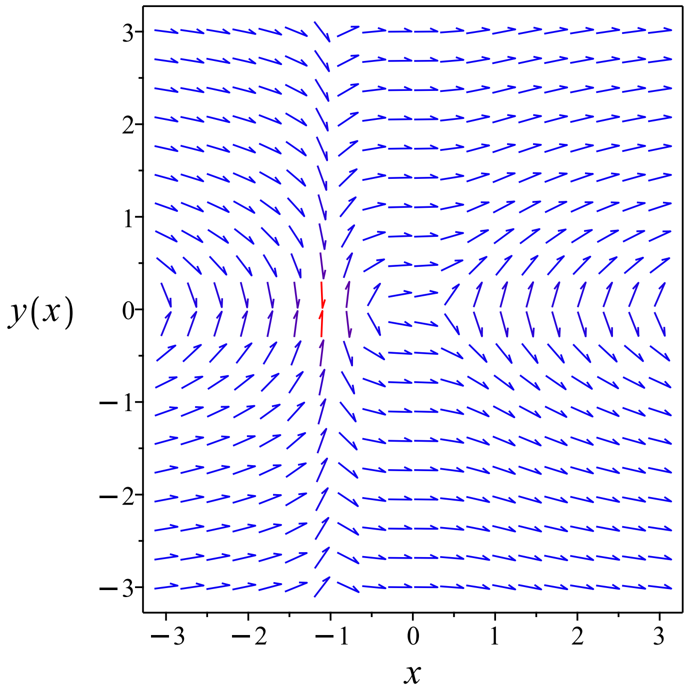


Figure 82: Slope field plot

Verification of solutions

$$\frac{\ln(x+1)}{3}+\frac{\ln(x^2-x+1)}{3}=\frac{y^2}{2}+c_1$$

Verified OK.

2.2.3 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x,y)+N(x,y)\frac{dy}{dx}=0 \tag{A}$$

We assume there exists a function $\phi(x,y)=c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y)=0$$

Hence

$$\frac{\partial\phi}{\partial x}+\frac{\partial\phi}{\partial y}\frac{dy}{dx}=0 \tag{B}$$

Comparing (A,B) shows that

$$\frac{\partial \phi}{\partial x} = M$$
$$\frac{\partial \phi}{\partial y} = N$$

But since $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x,y)$ but at least we know now that we can do that since the condition $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x,y)\,\mathrm{d}x + N(x,y)\,\mathrm{d}y = 0 \tag{1A}$$

Therefore

$$(y)\,\mathrm{d}y = \left(\frac{x^2}{x^3+1}\right)\mathrm{d}x$$
$$\left(-\frac{x^2}{x^3+1}\right)\mathrm{d}x + (y)\,\mathrm{d}y = 0 \tag{2A}$$

Comparing (1A) and (2A) shows that

$$M(x,y) = -\frac{x^2}{x^3+1}$$
$$N(x,y) = y$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Using result found above gives

$$\begin{aligned}\frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}\left(-\frac{x^2}{x^3+1}\right)\\ &= 0\end{aligned}$$

And

$$
\begin{aligned}
\frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}(y) \\
&= 0
\end{aligned}
$$

Since $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$, then the ODE is exact The following equations are now set up to solve for the function $\phi(x, y)$

$$
\frac{\partial \phi}{\partial x} = M \tag{1}
$$

$$
\frac{\partial \phi}{\partial y} = N \tag{2}
$$

Integrating (1) w.r.t. x gives

$$
\begin{aligned}
\int \frac{\partial \phi}{\partial x} \, \mathrm{d}x &= \int M \, \mathrm{d}x \\
\int \frac{\partial \phi}{\partial x} \, \mathrm{d}x &= \int -\frac{x^2}{x^3+1} \, \mathrm{d}x
\end{aligned}
$$

$$
\phi = -\frac{\ln(x^3+1)}{3} + f(y) \tag{3}
$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$
\frac{\partial \phi}{\partial y} = 0 + f'(y) \tag{4}
$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = y$. Therefore equation (4) becomes

$$
y = 0 + f'(y) \tag{5}
$$

Solving equation (5) for $f'(y)$ gives

$$
f'(y) = y
$$

Integrating the above w.r.t y gives

$$
\begin{aligned}
\int f'(y) \, \mathrm{d}y &= \int (y) \, \mathrm{d}y \\
f(y) &= \frac{y^2}{2} + c_1
\end{aligned}
$$

Where c_1 is constant of integration. Substituting result found above for $f(y)$ into equation (3) gives ϕ

$$\phi = -\frac{\ln(x^3+1)}{3} + \frac{y^2}{2} + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = -\frac{\ln(x^3+1)}{3} + \frac{y^2}{2}$$

Summary

The solution(s) found are the following

$$-\frac{\ln(x^3+1)}{3} + \frac{y^2}{2} = c_1 \tag{1}$$

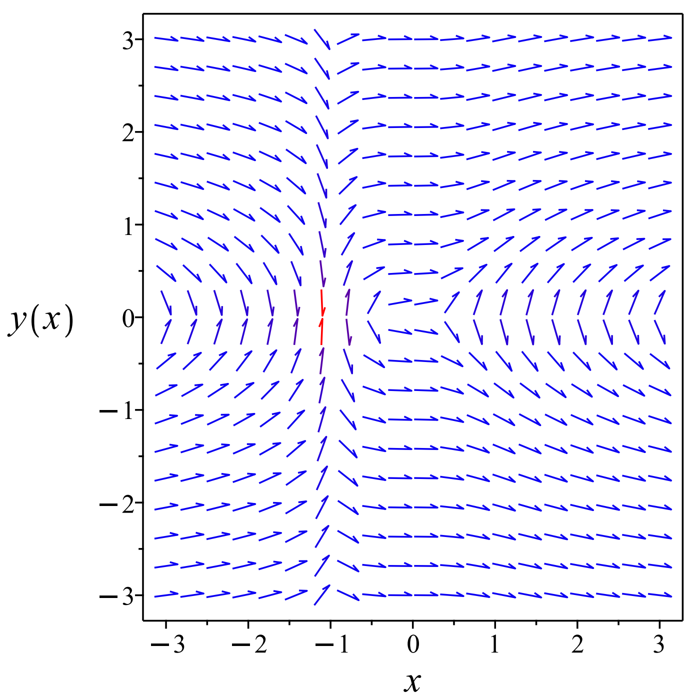


Figure 83: Slope field plot

Verification of solutions

$$-\frac{\ln(x^3+1)}{3} + \frac{y^2}{2} = c_1$$

Verified OK.

2.2.4 Maple step by step solution

Let's solve

$y' - \frac{x^2}{(x^3+1)y} = 0$

- Highest derivative means the order of the ODE is 1

  y'

- Separate variables

  $yy' = \frac{x^2}{x^3+1}$

- Integrate both sides with respect to x

  $\int yy'dx = \int \frac{x^2}{x^3+1}dx + c_1$

- Evaluate integral

  $\frac{y^2}{2} = \frac{\ln(x^3+1)}{3} + c_1$

- Solve for y

  $\left\{y = -\frac{\sqrt{6\ln(x^3+1)+18c_1}}{3}, y = \frac{\sqrt{6\ln(x^3+1)+18c_1}}{3}\right\}$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
<- Bernoulli successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 39

```
dsolve(diff(y(x),x) = x^2/(x^3+1)/y(x),y(x), singsol=all)
```

$$y(x) = -\frac{\sqrt{6\ln(x^3+1)+9c_1}}{3}$$
$$y(x) = \frac{\sqrt{6\ln(x^3+1)+9c_1}}{3}$$

✓ Solution by Mathematica

Time used: 0.12 (sec). Leaf size: 56

```
DSolve[y'[x] == x^2/(x^3+1)/y[x],y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to -\sqrt{\frac{2}{3}}\sqrt{\log\left(x^3+1\right)+3c_1}$$
$$y(x) \to \sqrt{\frac{2}{3}}\sqrt{\log\left(x^3+1\right)+3c_1}$$

2.3 problem 3

Internal problem ID [481]
Internal file name [`OUTPUT/481_Sunday_June_05_2022_01_42_10_AM_4186121/index.tex`]

Book: Elementary differential equations and boundary value problems, 10th ed., Boyce and DiPrima
Section: Section 2.2. Page 48
Problem number: 3.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"exact", "riccati", "separable", "first_order_ode_lie_symmetry_lookup"**

Maple gives the following as the ode type

`[_separable]`

$$\boxed{\sin(x)\, y^2 + y' = 0}$$

2.3.1 Solving as separable ode

In canonical form the ODE is

$$\begin{aligned} y' &= F(x,y) \\ &= f(x)g(y) \\ &= -\sin(x)\, y^2 \end{aligned}$$

Where $f(x) = -\sin(x)$ and $g(y) = y^2$. Integrating both sides gives

$$\frac{1}{y^2}\, dy = -\sin(x)\, dx$$

$$\int \frac{1}{y^2}\, dy = \int -\sin(x)\, dx$$

$$-\frac{1}{y} = \cos(x) + c_1$$

Which results in

$$y = -\frac{1}{\cos(x) + c_1}$$

Summary

The solution(s) found are the following

$$y = -\frac{1}{\cos(x) + c_1} \tag{1}$$

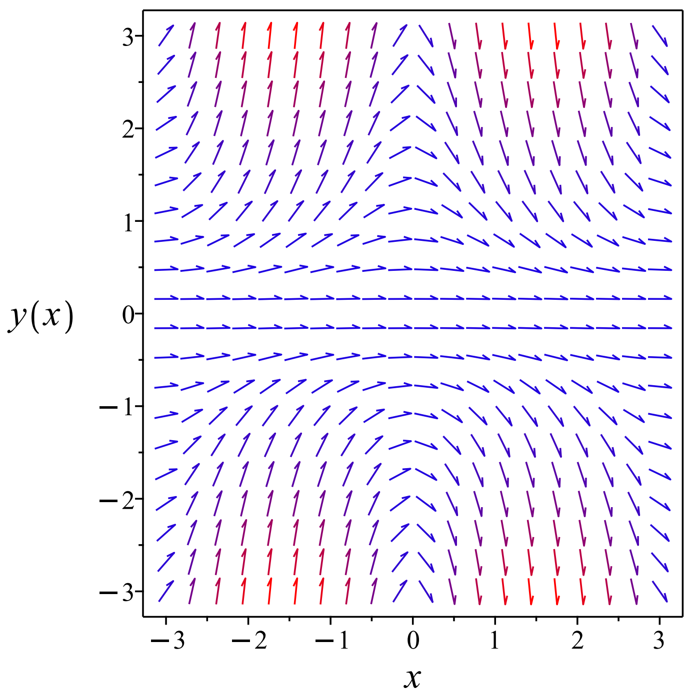


Figure 84: Slope field plot

Verification of solutions

$$y = -\frac{1}{\cos(x) + c_1}$$

Verified OK.

2.3.2 Solving as first order ode lie symmetry lookup ode

Writing the ode as

$$
\begin{aligned}
y' &= -\sin(x)\, y^2 \\
y' &= \omega(x, y)
\end{aligned}
$$

The condition of Lie symmetry is the linearized PDE given by

$$
\eta_x + \omega(\eta_y - \xi_x) - \omega^2 \xi_y - \omega_x \xi - \omega_y \eta = 0 \tag{A}
$$

The type of this ode is known. It is of type `separable`. Therefore we do not need to solve the PDE (A), and can just use the lookup table shown below to find ξ, η

Table 100: Lie symmetry infinitesimal lookup table for known first order ODE's

ODE class	Form	ξ	η
linear ode	$y' = f(x)y(x) + g(x)$	0	$e^{\int f dx}$
separable ode	$y' = f(x)\, g(y)$	$\frac{1}{f}$	0
quadrature ode	$y' = f(x)$	0	1
quadrature ode	$y' = g(y)$	1	0
homogeneous ODEs of Class A	$y' = f\left(\frac{y}{x}\right)$	x	y
homogeneous ODEs of Class C	$y' = (a + bx + cy)^{\frac{n}{m}}$	1	$-\frac{b}{c}$
homogeneous class D	$y' = \frac{y}{x} + g(x)\, F\left(\frac{y}{x}\right)$	x^2	xy
First order special form ID 1	$y' = g(x)\, e^{h(x)+by} + f(x)$	$\frac{e^{-\int bf(x)dx-h(x)}}{g(x)}$	$\frac{f(x)e^{-\int bf(x)dx-h(x)}}{g(x)}$
polynomial type ode	$y' = \frac{a_1x+b_1y+c_1}{a_2x+b_2y+c_2}$	$\frac{a_1b_2x-a_2b_1x-b_1c_2+b_2c_1}{a_1b_2-a_2b_1}$	$\frac{a_1b_2y-a_2b_1y-a_1c_2-a_2c_1}{a_1b_2-a_2b_1}$
Bernoulli ode	$y' = f(x)\, y + g(x)\, y^n$	0	$e^{-\int (n-1)f(x)dx} y^n$
Reduced Riccati	$y' = f_1(x)\, y + f_2(x)\, y^2$	0	$e^{-\int f_1 dx}$

The above table shows that

$$
\begin{aligned}
\xi(x,y) &= -\frac{1}{\sin(x)} \\
\eta(x,y) &= 0
\end{aligned} \tag{A1}
$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x,y) \to (R,S)$ where (R,S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$
\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}
$$

The above comes from the requirements that $\left(\xi\frac{\partial}{\partial x} + \eta\frac{\partial}{\partial y}\right) S(x,y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the canonical coordinates, where $S(R)$. Since $\eta = 0$ then in this special case

$$
R = y
$$

S is found from

$$
\begin{aligned}
S &= \int \frac{1}{\xi} dx \\
&= \int \frac{1}{-\frac{1}{\sin(x)}} dx
\end{aligned}
$$

Which results in

$$
S = \cos(x)
$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$
\frac{dS}{dR} = \frac{S_x + \omega(x,y)S_y}{R_x + \omega(x,y)R_y} \tag{2}
$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x,y)$ is the right hand side of the original ode given by

$$
\omega(x,y) = -\sin(x)\, y^2
$$

Evaluating all the partial derivatives gives

$$\begin{aligned}
R_x &= 0 \\
R_y &= 1 \\
S_x &= -\sin(x) \\
S_y &= 0
\end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = \frac{1}{y^2} \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = \frac{1}{R^2}$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$S(R) = -\frac{1}{R} + c_1 \tag{4}$$

To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$\cos(x) = -\frac{1}{y} + c_1$$

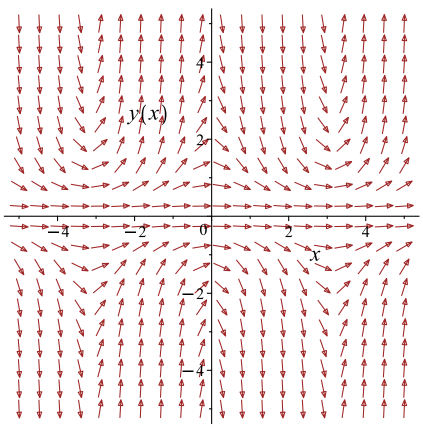
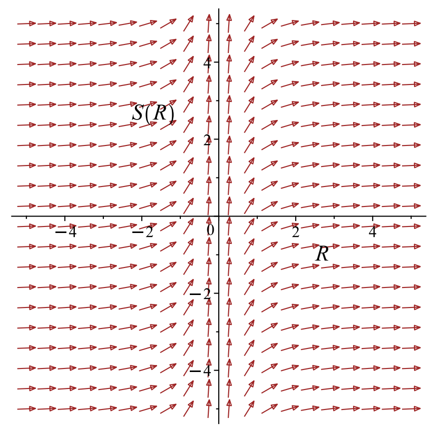
Which simplifies to

$$\cos(x) = -\frac{1}{y} + c_1$$

Which gives

$$y = -\frac{1}{\cos(x) - c_1}$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in x, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dx} = -\sin(x)\, y^2$ 	$R = y$ $S = \cos(x)$	$\frac{dS}{dR} = \frac{1}{R^2}$ 

Summary

The solution(s) found are the following

$$y = -\frac{1}{\cos(x) - c_1} \tag{1}$$

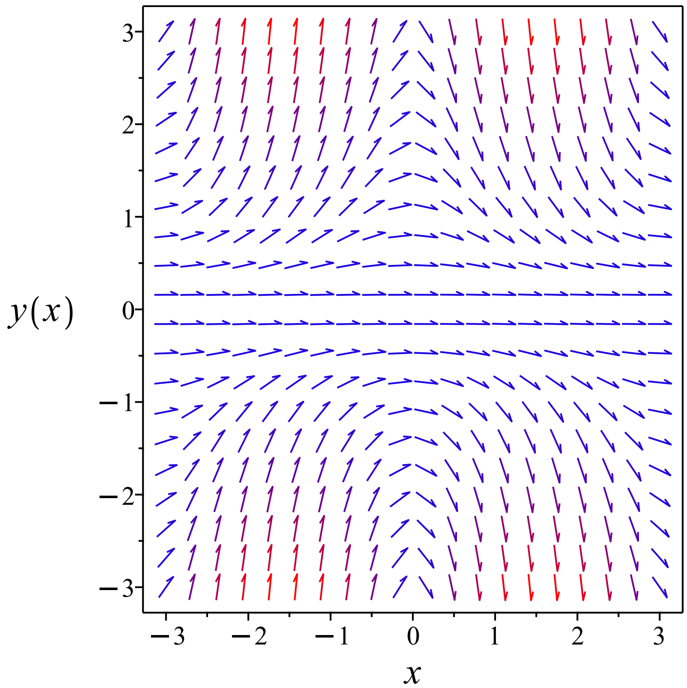


Figure 85: Slope field plot

Verification of solutions

$$y = -\frac{1}{\cos(x) - c_1}$$

Verified OK.

2.3.3 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x,y) + N(x,y)\frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x,y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y) = 0$$

Hence

$$\frac{\partial\phi}{\partial x} + \frac{\partial\phi}{\partial y}\frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$
\begin{aligned}
\frac{\partial \phi}{\partial x} &= M \\
\frac{\partial \phi}{\partial y} &= N
\end{aligned}
$$

But since $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ then for the above to be valid, we require that

$$
\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}
$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x, y)$ but at least we know now that we can do that since the condition $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$
M(x, y)\, \mathrm{d}x + N(x, y)\, \mathrm{d}y = 0 \tag{1A}
$$

Therefore

$$
\begin{aligned}
\left(-\frac{1}{y^2}\right) \mathrm{d}y &= (\sin(x))\, \mathrm{d}x \\
(-\sin(x))\, \mathrm{d}x + \left(-\frac{1}{y^2}\right) \mathrm{d}y &= 0 \qquad \text{(2A)}
\end{aligned}
$$

Comparing (1A) and (2A) shows that

$$
\begin{aligned}
M(x, y) &= -\sin(x) \\
N(x, y) &= -\frac{1}{y^2}
\end{aligned}
$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$
\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}
$$

Using result found above gives

$$
\begin{aligned}
\frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}(-\sin(x)) \\
&= 0
\end{aligned}
$$

And

$$\frac{\partial N}{\partial x} = \frac{\partial}{\partial x}\left(-\frac{1}{y^2}\right)$$
$$= 0$$

Since $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$, then the ODE is exact The following equations are now set up to solve for the function $\phi(x,y)$

$$\frac{\partial \phi}{\partial x} = M \tag{1}$$
$$\frac{\partial \phi}{\partial y} = N \tag{2}$$

Integrating (1) w.r.t. x gives

$$\int \frac{\partial \phi}{\partial x}\,\mathrm{d}x = \int M\,\mathrm{d}x$$
$$\int \frac{\partial \phi}{\partial x}\,\mathrm{d}x = \int -\sin(x)\,\mathrm{d}x$$
$$\phi = \cos(x) + f(y) \tag{3}$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$\frac{\partial \phi}{\partial y} = 0 + f'(y) \tag{4}$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = -\frac{1}{y^2}$. Therefore equation (4) becomes

$$-\frac{1}{y^2} = 0 + f'(y) \tag{5}$$

Solving equation (5) for $f'(y)$ gives

$$f'(y) = -\frac{1}{y^2}$$

Integrating the above w.r.t y gives

$$\int f'(y)\,\mathrm{d}y = \int \left(-\frac{1}{y^2}\right)\mathrm{d}y$$
$$f(y) = \frac{1}{y} + c_1$$

Where c_1 is constant of integration. Substituting result found above for $f(y)$ into equation (3) gives ϕ

$$\phi = \cos(x) + \frac{1}{y} + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = \cos(x) + \frac{1}{y}$$

The solution becomes

$$y = -\frac{1}{\cos(x) - c_1}$$

Summary

The solution(s) found are the following

$$y = -\frac{1}{\cos(x) - c_1} \tag{1}$$

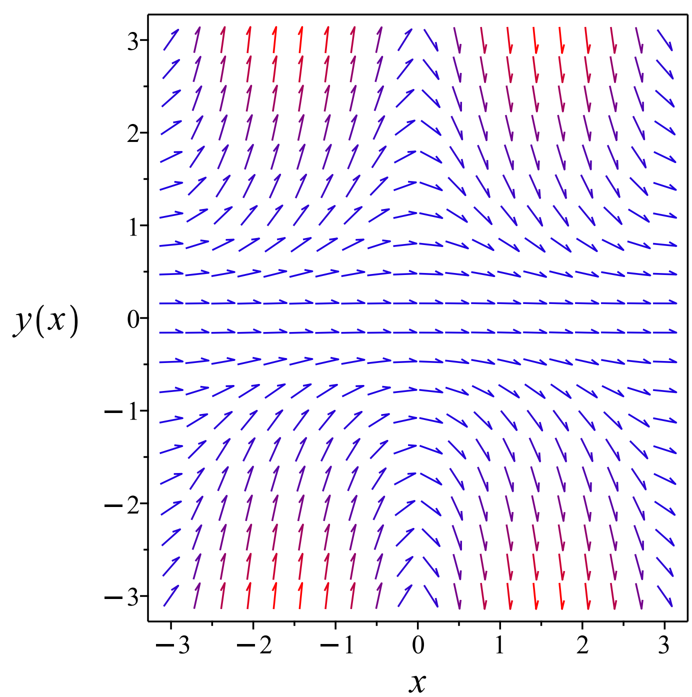


Figure 86: Slope field plot

Verification of solutions

$$y = -\frac{1}{\cos(x) - c_1}$$

Verified OK.

2.3.4 Solving as riccati ode

In canonical form the ODE is

$$\begin{aligned} y' &= F(x,y) \\ &= -\sin(x)\, y^2 \end{aligned}$$

This is a Riccati ODE. Comparing the ODE to solve

$$y' = -\sin(x)\, y^2$$

With Riccati ODE standard form

$$y' = f_0(x) + f_1(x)y + f_2(x)y^2$$

Shows that $f_0(x) = 0$, $f_1(x) = 0$ and $f_2(x) = -\sin(x)$. Let

$$\begin{aligned} y &= \frac{-u'}{f_2 u} \\ &= \frac{-u'}{-\sin(x)\, u} \end{aligned} \tag{1}$$

Using the above substitution in the given ODE results (after some simplification)in a second order ODE to solve for $u(x)$ which is

$$f_2 u''(x) - (f_2' + f_1 f_2)\, u'(x) + f_2^2 f_0 u(x) = 0 \tag{2}$$

But

$$\begin{aligned} f_2' &= -\cos(x) \\ f_1 f_2 &= 0 \\ f_2^2 f_0 &= 0 \end{aligned}$$

Substituting the above terms back in equation (2) gives

$$-\sin(x)\, u''(x) + \cos(x)\, u'(x) = 0$$

Solving the above ODE (this ode solved using Maple, not this program), gives

$$u(x) = c_1 + \cos(x) c_2$$

The above shows that

$$u'(x) = -c_2 \sin(x)$$

Using the above in (1) gives the solution

$$y = -\frac{c_2}{c_1 + \cos(x) c_2}$$

Dividing both numerator and denominator by c_1 gives, after renaming the constant $\frac{c_2}{c_1} = c_3$ the following solution

$$y = -\frac{1}{c_3 + \cos(x)}$$

Summary

The solution(s) found are the following

$$y = -\frac{1}{c_3 + \cos(x)} \tag{1}$$

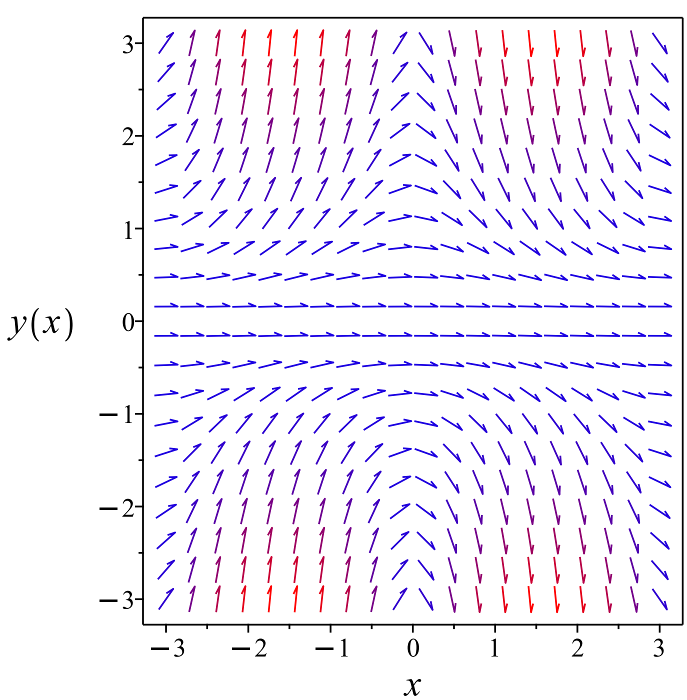


Figure 87: Slope field plot

Verification of solutions

$$y = -\frac{1}{c_3 + \cos(x)}$$

Verified OK.

2.3.5 Maple step by step solution

Let's solve

$\sin(x)\, y^2 + y' = 0$

- Highest derivative means the order of the ODE is 1

  y'

- Separate variables

  $\frac{y'}{y^2} = -\sin(x)$

- Integrate both sides with respect to x

  $\int \frac{y'}{y^2} dx = \int -\sin(x)\, dx + c_1$

- Evaluate integral

  $-\frac{1}{y} = \cos(x) + c_1$

- Solve for y

  $y = -\frac{1}{\cos(x) + c_1}$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
<- Bernoulli successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 12

```
dsolve(sin(x)*y(x)^2+diff(y(x),x) = 0,y(x), singsol=all)
```

$$y(x) = \frac{1}{-\cos(x) + c_1}$$

✓ Solution by Mathematica

Time used: 0.133 (sec). Leaf size: 19

```
DSolve[Sin[x]*y[x]^2+y'[x]== 0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to -\frac{1}{\cos(x) + c_1}$$
$$y(x) \to 0$$

2.4 problem 4

Internal problem ID [482]
Internal file name [`OUTPUT/482_Sunday_June_05_2022_01_42_11_AM_75746612/index.tex`]

Book: Elementary differential equations and boundary value problems, 10th ed., Boyce and DiPrima
Section: Section 2.2. Page 48
Problem number: 4.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"exact", "separable", "differentialType", "first_order_ode_lie_symmetry_lookup"**

Maple gives the following as the ode type

`[_separable]`

$$\boxed{y' - \frac{3x^2 - 1}{3 + 2y} = 0}$$

2.4.1 Solving as separable ode

In canonical form the ODE is

$$\begin{aligned} y' &= F(x,y) \\ &= f(x)g(y) \\ &= \frac{3x^2 - 1}{3 + 2y} \end{aligned}$$

Where $f(x) = 3x^2 - 1$ and $g(y) = \frac{1}{3+2y}$. Integrating both sides gives

$$\frac{1}{\frac{1}{3+2y}}\, dy = 3x^2 - 1\, dx$$

$$\int \frac{1}{\frac{1}{3+2y}} dy = \int 3x^2 - 1 \, dx$$
$$y^2 + 3y = x^3 + c_1 - x$$

Which results in

$$y = -\frac{3}{2} + \frac{\sqrt{4x^3 + 4c_1 - 4x + 9}}{2}$$
$$y = -\frac{3}{2} - \frac{\sqrt{4x^3 + 4c_1 - 4x + 9}}{2}$$

Summary

The solution(s) found are the following

$$y = -\frac{3}{2} + \frac{\sqrt{4x^3 + 4c_1 - 4x + 9}}{2} \tag{1}$$
$$y = -\frac{3}{2} - \frac{\sqrt{4x^3 + 4c_1 - 4x + 9}}{2} \tag{2}$$

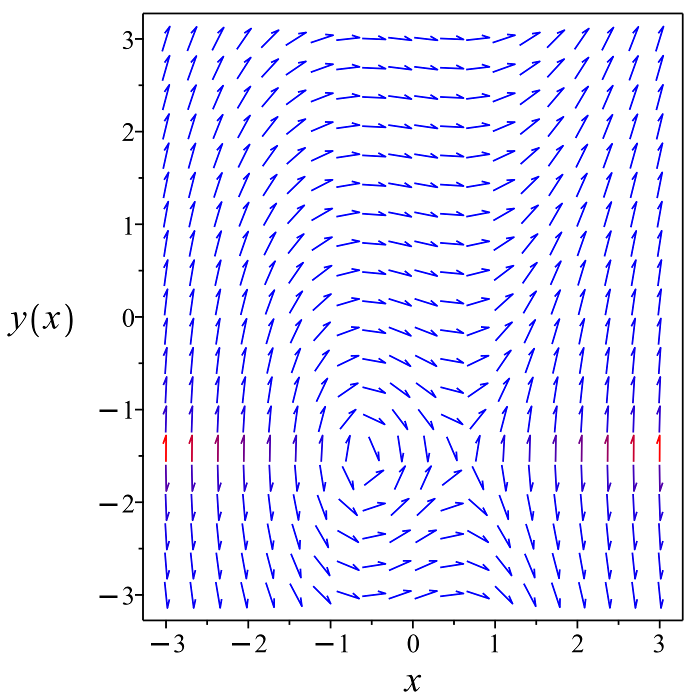


Figure 88: Slope field plot

Verification of solutions

$$y = -\frac{3}{2} + \frac{\sqrt{4x^3 + 4c_1 - 4x + 9}}{2}$$

Verified OK.

$$y = -\frac{3}{2} - \frac{\sqrt{4x^3 + 4c_1 - 4x + 9}}{2}$$

Verified OK.

2.4.2 Solving as differentialType ode

Writing the ode as

$$y' = \frac{3x^2 - 1}{3 + 2y} \tag{1}$$

Which becomes

$$(3 + 2y)\, dy = (3x^2 - 1)\, dx \tag{2}$$

But the RHS is complete differential because

$$(3x^2 - 1)\, dx = d(x^3 - x)$$

Hence (2) becomes

$$(3 + 2y)\, dy = d(x^3 - x)$$

Integrating both sides gives gives these solutions

$$y = -\frac{3}{2} + \frac{\sqrt{4x^3 + 4c_1 - 4x + 9}}{2} + c_1$$
$$y = -\frac{3}{2} - \frac{\sqrt{4x^3 + 4c_1 - 4x + 9}}{2} + c_1$$

Summary
The solution(s) found are the following

$$y = -\frac{3}{2} + \frac{\sqrt{4x^3 + 4c_1 - 4x + 9}}{2} + c_1 \tag{1}$$
$$y = -\frac{3}{2} - \frac{\sqrt{4x^3 + 4c_1 - 4x + 9}}{2} + c_1 \tag{2}$$

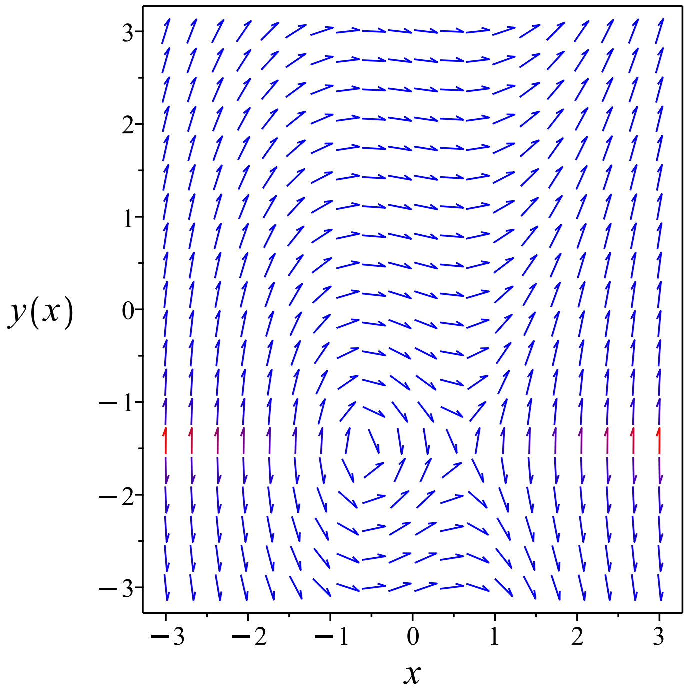


Figure 89: Slope field plot

Verification of solutions

$$y = -\frac{3}{2} + \frac{\sqrt{4x^3 + 4c_1 - 4x + 9}}{2} + c_1$$

Verified OK.

$$y = -\frac{3}{2} - \frac{\sqrt{4x^3 + 4c_1 - 4x + 9}}{2} + c_1$$

Verified OK.

2.4.3 Solving as first order ode lie symmetry lookup ode

Writing the ode as

$$y' = \frac{3x^2 - 1}{3 + 2y}$$
$$y' = \omega(x, y)$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x + \omega(\eta_y - \xi_x) - \omega^2\xi_y - \omega_x\xi - \omega_y\eta = 0 \tag{A}$$

The type of this ode is known. It is of type `separable`. Therefore we do not need to solve the PDE (A), and can just use the lookup table shown below to find ξ, η

Table 103: Lie symmetry infinitesimal lookup table for known first order ODE's

ODE class	Form	ξ	η
linear ode	$y' = f(x)y(x) + g(x)$	0	$e^{\int f dx}$
separable ode	$y' = f(x)\, g(y)$	$\frac{1}{f}$	0
quadrature ode	$y' = f(x)$	0	1
quadrature ode	$y' = g(y)$	1	0
homogeneous ODEs of Class A	$y' = f\left(\frac{y}{x}\right)$	x	y
homogeneous ODEs of Class C	$y' = (a + bx + cy)^{\frac{n}{m}}$	1	$-\frac{b}{c}$
homogeneous class D	$y' = \frac{y}{x} + g(x)\, F\left(\frac{y}{x}\right)$	x^2	xy
First order special form ID 1	$y' = g(x)\, e^{h(x)+by} + f(x)$	$\frac{e^{-\int bf(x)dx - h(x)}}{g(x)}$	$\frac{f(x)e^{-\int bf(x)dx - h(x)}}{g(x)}$
polynomial type ode	$y' = \frac{a_1x+b_1y+c_1}{a_2x+b_2y+c_2}$	$\frac{a_1b_2x - a_2b_1x - b_1c_2 + b_2c_1}{a_1b_2 - a_2b_1}$	$\frac{a_1b_2y - a_2b_1y - a_1c_2 - a_2c_1}{a_1b_2 - a_2b_1}$
Bernoulli ode	$y' = f(x)\, y + g(x)\, y^n$	0	$e^{-\int (n-1)f(x)dx} y^n$
Reduced Riccati	$y' = f_1(x)\, y + f_2(x)\, y^2$	0	$e^{-\int f_1 dx}$

The above table shows that

$$
\begin{aligned}
\xi(x, y) &= \frac{1}{3x^2 - 1} \\
\eta(x, y) &= 0
\end{aligned} \tag{A1}
$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x, y) \to (R, S)$ where (R, S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$
\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}
$$

The above comes from the requirements that $\left(\xi\frac{\partial}{\partial x}+\eta\frac{\partial}{\partial y}\right)S(x,y)=1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the canonical coordinates, where $S(R)$. Since $\eta=0$ then in this special case

$$R = y$$

S is found from

$$\begin{aligned}S &= \int \frac{1}{\xi}dx \\ &= \int \frac{1}{\frac{1}{3x^2-1}}dx\end{aligned}$$

Which results in

$$S = x^3 - x$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x,y)S_y}{R_x + \omega(x,y)R_y} \tag{2}$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x,y)$ is the right hand side of the original ode given by

$$\omega(x,y) = \frac{3x^2-1}{3+2y}$$

Evaluating all the partial derivatives gives

$$\begin{aligned}R_x &= 0 \\ R_y &= 1 \\ S_x &= 3x^2-1 \\ S_y &= 0\end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = 3 + 2y \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = 3 + 2R$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordinates R, S. Integrating the above gives

$$S(R) = R^2 + 3R + c_1 \tag{4}$$

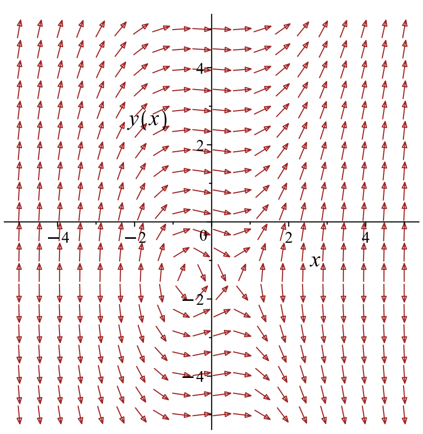
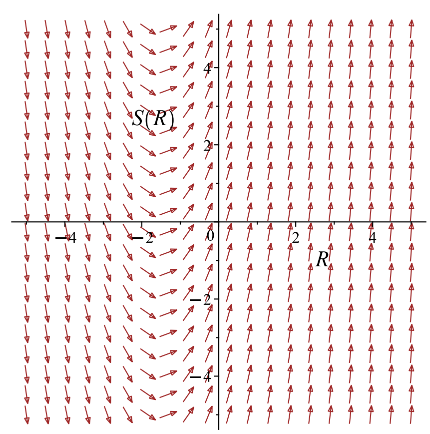
To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$x^3 - x = y^2 + c_1 + 3y$$

Which simplifies to

$$x^3 - x = y^2 + c_1 + 3y$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in x, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dx} = \frac{3x^2-1}{3+2y}$ 	$R = y$ $S = x^3 - x$	$\frac{dS}{dR} = 3 + 2R$ 

Summary

The solution(s) found are the following

$$x^3 - x = y^2 + c_1 + 3y \tag{1}$$

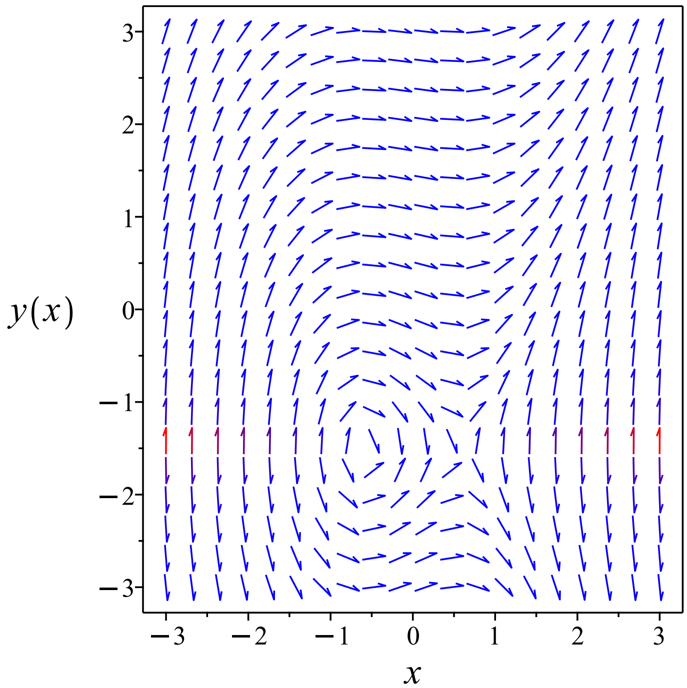


Figure 90: Slope field plot

Verification of solutions

$$x^3 - x = y^2 + c_1 + 3y$$

Verified OK.

2.4.4 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x, y) + N(x, y)\frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x, y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x, y) = 0$$

Hence

$$\frac{\partial \phi}{\partial x} + \frac{\partial \phi}{\partial y}\frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$\frac{\partial \phi}{\partial x} = M$$
$$\frac{\partial \phi}{\partial y} = N$$

But since $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x,y)$ but at least we know now that we can do that since the condition $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x,y)\,\mathrm{d}x + N(x,y)\,\mathrm{d}y = 0 \tag{1A}$$

Therefore

$$(3+2y)\,\mathrm{d}y = \left(3x^2-1\right)\mathrm{d}x$$
$$\left(-3x^2+1\right)\mathrm{d}x + (3+2y)\,\mathrm{d}y = 0 \tag{2A}$$

Comparing (1A) and (2A) shows that

$$M(x,y) = -3x^2+1$$
$$N(x,y) = 3+2y$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Using result found above gives

$$\frac{\partial M}{\partial y} = \frac{\partial}{\partial y}\left(-3x^2+1\right)$$
$$= 0$$

And

$$\frac{\partial N}{\partial x} = \frac{\partial}{\partial x}(3+2y)$$
$$= 0$$

Since $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$, then the ODE is exact The following equations are now set up to solve for the function $\phi(x, y)$

$$\frac{\partial \phi}{\partial x} = M \tag{1}$$
$$\frac{\partial \phi}{\partial y} = N \tag{2}$$

Integrating (1) w.r.t. x gives

$$\int \frac{\partial \phi}{\partial x} \, \mathrm{d}x = \int M \, \mathrm{d}x$$
$$\int \frac{\partial \phi}{\partial x} \, \mathrm{d}x = \int -3x^2 + 1 \, \mathrm{d}x$$
$$\phi = -x^3 + x + f(y) \tag{3}$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$\frac{\partial \phi}{\partial y} = 0 + f'(y) \tag{4}$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = 3 + 2y$. Therefore equation (4) becomes

$$3 + 2y = 0 + f'(y) \tag{5}$$

Solving equation (5) for $f'(y)$ gives

$$f'(y) = 3 + 2y$$

Integrating the above w.r.t y gives

$$\int f'(y) \, \mathrm{d}y = \int (3 + 2y) \, \mathrm{d}y$$
$$f(y) = y^2 + 3y + c_1$$

Where c_1 is constant of integration. Substituting result found above for $f(y)$ into equation (3) gives ϕ

$$\phi = -x^3 + y^2 + x + 3y + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = -x^3 + y^2 + x + 3y$$

Summary
The solution(s) found are the following

$$-x^3 + y^2 + 3y + x = c_1 \tag{1}$$

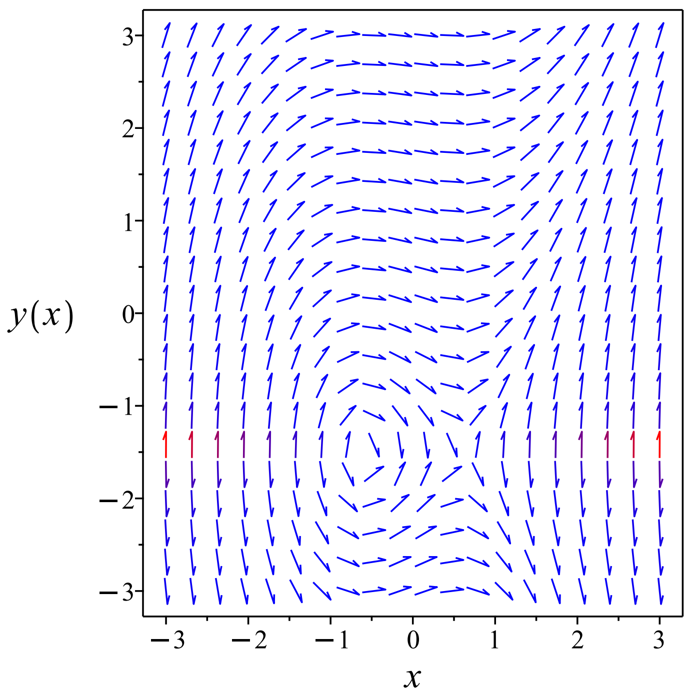


Figure 91: Slope field plot

Verification of solutions

$$-x^3 + y^2 + 3y + x = c_1$$

Verified OK.

2.4.5 Maple step by step solution

Let's solve

$y' - \frac{3x^2-1}{3+2y} = 0$

- Highest derivative means the order of the ODE is 1

  y'

- Separate variables

  $(3+2y)\, y' = 3x^2 - 1$

- Integrate both sides with respect to x

  $\int (3+2y)\, y' dx = \int (3x^2-1)\, dx + c_1$

- Evaluate integral

  $y^2 + 3y = x^3 + c_1 - x$

- Solve for y

  $\left\{y = -\frac{3}{2} - \frac{\sqrt{4x^3+4c_1-4x+9}}{2}, y = -\frac{3}{2} + \frac{\sqrt{4x^3+4c_1-4x+9}}{2}\right\}$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
<- separable successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 45

```
dsolve(diff(y(x),x) = (3*x^2-1)/(3+2*y(x)),y(x), singsol=all)
```

$$y(x) = -\frac{3}{2} - \frac{\sqrt{4x^3+4c_1-4x+9}}{2}$$
$$y(x) = -\frac{3}{2} + \frac{\sqrt{4x^3+4c_1-4x+9}}{2}$$

✓ Solution by Mathematica

Time used: 0.121 (sec). Leaf size: 59

```
DSolve[y'[x] == (3*x^2-1)/(3+2*y[x]),y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \frac{1}{2}\left(-3-\sqrt{4x^3-4x+9+4c_1}\right)$$
$$y(x) \to \frac{1}{2}\left(-3+\sqrt{4x^3-4x+9+4c_1}\right)$$

2.5 problem 5

Internal problem ID [483]
Internal file name [OUTPUT/483_Sunday_June_05_2022_01_42_12_AM_72792709/index.tex]

Book: Elementary differential equations and boundary value problems, 10th ed., Boyce and DiPrima
Section: Section 2.2. Page 48
Problem number: 5.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"exact", "separable", "first_order_ode_lie_symmetry_lookup"**

Maple gives the following as the ode type

`[_separable]`

$$\boxed{y' - \cos(x)^2 \cos(2y)^2 = 0}$$

2.5.1 Solving as separable ode

In canonical form the ODE is

$$\begin{aligned} y' &= F(x,y) \\ &= f(x)g(y) \\ &= \cos(x)^2 \cos(2y)^2 \end{aligned}$$

Where $f(x) = \cos(x)^2$ and $g(y) = \cos(2y)^2$. Integrating both sides gives

$$\frac{1}{\cos(2y)^2}\,dy = \cos(x)^2\,dx$$

$$\int \frac{1}{\cos(2y)^2}\,dy = \int \cos(x)^2\,dx$$

$$\frac{\tan(2y)}{2} = \frac{\cos(x)\sin(x)}{2} + \frac{x}{2} + c_1$$

Which results in

$$y = \frac{\arctan(\cos(x)\sin(x) + 2c_1 + x)}{2}$$

Summary

The solution(s) found are the following

$$y = \frac{\arctan(\cos(x)\sin(x) + 2c_1 + x)}{2} \tag{1}$$

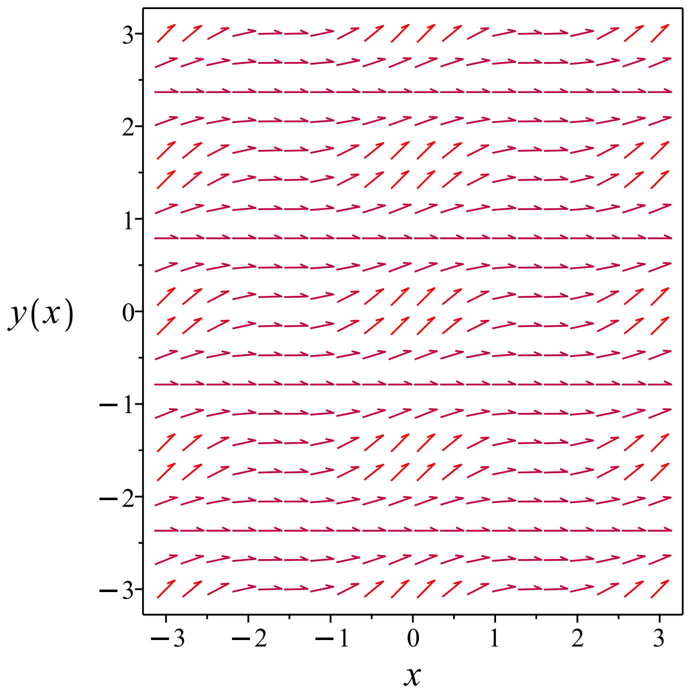


Figure 92: Slope field plot

Verification of solutions

$$y = \frac{\arctan(\cos(x)\sin(x) + 2c_1 + x)}{2}$$

Verified OK.

2.5.2 Solving as first order ode lie symmetry lookup ode

Writing the ode as

$$
\begin{aligned}
y' &= \cos(x)^2 \cos(2y)^2 \\
y' &= \omega(x, y)
\end{aligned}
$$

The condition of Lie symmetry is the linearized PDE given by

$$
\eta_x + \omega(\eta_y - \xi_x) - \omega^2 \xi_y - \omega_x \xi - \omega_y \eta = 0 \tag{A}
$$

The type of this ode is known. It is of type `separable`. Therefore we do not need to solve the PDE (A), and can just use the lookup table shown below to find ξ, η

Table 106: Lie symmetry infinitesimal lookup table for known first order ODE's

ODE class	Form	ξ	η
linear ode	$y' = f(x)y(x) + g(x)$	0	$e^{\int f dx}$
separable ode	$y' = f(x)\, g(y)$	$\frac{1}{f}$	0
quadrature ode	$y' = f(x)$	0	1
quadrature ode	$y' = g(y)$	1	0
homogeneous ODEs of Class A	$y' = f\left(\frac{y}{x}\right)$	x	y
homogeneous ODEs of Class C	$y' = (a + bx + cy)^{\frac{n}{m}}$	1	$-\frac{b}{c}$
homogeneous class D	$y' = \frac{y}{x} + g(x)\, F\left(\frac{y}{x}\right)$	x^2	xy
First order special form ID 1	$y' = g(x)\, e^{h(x)+by} + f(x)$	$\frac{e^{-\int bf(x)dx - h(x)}}{g(x)}$	$\frac{f(x)e^{-\int bf(x)dx - h(x)}}{g(x)}$
polynomial type ode	$y' = \frac{a_1x+b_1y+c_1}{a_2x+b_2y+c_2}$	$\frac{a_1b_2x - a_2b_1x - b_1c_2 + b_2c_1}{a_1b_2 - a_2b_1}$	$\frac{a_1b_2y - a_2b_1y - a_1c_2 - a_2c_1}{a_1b_2 - a_2b_1}$
Bernoulli ode	$y' = f(x)\, y + g(x)\, y^n$	0	$e^{-\int (n-1)f(x)dx} y^n$
Reduced Riccati	$y' = f_1(x)\, y + f_2(x)\, y^2$	0	$e^{-\int f_1 dx}$

The above table shows that

$$\xi(x, y) = \frac{1}{\cos(x)^2}$$
$$\eta(x, y) = 0 \tag{A1}$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x, y) \to (R, S)$ where (R, S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y}\right) S(x, y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the canonical coordinates, where $S(R)$. Since $\eta = 0$ then in this special case

$$R = y$$

S is found from

$$\begin{aligned} S &= \int \frac{1}{\xi} dx \\ &= \int \frac{1}{\frac{1}{\cos(x)^2}} dx \end{aligned}$$

Which results in

$$S = \frac{\cos(x)\sin(x)}{2} + \frac{x}{2}$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x, y) S_y}{R_x + \omega(x, y) R_y} \tag{2}$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x, y)$ is the right hand side of the original ode given by

$$\omega(x, y) = \cos(x)^2 \cos(2y)^2$$

Evaluating all the partial derivatives gives

$$
\begin{aligned}
R_x &= 0 \\
R_y &= 1 \\
S_x &= \frac{\cos(2x)}{2} + \frac{1}{2} \\
S_y &= 0
\end{aligned}
$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$
\frac{dS}{dR} = \sec(2y)^2 \tag{2A}
$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$
\frac{dS}{dR} = \sec(2R)^2
$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$
S(R) = \frac{\tan(2R)}{2} + c_1 \tag{4}
$$

To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$
\frac{\sin(2x)}{4} + \frac{x}{2} = \frac{\tan(2y)}{2} + c_1
$$

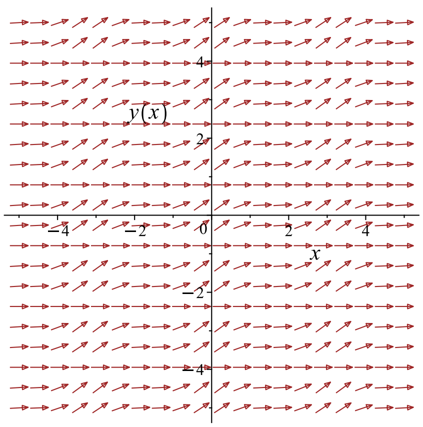
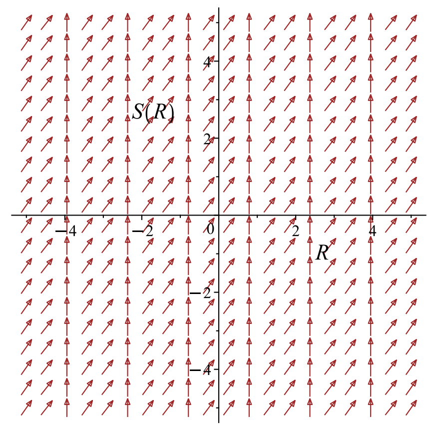
Which simplifies to

$$
\frac{\sin(2x)}{4} + \frac{x}{2} = \frac{\tan(2y)}{2} + c_1
$$

Which gives

$$
y = -\frac{\arctan\left(-\frac{\sin(2x)}{2} - x + 2c_1\right)}{2}
$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in x, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dx} = \cos(x)^2 \cos(2y)^2$ 	$R = y$ $S = \frac{\sin(2x)}{4} + \frac{x}{2}$	$\frac{dS}{dR} = \sec(2R)^2$ 

Summary

The solution(s) found are the following

$$y = -\frac{\arctan\left(-\frac{\sin(2x)}{2} - x + 2c_1\right)}{2} \tag{1}$$

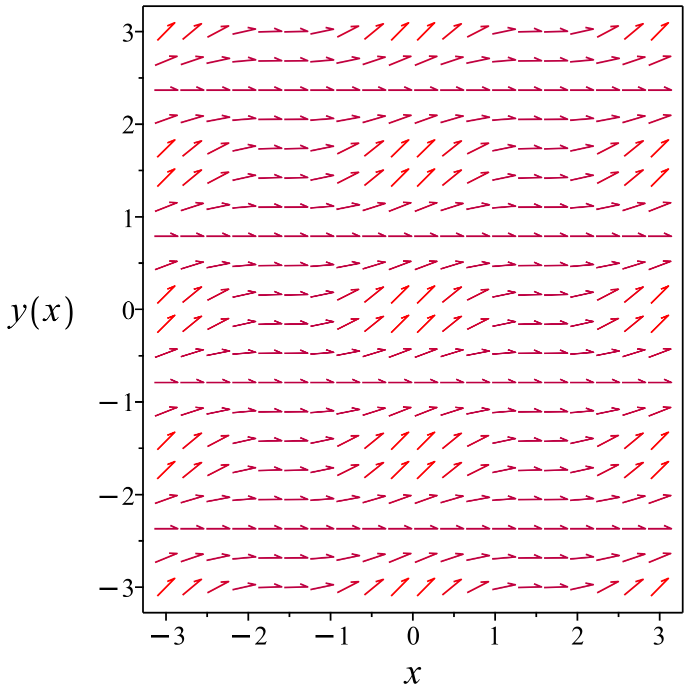


Figure 93: Slope field plot

Verification of solutions

$$y = -\frac{\arctan\left(-\frac{\sin(2x)}{2} - x + 2c_1\right)}{2}$$

Verified OK.

2.5.3 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x,y) + N(x,y)\frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x,y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y) = 0$$

Hence

$$\frac{\partial\phi}{\partial x} + \frac{\partial\phi}{\partial y}\frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$\frac{\partial \phi}{\partial x} = M$$
$$\frac{\partial \phi}{\partial y} = N$$

But since $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x, y)$ but at least we know now that we can do that since the condition $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x, y)\,\mathrm{d}x + N(x, y)\,\mathrm{d}y = 0 \tag{1A}$$

Therefore

$$\left(\frac{1}{\cos(2y)^2}\right)\mathrm{d}y = \left(\cos(x)^2\right)\mathrm{d}x$$
$$\left(-\cos(x)^2\right)\mathrm{d}x + \left(\frac{1}{\cos(2y)^2}\right)\mathrm{d}y = 0 \tag{2A}$$

Comparing (1A) and (2A) shows that

$$M(x, y) = -\cos(x)^2$$
$$N(x, y) = \frac{1}{\cos(2y)^2}$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Using result found above gives

$$\frac{\partial M}{\partial y} = \frac{\partial}{\partial y}\left(-\cos(x)^2\right)$$
$$= 0$$

And

$$
\begin{aligned}
\frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}\left(\frac{1}{\cos(2y)^2}\right) \\
&= 0
\end{aligned}
$$

Since $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$, then the ODE is exact The following equations are now set up to solve for the function $\phi(x,y)$

$$
\begin{aligned}
\frac{\partial \phi}{\partial x} &= M \qquad (1) \\
\frac{\partial \phi}{\partial y} &= N \qquad (2)
\end{aligned}
$$

Integrating (1) w.r.t. x gives

$$
\begin{aligned}
\int \frac{\partial \phi}{\partial x}\,\mathrm{d}x &= \int M\,\mathrm{d}x \\
\int \frac{\partial \phi}{\partial x}\,\mathrm{d}x &= \int -\cos(x)^2\,\mathrm{d}x \\
\phi &= -\frac{\sin(2x)}{4} - \frac{x}{2} + f(y) \qquad (3)
\end{aligned}
$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$
\frac{\partial \phi}{\partial y} = 0 + f'(y) \qquad (4)
$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = \frac{1}{\cos(2y)^2}$. Therefore equation (4) becomes

$$
\frac{1}{\cos(2y)^2} = 0 + f'(y) \qquad (5)
$$

Solving equation (5) for $f'(y)$ gives

$$
f'(y) = \frac{1}{\cos(2y)^2}
$$

Integrating the above w.r.t y gives

$$\int f'(y)\,\mathrm{d}y = \int \left(\sec(2y)^2\right) \mathrm{d}y$$

$$f(y) = \frac{\tan(2y)}{2} + c_1$$

Where c_1 is constant of integration. Substituting result found above for $f(y)$ into equation (3) gives ϕ

$$\phi = -\frac{\sin(2x)}{4} - \frac{x}{2} + \frac{\tan(2y)}{2} + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = -\frac{\sin(2x)}{4} - \frac{x}{2} + \frac{\tan(2y)}{2}$$

Summary

The solution(s) found are the following

$$-\frac{x}{2} - \frac{\sin(2x)}{4} + \frac{\tan(2y)}{2} = c_1 \tag{1}$$

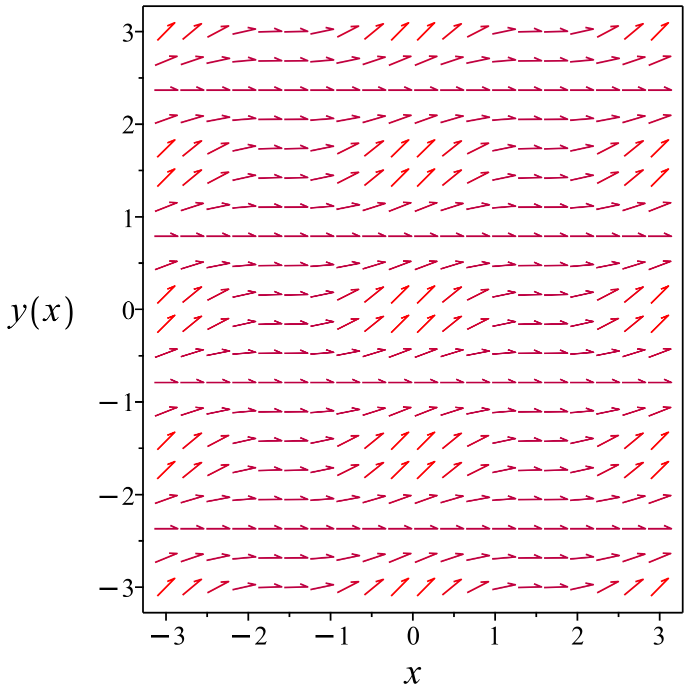


Figure 94: Slope field plot

Verification of solutions

$$-\frac{x}{2} - \frac{\sin(2x)}{4} + \frac{\tan(2y)}{2} = c_1$$

Verified OK.

2.5.4 Maple step by step solution

Let's solve

$y' - \cos(x)^2 \cos(2y)^2 = 0$

- Highest derivative means the order of the ODE is 1

  y'

- Separate variables

  $\frac{y'}{\cos(2y)^2} = \cos(x)^2$

- Integrate both sides with respect to x

  $\int \frac{y'}{\cos(2y)^2} dx = \int \cos(x)^2 dx + c_1$

- Evaluate integral

$$\frac{\tan(2y)}{2} = \frac{\cos(x)\sin(x)}{2} + \frac{x}{2} + c_1$$

- Solve for y

$$y = \frac{\arctan(\cos(x)\sin(x) + 2c_1 + x)}{2}$$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
<- separable successful`
```

✓ Solution by Maple

Time used: 0.015 (sec). Leaf size: 18

```
dsolve(diff(y(x),x) = cos(x)^2*cos(2*y(x))^2,y(x), singsol=all)
```

$$y(x) = \frac{\arctan\left(x + 2c_1 + \frac{\sin(2x)}{2}\right)}{2}$$

✓ Solution by Mathematica

Time used: 1.312 (sec). Leaf size: 63

```
DSolve[y'[x] == Cos[x]^2*Cos[2*y[x]]^2,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \frac{1}{2}\arctan\left(x + \sin(x)\cos(x) + \frac{c_1}{4}\right)$$
$$y(x) \to \frac{1}{2}\arctan\left(x + \sin(x)\cos(x) + \frac{c_1}{4}\right)$$
$$y(x) \to -\frac{\pi}{4}$$
$$y(x) \to \frac{\pi}{4}$$

2.6 problem 6

Internal problem ID [484]
Internal file name [`OUTPUT/484_Sunday_June_05_2022_01_42_13_AM_90091053/index.tex`]

Book: Elementary differential equations and boundary value problems, 10th ed., Boyce and DiPrima
Section: Section 2.2. Page 48
Problem number: 6.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"exact", "separable", "first_order_ode_lie_symmetry_lookup"**

Maple gives the following as the ode type

`[_separable]`

$$\boxed{y'x - \sqrt{1 - y^2} = 0}$$

2.6.1 Solving as separable ode

In canonical form the ODE is

$$\begin{aligned} y' &= F(x, y) \\ &= f(x)g(y) \\ &= \frac{\sqrt{-y^2 + 1}}{x} \end{aligned}$$

Where $f(x) = \frac{1}{x}$ and $g(y) = \sqrt{-y^2 + 1}$. Integrating both sides gives

$$\frac{1}{\sqrt{-y^2 + 1}}\, dy = \frac{1}{x}\, dx$$

$$\int \frac{1}{\sqrt{-y^2 + 1}}\, dy = \int \frac{1}{x}\, dx$$

$$\arcsin(y) = \ln(x) + c_1$$

Which results in

$$y = \sin(\ln(x) + c_1)$$

Summary

The solution(s) found are the following

$$y = \sin(\ln(x) + c_1) \tag{1}$$

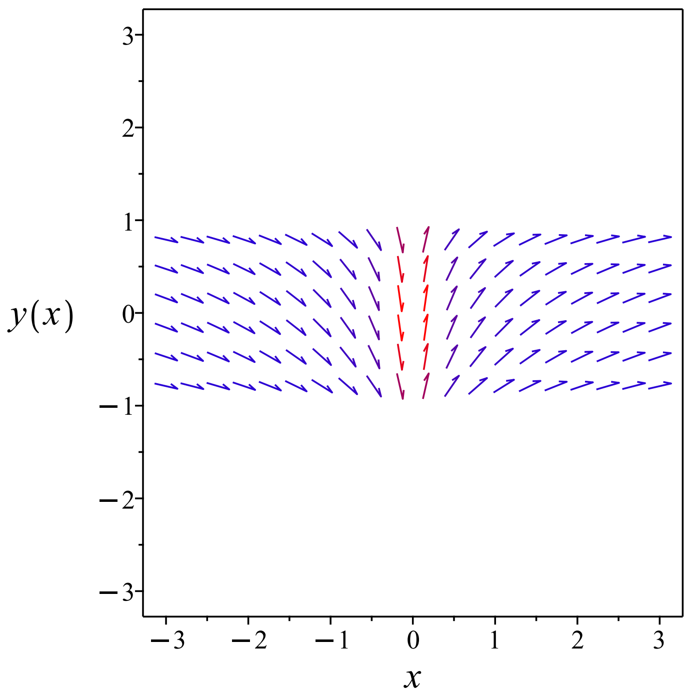


Figure 95: Slope field plot

Verification of solutions

$$y = \sin(\ln(x) + c_1)$$

Verified OK.

2.6.2 Solving as first order ode lie symmetry lookup ode

Writing the ode as

$$
\begin{aligned}
y' &= \frac{\sqrt{-y^2+1}}{x} \\
y' &= \omega(x,y)
\end{aligned}
$$

The condition of Lie symmetry is the linearized PDE given by

$$
\eta_x + \omega(\eta_y - \xi_x) - \omega^2\xi_y - \omega_x\xi - \omega_y\eta = 0 \tag{A}
$$

The type of this ode is known. It is of type `separable`. Therefore we do not need to solve the PDE (A), and can just use the lookup table shown below to find ξ, η

Table 109: Lie symmetry infinitesimal lookup table for known first order ODE's

ODE class	Form	ξ	η
linear ode	$y' = f(x)y(x) + g(x)$	0	$e^{\int f dx}$
separable ode	$y' = f(x)\, g(y)$	$\frac{1}{f}$	0
quadrature ode	$y' = f(x)$	0	1
quadrature ode	$y' = g(y)$	1	0
homogeneous ODEs of Class A	$y' = f\left(\frac{y}{x}\right)$	x	y
homogeneous ODEs of Class C	$y' = (a + bx + cy)^{\frac{n}{m}}$	1	$-\frac{b}{c}$
homogeneous class D	$y' = \frac{y}{x} + g(x)\, F\left(\frac{y}{x}\right)$	x^2	xy
First order special form ID 1	$y' = g(x)\, e^{h(x)+by} + f(x)$	$\frac{e^{-\int bf(x)dx - h(x)}}{g(x)}$	$\frac{f(x)e^{-\int bf(x)dx - h(x)}}{g(x)}$
polynomial type ode	$y' = \frac{a_1x+b_1y+c_1}{a_2x+b_2y+c_2}$	$\frac{a_1b_2x - a_2b_1x - b_1c_2 + b_2c_1}{a_1b_2 - a_2b_1}$	$\frac{a_1b_2y - a_2b_1y - a_1c_2 - a_2c_1}{a_1b_2 - a_2b_1}$
Bernoulli ode	$y' = f(x)\, y + g(x)\, y^n$	0	$e^{-\int (n-1)f(x)dx} y^n$
Reduced Riccati	$y' = f_1(x)\, y + f_2(x)\, y^2$	0	$e^{-\int f_1 dx}$

The above table shows that

$$
\begin{aligned}
\xi(x, y) &= x \\
\eta(x, y) &= 0
\end{aligned}
\tag{A1}
$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x, y) \to (R, S)$ where (R, S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$
\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}
$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y}\right) S(x, y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the canonical coordinates, where $S(R)$. Since $\eta = 0$ then in this special case

$$
R = y
$$

S is found from

$$
\begin{aligned}
S &= \int \frac{1}{\xi} dx \\
&= \int \frac{1}{x} dx
\end{aligned}
$$

Which results in

$$
S = \ln(x)
$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$
\frac{dS}{dR} = \frac{S_x + \omega(x, y) S_y}{R_x + \omega(x, y) R_y} \tag{2}
$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x, y)$ is the right hand side of the original ode given by

$$
\omega(x, y) = \frac{\sqrt{-y^2 + 1}}{x}
$$

Evaluating all the partial derivatives gives

$$
\begin{aligned}
R_x &= 0 \\
R_y &= 1 \\
S_x &= \frac{1}{x} \\
S_y &= 0
\end{aligned}
$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$
\frac{dS}{dR} = \frac{1}{\sqrt{-y^2 + 1}} \tag{2A}
$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$
\frac{dS}{dR} = \frac{1}{\sqrt{-R^2 + 1}}
$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$
S(R) = \arcsin(R) + c_1 \tag{4}
$$

To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$
\ln(x) = \arcsin(y) + c_1
$$

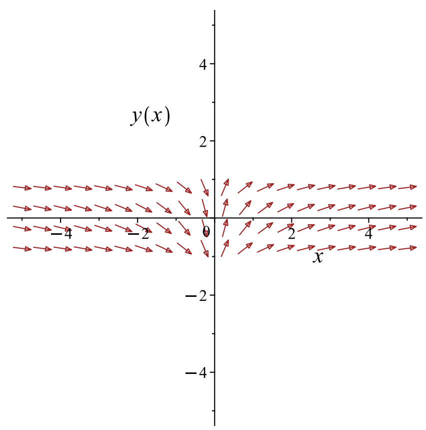
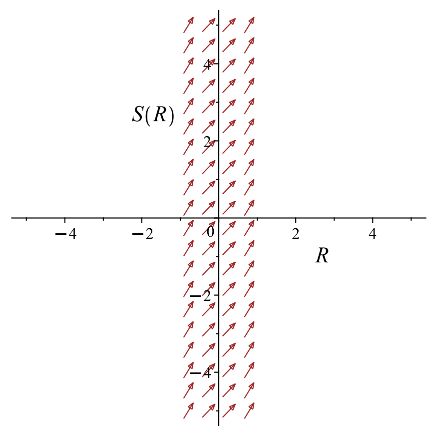
Which simplifies to

$$
\ln(x) = \arcsin(y) + c_1
$$

Which gives

$$
y = -\sin(-\ln(x) + c_1)
$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in x, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dx} = \frac{\sqrt{-y^2+1}}{x}$ 	$R = y$ $S = \ln(x)$	$\frac{dS}{dR} = \frac{1}{\sqrt{-R^2+1}}$ 

Summary

The solution(s) found are the following

$$y = -\sin(-\ln(x) + c_1) \tag{1}$$

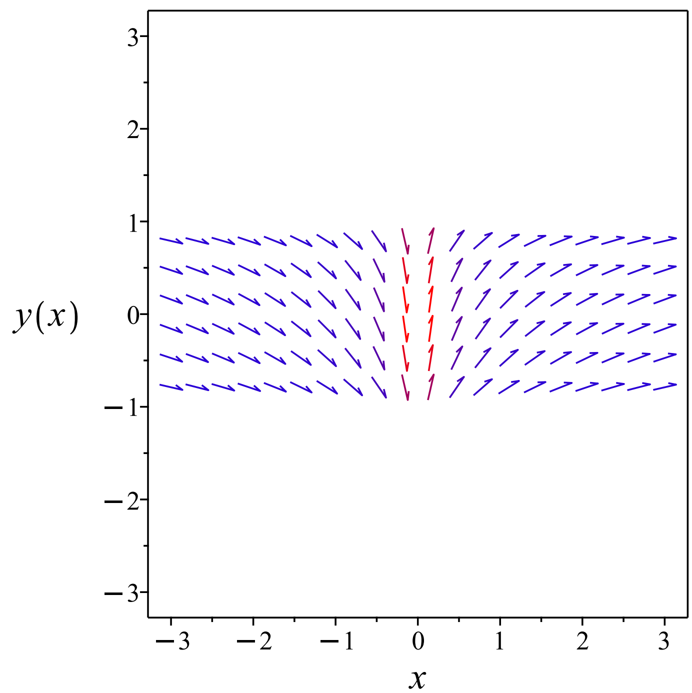


Figure 96: Slope field plot

Verification of solutions

$$y = -\sin(-\ln(x) + c_1)$$

Verified OK.

2.6.3 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x, y) + N(x, y)\frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x, y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x, y) = 0$$

Hence

$$\frac{\partial \phi}{\partial x} + \frac{\partial \phi}{\partial y}\frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$\frac{\partial \phi}{\partial x} = M$$
$$\frac{\partial \phi}{\partial y} = N$$

But since $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x, y)$ but at least we know now that we can do that since the condition $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x, y)\, \mathrm{d}x + N(x, y)\, \mathrm{d}y = 0 \tag{1A}$$

Therefore

$$\left(\frac{1}{\sqrt{-y^2 + 1}}\right) \mathrm{d}y = \left(\frac{1}{x}\right) \mathrm{d}x$$
$$\left(-\frac{1}{x}\right) \mathrm{d}x + \left(\frac{1}{\sqrt{-y^2 + 1}}\right) \mathrm{d}y = 0 \tag{2A}$$

Comparing (1A) and (2A) shows that

$$M(x, y) = -\frac{1}{x}$$
$$N(x, y) = \frac{1}{\sqrt{-y^2 + 1}}$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Using result found above gives

$$\frac{\partial M}{\partial y} = \frac{\partial}{\partial y}\left(-\frac{1}{x}\right)$$
$$= 0$$

And

$$\begin{aligned}\frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}\left(\frac{1}{\sqrt{-y^2+1}}\right)\\ &= 0\end{aligned}$$

Since $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$, then the ODE is <u>exact</u> The following equations are now set up to solve for the function $\phi(x, y)$

$$\frac{\partial \phi}{\partial x} = M \tag{1}$$

$$\frac{\partial \phi}{\partial y} = N \tag{2}$$

Integrating (1) w.r.t. x gives

$$\begin{aligned}\int \frac{\partial \phi}{\partial x}\, \mathrm{d}x &= \int M\, \mathrm{d}x\\ \int \frac{\partial \phi}{\partial x}\, \mathrm{d}x &= \int -\frac{1}{x}\, \mathrm{d}x\end{aligned}$$

$$\phi = -\ln(x) + f(y) \tag{3}$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$\frac{\partial \phi}{\partial y} = 0 + f'(y) \tag{4}$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = \frac{1}{\sqrt{-y^2+1}}$. Therefore equation (4) becomes

$$\frac{1}{\sqrt{-y^2+1}} = 0 + f'(y) \tag{5}$$

Solving equation (5) for $f'(y)$ gives

$$f'(y) = \frac{1}{\sqrt{-y^2+1}}$$

Integrating the above w.r.t y gives

$$\begin{aligned}\int f'(y)\, \mathrm{d}y &= \int \left(\frac{1}{\sqrt{-y^2+1}}\right) \mathrm{d}y\\ f(y) &= \arcsin(y) + c_1\end{aligned}$$

Where c_1 is constant of integration. Substituting result found above for $f(y)$ into equation (3) gives ϕ

$$\phi = -\ln(x) + \arcsin(y) + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = -\ln(x) + \arcsin(y)$$

The solution becomes

$$y = \sin(\ln(x) + c_1)$$

Summary

The solution(s) found are the following

$$y = \sin(\ln(x) + c_1) \tag{1}$$

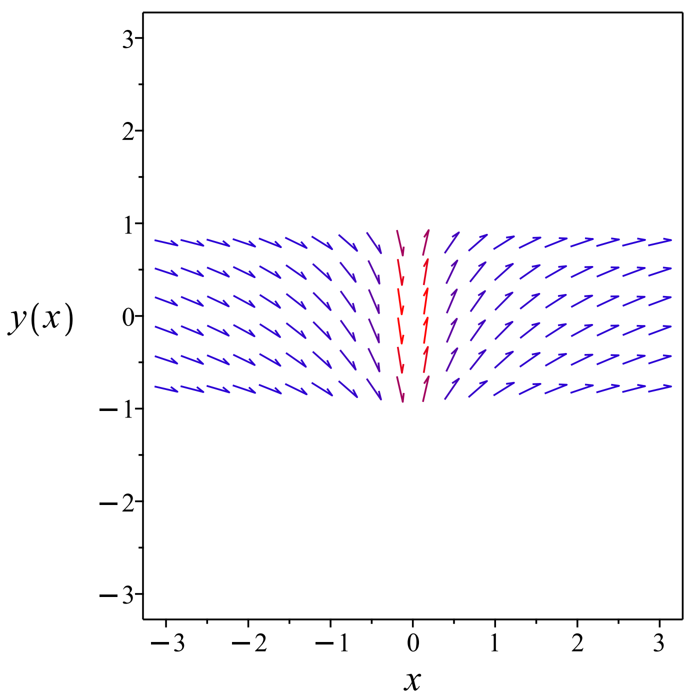


Figure 97: Slope field plot

Verification of solutions

$$y = \sin(\ln(x) + c_1)$$

Verified OK.

2.6.4 Maple step by step solution

Let's solve

$y'x - \sqrt{1 - y^2} = 0$

- Highest derivative means the order of the ODE is 1

  y'

- Separate variables

  $\frac{y'}{\sqrt{1-y^2}} = \frac{1}{x}$

- Integrate both sides with respect to x

  $\int \frac{y'}{\sqrt{1-y^2}} dx = \int \frac{1}{x} dx + c_1$

- Evaluate integral

  $\arcsin(y) = \ln(x) + c_1$

- Solve for y

  $y = \sin(\ln(x) + c_1)$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
<- separable successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 9

```
dsolve(x*diff(y(x),x) = (1-y(x)^2)^(1/2),y(x), singsol=all)
```

$$y(x) = \sin(\ln(x) + c_1)$$

✓ Solution by Mathematica

Time used: 0.206 (sec). Leaf size: 29

```
DSolve[x*y'[x] == (1-y[x]^2)^(1/2),y[x],x,IncludeSingularSolutions -> True]
```

$$
\begin{aligned}
y(x) &\to \cos(\log(x) + c_1) \\
y(x) &\to -1 \\
y(x) &\to 1 \\
y(x) &\to \text{Interval}[\{-1, 1\}]
\end{aligned}
$$

2.7 problem 7

Internal problem ID [485]
Internal file name [`OUTPUT/485_Sunday_June_05_2022_01_42_14_AM_30092807/index.tex`]

Book: Elementary differential equations and boundary value problems, 10th ed., Boyce and DiPrima
Section: Section 2.2. Page 48
Problem number: 7.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"unknown"**

Maple gives the following as the ode type

`` [`y=_G(x,y')`] ``

Unable to solve or complete the solution.

$$\boxed{y' - \frac{-\mathrm{e}^{-x} + x}{\mathrm{e}^{y} + x} = 0}$$

Unable to determine ODE type.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
trying homogeneous types:
trying Chini
differential order: 1; looking for linear symmetries
trying exact
Looking for potential symmetries
trying inverse_Riccati
trying an equivalence to an Abel ODE
differential order: 1; trying a linearization to 2nd order
--- trying a change of variables {x -> y(x), y(x) -> x}
differential order: 1; trying a linearization to 2nd order
trying 1st order ODE linearizable_by_differentiation
--- Trying Lie symmetry methods, 1st order ---
`, `-> Computing symmetries using: way = 3
`, `-> Computing symmetries using: way = 4
`, `-> Computing symmetries using: way = 5
trying symmetry patterns for 1st order ODEs
-> trying a symmetry pattern of the form [F(x)*G(y), 0]
-> trying a symmetry pattern of the form [0, F(x)*G(y)]
-> trying symmetry patterns of the forms [F(x),G(y)] and [G(y),F(x)]
`, `-> Computing symmetries using: way = HINT
   -> Calling odsolve with the ODE`, diff(y(x), x) = -y(x)/x, y(x)`      *** Sublevel 2 ***
      Methods for first order ODEs:
      --- Trying classification methods ---
      trying a quadrature
      trying 1st order linear
      <- 1st order linear successful
   -> Calling odsolve with the ODE`, diff(y(x), x), y(x)`      *** Sublevel 2 ***
      Methods for first order ODEs:
      --- Trying classification methods ---
      trying a quadrature
      trying 1st order linear
      <- 1st order linear successful
   -> Calling odsolve with the ODE`, diff(y(x), x)+y(x), y(x)`      *** Sublevel 2 ***
      Methods for first order ODEs:
      --- Trying classification methods ---
      trying a quadrature
      trying 1st order linear
```

✗ Solution by Maple

```
dsolve(diff(y(x),x) = (-exp(-x)+x)/(exp(y(x))+x),y(x), singsol=all)
```

No solution found

✗ Solution by Mathematica

Time used: 0.0 (sec). Leaf size: 0

```
DSolve[y'[x] == (-Exp[-x]+x)/(Exp[y[x]]+x),y[x],x,IncludeSingularSolutions -> True]
```

Not solved

2.8 problem 8

Internal problem ID [486]
Internal file name [`OUTPUT/486_Sunday_June_05_2022_01_42_16_AM_5493618/index.tex`]

Book: Elementary differential equations and boundary value problems, 10th ed., Boyce and DiPrima
Section: Section 2.2. Page 48
Problem number: 8.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"exact", "separable", "differentialType", "first_order_ode_lie_symmetry_lookup"**

Maple gives the following as the ode type

`[_separable]`

$$\boxed{y' - \frac{x^2}{1 + y^2} = 0}$$

2.8.1 Solving as separable ode

In canonical form the ODE is

$$\begin{aligned} y' &= F(x,y) \\ &= f(x)g(y) \\ &= \frac{x^2}{y^2+1} \end{aligned}$$

Where $f(x) = x^2$ and $g(y) = \frac{1}{y^2+1}$. Integrating both sides gives

$$\frac{1}{\frac{1}{y^2+1}}\, dy = x^2\, dx$$

$$\int \frac{1}{\frac{1}{y^2+1}}\,dy = \int x^2\,dx$$

$$\frac{1}{3}y^3 + y = \frac{x^3}{3} + c_1$$

Which results in

$$y = \frac{\left(4x^3 + 12c_1 + 4\sqrt{x^6 + 6c_1x^3 + 9c_1^2 + 4}\right)^{\frac{1}{3}}}{2} - \frac{2}{\left(4x^3 + 12c_1 + 4\sqrt{x^6 + 6c_1x^3 + 9c_1^2 + 4}\right)^{\frac{1}{3}}}$$

$$y = -\frac{\left(4x^3 + 12c_1 + 4\sqrt{x^6 + 6c_1x^3 + 9c_1^2 + 4}\right)^{\frac{1}{3}}}{4} + \frac{1}{\left(4x^3 + 12c_1 + 4\sqrt{x^6 + 6c_1x^3 + 9c_1^2 + 4}\right)^{\frac{1}{3}}} + \frac{i\sqrt{3}\left(\frac{\left(4x^3+12c_1+4\sqrt{x^6+6c_1x^3+9c_1^2+4}\right)^{\frac{1}{3}}}{2} + \frac{2}{\left(4x^3+12c_1+4\sqrt{x^6+6c_1x^3+9c_1^2+4}\right)^{\frac{1}{3}}}\right)}{2}$$

$$y = -\frac{\left(4x^3 + 12c_1 + 4\sqrt{x^6 + 6c_1x^3 + 9c_1^2 + 4}\right)^{\frac{1}{3}}}{4} + \frac{1}{\left(4x^3 + 12c_1 + 4\sqrt{x^6 + 6c_1x^3 + 9c_1^2 + 4}\right)^{\frac{1}{3}}} - \frac{i\sqrt{3}\left(\frac{\left(4x^3+12c_1+4\sqrt{x^6+6c_1x^3+9c_1^2+4}\right)^{\frac{1}{3}}}{2} + \frac{2}{\left(4x^3+12c_1+4\sqrt{x^6+6c_1x^3+9c_1^2+4}\right)^{\frac{1}{3}}}\right)}{2}$$

Summary
The solution(s) found are the following

$$y = \frac{\left(4x^3 + 12c_1 + 4\sqrt{x^6 + 6c_1x^3 + 9c_1^2 + 4}\right)^{\frac{1}{3}}}{2} - \frac{2}{\left(4x^3 + 12c_1 + 4\sqrt{x^6 + 6c_1x^3 + 9c_1^2 + 4}\right)^{\frac{1}{3}}} \tag{1}$$

$$y = -\frac{\left(4x^3 + 12c_1 + 4\sqrt{x^6 + 6c_1x^3 + 9c_1^2 + 4}\right)^{\frac{1}{3}}}{4} + \frac{1}{\left(4x^3 + 12c_1 + 4\sqrt{x^6 + 6c_1x^3 + 9c_1^2 + 4}\right)^{\frac{1}{3}}} + \frac{i\sqrt{3}\left(\frac{\left(4x^3+12c_1+4\sqrt{x^6+6c_1x^3+9c_1^2+4}\right)^{\frac{1}{3}}}{2} + \frac{2}{\left(4x^3+12c_1+4\sqrt{x^6+6c_1x^3+9c_1^2+4}\right)^{\frac{1}{3}}}\right)}{2} \tag{2}$$

$$y = -\frac{\left(4x^3 + 12c_1 + 4\sqrt{x^6 + 6c_1x^3 + 9c_1^2 + 4}\right)^{\frac{1}{3}}}{4} + \frac{1}{\left(4x^3 + 12c_1 + 4\sqrt{x^6 + 6c_1x^3 + 9c_1^2 + 4}\right)^{\frac{1}{3}}} - \frac{i\sqrt{3}\left(\frac{\left(4x^3+12c_1+4\sqrt{x^6+6c_1x^3+9c_1^2+4}\right)^{\frac{1}{3}}}{2} + \frac{2}{\left(4x^3+12c_1+4\sqrt{x^6+6c_1x^3+9c_1^2+4}\right)^{\frac{1}{3}}}\right)}{2} \tag{3}$$

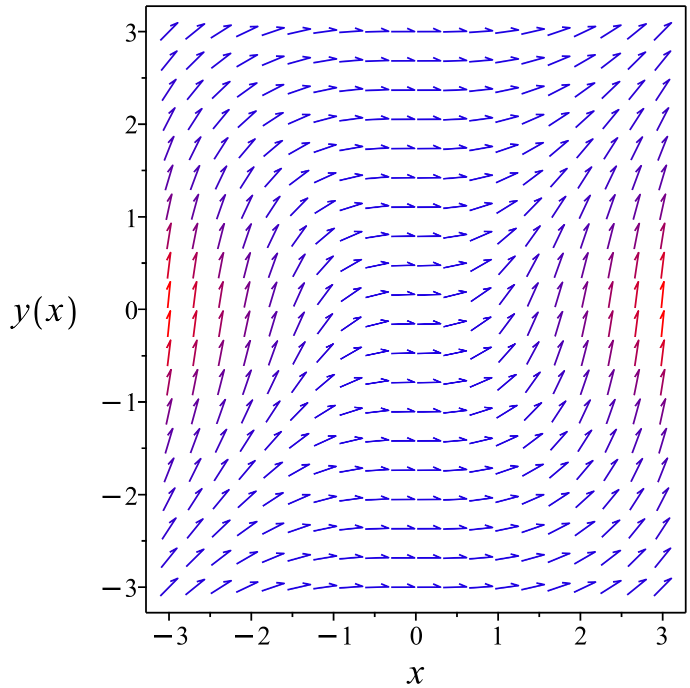


Figure 98: Slope field plot

Verification of solutions

$$y = \frac{\left(4x^3+12c_1+4\sqrt{x^6+6c_1x^3+9c_1^2+4}\right)^{\frac{1}{3}}}{2} - \frac{2}{\left(4x^3+12c_1+4\sqrt{x^6+6c_1x^3+9c_1^2+4}\right)^{\frac{1}{3}}}$$

Verified OK.

$$y = -\frac{\left(4x^3+12c_1+4\sqrt{x^6+6c_1x^3+9c_1^2+4}\right)^{\frac{1}{3}}}{4} + \frac{1}{\left(4x^3+12c_1+4\sqrt{x^6+6c_1x^3+9c_1^2+4}\right)^{\frac{1}{3}}} + \frac{i\sqrt{3}\left(\frac{\left(4x^3+12c_1+4\sqrt{x^6+6c_1x^3+9c_1^2+4}\right)^{\frac{1}{3}}}{2} + \frac{2}{\left(4x^3+12c_1+4\sqrt{x^6+6c_1x^3+9c_1^2+4}\right)^{\frac{1}{3}}}\right)}{2}$$

Verified OK.

$$y = -\frac{\left(4x^3+12c_1+4\sqrt{x^6+6c_1x^3+9c_1^2+4}\right)^{\frac{1}{3}}}{4} + \frac{1}{\left(4x^3+12c_1+4\sqrt{x^6+6c_1x^3+9c_1^2+4}\right)^{\frac{1}{3}}} - \frac{i\sqrt{3}\left(\frac{\left(4x^3+12c_1+4\sqrt{x^6+6c_1x^3+9c_1^2+4}\right)^{\frac{1}{3}}}{2} + \frac{2}{\left(4x^3+12c_1+4\sqrt{x^6+6c_1x^3+9c_1^2+4}\right)^{\frac{1}{3}}}\right)}{2}$$

Verified OK.

2.8.2 Solving as differentialType ode

Writing the ode as

$$y' = \frac{x^2}{1+y^2} \tag{1}$$

Which becomes

$$\left(y^2+1\right) dy = \left(x^2\right) dx \tag{2}$$

But the RHS is complete differential because

$$\left(x^{2}\right) dx = d\left(\frac{x^{3}}{3}\right)$$

Hence (2) becomes

$$\left(y^{2}+1\right) dy = d\left(\frac{x^{3}}{3}\right)$$

Integrating both sides gives gives these solutions

$$y = \frac{\left(4x^{3}+12c_{1}+4\sqrt{x^{6}+6c_{1}x^{3}+9c_{1}^{2}+4}\right)^{\frac{1}{3}}}{2} - \frac{2}{\left(4x^{3}+12c_{1}+4\sqrt{x^{6}+6c_{1}x^{3}+9c_{1}^{2}+4}\right)^{\frac{1}{3}}} + c_{1}$$

$$y = -\frac{\left(4x^{3}+12c_{1}+4\sqrt{x^{6}+6c_{1}x^{3}+9c_{1}^{2}+4}\right)^{\frac{1}{3}}}{4} + \frac{1}{\left(4x^{3}+12c_{1}+4\sqrt{x^{6}+6c_{1}x^{3}+9c_{1}^{2}+4}\right)^{\frac{1}{3}}} + \frac{i\sqrt{3}\,\Bigg(\Big(}{}$$

$$y = -\frac{\left(4x^{3}+12c_{1}+4\sqrt{x^{6}+6c_{1}x^{3}+9c_{1}^{2}+4}\right)^{\frac{1}{3}}}{4} + \frac{1}{\left(4x^{3}+12c_{1}+4\sqrt{x^{6}+6c_{1}x^{3}+9c_{1}^{2}+4}\right)^{\frac{1}{3}}} - \frac{i\sqrt{3}\,\Bigg(\Big(}{}$$

Summary

The solution(s) found are the following

$$y = \frac{\left(4x^3+12c_1+4\sqrt{x^6+6c_1x^3+9c_1^2+4}\right)^{\frac{1}{3}}}{2} - \frac{2}{\left(4x^3+12c_1+4\sqrt{x^6+6c_1x^3+9c_1^2+4}\right)^{\frac{1}{3}}} + c_1 \tag{1}$$

$$y = -\frac{\left(4x^3+12c_1+4\sqrt{x^6+6c_1x^3+9c_1^2+4}\right)^{\frac{1}{3}}}{4} + \frac{1}{\left(4x^3+12c_1+4\sqrt{x^6+6c_1x^3+9c_1^2+4}\right)^{\frac{1}{3}}} + \frac{i\sqrt{3}\left(\frac{\left(4x^3+12c_1+4\sqrt{x^6+6c_1x^3+9c_1^2+4}\right)^{\frac{1}{3}}}{2} + \frac{2}{\left(4x^3+12c_1+4\sqrt{x^6+6c_1x^3+9c_1^2+4}\right)^{\frac{1}{3}}}\right)}{2} + c_1 \tag{2}$$

$$y = -\frac{\left(4x^3+12c_1+4\sqrt{x^6+6c_1x^3+9c_1^2+4}\right)^{\frac{1}{3}}}{4} + \frac{1}{\left(4x^3+12c_1+4\sqrt{x^6+6c_1x^3+9c_1^2+4}\right)^{\frac{1}{3}}} - \frac{i\sqrt{3}\left(\frac{\left(4x^3+12c_1+4\sqrt{x^6+6c_1x^3+9c_1^2+4}\right)^{\frac{1}{3}}}{2} + \frac{2}{\left(4x^3+12c_1+4\sqrt{x^6+6c_1x^3+9c_1^2+4}\right)^{\frac{1}{3}}}\right)}{2} + c_1 \tag{3}$$

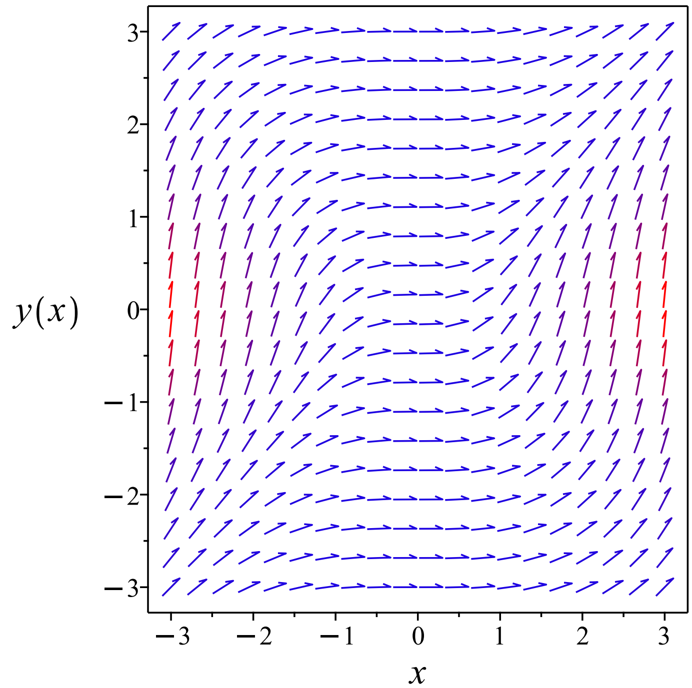


Figure 99: Slope field plot

Verification of solutions

$$y=\frac{\left(4x^{3}+12c_{1}+4\sqrt{x^{6}+6c_{1}x^{3}+9c_{1}^{2}+4}\right)^{\frac{1}{3}}}{2}-\frac{2}{\left(4x^{3}+12c_{1}+4\sqrt{x^{6}+6c_{1}x^{3}+9c_{1}^{2}+4}\right)^{\frac{1}{3}}}+c_{1}$$

Verified OK.

$$y=-\frac{\left(4x^{3}+12c_{1}+4\sqrt{x^{6}+6c_{1}x^{3}+9c_{1}^{2}+4}\right)^{\frac{1}{3}}}{4}+\frac{1}{\left(4x^{3}+12c_{1}+4\sqrt{x^{6}+6c_{1}x^{3}+9c_{1}^{2}+4}\right)^{\frac{1}{3}}}+\frac{i\sqrt{3}\left(\frac{\left(4x^{3}+12c_{1}+4\sqrt{x^{6}+6c_{1}x^{3}+9c_{1}^{2}+4}\right)^{\frac{1}{3}}}{2}+\frac{2}{\left(4x^{3}+12c_{1}+4\sqrt{x^{6}+6c_{1}x^{3}+9c_{1}^{2}+4}\right)^{\frac{1}{3}}}\right)}{2}+c_{1}$$

Verified OK.

$$y=-\frac{\left(4x^{3}+12c_{1}+4\sqrt{x^{6}+6c_{1}x^{3}+9c_{1}^{2}+4}\right)^{\frac{1}{3}}}{4}+\frac{1}{\left(4x^{3}+12c_{1}+4\sqrt{x^{6}+6c_{1}x^{3}+9c_{1}^{2}+4}\right)^{\frac{1}{3}}}-\frac{i\sqrt{3}\left(\frac{\left(4x^{3}+12c_{1}+4\sqrt{x^{6}+6c_{1}x^{3}+9c_{1}^{2}+4}\right)^{\frac{1}{3}}}{2}+\frac{2}{\left(4x^{3}+12c_{1}+4\sqrt{x^{6}+6c_{1}x^{3}+9c_{1}^{2}+4}\right)^{\frac{1}{3}}}\right)}{2}+c_{1}$$

Verified OK.

2.8.3 Solving as first order ode lie symmetry lookup ode

Writing the ode as

$$y'=\frac{x^{2}}{y^{2}+1}$$
$$y'=\omega(x,y)$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x + \omega(\eta_y - \xi_x) - \omega^2 \xi_y - \omega_x \xi - \omega_y \eta = 0 \tag{A}$$

The type of this ode is known. It is of type `separable`. Therefore we do not need to solve the PDE (A), and can just use the lookup table shown below to find ξ, η

Table 112: Lie symmetry infinitesimal lookup table for known first order ODE's

ODE class	Form	ξ	η
linear ode	$y' = f(x)y(x) + g(x)$	0	$e^{\int f dx}$
separable ode	$y' = f(x)\, g(y)$	$\frac{1}{f}$	0
quadrature ode	$y' = f(x)$	0	1
quadrature ode	$y' = g(y)$	1	0
homogeneous ODEs of Class A	$y' = f\left(\frac{y}{x}\right)$	x	y
homogeneous ODEs of Class C	$y' = (a + bx + cy)^{\frac{n}{m}}$	1	$-\frac{b}{c}$
homogeneous class D	$y' = \frac{y}{x} + g(x)\, F\left(\frac{y}{x}\right)$	x^2	xy
First order special form ID 1	$y' = g(x)\, e^{h(x)+by} + f(x)$	$\frac{e^{-\int bf(x)dx-h(x)}}{g(x)}$	$\frac{f(x)e^{-\int bf(x)dx-h(x)}}{g(x)}$
polynomial type ode	$y' = \frac{a_1x+b_1y+c_1}{a_2x+b_2y+c_2}$	$\frac{a_1b_2x-a_2b_1x-b_1c_2+b_2c_1}{a_1b_2-a_2b_1}$	$\frac{a_1b_2y-a_2b_1y-a_1c_2-a_2c_1}{a_1b_2-a_2b_1}$
Bernoulli ode	$y' = f(x)\, y + g(x)\, y^n$	0	$e^{-\int (n-1)f(x)dx} y^n$
Reduced Riccati	$y' = f_1(x)\, y + f_2(x)\, y^2$	0	$e^{-\int f_1 dx}$

The above table shows that

$$\begin{aligned} \xi(x,y) &= \frac{1}{x^2} \\ \eta(x,y) &= 0 \end{aligned} \tag{A1}$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x, y) \to (R, S)$ where (R, S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y}\right) S(x, y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the canonical coordinates, where $S(R)$. Since $\eta = 0$ then in this special case

$$R = y$$

S is found from

$$\begin{aligned} S &= \int \frac{1}{\xi} dx \\ &= \int \frac{1}{\frac{1}{x^2}} dx \end{aligned}$$

Which results in

$$S = \frac{x^3}{3}$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x, y) S_y}{R_x + \omega(x, y) R_y} \tag{2}$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x, y)$ is the right hand side of the original ode given by

$$\omega(x, y) = \frac{x^2}{y^2 + 1}$$

Evaluating all the partial derivatives gives

$$\begin{aligned} R_x &= 0 \\ R_y &= 1 \\ S_x &= x^2 \\ S_y &= 0 \end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = y^2 + 1 \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = R^2 + 1$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$S(R) = \frac{1}{3}R^3 + R + c_1 \tag{4}$$

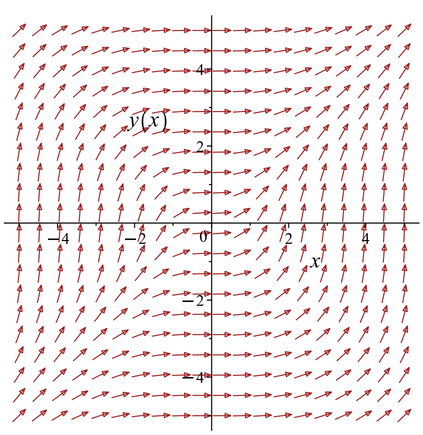
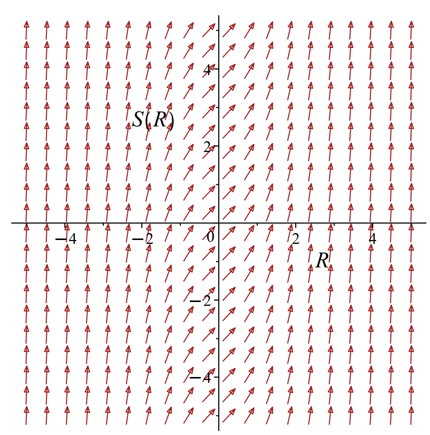
To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$\frac{x^3}{3} = \frac{y^3}{3} + y + c_1$$

Which simplifies to

$$\frac{x^3}{3} = \frac{y^3}{3} + y + c_1$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in x, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dx} = \frac{x^2}{y^2+1}$ 	$R = y$ $S = \frac{x^3}{3}$	$\frac{dS}{dR} = R^2 + 1$ 

Summary
The solution(s) found are the following

$$\frac{x^3}{3} = \frac{y^3}{3} + y + c_1 \tag{1}$$

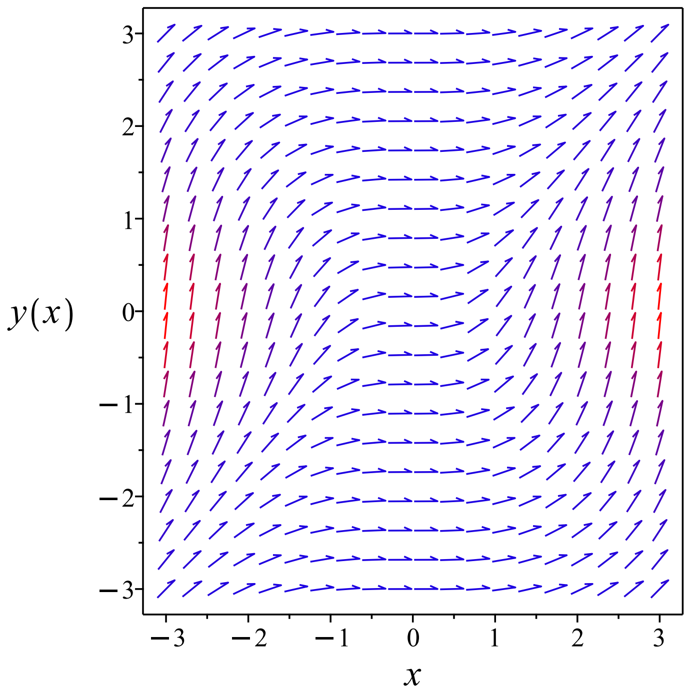


Figure 100: Slope field plot

Verification of solutions

$$\frac{x^3}{3} = \frac{y^3}{3} + y + c_1$$

Verified OK.

2.8.4 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x,y) + N(x,y)\frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x,y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y) = 0$$

Hence

$$\frac{\partial \phi}{\partial x} + \frac{\partial \phi}{\partial y}\frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$\begin{aligned}\frac{\partial \phi}{\partial x} &= M\\ \frac{\partial \phi}{\partial y} &= N\end{aligned}$$

But since $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x, y)$ but at least we know now that we can do that since the condition $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x, y)\,\mathrm{d}x + N(x, y)\,\mathrm{d}y = 0 \tag{1A}$$

Therefore

$$\begin{aligned}(y^2 + 1)\,\mathrm{d}y &= (x^2)\,\mathrm{d}x\\ (-x^2)\,\mathrm{d}x + (y^2 + 1)\,\mathrm{d}y &= 0 \qquad \text{(2A)}\end{aligned}$$

Comparing (1A) and (2A) shows that

$$\begin{aligned}M(x, y) &= -x^2\\ N(x, y) &= y^2 + 1\end{aligned}$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Using result found above gives

$$\begin{aligned}\frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}(-x^2)\\ &= 0\end{aligned}$$

And

$$
\begin{aligned}
\frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}(y^2+1) \\
&= 0
\end{aligned}
$$

Since $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$, then the ODE is exact The following equations are now set up to solve for the function $\phi(x,y)$

$$
\frac{\partial \phi}{\partial x} = M \tag{1}
$$

$$
\frac{\partial \phi}{\partial y} = N \tag{2}
$$

Integrating (1) w.r.t. x gives

$$
\begin{aligned}
\int \frac{\partial \phi}{\partial x} \,\mathrm{d}x &= \int M \,\mathrm{d}x \\
\int \frac{\partial \phi}{\partial x} \,\mathrm{d}x &= \int -x^2 \,\mathrm{d}x
\end{aligned}
$$

$$
\phi = -\frac{x^3}{3} + f(y) \tag{3}
$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$
\frac{\partial \phi}{\partial y} = 0 + f'(y) \tag{4}
$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = y^2 + 1$. Therefore equation (4) becomes

$$
y^2 + 1 = 0 + f'(y) \tag{5}
$$

Solving equation (5) for $f'(y)$ gives

$$
f'(y) = y^2 + 1
$$

Integrating the above w.r.t y gives

$$
\begin{aligned}
\int f'(y) \,\mathrm{d}y &= \int (y^2+1) \,\mathrm{d}y \\
f(y) &= \frac{1}{3}y^3 + y + c_1
\end{aligned}
$$

Where c_1 is constant of integration. Substituting result found above for $f(y)$ into equation (3) gives ϕ

$$\phi = -\frac{1}{3}x^3 + \frac{1}{3}y^3 + y + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = -\frac{1}{3}x^3 + \frac{1}{3}y^3 + y$$

Summary

The solution(s) found are the following

$$-\frac{x^3}{3} + \frac{y^3}{3} + y = c_1 \tag{1}$$

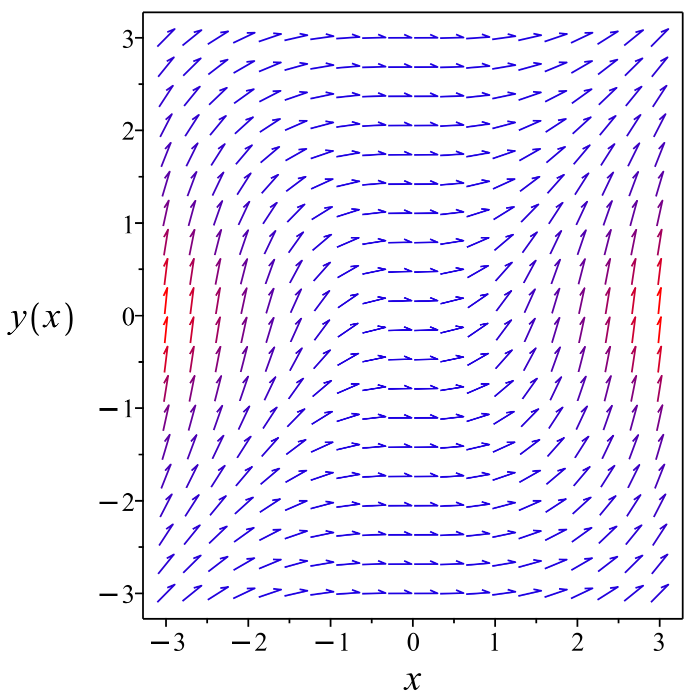


Figure 101: Slope field plot

Verification of solutions

$$-\frac{x^3}{3} + \frac{y^3}{3} + y = c_1$$

Verified OK.

2.8.5 Maple step by step solution

Let's solve

$y' - \frac{x^2}{1+y^2} = 0$

- Highest derivative means the order of the ODE is 1

  y'

- Separate variables

  $(1+y^2)\, y' = x^2$

- Integrate both sides with respect to x

  $\int (1+y^2)\, y' dx = \int x^2 dx + c_1$

- Evaluate integral

  $\frac{y^3}{3} + y = \frac{x^3}{3} + c_1$

- Solve for y

  $y = \frac{\left(4x^3+12c_1+4\sqrt{x^6+6c_1x^3+9c_1^2+4}\right)^{\frac{1}{3}}}{2} - \frac{2}{\left(4x^3+12c_1+4\sqrt{x^6+6c_1x^3+9c_1^2+4}\right)^{\frac{1}{3}}}$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
<- separable successful`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 268

```
dsolve(diff(y(x),x) = x^2/(1+y(x)^2),y(x), singsol=all)
```

$$y(x) = \frac{\left(4x^3+12c_1+4\sqrt{x^6+6c_1x^3+9c_1^2+4}\right)^{\frac{2}{3}}-4}{2\left(4x^3+12c_1+4\sqrt{x^6+6c_1x^3+9c_1^2+4}\right)^{\frac{1}{3}}}$$

$$y(x) = -\frac{(1+i\sqrt{3})\left(4x^3+12c_1+4\sqrt{x^6+6c_1x^3+9c_1^2+4}\right)^{\frac{2}{3}}+4i\sqrt{3}-4}{4\left(4x^3+12c_1+4\sqrt{x^6+6c_1x^3+9c_1^2+4}\right)^{\frac{1}{3}}}$$

$$y(x) = \frac{i\left(4x^3+12c_1+4\sqrt{x^6+6c_1x^3+9c_1^2+4}\right)^{\frac{2}{3}}\sqrt{3}+4i\sqrt{3}-\left(4x^3+12c_1+4\sqrt{x^6+6c_1x^3+9c_1^2+4}\right)^{\frac{2}{3}}+4}{4\left(4x^3+12c_1+4\sqrt{x^6+6c_1x^3+9c_1^2+4}\right)^{\frac{1}{3}}}$$

✓ Solution by Mathematica

Time used: 2.246 (sec). Leaf size: 307

```
DSolve[y'[x]== x^2/(1+y[x]^2),y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \frac{-2+\sqrt[3]{2}\left(x^3+\sqrt{x^6+6c_1x^3+4+9c_1{}^2}+3c_1\right)^{2/3}}{2^{2/3}\sqrt[3]{x^3+\sqrt{x^6+6c_1x^3+4+9c_1{}^2}+3c_1}}$$

$$y(x) \to \frac{i(\sqrt{3}+i)\sqrt[3]{x^3+\sqrt{x^6+6c_1x^3+4+9c_1{}^2}+3c_1}}{2\sqrt[3]{2}} + \frac{1+i\sqrt{3}}{2^{2/3}\sqrt[3]{x^3+\sqrt{x^6+6c_1x^3+4+9c_1{}^2}+3c_1}}$$

$$y(x) \to \frac{1-i\sqrt{3}}{2^{2/3}\sqrt[3]{x^3+\sqrt{x^6+6c_1x^3+4+9c_1{}^2}+3c_1}} - \frac{(1+i\sqrt{3})\sqrt[3]{x^3+\sqrt{x^6+6c_1x^3+4+9c_1{}^2}+3c_1}}{2\sqrt[3]{2}}$$

2.9 problem 9

Internal problem ID [487]
Internal file name [`OUTPUT/487_Sunday_June_05_2022_01_42_17_AM_24604466/index.tex`]

Book: Elementary differential equations and boundary value problems, 10th ed., Boyce and DiPrima
Section: Section 2.2. Page 48
Problem number: 9.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"exact", "riccati", "separable", "first_order_ode_lie_symmetry_lookup"**

Maple gives the following as the ode type

`[_separable]`

$$\boxed{y' - (1 - 2x)\, y^2 = 0}$$

With initial conditions

$$\left[y(0) = -\frac{1}{6}\right]$$

2.9.1 Existence and uniqueness analysis

This is non linear first order ODE. In canonical form it is written as

$$\begin{aligned} y' &= f(x, y) \\ &= -y^2(2x - 1) \end{aligned}$$

The x domain of $f(x, y)$ when $y = -\frac{1}{6}$ is

$$\{-\infty < x < \infty\}$$

And the point $x_0 = 0$ is inside this domain. The y domain of $f(x, y)$ when $x = 0$ is

$$\{-\infty < y < \infty\}$$

And the point $y_0 = -\frac{1}{6}$ is inside this domain. Now we will look at the continuity of

$$\begin{aligned}
\frac{\partial f}{\partial y} &= \frac{\partial}{\partial y}\left(-y^2(2x - 1)\right) \\
&= -2(2x - 1)\, y
\end{aligned}$$

The x domain of $\frac{\partial f}{\partial y}$ when $y = -\frac{1}{6}$ is

$$\{-\infty < x < \infty\}$$

And the point $x_0 = 0$ is inside this domain. The y domain of $\frac{\partial f}{\partial y}$ when $x = 0$ is

$$\{-\infty < y < \infty\}$$

And the point $y_0 = -\frac{1}{6}$ is inside this domain. Therefore solution exists and is unique.

2.9.2 Solving as separable ode

In canonical form the ODE is

$$\begin{aligned}
y' &= F(x, y) \\
&= f(x)g(y) \\
&= (1 - 2x)\, y^2
\end{aligned}$$

Where $f(x) = 1 - 2x$ and $g(y) = y^2$. Integrating both sides gives

$$\begin{aligned}
\frac{1}{y^2}\, dy &= 1 - 2x\, dx \\
\int \frac{1}{y^2}\, dy &= \int 1 - 2x\, dx \\
-\frac{1}{y} &= -x^2 + c_1 + x
\end{aligned}$$

Which results in

$$y = -\frac{1}{-x^2 + c_1 + x}$$

Initial conditions are used to solve for c_1. Substituting $x = 0$ and $y = -\frac{1}{6}$ in the above solution gives an equation to solve for the constant of integration.

$$-\frac{1}{6} = -\frac{1}{c_1}$$

$$c_1 = 6$$

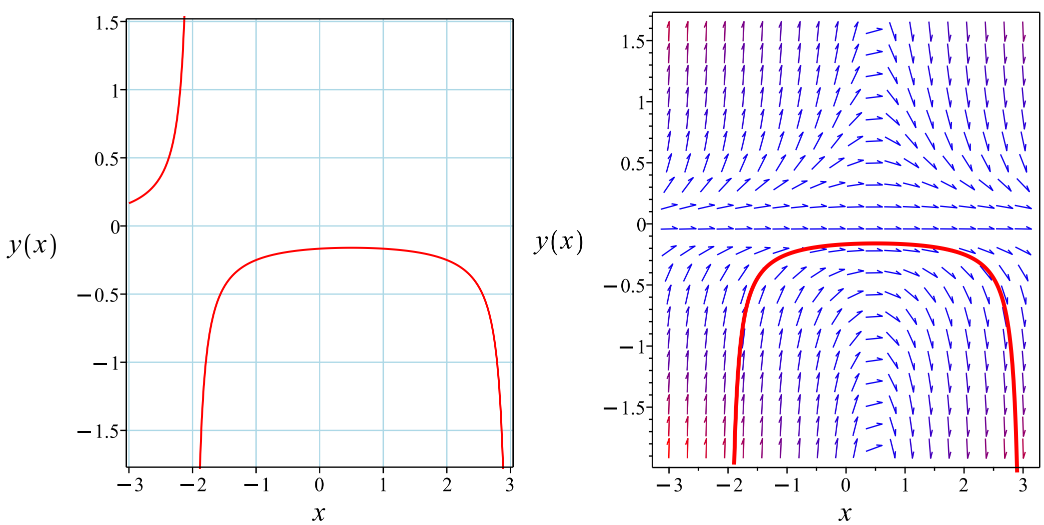
Substituting c_1 found above in the general solution gives

$$y = \frac{1}{x^2 - x - 6}$$

Summary

The solution(s) found are the following

$$y = \frac{1}{x^2 - x - 6} \tag{1}$$



(a) Solution plot (b) Slope field plot

Verification of solutions

$$y = \frac{1}{x^2 - x - 6}$$

Verified OK.

2.9.3 Solving as first order ode lie symmetry lookup ode

Writing the ode as

$$
\begin{aligned}
y' &= -y^2(2x-1) \\
y' &= \omega(x,y)
\end{aligned}
$$

The condition of Lie symmetry is the linearized PDE given by

$$
\eta_x + \omega(\eta_y - \xi_x) - \omega^2 \xi_y - \omega_x \xi - \omega_y \eta = 0 \tag{A}
$$

The type of this ode is known. It is of type `separable`. Therefore we do not need to solve the PDE (A), and can just use the lookup table shown below to find ξ, η

Table 115: Lie symmetry infinitesimal lookup table for known first order ODE's

ODE class	Form	ξ	η
linear ode	$y' = f(x)y(x) + g(x)$	0	$e^{\int f dx}$
separable ode	$y' = f(x)\, g(y)$	$\frac{1}{f}$	0
quadrature ode	$y' = f(x)$	0	1
quadrature ode	$y' = g(y)$	1	0
homogeneous ODEs of Class A	$y' = f\left(\frac{y}{x}\right)$	x	y
homogeneous ODEs of Class C	$y' = (a + bx + cy)^{\frac{n}{m}}$	1	$-\frac{b}{c}$
homogeneous class D	$y' = \frac{y}{x} + g(x)\, F\left(\frac{y}{x}\right)$	x^2	xy
First order special form ID 1	$y' = g(x)\, e^{h(x)+by} + f(x)$	$\frac{e^{-\int bf(x)dx - h(x)}}{g(x)}$	$\frac{f(x)e^{-\int bf(x)dx - h(x)}}{g(x)}$
polynomial type ode	$y' = \frac{a_1x+b_1y+c_1}{a_2x+b_2y+c_2}$	$\frac{a_1b_2x - a_2b_1x - b_1c_2 + b_2c_1}{a_1b_2 - a_2b_1}$	$\frac{a_1b_2y - a_2b_1y - a_1c_2 - a_2c_1}{a_1b_2 - a_2b_1}$
Bernoulli ode	$y' = f(x)\, y + g(x)\, y^n$	0	$e^{-\int (n-1)f(x)dx} y^n$
Reduced Riccati	$y' = f_1(x)\, y + f_2(x)\, y^2$	0	$e^{-\int f_1 dx}$

The above table shows that

$$\xi(x,y) = \frac{1}{1-2x}$$
$$\eta(x,y) = 0 \tag{A1}$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x,y) \to (R,S)$ where (R,S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y}\right) S(x,y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the canonical coordinates, where $S(R)$. Since $\eta = 0$ then in this special case

$$R = y$$

S is found from

$$\begin{aligned} S &= \int \frac{1}{\xi} dx \\ &= \int \frac{1}{\frac{1}{1-2x}} dx \end{aligned}$$

Which results in

$$S = -x^2 + x$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x,y)S_y}{R_x + \omega(x,y)R_y} \tag{2}$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x,y)$ is the right hand side of the original ode given by

$$\omega(x,y) = -y^2(2x-1)$$

Evaluating all the partial derivatives gives

$$\begin{aligned} R_x &= 0 \\ R_y &= 1 \\ S_x &= 1-2x \\ S_y &= 0 \end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = \frac{1}{y^2} \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = \frac{1}{R^2}$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$S(R) = -\frac{1}{R} + c_1 \tag{4}$$

To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$-x^2 + x = -\frac{1}{y} + c_1$$

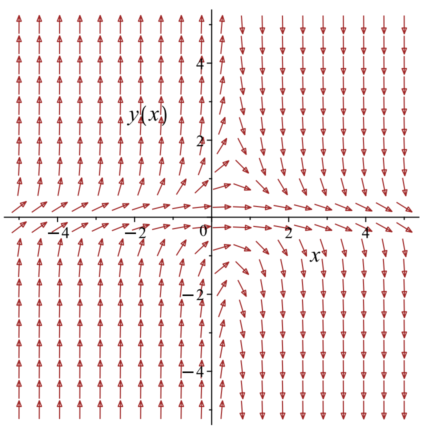
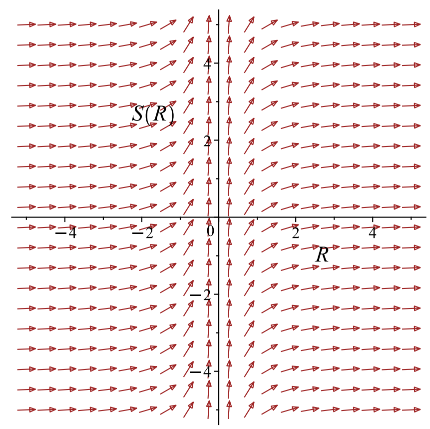
Which simplifies to

$$-x^2 + x = -\frac{1}{y} + c_1$$

Which gives

$$y = \frac{1}{x^2 + c_1 - x}$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in x, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dx} = -y^2(2x-1)$ 	$R = y$ $S = -x^2 + x$	$\frac{dS}{dR} = \frac{1}{R^2}$ 

Initial conditions are used to solve for c_1. Substituting $x = 0$ and $y = -\frac{1}{6}$ in the above solution gives an equation to solve for the constant of integration.

$$-\frac{1}{6} = \frac{1}{c_1}$$

$$c_1 = -6$$

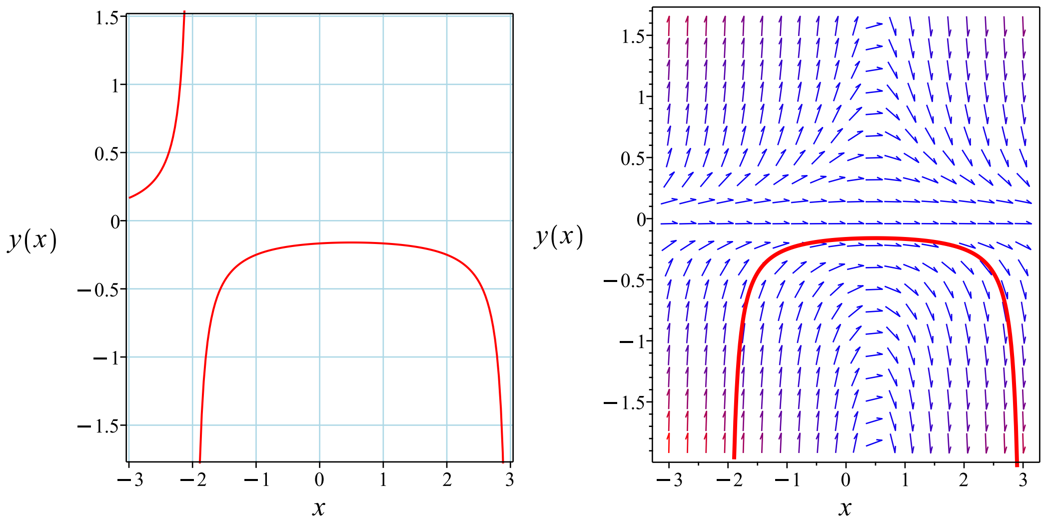
Substituting c_1 found above in the general solution gives

$$y = \frac{1}{x^2 - x - 6}$$

Summary

The solution(s) found are the following

$$y = \frac{1}{x^2 - x - 6} \tag{1}$$



(a) Solution plot (b) Slope field plot

<u>Verification of solutions</u>

$$y = \frac{1}{x^2 - x - 6}$$

Verified OK.

2.9.4 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x,y) + N(x,y)\frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x,y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y) = 0$$

Hence

$$\frac{\partial \phi}{\partial x} + \frac{\partial \phi}{\partial y}\frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$\frac{\partial \phi}{\partial x} = M$$
$$\frac{\partial \phi}{\partial y} = N$$

But since $\frac{\partial^2\phi}{\partial x\partial y} = \frac{\partial^2\phi}{\partial y\partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x, y)$ but at least we know now that we can do that since the condition $\frac{\partial^2\phi}{\partial x\partial y} = \frac{\partial^2\phi}{\partial y\partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x, y)\,\mathrm{d}x + N(x, y)\,\mathrm{d}y = 0 \tag{1A}$$

Therefore

$$\begin{aligned}\left(-\frac{1}{y^2}\right)\mathrm{d}y &= (2x-1)\,\mathrm{d}x\\ (1-2x)\,\mathrm{d}x + \left(-\frac{1}{y^2}\right)\mathrm{d}y &= 0 \qquad (2A)\end{aligned}$$

Comparing (1A) and (2A) shows that

$$\begin{aligned}M(x, y) &= 1-2x\\ N(x, y) &= -\frac{1}{y^2}\end{aligned}$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Using result found above gives

$$\begin{aligned}\frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}(1-2x)\\ &= 0\end{aligned}$$

And

$$\begin{aligned}\frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}\left(-\frac{1}{y^2}\right)\\ &= 0\end{aligned}$$

Since $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$, then the ODE is exact The following equations are now set up to solve for the function $\phi(x, y)$

$$\frac{\partial \phi}{\partial x} = M \tag{1}$$

$$\frac{\partial \phi}{\partial y} = N \tag{2}$$

Integrating (1) w.r.t. x gives

$$\int \frac{\partial \phi}{\partial x} \, \mathrm{d}x = \int M \, \mathrm{d}x$$

$$\int \frac{\partial \phi}{\partial x} \, \mathrm{d}x = \int 1 - 2x \, \mathrm{d}x$$

$$\phi = -x^2 + x + f(y) \tag{3}$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$\frac{\partial \phi}{\partial y} = 0 + f'(y) \tag{4}$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = -\frac{1}{y^2}$. Therefore equation (4) becomes

$$-\frac{1}{y^2} = 0 + f'(y) \tag{5}$$

Solving equation (5) for $f'(y)$ gives

$$f'(y) = -\frac{1}{y^2}$$

Integrating the above w.r.t y gives

$$\int f'(y) \, \mathrm{d}y = \int \left(-\frac{1}{y^2}\right) \mathrm{d}y$$

$$f(y) = \frac{1}{y} + c_1$$

Where c_1 is constant of integration. Substituting result found above for $f(y)$ into equation (3) gives ϕ

$$\phi = -x^2 + x + \frac{1}{y} + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = -x^2 + x + \frac{1}{y}$$

The solution becomes

$$y = \frac{1}{x^2 + c_1 - x}$$

Initial conditions are used to solve for c_1. Substituting $x = 0$ and $y = -\frac{1}{6}$ in the above solution gives an equation to solve for the constant of integration.

$$-\frac{1}{6} = \frac{1}{c_1}$$

$$c_1 = -6$$

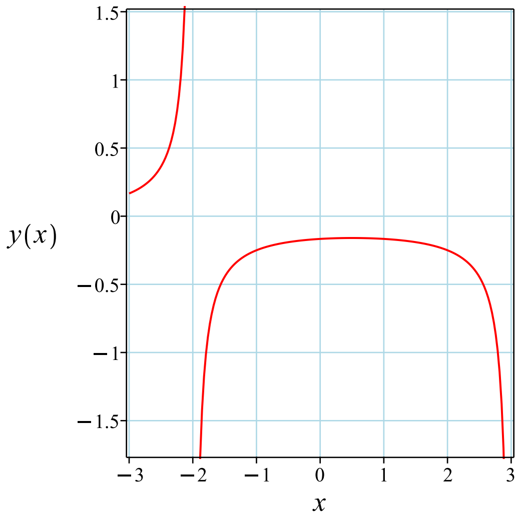
Substituting c_1 found above in the general solution gives

$$y = \frac{1}{x^2 - x - 6}$$

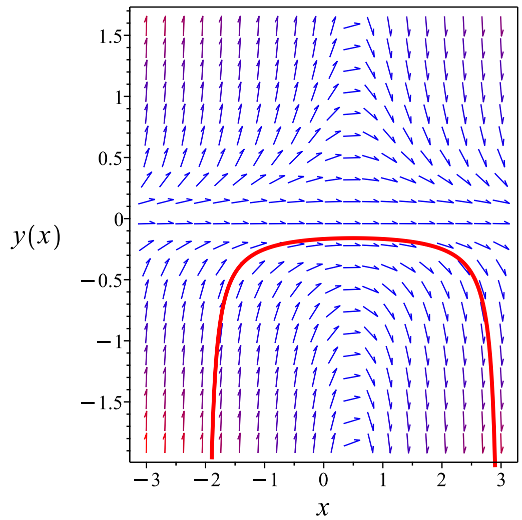
Summary

The solution(s) found are the following

$$y = \frac{1}{x^2 - x - 6} \tag{1}$$



(a) Solution plot



(b) Slope field plot

Verification of solutions

$$y = \frac{1}{x^2 - x - 6}$$

Verified OK.

2.9.5 Solving as riccati ode

In canonical form the ODE is

$$\begin{aligned} y' &= F(x, y) \\ &= -y^2(2x - 1) \end{aligned}$$

This is a Riccati ODE. Comparing the ODE to solve

$$y' = -2x\, y^2 + y^2$$

With Riccati ODE standard form

$$y' = f_0(x) + f_1(x)y + f_2(x)y^2$$

Shows that $f_0(x) = 0$, $f_1(x) = 0$ and $f_2(x) = 1 - 2x$. Let

$$\begin{aligned} y &= \frac{-u'}{f_2 u} \\ &= \frac{-u'}{(1 - 2x)\, u} \end{aligned} \tag{1}$$

Using the above substitution in the given ODE results (after some simplification)in a second order ODE to solve for $u(x)$ which is

$$f_2u''(x) - (f_2' + f_1f_2)\,u'(x) + f_2^2f_0u(x) = 0 \tag{2}$$

But

$$\begin{aligned} f_2' &= -2 \\ f_1f_2 &= 0 \\ f_2^2f_0 &= 0 \end{aligned}$$

Substituting the above terms back in equation (2) gives

$$(1-2x)\,u''(x) + 2u'(x) = 0$$

Solving the above ODE (this ode solved using Maple, not this program), gives

$$u(x) = c_1 + c_2\left(x - \frac{1}{2}\right)^2$$

The above shows that

$$u'(x) = c_2(2x-1)$$

Using the above in (1) gives the solution

$$y = -\frac{c_2(2x-1)}{(1-2x)\left(c_1 + c_2\left(x-\frac{1}{2}\right)^2\right)}$$

Dividing both numerator and denominator by c_1 gives, after renaming the constant $\frac{c_2}{c_1} = c_3$ the following solution

$$y = \frac{4}{4x^2 + 4c_3 - 4x + 1}$$

Initial conditions are used to solve for c_3. Substituting $x = 0$ and $y = -\frac{1}{6}$ in the above solution gives an equation to solve for the constant of integration.

$$-\frac{1}{6} = \frac{4}{4c_3 + 1}$$

$$c_3 = -\frac{25}{4}$$

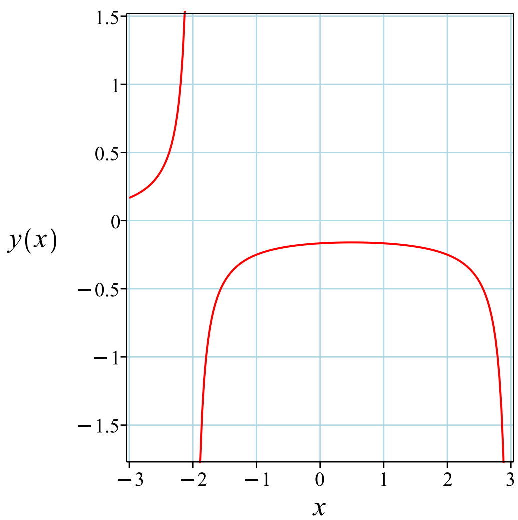
Substituting c_3 found above in the general solution gives

$$y = \frac{1}{x^2 - x - 6}$$

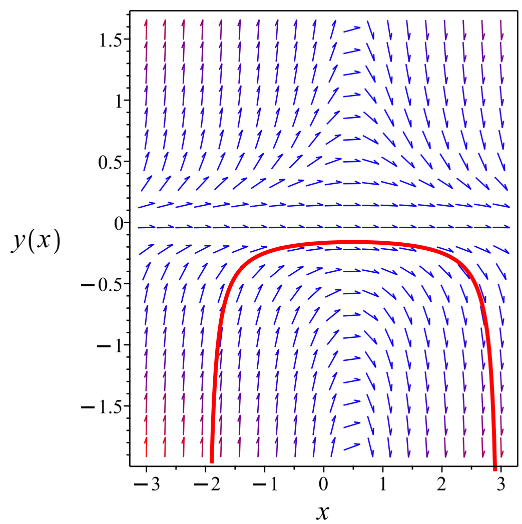
Summary

The solution(s) found are the following

$$y = \frac{1}{x^2 - x - 6} \tag{1}$$



(a) Solution plot



(b) Slope field plot

Verification of solutions

$$y = \frac{1}{x^2 - x - 6}$$

Verified OK.

2.9.6 Maple step by step solution

Let's solve

$\left[y' - (1 - 2x)\,y^2 = 0, y(0) = -\frac{1}{6}\right]$

- Highest derivative means the order of the ODE is 1

  y'

- Separate variables

  $\frac{y'}{y^2} = 1 - 2x$

- Integrate both sides with respect to x

  $\int \frac{y'}{y^2}dx = \int (1 - 2x)\,dx + c_1$

- Evaluate integral

  $-\frac{1}{y} = -x^2 + c_1 + x$

- Solve for y

  $y = -\frac{1}{-x^2+c_1+x}$

- Use initial condition $y(0) = -\frac{1}{6}$

  $-\frac{1}{6} = -\frac{1}{c_1}$

- Solve for c_1

  $c_1 = 6$

- Substitute $c_1 = 6$ into general solution and simplify

  $y = \frac{1}{x^2-x-6}$

- Solution to the IVP

  $y = \frac{1}{x^2-x-6}$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
<- Bernoulli successful`
```

✓ Solution by Maple

Time used: 0.062 (sec). Leaf size: 14

```
dsolve([diff(y(x),x) = (1-2*x)*y(x)^2,y(0) = -1/6],y(x), singsol=all)
```

$$y(x) = \frac{1}{x^2 - x - 6}$$

✓ Solution by Mathematica

Time used: 0.131 (sec). Leaf size: 15

```
DSolve[{y'[x] == (1-2*x)*y[x]^2,y[0]==-1/6},y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \frac{1}{x^2 - x - 6}$$

2.10 problem 10

Internal problem ID [488]
Internal file name [`OUTPUT/488_Sunday_June_05_2022_01_42_18_AM_32830898/index.tex`]

Book: Elementary differential equations and boundary value problems, 10th ed., Boyce and DiPrima
Section: Section 2.2. Page 48
Problem number: 10.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"exact", "separable", "differentialType", "homogeneousTypeMapleC", "first_order_ode_lie_symmetry_lookup"**

Maple gives the following as the ode type

```
[_separable]
```

$$\boxed{y' - \frac{1-2x}{y} = 0}$$

With initial conditions

$$[y(1) = -2]$$

2.10.1 Existence and uniqueness analysis

This is non linear first order ODE. In canonical form it is written as

$$
\begin{aligned}
y' &= f(x,y) \\
&= -\frac{2x-1}{y}
\end{aligned}
$$

The x domain of $f(x,y)$ when $y=-2$ is

$$
\{-\infty < x < \infty\}
$$

And the point $x_0 = 1$ is inside this domain. The y domain of $f(x,y)$ when $x = 1$ is

$$
\{y < 0 \lor 0 < y\}
$$

And the point $y_0 = -2$ is inside this domain. Now we will look at the continuity of

$$
\begin{aligned}
\frac{\partial f}{\partial y} &= \frac{\partial}{\partial y}\left(-\frac{2x-1}{y}\right) \\
&= \frac{2x-1}{y^2}
\end{aligned}
$$

The x domain of $\frac{\partial f}{\partial y}$ when $y=-2$ is

$$
\{-\infty < x < \infty\}
$$

And the point $x_0 = 1$ is inside this domain. The y domain of $\frac{\partial f}{\partial y}$ when $x = 1$ is

$$
\{y < 0 \lor 0 < y\}
$$

And the point $y_0 = -2$ is inside this domain. Therefore solution exists and is unique.

2.10.2 Solving as separable ode

In canonical form the ODE is

$$
\begin{aligned}
y' &= F(x,y) \\
&= f(x)g(y) \\
&= \frac{1-2x}{y}
\end{aligned}
$$

Where $f(x) = 1 - 2x$ and $g(y) = \frac{1}{y}$. Integrating both sides gives

$$\frac{1}{\frac{1}{y}}\,dy = 1 - 2x\,dx$$

$$\int \frac{1}{\frac{1}{y}}\,dy = \int 1 - 2x\,dx$$

$$\frac{y^2}{2} = -x^2 + c_1 + x$$

Which results in

$$y = \sqrt{-2x^2 + 2c_1 + 2x}$$

$$y = -\sqrt{-2x^2 + 2c_1 + 2x}$$

Initial conditions are used to solve for c_1. Substituting $x = 1$ and $y = -2$ in the above solution gives an equation to solve for the constant of integration.

$$-2 = -\sqrt{c_1}\,\sqrt{2}$$

$$c_1 = 2$$

Substituting c_1 found above in the general solution gives

$$y = -\sqrt{-2x^2 + 2x + 4}$$

Initial conditions are used to solve for c_1. Substituting $x = 1$ and $y = -2$ in the above solution gives an equation to solve for the constant of integration.

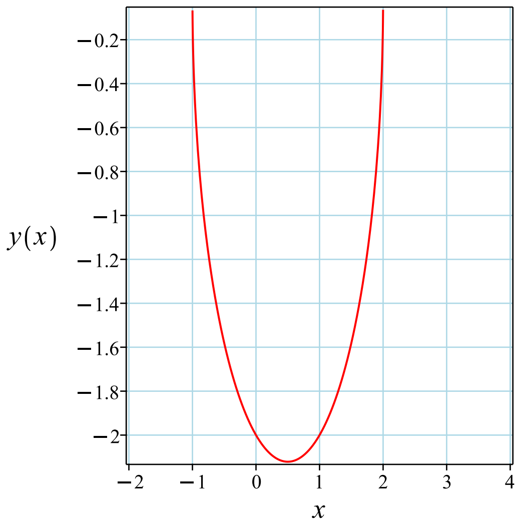
$$-2 = \sqrt{c_1}\,\sqrt{2}$$

Warning: Unable to solve for constant of integration.

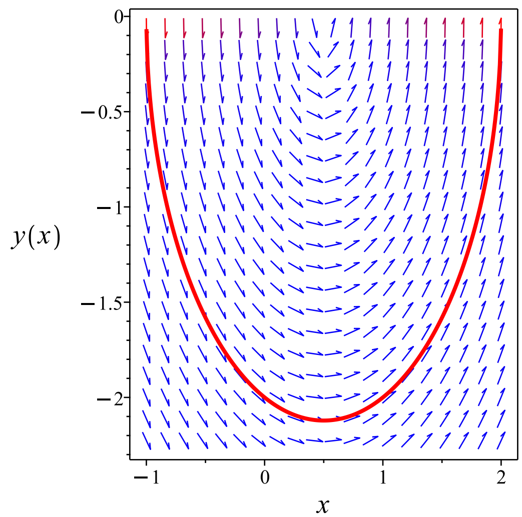
Summary

The solution(s) found are the following

$$y = -\sqrt{-2x^2 + 2x +$$



(a) Solution plot



(b) Slope field plot

Verification of solutions

$$y = -\sqrt{-2x^2 + 2x + 4}$$

Verified OK.

2.10.3 Solving as differentialType ode

Writing the ode as

$$y' = \frac{1 - 2x}{y} \tag{1}$$

Which becomes

$$(y)\, dy = (1 - 2x)\, dx \tag{2}$$

But the RHS is complete differential because

$$(1 - 2x)\, dx = d\left(-x^2 + x\right)$$

Hence (2) becomes

$$(y)\, dy = d\left(-x^2 + x\right)$$

Integrating both sides gives gives these solutions

$$y = \sqrt{-2x^2 + 2c_1 + 2x} + c_1$$
$$y = -\sqrt{-2x^2 + 2c_1 + 2x} + c_1$$

Initial conditions are used to solve for c_1. Substituting $x = 1$ and $y = -2$ in the above solution gives an equation to solve for the constant of integration.

$$-2 = -\sqrt{c_1}\,\sqrt{2} + c_1$$

$$c_1 = \sqrt{2}\left(\frac{\sqrt{2}}{2} + \frac{i\sqrt{6}}{2}\right) - 2$$

Substituting c_1 found above in the general solution gives

$$y = -\sqrt{-2x^2 - 2 + 2i\sqrt{3} + 2x} - 1 + i\sqrt{3}$$

Initial conditions are used to solve for c_1. Substituting $x = 1$ and $y = -2$ in the above solution gives an equation to solve for the constant of integration.

$$-2 = \sqrt{c_1}\,\sqrt{2} + c_1$$

Warning: Unable to solve for constant of integration.

Summary

The solution(s) found are the following

$$y = -\sqrt{-2x^2 - 2 + 2i\sqrt{3} + 2$$

Verification of solutions

$$y = -\sqrt{-2x^2 - 2 + 2i\sqrt{3} + 2x} - 1 + i\sqrt{3}$$

Verified OK.

2.10.4 Solving as homogeneousTypeMapleC ode

Let $Y = y + y_0$ and $X = x + x_0$ then the above is transformed to new ode in $Y(X)$

$$\frac{d}{dX}Y(X) = -\frac{2X + 2x_0 - 1}{Y(X) + y_0}$$

Solving for possible values of x_0 and y_0 which makes the above ode a homogeneous ode results in

$$x_0 = \frac{1}{2}$$
$$y_0 = 0$$

Using these values now it is possible to easily solve for $Y(X)$. The above ode now becomes

$$\frac{d}{dX}Y(X) = -\frac{2X}{Y(X)}$$

In canonical form, the ODE is

$$
\begin{aligned}
Y' &= F(X, Y) \\
&= -\frac{2X}{Y}
\end{aligned} \tag{1}
$$

An ode of the form $Y' = \frac{M(X,Y)}{N(X,Y)}$ is called homogeneous if the functions $M(X, Y)$ and $N(X, Y)$ are both homogeneous functions and of the same order. Recall that a function $f(X, Y)$ is homogeneous of order n if

$$
f(t^n X, t^n Y) = t^n f(X, Y)
$$

In this case, it can be seen that both $M = -2X$ and $N = Y$ are both homogeneous and of the same order $n = 1$. Therefore this is a homogeneous ode. Since this ode is homogeneous, it is converted to separable ODE using the substitution $u = \frac{Y}{X}$, or $Y = uX$. Hence

$$
\frac{\mathrm{d}Y}{\mathrm{d}X} = \frac{\mathrm{d}u}{\mathrm{d}X} X + u
$$

Applying the transformation $Y = uX$ to the above ODE in (1) gives

$$
\begin{aligned}
\frac{\mathrm{d}u}{\mathrm{d}X} X + u &= -\frac{2}{u} \\
\frac{\mathrm{d}u}{\mathrm{d}X} &= \frac{-\frac{2}{u(X)} - u(X)}{X}
\end{aligned}
$$

Or

$$
\frac{d}{dX} u(X) - \frac{-\frac{2}{u(X)} - u(X)}{X} = 0
$$

Or

$$
\left(\frac{d}{dX} u(X)\right) u(X) X + u(X)^2 + 2 = 0
$$

Which is now solved as separable in $u(X)$. Which is now solved in $u(X)$. In canonical form the ODE is

$$
\begin{aligned}
u' &= F(X, u) \\
&= f(X) g(u) \\
&= -\frac{u^2 + 2}{uX}
\end{aligned}
$$

Where $f(X) = -\frac{1}{X}$ and $g(u) = \frac{u^2+2}{u}$. Integrating both sides gives

$$\frac{1}{\frac{u^2+2}{u}}\,du = -\frac{1}{X}\,dX$$

$$\int \frac{1}{\frac{u^2+2}{u}}\,du = \int -\frac{1}{X}\,dX$$

$$\frac{\ln(u^2+2)}{2} = -\ln(X) + c_2$$

Raising both side to exponential gives

$$\sqrt{u^2+2} = e^{-\ln(X)+c_2}$$

Which simplifies to

$$\sqrt{u^2+2} = \frac{c_3}{X}$$

Which simplifies to

$$\sqrt{u(X)^2+2} = \frac{c_3 e^{c_2}}{X}$$

The solution is

$$\sqrt{u(X)^2+2} = \frac{c_3 e^{c_2}}{X}$$

Now u in the above solution is replaced back by Y using $u = \frac{Y}{X}$ which results in the solution

$$\sqrt{\frac{Y(X)^2}{X^2}+2} = \frac{c_3 e^{c_2}}{X}$$

Using the solution for $Y(X)$

$$\sqrt{\frac{Y(X)^2+2X^2}{X^2}} = \frac{c_3 e^{c_2}}{X}$$

And replacing back terms in the above solution using

$$Y = y + y_0$$
$$X = x + x_0$$

Or

$$Y = y$$
$$X = x + \frac{1}{2}$$

Then the solution in y becomes

$$\sqrt{\frac{y^2 + 2\left(x - \frac{1}{2}\right)^2}{\left(x - \frac{1}{2}\right)^2}} = \frac{c_3 e^{c_2}}{x - \frac{1}{2}}$$

Initial conditions are used to solve for c_2. Substituting $x = 1$ and $y = -2$ in the above solution gives an equation to solve for the constant of integration.

$$3\sqrt{2} = 2c_3 e^{c_2}$$

$$c_2 = \frac{\ln\left(\frac{9}{2c_3^2}\right)}{2}$$

Substituting c_2 found above in the general solution gives

$$\sqrt{\frac{y^2 + 2\left(x - \frac{1}{2}\right)^2}{\left(x - \frac{1}{2}\right)^2}} = \frac{3c_3\sqrt{2}\,\sqrt{\frac{1}{c_3^2}}}{2x - 1}$$

The above simplifies to

$$\sqrt{2}\left(2\sqrt{\frac{4x^2 + 2y^2 - 4x + 1}{(2x - 1)^2}}\, x - 3c_3\sqrt{\frac{1}{c_3^2}} - \sqrt{\frac{4x^2 + 2y^2 - 4x + 1}{(2x - 1)^2}}\right) = 0$$

Summary

The solution(s) found are the following

$$\sqrt{2}\left(2\sqrt{\frac{4x^2 + 2y^2 - 4x + 1}{(2x - 1)^2}}\, x - 3\,\mathrm{csgn}\left(\frac{1}{c_3}\right) - \sqrt{\frac{4x^2 + 2y^2 - 4x + 1}{(2x - 1)^2}}\right) = 0 \quad (1)$$

Verification of solutions

$$\sqrt{2}\left(2\sqrt{\frac{4x^2 + 2y^2 - 4x + 1}{(2x - 1)^2}}\, x - 3\,\mathrm{csgn}\left(\frac{1}{c_3}\right) - \sqrt{\frac{4x^2 + 2y^2 - 4x + 1}{(2x - 1)^2}}\right) = 0$$

Verified OK.

2.10.5 Solving as first order ode lie symmetry lookup ode

Writing the ode as

$$y' = -\frac{2x-1}{y}$$
$$y' = \omega(x,y)$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x + \omega(\eta_y - \xi_x) - \omega^2\xi_y - \omega_x\xi - \omega_y\eta = 0 \tag{A}$$

The type of this ode is known. It is of type `separable`. Therefore we do not need to solve the PDE (A), and can just use the lookup table shown below to find ξ, η

Table 118: Lie symmetry infinitesimal lookup table for known first order ODE's

ODE class	Form	ξ	η
linear ode	$y' = f(x)y(x) + g(x)$	0	$e^{\int f dx}$
separable ode	$y' = f(x)\, g(y)$	$\frac{1}{f}$	0
quadrature ode	$y' = f(x)$	0	1
quadrature ode	$y' = g(y)$	1	0
homogeneous ODEs of Class A	$y' = f\left(\frac{y}{x}\right)$	x	y
homogeneous ODEs of Class C	$y' = (a + bx + cy)^{\frac{n}{m}}$	1	$-\frac{b}{c}$
homogeneous class D	$y' = \frac{y}{x} + g(x)\, F\left(\frac{y}{x}\right)$	x^2	xy
First order special form ID 1	$y' = g(x)\, e^{h(x)+by} + f(x)$	$\frac{e^{-\int bf(x)dx - h(x)}}{g(x)}$	$\frac{f(x)e^{-\int bf(x)dx - h(x)}}{g(x)}$
polynomial type ode	$y' = \frac{a_1x+b_1y+c_1}{a_2x+b_2y+c_2}$	$\frac{a_1b_2x-a_2b_1x-b_1c_2+b_2c_1}{a_1b_2-a_2b_1}$	$\frac{a_1b_2y-a_2b_1y-a_1c_2-a_2c_1}{a_1b_2-a_2b_1}$
Bernoulli ode	$y' = f(x)\, y + g(x)\, y^n$	0	$e^{-\int (n-1)f(x)dx} y^n$
Reduced Riccati	$y' = f_1(x)\, y + f_2(x)\, y^2$	0	$e^{-\int f_1 dx}$

The above table shows that

$$
\begin{aligned}
\xi(x,y) &= \frac{1}{1-2x} \\
\eta(x,y) &= 0
\end{aligned} \tag{A1}
$$

The next step is to determine the canonical coordinates R,S. The canonical coordinates map $(x,y) \to (R,S)$ where (R,S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$
\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}
$$

The above comes from the requirements that $\left(\xi\frac{\partial}{\partial x} + \eta\frac{\partial}{\partial y}\right) S(x,y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the canonical coordinates, where $S(R)$. Since $\eta = 0$ then in this special case

$$
R = y
$$

S is found from

$$
\begin{aligned}
S &= \int \frac{1}{\xi} dx \\
&= \int \frac{1}{\frac{1}{1-2x}} dx
\end{aligned}
$$

Which results in

$$
S = -x^2 + x
$$

Now that R,S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$
\frac{dS}{dR} = \frac{S_x + \omega(x,y)S_y}{R_x + \omega(x,y)R_y} \tag{2}
$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x,y)$ is the right hand side of the original ode given by

$$
\omega(x,y) = -\frac{2x-1}{y}
$$

Evaluating all the partial derivatives gives

$$
\begin{aligned}
R_x &= 0 \\
R_y &= 1 \\
S_x &= 1 - 2x \\
S_y &= 0
\end{aligned}
$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$
\frac{dS}{dR} = y \tag{2A}
$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$
\frac{dS}{dR} = R
$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$
S(R) = \frac{R^2}{2} + c_1 \tag{4}
$$

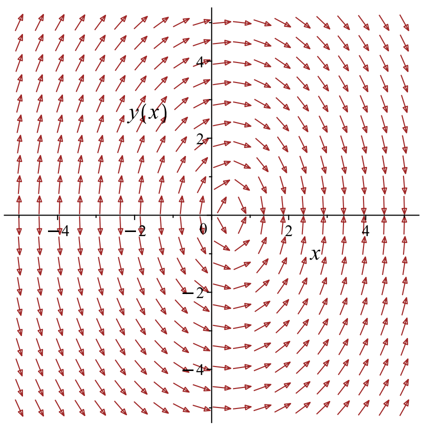
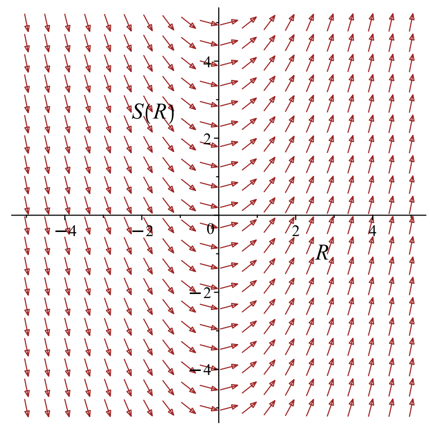
To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$
-x^2 + x = \frac{y^2}{2} + c_1
$$

Which simplifies to

$$
-x^2 + x = \frac{y^2}{2} + c_1
$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in x, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dx} = -\frac{2x-1}{y}$ 	$R = y$ $S = -x^2 + x$	$\frac{dS}{dR} = R$ 

Initial conditions are used to solve for c_1. Substituting $x = 1$ and $y = -2$ in the above solution gives an equation to solve for the constant of integration.

$$0 = c_1 + 2$$

$$c_1 = -2$$

Substituting c_1 found above in the general solution gives

$$-x^2 + x = \frac{y^2}{2} - 2$$

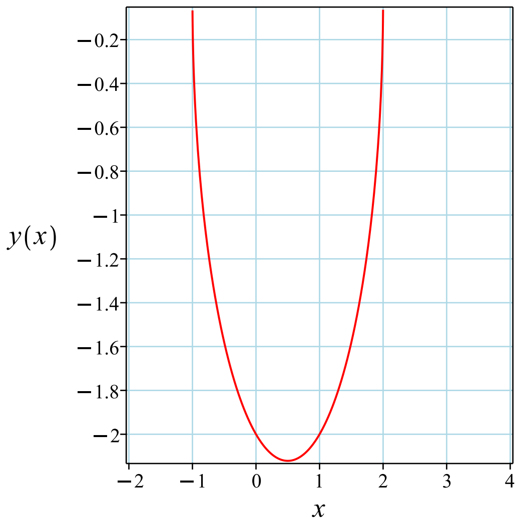
Solving for y from the above gives

$$y = -\sqrt{-2x^2 + 2x + 4}$$

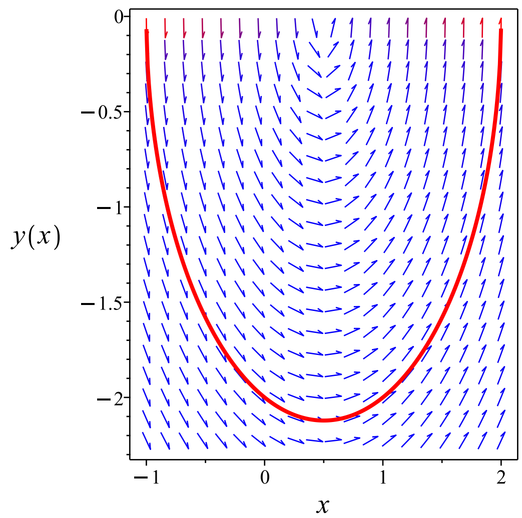
Summary

The solution(s) found are the following

$$y = -\sqrt{-2x^2 + 2x + 4} \tag{1}$$



(a) Solution plot



(b) Slope field plot

<u>Verification of solutions</u>

$$y = -\sqrt{-2x^2 + 2x + 4}$$

Verified OK. `{positive}`

2.10.6 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x,y) + N(x,y)\frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x,y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y) = 0$$

Hence

$$\frac{\partial \phi}{\partial x} + \frac{\partial \phi}{\partial y}\frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$\frac{\partial \phi}{\partial x} = M$$
$$\frac{\partial \phi}{\partial y} = N$$

But since $\frac{\partial^2\phi}{\partial x\partial y} = \frac{\partial^2\phi}{\partial y\partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x, y)$ but at least we know now that we can do that since the condition $\frac{\partial^2\phi}{\partial x\partial y} = \frac{\partial^2\phi}{\partial y\partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x, y)\,\mathrm{d}x + N(x, y)\,\mathrm{d}y = 0 \tag{1A}$$

Therefore

$$\begin{aligned}(-y)\,\mathrm{d}y &= (2x - 1)\,\mathrm{d}x\\ (1 - 2x)\,\mathrm{d}x + (-y)\,\mathrm{d}y &= 0 \end{aligned}\tag{2A}$$

Comparing (1A) and (2A) shows that

$$\begin{aligned}M(x, y) &= 1 - 2x\\ N(x, y) &= -y\end{aligned}$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Using result found above gives

$$\begin{aligned}\frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}(1 - 2x)\\ &= 0\end{aligned}$$

And

$$\begin{aligned}\frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}(-y)\\ &= 0\end{aligned}$$

Since $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$, then the ODE is exact The following equations are now set up to solve for the function $\phi(x, y)$

$$\frac{\partial \phi}{\partial x} = M \tag{1}$$

$$\frac{\partial \phi}{\partial y} = N \tag{2}$$

Integrating (1) w.r.t. x gives

$$\int \frac{\partial \phi}{\partial x} \, \mathrm{d}x = \int M \, \mathrm{d}x$$
$$\int \frac{\partial \phi}{\partial x} \, \mathrm{d}x = \int 1 - 2x \, \mathrm{d}x$$
$$\phi = -x^2 + x + f(y) \tag{3}$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$\frac{\partial \phi}{\partial y} = 0 + f'(y) \tag{4}$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = -y$. Therefore equation (4) becomes

$$-y = 0 + f'(y) \tag{5}$$

Solving equation (5) for $f'(y)$ gives

$$f'(y) = -y$$

Integrating the above w.r.t y gives

$$\int f'(y) \, \mathrm{d}y = \int (-y) \, \mathrm{d}y$$
$$f(y) = -\frac{y^2}{2} + c_1$$

Where c_1 is constant of integration. Substituting result found above for $f(y)$ into equation (3) gives ϕ

$$\phi = -x^2 - \frac{1}{2}y^2 + x + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = -x^2 - \frac{1}{2}y^2 + x$$

Initial conditions are used to solve for c_1. Substituting $x = 1$ and $y = -2$ in the above solution gives an equation to solve for the constant of integration.

$$-2 = c_1$$

$$c_1 = -2$$

Substituting c_1 found above in the general solution gives

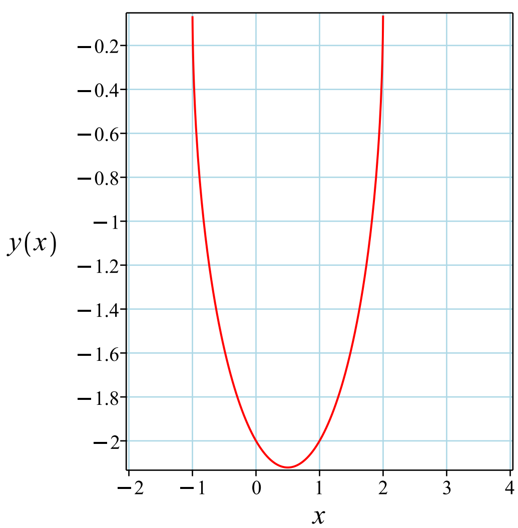
$$-x^2 - \frac{1}{2}y^2 + x = -2$$

Solving for y from the above gives

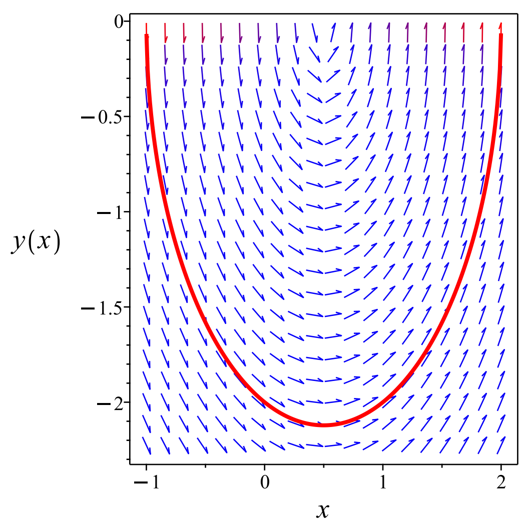
$$y = -\sqrt{-2x^2 + 2x + 4}$$

<u>Summary</u>
The solution(s) found are the following

$$y = -\sqrt{-2x^2 + 2x + 4} \tag{1}$$



(a) Solution plot



(b) Slope field plot

<u>Verification of solutions</u>

$$y = -\sqrt{-2x^2 + 2x + 4}$$

Verified OK. `{positive}`

2.10.7 Maple step by step solution

Let's solve

$\left[y' - \frac{1-2x}{y} = 0, y(1) = -2\right]$

- Highest derivative means the order of the ODE is 1

  y'
- Separate variables

  $yy' = 1 - 2x$
- Integrate both sides with respect to x

  $\int yy'dx = \int (1 - 2x)\,dx + c_1$
- Evaluate integral

  $\frac{y^2}{2} = -x^2 + c_1 + x$
- Solve for y

  $\{y = \sqrt{-2x^2 + 2c_1 + 2x}, y = -\sqrt{-2x^2 + 2c_1 + 2x}\}$
- Use initial condition $y(1) = -2$

  $-2 = \sqrt{c_1}\,\sqrt{2}$
- Solution does not satisfy initial condition
- Use initial condition $y(1) = -2$

  $-2 = -\sqrt{c_1}\,\sqrt{2}$
- Solve for c_1

  $c_1 = 2$
- Substitute $c_1 = 2$ into general solution and simplify

  $y = -\sqrt{-2x^2 + 2x + 4}$
- Solution to the IVP

  $y = -\sqrt{-2x^2 + 2x + 4}$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
<- Bernoulli successful`
```

✓ Solution by Maple

Time used: 0.047 (sec). Leaf size: 18

```
dsolve([diff(y(x),x) = (1-2*x)/y(x),y(1) = -2],y(x), singsol=all)
```

$$y(x) = -\sqrt{-2x^2 + 2x + 4}$$

✓ Solution by Mathematica

Time used: 0.094 (sec). Leaf size: 24

```
DSolve[{y'[x] == (1-2*x)/y[x],y[1]==-2},y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to -\sqrt{2}\sqrt{-x^2 + x + 2}$$

2.11 problem 11

Internal problem ID [489]
Internal file name [`OUTPUT/489_Sunday_June_05_2022_01_42_19_AM_46023698/index.tex`]

Book: Elementary differential equations and boundary value problems, 10th ed., Boyce and DiPrima
Section: Section 2.2. Page 48
Problem number: 11.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"exact", "separable", "first_order_ode_lie_symmetry_lookup"**

Maple gives the following as the ode type

`[_separable]`

$$\boxed{y' e^{-x} y = -x}$$

With initial conditions

$$[y(0) = 1]$$

2.11.1 Existence and uniqueness analysis

This is non linear first order ODE. In canonical form it is written as

$$\begin{aligned} y' &= f(x, y) \\ &= -\frac{x\,e^{x}}{y} \end{aligned}$$

The x domain of $f(x, y)$ when $y = 1$ is

$$\{-\infty < x < \infty\}$$

And the point $x_0 = 0$ is inside this domain. The y domain of $f(x, y)$ when $x = 0$ is

$$\{-\infty < y < \infty\}$$

And the point $y_0 = 1$ is inside this domain. Now we will look at the continuity of

$$\begin{aligned}\frac{\partial f}{\partial y} &= \frac{\partial}{\partial y}\left(-\frac{x\,\mathrm{e}^{x}}{y}\right)\\&= \frac{x\,\mathrm{e}^{x}}{y^2}\end{aligned}$$

The x domain of $\frac{\partial f}{\partial y}$ when $y = 1$ is

$$\{-\infty < x < \infty\}$$

And the point $x_0 = 0$ is inside this domain. The y domain of $\frac{\partial f}{\partial y}$ when $x = 0$ is

$$\{-\infty < y < \infty\}$$

And the point $y_0 = 1$ is inside this domain. Therefore solution exists and is unique.

2.11.2 Solving as separable ode

In canonical form the ODE is

$$\begin{aligned}y' &= F(x, y)\\&= f(x)g(y)\\&= -\frac{x\,\mathrm{e}^{x}}{y}\end{aligned}$$

Where $f(x) = -x\,\mathrm{e}^{x}$ and $g(y) = \frac{1}{y}$. Integrating both sides gives

$$\frac{1}{\frac{1}{y}}\,dy = -x\,\mathrm{e}^{x}\,dx$$

$$\int \frac{1}{\frac{1}{y}}\,dy = \int -x\,\mathrm{e}^{x}\,dx$$

$$\frac{y^2}{2} = -(x-1)\,\mathrm{e}^{x} + c_1$$

Which results in

$$y = \sqrt{-2x\,e^{x} + 2\,e^{x} + 2c_1}$$
$$y = -\sqrt{-2x\,e^{x} + 2\,e^{x} + 2c_1}$$

Initial conditions are used to solve for c_1. Substituting $x = 0$ and $y = 1$ in the above solution gives an equation to solve for the constant of integration.

$$1 = -\sqrt{2 + 2c_1}$$

Warning: Unable to solve for constant of integration. Initial conditions are used to solve for c_1. Substituting $x = 0$ and $y = 1$ in the above solution gives an equation to solve for the constant of integration.

$$1 = \sqrt{2 + 2c_1}$$

$$c_1 = -\frac{1}{2}$$

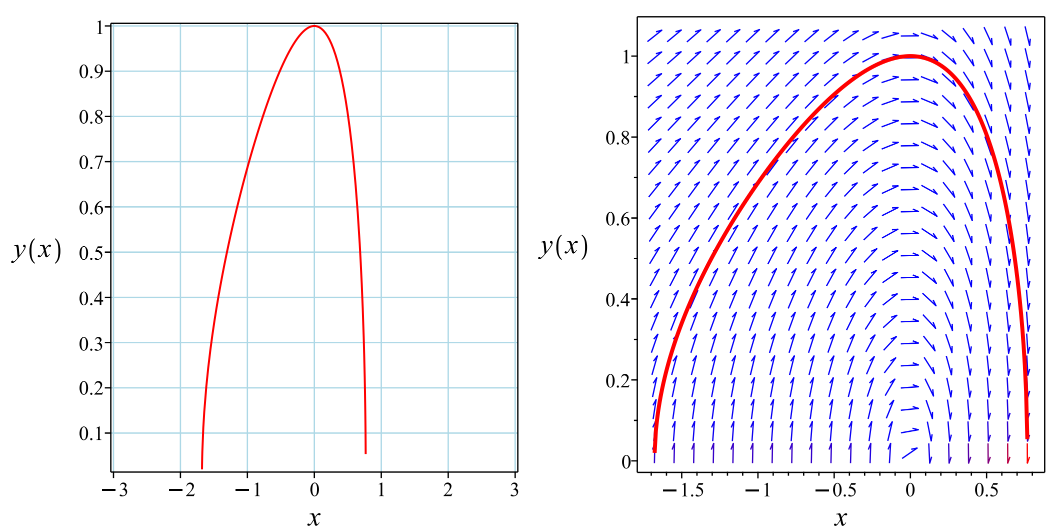
Substituting c_1 found above in the general solution gives

$$y = \sqrt{-2x\,e^{x} + 2\,e^{x} - 1}$$

Summary

The solution(s) found are the following

$$y = \sqrt{-2x\,e^{x} + 2\,e^{x} - 1} \tag{1}$$



(a) Solution plot

(b) Slope field plot

Verification of solutions

$$y = \sqrt{-2x\,\mathrm{e}^{x} + 2\,\mathrm{e}^{x} - 1}$$

Verified OK.

2.11.3 Solving as first order ode lie symmetry lookup ode

Writing the ode as

$$y' = -\frac{x\,\mathrm{e}^{x}}{y}$$
$$y' = \omega(x, y)$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x + \omega(\eta_y - \xi_x) - \omega^2 \xi_y - \omega_x \xi - \omega_y \eta = 0 \tag{A}$$

The type of this ode is known. It is of type `separable`. Therefore we do not need to solve the PDE (A), and can just use the lookup table shown below to find ξ, η

Table 121: Lie symmetry infinitesimal lookup table for known first order ODE's

ODE class	Form	ξ	η
linear ode	$y' = f(x)y(x) + g(x)$	0	$e^{\int f dx}$
separable ode	$y' = f(x)\, g(y)$	$\frac{1}{f}$	0
quadrature ode	$y' = f(x)$	0	1
quadrature ode	$y' = g(y)$	1	0
homogeneous ODEs of Class A	$y' = f\left(\frac{y}{x}\right)$	x	y
homogeneous ODEs of Class C	$y' = (a + bx + cy)^{\frac{n}{m}}$	1	$-\frac{b}{c}$
homogeneous class D	$y' = \frac{y}{x} + g(x)\, F\left(\frac{y}{x}\right)$	x^2	xy
First order special form ID 1	$y' = g(x)\, e^{h(x)+by} + f(x)$	$\frac{e^{-\int bf(x)dx - h(x)}}{g(x)}$	$\frac{f(x)e^{-\int bf(x)dx - h(x)}}{g(x)}$
polynomial type ode	$y' = \frac{a_1x+b_1y+c_1}{a_2x+b_2y+c_2}$	$\frac{a_1b_2x - a_2b_1x - b_1c_2 + b_2c_1}{a_1b_2 - a_2b_1}$	$\frac{a_1b_2y - a_2b_1y - a_1c_2 - a_2c_1}{a_1b_2 - a_2b_1}$
Bernoulli ode	$y' = f(x)\, y + g(x)\, y^n$	0	$e^{-\int (n-1)f(x)dx} y^n$
Reduced Riccati	$y' = f_1(x)\, y + f_2(x)\, y^2$	0	$e^{-\int f_1 dx}$

The above table shows that

$$
\begin{aligned}
\xi(x, y) &= -\frac{e^{-x}}{x} \\
\eta(x, y) &= 0
\end{aligned}
\tag{A1}
$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x, y) \to (R, S)$ where (R, S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$
\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}
$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y}\right) S(x, y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the

canonical coordinates, where $S(R)$. Since $\eta = 0$ then in this special case

$$R = y$$

S is found from

$$\begin{aligned} S &= \int \frac{1}{\xi} dx \\ &= \int \frac{1}{-\frac{e^{-x}}{x}} dx \end{aligned}$$

Which results in

$$S = -(x - 1)\, e^{x}$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x, y) S_y}{R_x + \omega(x, y) R_y} \tag{2}$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x, y)$ is the right hand side of the original ode given by

$$\omega(x, y) = -\frac{x\, e^{x}}{y}$$

Evaluating all the partial derivatives gives

$$\begin{aligned} R_x &= 0 \\ R_y &= 1 \\ S_x &= -x\, e^{x} \\ S_y &= 0 \end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = y \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = R$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by

integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$S(R) = \frac{R^2}{2} + c_1 \tag{4}$$

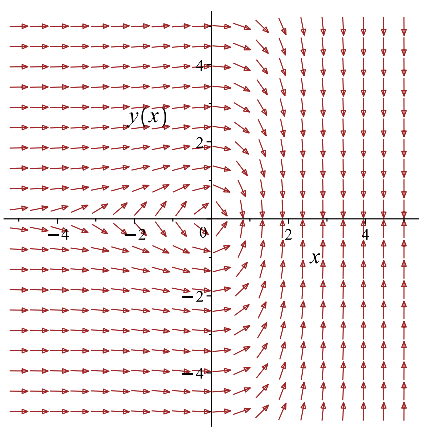
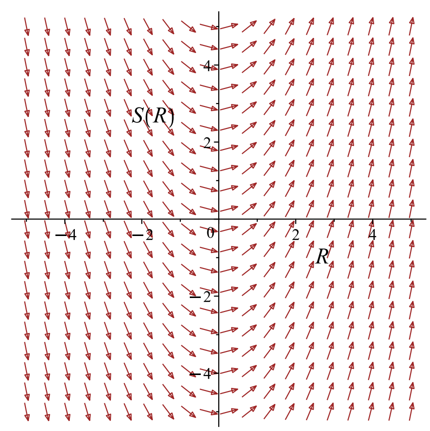
To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$-(x-1)\,\mathrm{e}^x = \frac{y^2}{2} + c_1$$

Which simplifies to

$$-(x-1)\,\mathrm{e}^x = \frac{y^2}{2} + c_1$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in x, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dx} = -\frac{x\,\mathrm{e}^x}{y}$ 	$R = y$ $S = -(x-1)\,\mathrm{e}^x$	$\frac{dS}{dR} = R$ 

Initial conditions are used to solve for c_1. Substituting $x = 0$ and $y = 1$ in the above solution gives an equation to solve for the constant of integration.

$$1 = \frac{1}{2} + c_1$$

$$c_1 = \frac{1}{2}$$

Substituting c_1 found above in the general solution gives

$$-(x-1)\,\mathrm{e}^{x} = \frac{y^2}{2} + \frac{1}{2}$$

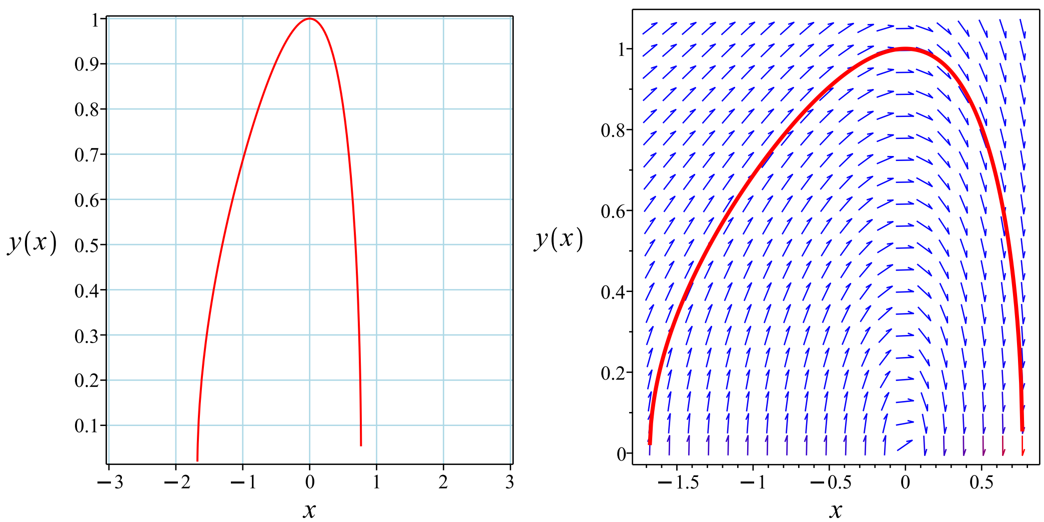
Solving for y from the above gives

$$y = \sqrt{-2x\,\mathrm{e}^{x} + 2\,\mathrm{e}^{x} - 1}$$

Summary

The solution(s) found are the following

$$y = \sqrt{-2x\,\mathrm{e}^{x} + 2\,\mathrm{e}^{x} - 1} \tag{1}$$



(a) Solution plot

(b) Slope field plot

Verification of solutions

$$y = \sqrt{-2x\,\mathrm{e}^{x} + 2\,\mathrm{e}^{x} - 1}$$

Verified OK.

2.11.4 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x,y) + N(x,y)\frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x,y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y) = 0$$

Hence

$$\frac{\partial \phi}{\partial x} + \frac{\partial \phi}{\partial y}\frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$\begin{aligned}\frac{\partial \phi}{\partial x} &= M\\ \frac{\partial \phi}{\partial y} &= N\end{aligned}$$

But since $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x,y)$ but at least we know now that we can do that since the condition $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x,y)\,\mathrm{d}x + N(x,y)\,\mathrm{d}y = 0 \tag{1A}$$

Therefore

$$\begin{aligned}(-y)\,\mathrm{d}y &= (x\,\mathrm{e}^{x})\,\mathrm{d}x\\ (-x\,\mathrm{e}^{x})\,\mathrm{d}x + (-y)\,\mathrm{d}y &= 0 \end{aligned} \tag{2A}$$

Comparing (1A) and (2A) shows that

$$\begin{aligned}M(x,y) &= -x\,\mathrm{e}^{x}\\ N(x,y) &= -y\end{aligned}$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Using result found above gives

$$\begin{aligned}\frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}(-x\,\mathrm{e}^{x}) \\ &= 0\end{aligned}$$

And

$$\begin{aligned}\frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}(-y) \\ &= 0\end{aligned}$$

Since $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$, then the ODE is <u>exact</u> The following equations are now set up to solve for the function $\phi(x, y)$

$$\frac{\partial \phi}{\partial x} = M \tag{1}$$

$$\frac{\partial \phi}{\partial y} = N \tag{2}$$

Integrating (1) w.r.t. x gives

$$\begin{aligned}\int \frac{\partial \phi}{\partial x}\,\mathrm{d}x &= \int M\,\mathrm{d}x \\ \int \frac{\partial \phi}{\partial x}\,\mathrm{d}x &= \int -x\,\mathrm{e}^{x}\,\mathrm{d}x \\ \phi &= -(x-1)\,\mathrm{e}^{x} + f(y)\end{aligned} \tag{3}$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$\frac{\partial \phi}{\partial y} = 0 + f'(y) \tag{4}$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = -y$. Therefore equation (4) becomes

$$-y = 0 + f'(y) \tag{5}$$

Solving equation (5) for $f'(y)$ gives

$$f'(y) = -y$$

Integrating the above w.r.t y gives

$$\int f'(y) \, \mathrm{d}y = \int (-y) \, \mathrm{d}y$$
$$f(y) = -\frac{y^2}{2} + c_1$$

Where c_1 is constant of integration. Substituting result found above for $f(y)$ into equation (3) gives ϕ

$$\phi = -(x-1)\,\mathrm{e}^x - \frac{y^2}{2} + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = -(x-1)\,\mathrm{e}^x - \frac{y^2}{2}$$

Initial conditions are used to solve for c_1. Substituting $x = 0$ and $y = 1$ in the above solution gives an equation to solve for the constant of integration.

$$\frac{1}{2} = c_1$$

$$c_1 = \frac{1}{2}$$

Substituting c_1 found above in the general solution gives

$$-(x-1)\,\mathrm{e}^x - \frac{y^2}{2} = \frac{1}{2}$$

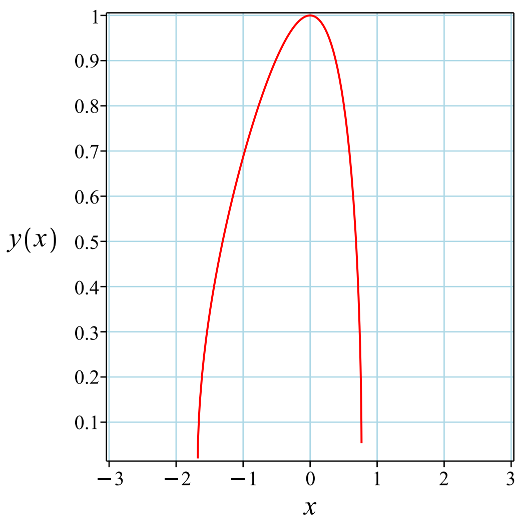
Solving for y from the above gives

$$y = \sqrt{-2x\,\mathrm{e}^x + 2\,\mathrm{e}^x - 1}$$

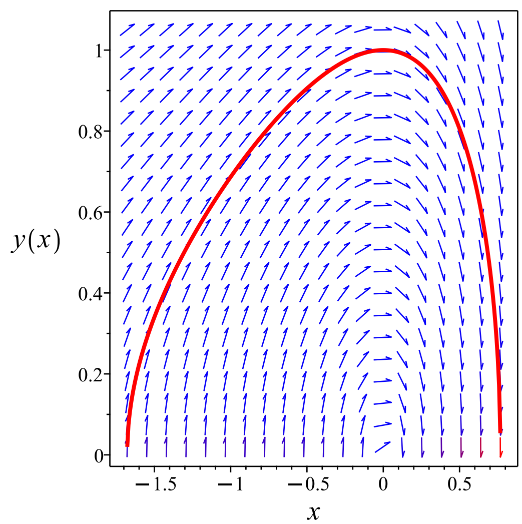
Summary

The solution(s) found are the following

$$y = \sqrt{-2x\,\mathrm{e}^{x} + 2\,\mathrm{e}^{x} - 1} \tag{1}$$



(a) Solution plot



(b) Slope field plot

Verification of solutions

$$y = \sqrt{-2x\,\mathrm{e}^{x} + 2\,\mathrm{e}^{x} - 1}$$

Verified OK.

2.11.5 Maple step by step solution

Let's solve

$\left[\frac{yy'}{\mathrm{e}^{x}} = -x, y(0) = 1\right]$

- Highest derivative means the order of the ODE is 1

  y'

- Separate variables

  $yy' = -x\,\mathrm{e}^{x}$

- Integrate both sides with respect to x

  $\int yy'dx = \int -x\,\mathrm{e}^{x}dx + c_1$

- Evaluate integral

$$\frac{y^2}{2} = -(x-1)\,e^x + c_1$$

- Solve for y

$$\left\{y = \sqrt{-2x\,e^x + 2\,e^x + 2c_1}, y = -\sqrt{-2x\,e^x + 2\,e^x + 2c_1}\right\}$$

- Use initial condition $y(0) = 1$

$$1 = \sqrt{2 + 2c_1}$$

- Solve for c_1

$$c_1 = -\frac{1}{2}$$

- Substitute $c_1 = -\frac{1}{2}$ into general solution and simplify

$$y = \sqrt{-2x\,e^x + 2\,e^x - 1}$$

- Use initial condition $y(0) = 1$

$$1 = -\sqrt{2 + 2c_1}$$

- Solution does not satisfy initial condition
- Solution to the IVP

$$y = \sqrt{-2x\,e^x + 2\,e^x - 1}$$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
<- Bernoulli successful`
```

✓ Solution by Maple

Time used: 0.078 (sec). Leaf size: 17

```
dsolve([x+y(x)*diff(y(x),x)/exp(x) = 0,y(0) = 1],y(x), singsol=all)
```

$$y(x) = \sqrt{-1 - 2x\,e^x + 2\,e^x}$$

✓ Solution by Mathematica

Time used: 1.763 (sec). Leaf size: 19

```
DSolve[{x+y[x]*y'[x]/Exp[x] == 0,y[0]==1},y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \sqrt{-2e^x(x-1)-1}$$

2.12 problem 12

Internal problem ID [490]
Internal file name [`OUTPUT/490_Sunday_June_05_2022_01_42_20_AM_13382328/index.tex`]

Book: Elementary differential equations and boundary value problems, 10th ed., Boyce and DiPrima
Section: Section 2.2. Page 48
Problem number: 12.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"exact", "riccati", "separable", "first_order_ode_lie_symmetry_lookup"**

Maple gives the following as the ode type

`[_separable]`

$$\boxed{r' - \frac{r^2}{x} = 0}$$

With initial conditions

$$[r(1) = 2]$$

2.12.1 Existence and uniqueness analysis

This is non linear first order ODE. In canonical form it is written as

$$\begin{aligned} r' &= f(x, r) \\ &= \frac{r^2}{x} \end{aligned}$$

The x domain of $f(x,r)$ when $r=2$ is

$$\{x<0 \lor 0<x\}$$

And the point $x_0=1$ is inside this domain. The r domain of $f(x,r)$ when $x=1$ is

$$\{-\infty<r<\infty\}$$

And the point $r_0=2$ is inside this domain. Now we will look at the continuity of

$$\begin{aligned}\frac{\partial f}{\partial r}&=\frac{\partial}{\partial r}\left(\frac{r^2}{x}\right)\\&=\frac{2r}{x}\end{aligned}$$

The x domain of $\frac{\partial f}{\partial r}$ when $r=2$ is

$$\{x<0 \lor 0<x\}$$

And the point $x_0=1$ is inside this domain. The r domain of $\frac{\partial f}{\partial r}$ when $x=1$ is

$$\{-\infty<r<\infty\}$$

And the point $r_0=2$ is inside this domain. Therefore solution exists and is unique.

2.12.2 Solving as separable ode

In canonical form the ODE is

$$\begin{aligned}r'&=F(x,r)\\&=f(x)g(r)\\&=\frac{r^2}{x}\end{aligned}$$

Where $f(x)=\frac{1}{x}$ and $g(r)=r^2$. Integrating both sides gives

$$\begin{aligned}\frac{1}{r^2}\,dr&=\frac{1}{x}\,dx\\\int\frac{1}{r^2}\,dr&=\int\frac{1}{x}\,dx\\-\frac{1}{r}&=\ln(x)+c_1\end{aligned}$$

Which results in

$$r = -\frac{1}{\ln(x) + c_1}$$

Initial conditions are used to solve for c_1. Substituting $x = 1$ and $r = 2$ in the above solution gives an equation to solve for the constant of integration.

$$2 = -\frac{1}{c_1}$$

$$c_1 = -\frac{1}{2}$$

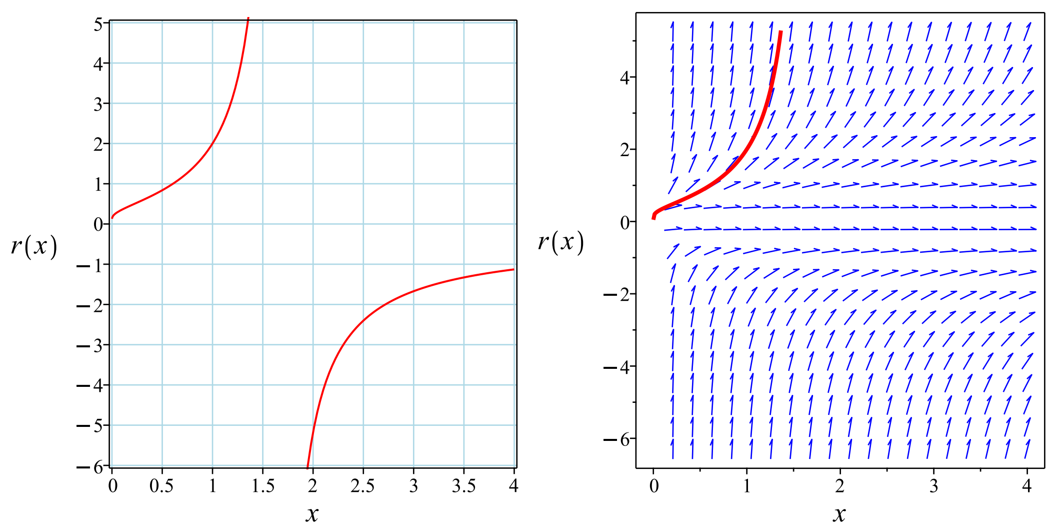
Substituting c_1 found above in the general solution gives

$$r = -\frac{2}{2\ln(x) - 1}$$

Summary

The solution(s) found are the following

$$r = -\frac{2}{2\ln(x) - 1} \tag{1}$$



(a) Solution plot

(b) Slope field plot

Verification of solutions

$$r = -\frac{2}{2\ln(x) - 1}$$

Verified OK.

2.12.3 Solving as first order ode lie symmetry lookup ode

Writing the ode as

$$
\begin{aligned}
r' &= \frac{r^2}{x} \\
r' &= \omega(x, r)
\end{aligned}
$$

The condition of Lie symmetry is the linearized PDE given by

$$
\eta_x + \omega(\eta_r - \xi_x) - \omega^2 \xi_r - \omega_x \xi - \omega_r \eta = 0 \tag{A}
$$

The type of this ode is known. It is of type `separable`. Therefore we do not need to solve the PDE (A), and can just use the lookup table shown below to find ξ, η

Table 124: Lie symmetry infinitesimal lookup table for known first order ODE's

ODE class	Form	ξ	η
linear ode	$y' = f(x)y(x) + g(x)$	0	$e^{\int f dx}$
separable ode	$y' = f(x)\, g(y)$	$\frac{1}{f}$	0
quadrature ode	$y' = f(x)$	0	1
quadrature ode	$y' = g(y)$	1	0
homogeneous ODEs of Class A	$y' = f\left(\frac{y}{x}\right)$	x	y
homogeneous ODEs of Class C	$y' = (a + bx + cy)^{\frac{n}{m}}$	1	$-\frac{b}{c}$
homogeneous class D	$y' = \frac{y}{x} + g(x)\, F\left(\frac{y}{x}\right)$	x^2	xy
First order special form ID 1	$y' = g(x)\, e^{h(x)+by} + f(x)$	$\frac{e^{-\int bf(x)dx - h(x)}}{g(x)}$	$\frac{f(x)e^{-\int bf(x)dx - h(x)}}{g(x)}$
polynomial type ode	$y' = \frac{a_1x + b_1y + c_1}{a_2x + b_2y + c_2}$	$\frac{a_1b_2x - a_2b_1x - b_1c_2 + b_2c_1}{a_1b_2 - a_2b_1}$	$\frac{a_1b_2y - a_2b_1y - a_1c_2 - a_2c_1}{a_1b_2 - a_2b_1}$
Bernoulli ode	$y' = f(x)\, y + g(x)\, y^n$	0	$e^{-\int (n-1)f(x)dx} y^n$
Reduced Riccati	$y' = f_1(x)\, y + f_2(x)\, y^2$	0	$e^{-\int f_1 dx}$

The above table shows that

$$\begin{aligned}\xi(x, r) &= x \\ \eta(x, r) &= 0\end{aligned} \tag{A1}$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x, r) \to (R, S)$ where (R, S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$\frac{dx}{\xi} = \frac{dr}{\eta} = dS \tag{1}$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial r}\right) S(x, r) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the canonical coordinates, where $S(R)$. Since $\eta = 0$ then in this special case

$$R = r$$

S is found from

$$\begin{aligned}S &= \int \frac{1}{\xi} dx \\ &= \int \frac{1}{x} dx\end{aligned}$$

Which results in

$$S = \ln(x)$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x, r) S_r}{R_x + \omega(x, r) R_r} \tag{2}$$

Where in the above R_x, R_r, S_x, S_r are all partial derivatives and $\omega(x, r)$ is the right hand side of the original ode given by

$$\omega(x, r) = \frac{r^2}{x}$$

Evaluating all the partial derivatives gives

$$\begin{aligned}R_x &= 0 \\ R_r &= 1 \\ S_x &= \frac{1}{x} \\ S_r &= 0\end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = \frac{1}{r^2} \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for x, r in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = \frac{1}{R^2}$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$S(R) = -\frac{1}{R} + c_1 \tag{4}$$

To complete the solution, we just need to transform (4) back to x, r coordinates. This results in

$$\ln(x) = -\frac{1}{r} + c_1$$

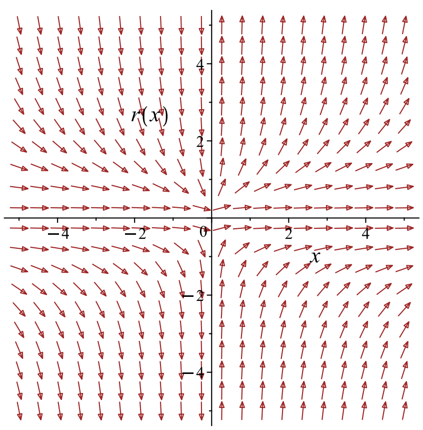
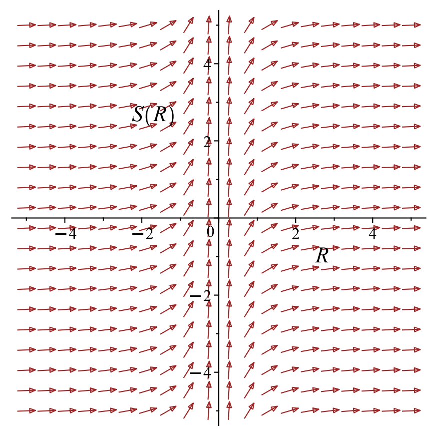
Which simplifies to

$$\ln(x) = -\frac{1}{r} + c_1$$

Which gives

$$r = -\frac{1}{\ln(x) - c_1}$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in x, r coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dr}{dx} = \frac{r^2}{x}$ 	$R = r$ $S = \ln(x)$	$\frac{dS}{dR} = \frac{1}{R^2}$ 

Initial conditions are used to solve for c_1. Substituting $x = 1$ and $r = 2$ in the above solution gives an equation to solve for the constant of integration.

$$2 = \frac{1}{c_1}$$

$$c_1 = \frac{1}{2}$$

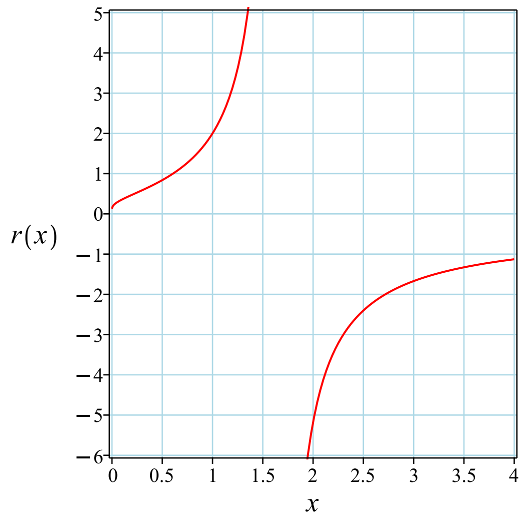
Substituting c_1 found above in the general solution gives

$$r = -\frac{2}{2\ln(x) - 1}$$

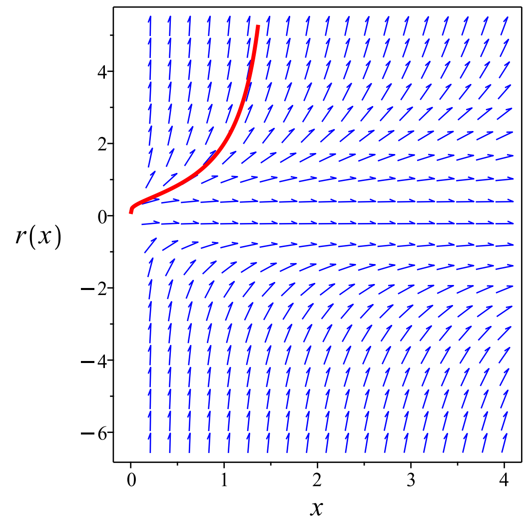
Summary

The solution(s) found are the following

$$r = -\frac{2}{2\ln(x) - 1} \tag{1}$$



(a) Solution plot



(b) Slope field plot

<u>Verification of solutions</u>

$$r = -\frac{2}{2\ln(x) - 1}$$

Verified OK.

2.12.4 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x,y) + N(x,y)\frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x,y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y) = 0$$

Hence

$$\frac{\partial \phi}{\partial x} + \frac{\partial \phi}{\partial y}\frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$\frac{\partial \phi}{\partial x} = M$$
$$\frac{\partial \phi}{\partial y} = N$$

But since $\frac{\partial^2\phi}{\partial x\partial y} = \frac{\partial^2\phi}{\partial y\partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x,y)$ but at least we know now that we can do that since the condition $\frac{\partial^2\phi}{\partial x\partial y} = \frac{\partial^2\phi}{\partial y\partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x,r)\,\mathrm{d}x + N(x,r)\,\mathrm{d}r = 0 \tag{1A}$$

Therefore

$$\left(\frac{1}{r^2}\right)\mathrm{d}r = \left(\frac{1}{x}\right)\mathrm{d}x$$
$$\left(-\frac{1}{x}\right)\mathrm{d}x + \left(\frac{1}{r^2}\right)\mathrm{d}r = 0 \tag{2A}$$

Comparing (1A) and (2A) shows that

$$M(x,r) = -\frac{1}{x}$$
$$N(x,r) = \frac{1}{r^2}$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial r} = \frac{\partial N}{\partial x}$$

Using result found above gives

$$\begin{aligned}\frac{\partial M}{\partial r} &= \frac{\partial}{\partial r}\left(-\frac{1}{x}\right)\\ &= 0\end{aligned}$$

And

$$\begin{aligned}\frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}\left(\frac{1}{r^2}\right)\\ &= 0\end{aligned}$$

Since $\frac{\partial M}{\partial r} = \frac{\partial N}{\partial x}$, then the ODE is exact The following equations are now set up to solve for the function $\phi(x, r)$

$$\frac{\partial \phi}{\partial x} = M \tag{1}$$

$$\frac{\partial \phi}{\partial r} = N \tag{2}$$

Integrating (1) w.r.t. x gives

$$\int \frac{\partial \phi}{\partial x} \mathrm{d}x = \int M \mathrm{d}x$$

$$\int \frac{\partial \phi}{\partial x} \mathrm{d}x = \int -\frac{1}{x} \mathrm{d}x$$

$$\phi = -\ln(x) + f(r) \tag{3}$$

Where $f(r)$ is used for the constant of integration since ϕ is a function of both x and r. Taking derivative of equation (3) w.r.t r gives

$$\frac{\partial \phi}{\partial r} = 0 + f'(r) \tag{4}$$

But equation (2) says that $\frac{\partial \phi}{\partial r} = \frac{1}{r^2}$. Therefore equation (4) becomes

$$\frac{1}{r^2} = 0 + f'(r) \tag{5}$$

Solving equation (5) for $f'(r)$ gives

$$f'(r) = \frac{1}{r^2}$$

Integrating the above w.r.t r gives

$$\int f'(r) \mathrm{d}r = \int \left(\frac{1}{r^2}\right) \mathrm{d}r$$

$$f(r) = -\frac{1}{r} + c_1$$

Where c_1 is constant of integration. Substituting result found above for $f(r)$ into equation (3) gives ϕ

$$\phi = -\ln(x) - \frac{1}{r} + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = -\ln(x) - \frac{1}{r}$$

The solution becomes

$$r = -\frac{1}{\ln(x) + c_1}$$

Initial conditions are used to solve for c_1. Substituting $x = 1$ and $r = 2$ in the above solution gives an equation to solve for the constant of integration.

$$2 = -\frac{1}{c_1}$$

$$c_1 = -\frac{1}{2}$$

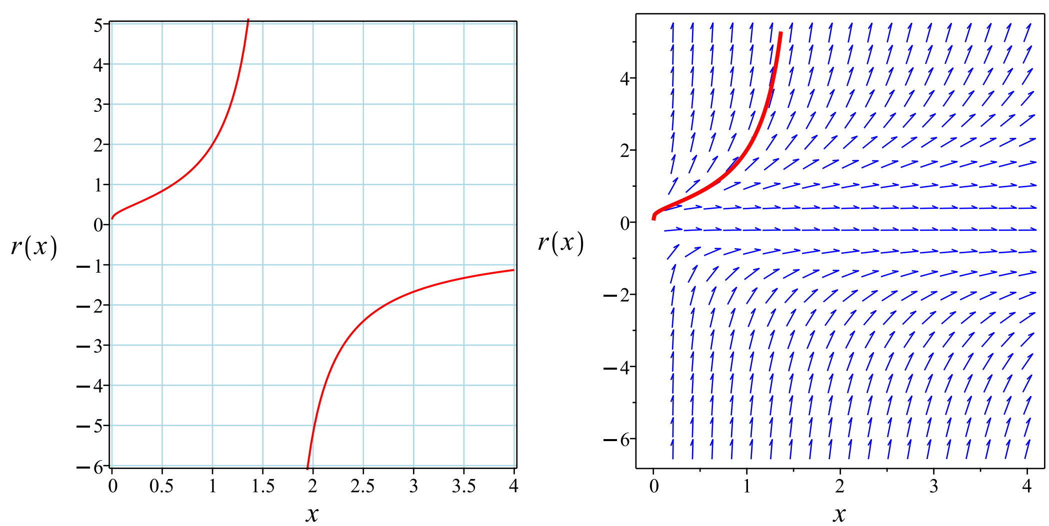
Substituting c_1 found above in the general solution gives

$$r = -\frac{2}{2\ln(x) - 1}$$

Summary

The solution(s) found are the following

$$r = -\frac{2}{2\ln(x) - 1} \tag{1}$$



(a) Solution plot

(b) Slope field plot

Verification of solutions

$$r = -\frac{2}{2\ln(x) - 1}$$

Verified OK.

2.12.5 Solving as riccati ode

In canonical form the ODE is

$$\begin{aligned} r' &= F(x, r) \\ &= \frac{r^2}{x} \end{aligned}$$

This is a Riccati ODE. Comparing the ODE to solve

$$r' = \frac{r^2}{x}$$

With Riccati ODE standard form

$$r' = f_0(x) + f_1(x)r + f_2(x)r^2$$

Shows that $f_0(x) = 0$, $f_1(x) = 0$ and $f_2(x) = \frac{1}{x}$. Let

$$\begin{aligned} r &= \frac{-u'}{f_2 u} \\ &= \frac{-u'}{\frac{u}{x}} \end{aligned} \tag{1}$$

Using the above substitution in the given ODE results (after some simplification)in a second order ODE to solve for $u(x)$ which is

$$f_2 u''(x) - (f_2' + f_1 f_2) u'(x) + f_2^2 f_0 u(x) = 0 \tag{2}$$

But

$$\begin{aligned} f_2' &= -\frac{1}{x^2} \\ f_1 f_2 &= 0 \\ f_2^2 f_0 &= 0 \end{aligned}$$

Substituting the above terms back in equation (2) gives

$$\frac{u''(x)}{x} + \frac{u'(x)}{x^2} = 0$$

Solving the above ODE (this ode solved using Maple, not this program), gives

$$u(x) = c_2 \ln(x) + c_1$$

The above shows that

$$u'(x) = \frac{c_2}{x}$$

Using the above in (1) gives the solution

$$r = -\frac{c_2}{c_2 \ln(x) + c_1}$$

Dividing both numerator and denominator by c_1 gives, after renaming the constant $\frac{c_2}{c_1} = c_3$ the following solution

$$r = -\frac{1}{\ln(x) + c_3}$$

Initial conditions are used to solve for c_3. Substituting $x = 1$ and $r = 2$ in the above solution gives an equation to solve for the constant of integration.

$$2 = -\frac{1}{c_3}$$

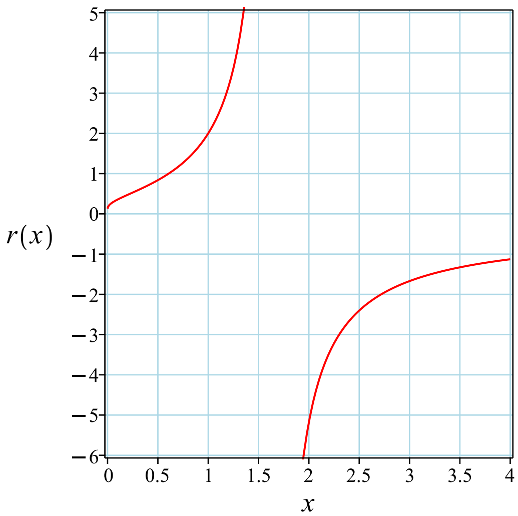
$$c_3 = -\frac{1}{2}$$

Substituting c_3 found above in the general solution gives

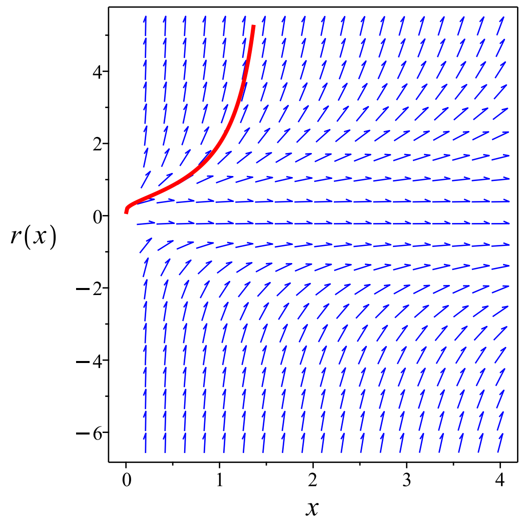
$$r = -\frac{2}{2 \ln(x) - 1}$$

Summary
The solution(s) found are the following

$$r = -\frac{2}{2 \ln(x) - 1} \tag{1}$$



(a) Solution plot



(b) Slope field plot

<u>Verification of solutions</u>

$$r = -\frac{2}{2\ln(x) - 1}$$

Verified OK.

2.12.6 Maple step by step solution

Let's solve

$\left[r' - \frac{r^2}{x} = 0, r(1) = 2\right]$

- Highest derivative means the order of the ODE is 1

  r'

- Separate variables

  $\frac{r'}{r^2} = \frac{1}{x}$

- Integrate both sides with respect to x

  $\int \frac{r'}{r^2} dx = \int \frac{1}{x} dx + c_1$

- Evaluate integral

  $-\frac{1}{r} = \ln(x) + c_1$

- Solve for r

$r = -\frac{1}{\ln(x)+c_1}$

- Use initial condition $r(1) = 2$

  $2 = -\frac{1}{c_1}$

- Solve for c_1

  $c_1 = -\frac{1}{2}$

- Substitute $c_1 = -\frac{1}{2}$ into general solution and simplify

  $r = -\frac{2}{2\ln(x)-1}$

- Solution to the IVP

  $r = -\frac{2}{2\ln(x)-1}$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
<- Bernoulli successful`
```

✓ Solution by Maple
Time used: 0.015 (sec). Leaf size: 14

```
dsolve([diff(r(x),x) = r(x)^2/x,r(1) = 2],r(x), singsol=all)
```

$$r(x) = -\frac{2}{2\ln(x) - 1}$$

✓ Solution by Mathematica
Time used: 0.128 (sec). Leaf size: 15

```
DSolve[{r'[x] == r[x]^2/x,r[1]==2},r[x],x,IncludeSingularSolutions -> True]
```

$$r(x) \to \frac{2}{1 - 2\log(x)}$$

2.13 problem 13

Internal problem ID [491]
Internal file name [`OUTPUT/491_Sunday_June_05_2022_01_42_21_AM_11702963/index.tex`]

Book: Elementary differential equations and boundary value problems, 10th ed., Boyce and DiPrima
Section: Section 2.2. Page 48
Problem number: 13.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"exact", "separable", "first_order_ode_lie_symmetry_lookup"**

Maple gives the following as the ode type

`[_separable]`

$$\boxed{y' - \frac{2x}{y + x^2 y} = 0}$$

With initial conditions

$$[y(0) = -2]$$

2.13.1 Existence and uniqueness analysis

This is non linear first order ODE. In canonical form it is written as

$$\begin{aligned} y' &= f(x,y) \\ &= \frac{2x}{y\,(x^2+1)} \end{aligned}$$

The x domain of $f(x,y)$ when $y=-2$ is

$$\{-\infty < x < \infty\}$$

And the point $x_0 = 0$ is inside this domain. The y domain of $f(x,y)$ when $x = 0$ is

$$\{-\infty < y < \infty\}$$

And the point $y_0 = -2$ is inside this domain. Now we will look at the continuity of

$$\begin{aligned}\frac{\partial f}{\partial y} &= \frac{\partial}{\partial y}\left(\frac{2x}{y\,(x^2+1)}\right)\\ &= -\frac{2x}{y^2\,(x^2+1)}\end{aligned}$$

The x domain of $\frac{\partial f}{\partial y}$ when $y = -2$ is

$$\{-\infty < x < \infty\}$$

And the point $x_0 = 0$ is inside this domain. The y domain of $\frac{\partial f}{\partial y}$ when $x = 0$ is

$$\{-\infty < y < \infty\}$$

And the point $y_0 = -2$ is inside this domain. Therefore solution exists and is unique.

2.13.2 Solving as separable ode

In canonical form the ODE is

$$\begin{aligned}y' &= F(x,y)\\ &= f(x)g(y)\\ &= \frac{2x}{y\,(x^2+1)}\end{aligned}$$

Where $f(x) = \frac{2x}{x^2+1}$ and $g(y) = \frac{1}{y}$. Integrating both sides gives

$$\begin{aligned}\frac{1}{\frac{1}{y}}\,dy &= \frac{2x}{x^2+1}\,dx\\ \int \frac{1}{\frac{1}{y}}\,dy &= \int \frac{2x}{x^2+1}\,dx\\ \frac{y^2}{2} &= \ln\left(x^2+1\right) + c_1\end{aligned}$$

Which results in

$$y = \sqrt{2\ln(x^2+1) + 2c_1}$$
$$y = -\sqrt{2\ln(x^2+1) + 2c_1}$$

Initial conditions are used to solve for c_1. Substituting $x = 0$ and $y = -2$ in the above solution gives an equation to solve for the constant of integration.

$$-2 = -\sqrt{c_1}\,\sqrt{2}$$

$$c_1 = 2$$

Substituting c_1 found above in the general solution gives

$$y = -\sqrt{2\ln(x^2+1) + 4}$$

Initial conditions are used to solve for c_1. Substituting $x = 0$ and $y = -2$ in the above solution gives an equation to solve for the constant of integration.

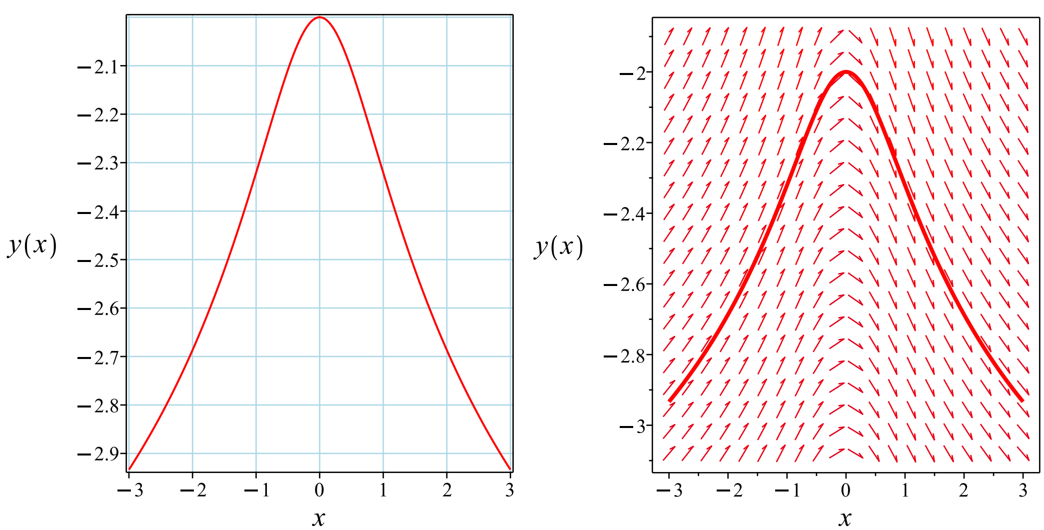
$$-2 = \sqrt{c_1}\,\sqrt{2}$$

Warning: Unable to solve for constant of integration.

Summary

The solution(s) found are the following

$$y = -\sqrt{2\ln(x^2+1)$$



(a) Solution plot

(b) Slope field plot

Verification of solutions

$$y = -\sqrt{2\ln\left(x^2+1\right)+4}$$

Verified OK.

2.13.3 Solving as first order ode lie symmetry lookup ode

Writing the ode as

$$\begin{aligned}y' &= \frac{2x}{y\left(x^2+1\right)}\\ y' &= \omega(x,y)\end{aligned}$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x + \omega(\eta_y - \xi_x) - \omega^2\xi_y - \omega_x\xi - \omega_y\eta = 0 \tag{A}$$

The type of this ode is known. It is of type `separable`. Therefore we do not need to solve the PDE (A), and can just use the lookup table shown below to find ξ, η

Table 127: Lie symmetry infinitesimal lookup table for known first order ODE's

ODE class	Form	ξ	η
linear ode	$y' = f(x)y(x) + g(x)$	0	$e^{\int f dx}$
separable ode	$y' = f(x)\, g(y)$	$\frac{1}{f}$	0
quadrature ode	$y' = f(x)$	0	1
quadrature ode	$y' = g(y)$	1	0
homogeneous ODEs of Class A	$y' = f\left(\frac{y}{x}\right)$	x	y
homogeneous ODEs of Class C	$y' = (a + bx + cy)^{\frac{n}{m}}$	1	$-\frac{b}{c}$
homogeneous class D	$y' = \frac{y}{x} + g(x)\, F\left(\frac{y}{x}\right)$	x^2	xy
First order special form ID 1	$y' = g(x)\, e^{h(x)+by} + f(x)$	$\frac{e^{-\int bf(x)dx - h(x)}}{g(x)}$	$\frac{f(x)e^{-\int bf(x)dx - h(x)}}{g(x)}$
polynomial type ode	$y' = \frac{a_1x+b_1y+c_1}{a_2x+b_2y+c_2}$	$\frac{a_1b_2x - a_2b_1x - b_1c_2 + b_2c_1}{a_1b_2 - a_2b_1}$	$\frac{a_1b_2y - a_2b_1y - a_1c_2 - a_2c_1}{a_1b_2 - a_2b_1}$
Bernoulli ode	$y' = f(x)\, y + g(x)\, y^n$	0	$e^{-\int (n-1)f(x)dx} y^n$
Reduced Riccati	$y' = f_1(x)\, y + f_2(x)\, y^2$	0	$e^{-\int f_1 dx}$

The above table shows that

$$
\begin{aligned}
\xi(x,y) &= \frac{x^2 + 1}{2x} \\
\eta(x,y) &= 0
\end{aligned} \tag{A1}
$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x,y) \to (R,S)$ where (R,S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$
\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}
$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y}\right) S(x,y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the

canonical coordinates, where $S(R)$. Since $\eta = 0$ then in this special case

$$R = y$$

S is found from

$$\begin{aligned} S &= \int \frac{1}{\xi} dx \\ &= \int \frac{1}{\frac{x^2+1}{2x}} dx \end{aligned}$$

Which results in

$$S = \ln\left(x^2 + 1\right)$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x, y) S_y}{R_x + \omega(x, y) R_y} \tag{2}$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x, y)$ is the right hand side of the original ode given by

$$\omega(x, y) = \frac{2x}{y\left(x^2 + 1\right)}$$

Evaluating all the partial derivatives gives

$$\begin{aligned} R_x &= 0 \\ R_y &= 1 \\ S_x &= \frac{2x}{x^2 + 1} \\ S_y &= 0 \end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = y \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = R$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordinates R, S. Integrating the above gives

$$S(R) = \frac{R^2}{2} + c_1 \tag{4}$$

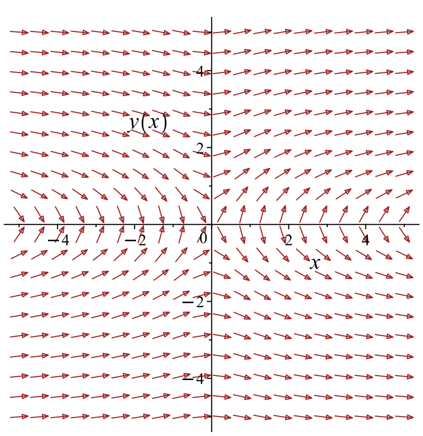
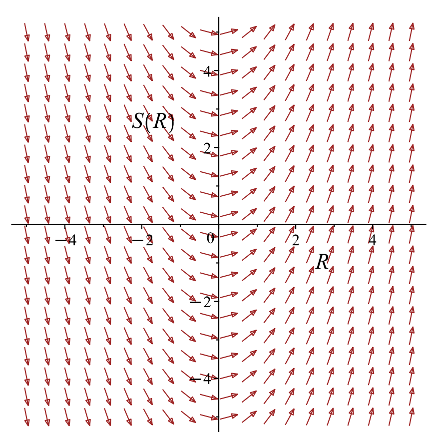
To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$\ln\left(x^2+1\right) = \frac{y^2}{2} + c_1$$

Which simplifies to

$$\ln\left(x^2+1\right) = \frac{y^2}{2} + c_1$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in x, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dx} = \frac{2x}{y(x^2+1)}$  	$R = y$ $S = \ln\left(x^2+1\right)$	$\frac{dS}{dR} = R$ 

Initial conditions are used to solve for c_1. Substituting $x = 0$ and $y = -2$ in the above solution gives an equation to solve for the constant of integration.

$$0 = c_1 + 2$$

$$c_1 = -2$$

Substituting c_1 found above in the general solution gives

$$\ln\left(x^2+1\right) = \frac{y^2}{2} - 2$$

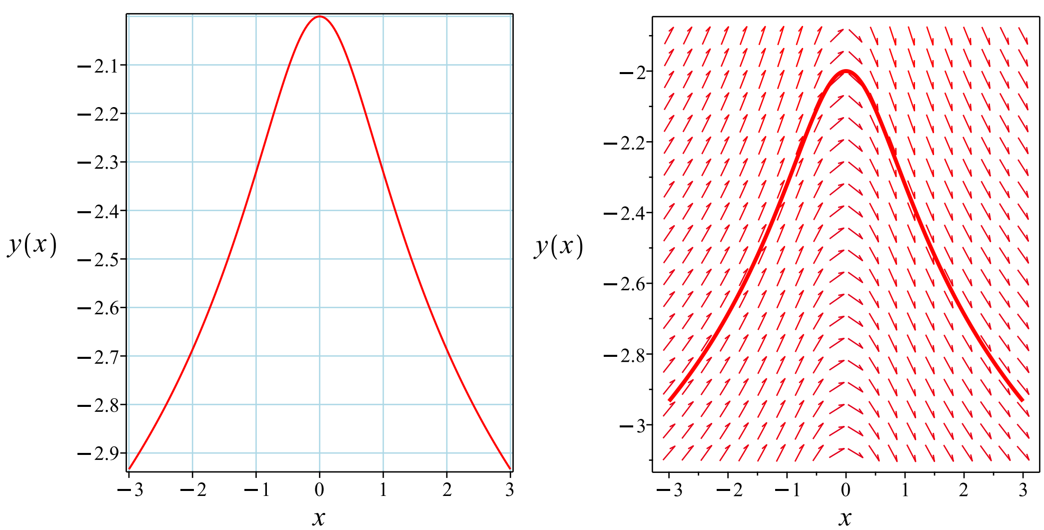
Solving for y from the above gives

$$y = -\sqrt{2\ln\left(x^2+1\right)+4}$$

Summary

The solution(s) found are the following

$$y = -\sqrt{2\ln\left(x^2+1\right)+4} \tag{1}$$



(a) Solution plot (b) Slope field plot

Verification of solutions

$$y = -\sqrt{2\ln\left(x^2+1\right)+4}$$

Verified OK.

2.13.4 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x,y)+N(x,y)\frac{dy}{dx}=0 \tag{A}$$

We assume there exists a function $\phi(x,y)=c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y)=0$$

Hence

$$\frac{\partial\phi}{\partial x}+\frac{\partial\phi}{\partial y}\frac{dy}{dx}=0 \tag{B}$$

Comparing (A,B) shows that

$$\begin{aligned}\frac{\partial\phi}{\partial x}&=M\\ \frac{\partial\phi}{\partial y}&=N\end{aligned}$$

But since $\frac{\partial^2\phi}{\partial x\partial y}=\frac{\partial^2\phi}{\partial y\partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y}=\frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x,y)$ but at least we know now that we can do that since the condition $\frac{\partial^2\phi}{\partial x\partial y}=\frac{\partial^2\phi}{\partial y\partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x,y)\,\mathrm{d}x+N(x,y)\,\mathrm{d}y=0 \tag{1A}$$

Therefore

$$\begin{aligned}\left(\frac{y}{2}\right)\mathrm{d}y&=\left(\frac{x}{x^2+1}\right)\mathrm{d}x\\ \left(-\frac{x}{x^2+1}\right)\mathrm{d}x+\left(\frac{y}{2}\right)\mathrm{d}y&=0 \qquad\text{(2A)}\end{aligned}$$

Comparing (1A) and (2A) shows that

$$
\begin{aligned}
M(x,y) &= -\frac{x}{x^2+1} \\
N(x,y) &= \frac{y}{2}
\end{aligned}
$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$
\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}
$$

Using result found above gives

$$
\begin{aligned}
\frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}\left(-\frac{x}{x^2+1}\right) \\
&= 0
\end{aligned}
$$

And

$$
\begin{aligned}
\frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}\left(\frac{y}{2}\right) \\
&= 0
\end{aligned}
$$

Since $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$, then the ODE is exact The following equations are now set up to solve for the function $\phi(x,y)$

$$
\frac{\partial \phi}{\partial x} = M \tag{1}
$$

$$
\frac{\partial \phi}{\partial y} = N \tag{2}
$$

Integrating (1) w.r.t. x gives

$$
\begin{aligned}
\int \frac{\partial \phi}{\partial x}\,\mathrm{d}x &= \int M\,\mathrm{d}x \\
\int \frac{\partial \phi}{\partial x}\,\mathrm{d}x &= \int -\frac{x}{x^2+1}\,\mathrm{d}x \\
\phi &= -\frac{\ln(x^2+1)}{2} + f(y)
\end{aligned} \tag{3}
$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$
\frac{\partial \phi}{\partial y} = 0 + f'(y) \tag{4}
$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = \frac{y}{2}$. Therefore equation (4) becomes

$$\frac{y}{2} = 0 + f'(y) \tag{5}$$

Solving equation (5) for $f'(y)$ gives

$$f'(y) = \frac{y}{2}$$

Integrating the above w.r.t y gives

$$\int f'(y) \, \mathrm{d}y = \int \left(\frac{y}{2}\right) \mathrm{d}y$$

$$f(y) = \frac{y^2}{4} + c_1$$

Where c_1 is constant of integration. Substituting result found above for $f(y)$ into equation (3) gives ϕ

$$\phi = -\frac{\ln(x^2+1)}{2} + \frac{y^2}{4} + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = -\frac{\ln(x^2+1)}{2} + \frac{y^2}{4}$$

Initial conditions are used to solve for c_1. Substituting $x = 0$ and $y = -2$ in the above solution gives an equation to solve for the constant of integration.

$$1 = c_1$$

$$c_1 = 1$$

Substituting c_1 found above in the general solution gives

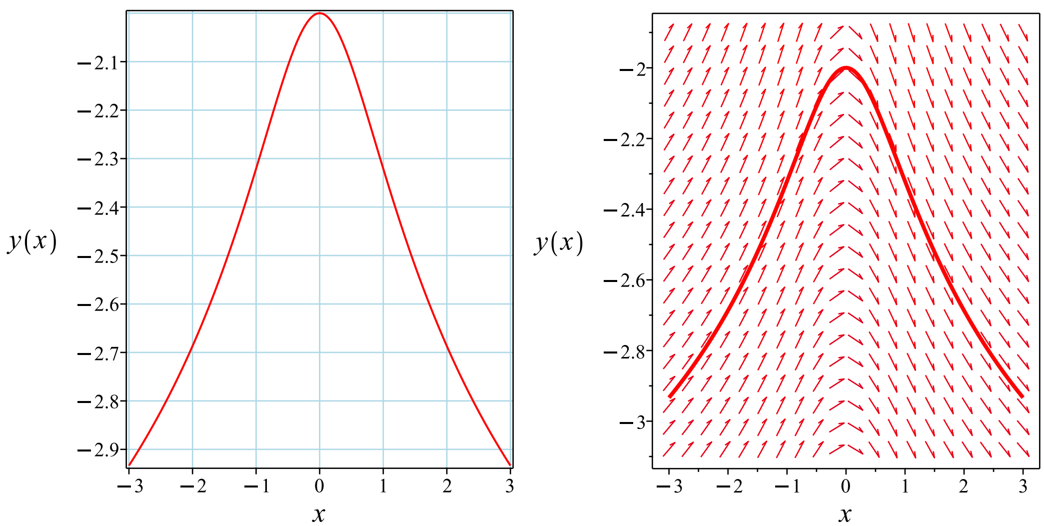
$$-\frac{\ln(x^2+1)}{2} + \frac{y^2}{4} = 1$$

Solving for y from the above gives

$$y = -\sqrt{2\ln(x^2+1)+4}$$

Summary
The solution(s) found are the following

$$y = -\sqrt{2\ln(x^2+1)+4} \tag{1}$$



(a) Solution plot (b) Slope field plot

Verification of solutions

$$y = -\sqrt{2\ln(x^2+1)+4}$$

Verified OK.

2.13.5 Maple step by step solution

Let's solve

$\left[y' - \frac{2x}{y+x^2y} = 0, y(0) = -2\right]$

- Highest derivative means the order of the ODE is 1

  y'

- Separate variables

  $yy' = \frac{2x}{x^2+1}$

- Integrate both sides with respect to x

  $\int yy'dx = \int \frac{2x}{x^2+1}dx + c_1$

- Evaluate integral

  $\frac{y^2}{2} = \ln(x^2+1) + c_1$

- Solve for y

  $\left\{y = \sqrt{2\ln(x^2+1)+2c_1}, y = -\sqrt{2\ln(x^2+1)+2c_1}\right\}$

- Use initial condition $y(0) = -2$

  $-2 = \sqrt{c_1}\sqrt{2}$

- Solution does not satisfy initial condition
- Use initial condition $y(0) = -2$

  $-2 = -\sqrt{c_1}\sqrt{2}$

- Solve for c_1

  $c_1 = 2$

- Substitute $c_1 = 2$ into general solution and simplify

  $y = -\sqrt{2\ln(x^2+1)+4}$

- Solution to the IVP

  $y = -\sqrt{2\ln(x^2+1)+4}$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
<- Bernoulli successful`
```

✓ Solution by Maple

Time used: 0.031 (sec). Leaf size: 18

```
dsolve([diff(y(x),x) = 2*x/(y(x)+x^2*y(x)),y(0) = -2],y(x), singsol=all)
```

$$y(x) = -\sqrt{2\ln\left(x^2+1\right)+4}$$

✓ Solution by Mathematica

Time used: 0.091 (sec). Leaf size: 24

```
DSolve[{y'[x] == 2*x/(y[x]+x^2*y[x]),y[0]==-2},y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to -\sqrt{2}\sqrt{\log\left(x^2+1\right)+2}$$

2.14 problem 14

Internal problem ID [492]
Internal file name [`OUTPUT/492_Sunday_June_05_2022_01_42_22_AM_57989922/index.tex`]

Book: Elementary differential equations and boundary value problems, 10th ed., Boyce and DiPrima
Section: Section 2.2. Page 48
Problem number: 14.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"exact", "riccati", "separable", "first_order_ode_lie_symmetry_lookup"**

Maple gives the following as the ode type

`[_separable]`

$$\boxed{y' - \frac{x y^2}{\sqrt{x^2+1}} = 0}$$

With initial conditions

$$[y(0) = 1]$$

2.14.1 Existence and uniqueness analysis

This is non linear first order ODE. In canonical form it is written as

$$\begin{aligned}
y' &= f(x,y)\\
&= \frac{x\,y^2}{\sqrt{x^2+1}}
\end{aligned}$$

The x domain of $f(x,y)$ when $y=1$ is

$$\{-\infty < x < \infty\}$$

And the point $x_0 = 0$ is inside this domain. The y domain of $f(x,y)$ when $x=0$ is

$$\{-\infty < y < \infty\}$$

And the point $y_0 = 1$ is inside this domain. Now we will look at the continuity of

$$\begin{aligned}
\frac{\partial f}{\partial y} &= \frac{\partial}{\partial y}\left(\frac{x\,y^2}{\sqrt{x^2+1}}\right)\\
&= \frac{2xy}{\sqrt{x^2+1}}
\end{aligned}$$

The x domain of $\frac{\partial f}{\partial y}$ when $y=1$ is

$$\{-\infty < x < \infty\}$$

And the point $x_0 = 0$ is inside this domain. The y domain of $\frac{\partial f}{\partial y}$ when $x=0$ is

$$\{-\infty < y < \infty\}$$

And the point $y_0 = 1$ is inside this domain. Therefore solution exists and is unique.

2.14.2 Solving as separable ode

In canonical form the ODE is

$$\begin{aligned}
y' &= F(x,y)\\
&= f(x)g(y)\\
&= \frac{x\,y^2}{\sqrt{x^2+1}}
\end{aligned}$$

Where $f(x) = \frac{x}{\sqrt{x^2+1}}$ and $g(y) = y^2$. Integrating both sides gives

$$\frac{1}{y^2}\,dy = \frac{x}{\sqrt{x^2+1}}\,dx$$

$$\int \frac{1}{y^2}\,dy = \int \frac{x}{\sqrt{x^2+1}}\,dx$$

$$-\frac{1}{y} = \sqrt{x^2+1} + c_1$$

Which results in

$$y = -\frac{1}{\sqrt{x^2+1} + c_1}$$

Initial conditions are used to solve for c_1. Substituting $x = 0$ and $y = 1$ in the above solution gives an equation to solve for the constant of integration.

$$1 = -\frac{1}{1 + c_1}$$

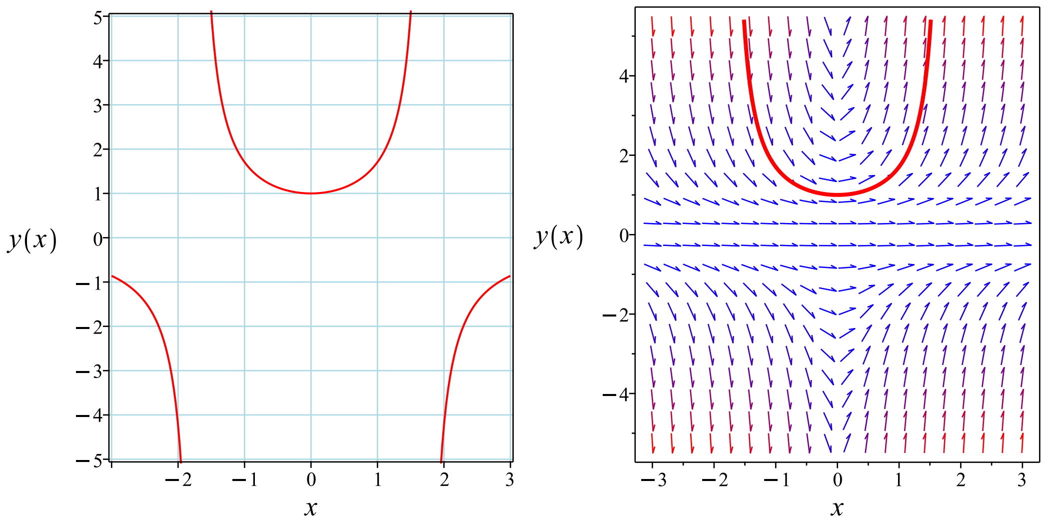
$$c_1 = -2$$

Substituting c_1 found above in the general solution gives

$$y = -\frac{1}{\sqrt{x^2+1} - 2}$$

Summary
The solution(s) found are the following

$$y = -\frac{1}{\sqrt{x^2+1} - 2} \tag{1}$$



(a) Solution plot

(b) Slope field plot

Verification of solutions

$$y = -\frac{1}{\sqrt{x^2+1}-2}$$

Verified OK.

2.14.3 Solving as first order ode lie symmetry lookup ode

Writing the ode as

$$y' = \frac{x\,y^2}{\sqrt{x^2+1}}$$
$$y' = \omega(x,y)$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x + \omega(\eta_y - \xi_x) - \omega^2\xi_y - \omega_x\xi - \omega_y\eta = 0 \tag{A}$$

The type of this ode is known. It is of type `separable`. Therefore we do not need to solve the PDE (A), and can just use the lookup table shown below to find ξ, η

Table 130: Lie symmetry infinitesimal lookup table for known first order ODE's

ODE class	Form	ξ	η
linear ode	$y' = f(x)y(x) + g(x)$	0	$e^{\int f dx}$
separable ode	$y' = f(x)\, g(y)$	$\frac{1}{f}$	0
quadrature ode	$y' = f(x)$	0	1
quadrature ode	$y' = g(y)$	1	0
homogeneous ODEs of Class A	$y' = f\left(\frac{y}{x}\right)$	x	y
homogeneous ODEs of Class C	$y' = (a + bx + cy)^{\frac{n}{m}}$	1	$-\frac{b}{c}$
homogeneous class D	$y' = \frac{y}{x} + g(x)\, F\left(\frac{y}{x}\right)$	x^2	xy
First order special form ID 1	$y' = g(x)\, e^{h(x)+by} + f(x)$	$\frac{e^{-\int b f(x) dx - h(x)}}{g(x)}$	$\frac{f(x) e^{-\int b f(x) dx - h(x)}}{g(x)}$
polynomial type ode	$y' = \frac{a_1 x + b_1 y + c_1}{a_2 x + b_2 y + c_2}$	$\frac{a_1 b_2 x - a_2 b_1 x - b_1 c_2 + b_2 c_1}{a_1 b_2 - a_2 b_1}$	$\frac{a_1 b_2 y - a_2 b_1 y - a_1 c_2 - a_2 c_1}{a_1 b_2 - a_2 b_1}$
Bernoulli ode	$y' = f(x)\, y + g(x)\, y^n$	0	$e^{-\int (n-1) f(x) dx} y^n$
Reduced Riccati	$y' = f_1(x)\, y + f_2(x)\, y^2$	0	$e^{-\int f_1 dx}$

The above table shows that

$$
\begin{aligned}
\xi(x, y) &= \frac{\sqrt{x^2 + 1}}{x} \\
\eta(x, y) &= 0
\end{aligned} \tag{A1}
$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x, y) \to (R, S)$ where (R, S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$
\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}
$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y}\right) S(x, y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the

canonical coordinates, where $S(R)$. Since $\eta = 0$ then in this special case

$$R = y$$

S is found from

$$\begin{aligned} S &= \int \frac{1}{\xi} dx \\ &= \int \frac{1}{\frac{\sqrt{x^2+1}}{x}} dx \end{aligned}$$

Which results in

$$S = \sqrt{x^2+1}$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x,y) S_y}{R_x + \omega(x,y) R_y} \tag{2}$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x, y)$ is the right hand side of the original ode given by

$$\omega(x,y) = \frac{x\, y^2}{\sqrt{x^2+1}}$$

Evaluating all the partial derivatives gives

$$\begin{aligned} R_x &= 0 \\ R_y &= 1 \\ S_x &= \frac{x}{\sqrt{x^2+1}} \\ S_y &= 0 \end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = \frac{1}{y^2} \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = \frac{1}{R^2}$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$S(R) = -\frac{1}{R} + c_1 \tag{4}$$

To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$\sqrt{x^2+1} = -\frac{1}{y} + c_1$$

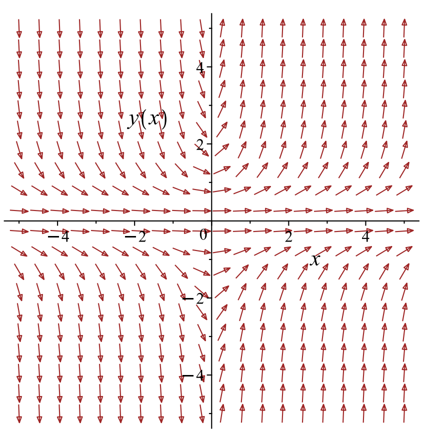
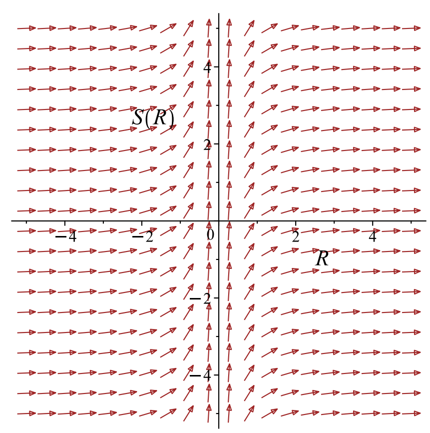
Which simplifies to

$$\sqrt{x^2+1} = -\frac{1}{y} + c_1$$

Which gives

$$y = -\frac{1}{\sqrt{x^2+1} - c_1}$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in x, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dx} = \frac{x\,y^2}{\sqrt{x^2+1}}$  	$R = y$ $S = \sqrt{x^2+1}$	$\frac{dS}{dR} = \frac{1}{R^2}$ 

Initial conditions are used to solve for c_1. Substituting $x = 0$ and $y = 1$ in the above solution gives an equation to solve for the constant of integration.

$$1 = \frac{1}{c_1 - 1}$$

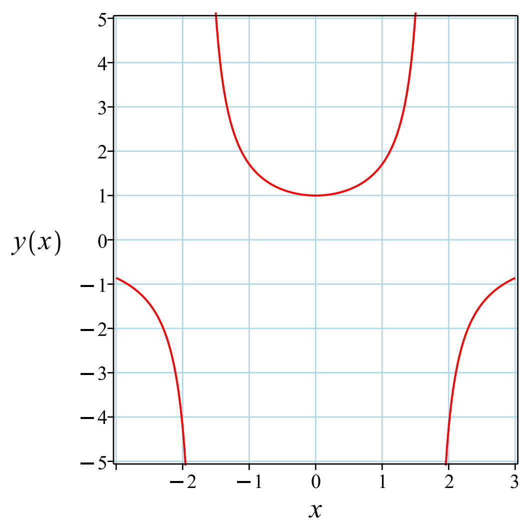
$$c_1 = 2$$

Substituting c_1 found above in the general solution gives

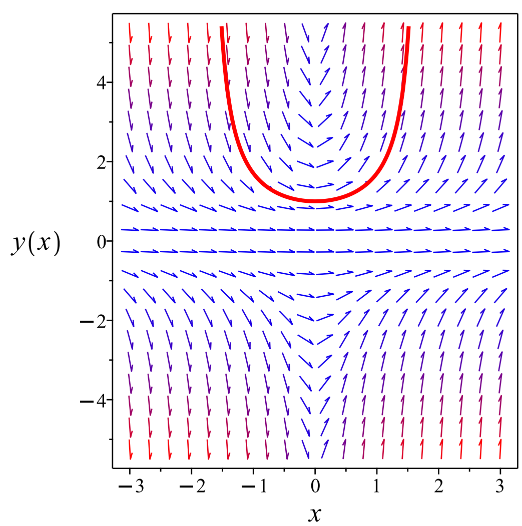
$$y = -\frac{1}{\sqrt{x^2+1} - 2}$$

Summary
The solution(s) found are the following

$$y = -\frac{1}{\sqrt{x^2+1} - 2} \tag{1}$$



(a) Solution plot



(b) Slope field plot

Verification of solutions

$$y = -\frac{1}{\sqrt{x^2+1} - 2}$$

Verified OK.

2.14.4 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x, y) + N(x, y)\frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x, y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x, y) = 0$$

Hence

$$\frac{\partial \phi}{\partial x} + \frac{\partial \phi}{\partial y}\frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$\begin{aligned}\frac{\partial \phi}{\partial x} &= M\\ \frac{\partial \phi}{\partial y} &= N\end{aligned}$$

But since $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x, y)$ but at least we know now that we can do that since the condition $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x, y)\,\mathrm{d}x + N(x, y)\,\mathrm{d}y = 0 \tag{1A}$$

Therefore

$$\begin{aligned}\left(\frac{1}{y^2}\right)\mathrm{d}y &= \left(\frac{x}{\sqrt{x^2+1}}\right)\mathrm{d}x\\ \left(-\frac{x}{\sqrt{x^2+1}}\right)\mathrm{d}x + \left(\frac{1}{y^2}\right)\mathrm{d}y &= 0 \end{aligned} \tag{2A}$$

Comparing (1A) and (2A) shows that

$$
\begin{aligned}
M(x,y) &= -\frac{x}{\sqrt{x^2+1}} \\
N(x,y) &= \frac{1}{y^2}
\end{aligned}
$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$
\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}
$$

Using result found above gives

$$
\begin{aligned}
\frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}\left(-\frac{x}{\sqrt{x^2+1}}\right) \\
&= 0
\end{aligned}
$$

And

$$
\begin{aligned}
\frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}\left(\frac{1}{y^2}\right) \\
&= 0
\end{aligned}
$$

Since $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$, then the ODE is exact The following equations are now set up to solve for the function $\phi(x,y)$

$$
\frac{\partial \phi}{\partial x} = M \tag{1}
$$

$$
\frac{\partial \phi}{\partial y} = N \tag{2}
$$

Integrating (1) w.r.t. x gives

$$
\begin{aligned}
\int \frac{\partial \phi}{\partial x} \, \mathrm{d}x &= \int M \, \mathrm{d}x \\
\int \frac{\partial \phi}{\partial x} \, \mathrm{d}x &= \int -\frac{x}{\sqrt{x^2+1}} \, \mathrm{d}x
\end{aligned}
$$

$$
\phi = -\sqrt{x^2+1} + f(y) \tag{3}
$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$
\frac{\partial \phi}{\partial y} = 0 + f'(y) \tag{4}
$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = \frac{1}{y^2}$. Therefore equation (4) becomes

$$\frac{1}{y^2} = 0 + f'(y) \tag{5}$$

Solving equation (5) for $f'(y)$ gives

$$f'(y) = \frac{1}{y^2}$$

Integrating the above w.r.t y gives

$$\int f'(y)\, \mathrm{d}y = \int \left(\frac{1}{y^2}\right) \mathrm{d}y$$

$$f(y) = -\frac{1}{y} + c_1$$

Where c_1 is constant of integration. Substituting result found above for $f(y)$ into equation (3) gives ϕ

$$\phi = -\sqrt{x^2+1} - \frac{1}{y} + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = -\sqrt{x^2+1} - \frac{1}{y}$$

The solution becomes

$$y = -\frac{1}{\sqrt{x^2+1} + c_1}$$

Initial conditions are used to solve for c_1. Substituting $x = 0$ and $y = 1$ in the above solution gives an equation to solve for the constant of integration.

$$1 = -\frac{1}{1 + c_1}$$

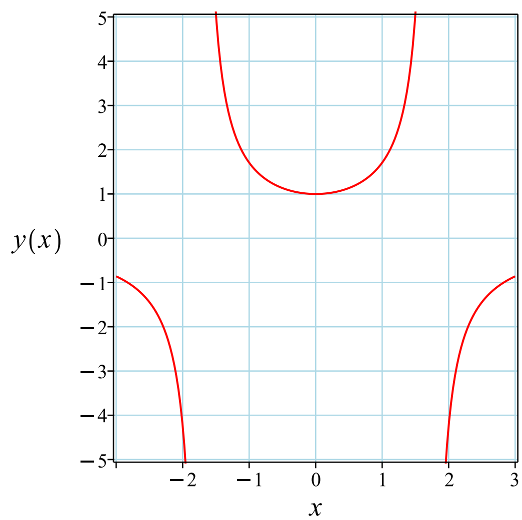
$$c_1 = -2$$

Substituting c_1 found above in the general solution gives

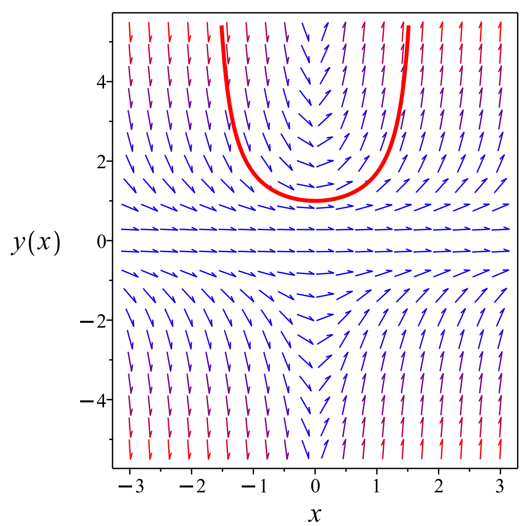
$$y = -\frac{1}{\sqrt{x^2+1}-2}$$

<u>Summary</u>
The solution(s) found are the following

$$y = -\frac{1}{\sqrt{x^2+1}-2} \tag{1}$$



(a) Solution plot



(b) Slope field plot

<u>Verification of solutions</u>

$$y = -\frac{1}{\sqrt{x^2+1}-2}$$

Verified OK.

2.14.5 Solving as riccati ode

In canonical form the ODE is

$$\begin{aligned} y' &= F(x,y) \\ &= \frac{x\,y^2}{\sqrt{x^2+1}} \end{aligned}$$

This is a Riccati ODE. Comparing the ODE to solve

$$y' = \frac{x\,y^2}{\sqrt{x^2+1}}$$

With Riccati ODE standard form

$$y' = f_0(x) + f_1(x)y + f_2(x)y^2$$

Shows that $f_0(x) = 0$, $f_1(x) = 0$ and $f_2(x) = \frac{x}{\sqrt{x^2+1}}$. Let

$$\begin{aligned} y &= \frac{-u'}{f_2 u} \\ &= \frac{-u'}{\frac{xu}{\sqrt{x^2+1}}} \end{aligned} \tag{1}$$

Using the above substitution in the given ODE results (after some simplification)in a second order ODE to solve for $u(x)$ which is

$$f_2 u''(x) - (f_2' + f_1 f_2) u'(x) + f_2^2 f_0 u(x) = 0 \tag{2}$$

But

$$\begin{aligned} f_2' &= \frac{1}{\sqrt{x^2+1}} - \frac{x^2}{(x^2+1)^{\frac{3}{2}}} \\ f_1 f_2 &= 0 \\ f_2^2 f_0 &= 0 \end{aligned}$$

Substituting the above terms back in equation (2) gives

$$\frac{x u''(x)}{\sqrt{x^2+1}} - \left(\frac{1}{\sqrt{x^2+1}} - \frac{x^2}{(x^2+1)^{\frac{3}{2}}}\right) u'(x) = 0$$

Solving the above ODE (this ode solved using Maple, not this program), gives

$$u(x) = c_1 + \sqrt{x^2+1}\, c_2$$

The above shows that

$$u'(x) = \frac{c_2 x}{\sqrt{x^2+1}}$$

Using the above in (1) gives the solution

$$y = -\frac{c_2}{c_1 + \sqrt{x^2+1}\, c_2}$$

Dividing both numerator and denominator by c_1 gives, after renaming the constant $\frac{c_2}{c_1} = c_3$ the following solution

$$y = -\frac{1}{c_3 + \sqrt{x^2 + 1}}$$

Initial conditions are used to solve for c_3. Substituting $x = 0$ and $y = 1$ in the above solution gives an equation to solve for the constant of integration.

$$1 = -\frac{1}{c_3 + 1}$$

$$c_3 = -2$$

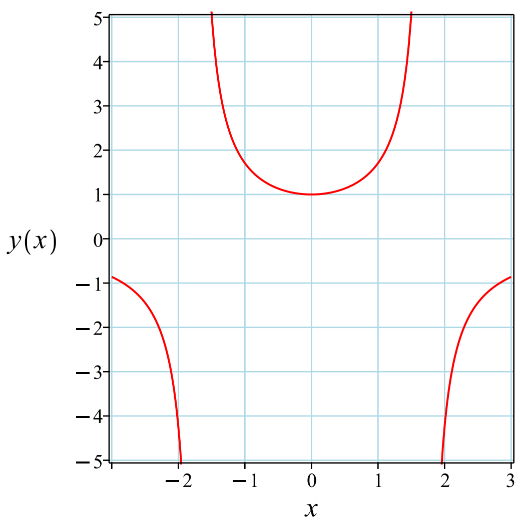
Substituting c_3 found above in the general solution gives

$$y = -\frac{1}{\sqrt{x^2 + 1} - 2}$$

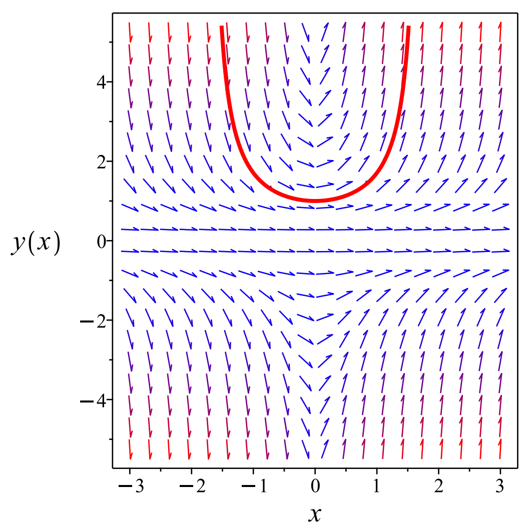
Summary

The solution(s) found are the following

$$y = -\frac{1}{\sqrt{x^2 + 1} - 2} \tag{1}$$



(a) Solution plot



(b) Slope field plot

Verification of solutions

$$y = -\frac{1}{\sqrt{x^2 + 1} - 2}$$

Verified OK.

2.14.6 Maple step by step solution

Let's solve

$\left[y' - \frac{x y^2}{\sqrt{x^2+1}} = 0, y(0) = 1\right]$

- Highest derivative means the order of the ODE is 1

  y'
- Separate variables

  $\frac{y'}{y^2} = \frac{x}{\sqrt{x^2+1}}$
- Integrate both sides with respect to x

  $\int \frac{y'}{y^2} dx = \int \frac{x}{\sqrt{x^2+1}} dx + c_1$
- Evaluate integral

  $-\frac{1}{y} = \sqrt{x^2+1} + c_1$
- Solve for y

  $y = -\frac{1}{\sqrt{x^2+1}+c_1}$
- Use initial condition $y(0) = 1$

  $1 = -\frac{1}{1+c_1}$
- Solve for c_1

  $c_1 = -2$
- Substitute $c_1 = -2$ into general solution and simplify

  $y = -\frac{1}{\sqrt{x^2+1}-2}$
- Solution to the IVP

  $y = -\frac{1}{\sqrt{x^2+1}-2}$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
<- Bernoulli successful`
```

✓ Solution by Maple

Time used: 0.032 (sec). Leaf size: 17

```
dsolve([diff(y(x),x) = x*y(x)^2/(x^2+1)^(1/2),y(0) = 1],y(x), singsol=all)
```

$$y(x) = -\frac{1}{\sqrt{x^2+1}-2}$$

✓ Solution by Mathematica

Time used: 0.183 (sec). Leaf size: 20

```
DSolve[{y'[x] == x*y[x]^2/(x^2+1)^(1/2),y[0]==1},y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \frac{1}{2-\sqrt{x^2+1}}$$

2.15 problem 15

Internal problem ID [493]
Internal file name [`OUTPUT/493_Sunday_June_05_2022_01_42_24_AM_81327021/index.tex`]

Book: Elementary differential equations and boundary value problems, 10th ed., Boyce and DiPrima
Section: Section 2.2. Page 48
Problem number: 15.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"exact", "separable", "differentialType", "homogeneousTypeMapleC", "first_order_ode_lie_symmetry_lookup"**

Maple gives the following as the ode type

`[_separable]`

$$\boxed{y' - \frac{2x}{1+2y} = 0}$$

With initial conditions

$$[y(2) = 0]$$

2.15.1 Existence and uniqueness analysis

This is non linear first order ODE. In canonical form it is written as

$$\begin{aligned} y' &= f(x,y) \\ &= \frac{2x}{1+2y} \end{aligned}$$

The x domain of $f(x,y)$ when $y=0$ is

$$\{-\infty < x < \infty\}$$

And the point $x_0 = 2$ is inside this domain. The y domain of $f(x,y)$ when $x=2$ is

$$\left\{y < -\frac{1}{2} \lor -\frac{1}{2} < y\right\}$$

And the point $y_0 = 0$ is inside this domain. Now we will look at the continuity of

$$\begin{aligned} \frac{\partial f}{\partial y} &= \frac{\partial}{\partial y}\left(\frac{2x}{1+2y}\right) \\ &= -\frac{4x}{(1+2y)^2} \end{aligned}$$

The x domain of $\frac{\partial f}{\partial y}$ when $y=0$ is

$$\{-\infty < x < \infty\}$$

And the point $x_0 = 2$ is inside this domain. The y domain of $\frac{\partial f}{\partial y}$ when $x=2$ is

$$\left\{y < -\frac{1}{2} \lor -\frac{1}{2} < y\right\}$$

And the point $y_0 = 0$ is inside this domain. Therefore solution exists and is unique.

2.15.2 Solving as separable ode

In canonical form the ODE is

$$\begin{aligned} y' &= F(x,y) \\ &= f(x)g(y) \\ &= \frac{2x}{1+2y} \end{aligned}$$

Where $f(x) = 2x$ and $g(y) = \frac{1}{1+2y}$. Integrating both sides gives

$$\begin{aligned} \frac{1}{\frac{1}{1+2y}}\,dy &= 2x\,dx \\ \int \frac{1}{\frac{1}{1+2y}}\,dy &= \int 2x\,dx \\ y^2 + y &= x^2 + c_1 \end{aligned}$$

Which results in

$$\begin{aligned} y &= -\frac{1}{2} + \frac{\sqrt{4x^2+4c_1+1}}{2} \\ y &= -\frac{1}{2} - \frac{\sqrt{4x^2+4c_1+1}}{2} \end{aligned}$$

Initial conditions are used to solve for c_1. Substituting $x = 2$ and $y = 0$ in the above solution gives an equation to solve for the constant of integration.

$$0 = -\frac{1}{2} - \frac{\sqrt{17+4c_1}}{2}$$

Warning: Unable to solve for constant of integration. Initial conditions are used to solve for c_1. Substituting $x = 2$ and $y = 0$ in the above solution gives an equation to solve for the constant of integration.

$$0 = -\frac{1}{2} + \frac{\sqrt{17+4c_1}}{2}$$

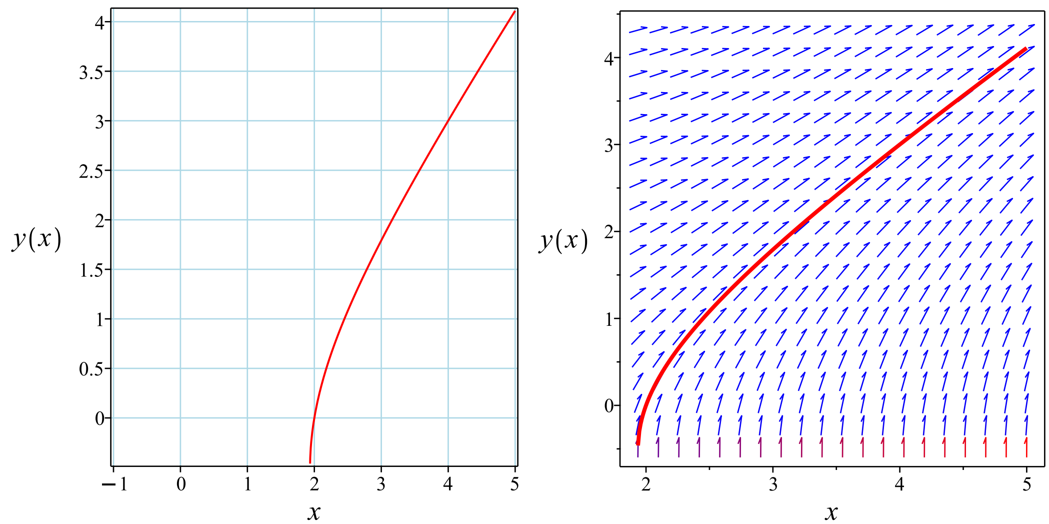
$$c_1 = -4$$

Substituting c_1 found above in the general solution gives

$$y = -\frac{1}{2} + \frac{\sqrt{4x^2-15}}{2}$$

Summary
The solution(s) found are the following

$$y = -\frac{1}{2} + \frac{\sqrt{4x^2 - 15}}{2} \tag{1}$$



(a) Solution plot (b) Slope field plot

Verification of solutions

$$y = -\frac{1}{2} + \frac{\sqrt{4x^2 - 15}}{2}$$

Verified OK.

2.15.3 Solving as differentialType ode

Writing the ode as

$$y' = \frac{2x}{1 + 2y} \tag{1}$$

Which becomes

$$(1 + 2y)\, dy = (2x)\, dx \tag{2}$$

But the RHS is complete differential because

$$(2x)\, dx = d(x^2)$$

Hence (2) becomes

$$(1+2y)\,dy = d(x^2)$$

Integrating both sides gives gives these solutions

$$y = -\frac{1}{2} + \frac{\sqrt{4x^2+4c_1+1}}{2} + c_1$$
$$y = -\frac{1}{2} - \frac{\sqrt{4x^2+4c_1+1}}{2} + c_1$$

Initial conditions are used to solve for c_1. Substituting $x = 2$ and $y = 0$ in the above solution gives an equation to solve for the constant of integration.

$$0 = -\frac{1}{2} - \frac{\sqrt{17+4c_1}}{2} + c_1$$

$$c_1 = \sqrt{5}+1$$

Substituting c_1 found above in the general solution gives

$$y = \frac{1}{2} - \frac{\sqrt{4x^2+4\sqrt{5}+5}}{2} + \sqrt{5}$$

Initial conditions are used to solve for c_1. Substituting $x = 2$ and $y = 0$ in the above solution gives an equation to solve for the constant of integration.

$$0 = -\frac{1}{2} + \frac{\sqrt{17+4c_1}}{2} + c_1$$

$$c_1 = -\sqrt{5}+1$$

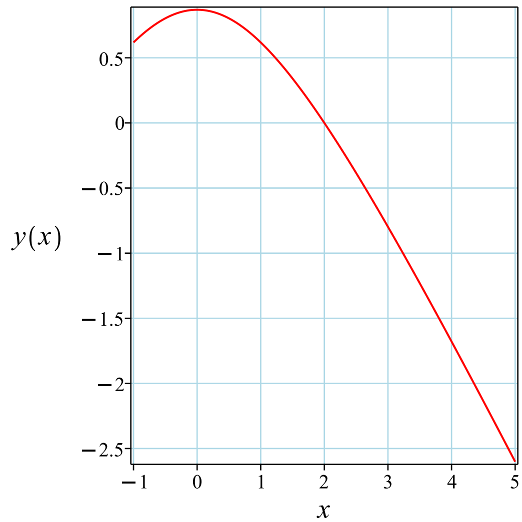
Substituting c_1 found above in the general solution gives

$$y = \frac{1}{2} + \frac{\sqrt{4x^2-4\sqrt{5}+5}}{2} - \sqrt{5}$$

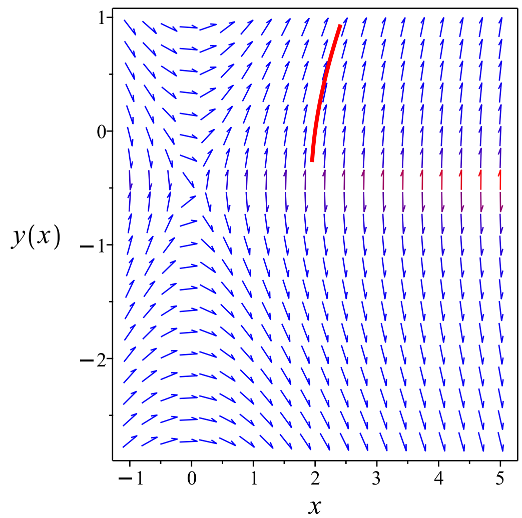
Summary

The solution(s) found are the following

$$y = \frac{1}{2} + \frac{\sqrt{4x^2-4\sqrt{5}+5}}{2} - \sqrt{5} \tag{1}$$
$$y = \frac{1}{2} - \frac{\sqrt{4x^2+4\sqrt{5}+5}}{2} + \sqrt{5} \tag{2}$$



(a) Solution plot



(b) Slope field plot

<u>Verification of solutions</u>

$$y = \frac{1}{2} + \frac{\sqrt{4x^2 - 4\sqrt{5} + 5}}{2} - \sqrt{5}$$

Verified OK.

$$y = \frac{1}{2} - \frac{\sqrt{4x^2 + 4\sqrt{5} + 5}}{2} + \sqrt{5}$$

Verified OK.

2.15.4 Solving as homogeneousTypeMapleC ode

Let $Y = y + y_0$ and $X = x + x_0$ then the above is transformed to new ode in $Y(X)$

$$\frac{d}{dX}Y(X) = \frac{2X + 2x_0}{1 + 2Y(X) + 2y_0}$$

Solving for possible values of x_0 and y_0 which makes the above ode a homogeneous ode results in

$$x_0 = 0$$
$$y_0 = -\frac{1}{2}$$

Using these values now it is possible to easily solve for $Y(X)$. The above ode now becomes

$$\frac{d}{dX}Y(X) = \frac{X}{Y(X)}$$

In canonical form, the ODE is

$$\begin{aligned} Y' &= F(X, Y) \\ &= \frac{X}{Y} \end{aligned} \tag{1}$$

An ode of the form $Y' = \frac{M(X,Y)}{N(X,Y)}$ is called homogeneous if the functions $M(X, Y)$ and $N(X, Y)$ are both homogeneous functions and of the same order. Recall that a function $f(X, Y)$ is homogeneous of order n if

$$f(t^n X, t^n Y) = t^n f(X, Y)$$

In this case, it can be seen that both $M = X$ and $N = Y$ are both homogeneous and of the same order $n = 1$. Therefore this is a homogeneous ode. Since this ode is homogeneous, it is converted to separable ODE using the substitution $u = \frac{Y}{X}$, or $Y = uX$. Hence

$$\frac{\mathrm{d}Y}{\mathrm{d}X} = \frac{\mathrm{d}u}{\mathrm{d}X}X + u$$

Applying the transformation $Y = uX$ to the above ODE in (1) gives

$$\begin{aligned} \frac{\mathrm{d}u}{\mathrm{d}X}X + u &= \frac{1}{u} \\ \frac{\mathrm{d}u}{\mathrm{d}X} &= \frac{\frac{1}{u(X)} - u(X)}{X} \end{aligned}$$

Or

$$\frac{d}{dX}u(X) - \frac{\frac{1}{u(X)} - u(X)}{X} = 0$$

Or

$$\left(\frac{d}{dX}u(X)\right) u(X) X + u(X)^2 - 1 = 0$$

Which is now solved as separable in $u(X)$. Which is now solved in $u(X)$. In canonical form the ODE is

$$\begin{aligned} u' &= F(X, u) \\ &= f(X)g(u) \\ &= -\frac{u^2 - 1}{uX} \end{aligned}$$

Where $f(X) = -\frac{1}{X}$ and $g(u) = \frac{u^2-1}{u}$. Integrating both sides gives

$$\frac{1}{\frac{u^2-1}{u}}\,du = -\frac{1}{X}\,dX$$

$$\int \frac{1}{\frac{u^2-1}{u}}\,du = \int -\frac{1}{X}\,dX$$

$$\frac{\ln(u-1)}{2} + \frac{\ln(u+1)}{2} = -\ln(X) + c_2$$

The above can be written as

$$\left(\frac{1}{2}\right)(\ln(u-1) + \ln(u+1)) = -\ln(X) + 2c_2$$

$$\begin{aligned}\ln(u-1) + \ln(u+1) &= (2)(-\ln(X) + 2c_2)\\ &= -2\ln(X) + 4c_2\end{aligned}$$

Raising both side to exponential gives

$$e^{\ln(u-1)+\ln(u+1)} = e^{-2\ln(X)+2c_2}$$

Which simplifies to

$$\begin{aligned}u^2 - 1 &= \frac{2c_2}{X^2}\\ &= \frac{c_3}{X^2}\end{aligned}$$

The solution is

$$u(X)^2 - 1 = \frac{c_3}{X^2}$$

Now u in the above solution is replaced back by Y using $u = \frac{Y}{X}$ which results in the solution

$$\frac{Y(X)^2}{X^2} - 1 = \frac{c_3}{X^2}$$

Which simplifies to

$$-(X - Y(X))(X + Y(X)) = c_3$$

Using the solution for $Y(X)$

$$-(X - Y(X))(X + Y(X)) = c_3$$

And replacing back terms in the above solution using

$$
\begin{aligned}
Y &= y + y_0 \\
X &= x + x_0
\end{aligned}
$$

Or

$$
\begin{aligned}
Y &= y - \frac{1}{2} \\
X &= x
\end{aligned}
$$

Then the solution in y becomes

$$
-\left(x - y - \frac{1}{2}\right)\left(x + y + \frac{1}{2}\right) = c_3
$$

Initial conditions are used to solve for c_3. Substituting $x = 2$ and $y = 0$ in the above solution gives an equation to solve for the constant of integration.

$$
-\frac{15}{4} = c_3
$$

$$
c_3 = -\frac{15}{4}
$$

Substituting c_3 found above in the general solution gives

$$
-\left(x - y - \frac{1}{2}\right)\left(x + y + \frac{1}{2}\right) = -\frac{15}{4}
$$

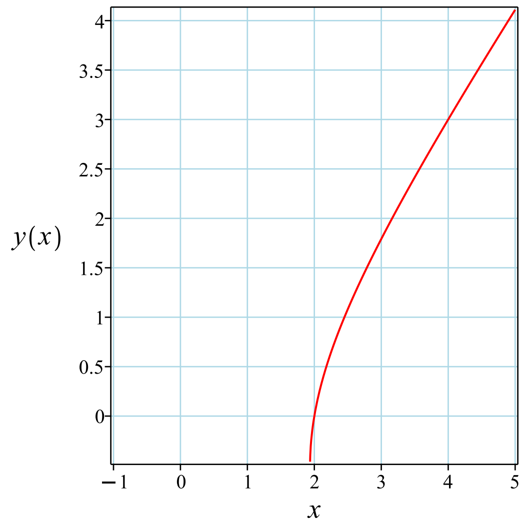
Solving for y from the above gives

$$
y = -\frac{1}{2} + \frac{\sqrt{4x^2 - 15}}{2}
$$

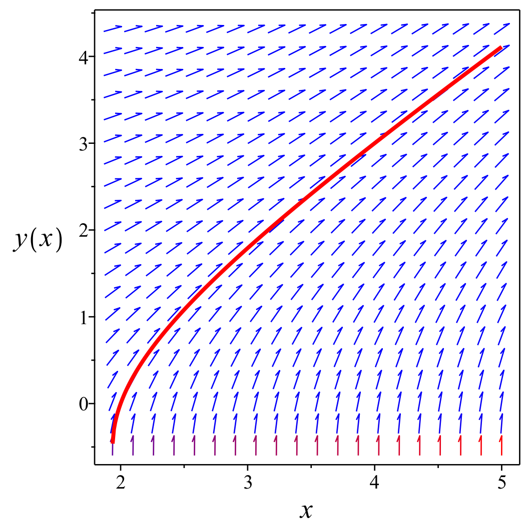
<u>Summary</u>

The solution(s) found are the following

$$
y = -\frac{1}{2} + \frac{\sqrt{4x^2 - 15}}{2} \tag{1}
$$



(a) Solution plot



(b) Slope field plot

<u>Verification of solutions</u>

$$y = -\frac{1}{2} + \frac{\sqrt{4x^2 - 15}}{2}$$

Verified OK.

2.15.5 Solving as first order ode lie symmetry lookup ode

Writing the ode as

$$y' = \frac{2x}{1 + 2y}$$
$$y' = \omega(x, y)$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x + \omega(\eta_y - \xi_x) - \omega^2 \xi_y - \omega_x \xi - \omega_y \eta = 0 \tag{A}$$

The type of this ode is known. It is of type `separable`. Therefore we do not need to solve the PDE (A), and can just use the lookup table shown below to find ξ, η

Table 133: Lie symmetry infinitesimal lookup table for known first order ODE's

ODE class	Form	ξ	η
linear ode	$y' = f(x)y(x) + g(x)$	0	$e^{\int f dx}$
separable ode	$y' = f(x)\, g(y)$	$\frac{1}{f}$	0
quadrature ode	$y' = f(x)$	0	1
quadrature ode	$y' = g(y)$	1	0
homogeneous ODEs of Class A	$y' = f\left(\frac{y}{x}\right)$	x	y
homogeneous ODEs of Class C	$y' = (a + bx + cy)^{\frac{n}{m}}$	1	$-\frac{b}{c}$
homogeneous class D	$y' = \frac{y}{x} + g(x)\, F\left(\frac{y}{x}\right)$	x^2	xy
First order special form ID 1	$y' = g(x)\, e^{h(x)+by} + f(x)$	$\frac{e^{-\int bf(x)dx - h(x)}}{g(x)}$	$\frac{f(x)e^{-\int bf(x)dx - h(x)}}{g(x)}$
polynomial type ode	$y' = \frac{a_1x+b_1y+c_1}{a_2x+b_2y+c_2}$	$\frac{a_1b_2x - a_2b_1x - b_1c_2 + b_2c_1}{a_1b_2 - a_2b_1}$	$\frac{a_1b_2y - a_2b_1y - a_1c_2 - a_2c_1}{a_1b_2 - a_2b_1}$
Bernoulli ode	$y' = f(x)\, y + g(x)\, y^n$	0	$e^{-\int (n-1)f(x)dx} y^n$
Reduced Riccati	$y' = f_1(x)\, y + f_2(x)\, y^2$	0	$e^{-\int f_1 dx}$

The above table shows that

$$
\begin{aligned}
\xi(x,y) &= \frac{1}{2x} \\
\eta(x,y) &= 0
\end{aligned} \tag{A1}
$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x,y) \to (R,S)$ where (R,S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$
\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}
$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y}\right) S(x,y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the

canonical coordinates, where $S(R)$. Since $\eta = 0$ then in this special case

$$R = y$$

S is found from

$$\begin{aligned} S &= \int \frac{1}{\xi} dx \\ &= \int \frac{1}{\frac{1}{2x}} dx \end{aligned}$$

Which results in

$$S = x^2$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x, y) S_y}{R_x + \omega(x, y) R_y} \tag{2}$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x, y)$ is the right hand side of the original ode given by

$$\omega(x, y) = \frac{2x}{1 + 2y}$$

Evaluating all the partial derivatives gives

$$\begin{aligned} R_x &= 0 \\ R_y &= 1 \\ S_x &= 2x \\ S_y &= 0 \end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = 1 + 2y \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = 1 + 2R$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by

integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$S(R) = R^2 + R + c_1 \tag{4}$$

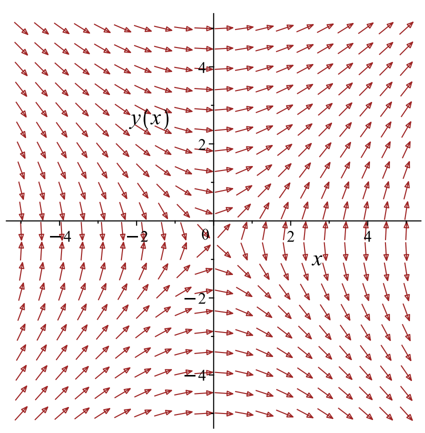
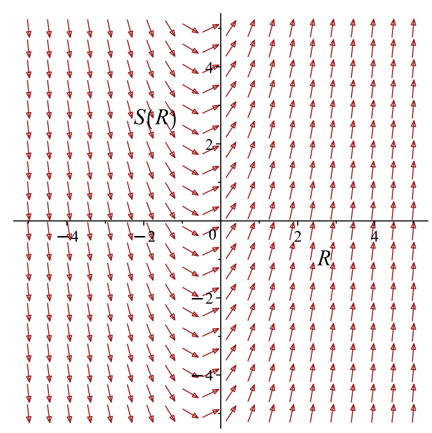
To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$x^2 = y^2 + c_1 + y$$

Which simplifies to

$$x^2 = y^2 + c_1 + y$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in x, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dx} = \frac{2x}{1+2y}$ 	$R = y$ $S = x^2$	$\frac{dS}{dR} = 1 + 2R$ 

Initial conditions are used to solve for c_1. Substituting $x = 2$ and $y = 0$ in the above solution gives an equation to solve for the constant of integration.

$$4 = c_1$$

$$c_1 = 4$$

Substituting c_1 found above in the general solution gives

$$x^2 = y^2 + y + 4$$

<u>Summary</u>
The solution(s) found are the following

$$x^2 = y^2 + y + 4 \tag{1}$$

<u>Verification of solutions</u>

$$x^2 = y^2 + y + 4$$

Verified OK.

2.15.6 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x,y) + N(x,y)\frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x,y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y) = 0$$

Hence

$$\frac{\partial \phi}{\partial x} + \frac{\partial \phi}{\partial y}\frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$\frac{\partial \phi}{\partial x} = M$$
$$\frac{\partial \phi}{\partial y} = N$$

But since $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x,y)$ but at least we know now that we can do that since the condition $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ is satisfied. If this condition is not satisfied then this method will not work

and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x,y)\,\mathrm{d}x + N(x,y)\,\mathrm{d}y = 0 \tag{1A}$$

Therefore

$$\begin{aligned}\left(y+\frac{1}{2}\right)\mathrm{d}y &= (x)\,\mathrm{d}x\\ (-x)\,\mathrm{d}x + \left(y+\frac{1}{2}\right)\mathrm{d}y &= 0 \end{aligned}\tag{2A}$$

Comparing (1A) and (2A) shows that

$$\begin{aligned}M(x,y) &= -x\\ N(x,y) &= y+\frac{1}{2}\end{aligned}$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Using result found above gives

$$\begin{aligned}\frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}(-x)\\ &= 0\end{aligned}$$

And

$$\begin{aligned}\frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}\left(y+\frac{1}{2}\right)\\ &= 0\end{aligned}$$

Since $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$, then the ODE is exact The following equations are now set up to solve for the function $\phi(x,y)$

$$\frac{\partial \phi}{\partial x} = M \tag{1}$$

$$\frac{\partial \phi}{\partial y} = N \tag{2}$$

Integrating (1) w.r.t. x gives

$$\int \frac{\partial \phi}{\partial x} \, dx = \int M \, dx$$
$$\int \frac{\partial \phi}{\partial x} \, dx = \int -x \, dx$$
$$\phi = -\frac{x^2}{2} + f(y) \tag{3}$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$\frac{\partial \phi}{\partial y} = 0 + f'(y) \tag{4}$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = y + \frac{1}{2}$. Therefore equation (4) becomes

$$y + \frac{1}{2} = 0 + f'(y) \tag{5}$$

Solving equation (5) for $f'(y)$ gives

$$f'(y) = y + \frac{1}{2}$$

Integrating the above w.r.t y gives

$$\int f'(y) \, dy = \int \left(y + \frac{1}{2}\right) dy$$
$$f(y) = \frac{1}{2}y^2 + \frac{1}{2}y + c_1$$

Where c_1 is constant of integration. Substituting result found above for $f(y)$ into equation (3) gives ϕ

$$\phi = -\frac{1}{2}x^2 + \frac{1}{2}y^2 + \frac{1}{2}y + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = -\frac{1}{2}x^2 + \frac{1}{2}y^2 + \frac{1}{2}y$$

Initial conditions are used to solve for c_1. Substituting $x = 2$ and $y = 0$ in the above solution gives an equation to solve for the constant of integration.

$$-2 = c_1$$

$$c_1 = -2$$

Substituting c_1 found above in the general solution gives

$$-\frac{1}{2}x^2 + \frac{1}{2}y^2 + \frac{1}{2}y = -2$$

Summary

The solution(s) found are the following

$$-\frac{x^2}{2} + \frac{y^2}{2} + \frac{y}{2} = -2 \tag{1}$$

Verification of solutions

$$-\frac{x^2}{2} + \frac{y^2}{2} + \frac{y}{2} = -2$$

Verified OK.

2.15.7 Maple step by step solution

Let's solve

$\left[y' - \frac{2x}{1+2y} = 0, y(2) = 0\right]$

- Highest derivative means the order of the ODE is 1

  y'

- Separate variables

  $y'(1 + 2y) = 2x$

- Integrate both sides with respect to x

  $\int y'(1 + 2y)\, dx = \int 2x dx + c_1$

- Evaluate integral

  $y^2 + y = x^2 + c_1$

- Solve for y

$$\left\{y=-\frac{1}{2}-\frac{\sqrt{4x^2+4c_1+1}}{2}, y=-\frac{1}{2}+\frac{\sqrt{4x^2+4c_1+1}}{2}\right\}$$

- Use initial condition $y(2)=0$

  $0=-\frac{1}{2}-\frac{\sqrt{17+4c_1}}{2}$
- Solution does not satisfy initial condition
- Use initial condition $y(2)=0$

  $0=-\frac{1}{2}+\frac{\sqrt{17+4c_1}}{2}$
- Solve for c_1

  $c_1=-4$
- Substitute $c_1=-4$ into general solution and simplify

  $y=-\frac{1}{2}+\frac{\sqrt{4x^2-15}}{2}$
- Solution to the IVP

  $y=-\frac{1}{2}+\frac{\sqrt{4x^2-15}}{2}$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
<- separable successful`
```

✓ Solution by Maple

Time used: 0.047 (sec). Leaf size: 17

```
dsolve([diff(y(x),x) = 2*x/(1+2*y(x)),y(2) = 0],y(x), singsol=all)
```

$$y(x)=-\frac{1}{2}+\frac{\sqrt{4x^2-15}}{2}$$

✓ Solution by Mathematica

Time used: 0.115 (sec). Leaf size: 22

```
DSolve[{y'[x] == 2*x/(1+2*y[x]),y[2]==0},y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \frac{1}{2}\left(\sqrt{4x^2-15}-1\right)$$

2.16 problem 16

Internal problem ID [494]
Internal file name [`OUTPUT/494_Sunday_June_05_2022_01_42_24_AM_46498650/index.tex`]

Book: Elementary differential equations and boundary value problems, 10th ed., Boyce and DiPrima
Section: Section 2.2. Page 48
Problem number: 16.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"exact", "separable", "differentialType", "first_order_ode_lie_symmetry_lookup"**

Maple gives the following as the ode type

`[_separable]`

$$\boxed{y' - \frac{x(x^2+1)}{4y^3} = 0}$$

With initial conditions

$$\left[y(0) = -\frac{\sqrt{2}}{2}\right]$$

2.16.1 Existence and uniqueness analysis

This is non linear first order ODE. In canonical form it is written as

$$\begin{aligned} y' &= f(x,y) \\ &= \frac{x(x^2+1)}{4y^3} \end{aligned}$$

The x domain of $f(x,y)$ when $y = -\frac{\sqrt{2}}{2}$ is

$$\{-\infty < x < \infty\}$$

And the point $x_0 = 0$ is inside this domain. The y domain of $f(x,y)$ when $x = 0$ is

$$\{-\infty < y < \infty\}$$

And the point $y_0 = -\frac{\sqrt{2}}{2}$ is inside this domain. Now we will look at the continuity of

$$\begin{aligned} \frac{\partial f}{\partial y} &= \frac{\partial}{\partial y}\left(\frac{x(x^2+1)}{4y^3}\right) \\ &= -\frac{3x(x^2+1)}{4y^4} \end{aligned}$$

The x domain of $\frac{\partial f}{\partial y}$ when $y = -\frac{\sqrt{2}}{2}$ is

$$\{-\infty < x < \infty\}$$

And the point $x_0 = 0$ is inside this domain. The y domain of $\frac{\partial f}{\partial y}$ when $x = 0$ is

$$\{-\infty < y < \infty\}$$

And the point $y_0 = -\frac{\sqrt{2}}{2}$ is inside this domain. Therefore solution exists and is unique.

2.16.2 Solving as separable ode

In canonical form the ODE is

$$\begin{aligned} y' &= F(x,y) \\ &= f(x)g(y) \\ &= \frac{x(x^2+1)}{4y^3} \end{aligned}$$

Where $f(x) = \frac{x(x^2+1)}{4}$ and $g(y) = \frac{1}{y^3}$. Integrating both sides gives

$$\frac{1}{\frac{1}{y^3}}\,dy = \frac{x(x^2+1)}{4}\,dx$$

$$\int \frac{1}{\frac{1}{y^3}}\,dy = \int \frac{x(x^2+1)}{4}\,dx$$

$$\frac{y^4}{4} = \frac{(x^2+1)^2}{16} + c_1$$

The solution is

$$\frac{y^4}{4} - \frac{(x^2+1)^2}{16} - c_1 = 0$$

Initial conditions are used to solve for c_1. Substituting $x = 0$ and $y = -\frac{\sqrt{2}}{2}$ in the above solution gives an equation to solve for the constant of integration.

$$-c_1 = 0$$

$$c_1 = 0$$

Substituting c_1 found above in the general solution gives

$$\frac{1}{4}y^4 - \frac{1}{16}x^4 - \frac{1}{8}x^2 - \frac{1}{16} = 0$$

<u>Summary</u>
The solution(s) found are the following

$$\frac{y^4}{4} - \frac{x^4}{16} - \frac{x^2}{8} - \frac{1}{16} = 0 \tag{1}$$

<u>Verification of solutions</u>

$$\frac{y^4}{4} - \frac{x^4}{16} - \frac{x^2}{8} - \frac{1}{16} = 0$$

Verified OK.

2.16.3 Solving as differentialType ode

Writing the ode as

$$y' = \frac{x(x^2+1)}{4y^3} \tag{1}$$

Which becomes

$$(4y^3)\, dy = (x(x^2+1))\, dx \tag{2}$$

But the RHS is complete differential because

$$(x(x^2+1))\, dx = d\left(\frac{1}{2}x^2 + \frac{1}{4}x^4\right)$$

Hence (2) becomes

$$(4y^3)\, dy = d\left(\frac{1}{2}x^2 + \frac{1}{4}x^4\right)$$

Integrating both sides gives gives these solutions

$$\begin{aligned}
y &= \left(\frac{1}{2}x^2 + \frac{1}{4}x^4 + c_1\right)^{\frac{1}{4}} + c_1 \\
y &= i\left(\frac{1}{2}x^2 + \frac{1}{4}x^4 + c_1\right)^{\frac{1}{4}} + c_1 \\
y &= -\left(\frac{1}{2}x^2 + \frac{1}{4}x^4 + c_1\right)^{\frac{1}{4}} + c_1 \\
y &= -i\left(\frac{1}{2}x^2 + \frac{1}{4}x^4 + c_1\right)^{\frac{1}{4}} + c_1
\end{aligned}$$

Initial conditions are used to solve for c_1. Substituting $x = 0$ and $y = -\frac{\sqrt{2}}{2}$ in the above solution gives an equation to solve for the constant of integration.

$$-\frac{\sqrt{2}}{2} = -ic_1^{\frac{1}{4}} + c_1$$

Warning: Unable to solve for constant of integration. Initial conditions are used to solve for c_1. Substituting $x = 0$ and $y = -\frac{\sqrt{2}}{2}$ in the above solution gives an equation to solve for the constant of integration.

$$-\frac{\sqrt{2}}{2} = -c_1^{\frac{1}{4}} + c_1$$

Warning: Unable to solve for constant of integration. Initial conditions are used to solve for c_1. Substituting $x = 0$ and $y = -\frac{\sqrt{2}}{2}$ in the above solution gives an equation to solve for the constant of integration.

$$-\frac{\sqrt{2}}{2} = i c_1^{\frac{1}{4}} + c_1$$

Warning: Unable to solve for constant of integration. Initial conditions are used to solve for c_1. Substituting $x = 0$ and $y = -\frac{\sqrt{2}}{2}$ in the above solution gives an equation to solve for the constant of integration.

$$-\frac{\sqrt{2}}{2} = c_1^{\frac{1}{4}} + c_1$$

Warning: Unable to solve for constant of integration.

Verification of solutions N/A

2.16.4 Solving as first order ode lie symmetry lookup ode

Writing the ode as

$$\begin{aligned} y' &= \frac{x(x^2+1)}{4y^3} \\ y' &= \omega(x,y) \end{aligned}$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x + \omega(\eta_y - \xi_x) - \omega^2 \xi_y - \omega_x \xi - \omega_y \eta = 0 \tag{A}$$

The type of this ode is known. It is of type `separable`. Therefore we do not need to solve the PDE (A), and can just use the lookup table shown below to find ξ, η

Table 136: Lie symmetry infinitesimal lookup table for known first order ODE's

ODE class	Form	ξ	η
linear ode	$y' = f(x)y(x) + g(x)$	0	$e^{\int f dx}$
separable ode	$y' = f(x)\, g(y)$	$\frac{1}{f}$	0
quadrature ode	$y' = f(x)$	0	1
quadrature ode	$y' = g(y)$	1	0
homogeneous ODEs of Class A	$y' = f\left(\frac{y}{x}\right)$	x	y
homogeneous ODEs of Class C	$y' = (a + bx + cy)^{\frac{n}{m}}$	1	$-\frac{b}{c}$
homogeneous class D	$y' = \frac{y}{x} + g(x)\, F\left(\frac{y}{x}\right)$	x^2	xy
First order special form ID 1	$y' = g(x)\, e^{h(x)+by} + f(x)$	$\frac{e^{-\int b f(x) dx - h(x)}}{g(x)}$	$\frac{f(x) e^{-\int b f(x) dx - h(x)}}{g(x)}$
polynomial type ode	$y' = \frac{a_1 x + b_1 y + c_1}{a_2 x + b_2 y + c_2}$	$\frac{a_1 b_2 x - a_2 b_1 x - b_1 c_2 + b_2 c_1}{a_1 b_2 - a_2 b_1}$	$\frac{a_1 b_2 y - a_2 b_1 y - a_1 c_2 - a_2 c_1}{a_1 b_2 - a_2 b_1}$
Bernoulli ode	$y' = f(x)\, y + g(x)\, y^n$	0	$e^{-\int (n-1) f(x) dx} y^n$
Reduced Riccati	$y' = f_1(x)\, y + f_2(x)\, y^2$	0	$e^{-\int f_1 dx}$

The above table shows that

$$
\begin{aligned}
\xi(x, y) &= \frac{4}{(x^2 + 1)\, x} \\
\eta(x, y) &= 0
\end{aligned} \tag{A1}
$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x, y) \to (R, S)$ where (R, S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$
\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}
$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y}\right) S(x, y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the

canonical coordinates, where $S(R)$. Since $\eta = 0$ then in this special case

$$R = y$$

S is found from

$$\begin{aligned} S &= \int \frac{1}{\xi} dx \\ &= \int \frac{1}{\frac{4}{(x^2+1)x}} dx \end{aligned}$$

Which results in

$$S = \frac{(x^2+1)^2}{16}$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x,y) S_y}{R_x + \omega(x,y) R_y} \tag{2}$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x,y)$ is the right hand side of the original ode given by

$$\omega(x,y) = \frac{x(x^2+1)}{4y^3}$$

Evaluating all the partial derivatives gives

$$\begin{aligned} R_x &= 0 \\ R_y &= 1 \\ S_x &= \frac{x(x^2+1)}{4} \\ S_y &= 0 \end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = y^3 \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = R^3$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordinates R, S. Integrating the above gives

$$S(R) = \frac{R^4}{4} + c_1 \tag{4}$$

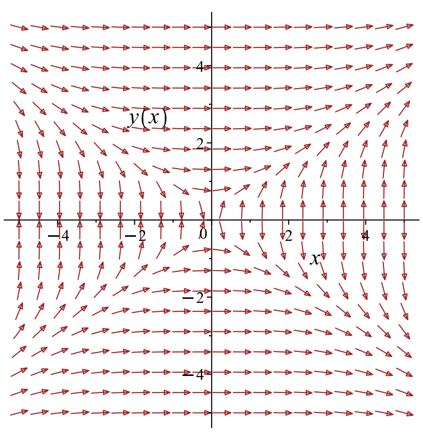
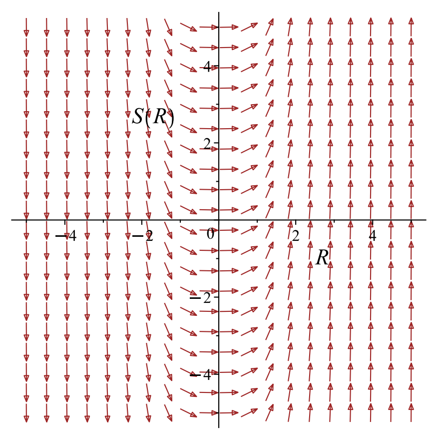
To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$\frac{(x^2+1)^2}{16} = \frac{y^4}{4} + c_1$$

Which simplifies to

$$\frac{(x^2+1)^2}{16} = \frac{y^4}{4} + c_1$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in x, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dx} = \frac{x(x^2+1)}{4y^3}$ 	$R = y$ $S = \frac{(x^2+1)^2}{16}$	$\frac{dS}{dR} = R^3$ 

Initial conditions are used to solve for c_1. Substituting $x = 0$ and $y = -\frac{\sqrt{2}}{2}$ in the above solution gives an equation to solve for the constant of integration.

$$\frac{1}{16} = \frac{1}{16} + c_1$$

$$c_1 = 0$$

Substituting c_1 found above in the general solution gives

$$\frac{(x^2+1)^2}{16} = \frac{y^4}{4}$$

Summary

The solution(s) found are the following

$$\frac{(x^2+1)^2}{16} = \frac{y^4}{4} \tag{1}$$

Verification of solutions

$$\frac{(x^2+1)^2}{16} = \frac{y^4}{4}$$

Verified OK.

2.16.5 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x,y) + N(x,y)\frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x,y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y) = 0$$

Hence

$$\frac{\partial\phi}{\partial x} + \frac{\partial\phi}{\partial y}\frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$\frac{\partial\phi}{\partial x} = M$$
$$\frac{\partial\phi}{\partial y} = N$$

But since $\frac{\partial^2\phi}{\partial x\partial y} = \frac{\partial^2\phi}{\partial y\partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x,y)$ but at least we know now that we can do that since the condition $\frac{\partial^2\phi}{\partial x\partial y} = \frac{\partial^2\phi}{\partial y\partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x,y)\,\mathrm{d}x + N(x,y)\,\mathrm{d}y = 0 \tag{1A}$$

Therefore

$$\begin{aligned}(4y^3)\,\mathrm{d}y &= \left(x\left(x^2+1\right)\right)\mathrm{d}x\\ \left(-x\left(x^2+1\right)\right)\mathrm{d}x + \left(4y^3\right)\mathrm{d}y &= 0 \end{aligned} \tag{2A}$$

Comparing (1A) and (2A) shows that

$$\begin{aligned}M(x,y) &= -x\left(x^2+1\right)\\ N(x,y) &= 4y^3\end{aligned}$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Using result found above gives

$$\begin{aligned}\frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}\left(-x\left(x^2+1\right)\right)\\ &= 0\end{aligned}$$

And

$$\begin{aligned}\frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}\left(4y^3\right)\\ &= 0\end{aligned}$$

Since $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$, then the ODE is exact The following equations are now set up to solve for the function $\phi(x,y)$

$$\frac{\partial\phi}{\partial x} = M \tag{1}$$

$$\frac{\partial\phi}{\partial y} = N \tag{2}$$

Integrating (1) w.r.t. x gives

$$\int \frac{\partial \phi}{\partial x}\,\mathrm{d}x = \int M\,\mathrm{d}x$$

$$\int \frac{\partial \phi}{\partial x}\,\mathrm{d}x = \int -x(x^2+1)\,\mathrm{d}x$$

$$\phi = -\frac{(x^2+1)^2}{4} + f(y) \tag{3}$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$\frac{\partial \phi}{\partial y} = 0 + f'(y) \tag{4}$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = 4y^3$. Therefore equation (4) becomes

$$4y^3 = 0 + f'(y) \tag{5}$$

Solving equation (5) for $f'(y)$ gives

$$f'(y) = 4y^3$$

Integrating the above w.r.t y gives

$$\int f'(y)\,\mathrm{d}y = \int (4y^3)\,\mathrm{d}y$$

$$f(y) = y^4 + c_1$$

Where c_1 is constant of integration. Substituting result found above for $f(y)$ into equation (3) gives ϕ

$$\phi = -\frac{(x^2+1)^2}{4} + y^4 + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = -\frac{(x^2+1)^2}{4} + y^4$$

Initial conditions are used to solve for c_1. Substituting $x = 0$ and $y = -\frac{\sqrt{2}}{2}$ in the above solution gives an equation to solve for the constant of integration.

$$0 = c_1$$

$$c_1 = 0$$

Substituting c_1 found above in the general solution gives

$$-\frac{(x^2+1)^2}{4} + y^4 = 0$$

Summary
The solution(s) found are the following

$$-\frac{(x^2+1)^2}{4} + y^4 = 0 \tag{1}$$

Verification of solutions

$$-\frac{(x^2+1)^2}{4} + y^4 = 0$$

Verified OK.

2.16.6 Maple step by step solution

Let's solve
$\left[y' - \frac{x(x^2+1)}{4y^3} = 0, y(0) = -\frac{\sqrt{2}}{2}\right]$

- Highest derivative means the order of the ODE is 1
  y'
- Separate variables
  $y'y^3 = \frac{x(x^2+1)}{4}$
- Integrate both sides with respect to x
  $\int y'y^3 dx = \int \frac{x(x^2+1)}{4} dx + c_1$
- Evaluate integral
  $\frac{y^4}{4} = \frac{(x^2+1)^2}{16} + c_1$
- Solve for y

$$\left\{y = \left(\frac{1}{4}x^4 + \frac{1}{2}x^2 + 4c_1 + \frac{1}{4}\right)^{\frac{1}{4}}, y = -\left(\frac{1}{4}x^4 + \frac{1}{2}x^2 + 4c_1 + \frac{1}{4}\right)^{\frac{1}{4}}\right\}$$

- Use initial condition $y(0) = -\frac{\sqrt{2}}{2}$

  $-\frac{\sqrt{2}}{2} = \left(\frac{1}{4} + 4c_1\right)^{\frac{1}{4}}$
- Solution does not satisfy initial condition
- Use initial condition $y(0) = -\frac{\sqrt{2}}{2}$

  $-\frac{\sqrt{2}}{2} = -\left(\frac{1}{4} + 4c_1\right)^{\frac{1}{4}}$
- Solve for c_1

  $c_1 = 0$
- Substitute $c_1 = 0$ into general solution and simplify

  $y = -\frac{\sqrt{2}\left((x^2+1)^2\right)^{\frac{1}{4}}}{2}$
- Solution to the IVP

  $y = -\frac{\sqrt{2}\left((x^2+1)^2\right)^{\frac{1}{4}}}{2}$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
<- Bernoulli successful`
```

✓ Solution by Maple

Time used: 0.078 (sec). Leaf size: 15

```
dsolve([diff(y(x),x) = 1/4*x*(x^2+1)/y(x)^3,y(0) = -1/sqrt(2)],y(x), singsol=all)
```

$$y(x) = -\frac{\sqrt{2x^2 + 2}}{2}$$

✓ Solution by Mathematica

Time used: 0.23 (sec). Leaf size: 23

```
DSolve[{y'[x] == 1/4*x*(x^2+1)/y[x]^3,y[0]==-(1/Sqrt[2])},y[x],x,IncludeSingularSolutions ->
```

$$y(x) \to -\frac{\sqrt[4]{(x^2+1)^2}}{\sqrt{2}}$$

2.17 problem 17

Internal problem ID [495]
Internal file name [`OUTPUT/495_Sunday_June_05_2022_01_42_26_AM_66277094/index.tex`]

Book: Elementary differential equations and boundary value problems, 10th ed., Boyce and DiPrima
Section: Section 2.2. Page 48
Problem number: 17.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"exact", "separable", "first_order_ode_lie_symmetry_lookup"**

Maple gives the following as the ode type

`[_separable]`

$$\boxed{y' - \frac{-e^{x} + 3x^{2}}{-5 + 2y} = 0}$$

With initial conditions

$$[y(0) = 1]$$

2.17.1 Existence and uniqueness analysis

This is non linear first order ODE. In canonical form it is written as

$$\begin{aligned} y' &= f(x, y) \\ &= -\frac{-3x^{2} + e^{x}}{-5 + 2y} \end{aligned}$$

The x domain of $f(x, y)$ when $y = 1$ is

$$\{-\infty < x < \infty\}$$

And the point $x_0 = 0$ is inside this domain. The y domain of $f(x, y)$ when $x = 0$ is

$$\left\{y < \frac{5}{2} \lor \frac{5}{2} < y\right\}$$

And the point $y_0 = 1$ is inside this domain. Now we will look at the continuity of

$$\begin{aligned}\frac{\partial f}{\partial y} &= \frac{\partial}{\partial y}\left(-\frac{-3x^2 + e^x}{-5 + 2y}\right) \\ &= \frac{-6x^2 + 2e^x}{(-5 + 2y)^2}\end{aligned}$$

The x domain of $\frac{\partial f}{\partial y}$ when $y = 1$ is

$$\{-\infty < x < \infty\}$$

And the point $x_0 = 0$ is inside this domain. The y domain of $\frac{\partial f}{\partial y}$ when $x = 0$ is

$$\left\{y < \frac{5}{2} \lor \frac{5}{2} < y\right\}$$

And the point $y_0 = 1$ is inside this domain. Therefore solution exists and is unique.

2.17.2 Solving as separable ode

In canonical form the ODE is

$$\begin{aligned}y' &= F(x, y) \\ &= f(x)g(y) \\ &= \frac{-e^x + 3x^2}{-5 + 2y}\end{aligned}$$

Where $f(x) = -e^x + 3x^2$ and $g(y) = \frac{1}{-5+2y}$. Integrating both sides gives

$$\begin{aligned}\frac{1}{\frac{1}{-5+2y}}\,dy &= -e^x + 3x^2\,dx \\ \int \frac{1}{\frac{1}{-5+2y}}\,dy &= \int -e^x + 3x^2\,dx \\ y^2 - 5y &= x^3 - e^x + c_1\end{aligned}$$

Which results in

$$y = \frac{5}{2} + \frac{\sqrt{25 + 4x^3 - 4\,\mathrm{e}^x + 4c_1}}{2}$$
$$y = \frac{5}{2} - \frac{\sqrt{25 + 4x^3 - 4\,\mathrm{e}^x + 4c_1}}{2}$$

Initial conditions are used to solve for c_1. Substituting $x = 0$ and $y = 1$ in the above solution gives an equation to solve for the constant of integration.

$$1 = \frac{5}{2} - \frac{\sqrt{21 + 4c_1}}{2}$$

$$c_1 = -3$$

Substituting c_1 found above in the general solution gives

$$y = \frac{5}{2} - \frac{\sqrt{13 + 4x^3 - 4\,\mathrm{e}^x}}{2}$$

Initial conditions are used to solve for c_1. Substituting $x = 0$ and $y = 1$ in the above solution gives an equation to solve for the constant of integration.

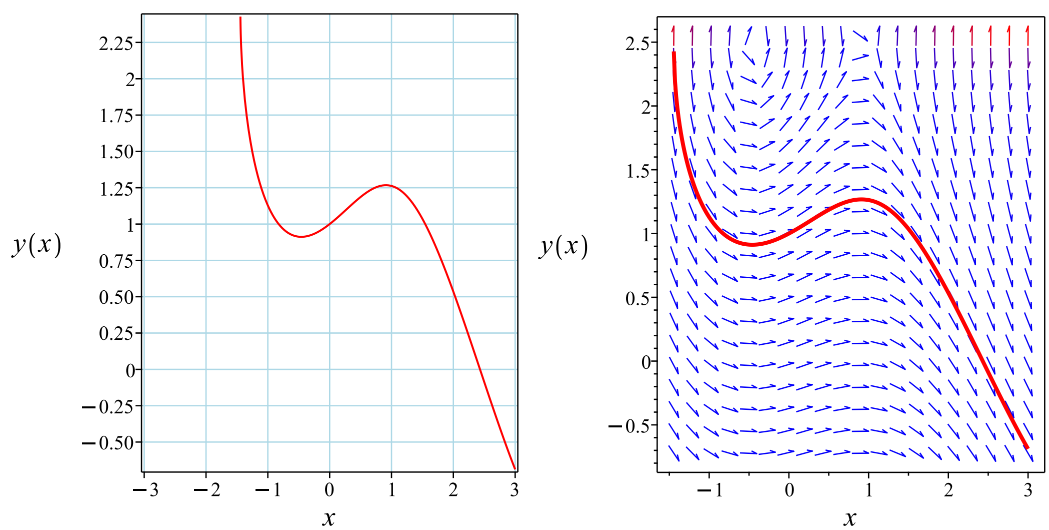
$$1 = \frac{5}{2} + \frac{\sqrt{21 + 4c_1}}{2}$$

Summary

The solution(s) found are the following

Warning: Unable to solve for constant of integration.

$$y = \frac{5}{2} - \frac{\sqrt{13 + 4x^3 -}}{2}$$



(a) Solution plot (b) Slope field plot

Verification of solutions

$$y = \frac{5}{2} - \frac{\sqrt{13 + 4x^3 - 4\,\mathrm{e}^x}}{2}$$

Verified OK.

2.17.3 Solving as first order ode lie symmetry lookup ode

Writing the ode as

$$y' = -\frac{-3x^2 + \mathrm{e}^x}{-5 + 2y}$$
$$y' = \omega(x, y)$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x + \omega(\eta_y - \xi_x) - \omega^2 \xi_y - \omega_x \xi - \omega_y \eta = 0 \tag{A}$$

The type of this ode is known. It is of type `separable`. Therefore we do not need to solve the PDE (A), and can just use the lookup table shown below to find ξ, η

Table 139: Lie symmetry infinitesimal lookup table for known first order ODE's

ODE class	Form	ξ	η
linear ode	$y' = f(x)y(x) + g(x)$	0	$e^{\int f dx}$
separable ode	$y' = f(x)\, g(y)$	$\frac{1}{f}$	0
quadrature ode	$y' = f(x)$	0	1
quadrature ode	$y' = g(y)$	1	0
homogeneous ODEs of Class A	$y' = f\left(\frac{y}{x}\right)$	x	y
homogeneous ODEs of Class C	$y' = (a + bx + cy)^{\frac{n}{m}}$	1	$-\frac{b}{c}$
homogeneous class D	$y' = \frac{y}{x} + g(x)\, F\left(\frac{y}{x}\right)$	x^2	xy
First order special form ID 1	$y' = g(x)\, e^{h(x)+by} + f(x)$	$\frac{e^{-\int bf(x)dx-h(x)}}{g(x)}$	$\frac{f(x)e^{-\int bf(x)dx-h(x)}}{g(x)}$
polynomial type ode	$y' = \frac{a_1x+b_1y+c_1}{a_2x+b_2y+c_2}$	$\frac{a_1b_2x-a_2b_1x-b_1c_2+b_2c_1}{a_1b_2-a_2b_1}$	$\frac{a_1b_2y-a_2b_1y-a_1c_2-a_2c_1}{a_1b_2-a_2b_1}$
Bernoulli ode	$y' = f(x)\, y + g(x)\, y^n$	0	$e^{-\int (n-1)f(x)dx} y^n$
Reduced Riccati	$y' = f_1(x)\, y + f_2(x)\, y^2$	0	$e^{-\int f_1 dx}$

The above table shows that

$$
\begin{aligned}
\xi(x,y) &= \frac{1}{-e^{x} + 3x^2} \\
\eta(x,y) &= 0 \qquad \text{(A1)}
\end{aligned}
$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x,y) \to (R,S)$ where (R,S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$
\frac{dx}{\xi} = \frac{dy}{\eta} = dS \qquad (1)
$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y}\right) S(x,y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the

canonical coordinates, where $S(R)$. Since $\eta = 0$ then in this special case

$$R = y$$

S is found from

$$\begin{aligned} S &= \int \frac{1}{\xi} dx \\ &= \int \frac{1}{\frac{1}{-e^x + 3x^2}} dx \end{aligned}$$

Which results in

$$S = x^3 - e^x$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x, y) S_y}{R_x + \omega(x, y) R_y} \tag{2}$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x, y)$ is the right hand side of the original ode given by

$$\omega(x, y) = -\frac{-3x^2 + e^x}{-5 + 2y}$$

Evaluating all the partial derivatives gives

$$\begin{aligned} R_x &= 0 \\ R_y &= 1 \\ S_x &= -e^x + 3x^2 \\ S_y &= 0 \end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = -5 + 2y \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = -5 + 2R$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordinates R, S. Integrating the above gives

$$S(R) = R^2 - 5R + c_1 \tag{4}$$

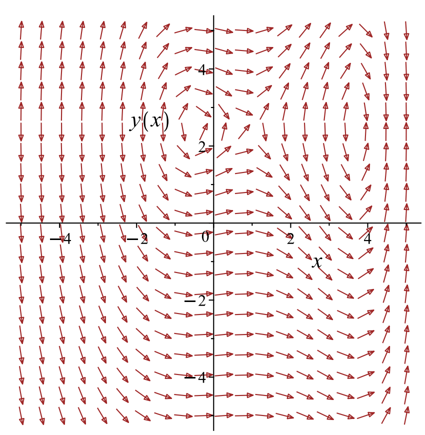
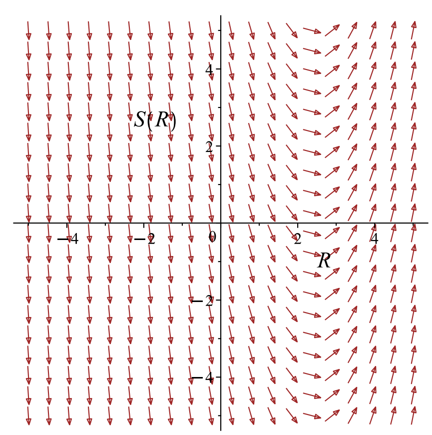
To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$x^3 - e^x = y^2 + c_1 - 5y$$

Which simplifies to

$$x^3 - e^x = y^2 + c_1 - 5y$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in x, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dx} = -\frac{-3x^2+e^x}{-5+2y}$ 	$R = y$ $S = x^3 - e^x$	$\frac{dS}{dR} = -5 + 2R$ 

Initial conditions are used to solve for c_1. Substituting $x = 0$ and $y = 1$ in the above solution gives an equation to solve for the constant of integration.

$$-1 = -4 + c_1$$

$$c_1 = 3$$

Substituting c_1 found above in the general solution gives

$$x^3 - e^x = y^2 - 5y + 3$$

Summary
The solution(s) found are the following

$$x^3 - e^x = y^2 - 5y + 3 \tag{1}$$

Verification of solutions

$$x^3 - e^x = y^2 - 5y + 3$$

Verified OK.

2.17.4 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x,y) + N(x,y)\frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x,y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y) = 0$$

Hence

$$\frac{\partial \phi}{\partial x} + \frac{\partial \phi}{\partial y}\frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$\frac{\partial \phi}{\partial x} = M$$
$$\frac{\partial \phi}{\partial y} = N$$

But since $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x,y)$ but at least we know now that we can do that since the condition $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ is satisfied. If this condition is not satisfied then this method will not work

and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x,y)\,\mathrm{d}x + N(x,y)\,\mathrm{d}y = 0 \tag{1A}$$

Therefore

$$\begin{aligned}(5-2y)\,\mathrm{d}y &= \left(-3x^2+\mathrm{e}^x\right)\mathrm{d}x\\ \left(-\mathrm{e}^x+3x^2\right)\mathrm{d}x + (5-2y)\,\mathrm{d}y &= 0 \qquad (2\mathrm{A})\end{aligned}$$

Comparing (1A) and (2A) shows that

$$\begin{aligned}M(x,y) &= -\mathrm{e}^x+3x^2\\ N(x,y) &= 5-2y\end{aligned}$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Using result found above gives

$$\begin{aligned}\frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}\left(-\mathrm{e}^x+3x^2\right)\\ &= 0\end{aligned}$$

And

$$\begin{aligned}\frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}(5-2y)\\ &= 0\end{aligned}$$

Since $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$, then the ODE is exact The following equations are now set up to solve for the function $\phi(x,y)$

$$\frac{\partial \phi}{\partial x} = M \tag{1}$$

$$\frac{\partial \phi}{\partial y} = N \tag{2}$$

Integrating (1) w.r.t. x gives

$$\begin{aligned}\int \frac{\partial \phi}{\partial x}\,\mathrm{d}x &= \int M\,\mathrm{d}x\\ \int \frac{\partial \phi}{\partial x}\,\mathrm{d}x &= \int -\mathrm{e}^x+3x^2\,\mathrm{d}x\\ \phi &= x^3-\mathrm{e}^x+f(y) \qquad (3)\end{aligned}$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$\frac{\partial \phi}{\partial y} = 0 + f'(y) \tag{4}$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = 5 - 2y$. Therefore equation (4) becomes

$$5 - 2y = 0 + f'(y) \tag{5}$$

Solving equation (5) for $f'(y)$ gives

$$f'(y) = 5 - 2y$$

Integrating the above w.r.t y gives

$$\int f'(y)\, \mathrm{d}y = \int (5 - 2y)\, \mathrm{d}y$$
$$f(y) = -y^2 + 5y + c_1$$

Where c_1 is constant of integration. Substituting result found above for $f(y)$ into equation (3) gives ϕ

$$\phi = x^3 - y^2 - \mathrm{e}^x + 5y + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = x^3 - y^2 - \mathrm{e}^x + 5y$$

Initial conditions are used to solve for c_1. Substituting $x = 0$ and $y = 1$ in the above solution gives an equation to solve for the constant of integration.

$$3 = c_1$$

$$c_1 = 3$$

Substituting c_1 found above in the general solution gives

$$x^3 - y^2 - \mathrm{e}^x + 5y = 3$$

Summary
The solution(s) found are the following

$$x^3 - y^2 - e^x + 5y = 3 \tag{1}$$

Verification of solutions

$$x^3 - y^2 - e^x + 5y = 3$$

Verified OK.

2.17.5 Maple step by step solution

Let's solve

$\left[y' - \frac{-e^x+3x^2}{-5+2y} = 0, y(0) = 1\right]$

- Highest derivative means the order of the ODE is 1

  y'

- Separate variables

  $y'(-5+2y) = -e^x + 3x^2$

- Integrate both sides with respect to x

  $\int y'(-5+2y)\,dx = \int (-e^x + 3x^2)\,dx + c_1$

- Evaluate integral

  $y^2 - 5y = x^3 - e^x + c_1$

- Solve for y

  $\left\{y = \frac{5}{2} - \frac{\sqrt{25+4x^3-4\,e^x+4c_1}}{2}, y = \frac{5}{2} + \frac{\sqrt{25+4x^3-4\,e^x+4c_1}}{2}\right\}$

- Use initial condition $y(0) = 1$

  $1 = \frac{5}{2} - \frac{\sqrt{21+4c_1}}{2}$

- Solve for c_1

  $c_1 = -3$

- Substitute $c_1 = -3$ into general solution and simplify

  $y = \frac{5}{2} - \frac{\sqrt{13+4x^3-4\,e^x}}{2}$

- Use initial condition $y(0) = 1$

  $1 = \frac{5}{2} + \frac{\sqrt{21+4c_1}}{2}$

- Solution does not satisfy initial condition

- Solution to the IVP

  $y = \frac{5}{2} - \frac{\sqrt{13+4x^3-4\,\mathrm{e}^x}}{2}$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
<- separable successful`
```

✓ Solution by Maple

Time used: 0.172 (sec). Leaf size: 21

```
dsolve([diff(y(x),x) = (-exp(x)+3*x^2)/(-5+2*y(x)),y(0) = 1],y(x), singsol=all)
```

$$y(x) = \frac{5}{2} - \frac{\sqrt{13+4x^3-4\,\mathrm{e}^x}}{2}$$

✓ Solution by Mathematica

Time used: 0.891 (sec). Leaf size: 29

```
DSolve[{y'[x] == (-Exp[x]+3*x^2)/(-5+2*y[x]),y[0]==1},y[x],x,IncludeSingularSolutions -> Tru
```

$$y(x) \to \frac{1}{2}\left(5 - \sqrt{4x^3 - 4e^x + 13}\right)$$

2.18 problem 18

Internal problem ID [496]
Internal file name [`OUTPUT/496_Sunday_June_05_2022_01_42_27_AM_16414608/index.tex`]

Book: Elementary differential equations and boundary value problems, 10th ed., Boyce and DiPrima
Section: Section 2.2. Page 48
Problem number: 18.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"exact", "separable", "first_order_ode_lie_symmetry_lookup"**

Maple gives the following as the ode type

`[_separable]`

$$\boxed{y' - \frac{-\mathrm{e}^{x} + \mathrm{e}^{-x}}{3 + 4y} = 0}$$

With initial conditions

$$[y(0) = 1]$$

2.18.1 Existence and uniqueness analysis

This is non linear first order ODE. In canonical form it is written as

$$\begin{aligned} y' &= f(x, y) \\ &= -\frac{\mathrm{e}^{x} - \mathrm{e}^{-x}}{3 + 4y} \end{aligned}$$

The x domain of $f(x, y)$ when $y = 1$ is

$$\{-\infty < x < \infty\}$$

And the point $x_0 = 0$ is inside this domain. The y domain of $f(x,y)$ when $x = 0$ is

$$\{-\infty < y < \infty\}$$

And the point $y_0 = 1$ is inside this domain. Now we will look at the continuity of

$$\begin{aligned}\frac{\partial f}{\partial y} &= \frac{\partial}{\partial y}\left(-\frac{e^{x} - e^{-x}}{3+4y}\right) \\ &= \frac{4\,e^{x} - 4\,e^{-x}}{(3+4y)^2}\end{aligned}$$

The x domain of $\frac{\partial f}{\partial y}$ when $y = 1$ is

$$\{-\infty < x < \infty\}$$

And the point $x_0 = 0$ is inside this domain. The y domain of $\frac{\partial f}{\partial y}$ when $x = 0$ is

$$\{-\infty < y < \infty\}$$

And the point $y_0 = 1$ is inside this domain. Therefore solution exists and is unique.

2.18.2 Solving as separable ode

In canonical form the ODE is

$$\begin{aligned}y' &= F(x,y) \\ &= f(x)g(y) \\ &= \frac{-e^{x} + e^{-x}}{3+4y}\end{aligned}$$

Where $f(x) = -e^{x} + e^{-x}$ and $g(y) = \frac{1}{3+4y}$. Integrating both sides gives

$$\begin{aligned}\frac{1}{\frac{1}{3+4y}}\,dy &= -e^{x} + e^{-x}\,dx \\ \int \frac{1}{\frac{1}{3+4y}}\,dy &= \int -e^{x} + e^{-x}\,dx \\ 2y^2 + 3y &= -e^{x} - e^{-x} + c_1\end{aligned}$$

Which results in

$$y = -\frac{\left(3\,\mathrm{e}^{x}-\sqrt{-8\,\mathrm{e}^{3x}+8c_1\mathrm{e}^{2x}+9\,\mathrm{e}^{2x}-8\,\mathrm{e}^{x}}\right)\mathrm{e}^{-x}}{4}$$

$$y = -\frac{\left(3\,\mathrm{e}^{x}+\sqrt{-8\,\mathrm{e}^{3x}+8c_1\mathrm{e}^{2x}+9\,\mathrm{e}^{2x}-8\,\mathrm{e}^{x}}\right)\mathrm{e}^{-x}}{4}$$

Initial conditions are used to solve for c_1. Substituting $x=0$ and $y=1$ in the above solution gives an equation to solve for the constant of integration.

$$1 = -\frac{3}{4}-\frac{\sqrt{-7+8c_1}}{4}$$

Warning: Unable to solve for constant of integration. Initial conditions are used to solve for c_1. Substituting $x=0$ and $y=1$ in the above solution gives an equation to solve for the constant of integration.

$$1 = -\frac{3}{4}+\frac{\sqrt{-7+8c_1}}{4}$$

$$c_1 = 7$$

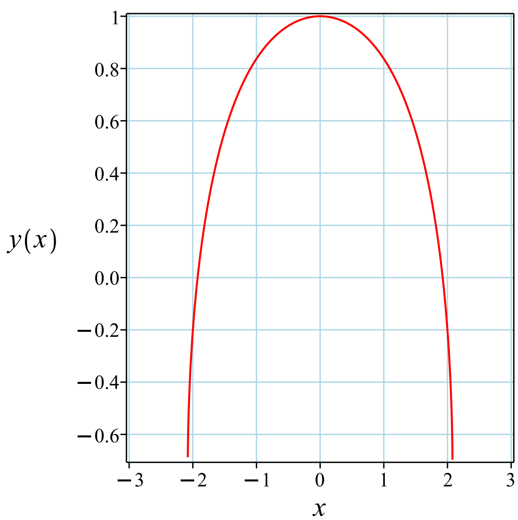
Substituting c_1 found above in the general solution gives

$$y = -\frac{3}{4}+\frac{\mathrm{e}^{-x}\sqrt{65\,\mathrm{e}^{2x}-8\,\mathrm{e}^{x}\mathrm{e}^{2x}-8\,\mathrm{e}^{x}}}{4}$$

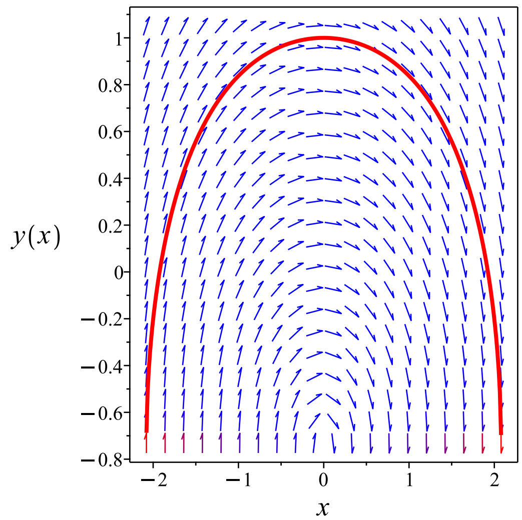
Summary

The solution(s) found are the following

$$y = -\frac{3}{4}+\frac{\mathrm{e}^{-x}\sqrt{65\,\mathrm{e}^{2x}-8\,\mathrm{e}^{x}\mathrm{e}^{2x}-8\,\mathrm{e}^{x}}}{4} \tag{1}$$



(a) Solution plot



(b) Slope field plot

Verification of solutions

$$y = -\frac{3}{4} + \frac{e^{-x}\sqrt{65\,e^{2x} - 8\,e^{x}e^{2x} - 8\,e^{x}}}{4}$$

Verified OK.

2.18.3 Solving as first order ode lie symmetry lookup ode

Writing the ode as

$$y' = -\frac{e^{x} - e^{-x}}{3 + 4y}$$
$$y' = \omega(x, y)$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x + \omega(\eta_y - \xi_x) - \omega^2\xi_y - \omega_x\xi - \omega_y\eta = 0 \tag{A}$$

The type of this ode is known. It is of type `separable`. Therefore we do not need to solve the PDE (A), and can just use the lookup table shown below to find ξ, η

Table 142: Lie symmetry infinitesimal lookup table for known first order ODE's

ODE class	Form	ξ	η
linear ode	$y' = f(x)y(x) + g(x)$	0	$e^{\int f dx}$
separable ode	$y' = f(x)\, g(y)$	$\frac{1}{f}$	0
quadrature ode	$y' = f(x)$	0	1
quadrature ode	$y' = g(y)$	1	0
homogeneous ODEs of Class A	$y' = f\left(\frac{y}{x}\right)$	x	y
homogeneous ODEs of Class C	$y' = (a + bx + cy)^{\frac{n}{m}}$	1	$-\frac{b}{c}$
homogeneous class D	$y' = \frac{y}{x} + g(x)\, F\left(\frac{y}{x}\right)$	x^2	xy
First order special form ID 1	$y' = g(x)\, e^{h(x)+by} + f(x)$	$\frac{e^{-\int bf(x)dx - h(x)}}{g(x)}$	$\frac{f(x)e^{-\int bf(x)dx - h(x)}}{g(x)}$
polynomial type ode	$y' = \frac{a_1x+b_1y+c_1}{a_2x+b_2y+c_2}$	$\frac{a_1b_2x - a_2b_1x - b_1c_2 + b_2c_1}{a_1b_2 - a_2b_1}$	$\frac{a_1b_2y - a_2b_1y - a_1c_2 - a_2c_1}{a_1b_2 - a_2b_1}$
Bernoulli ode	$y' = f(x)\, y + g(x)\, y^n$	0	$e^{-\int (n-1)f(x)dx} y^n$
Reduced Riccati	$y' = f_1(x)\, y + f_2(x)\, y^2$	0	$e^{-\int f_1 dx}$

The above table shows that

$$
\begin{aligned}
\xi(x, y) &= \frac{1}{-\mathrm{e}^{x} + \mathrm{e}^{-x}} \\
\eta(x, y) &= 0 \qquad \text{(A1)}
\end{aligned}
$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x, y) \to (R, S)$ where (R, S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$
\frac{dx}{\xi} = \frac{dy}{\eta} = dS \qquad (1)
$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y}\right) S(x, y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the

canonical coordinates, where $S(R)$. Since $\eta = 0$ then in this special case

$$R = y$$

S is found from

$$\begin{aligned} S &= \int \frac{1}{\xi} dx \\ &= \int \frac{1}{\frac{1}{-e^x + e^{-x}}} dx \end{aligned}$$

Which results in

$$S = -e^x - e^{-x}$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x, y) S_y}{R_x + \omega(x, y) R_y} \tag{2}$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x, y)$ is the right hand side of the original ode given by

$$\omega(x, y) = -\frac{e^x - e^{-x}}{3 + 4y}$$

Evaluating all the partial derivatives gives

$$\begin{aligned} R_x &= 0 \\ R_y &= 1 \\ S_x &= -e^x + e^{-x} \\ S_y &= 0 \end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = 3 + 4y \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = 3 + 4R$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by

integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$S(R) = 2R^2 + 3R + c_1 \tag{4}$$

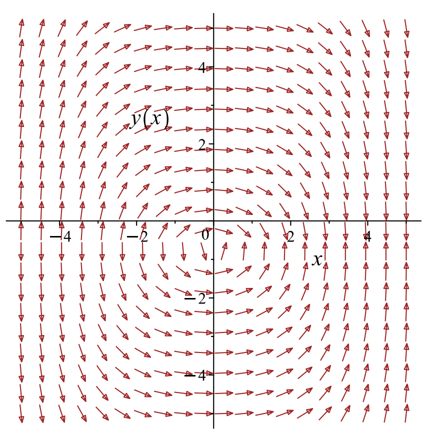
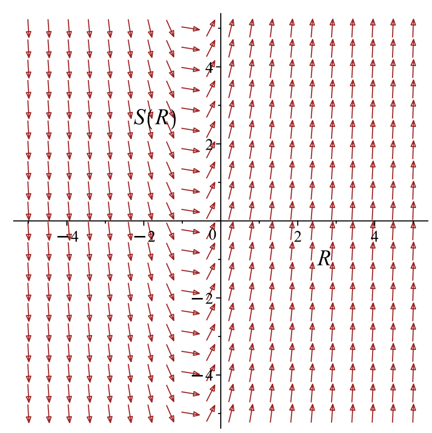
To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$-e^x - e^{-x} = 2y^2 + c_1 + 3y$$

Which simplifies to

$$-e^x - e^{-x} = 2y^2 + c_1 + 3y$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in x, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dx} = -\frac{e^x - e^{-x}}{3+4y}$ 	$R = y$ $S = -e^x - e^{-x}$	$\frac{dS}{dR} = 3 + 4R$ 

Initial conditions are used to solve for c_1. Substituting $x = 0$ and $y = 1$ in the above solution gives an equation to solve for the constant of integration.

$$-2 = 5 + c_1$$

$$c_1 = -7$$

Substituting c_1 found above in the general solution gives

$$-e^{x}-e^{-x}=2y^{2}+3y-7$$

Summary
The solution(s) found are the following

$$-e^{x}-e^{-x}=2y^{2}+3y-7 \tag{1}$$

Verification of solutions

$$-e^{x}-e^{-x}=2y^{2}+3y-7$$

Verified OK.

2.18.4 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x,y)+N(x,y)\frac{dy}{dx}=0 \tag{A}$$

We assume there exists a function $\phi(x,y)=c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y)=0$$

Hence

$$\frac{\partial\phi}{\partial x}+\frac{\partial\phi}{\partial y}\frac{dy}{dx}=0 \tag{B}$$

Comparing (A,B) shows that

$$\frac{\partial\phi}{\partial x}=M$$
$$\frac{\partial\phi}{\partial y}=N$$

But since $\frac{\partial^2\phi}{\partial x\partial y}=\frac{\partial^2\phi}{\partial y\partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y}=\frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x,y)$ but at least we know now that we can do that since the condition $\frac{\partial^2\phi}{\partial x\partial y}=\frac{\partial^2\phi}{\partial y\partial x}$ is satisfied. If this condition is not satisfied then this method will not work

and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x,y)\,\mathrm{d}x + N(x,y)\,\mathrm{d}y = 0 \tag{1A}$$

Therefore

$$\begin{aligned}(-3-4y)\,\mathrm{d}y &= \left(\mathrm{e}^{x}-\mathrm{e}^{-x}\right)\mathrm{d}x\\ \left(-\mathrm{e}^{x}+\mathrm{e}^{-x}\right)\mathrm{d}x + (-3-4y)\,\mathrm{d}y &= 0 \end{aligned}\tag{2A}$$

Comparing (1A) and (2A) shows that

$$\begin{aligned} M(x,y) &= -\mathrm{e}^{x}+\mathrm{e}^{-x}\\ N(x,y) &= -3-4y \end{aligned}$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Using result found above gives

$$\begin{aligned}\frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}\left(-\mathrm{e}^{x}+\mathrm{e}^{-x}\right)\\ &= 0\end{aligned}$$

And

$$\begin{aligned}\frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}(-3-4y)\\ &= 0\end{aligned}$$

Since $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$, then the ODE is exact The following equations are now set up to solve for the function $\phi(x,y)$

$$\frac{\partial \phi}{\partial x} = M \tag{1}$$

$$\frac{\partial \phi}{\partial y} = N \tag{2}$$

Integrating (1) w.r.t. x gives

$$\begin{aligned}\int \frac{\partial \phi}{\partial x}\,\mathrm{d}x &= \int M\,\mathrm{d}x\\ \int \frac{\partial \phi}{\partial x}\,\mathrm{d}x &= \int -\mathrm{e}^{x}+\mathrm{e}^{-x}\,\mathrm{d}x\\ \phi &= -\mathrm{e}^{x}-\mathrm{e}^{-x}+f(y)\end{aligned}\tag{3}$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$\frac{\partial \phi}{\partial y} = 0 + f'(y) \tag{4}$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = -3 - 4y$. Therefore equation (4) becomes

$$-3 - 4y = 0 + f'(y) \tag{5}$$

Solving equation (5) for $f'(y)$ gives

$$f'(y) = -3 - 4y$$

Integrating the above w.r.t y gives

$$\int f'(y)\,\mathrm{d}y = \int (-3 - 4y)\,\mathrm{d}y$$
$$f(y) = -2y^2 - 3y + c_1$$

Where c_1 is constant of integration. Substituting result found above for $f(y)$ into equation (3) gives ϕ

$$\phi = -2y^2 - \mathrm{e}^x - 3y - \mathrm{e}^{-x} + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = -2y^2 - \mathrm{e}^x - 3y - \mathrm{e}^{-x}$$

Initial conditions are used to solve for c_1. Substituting $x = 0$ and $y = 1$ in the above solution gives an equation to solve for the constant of integration.

$$-7 = c_1$$

$$c_1 = -7$$

Substituting c_1 found above in the general solution gives

$$-2y^2 - \mathrm{e}^x - 3y - \mathrm{e}^{-x} = -7$$

Summary

The solution(s) found are the following

$$-2y^2 - e^x - 3y - e^{-x} = -7 \tag{1}$$

Verification of solutions

$$-2y^2 - e^x - 3y - e^{-x} = -7$$

Verified OK.

2.18.5 Maple step by step solution

Let's solve

$\left[y' - \frac{-e^x + e^{-x}}{3+4y} = 0, y(0) = 1\right]$

- Highest derivative means the order of the ODE is 1

  y'

- Separate variables

  $y'(3+4y) = -e^x + e^{-x}$

- Integrate both sides with respect to x

  $\int y'(3+4y)\, dx = \int (-e^x + e^{-x})\, dx + c_1$

- Evaluate integral

  $2y^2 + 3y = -e^x - e^{-x} + c_1$

- Solve for y

  $\left\{y = -\frac{3\,e^x - \sqrt{-8(e^x)^3 + 8c_1(e^x)^2 + 9(e^x)^2 - 8\,e^x}}{4\,e^x}, y = -\frac{3\,e^x + \sqrt{-8(e^x)^3 + 8c_1(e^x)^2 + 9(e^x)^2 - 8\,e^x}}{4\,e^x}\right\}$

- Use initial condition $y(0) = 1$

  $1 = -\frac{3}{4} + \frac{\sqrt{-7+8c_1}}{4}$

- Solve for c_1

  $c_1 = 7$

- Substitute $c_1 = 7$ into general solution and simplify

  $y = -\frac{3}{4} + \frac{e^{-x}\sqrt{(-8\,e^{2x} + 65\,e^x - 8)e^x}}{4}$

- Use initial condition $y(0) = 1$

  $1 = -\frac{3}{4} - \frac{\sqrt{-7+8c_1}}{4}$

- Solution does not satisfy initial condition
- Solution to the IVP

  $y = -\frac{3}{4} + \frac{e^{-x}\sqrt{(-8\,e^{2x}+65\,e^{x}-8)e^{x}}}{4}$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
<- separable successful`
```

✓ Solution by Maple

Time used: 0.235 (sec). Leaf size: 29

```
dsolve([diff(y(x),x) = (exp(-x)-exp(x))/(3+4*y(x)),y(0) = 1],y(x), singsol=all)
```

$$y(x) = -\frac{3}{4} + \frac{\sqrt{e^{x}\left(-8\,e^{2x}+65\,e^{x}-8\right)}\,e^{-x}}{4}$$

✓ Solution by Mathematica

Time used: 1.347 (sec). Leaf size: 29

```
DSolve[{y'[x] == (Exp[-x]-Exp[x])/(3+4*y[x]),y[0]==1},y[x],x,IncludeSingularSolutions -> Tru
```

$$y(x) \to \frac{1}{4}\left(\sqrt{-8e^{-x}-8e^{x}+65}-3\right)$$

2.19 problem 19

Internal problem ID [497]
Internal file name [`OUTPUT/497_Sunday_June_05_2022_01_42_28_AM_60921212/index.tex`]

Book: Elementary differential equations and boundary value problems, 10th ed., Boyce and DiPrima
Section: Section 2.2. Page 48
Problem number: 19.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"exact", "separable", "first_order_ode_lie_symmetry_lookup"**

Maple gives the following as the ode type

`[_separable]`

$$\boxed{\cos (3y)\, y' = -\sin (2x)}$$

With initial conditions

$$\left[y\left(\frac{\pi}{2}\right) = 0\right]$$

2.19.1 Existence and uniqueness analysis

This is non linear first order ODE. In canonical form it is written as

$$\begin{aligned} y' &= f(x,y) \\ &= -\frac{\sin (2x)}{\cos (3y)} \end{aligned}$$

The x domain of $f(x,y)$ when $y=0$ is

$$\{-\infty < x < \infty\}$$

And the point $x_0 = \frac{\pi}{2}$ is inside this domain. The y domain of $f(x, y)$ when $x = \frac{\pi}{2}$ is

$$\{-\infty < y < \infty\}$$

And the point $y_0 = 0$ is inside this domain. Now we will look at the continuity of

$$\begin{aligned}\frac{\partial f}{\partial y} &= \frac{\partial}{\partial y}\left(-\frac{\sin(2x)}{\cos(3y)}\right) \\ &= -\frac{3\sin(2x)\sin(3y)}{\cos(3y)^2}\end{aligned}$$

The x domain of $\frac{\partial f}{\partial y}$ when $y = 0$ is

$$\{-\infty < x < \infty\}$$

And the point $x_0 = \frac{\pi}{2}$ is inside this domain. The y domain of $\frac{\partial f}{\partial y}$ when $x = \frac{\pi}{2}$ is

$$\{-\infty < y < \infty\}$$

And the point $y_0 = 0$ is inside this domain. Therefore solution exists and is unique.

2.19.2 Solving as separable ode

In canonical form the ODE is

$$\begin{aligned}y' &= F(x, y) \\ &= f(x)g(y) \\ &= -\sin(2x)\sec(3y)\end{aligned}$$

Where $f(x) = -\sin(2x)$ and $g(y) = \sec(3y)$. Integrating both sides gives

$$\begin{aligned}\frac{1}{\sec(3y)}\, dy &= -\sin(2x)\, dx \\ \int \frac{1}{\sec(3y)}\, dy &= \int -\sin(2x)\, dx \\ \frac{\sin(3y)}{3} &= \frac{\cos(2x)}{2} + c_1\end{aligned}$$

Which results in

$$y = \frac{\arcsin\left(\frac{3\cos(2x)}{2} + 3c_1\right)}{3}$$

Initial conditions are used to solve for c_1. Substituting $x = \frac{\pi}{2}$ and $y = 0$ in the above solution gives an equation to solve for the constant of integration.

$$0 = \frac{\arcsin\left(-\frac{3}{2} + 3c_1\right)}{3}$$

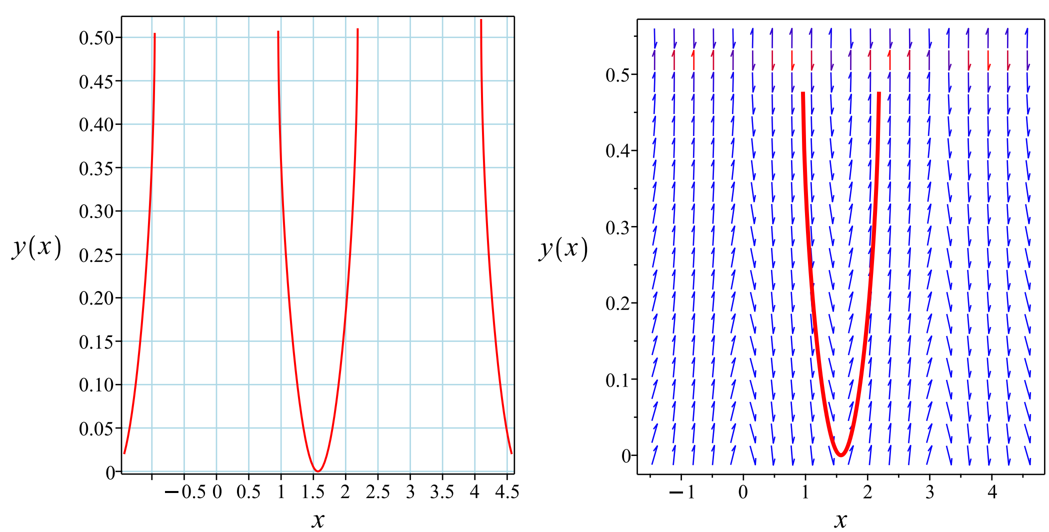
$$c_1 = \frac{1}{2}$$

Substituting c_1 found above in the general solution gives

$$y = \frac{\arcsin\left(\frac{3\cos(2x)}{2} + \frac{3}{2}\right)}{3}$$

<u>Summary</u>
The solution(s) found are the following

$$y = \frac{\arcsin\left(\frac{3\cos(2x)}{2} + \frac{3}{2}\right)}{3} \tag{1}$$



(a) Solution plot (b) Slope field plot

<u>Verification of solutions</u>

$$y = \frac{\arcsin\left(\frac{3\cos(2x)}{2} + \frac{3}{2}\right)}{3}$$

Verified OK.

2.19.3 Solving as first order ode lie symmetry lookup ode

Writing the ode as

$$y' = -\frac{\sin(2x)}{\cos(3y)}$$
$$y' = \omega(x, y)$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x + \omega(\eta_y - \xi_x) - \omega^2 \xi_y - \omega_x \xi - \omega_y \eta = 0 \tag{A}$$

The type of this ode is known. It is of type `separable`. Therefore we do not need to solve the PDE (A), and can just use the lookup table shown below to find ξ, η

Table 145: Lie symmetry infinitesimal lookup table for known first order ODE's

ODE class	Form	ξ	η
linear ode	$y' = f(x)y(x) + g(x)$	0	$e^{\int f dx}$
separable ode	$y' = f(x)\, g(y)$	$\frac{1}{f}$	0
quadrature ode	$y' = f(x)$	0	1
quadrature ode	$y' = g(y)$	1	0
homogeneous ODEs of Class A	$y' = f\left(\frac{y}{x}\right)$	x	y
homogeneous ODEs of Class C	$y' = (a + bx + cy)^{\frac{n}{m}}$	1	$-\frac{b}{c}$
homogeneous class D	$y' = \frac{y}{x} + g(x)\, F\left(\frac{y}{x}\right)$	x^2	xy
First order special form ID 1	$y' = g(x)\, e^{h(x)+by} + f(x)$	$\frac{e^{-\int bf(x)dx - h(x)}}{g(x)}$	$\frac{f(x)e^{-\int bf(x)dx - h(x)}}{g(x)}$
polynomial type ode	$y' = \frac{a_1x+b_1y+c_1}{a_2x+b_2y+c_2}$	$\frac{a_1b_2x - a_2b_1x - b_1c_2 + b_2c_1}{a_1b_2 - a_2b_1}$	$\frac{a_1b_2y - a_2b_1y - a_1c_2 - a_2c_1}{a_1b_2 - a_2b_1}$
Bernoulli ode	$y' = f(x)\, y + g(x)\, y^n$	0	$e^{-\int (n-1)f(x)dx} y^n$
Reduced Riccati	$y' = f_1(x)\, y + f_2(x)\, y^2$	0	$e^{-\int f_1 dx}$

The above table shows that

$$\begin{aligned} \xi(x,y) &= -\frac{1}{\sin(2x)} \\ \eta(x,y) &= 0 \end{aligned} \tag{A1}$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x,y) \to (R,S)$ where (R,S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}$$

The above comes from the requirements that $\left(\xi\frac{\partial}{\partial x} + \eta\frac{\partial}{\partial y}\right) S(x,y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the canonical coordinates, where $S(R)$. Since $\eta = 0$ then in this special case

$$R = y$$

S is found from

$$\begin{aligned} S &= \int \frac{1}{\xi} dx \\ &= \int \frac{1}{-\frac{1}{\sin(2x)}} dx \end{aligned}$$

Which results in

$$S = \frac{\cos(2x)}{2}$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x,y)S_y}{R_x + \omega(x,y)R_y} \tag{2}$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x,y)$ is the right hand side of the original ode given by

$$\omega(x,y) = -\frac{\sin(2x)}{\cos(3y)}$$

Evaluating all the partial derivatives gives

$$
\begin{aligned}
R_x &= 0 \\
R_y &= 1 \\
S_x &= -\sin(2x) \\
S_y &= 0
\end{aligned}
$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$
\frac{dS}{dR} = \cos(3y) \tag{2A}
$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$
\frac{dS}{dR} = \cos(3R)
$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$
S(R) = \frac{\sin(3R)}{3} + c_1 \tag{4}
$$

To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$
\frac{\cos(2x)}{2} = \frac{\sin(3y)}{3} + c_1
$$

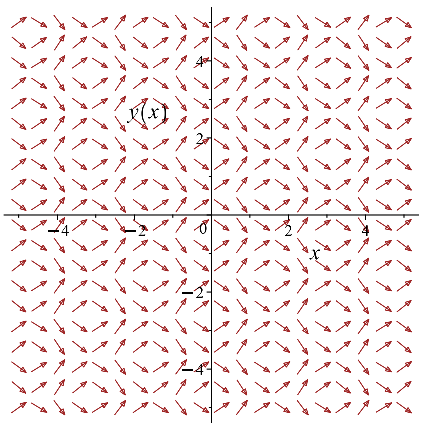
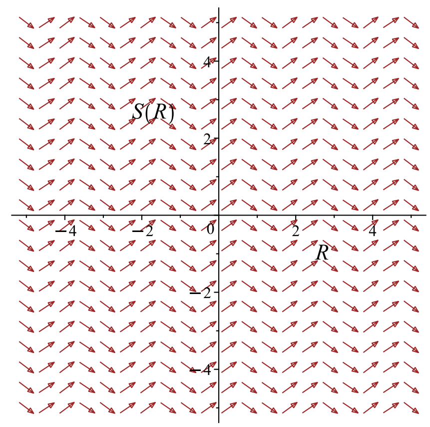
Which simplifies to

$$
\frac{\cos(2x)}{2} = \frac{\sin(3y)}{3} + c_1
$$

Which gives

$$
y = -\frac{\arcsin\left(-\frac{3\cos(2x)}{2} + 3c_1\right)}{3}
$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in x, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dx} = -\frac{\sin(2x)}{\cos(3y)}$ 	$R = y$ $S = \frac{\cos(2x)}{2}$	$\frac{dS}{dR} = \cos(3R)$ 

Initial conditions are used to solve for c_1. Substituting $x = \frac{\pi}{2}$ and $y = 0$ in the above solution gives an equation to solve for the constant of integration.

$$0 = -\frac{\arcsin\left(\frac{3}{2} + 3c_1\right)}{3}$$

$$c_1 = -\frac{1}{2}$$

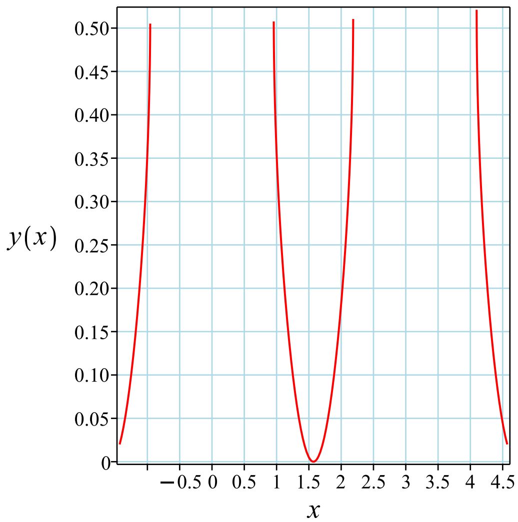
Substituting c_1 found above in the general solution gives

$$y = \frac{\arcsin\left(\frac{3\cos(2x)}{2} + \frac{3}{2}\right)}{3}$$

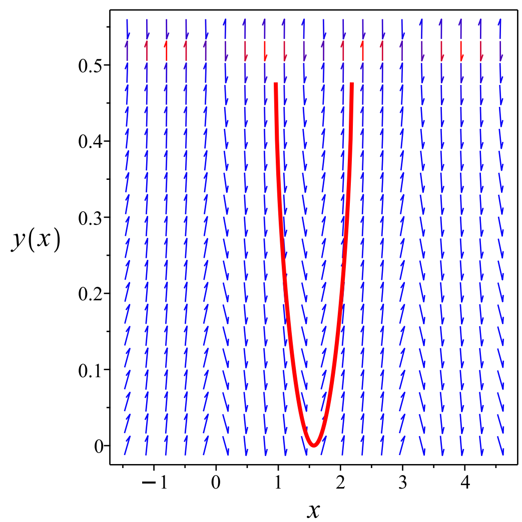
Summary

The solution(s) found are the following

$$y = \frac{\arcsin\left(\frac{3\cos(2x)}{2} + \frac{3}{2}\right)}{3} \tag{1}$$



(a) Solution plot



(b) Slope field plot

<u>Verification of solutions</u>

$$y = \frac{\arcsin\left(\frac{3\cos(2x)}{2} + \frac{3}{2}\right)}{3}$$

Verified OK.

2.19.4 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x,y) + N(x,y)\frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x,y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y) = 0$$

Hence

$$\frac{\partial\phi}{\partial x} + \frac{\partial\phi}{\partial y}\frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$\frac{\partial\phi}{\partial x} = M$$
$$\frac{\partial\phi}{\partial y} = N$$

But since $\frac{\partial^2\phi}{\partial x\partial y} = \frac{\partial^2\phi}{\partial y\partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x, y)$ but at least we know now that we can do that since the condition $\frac{\partial^2\phi}{\partial x\partial y} = \frac{\partial^2\phi}{\partial y\partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x, y)\,\mathrm{d}x + N(x, y)\,\mathrm{d}y = 0 \tag{1A}$$

Therefore

$$\begin{aligned}(-\cos(3y))\,\mathrm{d}y &= (\sin(2x))\,\mathrm{d}x \\ (-\sin(2x))\,\mathrm{d}x + (-\cos(3y))\,\mathrm{d}y &= 0 \end{aligned} \tag{2A}$$

Comparing (1A) and (2A) shows that

$$\begin{aligned} M(x, y) &= -\sin(2x) \\ N(x, y) &= -\cos(3y) \end{aligned}$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Using result found above gives

$$\begin{aligned} \frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}(-\sin(2x)) \\ &= 0 \end{aligned}$$

And

$$\begin{aligned} \frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}(-\cos(3y)) \\ &= 0 \end{aligned}$$

Since $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$, then the ODE is exact The following equations are now set up to solve for the function $\phi(x, y)$

$$\frac{\partial\phi}{\partial x} = M \tag{1}$$

$$\frac{\partial\phi}{\partial y} = N \tag{2}$$

Integrating (1) w.r.t. x gives

$$\int \frac{\partial \phi}{\partial x} \mathrm{d}x = \int M \mathrm{d}x$$
$$\int \frac{\partial \phi}{\partial x} \mathrm{d}x = \int -\sin(2x) \mathrm{d}x$$
$$\phi = \frac{\cos(2x)}{2} + f(y) \tag{3}$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$\frac{\partial \phi}{\partial y} = 0 + f'(y) \tag{4}$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = -\cos(3y)$. Therefore equation (4) becomes

$$-\cos(3y) = 0 + f'(y) \tag{5}$$

Solving equation (5) for $f'(y)$ gives

$$f'(y) = -\cos(3y)$$

Integrating the above w.r.t y gives

$$\int f'(y) \mathrm{d}y = \int (-\cos(3y)) \mathrm{d}y$$
$$f(y) = -\frac{\sin(3y)}{3} + c_1$$

Where c_1 is constant of integration. Substituting result found above for $f(y)$ into equation (3) gives ϕ

$$\phi = \frac{\cos(2x)}{2} - \frac{\sin(3y)}{3} + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = \frac{\cos(2x)}{2} - \frac{\sin(3y)}{3}$$

Initial conditions are used to solve for c_1. Substituting $x = \frac{\pi}{2}$ and $y = 0$ in the above solution gives an equation to solve for the constant of integration.

$$-\frac{1}{2} = c_1$$

$$c_1 = -\frac{1}{2}$$

Substituting c_1 found above in the general solution gives

$$\frac{\cos(2x)}{2} - \frac{\sin(3y)}{3} = -\frac{1}{2}$$

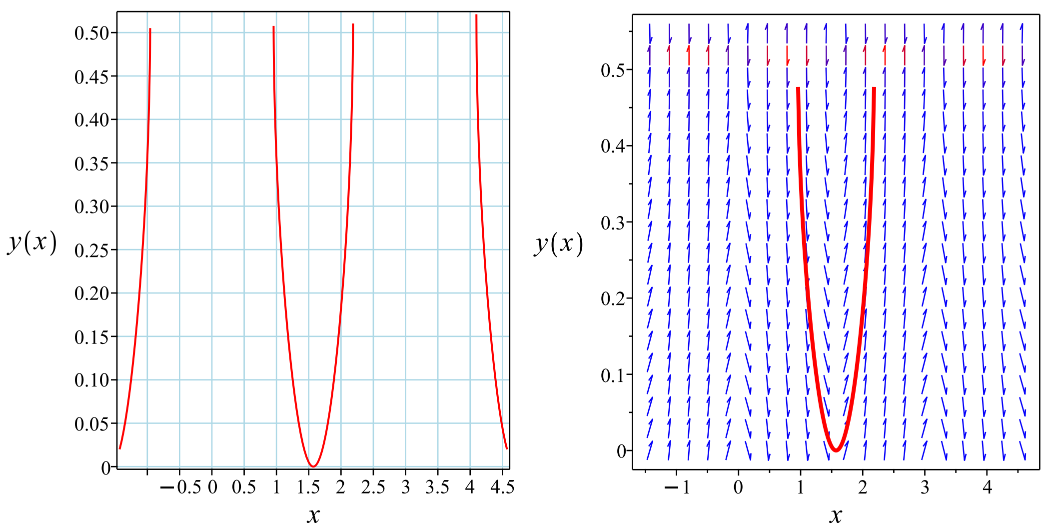
Solving for y from the above gives

$$y = \frac{\arcsin\left(\frac{3\cos(2x)}{2} + \frac{3}{2}\right)}{3}$$

Summary

The solution(s) found are the following

$$y = \frac{\arcsin\left(\frac{3\cos(2x)}{2} + \frac{3}{2}\right)}{3} \tag{1}$$



(a) Solution plot

(b) Slope field plot

Verification of solutions

$$y = \frac{\arcsin\left(\frac{3\cos(2x)}{2} + \frac{3}{2}\right)}{3}$$

Verified OK.

2.19.5 Maple step by step solution

Let's solve

$\left[\cos(3y)\, y' = -\sin(2x), y\left(\frac{\pi}{2}\right) = 0\right]$

- Highest derivative means the order of the ODE is 1

  y'

- Integrate both sides with respect to x

  $\int \cos(3y)\, y' dx = \int -\sin(2x)\, dx + c_1$

- Evaluate integral

  $\frac{\sin(3y)}{3} = \frac{\cos(2x)}{2} + c_1$

- Solve for y

  $y = \frac{\arcsin\left(\frac{3\cos(2x)}{2} + 3c_1\right)}{3}$

- Use initial condition $y\left(\frac{\pi}{2}\right) = 0$

  $0 = \frac{\arcsin\left(-\frac{3}{2} + 3c_1\right)}{3}$

- Solve for c_1

  $c_1 = \frac{1}{2}$

- Substitute $c_1 = \frac{1}{2}$ into general solution and simplify

  $y = \frac{\arcsin\left(\frac{3\cos(2x)}{2} + \frac{3}{2}\right)}{3}$

- Solution to the IVP

  $y = \frac{\arcsin\left(\frac{3\cos(2x)}{2} + \frac{3}{2}\right)}{3}$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
<- separable successful`
```

✓ Solution by Maple

Time used: 0.297 (sec). Leaf size: 15

```
dsolve([sin(2*x)+cos(3*y(x))*diff(y(x),x) = 0,y(1/2*Pi) = 0],y(x), singsol=all)
```

$$y(x) = \frac{\arcsin\left(\frac{3}{2} + \frac{3\cos(2x)}{2}\right)}{3}$$

✓ Solution by Mathematica

Time used: 0.614 (sec). Leaf size: 16

```
DSolve[{Sin[2*x]+Cos[3*y[x]]*y'[x] == 0,y[Pi/2]==0},y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \frac{1}{3}\arcsin\left(3\cos^2(x)\right)$$

2.20 problem 20

Internal problem ID [498]
Internal file name [`OUTPUT/498_Sunday_June_05_2022_01_42_30_AM_58411234/index.tex`]

Book: Elementary differential equations and boundary value problems, 10th ed., Boyce and DiPrima
Section: Section 2.2. Page 48
Problem number: 20.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"exact", "separable", "first_order_ode_lie_symmetry_lookup"**

Maple gives the following as the ode type

`[_separable]`

$$\boxed{\sqrt{-x^{2}+1}\, y^{2} y' = \arcsin\left(x\right)}$$

With initial conditions

$$[y(0) = 1]$$

2.20.1 Existence and uniqueness analysis

This is non linear first order ODE. In canonical form it is written as

$$\begin{aligned} y' &= f(x,y) \\ &= \frac{\arcsin\left(x\right)}{\sqrt{-x^{2}+1}\, y^{2}} \end{aligned}$$

The x domain of $f(x,y)$ when $y=1$ is

$$\{-1<x<1\}$$

And the point $x_0 = 0$ is inside this domain. The y domain of $f(x,y)$ when $x = 0$ is

$$\{-\infty < y < \infty\}$$

And the point $y_0 = 1$ is inside this domain. Now we will look at the continuity of

$$\begin{aligned}\frac{\partial f}{\partial y} &= \frac{\partial}{\partial y}\left(\frac{\arcsin(x)}{\sqrt{-x^2+1}\,y^2}\right)\\&= -\frac{2\arcsin(x)}{\sqrt{-x^2+1}\,y^3}\end{aligned}$$

The x domain of $\frac{\partial f}{\partial y}$ when $y = 1$ is

$$\{-1 < x < 1\}$$

And the point $x_0 = 0$ is inside this domain. The y domain of $\frac{\partial f}{\partial y}$ when $x = 0$ is

$$\{-\infty < y < \infty\}$$

And the point $y_0 = 1$ is inside this domain. Therefore solution exists and is unique.

2.20.2 Solving as separable ode

In canonical form the ODE is

$$\begin{aligned}y' &= F(x,y)\\&= f(x)g(y)\\&= \frac{\arcsin(x)}{\sqrt{-x^2+1}\,y^2}\end{aligned}$$

Where $f(x) = \frac{\arcsin(x)}{\sqrt{-x^2+1}}$ and $g(y) = \frac{1}{y^2}$. Integrating both sides gives

$$\frac{1}{\frac{1}{y^2}}\,dy = \frac{\arcsin(x)}{\sqrt{-x^2+1}}\,dx$$

$$\int \frac{1}{\frac{1}{y^2}}\,dy = \int \frac{\arcsin(x)}{\sqrt{-x^2+1}}\,dx$$

$$\frac{y^3}{3} = \frac{\arcsin(x)^2}{2} + c_1$$

Which results in

$$y = \frac{\left(12 \arcsin \left(x\right)^{2}+24 c_{1}\right)^{\frac{1}{3}}}{2}$$

$$y = -\frac{\left(12 \arcsin \left(x\right)^{2}+24 c_{1}\right)^{\frac{1}{3}}}{4}+\frac{i \sqrt{3}\, \left(12 \arcsin \left(x\right)^{2}+24 c_{1}\right)^{\frac{1}{3}}}{4}$$

$$y = -\frac{\left(12 \arcsin \left(x\right)^{2}+24 c_{1}\right)^{\frac{1}{3}}}{4}-\frac{i \sqrt{3}\, \left(12 \arcsin \left(x\right)^{2}+24 c_{1}\right)^{\frac{1}{3}}}{4}$$

Initial conditions are used to solve for c_1. Substituting $x = 0$ and $y = 1$ in the above solution gives an equation to solve for the constant of integration.

$$1 = -\frac{i c_{1}^{\frac{1}{3}} 24^{\frac{1}{3}} \sqrt{3}}{4}-\frac{c_{1}^{\frac{1}{3}} 24^{\frac{1}{3}}}{4}$$

Warning: Unable to solve for constant of integration. Initial conditions are used to solve for c_1. Substituting $x = 0$ and $y = 1$ in the above solution gives an equation to solve for the constant of integration.

$$1 = \frac{i c_{1}^{\frac{1}{3}} 24^{\frac{1}{3}} \sqrt{3}}{4}-\frac{c_{1}^{\frac{1}{3}} 24^{\frac{1}{3}}}{4}$$

Warning: Unable to solve for constant of integration. Initial conditions are used to solve for c_1. Substituting $x = 0$ and $y = 1$ in the above solution gives an equation to solve for the constant of integration.

$$1 = \frac{c_{1}^{\frac{1}{3}} 24^{\frac{1}{3}}}{2}$$

$$c_1 = \frac{1}{3}$$

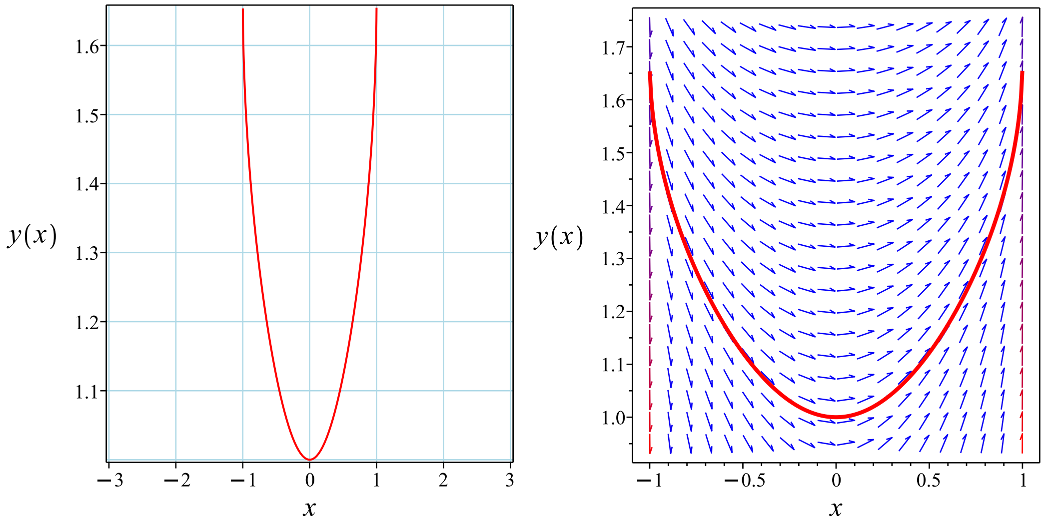
Substituting c_1 found above in the general solution gives

$$y = \frac{\left(12 \arcsin \left(x\right)^{2}+8\right)^{\frac{1}{3}}}{2}$$

Summary

The solution(s) found are the following

$$y = \frac{\left(12 \arcsin \left(x\right)^{2}+8\right)^{\frac{1}{3}}}{2} \tag{1}$$



(a) Solution plot

(b) Slope field plot

Verification of solutions

$$y = \frac{\left(12 \arcsin\left(x\right)^{2}+8\right)^{\frac{1}{3}}}{2}$$

Verified OK.

2.20.3 Solving as first order ode lie symmetry lookup ode

Writing the ode as

$$y' = \frac{\arcsin\left(x\right)}{\sqrt{-x^{2}+1}\, y^{2}}$$
$$y' = \omega(x, y)$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x + \omega(\eta_y - \xi_x) - \omega^2 \xi_y - \omega_x \xi - \omega_y \eta = 0 \tag{A}$$

The type of this ode is known. It is of type `separable`. Therefore we do not need to solve the PDE (A), and can just use the lookup table shown below to find ξ, η

Table 148: Lie symmetry infinitesimal lookup table for known first order ODE's

ODE class	Form	ξ	η
linear ode	$y' = f(x)y(x) + g(x)$	0	$e^{\int f dx}$
separable ode	$y' = f(x)\, g(y)$	$\frac{1}{f}$	0
quadrature ode	$y' = f(x)$	0	1
quadrature ode	$y' = g(y)$	1	0
homogeneous ODEs of Class A	$y' = f\left(\frac{y}{x}\right)$	x	y
homogeneous ODEs of Class C	$y' = (a + bx + cy)^{\frac{n}{m}}$	1	$-\frac{b}{c}$
homogeneous class D	$y' = \frac{y}{x} + g(x)\, F\left(\frac{y}{x}\right)$	x^2	xy
First order special form ID 1	$y' = g(x)\, e^{h(x)+by} + f(x)$	$\frac{e^{-\int bf(x)dx - h(x)}}{g(x)}$	$\frac{f(x)e^{-\int bf(x)dx - h(x)}}{g(x)}$
polynomial type ode	$y' = \frac{a_1x+b_1y+c_1}{a_2x+b_2y+c_2}$	$\frac{a_1b_2x - a_2b_1x - b_1c_2 + b_2c_1}{a_1b_2 - a_2b_1}$	$\frac{a_1b_2y - a_2b_1y - a_1c_2 - a_2c_1}{a_1b_2 - a_2b_1}$
Bernoulli ode	$y' = f(x)\, y + g(x)\, y^n$	0	$e^{-\int (n-1)f(x)dx} y^n$
Reduced Riccati	$y' = f_1(x)\, y + f_2(x)\, y^2$	0	$e^{-\int f_1 dx}$

The above table shows that

$$
\begin{aligned}
\xi(x, y) &= \frac{\sqrt{-x^2 + 1}}{\arcsin(x)} \\
\eta(x, y) &= 0
\end{aligned} \tag{A1}
$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x, y) \to (R, S)$ where (R, S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$
\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}
$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y}\right) S(x, y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the

canonical coordinates, where $S(R)$. Since $\eta = 0$ then in this special case

$$R = y$$

S is found from

$$\begin{aligned} S &= \int \frac{1}{\xi} dx \\ &= \int \frac{1}{\frac{\sqrt{-x^2+1}}{\arcsin(x)}} dx \end{aligned}$$

Which results in

$$S = \frac{\arcsin(x)^2}{2}$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x,y) S_y}{R_x + \omega(x,y) R_y} \tag{2}$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x,y)$ is the right hand side of the original ode given by

$$\omega(x,y) = \frac{\arcsin(x)}{\sqrt{-x^2+1}\, y^2}$$

Evaluating all the partial derivatives gives

$$\begin{aligned} R_x &= 0 \\ R_y &= 1 \\ S_x &= \frac{\arcsin(x)}{\sqrt{-x^2+1}} \\ S_y &= 0 \end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = y^2 \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = R^2$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordinates R, S. Integrating the above gives

$$S(R) = \frac{R^3}{3} + c_1 \tag{4}$$

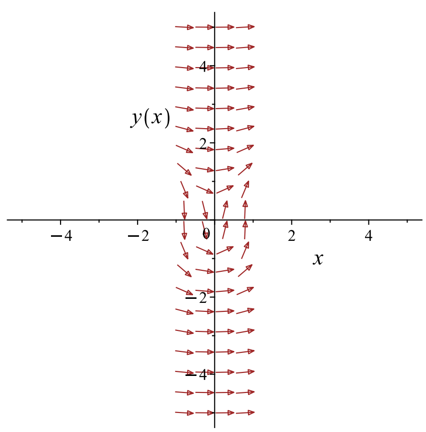
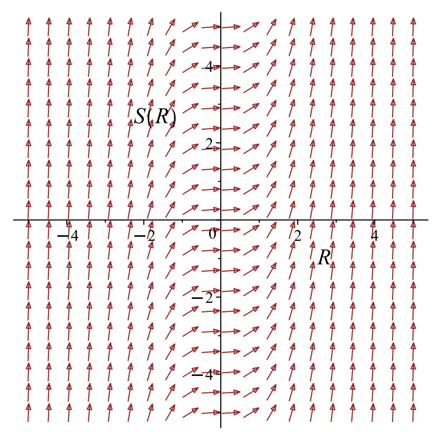
To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$\frac{\arcsin(x)^2}{2} = \frac{y^3}{3} + c_1$$

Which simplifies to

$$\frac{\arcsin(x)^2}{2} = \frac{y^3}{3} + c_1$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in x, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dx} = \frac{\arcsin(x)}{\sqrt{-x^2+1}\, y^2}$ 	$R = y$ $S = \frac{\arcsin(x)^2}{2}$	$\frac{dS}{dR} = R^2$ 

Initial conditions are used to solve for c_1. Substituting $x = 0$ and $y = 1$ in the above solution gives an equation to solve for the constant of integration.

$$0 = \frac{1}{3} + c_1$$

$$
c_1 = -\frac{1}{3}
$$

Substituting c_1 found above in the general solution gives

$$
\frac{\arcsin(x)^2}{2} = \frac{y^3}{3} - \frac{1}{3}
$$

Summary

The solution(s) found are the following

$$
\frac{\arcsin(x)^2}{2} = \frac{y^3}{3} - \frac{1}{3} \tag{1}
$$

Verification of solutions

$$
\frac{\arcsin(x)^2}{2} = \frac{y^3}{3} - \frac{1}{3}
$$

Verified OK.

2.20.4 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$
M(x,y) + N(x,y)\frac{dy}{dx} = 0 \tag{A}
$$

We assume there exists a function $\phi(x,y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$
\frac{d}{dx}\phi(x,y) = 0
$$

Hence

$$
\frac{\partial \phi}{\partial x} + \frac{\partial \phi}{\partial y}\frac{dy}{dx} = 0 \tag{B}
$$

Comparing (A,B) shows that

$$
\frac{\partial \phi}{\partial x} = M
$$
$$
\frac{\partial \phi}{\partial y} = N
$$

But since $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ then for the above to be valid, we require that

$$
\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}
$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x,y)$ but at least we know now that we can do that since the condition $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x,y)\,\mathrm{d}x + N(x,y)\,\mathrm{d}y = 0 \tag{1A}$$

Therefore

$$(y^2)\,\mathrm{d}y = \left(\frac{\arcsin(x)}{\sqrt{-x^2+1}}\right)\mathrm{d}x$$

$$\left(-\frac{\arcsin(x)}{\sqrt{-x^2+1}}\right)\mathrm{d}x + (y^2)\,\mathrm{d}y = 0 \tag{2A}$$

Comparing (1A) and (2A) shows that

$$M(x,y) = -\frac{\arcsin(x)}{\sqrt{-x^2+1}}$$
$$N(x,y) = y^2$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Using result found above gives

$$\frac{\partial M}{\partial y} = \frac{\partial}{\partial y}\left(-\frac{\arcsin(x)}{\sqrt{-x^2+1}}\right)$$
$$= 0$$

And

$$\frac{\partial N}{\partial x} = \frac{\partial}{\partial x}(y^2)$$
$$= 0$$

Since $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$, then the ODE is exact The following equations are now set up to solve for the function $\phi(x,y)$

$$\frac{\partial \phi}{\partial x} = M \tag{1}$$

$$\frac{\partial \phi}{\partial y} = N \tag{2}$$

Integrating (1) w.r.t. x gives

$$\int \frac{\partial \phi}{\partial x} \, \mathrm{d}x = \int M \, \mathrm{d}x$$

$$\int \frac{\partial \phi}{\partial x} \, \mathrm{d}x = \int -\frac{\arcsin(x)}{\sqrt{-x^2+1}} \, \mathrm{d}x$$

$$\phi = -\frac{\arcsin(x)^2}{2} + f(y) \tag{3}$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$\frac{\partial \phi}{\partial y} = 0 + f'(y) \tag{4}$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = y^2$. Therefore equation (4) becomes

$$y^2 = 0 + f'(y) \tag{5}$$

Solving equation (5) for $f'(y)$ gives

$$f'(y) = y^2$$

Integrating the above w.r.t y gives

$$\int f'(y) \, \mathrm{d}y = \int (y^2) \, \mathrm{d}y$$

$$f(y) = \frac{y^3}{3} + c_1$$

Where c_1 is constant of integration. Substituting result found above for $f(y)$ into equation (3) gives ϕ

$$\phi = -\frac{\arcsin(x)^2}{2} + \frac{y^3}{3} + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = -\frac{\arcsin(x)^2}{2} + \frac{y^3}{3}$$

Initial conditions are used to solve for c_1. Substituting $x = 0$ and $y = 1$ in the above solution gives an equation to solve for the constant of integration.

$$\frac{1}{3} = c_1$$

$$c_1 = \frac{1}{3}$$

Substituting c_1 found above in the general solution gives

$$-\frac{\arcsin(x)^2}{2} + \frac{y^3}{3} = \frac{1}{3}$$

Summary

The solution(s) found are the following

$$-\frac{\arcsin(x)^2}{2} + \frac{y^3}{3} = \frac{1}{3} \tag{1}$$

Verification of solutions

$$-\frac{\arcsin(x)^2}{2} + \frac{y^3}{3} = \frac{1}{3}$$

Verified OK.

2.20.5 Maple step by step solution

Let's solve

$\left[\sqrt{-x^2+1}\, y^2 y' = \arcsin(x), y(0) = 1\right]$

- Highest derivative means the order of the ODE is 1

  y'

- Separate variables

  $y' y^2 = \frac{\arcsin(x)}{\sqrt{-x^2+1}}$

- Integrate both sides with respect to x

  $\int y' y^2 dx = \int \frac{\arcsin(x)}{\sqrt{-x^2+1}} dx + c_1$

- Evaluate integral

  $\frac{y^3}{3} = \frac{\arcsin(x)^2}{2} + c_1$

- Solve for y

$$y = \frac{\left(12 \arcsin(x)^2+24c_1\right)^{\frac{1}{3}}}{2}$$

- Use initial condition $y(0) = 1$

$$1 = \frac{c_1^{\frac{1}{3}} 24^{\frac{1}{3}}}{2}$$

- Solve for c_1

$$c_1 = \frac{1}{3}$$

- Substitute $c_1 = \frac{1}{3}$ into general solution and simplify

$$y = \frac{\left(12 \arcsin(x)^2+8\right)^{\frac{1}{3}}}{2}$$

- Solution to the IVP

$$y = \frac{\left(12 \arcsin(x)^2+8\right)^{\frac{1}{3}}}{2}$$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
<- Bernoulli successful`
```

✓ Solution by Maple

Time used: 0.109 (sec). Leaf size: 16

```
dsolve([(-x^2+1)^(1/2)*y(x)^2*diff(y(x),x) = arcsin(x),y(0) = 1],y(x), singsol=all)
```

$$y(x) = \frac{\left(8 + 12 \arcsin\left(x\right)^2\right)^{\frac{1}{3}}}{2}$$

✓ Solution by Mathematica

Time used: 0.527 (sec). Leaf size: 19

```
DSolve[{(-x^2+1)^(1/2)*y[x]^2*y'[x] == ArcSin[x],y[0]==1},y[x],x,IncludeSingularSolutions ->
```

$$y(x) \to \sqrt[3]{\frac{3\arcsin(x)^2}{2}+1}$$

2.21 problem 21

Internal problem ID [499]
Internal file name [`OUTPUT/499_Sunday_June_05_2022_01_42_31_AM_97879552/index.tex`]

Book: Elementary differential equations and boundary value problems, 10th ed., Boyce and DiPrima
Section: Section 2.2. Page 48
Problem number: 21.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"exact", "separable", "differentialType", "first_order_ode_lie_symmetry_lookup"**

Maple gives the following as the ode type

`[_separable]`

$$\boxed{y' - \frac{3x^2 + 1}{-6y + 3y^2} = 0}$$

With initial conditions

$$[y(0) = 1]$$

2.21.1 Existence and uniqueness analysis

This is non linear first order ODE. In canonical form it is written as

$$\begin{aligned} y' &= f(x, y) \\ &= \frac{3x^2+1}{3y\,(y-2)} \end{aligned}$$

The x domain of $f(x, y)$ when $y = 1$ is

$$\{-\infty < x < \infty\}$$

And the point $x_0 = 0$ is inside this domain. The y domain of $f(x, y)$ when $x = 0$ is

$$\{-\infty \leq y < 0, 0 < y < 2, 2 < y \leq \infty\}$$

And the point $y_0 = 1$ is inside this domain. Now we will look at the continuity of

$$\begin{aligned} \frac{\partial f}{\partial y} &= \frac{\partial}{\partial y}\left(\frac{3x^2+1}{3y\,(y-2)}\right) \\ &= -\frac{3x^2+1}{3y^2\,(y-2)} - \frac{3x^2+1}{3y\,(y-2)^2} \end{aligned}$$

The x domain of $\frac{\partial f}{\partial y}$ when $y = 1$ is

$$\{-\infty < x < \infty\}$$

And the point $x_0 = 0$ is inside this domain. The y domain of $\frac{\partial f}{\partial y}$ when $x = 0$ is

$$\{-\infty \leq y < 0, 0 < y < 2, 2 < y \leq \infty\}$$

And the point $y_0 = 1$ is inside this domain. Therefore solution exists and is unique.

2.21.2 Solving as separable ode

In canonical form the ODE is

$$\begin{aligned} y' &= F(x, y) \\ &= f(x)g(y) \\ &= \frac{x^2+\frac{1}{3}}{y\,(y-2)} \end{aligned}$$

Where $f(x) = x^2 + \frac{1}{3}$ and $g(y) = \frac{1}{y(y-2)}$. Integrating both sides gives

$$\frac{1}{\frac{1}{y(y-2)}}\,dy = x^2 + \frac{1}{3}\,dx$$

$$\int \frac{1}{\frac{1}{y(y-2)}}\,dy = \int x^2 + \frac{1}{3}\,dx$$

$$\frac{1}{3}y^3 - y^2 = \frac{1}{3}x^3 + \frac{1}{3}x + c_1$$

Which results in

$$y = \frac{\left(8 + 4x^3 + 12c_1 + 4x + 4\sqrt{x^6 + 6c_1x^3 + 2x^4 + 4x^3 + 9c_1^2 + 6c_1x + x^2 + 12c_1 + 4x}\right)^{\frac{1}{3}}}{2} + \frac{2}{\left(8 + 4x^3 + 12c_1 + 4x + 4\sqrt{x^6 + 6c_1x^3 + 2x^4 + 4x^3 + 9c_1^2 + 6c_1x + x^2 + 12c_1 + 4x}\right)^{\frac{1}{3}}} + 1$$

$$y = -\frac{\left(8 + 4x^3 + 12c_1 + 4x + 4\sqrt{x^6 + 6c_1x^3 + 2x^4 + 4x^3 + 9c_1^2 + 6c_1x + x^2 + 12c_1 + 4x}\right)^{\frac{1}{3}}}{4} - \frac{1}{\left(8 + 4x^3 + 12c_1 + 4x + 4\sqrt{x^6 + 6c_1x^3 + 2x^4 + 4x^3 + 9c_1^2 + 6c_1x + x^2 + 12c_1 + 4x}\right)^{\frac{1}{3}}} + 1 + \frac{i\sqrt{3}\left(\frac{\left(8+4x^3+12c_1+4x+4\sqrt{x^6+6c_1x^3+2x^4+4x^3+9c_1^2+6c_1x+x^2+12c_1+4x}\right)^{\frac{1}{3}}}{2} - \frac{2}{\left(8+4x^3+12c_1+4x+4\sqrt{x^6+6c_1x^3+2x^4+4x^3+9c_1^2 \ldots\right.}}\right.}{2}$$

$$y = -\frac{\left(8+4x^3+12c_1+4x+4\sqrt{x^6+6c_1x^3+2x^4+4x^3+9c_1^2+6c_1x+x^2+12c_1+4x}\right)^{\frac{1}{3}}}{4} - \frac{1}{\left(8+4x^3+12c_1+4x+4\sqrt{x^6+6c_1x^3+2x^4+4x^3+9c_1^2+6c_1x+x^2+12c_1+4x}\right)^{\frac{1}{3}}} + 1 - \frac{i\sqrt{3}\left(\frac{\left(8+4x^3+12c_1+4x+4\sqrt{x^6+6c_1x^3+2x^4+4x^3+9c_1^2+6c_1x+x^2+12c_1+4x}\right)^{\frac{1}{3}}}{2} - \frac{2}{\left(8+4x^3+12c_1+4x+4\sqrt{x^6+6c_1x^3+2x^4+4x^3+9c\right.}}\right.}{2}$$

Initial conditions are used to solve for c_1. Substituting $x = 0$ and $y = 1$ in the above solution gives an equation to solve for the constant of integration.

$$1 = \frac{-i\sqrt{3}\left(8+12c_1+4\sqrt{3}\,\sqrt{3c_1^2+4c_1}\right)^{\frac{2}{3}} + 4i\sqrt{3} - \left(8+12c_1+4\sqrt{3}\,\sqrt{3c_1^2+4c_1}\right)^{\frac{2}{3}} + 4\left(8+12c_1+4\sqrt{\right.}}{4\left(8+12c_1+4\sqrt{3}\,\sqrt{3c_1^2+4c_1}\right)^{\frac{1}{3}}}$$

Warning: Unable to solve for constant of integration. Initial conditions are used to solve for c_1. Substituting $x = 0$ and $y = 1$ in the above solution gives an equation to solve for the constant of integration.

$$1 = \frac{i\sqrt{3}\left(8+12c_1+4\sqrt{3}\,\sqrt{3c_1^2+4c_1}\right)^{\frac{2}{3}} - 4i\sqrt{3} - \left(8+12c_1+4\sqrt{3}\,\sqrt{3c_1^2+4c_1}\right)^{\frac{2}{3}} + 4\left(8+12c_1+4\sqrt{3}\right.}{4\left(8+12c_1+4\sqrt{3}\,\sqrt{3c_1^2+4c_1}\right)^{\frac{1}{3}}}$$

Warning: Unable to solve for constant of integration. Initial conditions are used to solve for c_1. Substituting $x = 0$ and $y = 1$ in the above solution gives an equation to solve for the constant of integration.

$$1 = \frac{\left(8+12c_1+4\sqrt{3}\,\sqrt{3c_1^2+4c_1}\right)^{\frac{2}{3}} + 2\left(8+12c_1+4\sqrt{3}\,\sqrt{3c_1^2+4c_1}\right)^{\frac{1}{3}} + 4}{2\left(8+12c_1+4\sqrt{3}\,\sqrt{3c_1^2+4c_1}\right)^{\frac{1}{3}}}$$

Warning: Unable to solve for constant of integration.

Verification of solutions N/A

2.21.3 Solving as differentialType ode

Writing the ode as

$$y' = \frac{3x^2+1}{-6y+3y^2} \tag{1}$$

Which becomes

$$\left(3y^2-6y\right)dy = \left(3x^2+1\right)dx \tag{2}$$

But the RHS is complete differential because

$$\left(3x^2+1\right)dx = d\left(x^3+x\right)$$

Hence (2) becomes

$$\left(3y^2-6y\right)dy = d\left(x^3+x\right)$$

Integrating both sides gives gives these solutions

$$y = \frac{\left(8+4x^3+4c_1+4x+4\sqrt{x^6+2c_1x^3+2x^4+4x^3+c_1^2+2c_1x+x^2+4c_1+4x}\right)^{\frac{1}{3}}}{2} + \frac{}{\left(8+4x^3+4c_1\right.}$$

$$y = -\frac{\left(8+4x^3+4c_1+4x+4\sqrt{x^6+2c_1x^3+2x^4+4x^3+c_1^2+2c_1x+x^2+4c_1+4x}\right)^{\frac{1}{3}}}{4} - \frac{}{\left(8+4x^3+4c\right.}$$

$$y = -\frac{\left(8+4x^3+4c_1+4x+4\sqrt{x^6+2c_1x^3+2x^4+4x^3+c_1^2+2c_1x+x^2+4c_1+4x}\right)^{\frac{1}{3}}}{4} - \frac{}{\left(8+4x^3+4c\right.}$$

Initial conditions are used to solve for c_1. Substituting $x=0$ and $y=1$ in the above solution gives an equation to solve for the constant of integration.

$$1 = \frac{-i\sqrt{3}\left(8+4c_1+4\sqrt{c_1^2+4c_1}\right)^{\frac{2}{3}}+4i\sqrt{3}-\left(8+4c_1+4\sqrt{c_1^2+4c_1}\right)^{\frac{2}{3}}+4c_1\left(8+4c_1+4\sqrt{c_1^2+4c_1}\right)^{\frac{1}{3}}}{4\left(8+4c_1+4\sqrt{c_1^2+4c_1}\right)^{\frac{1}{3}}}$$

Warning: Unable to solve for constant of integration. Initial conditions are used to solve for c_1. Substituting $x=0$ and $y=1$ in the above solution gives an equation to solve for the constant of integration.

$$1 = \frac{i\sqrt{3}\left(8+4c_1+4\sqrt{c_1^2+4c_1}\right)^{\frac{2}{3}} - 4i\sqrt{3} - \left(8+4c_1+4\sqrt{c_1^2+4c_1}\right)^{\frac{2}{3}} + 4c_1\left(8+4c_1+4\sqrt{c_1^2+4c_1}\right)^{\frac{1}{3}} +}{4\left(8+4c_1+4\sqrt{c_1^2+4c_1}\right)^{\frac{1}{3}}}$$

Warning: Unable to solve for constant of integration. Initial conditions are used to solve for c_1. Substituting $x=0$ and $y=1$ in the above solution gives an equation to solve for the constant of integration.

$$1 = \frac{\left(8+4c_1+4\sqrt{c_1^2+4c_1}\right)^{\frac{2}{3}} + 2c_1\left(8+4c_1+4\sqrt{c_1^2+4c_1}\right)^{\frac{1}{3}} + 2\left(8+4c_1+4\sqrt{c_1^2+4c_1}\right)^{\frac{1}{3}} + 4}{2\left(8+4c_1+4\sqrt{c_1^2+4c_1}\right)^{\frac{1}{3}}}$$

Warning: Unable to solve for constant of integration.

Verification of solutions N/A

2.21.4 Solving as first order ode lie symmetry lookup ode

Writing the ode as

$$\begin{aligned} y' &= \frac{3x^2+1}{3y(y-2)} \\ y' &= \omega(x,y) \end{aligned}$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x + \omega(\eta_y - \xi_x) - \omega^2 \xi_y - \omega_x \xi - \omega_y \eta = 0 \tag{A}$$

The type of this ode is known. It is of type `separable`. Therefore we do not need to solve the PDE (A), and can just use the lookup table shown below to find ξ, η

Table 151: Lie symmetry infinitesimal lookup table for known first order ODE's

ODE class	Form	ξ	η
linear ode	$y' = f(x)y(x) + g(x)$	0	$e^{\int f dx}$
separable ode	$y' = f(x)\, g(y)$	$\frac{1}{f}$	0
quadrature ode	$y' = f(x)$	0	1
quadrature ode	$y' = g(y)$	1	0
homogeneous ODEs of Class A	$y' = f\left(\frac{y}{x}\right)$	x	y
homogeneous ODEs of Class C	$y' = (a + bx + cy)^{\frac{n}{m}}$	1	$-\frac{b}{c}$
homogeneous class D	$y' = \frac{y}{x} + g(x)\, F\left(\frac{y}{x}\right)$	x^2	xy
First order special form ID 1	$y' = g(x)\, e^{h(x)+by} + f(x)$	$\frac{e^{-\int bf(x)dx - h(x)}}{g(x)}$	$\frac{f(x)e^{-\int bf(x)dx - h(x)}}{g(x)}$
polynomial type ode	$y' = \frac{a_1x+b_1y+c_1}{a_2x+b_2y+c_2}$	$\frac{a_1b_2x-a_2b_1x-b_1c_2+b_2c_1}{a_1b_2-a_2b_1}$	$\frac{a_1b_2y-a_2b_1y-a_1c_2-a_2c_1}{a_1b_2-a_2b_1}$
Bernoulli ode	$y' = f(x)\, y + g(x)\, y^n$	0	$e^{-\int (n-1)f(x)dx} y^n$
Reduced Riccati	$y' = f_1(x)\, y + f_2(x)\, y^2$	0	$e^{-\int f_1 dx}$

The above table shows that

$$
\begin{aligned}
\xi(x, y) &= \frac{1}{x^2 + \frac{1}{3}} \\
\eta(x, y) &= 0
\end{aligned}
\tag{A1}
$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x, y) \to (R, S)$ where (R, S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$
\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}
$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y}\right) S(x, y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the

canonical coordinates, where $S(R)$. Since $\eta = 0$ then in this special case

$$R = y$$

S is found from

$$\begin{aligned} S &= \int \frac{1}{\xi} dx \\ &= \int \frac{1}{\frac{1}{x^2+\frac{1}{3}}} dx \end{aligned}$$

Which results in

$$S = \frac{1}{3}x^3 + \frac{1}{3}x$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x,y)S_y}{R_x + \omega(x,y)R_y} \tag{2}$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x,y)$ is the right hand side of the original ode given by

$$\omega(x,y) = \frac{3x^2+1}{3y(y-2)}$$

Evaluating all the partial derivatives gives

$$\begin{aligned} R_x &= 0 \\ R_y &= 1 \\ S_x &= x^2 + \frac{1}{3} \\ S_y &= 0 \end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = y(y-2) \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = R(R-2)$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordinates R, S. Integrating the above gives

$$S(R) = \frac{1}{3}R^3 - R^2 + c_1 \tag{4}$$

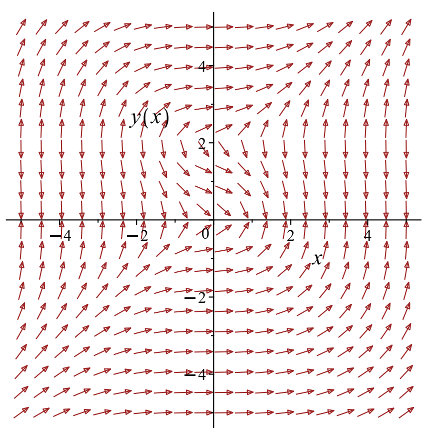
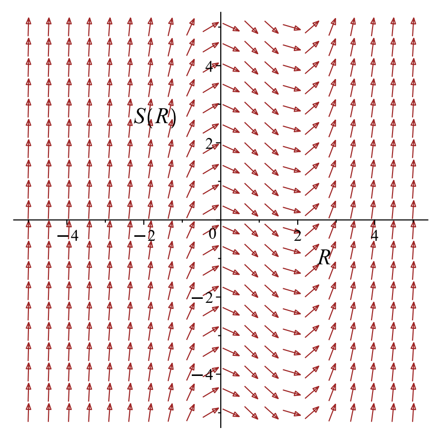
To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$\frac{1}{3}x^3 + \frac{1}{3}x = \frac{y^3}{3} - y^2 + c_1$$

Which simplifies to

$$\frac{1}{3}x^3 + \frac{1}{3}x = \frac{y^3}{3} - y^2 + c_1$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in x, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dx} = \frac{3x^2+1}{3y(y-2)}$ 	$R = y$ $S = \frac{1}{3}x^3 + \frac{1}{3}x$	$\frac{dS}{dR} = R(R-2)$ 

Initial conditions are used to solve for c_1. Substituting $x = 0$ and $y = 1$ in the above solution gives an equation to solve for the constant of integration.

$$0 = -\frac{2}{3} + c_1$$

$$c_1 = \frac{2}{3}$$

Substituting c_1 found above in the general solution gives

$$\frac{1}{3}x^3 + \frac{1}{3}x = \frac{1}{3}y^3 - y^2 + \frac{2}{3}$$

Summary

The solution(s) found are the following

$$\frac{1}{3}x^3 + \frac{1}{3}x = \frac{y^3}{3} - y^2 + \frac{2}{3} \tag{1}$$

Verification of solutions

$$\frac{1}{3}x^3 + \frac{1}{3}x = \frac{y^3}{3} - y^2 + \frac{2}{3}$$

Verified OK.

2.21.5 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x,y) + N(x,y)\frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x,y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y) = 0$$

Hence

$$\frac{\partial\phi}{\partial x} + \frac{\partial\phi}{\partial y}\frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$\begin{aligned}\frac{\partial\phi}{\partial x} &= M\\ \frac{\partial\phi}{\partial y} &= N\end{aligned}$$

But since $\frac{\partial^2\phi}{\partial x\partial y} = \frac{\partial^2\phi}{\partial y\partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x,y)$ but at least we know now that we can do that since the condition $\frac{\partial^2\phi}{\partial x\partial y}=\frac{\partial^2\phi}{\partial y\partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x,y)\,\mathrm{d}x+N(x,y)\,\mathrm{d}y=0 \tag{1A}$$

Therefore

$$\begin{aligned}(3y(y-2))\,\mathrm{d}y&=\left(3x^2+1\right)\mathrm{d}x\\ \left(-3x^2-1\right)\mathrm{d}x+(3y(y-2))\,\mathrm{d}y&=0 \end{aligned}\tag{2A}$$

Comparing (1A) and (2A) shows that

$$\begin{aligned}M(x,y)&=-3x^2-1\\ N(x,y)&=3y(y-2)\end{aligned}$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y}=\frac{\partial N}{\partial x}$$

Using result found above gives

$$\begin{aligned}\frac{\partial M}{\partial y}&=\frac{\partial}{\partial y}\left(-3x^2-1\right)\\ &=0\end{aligned}$$

And

$$\begin{aligned}\frac{\partial N}{\partial x}&=\frac{\partial}{\partial x}(3y(y-2))\\ &=0\end{aligned}$$

Since $\frac{\partial M}{\partial y}=\frac{\partial N}{\partial x}$, then the ODE is <u>exact</u> The following equations are now set up to solve for the function $\phi(x,y)$

$$\frac{\partial\phi}{\partial x}=M \tag{1}$$

$$\frac{\partial\phi}{\partial y}=N \tag{2}$$

Integrating (1) w.r.t. x gives

$$\int \frac{\partial \phi}{\partial x}\,\mathrm{d}x = \int M\,\mathrm{d}x$$

$$\int \frac{\partial \phi}{\partial x}\,\mathrm{d}x = \int -3x^2 - 1\,\mathrm{d}x$$

$$\phi = -x^3 - x + f(y) \tag{3}$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$\frac{\partial \phi}{\partial y} = 0 + f'(y) \tag{4}$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = 3y(y-2)$. Therefore equation (4) becomes

$$3y(y-2) = 0 + f'(y) \tag{5}$$

Solving equation (5) for $f'(y)$ gives

$$f'(y) = 3y(y-2)$$

Integrating the above w.r.t y gives

$$\int f'(y)\,\mathrm{d}y = \int (3y(y-2))\,\mathrm{d}y$$

$$f(y) = y^3 - 3y^2 + c_1$$

Where c_1 is constant of integration. Substituting result found above for $f(y)$ into equation (3) gives ϕ

$$\phi = -x^3 + y^3 - 3y^2 - x + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = -x^3 + y^3 - 3y^2 - x$$

Initial conditions are used to solve for c_1. Substituting $x = 0$ and $y = 1$ in the above solution gives an equation to solve for the constant of integration.

$$-2 = c_1$$

$$c_1 = -2$$

Substituting c_1 found above in the general solution gives

$$-x^3 + y^3 - 3y^2 - x = -2$$

Summary

The solution(s) found are the following

$$-x^3 + y^3 - 3y^2 - x = -2 \tag{1}$$

Verification of solutions

$$-x^3 + y^3 - 3y^2 - x = -2$$

Verified OK.

2.21.6 Maple step by step solution

Let's solve

$\left[y' - \frac{3x^2+1}{-6y+3y^2} = 0, y(0) = 1\right]$

- Highest derivative means the order of the ODE is 1

  y'

- Separate variables

  $y'(-6y + 3y^2) = 3x^2 + 1$

- Integrate both sides with respect to x

  $\int y'(-6y + 3y^2)\, dx = \int (3x^2 + 1)\, dx + c_1$

- Evaluate integral

  $y^3 - 3y^2 = x^3 + c_1 + x$

- Solve for y

  $y = \frac{\left(8+4x^3+4c_1+4x+4\sqrt{x^6+2c_1x^3+2x^4+4x^3+c_1^2+2c_1x+x^2+4c_1+4x}\right)^{\frac{1}{3}}}{2} + \frac{2}{\left(8+4x^3+4c_1+4x+4\sqrt{x^6+2c_1x^3+2x^4+4x^3+c_1^2+2c}\right.}$

- Use initial condition $y(0) = 1$

$$1 = \frac{\left(8+4c_1+4\sqrt{c_1^2+4c_1}\right)^{\frac{1}{3}}}{2} + \frac{2}{\left(8+4c_1+4\sqrt{c_1^2+4c_1}\right)^{\frac{1}{3}}} + 1$$

- Solve for c_1

$$c_1 = RootOf\left(\left(8+4\_Z+4\sqrt{\_Z^2+4\_Z}\right)^{\frac{2}{3}}+4\right)$$

- Substitute $c_1 = RootOf\left(\left(8+4\_Z+4\sqrt{\_Z^2+4\_Z}\right)^{\frac{2}{3}}+4\right)$ into general solution and simplify

$$y = \frac{\left(8+4x^3+4RootOf\left(\left(8+4\_Z+4\sqrt{\_Z(\_Z+4)}\right)^{\frac{2}{3}}+4\right)+4x+4\sqrt{\left(x^3+x+RootOf\left(\left(8+4\_Z+4\sqrt{\_Z(\_Z+4)}\right)^{\frac{2}{3}}+4\right)+\cdots\right.}\right.}{2\left(8+4x^3+4RootOf\left(\left(8+4\cdots\right.\right.\right.}$$

- Solution to the IVP

$$y = \frac{\left(8+4x^3+4RootOf\left(\left(8+4\_Z+4\sqrt{\_Z(\_Z+4)}\right)^{\frac{2}{3}}+4\right)+4x+4\sqrt{\left(x^3+x+RootOf\left(\left(8+4\_Z+4\sqrt{\_Z(\_Z+4)}\right)^{\frac{2}{3}}+4\right)+\cdots\right.}\right.}{2\left(8+4x^3+4RootOf\left(\left(8+4\cdots\right.\right.\right.}$$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
<- separable successful`
```

✓ Solution by Maple

Time used: 0.11 (sec). Leaf size: 109

```
dsolve([diff(y(x),x) = (3*x^2+1)/(-6*y(x)+3*y(x)^2),y(0) = 1],y(x), singsol=all)
```

$$y(x) = -\frac{\left(1+i\sqrt{3}\right)\left(4x^3+4x+4\sqrt{x^6+2x^4+x^2-4}\right)^{\frac{2}{3}}-4i\sqrt{3}-4\left(4x^3+4x+4\sqrt{x^6+2x^4+x^2-4}\right)^{\frac{1}{3}}+4}{4\left(4x^3+4x+4\sqrt{x^6+2x^4+x^2-4}\right)^{\frac{1}{3}}}$$

✓ Solution by Mathematica

Time used: 4.019 (sec). Leaf size: 158

```
DSolve[{y'[x] == (3*x^2+1)/(-6*y[x]+3*y[x]^2),y[0]==1},y[x],x,IncludeSingularSolutions -> Tr
```

$$y(x) \to \frac{-i2^{2/3}\sqrt{3}\left(x^3+\sqrt{x^6+2x^4+x^2-4}+x\right)^{2/3}-2^{2/3}\left(x^3+\sqrt{x^6+2x^4+x^2-4}+x\right)^{2/3}+4\sqrt[3]{x^3+\sqrt{x^6+}}}{4\sqrt[3]{x^3+\sqrt{x^6+2x^4+x^2-4}+x}}$$

2.22 problem 22

Internal problem ID [500]
Internal file name [`OUTPUT/500_Sunday_June_05_2022_01_42_32_AM_94693732/index.tex`]

Book: Elementary differential equations and boundary value problems, 10th ed., Boyce and DiPrima
Section: Section 2.2. Page 48
Problem number: 22.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"exact", "separable", "differential-Type", "first_order_ode_lie_symmetry_lookup"**

Maple gives the following as the ode type

`[_separable]`

$$\boxed{y' - \frac{3x^2}{-4 + 3y^2} = 0}$$

With initial conditions

$$[y(1) = 0]$$

2.22.1 Solving as separable ode

In canonical form the ODE is

$$\begin{aligned} y' &= F(x, y) \\ &= f(x)g(y) \\ &= \frac{3x^2}{3y^2 - 4} \end{aligned}$$

Where $f(x) = 3x^2$ and $g(y) = \frac{1}{3y^2-4}$. Integrating both sides gives

$$\frac{1}{\frac{1}{3y^2-4}}\,dy = 3x^2\,dx$$

$$\int \frac{1}{\frac{1}{3y^2-4}}\,dy = \int 3x^2\,dx$$

$$y^3 - 4y = x^3 + c_1$$

Which results in

$$y = \frac{\left(108x^3 + 108c_1 + 12\sqrt{81x^6 + 162c_1x^3 + 81c_1^2 - 768}\right)^{\frac{1}{3}}}{6} + \frac{8}{\left(108x^3 + 108c_1 + 12\sqrt{81x^6 + 162c_1x^3 + 81c_1^2 - 768}\right)^{\frac{1}{3}}}$$

$$y = -\frac{\left(108x^3 + 108c_1 + 12\sqrt{81x^6 + 162c_1x^3 + 81c_1^2 - 768}\right)^{\frac{1}{3}}}{12} - \frac{4}{\left(108x^3 + 108c_1 + 12\sqrt{81x^6 + 162c_1x^3 + 81c_1^2 - 768}\right)^{\frac{1}{3}}} + \frac{i\sqrt{3}\left(\frac{\left(108x^3+108c_1+12\sqrt{81x^6+162c_1x^3+81c_1^2-768}\right)^{\frac{1}{3}}}{6} - \frac{8}{\left(108x^3+108c_1+12\sqrt{81x^6+162c_1x^3+81c_1^2-768}\right)^{\frac{1}{3}}}\right)}{2}$$

$$y = -\frac{\left(108x^3 + 108c_1 + 12\sqrt{81x^6 + 162c_1x^3 + 81c_1^2 - 768}\right)^{\frac{1}{3}}}{12} - \frac{4}{\left(108x^3 + 108c_1 + 12\sqrt{81x^6 + 162c_1x^3 + 81c_1^2 - 768}\right)^{\frac{1}{3}}} - \frac{i\sqrt{3}\left(\frac{\left(108x^3+108c_1+12\sqrt{81x^6+162c_1x^3+81c_1^2-768}\right)^{\frac{1}{3}}}{6} - \frac{8}{\left(108x^3+108c_1+12\sqrt{81x^6+162c_1x^3+81c_1^2-768}\right)^{\frac{1}{3}}}\right)}{2}$$

Initial conditions are used to solve for c_1. Substituting $x = 1$ and $y = 0$ in the above solution gives an equation to solve for the constant of integration.

$$0 = \frac{-i\sqrt{3}\left(108+108c_1+12\sqrt{81c_1^2+162c_1-687}\right)^{\frac{2}{3}}+48i\sqrt{3}-\left(108+108c_1+12\sqrt{81c_1^2+162c_1-687\right.}}{12\left(108+108c_1+12\sqrt{81c_1^2+162c_1-687}\right)^{\frac{1}{3}}}$$

$$c_1 = -1$$

Substituting c_1 found above in the general solution gives

$$y = \frac{-i\sqrt{3}\left(108x^3-108+12\sqrt{81x^6-162x^3-687}\right)^{\frac{2}{3}}-\left(108x^3-108+12\sqrt{81x^6-162x^3-687}\right)^{\frac{2}{3}}+48i}{12\left(108x^3-108+12\sqrt{81x^6-162x^3-687}\right)^{\frac{1}{3}}}$$

Initial conditions are used to solve for c_1. Substituting $x = 1$ and $y = 0$ in the above solution gives an equation to solve for the constant of integration.

$$0 = \frac{i\sqrt{3}\left(108+108c_1+12\sqrt{81c_1^2+162c_1-687}\right)^{\frac{2}{3}}-48i\sqrt{3}-\left(108+108c_1+12\sqrt{81c_1^2+162c_1-687}\right)}{12\left(108+108c_1+12\sqrt{81c_1^2+162c_1-687}\right)^{\frac{1}{3}}}$$

Unable to solve for constant of integration. Warning: Unable to solve for constant of integration. Initial conditions are used to solve for c_1. Substituting $x = 1$ and $y = 0$ in the above solution gives an equation to solve for the constant of integration.

$$0 = \frac{48+\left(108+108c_1+12\sqrt{81c_1^2+162c_1-687}\right)^{\frac{2}{3}}}{6\left(108+108c_1+12\sqrt{81c_1^2+162c_1-687}\right)^{\frac{1}{3}}}$$

Summary

The solution(s) found are the following

Warning: Unable to solve for constant of integration.

$$y = \frac{-i\sqrt{3}\left(108x^3-108+12\sqrt{81x^6-162x^3-687}\right)^{\frac{2}{3}} -}{12\left(108x^3-108+\right.}$$

Verification of solutions

$$y = \frac{-i\sqrt{3}\left(108x^3-108+12\sqrt{81x^6-162x^3-687}\right)^{\frac{2}{3}}-\left(108x^3-108+12\sqrt{81x^6-162x^3-687}\right)^{\frac{2}{3}}+48i\sqrt{}}{12\left(108x^3-108+12\sqrt{81x^6-162x^3-687}\right)^{\frac{1}{3}}}$$

Verified OK.

2.22.2 Solving as differentialType ode

Writing the ode as

$$y' = \frac{3x^2}{-4 + 3y^2} \tag{1}$$

Which becomes

$$(3y^2 - 4)\, dy = (3x^2)\, dx \tag{2}$$

But the RHS is complete differential because

$$(3x^2)\, dx = d(x^3)$$

Hence (2) becomes

$$(3y^2 - 4)\, dy = d(x^3)$$

Integrating both sides gives gives these solutions

$$y = \frac{\left(108x^3 + 108c_1 + 12\sqrt{81x^6 + 162c_1x^3 + 81c_1^2 - 768}\right)^{\frac{1}{3}}}{6} + \frac{8}{\left(108x^3 + 108c_1 + 12\sqrt{81x^6 + 162c_1x^3 + 8\cdots}\right.}$$

$$y = -\frac{\left(108x^3 + 108c_1 + 12\sqrt{81x^6 + 162c_1x^3 + 81c_1^2 - 768}\right)^{\frac{1}{3}}}{12} - \frac{4}{\left(108x^3 + 108c_1 + 12\sqrt{81x^6 + 162c_1x^3 + \cdots}\right.}$$

$$y = -\frac{\left(108x^3 + 108c_1 + 12\sqrt{81x^6 + 162c_1x^3 + 81c_1^2 - 768}\right)^{\frac{1}{3}}}{12} - \frac{4}{\left(108x^3 + 108c_1 + 12\sqrt{81x^6 + 162c_1x^3 + \cdots}\right.}$$

Initial conditions are used to solve for c_1. Substituting $x = 1$ and $y = 0$ in the above solution gives an equation to solve for the constant of integration.

$$0 = \frac{-i\sqrt{3}\left(108 + 108c_1 + 12\sqrt{81c_1^2 + 162c_1 - 687}\right)^{\frac{2}{3}} + 48i\sqrt{3} - \left(108 + 108c_1 + 12\sqrt{81c_1^2 + 162c_1 - 687\cdots}\right.}{12\left(108 + 108c_1 + 12\sqrt{81c_1^2 + 162c_1 - 687}\right)\cdots}$$

Warning: Unable to solve for constant of integration. Initial conditions are used to solve for c_1. Substituting $x = 1$ and $y = 0$ in the above solution gives an equation to solve for the constant of integration.

$$
0 = \frac{i\sqrt{3}\left(108 + 108c_1 + 12\sqrt{81c_1^2 + 162c_1 - 687}\right)^{\frac{2}{3}} - 48i\sqrt{3} - \left(108 + 108c_1 + 12\sqrt{81c_1^2 + 162c_1 - 687}\right)}{12\left(108 + 108c_1 + 12\sqrt{81c_1^2 + 162c_1 - 687}\right)^{\frac{1}{3}}}
$$

Warning: Unable to solve for constant of integration. Initial conditions are used to solve for c_1. Substituting $x = 1$ and $y = 0$ in the above solution gives an equation to solve for the constant of integration.

$$
0 = \frac{\left(108 + 108c_1 + 12\sqrt{81c_1^2 + 162c_1 - 687}\right)^{\frac{2}{3}} + 6c_1\left(108 + 108c_1 + 12\sqrt{81c_1^2 + 162c_1 - 687}\right)^{\frac{1}{3}} + 48}{6\left(108 + 108c_1 + 12\sqrt{81c_1^2 + 162c_1 - 687}\right)^{\frac{1}{3}}}
$$

Warning: Unable to solve for constant of integration.

Verification of solutions N/A

2.22.3 Solving as first order ode lie symmetry lookup ode

Writing the ode as

$$
\begin{aligned}
y' &= \frac{3x^2}{3y^2 - 4} \\
y' &= \omega(x, y)
\end{aligned}
$$

The condition of Lie symmetry is the linearized PDE given by

$$
\eta_x + \omega(\eta_y - \xi_x) - \omega^2\xi_y - \omega_x\xi - \omega_y\eta = 0 \tag{A}
$$

The type of this ode is known. It is of type `separable`. Therefore we do not need to solve the PDE (A), and can just use the lookup table shown below to find ξ, η

Table 154: Lie symmetry infinitesimal lookup table for known first order ODE's

ODE class	Form	ξ	η
linear ode	$y' = f(x)y(x) + g(x)$	0	$e^{\int f dx}$
separable ode	$y' = f(x)\, g(y)$	$\frac{1}{f}$	0
quadrature ode	$y' = f(x)$	0	1
quadrature ode	$y' = g(y)$	1	0
homogeneous ODEs of Class A	$y' = f\left(\frac{y}{x}\right)$	x	y
homogeneous ODEs of Class C	$y' = (a + bx + cy)^{\frac{n}{m}}$	1	$-\frac{b}{c}$
homogeneous class D	$y' = \frac{y}{x} + g(x)\, F\left(\frac{y}{x}\right)$	x^2	xy
First order special form ID 1	$y' = g(x)\, e^{h(x)+by} + f(x)$	$\frac{e^{-\int bf(x)dx-h(x)}}{g(x)}$	$\frac{f(x)e^{-\int bf(x)dx-h(x)}}{g(x)}$
polynomial type ode	$y' = \frac{a_1x+b_1y+c_1}{a_2x+b_2y+c_2}$	$\frac{a_1b_2x-a_2b_1x-b_1c_2+b_2c_1}{a_1b_2-a_2b_1}$	$\frac{a_1b_2y-a_2b_1y-a_1c_2-a_2c_1}{a_1b_2-a_2b_1}$
Bernoulli ode	$y' = f(x)\, y + g(x)\, y^n$	0	$e^{-\int (n-1)f(x)dx} y^n$
Reduced Riccati	$y' = f_1(x)\, y + f_2(x)\, y^2$	0	$e^{-\int f_1 dx}$

The above table shows that

$$
\begin{aligned}
\xi(x,y) &= \frac{1}{3x^2} \\
\eta(x,y) &= 0 \qquad \text{(A1)}
\end{aligned}
$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x,y) \to (R,S)$ where (R,S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$
\frac{dx}{\xi} = \frac{dy}{\eta} = dS \qquad (1)
$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y}\right) S(x,y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the

canonical coordinates, where $S(R)$. Since $\eta = 0$ then in this special case

$$R = y$$

S is found from

$$\begin{aligned} S &= \int \frac{1}{\xi} dx \\ &= \int \frac{1}{\frac{1}{3x^2}} dx \end{aligned}$$

Which results in

$$S = x^3$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x,y) S_y}{R_x + \omega(x,y) R_y} \tag{2}$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x,y)$ is the right hand side of the original ode given by

$$\omega(x,y) = \frac{3x^2}{3y^2 - 4}$$

Evaluating all the partial derivatives gives

$$\begin{aligned} R_x &= 0 \\ R_y &= 1 \\ S_x &= 3x^2 \\ S_y &= 0 \end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = 3y^2 - 4 \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = 3R^2 - 4$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by

integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$S(R) = R^3 - 4R + c_1 \tag{4}$$

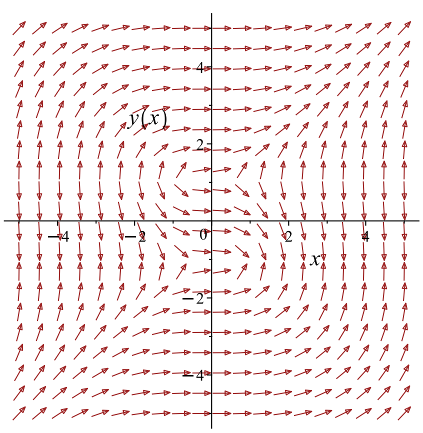
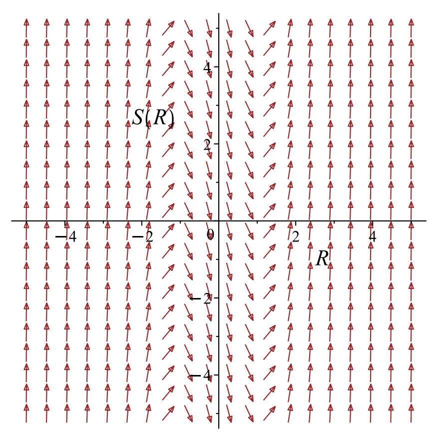
To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$x^3 = y^3 + c_1 - 4y$$

Which simplifies to

$$x^3 = y^3 + c_1 - 4y$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in x, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dx} = \frac{3x^2}{3y^2-4}$ 	$R = y$ $S = x^3$	$\frac{dS}{dR} = 3R^2 - 4$ 

Initial conditions are used to solve for c_1. Substituting $x = 1$ and $y = 0$ in the above solution gives an equation to solve for the constant of integration.

$$1 = c_1$$

$$c_1 = 1$$

Substituting c_1 found above in the general solution gives

$$x^3 = y^3 - 4y + 1$$

Summary

The solution(s) found are the following

$$x^3 = y^3 - 4y + 1 \tag{1}$$

Verification of solutions

$$x^3 = y^3 - 4y + 1$$

Verified OK.

2.22.4 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x,y) + N(x,y)\frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x,y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y) = 0$$

Hence

$$\frac{\partial \phi}{\partial x} + \frac{\partial \phi}{\partial y}\frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$\frac{\partial \phi}{\partial x} = M$$
$$\frac{\partial \phi}{\partial y} = N$$

But since $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x,y)$ but at least we know now that we can do that since the condition $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ is satisfied. If this condition is not satisfied then this method will not work

and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x,y)\,\mathrm{d}x + N(x,y)\,\mathrm{d}y = 0 \tag{1A}$$

Therefore

$$\begin{aligned}\left(y^2 - \frac{4}{3}\right)\mathrm{d}y &= \left(x^2\right)\mathrm{d}x\\ \left(-x^2\right)\mathrm{d}x + \left(y^2 - \frac{4}{3}\right)\mathrm{d}y &= 0 \end{aligned} \tag{2A}$$

Comparing (1A) and (2A) shows that

$$\begin{aligned} M(x,y) &= -x^2\\ N(x,y) &= y^2 - \frac{4}{3} \end{aligned}$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Using result found above gives

$$\begin{aligned} \frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}\left(-x^2\right)\\ &= 0 \end{aligned}$$

And

$$\begin{aligned} \frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}\left(y^2 - \frac{4}{3}\right)\\ &= 0 \end{aligned}$$

Since $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$, then the ODE is <u>exact</u> The following equations are now set up to solve for the function $\phi(x,y)$

$$\frac{\partial \phi}{\partial x} = M \tag{1}$$

$$\frac{\partial \phi}{\partial y} = N \tag{2}$$

Integrating (1) w.r.t. x gives

$$
\begin{aligned}
\int \frac{\partial \phi}{\partial x} \mathrm{d}x &= \int M \mathrm{d}x \\
\int \frac{\partial \phi}{\partial x} \mathrm{d}x &= \int -x^2 \mathrm{d}x \\
\phi &= -\frac{x^3}{3} + f(y) \qquad (3)
\end{aligned}
$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$
\frac{\partial \phi}{\partial y} = 0 + f'(y) \qquad (4)
$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = y^2 - \frac{4}{3}$. Therefore equation (4) becomes

$$
y^2 - \frac{4}{3} = 0 + f'(y) \qquad (5)
$$

Solving equation (5) for $f'(y)$ gives

$$
f'(y) = y^2 - \frac{4}{3}
$$

Integrating the above w.r.t y gives

$$
\begin{aligned}
\int f'(y) \mathrm{d}y &= \int \left(y^2 - \frac{4}{3}\right) \mathrm{d}y \\
f(y) &= \frac{1}{3}y^3 - \frac{4}{3}y + c_1
\end{aligned}
$$

Where c_1 is constant of integration. Substituting result found above for $f(y)$ into equation (3) gives ϕ

$$
\phi = -\frac{1}{3}x^3 + \frac{1}{3}y^3 - \frac{4}{3}y + c_1
$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$
c_1 = -\frac{1}{3}x^3 + \frac{1}{3}y^3 - \frac{4}{3}y
$$

Initial conditions are used to solve for c_1. Substituting $x = 1$ and $y = 0$ in the above solution gives an equation to solve for the constant of integration.

$$-\frac{1}{3} = c_1$$

$$c_1 = -\frac{1}{3}$$

Substituting c_1 found above in the general solution gives

$$-\frac{1}{3}x^3 + \frac{1}{3}y^3 - \frac{4}{3}y = -\frac{1}{3}$$

Summary

The solution(s) found are the following

$$-\frac{x^3}{3} + \frac{y^3}{3} - \frac{4y}{3} = -\frac{1}{3} \tag{1}$$

Verification of solutions

$$-\frac{x^3}{3} + \frac{y^3}{3} - \frac{4y}{3} = -\frac{1}{3}$$

Verified OK.

2.22.5 Maple step by step solution

Let's solve

$\left[y' - \frac{3x^2}{-4+3y^2} = 0, y(1) = 0\right]$

- Highest derivative means the order of the ODE is 1

  y'

- Separate variables

  $y'(-4 + 3y^2) = 3x^2$

- Integrate both sides with respect to x

  $\int y'(-4 + 3y^2)\, dx = \int 3x^2 dx + c_1$

- Evaluate integral

  $y^3 - 4y = x^3 + c_1$

- Solve for y

$$y=\frac{\left(108x^3+108c_1+12\sqrt{81x^6+162c_1x^3+81c_1^2-768}\right)^{\frac{1}{3}}}{6}+\frac{8}{\left(108x^3+108c_1+12\sqrt{81x^6+162c_1x^3+81c_1^2-768}\right)^{\frac{1}{3}}}$$

- Use initial condition $y(1)=0$

$$0=\frac{\left(108+108c_1+12\sqrt{81c_1^2+162c_1-687}\right)^{\frac{1}{3}}}{6}+\frac{8}{\left(108+108c_1+12\sqrt{81c_1^2+162c_1-687}\right)^{\frac{1}{3}}}$$

- Solution does not satisfy initial condition

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
<- separable successful`
```

✓ Solution by Maple

Time used: 0.094 (sec). Leaf size: 73

```
dsolve([diff(y(x),x) = 3*x^2/(-4+3*y(x)^2),y(1) = 0],y(x), singsol=all)
```

$$y(x)=-\frac{\left(1+i\sqrt{3}\right)\left(-108+108x^3+12\sqrt{81x^6-162x^3-687}\right)^{\frac{2}{3}}-48i\sqrt{3}+48}{12\left(-108+108x^3+12\sqrt{81x^6-162x^3-687}\right)^{\frac{1}{3}}}$$

✓ Solution by Mathematica

Time used: 9.526 (sec). Leaf size: 137

```
DSolve[{y'[x]== 3*x^2/(-4+3*y[x]^2),y[1]==0},y[x],x,IncludeSingularSolutions -> True]
```

$$y(x)\to\frac{-i\sqrt[3]{2}3^{2/3}\left(9x^3+\sqrt{81x^6-162x^3-687}-9\right)^{2/3}-\sqrt[3]{2}\sqrt[6]{3}\left(9x^3+\sqrt{81x^6-162x^3-687}-9\right)^{2/3}-8\sqrt{3}+}{2\,2^{2/3}3^{5/6}\sqrt[3]{9x^3+\sqrt{81x^6-162x^3-687}-9}}$$

2.23 problem 23

Internal problem ID [501]
Internal file name [`OUTPUT/501_Sunday_June_05_2022_01_42_33_AM_10464111/index.tex`]

Book: Elementary differential equations and boundary value problems, 10th ed., Boyce and DiPrima
Section: Section 2.2. Page 48
Problem number: 23.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"exact", "riccati", "separable", "first_order_ode_lie_symmetry_lookup"**

Maple gives the following as the ode type

`[_separable]`

$$\boxed{y' - 2y^2 - xy^2 = 0}$$

With initial conditions

$$[y(0) = 1]$$

2.23.1 Existence and uniqueness analysis

This is non linear first order ODE. In canonical form it is written as

$$\begin{aligned} y' &= f(x, y) \\ &= x\,y^2 + 2y^2 \end{aligned}$$

The x domain of $f(x, y)$ when $y = 1$ is

$$\{-\infty < x < \infty\}$$

And the point $x_0 = 0$ is inside this domain. The y domain of $f(x, y)$ when $x = 0$ is

$$\{-\infty < y < \infty\}$$

And the point $y_0 = 1$ is inside this domain. Now we will look at the continuity of

$$\begin{aligned}\frac{\partial f}{\partial y} &= \frac{\partial}{\partial y}\left(x\, y^2 + 2y^2\right) \\ &= 2yx + 4y\end{aligned}$$

The x domain of $\frac{\partial f}{\partial y}$ when $y = 1$ is

$$\{-\infty < x < \infty\}$$

And the point $x_0 = 0$ is inside this domain. The y domain of $\frac{\partial f}{\partial y}$ when $x = 0$ is

$$\{-\infty < y < \infty\}$$

And the point $y_0 = 1$ is inside this domain. Therefore solution exists and is unique.

2.23.2 Solving as separable ode

In canonical form the ODE is

$$\begin{aligned}y' &= F(x, y) \\ &= f(x)g(y) \\ &= y^2(2 + x)\end{aligned}$$

Where $f(x) = 2 + x$ and $g(y) = y^2$. Integrating both sides gives

$$\begin{aligned}\frac{1}{y^2}\, dy &= 2 + x\, dx \\ \int \frac{1}{y^2}\, dy &= \int 2 + x\, dx \\ -\frac{1}{y} &= 2x + \frac{1}{2}x^2 + c_1\end{aligned}$$

Which results in

$$y = -\frac{2}{x^2 + 2c_1 + 4x}$$

Initial conditions are used to solve for c_1. Substituting $x = 0$ and $y = 1$ in the above solution gives an equation to solve for the constant of integration.

$$1 = -\frac{1}{c_1}$$

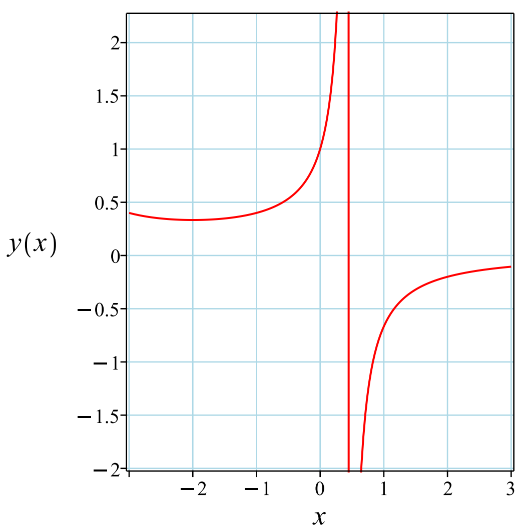
$$c_1 = -1$$

Substituting c_1 found above in the general solution gives

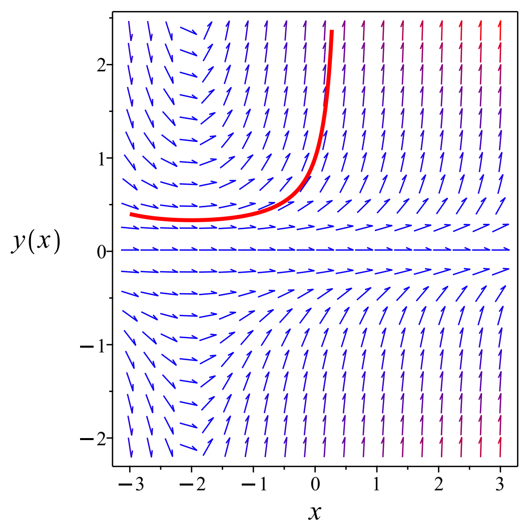
$$y = -\frac{2}{x^2 + 4x - 2}$$

Summary
The solution(s) found are the following

$$y = -\frac{2}{x^2 + 4x - 2} \tag{1}$$



(a) Solution plot



(b) Slope field plot

Verification of solutions

$$y = -\frac{2}{x^2 + 4x - 2}$$

Verified OK.

2.23.3 Solving as first order ode lie symmetry lookup ode

Writing the ode as

$$y' = x\,y^2 + 2y^2$$
$$y' = \omega(x, y)$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x + \omega(\eta_y - \xi_x) - \omega^2\xi_y - \omega_x\xi - \omega_y\eta = 0 \tag{A}$$

The type of this ode is known. It is of type `separable`. Therefore we do not need to solve the PDE (A), and can just use the lookup table shown below to find ξ, η

Table 157: Lie symmetry infinitesimal lookup table for known first order ODE's

ODE class	Form	ξ	η
linear ode	$y' = f(x)y(x) + g(x)$	0	$e^{\int f dx}$
separable ode	$y' = f(x)\,g(y)$	$\frac{1}{f}$	0
quadrature ode	$y' = f(x)$	0	1
quadrature ode	$y' = g(y)$	1	0
homogeneous ODEs of Class A	$y' = f\left(\frac{y}{x}\right)$	x	y
homogeneous ODEs of Class C	$y' = (a + bx + cy)^{\frac{n}{m}}$	1	$-\frac{b}{c}$
homogeneous class D	$y' = \frac{y}{x} + g(x)\,F\left(\frac{y}{x}\right)$	x^2	xy
First order special form ID 1	$y' = g(x)\,e^{h(x)+by} + f(x)$	$\frac{e^{-\int bf(x)dx-h(x)}}{g(x)}$	$\frac{f(x)e^{-\int bf(x)dx-h(x)}}{g(x)}$
polynomial type ode	$y' = \frac{a_1x+b_1y+c_1}{a_2x+b_2y+c_2}$	$\frac{a_1b_2x-a_2b_1x-b_1c_2+b_2c_1}{a_1b_2-a_2b_1}$	$\frac{a_1b_2y-a_2b_1y-a_1c_2-a_2c_1}{a_1b_2-a_2b_1}$
Bernoulli ode	$y' = f(x)\,y + g(x)\,y^n$	0	$e^{-\int(n-1)f(x)dx}y^n$
Reduced Riccati	$y' = f_1(x)\,y + f_2(x)\,y^2$	0	$e^{-\int f_1 dx}$

The above table shows that

$$\begin{aligned}\xi(x,y) &= \frac{1}{2+x} \\ \eta(x,y) &= 0\end{aligned} \tag{A1}$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x,y) \to (R,S)$ where (R,S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}$$

The above comes from the requirements that $\left(\xi\frac{\partial}{\partial x} + \eta\frac{\partial}{\partial y}\right) S(x,y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the canonical coordinates, where $S(R)$. Since $\eta = 0$ then in this special case

$$R = y$$

S is found from

$$\begin{aligned}S &= \int \frac{1}{\xi} dx \\ &= \int \frac{1}{\frac{1}{2+x}} dx\end{aligned}$$

Which results in

$$S = 2x + \frac{1}{2}x^2$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x,y)S_y}{R_x + \omega(x,y)R_y} \tag{2}$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x,y)$ is the right hand side of the original ode given by

$$\omega(x,y) = x\,y^2 + 2y^2$$

Evaluating all the partial derivatives gives

$$\begin{aligned}R_x &= 0 \\ R_y &= 1 \\ S_x &= 2+x \\ S_y &= 0\end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = \frac{1}{y^2} \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = \frac{1}{R^2}$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$S(R) = -\frac{1}{R} + c_1 \tag{4}$$

To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$2x + \frac{1}{2}x^2 = -\frac{1}{y} + c_1$$

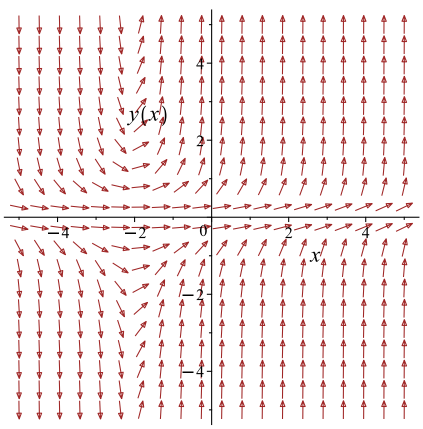
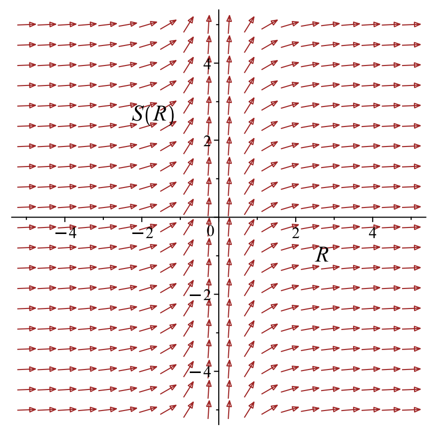
Which simplifies to

$$2x + \frac{1}{2}x^2 = -\frac{1}{y} + c_1$$

Which gives

$$y = \frac{2}{-x^2 + 2c_1 - 4x}$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in x, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dx} = x\,y^2 + 2y^2$ 	$R = y$ $S = 2x + \frac{1}{2}x^2$	$\frac{dS}{dR} = \frac{1}{R^2}$ 

Initial conditions are used to solve for c_1. Substituting $x = 0$ and $y = 1$ in the above solution gives an equation to solve for the constant of integration.

$$1 = \frac{1}{c_1}$$

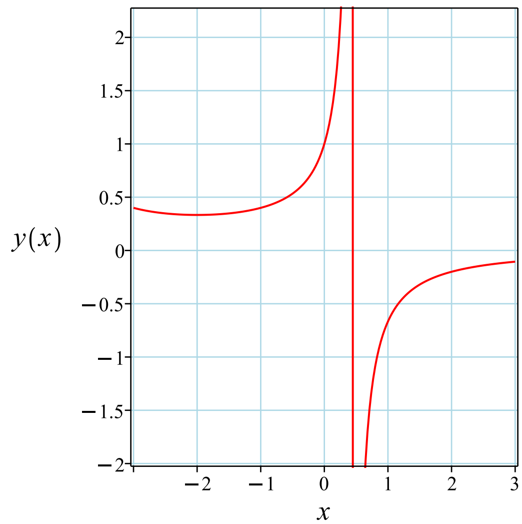
$$c_1 = 1$$

Substituting c_1 found above in the general solution gives

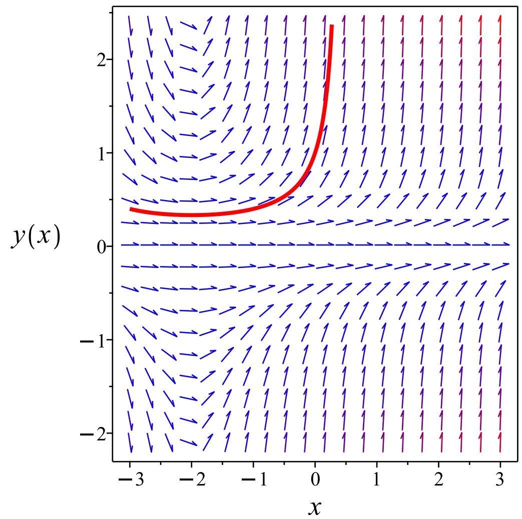
$$y = -\frac{2}{x^2 + 4x - 2}$$

<u>Summary</u>
The solution(s) found are the following

$$y = -\frac{2}{x^2 + 4x - 2} \tag{1}$$



(a) Solution plot



(b) Slope field plot

Verification of solutions

$$y = -\frac{2}{x^2 + 4x - 2}$$

Verified OK.

2.23.4 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x,y) + N(x,y)\frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x,y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y) = 0$$

Hence

$$\frac{\partial \phi}{\partial x} + \frac{\partial \phi}{\partial y}\frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$\frac{\partial \phi}{\partial x} = M$$
$$\frac{\partial \phi}{\partial y} = N$$

But since $\frac{\partial^2\phi}{\partial x\partial y} = \frac{\partial^2\phi}{\partial y\partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x, y)$ but at least we know now that we can do that since the condition $\frac{\partial^2\phi}{\partial x\partial y} = \frac{\partial^2\phi}{\partial y\partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x, y)\, \mathrm{d}x + N(x, y)\, \mathrm{d}y = 0 \tag{1A}$$

Therefore

$$\left(\frac{1}{y^2}\right) \mathrm{d}y = (2 + x)\, \mathrm{d}x$$
$$(-x - 2)\, \mathrm{d}x + \left(\frac{1}{y^2}\right) \mathrm{d}y = 0 \tag{2A}$$

Comparing (1A) and (2A) shows that

$$M(x, y) = -x - 2$$
$$N(x, y) = \frac{1}{y^2}$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Using result found above gives

$$\frac{\partial M}{\partial y} = \frac{\partial}{\partial y}(-x - 2)$$
$$= 0$$

And

$$\frac{\partial N}{\partial x} = \frac{\partial}{\partial x}\left(\frac{1}{y^2}\right)$$
$$= 0$$

Since $\frac{\partial M}{\partial y}=\frac{\partial N}{\partial x}$, then the ODE is exact The following equations are now set up to solve for the function $\phi(x,y)$

$$
\begin{aligned}
\frac{\partial \phi}{\partial x} &= M \qquad (1)\\
\frac{\partial \phi}{\partial y} &= N \qquad (2)
\end{aligned}
$$

Integrating (1) w.r.t. x gives

$$
\begin{aligned}
\int \frac{\partial \phi}{\partial x} \,\mathrm{d}x &= \int M \,\mathrm{d}x\\
\int \frac{\partial \phi}{\partial x} \,\mathrm{d}x &= \int -x-2 \,\mathrm{d}x\\
\phi &= -\frac{1}{2}x^2-2x+f(y) \qquad (3)
\end{aligned}
$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$
\frac{\partial \phi}{\partial y} = 0+f'(y) \qquad (4)
$$

But equation (2) says that $\frac{\partial \phi}{\partial y}=\frac{1}{y^2}$. Therefore equation (4) becomes

$$
\frac{1}{y^2} = 0+f'(y) \qquad (5)
$$

Solving equation (5) for $f'(y)$ gives

$$
f'(y) = \frac{1}{y^2}
$$

Integrating the above w.r.t y gives

$$
\begin{aligned}
\int f'(y) \,\mathrm{d}y &= \int \left(\frac{1}{y^2}\right) \mathrm{d}y\\
f(y) &= -\frac{1}{y}+c_1
\end{aligned}
$$

Where c_1 is constant of integration. Substituting result found above for $f(y)$ into equation (3) gives ϕ

$$\phi = -\frac{x^2}{2} - 2x - \frac{1}{y} + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = -\frac{x^2}{2} - 2x - \frac{1}{y}$$

The solution becomes

$$y = -\frac{2}{x^2 + 2c_1 + 4x}$$

Initial conditions are used to solve for c_1. Substituting $x = 0$ and $y = 1$ in the above solution gives an equation to solve for the constant of integration.

$$1 = -\frac{1}{c_1}$$

$$c_1 = -1$$

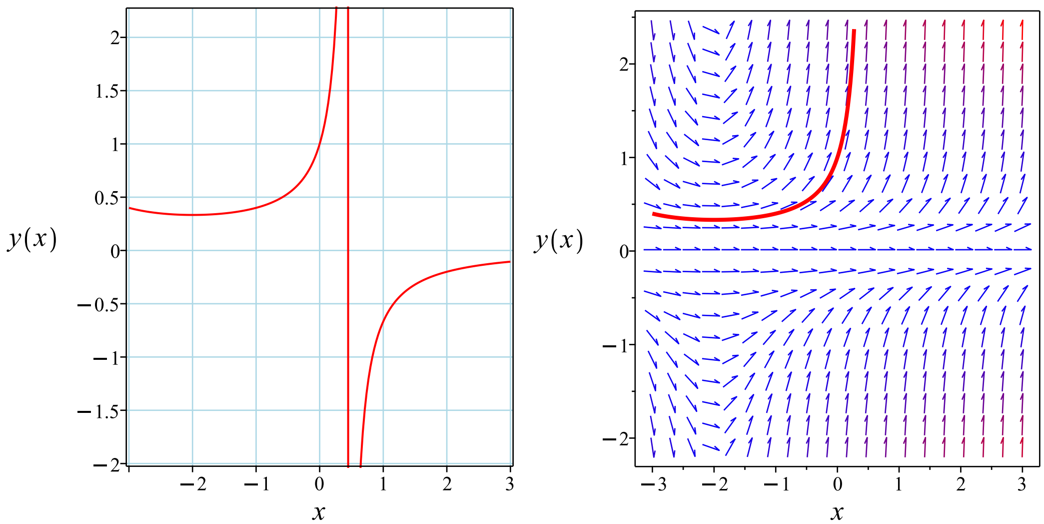
Substituting c_1 found above in the general solution gives

$$y = -\frac{2}{x^2 + 4x - 2}$$

Summary

The solution(s) found are the following

$$y = -\frac{2}{x^2 + 4x - 2} \tag{1}$$



(a) Solution plot

(b) Slope field plot

Verification of solutions

$$y = -\frac{2}{x^2 + 4x - 2}$$

Verified OK.

2.23.5 Solving as riccati ode

In canonical form the ODE is

$$\begin{aligned} y' &= F(x, y) \\ &= x\,y^2 + 2y^2 \end{aligned}$$

This is a Riccati ODE. Comparing the ODE to solve

$$y' = x\,y^2 + 2y^2$$

With Riccati ODE standard form

$$y' = f_0(x) + f_1(x)y + f_2(x)y^2$$

Shows that $f_0(x) = 0$, $f_1(x) = 0$ and $f_2(x) = 2 + x$. Let

$$\begin{aligned} y &= \frac{-u'}{f_2 u} \\ &= \frac{-u'}{(2 + x)\,u} \end{aligned} \tag{1}$$

Using the above substitution in the given ODE results (after some simplification)in a second order ODE to solve for $u(x)$ which is

$$f_2 u''(x) - (f_2' + f_1 f_2)\, u'(x) + f_2^2 f_0 u(x) = 0 \tag{2}$$

But

$$\begin{aligned} f_2' &= 1 \\ f_1 f_2 &= 0 \\ f_2^2 f_0 &= 0 \end{aligned}$$

Substituting the above terms back in equation (2) gives

$$(2 + x)\, u''(x) - u'(x) = 0$$

Solving the above ODE (this ode solved using Maple, not this program), gives

$$u(x) = c_1 + c_2(2 + x)^2$$

The above shows that

$$u'(x) = 2(2 + x)\, c_2$$

Using the above in (1) gives the solution

$$y = -\frac{2c_2}{c_1 + c_2\,(2 + x)^2}$$

Dividing both numerator and denominator by c_1 gives, after renaming the constant $\frac{c_2}{c_1} = c_3$ the following solution

$$y = -\frac{2}{x^2 + c_3 + 4x + 4}$$

Initial conditions are used to solve for c_3. Substituting $x = 0$ and $y = 1$ in the above solution gives an equation to solve for the constant of integration.

$$1 = -\frac{2}{c_3 + 4}$$

$$c_3 = -6$$

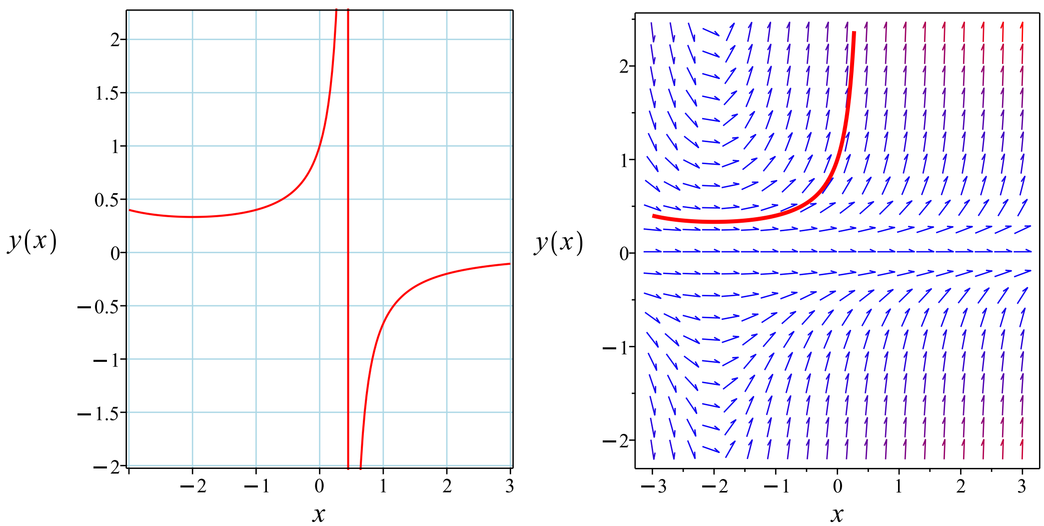
Substituting c_3 found above in the general solution gives

$$y = -\frac{2}{x^2 + 4x - 2}$$

Summary

The solution(s) found are the following

$$y = -\frac{2}{x^2 + 4x - 2} \tag{1}$$



(a) Solution plot

(b) Slope field plot

Verification of solutions

$$y = -\frac{2}{x^2 + 4x - 2}$$

Verified OK.

2.23.6 Maple step by step solution

Let's solve

$[y' - 2y^2 - xy^2 = 0, y(0) = 1]$

- Highest derivative means the order of the ODE is 1

  y'

- Separate variables

$\frac{y'}{y^2} = 2 + x$

- Integrate both sides with respect to x

  $\int \frac{y'}{y^2} dx = \int (2 + x)\, dx + c_1$

- Evaluate integral

  $-\frac{1}{y} = 2x + \frac{1}{2}x^2 + c_1$

- Solve for y

  $y = -\frac{2}{x^2 + 2c_1 + 4x}$

- Use initial condition $y(0) = 1$

  $1 = -\frac{1}{c_1}$

- Solve for c_1

  $c_1 = -1$

- Substitute $c_1 = -1$ into general solution and simplify

  $y = -\frac{2}{x^2 + 4x - 2}$

- Solution to the IVP

  $y = -\frac{2}{x^2 + 4x - 2}$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
<- Bernoulli successful`
```

✓ Solution by Maple

Time used: 0.063 (sec). Leaf size: 16

```
dsolve([diff(y(x),x) = 2*y(x)^2+x*y(x)^2,y(0) = 1],y(x), singsol=all)
```

$$y(x) = -\frac{2}{x^2 + 4x - 2}$$

✓ Solution by Mathematica

Time used: 0.127 (sec). Leaf size: 17

```
DSolve[{y'[x] == 2*y[x]^2+x*y[x]^2,y[0]==1},y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to -\frac{2}{x^2 + 4x - 2}$$

2.24 problem 24

Internal problem ID [502]
Internal file name [`OUTPUT/502_Sunday_June_05_2022_01_42_34_AM_42187153/index.tex`]

Book: Elementary differential equations and boundary value problems, 10th ed., Boyce and DiPrima
Section: Section 2.2. Page 48
Problem number: 24.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"exact", "separable", "first_order_ode_lie_symmetry_lookup"**

Maple gives the following as the ode type

`[_separable]`

$$\boxed{y' - \frac{2 - e^{x}}{3 + 2y} = 0}$$

With initial conditions

$$[y(0) = 0]$$

2.24.1 Existence and uniqueness analysis

This is non linear first order ODE. In canonical form it is written as

$$\begin{aligned} y' &= f(x, y) \\ &= -\frac{-2 + e^{x}}{3 + 2y} \end{aligned}$$

The x domain of $f(x, y)$ when $y = 0$ is

$$\{-\infty < x < \infty\}$$

And the point $x_0 = 0$ is inside this domain. The y domain of $f(x,y)$ when $x = 0$ is

$$\left\{y < -\frac{3}{2} \vee -\frac{3}{2} < y\right\}$$

And the point $y_0 = 0$ is inside this domain. Now we will look at the continuity of

$$\begin{aligned}\frac{\partial f}{\partial y} &= \frac{\partial}{\partial y}\left(-\frac{-2 + e^{x}}{3 + 2y}\right) \\ &= \frac{-4 + 2\,e^{x}}{(3 + 2y)^{2}}\end{aligned}$$

The x domain of $\frac{\partial f}{\partial y}$ when $y = 0$ is

$$\{-\infty < x < \infty\}$$

And the point $x_0 = 0$ is inside this domain. The y domain of $\frac{\partial f}{\partial y}$ when $x = 0$ is

$$\left\{y < -\frac{3}{2} \vee -\frac{3}{2} < y\right\}$$

And the point $y_0 = 0$ is inside this domain. Therefore solution exists and is unique.

2.24.2 Solving as separable ode

In canonical form the ODE is

$$\begin{aligned}y' &= F(x,y) \\ &= f(x)g(y) \\ &= \frac{2 - e^{x}}{3 + 2y}\end{aligned}$$

Where $f(x) = 2 - e^{x}$ and $g(y) = \frac{1}{3+2y}$. Integrating both sides gives

$$\begin{aligned}\frac{1}{\frac{1}{3+2y}}\,dy &= 2 - e^{x}\,dx \\ \int \frac{1}{\frac{1}{3+2y}}\,dy &= \int 2 - e^{x}\,dx \\ y^{2} + 3y &= 2x - e^{x} + c_1\end{aligned}$$

Which results in

$$y = -\frac{3}{2}+\frac{\sqrt{9-4\,\mathrm{e}^{x}+4c_1+8x}}{2}$$
$$y = -\frac{3}{2}-\frac{\sqrt{9-4\,\mathrm{e}^{x}+4c_1+8x}}{2}$$

Initial conditions are used to solve for c_1. Substituting $x = 0$ and $y = 0$ in the above solution gives an equation to solve for the constant of integration.

$$0 = -\frac{3}{2}-\frac{\sqrt{5+4c_1}}{2}$$

Warning: Unable to solve for constant of integration. Initial conditions are used to solve for c_1. Substituting $x = 0$ and $y = 0$ in the above solution gives an equation to solve for the constant of integration.

$$0 = -\frac{3}{2}+\frac{\sqrt{5+4c_1}}{2}$$

$$c_1 = 1$$

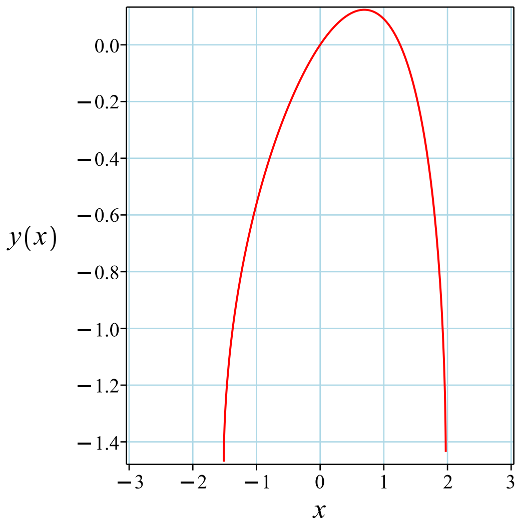
Substituting c_1 found above in the general solution gives

$$y = -\frac{3}{2}+\frac{\sqrt{13-4\,\mathrm{e}^{x}+8x}}{2}$$

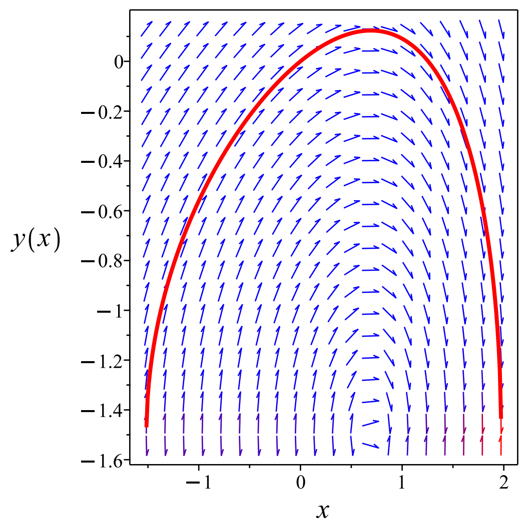
Summary

The solution(s) found are the following

$$y = -\frac{3}{2}+\frac{\sqrt{13-4\,\mathrm{e}^{x}+8x}}{2} \tag{1}$$



(a) Solution plot



(b) Slope field plot

Verification of solutions

$$y = -\frac{3}{2} + \frac{\sqrt{13 - 4\,e^{x} + 8x}}{2}$$

Verified OK.

2.24.3 Solving as first order ode lie symmetry lookup ode

Writing the ode as

$$y' = -\frac{-2 + e^{x}}{3 + 2y}$$
$$y' = \omega(x, y)$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x + \omega(\eta_y - \xi_x) - \omega^2 \xi_y - \omega_x \xi - \omega_y \eta = 0 \tag{A}$$

The type of this ode is known. It is of type `separable`. Therefore we do not need to solve the PDE (A), and can just use the lookup table shown below to find ξ, η

Table 160: Lie symmetry infinitesimal lookup table for known first order ODE's

ODE class	Form	ξ	η
linear ode	$y' = f(x)y(x) + g(x)$	0	$e^{\int f dx}$
separable ode	$y' = f(x)\, g(y)$	$\frac{1}{f}$	0
quadrature ode	$y' = f(x)$	0	1
quadrature ode	$y' = g(y)$	1	0
homogeneous ODEs of Class A	$y' = f\left(\frac{y}{x}\right)$	x	y
homogeneous ODEs of Class C	$y' = (a + bx + cy)^{\frac{n}{m}}$	1	$-\frac{b}{c}$
homogeneous class D	$y' = \frac{y}{x} + g(x)\, F\left(\frac{y}{x}\right)$	x^2	xy
First order special form ID 1	$y' = g(x)\, e^{h(x)+by} + f(x)$	$\frac{e^{-\int bf(x)dx - h(x)}}{g(x)}$	$\frac{f(x)e^{-\int bf(x)dx - h(x)}}{g(x)}$
polynomial type ode	$y' = \frac{a_1x+b_1y+c_1}{a_2x+b_2y+c_2}$	$\frac{a_1b_2x-a_2b_1x-b_1c_2+b_2c_1}{a_1b_2-a_2b_1}$	$\frac{a_1b_2y-a_2b_1y-a_1c_2-a_2c_1}{a_1b_2-a_2b_1}$
Bernoulli ode	$y' = f(x)\, y + g(x)\, y^n$	0	$e^{-\int (n-1)f(x)dx} y^n$
Reduced Riccati	$y' = f_1(x)\, y + f_2(x)\, y^2$	0	$e^{-\int f_1 dx}$

The above table shows that

$$
\begin{aligned}
\xi(x, y) &= \frac{1}{2 - \mathrm{e}^x} \\
\eta(x, y) &= 0
\end{aligned}
\tag{A1}
$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x, y) \to (R, S)$ where (R, S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$
\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}
$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y}\right) S(x, y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the

canonical coordinates, where $S(R)$. Since $\eta = 0$ then in this special case

$$R = y$$

S is found from

$$\begin{aligned} S &= \int \frac{1}{\xi} dx \\ &= \int \frac{1}{\frac{1}{2-e^x}} dx \end{aligned}$$

Which results in

$$S = 2x - e^x$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x,y)S_y}{R_x + \omega(x,y)R_y} \tag{2}$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x, y)$ is the right hand side of the original ode given by

$$\omega(x,y) = -\frac{-2 + e^x}{3 + 2y}$$

Evaluating all the partial derivatives gives

$$\begin{aligned} R_x &= 0 \\ R_y &= 1 \\ S_x &= 2 - e^x \\ S_y &= 0 \end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = 3 + 2y \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = 3 + 2R$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by

integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$S(R) = R^2 + 3R + c_1 \tag{4}$$

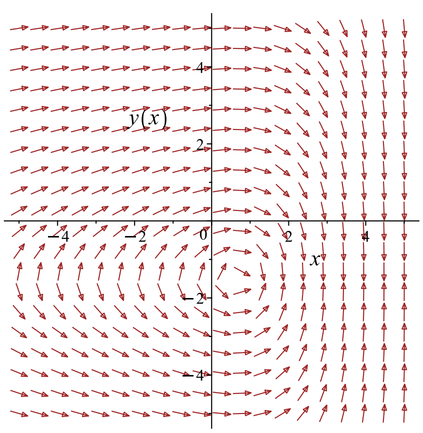
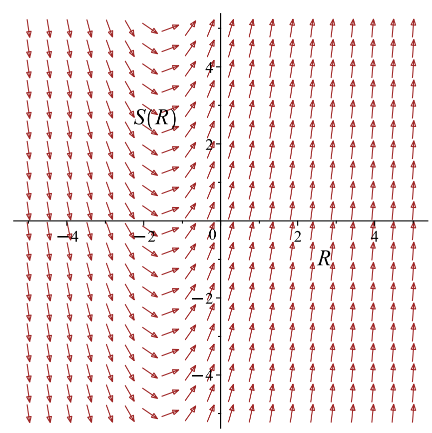
To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$2x - e^x = y^2 + c_1 + 3y$$

Which simplifies to

$$2x - e^x = y^2 + c_1 + 3y$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in x, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dx} = -\frac{-2+e^x}{3+2y}$ 	$R = y$ $S = 2x - e^x$	$\frac{dS}{dR} = 3 + 2R$ 

Initial conditions are used to solve for c_1. Substituting $x = 0$ and $y = 0$ in the above solution gives an equation to solve for the constant of integration.

$$-1 = c_1$$

$$c_1 = -1$$

Substituting c_1 found above in the general solution gives

$$2x - e^x = y^2 + 3y - 1$$

Summary

The solution(s) found are the following

$$2x - e^x = y^2 + 3y - 1 \tag{1}$$

Verification of solutions

$$2x - e^x = y^2 + 3y - 1$$

Verified OK.

2.24.4 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x,y) + N(x,y)\frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x,y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y) = 0$$

Hence

$$\frac{\partial\phi}{\partial x} + \frac{\partial\phi}{\partial y}\frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$\begin{aligned}\frac{\partial\phi}{\partial x} &= M\\ \frac{\partial\phi}{\partial y} &= N\end{aligned}$$

But since $\frac{\partial^2\phi}{\partial x\partial y} = \frac{\partial^2\phi}{\partial y\partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x,y)$ but at least we know now that we can do that since the condition $\frac{\partial^2\phi}{\partial x\partial y} = \frac{\partial^2\phi}{\partial y\partial x}$ is satisfied. If this condition is not satisfied then this method will not work

and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x,y)\,\mathrm{d}x + N(x,y)\,\mathrm{d}y = 0 \tag{1A}$$

Therefore

$$\begin{aligned}(-3-2y)\,\mathrm{d}y &= (-2+\mathrm{e}^{x})\,\mathrm{d}x\\ (2-\mathrm{e}^{x})\,\mathrm{d}x + (-3-2y)\,\mathrm{d}y &= 0 \end{aligned} \tag{2A}$$

Comparing (1A) and (2A) shows that

$$\begin{aligned}M(x,y) &= 2-\mathrm{e}^{x}\\ N(x,y) &= -3-2y\end{aligned}$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Using result found above gives

$$\begin{aligned}\frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}(2-\mathrm{e}^{x})\\ &= 0\end{aligned}$$

And

$$\begin{aligned}\frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}(-3-2y)\\ &= 0\end{aligned}$$

Since $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$, then the ODE is exact The following equations are now set up to solve for the function $\phi(x,y)$

$$\frac{\partial \phi}{\partial x} = M \tag{1}$$

$$\frac{\partial \phi}{\partial y} = N \tag{2}$$

Integrating (1) w.r.t. x gives

$$\begin{aligned}\int \frac{\partial \phi}{\partial x}\,\mathrm{d}x &= \int M\,\mathrm{d}x\\ \int \frac{\partial \phi}{\partial x}\,\mathrm{d}x &= \int 2-\mathrm{e}^{x}\,\mathrm{d}x\\ \phi &= 2x-\mathrm{e}^{x}+f(y)\end{aligned} \tag{3}$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$\frac{\partial \phi}{\partial y} = 0 + f'(y) \tag{4}$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = -3 - 2y$. Therefore equation (4) becomes

$$-3 - 2y = 0 + f'(y) \tag{5}$$

Solving equation (5) for $f'(y)$ gives

$$f'(y) = -3 - 2y$$

Integrating the above w.r.t y gives

$$\int f'(y)\,\mathrm{d}y = \int (-3 - 2y)\,\mathrm{d}y$$

$$f(y) = -y^2 - 3y + c_1$$

Where c_1 is constant of integration. Substituting result found above for $f(y)$ into equation (3) gives ϕ

$$\phi = -y^2 - \mathrm{e}^x + 2x - 3y + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = -y^2 - \mathrm{e}^x + 2x - 3y$$

Initial conditions are used to solve for c_1. Substituting $x = 0$ and $y = 0$ in the above solution gives an equation to solve for the constant of integration.

$$-1 = c_1$$

$$c_1 = -1$$

Substituting c_1 found above in the general solution gives

$$-y^2 - \mathrm{e}^x + 2x - 3y = -1$$

Summary

The solution(s) found are the following

$$-y^2 - e^x + 2x - 3y = -1 \tag{1}$$

Verification of solutions

$$-y^2 - e^x + 2x - 3y = -1$$

Verified OK.

2.24.5 Maple step by step solution

Let's solve

$\left[y' - \frac{2-e^x}{3+2y} = 0, y(0) = 0\right]$

- Highest derivative means the order of the ODE is 1

  y'

- Separate variables

  $(3+2y)\, y' = 2 - e^x$

- Integrate both sides with respect to x

  $\int (3+2y)\, y' dx = \int (2 - e^x)\, dx + c_1$

- Evaluate integral

  $y^2 + 3y = 2x - e^x + c_1$

- Solve for y

  $\left\{y = -\frac{3}{2} - \frac{\sqrt{9-4\,e^x+4c_1+8x}}{2}, y = -\frac{3}{2} + \frac{\sqrt{9-4\,e^x+4c_1+8x}}{2}\right\}$

- Use initial condition $y(0) = 0$

  $0 = -\frac{3}{2} - \frac{\sqrt{5+4c_1}}{2}$

- Solution does not satisfy initial condition
- Use initial condition $y(0) = 0$

  $0 = -\frac{3}{2} + \frac{\sqrt{5+4c_1}}{2}$

- Solve for c_1

  $c_1 = 1$

- Substitute $c_1 = 1$ into general solution and simplify

  $y = -\frac{3}{2} + \frac{\sqrt{13-4\,e^x+8x}}{2}$

- Solution to the IVP

  $y = -\frac{3}{2} + \frac{\sqrt{13-4\,\mathrm{e}^{x}+8x}}{2}$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
<- separable successful`
```

✓ Solution by Maple

Time used: 0.141 (sec). Leaf size: 19

```
dsolve([diff(y(x),x) = (2-exp(x))/(3+2*y(x)),y(0) = 0],y(x), singsol=all)
```

$$y(x) = -\frac{3}{2} + \frac{\sqrt{13-4\,\mathrm{e}^{x}+8x}}{2}$$

✓ Solution by Mathematica

Time used: 0.737 (sec). Leaf size: 25

```
DSolve[{y'[x] == (2-Exp[x])/(3+2*y[x]),y[0]==0},y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \rightarrow \frac{1}{2}\left(\sqrt{8x-4e^{x}+13}-3\right)$$

2.25 problem 25

Internal problem ID [503]
Internal file name [`OUTPUT/503_Sunday_June_05_2022_01_42_35_AM_21918699/index.tex`]

Book: Elementary differential equations and boundary value problems, 10th ed., Boyce and DiPrima
Section: Section 2.2. Page 48
Problem number: 25.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"exact", "separable", "first_order_ode_lie_symmetry_lookup"**

Maple gives the following as the ode type

`[_separable]`

$$\boxed{y' - \frac{2\cos(2x)}{3+2y} = 0}$$

With initial conditions

$$[y(0) = -1]$$

2.25.1 Existence and uniqueness analysis

This is non linear first order ODE. In canonical form it is written as

$$\begin{aligned} y' &= f(x,y) \\ &= \frac{2\cos(2x)}{3+2y} \end{aligned}$$

The x domain of $f(x,y)$ when $y=-1$ is

$$\{-\infty < x < \infty\}$$

And the point $x_0 = 0$ is inside this domain. The y domain of $f(x, y)$ when $x = 0$ is

$$\left\{y < -\frac{3}{2} \vee -\frac{3}{2} < y\right\}$$

And the point $y_0 = -1$ is inside this domain. Now we will look at the continuity of

$$\begin{aligned}\frac{\partial f}{\partial y} &= \frac{\partial}{\partial y}\left(\frac{2\cos(2x)}{3+2y}\right)\\ &= -\frac{4\cos(2x)}{(3+2y)^2}\end{aligned}$$

The x domain of $\frac{\partial f}{\partial y}$ when $y = -1$ is

$$\{-\infty < x < \infty\}$$

And the point $x_0 = 0$ is inside this domain. The y domain of $\frac{\partial f}{\partial y}$ when $x = 0$ is

$$\left\{y < -\frac{3}{2} \vee -\frac{3}{2} < y\right\}$$

And the point $y_0 = -1$ is inside this domain. Therefore solution exists and is unique.

2.25.2 Solving as separable ode

In canonical form the ODE is

$$\begin{aligned}y' &= F(x, y)\\ &= f(x)g(y)\\ &= \frac{2\cos(2x)}{3+2y}\end{aligned}$$

Where $f(x) = 2\cos(2x)$ and $g(y) = \frac{1}{3+2y}$. Integrating both sides gives

$$\begin{aligned}\frac{1}{\frac{1}{3+2y}}\,dy &= 2\cos(2x)\,dx\\ \int \frac{1}{\frac{1}{3+2y}}\,dy &= \int 2\cos(2x)\,dx\\ y^2 + 3y &= \sin(2x) + c_1\end{aligned}$$

Which results in

$$y = -\frac{3}{2}+\frac{\sqrt{9+4\sin\left(2x\right)+4c_1}}{2}$$
$$y = -\frac{3}{2}-\frac{\sqrt{9+4\sin\left(2x\right)+4c_1}}{2}$$

Initial conditions are used to solve for c_1. Substituting $x = 0$ and $y = -1$ in the above solution gives an equation to solve for the constant of integration.

$$-1 = -\frac{3}{2}-\frac{\sqrt{9+4c_1}}{2}$$

Warning: Unable to solve for constant of integration. Initial conditions are used to solve for c_1. Substituting $x = 0$ and $y = -1$ in the above solution gives an equation to solve for the constant of integration.

$$-1 = -\frac{3}{2}+\frac{\sqrt{9+4c_1}}{2}$$

$$c_1 = -2$$

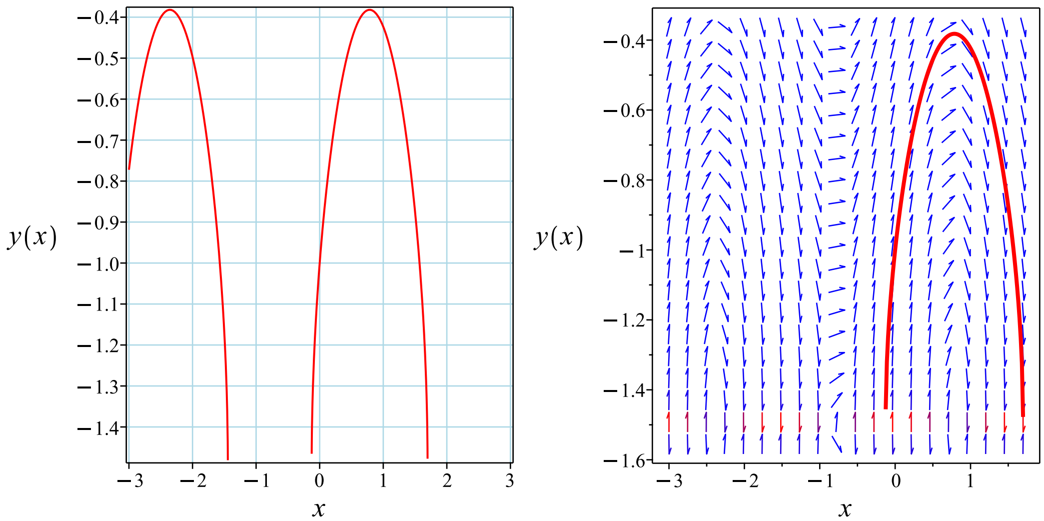
Substituting c_1 found above in the general solution gives

$$y = -\frac{3}{2}+\frac{\sqrt{1+4\sin\left(2x\right)}}{2}$$

Summary

The solution(s) found are the following

$$y = -\frac{3}{2}+\frac{\sqrt{1+4\sin\left(2x\right)}}{2} \tag{1}$$



(a) Solution plot (b) Slope field plot

Verification of solutions

$$y = -\frac{3}{2} + \frac{\sqrt{1 + 4\sin(2x)}}{2}$$

Verified OK.

2.25.3 Solving as first order ode lie symmetry lookup ode

Writing the ode as

$$y' = \frac{2\cos(2x)}{3 + 2y}$$
$$y' = \omega(x, y)$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x + \omega(\eta_y - \xi_x) - \omega^2 \xi_y - \omega_x \xi - \omega_y \eta = 0 \tag{A}$$

The type of this ode is known. It is of type `separable`. Therefore we do not need to solve the PDE (A), and can just use the lookup table shown below to find ξ, η

Table 163: Lie symmetry infinitesimal lookup table for known first order ODE's

ODE class	Form	ξ	η
linear ode	$y' = f(x)y(x) + g(x)$	0	$e^{\int f dx}$
separable ode	$y' = f(x)\, g(y)$	$\frac{1}{f}$	0
quadrature ode	$y' = f(x)$	0	1
quadrature ode	$y' = g(y)$	1	0
homogeneous ODEs of Class A	$y' = f\left(\frac{y}{x}\right)$	x	y
homogeneous ODEs of Class C	$y' = (a + bx + cy)^{\frac{n}{m}}$	1	$-\frac{b}{c}$
homogeneous class D	$y' = \frac{y}{x} + g(x)\, F\left(\frac{y}{x}\right)$	x^2	xy
First order special form ID 1	$y' = g(x)\, e^{h(x)+by} + f(x)$	$\frac{e^{-\int bf(x)dx - h(x)}}{g(x)}$	$\frac{f(x)e^{-\int bf(x)dx - h(x)}}{g(x)}$
polynomial type ode	$y' = \frac{a_1x+b_1y+c_1}{a_2x+b_2y+c_2}$	$\frac{a_1b_2x-a_2b_1x-b_1c_2+b_2c_1}{a_1b_2-a_2b_1}$	$\frac{a_1b_2y-a_2b_1y-a_1c_2-a_2c_1}{a_1b_2-a_2b_1}$
Bernoulli ode	$y' = f(x)\, y + g(x)\, y^n$	0	$e^{-\int (n-1)f(x)dx} y^n$
Reduced Riccati	$y' = f_1(x)\, y + f_2(x)\, y^2$	0	$e^{-\int f_1 dx}$

The above table shows that

$$
\begin{aligned}
\xi(x, y) &= \frac{1}{2\cos(2x)} \\
\eta(x, y) &= 0
\end{aligned}
\tag{A1}
$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x, y) \to (R, S)$ where (R, S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$
\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}
$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y}\right) S(x, y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the

canonical coordinates, where $S(R)$. Since $\eta = 0$ then in this special case

$$R = y$$

S is found from

$$\begin{aligned} S &= \int \frac{1}{\xi} dx \\ &= \int \frac{1}{\frac{1}{2\cos(2x)}} dx \end{aligned}$$

Which results in

$$S = \sin(2x)$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x,y)S_y}{R_x + \omega(x,y)R_y} \tag{2}$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x,y)$ is the right hand side of the original ode given by

$$\omega(x,y) = \frac{2\cos(2x)}{3+2y}$$

Evaluating all the partial derivatives gives

$$\begin{aligned} R_x &= 0 \\ R_y &= 1 \\ S_x &= 2\cos(2x) \\ S_y &= 0 \end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = 3 + 2y \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = 3 + 2R$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$S(R) = R^2 + 3R + c_1 \tag{4}$$

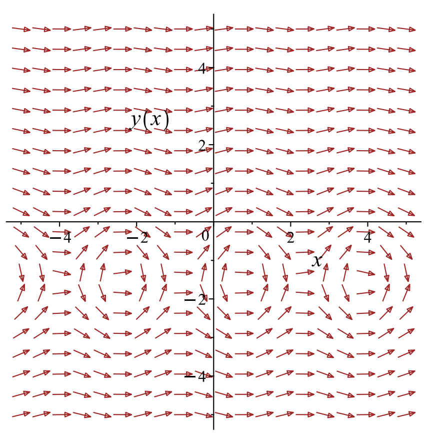
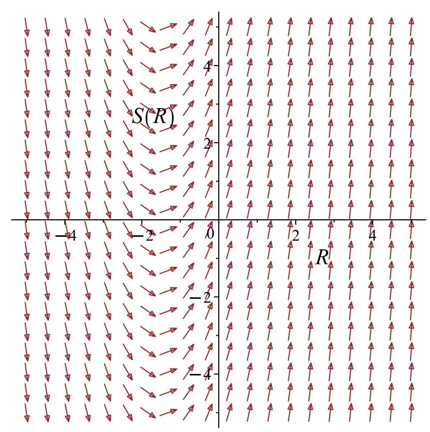
To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$\sin(2x) = y^2 + c_1 + 3y$$

Which simplifies to

$$\sin(2x) = y^2 + c_1 + 3y$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in x, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dx} = \frac{2\cos(2x)}{3+2y}$ 	$R = y$ $S = \sin(2x)$	$\frac{dS}{dR} = 3 + 2R$ 

Initial conditions are used to solve for c_1. Substituting $x = 0$ and $y = -1$ in the above solution gives an equation to solve for the constant of integration.

$$0 = c_1 - 2$$

$$c_1 = 2$$

Substituting c_1 found above in the general solution gives

$$\sin(2x) = y^2 + 3y + 2$$

Summary
The solution(s) found are the following

$$\sin(2x) = y^2 + 3y + 2 \tag{1}$$

Verification of solutions

$$\sin(2x) = y^2 + 3y + 2$$

Verified OK.

2.25.4 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x,y) + N(x,y)\frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x,y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y) = 0$$

Hence

$$\frac{\partial\phi}{\partial x} + \frac{\partial\phi}{\partial y}\frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$\frac{\partial\phi}{\partial x} = M$$
$$\frac{\partial\phi}{\partial y} = N$$

But since $\frac{\partial^2\phi}{\partial x\partial y} = \frac{\partial^2\phi}{\partial y\partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x,y)$ but at least we know now that we can do that since the condition $\frac{\partial^2\phi}{\partial x\partial y} = \frac{\partial^2\phi}{\partial y\partial x}$ is satisfied. If this condition is not satisfied then this method will not work

and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x,y)\,\mathrm{d}x + N(x,y)\,\mathrm{d}y = 0 \tag{1A}$$

Therefore

$$\begin{aligned}\left(\frac{3}{2}+y\right)\mathrm{d}y &= (\cos(2x))\,\mathrm{d}x\\(-\cos(2x))\,\mathrm{d}x + \left(\frac{3}{2}+y\right)\mathrm{d}y &= 0\end{aligned} \tag{2A}$$

Comparing (1A) and (2A) shows that

$$\begin{aligned}M(x,y) &= -\cos(2x)\\N(x,y) &= \frac{3}{2}+y\end{aligned}$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Using result found above gives

$$\begin{aligned}\frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}(-\cos(2x))\\&= 0\end{aligned}$$

And

$$\begin{aligned}\frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}\left(\frac{3}{2}+y\right)\\&= 0\end{aligned}$$

Since $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$, then the ODE is exact The following equations are now set up to solve for the function $\phi(x,y)$

$$\frac{\partial \phi}{\partial x} = M \tag{1}$$

$$\frac{\partial \phi}{\partial y} = N \tag{2}$$

Integrating (1) w.r.t. x gives

$$\int \frac{\partial \phi}{\partial x} \,\mathrm{d}x = \int M \,\mathrm{d}x$$

$$\int \frac{\partial \phi}{\partial x} \,\mathrm{d}x = \int -\cos(2x) \,\mathrm{d}x$$

$$\phi = -\frac{\sin(2x)}{2} + f(y) \tag{3}$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$\frac{\partial \phi}{\partial y} = 0 + f'(y) \tag{4}$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = \frac{3}{2} + y$. Therefore equation (4) becomes

$$\frac{3}{2} + y = 0 + f'(y) \tag{5}$$

Solving equation (5) for $f'(y)$ gives

$$f'(y) = \frac{3}{2} + y$$

Integrating the above w.r.t y gives

$$\int f'(y) \,\mathrm{d}y = \int \left(\frac{3}{2} + y\right) \mathrm{d}y$$

$$f(y) = \frac{3}{2}y + \frac{1}{2}y^2 + c_1$$

Where c_1 is constant of integration. Substituting result found above for $f(y)$ into equation (3) gives ϕ

$$\phi = -\frac{\sin(2x)}{2} + \frac{3y}{2} + \frac{y^2}{2} + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = -\frac{\sin(2x)}{2} + \frac{3y}{2} + \frac{y^2}{2}$$

Initial conditions are used to solve for c_1. Substituting $x = 0$ and $y = -1$ in the above solution gives an equation to solve for the constant of integration.

$$-1 = c_1$$

$$c_1 = -1$$

Substituting c_1 found above in the general solution gives

$$-\frac{\sin(2x)}{2} + \frac{3y}{2} + \frac{y^2}{2} = -1$$

Summary

The solution(s) found are the following

$$-\frac{\sin(2x)}{2} + \frac{3y}{2} + \frac{y^2}{2} = -1 \tag{1}$$

Verification of solutions

$$-\frac{\sin(2x)}{2} + \frac{3y}{2} + \frac{y^2}{2} = -1$$

Verified OK.

2.25.5 Maple step by step solution

Let's solve

$\left[y' - \frac{2\cos(2x)}{3+2y} = 0, y(0) = -1\right]$

- Highest derivative means the order of the ODE is 1

  y'

- Separate variables

  $(3 + 2y)\, y' = 2\cos(2x)$

- Integrate both sides with respect to x

  $\int (3 + 2y)\, y' dx = \int 2\cos(2x)\, dx + c_1$

- Evaluate integral

  $y^2 + 3y = \sin(2x) + c_1$

- Solve for y

$$\left\{y = -\frac{3}{2} - \frac{\sqrt{9+4\sin(2x)+4c_1}}{2}, y = -\frac{3}{2} + \frac{\sqrt{9+4\sin(2x)+4c_1}}{2}\right\}$$

- Use initial condition $y(0) = -1$

  $-1 = -\frac{3}{2} - \frac{\sqrt{9+4c_1}}{2}$

- Solution does not satisfy initial condition
- Use initial condition $y(0) = -1$

  $-1 = -\frac{3}{2} + \frac{\sqrt{9+4c_1}}{2}$

- Solve for c_1

  $c_1 = -2$

- Substitute $c_1 = -2$ into general solution and simplify

  $y = -\frac{3}{2} + \frac{\sqrt{1+4\sin(2x)}}{2}$

- Solution to the IVP

  $y = -\frac{3}{2} + \frac{\sqrt{1+4\sin(2x)}}{2}$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
<- separable successful`
```

✓ Solution by Maple

Time used: 0.25 (sec). Leaf size: 18

```
dsolve([diff(y(x),x) = 2*cos(2*x)/(3+2*y(x)),y(0) = -1],y(x), singsol=all)
```

$$y(x) = -\frac{3}{2} + \frac{\sqrt{1 + 4\sin(2x)}}{2}$$

✓ Solution by Mathematica

Time used: 0.175 (sec). Leaf size: 23

```
DSolve[{y'[x] == 2*Cos[2*x]/(3+2*y[x]),y[0]==-1},y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \frac{1}{2}\left(\sqrt{4\sin(2x)+1}-3\right)$$

2.26 problem 26

Internal problem ID [504]
Internal file name [`OUTPUT/504_Sunday_June_05_2022_01_42_37_AM_55464902/index.tex`]

Book: Elementary differential equations and boundary value problems, 10th ed., Boyce and DiPrima
Section: Section 2.2. Page 48
Problem number: 26.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"exact", "riccati", "separable", "first_order_ode_lie_symmetry_lookup"**

Maple gives the following as the ode type

`[_separable]`

$$\boxed{y' - 2(x+1)\left(1+y^2\right) = 0}$$

With initial conditions

$$[y(0) = 0]$$

2.26.1 Existence and uniqueness analysis

This is non linear first order ODE. In canonical form it is written as

$$\begin{aligned} y' &= f(x,y) \\ &= 2(x+1)\left(y^2+1\right) \end{aligned}$$

The x domain of $f(x,y)$ when $y=0$ is

$$\{-\infty < x < \infty\}$$

And the point $x_0 = 0$ is inside this domain. The y domain of $f(x, y)$ when $x = 0$ is

$$\{-\infty < y < \infty\}$$

And the point $y_0 = 0$ is inside this domain. Now we will look at the continuity of

$$\begin{aligned}\frac{\partial f}{\partial y} &= \frac{\partial}{\partial y}\left(2(x+1)\left(y^2+1\right)\right)\\ &= 4(x+1)\,y\end{aligned}$$

The x domain of $\frac{\partial f}{\partial y}$ when $y = 0$ is

$$\{-\infty < x < \infty\}$$

And the point $x_0 = 0$ is inside this domain. The y domain of $\frac{\partial f}{\partial y}$ when $x = 0$ is

$$\{-\infty < y < \infty\}$$

And the point $y_0 = 0$ is inside this domain. Therefore solution exists and is unique.

2.26.2 Solving as separable ode

In canonical form the ODE is

$$\begin{aligned}y' &= F(x, y)\\ &= f(x)g(y)\\ &= (x+1)\left(2y^2+2\right)\end{aligned}$$

Where $f(x) = x + 1$ and $g(y) = 2y^2 + 2$. Integrating both sides gives

$$\begin{aligned}\frac{1}{2y^2+2}\,dy &= x+1\,dx\\ \int \frac{1}{2y^2+2}\,dy &= \int x+1\,dx\\ \frac{\arctan(y)}{2} &= \frac{1}{2}x^2 + x + c_1\end{aligned}$$

Which results in

$$y = \tan\left(x^2 + 2c_1 + 2x\right)$$

Initial conditions are used to solve for c_1. Substituting $x = 0$ and $y = 0$ in the above solution gives an equation to solve for the constant of integration.

$$0 = \tan(2c_1)$$

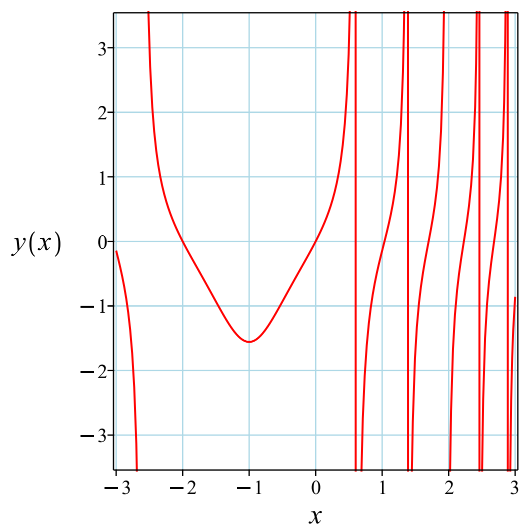
$$c_1 = 0$$

Substituting c_1 found above in the general solution gives

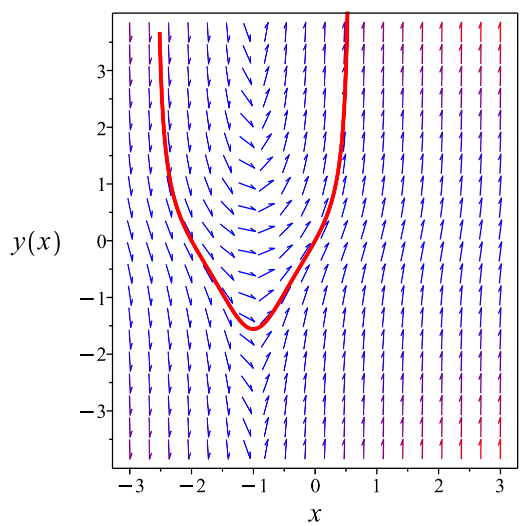
$$y = \tan(x^2 + 2x)$$

<u>Summary</u>
The solution(s) found are the following

$$y = \tan(x^2 + 2x) \tag{1}$$



(a) Solution plot



(b) Slope field plot

<u>Verification of solutions</u>

$$y = \tan(x^2 + 2x)$$

Verified OK.

2.26.3 Solving as first order ode lie symmetry lookup ode

Writing the ode as

$$
\begin{aligned}
y' &= 2(x+1)\left(y^2+1\right) \\
y' &= \omega(x,y)
\end{aligned}
$$

The condition of Lie symmetry is the linearized PDE given by

$$
\eta_x + \omega(\eta_y - \xi_x) - \omega^2 \xi_y - \omega_x \xi - \omega_y \eta = 0 \tag{A}
$$

The type of this ode is known. It is of type `separable`. Therefore we do not need to solve the PDE (A), and can just use the lookup table shown below to find ξ, η

Table 166: Lie symmetry infinitesimal lookup table for known first order ODE's

ODE class	Form	ξ	η
linear ode	$y' = f(x)y(x) + g(x)$	0	$e^{\int f dx}$
separable ode	$y' = f(x)\, g(y)$	$\frac{1}{f}$	0
quadrature ode	$y' = f(x)$	0	1
quadrature ode	$y' = g(y)$	1	0
homogeneous ODEs of Class A	$y' = f\left(\frac{y}{x}\right)$	x	y
homogeneous ODEs of Class C	$y' = (a + bx + cy)^{\frac{n}{m}}$	1	$-\frac{b}{c}$
homogeneous class D	$y' = \frac{y}{x} + g(x)\, F\left(\frac{y}{x}\right)$	x^2	xy
First order special form ID 1	$y' = g(x)\, e^{h(x)+by} + f(x)$	$\frac{e^{-\int b f(x) dx - h(x)}}{g(x)}$	$\frac{f(x) e^{-\int b f(x) dx - h(x)}}{g(x)}$
polynomial type ode	$y' = \frac{a_1 x + b_1 y + c_1}{a_2 x + b_2 y + c_2}$	$\frac{a_1 b_2 x - a_2 b_1 x - b_1 c_2 + b_2 c_1}{a_1 b_2 - a_2 b_1}$	$\frac{a_1 b_2 y - a_2 b_1 y - a_1 c_2 - a_2 c_1}{a_1 b_2 - a_2 b_1}$
Bernoulli ode	$y' = f(x)\, y + g(x)\, y^n$	0	$e^{-\int (n-1) f(x) dx} y^n$
Reduced Riccati	$y' = f_1(x)\, y + f_2(x)\, y^2$	0	$e^{-\int f_1 dx}$

The above table shows that

$$\xi(x, y) = \frac{1}{x+1}$$
$$\eta(x, y) = 0 \tag{A1}$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x, y) \to (R, S)$ where (R, S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y}\right) S(x, y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the canonical coordinates, where $S(R)$. Since $\eta = 0$ then in this special case

$$R = y$$

S is found from

$$S = \int \frac{1}{\xi} dx$$
$$= \int \frac{1}{\frac{1}{x+1}} dx$$

Which results in

$$S = \frac{1}{2}x^2 + x$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x, y)S_y}{R_x + \omega(x, y)R_y} \tag{2}$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x, y)$ is the right hand side of the original ode given by

$$\omega(x, y) = 2(x + 1)\left(y^2 + 1\right)$$

Evaluating all the partial derivatives gives

$$R_x = 0$$
$$R_y = 1$$
$$S_x = x + 1$$
$$S_y = 0$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = \frac{1}{2y^2 + 2} \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = \frac{1}{2R^2 + 2}$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$S(R) = \frac{\arctan(R)}{2} + c_1 \tag{4}$$

To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$\frac{1}{2}x^2 + x = \frac{\arctan(y)}{2} + c_1$$

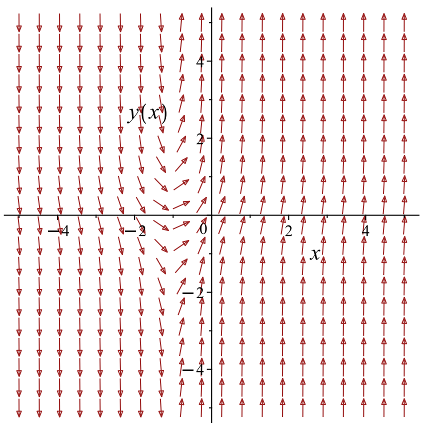
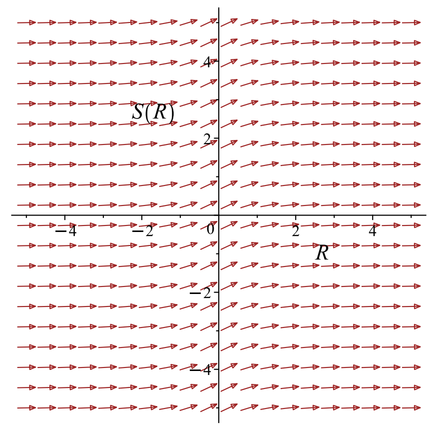
Which simplifies to

$$\frac{1}{2}x^2 + x = \frac{\arctan(y)}{2} + c_1$$

Which gives

$$y = -\tan\left(-x^2 + 2c_1 - 2x\right)$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in x, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dx} = 2(x+1)(y^2+1)$ 	$R = y$ $S = \frac{1}{2}x^2 + x$	$\frac{dS}{dR} = \frac{1}{2R^2+2}$ 

Initial conditions are used to solve for c_1. Substituting $x = 0$ and $y = 0$ in the above solution gives an equation to solve for the constant of integration.

$$0 = -\tan(2c_1)$$

$$c_1 = 0$$

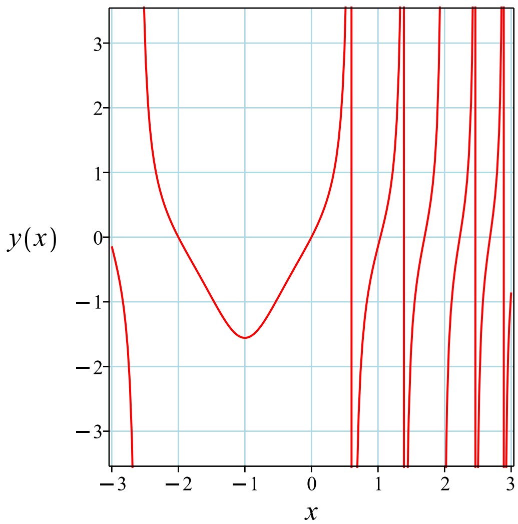
Substituting c_1 found above in the general solution gives

$$y = \tan(x^2 + 2x)$$

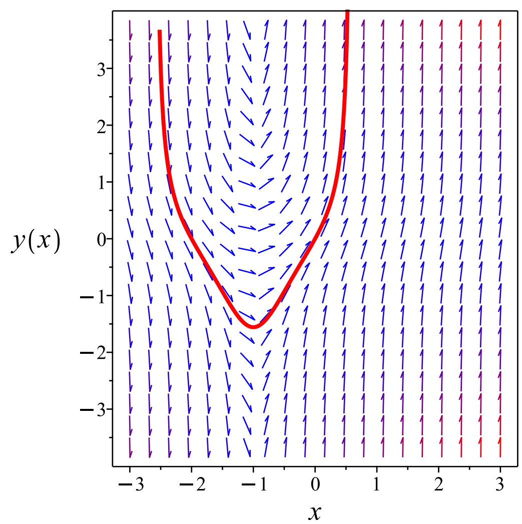
Summary

The solution(s) found are the following

$$y = \tan(x^2 + 2x) \tag{1}$$



(a) Solution plot



(b) Slope field plot

Verification of solutions

$$y = \tan\left(x^2 + 2x\right)$$

Verified OK.

2.26.4 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x,y) + N(x,y)\frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x,y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y) = 0$$

Hence

$$\frac{\partial \phi}{\partial x} + \frac{\partial \phi}{\partial y}\frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$\begin{aligned}\frac{\partial \phi}{\partial x} &= M \\ \frac{\partial \phi}{\partial y} &= N\end{aligned}$$

But since $\frac{\partial^2\phi}{\partial x\partial y} = \frac{\partial^2\phi}{\partial y\partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x, y)$ but at least we know now that we can do that since the condition $\frac{\partial^2\phi}{\partial x\partial y} = \frac{\partial^2\phi}{\partial y\partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x, y)\,\mathrm{d}x + N(x, y)\,\mathrm{d}y = 0 \tag{1A}$$

Therefore

$$\begin{aligned}\left(\frac{1}{2y^2+2}\right)\mathrm{d}y &= (x+1)\,\mathrm{d}x\\ (-x-1)\,\mathrm{d}x + \left(\frac{1}{2y^2+2}\right)\mathrm{d}y &= 0 \qquad (2\mathrm{A})\end{aligned}$$

Comparing (1A) and (2A) shows that

$$\begin{aligned}M(x, y) &= -x - 1\\ N(x, y) &= \frac{1}{2y^2+2}\end{aligned}$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Using result found above gives

$$\begin{aligned}\frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}(-x-1)\\ &= 0\end{aligned}$$

And

$$\begin{aligned}\frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}\left(\frac{1}{2y^2+2}\right)\\ &= 0\end{aligned}$$

Since $\frac{\partial M}{\partial y}=\frac{\partial N}{\partial x}$, then the ODE is exact The following equations are now set up to solve for the function $\phi(x,y)$

$$\frac{\partial \phi}{\partial x}=M \tag{1}$$
$$\frac{\partial \phi}{\partial y}=N \tag{2}$$

Integrating (1) w.r.t. x gives

$$\int \frac{\partial \phi}{\partial x}\,\mathrm{d}x=\int M\,\mathrm{d}x$$
$$\int \frac{\partial \phi}{\partial x}\,\mathrm{d}x=\int -x-1\,\mathrm{d}x$$
$$\phi=-\frac{1}{2}x^2-x+f(y) \tag{3}$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$\frac{\partial \phi}{\partial y}=0+f'(y) \tag{4}$$

But equation (2) says that $\frac{\partial \phi}{\partial y}=\frac{1}{2y^2+2}$. Therefore equation (4) becomes

$$\frac{1}{2y^2+2}=0+f'(y) \tag{5}$$

Solving equation (5) for $f'(y)$ gives

$$f'(y)=\frac{1}{2y^2+2}$$

Integrating the above w.r.t y gives

$$\int f'(y)\,\mathrm{d}y=\int \left(\frac{1}{2y^2+2}\right)\mathrm{d}y$$
$$f(y)=\frac{\arctan(y)}{2}+c_1$$

Where c_1 is constant of integration. Substituting result found above for $f(y)$ into equation (3) gives ϕ

$$\phi = -\frac{x^2}{2} - x + \frac{\arctan(y)}{2} + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = -\frac{x^2}{2} - x + \frac{\arctan(y)}{2}$$

The solution becomes

$$y = \tan\left(x^2 + 2c_1 + 2x\right)$$

Initial conditions are used to solve for c_1. Substituting $x = 0$ and $y = 0$ in the above solution gives an equation to solve for the constant of integration.

$$0 = \tan(2c_1)$$

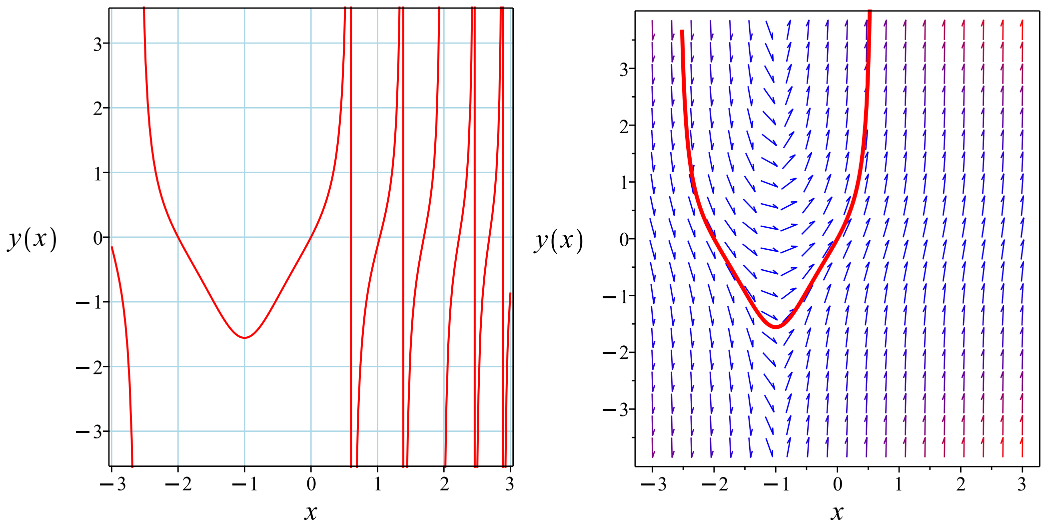
$$c_1 = 0$$

Substituting c_1 found above in the general solution gives

$$y = \tan\left(x^2 + 2x\right)$$

Summary
The solution(s) found are the following

$$y = \tan\left(x^2 + 2x\right) \tag{1}$$



(a) Solution plot (b) Slope field plot

Verification of solutions

$$y = \tan\left(x^2 + 2x\right)$$

Verified OK.

2.26.5 Solving as riccati ode

In canonical form the ODE is

$$\begin{aligned} y' &= F(x, y) \\ &= 2(x+1)\left(y^2+1\right) \end{aligned}$$

This is a Riccati ODE. Comparing the ODE to solve

$$y' = 2x\,y^2 + 2y^2 + 2x + 2$$

With Riccati ODE standard form

$$y' = f_0(x) + f_1(x)y + f_2(x)y^2$$

Shows that $f_0(x) = 2 + 2x$, $f_1(x) = 0$ and $f_2(x) = 2 + 2x$. Let

$$\begin{aligned} y &= \frac{-u'}{f_2 u} \\ &= \frac{-u'}{(2+2x)\,u} \end{aligned} \tag{1}$$

Using the above substitution in the given ODE results (after some simplification)in a second order ODE to solve for $u(x)$ which is

$$f_2u''(x) - (f_2' + f_1f_2)\,u'(x) + f_2^2f_0u(x) = 0 \tag{2}$$

But

$$\begin{aligned} f_2' &= 2 \\ f_1f_2 &= 0 \\ f_2^2f_0 &= (2+2x)^3 \end{aligned}$$

Substituting the above terms back in equation (2) gives

$$(2+2x)\,u''(x) - 2u'(x) + (2+2x)^3\,u(x) = 0$$

Solving the above ODE (this ode solved using Maple, not this program), gives

$$u(x) = c_1\sin\left(x^2+2x\right) + c_2\cos\left(x^2+2x\right)$$

The above shows that

$$u'(x) = -2(x+1)\left(c_2\sin\left(x^2+2x\right) - c_1\cos\left(x^2+2x\right)\right)$$

Using the above in (1) gives the solution

$$y = \frac{2(x+1)\,(c_2\sin(x^2+2x) - c_1\cos(x^2+2x))}{(2+2x)\,(c_1\sin(x^2+2x) + c_2\cos(x^2+2x))}$$

Dividing both numerator and denominator by c_1 gives, after renaming the constant $\frac{c_2}{c_1} = c_3$ the following solution

$$y = \frac{\sin(x^2+2x) - c_3\cos(x^2+2x)}{c_3\sin(x^2+2x) + \cos(x^2+2x)}$$

Initial conditions are used to solve for c_3. Substituting $x = 0$ and $y = 0$ in the above solution gives an equation to solve for the constant of integration.

$$0 = -c_3$$

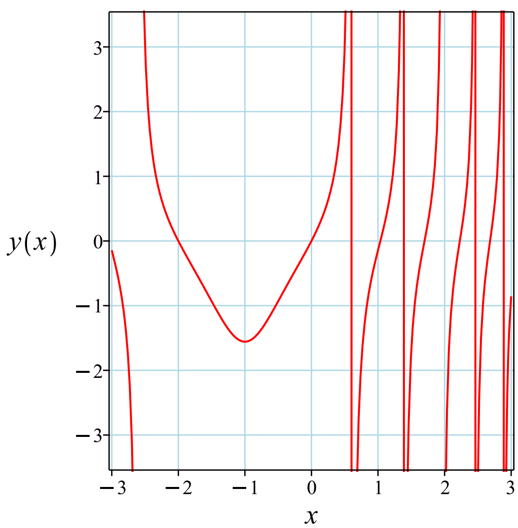
$$c_3 = 0$$

Substituting c_3 found above in the general solution gives

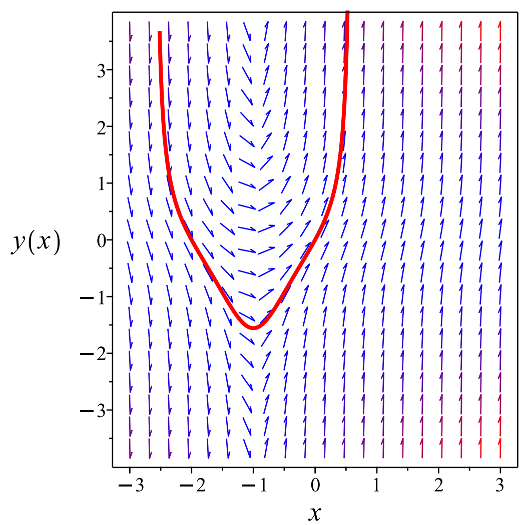
$$y = \frac{\sin(x(2+x))}{\cos(x(2+x))}$$

Summary
The solution(s) found are the following

$$y = \frac{\sin(x(2+x))}{\cos(x(2+x))} \tag{1}$$



(a) Solution plot



(b) Slope field plot

Verification of solutions

$$y = \frac{\sin(x(2+x))}{\cos(x(2+x))}$$

Verified OK.

2.26.6 Maple step by step solution

Let's solve

$[y' - 2(x+1)(1+y^2) = 0, y(0) = 0]$

- Highest derivative means the order of the ODE is 1

  y'

- Separate variables

  $\frac{y'}{1+y^2} = 2 + 2x$
- Integrate both sides with respect to x

  $\int \frac{y'}{1+y^2} dx = \int (2 + 2x) \, dx + c_1$
- Evaluate integral

  $\arctan(y) = x^2 + c_1 + 2x$
- Solve for y

  $y = \tan(x^2 + c_1 + 2x)$
- Use initial condition $y(0) = 0$

  $0 = \tan(c_1)$
- Solve for c_1

  $c_1 = 0$
- Substitute $c_1 = 0$ into general solution and simplify

  $y = \tan(x^2 + 2x)$
- Solution to the IVP

  $y = \tan(x^2 + 2x)$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
<- separable successful`
```

✓ Solution by Maple

Time used: 0.031 (sec). Leaf size: 12

```
dsolve([diff(y(x),x) = 2*(1+x)*(1+y(x)^2),y(0) = 0],y(x), singsol=all)
```

$$y(x) = \tan(x^2 + 2x)$$

✓ Solution by Mathematica

Time used: 0.199 (sec). Leaf size: 11

```
DSolve[{y'[x] == 2*(1+x)*(1+y[x]^2),y[0]==0},y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \tan(x(x+2))$$

2.27 problem 27

Internal problem ID [505]
Internal file name [`OUTPUT/505_Sunday_June_05_2022_01_42_38_AM_39711965/index.tex`]

Book: Elementary differential equations and boundary value problems, 10th ed., Boyce and DiPrima
Section: Section 2.2. Page 48
Problem number: 27.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"exact", "riccati", "bernoulli", "separable", "first_order_ode_lie_symmetry_lookup"**

Maple gives the following as the ode type

`[_separable]`

$$\boxed{y' - \frac{t(4-y)\,y}{3} = 0}$$

2.27.1 Solving as separable ode

In canonical form the ODE is

$$\begin{aligned} y' &= F(t,y)\\ &= f(t)g(y)\\ &= -\frac{ty(y-4)}{3} \end{aligned}$$

Where $f(t) = -\frac{t}{3}$ and $g(y) = y(y-4)$. Integrating both sides gives

$$\frac{1}{y\,(y-4)}\,dy = -\frac{t}{3}\,dt$$
$$\int \frac{1}{y\,(y-4)}\,dy = \int -\frac{t}{3}\,dt$$
$$\frac{\ln\,(y-4)}{4} - \frac{\ln\,(y)}{4} = -\frac{t^2}{6} + c_1$$

The above can be written as

$$\left(\frac{1}{4}\right)(\ln\,(y-4) - \ln\,(y)) = -\frac{t^2}{6} + 2c_1$$
$$\ln\,(y-4) - \ln\,(y) = (4)\left(-\frac{t^2}{6} + 2c_1\right)$$
$$= -\frac{2t^2}{3} + 8c_1$$

Raising both side to exponential gives

$$e^{\ln(y-4)-\ln(y)} = e^{-\frac{2t^2}{3}+4c_1}$$

Which simplifies to

$$\frac{y-4}{y} = 4c_1 e^{-\frac{2t^2}{3}}$$
$$= c_2 e^{-\frac{2t^2}{3}}$$

Summary

The solution(s) found are the following

$$y = -\frac{4}{-1 + c_2 e^{-\frac{2t^2}{3}}} \tag{1}$$

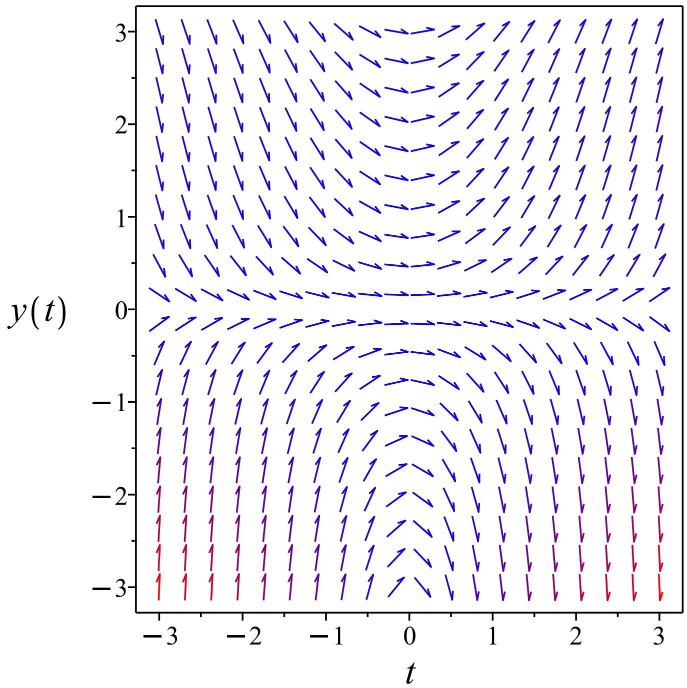


Figure 142: Slope field plot

Verification of solutions

$$y = -\frac{4}{-1 + c_2 e^{-\frac{2t^2}{3}}}$$

Verified OK.

2.27.2 Solving as first order ode lie symmetry lookup ode

Writing the ode as

$$y' = -\frac{ty(y-4)}{3}$$
$$y' = \omega(t,y)$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_t + \omega(\eta_y - \xi_t) - \omega^2 \xi_y - \omega_t \xi - \omega_y \eta = 0 \tag{A}$$

The type of this ode is known. It is of type `separable`. Therefore we do not need to solve the PDE (A), and can just use the lookup table shown below to find ξ, η

Table 169: Lie symmetry infinitesimal lookup table for known first order ODE's

ODE class	Form	ξ	η
linear ode	$y' = f(x)y(x) + g(x)$	0	$e^{\int f dx}$
separable ode	$y' = f(x)\, g(y)$	$\frac{1}{f}$	0
quadrature ode	$y' = f(x)$	0	1
quadrature ode	$y' = g(y)$	1	0
homogeneous ODEs of Class A	$y' = f\left(\frac{y}{x}\right)$	x	y
homogeneous ODEs of Class C	$y' = (a + bx + cy)^{\frac{n}{m}}$	1	$-\frac{b}{c}$
homogeneous class D	$y' = \frac{y}{x} + g(x)\, F\left(\frac{y}{x}\right)$	x^2	xy
First order special form ID 1	$y' = g(x)\, e^{h(x)+by} + f(x)$	$\frac{e^{-\int bf(x)dx-h(x)}}{g(x)}$	$\frac{f(x)e^{-\int bf(x)dx-h(x)}}{g(x)}$
polynomial type ode	$y' = \frac{a_1x+b_1y+c_1}{a_2x+b_2y+c_2}$	$\frac{a_1b_2x-a_2b_1x-b_1c_2+b_2c_1}{a_1b_2-a_2b_1}$	$\frac{a_1b_2y-a_2b_1y-a_1c_2-a_2c_1}{a_1b_2-a_2b_1}$
Bernoulli ode	$y' = f(x)\, y + g(x)\, y^n$	0	$e^{-\int(n-1)f(x)dx} y^n$
Reduced Riccati	$y' = f_1(x)\, y + f_2(x)\, y^2$	0	$e^{-\int f_1 dx}$

The above table shows that

$$
\begin{aligned}
\xi(t, y) &= -\frac{3}{t} \\
\eta(t, y) &= 0
\end{aligned} \tag{A1}
$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(t, y) \to (R, S)$ where (R, S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$
\frac{dt}{\xi} = \frac{dy}{\eta} = dS \tag{1}
$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial t} + \eta \frac{\partial}{\partial y}\right) S(t, y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the

canonical coordinates, where $S(R)$. Since $\eta = 0$ then in this special case

$$R = y$$

S is found from

$$\begin{aligned} S &= \int \frac{1}{\xi} dt \\ &= \int \frac{1}{-\frac{3}{t}} dt \end{aligned}$$

Which results in

$$S = -\frac{t^2}{6}$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_t + \omega(t, y) S_y}{R_t + \omega(t, y) R_y} \tag{2}$$

Where in the above R_t, R_y, S_t, S_y are all partial derivatives and $\omega(t, y)$ is the right hand side of the original ode given by

$$\omega(t, y) = -\frac{ty(y - 4)}{3}$$

Evaluating all the partial derivatives gives

$$\begin{aligned} R_t &= 0 \\ R_y &= 1 \\ S_t &= -\frac{t}{3} \\ S_y &= 0 \end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = \frac{1}{y\,(y - 4)} \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for t, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = \frac{1}{R\,(R - 4)}$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordinates R, S. Integrating the above gives

$$S(R) = \frac{\ln(R-4)}{4} - \frac{\ln(R)}{4} + c_1 \tag{4}$$

To complete the solution, we just need to transform (4) back to t, y coordinates. This results in

$$-\frac{t^2}{6} = \frac{\ln(y-4)}{4} - \frac{\ln(y)}{4} + c_1$$

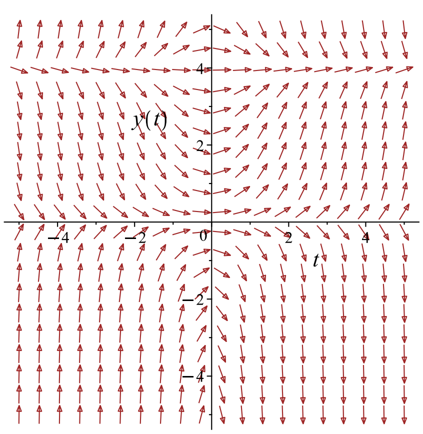
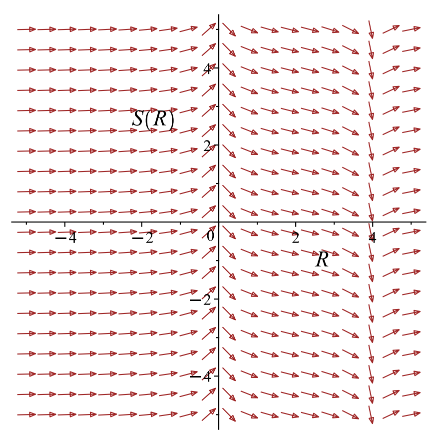
Which simplifies to

$$-\frac{t^2}{6} = \frac{\ln(y-4)}{4} - \frac{\ln(y)}{4} + c_1$$

Which gives

$$y = \frac{4\,\mathrm{e}^{\frac{2t^2}{3}+4c_1}}{-1+\mathrm{e}^{\frac{2t^2}{3}+4c_1}}$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in t, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dt} = -\frac{ty(y-4)}{3}$ 	$R = y$ $S = -\frac{t^2}{6}$	$\frac{dS}{dR} = \frac{1}{R(R-4)}$ 

Summary

The solution(s) found are the following

$$y = \frac{4\,\mathrm{e}^{\frac{2t^2}{3}+4c_1}}{-1+\mathrm{e}^{\frac{2t^2}{3}+4c_1}} \tag{1}$$

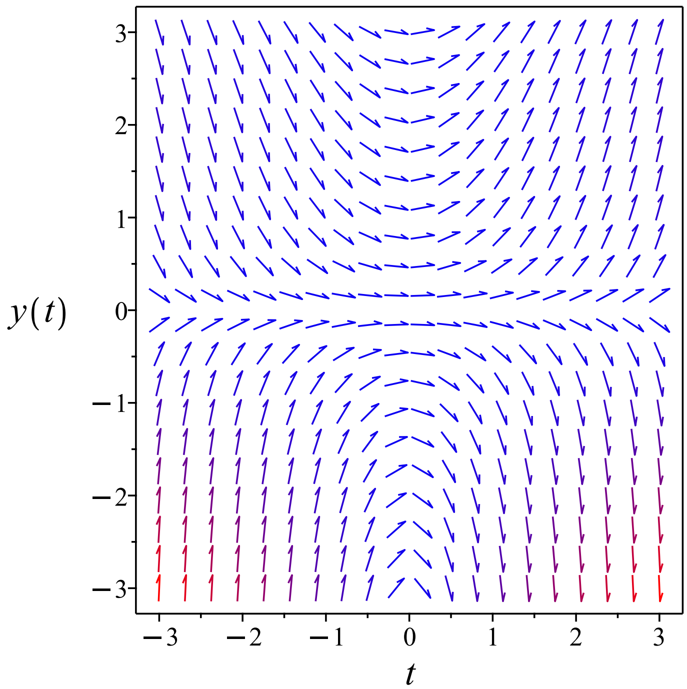


Figure 143: Slope field plot

Verification of solutions

$$y = \frac{4\,\mathrm{e}^{\frac{2t^2}{3}+4c_1}}{-1+\mathrm{e}^{\frac{2t^2}{3}+4c_1}}$$

Verified OK.

2.27.3 Solving as bernoulli ode

In canonical form, the ODE is

$$\begin{aligned} y' &= F(t,y) \\ &= -\frac{ty(y-4)}{3} \end{aligned}$$

This is a Bernoulli ODE.

$$y' = \frac{4t}{3}y - \frac{t}{3}y^2 \tag{1}$$

The standard Bernoulli ODE has the form

$$y' = f_0(t)y + f_1(t)y^n \tag{2}$$

The first step is to divide the above equation by y^n which gives

$$\frac{y'}{y^n} = f_0(t)y^{1-n} + f_1(t) \tag{3}$$

The next step is use the substitution $w = y^{1-n}$ in equation (3) which generates a new ODE in $w(t)$ which will be linear and can be easily solved using an integrating factor. Backsubstitution then gives the solution $y(t)$ which is what we want.

This method is now applied to the ODE at hand. Comparing the ODE (1) With (2) Shows that

$$\begin{aligned} f_0(t) &= \frac{4t}{3} \\ f_1(t) &= -\frac{t}{3} \\ n &= 2 \end{aligned}$$

Dividing both sides of ODE (1) by $y^n = y^2$ gives

$$y'\frac{1}{y^2} = \frac{4t}{3y} - \frac{t}{3} \tag{4}$$

Let

$$\begin{aligned} w &= y^{1-n} \\ &= \frac{1}{y} \end{aligned} \tag{5}$$

Taking derivative of equation (5) w.r.t t gives

$$w' = -\frac{1}{y^2}y' \tag{6}$$

Substituting equations (5) and (6) into equation (4) gives

$$\begin{aligned} -w'(t) &= \frac{4w(t)\,t}{3} - \frac{t}{3} \\ w' &= -\frac{4}{3}tw + \frac{1}{3}t \end{aligned} \tag{7}$$

The above now is a linear ODE in $w(t)$ which is now solved.

Entering Linear first order ODE solver. In canonical form a linear first order is

$$w'(t) + p(t)w(t) = q(t)$$

Where here

$$p(t) = \frac{4t}{3}$$
$$q(t) = \frac{t}{3}$$

Hence the ode is

$$w'(t) + \frac{4w(t)\,t}{3} = \frac{t}{3}$$

The integrating factor μ is

$$\mu = \mathrm{e}^{\int \frac{4t}{3}dt}$$
$$= \mathrm{e}^{\frac{2t^2}{3}}$$

The ode becomes

$$\frac{\mathrm{d}}{\mathrm{d}t}(\mu w) = (\mu)\left(\frac{t}{3}\right)$$
$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\mathrm{e}^{\frac{2t^2}{3}} w\right) = \left(\mathrm{e}^{\frac{2t^2}{3}}\right)\left(\frac{t}{3}\right)$$
$$\mathrm{d}\left(\mathrm{e}^{\frac{2t^2}{3}} w\right) = \left(\frac{t\,\mathrm{e}^{\frac{2t^2}{3}}}{3}\right)\,\mathrm{d}t$$

Integrating gives

$$\mathrm{e}^{\frac{2t^2}{3}} w = \int \frac{t\,\mathrm{e}^{\frac{2t^2}{3}}}{3}\,\mathrm{d}t$$
$$\mathrm{e}^{\frac{2t^2}{3}} w = \frac{\mathrm{e}^{\frac{2t^2}{3}}}{4} + c_1$$

Dividing both sides by the integrating factor $\mu = \mathrm{e}^{\frac{2t^2}{3}}$ results in

$$w(t) = \frac{\mathrm{e}^{-\frac{2t^2}{3}}\,\mathrm{e}^{\frac{2t^2}{3}}}{4} + c_1 \mathrm{e}^{-\frac{2t^2}{3}}$$

which simplifies to

$$w(t) = \frac{1}{4} + c_1 \mathrm{e}^{-\frac{2t^2}{3}}$$

Replacing w in the above by $\frac{1}{y}$ using equation (5) gives the final solution.

$$\frac{1}{y} = \frac{1}{4} + c_1 \mathrm{e}^{-\frac{2t^2}{3}}$$

Or

$$y = \frac{1}{\frac{1}{4} + c_1 \mathrm{e}^{-\frac{2t^2}{3}}}$$

Summary

The solution(s) found are the following

$$y = \frac{1}{\frac{1}{4} + c_1 \mathrm{e}^{-\frac{2t^2}{3}}} \tag{1}$$

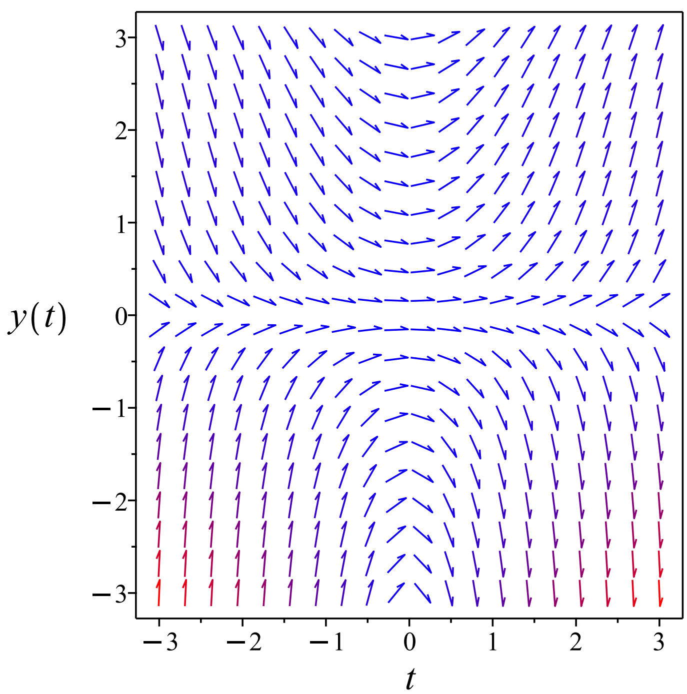


Figure 144: Slope field plot

Verification of solutions

$$y = \frac{1}{\frac{1}{4} + c_1 \mathrm{e}^{-\frac{2t^2}{3}}}$$

Verified OK.

2.27.4 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x,y)+N(x,y)\frac{dy}{dx}=0 \tag{A}$$

We assume there exists a function $\phi(x,y)=c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y)=0$$

Hence

$$\frac{\partial\phi}{\partial x}+\frac{\partial\phi}{\partial y}\frac{dy}{dx}=0 \tag{B}$$

Comparing (A,B) shows that

$$\begin{aligned}\frac{\partial\phi}{\partial x}&=M\\ \frac{\partial\phi}{\partial y}&=N\end{aligned}$$

But since $\frac{\partial^2\phi}{\partial x\partial y}=\frac{\partial^2\phi}{\partial y\partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y}=\frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x,y)$ but at least we know now that we can do that since the condition $\frac{\partial^2\phi}{\partial x\partial y}=\frac{\partial^2\phi}{\partial y\partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(t,y)\,\mathrm{d}t+N(t,y)\,\mathrm{d}y=0 \tag{1A}$$

Therefore

$$\begin{aligned}\left(-\frac{3}{y\,(y-4)}\right)\mathrm{d}y&=(t)\,\mathrm{d}t\\ (-t)\,\mathrm{d}t+\left(-\frac{3}{y\,(y-4)}\right)\mathrm{d}y&=0 \qquad (2A)\end{aligned}$$

Comparing (1A) and (2A) shows that

$$
\begin{aligned}
M(t, y) &= -t \\
N(t, y) &= -\frac{3}{y(y-4)}
\end{aligned}
$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$
\frac{\partial M}{\partial y} = \frac{\partial N}{\partial t}
$$

Using result found above gives

$$
\begin{aligned}
\frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}(-t) \\
&= 0
\end{aligned}
$$

And

$$
\begin{aligned}
\frac{\partial N}{\partial t} &= \frac{\partial}{\partial t}\left(-\frac{3}{y(y-4)}\right) \\
&= 0
\end{aligned}
$$

Since $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial t}$, then the ODE is <u>exact</u> The following equations are now set up to solve for the function $\phi(t, y)$

$$
\begin{aligned}
\frac{\partial \phi}{\partial t} &= M \qquad (1) \\
\frac{\partial \phi}{\partial y} &= N \qquad (2)
\end{aligned}
$$

Integrating (1) w.r.t. t gives

$$
\begin{aligned}
\int \frac{\partial \phi}{\partial t} \, \mathrm{d}t &= \int M \, \mathrm{d}t \\
\int \frac{\partial \phi}{\partial t} \, \mathrm{d}t &= \int -t \, \mathrm{d}t \\
\phi &= -\frac{t^2}{2} + f(y) \qquad (3)
\end{aligned}
$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both t and y. Taking derivative of equation (3) w.r.t y gives

$$
\frac{\partial \phi}{\partial y} = 0 + f'(y) \qquad (4)
$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = -\frac{3}{y(y-4)}$. Therefore equation (4) becomes

$$-\frac{3}{y\left(y-4\right)} = 0 + f'(y) \tag{5}$$

Solving equation (5) for $f'(y)$ gives

$$f'(y) = -\frac{3}{y\left(y-4\right)}$$

Integrating the above w.r.t y gives

$$\int f'(y)\,\mathrm{d}y = \int \left(-\frac{3}{y\left(y-4\right)}\right)\mathrm{d}y$$

$$f(y) = -\frac{3\ln\left(y-4\right)}{4} + \frac{3\ln\left(y\right)}{4} + c_1$$

Where c_1 is constant of integration. Substituting result found above for $f(y)$ into equation (3) gives ϕ

$$\phi = -\frac{t^2}{2} - \frac{3\ln\left(y-4\right)}{4} + \frac{3\ln\left(y\right)}{4} + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = -\frac{t^2}{2} - \frac{3\ln\left(y-4\right)}{4} + \frac{3\ln\left(y\right)}{4}$$

The solution becomes

$$y = \frac{4\,\mathrm{e}^{\frac{2t^2}{3}+\frac{4c_1}{3}}}{-1+\mathrm{e}^{\frac{2t^2}{3}+\frac{4c_1}{3}}}$$

Summary

The solution(s) found are the following

$$y = \frac{4\,\mathrm{e}^{\frac{2t^2}{3}+\frac{4c_1}{3}}}{-1+\mathrm{e}^{\frac{2t^2}{3}+\frac{4c_1}{3}}} \tag{1}$$

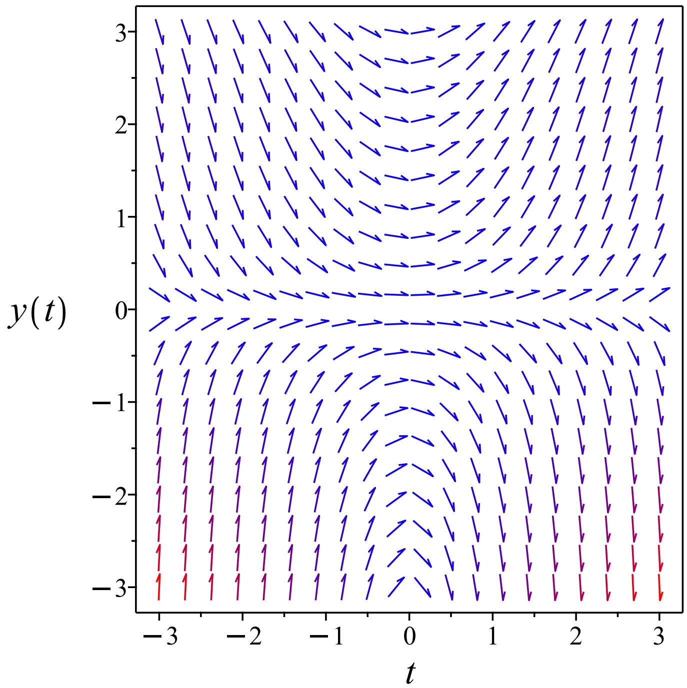


Figure 145: Slope field plot

Verification of solutions

$$y = \frac{4\,\mathrm{e}^{\frac{2t^2}{3}+\frac{4c_1}{3}}}{-1+\mathrm{e}^{\frac{2t^2}{3}+\frac{4c_1}{3}}}$$

Verified OK.

2.27.5 Solving as riccati ode

In canonical form the ODE is

$$\begin{aligned} y' &= F(t,y) \\ &= -\frac{ty(y-4)}{3} \end{aligned}$$

This is a Riccati ODE. Comparing the ODE to solve

$$y' = -\frac{1}{3}t\,y^2 + \frac{4}{3}ty$$

With Riccati ODE standard form

$$y' = f_0(t) + f_1(t)y + f_2(t)y^2$$

Shows that $f_0(t) = 0$, $f_1(t) = \frac{4t}{3}$ and $f_2(t) = -\frac{t}{3}$. Let

$$\begin{aligned} y &= \frac{-u'}{f_2 u} \\ &= \frac{-u'}{-\frac{tu}{3}} \end{aligned} \tag{1}$$

Using the above substitution in the given ODE results (after some simplification)in a second order ODE to solve for $u(x)$ which is

$$f_2 u''(t) - (f_2' + f_1 f_2) u'(t) + f_2^2 f_0 u(t) = 0 \tag{2}$$

But

$$\begin{aligned} f_2' &= -\frac{1}{3} \\ f_1 f_2 &= -\frac{4t^2}{9} \\ f_2^2 f_0 &= 0 \end{aligned}$$

Substituting the above terms back in equation (2) gives

$$-\frac{t u''(t)}{3} - \left(-\frac{1}{3} - \frac{4t^2}{9}\right) u'(t) = 0$$

Solving the above ODE (this ode solved using Maple, not this program), gives

$$u(t) = c_1 + \mathrm{e}^{\frac{2t^2}{3}} c_2$$

The above shows that

$$u'(t) = \frac{4t\,\mathrm{e}^{\frac{2t^2}{3}} c_2}{3}$$

Using the above in (1) gives the solution

$$y = \frac{4\,\mathrm{e}^{\frac{2t^2}{3}} c_2}{c_1 + \mathrm{e}^{\frac{2t^2}{3}} c_2}$$

Dividing both numerator and denominator by c_1 gives, after renaming the constant $\frac{c_2}{c_1} = c_3$ the following solution

$$y = \frac{4\,\mathrm{e}^{\frac{2t^2}{3}}}{c_3 + \mathrm{e}^{\frac{2t^2}{3}}}$$

Summary

The solution(s) found are the following

$$y = \frac{4\,\mathrm{e}^{\frac{2t^2}{3}}}{c_3 + \mathrm{e}^{\frac{2t^2}{3}}} \tag{1}$$

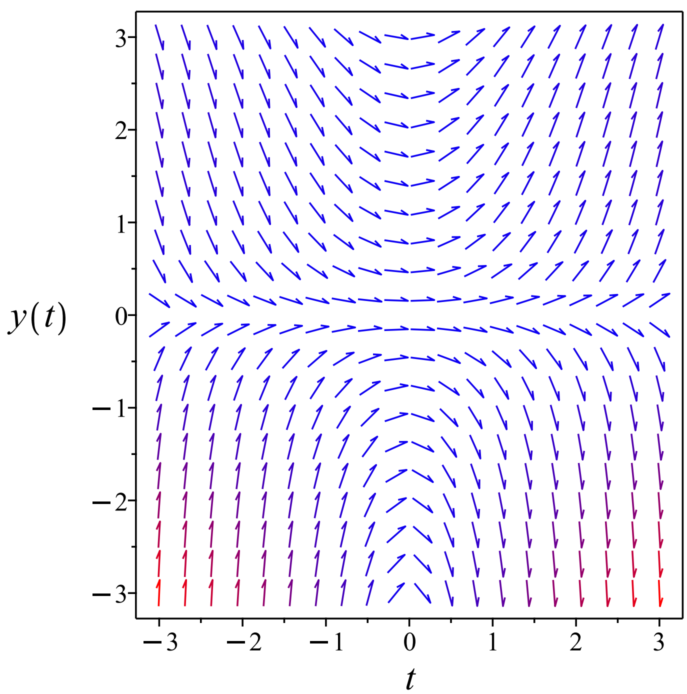


Figure 146: Slope field plot

Verification of solutions

$$y = \frac{4\,\mathrm{e}^{\frac{2t^2}{3}}}{c_3 + \mathrm{e}^{\frac{2t^2}{3}}}$$

Verified OK.

2.27.6 Maple step by step solution

Let's solve

$y' - \frac{t(4-y)y}{3} = 0$

- Highest derivative means the order of the ODE is 1

  y'

- Separate variables

  $\frac{y'}{(4-y)y} = \frac{t}{3}$

- Integrate both sides with respect to t

  $\int \frac{y'}{(4-y)y} dt = \int \frac{t}{3} dt + c_1$

- Evaluate integral

  $-\frac{\ln(y-4)}{4} + \frac{\ln(y)}{4} = \frac{t^2}{6} + c_1$

- Solve for y

  $y = \frac{4\,\mathrm{e}^{\frac{2t^2}{3}+4c_1}}{-1+\mathrm{e}^{\frac{2t^2}{3}+4c_1}}$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
<- Bernoulli successful`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 19

```
dsolve(diff(y(t),t) = 1/3*t*(4-y(t))*y(t),y(t), singsol=all)
```

$$y(t) = \frac{4}{1 + 4\,\mathrm{e}^{-\frac{2t^2}{3}} c_1}$$

✓ Solution by Mathematica

Time used: 0.248 (sec). Leaf size: 44

```
DSolve[y'[t]== 1/3*t*(4-y[t])*y[t],y[t],t,IncludeSingularSolutions -> True]
```

$$y(t) \to \frac{4e^{\frac{2t^2}{3}}}{e^{\frac{2t^2}{3}} + e^{4c_1}}$$
$$y(t) \to 0$$
$$y(t) \to 4$$

2.28 problem 28

Internal problem ID [506]
Internal file name [`OUTPUT/506_Sunday_June_05_2022_01_42_39_AM_99102865/index.tex`]

Book: Elementary differential equations and boundary value problems, 10th ed., Boyce and DiPrima
Section: Section 2.2. Page 48
Problem number: 28.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"exact", "riccati", "bernoulli", "separable", "first_order_ode_lie_symmetry_lookup"**

Maple gives the following as the ode type

`[_separable]`

$$\boxed{y' - \frac{ty(4-y)}{t+1} = 0}$$

2.28.1 Solving as separable ode

In canonical form the ODE is

$$\begin{aligned} y' &= F(t, y) \\ &= f(t)g(y) \\ &= -\frac{ty(y-4)}{t+1} \end{aligned}$$

Where $f(t) = -\frac{t}{t+1}$ and $g(y) = y(y-4)$. Integrating both sides gives

$$\frac{1}{y(y-4)}\,dy = -\frac{t}{t+1}\,dt$$
$$\int \frac{1}{y(y-4)}\,dy = \int -\frac{t}{t+1}\,dt$$
$$\frac{\ln(y-4)}{4} - \frac{\ln(y)}{4} = -t + \ln(t+1) + c_1$$

The above can be written as

$$\left(\frac{1}{4}\right)(\ln(y-4) - \ln(y)) = -t + \ln(t+1) + 2c_1$$
$$\ln(y-4) - \ln(y) = (4)(-t + \ln(t+1) + 2c_1)$$
$$= -4t + 4\ln(t+1) + 8c_1$$

Raising both side to exponential gives

$$e^{\ln(y-4)-\ln(y)} = e^{-4t+4\ln(t+1)+4c_1}$$

Which simplifies to

$$\frac{y-4}{y} = 4c_1 e^{-4t+4\ln(t+1)}$$
$$= c_2 e^{-4t+4\ln(t+1)}$$

Summary

The solution(s) found are the following

$$y = -\frac{4}{-1 + c_2 e^{-4t+4\ln(t+1)}} \tag{1}$$

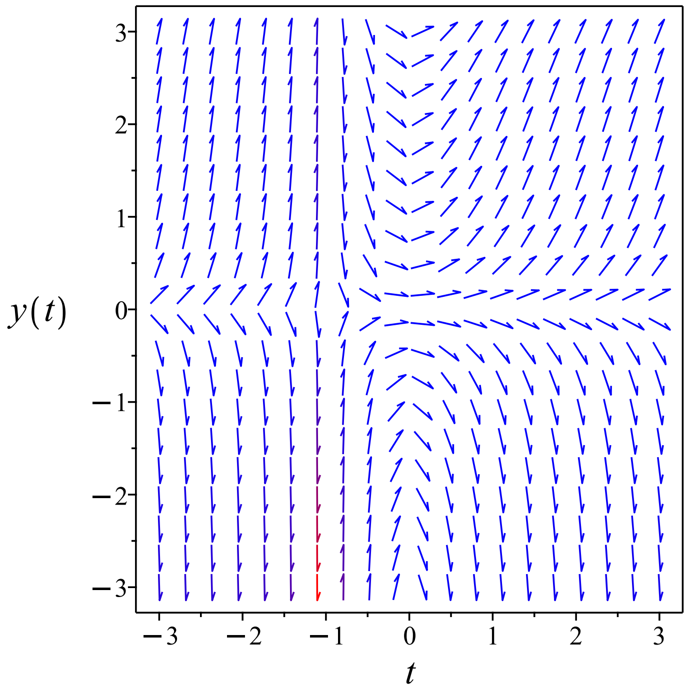


Figure 147: Slope field plot

Verification of solutions

$$y = -\frac{4}{-1 + c_2 \mathrm{e}^{-4t+4\ln(t+1)}}$$

Verified OK.

2.28.2 Solving as first order ode lie symmetry lookup ode

Writing the ode as

$$\begin{aligned} y' &= -\frac{ty(y-4)}{t+1} \\ y' &= \omega(t,y) \end{aligned}$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_t + \omega(\eta_y - \xi_t) - \omega^2 \xi_y - \omega_t \xi - \omega_y \eta = 0 \tag{A}$$

The type of this ode is known. It is of type `separable`. Therefore we do not need to solve the PDE (A), and can just use the lookup table shown below to find ξ, η

Table 172: Lie symmetry infinitesimal lookup table for known first order ODE's

ODE class	Form	ξ	η
linear ode	$y' = f(x)y(x) + g(x)$	0	$e^{\int f dx}$
separable ode	$y' = f(x)\, g(y)$	$\frac{1}{f}$	0
quadrature ode	$y' = f(x)$	0	1
quadrature ode	$y' = g(y)$	1	0
homogeneous ODEs of Class A	$y' = f\left(\frac{y}{x}\right)$	x	y
homogeneous ODEs of Class C	$y' = (a + bx + cy)^{\frac{n}{m}}$	1	$-\frac{b}{c}$
homogeneous class D	$y' = \frac{y}{x} + g(x)\, F\left(\frac{y}{x}\right)$	x^2	xy
First order special form ID 1	$y' = g(x)\, e^{h(x)+by} + f(x)$	$\frac{e^{-\int bf(x)dx - h(x)}}{g(x)}$	$\frac{f(x)e^{-\int bf(x)dx-h(x)}}{g(x)}$
polynomial type ode	$y' = \frac{a_1x+b_1y+c_1}{a_2x+b_2y+c_2}$	$\frac{a_1b_2x-a_2b_1x-b_1c_2+b_2c_1}{a_1b_2-a_2b_1}$	$\frac{a_1b_2y-a_2b_1y-a_1c_2-a_2c_1}{a_1b_2-a_2b_1}$
Bernoulli ode	$y' = f(x)\, y + g(x)\, y^n$	0	$e^{-\int (n-1)f(x)dx} y^n$
Reduced Riccati	$y' = f_1(x)\, y + f_2(x)\, y^2$	0	$e^{-\int f_1 dx}$

The above table shows that

$$
\begin{aligned}
\xi(t, y) &= -\frac{t+1}{t} \\
\eta(t, y) &= 0
\end{aligned}
\tag{A1}
$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(t, y) \to (R, S)$ where (R, S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$
\frac{dt}{\xi} = \frac{dy}{\eta} = dS \tag{1}
$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial t} + \eta \frac{\partial}{\partial y}\right) S(t, y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the

canonical coordinates, where $S(R)$. Since $\eta = 0$ then in this special case

$$R = y$$

S is found from

$$\begin{aligned} S &= \int \frac{1}{\xi} dt \\ &= \int \frac{1}{-\frac{t+1}{t}} dt \end{aligned}$$

Which results in

$$S = -t + \ln(t + 1)$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_t + \omega(t, y)S_y}{R_t + \omega(t, y)R_y} \tag{2}$$

Where in the above R_t, R_y, S_t, S_y are all partial derivatives and $\omega(t, y)$ is the right hand side of the original ode given by

$$\omega(t, y) = -\frac{ty(y - 4)}{t + 1}$$

Evaluating all the partial derivatives gives

$$\begin{aligned} R_t &= 0 \\ R_y &= 1 \\ S_t &= -\frac{t}{t + 1} \\ S_y &= 0 \end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = \frac{1}{y(y - 4)} \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for t, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = \frac{1}{R(R - 4)}$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordinates R, S. Integrating the above gives

$$S(R) = \frac{\ln (R-4)}{4} - \frac{\ln (R)}{4} + c_1 \tag{4}$$

To complete the solution, we just need to transform (4) back to t, y coordinates. This results in

$$-t + \ln (t+1) = \frac{\ln (y-4)}{4} - \frac{\ln (y)}{4} + c_1$$

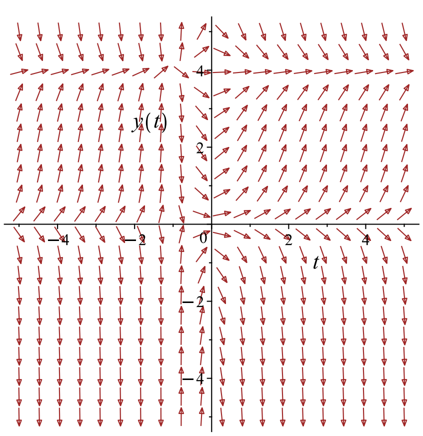
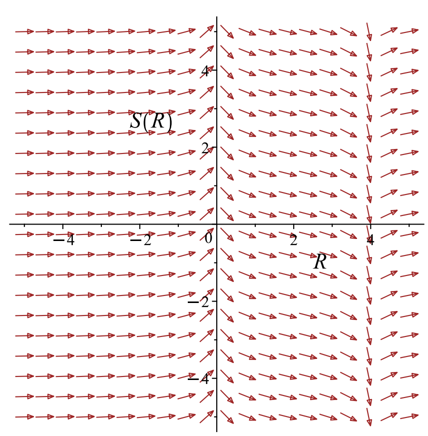
Which simplifies to

$$-t + \ln (t+1) = \frac{\ln (y-4)}{4} - \frac{\ln (y)}{4} + c_1$$

Which gives

$$y = \frac{4\,\mathrm{e}^{4t+4c_1}}{-1 - t^4 - 4t^3 - 6t^2 + \mathrm{e}^{4t+4c_1} - 4t}$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in t, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dt} = -\frac{ty(y-4)}{t+1}$ 	$R = y$ $S = -t + \ln (t+1)$	$\frac{dS}{dR} = \frac{1}{R(R-4)}$ 

Summary
The solution(s) found are the following

$$y = \frac{4\,\mathrm{e}^{4t+4c_1}}{-1-t^4-4t^3-6t^2+\mathrm{e}^{4t+4c_1}-4t} \tag{1}$$

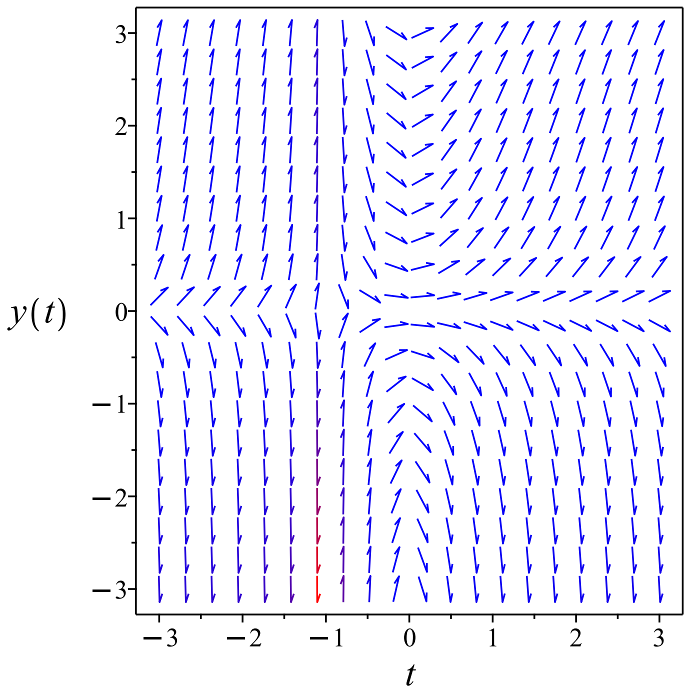


Figure 148: Slope field plot

Verification of solutions

$$y = \frac{4\,\mathrm{e}^{4t+4c_1}}{-1-t^4-4t^3-6t^2+\mathrm{e}^{4t+4c_1}-4t}$$

Verified OK.

2.28.3 Solving as bernoulli ode

In canonical form, the ODE is

$$\begin{aligned} y' &= F(t,y) \\ &= -\frac{ty(y-4)}{t+1} \end{aligned}$$

This is a Bernoulli ODE.

$$y' = \frac{4t}{t+1}y - \frac{t}{t+1}y^2 \tag{1}$$

The standard Bernoulli ODE has the form

$$y' = f_0(t)y + f_1(t)y^n \tag{2}$$

The first step is to divide the above equation by y^n which gives

$$\frac{y'}{y^n} = f_0(t)y^{1-n} + f_1(t) \tag{3}$$

The next step is use the substitution $w = y^{1-n}$ in equation (3) which generates a new ODE in $w(t)$ which will be linear and can be easily solved using an integrating factor. Backsubstitution then gives the solution $y(t)$ which is what we want.

This method is now applied to the ODE at hand. Comparing the ODE (1) With (2) Shows that

$$\begin{aligned}
f_0(t) &= \frac{4t}{t+1} \\
f_1(t) &= -\frac{t}{t+1} \\
n &= 2
\end{aligned}$$

Dividing both sides of ODE (1) by $y^n = y^2$ gives

$$y'\frac{1}{y^2} = \frac{4t}{(t+1)\,y} - \frac{t}{t+1} \tag{4}$$

Let

$$\begin{aligned}
w &= y^{1-n} \\
&= \frac{1}{y}
\end{aligned} \tag{5}$$

Taking derivative of equation (5) w.r.t t gives

$$w' = -\frac{1}{y^2}y' \tag{6}$$

Substituting equations (5) and (6) into equation (4) gives

$$\begin{aligned}
-w'(t) &= \frac{4tw(t)}{t+1} - \frac{t}{t+1} \\
w' &= -\frac{4tw}{t+1} + \frac{t}{t+1}
\end{aligned} \tag{7}$$

The above now is a linear ODE in $w(t)$ which is now solved.

Entering Linear first order ODE solver. In canonical form a linear first order is

$$w'(t) + p(t)w(t) = q(t)$$

Where here

$$\begin{aligned}
p(t) &= \frac{4t}{t+1} \\
q(t) &= \frac{t}{t+1}
\end{aligned}$$

Hence the ode is

$$w'(t) + \frac{4tw(t)}{t+1} = \frac{t}{t+1}$$

The integrating factor μ is

$$\begin{aligned}
\mu &= \mathrm{e}^{\int \frac{4t}{t+1}dt} \\
&= \mathrm{e}^{4t-4\ln(t+1)}
\end{aligned}$$

Which simplifies to

$$\mu = \frac{\mathrm{e}^{4t}}{(t+1)^4}$$

The ode becomes

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}(\mu w) &= (\mu)\left(\frac{t}{t+1}\right) \\
\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\mathrm{e}^{4t}w}{(t+1)^4}\right) &= \left(\frac{\mathrm{e}^{4t}}{(t+1)^4}\right)\left(\frac{t}{t+1}\right) \\
\mathrm{d}\left(\frac{\mathrm{e}^{4t}w}{(t+1)^4}\right) &= \left(\frac{t\,\mathrm{e}^{4t}}{(t+1)^5}\right)\mathrm{d}t
\end{aligned}$$

Integrating gives

$$\begin{aligned}
\frac{\mathrm{e}^{4t}w}{(t+1)^4} &= \int \frac{t\,\mathrm{e}^{4t}}{(t+1)^5}\,\mathrm{d}t \\
\frac{\mathrm{e}^{4t}w}{(t+1)^4} &= \frac{\mathrm{e}^{4t}}{4\,(t+1)^4} + c_1
\end{aligned}$$

Dividing both sides by the integrating factor $\mu = \frac{\mathrm{e}^{4t}}{(t+1)^4}$ results in

$$w(t) = \frac{\mathrm{e}^{-4t}\mathrm{e}^{4t}}{4} + c_1\mathrm{e}^{-4t}(t+1)^4$$

which simplifies to

$$w(t) = \frac{1}{4} + c_1 e^{-4t} (t+1)^4$$

Replacing w in the above by $\frac{1}{y}$ using equation (5) gives the final solution.

$$\frac{1}{y} = \frac{1}{4} + c_1 e^{-4t} (t+1)^4$$

Or

$$y = \frac{1}{\frac{1}{4} + c_1 e^{-4t} (t+1)^4}$$

Summary

The solution(s) found are the following

$$y = \frac{1}{\frac{1}{4} + c_1 e^{-4t} (t+1)^4} \tag{1}$$

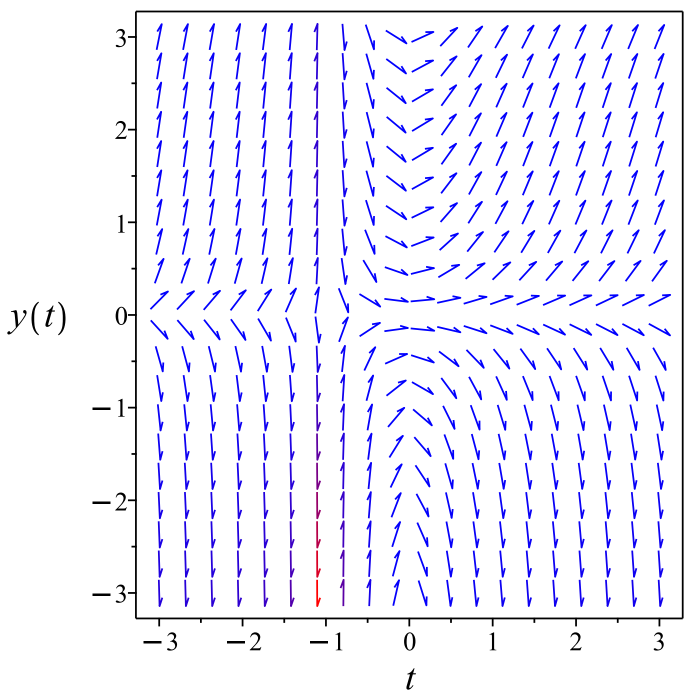


Figure 149: Slope field plot

Verification of solutions

$$y = \frac{1}{\frac{1}{4} + c_1 e^{-4t} (t+1)^4}$$

Verified OK.

2.28.4 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x,y)+N(x,y)\frac{dy}{dx}=0 \tag{A}$$

We assume there exists a function $\phi(x,y)=c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y)=0$$

Hence

$$\frac{\partial\phi}{\partial x}+\frac{\partial\phi}{\partial y}\frac{dy}{dx}=0 \tag{B}$$

Comparing (A,B) shows that

$$\begin{aligned}\frac{\partial\phi}{\partial x}&=M\\ \frac{\partial\phi}{\partial y}&=N\end{aligned}$$

But since $\frac{\partial^2\phi}{\partial x\partial y}=\frac{\partial^2\phi}{\partial y\partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y}=\frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x,y)$ but at least we know now that we can do that since the condition $\frac{\partial^2\phi}{\partial x\partial y}=\frac{\partial^2\phi}{\partial y\partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(t,y)\,\mathrm{d}t+N(t,y)\,\mathrm{d}y=0 \tag{1A}$$

Therefore

$$\begin{aligned}\left(-\frac{1}{y\,(y-4)}\right)\mathrm{d}y&=\left(\frac{t}{t+1}\right)\mathrm{d}t\\ \left(-\frac{t}{t+1}\right)\mathrm{d}t+\left(-\frac{1}{y\,(y-4)}\right)\mathrm{d}y&=0 \end{aligned}\tag{2A}$$

Comparing (1A) and (2A) shows that

$$
\begin{aligned}
M(t, y) &= -\frac{t}{t+1} \\
N(t, y) &= -\frac{1}{y\,(y-4)}
\end{aligned}
$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$
\frac{\partial M}{\partial y} = \frac{\partial N}{\partial t}
$$

Using result found above gives

$$
\begin{aligned}
\frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}\left(-\frac{t}{t+1}\right) \\
&= 0
\end{aligned}
$$

And

$$
\begin{aligned}
\frac{\partial N}{\partial t} &= \frac{\partial}{\partial t}\left(-\frac{1}{y\,(y-4)}\right) \\
&= 0
\end{aligned}
$$

Since $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial t}$, then the ODE is exact The following equations are now set up to solve for the function $\phi(t, y)$

$$
\frac{\partial \phi}{\partial t} = M \tag{1}
$$

$$
\frac{\partial \phi}{\partial y} = N \tag{2}
$$

Integrating (1) w.r.t. t gives

$$
\begin{aligned}
\int \frac{\partial \phi}{\partial t} \, \mathrm{d}t &= \int M \, \mathrm{d}t \\
\int \frac{\partial \phi}{\partial t} \, \mathrm{d}t &= \int -\frac{t}{t+1} \, \mathrm{d}t
\end{aligned}
$$

$$
\phi = -t + \ln(t+1) + f(y) \tag{3}
$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both t and y. Taking derivative of equation (3) w.r.t y gives

$$
\frac{\partial \phi}{\partial y} = 0 + f'(y) \tag{4}
$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = -\frac{1}{y(y-4)}$. Therefore equation (4) becomes

$$-\frac{1}{y\left(y-4\right)} = 0 + f'(y) \tag{5}$$

Solving equation (5) for $f'(y)$ gives

$$f'(y) = -\frac{1}{y\left(y-4\right)}$$

Integrating the above w.r.t y gives

$$\int f'(y)\,\mathrm{d}y = \int \left(-\frac{1}{y\left(y-4\right)}\right)\mathrm{d}y$$

$$f(y) = -\frac{\ln\left(y-4\right)}{4} + \frac{\ln\left(y\right)}{4} + c_1$$

Where c_1 is constant of integration. Substituting result found above for $f(y)$ into equation (3) gives ϕ

$$\phi = -t + \ln\left(t+1\right) - \frac{\ln\left(y-4\right)}{4} + \frac{\ln\left(y\right)}{4} + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = -t + \ln\left(t+1\right) - \frac{\ln\left(y-4\right)}{4} + \frac{\ln\left(y\right)}{4}$$

The solution becomes

$$y = \frac{4\,\mathrm{e}^{4t+4c_1}}{-1 - t^4 - 4t^3 - 6t^2 + \mathrm{e}^{4t+4c_1} - 4t}$$

Summary

The solution(s) found are the following

$$y = \frac{4\,\mathrm{e}^{4t+4c_1}}{-1 - t^4 - 4t^3 - 6t^2 + \mathrm{e}^{4t+4c_1} - 4t} \tag{1}$$

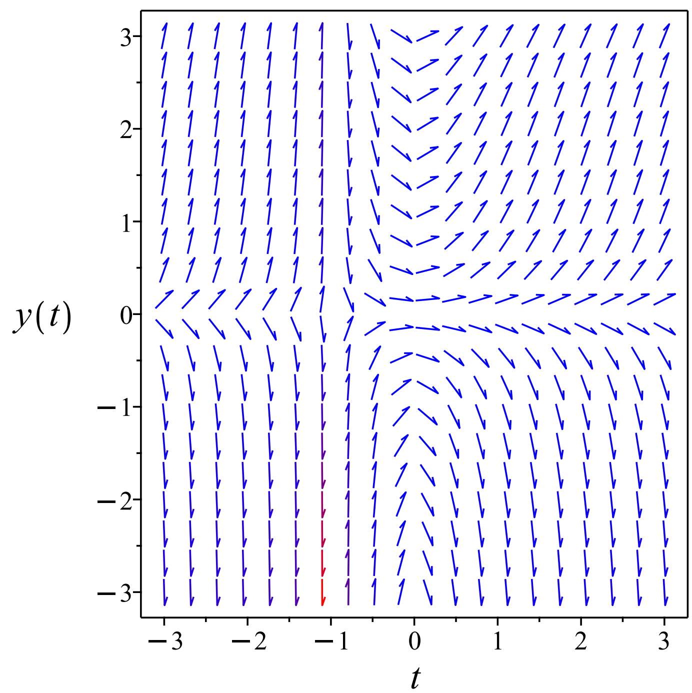


Figure 150: Slope field plot

<u>Verification of solutions</u>

$$y = \frac{4\,\mathrm{e}^{4t+4c_1}}{-1 - t^4 - 4t^3 - 6t^2 + \mathrm{e}^{4t+4c_1} - 4t}$$

Verified OK.

2.28.5 Solving as riccati ode

In canonical form the ODE is

$$\begin{aligned} y' &= F(t, y) \\ &= -\frac{ty(y-4)}{t+1} \end{aligned}$$

This is a Riccati ODE. Comparing the ODE to solve

$$y' = \frac{4ty}{t+1} - \frac{t\,y^2}{t+1}$$

With Riccati ODE standard form

$$y' = f_0(t) + f_1(t)y + f_2(t)y^2$$

Shows that $f_0(t) = 0$, $f_1(t) = \frac{4t}{t+1}$ and $f_2(t) = -\frac{t}{t+1}$. Let

$$\begin{aligned} y &= \frac{-u'}{f_2 u} \\ &= \frac{-u'}{-\frac{tu}{t+1}} \end{aligned} \tag{1}$$

Using the above substitution in the given ODE results (after some simplification)in a second order ODE to solve for $u(x)$ which is

$$f_2 u''(t) - (f_2' + f_1 f_2)\, u'(t) + f_2^2 f_0 u(t) = 0 \tag{2}$$

But

$$\begin{aligned} f_2' &= -\frac{1}{t+1} + \frac{t}{(t+1)^2} \\ f_1 f_2 &= -\frac{4t^2}{(t+1)^2} \\ f_2^2 f_0 &= 0 \end{aligned}$$

Substituting the above terms back in equation (2) gives

$$-\frac{tu''(t)}{t+1} - \left(-\frac{1}{t+1} + \frac{t}{(t+1)^2} - \frac{4t^2}{(t+1)^2}\right) u'(t) = 0$$

Solving the above ODE (this ode solved using Maple, not this program), gives

$$u(t) = \frac{c_2 e^{4t} + (t+1)^4 c_1}{(t+1)^4}$$

The above shows that

$$u'(t) = \frac{4\, e^{4t} c_2 t}{(t+1)^5}$$

Using the above in (1) gives the solution

$$y = \frac{4\, e^{4t} c_2}{c_2 e^{4t} + (t+1)^4 c_1}$$

Dividing both numerator and denominator by c_1 gives, after renaming the constant $\frac{c_2}{c_1} = c_3$ the following solution

$$y = \frac{4\,e^{4t}}{e^{4t} + c_3 \left(t^4 + 4t^3 + 6t^2 + 4t + 1\right)}$$

Summary

The solution(s) found are the following

$$y = \frac{4\,e^{4t}}{e^{4t} + c_3 \left(t^4 + 4t^3 + 6t^2 + 4t + 1\right)} \tag{1}$$

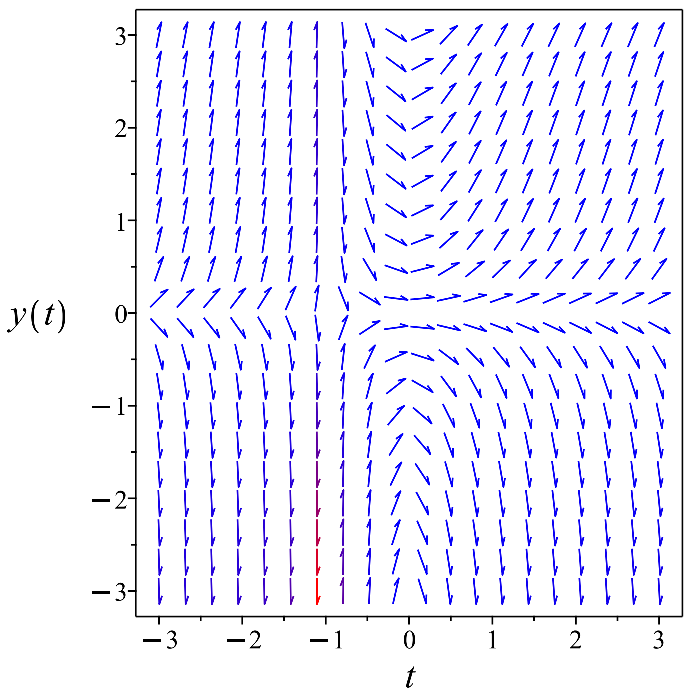


Figure 151: Slope field plot

Verification of solutions

$$y = \frac{4\,e^{4t}}{e^{4t} + c_3 \left(t^4 + 4t^3 + 6t^2 + 4t + 1\right)}$$

Verified OK.

2.28.6 Maple step by step solution

Let's solve

$y' - \frac{ty(4-y)}{t+1} = 0$

- Highest derivative means the order of the ODE is 1

  y'

- Separate variables

  $\frac{y'}{(4-y)y} = \frac{t}{t+1}$

- Integrate both sides with respect to t

  $\int \frac{y'}{(4-y)y} dt = \int \frac{t}{t+1} dt + c_1$

- Evaluate integral

  $-\frac{\ln(y-4)}{4} + \frac{\ln(y)}{4} = t - \ln(t+1) + c_1$

- Solve for y

  $y = \frac{4\,\mathrm{e}^{4t+4c_1}}{-1-t^4-4t^3-6t^2+\mathrm{e}^{4t+4c_1}-4t}$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
<- Bernoulli successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 22

```
dsolve(diff(y(t),t) = t*y(t)*(4-y(t))/(1+t),y(t), singsol=all)
```

$$y(t) = \frac{4}{1 + 4\,\mathrm{e}^{-4t}\,(t+1)^4\,c_1}$$

✓ Solution by Mathematica

Time used: 1.337 (sec). Leaf size: 42

```
DSolve[y'[t] == t*y[t]*(4-y[t])/(1+t),y[t],t,IncludeSingularSolutions -> True]
```

$$y(t) \to \frac{4e^{4t}}{e^{4t} + e^{4c_1}(t+1)^4}$$
$$y(t) \to 0$$
$$y(t) \to 4$$

2.29 problem 29

Internal problem ID [507]
Internal file name [`OUTPUT/507_Sunday_June_05_2022_01_42_40_AM_35330923/index.tex`]

Book: Elementary differential equations and boundary value problems, 10th ed., Boyce and DiPrima
Section: Section 2.2. Page 48
Problem number: 29.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"quadrature"**

Maple gives the following as the ode type

`[_quadrature]`

$$\boxed{y' - \frac{b + ay}{d + cy} = 0}$$

2.29.1 Solving as quadrature ode

Integrating both sides gives

$$\int \frac{cy + d}{ay + b} dy = x + c_1$$
$$\frac{cy}{a} + \frac{(ad - bc)\ln(ay + b)}{a^2} = x + c_1$$

Solving for y gives these solutions

Summary
The solution(s) found are the following

$$y = \frac{c_1 a^2 + x\,a^2 - \left(-\operatorname{LambertW}\left(-\frac{c\,\mathrm{e}^{\frac{c_1 a^2 + x\,a^2 + bc}{ad - bc}}}{-ad + bc}\right) + \frac{c_1 a^2 + x\,a^2 + bc}{ad - bc}\right) ad + \left(-\operatorname{LambertW}\left(-\frac{c\,\mathrm{e}^{\frac{c_1 a^2 + x\,a^2 + bc}{ad - bc}}}{-ad + bc}\right)\right.}{ac} \tag{1}$$

Verification of solutions

$$y = \frac{c_1 a^2 + x\,a^2 - \left(-\operatorname{LambertW}\left(-\frac{c\,\mathrm{e}^{\frac{c_1 a^2 + x\,a^2 + bc}{ad-bc}}}{-ad+bc}\right) + \frac{c_1 a^2 + x\,a^2 + bc}{ad-bc}\right) ad + \left(-\operatorname{LambertW}\left(-\frac{c\,\mathrm{e}^{\frac{c_1 a^2 + x\,a^2 + bc}{ad-bc}}}{-ad+bc}\right)\right.}{ac}$$

Verified OK.

2.29.2 Maple step by step solution

Let's solve

$y' - \frac{b+ay}{d+cy} = 0$

- Highest derivative means the order of the ODE is 1

  y'

- Separate variables

  $\frac{y'(d+cy)}{b+ay} = 1$

- Integrate both sides with respect to x

  $\int \frac{y'(d+cy)}{b+ay}dx = \int 1dx + c_1$

- Evaluate integral

  $\frac{cy}{a} + \frac{(ad-bc)\ln(b+ay)}{a^2} = x + c_1$

- Solve for y

  $$y = \frac{c_1 a^2 + x\,a^2 - \left(-\mathit{LambertW}\left(-\frac{c\,\mathrm{e}^{\frac{c_1 a^2 + x\,a^2 + bc}{ad-bc}}}{-ad+bc}\right) + \frac{c_1 a^2 + x\,a^2 + bc}{ad-bc}\right) ad + \left(-\mathit{LambertW}\left(-\frac{c\,\mathrm{e}^{\frac{c_1 a^2 + x\,a^2 + bc}{ad-bc}}}{-ad+bc}\right) + \frac{c_1 a^2 + x\,a^2 + bc}{ad-bc}\right) bc}{ac}$$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
<- separable successful`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 61

```
dsolve(diff(y(x),x) = (b+a*y(x))/(d+c*y(x)),y(x), singsol=all)
```

$$y(x) = \frac{(ad - bc)\,\mathrm{LambertW}\left(\frac{c\,e^{\frac{(c_1+x)a^2+bc}{ad-bc}}}{ad-bc}\right) - bc}{ac}$$

✓ Solution by Mathematica

Time used: 16.166 (sec). Leaf size: 83

```
DSolve[y'[x] == (b+a*y[x])/(d+c*y[x]),y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \frac{-bc + (ad - bc)W\left(-\frac{c\left(e^{-1-\frac{a^2(x+c_1)}{bc}}\right)^{\frac{bc}{bc-ad}}}{bc-ad}\right)}{ac}$$

$$y(x) \to -\frac{b}{a}$$

2.30 problem 31

Internal problem ID [508]
Internal file name [`OUTPUT/508_Sunday_June_05_2022_01_42_41_AM_3383851/index.tex`]

Book: Elementary differential equations and boundary value problems, 10th ed., Boyce and DiPrima
Section: Section 2.2. Page 48
Problem number: 31.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"riccati", "homogeneousTypeD2", "first_order_ode_lie_symmetry_calculated"**

Maple gives the following as the ode type

```
[[_homogeneous, `class A`], _rational, _Riccati]
```

$$\boxed{y' - \frac{x^2 + yx + y^2}{x^2} = 0}$$

2.30.1 Solving as homogeneousTypeD2 ode

Using the change of variables $y = u(x)\,x$ on the above ode results in new ode in $u(x)$

$$u'(x)\,x + u(x) - \frac{x^2 + u(x)\,x^2 + u(x)^2\,x^2}{x^2} = 0$$

In canonical form the ODE is

$$\begin{aligned} u' &= F(x,u) \\ &= f(x)g(u) \\ &= \frac{u^2+1}{x} \end{aligned}$$

Where $f(x) = \frac{1}{x}$ and $g(u) = u^2 + 1$. Integrating both sides gives

$$
\begin{aligned}
\frac{1}{u^2+1}\,du &= \frac{1}{x}\,dx \\
\int \frac{1}{u^2+1}\,du &= \int \frac{1}{x}\,dx \\
\arctan(u) &= \ln(x) + c_2
\end{aligned}
$$

The solution is

$$
\arctan(u(x)) - \ln(x) - c_2 = 0
$$

Replacing $u(x)$ in the above solution by $\frac{y}{x}$ results in the solution for y in implicit form

$$
\begin{aligned}
\arctan\left(\frac{y}{x}\right) - \ln(x) - c_2 &= 0 \\
\arctan\left(\frac{y}{x}\right) - \ln(x) - c_2 &= 0
\end{aligned}
$$

Summary

The solution(s) found are the following

$$
\arctan\left(\frac{y}{x}\right) - \ln(x) - c_2 = 0 \tag{1}
$$

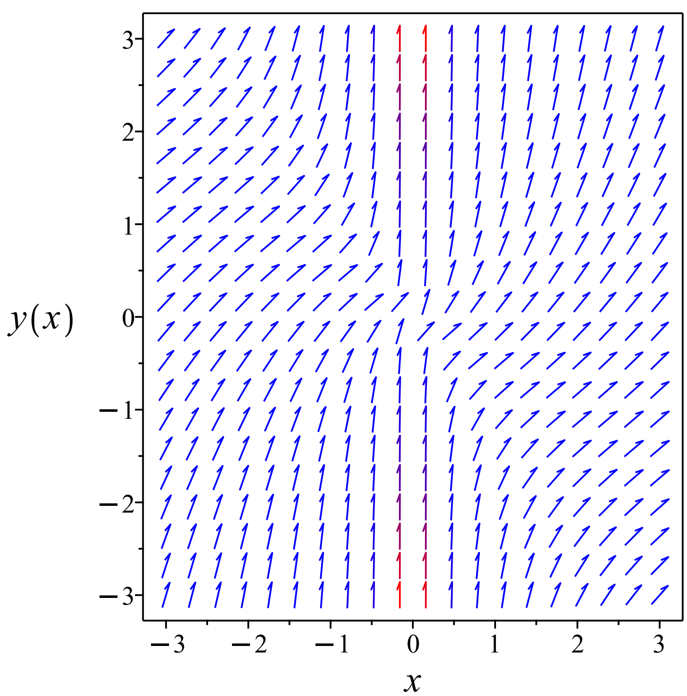


Figure 152: Slope field plot

Verification of solutions

$$\arctan\left(\frac{y}{x}\right) - \ln(x) - c_2 = 0$$

Verified OK.

2.30.2 Solving as first order ode lie symmetry calculated ode

Writing the ode as

$$\begin{aligned} y' &= \frac{x^2 + yx + y^2}{x^2} \\ y' &= \omega(x, y) \end{aligned}$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x + \omega(\eta_y - \xi_x) - \omega^2 \xi_y - \omega_x \xi - \omega_y \eta = 0 \tag{A}$$

The type of this ode is not in the lookup table. To determine ξ, η then (A) is solved using ansatz. Making bivariate polynomials of degree 1 to use as anstaz gives

$$\xi = x a_2 + y a_3 + a_1 \tag{1E}$$

$$\eta = x b_2 + y b_3 + b_1 \tag{2E}$$

Where the unknown coefficients are

$$\{a_1, a_2, a_3, b_1, b_2, b_3\}$$

Substituting equations (1E,2E) and ω into (A) gives

$$\begin{aligned} & b_2 + \frac{(x^2 + yx + y^2)(b_3 - a_2)}{x^2} - \frac{(x^2 + yx + y^2)^2 a_3}{x^4} \\ & - \left(\frac{2x + y}{x^2} - \frac{2(x^2 + yx + y^2)}{x^3}\right)(x a_2 + y a_3 + a_1) \\ & - \frac{(x + 2y)(x b_2 + y b_3 + b_1)}{x^2} = 0 \end{aligned} \tag{5E}$$

Putting the above in normal form gives

$$-\frac{x^4 a_2 + x^4 a_3 - x^4 b_3 + 2x^3 y a_3 + 2x^3 y b_2 - x^2 y^2 a_2 + 2x^2 y^2 a_3 + x^2 y^2 b_3 + y^4 a_3 + x^3 b_1 - x^2 y a_1 + 2x^2 y b_1 -}{x^4}$$
$$= 0$$

Setting the numerator to zero gives

$$-x^4a_2 - x^4a_3 + x^4b_3 - 2x^3ya_3 - 2x^3yb_2 + x^2y^2a_2 - 2x^2y^2a_3 - x^2y^2b_3 - y^4a_3 - x^3b_1 + x^2ya_1 - 2x^2yb_1 + 2x\,y^2a_1 = 0 \tag{6E}$$

Looking at the above PDE shows the following are all the terms with $\{x, y\}$ in them.

$$\{x, y\}$$

The following substitution is now made to be able to collect on all terms with $\{x, y\}$ in them

$$\{x = v_1, y = v_2\}$$

The above PDE (6E) now becomes

$$-a_2v_1^4 + a_2v_1^2v_2^2 - a_3v_1^4 - 2a_3v_1^3v_2 - 2a_3v_1^2v_2^2 - a_3v_2^4 - 2b_2v_1^3v_2 + b_3v_1^4 - b_3v_1^2v_2^2 + a_1v_1^2v_2 + 2a_1v_1v_2^2 - b_1v_1^3 - 2b_1v_1^2v_2 = 0 \tag{7E}$$

Collecting the above on the terms v_i introduced, and these are

$$\{v_1, v_2\}$$

Equation (7E) now becomes

$$(-a_2 - a_3 + b_3)\,v_1^4 + (-2a_3 - 2b_2)\,v_1^3v_2 - b_1v_1^3 + (a_2 - 2a_3 - b_3)\,v_1^2v_2^2 + (a_1 - 2b_1)\,v_1^2v_2 + 2a_1v_1v_2^2 - a_3v_2^4 = 0 \tag{8E}$$

Setting each coefficients in (8E) to zero gives the following equations to solve

$$\begin{aligned}
2a_1 &= 0 \\
-a_3 &= 0 \\
-b_1 &= 0 \\
a_1 - 2b_1 &= 0 \\
-2a_3 - 2b_2 &= 0 \\
-a_2 - a_3 + b_3 &= 0 \\
a_2 - 2a_3 - b_3 &= 0
\end{aligned}$$

Solving the above equations for the unknowns gives

$$
\begin{aligned}
a_1 &= 0 \\
a_2 &= b_3 \\
a_3 &= 0 \\
b_1 &= 0 \\
b_2 &= 0 \\
b_3 &= b_3
\end{aligned}
$$

Substituting the above solution in the anstaz (1E,2E) (using 1 as arbitrary value for any unknown in the RHS) gives

$$
\begin{aligned}
\xi &= x \\
\eta &= y
\end{aligned}
$$

Shifting is now applied to make $\xi = 0$ in order to simplify the rest of the computation

$$
\begin{aligned}
\eta &= \eta - \omega(x, y)\, \xi \\
&= y - \left(\frac{x^2 + yx + y^2}{x^2}\right)(x) \\
&= \frac{-x^2 - y^2}{x} \\
\xi &= 0
\end{aligned}
$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x, y) \to (R, S)$ where (R, S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$
\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}
$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y}\right) S(x, y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$
R = x
$$

S is found from

$$
\begin{aligned}
S &= \int \frac{1}{\eta} dy \\
&= \int \frac{1}{\frac{-x^2 - y^2}{x}} dy
\end{aligned}
$$

Which results in

$$S = -\arctan\left(\frac{y}{x}\right)$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x, y)S_y}{R_x + \omega(x, y)R_y} \tag{2}$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x, y)$ is the right hand side of the original ode given by

$$\omega(x, y) = \frac{x^2 + yx + y^2}{x^2}$$

Evaluating all the partial derivatives gives

$$\begin{aligned}
R_x &= 1 \\
R_y &= 0 \\
S_x &= \frac{y}{x^2 + y^2} \\
S_y &= -\frac{x}{x^2 + y^2}
\end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = -\frac{1}{x} \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = -\frac{1}{R}$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$S(R) = -\ln(R) + c_1 \tag{4}$$

To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$-\arctan\left(\frac{y}{x}\right) = -\ln(x) + c_1$$

Which simplifies to

$$-\arctan\left(\frac{y}{x}\right) = -\ln(x) + c_1$$

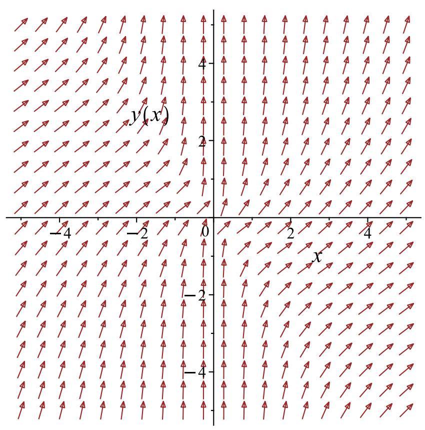
Which gives

$$y = -\tan(-\ln(x) + c_1)\, x$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

<table>
<tr>
<th>Original ode in x, y coordinates</th>
<th>Canonical coordinates transformation</th>
<th>ODE in canonical coordinates (R, S)</th>
</tr>
<tr>
<td>

$\frac{dy}{dx} = \frac{x^2+yx+y^2}{x^2}$



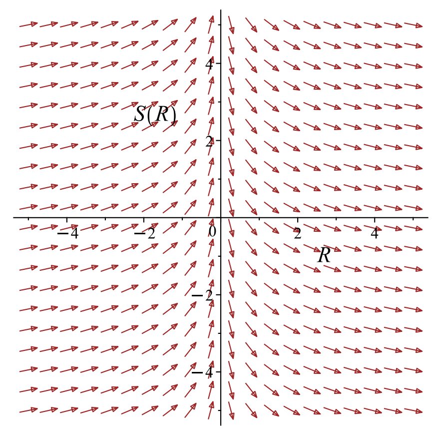
</td>
<td>

$R = x$

$S = -\arctan\left(\frac{y}{x}\right)$

</td>
<td>

$\frac{dS}{dR} = -\frac{1}{R}$



</td>
</tr>
</table>

Summary

The solution(s) found are the following

$$y = -\tan(-\ln(x) + c_1)\, x \tag{1}$$

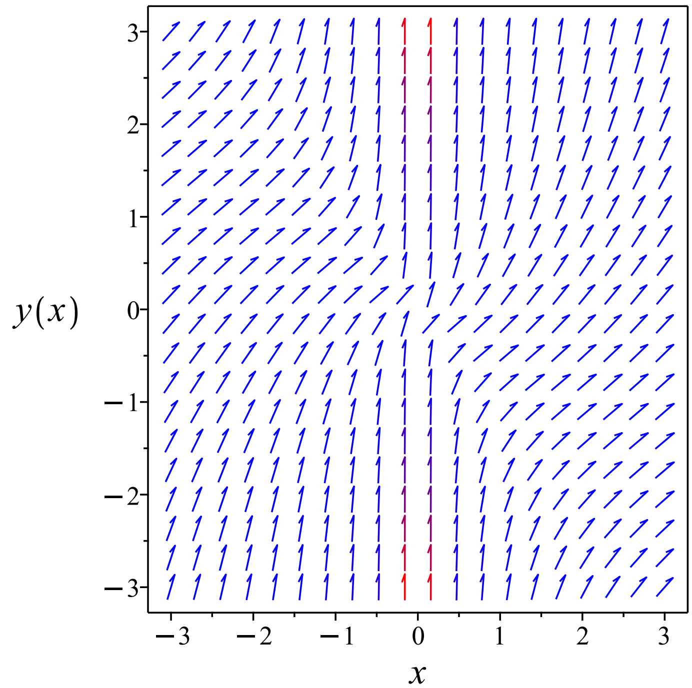


Figure 153: Slope field plot

Verification of solutions

$$y = -\tan(-\ln(x) + c_1)\, x$$

Verified OK.

2.30.3 Solving as riccati ode

In canonical form the ODE is

$$\begin{aligned} y' &= F(x,y) \\ &= \frac{x^2 + yx + y^2}{x^2} \end{aligned}$$

This is a Riccati ODE. Comparing the ODE to solve

$$y' = 1 + \frac{y}{x} + \frac{y^2}{x^2}$$

With Riccati ODE standard form

$$y' = f_0(x) + f_1(x)y + f_2(x)y^2$$

Shows that $f_0(x) = 1$, $f_1(x) = \frac{1}{x}$ and $f_2(x) = \frac{1}{x^2}$. Let

$$
\begin{aligned}
y &= \frac{-u'}{f_2 u} \\
&= \frac{-u'}{\frac{u}{x^2}}
\end{aligned} \tag{1}
$$

Using the above substitution in the given ODE results (after some simplification)in a second order ODE to solve for $u(x)$ which is

$$
f_2 u''(x) - (f_2' + f_1 f_2) u'(x) + f_2^2 f_0 u(x) = 0 \tag{2}
$$

But

$$
\begin{aligned}
f_2' &= -\frac{2}{x^3} \\
f_1 f_2 &= \frac{1}{x^3} \\
f_2^2 f_0 &= \frac{1}{x^4}
\end{aligned}
$$

Substituting the above terms back in equation (2) gives

$$
\frac{u''(x)}{x^2} + \frac{u'(x)}{x^3} + \frac{u(x)}{x^4} = 0
$$

Solving the above ODE (this ode solved using Maple, not this program), gives

$$
u(x) = c_1 \sin(\ln(x)) + c_2 \cos(\ln(x))
$$

The above shows that

$$
u'(x) = \frac{c_1 \cos(\ln(x)) - c_2 \sin(\ln(x))}{x}
$$

Using the above in (1) gives the solution

$$
y = -\frac{x(c_1 \cos(\ln(x)) - c_2 \sin(\ln(x)))}{c_1 \sin(\ln(x)) + c_2 \cos(\ln(x))}
$$

Dividing both numerator and denominator by c_1 gives, after renaming the constant $\frac{c_2}{c_1} = c_3$ the following solution

$$y = \frac{(-c_3 \cos(\ln(x)) + \sin(\ln(x)))\, x}{c_3 \sin(\ln(x)) + \cos(\ln(x))}$$

Summary

The solution(s) found are the following

$$y = \frac{(-c_3 \cos(\ln(x)) + \sin(\ln(x)))\, x}{c_3 \sin(\ln(x)) + \cos(\ln(x))} \tag{1}$$

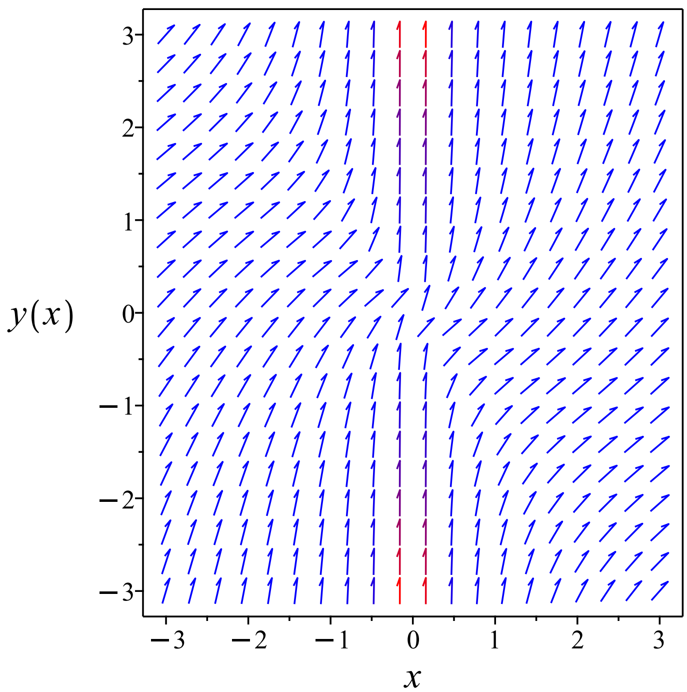


Figure 154: Slope field plot

Verification of solutions

$$y = \frac{(-c_3 \cos(\ln(x)) + \sin(\ln(x)))\, x}{c_3 \sin(\ln(x)) + \cos(\ln(x))}$$

Verified OK.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
trying homogeneous types:
trying homogeneous D
<- homogeneous successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 11

```
dsolve(diff(y(x),x) = (x^2+x*y(x)+y(x)^2)/x^2,y(x), singsol=all)
```

$$y(x) = \tan\left(\ln\left(x\right) + c_1\right) x$$

✓ Solution by Mathematica

Time used: 0.199 (sec). Leaf size: 13

```
DSolve[y'[x] == (x^2+x*y[x]+y[x]^2)/x^2,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to x \tan(\log(x) + c_1)$$

2.31 problem 32

Internal problem ID [509]
Internal file name [`OUTPUT/509_Sunday_June_05_2022_01_42_42_AM_58248688/index.tex`]

Book: Elementary differential equations and boundary value problems, 10th ed., Boyce and DiPrima
Section: Section 2.2. Page 48
Problem number: 32.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"bernoulli", "homogeneousTypeD2", "exactWithIntegrationFactor", "first_order_ode_lie_symmetry_lookup"**

Maple gives the following as the ode type

```
[[_homogeneous, `class A`], _rational, _Bernoulli]
```

$$\boxed{y' - \frac{x^2 + 3y^2}{2xy} = 0}$$

2.31.1 Solving as homogeneousTypeD2 ode

Using the change of variables $y = u(x)\,x$ on the above ode results in new ode in $u(x)$

$$u'(x)\,x + u(x) - \frac{x^2 + 3u(x)^2\,x^2}{2x^2u\,(x)} = 0$$

In canonical form the ODE is

$$\begin{aligned} u' &= F(x,u) \\ &= f(x)g(u) \\ &= \frac{u^2+1}{2xu} \end{aligned}$$

Where $f(x) = \frac{1}{2x}$ and $g(u) = \frac{u^2+1}{u}$. Integrating both sides gives

$$\frac{1}{\frac{u^2+1}{u}}\, du = \frac{1}{2x}\, dx$$

$$\int \frac{1}{\frac{u^2+1}{u}}\, du = \int \frac{1}{2x}\, dx$$

$$\frac{\ln\left(u^2+1\right)}{2} = \frac{\ln\left(x\right)}{2} + c_2$$

Raising both side to exponential gives

$$\sqrt{u^2+1} = \mathrm{e}^{\frac{\ln(x)}{2}+c_2}$$

Which simplifies to

$$\sqrt{u^2+1} = c_3\sqrt{x}$$

Which simplifies to

$$\sqrt{u\left(x\right)^2+1} = c_3\sqrt{x}\,\mathrm{e}^{c_2}$$

The solution is

$$\sqrt{u\left(x\right)^2+1} = c_3\sqrt{x}\,\mathrm{e}^{c_2}$$

Replacing $u(x)$ in the above solution by $\frac{y}{x}$ results in the solution for y in implicit form

$$\sqrt{\frac{y^2}{x^2}+1} = c_3\sqrt{x}\,\mathrm{e}^{c_2}$$

$$\sqrt{\frac{x^2+y^2}{x^2}} = c_3\sqrt{x}\,\mathrm{e}^{c_2}$$

Summary

The solution(s) found are the following

$$\sqrt{\frac{x^2+y^2}{x^2}} = c_3\sqrt{x}\,\mathrm{e}^{c_2} \tag{1}$$

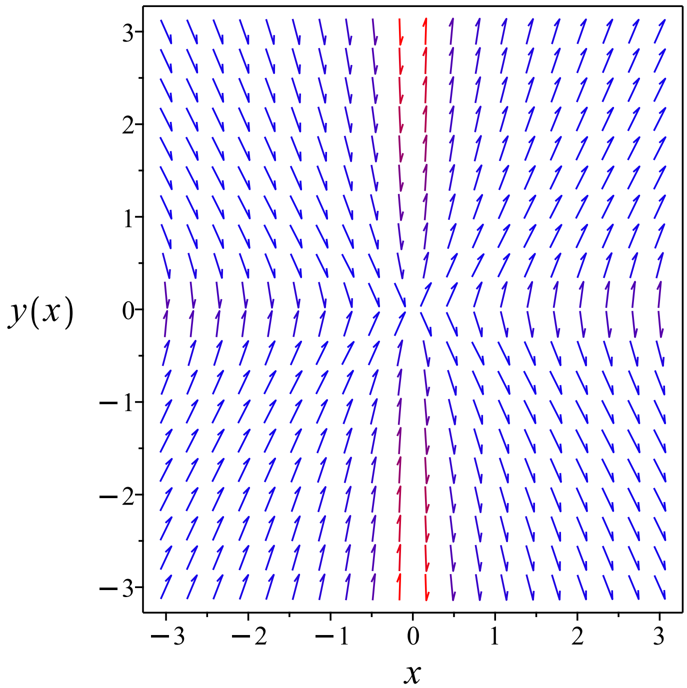


Figure 155: Slope field plot

Verification of solutions

$$\sqrt{\frac{x^2+y^2}{x^2}} = c_3\sqrt{x}\,\mathrm{e}^{c_2}$$

Verified OK.

2.31.2 Solving as first order ode lie symmetry lookup ode

Writing the ode as

$$y' = \frac{x^2+3y^2}{2xy}$$
$$y' = \omega(x,y)$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x + \omega(\eta_y - \xi_x) - \omega^2\xi_y - \omega_x\xi - \omega_y\eta = 0 \tag{A}$$

The type of this ode is known. It is of type `Bernoulli`. Therefore we do not need to solve the PDE (A), and can just use the lookup table shown below to find ξ, η

Table 176: Lie symmetry infinitesimal lookup table for known first order ODE's

ODE class	Form	ξ	η
linear ode	$y' = f(x)y(x) + g(x)$	0	$e^{\int f dx}$
separable ode	$y' = f(x)\, g(y)$	$\frac{1}{f}$	0
quadrature ode	$y' = f(x)$	0	1
quadrature ode	$y' = g(y)$	1	0
homogeneous ODEs of Class A	$y' = f\left(\frac{y}{x}\right)$	x	y
homogeneous ODEs of Class C	$y' = (a + bx + cy)^{\frac{n}{m}}$	1	$-\frac{b}{c}$
homogeneous class D	$y' = \frac{y}{x} + g(x)\, F\left(\frac{y}{x}\right)$	x^2	xy
First order special form ID 1	$y' = g(x)\, e^{h(x)+by} + f(x)$	$\frac{e^{-\int bf(x)dx - h(x)}}{g(x)}$	$\frac{f(x)e^{-\int bf(x)dx - h(x)}}{g(x)}$
polynomial type ode	$y' = \frac{a_1x+b_1y+c_1}{a_2x+b_2y+c_2}$	$\frac{a_1b_2x - a_2b_1x - b_1c_2 + b_2c_1}{a_1b_2 - a_2b_1}$	$\frac{a_1b_2y - a_2b_1y - a_1c_2 - a_2c_1}{a_1b_2 - a_2b_1}$
Bernoulli ode	$y' = f(x)\, y + g(x)\, y^n$	0	$e^{-\int (n-1)f(x)dx} y^n$
Reduced Riccati	$y' = f_1(x)\, y + f_2(x)\, y^2$	0	$e^{-\int f_1 dx}$

The above table shows that

$$
\begin{aligned}
\xi(x, y) &= 0 \\
\eta(x, y) &= \frac{x^3}{y} \qquad \text{(A1)}
\end{aligned}
$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x, y) \to (R, S)$ where (R, S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$
\frac{dx}{\xi} = \frac{dy}{\eta} = dS \qquad (1)
$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y}\right) S(x, y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the

canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$R = x$$

S is found from

$$\begin{aligned} S &= \int \frac{1}{\eta} dy \\ &= \int \frac{1}{\frac{x^3}{y}} dy \end{aligned}$$

Which results in

$$S = \frac{y^2}{2x^3}$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x,y) S_y}{R_x + \omega(x,y) R_y} \tag{2}$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x,y)$ is the right hand side of the original ode given by

$$\omega(x,y) = \frac{x^2 + 3y^2}{2xy}$$

Evaluating all the partial derivatives gives

$$\begin{aligned} R_x &= 1 \\ R_y &= 0 \\ S_x &= -\frac{3y^2}{2x^4} \\ S_y &= \frac{y}{x^3} \end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = \frac{1}{2x^2} \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = \frac{1}{2R^2}$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordinates R, S. Integrating the above gives

$$S(R) = -\frac{1}{2R} + c_1 \tag{4}$$

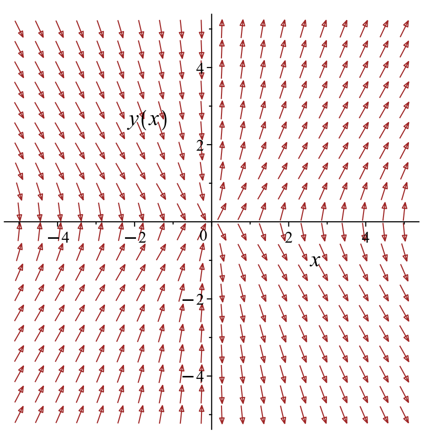
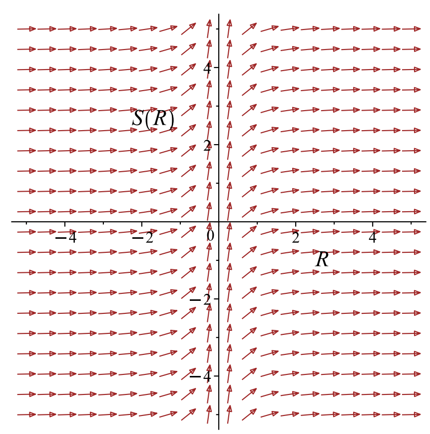
To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$\frac{y^2}{2x^3} = -\frac{1}{2x} + c_1$$

Which simplifies to

$$\frac{y^2}{2x^3} = -\frac{1}{2x} + c_1$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in x, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dx} = \frac{x^2+3y^2}{2xy}$ 	$R = x$ $S = \frac{y^2}{2x^3}$	$\frac{dS}{dR} = \frac{1}{2R^2}$ 

Summary

The solution(s) found are the following

$$\frac{y^2}{2x^3} = -\frac{1}{2x} + c_1 \tag{1}$$

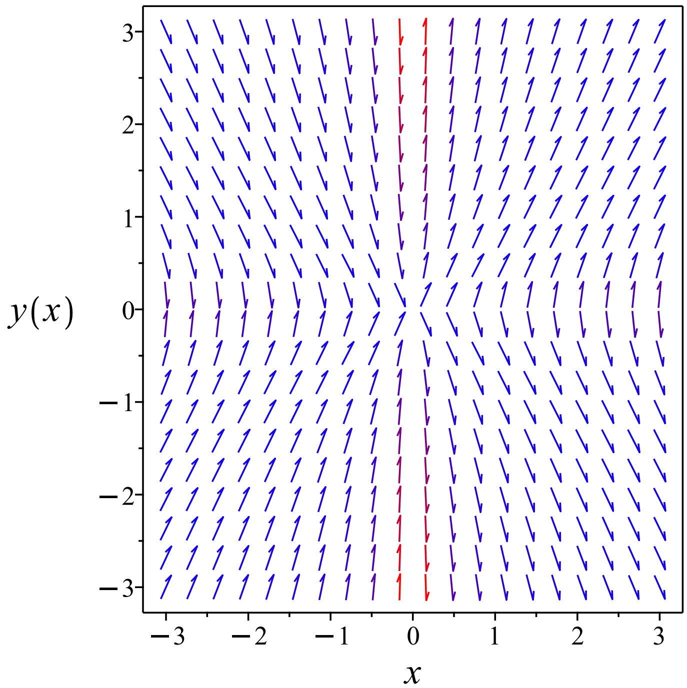


Figure 156: Slope field plot

Verification of solutions

$$\frac{y^2}{2x^3} = -\frac{1}{2x} + c_1$$

Verified OK.

2.31.3 Solving as bernoulli ode

In canonical form, the ODE is

$$\begin{aligned} y' &= F(x, y) \\ &= \frac{x^2 + 3y^2}{2xy} \end{aligned}$$

This is a Bernoulli ODE.

$$y' = \frac{3}{2x}y + \frac{x}{2}\frac{1}{y} \tag{1}$$

The standard Bernoulli ODE has the form

$$y' = f_0(x)y + f_1(x)y^n \tag{2}$$

The first step is to divide the above equation by y^n which gives

$$\frac{y'}{y^n} = f_0(x)y^{1-n} + f_1(x) \tag{3}$$

The next step is use the substitution $w = y^{1-n}$ in equation (3) which generates a new ODE in $w(x)$ which will be linear and can be easily solved using an integrating factor. Backsubstitution then gives the solution $y(x)$ which is what we want.

This method is now applied to the ODE at hand. Comparing the ODE (1) With (2) Shows that

$$
\begin{aligned}
f_0(x) &= \frac{3}{2x} \\
f_1(x) &= \frac{x}{2} \\
n &= -1
\end{aligned}
$$

Dividing both sides of ODE (1) by $y^n = \frac{1}{y}$ gives

$$
y'y = \frac{3y^2}{2x} + \frac{x}{2} \tag{4}
$$

Let

$$
\begin{aligned}
w &= y^{1-n} \\
&= y^2
\end{aligned} \tag{5}
$$

Taking derivative of equation (5) w.r.t x gives

$$
w' = 2yy' \tag{6}
$$

Substituting equations (5) and (6) into equation (4) gives

$$
\begin{aligned}
\frac{w'(x)}{2} &= \frac{3w(x)}{2x} + \frac{x}{2} \\
w' &= \frac{3w}{x} + x
\end{aligned} \tag{7}
$$

The above now is a linear ODE in $w(x)$ which is now solved.

Entering Linear first order ODE solver. In canonical form a linear first order is

$$
w'(x) + p(x)w(x) = q(x)
$$

Where here

$$
\begin{aligned}
p(x) &= -\frac{3}{x} \\
q(x) &= x
\end{aligned}
$$

Hence the ode is

$$w'(x) - \frac{3w(x)}{x} = x$$

The integrating factor μ is

$$\begin{aligned}\mu &= e^{\int -\frac{3}{x}dx} \\ &= \frac{1}{x^3}\end{aligned}$$

The ode becomes

$$\begin{aligned}\frac{d}{dx}(\mu w) &= (\mu)(x) \\ \frac{d}{dx}\left(\frac{w}{x^3}\right) &= \left(\frac{1}{x^3}\right)(x) \\ d\left(\frac{w}{x^3}\right) &= \frac{1}{x^2}\, dx\end{aligned}$$

Integrating gives

$$\begin{aligned}\frac{w}{x^3} &= \int \frac{1}{x^2}\, dx \\ \frac{w}{x^3} &= -\frac{1}{x} + c_1\end{aligned}$$

Dividing both sides by the integrating factor $\mu = \frac{1}{x^3}$ results in

$$w(x) = c_1 x^3 - x^2$$

Replacing w in the above by y^2 using equation (5) gives the final solution.

$$y^2 = c_1 x^3 - x^2$$

Solving for y gives

$$\begin{aligned}y(x) &= \sqrt{c_1 x - 1}\, x \\ y(x) &= -\sqrt{c_1 x - 1}\, x\end{aligned}$$

<u>Summary</u>

The solution(s) found are the following

$$y = \sqrt{c_1 x - 1}\, x \tag{1}$$

$$y = -\sqrt{c_1 x - 1}\, x \tag{2}$$

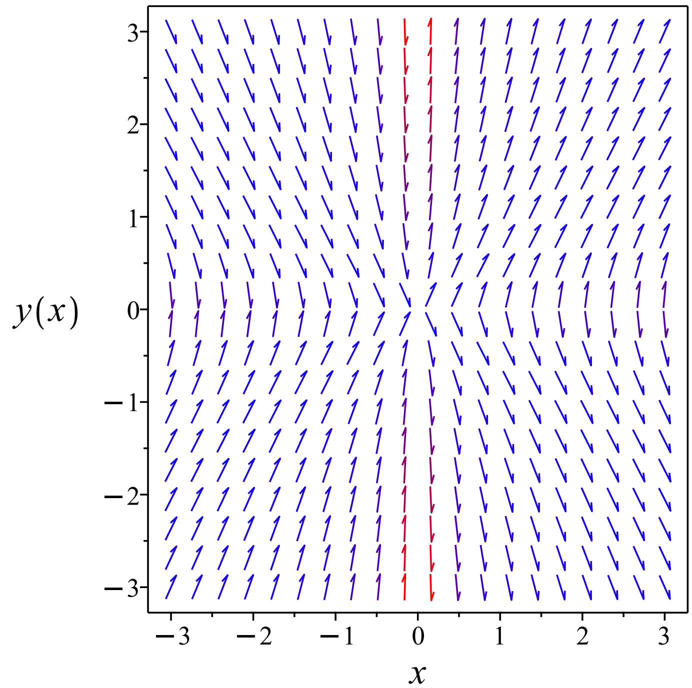


Figure 157: Slope field plot

Verification of solutions

$$y = \sqrt{c_1 x - 1}\, x$$

Verified OK.

$$y = -\sqrt{c_1 x - 1}\, x$$

Verified OK.

2.31.4 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x, y) + N(x, y) \frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x, y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x, y) = 0$$

Hence

$$\frac{\partial \phi}{\partial x} + \frac{\partial \phi}{\partial y}\frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$\begin{aligned}\frac{\partial \phi}{\partial x} &= M \\ \frac{\partial \phi}{\partial y} &= N\end{aligned}$$

But since $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x,y)$ but at least we know now that we can do that since the condition $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x,y)\,\mathrm{d}x + N(x,y)\,\mathrm{d}y = 0 \tag{1A}$$

Therefore

$$\begin{aligned}(2yx)\,\mathrm{d}y &= \left(x^2 + 3y^2\right)\mathrm{d}x \\ \left(-x^2 - 3y^2\right)\mathrm{d}x + (2yx)\,\mathrm{d}y &= 0 \qquad \text{(2A)}\end{aligned}$$

Comparing (1A) and (2A) shows that

$$\begin{aligned}M(x,y) &= -x^2 - 3y^2 \\ N(x,y) &= 2yx\end{aligned}$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Using result found above gives

$$\begin{aligned}\frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}\left(-x^2 - 3y^2\right) \\ &= -6y\end{aligned}$$

And

$$\begin{aligned}\frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}(2yx)\\ &= 2y\end{aligned}$$

Since $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$, then the ODE is not exact. Since the ODE is not exact, we will try to find an integrating factor to make it exact. Let

$$\begin{aligned}A &= \frac{1}{N}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right)\\ &= \frac{1}{2yx}((-6y) - (2y))\\ &= -\frac{4}{x}\end{aligned}$$

Since A does not depend on y, then it can be used to find an integrating factor. The integrating factor μ is

$$\begin{aligned}\mu &= e^{\int A\,\mathrm{d}x}\\ &= e^{\int -\frac{4}{x}\,\mathrm{d}x}\end{aligned}$$

The result of integrating gives

$$\begin{aligned}\mu &= e^{-4\ln(x)}\\ &= \frac{1}{x^4}\end{aligned}$$

M and N are multiplied by this integrating factor, giving new M and new N which are called $\overline{M}$ and $\overline{N}$ for now so not to confuse them with the original M and N.

$$\begin{aligned}\overline{M} &= \mu M\\ &= \frac{1}{x^4}(-x^2 - 3y^2)\\ &= \frac{-x^2 - 3y^2}{x^4}\end{aligned}$$

And

$$\begin{aligned}\overline{N} &= \mu N\\ &= \frac{1}{x^4}(2yx)\\ &= \frac{2y}{x^3}\end{aligned}$$

Now a modified ODE is ontained from the original ODE, which is exact and can be solved. The modified ODE is

$$\overline{M} + \overline{N}\frac{\mathrm{d}y}{\mathrm{d}x} = 0$$
$$\left(\frac{-x^2 - 3y^2}{x^4}\right) + \left(\frac{2y}{x^3}\right)\frac{\mathrm{d}y}{\mathrm{d}x} = 0$$

The following equations are now set up to solve for the function $\phi(x, y)$

$$\frac{\partial \phi}{\partial x} = \overline{M} \tag{1}$$
$$\frac{\partial \phi}{\partial y} = \overline{N} \tag{2}$$

Integrating (1) w.r.t. x gives

$$\int \frac{\partial \phi}{\partial x}\,\mathrm{d}x = \int \overline{M}\,\mathrm{d}x$$
$$\int \frac{\partial \phi}{\partial x}\,\mathrm{d}x = \int \frac{-x^2 - 3y^2}{x^4}\,\mathrm{d}x$$
$$\phi = \frac{x^2 + y^2}{x^3} + f(y) \tag{3}$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$\frac{\partial \phi}{\partial y} = \frac{2y}{x^3} + f'(y) \tag{4}$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = \frac{2y}{x^3}$. Therefore equation (4) becomes

$$\frac{2y}{x^3} = \frac{2y}{x^3} + f'(y) \tag{5}$$

Solving equation (5) for $f'(y)$ gives

$$f'(y) = 0$$

Therefore

$$f(y) = c_1$$

Where c_1 is constant of integration. Substituting this result for $f(y)$ into equation (3) gives ϕ

$$\phi = \frac{x^2 + y^2}{x^3} + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = \frac{x^2 + y^2}{x^3}$$

Summary

The solution(s) found are the following

$$\frac{x^2 + y^2}{x^3} = c_1 \tag{1}$$

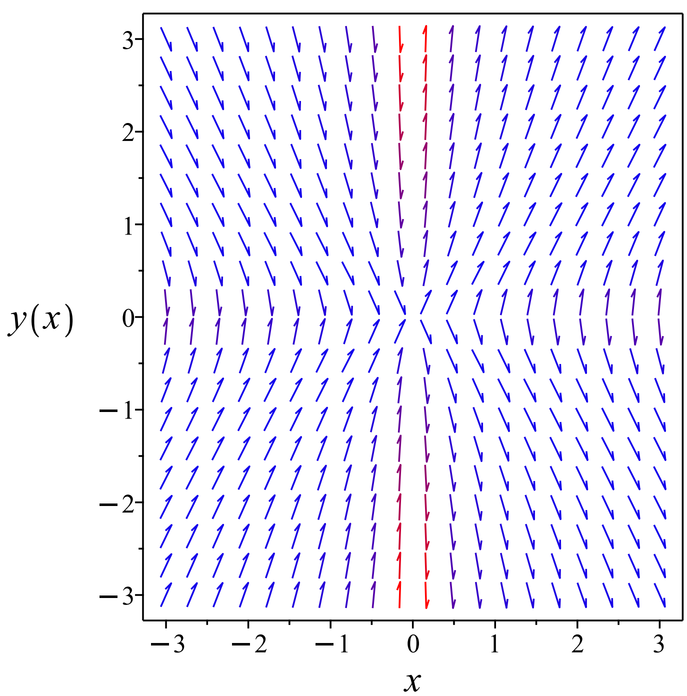


Figure 158: Slope field plot

Verification of solutions

$$\frac{x^2 + y^2}{x^3} = c_1$$

Verified OK.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
<- Bernoulli successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 26

```
dsolve(diff(y(x),x) = (x^2+3*y(x)^2)/(2*x*y(x)),y(x), singsol=all)
```

$$y(x) = \sqrt{c_1 x - 1}\, x$$
$$y(x) = -\sqrt{c_1 x - 1}\, x$$

✓ Solution by Mathematica

Time used: 0.251 (sec). Leaf size: 34

```
DSolve[y'[x] == (x^2+3*y[x]^2)/(2*x*y[x]),y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to -x\sqrt{-1 + c_1 x}$$
$$y(x) \to x\sqrt{-1 + c_1 x}$$

2.32 problem 33

Internal problem ID [510]
Internal file name [`OUTPUT/510_Sunday_June_05_2022_01_42_44_AM_10559292/index.tex`]

Book: Elementary differential equations and boundary value problems, 10th ed., Boyce and DiPrima
Section: Section 2.2. Page 48
Problem number: 33.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"homogeneousTypeD2", "first_order_ode_lie_symmetry_calculated"**

Maple gives the following as the ode type

```
[[_homogeneous, `class A`], _rational, [_Abel, `2nd type`, `
   class A`]]
```

$$\boxed{y' - \frac{4y - 3x}{2x - y} = 0}$$

2.32.1 Solving as homogeneousTypeD2 ode

Using the change of variables $y = u(x)\,x$ on the above ode results in new ode in $u(x)$

$$u'(x)\,x + u(x) - \frac{4u(x)\,x - 3x}{2x - u(x)\,x} = 0$$

In canonical form the ODE is

$$\begin{aligned}
u' &= F(x, u) \\
&= f(x)g(u) \\
&= -\frac{u^2 + 2u - 3}{x\,(u - 2)}
\end{aligned}$$

Where $f(x) = -\frac{1}{x}$ and $g(u) = \frac{u^2+2u-3}{u-2}$. Integrating both sides gives

$$\frac{1}{\frac{u^2+2u-3}{u-2}}\,du = -\frac{1}{x}\,dx$$
$$\int \frac{1}{\frac{u^2+2u-3}{u-2}}\,du = \int -\frac{1}{x}\,dx$$
$$-\frac{\ln(u-1)}{4} + \frac{5\ln(u+3)}{4} = -\ln(x) + c_2$$

The above can be written as

$$\frac{-\ln(u-1) + 5\ln(u+3)}{4} = -\ln(x) + c_2$$
$$-\ln(u-1) + 5\ln(u+3) = (4)(-\ln(x) + c_2)$$
$$= -4\ln(x) + 4c_2$$

Raising both side to exponential gives

$$e^{-\ln(u-1)+5\ln(u+3)} = e^{-4\ln(x)+4c_2}$$

Which simplifies to

$$\frac{(u+3)^5}{u-1} = \frac{4c_2}{x^4}$$
$$= \frac{c_3}{x^4}$$

Which simplifies to

$$u(x) = \operatorname{RootOf}\left(\_Z^5 + 15\_Z^4 + 90\_Z^3 + 270\_Z^2 + \left(-\frac{c_3 e^{4c_2}}{x^4} + 405\right)\_Z + \frac{c_3 e^{4c_2}}{x^4} + 243\right)$$

Therefore the solution y is

$$y = ux$$
$$= x \operatorname{RootOf}\left(\_Z^5 x^4 + 15x^4\_Z^4 + 90\_Z^3 x^4 + 270x^4\_Z^2 + \left(-c_3 e^{4c_2} + 405x^4\right)\_Z + c_3 e^{4c_2} + 243x^4\right)$$

Summary

The solution(s) found are the following

$$y = x \operatorname{RootOf}\left(\_Z^5 x^4 + 15x^4\_Z^4 + 90\_Z^3 x^4 + 270x^4\_Z^2 + \left(-c_3 e^{4c_2} + 405x^4\right)\_Z + c_3 e^{4c_2} + 243x^4\right) \tag{1}$$

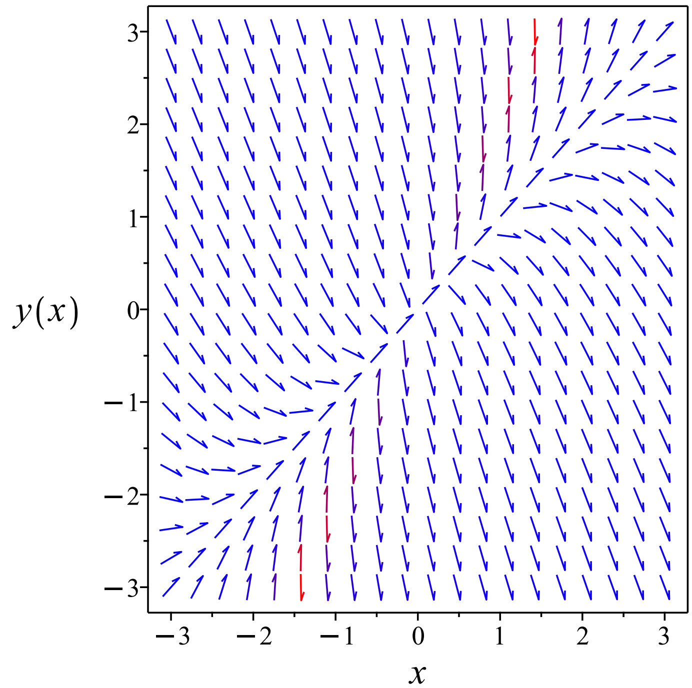


Figure 159: Slope field plot

Verification of solutions

$$y = x \operatorname{RootOf}\left(\_Z^5x^4 + 15x^4\_Z^4 + 90\_Z^3x^4 + 270x^4\_Z^2 + \left(-c_3 e^{4c_2} + 405x^4\right)\_Z + c_3 e^{4c_2} + 243x^4\right)$$

Verified OK.

2.32.2 Solving as first order ode lie symmetry calculated ode

Writing the ode as

$$y' = -\frac{4y - 3x}{-2x + y}$$
$$y' = \omega(x, y)$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x + \omega(\eta_y - \xi_x) - \omega^2\xi_y - \omega_x\xi - \omega_y\eta = 0 \tag{A}$$

The type of this ode is not in the lookup table. To determine ξ, η then (A) is solved using ansatz. Making bivariate polynomials of degree 1 to use as anstaz gives

$$\xi = xa_2 + ya_3 + a_1 \tag{1E}$$
$$\eta = xb_2 + yb_3 + b_1 \tag{2E}$$

Where the unknown coefficients are

$$\{a_1, a_2, a_3, b_1, b_2, b_3\}$$

Substituting equations (1E,2E) and ω into (A) gives

$$\begin{aligned} & b_2 - \frac{(4y-3x)(b_3-a_2)}{-2x+y} - \frac{(4y-3x)^2 a_3}{(-2x+y)^2} \\ & - \left(\frac{3}{-2x+y} - \frac{2(4y-3x)}{(-2x+y)^2}\right)(xa_2 + ya_3 + a_1) \\ & - \left(-\frac{4}{-2x+y} + \frac{4y-3x}{(-2x+y)^2}\right)(xb_2 + yb_3 + b_1) = 0 \end{aligned} \tag{5E}$$

Putting the above in normal form gives

$$\frac{6x^2a_2 - 9x^2a_3 - x^2b_2 - 6x^2b_3 - 6xya_2 + 24xya_3 - 4xyb_2 + 6xyb_3 + 4y^2a_2 - 11y^2a_3 + y^2b_2 - 4y^2b_3 - 5xb}{(2x-y)^2} = 0$$

Setting the numerator to zero gives

$$\begin{aligned} & 6x^2a_2 - 9x^2a_3 - x^2b_2 - 6x^2b_3 - 6xya_2 + 24xya_3 - 4xyb_2 \\ & + 6xyb_3 + 4y^2a_2 - 11y^2a_3 + y^2b_2 - 4y^2b_3 - 5xb_1 + 5ya_1 = 0 \end{aligned} \tag{6E}$$

Looking at the above PDE shows the following are all the terms with $\{x, y\}$ in them.

$$\{x, y\}$$

The following substitution is now made to be able to collect on all terms with $\{x, y\}$ in them

$$\{x = v_1, y = v_2\}$$

The above PDE (6E) now becomes

$$\begin{aligned} & 6a_2v_1^2 - 6a_2v_1v_2 + 4a_2v_2^2 - 9a_3v_1^2 + 24a_3v_1v_2 - 11a_3v_2^2 - b_2v_1^2 \\ & - 4b_2v_1v_2 + b_2v_2^2 - 6b_3v_1^2 + 6b_3v_1v_2 - 4b_3v_2^2 + 5a_1v_2 - 5b_1v_1 = 0 \end{aligned} \tag{7E}$$

Collecting the above on the terms v_i introduced, and these are

$$\{v_1, v_2\}$$

Equation (7E) now becomes

$$(6a_2 - 9a_3 - b_2 - 6b_3) v_1^2 + (-6a_2 + 24a_3 - 4b_2 + 6b_3) v_1 v_2 \\ - 5b_1 v_1 + (4a_2 - 11a_3 + b_2 - 4b_3) v_2^2 + 5a_1 v_2 = 0 \tag{8E}$$

Setting each coefficients in (8E) to zero gives the following equations to solve

$$\begin{aligned} 5a_1 &= 0 \\ -5b_1 &= 0 \\ -6a_2 + 24a_3 - 4b_2 + 6b_3 &= 0 \\ 4a_2 - 11a_3 + b_2 - 4b_3 &= 0 \\ 6a_2 - 9a_3 - b_2 - 6b_3 &= 0 \end{aligned}$$

Solving the above equations for the unknowns gives

$$\begin{aligned} a_1 &= 0 \\ a_2 &= 2a_3 + b_3 \\ a_3 &= a_3 \\ b_1 &= 0 \\ b_2 &= 3a_3 \\ b_3 &= b_3 \end{aligned}$$

Substituting the above solution in the anstaz (1E,2E) (using 1 as arbitrary value for any unknown in the RHS) gives

$$\begin{aligned} \xi &= x \\ \eta &= y \end{aligned}$$

Shifting is now applied to make $\xi = 0$ in order to simplify the rest of the computation

$$\begin{aligned} \eta &= \eta - \omega(x, y) \xi \\ &= y - \left(-\frac{4y - 3x}{-2x + y}\right)(x) \\ &= \frac{3x^2 - 2yx - y^2}{2x - y} \\ \xi &= 0 \end{aligned}$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x, y) \to (R, S)$ where (R, S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y}\right) S(x, y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$R = x$$

S is found from

$$\begin{aligned} S &= \int \frac{1}{\eta} dy \\ &= \int \frac{1}{\frac{3x^2 - 2yx - y^2}{2x - y}} dy \end{aligned}$$

Which results in

$$S = \frac{5 \ln (3x + y)}{4} - \frac{\ln (-x + y)}{4}$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x, y) S_y}{R_x + \omega(x, y) R_y} \tag{2}$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x, y)$ is the right hand side of the original ode given by

$$\omega(x, y) = -\frac{4y - 3x}{-2x + y}$$

Evaluating all the partial derivatives gives

$$\begin{aligned} R_x &= 1 \\ R_y &= 0 \\ S_x &= \frac{-4y + 3x}{(3x + y)(x - y)} \\ S_y &= \frac{2x - y}{(3x + y)(x - y)} \end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = 0 \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = 0$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$S(R) = c_1 \tag{4}$$

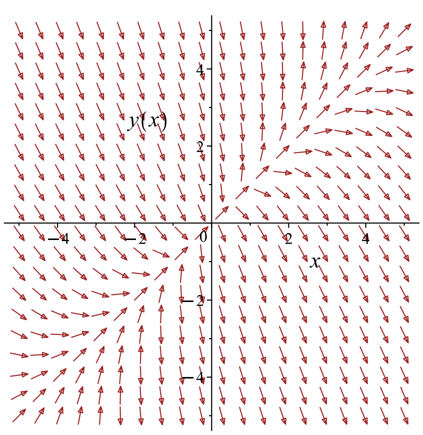
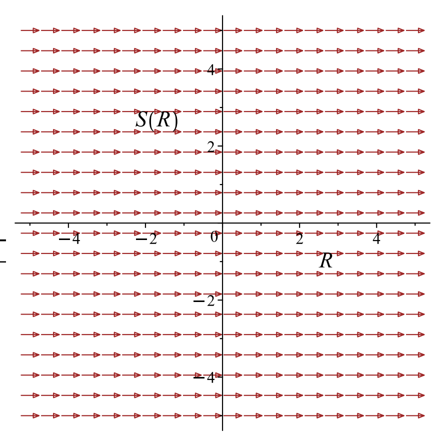
To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$\frac{5\ln(3x+y)}{4} - \frac{\ln(-x+y)}{4} = c_1$$

Which simplifies to

$$\frac{5\ln(3x+y)}{4} - \frac{\ln(-x+y)}{4} = c_1$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in x, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dx} = -\frac{4y-3x}{-2x+y}$ 	$R = x$ $S = \frac{5\ln(3x+y)}{4} - \ln(-$	$\frac{dS}{dR} = 0$ 

Summary

The solution(s) found are the following

$$\frac{5 \ln(3x+y)}{4} - \frac{\ln(-x+y)}{4} = c_1 \tag{1}$$

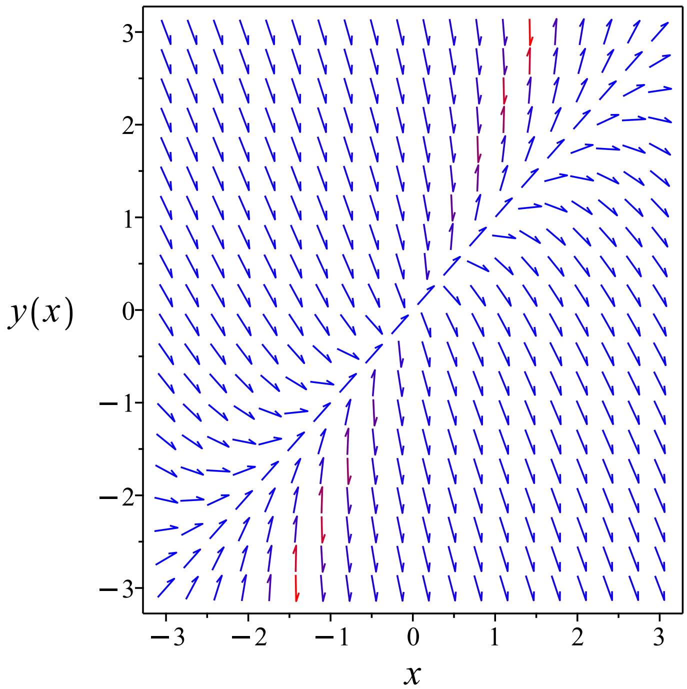


Figure 160: Slope field plot

Verification of solutions

$$\frac{5 \ln(3x+y)}{4} - \frac{\ln(-x+y)}{4} = c_1$$

Verified OK.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
trying homogeneous types:
trying homogeneous D
<- homogeneous successful`
```

✓ Solution by Maple

Time used: 0.375 (sec). Leaf size: 26

```
dsolve(diff(y(x),x) = (4*y(x)-3*x)/(2*x-y(x)),y(x), singsol=all)
```

$$y(x) = x\left(-3 + \text{RootOf}\left(\_Z^{20}c_1x^4 - \_Z^4 + 4\right)^4\right)$$

✓ Solution by Mathematica

Time used: 3.328 (sec). Leaf size: 336

```
DSolve[y'[x] == (4*y[x]-3*x)/(2*x-y[x]),y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \text{Root}\left[\#1^5 + 15\#1^4x + 90\#1^3x^2 + 270\#1^2x^3 + \#1\left(405x^4 - e^{4c_1}\right) + 243x^5 + e^{4c_1}x\&, 1\right]$$
$$y(x) \to \text{Root}\left[\#1^5 + 15\#1^4x + 90\#1^3x^2 + 270\#1^2x^3 + \#1\left(405x^4 - e^{4c_1}\right) + 243x^5 + e^{4c_1}x\&, 2\right]$$
$$y(x) \to \text{Root}\left[\#1^5 + 15\#1^4x + 90\#1^3x^2 + 270\#1^2x^3 + \#1\left(405x^4 - e^{4c_1}\right) + 243x^5 + e^{4c_1}x\&, 3\right]$$
$$y(x) \to \text{Root}\left[\#1^5 + 15\#1^4x + 90\#1^3x^2 + 270\#1^2x^3 + \#1\left(405x^4 - e^{4c_1}\right) + 243x^5 + e^{4c_1}x\&, 4\right]$$
$$y(x) \to \text{Root}\left[\#1^5 + 15\#1^4x + 90\#1^3x^2 + 270\#1^2x^3 + \#1\left(405x^4 - e^{4c_1}\right) + 243x^5 + e^{4c_1}x\&, 5\right]$$

2.33 problem 34

Internal problem ID [511]
Internal file name [`OUTPUT/511_Sunday_June_05_2022_01_42_45_AM_86261302/index.tex`]

Book: Elementary differential equations and boundary value problems, 10th ed., Boyce and DiPrima
Section: Section 2.2. Page 48
Problem number: 34.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"homogeneousTypeD2", "first_order_ode_lie_symmetry_calculated"**

Maple gives the following as the ode type

```
[[_homogeneous, `class A`], _rational, [_Abel, `2nd type`, `
   class A`]]
```

$$\boxed{y' + \frac{4x+3y}{2x+y} = 0}$$

2.33.1 Solving as homogeneousTypeD2 ode

Using the change of variables $y = u(x)\,x$ on the above ode results in new ode in $u(x)$

$$u'(x)\,x + u(x) + \frac{4x+3u(x)\,x}{2x+u(x)\,x} = 0$$

In canonical form the ODE is

$$\begin{aligned} u' &= F(x,u) \\ &= f(x)g(u) \\ &= -\frac{u^2+5u+4}{x\,(u+2)} \end{aligned}$$

Where $f(x) = -\frac{1}{x}$ and $g(u) = \frac{u^2+5u+4}{u+2}$. Integrating both sides gives

$$
\begin{aligned}
\frac{1}{\frac{u^2+5u+4}{u+2}}\,du &= -\frac{1}{x}\,dx \\
\int \frac{1}{\frac{u^2+5u+4}{u+2}}\,du &= \int -\frac{1}{x}\,dx \\
\frac{2\ln(u+4)}{3} + \frac{\ln(u+1)}{3} &= -\ln(x) + c_2
\end{aligned}
$$

The above can be written as

$$
\begin{aligned}
\frac{2\ln(u+4) + \ln(u+1)}{3} &= -\ln(x) + c_2 \\
2\ln(u+4) + \ln(u+1) &= (3)(-\ln(x) + c_2) \\
&= -3\ln(x) + 3c_2
\end{aligned}
$$

Raising both side to exponential gives

$$
e^{2\ln(u+4)+\ln(u+1)} = e^{-3\ln(x)+3c_2}
$$

Which simplifies to

$$
\begin{aligned}
(u+4)^2(u+1) &= \frac{3c_2}{x^3} \\
&= \frac{c_3}{x^3}
\end{aligned}
$$

Which simplifies to

$$
(u(x)+4)^2(u(x)+1) = \frac{c_3 e^{3c_2}}{x^3}
$$

The solution is

$$
(u(x)+4)^2(u(x)+1) = \frac{c_3 e^{3c_2}}{x^3}
$$

Replacing $u(x)$ in the above solution by $\frac{y}{x}$ results in the solution for y in implicit form

$$
\begin{aligned}
\left(\frac{y}{x}+4\right)^2\left(1+\frac{y}{x}\right) &= \frac{c_3 e^{3c_2}}{x^3} \\
\frac{(4x+y)^2(x+y)}{x^3} &= \frac{c_3 e^{3c_2}}{x^3}
\end{aligned}
$$

Which simplifies to

$$(4x+y)^2 (x+y) = c_3 e^{3c_2}$$

Summary

The solution(s) found are the following

$$(4x+y)^2 (x+y) = c_3 e^{3c_2} \tag{1}$$

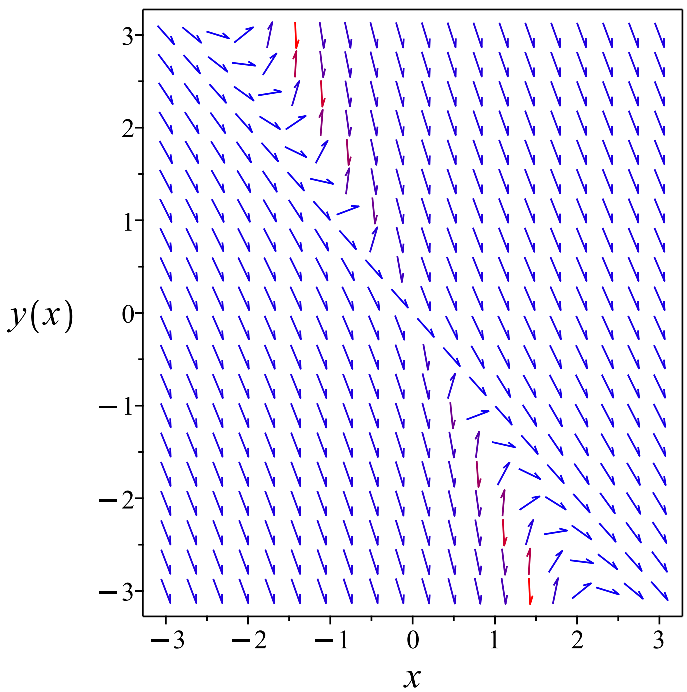


Figure 161: Slope field plot

Verification of solutions

$$(4x+y)^2 (x+y) = c_3 e^{3c_2}$$

Verified OK.

2.33.2 Solving as first order ode lie symmetry calculated ode

Writing the ode as

$$
\begin{aligned}
y' &= -\frac{4x+3y}{2x+y} \\
y' &= \omega(x,y)
\end{aligned}
$$

The condition of Lie symmetry is the linearized PDE given by

$$
\eta_x + \omega(\eta_y - \xi_x) - \omega^2 \xi_y - \omega_x \xi - \omega_y \eta = 0 \tag{A}
$$

The type of this ode is not in the lookup table. To determine ξ, η then (A) is solved using ansatz. Making bivariate polynomials of degree 1 to use as anstaz gives

$$
\xi = x a_2 + y a_3 + a_1 \tag{1E}
$$

$$
\eta = x b_2 + y b_3 + b_1 \tag{2E}
$$

Where the unknown coefficients are

$$
\{a_1, a_2, a_3, b_1, b_2, b_3\}
$$

Substituting equations (1E,2E) and ω into (A) gives

$$
\begin{aligned}
&b_2 - \frac{(4x+3y)(b_3-a_2)}{2x+y} - \frac{(4x+3y)^2 a_3}{(2x+y)^2} \\
&- \left(-\frac{4}{2x+y} + \frac{8x+6y}{(2x+y)^2}\right)(x a_2 + y a_3 + a_1) \\
&- \left(-\frac{3}{2x+y} + \frac{4x+3y}{(2x+y)^2}\right)(x b_2 + y b_3 + b_1) = 0
\end{aligned} \tag{5E}
$$

Putting the above in normal form gives

$$
\frac{8x^2 a_2 - 16x^2 a_3 + 6x^2 b_2 - 8x^2 b_3 + 8xy a_2 - 24xy a_3 + 4xy b_2 - 8xy b_3 + 3y^2 a_2 - 11y^2 a_3 + y^2 b_2 - 3y^2 b_3 + 2}{(2x+y)^2}
$$

$$
= 0
$$

Setting the numerator to zero gives

$$
\begin{aligned}
&8x^2 a_2 - 16x^2 a_3 + 6x^2 b_2 - 8x^2 b_3 + 8xy a_2 - 24xy a_3 + 4xy b_2 \\
&- 8xy b_3 + 3y^2 a_2 - 11y^2 a_3 + y^2 b_2 - 3y^2 b_3 + 2x b_1 - 2y a_1 = 0
\end{aligned} \tag{6E}
$$

Looking at the above PDE shows the following are all the terms with $\{x, y\}$ in them.

$$\{x, y\}$$

The following substitution is now made to be able to collect on all terms with $\{x, y\}$ in them

$$\{x = v_1, y = v_2\}$$

The above PDE (6E) now becomes

$$\begin{aligned} &8a_2v_1^2 + 8a_2v_1v_2 + 3a_2v_2^2 - 16a_3v_1^2 - 24a_3v_1v_2 - 11a_3v_2^2 + 6b_2v_1^2 \\ &+ 4b_2v_1v_2 + b_2v_2^2 - 8b_3v_1^2 - 8b_3v_1v_2 - 3b_3v_2^2 - 2a_1v_2 + 2b_1v_1 = 0 \end{aligned} \tag{7E}$$

Collecting the above on the terms v_i introduced, and these are

$$\{v_1, v_2\}$$

Equation (7E) now becomes

$$\begin{aligned} &(8a_2 - 16a_3 + 6b_2 - 8b_3)\, v_1^2 + (8a_2 - 24a_3 + 4b_2 - 8b_3)\, v_1v_2 \\ &+ 2b_1v_1 + (3a_2 - 11a_3 + b_2 - 3b_3)\, v_2^2 - 2a_1v_2 = 0 \end{aligned} \tag{8E}$$

Setting each coefficients in (8E) to zero gives the following equations to solve

$$\begin{aligned} -2a_1 &= 0 \\ 2b_1 &= 0 \\ 3a_2 - 11a_3 + b_2 - 3b_3 &= 0 \\ 8a_2 - 24a_3 + 4b_2 - 8b_3 &= 0 \\ 8a_2 - 16a_3 + 6b_2 - 8b_3 &= 0 \end{aligned}$$

Solving the above equations for the unknowns gives

$$\begin{aligned} a_1 &= 0 \\ a_2 &= 5a_3 + b_3 \\ a_3 &= a_3 \\ b_1 &= 0 \\ b_2 &= -4a_3 \\ b_3 &= b_3 \end{aligned}$$

Substituting the above solution in the anstaz (1E,2E) (using 1 as arbitrary value for any unknown in the RHS) gives

$$\xi = x$$
$$\eta = y$$

Shifting is now applied to make $\xi = 0$ in order to simplify the rest of the computation

$$\begin{aligned}\eta &= \eta - \omega(x, y)\,\xi \\ &= y - \left(-\frac{4x + 3y}{2x + y}\right)(x) \\ &= \frac{4x^2 + 5yx + y^2}{2x + y} \\ \xi &= 0\end{aligned}$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x, y) \to (R, S)$ where (R, S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}$$

The above comes from the requirements that $\left(\xi\frac{\partial}{\partial x} + \eta\frac{\partial}{\partial y}\right) S(x, y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$R = x$$

S is found from

$$\begin{aligned}S &= \int \frac{1}{\eta} dy \\ &= \int \frac{1}{\frac{4x^2+5yx+y^2}{2x+y}} dy\end{aligned}$$

Which results in

$$S = \frac{2\ln(4x + y)}{3} + \frac{\ln(x + y)}{3}$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x, y)S_y}{R_x + \omega(x, y)R_y} \tag{2}$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x, y)$ is the right hand side of the original ode given by

$$\omega(x, y) = -\frac{4x + 3y}{2x + y}$$

Evaluating all the partial derivatives gives

$$\begin{aligned} R_x &= 1 \\ R_y &= 0 \\ S_x &= \frac{4x + 3y}{(x + y)(4x + y)} \\ S_y &= \frac{2x + y}{(x + y)(4x + y)} \end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = 0 \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = 0$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$S(R) = c_1 \tag{4}$$

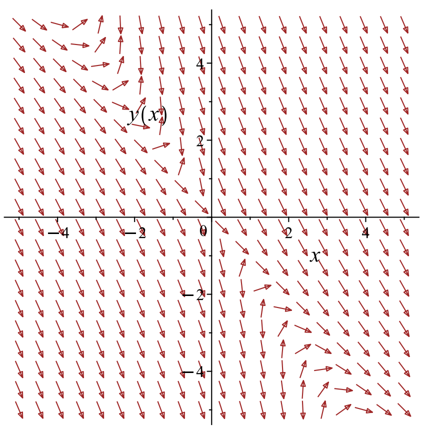
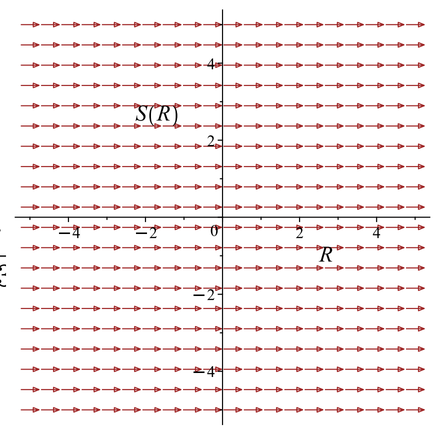
To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$\frac{2\ln(4x + y)}{3} + \frac{\ln(x + y)}{3} = c_1$$

Which simplifies to

$$\frac{2\ln(4x + y)}{3} + \frac{\ln(x + y)}{3} = c_1$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in x, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dx} = -\frac{4x+3y}{2x+y}$ 	$R = x$ $S = \frac{2\ln(4x+y)}{3} + \frac{\ln(x}{3}$	$\frac{dS}{dR} = 0$ 

Summary

The solution(s) found are the following

$$\frac{2\ln(4x+y)}{3} + \frac{\ln(x+y)}{3} = c_1 \tag{1}$$

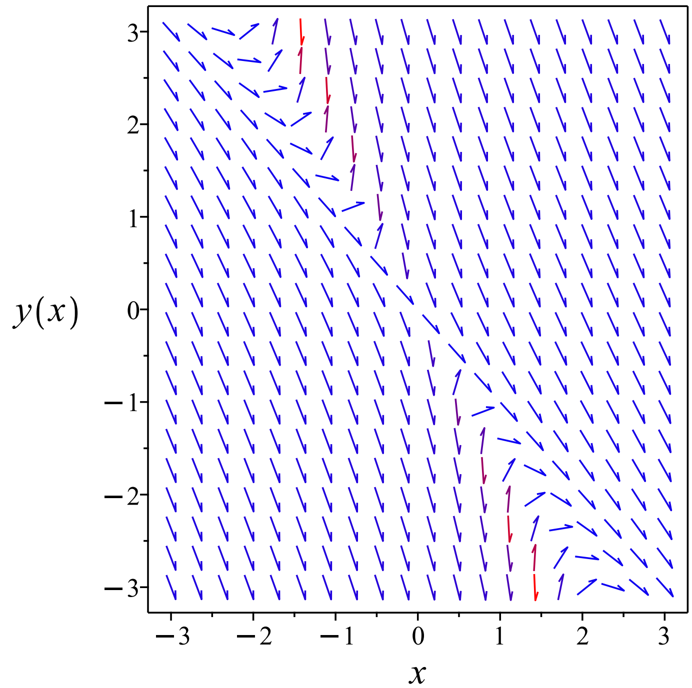


Figure 162: Slope field plot

Verification of solutions

$$\frac{2\ln(4x+y)}{3}+\frac{\ln(x+y)}{3}=c_1$$

Verified OK.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
trying homogeneous types:
trying homogeneous D
<- homogeneous successful`
```

✓ Solution by Maple

Time used: 0.125 (sec). Leaf size: 1228

```
dsolve(diff(y(x),x) = - (4*x+3*y(x))/(2*x+y(x)),y(x), singsol=all)
```

$$y(x) = \frac{\frac{\left(-4c_1x^3+\left(4c_1x^3+4\sqrt{x^6c_1^2(4c_1x^3+1)}\right)^{\frac{2}{3}}\right)^2}{4\left(4c_1x^3+4\sqrt{x^6c_1^2(4c_1x^3+1)}\right)^{\frac{2}{3}}c_1} - x^3}{x^2}$$

$$y(x) = \frac{-3x^3\left(4c_1x^3+4\sqrt{4c_1^3x^9+x^6c_1^2}\right)^{\frac{2}{3}}c_1 + \left(c_1x^3+\sqrt{4c_1^3x^9+x^6c_1^2}\right)\left(4c_1x^3+4\sqrt{4c_1^3x^9+x^6c_1^2}\right)^{\frac{1}{3}} + 4x^6c_1^2}{\left(4c_1x^3+4\sqrt{4c_1^3x^9+x^6c_1^2}\right)^{\frac{2}{3}}c_1x^2}$$

$$y(x) = \frac{-3x^3\left(4c_1x^3+4\sqrt{4c_1^3x^9+x^6c_1^2}\right)^{\frac{2}{3}}c_1 + \left(c_1x^3+\sqrt{4c_1^3x^9+x^6c_1^2}\right)\left(4c_1x^3+4\sqrt{4c_1^3x^9+x^6c_1^2}\right)^{\frac{1}{3}} + 4x^6c_1^2}{\left(4c_1x^3+4\sqrt{4c_1^3x^9+x^6c_1^2}\right)^{\frac{2}{3}}c_1x^2}$$

$$y(x) = -\frac{\frac{\left(4\sqrt{3}\,c_1x^3+\sqrt{3}\left(4c_1x^3+4\sqrt{x^6c_1^2(4c_1x^3+1)}\right)^{\frac{2}{3}}+4ic_1x^3-i\left(4c_1x^3+4\sqrt{x^6c_1^2(4c_1x^3+1)}\right)^{\frac{2}{3}}\right)^2}{16\left(4c_1x^3+4\sqrt{x^6c_1^2(4c_1x^3+1)}\right)^{\frac{2}{3}}c_1} + x^3}{x^2}$$

$$y(x) = -\frac{\frac{\left(4\sqrt{3}\,c_1x^3+\sqrt{3}\left(4c_1x^3+4\sqrt{x^6c_1^2(4c_1x^3+1)}\right)^{\frac{2}{3}}-4ic_1x^3+i\left(4c_1x^3+4\sqrt{x^6c_1^2(4c_1x^3+1)}\right)^{\frac{2}{3}}\right)^2}{16\left(4c_1x^3+4\sqrt{x^6c_1^2(4c_1x^3+1)}\right)^{\frac{2}{3}}c_1} + x^3}{x^2}$$

$$y(x) = -\frac{2\left(\frac{3x^3\left(4c_1x^3+4\sqrt{4c_1^3x^9+x^6c_1^2}\right)^{\frac{2}{3}}c_1}{2} - \frac{\left(c_1x^3+\sqrt{4c_1^3x^9+x^6c_1^2}\right)\left(i\sqrt{3}-1\right)\left(4c_1x^3+4\sqrt{4c_1^3x^9+x^6c_1^2}\right)^{\frac{1}{3}}}{4} + x^6\left(1+i\sqrt{3}\right)c_1^2\right)}{\left(4c_1x^3+4\sqrt{4c_1^3x^9+x^6c_1^2}\right)^{\frac{2}{3}}c_1x^2}$$

$$y(x) = \frac{-3x^3\left(4c_1x^3+4\sqrt{4c_1^3x^9+x^6c_1^2}\right)^{\frac{2}{3}}c_1 - \frac{\left(c_1x^3+\sqrt{4c_1^3x^9+x^6c_1^2}\right)\left(1+i\sqrt{3}\right)\left(4c_1x^3+4\sqrt{4c_1^3x^9+x^6c_1^2}\right)^{\frac{1}{3}}}{2} + 2x^6\left(i\sqrt{3}-1\right)c}{\left(4c_1x^3+4\sqrt{4c_1^3x^9+x^6c_1^2}\right)^{\frac{2}{3}}c_1x^2}$$

$$y(x) = -\frac{2\left(\frac{3x^3\left(4c_1x^3+4\sqrt{4c_1^3x^9+x^6c_1^2}\right)^{\frac{2}{3}}c_1}{2} - \frac{\left(c_1x^3+\sqrt{4c_1^3x^9+x^6c_1^2}\right)\left(i\sqrt{3}-1\right)\left(4c_1x^3+4\sqrt{4c_1^3x^9+x^6c_1^2}\right)^{\frac{1}{3}}}{4} + x^6\left(1+i\sqrt{3}\right)c_1^2\right)}{\left(4c_1x^3+4\sqrt{4c_1^3x^9+x^6c_1^2}\right)^{\frac{2}{3}}c_1x^2}$$

✓ Solution by Mathematica

Time used: 20.375 (sec). Leaf size: 484

```
DSolve[y'[x] == - (4*x+3*y[x])/(2*x+y[x]),y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \frac{\sqrt[3]{2x^3 + \sqrt{4e^{3c_1}x^3 + e^{6c_1}} + e^{3c_1}}}{\sqrt[3]{2}} + \frac{\sqrt[3]{2}x^2}{\sqrt[3]{2x^3 + \sqrt{4e^{3c_1}x^3 + e^{6c_1}} + e^{3c_1}}} - 3x$$

$$y(x) \to \frac{i(\sqrt{3}+i)\sqrt[3]{2x^3 + \sqrt{4e^{3c_1}x^3 + e^{6c_1}} + e^{3c_1}}}{2\sqrt[3]{2}} - \frac{(1+i\sqrt{3})x^2}{2^{2/3}\sqrt[3]{2x^3 + \sqrt{4e^{3c_1}x^3 + e^{6c_1}} + e^{3c_1}}} - 3x$$

$$y(x) \to -\frac{(1+i\sqrt{3})\sqrt[3]{2x^3 + \sqrt{4e^{3c_1}x^3 + e^{6c_1}} + e^{3c_1}}}{2\sqrt[3]{2}} + \frac{i(\sqrt{3}+i)x^2}{2^{2/3}\sqrt[3]{2x^3 + \sqrt{4e^{3c_1}x^3 + e^{6c_1}} + e^{3c_1}}} - 3x$$

$$y(x) \to \sqrt[3]{x^3} + \frac{(x^3)^{2/3}}{x} - 3x$$

$$y(x) \to \frac{1}{2}\left(i\left(\sqrt{3}+i\right)\sqrt[3]{x^3} + \frac{(-1-i\sqrt{3})(x^3)^{2/3}}{x} - 6x\right)$$

$$y(x) \to \frac{1}{2}\left(\left(-1-i\sqrt{3}\right)\sqrt[3]{x^3} + \frac{i(\sqrt{3}+i)(x^3)^{2/3}}{x} - 6x\right)$$

2.34 problem 35

Internal problem ID [512]
Internal file name [`OUTPUT/512_Sunday_June_05_2022_01_42_48_AM_92797294/index.tex`]

Book: Elementary differential equations and boundary value problems, 10th ed., Boyce and DiPrima
Section: Section 2.2. Page 48
Problem number: 35.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"homogeneousTypeD2", "first_order_ode_lie_symmetry_calculated"**

Maple gives the following as the ode type

```
[[_homogeneous, `class A`], _rational, [_Abel, `2nd type`, `
   class A`]]
```

$$y' - \frac{x + 3y}{x - y} = 0$$

2.34.1 Solving as homogeneousTypeD2 ode

Using the change of variables $y = u(x)\,x$ on the above ode results in new ode in $u(x)$

$$u'(x)\,x + u(x) - \frac{x + 3u(x)\,x}{x - u(x)\,x} = 0$$

In canonical form the ODE is

$$\begin{aligned} u' &= F(x,u) \\ &= f(x)g(u) \\ &= -\frac{(u+1)^2}{x\,(u-1)} \end{aligned}$$

Where $f(x) = -\frac{1}{x}$ and $g(u) = \frac{(u+1)^2}{u-1}$. Integrating both sides gives

$$\frac{1}{\frac{(u+1)^2}{u-1}}\,du = -\frac{1}{x}\,dx$$

$$\int \frac{1}{\frac{(u+1)^2}{u-1}}\,du = \int -\frac{1}{x}\,dx$$

$$\ln(u+1) + \frac{2}{u+1} = -\ln(x) + c_2$$

The solution is

$$\ln(u(x)+1) + \frac{2}{u(x)+1} + \ln(x) - c_2 = 0$$

Replacing $u(x)$ in the above solution by $\frac{y}{x}$ results in the solution for y in implicit form

$$\ln\left(1+\frac{y}{x}\right) + \frac{2}{1+\frac{y}{x}} + \ln(x) - c_2 = 0$$

$$\ln\left(\frac{x+y}{x}\right) + \frac{2x}{x+y} + \ln(x) - c_2 = 0$$

Summary

The solution(s) found are the following

$$\ln\left(\frac{x+y}{x}\right) + \frac{2x}{x+y} + \ln(x) - c_2 = 0 \tag{1}$$

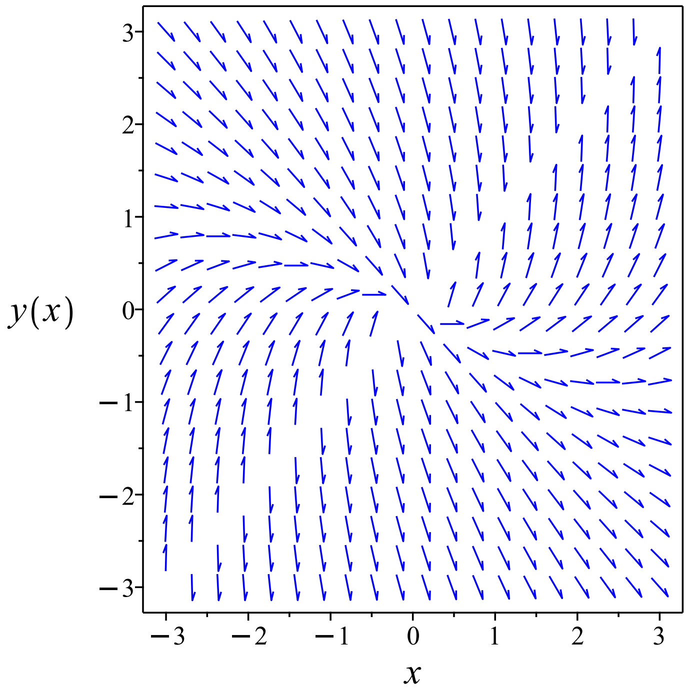


Figure 163: Slope field plot

Verification of solutions

$$\ln\left(\frac{x+y}{x}\right)+\frac{2x}{x+y}+\ln(x)-c_2=0$$

Verified OK.

2.34.2 Solving as first order ode lie symmetry calculated ode

Writing the ode as

$$y'=-\frac{3y+x}{-x+y}$$
$$y'=\omega(x,y)$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x+\omega(\eta_y-\xi_x)-\omega^2\xi_y-\omega_x\xi-\omega_y\eta=0 \tag{A}$$

The type of this ode is not in the lookup table. To determine ξ,η then (A) is solved using ansatz. Making bivariate polynomials of degree 1 to use as anstaz gives

$$\xi=xa_2+ya_3+a_1 \tag{1E}$$
$$\eta=xb_2+yb_3+b_1 \tag{2E}$$

Where the unknown coefficients are

$$\{a_1, a_2, a_3, b_1, b_2, b_3\}$$

Substituting equations (1E,2E) and ω into (A) gives

$$b_2 - \frac{(3y+x)(b_3-a_2)}{-x+y} - \frac{(3y+x)^2 a_3}{(-x+y)^2} - \left(-\frac{1}{-x+y} - \frac{3y+x}{(-x+y)^2}\right)(xa_2+ya_3+a_1) - \left(-\frac{3}{-x+y} + \frac{3y+x}{(-x+y)^2}\right)(xb_2+yb_3+b_1) = 0 \tag{5E}$$

Putting the above in normal form gives

$$-\frac{x^2a_2 + x^2a_3 + 3x^2b_2 - x^2b_3 - 2xya_2 + 6xya_3 + 2xyb_2 + 2xyb_3 - 3y^2a_2 + 5y^2a_3 - y^2b_2 + 3y^2b_3 + 4xb_1 -}{(x-y)^2} = 0$$

Setting the numerator to zero gives

$$-x^2a_2 - x^2a_3 - 3x^2b_2 + x^2b_3 + 2xya_2 - 6xya_3 - 2xyb_2 - 2xyb_3 + 3y^2a_2 - 5y^2a_3 + y^2b_2 - 3y^2b_3 - 4xb_1 + 4ya_1 = 0 \tag{6E}$$

Looking at the above PDE shows the following are all the terms with $\{x, y\}$ in them.

$$\{x, y\}$$

The following substitution is now made to be able to collect on all terms with $\{x, y\}$ in them

$$\{x = v_1, y = v_2\}$$

The above PDE (6E) now becomes

$$-a_2v_1^2 + 2a_2v_1v_2 + 3a_2v_2^2 - a_3v_1^2 - 6a_3v_1v_2 - 5a_3v_2^2 - 3b_2v_1^2 - 2b_2v_1v_2 + b_2v_2^2 + b_3v_1^2 - 2b_3v_1v_2 - 3b_3v_2^2 + 4a_1v_2 - 4b_1v_1 = 0 \tag{7E}$$

Collecting the above on the terms v_i introduced, and these are

$$\{v_1, v_2\}$$

Equation (7E) now becomes

$$\begin{aligned}(-a_2 - a_3 - 3b_2 + b_3) v_1^2 + (2a_2 - 6a_3 - 2b_2 - 2b_3) v_1 v_2 \\ - 4b_1 v_1 + (3a_2 - 5a_3 + b_2 - 3b_3) v_2^2 + 4a_1 v_2 = 0\end{aligned} \tag{8E}$$

Setting each coefficients in (8E) to zero gives the following equations to solve

$$\begin{aligned}4a_1 &= 0\\ -4b_1 &= 0\\ -a_2 - a_3 - 3b_2 + b_3 &= 0\\ 2a_2 - 6a_3 - 2b_2 - 2b_3 &= 0\\ 3a_2 - 5a_3 + b_2 - 3b_3 &= 0\end{aligned}$$

Solving the above equations for the unknowns gives

$$\begin{aligned}a_1 &= 0\\ a_2 &= -2b_2 + b_3\\ a_3 &= -b_2\\ b_1 &= 0\\ b_2 &= b_2\\ b_3 &= b_3\end{aligned}$$

Substituting the above solution in the anstaz (1E,2E) (using 1 as arbitrary value for any unknown in the RHS) gives

$$\begin{aligned}\xi &= x\\ \eta &= y\end{aligned}$$

Shifting is now applied to make $\xi = 0$ in order to simplify the rest of the computation

$$\begin{aligned}\eta &= \eta - \omega(x, y)\, \xi\\ &= y - \left(-\frac{3y + x}{-x + y}\right)(x)\\ &= \frac{-x^2 - 2yx - y^2}{x - y}\\ \xi &= 0\end{aligned}$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x, y) \to (R, S)$ where (R, S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y}\right) S(x, y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$R = x$$

S is found from

$$\begin{aligned} S &= \int \frac{1}{\eta} dy \\ &= \int \frac{1}{\frac{-x^2-2yx-y^2}{x-y}} dy \end{aligned}$$

Which results in

$$S = \frac{2x}{x+y} + \ln(x+y)$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x, y) S_y}{R_x + \omega(x, y) R_y} \tag{2}$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x, y)$ is the right hand side of the original ode given by

$$\omega(x, y) = -\frac{3y + x}{-x + y}$$

Evaluating all the partial derivatives gives

$$\begin{aligned} R_x &= 1 \\ R_y &= 0 \\ S_x &= \frac{3y + x}{(x+y)^2} \\ S_y &= \frac{-x + y}{(x+y)^2} \end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = 0 \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = 0$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$S(R) = c_1 \tag{4}$$

To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$\frac{(x+y)\ln(x+y)+2x}{x+y} = c_1$$

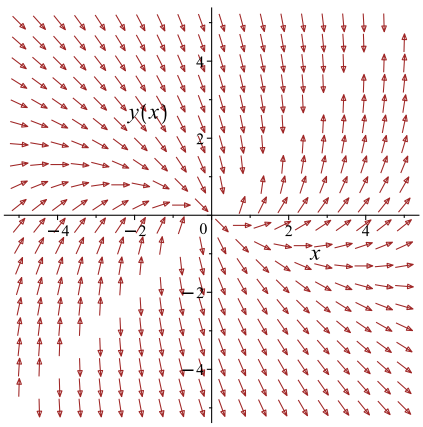
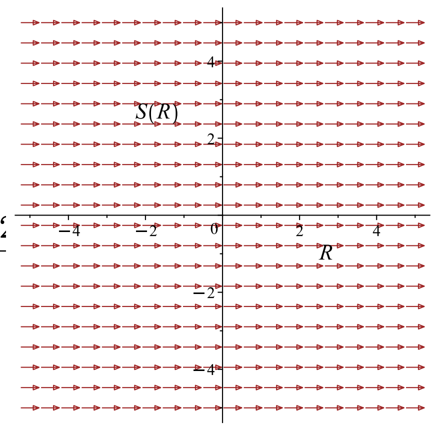
Which simplifies to

$$\frac{(x+y)\ln(x+y)+2x}{x+y} = c_1$$

Which gives

$$y = e^{\text{LambertW}\left(-2x\,e^{-c_1}\right)+c_1} - x$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in x, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dx} = -\frac{3y+x}{-x+y}$ 	$R = x$ $S = \frac{(x+y)\ln(x+y) + 2}{x+y}$	$\frac{dS}{dR} = 0$ 

Summary

The solution(s) found are the following

$$y = \mathrm{e}^{\mathrm{LambertW}\left(-2x\,\mathrm{e}^{-c_1}\right)+c_1} - x \tag{1}$$

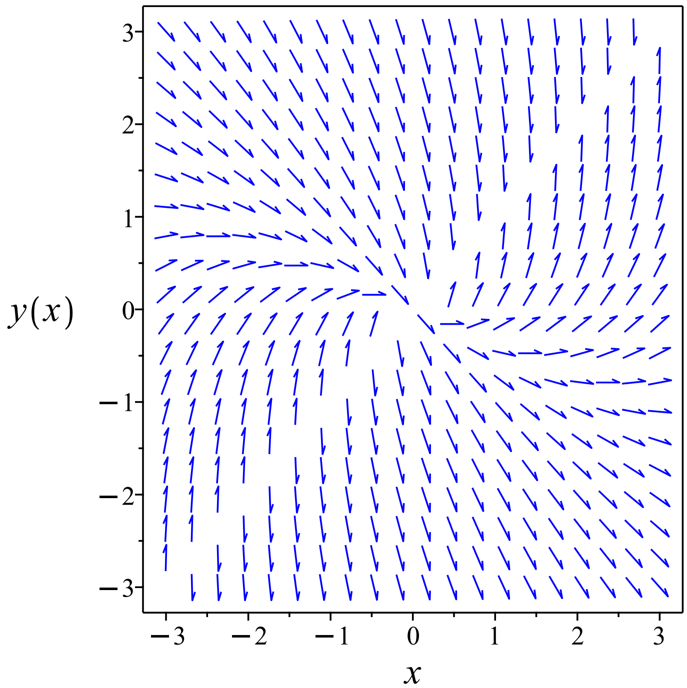


Figure 164: Slope field plot

Verification of solutions

$$y = e^{\text{LambertW}\left(-2x\,e^{-c_1}\right)+c_1} - x$$

Verified OK.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
trying homogeneous types:
trying homogeneous D
<- homogeneous successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 21

```
dsolve(diff(y(x),x) = (x+3*y(x))/(x-y(x)),y(x), singsol=all)
```

$$y(x) = -\frac{x(\operatorname{LambertW}(-2c_1x) + 2)}{\operatorname{LambertW}(-2c_1x)}$$

✓ Solution by Mathematica

Time used: 0.113 (sec). Leaf size: 33

```
DSolve[y'[x] == (x+3*y[x])/(x-y[x]),y[x],x,IncludeSingularSolutions -> True]
```

$$\text{Solve}\left[\frac{2}{\frac{y(x)}{x}+1} + \log\left(\frac{y(x)}{x}+1\right) = -\log(x) + c_1, y(x)\right]$$

2.35 problem 36

Internal problem ID [513]
Internal file name [`OUTPUT/513_Sunday_June_05_2022_01_42_49_AM_6394563/index.tex`]

Book: Elementary differential equations and boundary value problems, 10th ed., Boyce and DiPrima
Section: Section 2.2. Page 48
Problem number: 36.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"riccati", "homogeneousTypeD2", "first_order_ode_lie_symmetry_calculated"**

Maple gives the following as the ode type

```
[[_homogeneous, `class A`], _rational, _Riccati]
```

$$\boxed{3yx + y^2 - y'x^2 = -x^2}$$

2.35.1 Solving as homogeneousTypeD2 ode

Using the change of variables $y = u(x)\,x$ on the above ode results in new ode in $u(x)$

$$3u(x)\,x^2 + u(x)^2\,x^2 - (u'(x)\,x + u(x))\,x^2 = -x^2$$

In canonical form the ODE is

$$\begin{aligned} u' &= F(x,u) \\ &= f(x)g(u) \\ &= \frac{u^2 + 2u + 1}{x} \end{aligned}$$

Where $f(x) = \frac{1}{x}$ and $g(u) = u^2 + 2u + 1$. Integrating both sides gives

$$\frac{1}{u^2 + 2u + 1}\,du = \frac{1}{x}\,dx$$
$$\int \frac{1}{u^2 + 2u + 1}\,du = \int \frac{1}{x}\,dx$$
$$-\frac{1}{u+1} = \ln(x) + c_2$$

The solution is

$$-\frac{1}{u(x)+1} - \ln(x) - c_2 = 0$$

Replacing $u(x)$ in the above solution by $\frac{y}{x}$ results in the solution for y in implicit form

$$-\frac{1}{1+\frac{y}{x}} - \ln(x) - c_2 = 0$$
$$\frac{(-c_2 - \ln(x))\,y - x(c_2 + \ln(x) + 1)}{x+y} = 0$$

Which simplifies to

$$-\frac{\ln(x)\,y + c_2 y + x\ln(x) + c_2 x + x}{x+y} = 0$$

Summary

The solution(s) found are the following

$$-\frac{\ln(x)\,y + c_2 y + x\ln(x) + c_2 x + x}{x+y} = 0 \tag{1}$$

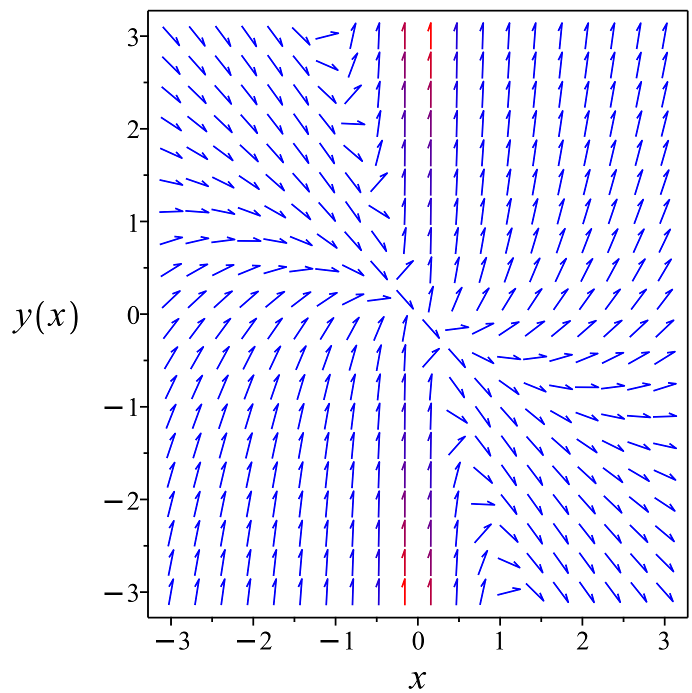


Figure 165: Slope field plot

Verification of solutions

$$-\frac{\ln (x) y+c_2 y+x \ln (x)+c_2 x+x}{x+y}=0$$

Verified OK.

2.35.2 Solving as first order ode lie symmetry calculated ode

Writing the ode as

$$\begin{aligned} y' &= \frac{x^2+3yx+y^2}{x^2} \\ y' &= \omega(x,y) \end{aligned}$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x+\omega(\eta_y-\xi_x)-\omega^2\xi_y-\omega_x\xi-\omega_y\eta=0 \tag{A}$$

The type of this ode is not in the lookup table. To determine ξ,η then (A) is solved using ansatz. Making bivariate polynomials of degree 1 to use as anstaz gives

$$\xi = xa_2+ya_3+a_1 \tag{1E}$$

$$\eta = xb_2+yb_3+b_1 \tag{2E}$$

Where the unknown coefficients are

$$\{a_1, a_2, a_3, b_1, b_2, b_3\}$$

Substituting equations (1E,2E) and ω into (A) gives

$$\begin{aligned} &b_2 + \frac{(x^2 + 3yx + y^2)(b_3 - a_2)}{x^2} - \frac{(x^2 + 3yx + y^2)^2 a_3}{x^4} \\ &- \left(\frac{2x + 3y}{x^2} - \frac{2(x^2 + 3yx + y^2)}{x^3}\right)(xa_2 + ya_3 + a_1) \\ &- \frac{(3x + 2y)(xb_2 + yb_3 + b_1)}{x^2} = 0 \end{aligned} \tag{5E}$$

Putting the above in normal form gives

$$-\frac{x^4 a_2 + x^4 a_3 + 2b_2 x^4 - x^4 b_3 + 6x^3 y a_3 + 2x^3 y b_2 - x^2 y^2 a_2 + 8x^2 y^2 a_3 + x^2 y^2 b_3 + 4x\,y^3 a_3 + y^4 a_3 + 3x^3 b_1 -}{x^4}$$
$$= 0$$

Setting the numerator to zero gives

$$\begin{aligned} &-x^4 a_2 - x^4 a_3 - 2b_2 x^4 + x^4 b_3 - 6x^3 y a_3 - 2x^3 y b_2 + x^2 y^2 a_2 - 8x^2 y^2 a_3 \\ &- x^2 y^2 b_3 - 4x\,y^3 a_3 - y^4 a_3 - 3x^3 b_1 + 3x^2 y a_1 - 2x^2 y b_1 + 2x\,y^2 a_1 = 0 \end{aligned} \tag{6E}$$

Looking at the above PDE shows the following are all the terms with $\{x, y\}$ in them.

$$\{x, y\}$$

The following substitution is now made to be able to collect on all terms with $\{x, y\}$ in them

$$\{x = v_1, y = v_2\}$$

The above PDE (6E) now becomes

$$\begin{aligned} &-a_2 v_1^4 + a_2 v_1^2 v_2^2 - a_3 v_1^4 - 6a_3 v_1^3 v_2 - 8a_3 v_1^2 v_2^2 - 4a_3 v_1 v_2^3 - a_3 v_2^4 - 2b_2 v_1^4 \\ &- 2b_2 v_1^3 v_2 + b_3 v_1^4 - b_3 v_1^2 v_2^2 + 3a_1 v_1^2 v_2 + 2a_1 v_1 v_2^2 - 3b_1 v_1^3 - 2b_1 v_1^2 v_2 = 0 \end{aligned} \tag{7E}$$

Collecting the above on the terms v_i introduced, and these are

$$\{v_1, v_2\}$$

Equation (7E) now becomes

$$(-a_2 - a_3 - 2b_2 + b_3) v_1^4 + (-6a_3 - 2b_2) v_1^3 v_2 - 3b_1 v_1^3 + (a_2 - 8a_3 - b_3) v_1^2 v_2^2 \\ + (3a_1 - 2b_1) v_1^2 v_2 - 4a_3 v_1 v_2^3 + 2a_1 v_1 v_2^2 - a_3 v_2^4 = 0 \tag{8E}$$

Setting each coefficients in (8E) to zero gives the following equations to solve

$$\begin{aligned} 2a_1 &= 0 \\ -4a_3 &= 0 \\ -a_3 &= 0 \\ -3b_1 &= 0 \\ 3a_1 - 2b_1 &= 0 \\ -6a_3 - 2b_2 &= 0 \\ a_2 - 8a_3 - b_3 &= 0 \\ -a_2 - a_3 - 2b_2 + b_3 &= 0 \end{aligned}$$

Solving the above equations for the unknowns gives

$$\begin{aligned} a_1 &= 0 \\ a_2 &= b_3 \\ a_3 &= 0 \\ b_1 &= 0 \\ b_2 &= 0 \\ b_3 &= b_3 \end{aligned}$$

Substituting the above solution in the anstaz (1E,2E) (using 1 as arbitrary value for any unknown in the RHS) gives

$$\begin{aligned} \xi &= x \\ \eta &= y \end{aligned}$$

Shifting is now applied to make $\xi = 0$ in order to simplify the rest of the computation

$$
\begin{aligned}
\eta &= \eta - \omega(x, y)\,\xi \\
&= y - \left(\frac{x^2 + 3yx + y^2}{x^2}\right)(x) \\
&= \frac{-x^2 - 2yx - y^2}{x} \\
\xi &= 0
\end{aligned}
$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x, y) \to (R, S)$ where (R, S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$
\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}
$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y}\right) S(x, y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$
R = x
$$

S is found from

$$
\begin{aligned}
S &= \int \frac{1}{\eta} dy \\
&= \int \frac{1}{\frac{-x^2 - 2yx - y^2}{x}} dy
\end{aligned}
$$

Which results in

$$
S = \frac{x}{x + y}
$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$
\frac{dS}{dR} = \frac{S_x + \omega(x, y) S_y}{R_x + \omega(x, y) R_y} \tag{2}
$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x, y)$ is the right hand side of the original ode given by

$$
\omega(x, y) = \frac{x^2 + 3yx + y^2}{x^2}
$$

Evaluating all the partial derivatives gives

$$
\begin{aligned}
R_x &= 1 \\
R_y &= 0 \\
S_x &= \frac{y}{(x+y)^2} \\
S_y &= -\frac{x}{(x+y)^2}
\end{aligned}
$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = -\frac{1}{x} \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = -\frac{1}{R}$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$S(R) = -\ln(R) + c_1 \tag{4}$$

To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$\frac{x}{x+y} = -\ln(x) + c_1$$

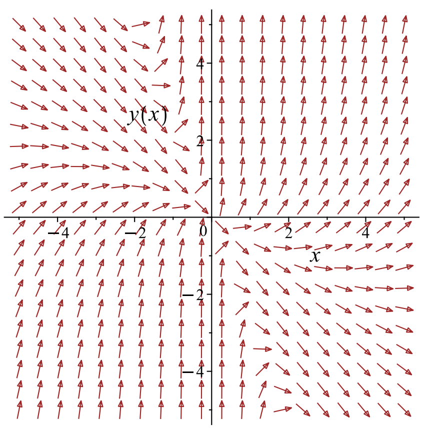
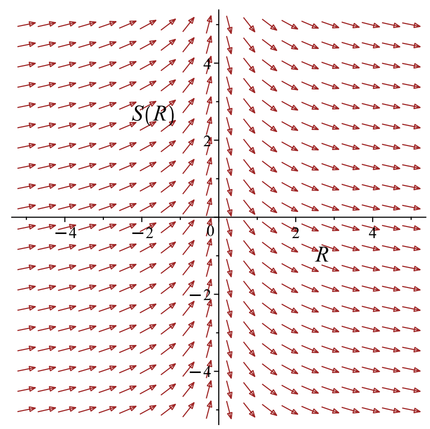
Which simplifies to

$$\frac{x}{x+y} = -\ln(x) + c_1$$

Which gives

$$y = -\frac{x(\ln(x) - c_1 + 1)}{\ln(x) - c_1}$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in x, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dx} = \frac{x^2+3yx+y^2}{x^2}$  	$R = x$ $S = \frac{x}{x+y}$	$\frac{dS}{dR} = -\frac{1}{R}$ 

Summary

The solution(s) found are the following

$$y = -\frac{x(\ln(x) - c_1 + 1)}{\ln(x) - c_1} \tag{1}$$

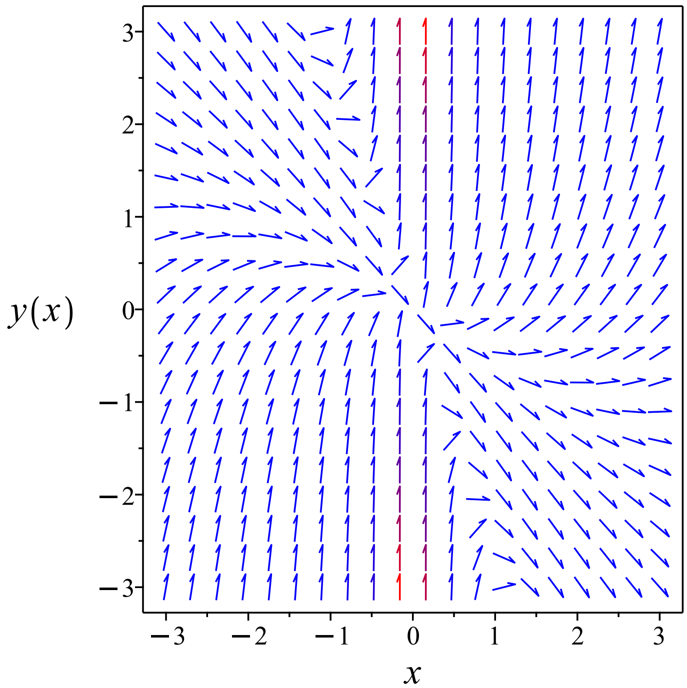


Figure 166: Slope field plot

Verification of solutions

$$y = -\frac{x(\ln(x) - c_1 + 1)}{\ln(x) - c_1}$$

Verified OK.

2.35.3 Solving as riccati ode

In canonical form the ODE is

$$\begin{aligned} y' &= F(x, y) \\ &= \frac{x^2 + 3yx + y^2}{x^2} \end{aligned}$$

This is a Riccati ODE. Comparing the ODE to solve

$$y' = 1 + \frac{3y}{x} + \frac{y^2}{x^2}$$

With Riccati ODE standard form

$$y' = f_0(x) + f_1(x)y + f_2(x)y^2$$

Shows that $f_0(x) = 1$, $f_1(x) = \frac{3}{x}$ and $f_2(x) = \frac{1}{x^2}$. Let

$$
\begin{aligned}
y &= \frac{-u'}{f_2 u} \\
&= \frac{-u'}{\frac{u}{x^2}}
\end{aligned} \tag{1}
$$

Using the above substitution in the given ODE results (after some simplification)in a second order ODE to solve for $u(x)$ which is

$$
f_2 u''(x) - (f_2' + f_1 f_2)\, u'(x) + f_2^2 f_0 u(x) = 0 \tag{2}
$$

But

$$
\begin{aligned}
f_2' &= -\frac{2}{x^3} \\
f_1 f_2 &= \frac{3}{x^3} \\
f_2^2 f_0 &= \frac{1}{x^4}
\end{aligned}
$$

Substituting the above terms back in equation (2) gives

$$
\frac{u''(x)}{x^2} - \frac{u'(x)}{x^3} + \frac{u(x)}{x^4} = 0
$$

Solving the above ODE (this ode solved using Maple, not this program), gives

$$
u(x) = x(c_2 \ln(x) + c_1)
$$

The above shows that

$$
u'(x) = c_2 \ln(x) + c_1 + c_2
$$

Using the above in (1) gives the solution

$$
y = -\frac{(c_2 \ln(x) + c_1 + c_2)\, x}{c_2 \ln(x) + c_1}
$$

Dividing both numerator and denominator by c_1 gives, after renaming the constant $\frac{c_2}{c_1} = c_3$ the following solution

$$
y = -\frac{(\ln(x) + c_3 + 1)\, x}{\ln(x) + c_3}
$$

Summary

The solution(s) found are the following

$$y = -\frac{(\ln(x) + c_3 + 1)\, x}{\ln(x) + c_3} \tag{1}$$

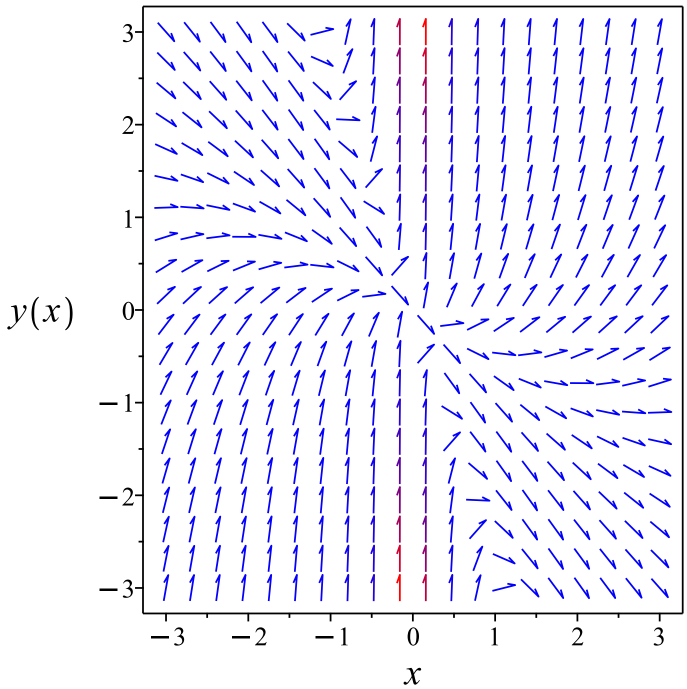


Figure 167: Slope field plot

Verification of solutions

$$y = -\frac{(\ln(x) + c_3 + 1)\, x}{\ln(x) + c_3}$$

Verified OK.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
trying homogeneous types:
trying homogeneous D
<- homogeneous successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 18

```
dsolve((x^2+3*x*y(x)+y(x)^2)-x^2* diff(y(x),x)=0,y(x), singsol=all)
```

$$y(x) = -\frac{x(\ln(x) + c_1 + 1)}{\ln(x) + c_1}$$

✓ Solution by Mathematica

Time used: 0.141 (sec). Leaf size: 28

```
DSolve[(x^2+3*x*y[x]+y[x]^2)-x^2* y'[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to -\frac{x(\log(x) + 1 + c_1)}{\log(x) + c_1}$$
$$y(x) \to -x$$

2.36 problem 37

Internal problem ID [514]
Internal file name [`OUTPUT/514_Sunday_June_05_2022_01_42_50_AM_58581911/index.tex`]

Book: Elementary differential equations and boundary value problems, 10th ed., Boyce and DiPrima
Section: Section 2.2. Page 48
Problem number: 37.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"bernoulli", "homogeneousTypeD2", "exactWithIntegrationFactor", "first_order_ode_lie_symmetry_lookup"**

Maple gives the following as the ode type

```
[[_homogeneous, `class A`], _rational, _Bernoulli]
```

$$\boxed{y' - \frac{x^2 - 3y^2}{2yx} = 0}$$

2.36.1 Solving as homogeneousTypeD2 ode

Using the change of variables $y = u(x)\,x$ on the above ode results in new ode in $u(x)$

$$u'(x)\,x + u(x) - \frac{x^2 - 3u(x)^2\,x^2}{2u\,(x)\,x^2} = 0$$

In canonical form the ODE is

$$\begin{aligned} u' &= F(x,u) \\ &= f(x)g(u) \\ &= -\frac{5u^2 - 1}{2xu} \end{aligned}$$

Where $f(x) = -\frac{1}{2x}$ and $g(u) = \frac{5u^2-1}{u}$. Integrating both sides gives

$$\frac{1}{\frac{5u^2-1}{u}}\, du = -\frac{1}{2x}\, dx$$
$$\int \frac{1}{\frac{5u^2-1}{u}}\, du = \int -\frac{1}{2x}\, dx$$
$$\frac{\ln(5u^2-1)}{10} = -\frac{\ln(x)}{2} + c_2$$

Raising both side to exponential gives

$$\left(5u^2-1\right)^{\frac{1}{10}} = \mathrm{e}^{-\frac{\ln(x)}{2}+c_2}$$

Which simplifies to

$$\left(5u^2-1\right)^{\frac{1}{10}} = \frac{c_3}{\sqrt{x}}$$

Which simplifies to

$$\left(5u(x)^2-1\right)^{\frac{1}{10}} = \frac{c_3\mathrm{e}^{c_2}}{\sqrt{x}}$$

The solution is

$$\left(5u(x)^2-1\right)^{\frac{1}{10}} = \frac{c_3\mathrm{e}^{c_2}}{\sqrt{x}}$$

Replacing $u(x)$ in the above solution by $\frac{y}{x}$ results in the solution for y in implicit form

$$\left(\frac{5y^2}{x^2}-1\right)^{\frac{1}{10}} = \frac{c_3\mathrm{e}^{c_2}}{\sqrt{x}}$$
$$\left(\frac{5y^2-x^2}{x^2}\right)^{\frac{1}{10}} = \frac{c_3\mathrm{e}^{c_2}}{\sqrt{x}}$$

Which simplifies to

$$\left(-\frac{-5y^2+x^2}{x^2}\right)^{\frac{1}{10}} = \frac{c_3\mathrm{e}^{c_2}}{\sqrt{x}}$$

Summary

The solution(s) found are the following

$$\left(-\frac{-5y^2+x^2}{x^2}\right)^{\frac{1}{10}} = \frac{c_3\mathrm{e}^{c_2}}{\sqrt{x}} \tag{1}$$

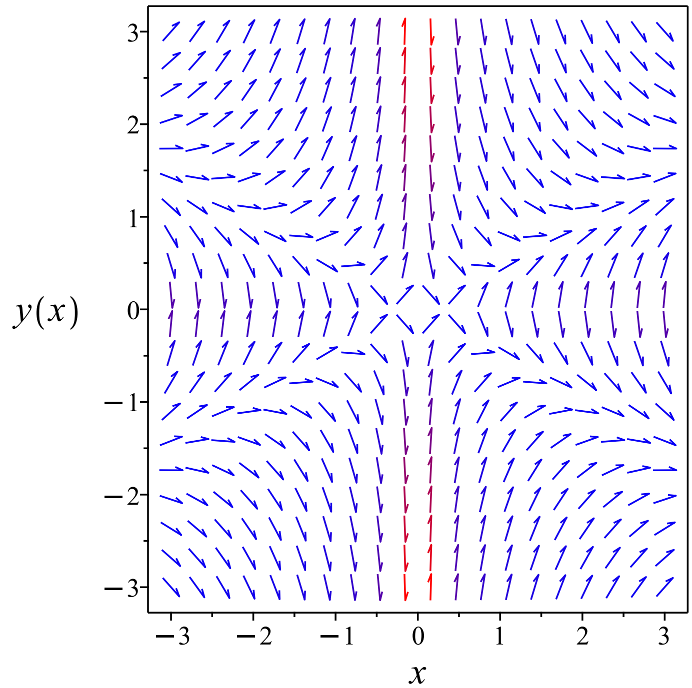


Figure 168: Slope field plot

Verification of solutions

$$
\left(-\frac{-5y^2+x^2}{x^2}\right)^{\frac{1}{10}} = \frac{c_3 e^{c_2}}{\sqrt{x}}
$$

Verified OK.

2.36.2 Solving as first order ode lie symmetry lookup ode

Writing the ode as

$$
\begin{aligned}
y' &= -\frac{-x^2+3y^2}{2yx} \\
y' &= \omega(x,y)
\end{aligned}
$$

The condition of Lie symmetry is the linearized PDE given by

$$
\eta_x + \omega(\eta_y - \xi_x) - \omega^2 \xi_y - \omega_x \xi - \omega_y \eta = 0 \tag{A}
$$

The type of this ode is known. It is of type `Bernoulli`. Therefore we do not need to solve the PDE (A), and can just use the lookup table shown below to find ξ, η

Table 178: Lie symmetry infinitesimal lookup table for known first order ODE's

ODE class	Form	ξ	η
linear ode	$y' = f(x)y(x) + g(x)$	0	$e^{\int f dx}$
separable ode	$y' = f(x)\, g(y)$	$\frac{1}{f}$	0
quadrature ode	$y' = f(x)$	0	1
quadrature ode	$y' = g(y)$	1	0
homogeneous ODEs of Class A	$y' = f\left(\frac{y}{x}\right)$	x	y
homogeneous ODEs of Class C	$y' = (a + bx + cy)^{\frac{n}{m}}$	1	$-\frac{b}{c}$
homogeneous class D	$y' = \frac{y}{x} + g(x)\, F\left(\frac{y}{x}\right)$	x^2	xy
First order special form ID 1	$y' = g(x)\, e^{h(x)+by} + f(x)$	$\frac{e^{-\int bf(x)dx - h(x)}}{g(x)}$	$\frac{f(x)e^{-\int bf(x)dx - h(x)}}{g(x)}$
polynomial type ode	$y' = \frac{a_1x+b_1y+c_1}{a_2x+b_2y+c_2}$	$\frac{a_1b_2x - a_2b_1x - b_1c_2 + b_2c_1}{a_1b_2 - a_2b_1}$	$\frac{a_1b_2y - a_2b_1y - a_1c_2 - a_2c_1}{a_1b_2 - a_2b_1}$
Bernoulli ode	$y' = f(x)\, y + g(x)\, y^n$	0	$e^{-\int (n-1)f(x)dx} y^n$
Reduced Riccati	$y' = f_1(x)\, y + f_2(x)\, y^2$	0	$e^{-\int f_1 dx}$

The above table shows that

$$
\begin{aligned}
\xi(x,y) &= 0 \\
\eta(x,y) &= \frac{1}{x^3 y}
\end{aligned} \tag{A1}
$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x,y) \to (R,S)$ where (R,S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$
\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}
$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y}\right) S(x,y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the

canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$R = x$$

S is found from

$$\begin{aligned} S &= \int \frac{1}{\eta} dy \\ &= \int \frac{1}{\frac{1}{x^3 y}} dy \end{aligned}$$

Which results in

$$S = \frac{x^3 y^2}{2}$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x, y) S_y}{R_x + \omega(x, y) R_y} \tag{2}$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x, y)$ is the right hand side of the original ode given by

$$\omega(x, y) = -\frac{-x^2 + 3y^2}{2yx}$$

Evaluating all the partial derivatives gives

$$\begin{aligned} R_x &= 1 \\ R_y &= 0 \\ S_x &= \frac{3x^2 y^2}{2} \\ S_y &= y\, x^3 \end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = \frac{x^4}{2} \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = \frac{R^4}{2}$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$S(R) = \frac{R^5}{10} + c_1 \tag{4}$$

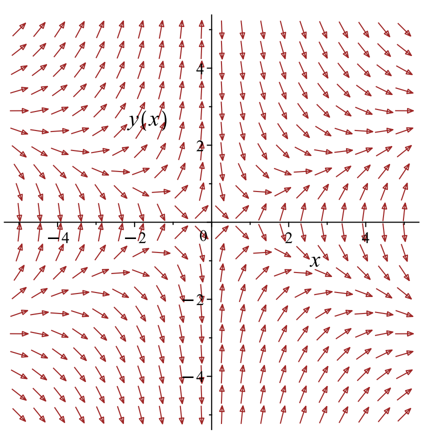
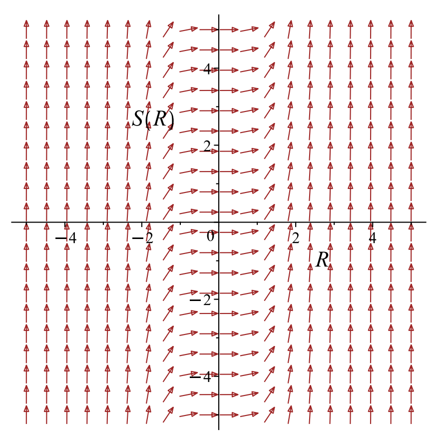
To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$\frac{x^3y^2}{2} = \frac{x^5}{10} + c_1$$

Which simplifies to

$$\frac{x^3y^2}{2} = \frac{x^5}{10} + c_1$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in x, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dx} = -\frac{-x^2+3y^2}{2yx}$ 	$R = x$ $S = \frac{x^3y^2}{2}$	$\frac{dS}{dR} = \frac{R^4}{2}$ 

Summary

The solution(s) found are the following

$$\frac{x^3y^2}{2} = \frac{x^5}{10} + c_1 \tag{1}$$

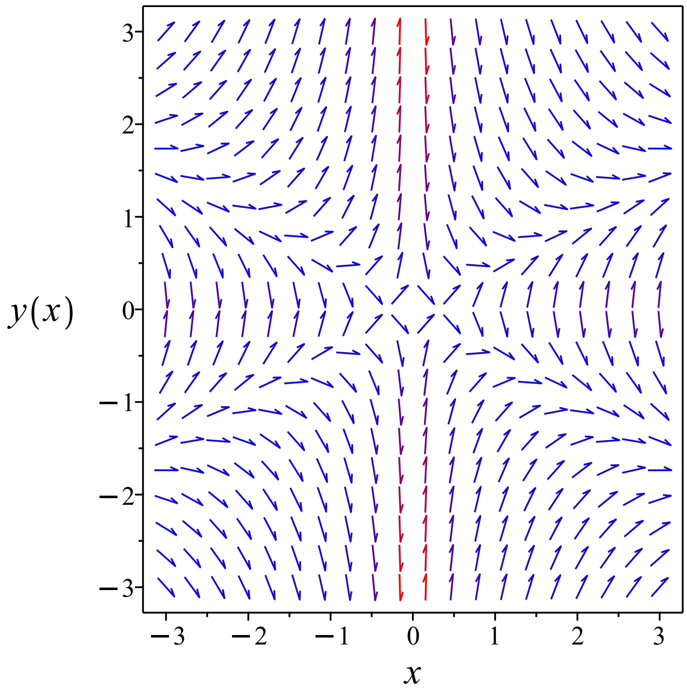


Figure 169: Slope field plot

Verification of solutions

$$\frac{x^3y^2}{2} = \frac{x^5}{10} + c_1$$

Verified OK.

2.36.3 Solving as bernoulli ode

In canonical form, the ODE is

$$\begin{aligned} y' &= F(x, y) \\ &= -\frac{-x^2 + 3y^2}{2yx} \end{aligned}$$

This is a Bernoulli ODE.

$$y' = -\frac{3}{2x}y + \frac{x}{2}\frac{1}{y} \tag{1}$$

The standard Bernoulli ODE has the form

$$y' = f_0(x)y + f_1(x)y^n \tag{2}$$

The first step is to divide the above equation by y^n which gives

$$\frac{y'}{y^n} = f_0(x)y^{1-n} + f_1(x) \tag{3}$$

The next step is use the substitution $w = y^{1-n}$ in equation (3) which generates a new ODE in $w(x)$ which will be linear and can be easily solved using an integrating factor. Backsubstitution then gives the solution $y(x)$ which is what we want.

This method is now applied to the ODE at hand. Comparing the ODE (1) With (2) Shows that

$$\begin{aligned} f_0(x) &= -\frac{3}{2x} \\ f_1(x) &= \frac{x}{2} \\ n &= -1 \end{aligned}$$

Dividing both sides of ODE (1) by $y^n = \frac{1}{y}$ gives

$$y'y = -\frac{3y^2}{2x} + \frac{x}{2} \tag{4}$$

Let

$$\begin{aligned} w &= y^{1-n} \\ &= y^2 \end{aligned} \tag{5}$$

Taking derivative of equation (5) w.r.t x gives

$$w' = 2yy' \tag{6}$$

Substituting equations (5) and (6) into equation (4) gives

$$\begin{aligned} \frac{w'(x)}{2} &= -\frac{3w(x)}{2x} + \frac{x}{2} \\ w' &= -\frac{3w}{x} + x \end{aligned} \tag{7}$$

The above now is a linear ODE in $w(x)$ which is now solved.

Entering Linear first order ODE solver. In canonical form a linear first order is

$$w'(x) + p(x)w(x) = q(x)$$

Where here

$$\begin{aligned} p(x) &= \frac{3}{x} \\ q(x) &= x \end{aligned}$$

Hence the ode is

$$w'(x) + \frac{3w(x)}{x} = x$$

The integrating factor μ is

$$\begin{aligned}\mu &= e^{\int \frac{3}{x}dx} \\ &= x^3\end{aligned}$$

The ode becomes

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}x}(\mu w) &= (\mu)(x) \\ \frac{\mathrm{d}}{\mathrm{d}x}(x^3 w) &= (x^3)(x) \\ \mathrm{d}(x^3 w) &= x^4 \,\mathrm{d}x\end{aligned}$$

Integrating gives

$$\begin{aligned}x^3 w &= \int x^4 \,\mathrm{d}x \\ x^3 w &= \frac{x^5}{5} + c_1\end{aligned}$$

Dividing both sides by the integrating factor $\mu = x^3$ results in

$$w(x) = \frac{x^2}{5} + \frac{c_1}{x^3}$$

Replacing w in the above by y^2 using equation (5) gives the final solution.

$$y^2 = \frac{x^2}{5} + \frac{c_1}{x^3}$$

Solving for y gives

$$\begin{aligned}y(x) &= \frac{\sqrt{5}\,\sqrt{x\,(x^5 + 5c_1)}}{5x^2} \\ y(x) &= -\frac{\sqrt{5}\,\sqrt{x\,(x^5 + 5c_1)}}{5x^2}\end{aligned}$$

Summary

The solution(s) found are the following

$$y = \frac{\sqrt{5}\,\sqrt{x\,(x^5 + 5c_1)}}{5x^2} \tag{1}$$

$$y = -\frac{\sqrt{5}\,\sqrt{x\,(x^5 + 5c_1)}}{5x^2} \tag{2}$$

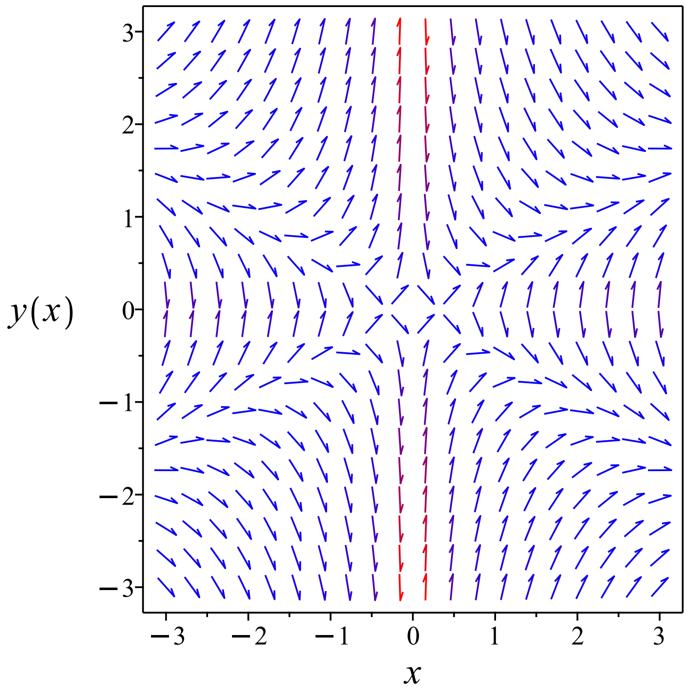


Figure 170: Slope field plot

Verification of solutions

$$y = \frac{\sqrt{5}\,\sqrt{x\,(x^5+5c_1)}}{5x^2}$$

Verified OK.

$$y = -\frac{\sqrt{5}\,\sqrt{x\,(x^5+5c_1)}}{5x^2}$$

Verified OK.

2.36.4 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x,y) + N(x,y)\,\frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x,y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y) = 0$$

Hence

$$\frac{\partial \phi}{\partial x} + \frac{\partial \phi}{\partial y}\frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$\frac{\partial \phi}{\partial x} = M$$
$$\frac{\partial \phi}{\partial y} = N$$

But since $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x, y)$ but at least we know now that we can do that since the condition $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x, y)\,\mathrm{d}x + N(x, y)\,\mathrm{d}y = 0 \tag{1A}$$

Therefore

$$(2yx)\,\mathrm{d}y = (x^2 - 3y^2)\,\mathrm{d}x$$
$$(-x^2 + 3y^2)\,\mathrm{d}x + (2yx)\,\mathrm{d}y = 0 \tag{2A}$$

Comparing (1A) and (2A) shows that

$$M(x, y) = -x^2 + 3y^2$$
$$N(x, y) = 2yx$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Using result found above gives

$$\frac{\partial M}{\partial y} = \frac{\partial}{\partial y}(-x^2 + 3y^2)$$
$$= 6y$$

And

$$
\begin{aligned}
\frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}(2yx) \\
&= 2y
\end{aligned}
$$

Since $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$, then the ODE is not exact. Since the ODE is not exact, we will try to find an integrating factor to make it exact. Let

$$
\begin{aligned}
A &= \frac{1}{N}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right) \\
&= \frac{1}{2yx}((6y) - (2y)) \\
&= \frac{2}{x}
\end{aligned}
$$

Since A does not depend on y, then it can be used to find an integrating factor. The integrating factor μ is

$$
\begin{aligned}
\mu &= e^{\int A \, \mathrm{d}x} \\
&= e^{\int \frac{2}{x} \, \mathrm{d}x}
\end{aligned}
$$

The result of integrating gives

$$
\begin{aligned}
\mu &= e^{2\ln(x)} \\
&= x^2
\end{aligned}
$$

M and N are multiplied by this integrating factor, giving new M and new N which are called $\overline{M}$ and $\overline{N}$ for now so not to confuse them with the original M and N.

$$
\begin{aligned}
\overline{M} &= \mu M \\
&= x^2(-x^2 + 3y^2) \\
&= -x^2(x^2 - 3y^2)
\end{aligned}
$$

And

$$
\begin{aligned}
\overline{N} &= \mu N \\
&= x^2(2yx) \\
&= 2y\,x^3
\end{aligned}
$$

Now a modified ODE is ontained from the original ODE, which is exact and can be solved. The modified ODE is

$$
\begin{aligned}
\overline{M} + \overline{N}\frac{\mathrm{d}y}{\mathrm{d}x} &= 0 \\
\left(-x^2(x^2 - 3y^2)\right) + \left(2y\,x^3\right)\frac{\mathrm{d}y}{\mathrm{d}x} &= 0
\end{aligned}
$$

The following equations are now set up to solve for the function $\phi(x,y)$

$$
\begin{aligned}
\frac{\partial \phi}{\partial x} &= \overline{M} && (1)\\
\frac{\partial \phi}{\partial y} &= \overline{N} && (2)
\end{aligned}
$$

Integrating (1) w.r.t. x gives

$$
\begin{aligned}
\int \frac{\partial \phi}{\partial x} \mathrm{d}x &= \int \overline{M} \,\mathrm{d}x\\
\int \frac{\partial \phi}{\partial x} \mathrm{d}x &= \int -x^2(x^2 - 3y^2) \,\mathrm{d}x\\
\phi &= -\frac{1}{5}x^5 + x^3y^2 + f(y) && (3)
\end{aligned}
$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$
\frac{\partial \phi}{\partial y} = 2y\,x^3 + f'(y) \tag{4}
$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = 2y\,x^3$. Therefore equation (4) becomes

$$
2y\,x^3 = 2y\,x^3 + f'(y) \tag{5}
$$

Solving equation (5) for $f'(y)$ gives

$$
f'(y) = 0
$$

Therefore

$$
f(y) = c_1
$$

Where c_1 is constant of integration. Substituting this result for $f(y)$ into equation (3) gives ϕ

$$
\phi = -\frac{1}{5}x^5 + x^3y^2 + c_1
$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$
c_1 = -\frac{1}{5}x^5 + x^3y^2
$$

Summary

The solution(s) found are the following

$$-\frac{x^5}{5}+x^3y^2=c_1 \tag{1}$$

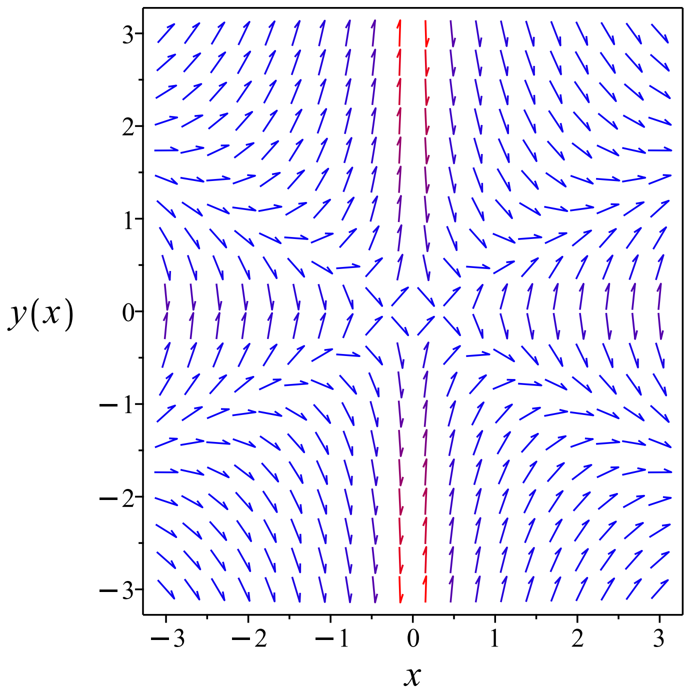


Figure 171: Slope field plot

Verification of solutions

$$-\frac{x^5}{5}+x^3y^2=c_1$$

Verified OK.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
<- Bernoulli successful`
```

✓ Solution by Maple

Time used: 0.015 (sec). Leaf size: 45

```
dsolve(diff(y(x),x) = (x^2-3*y(x)^2)/(2*x*y(x)),y(x), singsol=all)
```

$$y(x) = -\frac{\sqrt{5}\,\sqrt{x\,(x^5+5c_1)}}{5x^2}$$
$$y(x) = \frac{\sqrt{5}\,\sqrt{x\,(x^5+5c_1)}}{5x^2}$$

✓ Solution by Mathematica

Time used: 0.211 (sec). Leaf size: 50

```
DSolve[y'[x] == (x^2-3*y[x]^2)/(2*x*y[x]),y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to -\frac{\sqrt{\frac{x^5}{5}+c_1}}{x^{3/2}}$$
$$y(x) \to \frac{\sqrt{\frac{x^5}{5}+c_1}}{x^{3/2}}$$

2.37 problem 38

Internal problem ID [515]
Internal file name [`OUTPUT/515_Sunday_June_05_2022_01_42_52_AM_69066053/index.tex`]

Book: Elementary differential equations and boundary value problems, 10th ed., Boyce and DiPrima
Section: Section 2.2. Page 48
Problem number: 38.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"bernoulli", "homogeneousTypeD2", "exactWithIntegrationFactor", "first_order_ode_lie_symmetry_lookup"**

Maple gives the following as the ode type

```
[[_homogeneous, `class A`], _rational, _Bernoulli]
```

$$\boxed{y' - \frac{3y^2 - x^2}{2yx} = 0}$$

2.37.1 Solving as homogeneousTypeD2 ode

Using the change of variables $y = u(x)\,x$ on the above ode results in new ode in $u(x)$

$$u'(x)\,x + u(x) - \frac{3u(x)^2\,x^2 - x^2}{2u\,(x)\,x^2} = 0$$

In canonical form the ODE is

$$\begin{aligned} u' &= F(x,u) \\ &= f(x)g(u) \\ &= \frac{u^2-1}{2xu} \end{aligned}$$

Where $f(x) = \frac{1}{2x}$ and $g(u) = \frac{u^2-1}{u}$. Integrating both sides gives

$$\frac{1}{\frac{u^2-1}{u}}\,du = \frac{1}{2x}\,dx$$
$$\int \frac{1}{\frac{u^2-1}{u}}\,du = \int \frac{1}{2x}\,dx$$
$$\frac{\ln(u-1)}{2} + \frac{\ln(u+1)}{2} = \frac{\ln(x)}{2} + c_2$$

The above can be written as

$$\left(\frac{1}{2}\right)(\ln(u-1) + \ln(u+1)) = \frac{\ln(x)}{2} + 2c_2$$
$$\ln(u-1) + \ln(u+1) = (2)\left(\frac{\ln(x)}{2} + 2c_2\right)$$
$$= \ln(x) + 4c_2$$

Raising both side to exponential gives

$$e^{\ln(u-1)+\ln(u+1)} = e^{\ln(x)+2c_2}$$

Which simplifies to

$$u^2 - 1 = 2c_2 x$$
$$= c_3 x$$

The solution is

$$u(x)^2 - 1 = c_3 x$$

Replacing $u(x)$ in the above solution by $\frac{y}{x}$ results in the solution for y in implicit form

$$\frac{y^2}{x^2} - 1 = c_3 x$$
$$\frac{y^2}{x^2} - 1 = c_3 x$$

Summary
The solution(s) found are the following

$$\frac{y^2}{x^2} - 1 = c_3 x \tag{1}$$

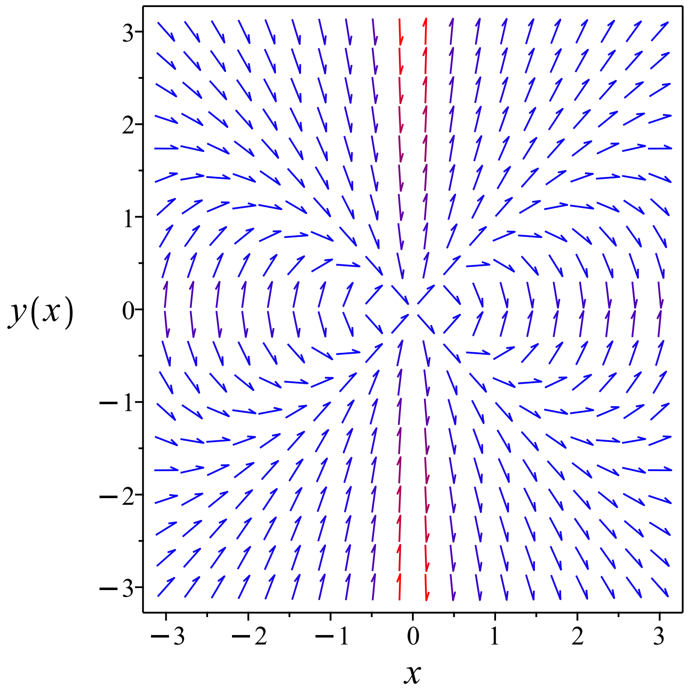


Figure 172: Slope field plot

Verification of solutions

$$\frac{y^2}{x^2} - 1 = c_3 x$$

Verified OK.

2.37.2 Solving as first order ode lie symmetry lookup ode

Writing the ode as

$$y' = \frac{-x^2 + 3y^2}{2yx}$$
$$y' = \omega(x, y)$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x + \omega(\eta_y - \xi_x) - \omega^2 \xi_y - \omega_x \xi - \omega_y \eta = 0 \tag{A}$$

The type of this ode is known. It is of type `Bernoulli`. Therefore we do not need to solve the PDE (A), and can just use the lookup table shown below to find ξ, η

Table 180: Lie symmetry infinitesimal lookup table for known first order ODE's

ODE class	Form	ξ	η
linear ode	$y' = f(x)y(x) + g(x)$	0	$e^{\int f dx}$
separable ode	$y' = f(x)\, g(y)$	$\frac{1}{f}$	0
quadrature ode	$y' = f(x)$	0	1
quadrature ode	$y' = g(y)$	1	0
homogeneous ODEs of Class A	$y' = f\left(\frac{y}{x}\right)$	x	y
homogeneous ODEs of Class C	$y' = (a + bx + cy)^{\frac{n}{m}}$	1	$-\frac{b}{c}$
homogeneous class D	$y' = \frac{y}{x} + g(x)\, F\left(\frac{y}{x}\right)$	x^2	xy
First order special form ID 1	$y' = g(x)\, e^{h(x)+by} + f(x)$	$\frac{e^{-\int bf(x)dx - h(x)}}{g(x)}$	$\frac{f(x)e^{-\int bf(x)dx - h(x)}}{g(x)}$
polynomial type ode	$y' = \frac{a_1x+b_1y+c_1}{a_2x+b_2y+c_2}$	$\frac{a_1b_2x - a_2b_1x - b_1c_2 + b_2c_1}{a_1b_2 - a_2b_1}$	$\frac{a_1b_2y - a_2b_1y - a_1c_2 - a_2c_1}{a_1b_2 - a_2b_1}$
Bernoulli ode	$y' = f(x)\, y + g(x)\, y^n$	0	$e^{-\int (n-1)f(x)dx} y^n$
Reduced Riccati	$y' = f_1(x)\, y + f_2(x)\, y^2$	0	$e^{-\int f_1 dx}$

The above table shows that

$$
\begin{aligned}
\xi(x,y) &= 0 \\
\eta(x,y) &= \frac{x^3}{y}
\end{aligned} \tag{A1}
$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x,y) \to (R,S)$ where (R,S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$
\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}
$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y}\right) S(x,y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the

canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$R = x$$

S is found from

$$\begin{aligned} S &= \int \frac{1}{\eta} dy \\ &= \int \frac{1}{\frac{x^3}{y}} dy \end{aligned}$$

Which results in

$$S = \frac{y^2}{2x^3}$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x,y) S_y}{R_x + \omega(x,y) R_y} \tag{2}$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x,y)$ is the right hand side of the original ode given by

$$\omega(x,y) = \frac{-x^2 + 3y^2}{2yx}$$

Evaluating all the partial derivatives gives

$$\begin{aligned} R_x &= 1 \\ R_y &= 0 \\ S_x &= -\frac{3y^2}{2x^4} \\ S_y &= \frac{y}{x^3} \end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = -\frac{1}{2x^2} \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = -\frac{1}{2R^2}$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordinates R, S. Integrating the above gives

$$S(R) = \frac{1}{2R} + c_1 \tag{4}$$

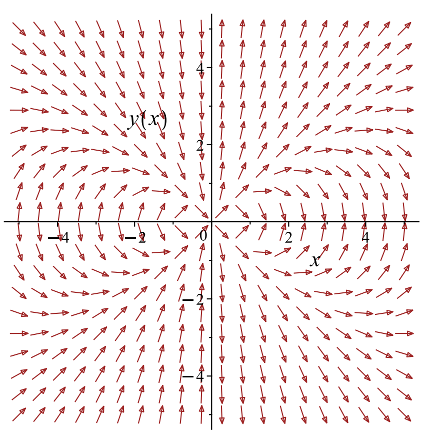
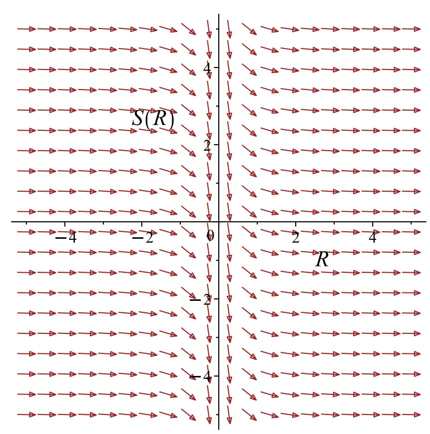
To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$\frac{y^2}{2x^3} = \frac{1}{2x} + c_1$$

Which simplifies to

$$\frac{y^2}{2x^3} = \frac{1}{2x} + c_1$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in x, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dx} = \frac{-x^2+3y^2}{2yx}$ 	$R = x$ $S = \frac{y^2}{2x^3}$	$\frac{dS}{dR} = -\frac{1}{2R^2}$ 

Summary

The solution(s) found are the following

$$\frac{y^2}{2x^3} = \frac{1}{2x} + c_1 \tag{1}$$

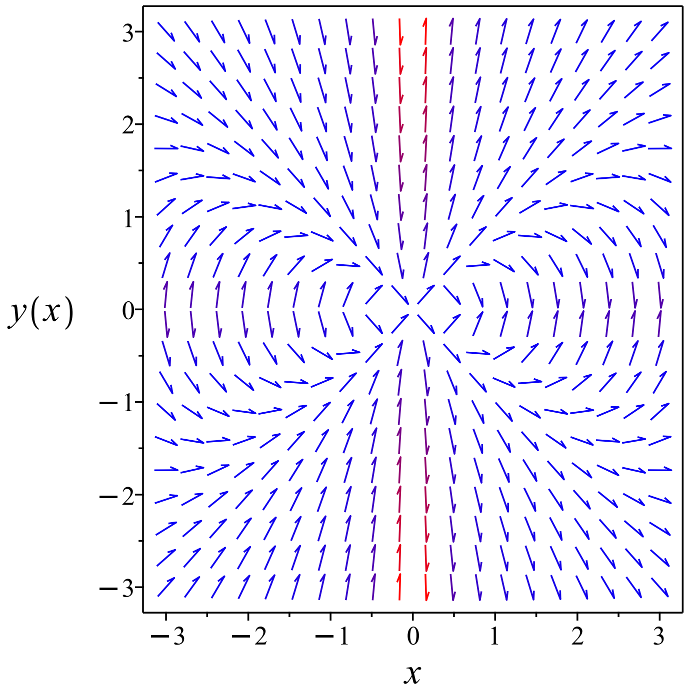


Figure 173: Slope field plot

Verification of solutions

$$\frac{y^2}{2x^3} = \frac{1}{2x} + c_1$$

Verified OK.

2.37.3 Solving as bernoulli ode

In canonical form, the ODE is

$$\begin{aligned} y' &= F(x,y) \\ &= \frac{-x^2+3y^2}{2yx} \end{aligned}$$

This is a Bernoulli ODE.

$$y' = \frac{3}{2x}y - \frac{x}{2}\frac{1}{y} \tag{1}$$

The standard Bernoulli ODE has the form

$$y' = f_0(x)y + f_1(x)y^n \tag{2}$$

The first step is to divide the above equation by y^n which gives

$$\frac{y'}{y^n} = f_0(x)y^{1-n} + f_1(x) \tag{3}$$

The next step is use the substitution $w = y^{1-n}$ in equation (3) which generates a new ODE in $w(x)$ which will be linear and can be easily solved using an integrating factor. Backsubstitution then gives the solution $y(x)$ which is what we want.

This method is now applied to the ODE at hand. Comparing the ODE (1) With (2) Shows that

$$
\begin{aligned}
f_0(x) &= \frac{3}{2x} \\
f_1(x) &= -\frac{x}{2} \\
n &= -1
\end{aligned}
$$

Dividing both sides of ODE (1) by $y^n = \frac{1}{y}$ gives

$$
y'y = \frac{3y^2}{2x} - \frac{x}{2} \tag{4}
$$

Let

$$
\begin{aligned}
w &= y^{1-n} \\
&= y^2
\end{aligned} \tag{5}
$$

Taking derivative of equation (5) w.r.t x gives

$$
w' = 2yy' \tag{6}
$$

Substituting equations (5) and (6) into equation (4) gives

$$
\begin{aligned}
\frac{w'(x)}{2} &= \frac{3w(x)}{2x} - \frac{x}{2} \\
w' &= \frac{3w}{x} - x
\end{aligned} \tag{7}
$$

The above now is a linear ODE in $w(x)$ which is now solved.

Entering Linear first order ODE solver. In canonical form a linear first order is

$$
w'(x) + p(x)w(x) = q(x)
$$

Where here

$$
\begin{aligned}
p(x) &= -\frac{3}{x} \\
q(x) &= -x
\end{aligned}
$$

Hence the ode is

$$w'(x) - \frac{3w(x)}{x} = -x$$

The integrating factor μ is

$$\begin{aligned}\mu &= e^{\int -\frac{3}{x}dx} \\ &= \frac{1}{x^3}\end{aligned}$$

The ode becomes

$$\begin{aligned}\frac{d}{dx}(\mu w) &= (\mu)(-x) \\ \frac{d}{dx}\left(\frac{w}{x^3}\right) &= \left(\frac{1}{x^3}\right)(-x) \\ d\left(\frac{w}{x^3}\right) &= \left(-\frac{1}{x^2}\right) dx\end{aligned}$$

Integrating gives

$$\begin{aligned}\frac{w}{x^3} &= \int -\frac{1}{x^2}\, dx \\ \frac{w}{x^3} &= \frac{1}{x} + c_1\end{aligned}$$

Dividing both sides by the integrating factor $\mu = \frac{1}{x^3}$ results in

$$w(x) = c_1 x^3 + x^2$$

Replacing w in the above by y^2 using equation (5) gives the final solution.

$$y^2 = c_1 x^3 + x^2$$

Solving for y gives

$$\begin{aligned}y(x) &= \sqrt{c_1 x + 1}\, x \\ y(x) &= -\sqrt{c_1 x + 1}\, x\end{aligned}$$

Summary

The solution(s) found are the following

$$y = \sqrt{c_1 x + 1}\, x \tag{1}$$

$$y = -\sqrt{c_1 x + 1}\, x \tag{2}$$

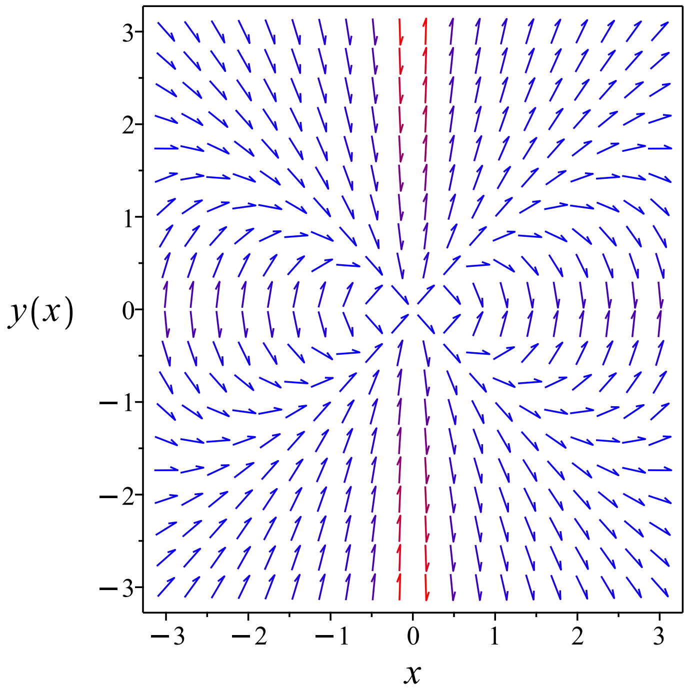


Figure 174: Slope field plot

<u>Verification of solutions</u>

$$y = \sqrt{c_1 x + 1}\, x$$

Verified OK.

$$y = -\sqrt{c_1 x + 1}\, x$$

Verified OK.

2.37.4 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x, y) + N(x, y) \frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x, y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx} \phi(x, y) = 0$$

Hence

$$\frac{\partial \phi}{\partial x} + \frac{\partial \phi}{\partial y}\frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$\begin{aligned}\frac{\partial \phi}{\partial x} &= M \\ \frac{\partial \phi}{\partial y} &= N\end{aligned}$$

But since $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x, y)$ but at least we know now that we can do that since the condition $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x, y)\,\mathrm{d}x + N(x, y)\,\mathrm{d}y = 0 \tag{1A}$$

Therefore

$$\begin{aligned}(2yx)\,\mathrm{d}y &= \left(-x^2 + 3y^2\right)\mathrm{d}x \\ \left(x^2 - 3y^2\right)\mathrm{d}x + (2yx)\,\mathrm{d}y &= 0 \qquad (2A)\end{aligned}$$

Comparing (1A) and (2A) shows that

$$\begin{aligned}M(x, y) &= x^2 - 3y^2 \\ N(x, y) &= 2yx\end{aligned}$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Using result found above gives

$$\begin{aligned}\frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}\left(x^2 - 3y^2\right) \\ &= -6y\end{aligned}$$

And

$$
\begin{aligned}
\frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}(2yx) \\
&= 2y
\end{aligned}
$$

Since $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$, then the ODE is not exact. Since the ODE is not exact, we will try to find an integrating factor to make it exact. Let

$$
\begin{aligned}
A &= \frac{1}{N}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right) \\
&= \frac{1}{2yx}((-6y) - (2y)) \\
&= -\frac{4}{x}
\end{aligned}
$$

Since A does not depend on y, then it can be used to find an integrating factor. The integrating factor μ is

$$
\begin{aligned}
\mu &= e^{\int A \, \mathrm{d}x} \\
&= e^{\int -\frac{4}{x} \, \mathrm{d}x}
\end{aligned}
$$

The result of integrating gives

$$
\begin{aligned}
\mu &= e^{-4\ln(x)} \\
&= \frac{1}{x^4}
\end{aligned}
$$

M and N are multiplied by this integrating factor, giving new M and new N which are called $\overline{M}$ and $\overline{N}$ for now so not to confuse them with the original M and N.

$$
\begin{aligned}
\overline{M} &= \mu M \\
&= \frac{1}{x^4}(x^2 - 3y^2) \\
&= \frac{x^2 - 3y^2}{x^4}
\end{aligned}
$$

And

$$
\begin{aligned}
\overline{N} &= \mu N \\
&= \frac{1}{x^4}(2yx) \\
&= \frac{2y}{x^3}
\end{aligned}
$$

Now a modified ODE is ontained from the original ODE, which is exact and can be solved. The modified ODE is

$$\overline{M} + \overline{N}\frac{\mathrm{d}y}{\mathrm{d}x} = 0$$
$$\left(\frac{x^2 - 3y^2}{x^4}\right) + \left(\frac{2y}{x^3}\right)\frac{\mathrm{d}y}{\mathrm{d}x} = 0$$

The following equations are now set up to solve for the function $\phi(x, y)$

$$\frac{\partial \phi}{\partial x} = \overline{M} \tag{1}$$
$$\frac{\partial \phi}{\partial y} = \overline{N} \tag{2}$$

Integrating (1) w.r.t. x gives

$$\int \frac{\partial \phi}{\partial x}\,\mathrm{d}x = \int \overline{M}\,\mathrm{d}x$$
$$\int \frac{\partial \phi}{\partial x}\,\mathrm{d}x = \int \frac{x^2 - 3y^2}{x^4}\,\mathrm{d}x$$
$$\phi = -\frac{1}{x} + \frac{y^2}{x^3} + f(y) \tag{3}$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$\frac{\partial \phi}{\partial y} = \frac{2y}{x^3} + f'(y) \tag{4}$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = \frac{2y}{x^3}$. Therefore equation (4) becomes

$$\frac{2y}{x^3} = \frac{2y}{x^3} + f'(y) \tag{5}$$

Solving equation (5) for $f'(y)$ gives

$$f'(y) = 0$$

Therefore

$$f(y) = c_1$$

Where c_1 is constant of integration. Substituting this result for $f(y)$ into equation (3) gives ϕ

$$\phi = -\frac{1}{x} + \frac{y^2}{x^3} + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = -\frac{1}{x} + \frac{y^2}{x^3}$$

Summary

The solution(s) found are the following

$$-\frac{1}{x} + \frac{y^2}{x^3} = c_1 \tag{1}$$

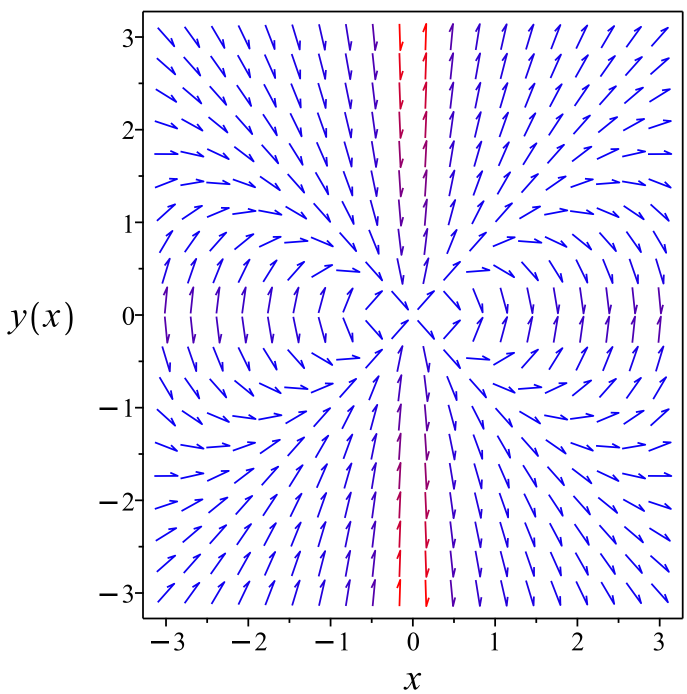


Figure 175: Slope field plot

Verification of solutions

$$-\frac{1}{x} + \frac{y^2}{x^3} = c_1$$

Verified OK.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
<- Bernoulli successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 26

```
dsolve(diff(y(x),x) = (3*y(x)^2-x^2)/(2*x*y(x)),y(x), singsol=all)
```

$$y(x) = \sqrt{c_1 x + 1}\, x$$
$$y(x) = -\sqrt{c_1 x + 1}\, x$$

✓ Solution by Mathematica

Time used: 0.209 (sec). Leaf size: 34

```
DSolve[y'[x] == (3*y[x]^2-x^2)/(2*x*y[x]),y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to -x\sqrt{1 + c_1 x}$$
$$y(x) \to x\sqrt{1 + c_1 x}$$

3 Section 2.4. Page 76

3.1 problem 1

Internal problem ID [516]
Internal file name [`OUTPUT/516_Sunday_June_05_2022_01_42_53_AM_57257413/index.tex`]

Book: Elementary differential equations and boundary value problems, 10th ed., Boyce and DiPrima
Section: Section 2.4. Page 76
Problem number: 1.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"linear", "exactWithIntegrationFactor", "first_order_ode_lie_symmetry_lookup"**

Maple gives the following as the ode type

`[_linear]`

$$\boxed{\ln(t)\, y + (t-3)\, y' = 2t}$$

3.1.1 Solving as linear ode

Entering Linear first order ODE solver. In canonical form a linear first order is

$$y' + p(t)y = q(t)$$

Where here

$$p(t) = \frac{\ln(t)}{t-3}$$
$$q(t) = \frac{2t}{t-3}$$

Hence the ode is

$$y' + \frac{\ln(t)\, y}{t-3} = \frac{2t}{t-3}$$

The integrating factor μ is

$$\mu = \mathrm{e}^{\int \frac{\ln(t)}{t-3}dt}$$

The ode becomes

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}(\mu y) &= (\mu)\left(\frac{2t}{t-3}\right) \\
\frac{\mathrm{d}}{\mathrm{d}t}\left(\mathrm{e}^{\int \frac{\ln(t)}{t-3}dt} y\right) &= \left(\mathrm{e}^{\int \frac{\ln(t)}{t-3}dt}\right)\left(\frac{2t}{t-3}\right) \\
\mathrm{d}\left(\mathrm{e}^{\int \frac{\ln(t)}{t-3}dt} y\right) &= \left(\frac{2t\,\mathrm{e}^{\int \frac{\ln(t)}{t-3}dt}}{t-3}\right)\mathrm{d}t
\end{aligned}$$

Integrating gives

$$\begin{aligned}
\mathrm{e}^{\int \frac{\ln(t)}{t-3}dt} y &= \int \frac{2t\,\mathrm{e}^{\int \frac{\ln(t)}{t-3}dt}}{t-3}\,\mathrm{d}t \\
\mathrm{e}^{\int \frac{\ln(t)}{t-3}dt} y &= \int \frac{2t\,\mathrm{e}^{\int \frac{\ln(t)}{t-3}dt}}{t-3}dt + c_1
\end{aligned}$$

Dividing both sides by the integrating factor $\mu = \mathrm{e}^{\int \frac{\ln(t)}{t-3}dt}$ results in

$$y = \mathrm{e}^{-\left(\int \frac{\ln(t)}{t-3}dt\right)}\left(\int \frac{2t\,\mathrm{e}^{\int \frac{\ln(t)}{t-3}dt}}{t-3}dt\right) + c_1 \mathrm{e}^{-\left(\int \frac{\ln(t)}{t-3}dt\right)}$$

which simplifies to

$$y = \mathrm{e}^{-\left(\int \frac{\ln(t)}{t-3}dt\right)}\left(2\left(\int \frac{t\,\mathrm{e}^{\int \frac{\ln(t)}{t-3}dt}}{t-3}dt\right) + c_1\right)$$

Which can be simplified to become

$$y = \mathrm{e}^{\int -\frac{\ln(t)}{t-3}dt}\left(2\left(\int \frac{t\,\mathrm{e}^{\int \frac{\ln(t)}{t-3}dt}}{t-3}dt\right) + c_1\right)$$

<u>Summary</u>

The solution(s) found are the following

$$y = \mathrm{e}^{\int -\frac{\ln(t)}{t-3}dt}\left(2\left(\int \frac{t\,\mathrm{e}^{\int \frac{\ln(t)}{t-3}dt}}{t-3}dt\right) + c_1\right) \tag{1}$$

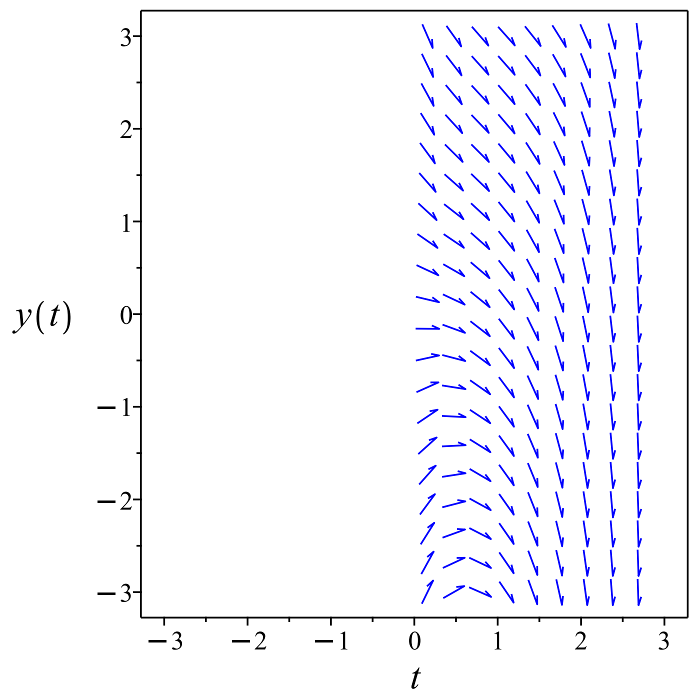


Figure 176: Slope field plot

Verification of solutions

$$y = e^{\int -\frac{\ln(t)}{t-3}dt} \left(2\left(\int \frac{t\, e^{\int \frac{\ln(t)}{t-3}dt}}{t-3}dt\right) + c_1\right)$$

Verified OK.

3.1.2 Solving as first order ode lie symmetry lookup ode

Writing the ode as

$$y' = -\frac{\ln(t)\, y - 2t}{t-3}$$
$$y' = \omega(t, y)$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_t + \omega(\eta_y - \xi_t) - \omega^2 \xi_y - \omega_t \xi - \omega_y \eta = 0 \tag{A}$$

The type of this ode is known. It is of type `linear`. Therefore we do not need to solve the PDE (A), and can just use the lookup table shown below to find ξ, η

Table 182: Lie symmetry infinitesimal lookup table for known first order ODE's

ODE class	Form	ξ	η
linear ode	$y' = f(x)y(x) + g(x)$	0	$e^{\int f dx}$
separable ode	$y' = f(x)\, g(y)$	$\frac{1}{f}$	0
quadrature ode	$y' = f(x)$	0	1
quadrature ode	$y' = g(y)$	1	0
homogeneous ODEs of Class A	$y' = f\left(\frac{y}{x}\right)$	x	y
homogeneous ODEs of Class C	$y' = (a + bx + cy)^{\frac{n}{m}}$	1	$-\frac{b}{c}$
homogeneous class D	$y' = \frac{y}{x} + g(x)\, F\left(\frac{y}{x}\right)$	x^2	xy
First order special form ID 1	$y' = g(x)\, e^{h(x)+by} + f(x)$	$\frac{e^{-\int bf(x)dx - h(x)}}{g(x)}$	$\frac{f(x)e^{-\int bf(x)dx - h(x)}}{g(x)}$
polynomial type ode	$y' = \frac{a_1 x + b_1 y + c_1}{a_2 x + b_2 y + c_2}$	$\frac{a_1 b_2 x - a_2 b_1 x - b_1 c_2 + b_2 c_1}{a_1 b_2 - a_2 b_1}$	$\frac{a_1 b_2 y - a_2 b_1 y - a_1 c_2 - a_2 c_1}{a_1 b_2 - a_2 b_1}$
Bernoulli ode	$y' = f(x)\, y + g(x)\, y^n$	0	$e^{-\int (n-1) f(x) dx} y^n$
Reduced Riccati	$y' = f_1(x)\, y + f_2(x)\, y^2$	0	$e^{-\int f_1 dx}$

The above table shows that

$$
\begin{aligned}
\xi(t, y) &= 0 \\
\eta(t, y) &= \mathrm{e}^{-\left(\ln(t) - \ln\left(\frac{t}{3}\right)\right) \ln\left(-\frac{t}{3}+1\right) + \operatorname{dilog}\left(\frac{t}{3}\right)}
\end{aligned} \tag{A1}
$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(t, y) \to (R, S)$ where (R, S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$
\frac{dt}{\xi} = \frac{dy}{\eta} = dS \tag{1}
$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial t} + \eta \frac{\partial}{\partial y}\right) S(t, y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the

canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$R = t$$

S is found from

$$\begin{aligned} S &= \int \frac{1}{\eta} dy \\ &= \int \frac{1}{\mathrm{e}^{-\left(\ln(t)-\ln\left(\frac{t}{3}\right)\right)\ln\left(-\frac{t}{3}+1\right)+\mathrm{dilog}\left(\frac{t}{3}\right)}} dy \end{aligned}$$

Which results in

$$S = \mathrm{e}^{-\ln\left(\frac{1}{3}\right)\ln\left(-\frac{t}{3}+1\right)-\mathrm{dilog}\left(\frac{t}{3}\right)} y$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_t + \omega(t,y) S_y}{R_t + \omega(t,y) R_y} \tag{2}$$

Where in the above R_t, R_y, S_t, S_y are all partial derivatives and $\omega(t,y)$ is the right hand side of the original ode given by

$$\omega(t,y) = -\frac{\ln(t)\, y - 2t}{t-3}$$

Evaluating all the partial derivatives gives

$$\begin{aligned} R_t &= 1 \\ R_y &= 0 \\ S_t &= -(-t+3)^{-1+\ln(3)}\, y \ln(t)\, \mathrm{e}^{-\ln(3)^2-\mathrm{dilog}\left(\frac{t}{3}\right)} \\ S_y &= (-t+3)^{\ln(3)}\, \mathrm{e}^{-\ln(3)^2-\mathrm{dilog}\left(\frac{t}{3}\right)} \end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = -2\,\mathrm{e}^{-\ln(3)^2-\mathrm{dilog}\left(\frac{t}{3}\right)} (-t+3)^{-1+\ln(3)}\, t \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for t, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = -2\,\mathrm{e}^{-\ln(3)^2-\mathrm{dilog}\left(\frac{R}{3}\right)} (-R+3)^{-1+\ln(3)}\, R$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by

integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$S(R) = \int -2\,\mathrm{e}^{-\ln(3)^2 - \mathrm{dilog}\left(\frac{R}{3}\right)} (-R+3)^{-1+\ln(3)} R dR + c_1 \tag{4}$$

To complete the solution, we just need to transform (4) back to t, y coordinates. This results in

$$y(-t+3)^{\ln(3)}\,\mathrm{e}^{-\ln(3)^2 - \mathrm{dilog}\left(\frac{t}{3}\right)} = \int -2\,\mathrm{e}^{-\ln(3)^2 - \mathrm{dilog}\left(\frac{t}{3}\right)} (-t+3)^{-1+\ln(3)} t dt + c_1$$

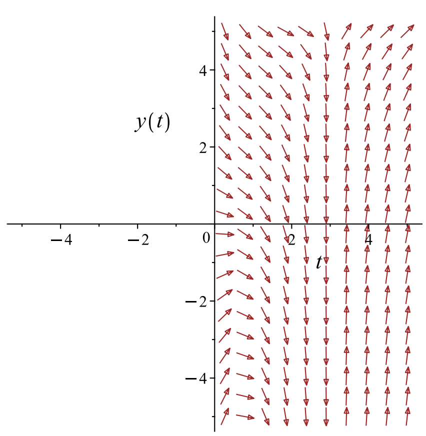
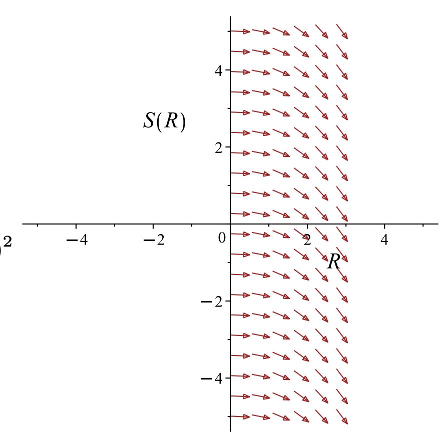
Which simplifies to

$$y(-t+3)^{\ln(3)}\,\mathrm{e}^{-\ln(3)^2 - \mathrm{dilog}\left(\frac{t}{3}\right)} = \int -2\,\mathrm{e}^{-\ln(3)^2 - \mathrm{dilog}\left(\frac{t}{3}\right)} (-t+3)^{-1+\ln(3)} t dt + c_1$$

Which gives

$$y = \left(\int -2\,\mathrm{e}^{-\ln(3)^2 - \mathrm{dilog}\left(\frac{t}{3}\right)} (-t+3)^{-1+\ln(3)} t dt + c_1\right) (-t+3)^{-\ln(3)}\,\mathrm{e}^{\ln(3)^2 + \mathrm{dilog}\left(\frac{t}{3}\right)}$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in t, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dt} = -\frac{\ln(t)y-2t}{t-3}$ 	$R = t$ $S = y(-t+3)^{\ln(3)}\,\mathrm{e}^{-\ln(3)^2}$	$\frac{dS}{dR} = -2\,\mathrm{e}^{-\ln(3)^2 - \mathrm{dilog}\left(\frac{R}{3}\right)} (-R+3)^{-1+\ln(3)} R$ 

Summary
The solution(s) found are the following

$$y = \left(\int -2\,\mathrm{e}^{-\ln(3)^2-\operatorname{dilog}\left(\frac{t}{3}\right)}(-t+3)^{-1+\ln(3)}\,t\,dt + c_1\right)(-t+3)^{-\ln(3)}\,\mathrm{e}^{\ln(3)^2+\operatorname{dilog}\left(\frac{t}{3}\right)} \tag{1}$$

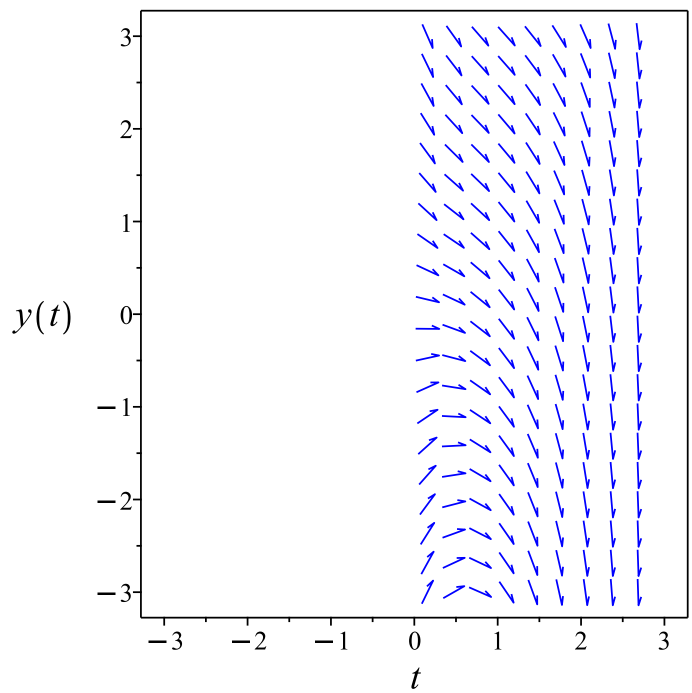


Figure 177: Slope field plot

Verification of solutions

$$y = \left(\int -2\,\mathrm{e}^{-\ln(3)^2-\operatorname{dilog}\left(\frac{t}{3}\right)}(-t+3)^{-1+\ln(3)}\,t\,dt + c_1\right)(-t+3)^{-\ln(3)}\,\mathrm{e}^{\ln(3)^2+\operatorname{dilog}\left(\frac{t}{3}\right)}$$

Verified OK.

3.1.3 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x,y) + N(x,y)\frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x,y) = c$ where c is constant, that satisfies the

ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y)=0$$

Hence

$$\frac{\partial \phi}{\partial x}+\frac{\partial \phi}{\partial y}\frac{dy}{dx}=0 \tag{B}$$

Comparing (A,B) shows that

$$\begin{aligned}\frac{\partial \phi}{\partial x}&=M\\ \frac{\partial \phi}{\partial y}&=N\end{aligned}$$

But since $\frac{\partial^2\phi}{\partial x\partial y}=\frac{\partial^2\phi}{\partial y\partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y}=\frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x,y)$ but at least we know now that we can do that since the condition $\frac{\partial^2\phi}{\partial x\partial y}=\frac{\partial^2\phi}{\partial y\partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(t,y)\,\mathrm{d}t+N(t,y)\,\mathrm{d}y=0 \tag{1A}$$

Therefore

$$\begin{aligned}(t-3)\,\mathrm{d}y&=(-\ln(t)\,y+2t)\,\mathrm{d}t\\ (\ln(t)\,y-2t)\,\mathrm{d}t+(t-3)\,\mathrm{d}y&=0 \qquad \text{(2A)}\end{aligned}$$

Comparing (1A) and (2A) shows that

$$\begin{aligned}M(t,y)&=\ln(t)\,y-2t\\ N(t,y)&=t-3\end{aligned}$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y}=\frac{\partial N}{\partial t}$$

Using result found above gives

$$
\begin{aligned}
\frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}(\ln(t)\, y - 2t) \\
&= \ln(t)
\end{aligned}
$$

And

$$
\begin{aligned}
\frac{\partial N}{\partial t} &= \frac{\partial}{\partial t}(t - 3) \\
&= 1
\end{aligned}
$$

Since $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial t}$, then the ODE is not exact. Since the ODE is not exact, we will try to find an integrating factor to make it exact. Let

$$
\begin{aligned}
A &= \frac{1}{N}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial t}\right) \\
&= \frac{1}{t-3}((\ln(t)) - (1)) \\
&= \frac{\ln(t) - 1}{t-3}
\end{aligned}
$$

Since A does not depend on y, then it can be used to find an integrating factor. The integrating factor μ is

$$
\begin{aligned}
\mu &= e^{\int A \, \mathrm{d}t} \\
&= e^{\int \frac{\ln(t)-1}{t-3} \, \mathrm{d}t}
\end{aligned}
$$

The result of integrating gives

$$
\begin{aligned}
\mu &= e^{(\ln(t)-\ln(\frac{t}{3}))\ln(-\frac{t}{3}+1)-\mathrm{dilog}(\frac{t}{3})-\ln(t-3)} \\
&= -(-t+3)^{-1+\ln(3)}\, \mathrm{e}^{-\ln(3)^2-\mathrm{dilog}(\frac{t}{3})}
\end{aligned}
$$

M and N are multiplied by this integrating factor, giving new M and new N which are called $\overline{M}$ and $\overline{N}$ for now so not to confuse them with the original M and N.

$$
\begin{aligned}
\overline{M} &= \mu M \\
&= -(-t+3)^{-1+\ln(3)}\, \mathrm{e}^{-\ln(3)^2-\mathrm{dilog}(\frac{t}{3})}(\ln(t)\, y - 2t) \\
&= (-t+3)^{-1+\ln(3)}\,(-\ln(t)\, y + 2t)\, \mathrm{e}^{-\ln(3)^2-\mathrm{dilog}(\frac{t}{3})}
\end{aligned}
$$

And

$$
\begin{aligned}
\overline{N} &= \mu N \\
&= -(-t+3)^{-1+\ln(3)}\, \mathrm{e}^{-\ln(3)^2-\mathrm{dilog}(\frac{t}{3})}(t-3) \\
&= (-t+3)^{\ln(3)}\, \mathrm{e}^{-\ln(3)^2-\mathrm{dilog}(\frac{t}{3})}
\end{aligned}
$$

Now a modified ODE is ontained from the original ODE, which is exact and can be solved. The modified ODE is

$$\overline{M} + \overline{N}\frac{\mathrm{d}y}{\mathrm{d}t} = 0$$

$$\left(\left(-t+3\right)^{-1+\ln(3)} \left(-\ln\left(t\right) y + 2t\right) \mathrm{e}^{-\ln(3)^{2}-\mathrm{dilog}\left(\frac{t}{3}\right)}\right) + \left(\left(-t+3\right)^{\ln(3)} \mathrm{e}^{-\ln(3)^{2}-\mathrm{dilog}\left(\frac{t}{3}\right)}\right) \frac{\mathrm{d}y}{\mathrm{d}t} = 0$$

The following equations are now set up to solve for the function $\phi(t, y)$

$$\frac{\partial\phi}{\partial t} = \overline{M} \tag{1}$$

$$\frac{\partial\phi}{\partial y} = \overline{N} \tag{2}$$

Integrating (1) w.r.t. t gives

$$\int \frac{\partial\phi}{\partial t}\,\mathrm{d}t = \int \overline{M}\,\mathrm{d}t$$

$$\int \frac{\partial\phi}{\partial t}\,\mathrm{d}t = \int \left(-t+3\right)^{-1+\ln(3)} \left(-\ln\left(t\right) y + 2t\right) \mathrm{e}^{-\ln(3)^{2}-\mathrm{dilog}\left(\frac{t}{3}\right)}\,\mathrm{d}t$$

$$\phi = \int^{t} \left(-\_a+3\right)^{-1+\ln(3)} \left(-\ln\left(\_a\right) y + 2\_a\right) \mathrm{e}^{-\ln(3)^{2}-\mathrm{dilog}\left(\frac{\_a}{3}\right)} d\_a + f(y) \tag{3}$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both t and y. Taking derivative of equation (3) w.r.t y gives

$$\frac{\partial\phi}{\partial y} = -\left(\int^{t} \left(-\_a+3\right)^{-1+\ln(3)} \ln\left(\_a\right) \mathrm{e}^{-\ln(3)^{2}-\mathrm{dilog}\left(\frac{\_a}{3}\right)} d\_a\right) + f'(y) \tag{4}$$

But equation (2) says that $\frac{\partial\phi}{\partial y} = \left(-t+3\right)^{\ln(3)} \mathrm{e}^{-\ln(3)^{2}-\mathrm{dilog}\left(\frac{t}{3}\right)}$. Therefore equation (4) becomes

$$\left(-t+3\right)^{\ln(3)} \mathrm{e}^{-\ln(3)^{2}-\mathrm{dilog}\left(\frac{t}{3}\right)} = -\left(\int^{t} \left(-\_a+3\right)^{-1+\ln(3)} \ln\left(\_a\right) \mathrm{e}^{-\ln(3)^{2}-\mathrm{dilog}\left(\frac{\_a}{3}\right)} d\_a\right) + f'(y) \tag{5}$$

Solving equation (5) for $f'(y)$ gives

$$f'(y) = \left(-t+3\right)^{\ln(3)} \mathrm{e}^{-\ln(3)^{2}-\mathrm{dilog}\left(\frac{t}{3}\right)} + \int^{t} \left(-\_a+3\right)^{-1+\ln(3)} \ln\left(\_a\right) \mathrm{e}^{-\ln(3)^{2}-\mathrm{dilog}\left(\frac{\_a}{3}\right)} d\_a$$

Integrating the above w.r.t y gives

$$\int f'(y)\,\mathrm{d}y = \int \left(\left(-t+3\right)^{\ln(3)} \mathrm{e}^{-\ln(3)^2-\mathrm{dilog}\left(\frac{t}{3}\right)} + \int^{t} \left(-\_a+3\right)^{-1+\ln(3)} \ln\left(\_a\right) \mathrm{e}^{-\ln(3)^2-\mathrm{dilog}\left(\frac{\_a}{3}\right)} d\_a\right) \mathrm{d}y$$

$$f(y) = \left(\left(-t+3\right)^{\ln(3)} \mathrm{e}^{-\ln(3)^2-\mathrm{dilog}\left(\frac{t}{3}\right)} + \int^{t} \left(-\_a+3\right)^{-1+\ln(3)} \ln\left(\_a\right) \mathrm{e}^{-\ln(3)^2-\mathrm{dilog}\left(\frac{\_a}{3}\right)} d\_a\right) y + c_1$$

Where c_1 is constant of integration. Substituting result found above for $f(y)$ into equation (3) gives ϕ

$$\phi = \int^{t} \left(-\_a+3\right)^{-1+\ln(3)} \left(-\ln\left(\_a\right) y + 2\_a\right) \mathrm{e}^{-\ln(3)^2-\mathrm{dilog}\left(\frac{\_a}{3}\right)} d\_a + \left(\left(-t+3\right)^{\ln(3)} \mathrm{e}^{-\ln(3)^2-\mathrm{dilog}\left(\frac{t}{3}\right)} + \int^{t} \left(-\_a+3\right)^{-1+\ln(3)} \ln\left(\_a\right) \mathrm{e}^{-\ln(3)^2-\mathrm{dilog}\left(\frac{\_a}{3}\right)} d\_a\right) y + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = \int^{t} \left(-\_a+3\right)^{-1+\ln(3)} \left(-\ln\left(\_a\right) y + 2\_a\right) \mathrm{e}^{-\ln(3)^2-\mathrm{dilog}\left(\frac{\_a}{3}\right)} d\_a + \left(\left(-t+3\right)^{\ln(3)} \mathrm{e}^{-\ln(3)^2-\mathrm{dilog}\left(\frac{t}{3}\right)} + \int^{t} \left(-\_a+3\right)^{-1+\ln(3)} \ln\left(\_a\right) \mathrm{e}^{-\ln(3)^2-\mathrm{dilog}\left(\frac{\_a}{3}\right)} d\_a\right) y$$

Summary

The solution(s) found are the following

$$\int^{t} \left(-\_a+3\right)^{-1+\ln(3)} \left(-\ln\left(\_a\right) y + 2\_a\right) \mathrm{e}^{-\ln(3)^2-\mathrm{dilog}\left(\frac{\_a}{3}\right)} d\_a + \left(\left(-t+3\right)^{\ln(3)} \mathrm{e}^{-\ln(3)^2-\mathrm{dilog}\left(\frac{t}{3}\right)} + \int^{t} \left(-\_a+3\right)^{-1+\ln(3)} \ln\left(\_a\right) \mathrm{e}^{-\ln(3)^2-\mathrm{dilog}\left(\frac{\_a}{3}\right)} d\_a\right) y = c_1 \tag{1}$$

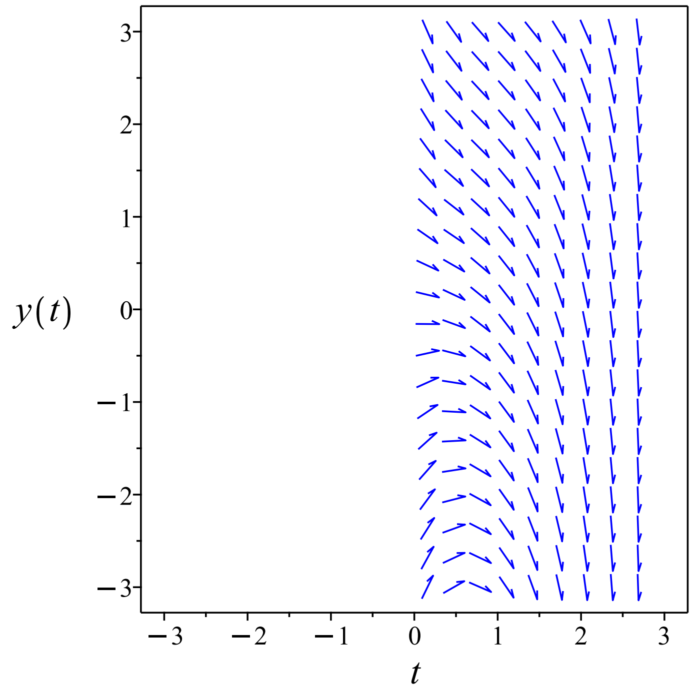


Figure 178: Slope field plot

Verification of solutions

$$\int^{t} (-\_a+3)^{-1+\ln(3)} \left(-\ln(\_a)\, y+2\_a\right) \mathrm{e}^{-\ln(3)^{2}-\operatorname{dilog}\left(\frac{\_a}{3}\right)} d\_a + \left((-t+3)^{\ln(3)} \mathrm{e}^{-\ln(3)^{2}-\operatorname{dilog}\left(\frac{t}{3}\right)} + \int^{t} (-\_a+3)^{-1+\ln(3)} \ln(\_a)\, \mathrm{e}^{-\ln(3)^{2}-\operatorname{dilog}\left(\frac{\_a}{3}\right)} d\_a\right) y = c_1$$

Verified OK.

3.1.4 Maple step by step solution

Let's solve

$\ln(t)\, y+(t-3)\, y' = 2t$

- Highest derivative means the order of the ODE is 1

  y'

- Isolate the derivative

  $y' = -\frac{\ln(t) y}{t-3}+\frac{2t}{t-3}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE

  $$y' + \frac{\ln(t)y}{t-3} = \frac{2t}{t-3}$$

- The ODE is linear; multiply by an integrating factor $\mu(t)$

  $$\mu(t)\left(y' + \frac{\ln(t)y}{t-3}\right) = \frac{2\mu(t)t}{t-3}$$

- Assume the lhs of the ODE is the total derivative $\frac{d}{dt}(\mu(t)\,y)$

  $$\mu(t)\left(y' + \frac{\ln(t)y}{t-3}\right) = \mu'(t)\,y + \mu(t)\,y'$$

- Isolate $\mu'(t)$

  $$\mu'(t) = \frac{\mu(t)\ln(t)}{t-3}$$

- Solve to find the integrating factor

  $$\mu(t) = 3^{\ln(-t+3)}\mathrm{e}^{-\ln(3)^2 - \mathit{dilog}\left(\frac{t}{3}\right)}$$

- Integrate both sides with respect to t

  $$\int \left(\frac{d}{dt}(\mu(t)\,y)\right) dt = \int \frac{2\mu(t)t}{t-3} dt + c_1$$

- Evaluate the integral on the lhs

  $$\mu(t)\,y = \int \frac{2\mu(t)t}{t-3} dt + c_1$$

- Solve for y

  $$y = \frac{\int \frac{2\mu(t)t}{t-3} dt + c_1}{\mu(t)}$$

- Substitute $\mu(t) = 3^{\ln(-t+3)}\mathrm{e}^{-\ln(3)^2 - \mathit{dilog}\left(\frac{t}{3}\right)}$

  $$y = \frac{\int \frac{2\,3^{\ln(-t+3)}\mathrm{e}^{-\ln(3)^2 - \mathit{dilog}\left(\frac{t}{3}\right)}t}{t-3} dt + c_1}{3^{\ln(-t+3)}\mathrm{e}^{-\ln(3)^2 - \mathit{dilog}\left(\frac{t}{3}\right)}}$$

- Simplify

  $$y = \left(-2\left(\int \mathrm{e}^{-\ln(3)^2 - \mathit{dilog}\left(\frac{t}{3}\right)}(-t+3)^{-1+\ln(3)}\,t dt\right) + c_1\right)(-t+3)^{-\ln(3)}\,\mathrm{e}^{\ln(3)^2 + \mathit{dilog}\left(\frac{t}{3}\right)}$$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
<- 1st order linear successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 57

```
dsolve(ln(t)*y(t)+(-3+t)*diff(y(t),t) = 2*t,y(t), singsol=all)
```

$$y(t) = \mathrm{e}^{\ln(3)^2+\mathrm{dilog}\left(\frac{t}{3}\right)}(-t+3)^{-\ln(3)}\left(-2\left(\int t(-t+3)^{-1+\ln(3)}\,\mathrm{e}^{-\ln(3)^2-\mathrm{dilog}\left(\frac{t}{3}\right)}dt\right)+c_1\right)$$

✓ Solution by Mathematica

Time used: 0.188 (sec). Leaf size: 69

```
DSolve[Log[t]*y[t]+(-3+t)*y'[t] == 2*t,y[t],t,IncludeSingularSolutions -> True]
```

$$y(t) \to e^{\mathrm{PolyLog}\left(2,1-\frac{t}{3}\right)-\log(3)\log(t-3)}\left(\int_1^t \frac{2e^{\log(3)\log(K[1]-3)-\mathrm{PolyLog}\left(2,1-\frac{K[1]}{3}\right)}K[1]}{K[1]-3}dK[1]+c_1\right)$$

3.2 problem 2

Internal problem ID [517]
Internal file name [`OUTPUT/517_Sunday_June_05_2022_01_42_55_AM_26177264/index.tex`]

Book: Elementary differential equations and boundary value problems, 10th ed., Boyce and DiPrima
Section: Section 2.4. Page 76
Problem number: 2.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"exact", "linear", "separable", "homogeneousTypeD2", "first_order_ode_lie_symmetry_lookup"**

Maple gives the following as the ode type

`[_separable]`

$$\boxed{y + (t-4)\, t y' = 0}$$

With initial conditions

$$[y(2) = 1]$$

3.2.1 Existence and uniqueness analysis

This is a linear ODE. In canonical form it is written as

$$y' + p(t)y = q(t)$$

Where here

$$\begin{aligned} p(t) &= \frac{1}{(t-4)\,t} \\ q(t) &= 0 \end{aligned}$$

Hence the ode is

$$y' + \frac{y}{(t-4)\,t} = 0$$

The domain of $p(t) = \frac{1}{(t-4)t}$ is

$$\{-\infty \leq t < 0, 0 < t < 4, 4 < t \leq \infty\}$$

And the point $t_0 = 2$ is inside this domain. Hence solution exists and is unique.

3.2.2 Solving as separable ode

In canonical form the ODE is

$$\begin{aligned} y' &= F(t,y) \\ &= f(t)g(y) \\ &= -\frac{y}{(t-4)\,t} \end{aligned}$$

Where $f(t) = -\frac{1}{(t-4)t}$ and $g(y) = y$. Integrating both sides gives

$$\begin{aligned} \frac{1}{y}\,dy &= -\frac{1}{(t-4)\,t}\,dt \\ \int \frac{1}{y}\,dy &= \int -\frac{1}{(t-4)\,t}\,dt \\ \ln(y) &= -\frac{\ln(t-4)}{4} + \frac{\ln(t)}{4} + c_1 \\ y &= \mathrm{e}^{-\frac{\ln(t-4)}{4} + \frac{\ln(t)}{4} + c_1} \\ &= c_1 \mathrm{e}^{-\frac{\ln(t-4)}{4} + \frac{\ln(t)}{4}} \end{aligned}$$

Which can be simplified to become

$$y = \frac{c_1 t^{\frac{1}{4}}}{(t-4)^{\frac{1}{4}}}$$

Initial conditions are used to solve for c_1. Substituting $t = 2$ and $y = 1$ in the above solution gives an equation to solve for the constant of integration.

$$1 = \frac{\sqrt{2}\, c_1}{2} - \frac{i\sqrt{2}\, c_1}{2}$$

$$c_1 = \left(\frac{1}{2} + \frac{i}{2}\right)\sqrt{2}$$

Substituting c_1 found above in the general solution gives

$$y = \frac{\sqrt{2}\, t^{\frac{1}{4}} + i\sqrt{2}\, t^{\frac{1}{4}}}{2\,(t-4)^{\frac{1}{4}}}$$

Summary

The solution(s) found are the following

$$y = \frac{\sqrt{2}\, t^{\frac{1}{4}} + i\sqrt{2}\, t^{\frac{1}{4}}}{2\,(t-4)^{\frac{1}{4}}} \tag{1}$$

Verification of solutions

$$y = \frac{\sqrt{2}\, t^{\frac{1}{4}} + i\sqrt{2}\, t^{\frac{1}{4}}}{2\,(t-4)^{\frac{1}{4}}}$$

Verified OK.

3.2.3 Solving as linear ode

Entering Linear first order ODE solver. The integrating factor μ is

$$\begin{aligned}\mu &= e^{\int \frac{1}{(t-4)t} dt} \\ &= e^{\frac{\ln(t-4)}{4} - \frac{\ln(t)}{4}}\end{aligned}$$

Which simplifies to

$$\mu = \frac{(t-4)^{\frac{1}{4}}}{t^{\frac{1}{4}}}$$

The ode becomes

$$\frac{d}{dt}\mu y = 0$$

$$\frac{d}{dt}\left(\frac{(t-4)^{\frac{1}{4}} y}{t^{\frac{1}{4}}}\right) = 0$$

Integrating gives

$$\frac{(t-4)^{\frac{1}{4}} y}{t^{\frac{1}{4}}} = c_1$$

Dividing both sides by the integrating factor $\mu = \frac{(t-4)^{\frac{1}{4}}}{t^{\frac{1}{4}}}$ results in

$$y = \frac{c_1 t^{\frac{1}{4}}}{(t-4)^{\frac{1}{4}}}$$

Initial conditions are used to solve for c_1. Substituting $t = 2$ and $y = 1$ in the above solution gives an equation to solve for the constant of integration.

$$1 = \frac{\sqrt{2}\, c_1}{2} - \frac{i\sqrt{2}\, c_1}{2}$$

$$c_1 = \left(\frac{1}{2} + \frac{i}{2}\right)\sqrt{2}$$

Substituting c_1 found above in the general solution gives

$$y = \frac{\sqrt{2}\, t^{\frac{1}{4}} + i\sqrt{2}\, t^{\frac{1}{4}}}{2\,(t-4)^{\frac{1}{4}}}$$

Summary

The solution(s) found are the following

$$y = \frac{\sqrt{2}\, t^{\frac{1}{4}} + i\sqrt{2}\, t^{\frac{1}{4}}}{2\,(t-4)^{\frac{1}{4}}} \tag{1}$$

Verification of solutions

$$y = \frac{\sqrt{2}\, t^{\frac{1}{4}} + i\sqrt{2}\, t^{\frac{1}{4}}}{2\,(t-4)^{\frac{1}{4}}}$$

Verified OK.

3.2.4 Solving as homogeneousTypeD2 ode

Using the change of variables $y = u(t)\,t$ on the above ode results in new ode in $u(t)$

$$u(t)\,t + (t-4)\,t(u'(t)\,t + u(t)) = 0$$

In canonical form the ODE is

$$\begin{aligned} u' &= F(t,u) \\ &= f(t)g(u) \\ &= -\frac{u(t-3)}{t\,(t-4)} \end{aligned}$$

Where $f(t) = -\frac{t-3}{(t-4)t}$ and $g(u) = u$. Integrating both sides gives

$$\begin{aligned} \frac{1}{u}\,du &= -\frac{t-3}{(t-4)\,t}\,dt \\ \int \frac{1}{u}\,du &= \int -\frac{t-3}{(t-4)\,t}\,dt \\ \ln(u) &= -\frac{\ln(t-4)}{4} - \frac{3\ln(t)}{4} + c_2 \\ u &= \mathrm{e}^{-\frac{\ln(t-4)}{4} - \frac{3\ln(t)}{4} + c_2} \\ &= c_2 \mathrm{e}^{-\frac{\ln(t-4)}{4} - \frac{3\ln(t)}{4}} \end{aligned}$$

Which simplifies to

$$u(t) = \frac{c_2}{(t-4)^{\frac{1}{4}}\,t^{\frac{3}{4}}}$$

Therefore the solution y is

$$\begin{aligned} y &= ut \\ &= \frac{t^{\frac{1}{4}} c_2}{(t-4)^{\frac{1}{4}}} \end{aligned}$$

Initial conditions are used to solve for c_2. Substituting $t = 2$ and $y = 1$ in the above solution gives an equation to solve for the constant of integration.

$$1 = \frac{\sqrt{2}\,c_2}{2} - \frac{i\sqrt{2}\,c_2}{2}$$

$$c_2 = \left(\frac{1}{2} + \frac{i}{2}\right)\sqrt{2}$$

Substituting c_2 found above in the general solution gives

$$y = \frac{\sqrt{2}\, t^{\frac{1}{4}} + i\sqrt{2}\, t^{\frac{1}{4}}}{2\left(t-4\right)^{\frac{1}{4}}}$$

Summary

The solution(s) found are the following

$$y = \frac{\sqrt{2}\, t^{\frac{1}{4}} + i\sqrt{2}\, t^{\frac{1}{4}}}{2\left(t-4\right)^{\frac{1}{4}}} \tag{1}$$

Verification of solutions

$$y = \frac{\sqrt{2}\, t^{\frac{1}{4}} + i\sqrt{2}\, t^{\frac{1}{4}}}{2\left(t-4\right)^{\frac{1}{4}}}$$

Verified OK.

3.2.5 Solving as first order ode lie symmetry lookup ode

Writing the ode as

$$\begin{aligned} y' &= -\frac{y}{\left(t-4\right)t} \\ y' &= \omega(t, y) \end{aligned}$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_t + \omega(\eta_y - \xi_t) - \omega^2\xi_y - \omega_t\xi - \omega_y\eta = 0 \tag{A}$$

The type of this ode is known. It is of type `linear`. Therefore we do not need to solve the PDE (A), and can just use the lookup table shown below to find ξ, η

Table 185: Lie symmetry infinitesimal lookup table for known first order ODE's

ODE class	Form	ξ	η
linear ode	$y' = f(x)y(x) + g(x)$	0	$e^{\int f dx}$
separable ode	$y' = f(x)\, g(y)$	$\frac{1}{f}$	0
quadrature ode	$y' = f(x)$	0	1
quadrature ode	$y' = g(y)$	1	0
homogeneous ODEs of Class A	$y' = f\left(\frac{y}{x}\right)$	x	y
homogeneous ODEs of Class C	$y' = (a + bx + cy)^{\frac{n}{m}}$	1	$-\frac{b}{c}$
homogeneous class D	$y' = \frac{y}{x} + g(x)\, F\left(\frac{y}{x}\right)$	x^2	xy
First order special form ID 1	$y' = g(x)\, e^{h(x)+by} + f(x)$	$\frac{e^{-\int bf(x)dx - h(x)}}{g(x)}$	$\frac{f(x)e^{-\int bf(x)dx - h(x)}}{g(x)}$
polynomial type ode	$y' = \frac{a_1x+b_1y+c_1}{a_2x+b_2y+c_2}$	$\frac{a_1b_2x - a_2b_1x - b_1c_2 + b_2c_1}{a_1b_2 - a_2b_1}$	$\frac{a_1b_2y - a_2b_1y - a_1c_2 - a_2c_1}{a_1b_2 - a_2b_1}$
Bernoulli ode	$y' = f(x)\, y + g(x)\, y^n$	0	$e^{-\int (n-1)f(x)dx} y^n$
Reduced Riccati	$y' = f_1(x)\, y + f_2(x)\, y^2$	0	$e^{-\int f_1 dx}$

The above table shows that

$$
\begin{aligned}
\xi(t, y) &= 0 \\
\eta(t, y) &= \mathrm{e}^{-\frac{\ln(t-4)}{4} + \frac{\ln(t)}{4}}
\end{aligned}
\tag{A1}
$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(t, y) \to (R, S)$ where (R, S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$
\frac{dt}{\xi} = \frac{dy}{\eta} = dS \tag{1}
$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial t} + \eta \frac{\partial}{\partial y}\right) S(t, y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the

canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$R = t$$

S is found from

$$\begin{aligned} S &= \int \frac{1}{\eta} dy \\ &= \int \frac{1}{\mathrm{e}^{-\frac{\ln(t-4)}{4}+\frac{\ln(t)}{4}}} dy \end{aligned}$$

Which results in

$$S = \mathrm{e}^{\ln\left((t-4)^{\frac{1}{4}}\right)+\ln\left(\frac{1}{t^{\frac{1}{4}}}\right)} y$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_t + \omega(t,y) S_y}{R_t + \omega(t,y) R_y} \tag{2}$$

Where in the above R_t, R_y, S_t, S_y are all partial derivatives and $\omega(t, y)$ is the right hand side of the original ode given by

$$\omega(t,y) = -\frac{y}{(t-4)\,t}$$

Evaluating all the partial derivatives gives

$$\begin{aligned} R_t &= 1 \\ R_y &= 0 \\ S_t &= \frac{y}{(t-4)^{\frac{3}{4}}\, t^{\frac{5}{4}}} \\ S_y &= \frac{(t-4)^{\frac{1}{4}}}{t^{\frac{1}{4}}} \end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = 0 \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for t, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = 0$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordinates R, S. Integrating the above gives

$$S(R) = c_1 \tag{4}$$

To complete the solution, we just need to transform (4) back to t, y coordinates. This results in

$$\frac{(t-4)^{\frac{1}{4}} y}{t^{\frac{1}{4}}} = c_1$$

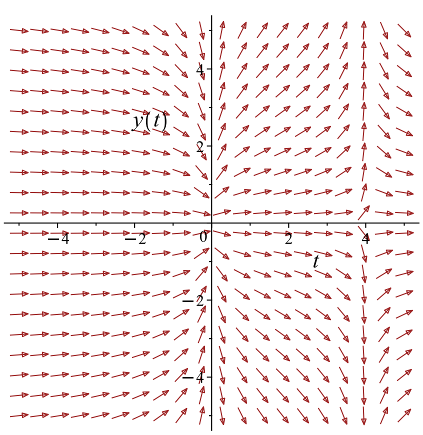
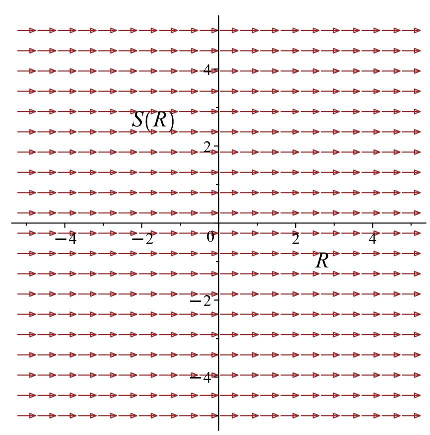
Which simplifies to

$$\frac{(t-4)^{\frac{1}{4}} y}{t^{\frac{1}{4}}} = c_1$$

Which gives

$$y = \frac{c_1 t^{\frac{1}{4}}}{(t-4)^{\frac{1}{4}}}$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in t, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dt} = -\frac{y}{(t-4)t}$ 	$R = t$ $S = \frac{(t-4)^{\frac{1}{4}} y}{t^{\frac{1}{4}}}$	$\frac{dS}{dR} = 0$ 

Initial conditions are used to solve for c_1. Substituting $t = 2$ and $y = 1$ in the above solution gives an equation to solve for the constant of integration.

$$1 = \frac{\sqrt{2}\, c_1}{2} - \frac{i\sqrt{2}\, c_1}{2}$$

$$c_1 = \left(\frac{1}{2} + \frac{i}{2}\right)\sqrt{2}$$

Substituting c_1 found above in the general solution gives

$$y = \frac{\sqrt{2}\, t^{\frac{1}{4}} + i\sqrt{2}\, t^{\frac{1}{4}}}{2\,(t-4)^{\frac{1}{4}}}$$

Summary
The solution(s) found are the following

$$y = \frac{\sqrt{2}\, t^{\frac{1}{4}} + i\sqrt{2}\, t^{\frac{1}{4}}}{2\,(t-4)^{\frac{1}{4}}} \tag{1}$$

Verification of solutions

$$y = \frac{\sqrt{2}\, t^{\frac{1}{4}} + i\sqrt{2}\, t^{\frac{1}{4}}}{2\,(t-4)^{\frac{1}{4}}}$$

Verified OK.

3.2.6 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x,y) + N(x,y)\,\frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x,y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y) = 0$$

Hence

$$\frac{\partial \phi}{\partial x} + \frac{\partial \phi}{\partial y}\frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$\frac{\partial \phi}{\partial x} = M$$
$$\frac{\partial \phi}{\partial y} = N$$

But since $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x, y)$ but at least we know now that we can do that since the condition $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(t, y)\,\mathrm{d}t + N(t, y)\,\mathrm{d}y = 0 \tag{1A}$$

Therefore

$$\left(-\frac{1}{y}\right)\mathrm{d}y = \left(\frac{1}{(t-4)\,t}\right)\mathrm{d}t$$
$$\left(-\frac{1}{(t-4)\,t}\right)\mathrm{d}t + \left(-\frac{1}{y}\right)\mathrm{d}y = 0 \tag{2A}$$

Comparing (1A) and (2A) shows that

$$M(t, y) = -\frac{1}{(t-4)\,t}$$
$$N(t, y) = -\frac{1}{y}$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial t}$$

Using result found above gives

$$\begin{aligned}\frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}\left(-\frac{1}{(t-4)\,t}\right)\\ &= 0\end{aligned}$$

And

$$
\begin{aligned}
\frac{\partial N}{\partial t} &= \frac{\partial}{\partial t}\left(-\frac{1}{y}\right) \\
&= 0
\end{aligned}
$$

Since $\frac{\partial M}{\partial y}=\frac{\partial N}{\partial t}$, then the ODE is exact The following equations are now set up to solve for the function $\phi(t,y)$

$$
\frac{\partial \phi}{\partial t} = M \tag{1}
$$

$$
\frac{\partial \phi}{\partial y} = N \tag{2}
$$

Integrating (1) w.r.t. t gives

$$
\begin{aligned}
\int \frac{\partial \phi}{\partial t}\, \mathrm{d}t &= \int M\, \mathrm{d}t \\
\int \frac{\partial \phi}{\partial t}\, \mathrm{d}t &= \int -\frac{1}{(t-4)\,t}\, \mathrm{d}t
\end{aligned}
$$

$$
\phi = -\frac{\ln(t-4)}{4} + \frac{\ln(t)}{4} + f(y) \tag{3}
$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both t and y. Taking derivative of equation (3) w.r.t y gives

$$
\frac{\partial \phi}{\partial y} = 0 + f'(y) \tag{4}
$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = -\frac{1}{y}$. Therefore equation (4) becomes

$$
-\frac{1}{y} = 0 + f'(y) \tag{5}
$$

Solving equation (5) for $f'(y)$ gives

$$
f'(y) = -\frac{1}{y}
$$

Integrating the above w.r.t y gives

$$
\begin{aligned}
\int f'(y)\, \mathrm{d}y &= \int \left(-\frac{1}{y}\right) \mathrm{d}y \\
f(y) &= -\ln(y) + c_1
\end{aligned}
$$

Where c_1 is constant of integration. Substituting result found above for $f(y)$ into equation (3) gives ϕ

$$\phi = -\frac{\ln(t-4)}{4} + \frac{\ln(t)}{4} - \ln(y) + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = -\frac{\ln(t-4)}{4} + \frac{\ln(t)}{4} - \ln(y)$$

The solution becomes

$$y = \mathrm{e}^{-\frac{\ln(t-4)}{4} + \frac{\ln(t)}{4} - c_1}$$

Initial conditions are used to solve for c_1. Substituting $t = 2$ and $y = 1$ in the above solution gives an equation to solve for the constant of integration.

$$1 = \frac{\sqrt{2}\,\mathrm{e}^{-c_1}}{2} - \frac{i\sqrt{2}\,\mathrm{e}^{-c_1}}{2}$$

$$c_1 = -\frac{i\pi}{4}$$

Substituting c_1 found above in the general solution gives

$$y = \frac{\sqrt{2}\,t^{\frac{1}{4}} + i\sqrt{2}\,t^{\frac{1}{4}}}{2\,(t-4)^{\frac{1}{4}}}$$

Summary

The solution(s) found are the following

$$y = \frac{\sqrt{2}\,t^{\frac{1}{4}} + i\sqrt{2}\,t^{\frac{1}{4}}}{2\,(t-4)^{\frac{1}{4}}} \tag{1}$$

Verification of solutions

$$y = \frac{\sqrt{2}\,t^{\frac{1}{4}} + i\sqrt{2}\,t^{\frac{1}{4}}}{2\,(t-4)^{\frac{1}{4}}}$$

Verified OK.

3.2.7 Maple step by step solution

Let's solve

$[y+(t-4)\,ty'=0, y(2)=1]$

- Highest derivative means the order of the ODE is 1

  y'

- Separate variables

  $\frac{y'}{y}=-\frac{1}{(t-4)t}$

- Integrate both sides with respect to t

  $\int \frac{y'}{y}dt=\int -\frac{1}{(t-4)t}dt+c_1$

- Evaluate integral

  $\ln(y)=-\frac{\ln(t-4)}{4}+\frac{\ln(t)}{4}+c_1$

- Use initial condition $y(2)=1$

  $0=-\frac{\mathrm{I}\pi}{4}+c_1$

- Solve for c_1

  $c_1=\frac{\mathrm{I}}{4}\pi$

- Substitute $c_1=\frac{\mathrm{I}}{4}\pi$ into general solution and simplify

  $\ln(y)=-\frac{\ln(t-4)}{4}+\frac{\ln(t)}{4}+\frac{\mathrm{I}\pi}{4}$

- Solution to the IVP

  $\ln(y)=-\frac{\ln(t-4)}{4}+\frac{\ln(t)}{4}+\frac{\mathrm{I}\pi}{4}$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
<- 1st order linear successful`
```

✓ Solution by Maple

Time used: 0.015 (sec). Leaf size: 19

```
dsolve([y(t)+(-4+t)*t*diff(y(t),t) = 0,y(2) = 1],y(t), singsol=all)
```

$$y(t) = \frac{\left(\frac{1}{2}+\frac{i}{2}\right)\sqrt{2}\, t^{\frac{1}{4}}}{\left(-4+t\right)^{\frac{1}{4}}}$$

✓ Solution by Mathematica

Time used: 0.033 (sec). Leaf size: 20

```
DSolve[{y[t]+(-4+t)*t*y'[t] == 0,y[2]==1},y[t],t,IncludeSingularSolutions -> True]
```

$$y(t) \to \frac{\sqrt[4]{t}}{\sqrt[4]{4-t}}$$

3.3 problem 3

Internal problem ID [518]
Internal file name [`OUTPUT/518_Sunday_June_05_2022_01_42_55_AM_79033667/index.tex`]

Book: Elementary differential equations and boundary value problems, 10th ed., Boyce and DiPrima
Section: Section 2.4. Page 76
Problem number: 3.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"linear", "exactWithIntegrationFactor", "first_order_ode_lie_symmetry_lookup"**

Maple gives the following as the ode type

`[_linear]`

$$\boxed{\tan(t)\, y + y' = \sin(t)}$$

With initial conditions

$$[y(\pi) = 0]$$

3.3.1 Existence and uniqueness analysis

This is a linear ODE. In canonical form it is written as

$$y' + p(t)y = q(t)$$

Where here

$$p(t) = \tan(t)$$
$$q(t) = \sin(t)$$

Hence the ode is

$$\tan(t)\, y + y' = \sin(t)$$

The domain of $p(t) = \tan(t)$ is

$$\left\{t < \frac{1}{2}\pi + \pi\_Z8 \vee \frac{1}{2}\pi + \pi\_Z8 < t\right\}$$

And the point $t_0 = \pi$ is inside this domain. The domain of $q(t) = \sin(t)$ is

$$\{-\infty < t < \infty\}$$

And the point $t_0 = \pi$ is also inside this domain. Hence solution exists and is unique.

3.3.2 Solving as linear ode

Entering Linear first order ODE solver. The integrating factor μ is

$$\begin{aligned}\mu &= \mathrm{e}^{\int \tan(t)dt} \\ &= \frac{1}{\cos(t)}\end{aligned}$$

Which simplifies to

$$\mu = \sec(t)$$

The ode becomes

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}(\mu y) &= (\mu)(\sin(t)) \\ \frac{\mathrm{d}}{\mathrm{d}t}(\sec(t)\, y) &= (\sec(t))(\sin(t)) \\ \mathrm{d}(\sec(t)\, y) &= \tan(t)\, \mathrm{d}t\end{aligned}$$

Integrating gives

$$\begin{aligned}\sec(t)\, y &= \int \tan(t)\, \mathrm{d}t \\ \sec(t)\, y &= -\ln(\cos(t)) + c_1\end{aligned}$$

Dividing both sides by the integrating factor $\mu = \sec(t)$ results in

$$y = -\cos(t)\ln(\cos(t)) + c_1\cos(t)$$

which simplifies to

$$y = \cos(t)\left(-\ln(\cos(t)) + c_1\right)$$

Initial conditions are used to solve for c_1. Substituting $t = \pi$ and $y = 0$ in the above solution gives an equation to solve for the constant of integration.

$$0 = i\pi - c_1$$

$$c_1 = i\pi$$

Substituting c_1 found above in the general solution gives

$$y = i\cos(t)\,\pi - \cos(t)\ln(\cos(t))$$

Summary
The solution(s) found are the following

$$y = i\cos(t)\,\pi - \cos(t)\ln(\cos(t)) \tag{1}$$

Verification of solutions

$$y = i\cos(t)\,\pi - \cos(t)\ln(\cos(t))$$

Verified OK.

3.3.3 Solving as first order ode lie symmetry lookup ode

Writing the ode as

$$\begin{aligned} y' &= -\tan(t)\,y + \sin(t) \\ y' &= \omega(t, y) \end{aligned}$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_t + \omega(\eta_y - \xi_t) - \omega^2\xi_y - \omega_t\xi - \omega_y\eta = 0 \tag{A}$$

The type of this ode is known. It is of type `linear`. Therefore we do not need to solve the PDE (A), and can just use the lookup table shown below to find ξ, η

Table 188: Lie symmetry infinitesimal lookup table for known first order ODE's

ODE class	Form	ξ	η
linear ode	$y' = f(x)y(x) + g(x)$	0	$e^{\int f dx}$
separable ode	$y' = f(x)\, g(y)$	$\frac{1}{f}$	0
quadrature ode	$y' = f(x)$	0	1
quadrature ode	$y' = g(y)$	1	0
homogeneous ODEs of Class A	$y' = f\left(\frac{y}{x}\right)$	x	y
homogeneous ODEs of Class C	$y' = (a + bx + cy)^{\frac{n}{m}}$	1	$-\frac{b}{c}$
homogeneous class D	$y' = \frac{y}{x} + g(x)\, F\left(\frac{y}{x}\right)$	x^2	xy
First order special form ID 1	$y' = g(x)\, e^{h(x)+by} + f(x)$	$\frac{e^{-\int bf(x)dx - h(x)}}{g(x)}$	$\frac{f(x)e^{-\int bf(x)dx - h(x)}}{g(x)}$
polynomial type ode	$y' = \frac{a_1x+b_1y+c_1}{a_2x+b_2y+c_2}$	$\frac{a_1b_2x - a_2b_1x - b_1c_2 + b_2c_1}{a_1b_2 - a_2b_1}$	$\frac{a_1b_2y - a_2b_1y - a_1c_2 - a_2c_1}{a_1b_2 - a_2b_1}$
Bernoulli ode	$y' = f(x)\, y + g(x)\, y^n$	0	$e^{-\int (n-1)f(x)dx} y^n$
Reduced Riccati	$y' = f_1(x)\, y + f_2(x)\, y^2$	0	$e^{-\int f_1 dx}$

The above table shows that

$$
\begin{aligned}
\xi(t, y) &= 0 \\
\eta(t, y) &= \cos(t) \qquad \text{(A1)}
\end{aligned}
$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(t, y) \to (R, S)$ where (R, S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$
\frac{dt}{\xi} = \frac{dy}{\eta} = dS \qquad (1)
$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial t} + \eta \frac{\partial}{\partial y}\right) S(t, y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the

canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$R = t$$

S is found from

$$\begin{aligned} S &= \int \frac{1}{\eta} dy \\ &= \int \frac{1}{\cos(t)} dy \end{aligned}$$

Which results in

$$S = \frac{y}{\cos(t)}$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_t + \omega(t,y) S_y}{R_t + \omega(t,y) R_y} \tag{2}$$

Where in the above R_t, R_y, S_t, S_y are all partial derivatives and $\omega(t,y)$ is the right hand side of the original ode given by

$$\omega(t,y) = -\tan(t)\, y + \sin(t)$$

Evaluating all the partial derivatives gives

$$\begin{aligned} R_t &= 1 \\ R_y &= 0 \\ S_t &= \sec(t)\tan(t)\, y \\ S_y &= \sec(t) \end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = \tan(t) \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for t, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = \tan(R)$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by

integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$S(R) = -\ln(\cos(R)) + c_1 \tag{4}$$

To complete the solution, we just need to transform (4) back to t, y coordinates. This results in

$$\sec(t)\, y = -\ln(\cos(t)) + c_1$$

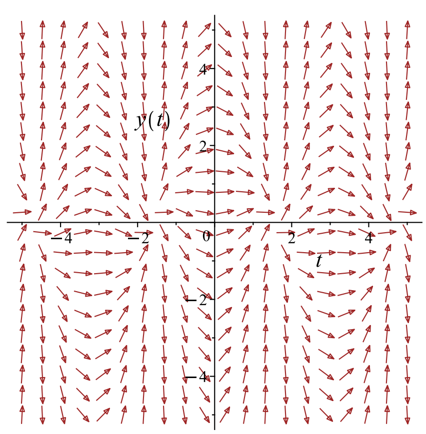
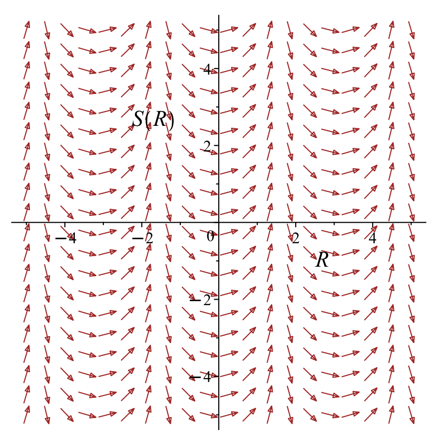
Which simplifies to

$$\sec(t)\, y = -\ln(\cos(t)) + c_1$$

Which gives

$$y = -\frac{\ln(\cos(t)) - c_1}{\sec(t)}$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in t, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dt} = -\tan(t)\, y + \sin(t)$ 	$R = t$ $S = \sec(t)\, y$	$\frac{dS}{dR} = \tan(R)$ 

Initial conditions are used to solve for c_1. Substituting $t = \pi$ and $y = 0$ in the above solution gives an equation to solve for the constant of integration.

$$0 = i\pi - c_1$$

$$c_1 = i\pi$$

Substituting c_1 found above in the general solution gives

$$y = i\cos(t)\,\pi - \cos(t)\ln(\cos(t))$$

Summary

The solution(s) found are the following

$$y = i\cos(t)\,\pi - \cos(t)\ln(\cos(t)) \tag{1}$$

Verification of solutions

$$y = i\cos(t)\,\pi - \cos(t)\ln(\cos(t))$$

Verified OK.

3.3.4 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x,y) + N(x,y)\frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x,y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y) = 0$$

Hence

$$\frac{\partial\phi}{\partial x} + \frac{\partial\phi}{\partial y}\frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$\begin{aligned}\frac{\partial\phi}{\partial x} &= M\\ \frac{\partial\phi}{\partial y} &= N\end{aligned}$$

But since $\frac{\partial^2\phi}{\partial x\partial y} = \frac{\partial^2\phi}{\partial y\partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x,y)$ but at least we know now that we can do that since the condition

$\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(t, y)\,\mathrm{d}t + N(t, y)\,\mathrm{d}y = 0 \tag{1A}$$

Therefore

$$\mathrm{d}y = (-\tan(t)\,y + \sin(t))\,\mathrm{d}t$$
$$(\tan(t)\,y - \sin(t))\,\mathrm{d}t + \mathrm{d}y = 0 \tag{2A}$$

Comparing (1A) and (2A) shows that

$$M(t, y) = \tan(t)\,y - \sin(t)$$
$$N(t, y) = 1$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial t}$$

Using result found above gives

$$\begin{aligned}\frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}(\tan(t)\,y - \sin(t)) \\ &= \tan(t)\end{aligned}$$

And

$$\begin{aligned}\frac{\partial N}{\partial t} &= \frac{\partial}{\partial t}(1) \\ &= 0\end{aligned}$$

Since $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial t}$, then the ODE is not exact. Since the ODE is not exact, we will try to find an integrating factor to make it exact. Let

$$\begin{aligned}A &= \frac{1}{N}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial t}\right) \\ &= 1((\tan(t)) - (0)) \\ &= \tan(t)\end{aligned}$$

Since A does not depend on y, then it can be used to find an integrating factor. The integrating factor μ is

$$\begin{aligned}\mu &= e^{\int A\,\mathrm{d}t} \\ &= e^{\int \tan(t)\,\mathrm{d}t}\end{aligned}$$

The result of integrating gives

$$
\begin{aligned}
\mu &= e^{-\ln(\cos(t))} \\
&= \sec(t)
\end{aligned}
$$

M and N are multiplied by this integrating factor, giving new M and new N which are called $\overline{M}$ and $\overline{N}$ for now so not to confuse them with the original M and N.

$$
\begin{aligned}
\overline{M} &= \mu M \\
&= \sec(t)(\tan(t)\, y - \sin(t)) \\
&= \tan(t)(\sec(t)\, y - 1)
\end{aligned}
$$

And

$$
\begin{aligned}
\overline{N} &= \mu N \\
&= \sec(t)\,(1) \\
&= \sec(t)
\end{aligned}
$$

Now a modified ODE is ontained from the original ODE, which is exact and can be solved. The modified ODE is

$$
\begin{aligned}
\overline{M} + \overline{N}\frac{\mathrm{d}y}{\mathrm{d}t} &= 0 \\
(\tan(t)(\sec(t)\, y - 1)) + (\sec(t))\frac{\mathrm{d}y}{\mathrm{d}t} &= 0
\end{aligned}
$$

The following equations are now set up to solve for the function $\phi(t, y)$

$$
\frac{\partial \phi}{\partial t} = \overline{M} \tag{1}
$$

$$
\frac{\partial \phi}{\partial y} = \overline{N} \tag{2}
$$

Integrating (1) w.r.t. t gives

$$
\begin{aligned}
\int \frac{\partial \phi}{\partial t}\,\mathrm{d}t &= \int \overline{M}\,\mathrm{d}t \\
\int \frac{\partial \phi}{\partial t}\,\mathrm{d}t &= \int \tan(t)(\sec(t)\, y - 1)\,\mathrm{d}t
\end{aligned}
$$

$$
\phi = \sec(t)\, y - \ln(\sec(t)) + f(y) \tag{3}
$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both t and y. Taking derivative of equation (3) w.r.t y gives

$$\frac{\partial \phi}{\partial y} = \sec(t) + f'(y) \tag{4}$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = \sec(t)$. Therefore equation (4) becomes

$$\sec(t) = \sec(t) + f'(y) \tag{5}$$

Solving equation (5) for $f'(y)$ gives

$$f'(y) = 0$$

Therefore

$$f(y) = c_1$$

Where c_1 is constant of integration. Substituting this result for $f(y)$ into equation (3) gives ϕ

$$\phi = \sec(t)\, y - \ln(\sec(t)) + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = \sec(t)\, y - \ln(\sec(t))$$

The solution becomes

$$y = \frac{\ln(\sec(t)) + c_1}{\sec(t)}$$

Initial conditions are used to solve for c_1. Substituting $t = \pi$ and $y = 0$ in the above solution gives an equation to solve for the constant of integration.

$$0 = -i\pi - c_1$$

$$c_1 = -i\pi$$

Substituting c_1 found above in the general solution gives

$$y = -i\cos(t)\,\pi + \ln(\sec(t))\cos(t)$$

Summary
The solution(s) found are the following

$$y = -i\cos(t)\,\pi + \ln(\sec(t))\cos(t) \tag{1}$$

Verification of solutions

$$y = -i\cos(t)\,\pi + \ln(\sec(t))\cos(t)$$

Verified OK.

3.3.5 Maple step by step solution

Let's solve

$[\tan(t)\,y + y' = \sin(t)\,, y(\pi) = 0]$

- Highest derivative means the order of the ODE is 1

  y'

- Isolate the derivative

  $y' = -\tan(t)\,y + \sin(t)$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE

  $\tan(t)\,y + y' = \sin(t)$

- The ODE is linear; multiply by an integrating factor $\mu(t)$

  $\mu(t)\,(\tan(t)\,y + y') = \mu(t)\sin(t)$

- Assume the lhs of the ODE is the total derivative $\frac{d}{dt}(\mu(t)\,y)$

  $\mu(t)\,(\tan(t)\,y + y') = \mu'(t)\,y + \mu(t)\,y'$

- Isolate $\mu'(t)$

  $\mu'(t) = \mu(t)\tan(t)$

- Solve to find the integrating factor

  $\mu(t) = \frac{1}{\cos(t)}$

- Integrate both sides with respect to t

  $\int\left(\frac{d}{dt}(\mu(t)\,y)\right)dt = \int \mu(t)\sin(t)\,dt + c_1$

- Evaluate the integral on the lhs

  $\mu(t)\,y = \int \mu(t)\sin(t)\,dt + c_1$

- Solve for y

$y = \frac{\int \mu(t) \sin(t) dt + c_1}{\mu(t)}$

- Substitute $\mu(t) = \frac{1}{\cos(t)}$

  $y = \cos(t) \left(\int \frac{\sin(t)}{\cos(t)} dt + c_1\right)$

- Evaluate the integrals on the rhs

  $y = \cos(t) \left(-\ln(\cos(t)) + c_1\right)$

- Use initial condition $y(\pi) = 0$

  $0 = \mathrm{I}\pi - c_1$

- Solve for c_1

  $c_1 = \mathrm{I}\pi$

- Substitute $c_1 = \mathrm{I}\pi$ into general solution and simplify

  $y = (-\ln(\cos(t)) + \mathrm{I}\pi) \cos(t)$

- Solution to the IVP

  $y = (-\ln(\cos(t)) + \mathrm{I}\pi) \cos(t)$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
<- 1st order linear successful`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 17

```
dsolve([tan(t)*y(t)+diff(y(t),t) = sin(t),y(Pi) = 0],y(t), singsol=all)
```

$$y(t) = (-\ln(\cos(t)) + i\pi) \cos(t)$$

✓ Solution by Mathematica

Time used: 0.073 (sec). Leaf size: 20

```
DSolve[{Tan[t]*y[t]+y'[t] == Sin[t],y[Pi]==0},y[t],t,IncludeSingularSolutions -> True]
```

$$y(t) \to i\cos(t)(\pi + i\log(\cos(t)))$$

3.4 problem 4

Internal problem ID [519]
Internal file name [`OUTPUT/519_Sunday_June_05_2022_01_42_56_AM_17528492/index.tex`]

Book: Elementary differential equations and boundary value problems, 10th ed., Boyce and DiPrima
Section: Section 2.4. Page 76
Problem number: 4.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"linear", "exactWithIntegrationFactor", "first_order_ode_lie_symmetry_lookup"**

Maple gives the following as the ode type

```
[_linear]
```

$$\boxed{2yt + \left(-t^2 + 4\right) y' = 3t^2}$$

With initial conditions

$$[y(-3) = 1]$$

3.4.1 Existence and uniqueness analysis

This is a linear ODE. In canonical form it is written as

$$y' + p(t)y = q(t)$$

Where here

$$p(t) = -\frac{2t}{t^2 - 4}$$
$$q(t) = -\frac{3t^2}{t^2 - 4}$$

Hence the ode is

$$y' - \frac{2ty}{t^2-4} = -\frac{3t^2}{t^2-4}$$

The domain of $p(t) = -\frac{2t}{t^2-4}$ is

$$\{-\infty \le t < -2, -2 < t < 2, 2 < t \le \infty\}$$

And the point $t_0 = -3$ is inside this domain. The domain of $q(t) = -\frac{3t^2}{t^2-4}$ is

$$\{-\infty \le t < -2, -2 < t < 2, 2 < t \le \infty\}$$

And the point $t_0 = -3$ is also inside this domain. Hence solution exists and is unique.

3.4.2 Solving as linear ode

Entering Linear first order ODE solver. The integrating factor μ is

$$\begin{aligned}\mu &= e^{\int -\frac{2t}{t^2-4}dt} \\ &= \frac{1}{t^2-4}\end{aligned}$$

The ode becomes

$$\begin{aligned}\frac{d}{dt}(\mu y) &= (\mu)\left(-\frac{3t^2}{t^2-4}\right) \\ \frac{d}{dt}\left(\frac{y}{t^2-4}\right) &= \left(\frac{1}{t^2-4}\right)\left(-\frac{3t^2}{t^2-4}\right) \\ d\left(\frac{y}{t^2-4}\right) &= \left(-\frac{3t^2}{(t^2-4)^2}\right) dt\end{aligned}$$

Integrating gives

$$\begin{aligned}\frac{y}{t^2-4} &= \int -\frac{3t^2}{(t^2-4)^2} dt \\ \frac{y}{t^2-4} &= \frac{3}{4(t-2)} - \frac{3\ln(t-2)}{8} + \frac{3}{4(2+t)} + \frac{3\ln(2+t)}{8} + c_1\end{aligned}$$

Dividing both sides by the integrating factor $\mu = \frac{1}{t^2-4}$ results in

$$y = (t^2-4)\left(\frac{3}{4(t-2)} - \frac{3\ln(t-2)}{8} + \frac{3}{4(2+t)} + \frac{3\ln(2+t)}{8}\right) + c_1(t^2-4)$$

which simplifies to

$$y = \frac{3(t^2-4)\ln(2+t)}{8} + c_1 t^2 - \frac{3\ln(t-2)\,t^2}{8} + \frac{3t}{2} - 4c_1 + \frac{3\ln(t-2)}{2}$$

Initial conditions are used to solve for c_1. Substituting $t = -3$ and $y = 1$ in the above solution gives an equation to solve for the constant of integration.

$$1 = -\frac{9}{2} + 5c_1 - \frac{15\ln(5)}{8}$$

$$c_1 = \frac{11}{10} + \frac{3\ln(5)}{8}$$

Substituting c_1 found above in the general solution gives

$$y = \frac{3\ln(2+t)\,t^2}{8} - \frac{3\ln(2+t)}{2} + \frac{11t^2}{10} + \frac{3t^2\ln(5)}{8} - \frac{3\ln(t-2)\,t^2}{8} + \frac{3t}{2} - \frac{22}{5} - \frac{3\ln(5)}{2} + \frac{3\ln(t-2)}{2}$$

Summary

The solution(s) found are the following

$$\begin{aligned} y = {} & \frac{3\ln(2+t)\,t^2}{8} - \frac{3\ln(2+t)}{2} + \frac{11t^2}{10} + \frac{3t^2\ln(5)}{8} \\ & - \frac{3\ln(t-2)\,t^2}{8} + \frac{3t}{2} - \frac{22}{5} - \frac{3\ln(5)}{2} + \frac{3\ln(t-2)}{2} \end{aligned} \tag{1}$$

Verification of solutions

$$\begin{aligned} y = {} & \frac{3\ln(2+t)\,t^2}{8} - \frac{3\ln(2+t)}{2} + \frac{11t^2}{10} + \frac{3t^2\ln(5)}{8} \\ & - \frac{3\ln(t-2)\,t^2}{8} + \frac{3t}{2} - \frac{22}{5} - \frac{3\ln(5)}{2} + \frac{3\ln(t-2)}{2} \end{aligned}$$

Verified OK.

3.4.3 Solving as first order ode lie symmetry lookup ode

Writing the ode as

$$y' = \frac{t(-3t+2y)}{t^2-4}$$
$$y' = \omega(t, y)$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_t + \omega(\eta_y - \xi_t) - \omega^2\xi_y - \omega_t\xi - \omega_y\eta = 0 \tag{A}$$

The type of this ode is known. It is of type `linear`. Therefore we do not need to solve the PDE (A), and can just use the lookup table shown below to find ξ, η

Table 191: Lie symmetry infinitesimal lookup table for known first order ODE's

ODE class	Form	ξ	η
linear ode	$y' = f(x)y(x) + g(x)$	0	$e^{\int f dx}$
separable ode	$y' = f(x)\, g(y)$	$\frac{1}{f}$	0
quadrature ode	$y' = f(x)$	0	1
quadrature ode	$y' = g(y)$	1	0
homogeneous ODEs of Class A	$y' = f\left(\frac{y}{x}\right)$	x	y
homogeneous ODEs of Class C	$y' = (a + bx + cy)^{\frac{n}{m}}$	1	$-\frac{b}{c}$
homogeneous class D	$y' = \frac{y}{x} + g(x)\, F\left(\frac{y}{x}\right)$	x^2	xy
First order special form ID 1	$y' = g(x)\, e^{h(x)+by} + f(x)$	$\frac{e^{-\int bf(x)dx-h(x)}}{g(x)}$	$\frac{f(x)e^{-\int bf(x)dx-h(x)}}{g(x)}$
polynomial type ode	$y' = \frac{a_1x+b_1y+c_1}{a_2x+b_2y+c_2}$	$\frac{a_1b_2x-a_2b_1x-b_1c_2+b_2c_1}{a_1b_2-a_2b_1}$	$\frac{a_1b_2y-a_2b_1y-a_1c_2-a_2c_1}{a_1b_2-a_2b_1}$
Bernoulli ode	$y' = f(x)\, y + g(x)\, y^n$	0	$e^{-\int (n-1)f(x)dx} y^n$
Reduced Riccati	$y' = f_1(x)\, y + f_2(x)\, y^2$	0	$e^{-\int f_1 dx}$

The above table shows that

$$
\begin{aligned}
\xi(t, y) &= 0 \\
\eta(t, y) &= t^2 - 4 \qquad \text{(A1)}
\end{aligned}
$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(t, y) \to (R, S)$ where (R, S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$
\frac{dt}{\xi} = \frac{dy}{\eta} = dS \qquad (1)
$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial t}+\eta \frac{\partial}{\partial y}\right) S(t, y)=1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the canonical coordinates, where $S(R)$. Since $\xi=0$ then in this special case

$$R=t$$

S is found from

$$\begin{aligned}
S &= \int \frac{1}{\eta} d y \\
&= \int \frac{1}{t^{2}-4} d y
\end{aligned}$$

Which results in

$$S=\frac{y}{t^{2}-4}$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{d S}{d R}=\frac{S_{t}+\omega(t, y) S_{y}}{R_{t}+\omega(t, y) R_{y}} \tag{2}$$

Where in the above $R_{t}, R_{y}, S_{t}, S_{y}$ are all partial derivatives and $\omega(t, y)$ is the right hand side of the original ode given by

$$\omega(t, y)=\frac{t(-3 t+2 y)}{t^{2}-4}$$

Evaluating all the partial derivatives gives

$$\begin{aligned}
R_{t} &= 1 \\
R_{y} &= 0 \\
S_{t} &= -\frac{2 y t}{\left(t^{2}-4\right)^{2}} \\
S_{y} &= \frac{1}{t^{2}-4}
\end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{d S}{d R}=-\frac{3 t^{2}}{\left(t^{2}-4\right)^{2}} \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for t, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{d S}{d R}=-\frac{3 R^{2}}{\left(R^{2}-4\right)^{2}}$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordinates R, S. Integrating the above gives

$$S(R) = \frac{3}{4(R-2)} - \frac{3\ln(R-2)}{8} + \frac{3}{4(R+2)} + \frac{3\ln(R+2)}{8} + c_1 \tag{4}$$

To complete the solution, we just need to transform (4) back to t, y coordinates. This results in

$$\frac{y}{t^2-4} = \frac{3}{4(t-2)} - \frac{3\ln(t-2)}{8} + \frac{3}{4(2+t)} + \frac{3\ln(2+t)}{8} + c_1$$

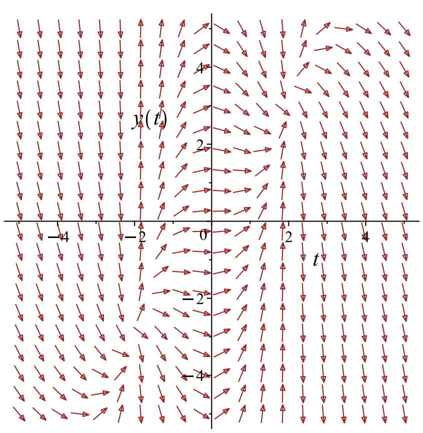
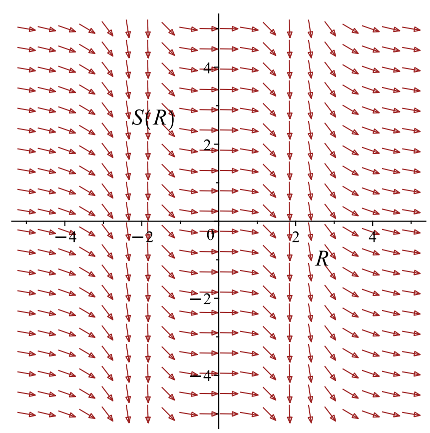
Which simplifies to

$$\frac{y}{t^2-4} = \frac{3}{4(t-2)} - \frac{3\ln(t-2)}{8} + \frac{3}{4(2+t)} + \frac{3\ln(2+t)}{8} + c_1$$

Which gives

$$y = -\frac{3\ln(t-2)\,t^2}{8} + \frac{3\ln(2+t)\,t^2}{8} + c_1 t^2 + \frac{3\ln(t-2)}{2} - \frac{3\ln(2+t)}{2} - 4c_1 + \frac{3t}{2}$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in t, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dt} = \frac{t(-3t+2y)}{t^2-4}$ 	$R = t$ $S = \frac{y}{t^2-4}$	$\frac{dS}{dR} = -\frac{3R^2}{(R^2-4)^2}$ 

Initial conditions are used to solve for c_1. Substituting $t = -3$ and $y = 1$ in the above solution gives an equation to solve for the constant of integration.

$$1 = -\frac{9}{2} + 5c_1 - \frac{15\ln(5)}{8}$$

$$c_1 = \frac{11}{10} + \frac{3\ln(5)}{8}$$

Substituting c_1 found above in the general solution gives

$$y = \frac{3\ln(2+t)\,t^2}{8} - \frac{3\ln(2+t)}{2} + \frac{11t^2}{10} + \frac{3t^2\ln(5)}{8} - \frac{3\ln(t-2)\,t^2}{8} + \frac{3t}{2} - \frac{22}{5} - \frac{3\ln(5)}{2} + \frac{3\ln(t-2)}{2}$$

Summary

The solution(s) found are the following

$$y = \frac{3\ln(2+t)\,t^2}{8} - \frac{3\ln(2+t)}{2} + \frac{11t^2}{10} + \frac{3t^2\ln(5)}{8} - \frac{3\ln(t-2)\,t^2}{8} + \frac{3t}{2} - \frac{22}{5} - \frac{3\ln(5)}{2} + \frac{3\ln(t-2)}{2} \tag{1}$$

Verification of solutions

$$y = \frac{3\ln(2+t)\,t^2}{8} - \frac{3\ln(2+t)}{2} + \frac{11t^2}{10} + \frac{3t^2\ln(5)}{8} - \frac{3\ln(t-2)\,t^2}{8} + \frac{3t}{2} - \frac{22}{5} - \frac{3\ln(5)}{2} + \frac{3\ln(t-2)}{2}$$

Verified OK.

3.4.4 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x,y) + N(x,y)\frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x,y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y) = 0$$

Hence

$$\frac{\partial\phi}{\partial x} + \frac{\partial\phi}{\partial y}\frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$\frac{\partial \phi}{\partial x} = M$$
$$\frac{\partial \phi}{\partial y} = N$$

But since $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x,y)$ but at least we know now that we can do that since the condition $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(t,y)\,\mathrm{d}t + N(t,y)\,\mathrm{d}y = 0 \tag{1A}$$

Therefore

$$\begin{aligned}(-t^2+4)\,\mathrm{d}y &= (3t^2-2ty)\,\mathrm{d}t\\ (-3t^2+2ty)\,\mathrm{d}t + (-t^2+4)\,\mathrm{d}y &= 0 \qquad (2A)\end{aligned}$$

Comparing (1A) and (2A) shows that

$$\begin{aligned}M(t,y) &= -3t^2+2ty\\ N(t,y) &= -t^2+4\end{aligned}$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial t}$$

Using result found above gives

$$\begin{aligned}\frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}(-3t^2+2ty)\\ &= 2t\end{aligned}$$

And

$$\begin{aligned}\frac{\partial N}{\partial t} &= \frac{\partial}{\partial t}(-t^2+4)\\ &= -2t\end{aligned}$$

Since $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial t}$, then the ODE is not exact. Since the ODE is not exact, we will try to find an integrating factor to make it exact. Let

$$
\begin{aligned}
A &= \frac{1}{N}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial t}\right) \\
&= -\frac{1}{t^2-4}((2t)-(-2t)) \\
&= -\frac{4t}{t^2-4}
\end{aligned}
$$

Since A does not depend on y, then it can be used to find an integrating factor. The integrating factor μ is

$$
\begin{aligned}
\mu &= e^{\int A\,\mathrm{d}t} \\
&= e^{\int -\frac{4t}{t^2-4}\,\mathrm{d}t}
\end{aligned}
$$

The result of integrating gives

$$
\begin{aligned}
\mu &= e^{-2\ln(t^2-4)} \\
&= \frac{1}{(t^2-4)^2}
\end{aligned}
$$

M and N are multiplied by this integrating factor, giving new M and new N which are called $\overline{M}$ and $\overline{N}$ for now so not to confuse them with the original M and N.

$$
\begin{aligned}
\overline{M} &= \mu M \\
&= \frac{1}{(t^2-4)^2}(-3t^2+2ty) \\
&= \frac{-3t^2+2ty}{(t^2-4)^2}
\end{aligned}
$$

And

$$
\begin{aligned}
\overline{N} &= \mu N \\
&= \frac{1}{(t^2-4)^2}(-t^2+4) \\
&= -\frac{1}{t^2-4}
\end{aligned}
$$

Now a modified ODE is ontained from the original ODE, which is exact and can be solved. The modified ODE is

$$
\begin{aligned}
\overline{M} + \overline{N}\frac{\mathrm{d}y}{\mathrm{d}t} &= 0 \\
\left(\frac{-3t^2+2ty}{(t^2-4)^2}\right) + \left(-\frac{1}{t^2-4}\right)\frac{\mathrm{d}y}{\mathrm{d}t} &= 0
\end{aligned}
$$

The following equations are now set up to solve for the function $\phi(t, y)$

$$\frac{\partial \phi}{\partial t} = \overline{M} \tag{1}$$
$$\frac{\partial \phi}{\partial y} = \overline{N} \tag{2}$$

Integrating (1) w.r.t. t gives

$$\int \frac{\partial \phi}{\partial t} \mathrm{d}t = \int \overline{M} \mathrm{d}t$$
$$\int \frac{\partial \phi}{\partial t} \mathrm{d}t = \int \frac{-3t^2 + 2ty}{(t^2-4)^2} \mathrm{d}t$$
$$\phi = -\frac{3\ln(t-2)}{8} + \frac{3-y}{4t-8} + \frac{3\ln(2+t)}{8} + \frac{3+y}{8+4t} + f(y) \tag{3}$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both t and y. Taking derivative of equation (3) w.r.t y gives

$$\frac{\partial \phi}{\partial y} = -\frac{1}{4t-8} + \frac{1}{8+4t} + f'(y) \tag{4}$$
$$= -\frac{1}{t^2-4} + f'(y)$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = -\frac{1}{t^2-4}$. Therefore equation (4) becomes

$$-\frac{1}{t^2-4} = -\frac{1}{t^2-4} + f'(y) \tag{5}$$

Solving equation (5) for $f'(y)$ gives

$$f'(y) = 0$$

Therefore

$$f(y) = c_1$$

Where c_1 is constant of integration. Substituting this result for $f(y)$ into equation (3) gives ϕ

$$\phi = -\frac{3\ln(t-2)}{8} + \frac{3-y}{4t-8} + \frac{3\ln(2+t)}{8} + \frac{3+y}{8+4t} + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = -\frac{3\ln(t-2)}{8} + \frac{3-y}{4t-8} + \frac{3\ln(2+t)}{8} + \frac{3+y}{8+4t}$$

The solution becomes

$$y = -\frac{3\ln(t-2)\,t^2}{8} + \frac{3\ln(2+t)\,t^2}{8} - c_1 t^2 + \frac{3\ln(t-2)}{2} - \frac{3\ln(2+t)}{2} + 4c_1 + \frac{3t}{2}$$

Initial conditions are used to solve for c_1. Substituting $t = -3$ and $y = 1$ in the above solution gives an equation to solve for the constant of integration.

$$1 = -\frac{15\ln(5)}{8} - \frac{9}{2} - 5c_1$$

$$c_1 = -\frac{11}{10} - \frac{3\ln(5)}{8}$$

Substituting c_1 found above in the general solution gives

$$y = \frac{3\ln(2+t)\,t^2}{8} - \frac{3\ln(2+t)}{2} + \frac{11t^2}{10} + \frac{3t^2\ln(5)}{8} - \frac{3\ln(t-2)\,t^2}{8} + \frac{3t}{2} - \frac{22}{5} - \frac{3\ln(5)}{2} + \frac{3\ln(t-2)}{2}$$

Summary

The solution(s) found are the following

$$y = \frac{3\ln(2+t)\,t^2}{8} - \frac{3\ln(2+t)}{2} + \frac{11t^2}{10} + \frac{3t^2\ln(5)}{8} - \frac{3\ln(t-2)\,t^2}{8} + \frac{3t}{2} - \frac{22}{5} - \frac{3\ln(5)}{2} + \frac{3\ln(t-2)}{2} \tag{1}$$

Verification of solutions

$$y = \frac{3\ln(2+t)\,t^2}{8} - \frac{3\ln(2+t)}{2} + \frac{11t^2}{10} + \frac{3t^2\ln(5)}{8} - \frac{3\ln(t-2)\,t^2}{8} + \frac{3t}{2} - \frac{22}{5} - \frac{3\ln(5)}{2} + \frac{3\ln(t-2)}{2}$$

Verified OK.

3.4.5 Maple step by step solution

Let's solve

$\left[2yt+\left(-t^{2}+4\right) y'=3t^{2}, y(-3)=1\right]$

- Highest derivative means the order of the ODE is 1

  y'

- Isolate the derivative

  $y'=\frac{2ty}{t^{2}-4}-\frac{3t^{2}}{t^{2}-4}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE

  $y'-\frac{2ty}{t^{2}-4}=-\frac{3t^{2}}{t^{2}-4}$

- The ODE is linear; multiply by an integrating factor $\mu(t)$

  $\mu(t)\left(y'-\frac{2ty}{t^{2}-4}\right)=-\frac{3\mu(t)t^{2}}{t^{2}-4}$

- Assume the lhs of the ODE is the total derivative $\frac{d}{dt}(\mu(t)\, y)$

  $\mu(t)\left(y'-\frac{2ty}{t^{2}-4}\right)=\mu'(t)\, y+\mu(t)\, y'$

- Isolate $\mu'(t)$

  $\mu'(t)=-\frac{2\mu(t)t}{t^{2}-4}$

- Solve to find the integrating factor

  $\mu(t)=\frac{1}{(t-2)(2+t)}$

- Integrate both sides with respect to t

  $\int\left(\frac{d}{dt}(\mu(t)\, y)\right)dt=\int-\frac{3\mu(t)t^{2}}{t^{2}-4}dt+c_{1}$

- Evaluate the integral on the lhs

  $\mu(t)\, y=\int-\frac{3\mu(t)t^{2}}{t^{2}-4}dt+c_{1}$

- Solve for y

  $y=\frac{\int-\frac{3\mu(t)t^{2}}{t^{2}-4}dt+c_{1}}{\mu(t)}$

- Substitute $\mu(t)=\frac{1}{(t-2)(2+t)}$

  $y=(t-2)\,(2+t)\left(\int-\frac{3t^{2}}{(t-2)(2+t)(t^{2}-4)}dt+c_{1}\right)$

- Evaluate the integrals on the rhs

  $y=(t-2)\,(2+t)\left(\frac{3}{4(t-2)}-\frac{3\ln(t-2)}{8}+\frac{3}{4(2+t)}+\frac{3\ln(2+t)}{8}+c_{1}\right)$

- Simplify

  $y = \frac{3(t^2-4)\ln(2+t)}{8} + c_1t^2 - \frac{3\ln(t-2)t^2}{8} + \frac{3t}{2} - 4c_1 + \frac{3\ln(t-2)}{2}$
- Use initial condition $y(-3) = 1$

  $1 = -\frac{9}{2} + 5c_1 - \frac{15\ln(5)}{8}$
- Solve for c_1

  $c_1 = \frac{11}{10} + \frac{3\ln(5)}{8}$
- Substitute $c_1 = \frac{11}{10} + \frac{3\ln(5)}{8}$ into general solution and simplify

  $y = -\frac{22}{5} + \frac{3(t^2-4)\ln(2+t)}{8} + \frac{(44+15\ln(5))t^2}{40} - \frac{3\ln(t-2)t^2}{8} + \frac{3t}{2} - \frac{3\ln(5)}{2} + \frac{3\ln(t-2)}{2}$
- Solution to the IVP

  $y = -\frac{22}{5} + \frac{3(t^2-4)\ln(2+t)}{8} + \frac{(44+15\ln(5))t^2}{40} - \frac{3\ln(t-2)t^2}{8} + \frac{3t}{2} - \frac{3\ln(5)}{2} + \frac{3\ln(t-2)}{2}$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
<- 1st order linear successful`
```

✓ Solution by Maple

Time used: 0.032 (sec). Leaf size: 42

```
dsolve([2*t*y(t)+(-t^2+4)*diff(y(t),t) = 3*t^2,y(-3) = 1],y(t), singsol=all)
```

$$y(t) = \frac{3t}{2} + \frac{3\ln(2+t)\,t^2}{8} - \frac{3\ln(2+t)}{2} - \frac{3\ln(t-2)\,t^2}{8} + \frac{3\ln(t-2)}{2} + \frac{11t^2}{10} - \frac{22}{5} + \frac{3\ln(5)\,t^2}{8} - \frac{3\ln(5)}{2}$$

✓ Solution by Mathematica

Time used: 0.049 (sec). Leaf size: 67

```
DSolve[{2*t*y[t]+(-t^2+4)*y'[t] == 3*t^2,y[-3]==1},y[t],t,IncludeSingularSolutions -> True]
```

$$y(t) \to \frac{1}{40}\left(-15i\pi t^2 + 44t^2 + 15t^2\log(5) - 15\left(t^2-4\right)\log(2-t) + 15\left(t^2-4\right)\log(t+2) + 60t + 60i\pi - 176 - 60\log(5)\right)$$

3.5 problem 5

Internal problem ID [520]
Internal file name [`OUTPUT/520_Sunday_June_05_2022_01_42_58_AM_84257707/index.tex`]

Book: Elementary differential equations and boundary value problems, 10th ed., Boyce and DiPrima
Section: Section 2.4. Page 76
Problem number: 5.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"linear", "exactWithIntegrationFactor", "first_order_ode_lie_symmetry_lookup"**

Maple gives the following as the ode type

`[_linear]`

$$\boxed{2yt + \left(-t^2+4\right) y' = 3t^2}$$

With initial conditions

$$[y(1) = -3]$$

3.5.1 Existence and uniqueness analysis

This is a linear ODE. In canonical form it is written as

$$y' + p(t)y = q(t)$$

Where here

$$p(t) = -\frac{2t}{t^2-4}$$
$$q(t) = -\frac{3t^2}{t^2-4}$$

Hence the ode is

$$y' - \frac{2ty}{t^2-4} = -\frac{3t^2}{t^2-4}$$

The domain of $p(t) = -\frac{2t}{t^2-4}$ is

$$\{-\infty \le t < -2, -2 < t < 2, 2 < t \le \infty\}$$

And the point $t_0 = 1$ is inside this domain. The domain of $q(t) = -\frac{3t^2}{t^2-4}$ is

$$\{-\infty \le t < -2, -2 < t < 2, 2 < t \le \infty\}$$

And the point $t_0 = 1$ is also inside this domain. Hence solution exists and is unique.

3.5.2 Solving as linear ode

Entering Linear first order ODE solver. The integrating factor μ is

$$\begin{aligned}
\mu &= e^{\int -\frac{2t}{t^2-4}dt} \\
&= \frac{1}{t^2-4}
\end{aligned}$$

The ode becomes

$$\begin{aligned}
\frac{d}{dt}(\mu y) &= (\mu)\left(-\frac{3t^2}{t^2-4}\right) \\
\frac{d}{dt}\left(\frac{y}{t^2-4}\right) &= \left(\frac{1}{t^2-4}\right)\left(-\frac{3t^2}{t^2-4}\right) \\
d\left(\frac{y}{t^2-4}\right) &= \left(-\frac{3t^2}{(t^2-4)^2}\right) dt
\end{aligned}$$

Integrating gives

$$\begin{aligned}
\frac{y}{t^2-4} &= \int -\frac{3t^2}{(t^2-4)^2} dt \\
\frac{y}{t^2-4} &= \frac{3}{4(t-2)} - \frac{3\ln(t-2)}{8} + \frac{3}{4(2+t)} + \frac{3\ln(2+t)}{8} + c_1
\end{aligned}$$

Dividing both sides by the integrating factor $\mu = \frac{1}{t^2-4}$ results in

$$y = (t^2-4)\left(\frac{3}{4(t-2)} - \frac{3\ln(t-2)}{8} + \frac{3}{4(2+t)} + \frac{3\ln(2+t)}{8}\right) + c_1(t^2-4)$$

which simplifies to

$$y = \frac{3(t^2-4)\ln(2+t)}{8} + c_1 t^2 - \frac{3\ln(t-2)\,t^2}{8} + \frac{3t}{2} - 4c_1 + \frac{3\ln(t-2)}{2}$$

Initial conditions are used to solve for c_1. Substituting $t = 1$ and $y = -3$ in the above solution gives an equation to solve for the constant of integration.

$$-3 = -\frac{9\ln(3)}{8} - 3c_1 + \frac{9i\pi}{8} + \frac{3}{2}$$

$$c_1 = -\frac{3\ln(3)}{8} + \frac{3i\pi}{8} + \frac{3}{2}$$

Substituting c_1 found above in the general solution gives

$$y = \frac{3\ln(2+t)\,t^2}{8} - \frac{3\ln(2+t)}{2} - \frac{3t^2\ln(3)}{8} + \frac{3it^2\pi}{8} + \frac{3t^2}{2} - \frac{3\ln(t-2)\,t^2}{8} + \frac{3t}{2} + \frac{3\ln(3)}{2} - \frac{3i\pi}{2} - 6 +$$

Summary

The solution(s) found are the following

$$\begin{aligned} y = {} & \frac{3\ln(2+t)\,t^2}{8} - \frac{3\ln(2+t)}{2} - \frac{3t^2\ln(3)}{8} + \frac{3it^2\pi}{8} + \frac{3t^2}{2} \\ & - \frac{3\ln(t-2)\,t^2}{8} + \frac{3t}{2} + \frac{3\ln(3)}{2} - \frac{3i\pi}{2} - 6 + \frac{3\ln(t-2)}{2} \end{aligned} \tag{1}$$

Verification of solutions

$$\begin{aligned} y = {} & \frac{3\ln(2+t)\,t^2}{8} - \frac{3\ln(2+t)}{2} - \frac{3t^2\ln(3)}{8} + \frac{3it^2\pi}{8} + \frac{3t^2}{2} \\ & - \frac{3\ln(t-2)\,t^2}{8} + \frac{3t}{2} + \frac{3\ln(3)}{2} - \frac{3i\pi}{2} - 6 + \frac{3\ln(t-2)}{2} \end{aligned}$$

Verified OK.

3.5.3 Solving as first order ode lie symmetry lookup ode

Writing the ode as

$$\begin{aligned} y' &= \frac{t(-3t+2y)}{t^2-4} \\ y' &= \omega(t,y) \end{aligned}$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_t + \omega(\eta_y - \xi_t) - \omega^2 \xi_y - \omega_t \xi - \omega_y \eta = 0 \tag{A}$$

The type of this ode is known. It is of type `linear`. Therefore we do not need to solve the PDE (A), and can just use the lookup table shown below to find ξ, η

Table 194: Lie symmetry infinitesimal lookup table for known first order ODE's

ODE class	Form	ξ	η
linear ode	$y' = f(x)y(x) + g(x)$	0	$e^{\int f dx}$
separable ode	$y' = f(x)\, g(y)$	$\frac{1}{f}$	0
quadrature ode	$y' = f(x)$	0	1
quadrature ode	$y' = g(y)$	1	0
homogeneous ODEs of Class A	$y' = f\left(\frac{y}{x}\right)$	x	y
homogeneous ODEs of Class C	$y' = (a + bx + cy)^{\frac{n}{m}}$	1	$-\frac{b}{c}$
homogeneous class D	$y' = \frac{y}{x} + g(x)\, F\left(\frac{y}{x}\right)$	x^2	xy
First order special form ID 1	$y' = g(x)\, e^{h(x)+by} + f(x)$	$\frac{e^{-\int bf(x)dx-h(x)}}{g(x)}$	$\frac{f(x)e^{-\int bf(x)dx-h(x)}}{g(x)}$
polynomial type ode	$y' = \frac{a_1x+b_1y+c_1}{a_2x+b_2y+c_2}$	$\frac{a_1b_2x-a_2b_1x-b_1c_2+b_2c_1}{a_1b_2-a_2b_1}$	$\frac{a_1b_2y-a_2b_1y-a_1c_2-a_2c_1}{a_1b_2-a_2b_1}$
Bernoulli ode	$y' = f(x)\, y + g(x)\, y^n$	0	$e^{-\int (n-1)f(x)dx} y^n$
Reduced Riccati	$y' = f_1(x)\, y + f_2(x)\, y^2$	0	$e^{-\int f_1 dx}$

The above table shows that

$$
\begin{aligned}
\xi(t, y) &= 0 \\
\eta(t, y) &= t^2 - 4
\end{aligned} \tag{A1}
$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(t, y) \to (R, S)$ where (R, S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$
\frac{dt}{\xi} = \frac{dy}{\eta} = dS \tag{1}
$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial t} + \eta \frac{\partial}{\partial y}\right) S(t,y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$R = t$$

S is found from

$$\begin{aligned} S &= \int \frac{1}{\eta} dy \\ &= \int \frac{1}{t^2 - 4} dy \end{aligned}$$

Which results in

$$S = \frac{y}{t^2 - 4}$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_t + \omega(t,y) S_y}{R_t + \omega(t,y) R_y} \tag{2}$$

Where in the above R_t, R_y, S_t, S_y are all partial derivatives and $\omega(t,y)$ is the right hand side of the original ode given by

$$\omega(t,y) = \frac{t(-3t + 2y)}{t^2 - 4}$$

Evaluating all the partial derivatives gives

$$\begin{aligned} R_t &= 1 \\ R_y &= 0 \\ S_t &= -\frac{2yt}{(t^2 - 4)^2} \\ S_y &= \frac{1}{t^2 - 4} \end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = -\frac{3t^2}{(t^2 - 4)^2} \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for t, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = -\frac{3R^2}{(R^2 - 4)^2}$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordinates R, S. Integrating the above gives

$$S(R) = \frac{3}{4(R-2)} - \frac{3\ln(R-2)}{8} + \frac{3}{4(R+2)} + \frac{3\ln(R+2)}{8} + c_1 \tag{4}$$

To complete the solution, we just need to transform (4) back to t, y coordinates. This results in

$$\frac{y}{t^2-4} = \frac{3}{4(t-2)} - \frac{3\ln(t-2)}{8} + \frac{3}{4(2+t)} + \frac{3\ln(2+t)}{8} + c_1$$

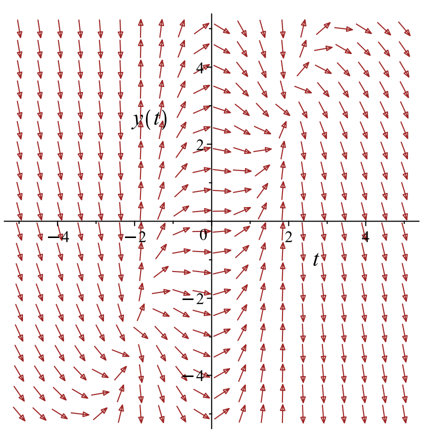
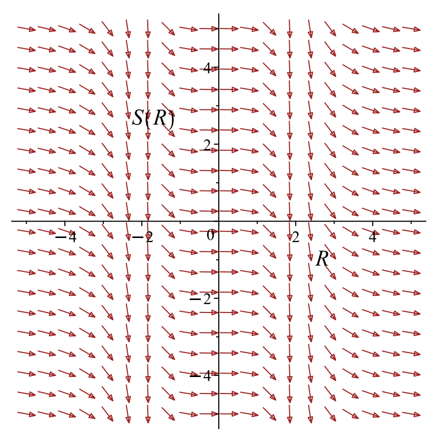
Which simplifies to

$$\frac{y}{t^2-4} = \frac{3}{4(t-2)} - \frac{3\ln(t-2)}{8} + \frac{3}{4(2+t)} + \frac{3\ln(2+t)}{8} + c_1$$

Which gives

$$y = -\frac{3\ln(t-2)\,t^2}{8} + \frac{3\ln(2+t)\,t^2}{8} + c_1 t^2 + \frac{3\ln(t-2)}{2} - \frac{3\ln(2+t)}{2} - 4c_1 + \frac{3t}{2}$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in t, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dt} = \frac{t(-3t+2y)}{t^2-4}$ 	$R = t$ $S = \frac{y}{t^2-4}$	$\frac{dS}{dR} = -\frac{3R^2}{(R^2-4)^2}$ 

Initial conditions are used to solve for c_1. Substituting $t = 1$ and $y = -3$ in the above solution gives an equation to solve for the constant of integration.

$$-3 = -\frac{9 \ln (3)}{8} - 3c_1 + \frac{9i\pi}{8} + \frac{3}{2}$$

$$c_1 = -\frac{3 \ln (3)}{8} + \frac{3i\pi}{8} + \frac{3}{2}$$

Substituting c_1 found above in the general solution gives

$$y = \frac{3 \ln (2+t) t^2}{8} - \frac{3 \ln (2+t)}{2} - \frac{3t^2 \ln (3)}{8} + \frac{3it^2\pi}{8} + \frac{3t^2}{2} - \frac{3 \ln (t-2) t^2}{8} + \frac{3t}{2} + \frac{3 \ln (3)}{2} - \frac{3i\pi}{2} - 6 +$$

Summary

The solution(s) found are the following

$$\begin{aligned} y = {} & \frac{3 \ln (2+t) t^2}{8} - \frac{3 \ln (2+t)}{2} - \frac{3t^2 \ln (3)}{8} + \frac{3it^2\pi}{8} + \frac{3t^2}{2} \\ & - \frac{3 \ln (t-2) t^2}{8} + \frac{3t}{2} + \frac{3 \ln (3)}{2} - \frac{3i\pi}{2} - 6 + \frac{3 \ln (t-2)}{2} \end{aligned} \tag{1}$$

Verification of solutions

$$\begin{aligned} y = {} & \frac{3 \ln (2+t) t^2}{8} - \frac{3 \ln (2+t)}{2} - \frac{3t^2 \ln (3)}{8} + \frac{3it^2\pi}{8} + \frac{3t^2}{2} \\ & - \frac{3 \ln (t-2) t^2}{8} + \frac{3t}{2} + \frac{3 \ln (3)}{2} - \frac{3i\pi}{2} - 6 + \frac{3 \ln (t-2)}{2} \end{aligned}$$

Verified OK.

3.5.4 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x, y) + N(x, y) \frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x, y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx} \phi(x, y) = 0$$

Hence

$$\frac{\partial \phi}{\partial x} + \frac{\partial \phi}{\partial y} \frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$\frac{\partial \phi}{\partial x} = M$$
$$\frac{\partial \phi}{\partial y} = N$$

But since $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x, y)$ but at least we know now that we can do that since the condition $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(t, y)\,\mathrm{d}t + N(t, y)\,\mathrm{d}y = 0 \tag{1A}$$

Therefore

$$\begin{aligned}(-t^2 + 4)\,\mathrm{d}y &= (3t^2 - 2ty)\,\mathrm{d}t \\ (-3t^2 + 2ty)\,\mathrm{d}t + (-t^2 + 4)\,\mathrm{d}y &= 0 \qquad (2A)\end{aligned}$$

Comparing (1A) and (2A) shows that

$$M(t, y) = -3t^2 + 2ty$$
$$N(t, y) = -t^2 + 4$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial t}$$

Using result found above gives

$$\begin{aligned}\frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}(-3t^2 + 2ty) \\ &= 2t\end{aligned}$$

And

$$\begin{aligned}\frac{\partial N}{\partial t} &= \frac{\partial}{\partial t}(-t^2 + 4) \\ &= -2t\end{aligned}$$

Since $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial t}$, then the ODE is <u>not exact</u>. Since the ODE is not exact, we will try to find an integrating factor to make it exact. Let

$$\begin{aligned} A &= \frac{1}{N}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial t}\right) \\ &= -\frac{1}{t^2-4}((2t) - (-2t)) \\ &= -\frac{4t}{t^2-4} \end{aligned}$$

Since A does not depend on y, then it can be used to find an integrating factor. The integrating factor μ is

$$\begin{aligned} \mu &= e^{\int A\, \mathrm{d}t} \\ &= e^{\int -\frac{4t}{t^2-4}\, \mathrm{d}t} \end{aligned}$$

The result of integrating gives

$$\begin{aligned} \mu &= e^{-2\ln(t^2-4)} \\ &= \frac{1}{(t^2-4)^2} \end{aligned}$$

M and N are multiplied by this integrating factor, giving new M and new N which are called $\overline{M}$ and $\overline{N}$ for now so not to confuse them with the original M and N.

$$\begin{aligned} \overline{M} &= \mu M \\ &= \frac{1}{(t^2-4)^2}(-3t^2+2ty) \\ &= \frac{-3t^2+2ty}{(t^2-4)^2} \end{aligned}$$

And

$$\begin{aligned} \overline{N} &= \mu N \\ &= \frac{1}{(t^2-4)^2}(-t^2+4) \\ &= -\frac{1}{t^2-4} \end{aligned}$$

Now a modified ODE is ontained from the original ODE, which is exact and can be solved. The modified ODE is

$$\begin{aligned} \overline{M} + \overline{N}\frac{\mathrm{d}y}{\mathrm{d}t} &= 0 \\ \left(\frac{-3t^2+2ty}{(t^2-4)^2}\right) + \left(-\frac{1}{t^2-4}\right)\frac{\mathrm{d}y}{\mathrm{d}t} &= 0 \end{aligned}$$

The following equations are now set up to solve for the function $\phi(t, y)$

$$\frac{\partial \phi}{\partial t} = \overline{M} \tag{1}$$

$$\frac{\partial \phi}{\partial y} = \overline{N} \tag{2}$$

Integrating (1) w.r.t. t gives

$$\int \frac{\partial \phi}{\partial t} \, \mathrm{d}t = \int \overline{M} \, \mathrm{d}t$$

$$\int \frac{\partial \phi}{\partial t} \, \mathrm{d}t = \int \frac{-3t^2 + 2ty}{(t^2 - 4)^2} \, \mathrm{d}t$$

$$\phi = -\frac{3 \ln (t - 2)}{8} + \frac{3 - y}{4t - 8} + \frac{3 \ln (2 + t)}{8} + \frac{3 + y}{8 + 4t} + f(y) \tag{3}$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both t and y. Taking derivative of equation (3) w.r.t y gives

$$\frac{\partial \phi}{\partial y} = -\frac{1}{4t - 8} + \frac{1}{8 + 4t} + f'(y) \tag{4}$$

$$= -\frac{1}{t^2 - 4} + f'(y)$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = -\frac{1}{t^2 - 4}$. Therefore equation (4) becomes

$$-\frac{1}{t^2 - 4} = -\frac{1}{t^2 - 4} + f'(y) \tag{5}$$

Solving equation (5) for $f'(y)$ gives

$$f'(y) = 0$$

Therefore

$$f(y) = c_1$$

Where c_1 is constant of integration. Substituting this result for $f(y)$ into equation (3) gives ϕ

$$\phi = -\frac{3 \ln (t - 2)}{8} + \frac{3 - y}{4t - 8} + \frac{3 \ln (2 + t)}{8} + \frac{3 + y}{8 + 4t} + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = -\frac{3\ln\left(t-2\right)}{8} + \frac{3-y}{4t-8} + \frac{3\ln\left(2+t\right)}{8} + \frac{3+y}{8+4t}$$

The solution becomes

$$y = -\frac{3\ln\left(t-2\right)t^2}{8} + \frac{3\ln\left(2+t\right)t^2}{8} - c_1 t^2 + \frac{3\ln\left(t-2\right)}{2} - \frac{3\ln\left(2+t\right)}{2} + 4c_1 + \frac{3t}{2}$$

Initial conditions are used to solve for c_1. Substituting $t = 1$ and $y = -3$ in the above solution gives an equation to solve for the constant of integration.

$$-3 = \frac{9i\pi}{8} - \frac{9\ln\left(3\right)}{8} + 3c_1 + \frac{3}{2}$$

$$c_1 = \frac{3\ln\left(3\right)}{8} - \frac{3i\pi}{8} - \frac{3}{2}$$

Substituting c_1 found above in the general solution gives

$$y = \frac{3\ln\left(2+t\right)t^2}{8} - \frac{3\ln\left(2+t\right)}{2} - \frac{3t^2\ln\left(3\right)}{8} + \frac{3it^2\pi}{8} + \frac{3t^2}{2} - \frac{3\ln\left(t-2\right)t^2}{8} + \frac{3t}{2} + \frac{3\ln\left(3\right)}{2} - \frac{3i\pi}{2} - 6 +$$

Summary

The solution(s) found are the following

$$\begin{aligned} y = {} & \frac{3\ln\left(2+t\right)t^2}{8} - \frac{3\ln\left(2+t\right)}{2} - \frac{3t^2\ln\left(3\right)}{8} + \frac{3it^2\pi}{8} + \frac{3t^2}{2} \\ & - \frac{3\ln\left(t-2\right)t^2}{8} + \frac{3t}{2} + \frac{3\ln\left(3\right)}{2} - \frac{3i\pi}{2} - 6 + \frac{3\ln\left(t-2\right)}{2} \end{aligned} \tag{1}$$

Verification of solutions

$$\begin{aligned} y = {} & \frac{3\ln\left(2+t\right)t^2}{8} - \frac{3\ln\left(2+t\right)}{2} - \frac{3t^2\ln\left(3\right)}{8} + \frac{3it^2\pi}{8} + \frac{3t^2}{2} \\ & - \frac{3\ln\left(t-2\right)t^2}{8} + \frac{3t}{2} + \frac{3\ln\left(3\right)}{2} - \frac{3i\pi}{2} - 6 + \frac{3\ln\left(t-2\right)}{2} \end{aligned}$$

Verified OK.

3.5.5 Maple step by step solution

Let's solve

$[2yt + (-t^2 + 4) y' = 3t^2, y(1) = -3]$

- Highest derivative means the order of the ODE is 1

  y'

- Isolate the derivative

  $y' = \frac{2ty}{t^2-4} - \frac{3t^2}{t^2-4}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE

  $y' - \frac{2ty}{t^2-4} = -\frac{3t^2}{t^2-4}$

- The ODE is linear; multiply by an integrating factor $\mu(t)$

  $\mu(t) \left(y' - \frac{2ty}{t^2-4}\right) = -\frac{3\mu(t)t^2}{t^2-4}$

- Assume the lhs of the ODE is the total derivative $\frac{d}{dt}(\mu(t) y)$

  $\mu(t) \left(y' - \frac{2ty}{t^2-4}\right) = \mu'(t) y + \mu(t) y'$

- Isolate $\mu'(t)$

  $\mu'(t) = -\frac{2\mu(t)t}{t^2-4}$

- Solve to find the integrating factor

  $\mu(t) = \frac{1}{(t-2)(2+t)}$

- Integrate both sides with respect to t

  $\int \left(\frac{d}{dt}(\mu(t) y)\right) dt = \int -\frac{3\mu(t)t^2}{t^2-4} dt + c_1$

- Evaluate the integral on the lhs

  $\mu(t) y = \int -\frac{3\mu(t)t^2}{t^2-4} dt + c_1$

- Solve for y

  $y = \frac{\int -\frac{3\mu(t)t^2}{t^2-4} dt + c_1}{\mu(t)}$

- Substitute $\mu(t) = \frac{1}{(t-2)(2+t)}$

  $y = (t-2)(2+t)\left(\int -\frac{3t^2}{(t-2)(2+t)(t^2-4)} dt + c_1\right)$

- Evaluate the integrals on the rhs

  $y = (t-2)(2+t)\left(\frac{3}{4(t-2)} - \frac{3\ln(t-2)}{8} + \frac{3}{4(2+t)} + \frac{3\ln(2+t)}{8} + c_1\right)$

- Simplify

  $y = \frac{3(t^2-4)\ln(2+t)}{8} + c_1 t^2 - \frac{3\ln(t-2)t^2}{8} + \frac{3t}{2} - 4c_1 + \frac{3\ln(t-2)}{2}$
- Use initial condition $y(1) = -3$

  $-3 = -\frac{9\ln(3)}{8} - 3c_1 + \frac{9\,\mathrm{I}\pi}{8} + \frac{3}{2}$
- Solve for c_1

  $c_1 = -\frac{3\ln(3)}{8} + \frac{3\,\mathrm{I}\pi}{8} + \frac{3}{2}$
- Substitute $c_1 = -\frac{3\ln(3)}{8} + \frac{3\,\mathrm{I}\pi}{8} + \frac{3}{2}$ into general solution and simplify

  $y = -6 + \frac{3(t^2-4)\ln(2+t)}{8} + \frac{3\,\mathrm{I}t^2\pi}{8} - \frac{3t^2\ln(3)}{8} - \frac{3\ln(t-2)t^2}{8} - \frac{3\,\mathrm{I}\pi}{2} + \frac{3t^2}{2} + \frac{3t}{2} + \frac{3\ln(3)}{2} + \frac{3\ln(t-2)}{2}$
- Solution to the IVP

  $y = -6 + \frac{3(t^2-4)\ln(2+t)}{8} + \frac{3\,\mathrm{I}t^2\pi}{8} - \frac{3t^2\ln(3)}{8} - \frac{3\ln(t-2)t^2}{8} - \frac{3\,\mathrm{I}\pi}{2} + \frac{3t^2}{2} + \frac{3t}{2} + \frac{3\ln(3)}{2} + \frac{3\ln(t-2)}{2}$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
<- 1st order linear successful`
```

✓ Solution by Maple

Time used: 0.047 (sec). Leaf size: 46

```
dsolve([2*t*y(t)+(-t^2+4)*diff(y(t),t) = 3*t^2,y(1) = -3],y(t), singsol=all)
```

$$y(t) = -6 + \frac{3(t^2-4)\ln(2+t)}{8} + \frac{3i\pi\, t^2}{8} - \frac{3\ln(3)\, t^2}{8} - \frac{3\ln(t-2)\, t^2}{8} - \frac{3i\pi}{2} + \frac{3t^2}{2} + \frac{3t}{2} + \frac{3\ln(3)}{2} + \frac{3\ln(t-2)}{2}$$

✓ Solution by Mathematica

Time used: 0.043 (sec). Leaf size: 52

```
DSolve[{2*t*y[t]+(-t^2+4)*y'[t] == 3*t^2,y[1]==-3},y[t],t,IncludeSingularSolutions -> True]
```

$$y(t) \to -\frac{3}{8}\left(-4t^2 + t^2 \log(3) + \left(t^2 - 4\right)\log(2-t) - \left(t^2 - 4\right)\log(t+2) - 4t + 16 - 4\log(3)\right)$$

3.6 problem 6

Internal problem ID [521]
Internal file name [`OUTPUT/521_Sunday_June_05_2022_01_42_59_AM_5852861/index.tex`]

Book: Elementary differential equations and boundary value problems, 10th ed., Boyce and DiPrima
Section: Section 2.4. Page 76
Problem number: 6.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"linear", "exactWithIntegrationFactor", "first_order_ode_lie_symmetry_lookup"**

Maple gives the following as the ode type

```
[_linear]
```

$$\boxed{y + \ln(t)\, y' = \cot(t)}$$

3.6.1 Solving as linear ode

Entering Linear first order ODE solver. In canonical form a linear first order is

$$y' + p(t)y = q(t)$$

Where here

$$p(t) = \frac{1}{\ln(t)}$$
$$q(t) = \frac{\cot(t)}{\ln(t)}$$

Hence the ode is

$$y' + \frac{y}{\ln(t)} = \frac{\cot(t)}{\ln(t)}$$

The integrating factor μ is

$$\mu = e^{\int \frac{1}{\ln(t)}dt}$$

The ode becomes

$$\frac{d}{dt}(\mu y) = (\mu)\left(\frac{\cot(t)}{\ln(t)}\right)$$

$$\frac{d}{dt}\left(e^{\int \frac{1}{\ln(t)}dt} y\right) = \left(e^{\int \frac{1}{\ln(t)}dt}\right)\left(\frac{\cot(t)}{\ln(t)}\right)$$

$$d\left(e^{\int \frac{1}{\ln(t)}dt} y\right) = \left(\frac{\cot(t)\, e^{\int \frac{1}{\ln(t)}dt}}{\ln(t)}\right) dt$$

Integrating gives

$$e^{\int \frac{1}{\ln(t)}dt} y = \int \frac{\cot(t)\, e^{\int \frac{1}{\ln(t)}dt}}{\ln(t)}\, dt$$

$$e^{\int \frac{1}{\ln(t)}dt} y = \int \frac{\cot(t)\, e^{\int \frac{1}{\ln(t)}dt}}{\ln(t)} dt + c_1$$

Dividing both sides by the integrating factor $\mu = e^{\int \frac{1}{\ln(t)}dt}$ results in

$$y = e^{-\left(\int \frac{1}{\ln(t)}dt\right)}\left(\int \frac{\cot(t)\, e^{\int \frac{1}{\ln(t)}dt}}{\ln(t)} dt\right) + c_1 e^{-\left(\int \frac{1}{\ln(t)}dt\right)}$$

which simplifies to

$$y = e^{-\left(\int \frac{1}{\ln(t)}dt\right)}\left(\int \frac{\cot(t)\, e^{\int \frac{1}{\ln(t)}dt}}{\ln(t)} dt + c_1\right)$$

Which can be simplified to become

$$y = e^{\int -\frac{1}{\ln(t)}dt}\left(\int \frac{\cot(t)\, e^{\int \frac{1}{\ln(t)}dt}}{\ln(t)} dt + c_1\right)$$

Summary

The solution(s) found are the following

$$y = e^{\int -\frac{1}{\ln(t)}dt}\left(\int \frac{\cot(t)\, e^{\int \frac{1}{\ln(t)}dt}}{\ln(t)} dt + c_1\right) \tag{1}$$

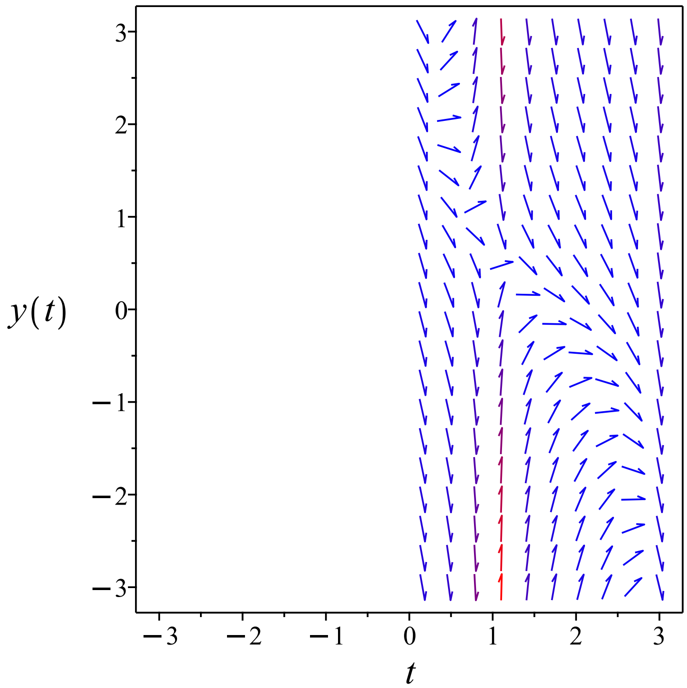


Figure 179: Slope field plot

Verification of solutions

$$y = e^{\int -\frac{1}{\ln(t)}dt} \left(\int \frac{\cot(t)\, e^{\int \frac{1}{\ln(t)}dt}}{\ln(t)} dt + c_1 \right)$$

Verified OK.

3.6.2 Solving as first order ode lie symmetry lookup ode

Writing the ode as

$$y' = \frac{-y + \cot(t)}{\ln(t)}$$
$$y' = \omega(t, y)$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_t + \omega(\eta_y - \xi_t) - \omega^2 \xi_y - \omega_t \xi - \omega_y \eta = 0 \tag{A}$$

The type of this ode is known. It is of type `linear`. Therefore we do not need to solve the PDE (A), and can just use the lookup table shown below to find ξ, η

Table 197: Lie symmetry infinitesimal lookup table for known first order ODE's

ODE class	Form	ξ	η
linear ode	$y' = f(x)y(x) + g(x)$	0	$e^{\int f dx}$
separable ode	$y' = f(x)\, g(y)$	$\frac{1}{f}$	0
quadrature ode	$y' = f(x)$	0	1
quadrature ode	$y' = g(y)$	1	0
homogeneous ODEs of Class A	$y' = f\left(\frac{y}{x}\right)$	x	y
homogeneous ODEs of Class C	$y' = (a + bx + cy)^{\frac{n}{m}}$	1	$-\frac{b}{c}$
homogeneous class D	$y' = \frac{y}{x} + g(x)\, F\left(\frac{y}{x}\right)$	x^2	xy
First order special form ID 1	$y' = g(x)\, e^{h(x)+by} + f(x)$	$\frac{e^{-\int bf(x)dx-h(x)}}{g(x)}$	$\frac{f(x)e^{-\int bf(x)dx-h(x)}}{g(x)}$
polynomial type ode	$y' = \frac{a_1x+b_1y+c_1}{a_2x+b_2y+c_2}$	$\frac{a_1b_2x-a_2b_1x-b_1c_2+b_2c_1}{a_1b_2-a_2b_1}$	$\frac{a_1b_2y-a_2b_1y-a_1c_2-a_2c_1}{a_1b_2-a_2b_1}$
Bernoulli ode	$y' = f(x)\, y + g(x)\, y^n$	0	$e^{-\int (n-1)f(x)dx} y^n$
Reduced Riccati	$y' = f_1(x)\, y + f_2(x)\, y^2$	0	$e^{-\int f_1 dx}$

The above table shows that

$$
\begin{aligned}
\xi(t,y) &= 0 \\
\eta(t,y) &= \mathrm{e}^{\mathrm{expIntegral}_1(-\ln(t))}
\end{aligned} \tag{A1}
$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(t, y) \to (R, S)$ where (R, S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$
\frac{dt}{\xi} = \frac{dy}{\eta} = dS \tag{1}
$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial t} + \eta \frac{\partial}{\partial y}\right) S(t, y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the

canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$R = t$$

S is found from

$$\begin{aligned} S &= \int \frac{1}{\eta} dy \\ &= \int \frac{1}{e^{\text{expIntegral}_1(-\ln(t))}} dy \end{aligned}$$

Which results in

$$S = e^{-\text{expIntegral}_1(-\ln(t))} y$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_t + \omega(t, y) S_y}{R_t + \omega(t, y) R_y} \tag{2}$$

Where in the above R_t, R_y, S_t, S_y are all partial derivatives and $\omega(t, y)$ is the right hand side of the original ode given by

$$\omega(t, y) = \frac{-y + \cot(t)}{\ln(t)}$$

Evaluating all the partial derivatives gives

$$\begin{aligned} R_t &= 1 \\ R_y &= 0 \\ S_t &= \frac{e^{-\text{expIntegral}_1(-\ln(t))} y}{\ln(t)} \\ S_y &= e^{-\text{expIntegral}_1(-\ln(t))} \end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = \frac{e^{-\text{expIntegral}_1(-\ln(t))} \cot(t)}{\ln(t)} \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for t, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = \frac{e^{-\text{expIntegral}_1(-\ln(R))} \cot(R)}{\ln(R)}$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordinates R, S. Integrating the above gives

$$S(R) = \int \frac{\mathrm{e}^{-\mathrm{expIntegral}_1(-\ln(R))} \cot(R)}{\ln(R)} dR + c_1 \tag{4}$$

To complete the solution, we just need to transform (4) back to t, y coordinates. This results in

$$\mathrm{e}^{-\mathrm{expIntegral}_1(-\ln(t))} y = \int \frac{\mathrm{e}^{-\mathrm{expIntegral}_1(-\ln(t))} \cot(t)}{\ln(t)} dt + c_1$$

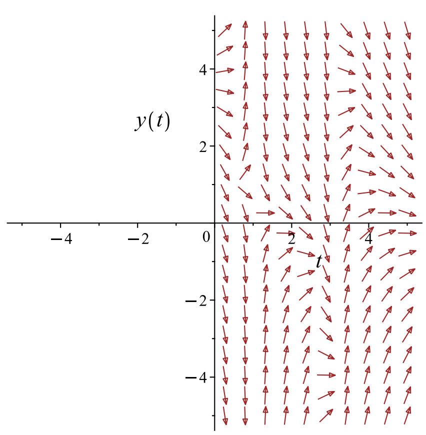
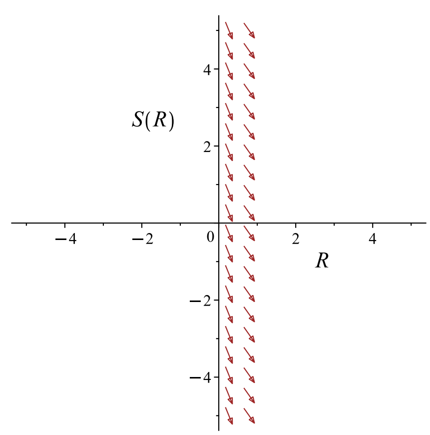
Which simplifies to

$$\mathrm{e}^{-\mathrm{expIntegral}_1(-\ln(t))} y = \int \frac{\mathrm{e}^{-\mathrm{expIntegral}_1(-\ln(t))} \cot(t)}{\ln(t)} dt + c_1$$

Which gives

$$y = \left(\int \frac{\mathrm{e}^{-\mathrm{expIntegral}_1(-\ln(t))} \cot(t)}{\ln(t)} dt + c_1\right) \mathrm{e}^{\mathrm{expIntegral}_1(-\ln(t))}$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in t, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dt} = \frac{-y+\cot(t)}{\ln(t)}$ 	$R = t$ $S = \mathrm{e}^{-\mathrm{expIntegral}_1(-\ln(t))} y$	$\frac{dS}{dR} = \frac{\mathrm{e}^{-\mathrm{expIntegral}_1(-\ln(R))} \cot(R)}{\ln(R)}$ 

Summary

The solution(s) found are the following

$$y = \left(\int \frac{e^{-\operatorname{expIntegral}_1(-\ln(t))} \cot(t)}{\ln(t)} dt + c_1\right) e^{\operatorname{expIntegral}_1(-\ln(t))} \tag{1}$$

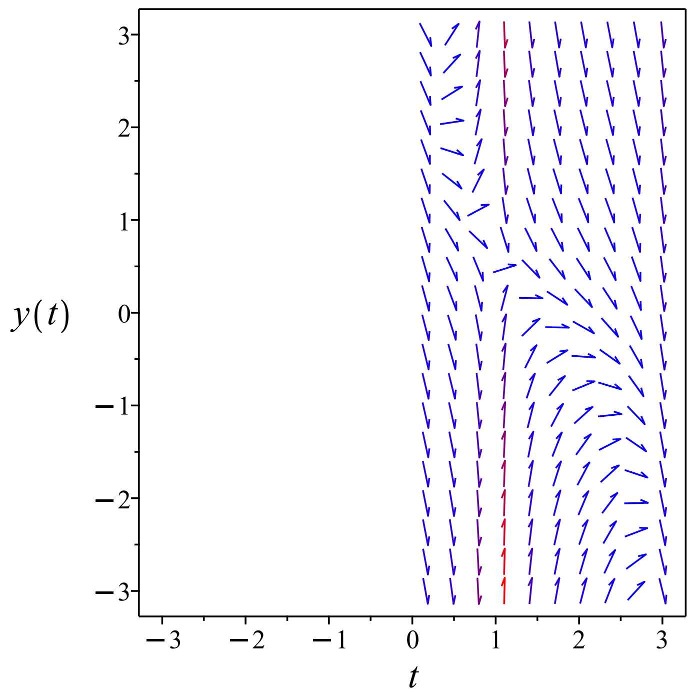


Figure 180: Slope field plot

Verification of solutions

$$y = \left(\int \frac{e^{-\operatorname{expIntegral}_1(-\ln(t))} \cot(t)}{\ln(t)} dt + c_1\right) e^{\operatorname{expIntegral}_1(-\ln(t))}$$

Verified OK.

3.6.3 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x, y) + N(x, y) \frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x, y) = c$ where c is constant, that satisfies the

ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x, y) = 0$$

Hence

$$\frac{\partial \phi}{\partial x} + \frac{\partial \phi}{\partial y}\frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$\frac{\partial \phi}{\partial x} = M$$
$$\frac{\partial \phi}{\partial y} = N$$

But since $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x, y)$ but at least we know now that we can do that since the condition $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(t, y)\,\mathrm{d}t + N(t, y)\,\mathrm{d}y = 0 \tag{1A}$$

Therefore

$$(\ln(t))\,\mathrm{d}y = (-y + \cot(t))\,\mathrm{d}t$$
$$(y - \cot(t))\,\mathrm{d}t + (\ln(t))\,\mathrm{d}y = 0 \tag{2A}$$

Comparing (1A) and (2A) shows that

$$M(t, y) = y - \cot(t)$$
$$N(t, y) = \ln(t)$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial t}$$

Using result found above gives

$$\begin{aligned}\frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}(y-\cot(t)) \\ &= 1\end{aligned}$$

And

$$\begin{aligned}\frac{\partial N}{\partial t} &= \frac{\partial}{\partial t}(\ln(t)) \\ &= \frac{1}{t}\end{aligned}$$

Since $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial t}$, then the ODE is not exact. Since the ODE is not exact, we will try to find an integrating factor to make it exact. Let

$$\begin{aligned}A &= \frac{1}{N}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial t}\right) \\ &= \frac{1}{\ln(t)}\left((1) - \left(\frac{1}{t}\right)\right) \\ &= \frac{-1+t}{t\ln(t)}\end{aligned}$$

Since A does not depend on y, then it can be used to find an integrating factor. The integrating factor μ is

$$\begin{aligned}\mu &= e^{\int A \, \mathrm{d}t} \\ &= e^{\int \frac{-1+t}{t\ln(t)} \, \mathrm{d}t}\end{aligned}$$

The result of integrating gives

$$\begin{aligned}\mu &= e^{-\operatorname{expIntegral}_1(-\ln(t)) - \ln(\ln(t))} \\ &= \frac{\mathrm{e}^{-\operatorname{expIntegral}_1(-\ln(t))}}{\ln(t)}\end{aligned}$$

M and N are multiplied by this integrating factor, giving new M and new N which are called $\overline{M}$ and $\overline{N}$ for now so not to confuse them with the original M and N.

$$\begin{aligned}\overline{M} &= \mu M \\ &= \frac{\mathrm{e}^{-\operatorname{expIntegral}_1(-\ln(t))}}{\ln(t)}(y-\cot(t)) \\ &= \frac{(y-\cot(t))\,\mathrm{e}^{-\operatorname{expIntegral}_1(-\ln(t))}}{\ln(t)}\end{aligned}$$

And

$$
\begin{aligned}
\overline{N} &= \mu N \\
&= \frac{e^{-\operatorname{expIntegral}_1(-\ln(t))}}{\ln(t)}(\ln(t)) \\
&= e^{-\operatorname{expIntegral}_1(-\ln(t))}
\end{aligned}
$$

Now a modified ODE is ontained from the original ODE, which is exact and can be solved. The modified ODE is

$$
\begin{aligned}
\overline{M} + \overline{N}\frac{dy}{dt} &= 0 \\
\left(\frac{(y - \cot(t))\, e^{-\operatorname{expIntegral}_1(-\ln(t))}}{\ln(t)}\right) + \left(e^{-\operatorname{expIntegral}_1(-\ln(t))}\right)\frac{dy}{dt} &= 0
\end{aligned}
$$

The following equations are now set up to solve for the function $\phi(t, y)$

$$
\frac{\partial \phi}{\partial t} = \overline{M} \tag{1}
$$

$$
\frac{\partial \phi}{\partial y} = \overline{N} \tag{2}
$$

Integrating (1) w.r.t. t gives

$$
\begin{aligned}
\int \frac{\partial \phi}{\partial t}\, dt &= \int \overline{M}\, dt \\
\int \frac{\partial \phi}{\partial t}\, dt &= \int \frac{(y - \cot(t))\, e^{-\operatorname{expIntegral}_1(-\ln(t))}}{\ln(t)}\, dt \\
\phi &= \int^t \frac{(y - \cot(\_a))\, e^{-\operatorname{expIntegral}_1(-\ln(\_a))}}{\ln(\_a)} d\_a + f(y)
\end{aligned} \tag{3}
$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both t and y. Taking derivative of equation (3) w.r.t y gives

$$
\frac{\partial \phi}{\partial y} = e^{-\operatorname{expIntegral}_1(-\ln(t))} + f'(y) \tag{4}
$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = e^{-\operatorname{expIntegral}_1(-\ln(t))}$. Therefore equation (4) becomes

$$
e^{-\operatorname{expIntegral}_1(-\ln(t))} = e^{-\operatorname{expIntegral}_1(-\ln(t))} + f'(y) \tag{5}
$$

Solving equation (5) for $f'(y)$ gives

$$f'(y) = 0$$

Therefore

$$f(y) = c_1$$

Where c_1 is constant of integration. Substituting this result for $f(y)$ into equation (3) gives ϕ

$$\phi = \int^t \frac{(y - \cot(\_a))\, e^{-\operatorname{expIntegral}_1(-\ln(\_a))}}{\ln(\_a)} d\_a + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = \int^t \frac{(y - \cot(\_a))\, e^{-\operatorname{expIntegral}_1(-\ln(\_a))}}{\ln(\_a)} d\_a$$

Summary

The solution(s) found are the following

$$\int^t \frac{(y - \cot(\_a))\, e^{-\operatorname{expIntegral}_1(-\ln(\_a))}}{\ln(\_a)} d\_a = c_1 \tag{1}$$

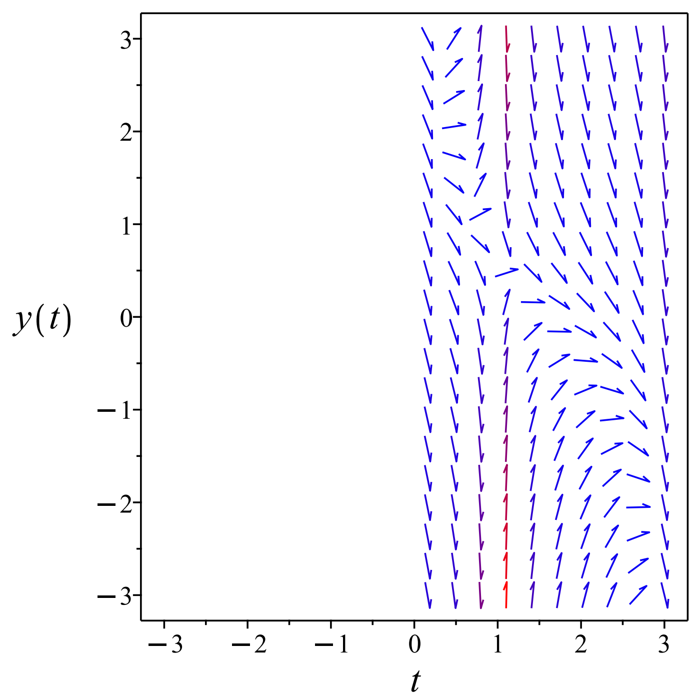


Figure 181: Slope field plot

Verification of solutions

$$\int^{t} \frac{(y-\cot(\_a))\, \mathrm{e}^{-\operatorname{expIntegral}_1(-\ln(\_a))}}{\ln(\_a)} d\_a = c_1$$

Verified OK.

3.6.4 Maple step by step solution

Let's solve

$y + \ln(t)\, y' = \cot(t)$

- Highest derivative means the order of the ODE is 1

  y'

- Isolate the derivative

  $y' = -\frac{y}{\ln(t)} + \frac{\cot(t)}{\ln(t)}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE

  $y' + \frac{y}{\ln(t)} = \frac{\cot(t)}{\ln(t)}$

- The ODE is linear; multiply by an integrating factor $\mu(t)$

  $\mu(t)\left(y' + \frac{y}{\ln(t)}\right) = \frac{\mu(t)\cot(t)}{\ln(t)}$

- Assume the lhs of the ODE is the total derivative $\frac{d}{dt}(\mu(t)\, y)$

  $\mu(t)\left(y' + \frac{y}{\ln(t)}\right) = \mu'(t)\, y + \mu(t)\, y'$

- Isolate $\mu'(t)$

  $\mu'(t) = \frac{\mu(t)}{\ln(t)}$

- Solve to find the integrating factor

  $\mu(t) = \mathrm{e}^{-\mathrm{Ei}_1(-\ln(t))}$

- Integrate both sides with respect to t

  $\int \left(\frac{d}{dt}(\mu(t)\, y)\right) dt = \int \frac{\mu(t)\cot(t)}{\ln(t)} dt + c_1$

- Evaluate the integral on the lhs

  $\mu(t)\, y = \int \frac{\mu(t)\cot(t)}{\ln(t)} dt + c_1$

- Solve for y

  $y = \frac{\int \frac{\mu(t)\cot(t)}{\ln(t)} dt + c_1}{\mu(t)}$

- Substitute $\mu(t) = e^{-\mathrm{Ei}_1(-\ln(t))}$

$$y = \frac{\int \frac{e^{-\mathrm{Ei}_1(-\ln(t))}\cot(t)}{\ln(t)}dt + c_1}{e^{-\mathrm{Ei}_1(-\ln(t))}}$$

- Simplify

$$y = \left(\int \frac{e^{-\mathrm{Ei}_1(-\ln(t))}\cot(t)}{\ln(t)}dt + c_1\right) e^{\mathrm{Ei}_1(-\ln(t))}$$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
<- 1st order linear successful`
```

✓ Solution by Maple

Time used: 0.015 (sec). Leaf size: 32

```
dsolve(y(t)+ln(t)*diff(y(t),t) = cot(t),y(t), singsol=all)
```

$$y(t) = \left(\int \frac{\cot(t)\, e^{-\mathrm{expIntegral}_1(-\ln(t))}}{\ln(t)}dt + c_1\right) e^{\mathrm{expIntegral}_1(-\ln(t))}$$

✓ Solution by Mathematica

Time used: 0.142 (sec). Leaf size: 36

```
DSolve[y[t]+Log[t]*y'[t] == Cot[t],y[t],t,IncludeSingularSolutions -> True]
```

$$y(t) \to e^{-\mathrm{LogIntegral}(t)}\left(\int_1^t \frac{e^{\mathrm{LogIntegral}(K[1])}\cot(K[1])}{\log(K[1])}dK[1] + c_1\right)$$

3.7 problem 11

Internal problem ID [522]
Internal file name [`OUTPUT/522_Sunday_June_05_2022_01_43_00_AM_26088981/index.tex`]

Book: Elementary differential equations and boundary value problems, 10th ed., Boyce and DiPrima
Section: Section 2.4. Page 76
Problem number: 11.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"exact", "separable", "differentialType", "first_order_ode_lie_symmetry_lookup"**

Maple gives the following as the ode type

`[_separable]`

$$\boxed{y' - \frac{t^2+1}{3y-y^2} = 0}$$

3.7.1 Solving as separable ode

In canonical form the ODE is

$$\begin{aligned} y' &= F(t,y) \\ &= f(t)g(y) \\ &= \frac{-t^2-1}{y\,(-3+y)} \end{aligned}$$

Where $f(t) = -t^2 - 1$ and $g(y) = \frac{1}{y(-3+y)}$. Integrating both sides gives

$$\frac{1}{\frac{1}{y(-3+y)}}\,dy = -t^2 - 1\,dt$$

$$\int \frac{1}{\frac{1}{y(-3+y)}}\, dy = \int -t^2 - 1\, dt$$

$$\frac{1}{3}y^3 - \frac{3}{2}y^2 = -\frac{1}{3}t^3 - t + c_1$$

Which results in

$$y = \frac{\left(27 - 4t^3 + 12c_1 - 12t + 2\sqrt{4t^6 - 24c_1t^3 + 24t^4 - 54t^3 + 36c_1^2 - 72c_1t + 36t^2 + 162c_1 - 162t}\right)^{\frac{1}{3}}}{2} + \frac{9}{2\left(27 - 4t^3 + 12c_1 - 12t + 2\sqrt{4t^6 - 24c_1t^3 + 24t^4 - 54t^3 + 36c_1^2 - 72c_1t + 36t^2 + 162c_1 - 162t}\right)^{\frac{1}{3}}} + \frac{3}{2}$$

$$y = -\frac{\left(27 - 4t^3 + 12c_1 - 12t + 2\sqrt{4t^6 - 24c_1t^3 + 24t^4 - 54t^3 + 36c_1^2 - 72c_1t + 36t^2 + 162c_1 - 162t}\right)^{\frac{1}{3}}}{4} - \frac{9}{4\left(27 - 4t^3 + 12c_1 - 12t + 2\sqrt{4t^6 - 24c_1t^3 + 24t^4 - 54t^3 + 36c_1^2 - 72c_1t + 36t^2 + 162c_1 - 162t}\right)^{\frac{1}{3}}} + \frac{3}{2} + \frac{i\sqrt{3}\left(\frac{\left(27-4t^3+12c_1-12t+2\sqrt{4t^6-24c_1t^3+24t^4-54t^3+36c_1^2-72c_1t+36t^2+162c_1-162t}\right)^{\frac{1}{3}}}{2} - \frac{}{2\left(27-4t^3+12c_1-12t+2\sqrt{4t^6-24c_1t^3}\right.}\right.}{2}$$

$$y = -\frac{\left(27 - 4t^3 + 12c_1 - 12t + 2\sqrt{4t^6 - 24c_1t^3 + 24t^4 - 54t^3 + 36c_1^2 - 72c_1t + 36t^2 + 162c_1 - 162t}\right)^{\frac{1}{3}}}{4} - \frac{9}{4\left(27 - 4t^3 + 12c_1 - 12t + 2\sqrt{4t^6 - 24c_1t^3 + 24t^4 - 54t^3 + 36c_1^2 - 72c_1t + 36t^2 + 162c_1 - 162t}\right)^{\frac{1}{3}}} + \frac{3}{2} - \frac{i\sqrt{3}\left(\frac{\left(27-4t^3+12c_1-12t+2\sqrt{4t^6-24c_1t^3+24t^4-54t^3+36c_1^2-72c_1t+36t^2+162c_1-162t}\right)^{\frac{1}{3}}}{2} - \frac{}{2\left(27-4t^3+12c_1-12t+2\sqrt{4t^6-24c_1t^3}\right.}\right.}{2}$$

Summary

The solution(s) found are the following

$$y = \frac{\left(27-4t^3+12c_1-12t+2\sqrt{4t^6-24c_1t^3+24t^4-54t^3+36c_1^2-72c_1t+36t^2+162c_1-162t}\right)^{\frac{1}{3}}}{2} + \frac{9}{2\left(27-4t^3+12c_1-12t+2\sqrt{4t^6-24c_1t^3+24t^4-54t^3+36c_1^2-72c_1t+36t^2+162c_1-162t}\right)^{\frac{1}{3}}} + \frac{3}{2} \tag{1}$$

$$y = -\frac{\left(27-4t^3+12c_1-12t+2\sqrt{4t^6-24c_1t^3+24t^4-54t^3+36c_1^2-72c_1t+36t^2+162c_1-162t}\right)^{\frac{1}{3}}}{4} - \frac{9}{4\left(27-4t^3+12c_1-12t+2\sqrt{4t^6-24c_1t^3+24t^4-54t^3+36c_1^2-72c_1t+36t^2+162c_1-162t}\right)^{\frac{1}{3}}} + \frac{3}{2} + \frac{i\sqrt{3}\left(\frac{\left(27-4t^3+12c_1-12t+2\sqrt{4t^6-24c_1t^3+24t^4-54t^3+36c_1^2-72c_1t+36t^2+162c_1-162t}\right)^{\frac{1}{3}}}{2} - \frac{\cdots}{2\left(27-4t^3+12c_1-12t+2\sqrt{4t^6-24c_1t^3\cdots}\right)}\cdots\right)}{2} \tag{2}$$

$$y = -\frac{\left(27-4t^3+12c_1-12t+2\sqrt{4t^6-24c_1t^3+24t^4-54t^3+36c_1^2-72c_1t+36t^2+162c_1-162t}\right)^{\frac{1}{3}}}{4} - \frac{9}{4\left(27-4t^3+12c_1-12t+2\sqrt{4t^6-24c_1t^3+24t^4-54t^3+36c_1^2-72c_1t+36t^2+162c_1-162t}\right)^{\frac{1}{3}}} + \frac{3}{2} - \frac{i\sqrt{3}\left(\frac{\left(27-4t^3+12c_1-12t+2\sqrt{4t^6-24c_1t^3+24t^4-54t^3+36c_1^2-72c_1t+36t^2+162c_1-162t}\right)^{\frac{1}{3}}}{2} - \frac{\cdots}{2\left(27-4t^3+12c_1-12t+2\sqrt{4t^6-24c_1t^3\cdots}\right)}\cdots\right)}{2} \tag{3}$$

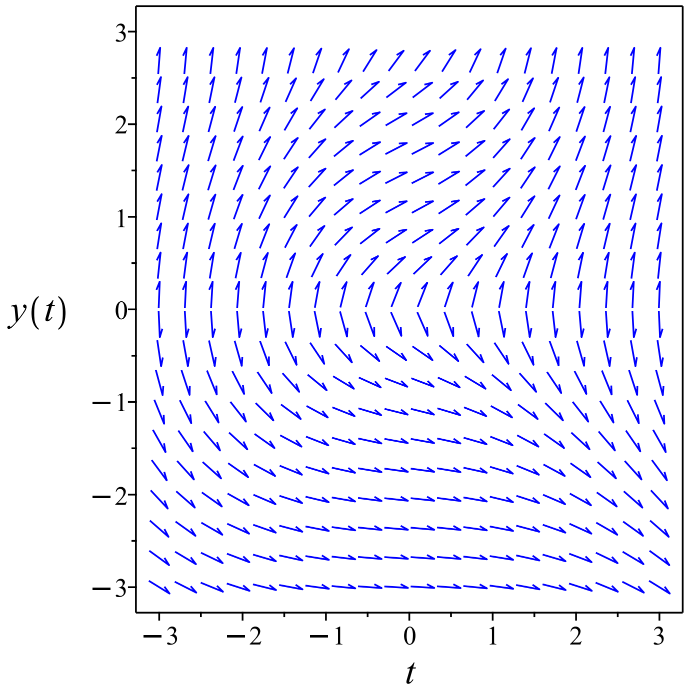


Figure 182: Slope field plot

Verification of solutions

$$y = \frac{\left(27-4t^3+12c_1-12t+2\sqrt{4t^6-24c_1t^3+24t^4-54t^3+36c_1^2-72c_1t+36t^2+162c_1-162t}\right)^{\frac{1}{3}}}{2} + \frac{9}{2\left(27-4t^3+12c_1-12t+2\sqrt{4t^6-24c_1t^3+24t^4-54t^3+36c_1^2-72c_1t+36t^2+162c_1-162t}\right)^{\frac{1}{3}}} + \frac{3}{2}$$

Verified OK.

$$y = -\frac{\left(27-4t^3+12c_1-12t+2\sqrt{4t^6-24c_1t^3+24t^4-54t^3+36c_1^2-72c_1t+36t^2+162c_1-162t}\right)^{\frac{1}{3}}}{4} - \frac{9}{4\left(27-4t^3+12c_1-12t+2\sqrt{4t^6-24c_1t^3+24t^4-54t^3+36c_1^2-72c_1t+36t^2+162c_1-162t}\right)^{\frac{1}{3}}} + \frac{3}{2} + \frac{i\sqrt{3}\left(\frac{\left(27-4t^3+12c_1-12t+2\sqrt{4t^6-24c_1t^3+24t^4-54t^3+36c_1^2-72c_1t+36t^2+162c_1-162t}\right)^{\frac{1}{3}}}{2} - \frac{}{2\left(27-4t^3+12c_1-12t+2\sqrt{4t^6-24c_1t^3}\right.}\right.}{2}$$

Verified OK.

$$y = -\frac{\left(27-4t^3+12c_1-12t+2\sqrt{4t^6-24c_1t^3+24t^4-54t^3+36c_1^2-72c_1t+36t^2+162c_1-162t}\right)^{\frac{1}{3}}}{4} - \frac{9}{4\left(27-4t^3+12c_1-12t+2\sqrt{4t^6-24c_1t^3+24t^4-54t^3+36c_1^2-72c_1t+36t^2+162c_1-162t}\right)^{\frac{1}{3}}} + \frac{3}{2} - \frac{i\sqrt{3}\left(\frac{\left(27-4t^3+12c_1-12t+2\sqrt{4t^6-24c_1t^3+24t^4-54t^3+36c_1^2-72c_1t+36t^2+162c_1-162t}\right)^{\frac{1}{3}}}{2} - \frac{}{2\left(27-4t^3+12c_1-12t+2\sqrt{4t^6-24c_1t^3}\right.}\right.}{2}$$

Verified OK.

3.7.2 Solving as differentialType ode

Writing the ode as

$$y' = \frac{t^2+1}{3y-y^2} \tag{1}$$

Which becomes

$$(y^2-3y)\,dy = (-t^2-1)\,dt \tag{2}$$

But the RHS is complete differential because

$$(-t^2-1)\,dt = d\left(-\frac{1}{3}t^3-t\right)$$

Hence (2) becomes

$$(y^2-3y)\,dy = d\left(-\frac{1}{3}t^3-t\right)$$

Integrating both sides gives gives these solutions

$$y = \frac{\left(27-4t^3+12c_1-12t+2\sqrt{4t^6-24c_1t^3+24t^4-54t^3+36c_1^2-72c_1t+36t^2+162c_1-162t}\right)^{\frac{1}{3}}}{2} + \frac{}{2}$$

$$y = -\frac{\left(27-4t^3+12c_1-12t+2\sqrt{4t^6-24c_1t^3+24t^4-54t^3+36c_1^2-72c_1t+36t^2+162c_1-162t}\right)^{\frac{1}{3}}}{4} -$$

$$y = -\frac{\left(27-4t^3+12c_1-12t+2\sqrt{4t^6-24c_1t^3+24t^4-54t^3+36c_1^2-72c_1t+36t^2+162c_1-162t}\right)^{\frac{1}{3}}}{4} -$$

Summary

The solution(s) found are the following

$$y = \frac{\left(27-4t^3+12c_1-12t+2\sqrt{4t^6-24c_1t^3+24t^4-54t^3+36c_1^2-72c_1t+36t^2+162c_1-162t}\right)^{\frac{1}{3}}}{2} + \frac{9}{2\left(27-4t^3+12c_1-12t+2\sqrt{4t^6-24c_1t^3+24t^4-54t^3+36c_1^2-72c_1t+36t^2+162c_1-162t}\right)^{\frac{1}{3}}} + \frac{3}{2} + c_1 \tag{1}$$

$$y = -\frac{\left(27-4t^3+12c_1-12t+2\sqrt{4t^6-24c_1t^3+24t^4-54t^3+36c_1^2-72c_1t+36t^2+162c_1-162t}\right)^{\frac{1}{3}}}{4} - \frac{9}{4\left(27-4t^3+12c_1-12t+2\sqrt{4t^6-24c_1t^3+24t^4-54t^3+36c_1^2-72c_1t+36t^2+162c_1-162t}\right)^{\frac{1}{3}}} + \frac{3}{2} + \frac{i\sqrt{3}\left(\frac{\left(27-4t^3+12c_1-12t+2\sqrt{4t^6-24c_1t^3+24t^4-54t^3+36c_1^2-72c_1t+36t^2+162c_1-162t}\right)^{\frac{1}{3}}}{2} - \frac{}{2\left(27-4t^3+12c_1-12t+2\sqrt{4t^6-24c_1t^3\cdots}\right.}\right.}{2} + c_1 \tag{2}$$

$$y = -\frac{\left(27-4t^3+12c_1-12t+2\sqrt{4t^6-24c_1t^3+24t^4-54t^3+36c_1^2-72c_1t+36t^2+162c_1-162t}\right)^{\frac{1}{3}}}{4} - \frac{9}{4\left(27-4t^3+12c_1-12t+2\sqrt{4t^6-24c_1t^3+24t^4-54t^3+36c_1^2-72c_1t+36t^2+162c_1-162t}\right)^{\frac{1}{3}}} + \frac{3}{2} - \frac{i\sqrt{3}\left(\frac{\left(27-4t^3+12c_1-12t+2\sqrt{4t^6-24c_1t^3+24t^4-54t^3+36c_1^2-72c_1t+36t^2+162c_1-162t}\right)^{\frac{1}{3}}}{2} - \frac{}{2\left(27-4t^3+12c_1-12t+2\sqrt{4t^6-24c_1t^3\cdots}\right.}\right.}{2} + c_1 \tag{3}$$

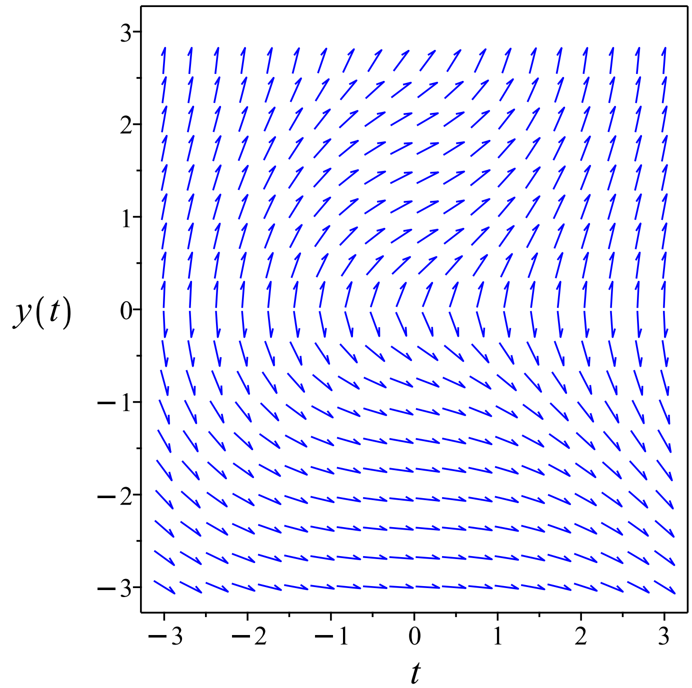


Figure 183: Slope field plot

Verification of solutions

$$y = \frac{\left(27-4t^3+12c_1-12t+2\sqrt{4t^6-24c_1t^3+24t^4-54t^3+36c_1^2-72c_1t+36t^2+162c_1-162t}\right)^{\frac{1}{3}}}{2} + \frac{9}{2\left(27-4t^3+12c_1-12t+2\sqrt{4t^6-24c_1t^3+24t^4-54t^3+36c_1^2-72c_1t+36t^2+162c_1-162t}\right)^{\frac{1}{3}}} + \frac{3}{2} + c_1$$

Verified OK.

$$y = -\frac{\left(27-4t^3+12c_1-12t+2\sqrt{4t^6-24c_1t^3+24t^4-54t^3+36c_1^2-72c_1t+36t^2+162c_1-162t}\right)^{\frac{1}{3}}}{4} - \frac{9}{4\left(27-4t^3+12c_1-12t+2\sqrt{4t^6-24c_1t^3+24t^4-54t^3+36c_1^2-72c_1t+36t^2+162c_1-162t}\right)^{\frac{1}{3}}} + \frac{3}{2} + \frac{i\sqrt{3}\left(\frac{\left(27-4t^3+12c_1-12t+2\sqrt{4t^6-24c_1t^3+24t^4-54t^3+36c_1^2-72c_1t+36t^2+162c_1-162t}\right)^{\frac{1}{3}}}{2} - \frac{\cdots}{2\left(27-4t^3+12c_1-12t+2\sqrt{4t^6-24c_1t^3 \cdots}\right.}\right.}{2} + c_1$$

Verified OK.

$$y = -\frac{\left(27-4t^3+12c_1-12t+2\sqrt{4t^6-24c_1t^3+24t^4-54t^3+36c_1^2-72c_1t+36t^2+162c_1-162t}\right)^{\frac{1}{3}}}{4} - \frac{9}{4\left(27-4t^3+12c_1-12t+2\sqrt{4t^6-24c_1t^3+24t^4-54t^3+36c_1^2-72c_1t+36t^2+162c_1-162t}\right)^{\frac{1}{3}}} + \frac{3}{2} - \frac{i\sqrt{3}\left(\frac{\left(27-4t^3+12c_1-12t+2\sqrt{4t^6-24c_1t^3+24t^4-54t^3+36c_1^2-72c_1t+36t^2+162c_1-162t}\right)^{\frac{1}{3}}}{2} - \frac{\cdots}{2\left(27-4t^3+12c_1-12t+2\sqrt{4t^6-24c_1t^3 \cdots}\right.}\right.}{2} + c_1$$

Verified OK.

3.7.3 Solving as first order ode lie symmetry lookup ode

Writing the ode as

$$y' = -\frac{t^2 + 1}{y(-3 + y)}$$
$$y' = \omega(t, y)$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_t + \omega(\eta_y - \xi_t) - \omega^2 \xi_y - \omega_t \xi - \omega_y \eta = 0 \tag{A}$$

The type of this ode is known. It is of type `separable`. Therefore we do not need to solve the PDE (A), and can just use the lookup table shown below to find ξ, η

Table 200: Lie symmetry infinitesimal lookup table for known first order ODE's

ODE class	Form	ξ	η
linear ode	$y' = f(x)y(x) + g(x)$	0	$e^{\int f dx}$
separable ode	$y' = f(x)\, g(y)$	$\frac{1}{f}$	0
quadrature ode	$y' = f(x)$	0	1
quadrature ode	$y' = g(y)$	1	0
homogeneous ODEs of Class A	$y' = f\left(\frac{y}{x}\right)$	x	y
homogeneous ODEs of Class C	$y' = (a + bx + cy)^{\frac{n}{m}}$	1	$-\frac{b}{c}$
homogeneous class D	$y' = \frac{y}{x} + g(x)\, F\left(\frac{y}{x}\right)$	x^2	xy
First order special form ID 1	$y' = g(x)\, e^{h(x)+by} + f(x)$	$\frac{e^{-\int bf(x)dx - h(x)}}{g(x)}$	$\frac{f(x)e^{-\int bf(x)dx - h(x)}}{g(x)}$
polynomial type ode	$y' = \frac{a_1x+b_1y+c_1}{a_2x+b_2y+c_2}$	$\frac{a_1b_2x - a_2b_1x - b_1c_2 + b_2c_1}{a_1b_2 - a_2b_1}$	$\frac{a_1b_2y - a_2b_1y - a_1c_2 - a_2c_1}{a_1b_2 - a_2b_1}$
Bernoulli ode	$y' = f(x)\, y + g(x)\, y^n$	0	$e^{-\int (n-1)f(x)dx} y^n$
Reduced Riccati	$y' = f_1(x)\, y + f_2(x)\, y^2$	0	$e^{-\int f_1 dx}$

The above table shows that

$$\xi(t, y) = \frac{1}{-t^2 - 1}$$
$$\eta(t, y) = 0 \tag{A1}$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(t, y) \to (R, S)$ where (R, S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$\frac{dt}{\xi} = \frac{dy}{\eta} = dS \tag{1}$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial t} + \eta \frac{\partial}{\partial y}\right) S(t, y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the canonical coordinates, where $S(R)$. Since $\eta = 0$ then in this special case

$$R = y$$

S is found from

$$\begin{aligned} S &= \int \frac{1}{\xi} dt \\ &= \int \frac{1}{\frac{1}{-t^2-1}} dt \end{aligned}$$

Which results in

$$S = -\frac{1}{3}t^3 - t$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_t + \omega(t, y)S_y}{R_t + \omega(t, y)R_y} \tag{2}$$

Where in the above R_t, R_y, S_t, S_y are all partial derivatives and $\omega(t, y)$ is the right hand side of the original ode given by

$$\omega(t, y) = -\frac{t^2 + 1}{y(-3 + y)}$$

Evaluating all the partial derivatives gives

$$
\begin{aligned}
R_t &= 0 \\
R_y &= 1 \\
S_t &= -t^2 - 1 \\
S_y &= 0
\end{aligned}
$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$
\frac{dS}{dR} = y(-3 + y) \tag{2A}
$$

We now need to express the RHS as function of R only. This is done by solving for t, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$
\frac{dS}{dR} = R(-3 + R)
$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$
S(R) = \frac{1}{3}R^3 - \frac{3}{2}R^2 + c_1 \tag{4}
$$

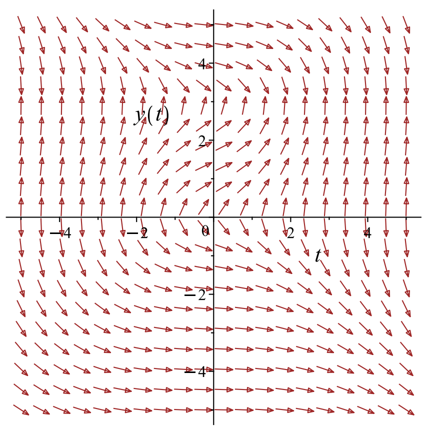
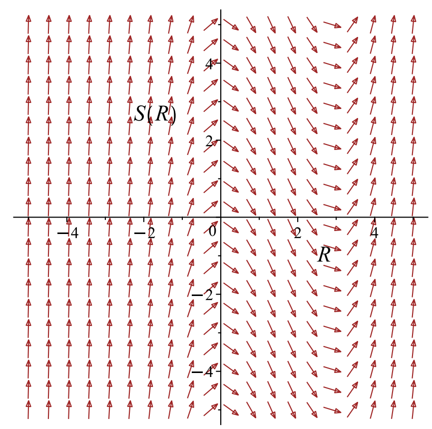
To complete the solution, we just need to transform (4) back to t, y coordinates. This results in

$$
-\frac{1}{3}t^3 - t = \frac{y^3}{3} - \frac{3y^2}{2} + c_1
$$

Which simplifies to

$$
-\frac{1}{3}t^3 - t = \frac{y^3}{3} - \frac{3y^2}{2} + c_1
$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in t, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dt} = -\frac{t^2+1}{y(-3+y)}$ 	$R = y$ $S = -\frac{1}{3}t^3 - t$	$\frac{dS}{dR} = R(-3+R)$ 

Summary

The solution(s) found are the following

$$-\frac{1}{3}t^3 - t = \frac{y^3}{3} - \frac{3y^2}{2} + c_1 \tag{1}$$

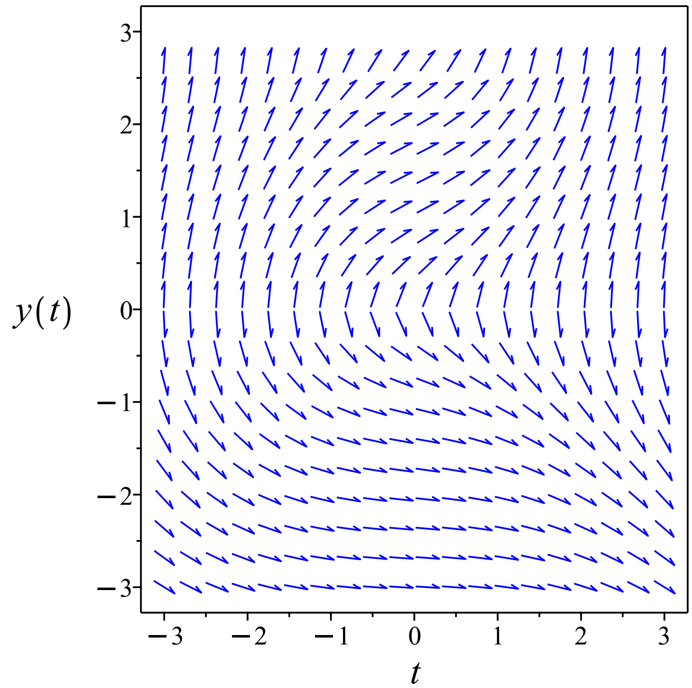


Figure 184: Slope field plot

Verification of solutions

$$-\frac{1}{3}t^3 - t = \frac{y^3}{3} - \frac{3y^2}{2} + c_1$$

Verified OK.

3.7.4 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x,y) + N(x,y)\frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x,y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y) = 0$$

Hence

$$\frac{\partial \phi}{\partial x} + \frac{\partial \phi}{\partial y}\frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$\begin{aligned}\frac{\partial\phi}{\partial x} &= M\\ \frac{\partial\phi}{\partial y} &= N\end{aligned}$$

But since $\frac{\partial^2\phi}{\partial x\partial y} = \frac{\partial^2\phi}{\partial y\partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x,y)$ but at least we know now that we can do that since the condition $\frac{\partial^2\phi}{\partial x\partial y} = \frac{\partial^2\phi}{\partial y\partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(t,y)\,\mathrm{d}t + N(t,y)\,\mathrm{d}y = 0 \tag{1A}$$

Therefore

$$\begin{aligned}(-y(-3+y))\,\mathrm{d}y &= \left(t^2+1\right)\mathrm{d}t\\ \left(-t^2-1\right)\mathrm{d}t + (-y(-3+y))\,\mathrm{d}y &= 0 \qquad (2A)\end{aligned}$$

Comparing (1A) and (2A) shows that

$$\begin{aligned}M(t,y) &= -t^2-1\\ N(t,y) &= -y(-3+y)\end{aligned}$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial t}$$

Using result found above gives

$$\begin{aligned}\frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}\left(-t^2-1\right)\\ &= 0\end{aligned}$$

And

$$\begin{aligned}\frac{\partial N}{\partial t} &= \frac{\partial}{\partial t}(-y(-3+y))\\ &= 0\end{aligned}$$

Since $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial t}$, then the ODE is exact The following equations are now set up to solve for the function $\phi(t, y)$

$$\frac{\partial \phi}{\partial t} = M \tag{1}$$

$$\frac{\partial \phi}{\partial y} = N \tag{2}$$

Integrating (1) w.r.t. t gives

$$\int \frac{\partial \phi}{\partial t} \,\mathrm{d}t = \int M \,\mathrm{d}t$$

$$\int \frac{\partial \phi}{\partial t} \,\mathrm{d}t = \int -t^2 - 1 \,\mathrm{d}t$$

$$\phi = -\frac{1}{3}t^3 - t + f(y) \tag{3}$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both t and y. Taking derivative of equation (3) w.r.t y gives

$$\frac{\partial \phi}{\partial y} = 0 + f'(y) \tag{4}$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = -y(-3 + y)$. Therefore equation (4) becomes

$$-y(-3 + y) = 0 + f'(y) \tag{5}$$

Solving equation (5) for $f'(y)$ gives

$$f'(y) = -y(-3 + y)$$

Integrating the above w.r.t y gives

$$\int f'(y) \,\mathrm{d}y = \int (-y(-3 + y)) \,\mathrm{d}y$$

$$f(y) = -\frac{1}{3}y^3 + \frac{3}{2}y^2 + c_1$$

Where c_1 is constant of integration. Substituting result found above for $f(y)$ into equation (3) gives ϕ

$$\phi = -\frac{1}{3}t^3 - t - \frac{1}{3}y^3 + \frac{3}{2}y^2 + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = -\frac{1}{3}t^3 - t - \frac{1}{3}y^3 + \frac{3}{2}y^2$$

Summary

The solution(s) found are the following

$$-\frac{t^3}{3} - \frac{y^3}{3} + \frac{3y^2}{2} - t = c_1 \tag{1}$$

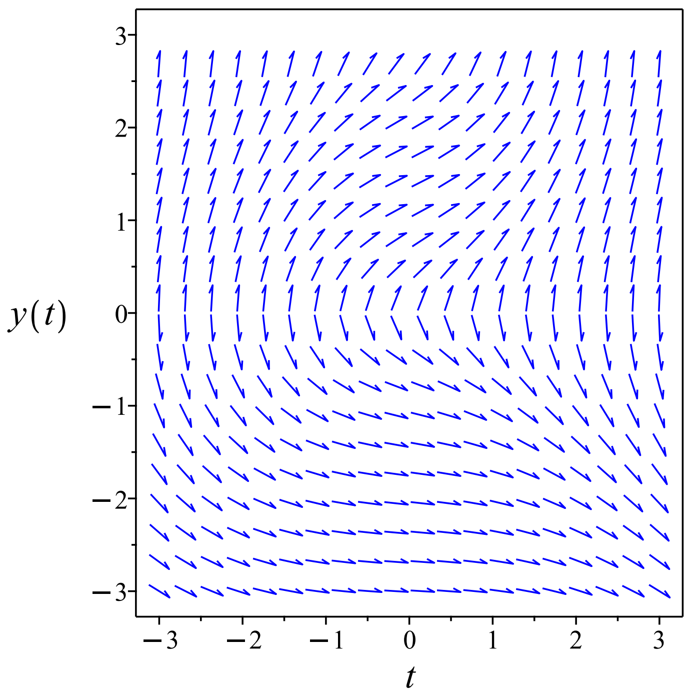


Figure 185: Slope field plot

Verification of solutions

$$-\frac{t^3}{3} - \frac{y^3}{3} + \frac{3y^2}{2} - t = c_1$$

Verified OK.

3.7.5 Maple step by step solution

Let's solve

$y' - \frac{t^2+1}{3y-y^2} = 0$

- Highest derivative means the order of the ODE is 1

  y'

- Separate variables

  $(3y - y^2)\, y' = t^2 + 1$

- Integrate both sides with respect to t

  $\int (3y - y^2)\, y' dt = \int (t^2 + 1)\, dt + c_1$

- Evaluate integral

  $-\frac{y^3}{3} + \frac{3y^2}{2} = \frac{1}{3}t^3 + c_1 + t$

- Solve for y

  $y = \frac{\left(27-4t^3-12c_1-12t+2\sqrt{4t^6+24c_1t^3+24t^4-54t^3+36c_1^2+72c_1t+36t^2-162c_1-162t}\right)^{\frac{1}{3}}}{2} + \frac{}{2\left(27-4t^3-12c_1-12t+2\sqrt{4t^6+24c_1t^3}\right.}$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
<- separable successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 444

```
dsolve(diff(y(t),t) = (t^2+1)/(3*y(t)-y(t)^2),y(t), singsol=all)
```

$$y(t) = \frac{\left(27-4t^3-12c_1-12t+2\sqrt{4t^6+24c_1t^3+24t^4-54t^3+36c_1^2+72c_1t+36t^2-162c_1-162t}\right)^{\frac{1}{3}}}{2} + \frac{9}{2\left(27-4t^3-12c_1-12t+2\sqrt{4t^6+24c_1t^3+24t^4-54t^3+36c_1^2+72c_1t+36t^2-162c_1-162t}\right)^{\frac{1}{3}}} + \frac{3}{2}$$

$$y(t) = -\frac{\left(1+i\sqrt{3}\right)\left(27-4t^3-12c_1-12t+2\sqrt{4}\sqrt{\left(t^3+3t+3c_1-\frac{27}{2}\right)\left(t^3+3c_1+3t\right)}\right)^{\frac{2}{3}} - 9i\sqrt{3} - 6\left(27-\right.}{4\left(27-4t^3-12c_1-12t+2\sqrt{4}\sqrt{\left(t^3+3t+3c_1-\frac{2}{2}\right.}\right.}$$

$$y(t) = \frac{\left(i\sqrt{3}-1\right)\left(27-4t^3-12c_1-12t+2\sqrt{4}\sqrt{\left(t^3+3t+3c_1-\frac{27}{2}\right)\left(t^3+3c_1+3t\right)}\right)^{\frac{2}{3}} - 9i\sqrt{3} + 6\left(27-4t^3\right.}{4\left(27-4t^3-12c_1-12t+2\sqrt{4}\sqrt{\left(t^3+3t+3c_1-\frac{27}{2}\right)}\right.}$$

✓ Solution by Mathematica

Time used: 3.185 (sec). Leaf size: 343

```
DSolve[y'[t] == (t^2+1)/(3*y[t]-y[t]^2),y[t],t,IncludeSingularSolutions -> True]
```

$$y(t) \to \frac{1}{2}\left(\sqrt[3]{-4t^3+\sqrt{-729+(4t^3+12t-3(9+4c_1))^2}-12t+27+12c_1} + \frac{9}{\sqrt[3]{-4t^3+\sqrt{-729+(4t^3+12t-3(9+4c_1))^2}-12t+27+12c_1}}+3\right)$$

$$y(t) \to \frac{1}{4}\left(i\left(\sqrt{3}+i\right)\sqrt[3]{-4t^3+\sqrt{-729+(4t^3+12t-3(9+4c_1))^2}-12t+27+12c_1} - \frac{9\left(1+i\sqrt{3}\right)}{\sqrt[3]{-4t^3+\sqrt{-729+(4t^3+12t-3(9+4c_1))^2}-12t+27+12c_1}}+6\right)$$

$$y(t) \to \frac{1}{4}\left(-\left(\left(1+i\sqrt{3}\right)\sqrt[3]{-4t^3+\sqrt{-729+(4t^3+12t-3(9+4c_1))^2}-12t+27+12c_1}\right) + \frac{9i\left(\sqrt{3}+i\right)}{\sqrt[3]{-4t^3+\sqrt{-729+(4t^3+12t-3(9+4c_1))^2}-12t+27+12c_1}}+6\right)$$

3.8 problem 12

Internal problem ID [523]
Internal file name [`OUTPUT/523_Sunday_June_05_2022_01_43_01_AM_79042422/index.tex`]

Book: Elementary differential equations and boundary value problems, 10th ed., Boyce and DiPrima
Section: Section 2.4. Page 76
Problem number: 12.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"exact", "separable", "first_order_ode_lie_symmetry_lookup"**

Maple gives the following as the ode type

`[_separable]`

$$\boxed{y' - \frac{\cot(t)\, y}{1+y} = 0}$$

3.8.1 Solving as separable ode

In canonical form the ODE is

$$\begin{aligned} y' &= F(t,y) \\ &= f(t)g(y) \\ &= \frac{\cot(t)\, y}{y+1} \end{aligned}$$

Where $f(t) = \cot(t)$ and $g(y) = \frac{y}{y+1}$. Integrating both sides gives

$$\frac{1}{\frac{y}{y+1}}\, dy = \cot(t)\, dt$$

$$\int \frac{1}{\frac{y}{y+1}}dy = \int \cot(t)\, dt$$

$$y + \ln(y) = \ln(\sin(t)) + c_1$$

Which results in

$$y = \text{LambertW}\left(e^{c_1} \sin(t)\right)$$

Since c_1 is constant, then exponential powers of this constant are constants also, and these can be simplified to just c_1 in the above solution. Which simplifies to

$$y = \text{LambertW}\left(e^{c_1} \sin(t)\right)$$

gives

$$y = \text{LambertW}(c_1 \sin(t))$$

Summary

The solution(s) found are the following

$$y = \text{LambertW}(c_1 \sin(t)) \tag{1}$$

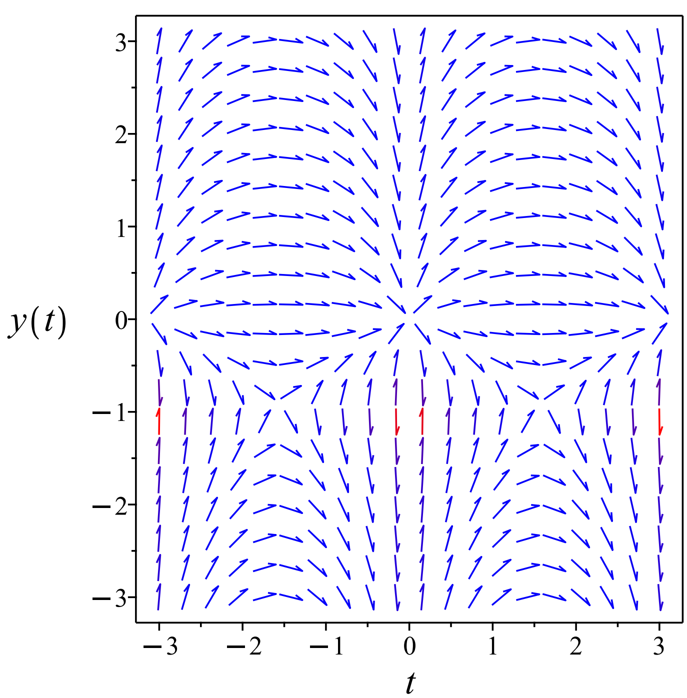


Figure 186: Slope field plot

Verification of solutions

$$
y = \text{LambertW}\left(c_1 \sin\left(t\right)\right)
$$

Verified OK.

3.8.2 Solving as first order ode lie symmetry lookup ode

Writing the ode as

$$
y' = \frac{\cot\left(t\right) y}{y+1}
$$
$$
y' = \omega(t, y)
$$

The condition of Lie symmetry is the linearized PDE given by

$$
\eta_t + \omega(\eta_y - \xi_t) - \omega^2 \xi_y - \omega_t \xi - \omega_y \eta = 0 \tag{A}
$$

The type of this ode is known. It is of type `separable`. Therefore we do not need to solve the PDE (A), and can just use the lookup table shown below to find ξ, η

Table 203: Lie symmetry infinitesimal lookup table for known first order ODE's

ODE class	Form	ξ	η
linear ode	$y' = f(x)y(x) + g(x)$	0	$e^{\int f dx}$
separable ode	$y' = f(x)\, g(y)$	$\frac{1}{f}$	0
quadrature ode	$y' = f(x)$	0	1
quadrature ode	$y' = g(y)$	1	0
homogeneous ODEs of Class A	$y' = f\left(\frac{y}{x}\right)$	x	y
homogeneous ODEs of Class C	$y' = (a + bx + cy)^{\frac{n}{m}}$	1	$-\frac{b}{c}$
homogeneous class D	$y' = \frac{y}{x} + g(x)\, F\left(\frac{y}{x}\right)$	x^2	xy
First order special form ID 1	$y' = g(x)\, e^{h(x)+by} + f(x)$	$\frac{e^{-\int bf(x)dx - h(x)}}{g(x)}$	$\frac{f(x)e^{-\int bf(x)dx - h(x)}}{g(x)}$
polynomial type ode	$y' = \frac{a_1 x + b_1 y + c_1}{a_2 x + b_2 y + c_2}$	$\frac{a_1 b_2 x - a_2 b_1 x - b_1 c_2 + b_2 c_1}{a_1 b_2 - a_2 b_1}$	$\frac{a_1 b_2 y - a_2 b_1 y - a_1 c_2 - a_2 c_1}{a_1 b_2 - a_2 b_1}$
Bernoulli ode	$y' = f(x)\, y + g(x)\, y^n$	0	$e^{-\int (n-1) f(x) dx} y^n$
Reduced Riccati	$y' = f_1(x)\, y + f_2(x)\, y^2$	0	$e^{-\int f_1 dx}$

The above table shows that

$$
\begin{aligned}
\xi(t, y) &= \frac{1}{\cot(t)} \\
\eta(t, y) &= 0
\end{aligned} \tag{A1}
$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(t, y) \to (R, S)$ where (R, S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$
\frac{dt}{\xi} = \frac{dy}{\eta} = dS \tag{1}
$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial t} + \eta \frac{\partial}{\partial y}\right) S(t, y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the

canonical coordinates, where $S(R)$. Since $\eta = 0$ then in this special case

$$R = y$$

S is found from

$$\begin{aligned} S &= \int \frac{1}{\xi} dt \\ &= \int \frac{1}{\frac{1}{\cot(t)}} dt \end{aligned}$$

Which results in

$$S = \ln (\sin (t))$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_t + \omega(t, y) S_y}{R_t + \omega(t, y) R_y} \tag{2}$$

Where in the above R_t, R_y, S_t, S_y are all partial derivatives and $\omega(t, y)$ is the right hand side of the original ode given by

$$\omega(t, y) = \frac{\cot (t) \, y}{y + 1}$$

Evaluating all the partial derivatives gives

$$\begin{aligned} R_t &= 0 \\ R_y &= 1 \\ S_t &= \cot (t) \\ S_y &= 0 \end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = \frac{y + 1}{y} \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for t, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = \frac{R + 1}{R}$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordinates R, S. Integrating the above gives

$$S(R) = R + \ln(R) + c_1 \tag{4}$$

To complete the solution, we just need to transform (4) back to t, y coordinates. This results in

$$\ln(\sin(t)) = y + \ln(y) + c_1$$

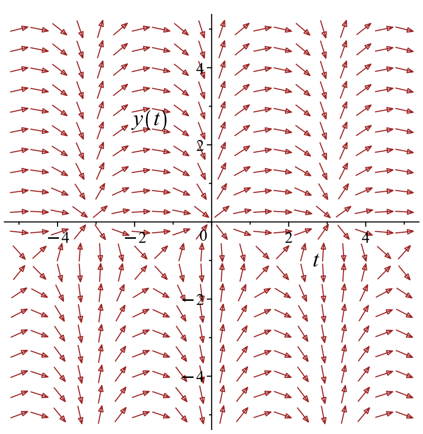
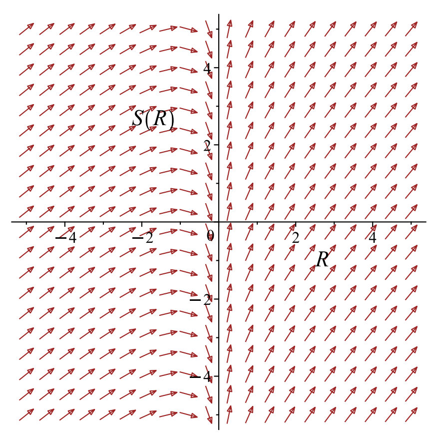
Which simplifies to

$$\ln(\sin(t)) = y + \ln(y) + c_1$$

Which gives

$$y = \text{LambertW}\left(e^{-c_1} \sin(t)\right)$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

<table>
<tr><th>Original ode in t, y coordinates</th><th>Canonical coordinates transformation</th><th>ODE in canonical coordinates (R, S)</th></tr>
<tr><td>$\frac{dy}{dt} = \frac{\cot(t)y}{y+1}$
</td><td>$R = y$
$S = \ln(\sin(t))$</td><td>$\frac{dS}{dR} = \frac{R+1}{R}$
</td></tr>
</table>

Summary

The solution(s) found are the following

$$y = \text{LambertW}\left(e^{-c_1} \sin(t)\right) \tag{1}$$

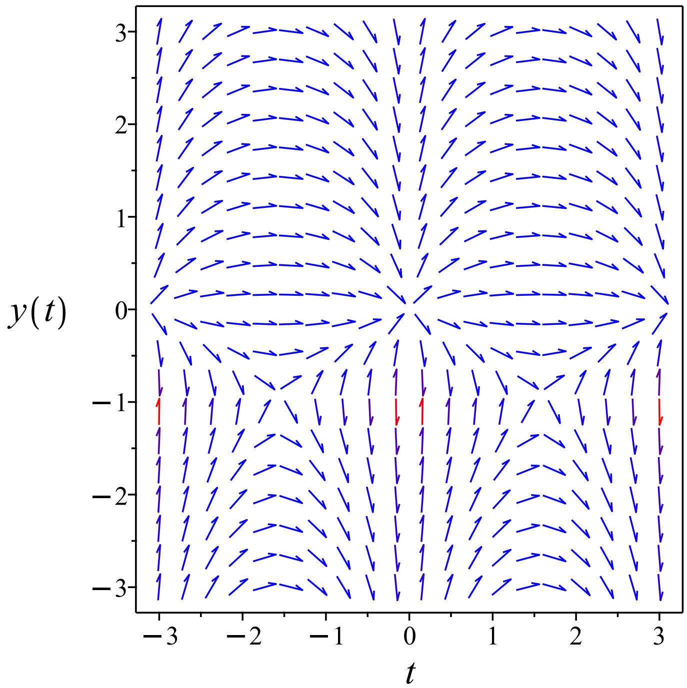


Figure 187: Slope field plot

Verification of solutions

$$y = \text{LambertW}\left(\mathrm{e}^{-c_1} \sin\left(t\right)\right)$$

Verified OK.

3.8.3 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x,y) + N(x,y)\frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x,y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y) = 0$$

Hence

$$\frac{\partial\phi}{\partial x} + \frac{\partial\phi}{\partial y}\frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$\begin{aligned}\frac{\partial \phi}{\partial x} &= M\\ \frac{\partial \phi}{\partial y} &= N\end{aligned}$$

But since $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x, y)$ but at least we know now that we can do that since the condition $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(t, y)\, \mathrm{d}t + N(t, y)\, \mathrm{d}y = 0 \tag{1A}$$

Therefore

$$\begin{aligned}\left(\frac{y+1}{y}\right) \mathrm{d}y &= (\cot(t))\, \mathrm{d}t\\ (-\cot(t))\, \mathrm{d}t + \left(\frac{y+1}{y}\right) \mathrm{d}y &= 0 \qquad (2A)\end{aligned}$$

Comparing (1A) and (2A) shows that

$$\begin{aligned}M(t, y) &= -\cot(t)\\ N(t, y) &= \frac{y+1}{y}\end{aligned}$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial t}$$

Using result found above gives

$$\begin{aligned}\frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}(-\cot(t))\\ &= 0\end{aligned}$$

And

$$\begin{aligned}\frac{\partial N}{\partial t} &= \frac{\partial}{\partial t}\left(\frac{y+1}{y}\right)\\ &= 0\end{aligned}$$

Since $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial t}$, then the ODE is exact The following equations are now set up to solve for the function $\phi(t, y)$

$$\frac{\partial \phi}{\partial t} = M \tag{1}$$

$$\frac{\partial \phi}{\partial y} = N \tag{2}$$

Integrating (1) w.r.t. t gives

$$\begin{aligned}\int \frac{\partial \phi}{\partial t}\,\mathrm{d}t &= \int M\,\mathrm{d}t\\ \int \frac{\partial \phi}{\partial t}\,\mathrm{d}t &= \int -\cot(t)\,\mathrm{d}t\\ \phi &= -\ln(\sin(t)) + f(y)\end{aligned} \tag{3}$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both t and y. Taking derivative of equation (3) w.r.t y gives

$$\frac{\partial \phi}{\partial y} = 0 + f'(y) \tag{4}$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = \frac{y+1}{y}$. Therefore equation (4) becomes

$$\frac{y+1}{y} = 0 + f'(y) \tag{5}$$

Solving equation (5) for $f'(y)$ gives

$$f'(y) = \frac{y+1}{y}$$

Integrating the above w.r.t y gives

$$\begin{aligned}\int f'(y)\,\mathrm{d}y &= \int \left(\frac{y+1}{y}\right)\mathrm{d}y\\ f(y) &= y + \ln(y) + c_1\end{aligned}$$

Where c_1 is constant of integration. Substituting result found above for $f(y)$ into equation (3) gives ϕ

$$\phi = -\ln(\sin(t)) + y + \ln(y) + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = -\ln(\sin(t)) + y + \ln(y)$$

The solution becomes

$$y = \text{LambertW}\left(e^{c_1} \sin(t)\right)$$

Summary

The solution(s) found are the following

$$y = \text{LambertW}\left(e^{c_1} \sin(t)\right) \tag{1}$$

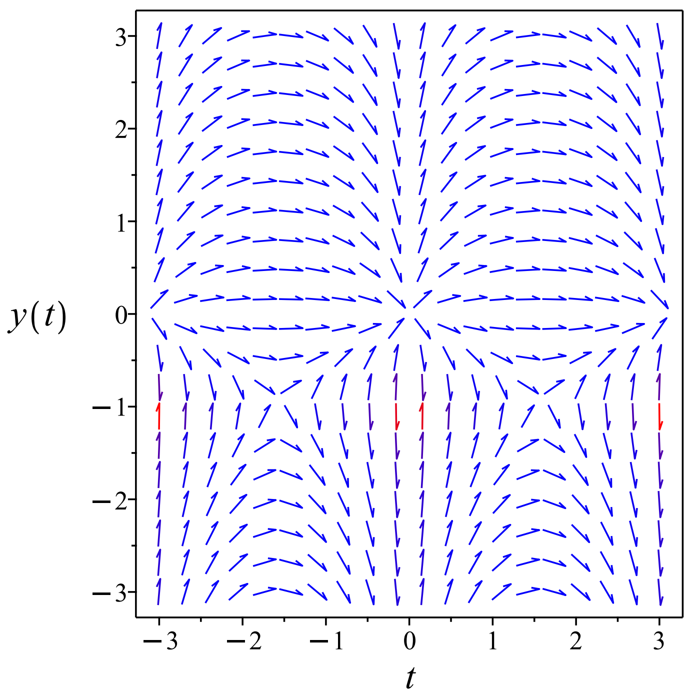


Figure 188: Slope field plot

Verification of solutions

$$y = \text{LambertW}\left(e^{c_1} \sin(t)\right)$$

Verified OK.

3.8.4 Maple step by step solution

Let's solve

$y' - \frac{\cot(t)y}{1+y} = 0$

- Highest derivative means the order of the ODE is 1

  y'

- Separate variables

  $\frac{y'(1+y)}{y} = \cot(t)$

- Integrate both sides with respect to t

  $\int \frac{y'(1+y)}{y} dt = \int \cot(t)\, dt + c_1$

- Evaluate integral

  $y + \ln(y) = \ln(\sin(t)) + c_1$

- Solve for y

  $y = \mathit{LambertW}(e^{c_1} \sin(t))$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
<- separable successful`
```

✓ Solution by Maple

Time used: 0.032 (sec). Leaf size: 9

```
dsolve(diff(y(t),t) = cot(t)*y(t)/(1+y(t)),y(t), singsol=all)
```

$$y(t) = \text{LambertW}(c_1 \sin(t))$$

✓ <u>Solution by Mathematica</u>
Time used: 1.602 (sec). Leaf size: 18

```
DSolve[y'[t] == Cot[t]*y[t]/(1+y[t]),y[t],t,IncludeSingularSolutions -> True]
```

$$y(t) \rightarrow W(e^{c_1} \sin(t))$$
$$y(t) \rightarrow 0$$

3.9 problem 13

Internal problem ID [524]
Internal file name [`OUTPUT/524_Sunday_June_05_2022_01_43_02_AM_61510835/index.tex`]

Book: Elementary differential equations and boundary value problems, 10th ed., Boyce and DiPrima
Section: Section 2.4. Page 76
Problem number: 13.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"exact", "separable", "differentialType", "homogeneousTypeD2", "first_order_ode_lie_symmetry_lookup"**

Maple gives the following as the ode type

`[_separable]`

$$\boxed{y' + \frac{4t}{y} = 0}$$

3.9.1 Solving as separable ode

In canonical form the ODE is

$$\begin{aligned} y' &= F(t, y) \\ &= f(t)g(y) \\ &= -\frac{4t}{y} \end{aligned}$$

Where $f(t) = -4t$ and $g(y) = \frac{1}{y}$. Integrating both sides gives

$$\frac{1}{\frac{1}{y}}\,dy = -4t\,dt$$

$$\int \frac{1}{\frac{1}{y}}\,dy = \int -4t\,dt$$

$$\frac{y^2}{2} = -2t^2 + c_1$$

Which results in

$$y = \sqrt{-4t^2 + 2c_1}$$

$$y = -\sqrt{-4t^2 + 2c_1}$$

<u>Summary</u>
The solution(s) found are the following

$$y = \sqrt{-4t^2 + 2c_1} \tag{1}$$

$$y = -\sqrt{-4t^2 + 2c_1} \tag{2}$$

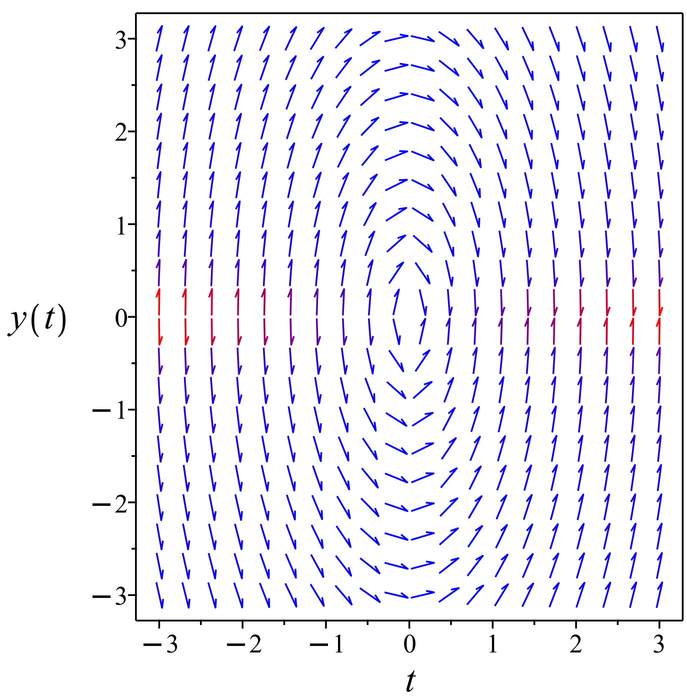


Figure 189: Slope field plot

Verification of solutions

$$y = \sqrt{-4t^2 + 2c_1}$$

Verified OK.

$$y = -\sqrt{-4t^2 + 2c_1}$$

Verified OK.

3.9.2 Solving as homogeneousTypeD2 ode

Using the change of variables $y = u(t)\, t$ on the above ode results in new ode in $u(t)$

$$u'(t)\, t + u(t) + \frac{4}{u\,(t)} = 0$$

In canonical form the ODE is

$$\begin{aligned} u' &= F(t, u) \\ &= f(t) g(u) \\ &= -\frac{u^2 + 4}{tu} \end{aligned}$$

Where $f(t) = -\frac{1}{t}$ and $g(u) = \frac{u^2+4}{u}$. Integrating both sides gives

$$\begin{aligned} \frac{1}{\frac{u^2+4}{u}}\, du &= -\frac{1}{t}\, dt \\ \int \frac{1}{\frac{u^2+4}{u}}\, du &= \int -\frac{1}{t}\, dt \\ \frac{\ln(u^2 + 4)}{2} &= -\ln(t) + c_2 \end{aligned}$$

Raising both side to exponential gives

$$\sqrt{u^2 + 4} = \mathrm{e}^{-\ln(t) + c_2}$$

Which simplifies to

$$\sqrt{u^2 + 4} = \frac{c_3}{t}$$

Which simplifies to

$$\sqrt{u\,(t)^2 + 4} = \frac{c_3 \mathrm{e}^{c_2}}{t}$$

The solution is

$$\sqrt{u(t)^2 + 4} = \frac{c_3 e^{c_2}}{t}$$

Replacing $u(t)$ in the above solution by $\frac{y}{t}$ results in the solution for y in implicit form

$$\sqrt{\frac{y^2}{t^2} + 4} = \frac{c_3 e^{c_2}}{t}$$
$$\sqrt{\frac{y^2 + 4t^2}{t^2}} = \frac{c_3 e^{c_2}}{t}$$

Summary

The solution(s) found are the following

$$\sqrt{\frac{y^2 + 4t^2}{t^2}} = \frac{c_3 e^{c_2}}{t} \tag{1}$$

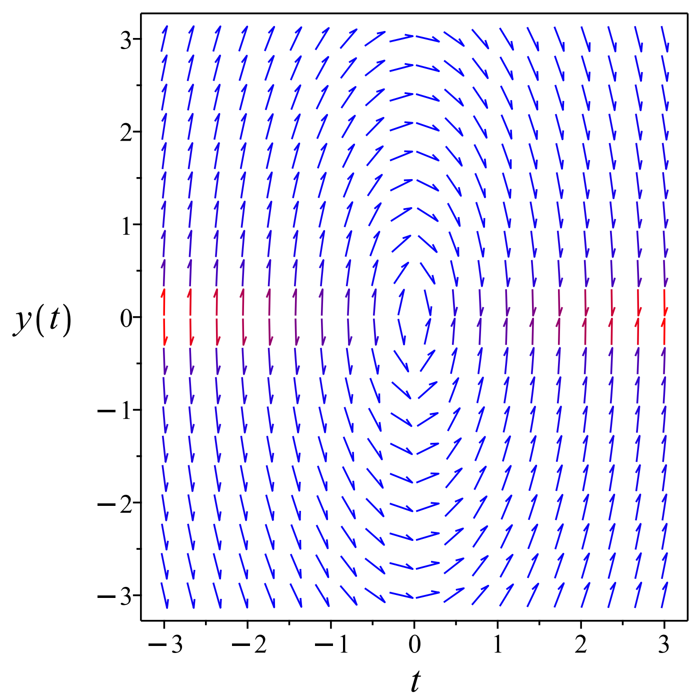


Figure 190: Slope field plot

Verification of solutions

$$\sqrt{\frac{y^2 + 4t^2}{t^2}} = \frac{c_3 e^{c_2}}{t}$$

Verified OK.

3.9.3 Solving as differentialType ode

Writing the ode as

$$y' = -\frac{4t}{y} \tag{1}$$

Which becomes

$$(y)\, dy = (-4t)\, dt \tag{2}$$

But the RHS is complete differential because

$$(-4t)\, dt = d\left(-2t^2\right)$$

Hence (2) becomes

$$(y)\, dy = d\left(-2t^2\right)$$

Integrating both sides gives gives these solutions

$$y = \sqrt{-4t^2 + 2c_1} + c_1$$
$$y = -\sqrt{-4t^2 + 2c_1} + c_1$$

Summary

The solution(s) found are the following

$$y = \sqrt{-4t^2 + 2c_1} + c_1 \tag{1}$$
$$y = -\sqrt{-4t^2 + 2c_1} + c_1 \tag{2}$$

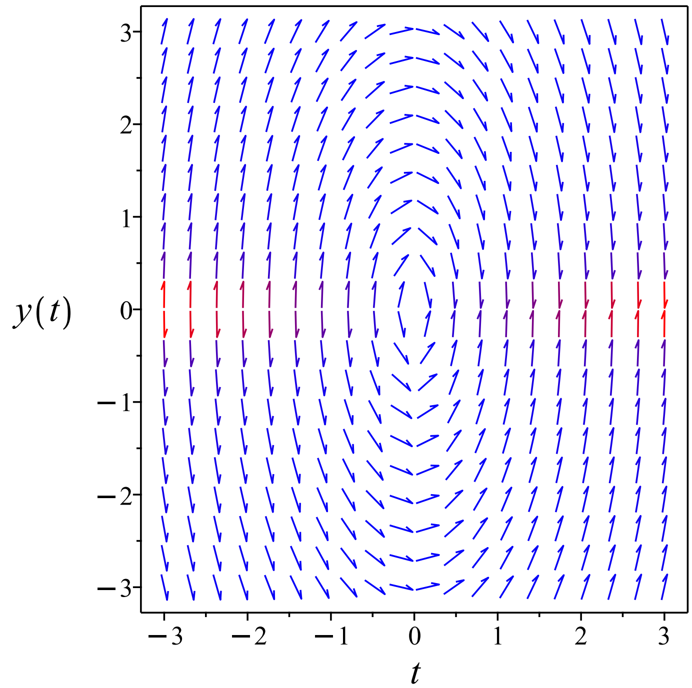


Figure 191: Slope field plot

Verification of solutions

$$y = \sqrt{-4t^2 + 2c_1} + c_1$$

Verified OK.

$$y = -\sqrt{-4t^2 + 2c_1} + c_1$$

Verified OK.

3.9.4 Solving as first order ode lie symmetry lookup ode

Writing the ode as

$$y' = -\frac{4t}{y}$$
$$y' = \omega(t, y)$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_t + \omega(\eta_y - \xi_t) - \omega^2 \xi_y - \omega_t \xi - \omega_y \eta = 0 \tag{A}$$

The type of this ode is known. It is of type `separable`. Therefore we do not need to solve the PDE (A), and can just use the lookup table shown below to find ξ, η

Table 206: Lie symmetry infinitesimal lookup table for known first order ODE's

ODE class	Form	ξ	η
linear ode	$y' = f(x)y(x) + g(x)$	0	$e^{\int f dx}$
separable ode	$y' = f(x)\, g(y)$	$\frac{1}{f}$	0
quadrature ode	$y' = f(x)$	0	1
quadrature ode	$y' = g(y)$	1	0
homogeneous ODEs of Class A	$y' = f\left(\frac{y}{x}\right)$	x	y
homogeneous ODEs of Class C	$y' = (a + bx + cy)^{\frac{n}{m}}$	1	$-\frac{b}{c}$
homogeneous class D	$y' = \frac{y}{x} + g(x)\, F\left(\frac{y}{x}\right)$	x^2	xy
First order special form ID 1	$y' = g(x)\, e^{h(x)+by} + f(x)$	$\frac{e^{-\int bf(x)dx - h(x)}}{g(x)}$	$\frac{f(x)e^{-\int bf(x)dx - h(x)}}{g(x)}$
polynomial type ode	$y' = \frac{a_1x+b_1y+c_1}{a_2x+b_2y+c_2}$	$\frac{a_1b_2x - a_2b_1x - b_1c_2 + b_2c_1}{a_1b_2 - a_2b_1}$	$\frac{a_1b_2y - a_2b_1y - a_1c_2 - a_2c_1}{a_1b_2 - a_2b_1}$
Bernoulli ode	$y' = f(x)\, y + g(x)\, y^n$	0	$e^{-\int (n-1)f(x)dx} y^n$
Reduced Riccati	$y' = f_1(x)\, y + f_2(x)\, y^2$	0	$e^{-\int f_1 dx}$

The above table shows that

$$
\begin{aligned}
\xi(t, y) &= -\frac{1}{4t} \\
\eta(t, y) &= 0
\end{aligned} \tag{A1}
$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(t, y) \to (R, S)$ where (R, S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$
\frac{dt}{\xi} = \frac{dy}{\eta} = dS \tag{1}
$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial t} + \eta \frac{\partial}{\partial y}\right) S(t, y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the

canonical coordinates, where $S(R)$. Since $\eta = 0$ then in this special case

$$R = y$$

S is found from

$$\begin{aligned} S &= \int \frac{1}{\xi} dt \\ &= \int \frac{1}{-\frac{1}{4t}} dt \end{aligned}$$

Which results in

$$S = -2t^2$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_t + \omega(t, y) S_y}{R_t + \omega(t, y) R_y} \tag{2}$$

Where in the above R_t, R_y, S_t, S_y are all partial derivatives and $\omega(t, y)$ is the right hand side of the original ode given by

$$\omega(t, y) = -\frac{4t}{y}$$

Evaluating all the partial derivatives gives

$$\begin{aligned} R_t &= 0 \\ R_y &= 1 \\ S_t &= -4t \\ S_y &= 0 \end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = y \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for t, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = R$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by

integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$S(R) = \frac{R^2}{2} + c_1 \tag{4}$$

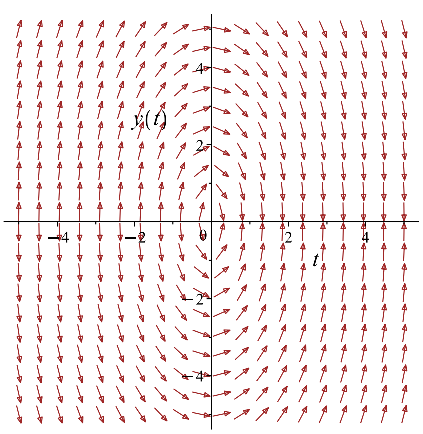
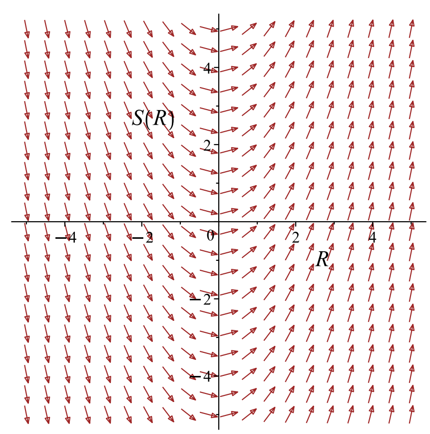
To complete the solution, we just need to transform (4) back to t, y coordinates. This results in

$$-2t^2 = \frac{y^2}{2} + c_1$$

Which simplifies to

$$-2t^2 = \frac{y^2}{2} + c_1$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in t, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dt} = -\frac{4t}{y}$ 	$R = y$ $S = -2t^2$	$\frac{dS}{dR} = R$ 

Summary

The solution(s) found are the following

$$-2t^2 = \frac{y^2}{2} + c_1 \tag{1}$$

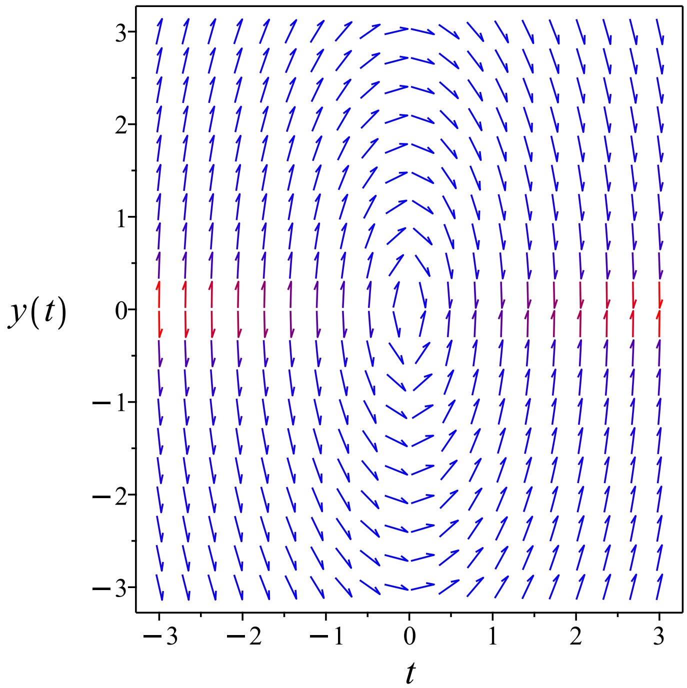


Figure 192: Slope field plot

Verification of solutions

$$-2t^2 = \frac{y^2}{2} + c_1$$

Verified OK.

3.9.5 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x,y) + N(x,y)\frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x,y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y) = 0$$

Hence

$$\frac{\partial\phi}{\partial x} + \frac{\partial\phi}{\partial y}\frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$\frac{\partial \phi}{\partial x} = M$$
$$\frac{\partial \phi}{\partial y} = N$$

But since $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x, y)$ but at least we know now that we can do that since the condition $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(t, y)\,\mathrm{d}t + N(t, y)\,\mathrm{d}y = 0 \tag{1A}$$

Therefore

$$\left(-\frac{y}{4}\right)\mathrm{d}y = (t)\,\mathrm{d}t$$
$$(-t)\,\mathrm{d}t + \left(-\frac{y}{4}\right)\mathrm{d}y = 0 \tag{2A}$$

Comparing (1A) and (2A) shows that

$$M(t, y) = -t$$
$$N(t, y) = -\frac{y}{4}$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial t}$$

Using result found above gives

$$\begin{aligned}\frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}(-t)\\ &= 0\end{aligned}$$

And

$$
\begin{aligned}
\frac{\partial N}{\partial t} &= \frac{\partial}{\partial t}\left(-\frac{y}{4}\right) \\
&= 0
\end{aligned}
$$

Since $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial t}$, then the ODE is exact The following equations are now set up to solve for the function $\phi(t, y)$

$$
\frac{\partial \phi}{\partial t} = M \tag{1}
$$

$$
\frac{\partial \phi}{\partial y} = N \tag{2}
$$

Integrating (1) w.r.t. t gives

$$
\int \frac{\partial \phi}{\partial t} \, \mathrm{d}t = \int M \, \mathrm{d}t
$$

$$
\int \frac{\partial \phi}{\partial t} \, \mathrm{d}t = \int -t \, \mathrm{d}t
$$

$$
\phi = -\frac{t^2}{2} + f(y) \tag{3}
$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both t and y. Taking derivative of equation (3) w.r.t y gives

$$
\frac{\partial \phi}{\partial y} = 0 + f'(y) \tag{4}
$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = -\frac{y}{4}$. Therefore equation (4) becomes

$$
-\frac{y}{4} = 0 + f'(y) \tag{5}
$$

Solving equation (5) for $f'(y)$ gives

$$
f'(y) = -\frac{y}{4}
$$

Integrating the above w.r.t y gives

$$
\int f'(y) \, \mathrm{d}y = \int \left(-\frac{y}{4}\right) \mathrm{d}y
$$

$$
f(y) = -\frac{y^2}{8} + c_1
$$

Where c_1 is constant of integration. Substituting result found above for $f(y)$ into equation (3) gives ϕ

$$\phi = -\frac{t^2}{2} - \frac{y^2}{8} + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = -\frac{t^2}{2} - \frac{y^2}{8}$$

Summary

The solution(s) found are the following

$$-\frac{t^2}{2} - \frac{y^2}{8} = c_1 \tag{1}$$

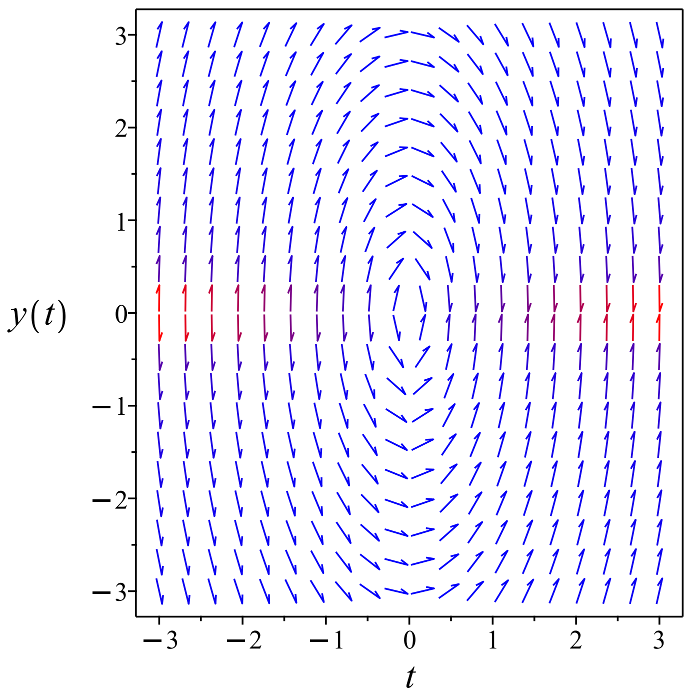


Figure 193: Slope field plot

Verification of solutions

$$-\frac{t^2}{2} - \frac{y^2}{8} = c_1$$

Verified OK.

3.9.6 Maple step by step solution

Let's solve

$y' + \frac{4t}{y} = 0$

- Highest derivative means the order of the ODE is 1

  y'

- Separate variables

  $y'y = -4t$

- Integrate both sides with respect to t

  $\int y'ydt = \int -4tdt + c_1$

- Evaluate integral

  $\frac{y^2}{2} = -2t^2 + c_1$

- Solve for y

  $\left\{y = \sqrt{-4t^2 + 2c_1}, y = -\sqrt{-4t^2 + 2c_1}\right\}$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
<- Bernoulli successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 27

```
dsolve(diff(y(t),t) = -4*t/y(t),y(t), singsol=all)
```

$$y(t) = \sqrt{-4t^2 + c_1}$$
$$y(t) = -\sqrt{-4t^2 + c_1}$$

✓ Solution by Mathematica

Time used: 0.087 (sec). Leaf size: 46

```
DSolve[y'[t]== -4*t/y[t],y[t],t,IncludeSingularSolutions -> True]
```

$$y(t) \to -\sqrt{2}\sqrt{-2t^2 + c_1}$$
$$y(t) \to \sqrt{2}\sqrt{-2t^2 + c_1}$$

3.10 problem 14

Internal problem ID [525]
Internal file name [`OUTPUT/525_Sunday_June_05_2022_01_43_03_AM_244180/index.tex`]

Book: Elementary differential equations and boundary value problems, 10th ed., Boyce and DiPrima
Section: Section 2.4. Page 76
Problem number: 14.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"exact", "riccati", "separable", "first_order_ode_lie_symmetry_lookup"**

Maple gives the following as the ode type

`[_separable]`

$$\boxed{y' - 2ty^2 = 0}$$

3.10.1 Solving as separable ode

In canonical form the ODE is

$$\begin{aligned} y' &= F(t, y) \\ &= f(t)g(y) \\ &= 2t\, y^2 \end{aligned}$$

Where $f(t) = 2t$ and $g(y) = y^2$. Integrating both sides gives

$$\frac{1}{y^2}\, dy = 2t\, dt$$

$$\int \frac{1}{y^2}\, dy = \int 2t\, dt$$

$$-\frac{1}{y} = t^2 + c_1$$

Which results in

$$y = -\frac{1}{t^2 + c_1}$$

Summary
The solution(s) found are the following

$$y = -\frac{1}{t^2 + c_1} \tag{1}$$

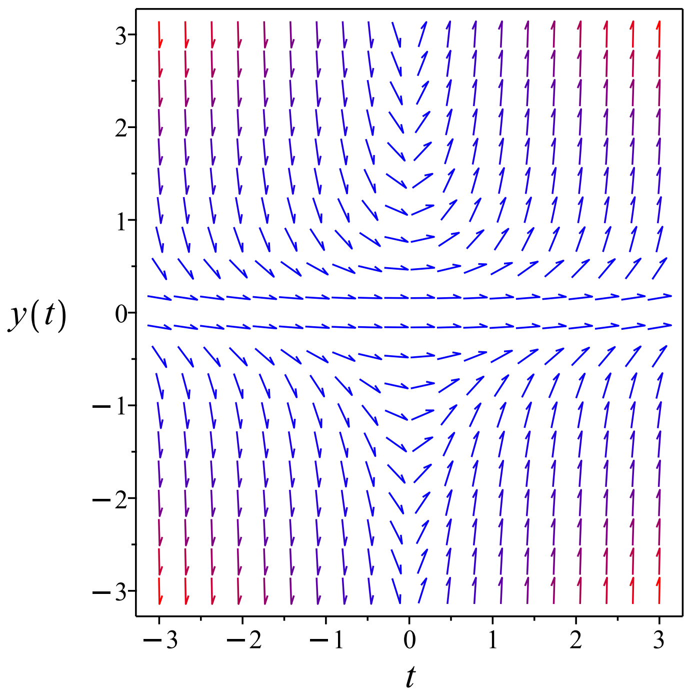


Figure 194: Slope field plot

Verification of solutions

$$y = -\frac{1}{t^2 + c_1}$$

Verified OK.

3.10.2 Solving as first order ode lie symmetry lookup ode

Writing the ode as

$$y' = 2t\,y^2$$
$$y' = \omega(t, y)$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_t + \omega(\eta_y - \xi_t) - \omega^2 \xi_y - \omega_t \xi - \omega_y \eta = 0 \tag{A}$$

The type of this ode is known. It is of type `separable`. Therefore we do not need to solve the PDE (A), and can just use the lookup table shown below to find ξ, η

Table 209: Lie symmetry infinitesimal lookup table for known first order ODE's

ODE class	Form	ξ	η
linear ode	$y' = f(x)y(x) + g(x)$	0	$e^{\int f dx}$
separable ode	$y' = f(x)\,g(y)$	$\frac{1}{f}$	0
quadrature ode	$y' = f(x)$	0	1
quadrature ode	$y' = g(y)$	1	0
homogeneous ODEs of Class A	$y' = f\left(\frac{y}{x}\right)$	x	y
homogeneous ODEs of Class C	$y' = (a + bx + cy)^{\frac{n}{m}}$	1	$-\frac{b}{c}$
homogeneous class D	$y' = \frac{y}{x} + g(x)\,F\left(\frac{y}{x}\right)$	x^2	xy
First order special form ID 1	$y' = g(x)\,e^{h(x)+by} + f(x)$	$\frac{e^{-\int bf(x)dx - h(x)}}{g(x)}$	$\frac{f(x)e^{-\int bf(x)dx - h(x)}}{g(x)}$
polynomial type ode	$y' = \frac{a_1x + b_1y + c_1}{a_2x + b_2y + c_2}$	$\frac{a_1b_2x - a_2b_1x - b_1c_2 + b_2c_1}{a_1b_2 - a_2b_1}$	$\frac{a_1b_2y - a_2b_1y - a_1c_2 - a_2c_1}{a_1b_2 - a_2b_1}$
Bernoulli ode	$y' = f(x)\,y + g(x)\,y^n$	0	$e^{-\int (n-1)f(x)dx}y^n$
Reduced Riccati	$y' = f_1(x)\,y + f_2(x)\,y^2$	0	$e^{-\int f_1 dx}$

The above table shows that

$$\begin{aligned}\xi(t,y) &= \frac{1}{2t}\\ \eta(t,y) &= 0\end{aligned} \tag{A1}$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(t,y) \to (R,S)$ where (R,S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$\frac{dt}{\xi} = \frac{dy}{\eta} = dS \tag{1}$$

The above comes from the requirements that $\left(\xi\frac{\partial}{\partial t} + \eta\frac{\partial}{\partial y}\right) S(t,y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the canonical coordinates, where $S(R)$. Since $\eta = 0$ then in this special case

$$R = y$$

S is found from

$$\begin{aligned}S &= \int \frac{1}{\xi} dt\\ &= \int \frac{1}{\frac{1}{2t}} dt\end{aligned}$$

Which results in

$$S = t^2$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_t + \omega(t,y)S_y}{R_t + \omega(t,y)R_y} \tag{2}$$

Where in the above R_t, R_y, S_t, S_y are all partial derivatives and $\omega(t,y)$ is the right hand side of the original ode given by

$$\omega(t,y) = 2t\, y^2$$

Evaluating all the partial derivatives gives

$$\begin{aligned}R_t &= 0\\ R_y &= 1\\ S_t &= 2t\\ S_y &= 0\end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = \frac{1}{y^2} \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for t, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = \frac{1}{R^2}$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$S(R) = -\frac{1}{R} + c_1 \tag{4}$$

To complete the solution, we just need to transform (4) back to t, y coordinates. This results in

$$t^2 = -\frac{1}{y} + c_1$$

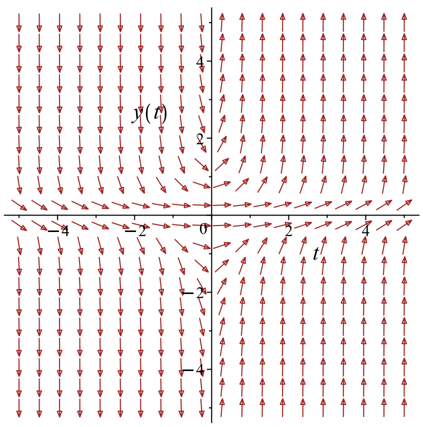
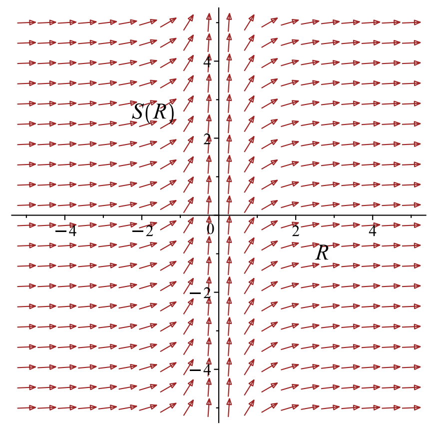
Which simplifies to

$$t^2 = -\frac{1}{y} + c_1$$

Which gives

$$y = \frac{1}{-t^2 + c_1}$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in t, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dt} = 2t\,y^2$ 	$R = y$ $S = t^2$	$\frac{dS}{dR} = \frac{1}{R^2}$ 

Summary

The solution(s) found are the following

$$y = \frac{1}{-t^2 + c_1} \tag{1}$$

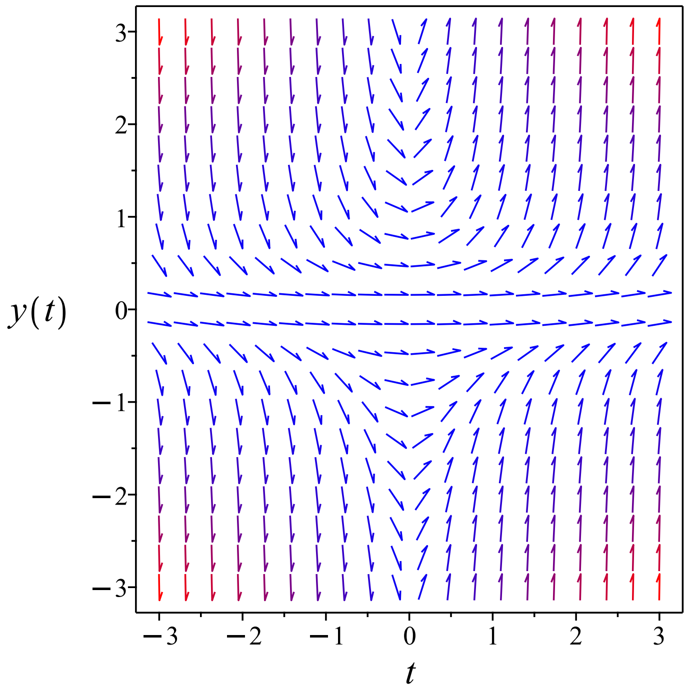


Figure 195: Slope field plot

Verification of solutions

$$y = \frac{1}{-t^2 + c_1}$$

Verified OK.

3.10.3 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x,y) + N(x,y)\frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x,y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y) = 0$$

Hence

$$\frac{\partial \phi}{\partial x} + \frac{\partial \phi}{\partial y}\frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$
\begin{aligned}
\frac{\partial \phi}{\partial x} &= M \\
\frac{\partial \phi}{\partial y} &= N
\end{aligned}
$$

But since $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ then for the above to be valid, we require that

$$
\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}
$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x, y)$ but at least we know now that we can do that since the condition $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$
M(t, y)\,\mathrm{d}t + N(t, y)\,\mathrm{d}y = 0 \tag{1A}
$$

Therefore

$$
\begin{aligned}
\left(\frac{1}{2y^2}\right) \mathrm{d}y &= (t)\,\mathrm{d}t \\
(-t)\,\mathrm{d}t + \left(\frac{1}{2y^2}\right) \mathrm{d}y &= 0 \qquad \text{(2A)}
\end{aligned}
$$

Comparing (1A) and (2A) shows that

$$
\begin{aligned}
M(t, y) &= -t \\
N(t, y) &= \frac{1}{2y^2}
\end{aligned}
$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$
\frac{\partial M}{\partial y} = \frac{\partial N}{\partial t}
$$

Using result found above gives

$$
\begin{aligned}
\frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}(-t) \\
&= 0
\end{aligned}
$$

And

$$
\begin{aligned}
\frac{\partial N}{\partial t} &= \frac{\partial}{\partial t}\left(\frac{1}{2y^2}\right) \\
&= 0
\end{aligned}
$$

Since $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial t}$, then the ODE is <u>exact</u> The following equations are now set up to solve for the function $\phi(t, y)$

$$
\frac{\partial \phi}{\partial t} = M \tag{1}
$$

$$
\frac{\partial \phi}{\partial y} = N \tag{2}
$$

Integrating (1) w.r.t. t gives

$$
\begin{aligned}
\int \frac{\partial \phi}{\partial t} \, \mathrm{d}t &= \int M \, \mathrm{d}t \\
\int \frac{\partial \phi}{\partial t} \, \mathrm{d}t &= \int -t \, \mathrm{d}t \\
\phi &= -\frac{t^2}{2} + f(y) \qquad (3)
\end{aligned}
$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both t and y. Taking derivative of equation (3) w.r.t y gives

$$
\frac{\partial \phi}{\partial y} = 0 + f'(y) \tag{4}
$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = \frac{1}{2y^2}$. Therefore equation (4) becomes

$$
\frac{1}{2y^2} = 0 + f'(y) \tag{5}
$$

Solving equation (5) for $f'(y)$ gives

$$
f'(y) = \frac{1}{2y^2}
$$

Integrating the above w.r.t y gives

$$\int f'(y)\,\mathrm{d}y = \int \left(\frac{1}{2y^2}\right) \mathrm{d}y$$
$$f(y) = -\frac{1}{2y} + c_1$$

Where c_1 is constant of integration. Substituting result found above for $f(y)$ into equation (3) gives ϕ

$$\phi = -\frac{t^2}{2} - \frac{1}{2y} + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = -\frac{t^2}{2} - \frac{1}{2y}$$

The solution becomes

$$y = -\frac{1}{t^2 + 2c_1}$$

Summary

The solution(s) found are the following

$$y = -\frac{1}{t^2 + 2c_1} \tag{1}$$

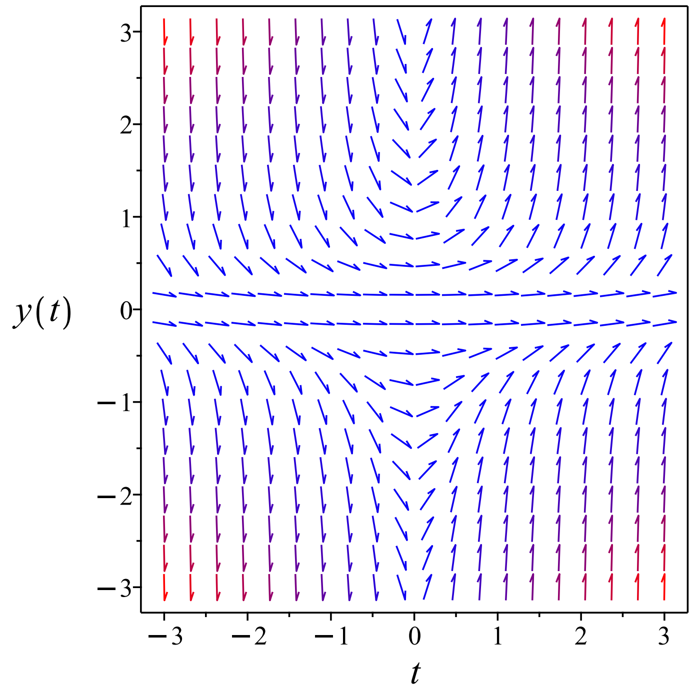


Figure 196: Slope field plot

Verification of solutions

$$y = -\frac{1}{t^2 + 2c_1}$$

Verified OK.

3.10.4 Solving as riccati ode

In canonical form the ODE is

$$\begin{aligned}
y' &= F(t, y) \\
&= 2t\,y^2
\end{aligned}$$

This is a Riccati ODE. Comparing the ODE to solve

$$y' = 2t\,y^2$$

With Riccati ODE standard form

$$y' = f_0(t) + f_1(t)y + f_2(t)y^2$$

Shows that $f_0(t) = 0$, $f_1(t) = 0$ and $f_2(t) = 2t$. Let

$$\begin{aligned} y &= \frac{-u'}{f_2 u} \\ &= \frac{-u'}{2tu} \end{aligned} \tag{1}$$

Using the above substitution in the given ODE results (after some simplification)in a second order ODE to solve for $u(x)$ which is

$$f_2 u''(t) - (f_2' + f_1 f_2) u'(t) + f_2^2 f_0 u(t) = 0 \tag{2}$$

But

$$\begin{aligned} f_2' &= 2 \\ f_1 f_2 &= 0 \\ f_2^2 f_0 &= 0 \end{aligned}$$

Substituting the above terms back in equation (2) gives

$$2tu''(t) - 2u'(t) = 0$$

Solving the above ODE (this ode solved using Maple, not this program), gives

$$u(t) = c_2 t^2 + c_1$$

The above shows that

$$u'(t) = 2c_2 t$$

Using the above in (1) gives the solution

$$y = -\frac{c_2}{c_2 t^2 + c_1}$$

Dividing both numerator and denominator by c_1 gives, after renaming the constant $\frac{c_2}{c_1} = c_3$ the following solution

$$y = -\frac{1}{t^2 + c_3}$$

Summary

The solution(s) found are the following

$$y = -\frac{1}{t^2 + c_3} \tag{1}$$

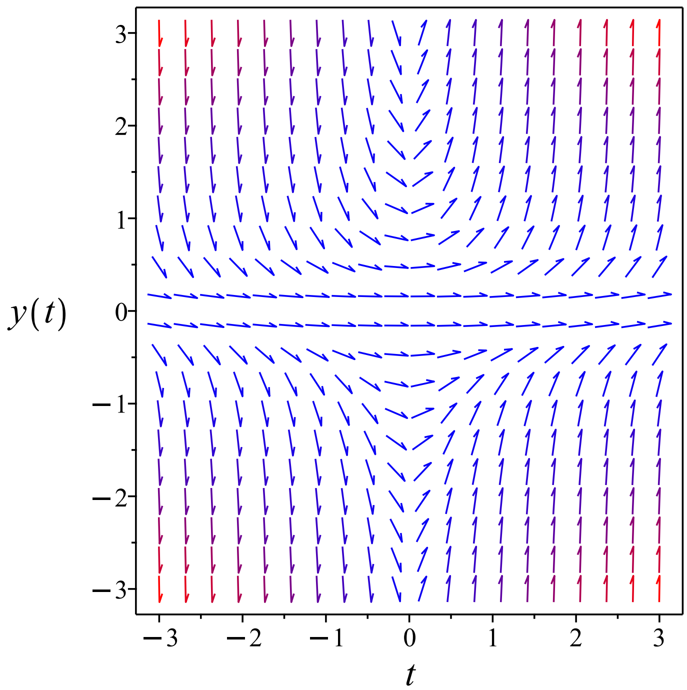


Figure 197: Slope field plot

Verification of solutions

$$y = -\frac{1}{t^2 + c_3}$$

Verified OK.

3.10.5 Maple step by step solution

Let's solve

$y' - 2ty^2 = 0$

- Highest derivative means the order of the ODE is 1

  y'

- Separate variables

$\frac{y'}{y^2} = 2t$

- Integrate both sides with respect to t

  $\int \frac{y'}{y^2}dt = \int 2tdt + c_1$

- Evaluate integral

  $-\frac{1}{y} = t^2 + c_1$

- Solve for y

  $y = -\frac{1}{t^2+c_1}$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
<- Bernoulli successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 13

```
dsolve(diff(y(t),t) = 2*t*y(t)^2,y(t), singsol=all)
```

$$y(t) = \frac{1}{-t^2 + c_1}$$

✓ Solution by Mathematica

Time used: 0.111 (sec). Leaf size: 20

```
DSolve[y'[t] == 2*t*y[t]^2,y[t],t,IncludeSingularSolutions -> True]
```

$$y(t) \to -\frac{1}{t^2 + c_1}$$
$$y(t) \to 0$$

3.11 problem 15

Internal problem ID [526]
Internal file name [`OUTPUT/526_Sunday_June_05_2022_01_43_04_AM_90328302/index.tex`]

Book: Elementary differential equations and boundary value problems, 10th ed., Boyce and DiPrima
Section: Section 2.4. Page 76
Problem number: 15.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"quadrature"**

Maple gives the following as the ode type

`[_quadrature]`

$$\boxed{y^3 + y' = 0}$$

3.11.1 Solving as quadrature ode

Integrating both sides gives

$$\int -\frac{1}{y^3} dy = t + c_1$$
$$\frac{1}{2y^2} = t + c_1$$

Solving for y gives these solutions

$$y_1 = \frac{1}{\sqrt{2c_1 + 2t}}$$
$$y_2 = -\frac{1}{\sqrt{2c_1 + 2t}}$$

Summary
The solution(s) found are the following

$$y = \frac{1}{\sqrt{2c_1 + 2t}} \tag{1}$$
$$y = -\frac{1}{\sqrt{2c_1 + 2t}} \tag{2}$$

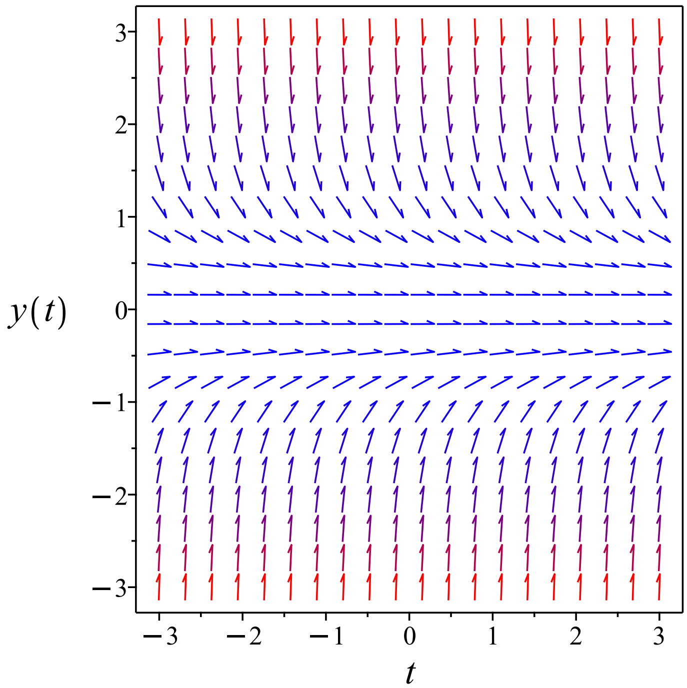


Figure 198: Slope field plot

Verification of solutions

$$y = \frac{1}{\sqrt{2c_1 + 2t}}$$

Verified OK.

$$y = -\frac{1}{\sqrt{2c_1 + 2t}}$$

Verified OK.

3.11.2 Maple step by step solution

Let's solve

$y^3 + y' = 0$

- Highest derivative means the order of the ODE is 1

  y'

- Separate variables

  $\frac{y'}{y^3} = -1$

- Integrate both sides with respect to t

  $\int \frac{y'}{y^3} dt = \int (-1)\, dt + c_1$

- Evaluate integral

  $-\frac{1}{2y^2} = -t + c_1$

- Solve for y

  $\left\{y = \frac{1}{\sqrt{-2c_1+2t}}, y = -\frac{1}{\sqrt{-2c_1+2t}}\right\}$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
<- Bernoulli successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 23

```
dsolve(y(t)^3+diff(y(t),t) = 0,y(t), singsol=all)
```

$$y(t) = \frac{1}{\sqrt{2t + c_1}}$$
$$y(t) = -\frac{1}{\sqrt{2t + c_1}}$$

✓ Solution by Mathematica

Time used: 0.124 (sec). Leaf size: 40

```
DSolve[y[t]^3+y'[t] == 0,y[t],t,IncludeSingularSolutions -> True]
```

$$y(t) \to -\frac{1}{\sqrt{2t - 2c_1}}$$
$$y(t) \to \frac{1}{\sqrt{2t - 2c_1}}$$
$$y(t) \to 0$$

3.12 problem 16

Internal problem ID [527]
Internal file name [`OUTPUT/527_Sunday_June_05_2022_01_43_05_AM_3895157/index.tex`]

Book: Elementary differential equations and boundary value problems, 10th ed., Boyce and DiPrima
Section: Section 2.4. Page 76
Problem number: 16.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"exact", "separable", "first_order_ode_lie_symmetry_lookup"**

Maple gives the following as the ode type

```
[_separable]
```

$$\boxed{y' - \frac{t^2}{(t^3+1)\,y} = 0}$$

3.12.1 Solving as separable ode

In canonical form the ODE is

$$\begin{aligned} y' &= F(t,y) \\ &= f(t)g(y) \\ &= \frac{t^2}{(t^3+1)\,y} \end{aligned}$$

Where $f(t) = \frac{t^2}{t^3+1}$ and $g(y) = \frac{1}{y}$. Integrating both sides gives

$$\frac{1}{\frac{1}{y}}\,dy = \frac{t^2}{t^3+1}\,dt$$

$$\int \frac{1}{\frac{1}{y}}\,dy = \int \frac{t^2}{t^3+1}\,dt$$
$$\frac{y^2}{2} = \frac{\ln\left(t^3+1\right)}{3} + c_1$$

Which results in

$$y = \frac{\sqrt{6\ln\left(t^3+1\right)+18c_1}}{3}$$
$$y = -\frac{\sqrt{6\ln\left(t^3+1\right)+18c_1}}{3}$$

<u>Summary</u>
The solution(s) found are the following

$$y = \frac{\sqrt{6\ln\left(t^3+1\right)+18c_1}}{3} \tag{1}$$
$$y = -\frac{\sqrt{6\ln\left(t^3+1\right)+18c_1}}{3} \tag{2}$$

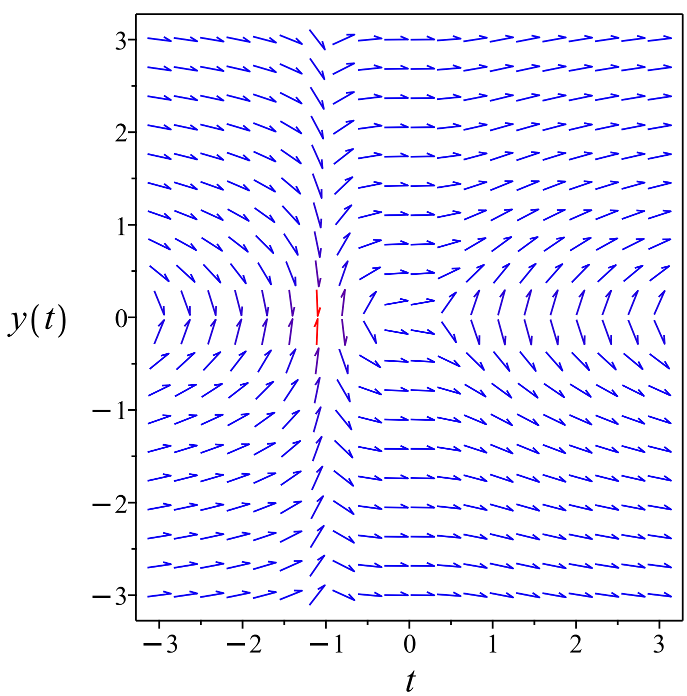


Figure 199: Slope field plot

Verification of solutions

$$y = \frac{\sqrt{6 \ln \left(t^3+1\right)+18 c_1}}{3}$$

Verified OK.

$$y = -\frac{\sqrt{6 \ln \left(t^3+1\right)+18 c_1}}{3}$$

Verified OK.

3.12.2 Solving as first order ode lie symmetry lookup ode

Writing the ode as

$$y' = \frac{t^2}{\left(t^3+1\right) y}$$
$$y' = \omega(t, y)$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_t + \omega(\eta_y - \xi_t) - \omega^2 \xi_y - \omega_t \xi - \omega_y \eta = 0 \tag{A}$$

The type of this ode is known. It is of type `separable`. Therefore we do not need to solve the PDE (A), and can just use the lookup table shown below to find ξ, η

Table 213: Lie symmetry infinitesimal lookup table for known first order ODE's

ODE class	Form	ξ	η
linear ode	$y' = f(x)y(x) + g(x)$	0	$e^{\int f dx}$
separable ode	$y' = f(x)\, g(y)$	$\frac{1}{f}$	0
quadrature ode	$y' = f(x)$	0	1
quadrature ode	$y' = g(y)$	1	0
homogeneous ODEs of Class A	$y' = f\left(\frac{y}{x}\right)$	x	y
homogeneous ODEs of Class C	$y' = (a + bx + cy)^{\frac{n}{m}}$	1	$-\frac{b}{c}$
homogeneous class D	$y' = \frac{y}{x} + g(x)\, F\left(\frac{y}{x}\right)$	x^2	xy
First order special form ID 1	$y' = g(x)\, e^{h(x)+by} + f(x)$	$\frac{e^{-\int b f(x) dx - h(x)}}{g(x)}$	$\frac{f(x) e^{-\int b f(x) dx - h(x)}}{g(x)}$
polynomial type ode	$y' = \frac{a_1 x + b_1 y + c_1}{a_2 x + b_2 y + c_2}$	$\frac{a_1 b_2 x - a_2 b_1 x - b_1 c_2 + b_2 c_1}{a_1 b_2 - a_2 b_1}$	$\frac{a_1 b_2 y - a_2 b_1 y - a_1 c_2 - a_2 c_1}{a_1 b_2 - a_2 b_1}$
Bernoulli ode	$y' = f(x)\, y + g(x)\, y^n$	0	$e^{-\int (n-1) f(x) dx} y^n$
Reduced Riccati	$y' = f_1(x)\, y + f_2(x)\, y^2$	0	$e^{-\int f_1 dx}$

The above table shows that

$$
\begin{aligned}
\xi(t, y) &= \frac{t^3 + 1}{t^2} \\
\eta(t, y) &= 0
\end{aligned}
\tag{A1}
$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(t, y) \to (R, S)$ where (R, S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$
\frac{dt}{\xi} = \frac{dy}{\eta} = dS \tag{1}
$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial t} + \eta \frac{\partial}{\partial y}\right) S(t, y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the

canonical coordinates, where $S(R)$. Since $\eta = 0$ then in this special case

$$R = y$$

S is found from

$$\begin{aligned} S &= \int \frac{1}{\xi} dt \\ &= \int \frac{1}{\frac{t^3+1}{t^2}} dt \end{aligned}$$

Which results in

$$S = \frac{\ln(t^3+1)}{3}$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_t + \omega(t,y)S_y}{R_t + \omega(t,y)R_y} \tag{2}$$

Where in the above R_t, R_y, S_t, S_y are all partial derivatives and $\omega(t,y)$ is the right hand side of the original ode given by

$$\omega(t,y) = \frac{t^2}{(t^3+1)\,y}$$

Evaluating all the partial derivatives gives

$$\begin{aligned} R_t &= 0 \\ R_y &= 1 \\ S_t &= \frac{t^2}{(t^2-t+1)(t+1)} \\ S_y &= 0 \end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = y \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for t, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = R$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordinates R, S. Integrating the above gives

$$S(R) = \frac{R^2}{2} + c_1 \tag{4}$$

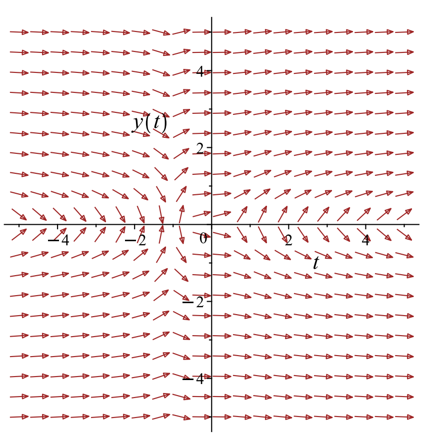
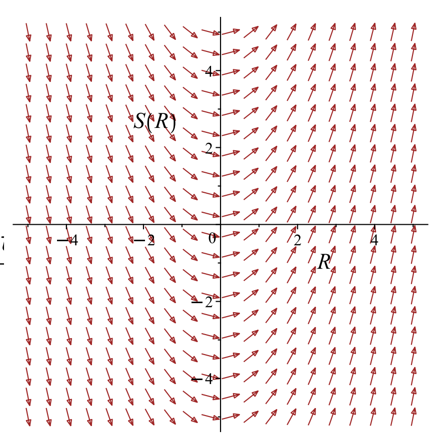
To complete the solution, we just need to transform (4) back to t, y coordinates. This results in

$$\frac{\ln(t+1)}{3} + \frac{\ln(t^2 - t + 1)}{3} = \frac{y^2}{2} + c_1$$

Which simplifies to

$$\frac{\ln(t+1)}{3} + \frac{\ln(t^2 - t + 1)}{3} = \frac{y^2}{2} + c_1$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in t, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dt} = \frac{t^2}{(t^3+1)y}$ 	$R = y$ $S = \frac{\ln(t+1)}{3} + \frac{\ln(t^2 -}{3}$	$\frac{dS}{dR} = R$ 

Summary

The solution(s) found are the following

$$\frac{\ln(t+1)}{3} + \frac{\ln(t^2 - t + 1)}{3} = \frac{y^2}{2} + c_1 \tag{1}$$

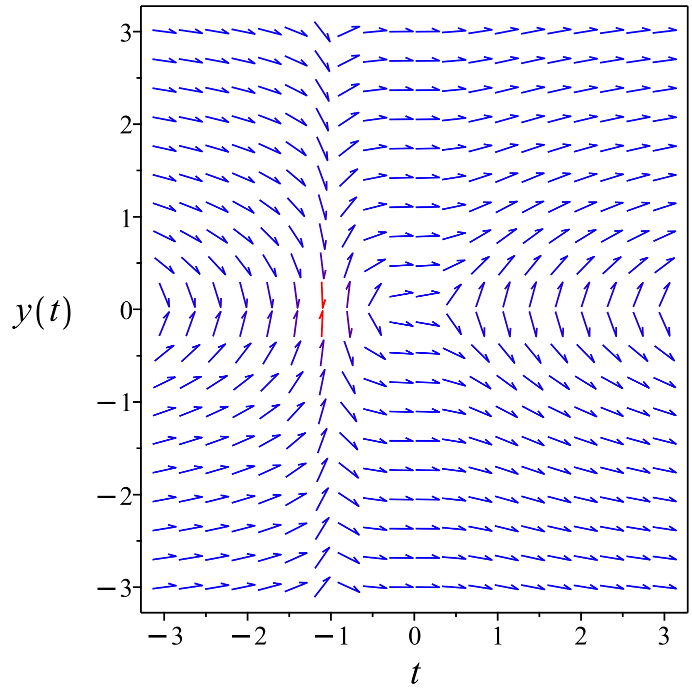


Figure 200: Slope field plot

Verification of solutions

$$\frac{\ln(t+1)}{3}+\frac{\ln(t^2-t+1)}{3}=\frac{y^2}{2}+c_1$$

Verified OK.

3.12.3 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x,y)+N(x,y)\frac{dy}{dx}=0 \tag{A}$$

We assume there exists a function $\phi(x,y)=c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y)=0$$

Hence

$$\frac{\partial\phi}{\partial x}+\frac{\partial\phi}{\partial y}\frac{dy}{dx}=0 \tag{B}$$

Comparing (A,B) shows that

$$\frac{\partial \phi}{\partial x} = M$$
$$\frac{\partial \phi}{\partial y} = N$$

But since $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x,y)$ but at least we know now that we can do that since the condition $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(t,y)\,\mathrm{d}t + N(t,y)\,\mathrm{d}y = 0 \tag{1A}$$

Therefore

$$(y)\,\mathrm{d}y = \left(\frac{t^2}{t^3+1}\right)\mathrm{d}t$$
$$\left(-\frac{t^2}{t^3+1}\right)\mathrm{d}t + (y)\,\mathrm{d}y = 0 \tag{2A}$$

Comparing (1A) and (2A) shows that

$$M(t,y) = -\frac{t^2}{t^3+1}$$
$$N(t,y) = y$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial t}$$

Using result found above gives

$$\begin{aligned}\frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}\left(-\frac{t^2}{t^3+1}\right) \\ &= 0\end{aligned}$$

And

$$
\begin{aligned}
\frac{\partial N}{\partial t} &= \frac{\partial}{\partial t}(y) \\
&= 0
\end{aligned}
$$

Since $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial t}$, then the ODE is <u>exact</u> The following equations are now set up to solve for the function $\phi(t, y)$

$$
\frac{\partial \phi}{\partial t} = M \tag{1}
$$

$$
\frac{\partial \phi}{\partial y} = N \tag{2}
$$

Integrating (1) w.r.t. t gives

$$
\begin{aligned}
\int \frac{\partial \phi}{\partial t} \, \mathrm{d}t &= \int M \, \mathrm{d}t \\
\int \frac{\partial \phi}{\partial t} \, \mathrm{d}t &= \int -\frac{t^2}{t^3+1} \, \mathrm{d}t
\end{aligned}
$$

$$
\phi = -\frac{\ln(t^3+1)}{3} + f(y) \tag{3}
$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both t and y. Taking derivative of equation (3) w.r.t y gives

$$
\frac{\partial \phi}{\partial y} = 0 + f'(y) \tag{4}
$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = y$. Therefore equation (4) becomes

$$
y = 0 + f'(y) \tag{5}
$$

Solving equation (5) for $f'(y)$ gives

$$
f'(y) = y
$$

Integrating the above w.r.t y gives

$$
\begin{aligned}
\int f'(y) \, \mathrm{d}y &= \int (y) \, \mathrm{d}y \\
f(y) &= \frac{y^2}{2} + c_1
\end{aligned}
$$

Where c_1 is constant of integration. Substituting result found above for $f(y)$ into equation (3) gives ϕ

$$\phi = -\frac{\ln(t^3+1)}{3} + \frac{y^2}{2} + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = -\frac{\ln(t^3+1)}{3} + \frac{y^2}{2}$$

Summary

The solution(s) found are the following

$$-\frac{\ln(t^3+1)}{3} + \frac{y^2}{2} = c_1 \tag{1}$$

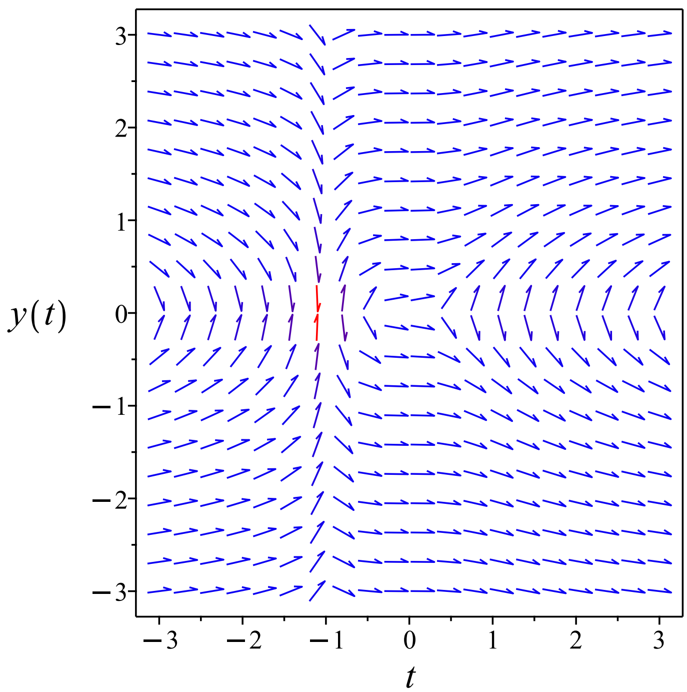


Figure 201: Slope field plot

Verification of solutions

$$-\frac{\ln(t^3+1)}{3} + \frac{y^2}{2} = c_1$$

Verified OK.

3.12.4 Maple step by step solution

Let's solve

$y' - \frac{t^2}{(t^3+1)y} = 0$

- Highest derivative means the order of the ODE is 1

  y'

- Separate variables

  $y'y = \frac{t^2}{t^3+1}$

- Integrate both sides with respect to t

  $\int y'ydt = \int \frac{t^2}{t^3+1}dt + c_1$

- Evaluate integral

  $\frac{y^2}{2} = \frac{\ln(t^3+1)}{3} + c_1$

- Solve for y

  $\left\{y = -\frac{\sqrt{6\ln(t^3+1)+18c_1}}{3}, y = \frac{\sqrt{6\ln(t^3+1)+18c_1}}{3}\right\}$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
<- Bernoulli successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 39

```
dsolve(diff(y(t),t) = t^2/(t^3+1)/y(t),y(t), singsol=all)
```

$$y(t) = -\frac{\sqrt{6\ln(t^3+1)+9c_1}}{3}$$
$$y(t) = \frac{\sqrt{6\ln(t^3+1)+9c_1}}{3}$$

✓ Solution by Mathematica

Time used: 0.123 (sec). Leaf size: 56

```
DSolve[y'[t] == t^2/(t^3+1)/y[t],y[t],t,IncludeSingularSolutions -> True]
```

$$y(t) \to -\sqrt{\frac{2}{3}}\sqrt{\log\left(t^3+1\right)+3c_1}$$
$$y(t) \to \sqrt{\frac{2}{3}}\sqrt{\log\left(t^3+1\right)+3c_1}$$

3.13 problem 17

Internal problem ID [528]
Internal file name [`OUTPUT/528_Sunday_June_05_2022_01_43_06_AM_89383031/index.tex`]

Book: Elementary differential equations and boundary value problems, 10th ed., Boyce and DiPrima
Section: Section 2.4. Page 76
Problem number: 17.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"exact", "riccati", "bernoulli", "separable", "first_order_ode_lie_symmetry_lookup"**

Maple gives the following as the ode type

`[_separable]`

$$\boxed{y' - t(3 - y)\, y = 0}$$

3.13.1 Solving as separable ode

In canonical form the ODE is

$$\begin{aligned} y' &= F(t, y) \\ &= f(t)g(y) \\ &= -ty(-3 + y) \end{aligned}$$

Where $f(t) = -t$ and $g(y) = y(-3 + y)$. Integrating both sides gives

$$\frac{1}{y\,(-3 + y)}\, dy = -t\, dt$$

$$\int \frac{1}{y(-3+y)}\,dy = \int -t\,dt$$

$$-\frac{\ln(y)}{3} + \frac{\ln(-3+y)}{3} = -\frac{t^2}{2} + c_1$$

The above can be written as

$$\left(-\frac{1}{3}\right)(\ln(y) - \ln(-3+y)) = -\frac{t^2}{2} + 2c_1$$

$$\begin{aligned}\ln(y) - \ln(-3+y) &= (-3)\left(-\frac{t^2}{2} + 2c_1\right)\\ &= \frac{3t^2}{2} - 6c_1\end{aligned}$$

Raising both side to exponential gives

$$e^{\ln(y)-\ln(-3+y)} = e^{\frac{3t^2}{2}-3c_1}$$

Which simplifies to

$$\begin{aligned}\frac{y}{-3+y} &= -3c_1 e^{\frac{3t^2}{2}}\\ &= c_2 e^{\frac{3t^2}{2}}\end{aligned}$$

Summary

The solution(s) found are the following

$$y = \frac{3c_2 e^{\frac{3t^2}{2}}}{-1 + c_2 e^{\frac{3t^2}{2}}} \tag{1}$$

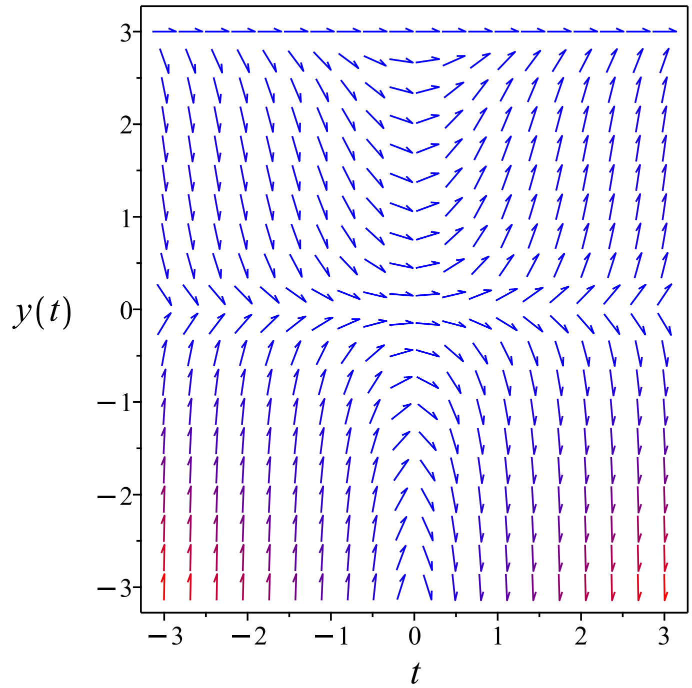


Figure 202: Slope field plot

Verification of solutions

$$y = \frac{3c_2 e^{\frac{3t^2}{2}}}{-1 + c_2 e^{\frac{3t^2}{2}}}$$

Verified OK.

3.13.2 Solving as first order ode lie symmetry lookup ode

Writing the ode as

$$y' = -ty(-3 + y)$$
$$y' = \omega(t, y)$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_t + \omega(\eta_y - \xi_t) - \omega^2 \xi_y - \omega_t \xi - \omega_y \eta = 0 \tag{A}$$

The type of this ode is known. It is of type `separable`. Therefore we do not need to solve the PDE (A), and can just use the lookup table shown below to find ξ, η

Table 216: Lie symmetry infinitesimal lookup table for known first order ODE's

ODE class	Form	ξ	η
linear ode	$y' = f(x)y(x) + g(x)$	0	$e^{\int f dx}$
separable ode	$y' = f(x)\, g(y)$	$\frac{1}{f}$	0
quadrature ode	$y' = f(x)$	0	1
quadrature ode	$y' = g(y)$	1	0
homogeneous ODEs of Class A	$y' = f\left(\frac{y}{x}\right)$	x	y
homogeneous ODEs of Class C	$y' = (a + bx + cy)^{\frac{n}{m}}$	1	$-\frac{b}{c}$
homogeneous class D	$y' = \frac{y}{x} + g(x)\, F\left(\frac{y}{x}\right)$	x^2	xy
First order special form ID 1	$y' = g(x)\, e^{h(x)+by} + f(x)$	$\frac{e^{-\int bf(x)dx - h(x)}}{g(x)}$	$\frac{f(x)e^{-\int bf(x)dx - h(x)}}{g(x)}$
polynomial type ode	$y' = \frac{a_1x+b_1y+c_1}{a_2x+b_2y+c_2}$	$\frac{a_1b_2x - a_2b_1x - b_1c_2 + b_2c_1}{a_1b_2 - a_2b_1}$	$\frac{a_1b_2y - a_2b_1y - a_1c_2 - a_2c_1}{a_1b_2 - a_2b_1}$
Bernoulli ode	$y' = f(x)\, y + g(x)\, y^n$	0	$e^{-\int (n-1)f(x)dx} y^n$
Reduced Riccati	$y' = f_1(x)\, y + f_2(x)\, y^2$	0	$e^{-\int f_1 dx}$

The above table shows that

$$
\begin{aligned}
\xi(t, y) &= -\frac{1}{t} \\
\eta(t, y) &= 0
\end{aligned}
\tag{A1}
$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(t, y) \to (R, S)$ where (R, S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$
\frac{dt}{\xi} = \frac{dy}{\eta} = dS \tag{1}
$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial t} + \eta \frac{\partial}{\partial y}\right) S(t, y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the

canonical coordinates, where $S(R)$. Since $\eta = 0$ then in this special case

$$R = y$$

S is found from

$$\begin{aligned} S &= \int \frac{1}{\xi} dt \\ &= \int \frac{1}{-\frac{1}{t}} dt \end{aligned}$$

Which results in

$$S = -\frac{t^2}{2}$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_t + \omega(t,y) S_y}{R_t + \omega(t,y) R_y} \tag{2}$$

Where in the above R_t, R_y, S_t, S_y are all partial derivatives and $\omega(t, y)$ is the right hand side of the original ode given by

$$\omega(t,y) = -ty(-3+y)$$

Evaluating all the partial derivatives gives

$$\begin{aligned} R_t &= 0 \\ R_y &= 1 \\ S_t &= -t \\ S_y &= 0 \end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = \frac{1}{y(-3+y)} \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for t, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = \frac{1}{R(-3+R)}$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordinates R, S. Integrating the above gives

$$S(R) = -\frac{\ln(R)}{3} + \frac{\ln(-3+R)}{3} + c_1 \tag{4}$$

To complete the solution, we just need to transform (4) back to t, y coordinates. This results in

$$-\frac{t^2}{2} = -\frac{\ln(y)}{3} + \frac{\ln(-3+y)}{3} + c_1$$

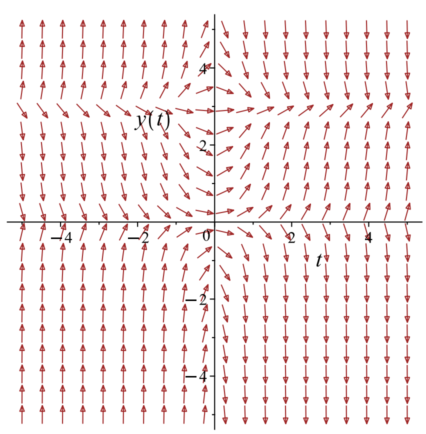
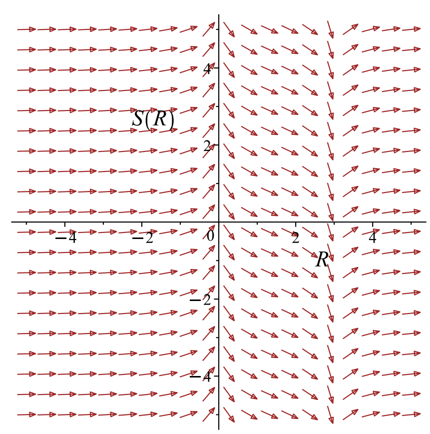
Which simplifies to

$$-\frac{t^2}{2} = -\frac{\ln(y)}{3} + \frac{\ln(-3+y)}{3} + c_1$$

Which gives

$$y = \frac{3\,e^{\frac{3t^2}{2}+3c_1}}{-1+e^{\frac{3t^2}{2}+3c_1}}$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in t, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dt} = -ty(-3+y)$ 	$R = y$ $S = -\frac{t^2}{2}$	$\frac{dS}{dR} = \frac{1}{R(-3+R)}$ 

Summary
The solution(s) found are the following

$$y = \frac{3\,\mathrm{e}^{\frac{3t^2}{2}+3c_1}}{-1+\mathrm{e}^{\frac{3t^2}{2}+3c_1}} \tag{1}$$

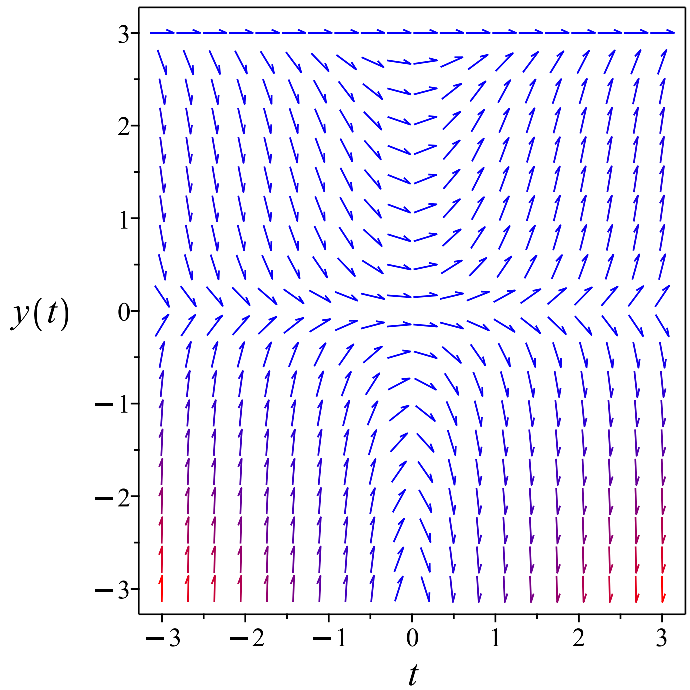


Figure 203: Slope field plot

Verification of solutions

$$y = \frac{3\,\mathrm{e}^{\frac{3t^2}{2}+3c_1}}{-1+\mathrm{e}^{\frac{3t^2}{2}+3c_1}}$$

Verified OK.

3.13.3 Solving as bernoulli ode

In canonical form, the ODE is

$$\begin{aligned} y' &= F(t,y) \\ &= -ty(-3+y) \end{aligned}$$

This is a Bernoulli ODE.

$$y' = 3ty - ty^2 \tag{1}$$

The standard Bernoulli ODE has the form

$$y' = f_0(t)y + f_1(t)y^n \tag{2}$$

The first step is to divide the above equation by y^n which gives

$$\frac{y'}{y^n} = f_0(t)y^{1-n} + f_1(t) \tag{3}$$

The next step is use the substitution $w = y^{1-n}$ in equation (3) which generates a new ODE in $w(t)$ which will be linear and can be easily solved using an integrating factor. Backsubstitution then gives the solution $y(t)$ which is what we want.

This method is now applied to the ODE at hand. Comparing the ODE (1) With (2) Shows that

$$\begin{aligned} f_0(t) &= 3t \\ f_1(t) &= -t \\ n &= 2 \end{aligned}$$

Dividing both sides of ODE (1) by $y^n = y^2$ gives

$$y'\frac{1}{y^2} = \frac{3t}{y} - t \tag{4}$$

Let

$$\begin{aligned} w &= y^{1-n} \\ &= \frac{1}{y} \end{aligned} \tag{5}$$

Taking derivative of equation (5) w.r.t t gives

$$w' = -\frac{1}{y^2}y' \tag{6}$$

Substituting equations (5) and (6) into equation (4) gives

$$\begin{aligned} -w'(t) &= 3w(t)\, t - t \\ w' &= -3tw + t \end{aligned} \tag{7}$$

The above now is a linear ODE in $w(t)$ which is now solved.

Entering Linear first order ODE solver. In canonical form a linear first order is

$$w'(t) + p(t)w(t) = q(t)$$

Where here

$$
\begin{aligned}
p(t) &= 3t \\
q(t) &= t
\end{aligned}
$$

Hence the ode is

$$
w'(t) + 3w(t)\,t = t
$$

The integrating factor μ is

$$
\begin{aligned}
\mu &= \mathrm{e}^{\int 3t dt} \\
&= \mathrm{e}^{\frac{3t^2}{2}}
\end{aligned}
$$

The ode becomes

$$
\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}(\mu w) &= (\mu)\,(t) \\
\frac{\mathrm{d}}{\mathrm{d}t}\left(\mathrm{e}^{\frac{3t^2}{2}} w\right) &= \left(\mathrm{e}^{\frac{3t^2}{2}}\right)(t) \\
\mathrm{d}\left(\mathrm{e}^{\frac{3t^2}{2}} w\right) &= \left(t\,\mathrm{e}^{\frac{3t^2}{2}}\right)\,\mathrm{d}t
\end{aligned}
$$

Integrating gives

$$
\begin{aligned}
\mathrm{e}^{\frac{3t^2}{2}} w &= \int t\,\mathrm{e}^{\frac{3t^2}{2}}\,\mathrm{d}t \\
\mathrm{e}^{\frac{3t^2}{2}} w &= \frac{\mathrm{e}^{\frac{3t^2}{2}}}{3} + c_1
\end{aligned}
$$

Dividing both sides by the integrating factor $\mu = \mathrm{e}^{\frac{3t^2}{2}}$ results in

$$
w(t) = \frac{\mathrm{e}^{-\frac{3t^2}{2}}\,\mathrm{e}^{\frac{3t^2}{2}}}{3} + c_1 \mathrm{e}^{-\frac{3t^2}{2}}
$$

which simplifies to

$$
w(t) = \frac{1}{3} + c_1 \mathrm{e}^{-\frac{3t^2}{2}}
$$

Replacing w in the above by $\frac{1}{y}$ using equation (5) gives the final solution.

$$
\frac{1}{y} = \frac{1}{3} + c_1 \mathrm{e}^{-\frac{3t^2}{2}}
$$

Or

$$y = \frac{1}{\frac{1}{3} + c_1 \mathrm{e}^{-\frac{3t^2}{2}}}$$

Summary

The solution(s) found are the following

$$y = \frac{1}{\frac{1}{3} + c_1 \mathrm{e}^{-\frac{3t^2}{2}}} \tag{1}$$

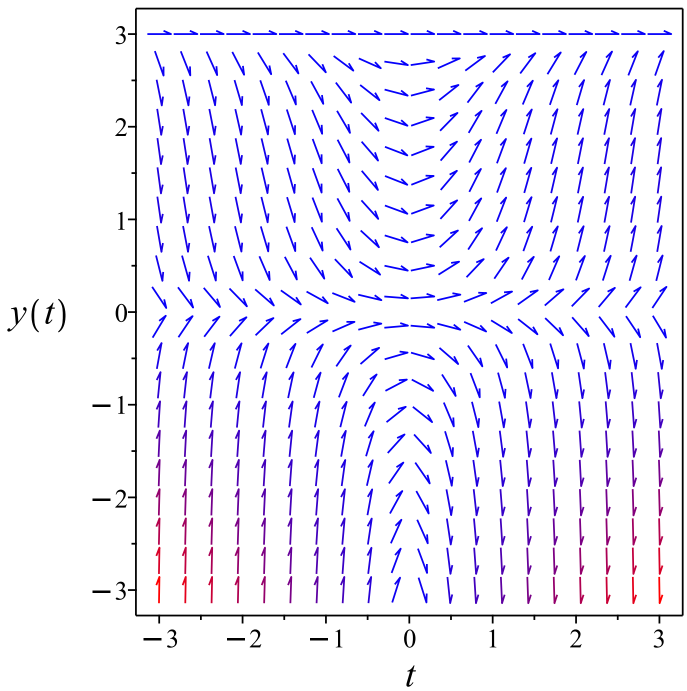


Figure 204: Slope field plot

Verification of solutions

$$y = \frac{1}{\frac{1}{3} + c_1 \mathrm{e}^{-\frac{3t^2}{2}}}$$

Verified OK.

3.13.4 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x,y)+N(x,y)\frac{dy}{dx}=0 \tag{A}$$

We assume there exists a function $\phi(x,y)=c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y)=0$$

Hence

$$\frac{\partial\phi}{\partial x}+\frac{\partial\phi}{\partial y}\frac{dy}{dx}=0 \tag{B}$$

Comparing (A,B) shows that

$$\begin{aligned}\frac{\partial\phi}{\partial x}&=M\\ \frac{\partial\phi}{\partial y}&=N\end{aligned}$$

But since $\frac{\partial^2\phi}{\partial x\partial y}=\frac{\partial^2\phi}{\partial y\partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y}=\frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x,y)$ but at least we know now that we can do that since the condition $\frac{\partial^2\phi}{\partial x\partial y}=\frac{\partial^2\phi}{\partial y\partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(t,y)\,\mathrm{d}t+N(t,y)\,\mathrm{d}y=0 \tag{1A}$$

Therefore

$$\begin{aligned}\left(-\frac{1}{y\,(-3+y)}\right)\mathrm{d}y&=(t)\,\mathrm{d}t\\ (-t)\,\mathrm{d}t+\left(-\frac{1}{y\,(-3+y)}\right)\mathrm{d}y&=0 \end{aligned}\tag{2A}$$

Comparing (1A) and (2A) shows that

$$
\begin{aligned}
M(t, y) &= -t \\
N(t, y) &= -\frac{1}{y(-3+y)}
\end{aligned}
$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$
\frac{\partial M}{\partial y} = \frac{\partial N}{\partial t}
$$

Using result found above gives

$$
\begin{aligned}
\frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}(-t) \\
&= 0
\end{aligned}
$$

And

$$
\begin{aligned}
\frac{\partial N}{\partial t} &= \frac{\partial}{\partial t}\left(-\frac{1}{y(-3+y)}\right) \\
&= 0
\end{aligned}
$$

Since $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial t}$, then the ODE is exact The following equations are now set up to solve for the function $\phi(t, y)$

$$
\frac{\partial \phi}{\partial t} = M \tag{1}
$$

$$
\frac{\partial \phi}{\partial y} = N \tag{2}
$$

Integrating (1) w.r.t. t gives

$$
\begin{aligned}
\int \frac{\partial \phi}{\partial t} \, \mathrm{d}t &= \int M \, \mathrm{d}t \\
\int \frac{\partial \phi}{\partial t} \, \mathrm{d}t &= \int -t \, \mathrm{d}t
\end{aligned}
$$

$$
\phi = -\frac{t^2}{2} + f(y) \tag{3}
$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both t and y. Taking derivative of equation (3) w.r.t y gives

$$
\frac{\partial \phi}{\partial y} = 0 + f'(y) \tag{4}
$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = -\frac{1}{y(-3+y)}$. Therefore equation (4) becomes

$$-\frac{1}{y\left(-3+y\right)} = 0 + f'(y) \tag{5}$$

Solving equation (5) for $f'(y)$ gives

$$f'(y) = -\frac{1}{y\left(-3+y\right)}$$

Integrating the above w.r.t y gives

$$\int f'(y)\,\mathrm{d}y = \int \left(-\frac{1}{y\left(-3+y\right)}\right) \mathrm{d}y$$

$$f(y) = \frac{\ln\left(y\right)}{3} - \frac{\ln\left(-3+y\right)}{3} + c_1$$

Where c_1 is constant of integration. Substituting result found above for $f(y)$ into equation (3) gives ϕ

$$\phi = -\frac{t^2}{2} + \frac{\ln\left(y\right)}{3} - \frac{\ln\left(-3+y\right)}{3} + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = -\frac{t^2}{2} + \frac{\ln\left(y\right)}{3} - \frac{\ln\left(-3+y\right)}{3}$$

The solution becomes

$$y = \frac{3\,\mathrm{e}^{\frac{3t^2}{2}+3c_1}}{-1+\mathrm{e}^{\frac{3t^2}{2}+3c_1}}$$

Summary

The solution(s) found are the following

$$y = \frac{3\,\mathrm{e}^{\frac{3t^2}{2}+3c_1}}{-1+\mathrm{e}^{\frac{3t^2}{2}+3c_1}} \tag{1}$$

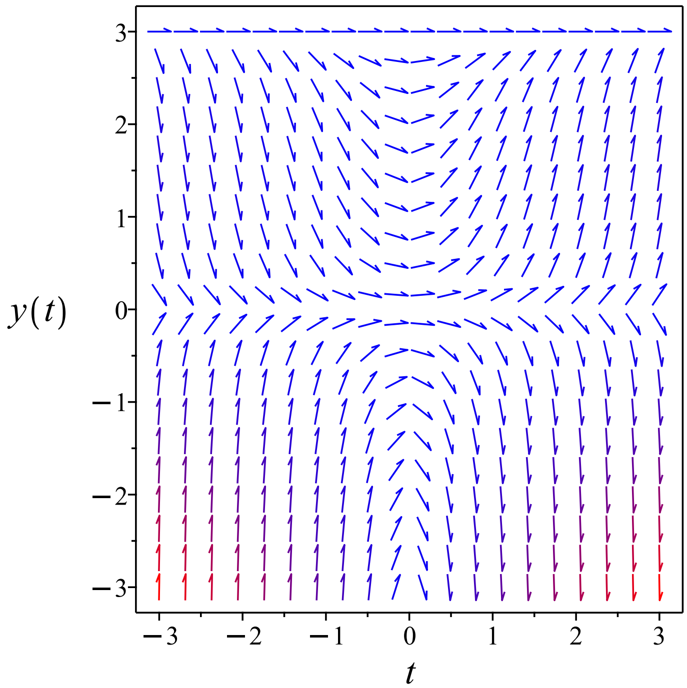


Figure 205: Slope field plot

Verification of solutions

$$y = \frac{3\,\mathrm{e}^{\frac{3t^2}{2}+3c_1}}{-1+\mathrm{e}^{\frac{3t^2}{2}+3c_1}}$$

Verified OK.

3.13.5 Solving as riccati ode

In canonical form the ODE is

$$\begin{aligned} y' &= F(t,y) \\ &= -ty(-3+y) \end{aligned}$$

This is a Riccati ODE. Comparing the ODE to solve

$$y' = -t\,y^2 + 3ty$$

With Riccati ODE standard form

$$y' = f_0(t) + f_1(t)y + f_2(t)y^2$$

Shows that $f_0(t) = 0$, $f_1(t) = 3t$ and $f_2(t) = -t$. Let

$$\begin{aligned} y &= \frac{-u'}{f_2 u} \\ &= \frac{-u'}{-tu} \end{aligned} \tag{1}$$

Using the above substitution in the given ODE results (after some simplification)in a second order ODE to solve for $u(x)$ which is

$$f_2 u''(t) - (f_2' + f_1 f_2) u'(t) + f_2^2 f_0 u(t) = 0 \tag{2}$$

But

$$\begin{aligned} f_2' &= -1 \\ f_1 f_2 &= -3t^2 \\ f_2^2 f_0 &= 0 \end{aligned}$$

Substituting the above terms back in equation (2) gives

$$-tu''(t) - (-3t^2 - 1) u'(t) = 0$$

Solving the above ODE (this ode solved using Maple, not this program), gives

$$u(t) = c_1 + c_2 e^{\frac{3t^2}{2}}$$

The above shows that

$$u'(t) = 3c_2 t\, e^{\frac{3t^2}{2}}$$

Using the above in (1) gives the solution

$$y = \frac{3c_2 e^{\frac{3t^2}{2}}}{c_1 + c_2 e^{\frac{3t^2}{2}}}$$

Dividing both numerator and denominator by c_1 gives, after renaming the constant $\frac{c_2}{c_1} = c_3$ the following solution

$$y = \frac{3\, e^{\frac{3t^2}{2}}}{c_3 + e^{\frac{3t^2}{2}}}$$

Summary
The solution(s) found are the following

$$y = \frac{3\,\mathrm{e}^{\frac{3t^2}{2}}}{c_3 + \mathrm{e}^{\frac{3t^2}{2}}} \tag{1}$$

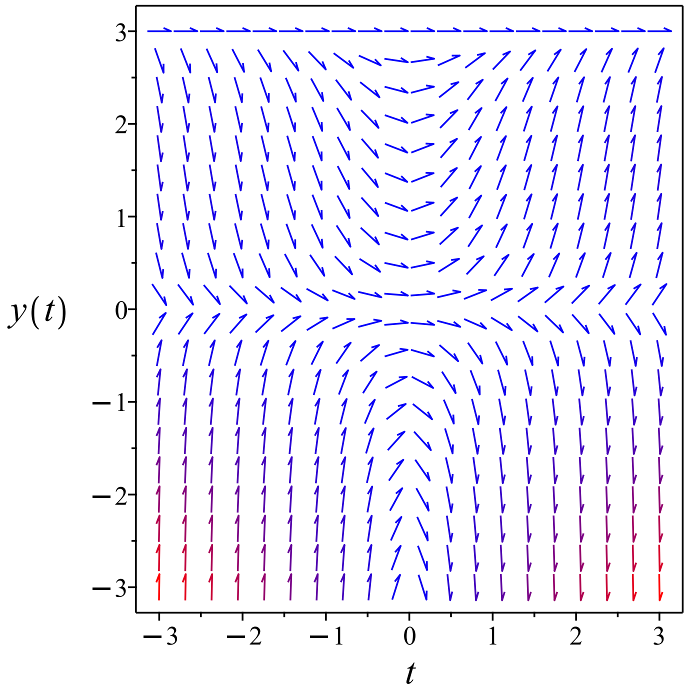


Figure 206: Slope field plot

Verification of solutions

$$y = \frac{3\,\mathrm{e}^{\frac{3t^2}{2}}}{c_3 + \mathrm{e}^{\frac{3t^2}{2}}}$$

Verified OK.

3.13.6 Maple step by step solution

Let's solve

$y' - t(3 - y)\,y = 0$

- Highest derivative means the order of the ODE is 1

y'

- Separate variables

  $\frac{y'}{(3-y)y} = t$

- Integrate both sides with respect to t

  $\int \frac{y'}{(3-y)y} dt = \int t dt + c_1$

- Evaluate integral

  $\frac{\ln(y)}{3} - \frac{\ln(-3+y)}{3} = \frac{t^2}{2} + c_1$

- Solve for y

  $y = \frac{3\,\mathrm{e}^{\frac{3t^2}{2}+3c_1}}{-1+\mathrm{e}^{\frac{3t^2}{2}+3c_1}}$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
<- Bernoulli successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 19

```
dsolve(diff(y(t),t) = t*(3-y(t))*y(t),y(t), singsol=all)
```

$$y(t) = \frac{3}{1 + 3\,\mathrm{e}^{-\frac{3t^2}{2}} c_1}$$

✓ Solution by Mathematica

Time used: 0.266 (sec). Leaf size: 44

```
DSolve[y'[t] == t*(3-y[t])*y[t],y[t],t,IncludeSingularSolutions -> True]
```

$$y(t) \to \frac{3e^{\frac{3t^2}{2}}}{e^{\frac{3t^2}{2}} + e^{3c_1}}$$
$$y(t) \to 0$$
$$y(t) \to 3$$

3.14 problem 18

Internal problem ID [529]
Internal file name [`OUTPUT/529_Sunday_June_05_2022_01_43_07_AM_62544468/index.tex`]

Book: Elementary differential equations and boundary value problems, 10th ed., Boyce and DiPrima
Section: Section 2.4. Page 76
Problem number: 18.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"riccati", "bernoulli", "first_order_ode_lie_symmetry_lookup"**

Maple gives the following as the ode type

`[_Bernoulli]`

$$\boxed{y' - y(3 - yt) = 0}$$

3.14.1 Solving as first order ode lie symmetry lookup ode

Writing the ode as

$$y' = -y(ty - 3)$$
$$y' = \omega(t, y)$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_t + \omega(\eta_y - \xi_t) - \omega^2 \xi_y - \omega_t \xi - \omega_y \eta = 0 \tag{A}$$

The type of this ode is known. It is of type `Bernoulli`. Therefore we do not need to solve the PDE (A), and can just use the lookup table shown below to find ξ, η

Table 219: Lie symmetry infinitesimal lookup table for known first order ODE's

ODE class	Form	ξ	η
linear ode	$y' = f(x)y(x) + g(x)$	0	$e^{\int f dx}$
separable ode	$y' = f(x)\, g(y)$	$\frac{1}{f}$	0
quadrature ode	$y' = f(x)$	0	1
quadrature ode	$y' = g(y)$	1	0
homogeneous ODEs of Class A	$y' = f\left(\frac{y}{x}\right)$	x	y
homogeneous ODEs of Class C	$y' = (a + bx + cy)^{\frac{n}{m}}$	1	$-\frac{b}{c}$
homogeneous class D	$y' = \frac{y}{x} + g(x)\, F\left(\frac{y}{x}\right)$	x^2	xy
First order special form ID 1	$y' = g(x)\, e^{h(x)+by} + f(x)$	$\frac{e^{-\int bf(x)dx-h(x)}}{g(x)}$	$\frac{f(x)e^{-\int bf(x)dx-h(x)}}{g(x)}$
polynomial type ode	$y' = \frac{a_1x+b_1y+c_1}{a_2x+b_2y+c_2}$	$\frac{a_1b_2x-a_2b_1x-b_1c_2+b_2c_1}{a_1b_2-a_2b_1}$	$\frac{a_1b_2y-a_2b_1y-a_1c_2-a_2c_1}{a_1b_2-a_2b_1}$
Bernoulli ode	$y' = f(x)\, y + g(x)\, y^n$	0	$e^{-\int (n-1)f(x)dx} y^n$
Reduced Riccati	$y' = f_1(x)\, y + f_2(x)\, y^2$	0	$e^{-\int f_1 dx}$

The above table shows that

$$\xi(t, y) = 0$$
$$\eta(t, y) = y^2 \mathrm{e}^{-3t} \tag{A1}$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(t, y) \to (R, S)$ where (R, S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$\frac{dt}{\xi} = \frac{dy}{\eta} = dS \tag{1}$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial t} + \eta \frac{\partial}{\partial y}\right) S(t, y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the

canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$R = t$$

S is found from

$$\begin{aligned}
S &= \int \frac{1}{\eta} dy \\
&= \int \frac{1}{y^2 e^{-3t}} dy
\end{aligned}$$

Which results in

$$S = -\frac{e^{3t}}{y}$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_t + \omega(t,y) S_y}{R_t + \omega(t,y) R_y} \tag{2}$$

Where in the above R_t, R_y, S_t, S_y are all partial derivatives and $\omega(t,y)$ is the right hand side of the original ode given by

$$\omega(t,y) = -y(ty - 3)$$

Evaluating all the partial derivatives gives

$$\begin{aligned}
R_t &= 1 \\
R_y &= 0 \\
S_t &= -\frac{3 e^{3t}}{y} \\
S_y &= \frac{e^{3t}}{y^2}
\end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = -e^{3t} t \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for t, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = -e^{3R} R$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordinates R, S. Integrating the above gives

$$S(R) = -\frac{(3R-1)\,\mathrm{e}^{3R}}{9} + c_1 \tag{4}$$

To complete the solution, we just need to transform (4) back to t, y coordinates. This results in

$$-\frac{\mathrm{e}^{3t}}{y} = -\frac{(-1+3t)\,\mathrm{e}^{3t}}{9} + c_1$$

Which simplifies to

$$-\frac{\mathrm{e}^{3t}}{y} = -\frac{(-1+3t)\,\mathrm{e}^{3t}}{9} + c_1$$

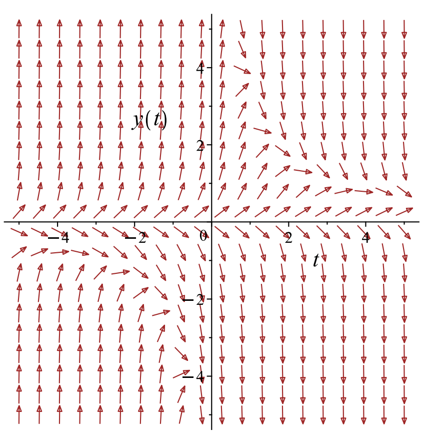
Which gives

$$y = \frac{9\,\mathrm{e}^{3t}}{3\,\mathrm{e}^{3t}t - \mathrm{e}^{3t} - 9c_1}$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

<table>
<tr><th>Original ode in t, y coordinates</th><th>Canonical coordinates transformation</th><th>ODE in canonical coordinates (R, S)</th></tr>
<tr><td>

$\frac{dy}{dt} = -y(ty-3)$



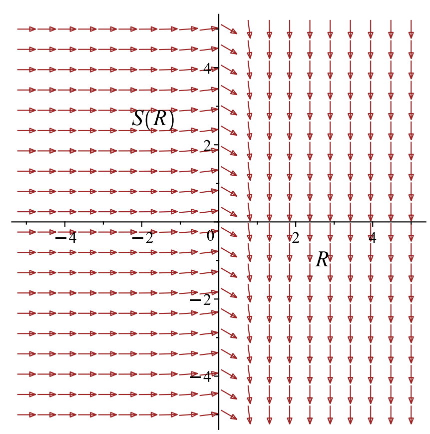
</td><td>

$R = t$

$S = -\frac{\mathrm{e}^{3t}}{y}$

</td><td>

$\frac{dS}{dR} = -\mathrm{e}^{3R}R$



</td></tr>
</table>

Summary
The solution(s) found are the following

$$y = \frac{9\,e^{3t}}{3\,e^{3t}t - e^{3t} - 9c_1} \tag{1}$$

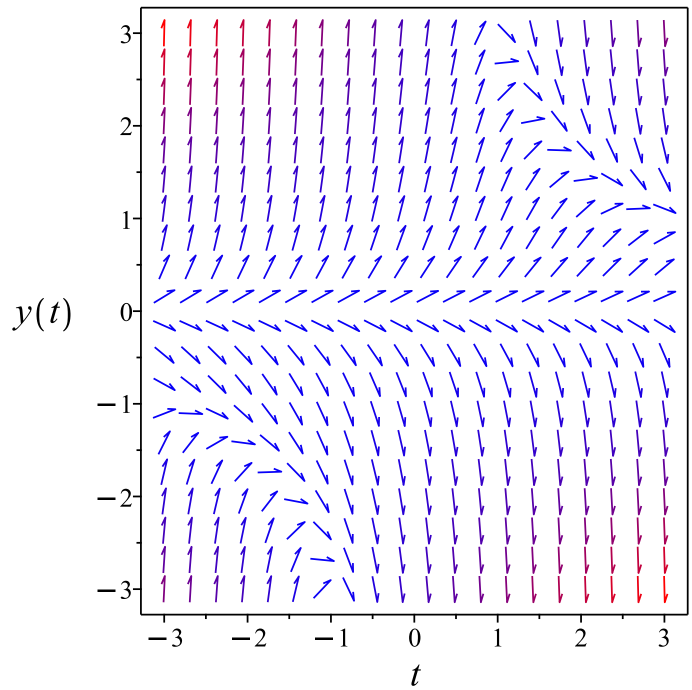


Figure 207: Slope field plot

Verification of solutions

$$y = \frac{9\,e^{3t}}{3\,e^{3t}t - e^{3t} - 9c_1}$$

Verified OK.

3.14.2 Solving as bernoulli ode

In canonical form, the ODE is

$$\begin{aligned} y' &= F(t, y) \\ &= -y(ty - 3) \end{aligned}$$

This is a Bernoulli ODE.

$$y' = 3y - t y^2 \tag{1}$$

The standard Bernoulli ODE has the form

$$y' = f_0(t)y + f_1(t)y^n \tag{2}$$

The first step is to divide the above equation by y^n which gives

$$\frac{y'}{y^n} = f_0(t)y^{1-n} + f_1(t) \tag{3}$$

The next step is use the substitution $w = y^{1-n}$ in equation (3) which generates a new ODE in $w(t)$ which will be linear and can be easily solved using an integrating factor. Backsubstitution then gives the solution $y(t)$ which is what we want.

This method is now applied to the ODE at hand. Comparing the ODE (1) With (2) Shows that

$$\begin{aligned} f_0(t) &= 3 \\ f_1(t) &= -t \\ n &= 2 \end{aligned}$$

Dividing both sides of ODE (1) by $y^n = y^2$ gives

$$y'\frac{1}{y^2} = \frac{3}{y} - t \tag{4}$$

Let

$$\begin{aligned} w &= y^{1-n} \\ &= \frac{1}{y} \end{aligned} \tag{5}$$

Taking derivative of equation (5) w.r.t t gives

$$w' = -\frac{1}{y^2}y' \tag{6}$$

Substituting equations (5) and (6) into equation (4) gives

$$\begin{aligned} -w'(t) &= 3w(t) - t \\ w' &= -3w + t \end{aligned} \tag{7}$$

The above now is a linear ODE in $w(t)$ which is now solved.

Entering Linear first order ODE solver. In canonical form a linear first order is

$$w'(t) + p(t)w(t) = q(t)$$

Where here

$$p(t) = 3$$
$$q(t) = t$$

Hence the ode is

$$w'(t) + 3w(t) = t$$

The integrating factor μ is

$$\mu = e^{\int 3dt}$$
$$= e^{3t}$$

The ode becomes

$$\frac{d}{dt}(\mu w) = (\mu)(t)$$
$$\frac{d}{dt}(e^{3t}w) = (e^{3t})(t)$$
$$d(e^{3t}w) = (e^{3t}t)\ dt$$

Integrating gives

$$e^{3t}w = \int e^{3t}t\,dt$$
$$e^{3t}w = \frac{(-1+3t)\,e^{3t}}{9} + c_1$$

Dividing both sides by the integrating factor $\mu = e^{3t}$ results in

$$w(t) = \frac{e^{-3t}(-1+3t)\,e^{3t}}{9} + c_1e^{-3t}$$

which simplifies to

$$w(t) = \frac{t}{3} - \frac{1}{9} + c_1e^{-3t}$$

Replacing w in the above by $\frac{1}{y}$ using equation (5) gives the final solution.

$$\frac{1}{y} = \frac{t}{3} - \frac{1}{9} + c_1e^{-3t}$$

Or

$$y = \frac{1}{\frac{t}{3} - \frac{1}{9} + c_1e^{-3t}}$$

Which is simplified to

$$y = \frac{9}{9c_1 e^{-3t} + 3t - 1}$$

Summary

The solution(s) found are the following

$$y = \frac{9}{9c_1 e^{-3t} + 3t - 1} \tag{1}$$

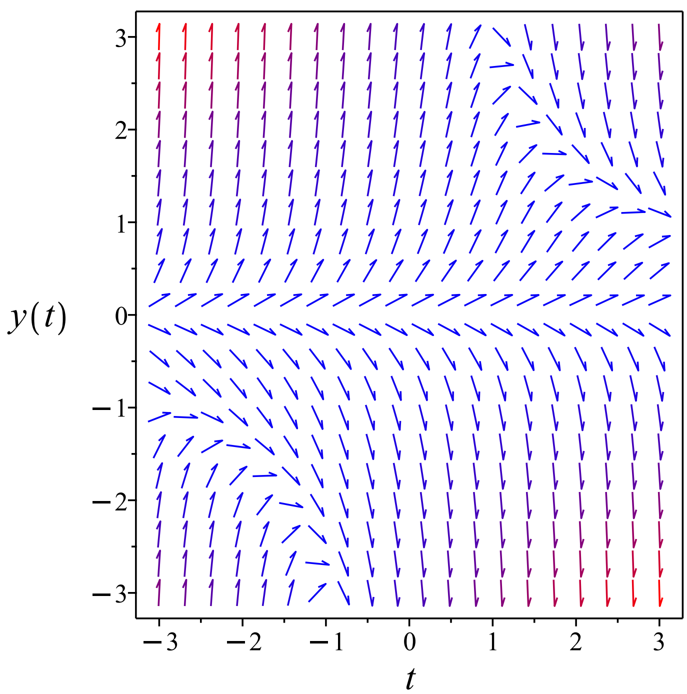


Figure 208: Slope field plot

Verification of solutions

$$y = \frac{9}{9c_1 e^{-3t} + 3t - 1}$$

Verified OK.

3.14.3 Solving as riccati ode

In canonical form the ODE is

$$
\begin{aligned}
y' &= F(t, y) \\
&= -y(ty - 3)
\end{aligned}
$$

This is a Riccati ODE. Comparing the ODE to solve

$$
y' = -t\,y^2 + 3y
$$

With Riccati ODE standard form

$$
y' = f_0(t) + f_1(t)y + f_2(t)y^2
$$

Shows that $f_0(t) = 0$, $f_1(t) = 3$ and $f_2(t) = -t$. Let

$$
\begin{aligned}
y &= \frac{-u'}{f_2 u} \\
&= \frac{-u'}{-tu}
\end{aligned} \tag{1}
$$

Using the above substitution in the given ODE results (after some simplification)in a second order ODE to solve for $u(x)$ which is

$$
f_2 u''(t) - (f_2' + f_1 f_2)\,u'(t) + f_2^2 f_0 u(t) = 0 \tag{2}
$$

But

$$
\begin{aligned}
f_2' &= -1 \\
f_1 f_2 &= -3t \\
f_2^2 f_0 &= 0
\end{aligned}
$$

Substituting the above terms back in equation (2) gives

$$
-tu''(t) - (-3t - 1)\,u'(t) = 0
$$

Solving the above ODE (this ode solved using Maple, not this program), gives

$$
u(t) = c_1 + (-1 + 3t)\,\mathrm{e}^{3t} c_2
$$

The above shows that

$$
u'(t) = 9c_2 \mathrm{e}^{3t} t
$$

Using the above in (1) gives the solution

$$y = \frac{9c_2 e^{3t}}{c_1 + (-1+3t)\,e^{3t} c_2}$$

Dividing both numerator and denominator by c_1 gives, after renaming the constant $\frac{c_2}{c_1} = c_3$ the following solution

$$y = \frac{9\,e^{3t}}{c_3 + (-1+3t)\,e^{3t}}$$

Summary

The solution(s) found are the following

$$y = \frac{9\,e^{3t}}{c_3 + (-1+3t)\,e^{3t}} \tag{1}$$

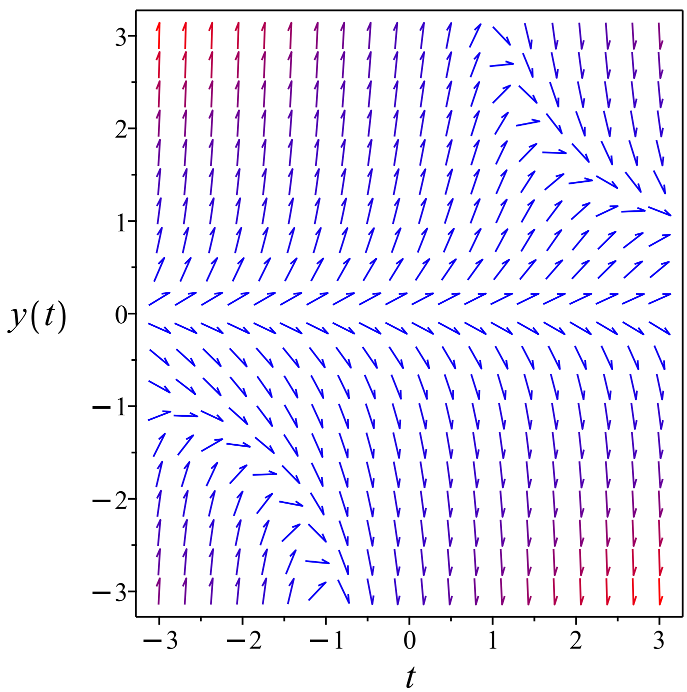


Figure 209: Slope field plot

Verification of solutions

$$y = \frac{9\,e^{3t}}{c_3 + (-1+3t)\,e^{3t}}$$

Verified OK.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
<- Bernoulli successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 20

```
dsolve(diff(y(t),t) = y(t)*(3-t*y(t)),y(t), singsol=all)
```

$$y(t) = \frac{9}{-1 + 9c_1 e^{-3t} + 3t}$$

✓ Solution by Mathematica

Time used: 0.118 (sec). Leaf size: 35

```
DSolve[y'[t] == y[t]*(3-t*y[t]),y[t],t,IncludeSingularSolutions -> True]
```

$$y(t) \to \frac{9e^{3t}}{e^{3t}(3t - 1) + 9c_1}$$
$$y(t) \to 0$$

3.15 problem 19

Internal problem ID [530]
Internal file name [`OUTPUT/530_Sunday_June_05_2022_01_43_08_AM_17571426/index.tex`]

Book: Elementary differential equations and boundary value problems, 10th ed., Boyce and DiPrima
Section: Section 2.4. Page 76
Problem number: 19.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"riccati", "bernoulli", "first_order_ode_lie_symmetry_lookup"**

Maple gives the following as the ode type

`[_Bernoulli]`

$$\boxed{y' + y(3 - yt) = 0}$$

3.15.1 Solving as first order ode lie symmetry lookup ode

Writing the ode as

$$\begin{aligned} y' &= y(ty - 3) \\ y' &= \omega(t, y) \end{aligned}$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_t + \omega(\eta_y - \xi_t) - \omega^2 \xi_y - \omega_t \xi - \omega_y \eta = 0 \tag{A}$$

The type of this ode is known. It is of type `Bernoulli`. Therefore we do not need to solve the PDE (A), and can just use the lookup table shown below to find ξ, η

Table 221: Lie symmetry infinitesimal lookup table for known first order ODE's

ODE class	Form	ξ	η
linear ode	$y' = f(x)y(x) + g(x)$	0	$e^{\int f dx}$
separable ode	$y' = f(x)\, g(y)$	$\frac{1}{f}$	0
quadrature ode	$y' = f(x)$	0	1
quadrature ode	$y' = g(y)$	1	0
homogeneous ODEs of Class A	$y' = f\left(\frac{y}{x}\right)$	x	y
homogeneous ODEs of Class C	$y' = (a + bx + cy)^{\frac{n}{m}}$	1	$-\frac{b}{c}$
homogeneous class D	$y' = \frac{y}{x} + g(x)\, F\left(\frac{y}{x}\right)$	x^2	xy
First order special form ID 1	$y' = g(x)\, e^{h(x)+by} + f(x)$	$\frac{e^{-\int bf(x)dx - h(x)}}{g(x)}$	$\frac{f(x)e^{-\int bf(x)dx - h(x)}}{g(x)}$
polynomial type ode	$y' = \frac{a_1x+b_1y+c_1}{a_2x+b_2y+c_2}$	$\frac{a_1b_2x - a_2b_1x - b_1c_2 + b_2c_1}{a_1b_2 - a_2b_1}$	$\frac{a_1b_2y - a_2b_1y - a_1c_2 - a_2c_1}{a_1b_2 - a_2b_1}$
Bernoulli ode	$y' = f(x)\, y + g(x)\, y^n$	0	$e^{-\int (n-1)f(x)dx} y^n$
Reduced Riccati	$y' = f_1(x)\, y + f_2(x)\, y^2$	0	$e^{-\int f_1 dx}$

The above table shows that

$$
\begin{aligned}
\xi(t, y) &= 0 \\
\eta(t, y) &= y^2 \mathrm{e}^{3t}
\end{aligned} \tag{A1}
$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(t, y) \to (R, S)$ where (R, S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$
\frac{dt}{\xi} = \frac{dy}{\eta} = dS \tag{1}
$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial t} + \eta \frac{\partial}{\partial y}\right) S(t, y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the

canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$R = t$$

S is found from

$$\begin{aligned} S &= \int \frac{1}{\eta} dy \\ &= \int \frac{1}{y^2 e^{3t}} dy \end{aligned}$$

Which results in

$$S = -\frac{e^{-3t}}{y}$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_t + \omega(t, y) S_y}{R_t + \omega(t, y) R_y} \tag{2}$$

Where in the above R_t, R_y, S_t, S_y are all partial derivatives and $\omega(t, y)$ is the right hand side of the original ode given by

$$\omega(t, y) = y(ty - 3)$$

Evaluating all the partial derivatives gives

$$\begin{aligned} R_t &= 1 \\ R_y &= 0 \\ S_t &= \frac{3 e^{-3t}}{y} \\ S_y &= \frac{e^{-3t}}{y^2} \end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = t e^{-3t} \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for t, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = R e^{-3R}$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordinates R, S. Integrating the above gives

$$S(R) = -\frac{(3R+1)\,e^{-3R}}{9} + c_1 \tag{4}$$

To complete the solution, we just need to transform (4) back to t, y coordinates. This results in

$$-\frac{e^{-3t}}{y} = -\frac{(1+3t)\,e^{-3t}}{9} + c_1$$

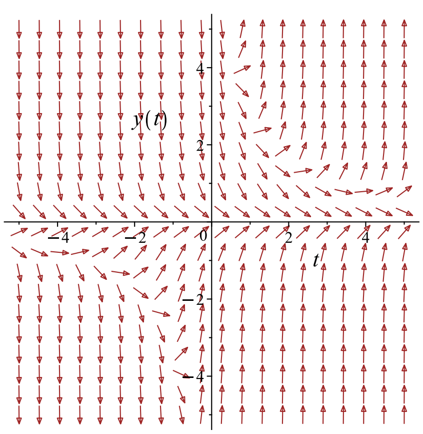
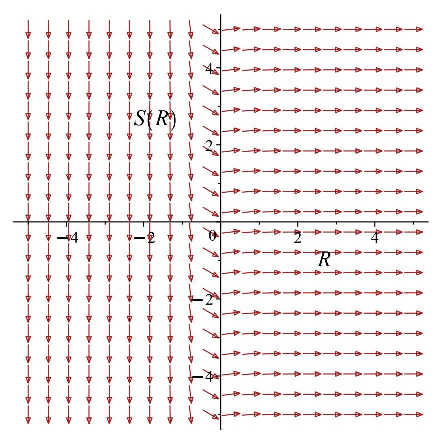
Which simplifies to

$$-\frac{e^{-3t}}{y} = -\frac{(1+3t)\,e^{-3t}}{9} + c_1$$

Which gives

$$y = \frac{9\,e^{-3t}}{3t\,e^{-3t} + e^{-3t} - 9c_1}$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in t, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dt} = y(ty-3)$ 	$R = t$ $S = -\frac{e^{-3t}}{y}$	$\frac{dS}{dR} = R\,e^{-3R}$ 

Summary
The solution(s) found are the following

$$y = \frac{9\,e^{-3t}}{3t\,e^{-3t} + e^{-3t} - 9c_1} \tag{1}$$

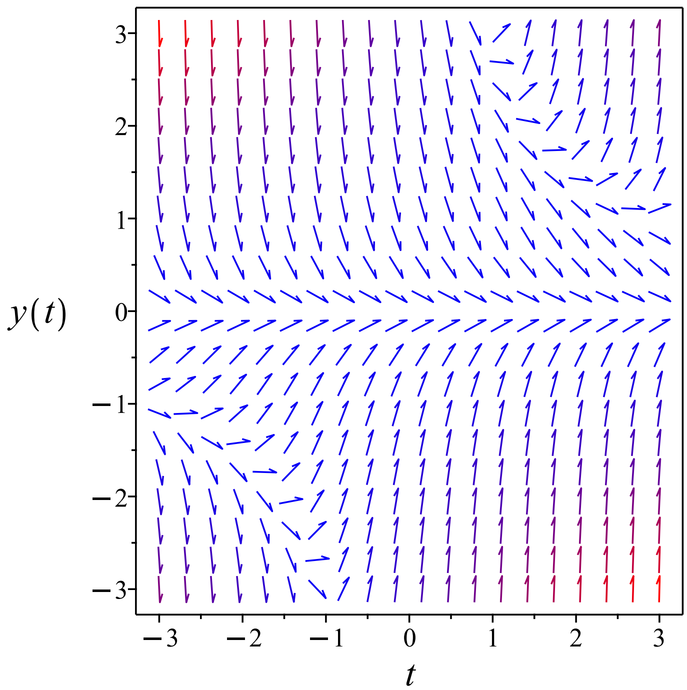


Figure 210: Slope field plot

Verification of solutions

$$y = \frac{9\,e^{-3t}}{3t\,e^{-3t} + e^{-3t} - 9c_1}$$

Verified OK.

3.15.2 Solving as bernoulli ode

In canonical form, the ODE is

$$\begin{aligned} y' &= F(t,y) \\ &= y(ty-3) \end{aligned}$$

This is a Bernoulli ODE.

$$y' = -3y + ty^2 \tag{1}$$

The standard Bernoulli ODE has the form

$$y' = f_0(t)y + f_1(t)y^n \tag{2}$$

The first step is to divide the above equation by y^n which gives

$$\frac{y'}{y^n} = f_0(t)y^{1-n} + f_1(t) \tag{3}$$

The next step is use the substitution $w = y^{1-n}$ in equation (3) which generates a new ODE in $w(t)$ which will be linear and can be easily solved using an integrating factor. Backsubstitution then gives the solution $y(t)$ which is what we want.

This method is now applied to the ODE at hand. Comparing the ODE (1) With (2) Shows that

$$\begin{aligned} f_0(t) &= -3 \\ f_1(t) &= t \\ n &= 2 \end{aligned}$$

Dividing both sides of ODE (1) by $y^n = y^2$ gives

$$y'\frac{1}{y^2} = -\frac{3}{y} + t \tag{4}$$

Let

$$\begin{aligned} w &= y^{1-n} \\ &= \frac{1}{y} \end{aligned} \tag{5}$$

Taking derivative of equation (5) w.r.t t gives

$$w' = -\frac{1}{y^2}y' \tag{6}$$

Substituting equations (5) and (6) into equation (4) gives

$$\begin{aligned} -w'(t) &= -3w(t) + t \\ w' &= 3w - t \end{aligned} \tag{7}$$

The above now is a linear ODE in $w(t)$ which is now solved.

Entering Linear first order ODE solver. In canonical form a linear first order is

$$w'(t) + p(t)w(t) = q(t)$$

Where here

$$
\begin{aligned}
p(t) &= -3 \\
q(t) &= -t
\end{aligned}
$$

Hence the ode is

$$
w'(t) - 3w(t) = -t
$$

The integrating factor μ is

$$
\begin{aligned}
\mu &= \mathrm{e}^{\int (-3)dt} \\
&= \mathrm{e}^{-3t}
\end{aligned}
$$

The ode becomes

$$
\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}(\mu w) &= (\mu)\,(-t) \\
\frac{\mathrm{d}}{\mathrm{d}t}\left(\mathrm{e}^{-3t} w\right) &= \left(\mathrm{e}^{-3t}\right)(-t) \\
\mathrm{d}\left(\mathrm{e}^{-3t} w\right) &= \left(-t\,\mathrm{e}^{-3t}\right)\,\mathrm{d}t
\end{aligned}
$$

Integrating gives

$$
\begin{aligned}
\mathrm{e}^{-3t} w &= \int -t\,\mathrm{e}^{-3t}\,\mathrm{d}t \\
\mathrm{e}^{-3t} w &= \frac{(1+3t)\,\mathrm{e}^{-3t}}{9} + c_1
\end{aligned}
$$

Dividing both sides by the integrating factor $\mu = \mathrm{e}^{-3t}$ results in

$$
w(t) = \frac{\mathrm{e}^{3t}(1+3t)\,\mathrm{e}^{-3t}}{9} + c_1 \mathrm{e}^{3t}
$$

which simplifies to

$$
w(t) = \frac{1}{9} + \frac{t}{3} + c_1 \mathrm{e}^{3t}
$$

Replacing w in the above by $\frac{1}{y}$ using equation (5) gives the final solution.

$$
\frac{1}{y} = \frac{1}{9} + \frac{t}{3} + c_1 \mathrm{e}^{3t}
$$

Or

$$
y = \frac{1}{\frac{1}{9} + \frac{t}{3} + c_1 \mathrm{e}^{3t}}
$$

Which is simplified to

$$y = \frac{9}{9c_1 e^{3t} + 3t + 1}$$

Summary
The solution(s) found are the following

$$y = \frac{9}{9c_1 e^{3t} + 3t + 1} \tag{1}$$

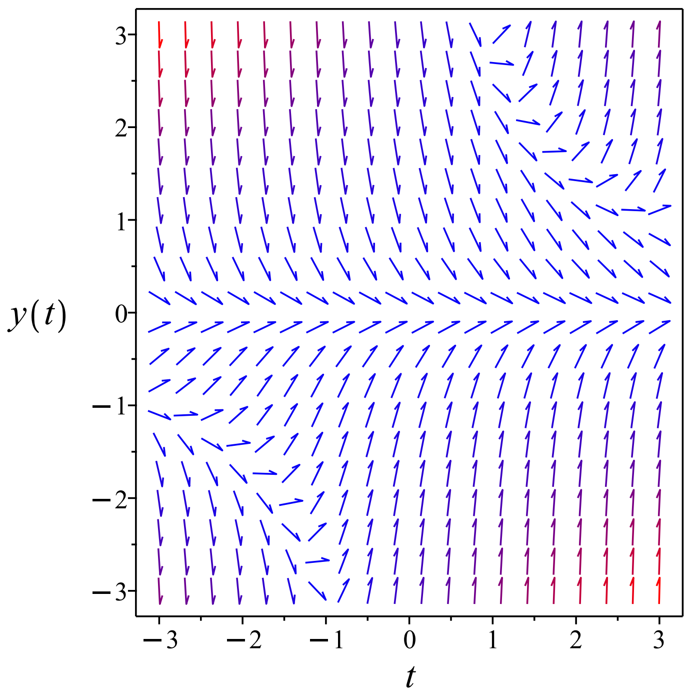


Figure 211: Slope field plot

Verification of solutions

$$y = \frac{9}{9c_1 e^{3t} + 3t + 1}$$

Verified OK.

3.15.3 Solving as riccati ode

In canonical form the ODE is

$$
\begin{aligned}
y' &= F(t, y) \\
&= y(ty - 3)
\end{aligned}
$$

This is a Riccati ODE. Comparing the ODE to solve

$$
y' = t\,y^2 - 3y
$$

With Riccati ODE standard form

$$
y' = f_0(t) + f_1(t)y + f_2(t)y^2
$$

Shows that $f_0(t) = 0$, $f_1(t) = -3$ and $f_2(t) = t$. Let

$$
\begin{aligned}
y &= \frac{-u'}{f_2 u} \\
&= \frac{-u'}{tu} \qquad (1)
\end{aligned}
$$

Using the above substitution in the given ODE results (after some simplification)in a second order ODE to solve for $u(x)$ which is

$$
f_2 u''(t) - (f_2' + f_1 f_2)\,u'(t) + f_2^2 f_0 u(t) = 0 \qquad (2)
$$

But

$$
\begin{aligned}
f_2' &= 1 \\
f_1 f_2 &= -3t \\
f_2^2 f_0 &= 0
\end{aligned}
$$

Substituting the above terms back in equation (2) gives

$$
tu''(t) - (1 - 3t)\,u'(t) = 0
$$

Solving the above ODE (this ode solved using Maple, not this program), gives

$$
u(t) = c_1 + (1 + 3t)\,e^{-3t} c_2
$$

The above shows that

$$
u'(t) = -9c_2 e^{-3t} t
$$

Using the above in (1) gives the solution

$$y = \frac{9c_2 e^{-3t}}{c_1 + (1+3t)\, e^{-3t} c_2}$$

Dividing both numerator and denominator by c_1 gives, after renaming the constant $\frac{c_2}{c_1} = c_3$ the following solution

$$y = \frac{9\, e^{-3t}}{3t\, e^{-3t} + e^{-3t} + c_3}$$

Summary

The solution(s) found are the following

$$y = \frac{9\, e^{-3t}}{3t\, e^{-3t} + e^{-3t} + c_3} \tag{1}$$

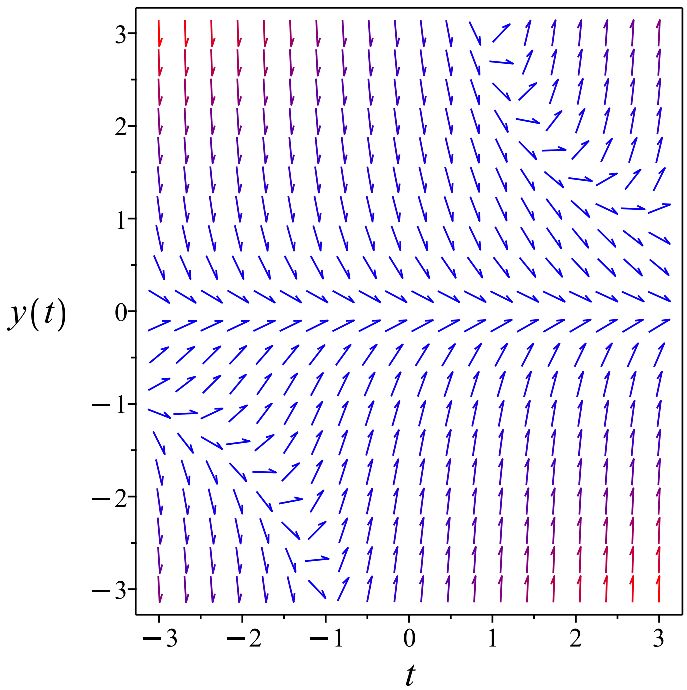


Figure 212: Slope field plot

Verification of solutions

$$y = \frac{9\, e^{-3t}}{3t\, e^{-3t} + e^{-3t} + c_3}$$

Verified OK.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
<- Bernoulli successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 20

```
dsolve(diff(y(t),t) = -y(t)*(3-t*y(t)),y(t), singsol=all)
```

$$y(t) = \frac{9}{1 + 9c_1 e^{3t} + 3t}$$

✓ Solution by Mathematica

Time used: 0.126 (sec). Leaf size: 28

```
DSolve[y'[t] == -y[t]*(3-t*y[t]),y[t],t,IncludeSingularSolutions -> True]
```

$$y(t) \to \frac{9}{3t + 9c_1 e^{3t} + 1}$$
$$y(t) \to 0$$

3.16 problem 20

Internal problem ID [531]
Internal file name [`OUTPUT/531_Sunday_June_05_2022_01_43_09_AM_59273030/index.tex`]

Book: Elementary differential equations and boundary value problems, 10th ed., Boyce and DiPrima
Section: Section 2.4. Page 76
Problem number: 20.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"riccati"**

Maple gives the following as the ode type

`[_Riccati]`

$$\boxed{y' + y^2 = -1 + t}$$

3.16.1 Solving as riccati ode

In canonical form the ODE is

$$\begin{aligned} y' &= F(t, y) \\ &= -y^2 + t - 1 \end{aligned}$$

This is a Riccati ODE. Comparing the ODE to solve

$$y' = -y^2 + t - 1$$

With Riccati ODE standard form

$$y' = f_0(t) + f_1(t)y + f_2(t)y^2$$

Shows that $f_0(t) = -1 + t$, $f_1(t) = 0$ and $f_2(t) = -1$. Let

$$\begin{aligned} y &= \frac{-u'}{f_2 u} \\ &= \frac{-u'}{-u} \end{aligned} \tag{1}$$

Using the above substitution in the given ODE results (after some simplification)in a second order ODE to solve for $u(x)$ which is

$$f_2 u''(t) - (f_2' + f_1 f_2)\, u'(t) + f_2^2 f_0 u(t) = 0 \tag{2}$$

But

$$\begin{aligned} f_2' &= 0 \\ f_1 f_2 &= 0 \\ f_2^2 f_0 &= -1 + t \end{aligned}$$

Substituting the above terms back in equation (2) gives

$$-u''(t) + (-1 + t)\, u(t) = 0$$

Solving the above ODE (this ode solved using Maple, not this program), gives

$$u(t) = c_1 \operatorname{AiryAi}(-1 + t) + c_2 \operatorname{AiryBi}(-1 + t)$$

The above shows that

$$u'(t) = c_1 \operatorname{AiryAi}(1, -1 + t) + c_2 \operatorname{AiryBi}(1, -1 + t)$$

Using the above in (1) gives the solution

$$y = \frac{c_1 \operatorname{AiryAi}(1, -1 + t) + c_2 \operatorname{AiryBi}(1, -1 + t)}{c_1 \operatorname{AiryAi}(-1 + t) + c_2 \operatorname{AiryBi}(-1 + t)}$$

Dividing both numerator and denominator by c_1 gives, after renaming the constant $\frac{c_2}{c_1} = c_3$ the following solution

$$y = \frac{c_3 \operatorname{AiryAi}(1, -1 + t) + \operatorname{AiryBi}(1, -1 + t)}{c_3 \operatorname{AiryAi}(-1 + t) + \operatorname{AiryBi}(-1 + t)}$$

Summary
The solution(s) found are the following

$$y = \frac{c_3 \operatorname{AiryAi}(1, -1 + t) + \operatorname{AiryBi}(1, -1 + t)}{c_3 \operatorname{AiryAi}(-1 + t) + \operatorname{AiryBi}(-1 + t)} \tag{1}$$

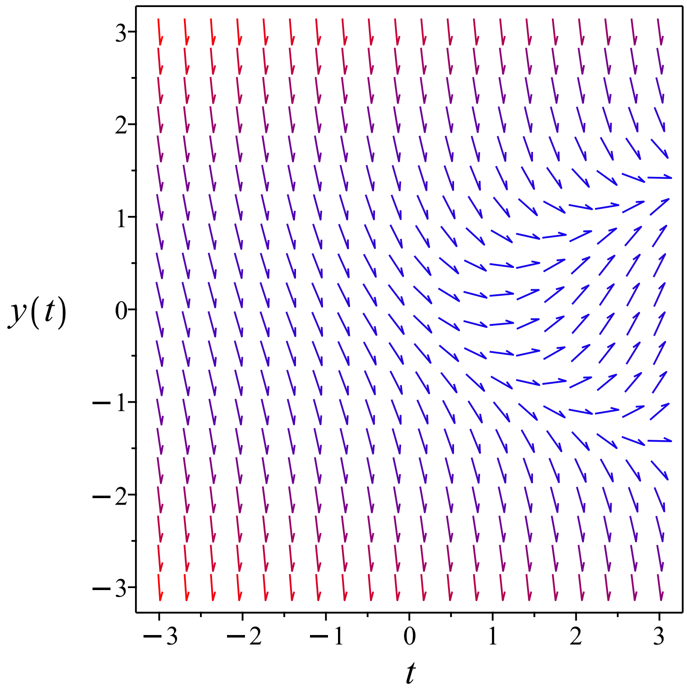


Figure 213: Slope field plot

<u>Verification of solutions</u>

$$y = \frac{c_3 \operatorname{AiryAi}(1, -1+t) + \operatorname{AiryBi}(1, -1+t)}{c_3 \operatorname{AiryAi}(-1+t) + \operatorname{AiryBi}(-1+t)}$$

Verified OK.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
trying homogeneous types:
trying Chini
differential order: 1; looking for linear symmetries
trying exact
Looking for potential symmetries
trying Riccati
trying Riccati Special
trying Riccati sub-methods:
   <- Abel AIR successful: ODE belongs to the 0F1 0-parameter (Airy type) class`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 31

```
dsolve(diff(y(t),t) = t-1-y(t)^2,y(t), singsol=all)
```

$$y(t) = \frac{\operatorname{AiryAi}(1, t-1) c_1 + \operatorname{AiryBi}(1, t-1)}{\operatorname{AiryAi}(t-1) c_1 + \operatorname{AiryBi}(t-1)}$$

✓ Solution by Mathematica

Time used: 0.123 (sec). Leaf size: 47

```
DSolve[y'[t] == t-1-y[t]^2,y[t],t,IncludeSingularSolutions -> True]
```

$$y(t) \to \frac{\text{AiryBiPrime}(t-1) + c_1 \,\text{AiryAiPrime}(t-1)}{\text{AiryBi}(t-1) + c_1 \,\text{AiryAi}(t-1)}$$
$$y(t) \to \frac{\text{AiryAiPrime}(t-1)}{\text{AiryAi}(t-1)}$$

4 Section 2.5. Page 88

4.1 problem 1

Internal problem ID [532]
Internal file name [`OUTPUT/532_Sunday_June_05_2022_01_43_10_AM_54801064/index.tex`]

Book: Elementary differential equations and boundary value problems, 10th ed., Boyce and DiPrima
Section: Section 2.5. Page 88
Problem number: 1.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"quadrature"**

Maple gives the following as the ode type

`[_quadrature]`

$$\boxed{y' - ay - by^2 = 0}$$

4.1.1 Solving as quadrature ode

Integrating both sides gives

$$\int \frac{1}{b\,y^2 + ay} dy = \int dx$$
$$\frac{\ln(y)}{a} - \frac{\ln(by + a)}{a} = x + c_1$$

The above can be written as

$$\left(\frac{1}{a}\right)(\ln(y) - \ln(by + a)) = x + c_1$$
$$\ln(y) - \ln(by + a) = (a)(x + c_1)$$
$$= a(x + c_1)$$

Raising both side to exponential gives

$$e^{\ln(y) - \ln(by + a)} = ac_1 e^{ax}$$

Which simplifies to

$$\frac{y}{by+a} = c_2 e^{ax}$$

Summary

The solution(s) found are the following

$$y = -\frac{e^{ax} c_2 a}{-1 + e^{ax} c_2 b} \tag{1}$$

Verification of solutions

$$y = -\frac{e^{ax} c_2 a}{-1 + e^{ax} c_2 b}$$

Verified OK.

4.1.2 Maple step by step solution

Let's solve

$y' - ay - by^2 = 0$

- Highest derivative means the order of the ODE is 1

  y'

- Separate variables

  $\frac{y'}{ay+by^2} = 1$

- Integrate both sides with respect to x

  $\int \frac{y'}{ay+by^2} dx = \int 1 dx + c_1$

- Evaluate integral

  $\frac{\ln(y)}{a} - \frac{\ln(by+a)}{a} = x + c_1$

- Solve for y

  $y = -\frac{e^{c_1 a + ax} a}{-1 + b\, e^{c_1 a + ax}}$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
<- Bernoulli successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 20

```
dsolve(diff(y(x),x) = a*y(x)+b*y(x)^2,y(x), singsol=all)
```

$$y(x) = \frac{a}{\mathrm{e}^{-ax} c_1 a - b}$$

✓ Solution by Mathematica

Time used: 0.888 (sec). Leaf size: 45

```
DSolve[y'[x]== a*y[x]+b*y[x]^2,y[x],x,IncludeSingularSolutions -> True]
```

$$\begin{aligned} y(x) &\to -\frac{a e^{a(x+c_1)}}{-1 + b e^{a(x+c_1)}} \\ y(x) &\to 0 \\ y(x) &\to -\frac{a}{b} \end{aligned}$$

4.2 problem 3

Internal problem ID [533]
Internal file name [`OUTPUT/533_Sunday_June_05_2022_01_43_11_AM_86686196/index.tex`]

Book: Elementary differential equations and boundary value problems, 10th ed., Boyce and DiPrima
Section: Section 2.5. Page 88
Problem number: 3.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"quadrature"**

Maple gives the following as the ode type

`[_quadrature]`

$$\boxed{y' - y(-2+y)(-1+y) = 0}$$

4.2.1 Solving as quadrature ode

Integrating both sides gives

$$\int \frac{1}{y(y-2)(y-1)} dy = \int dt$$
$$\frac{\ln(y)}{2} - \ln(y-1) + \frac{\ln(y-2)}{2} = t + c_1$$

Raising both side to exponential gives

$$e^{\frac{\ln(y)}{2} - \ln(y-1) + \frac{\ln(y-2)}{2}} = e^{t+c_1}$$

Which simplifies to

$$\frac{\sqrt{y}\,\sqrt{y-2}}{y-1} = c_2 e^t$$

Summary
The solution(s) found are the following

$$y = \frac{\sqrt{-c_2^2 e^{2t} + 1} - 1}{\sqrt{-c_2^2 e^{2t} + 1}} \tag{1}$$

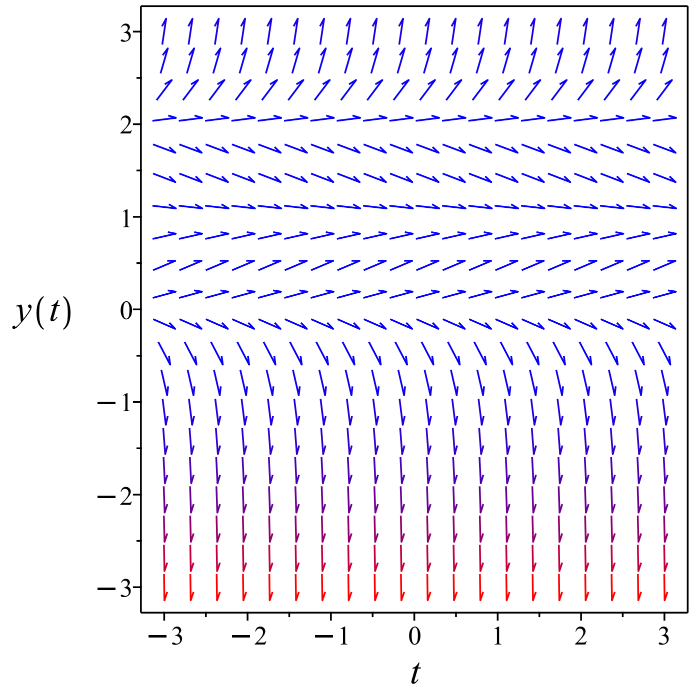


Figure 214: Slope field plot

Verification of solutions

$$y = \frac{\sqrt{-c_2^2 e^{2t} + 1} - 1}{\sqrt{-c_2^2 e^{2t} + 1}}$$

Verified OK.

4.2.2 Maple step by step solution

Let's solve

$y' - y(-2+y)(-1+y) = 0$

- Highest derivative means the order of the ODE is 1

  y'

- Separate variables

  $\frac{y'}{y(-2+y)(-1+y)} = 1$

- Integrate both sides with respect to t

  $\int \frac{y'}{y(-2+y)(-1+y)} dt = \int 1 dt + c_1$

- Evaluate integral

  $\frac{\ln(y)}{2} - \ln(-1+y) + \frac{\ln(-2+y)}{2} = t + c_1$

- Solve for y

  $\left\{y = \frac{(e^{c_1})^2 (e^t)^2}{(e^{c_1})^2 (e^t)^2 - \sqrt{-(e^{c_1})^2 (e^t)^2 + 1} - 1}, y = \frac{(e^{c_1})^2 (e^t)^2}{(e^{c_1})^2 (e^t)^2 + \sqrt{-(e^{c_1})^2 (e^t)^2 + 1} - 1}\right\}$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
<- separable successful`
```

✓ Solution by Maple

Time used: 0.125 (sec). Leaf size: 75

```
dsolve(diff(y(t),t) = y(t)*(-2+y(t))*(-1+y(t)),y(t), singsol=all)
```

$$y(t) = \frac{e^{2t} c_1}{\left(-1-\sqrt{-c_1 e^{2t}+1}\right)\sqrt{-c_1 e^{2t}+1}}$$
$$y(t) = \frac{e^{2t} c_1}{\left(1-\sqrt{-c_1 e^{2t}+1}\right)\sqrt{-c_1 e^{2t}+1}}$$

✓ Solution by Mathematica

Time used: 11.055 (sec). Leaf size: 100

```
DSolve[y'[t] == y[t]*(-2+y[t])*(-1+y[t]),y[t],t,IncludeSingularSolutions -> True]
```

$$y(t) \to \frac{-\sqrt{1+e^{2(t+c_1)}}+e^{2(t+c_1)}+1}{1+e^{2(t+c_1)}}$$
$$y(t) \to \frac{\sqrt{1+e^{2(t+c_1)}}+e^{2(t+c_1)}+1}{1+e^{2(t+c_1)}}$$
$$y(t) \to 0$$
$$y(t) \to 1$$
$$y(t) \to 2$$

4.3 problem 4

Internal problem ID [534]
Internal file name [`OUTPUT/534_Sunday_June_05_2022_01_43_47_AM_54468129/index.tex`]

Book: Elementary differential equations and boundary value problems, 10th ed., Boyce and DiPrima
Section: Section 2.5. Page 88
Problem number: 4.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"quadrature"**

Maple gives the following as the ode type

`[_quadrature]`

$$\boxed{y' - e^y = -1}$$

4.3.1 Solving as quadrature ode

Integrating both sides gives

$$\int \frac{1}{-1 + e^y} dy = \int dt$$
$$-\ln(e^y) + \ln(-1 + e^y) = t + c_1$$

Raising both side to exponential gives

$$e^{-\ln(e^y) + \ln(-1+e^y)} = e^{t+c_1}$$

Which simplifies to

$$-e^{-y} + 1 = c_2 e^t$$

Summary
The solution(s) found are the following

$$y = -\ln\left(-c_2 e^t + 1\right) \tag{1}$$

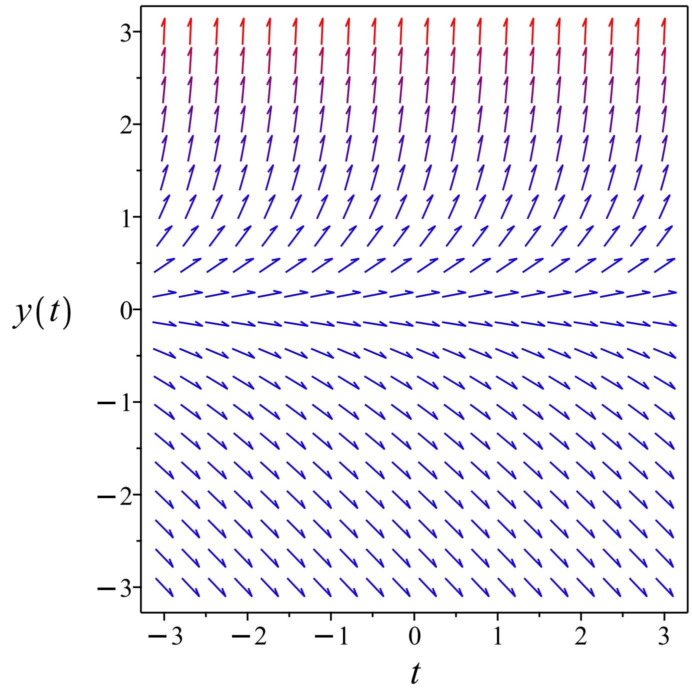


Figure 215: Slope field plot

Verification of solutions

$$y = -\ln\left(-c_2 e^t + 1\right)$$

Verified OK.

4.3.2 Maple step by step solution

Let's solve

$y' - e^y = -1$

- Highest derivative means the order of the ODE is 1

  y'

- Separate variables

  $\frac{y'}{-1+e^y} = 1$

- Integrate both sides with respect to t

  $\int \frac{y'}{-1+e^y} dt = \int 1 dt + c_1$

- Evaluate integral

$$-\ln(e^y) + \ln(-1 + e^y) = t + c_1$$

- Solve for y

$$y = \ln\left(-\frac{1}{e^{t+c_1} - 1}\right)$$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
<- separable successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 15

```
dsolve(diff(y(t),t) = -1+exp(y(t)),y(t), singsol=all)
```

$$y(t) = \ln\left(-\frac{1}{e^t c_1 - 1}\right)$$

✓ Solution by Mathematica

Time used: 0.817 (sec). Leaf size: 28

```
DSolve[y'[t]== -1+Exp[y[t]],y[t],t,IncludeSingularSolutions -> True]
```

$$y(t) \to \log\left(\frac{1}{2}\left(1 - \tanh\left(\frac{t + c_1}{2}\right)\right)\right)$$
$$y(t) \to 0$$

4.4 problem 5

Internal problem ID [535]
Internal file name [`OUTPUT/535_Sunday_June_05_2022_01_43_49_AM_20194512/index.tex`]

Book: Elementary differential equations and boundary value problems, 10th ed., Boyce and DiPrima
Section: Section 2.5. Page 88
Problem number: 5.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"quadrature"**

Maple gives the following as the ode type

`[_quadrature]`

$$\boxed{y' - e^{-y} = -1}$$

4.4.1 Solving as quadrature ode

Integrating both sides gives

$$\int \frac{1}{-1 + e^{-y}} dy = \int dt$$
$$\ln\left(e^{-y}\right) - \ln\left(-1 + e^{-y}\right) = t + c_1$$

Raising both side to exponential gives

$$e^{\ln\left(e^{-y}\right) - \ln\left(-1+e^{-y}\right)} = e^{t+c_1}$$

Which simplifies to

$$\frac{e^{-y}}{-1 + e^{-y}} = c_2 e^t$$

Summary
The solution(s) found are the following

$$y = -\ln\left(\frac{c_2}{c_2 e^t - 1}\right) - t \tag{1}$$

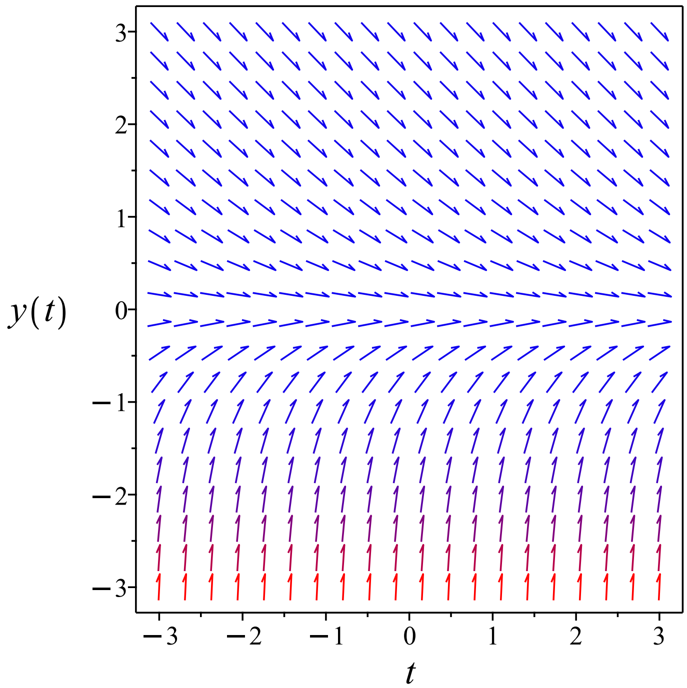


Figure 216: Slope field plot

Verification of solutions

$$y = -\ln\left(\frac{c_2}{c_2 e^t - 1}\right) - t$$

Verified OK.

4.4.2 Maple step by step solution

Let's solve

$y' - e^{-y} = -1$

- Highest derivative means the order of the ODE is 1

  y'

- Separate variables

  $\frac{y'}{-1+e^{-y}} = 1$

- Integrate both sides with respect to t

  $\int \frac{y'}{-1+e^{-y}} dt = \int 1 dt + c_1$

- Evaluate integral

$$\ln \left(\mathrm{e}^{-y}\right)-\ln \left(-1+\mathrm{e}^{-y}\right)=t+c_1$$

- Solve for y

$$y=\ln \left(\mathrm{e}^{t+c_1}-1\right)-t-c_1$$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
<- separable successful`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 18

```
dsolve(diff(y(t),t) = -1+exp(-y(t)),y(t), singsol=all)
```

$$y(t)=-t+\ln \left(\mathrm{e}^{t+c_1}-1\right)-c_1$$

✓ Solution by Mathematica

Time used: 0.853 (sec). Leaf size: 21

```
DSolve[y'[t] == -1+Exp[-y[t]],y[t],t,IncludeSingularSolutions -> True]
```

$$y(t) \to \log \left(1+e^{-t+c_1}\right)$$
$$y(t) \to 0$$

4.5 problem 6

Internal problem ID [536]
Internal file name [OUTPUT/536_Sunday_June_05_2022_01_43_50_AM_22554720/index.tex]

Book: Elementary differential equations and boundary value problems, 10th ed., Boyce and DiPrima
Section: Section 2.5. Page 88
Problem number: 6.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"quadrature"**

Maple gives the following as the ode type

`[_quadrature]`

$$\boxed{y' + \frac{2\arctan(y)}{1+y^2} = 0}$$

4.5.1 Solving as quadrature ode

Integrating both sides gives

$$\int -\frac{y^2+1}{2\arctan(y)}dy = \int dt$$
$$\int^{y} -\frac{\_a^2+1}{2\arctan(\_a)}d\_a = t + c_1$$

Summary
The solution(s) found are the following

$$\int^{y} -\frac{\_a^2+1}{2\arctan(\_a)}d\_a = t + c_1 \tag{1}$$

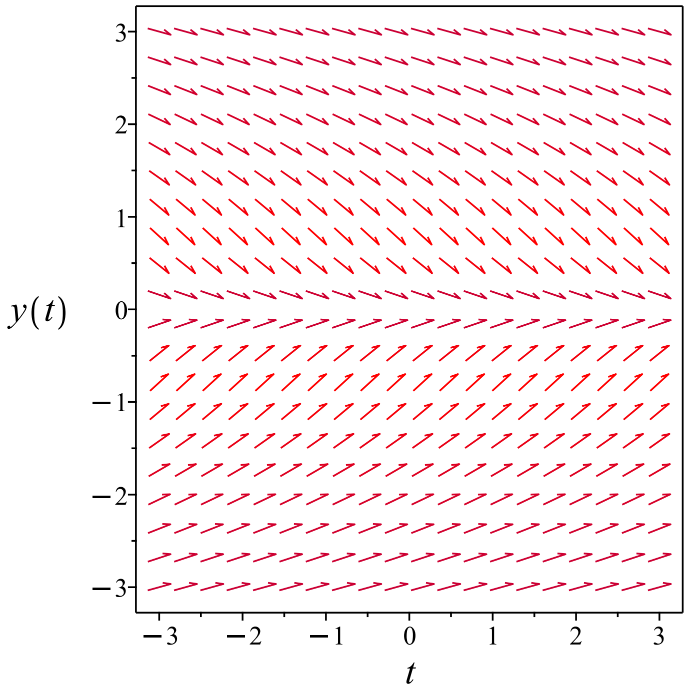


Figure 217: Slope field plot

Verification of solutions

$$\int^{y} -\frac{\_a^2 + 1}{2 \arctan(\_a)} d\_a = t + c_1$$

Verified OK.

4.5.2 Maple step by step solution

Let's solve

$y' + \frac{2\arctan(y)}{1+y^2} = 0$

- Highest derivative means the order of the ODE is 1

  y'

- Separate variables

  $\frac{(1+y^2)y'}{\arctan(y)} = -2$

- Integrate both sides with respect to t

  $\int \frac{(1+y^2)y'}{\arctan(y)} dt = \int (-2)\, dt + c_1$

- Cannot compute integral
  $\int \frac{(1+y^2)y'}{\arctan(y)}dt = -2t + c_1$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
<- separable successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 23

```
dsolve(diff(y(t),t) = -2*arctan(y(t))/(1+y(t)^2),y(t), singsol=all)
```

$$t + \frac{\left(\int^{y(t)} \frac{\_a^2+1}{\arctan(\_a)}d\_a\right)}{2} + c_1 = 0$$

✓ Solution by Mathematica

Time used: 1.013 (sec). Leaf size: 38

```
DSolve[y'[t] == -2*ArcTan[y[t]]/(1+y[t]^2),y[t],t,IncludeSingularSolutions -> True]
```

$$y(t) \to \text{InverseFunction}\left[\int_1^{\#1} \frac{K[1]^2+1}{\arctan(K[1])}dK[1]\&\right][-2t+c_1]$$
$$y(t) \to 0$$

4.6 problem 7

Internal problem ID [537]
Internal file name [`OUTPUT/537_Sunday_June_05_2022_01_43_52_AM_26451284/index.tex`]

Book: Elementary differential equations and boundary value problems, 10th ed., Boyce and DiPrima
Section: Section 2.5. Page 88
Problem number: 7.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"quadrature"**

Maple gives the following as the ode type

`[_quadrature]`

$$\boxed{y' + k(-1+y)^2 = 0}$$

4.6.1 Solving as quadrature ode

Integrating both sides gives

$$\int -\frac{1}{k\left(y-1\right)^{2}}dy = t + c_1$$
$$\frac{1}{k\left(y-1\right)} = t + c_1$$

Solving for y gives these solutions

$$y_1 = \frac{c_1 k + tk + 1}{k\left(t+c_1\right)}$$

Summary
The solution(s) found are the following

$$y = \frac{c_1 k + tk + 1}{k\left(t+c_1\right)} \tag{1}$$

Verification of solutions

$$y = \frac{c_1 k + tk + 1}{k (t + c_1)}$$

Verified OK.

4.6.2 Maple step by step solution

Let's solve

$y' + k(-1+y)^2 = 0$

- Highest derivative means the order of the ODE is 1

  y'

- Separate variables

  $\frac{y'}{(-1+y)^2} = -k$

- Integrate both sides with respect to t

  $\int \frac{y'}{(-1+y)^2} dt = \int -k dt + c_1$

- Evaluate integral

  $-\frac{1}{-1+y} = -tk + c_1$

- Solve for y

  $y = \frac{-tk + c_1 - 1}{-tk + c_1}$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
<- separable successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 20

```
dsolve(diff(y(t),t) = -k*(-1+y(t))^2,y(t), singsol=all)
```

$$y(t) = \frac{1 + k(t + c_1)}{k\,(t + c_1)}$$

✓ Solution by Mathematica

Time used: 0.139 (sec). Leaf size: 30

```
DSolve[y'[t]== -k*(-1+y[t])^2,y[t],t,IncludeSingularSolutions -> True]
```

$$y(t) \to \frac{kt + 1 - c_1}{kt - c_1}$$
$$y(t) \to 1$$

4.7 problem 9

Internal problem ID [538]
Internal file name [`OUTPUT/538_Sunday_June_05_2022_01_43_53_AM_86819030/index.tex`]

Book: Elementary differential equations and boundary value problems, 10th ed., Boyce and DiPrima
Section: Section 2.5. Page 88
Problem number: 9.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"quadrature"**

Maple gives the following as the ode type

```
[_quadrature]
```

$$\boxed{y' - y^2(y^2 - 1) = 0}$$

4.7.1 Solving as quadrature ode

Integrating both sides gives

$$\int \frac{1}{y^2(y^2-1)}dy = \int dt$$
$$\int^y \frac{1}{\_a^2(\_a^2-1)}d\_a = t + c_1$$

Summary
The solution(s) found are the following

$$\int^y \frac{1}{\_a^2(\_a^2-1)}d\_a = t + c_1 \tag{1}$$

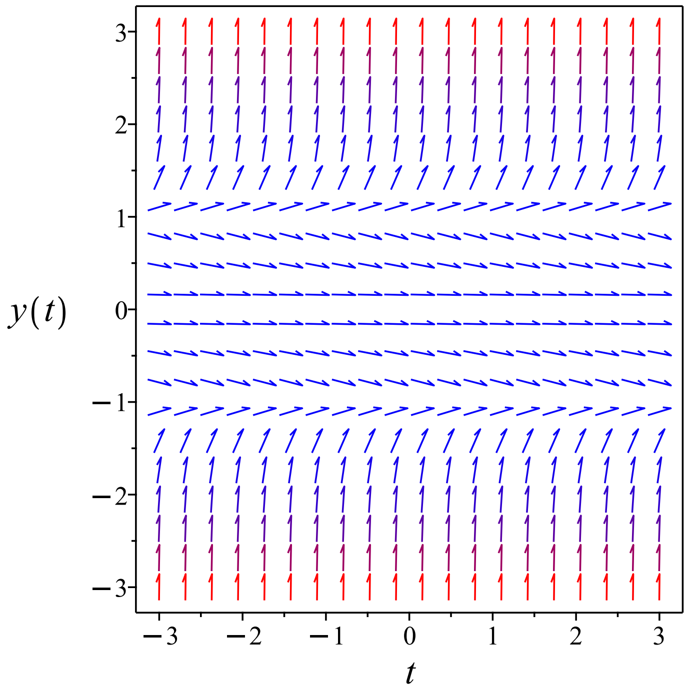


Figure 218: Slope field plot

Verification of solutions

$$\int^{y} \frac{1}{\_a^{2}\left(\_a^{2}-1\right)}d\_a = t + c_1$$

Verified OK.

4.7.2 Maple step by step solution

Let's solve

$y' - y^2(y^2 - 1) = 0$

- Highest derivative means the order of the ODE is 1

  y'

- Separate variables

  $\frac{y'}{y^2(y^2-1)} = 1$

- Integrate both sides with respect to t

  $\int \frac{y'}{y^2(y^2-1)}dt = \int 1dt + c_1$

- Evaluate integral

$$-\frac{\ln(1+y)}{2}+\frac{1}{y}+\frac{\ln(-1+y)}{2}=t+c_1$$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
<- separable successful`
```

✓ Solution by Maple

Time used: 0.282 (sec). Leaf size: 47

```
dsolve(diff(y(t),t) = y(t)^2*(y(t)^2-1),y(t), singsol=all)
```

$$y(t)=\mathrm{e}^{\mathrm{RootOf}\left(-\ln\left(\mathrm{e}^{\_Z}-2\right)\mathrm{e}^{\_Z}+2c_1\mathrm{e}^{\_Z}+\_Z\mathrm{e}^{\_Z}+2t\,\mathrm{e}^{\_Z}+\ln\left(\mathrm{e}^{\_Z}-2\right)-2c_1-\_Z-2t-2\right)}-1$$

✓ Solution by Mathematica

Time used: 0.246 (sec). Leaf size: 51

```
DSolve[y'[t] == y[t]^2*(y[t]^2-1),y[t],t,IncludeSingularSolutions -> True]
```

$$y(t)\to \text{InverseFunction}\left[\frac{1}{\#1}+\frac{1}{2}\log(1-\#1)-\frac{1}{2}\log(\#1+1)\&\right][t+c_1]$$
$$y(t)\to -1$$
$$y(t)\to 0$$
$$y(t)\to 1$$

4.8 problem 10

Internal problem ID [539]
Internal file name [`OUTPUT/539_Sunday_June_05_2022_01_43_54_AM_75461840/index.tex`]

Book: Elementary differential equations and boundary value problems, 10th ed., Boyce and DiPrima
Section: Section 2.5. Page 88
Problem number: 10.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"quadrature"**

Maple gives the following as the ode type

`[_quadrature]`

$$\boxed{y' - y(1 - y^2) = 0}$$

4.8.1 Solving as quadrature ode

Integrating both sides gives

$$\int -\frac{1}{y\,(y^2-1)}dy = \int dt$$
$$\ln (y) - \frac{\ln (y+1)}{2} - \frac{\ln (y-1)}{2} = t + c_1$$

Raising both side to exponential gives

$$\mathrm{e}^{\ln(y) - \frac{\ln(y+1)}{2} - \frac{\ln(y-1)}{2}} = \mathrm{e}^{t+c_1}$$

Which simplifies to

$$\frac{y}{\sqrt{y+1}\,\sqrt{y-1}} = c_2 \mathrm{e}^{t}$$

Summary
The solution(s) found are the following

$$y = \frac{-c_2^2\mathrm{e}^{2t} + \sqrt{\mathrm{e}^{4t}c_2^4 - c_2^2\mathrm{e}^{2t}} + 1}{c_2^2\mathrm{e}^{2t} - 1} + 1 \tag{1}$$

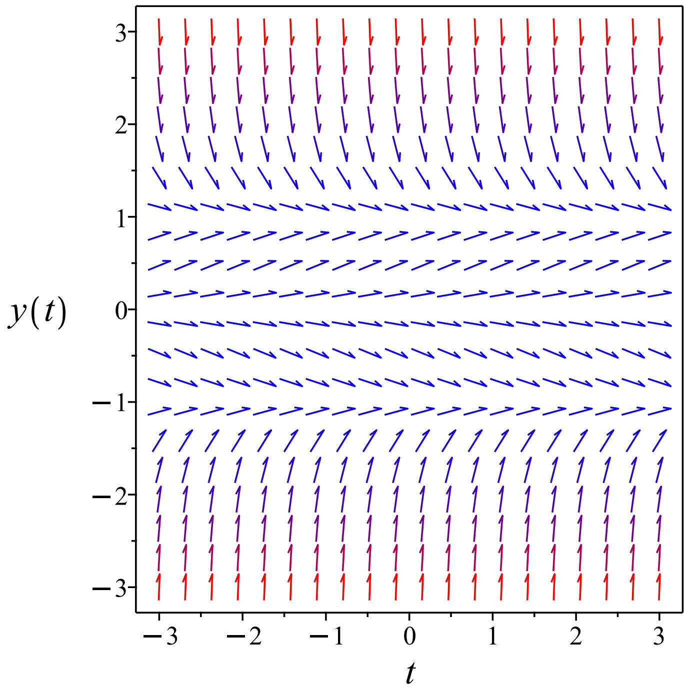


Figure 219: Slope field plot

Verification of solutions

$$y = \frac{-c_2^2 e^{2t} + \sqrt{e^{4t} c_2^4 - c_2^2 e^{2t}} + 1}{c_2^2 e^{2t} - 1} + 1$$

Verified OK.

4.8.2 Maple step by step solution

Let's solve

$y' - y(1 - y^2) = 0$

- Highest derivative means the order of the ODE is 1

  y'

- Separate variables

  $\frac{y'}{y(1-y^2)} = 1$

- Integrate both sides with respect to t

  $\int \frac{y'}{y(1-y^2)} dt = \int 1 dt + c_1$

- Evaluate integral

$$\ln(y) - \frac{\ln(1+y)}{2} - \frac{\ln(-1+y)}{2} = t + c_1$$

- Solve for y

$$\left\{y = \frac{\sqrt{(e^{2c_1+2t}-1)\,e^{2c_1+2t}}}{e^{2c_1+2t}-1}, y = -\frac{\sqrt{(e^{2c_1+2t}-1)\,e^{2c_1+2t}}}{e^{2c_1+2t}-1}\right\}$$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
<- Bernoulli successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 29

```
dsolve(diff(y(t),t) = y(t)*(1-y(t)^2),y(t), singsol=all)
```

$$
\begin{aligned}
y(t) &= \frac{1}{\sqrt{c_1 \mathrm{e}^{-2t}+1}} \\
y(t) &= -\frac{1}{\sqrt{c_1 \mathrm{e}^{-2t}+1}}
\end{aligned}
$$

✓ Solution by Mathematica

Time used: 0.676 (sec). Leaf size: 100

```
DSolve[y'[t]== y[t]*(1-y[t]^2),y[t],t,IncludeSingularSolutions -> True]
```

$$
\begin{aligned}
y(t) &\to -\frac{e^t}{\sqrt{e^{2t}+e^{2c_1}}} \\
y(t) &\to \frac{e^t}{\sqrt{e^{2t}+e^{2c_1}}} \\
y(t) &\to -1 \\
y(t) &\to 0 \\
y(t) &\to 1 \\
y(t) &\to -\frac{e^t}{\sqrt{e^{2t}}} \\
y(t) &\to \frac{e^t}{\sqrt{e^{2t}}}
\end{aligned}
$$

4.9 problem 11

Internal problem ID [540]
Internal file name [`OUTPUT/540_Sunday_June_05_2022_01_43_57_AM_7870853/index.tex`]

Book: Elementary differential equations and boundary value problems, 10th ed., Boyce and DiPrima
Section: Section 2.5. Page 88
Problem number: 11.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"quadrature"**

Maple gives the following as the ode type

```
[_quadrature]
```

$$\boxed{y' + b\sqrt{y} - ay = 0}$$

4.9.1 Solving as quadrature ode

Integrating both sides gives

$$\int \frac{1}{-b\sqrt{y} + ay} dy = \int dt$$
$$\frac{\ln(a^2y - b^2)}{a} - \frac{\ln(a\sqrt{y} + b)}{a} + \frac{\ln(a\sqrt{y} - b)}{a} = t + c_1$$

The above can be written as

$$\left(\frac{1}{a}\right)\left(\ln(a^2y - b^2) - \ln(a\sqrt{y} + b) + \ln(a\sqrt{y} - b)\right) = t + c_1$$
$$\ln(a^2y - b^2) - \ln(a\sqrt{y} + b) + \ln(a\sqrt{y} - b) = (a)(t + c_1)$$
$$= a(t + c_1)$$

Raising both side to exponential gives

$$e^{\ln(a^2y - b^2) - \ln(a\sqrt{y} + b) + \ln(a\sqrt{y} - b)} = ac_1 e^{at}$$

Which simplifies to

$$\frac{(a^2y - b^2)\left(a\sqrt{y} - b\right)}{a\sqrt{y} + b} = c_2 e^{at}$$

Summary

The solution(s) found are the following

$$y = \frac{2b\left(b - \sqrt{c_2 e^{at}}\right) + c_2 e^{at} - b^2}{a^2} \tag{1}$$

Verification of solutions

$$y = \frac{2b\left(b - \sqrt{c_2 e^{at}}\right) + c_2 e^{at} - b^2}{a^2}$$

Verified OK.

4.9.2 Maple step by step solution

Let's solve

$y' + b\sqrt{y} - ay = 0$

- Highest derivative means the order of the ODE is 1

  y'

- Separate variables

  $\frac{y'}{-b\sqrt{y}+ay} = 1$

- Integrate both sides with respect to t

  $\int \frac{y'}{-b\sqrt{y}+ay} dt = \int 1 dt + c_1$

- Evaluate integral

  $\frac{\ln(ya^2-b^2)}{a} - \frac{\ln(b+a\sqrt{y})}{a} + \frac{\ln(a\sqrt{y}-b)}{a} = t + c_1$

- Solve for y

  $\left\{y = \frac{2b\left(b-e^{\frac{1}{2}c_1 a+\frac{1}{2}at}\right)-b^2+e^{c_1 a+at}}{a^2}, y = \frac{2b\left(b+e^{\frac{1}{2}c_1 a+\frac{1}{2}at}\right)-b^2+e^{c_1 a+at}}{a^2}\right\}$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
<- Bernoulli successful`
```

✓ Solution by Maple

Time used: 0.015 (sec). Leaf size: 26

```
dsolve(diff(y(t),t) = -b*y(t)^(1/2)+a*y(t),y(t), singsol=all)
```

$$\frac{-\mathrm{e}^{\frac{at}{2}} c_1 a + \sqrt{y(t)}\, a - b}{a} = 0$$

✓ Solution by Mathematica

Time used: 0.844 (sec). Leaf size: 55

```
DSolve[y'[t] == -b*y[t]^(1/2)+a*y[t],y[t],t,IncludeSingularSolutions -> True]
```

$$y(t) \rightarrow \frac{e^{-ac_1}\left(e^{\frac{at}{2}} - be^{\frac{ac_1}{2}}\right)^2}{a^2}$$
$$y(t) \rightarrow 0$$
$$y(t) \rightarrow \frac{b^2}{a^2}$$

4.10 problem 12

Internal problem ID [541]
Internal file name [`OUTPUT/541_Sunday_June_05_2022_01_43_58_AM_15077121/index.tex`]

Book: Elementary differential equations and boundary value problems, 10th ed., Boyce and DiPrima
Section: Section 2.5. Page 88
Problem number: 12.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"quadrature"**

Maple gives the following as the ode type

```
[_quadrature]
```

$$\boxed{y' - y^2(4 - y^2) = 0}$$

4.10.1 Solving as quadrature ode

Integrating both sides gives

$$\int -\frac{1}{y^2 (y^2 - 4)} dy = \int dt$$

$$\int^y -\frac{1}{\_a^2 (\_a^2 - 4)} d\_a = t + c_1$$

Summary
The solution(s) found are the following

$$\int^y -\frac{1}{\_a^2 (\_a^2 - 4)} d\_a = t + c_1 \tag{1}$$

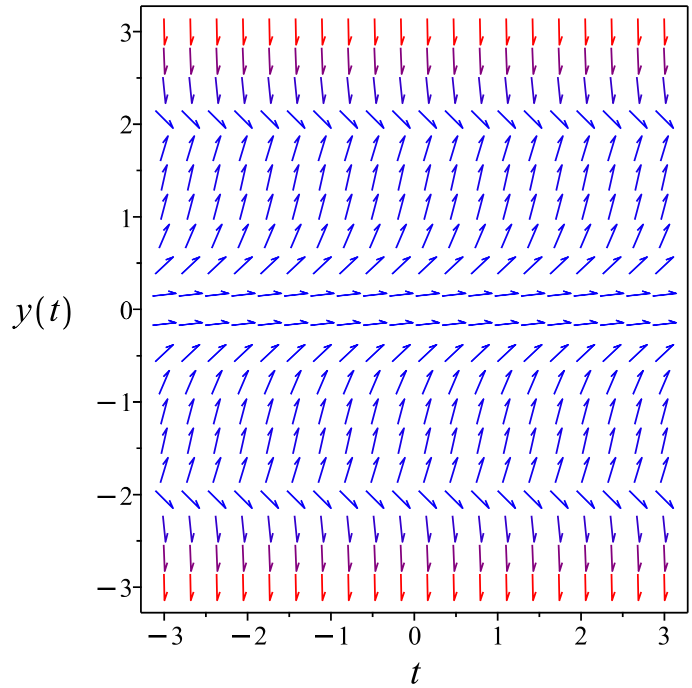


Figure 220: Slope field plot

Verification of solutions

$$\int^{y} -\frac{1}{\_a^{2}\left(\_a^{2}-4\right)}d\_a = t + c_1$$

Verified OK.

4.10.2 Maple step by step solution

Let's solve

$y' - y^2(4 - y^2) = 0$

- Highest derivative means the order of the ODE is 1

  y'

- Separate variables

  $\frac{y'}{y^2(4-y^2)} = 1$

- Integrate both sides with respect to t

  $\int \frac{y'}{y^2(4-y^2)}dt = \int 1dt + c_1$

- Evaluate integral

$$-\frac{1}{4y}-\frac{\ln(-2+y)}{16}+\frac{\ln(2+y)}{16}=t+c_1$$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
<- separable successful`
```

✓ Solution by Maple

Time used: 0.032 (sec). Leaf size: 49

```
dsolve(diff(y(t),t) = y(t)^2*(4-y(t)^2),y(t), singsol=all)
```

$$y(t)=\mathrm{e}^{\mathrm{RootOf}\left(\ln\left(\mathrm{e}^{\_Z}-4\right)\mathrm{e}^{\_Z}+16c_1\mathrm{e}^{\_Z}-\_Z\,\mathrm{e}^{\_Z}+16t\,\mathrm{e}^{\_Z}-2\ln\left(\mathrm{e}^{\_Z}-4\right)-32c_1+2\_Z-32t+4\right)}-2$$

✓ Solution by Mathematica

Time used: 0.247 (sec). Leaf size: 57

```
DSolve[y'[t] == y[t]^2*(4-y[t]^2),y[t],t,IncludeSingularSolutions -> True]
```

$$y(t)\to\text{InverseFunction}\left[\frac{1}{4\#1}+\frac{1}{16}\log(2-\#1)-\frac{1}{16}\log(\#1+2)\&\right][-t+c_1]$$
$$y(t)\to -2$$
$$y(t)\to 0$$
$$y(t)\to 2$$

4.11 problem 13

Internal problem ID [542]
Internal file name [`OUTPUT/542_Sunday_June_05_2022_01_43_59_AM_906908/index.tex`]

Book: Elementary differential equations and boundary value problems, 10th ed., Boyce and DiPrima
Section: Section 2.5. Page 88
Problem number: 13.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"quadrature"**

Maple gives the following as the ode type

```
[_quadrature]
```

$$\boxed{y' - (1-y)^2 y^2 = 0}$$

4.11.1 Solving as quadrature ode

Integrating both sides gives

$$\int \frac{1}{y^2 (y-1)^2} dy = \int dt$$
$$\int^y \frac{1}{\_a^2 (\_a-1)^2} d\_a = t + c_1$$

Summary
The solution(s) found are the following

$$\int^y \frac{1}{\_a^2 (\_a-1)^2} d\_a = t + c_1 \tag{1}$$

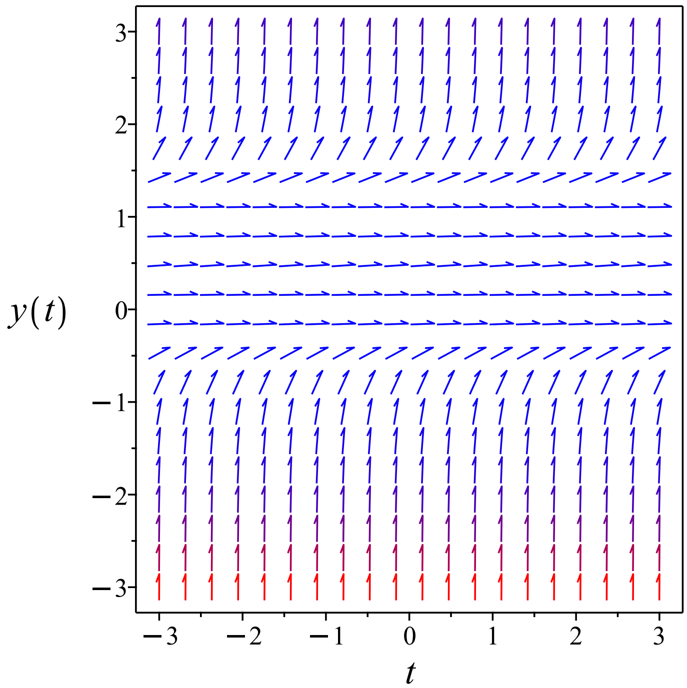


Figure 221: Slope field plot

Verification of solutions

$$\int^{y} \frac{1}{\_a^2 (\_a - 1)^2} d\_a = t + c_1$$

Verified OK.

4.11.2 Maple step by step solution

Let's solve

$y' - (1-y)^2 y^2 = 0$

- Highest derivative means the order of the ODE is 1

  y'

- Separate variables

  $\frac{y'}{(1-y)^2 y^2} = 1$

- Integrate both sides with respect to t

  $\int \frac{y'}{(1-y)^2 y^2} dt = \int 1 dt + c_1$

- Evaluate integral

$$-\frac{1}{y}+2\ln(y)-\frac{1}{-1+y}-2\ln(-1+y)=t+c_1$$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
<- separable successful`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 66

```
dsolve(diff(y(t),t) = (1-y(t))^2*y(t)^2,y(t), singsol=all)
```

$$y(t)=\mathrm{e}^{\mathrm{RootOf}\left(-2\ln\left(\mathrm{e}^{\_Z}+1\right)\mathrm{e}^{2\_Z}+c_1\mathrm{e}^{2\_Z}+2\_Z\mathrm{e}^{2\_Z}+t\,\mathrm{e}^{2\_Z}-2\ln\left(\mathrm{e}^{\_Z}+1\right)\mathrm{e}^{\_Z}+c_1\mathrm{e}^{\_Z}+2\_Z\mathrm{e}^{\_Z}+t\,\mathrm{e}^{\_Z}+2\,\mathrm{e}^{\_Z}+1\right)}+1$$

✓ Solution by Mathematica

Time used: 0.365 (sec). Leaf size: 50

```
DSolve[y'[t] == (1-y[t])^2*y[t]^2,y[t],t,IncludeSingularSolutions -> True]
```

$$y(t)\to\text{InverseFunction}\left[-\frac{1}{\#1-1}-\frac{1}{\#1}-2\log(1-\#1)+2\log(\#1)\&\right][t+c_1]$$
$$y(t)\to 0$$
$$y(t)\to 1$$

5 Section 2.6. Page 100

5.1 problem 1

Internal problem ID [543]
Internal file name [`OUTPUT/543_Sunday_June_05_2022_01_44_01_AM_47710901/index.tex`]

Book: Elementary differential equations and boundary value problems, 10th ed., Boyce and DiPrima
Section: Section 2.6. Page 100
Problem number: 1.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"exact", "separable", "differentialType", "homogeneousTypeMapleC", "first_order_ode_lie_symmetry_lookup"**

Maple gives the following as the ode type

`[_separable]`

$$\boxed{(-2+2y)\,y' = -3-2x}$$

5.1.1 Solving as separable ode

In canonical form the ODE is

$$\begin{aligned} y' &= F(x,y) \\ &= f(x)g(y) \\ &= \frac{-\frac{3}{2}-x}{y-1} \end{aligned}$$

Where $f(x) = -\frac{3}{2}-x$ and $g(y) = \frac{1}{y-1}$. Integrating both sides gives

$$\frac{1}{\frac{1}{y-1}}\,dy = -\frac{3}{2}-x\,dx$$

$$\int \frac{1}{\frac{1}{y-1}} dy = \int -\frac{3}{2} - x \, dx$$

$$\frac{1}{2} y^2 - y = -\frac{3}{2}x - \frac{1}{2}x^2 + c_1$$

Which results in

$$y = 1 + \sqrt{-x^2 + 2c_1 - 3x + 1}$$

$$y = 1 - \sqrt{-x^2 + 2c_1 - 3x + 1}$$

<u>Summary</u>

The solution(s) found are the following

$$y = 1 + \sqrt{-x^2 + 2c_1 - 3x + 1} \tag{1}$$

$$y = 1 - \sqrt{-x^2 + 2c_1 - 3x + 1} \tag{2}$$

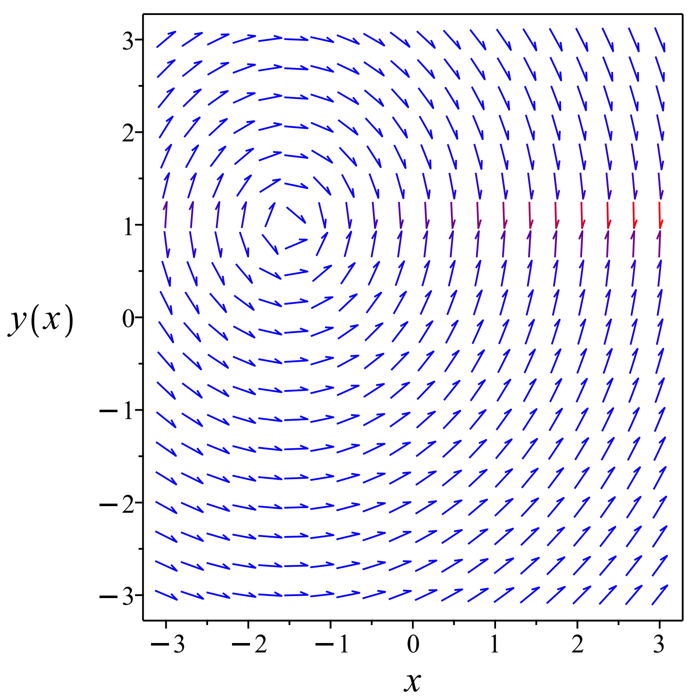


Figure 222: Slope field plot

Verification of solutions

$$y = 1 + \sqrt{-x^2 + 2c_1 - 3x + 1}$$

Verified OK.

$$y = 1 - \sqrt{-x^2 + 2c_1 - 3x + 1}$$

Verified OK.

5.1.2 Solving as differentialType ode

Writing the ode as

$$y' = \frac{-3 - 2x}{-2 + 2y} \tag{1}$$

Which becomes

$$(2y - 2)\, dy = (-3 - 2x)\, dx \tag{2}$$

But the RHS is complete differential because

$$(-3 - 2x)\, dx = d\left(-x^2 - 3x\right)$$

Hence (2) becomes

$$(2y - 2)\, dy = d\left(-x^2 - 3x\right)$$

Integrating both sides gives gives these solutions

$$y = 1 + \sqrt{-x^2 + c_1 - 3x + 1} + c_1$$
$$y = 1 - \sqrt{-x^2 + c_1 - 3x + 1} + c_1$$

Summary
The solution(s) found are the following

$$y = 1 + \sqrt{-x^2 + c_1 - 3x + 1} + c_1 \tag{1}$$
$$y = 1 - \sqrt{-x^2 + c_1 - 3x + 1} + c_1 \tag{2}$$

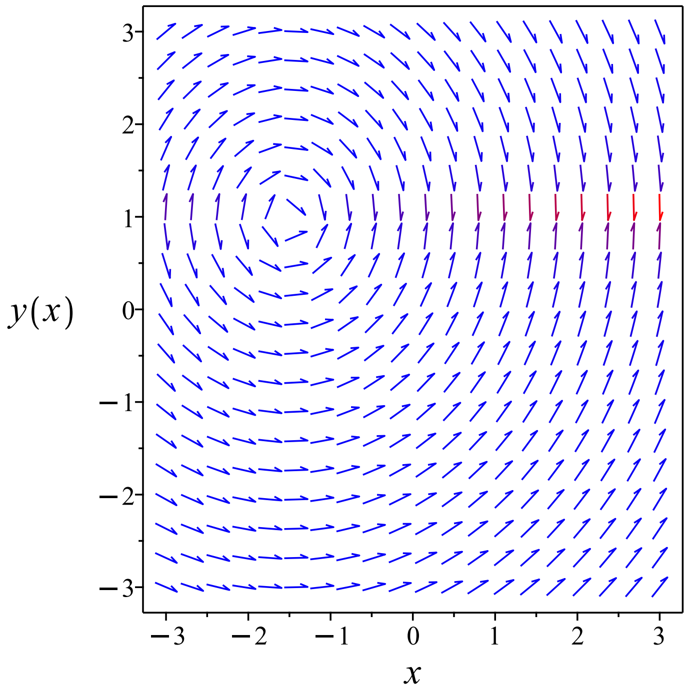


Figure 223: Slope field plot

Verification of solutions

$$y = 1 + \sqrt{-x^2 + c_1 - 3x + 1} + c_1$$

Verified OK.

$$y = 1 - \sqrt{-x^2 + c_1 - 3x + 1} + c_1$$

Verified OK.

5.1.3 Solving as homogeneousTypeMapleC ode

Let $Y = y + y_0$ and $X = x + x_0$ then the above is transformed to new ode in $Y(X)$

$$\frac{d}{dX}Y(X) = -\frac{2X + 2x_0 + 3}{2\left(Y\left(X\right) + y_0 - 1\right)}$$

Solving for possible values of x_0 and y_0 which makes the above ode a homogeneous ode results in

$$x_0 = -\frac{3}{2}$$
$$y_0 = 1$$

Using these values now it is possible to easily solve for $Y(X)$. The above ode now becomes

$$\frac{d}{dX}Y(X) = -\frac{X}{Y(X)}$$

In canonical form, the ODE is

$$\begin{aligned} Y' &= F(X, Y) \\ &= -\frac{X}{Y} \end{aligned} \tag{1}$$

An ode of the form $Y' = \frac{M(X,Y)}{N(X,Y)}$ is called homogeneous if the functions $M(X, Y)$ and $N(X, Y)$ are both homogeneous functions and of the same order. Recall that a function $f(X, Y)$ is homogeneous of order n if

$$f(t^n X, t^n Y) = t^n f(X, Y)$$

In this case, it can be seen that both $M = -X$ and $N = Y$ are both homogeneous and of the same order $n = 1$. Therefore this is a homogeneous ode. Since this ode is homogeneous, it is converted to separable ODE using the substitution $u = \frac{Y}{X}$, or $Y = uX$. Hence

$$\frac{\mathrm{d}Y}{\mathrm{d}X} = \frac{\mathrm{d}u}{\mathrm{d}X}X + u$$

Applying the transformation $Y = uX$ to the above ODE in (1) gives

$$\begin{aligned} \frac{\mathrm{d}u}{\mathrm{d}X}X + u &= -\frac{1}{u} \\ \frac{\mathrm{d}u}{\mathrm{d}X} &= \frac{-\frac{1}{u(X)} - u(X)}{X} \end{aligned}$$

Or

$$\frac{d}{dX}u(X) - \frac{-\frac{1}{u(X)} - u(X)}{X} = 0$$

Or

$$\left(\frac{d}{dX}u(X)\right)u(X)\,X + u(X)^2 + 1 = 0$$

Which is now solved as separable in $u(X)$. Which is now solved in $u(X)$. In canonical form the ODE is

$$\begin{aligned} u' &= F(X, u) \\ &= f(X)g(u) \\ &= -\frac{u^2 + 1}{uX} \end{aligned}$$

Where $f(X) = -\frac{1}{X}$ and $g(u) = \frac{u^2+1}{u}$. Integrating both sides gives

$$\frac{1}{\frac{u^2+1}{u}}\,du = -\frac{1}{X}\,dX$$
$$\int \frac{1}{\frac{u^2+1}{u}}\,du = \int -\frac{1}{X}\,dX$$
$$\frac{\ln(u^2+1)}{2} = -\ln(X) + c_2$$

Raising both side to exponential gives

$$\sqrt{u^2+1} = \mathrm{e}^{-\ln(X)+c_2}$$

Which simplifies to

$$\sqrt{u^2+1} = \frac{c_3}{X}$$

Which simplifies to

$$\sqrt{u(X)^2+1} = \frac{c_3\,\mathrm{e}^{c_2}}{X}$$

The solution is

$$\sqrt{u(X)^2+1} = \frac{c_3\,\mathrm{e}^{c_2}}{X}$$

Now u in the above solution is replaced back by Y using $u = \frac{Y}{X}$ which results in the solution

$$\sqrt{\frac{Y(X)^2}{X^2}+1} = \frac{c_3\,\mathrm{e}^{c_2}}{X}$$

Using the solution for $Y(X)$

$$\sqrt{\frac{Y(X)^2+X^2}{X^2}} = \frac{c_3\,\mathrm{e}^{c_2}}{X}$$

And replacing back terms in the above solution using

$$Y = y + y_0$$
$$X = x + x_0$$

Or

$$Y = y + 1$$
$$X = x - \frac{3}{2}$$

Then the solution in y becomes

$$\sqrt{\frac{(y-1)^2 + \left(x+\frac{3}{2}\right)^2}{\left(x+\frac{3}{2}\right)^2}} = \frac{c_3 \mathrm{e}^{c_2}}{x+\frac{3}{2}}$$

Summary
The solution(s) found are the following

$$\sqrt{\frac{(y-1)^2 + \left(x+\frac{3}{2}\right)^2}{\left(x+\frac{3}{2}\right)^2}} = \frac{c_3 \mathrm{e}^{c_2}}{x+\frac{3}{2}} \tag{1}$$

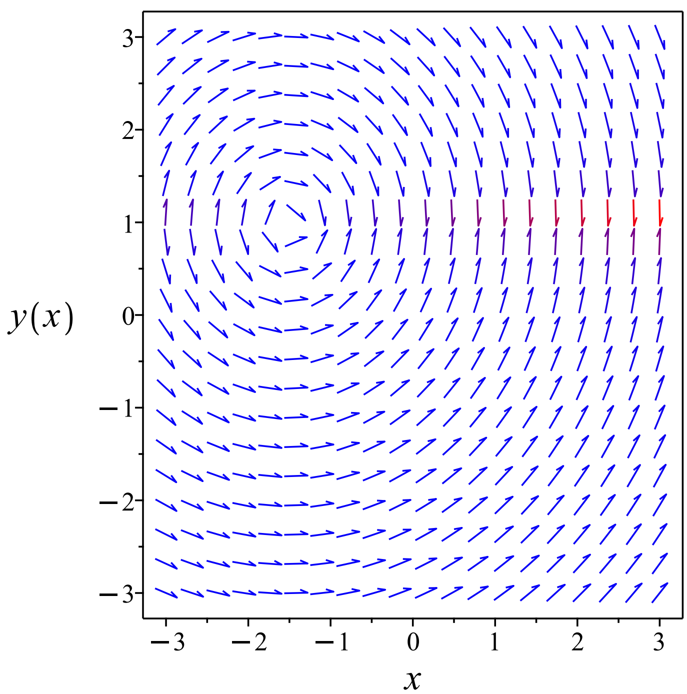


Figure 224: Slope field plot

Verification of solutions

$$
\sqrt{\frac{(y-1)^2+\left(x+\frac{3}{2}\right)^2}{\left(x+\frac{3}{2}\right)^2}}=\frac{c_3\mathrm{e}^{c_2}}{x+\frac{3}{2}}
$$

Verified OK.

5.1.4 Solving as first order ode lie symmetry lookup ode

Writing the ode as

$$
\begin{aligned}
y' &= -\frac{2x+3}{2\,(y-1)} \\
y' &= \omega(x,y)
\end{aligned}
$$

The condition of Lie symmetry is the linearized PDE given by

$$
\eta_x+\omega(\eta_y-\xi_x)-\omega^2\xi_y-\omega_x\xi-\omega_y\eta=0 \tag{A}
$$

The type of this ode is known. It is of type `separable`. Therefore we do not need to solve the PDE (A), and can just use the lookup table shown below to find ξ,η

Table 234: Lie symmetry infinitesimal lookup table for known first order ODE's

ODE class	Form	ξ	η
linear ode	$y' = f(x)y(x) + g(x)$	0	$e^{\int f dx}$
separable ode	$y' = f(x)\,g(y)$	$\frac{1}{f}$	0
quadrature ode	$y' = f(x)$	0	1
quadrature ode	$y' = g(y)$	1	0
homogeneous ODEs of Class A	$y' = f\left(\frac{y}{x}\right)$	x	y
homogeneous ODEs of Class C	$y' = (a + bx + cy)^{\frac{n}{m}}$	1	$-\frac{b}{c}$
homogeneous class D	$y' = \frac{y}{x} + g(x)\,F\left(\frac{y}{x}\right)$	x^2	xy
First order special form ID 1	$y' = g(x)\,e^{h(x)+by} + f(x)$	$\frac{e^{-\int bf(x)dx-h(x)}}{g(x)}$	$\frac{f(x)e^{-\int bf(x)dx-h(x)}}{g(x)}$
polynomial type ode	$y' = \frac{a_1x+b_1y+c_1}{a_2x+b_2y+c_2}$	$\frac{a_1b_2x-a_2b_1x-b_1c_2+b_2c_1}{a_1b_2-a_2b_1}$	$\frac{a_1b_2y-a_2b_1y-a_1c_2-a_2c_1}{a_1b_2-a_2b_1}$
Bernoulli ode	$y' = f(x)\,y + g(x)\,y^n$	0	$e^{-\int (n-1)f(x)dx}y^n$
Reduced Riccati	$y' = f_1(x)\,y + f_2(x)\,y^2$	0	$e^{-\int f_1 dx}$

The above table shows that

$$
\begin{aligned}
\xi(x,y) &= \frac{1}{-\frac{3}{2} - x} \\
\eta(x,y) &= 0
\end{aligned} \tag{A1}
$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x,y) \to (R,S)$ where (R,S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$
\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}
$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y}\right) S(x,y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the

canonical coordinates, where $S(R)$. Since $\eta = 0$ then in this special case

$$R = y$$

S is found from

$$\begin{aligned} S &= \int \frac{1}{\xi} dx \\ &= \int \frac{1}{\frac{1}{-\frac{3}{2} - x}} dx \end{aligned}$$

Which results in

$$S = -\frac{3}{2}x - \frac{1}{2}x^2$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x,y)S_y}{R_x + \omega(x,y)R_y} \tag{2}$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x,y)$ is the right hand side of the original ode given by

$$\omega(x,y) = -\frac{2x+3}{2(y-1)}$$

Evaluating all the partial derivatives gives

$$\begin{aligned} R_x &= 0 \\ R_y &= 1 \\ S_x &= -\frac{3}{2} - x \\ S_y &= 0 \end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = y - 1 \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = R - 1$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$S(R) = \frac{1}{2}R^2 - R + c_1 \tag{4}$$

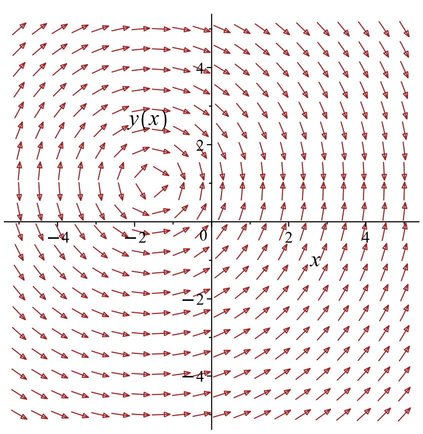
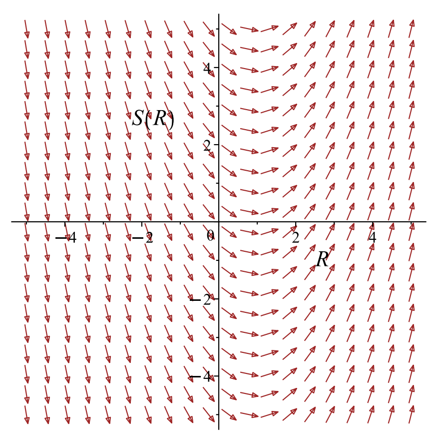
To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$-\frac{3}{2}x - \frac{1}{2}x^2 = \frac{y^2}{2} - y + c_1$$

Which simplifies to

$$-\frac{3}{2}x - \frac{1}{2}x^2 = \frac{y^2}{2} - y + c_1$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in x, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dx} = -\frac{2x+3}{2(y-1)}$ 	$R = y$ $S = -\frac{3}{2}x - \frac{1}{2}x^2$	$\frac{dS}{dR} = R - 1$ 

Summary

The solution(s) found are the following

$$-\frac{3}{2}x - \frac{1}{2}x^2 = \frac{y^2}{2} - y + c_1 \tag{1}$$

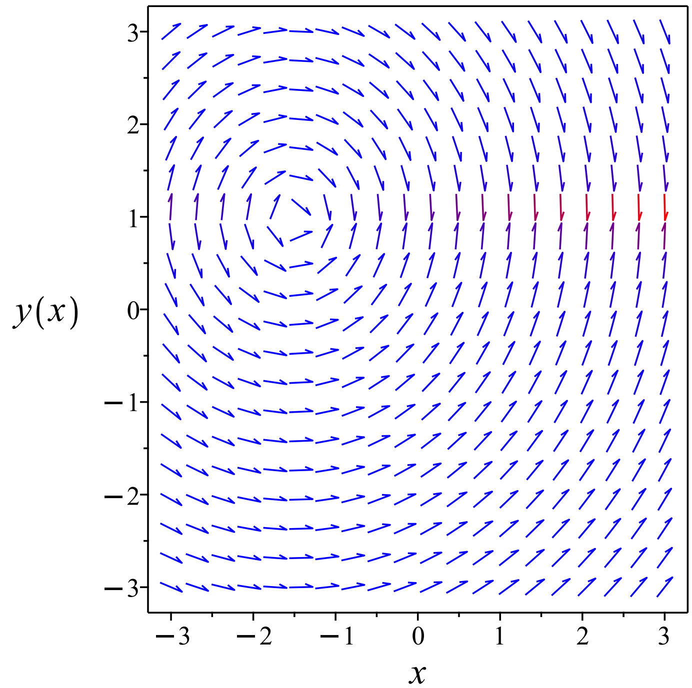


Figure 225: Slope field plot

Verification of solutions

$$-\frac{3}{2}x - \frac{1}{2}x^2 = \frac{y^2}{2} - y + c_1$$

Verified OK.

5.1.5 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x, y) + N(x, y)\frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x, y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x, y) = 0$$

Hence

$$\frac{\partial \phi}{\partial x} + \frac{\partial \phi}{\partial y}\frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$\frac{\partial \phi}{\partial x} = M$$
$$\frac{\partial \phi}{\partial y} = N$$

But since $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x, y)$ but at least we know now that we can do that since the condition $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x, y) \, \mathrm{d}x + N(x, y) \, \mathrm{d}y = 0 \tag{1A}$$

Therefore

$$(-2y + 2) \, \mathrm{d}y = (2x + 3) \, \mathrm{d}x$$
$$(-3 - 2x) \, \mathrm{d}x + (-2y + 2) \, \mathrm{d}y = 0 \tag{2A}$$

Comparing (1A) and (2A) shows that

$$M(x, y) = -3 - 2x$$
$$N(x, y) = -2y + 2$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Using result found above gives

$$\begin{aligned}\frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}(-3 - 2x) \\ &= 0\end{aligned}$$

And

$$\begin{aligned}\frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}(-2y + 2) \\ &= 0\end{aligned}$$

Since $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$, then the ODE is exact The following equations are now set up to solve for the function $\phi(x, y)$

$$\frac{\partial \phi}{\partial x} = M \tag{1}$$
$$\frac{\partial \phi}{\partial y} = N \tag{2}$$

Integrating (1) w.r.t. x gives

$$\int \frac{\partial \phi}{\partial x} \, \mathrm{d}x = \int M \, \mathrm{d}x$$
$$\int \frac{\partial \phi}{\partial x} \, \mathrm{d}x = \int -3 - 2x \, \mathrm{d}x$$
$$\phi = -x^2 - 3x + f(y) \tag{3}$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$\frac{\partial \phi}{\partial y} = 0 + f'(y) \tag{4}$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = -2y + 2$. Therefore equation (4) becomes

$$-2y + 2 = 0 + f'(y) \tag{5}$$

Solving equation (5) for $f'(y)$ gives

$$f'(y) = -2y + 2$$

Integrating the above w.r.t y gives

$$\int f'(y) \, \mathrm{d}y = \int (-2y + 2) \, \mathrm{d}y$$
$$f(y) = -y^2 + 2y + c_1$$

Where c_1 is constant of integration. Substituting result found above for $f(y)$ into equation (3) gives ϕ

$$\phi = -x^2 - y^2 - 3x + 2y + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = -x^2 - y^2 - 3x + 2y$$

Summary

The solution(s) found are the following

$$-y^2 - x^2 + 2y - 3x = c_1 \tag{1}$$

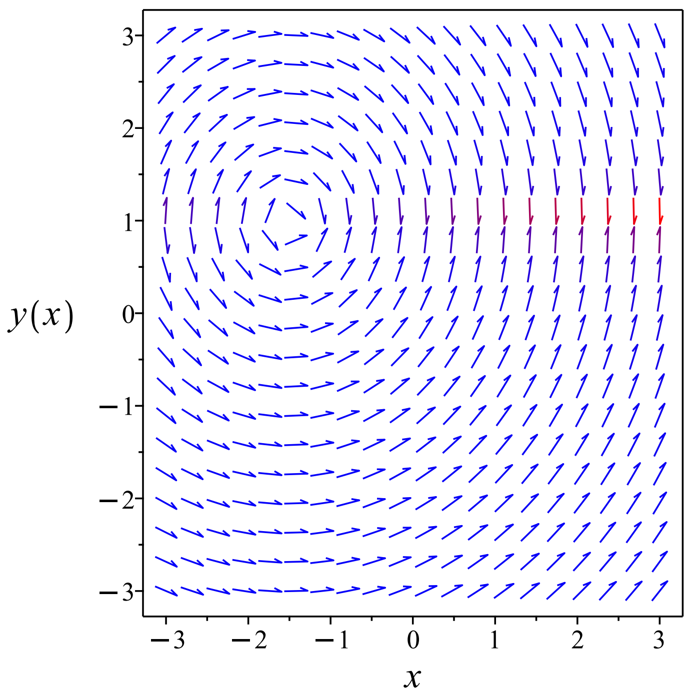


Figure 226: Slope field plot

Verification of solutions

$$-y^2 - x^2 + 2y - 3x = c_1$$

Verified OK.

5.1.6 Maple step by step solution

Let's solve

$(-2+2y)\,y' = -3-2x$

- Highest derivative means the order of the ODE is 1

  y'

- Integrate both sides with respect to x

  $\int (-2+2y)\,y' dx = \int (-3-2x)\,dx + c_1$

- Evaluate integral

  $y^2 - 2y = -x^2 + c_1 - 3x$

- Solve for y

  $\left\{y = 1-\sqrt{-x^2+c_1-3x+1}, y = 1+\sqrt{-x^2+c_1-3x+1}\right\}$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
<- separable successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 43

```
dsolve(3+2*x+(-2+2*y(x))*diff(y(x),x) = 0,y(x), singsol=all)
```

$$y(x) = 1-\sqrt{-x^2-c_1-3x+1}$$
$$y(x) = 1+\sqrt{-x^2-c_1-3x+1}$$

✓ Solution by Mathematica

Time used: 0.115 (sec). Leaf size: 51

```
DSolve[3+2*x+(-2+2*y[x])*y'[x] == 0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to 1 - \sqrt{-x^2 - 3x + 1 + 2c_1}$$
$$y(x) \to 1 + \sqrt{-x^2 - 3x + 1 + 2c_1}$$

5.2 problem 2

Internal problem ID [544]
Internal file name [`OUTPUT/544_Sunday_June_05_2022_01_44_02_AM_13505422/index.tex`]

Book: Elementary differential equations and boundary value problems, 10th ed., Boyce and DiPrima
Section: Section 2.6. Page 100
Problem number: 2.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"homogeneousTypeD2", "first_order_ode_lie_symmetry_calculated"**

Maple gives the following as the ode type

```
[[_homogeneous, `class A`], _rational, [_Abel, `2nd type`, `
   class A`]]
```

$$\boxed{4y + (2x - 2y)\, y' = -2x}$$

5.2.1 Solving as homogeneousTypeD2 ode

Using the change of variables $y = u(x)\, x$ on the above ode results in new ode in $u(x)$

$$4u(x)\, x + (2x - 2u(x)\, x)\, (u'(x)\, x + u(x)) = -2x$$

In canonical form the ODE is

$$\begin{aligned} u' &= F(x,u) \\ &= f(x)g(u) \\ &= -\frac{u^2 - 3u - 1}{(u-1)\, x} \end{aligned}$$

Where $f(x) = -\frac{1}{x}$ and $g(u) = \frac{u^2-3u-1}{u-1}$. Integrating both sides gives

$$\frac{1}{\frac{u^2-3u-1}{u-1}}\,du = -\frac{1}{x}\,dx$$

$$\int \frac{1}{\frac{u^2-3u-1}{u-1}}\,du = \int -\frac{1}{x}\,dx$$

$$\frac{\ln(u^2-3u-1)}{2} - \frac{\sqrt{13}\,\operatorname{arctanh}\left(\frac{(2u-3)\sqrt{13}}{13}\right)}{13} = -\ln(x) + c_2$$

The solution is

$$\frac{\ln\left(u(x)^2-3u(x)-1\right)}{2} - \frac{\sqrt{13}\,\operatorname{arctanh}\left(\frac{(2u(x)-3)\sqrt{13}}{13}\right)}{13} + \ln(x) - c_2 = 0$$

Replacing $u(x)$ in the above solution by $\frac{y}{x}$ results in the solution for y in implicit form

$$\frac{\ln\left(\frac{y^2}{x^2}-\frac{3y}{x}-1\right)}{2} - \frac{\sqrt{13}\,\operatorname{arctanh}\left(\frac{\left(\frac{2y}{x}-3\right)\sqrt{13}}{13}\right)}{13} + \ln(x) - c_2 = 0$$

$$\frac{\ln\left(\frac{y^2}{x^2}-\frac{3y}{x}-1\right)}{2} - \frac{\sqrt{13}\,\operatorname{arctanh}\left(\frac{(2y-3x)\sqrt{13}}{13x}\right)}{13} + \ln(x) - c_2 = 0$$

Summary

The solution(s) found are the following

$$\frac{\ln\left(\frac{y^2}{x^2}-\frac{3y}{x}-1\right)}{2} - \frac{\sqrt{13}\,\operatorname{arctanh}\left(\frac{(2y-3x)\sqrt{13}}{13x}\right)}{13} + \ln(x) - c_2 = 0 \tag{1}$$

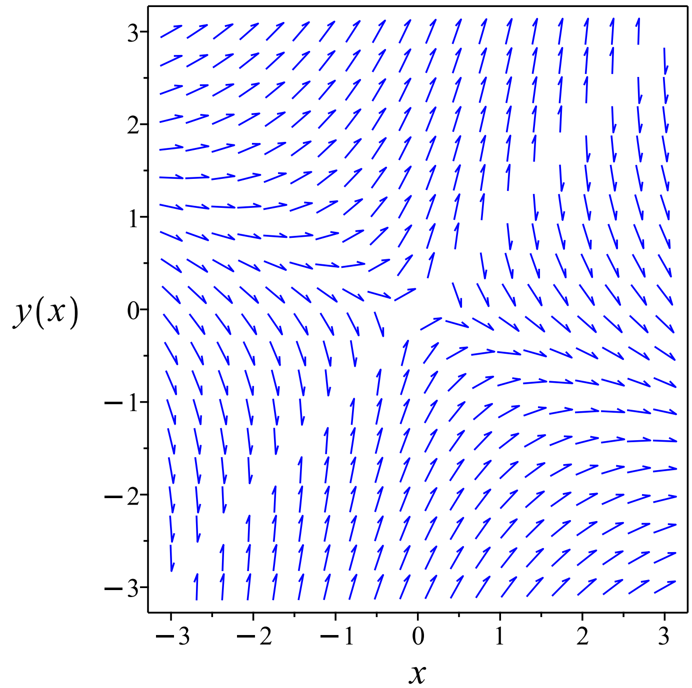


Figure 227: Slope field plot

Verification of solutions

$$\frac{\ln\left(\frac{y^2}{x^2}-\frac{3y}{x}-1\right)}{2}-\frac{\sqrt{13}\,\operatorname{arctanh}\left(\frac{(2y-3x)\sqrt{13}}{13x}\right)}{13}+\ln(x)-c_2=0$$

Verified OK.

5.2.2 Solving as first order ode lie symmetry calculated ode

Writing the ode as

$$y'=\frac{x+2y}{-x+y}$$
$$y'=\omega(x,y)$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x+\omega(\eta_y-\xi_x)-\omega^2\xi_y-\omega_x\xi-\omega_y\eta=0 \tag{A}$$

The type of this ode is not in the lookup table. To determine ξ,η then (A) is solved using ansatz. Making bivariate polynomials of degree 1 to use as anstaz gives

$$\xi=xa_2+ya_3+a_1 \tag{1E}$$

$$\eta=xb_2+yb_3+b_1 \tag{2E}$$

Where the unknown coefficients are

$$\{a_1, a_2, a_3, b_1, b_2, b_3\}$$

Substituting equations (1E,2E) and ω into (A) gives

$$\begin{aligned} & b_2 + \frac{(x+2y)(b_3 - a_2)}{-x+y} - \frac{(x+2y)^2 a_3}{(-x+y)^2} \\ & - \left(\frac{1}{-x+y} + \frac{x+2y}{(-x+y)^2}\right)(xa_2 + ya_3 + a_1) \\ & - \left(\frac{2}{-x+y} - \frac{x+2y}{(-x+y)^2}\right)(xb_2 + yb_3 + b_1) = 0 \end{aligned} \tag{5E}$$

Putting the above in normal form gives

$$\frac{x^2a_2 - x^2a_3 + 4x^2b_2 - x^2b_3 - 2xya_2 - 4xya_3 - 2xyb_2 + 2xyb_3 - 2y^2a_2 - 7y^2a_3 + y^2b_2 + 2y^2b_3 + 3xb_1 - 3}{(x-y)^2}$$
$$= 0$$

Setting the numerator to zero gives

$$\begin{aligned} & x^2a_2 - x^2a_3 + 4x^2b_2 - x^2b_3 - 2xya_2 - 4xya_3 - 2xyb_2 \\ & + 2xyb_3 - 2y^2a_2 - 7y^2a_3 + y^2b_2 + 2y^2b_3 + 3xb_1 - 3ya_1 = 0 \end{aligned} \tag{6E}$$

Looking at the above PDE shows the following are all the terms with $\{x, y\}$ in them.

$$\{x, y\}$$

The following substitution is now made to be able to collect on all terms with $\{x, y\}$ in them

$$\{x = v_1, y = v_2\}$$

The above PDE (6E) now becomes

$$\begin{aligned} & a_2v_1^2 - 2a_2v_1v_2 - 2a_2v_2^2 - a_3v_1^2 - 4a_3v_1v_2 - 7a_3v_2^2 + 4b_2v_1^2 \\ & - 2b_2v_1v_2 + b_2v_2^2 - b_3v_1^2 + 2b_3v_1v_2 + 2b_3v_2^2 - 3a_1v_2 + 3b_1v_1 = 0 \end{aligned} \tag{7E}$$

Collecting the above on the terms v_i introduced, and these are

$$\{v_1, v_2\}$$

Equation (7E) now becomes

$$(a_2 - a_3 + 4b_2 - b_3) v_1^2 + (-2a_2 - 4a_3 - 2b_2 + 2b_3) v_1 v_2 + 3b_1 v_1 + (-2a_2 - 7a_3 + b_2 + 2b_3) v_2^2 - 3a_1 v_2 = 0 \tag{8E}$$

Setting each coefficients in (8E) to zero gives the following equations to solve

$$\begin{aligned} -3a_1 &= 0 \\ 3b_1 &= 0 \\ -2a_2 - 7a_3 + b_2 + 2b_3 &= 0 \\ -2a_2 - 4a_3 - 2b_2 + 2b_3 &= 0 \\ a_2 - a_3 + 4b_2 - b_3 &= 0 \end{aligned}$$

Solving the above equations for the unknowns gives

$$\begin{aligned} a_1 &= 0 \\ a_2 &= -3a_3 + b_3 \\ a_3 &= a_3 \\ b_1 &= 0 \\ b_2 &= a_3 \\ b_3 &= b_3 \end{aligned}$$

Substituting the above solution in the anstaz (1E,2E) (using 1 as arbitrary value for any unknown in the RHS) gives

$$\begin{aligned} \xi &= x \\ \eta &= y \end{aligned}$$

Shifting is now applied to make $\xi = 0$ in order to simplify the rest of the computation

$$\begin{aligned} \eta &= \eta - \omega(x, y)\, \xi \\ &= y - \left(\frac{x + 2y}{-x + y}\right)(x) \\ &= \frac{x^2 + 3yx - y^2}{x - y} \\ \xi &= 0 \end{aligned}$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x, y) \to (R, S)$ where (R, S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y}\right) S(x, y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$R = x$$

S is found from

$$\begin{aligned} S &= \int \frac{1}{\eta} dy \\ &= \int \frac{1}{\frac{x^2+3yx-y^2}{x-y}} dy \end{aligned}$$

Which results in

$$S = \frac{\ln\left(-x^2 - 3yx + y^2\right)}{2} - \frac{\sqrt{13}\, \operatorname{arctanh}\left(\frac{(-3x+2y)\sqrt{13}}{13x}\right)}{13}$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x, y) S_y}{R_x + \omega(x, y) R_y} \tag{2}$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x, y)$ is the right hand side of the original ode given by

$$\omega(x, y) = \frac{x + 2y}{-x + y}$$

Evaluating all the partial derivatives gives

$$\begin{aligned} R_x &= 1 \\ R_y &= 0 \\ S_x &= \frac{x + 2y}{x^2 + 3yx - y^2} \\ S_y &= \frac{x - y}{x^2 + 3yx - y^2} \end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = 0 \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = 0$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$S(R) = c_1 \tag{4}$$

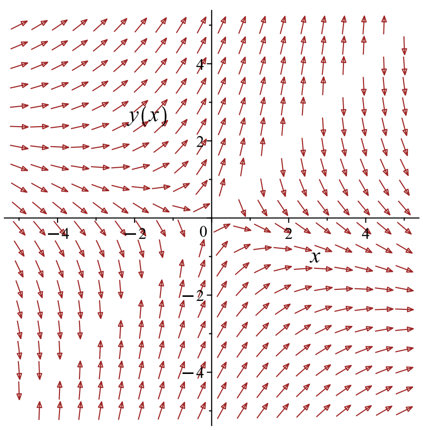
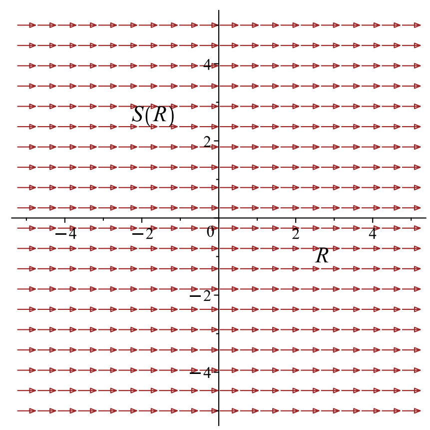
To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$\frac{\ln (y^2 - 3yx - x^2)}{2} + \frac{\sqrt{13}\, \operatorname{arctanh}\left(\frac{(-2y+3x)\sqrt{13}}{13x}\right)}{13} = c_1$$

Which simplifies to

$$\frac{\ln (y^2 - 3yx - x^2)}{2} + \frac{\sqrt{13}\, \operatorname{arctanh}\left(\frac{(-2y+3x)\sqrt{13}}{13x}\right)}{13} = c_1$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in x, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dx} = \frac{x+2y}{-x+y}$ 	$R = x$ $S = \frac{\ln(-x^2 - 3yx + y^2)}{2}$	$\frac{dS}{dR} = 0$ 

Summary
The solution(s) found are the following

$$\frac{\ln(y^2 - 3yx - x^2)}{2} + \frac{\sqrt{13}\, \operatorname{arctanh}\left(\frac{(-2y+3x)\sqrt{13}}{13x}\right)}{13} = c_1 \tag{1}$$

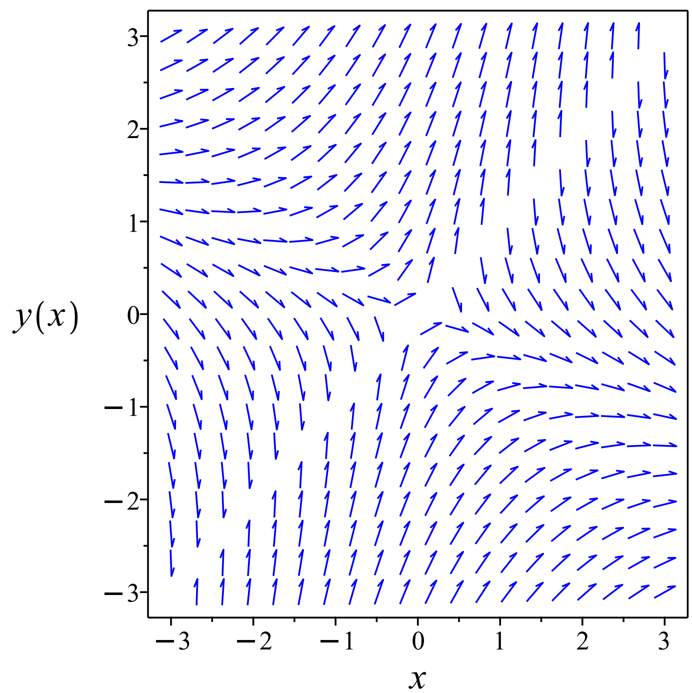


Figure 228: Slope field plot

Verification of solutions

$$\frac{\ln \left(y^{2}-3yx-x^{2}\right)}{2}+\frac{\sqrt{13}\, \operatorname{arctanh}\left(\frac{(-2y+3x)\sqrt{13}}{13x}\right)}{13}=c_{1}$$

Verified OK.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
trying homogeneous types:
trying homogeneous D
<- homogeneous successful`
```

✓ Solution by Maple

Time used: 0.14 (sec). Leaf size: 55

```
dsolve(2*x+4*y(x)+(2*x-2*y(x))*diff(y(x),x) = 0,y(x), singsol=all)
```

$$-\frac{\ln\left(\frac{-x^2-3xy(x)+y(x)^2}{x^2}\right)}{2}+\frac{\sqrt{13}\,\operatorname{arctanh}\left(\frac{(2y(x)-3x)\sqrt{13}}{13x}\right)}{13}-\ln(x)-c_1=0$$

✓ Solution by Mathematica

Time used: 0.065 (sec). Leaf size: 63

```
DSolve[2*x+4*y[x]+(2*x-2*y[x])*y'[x]== 0,y[x],x,IncludeSingularSolutions -> True]
```

$$\text{Solve}\left[\frac{1}{26}\left(\left(13+\sqrt{13}\right)\log\left(-\frac{2y(x)}{x}+\sqrt{13}+3\right)-\left(\sqrt{13}-13\right)\log\left(\frac{2y(x)}{x}+\sqrt{13}-3\right)\right)=-\log(x)+c_1,y(x)\right]$$

5.3 problem 3

Internal problem ID [545]
Internal file name [`OUTPUT/545_Sunday_June_05_2022_01_44_04_AM_28395296/index.tex`]

Book: Elementary differential equations and boundary value problems, 10th ed., Boyce and DiPrima
Section: Section 2.6. Page 100
Problem number: 3.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"exact", "differentialType"**

Maple gives the following as the ode type

```
[_exact, _rational]
```

$$\boxed{-2yx + (3 - x^2 + 6y^2)\, y' = -3x^2 - 2}$$

5.3.1 Solving as differentialType ode

Writing the ode as

$$y' = \frac{-2 - 3x^2 + 2yx}{3 - x^2 + 6y^2} \tag{1}$$

Which becomes

$$(-6y^2 - 3)\, dy = (-x^2)\, dy + (3x^2 - 2yx + 2)\, dx \tag{2}$$

But the RHS is complete differential because

$$(-x^2)\, dy + (3x^2 - 2yx + 2)\, dx = d(x^3 - y\, x^2 + 2x)$$

Hence (2) becomes

$$(-6y^2 - 3)\, dy = d(x^3 - y\, x^2 + 2x)$$

Integrating both sides gives gives these solutions

$$y = \frac{\left(-54x^3 - 54c_1 - 108x + 6\sqrt{75x^6 + 162c_1x^3 + 378x^4 + 81c_1^2 + 324c_1x + 162x^2 + 162}\right)^{\frac{1}{3}}}{6} - \frac{}{\left(-54x^3 -\right.}$$

$$y = -\frac{\left(-54x^3 - 54c_1 - 108x + 6\sqrt{75x^6 + 162c_1x^3 + 378x^4 + 81c_1^2 + 324c_1x + 162x^2 + 162}\right)^{\frac{1}{3}}}{12} + \frac{}{\left(-54x^3\right.}$$

$$y = -\frac{\left(-54x^3 - 54c_1 - 108x + 6\sqrt{75x^6 + 162c_1x^3 + 378x^4 + 81c_1^2 + 324c_1x + 162x^2 + 162}\right)^{\frac{1}{3}}}{12} + \frac{}{\left(-54x^3\right.}$$

Summary

The solution(s) found are the following

$$y = \frac{\left(-54x^3-54c_1-108x+6\sqrt{75x^6+162c_1x^3+378x^4+81c_1^2+324c_1x+162x^2+162}\right)^{\frac{1}{3}}}{6} - \frac{6\left(-\frac{x^2}{6}+\frac{1}{2}\right)}{\left(-54x^3-54c_1-108x+6\sqrt{75x^6+162c_1x^3+378x^4+81c_1^2+324c_1x+162x^2+162}\right)^{\frac{1}{3}}} + c_1 \tag{1}$$

$$y = -\frac{\left(-54x^3-54c_1-108x+6\sqrt{75x^6+162c_1x^3+378x^4+81c_1^2+324c_1x+162x^2+162}\right)^{\frac{1}{3}}}{12} + \frac{-\frac{x^2}{2}+\frac{3}{2}}{\left(-54x^3-54c_1-108x+6\sqrt{75x^6+162c_1x^3+378x^4+81c_1^2+324c_1x+162x^2+162}\right)^{\frac{1}{3}}} + \frac{i\sqrt{3}\left(\frac{\left(-54x^3-54c_1-108x+6\sqrt{75x^6+162c_1x^3+378x^4+81c_1^2+324c_1x+162x^2+162}\right)^{\frac{1}{3}}}{6} + \frac{-x^2+3\cdots}{\left(-54x^3-54c_1-108x+6\sqrt{75x^6+162c_1x^3+3\cdots}\right.}\right.}{2} + c_1 \tag{2}$$

$$y = -\frac{\left(-54x^3-54c_1-108x+6\sqrt{75x^6+162c_1x^3+378x^4+81c_1^2+324c_1x+162x^2+162}\right)^{\frac{1}{3}}}{12} + \frac{-\frac{x^2}{2}+\frac{3}{2}}{\left(-54x^3-54c_1-108x+6\sqrt{75x^6+162c_1x^3+378x^4+81c_1^2+324c_1x+162x^2+162}\right)^{\frac{1}{3}}} - \frac{i\sqrt{3}\left(\frac{\left(-54x^3-54c_1-108x+6\sqrt{75x^6+162c_1x^3+378x^4+81c_1^2+324c_1x+162x^2+162}\right)^{\frac{1}{3}}}{6} + \frac{-x^2+3\cdots}{\left(-54x^3-54c_1-108x+6\sqrt{75x^6+162c_1x^3+3\cdots}\right.}\right.}{2} + c_1 \tag{3}$$

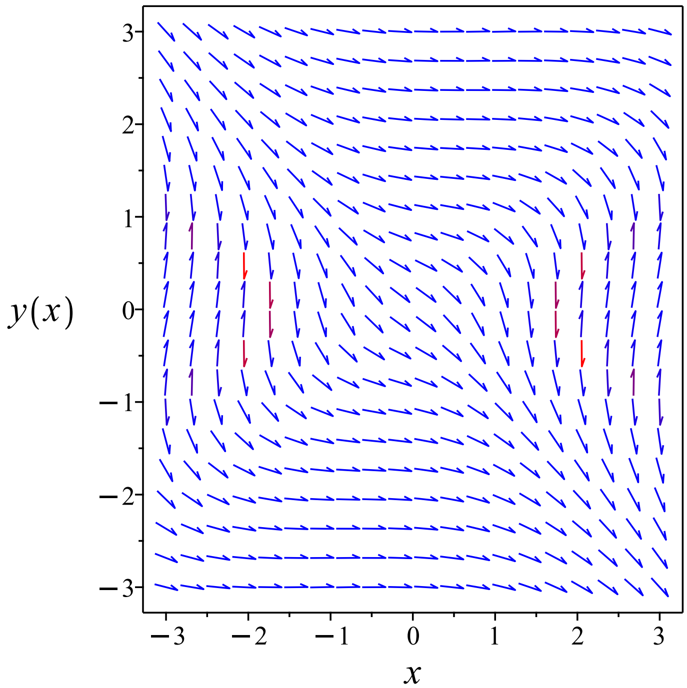


Figure 229: Slope field plot

Verification of solutions

$$y = \frac{\left(-54x^3 - 54c_1 - 108x + 6\sqrt{75x^6 + 162c_1x^3 + 378x^4 + 81c_1^2 + 324c_1x + 162x^2 + 162}\right)^{\frac{1}{3}}}{6} - \frac{6\left(-\frac{x^2}{6} + \frac{1}{2}\right)}{\left(-54x^3 - 54c_1 - 108x + 6\sqrt{75x^6 + 162c_1x^3 + 378x^4 + 81c_1^2 + 324c_1x + 162x^2 + 162}\right)^{\frac{1}{3}}} + c_1$$

Verified OK.

$$y = -\frac{\left(-54x^3 - 54c_1 - 108x + 6\sqrt{75x^6 + 162c_1x^3 + 378x^4 + 81c_1^2 + 324c_1x + 162x^2 + 162}\right)^{\frac{1}{3}}}{12} + \frac{-\frac{x^2}{2} + \frac{3}{2}}{\left(-54x^3 - 54c_1 - 108x + 6\sqrt{75x^6 + 162c_1x^3 + 378x^4 + 81c_1^2 + 324c_1x + 162x^2 + 162}\right)^{\frac{1}{3}}} + \frac{i\sqrt{3}\left(\frac{\left(-54x^3 - 54c_1 - 108x + 6\sqrt{75x^6 + 162c_1x^3 + 378x^4 + 81c_1^2 + 324c_1x + 162x^2 + 162}\right)^{\frac{1}{3}}}{6} + \frac{-x^2 + 3}{\left(-54x^3 - 54c_1 - 108x + 6\sqrt{75x^6 + 162c_1x^3 + 3}\right.}\right.}{2} + c_1$$

Verified OK.

$$y = -\frac{\left(-54x^3 - 54c_1 - 108x + 6\sqrt{75x^6 + 162c_1x^3 + 378x^4 + 81c_1^2 + 324c_1x + 162x^2 + 162}\right)^{\frac{1}{3}}}{12} + \frac{-\frac{x^2}{2} + \frac{3}{2}}{\left(-54x^3 - 54c_1 - 108x + 6\sqrt{75x^6 + 162c_1x^3 + 378x^4 + 81c_1^2 + 324c_1x + 162x^2 + 162}\right)^{\frac{1}{3}}} - \frac{i\sqrt{3}\left(\frac{\left(-54x^3 - 54c_1 - 108x + 6\sqrt{75x^6 + 162c_1x^3 + 378x^4 + 81c_1^2 + 324c_1x + 162x^2 + 162}\right)^{\frac{1}{3}}}{6} + \frac{-x^2 + 3}{\left(-54x^3 - 54c_1 - 108x + 6\sqrt{75x^6 + 162c_1x^3 + 3}\right.}\right.}{2} + c_1$$

Verified OK.

5.3.2 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x,y)+N(x,y)\frac{dy}{dx}=0 \tag{A}$$

We assume there exists a function $\phi(x,y)=c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y)=0$$

Hence

$$\frac{\partial\phi}{\partial x}+\frac{\partial\phi}{\partial y}\frac{dy}{dx}=0 \tag{B}$$

Comparing (A,B) shows that

$$\begin{aligned}\frac{\partial\phi}{\partial x}&=M\\ \frac{\partial\phi}{\partial y}&=N\end{aligned}$$

But since $\frac{\partial^2\phi}{\partial x\partial y}=\frac{\partial^2\phi}{\partial y\partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y}=\frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x,y)$ but at least we know now that we can do that since the condition $\frac{\partial^2\phi}{\partial x\partial y}=\frac{\partial^2\phi}{\partial y\partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x,y)\,\mathrm{d}x+N(x,y)\,\mathrm{d}y=0 \tag{1A}$$

Therefore

$$\begin{aligned}\left(-x^2+6y^2+3\right)\mathrm{d}y&=\left(-3x^2+2yx-2\right)\mathrm{d}x\\ \left(3x^2-2yx+2\right)\mathrm{d}x+\left(-x^2+6y^2+3\right)\mathrm{d}y&=0 \qquad (2A)\end{aligned}$$

Comparing (1A) and (2A) shows that

$$\begin{aligned}M(x,y)&=3x^2-2yx+2\\ N(x,y)&=-x^2+6y^2+3\end{aligned}$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Using result found above gives

$$\begin{aligned}\frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}(3x^2 - 2yx + 2)\\ &= -2x\end{aligned}$$

And

$$\begin{aligned}\frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}(-x^2 + 6y^2 + 3)\\ &= -2x\end{aligned}$$

Since $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$, then the ODE is exact The following equations are now set up to solve for the function $\phi(x, y)$

$$\frac{\partial \phi}{\partial x} = M \tag{1}$$

$$\frac{\partial \phi}{\partial y} = N \tag{2}$$

Integrating (1) w.r.t. x gives

$$\begin{aligned}\int \frac{\partial \phi}{\partial x}\, \mathrm{d}x &= \int M\, \mathrm{d}x\\ \int \frac{\partial \phi}{\partial x}\, \mathrm{d}x &= \int 3x^2 - 2yx + 2\, \mathrm{d}x\end{aligned}$$

$$\phi = x^3 - y\, x^2 + 2x + f(y) \tag{3}$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$\frac{\partial \phi}{\partial y} = -x^2 + f'(y) \tag{4}$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = -x^2 + 6y^2 + 3$. Therefore equation (4) becomes

$$-x^2 + 6y^2 + 3 = -x^2 + f'(y) \tag{5}$$

Solving equation (5) for $f'(y)$ gives

$$f'(y) = 6y^2 + 3$$

Integrating the above w.r.t y gives

$$\int f'(y)\, \mathrm{d}y = \int (6y^2 + 3)\, \mathrm{d}y$$
$$f(y) = 2y^3 + 3y + c_1$$

Where c_1 is constant of integration. Substituting result found above for $f(y)$ into equation (3) gives ϕ

$$\phi = x^3 - y\,x^2 + 2y^3 + 2x + 3y + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = x^3 - y\,x^2 + 2y^3 + 2x + 3y$$

Summary
The solution(s) found are the following

$$x^3 - x^2y + 2y^3 + 2x + 3y = c_1 \tag{1}$$

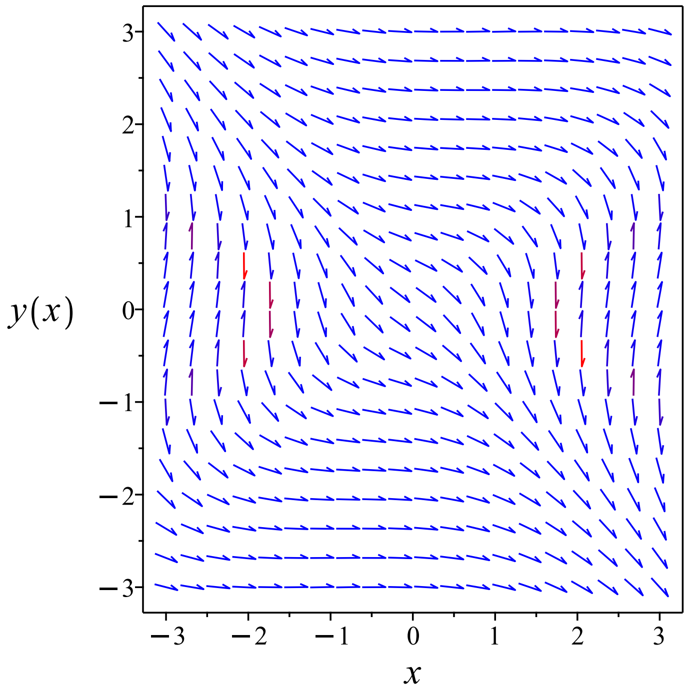


Figure 230: Slope field plot

Verification of solutions

$$x^3 - x^2y + 2y^3 + 2x + 3y = c_1$$

Verified OK.

5.3.3 Maple step by step solution

Let's solve

$-2yx + (3 - x^2 + 6y^2)\, y' = -3x^2 - 2$

- Highest derivative means the order of the ODE is 1

  y'

□ Check if ODE is exact

  ◦ ODE is exact if the lhs is the total derivative of a C^2 function

    $F'(x, y) = 0$

  ◦ Compute derivative of lhs

    $F'(x, y) + \left(\frac{\partial}{\partial y} F(x, y)\right) y' = 0$

- Evaluate derivatives

  $-2x = -2x$
- Condition met, ODE is exact

- Exact ODE implies solution will be of this form

  $\left[F(x,y) = c_1, M(x,y) = F'(x,y), N(x,y) = \frac{\partial}{\partial y}F(x,y)\right]$
- Solve for $F(x,y)$ by integrating $M(x,y)$ with respect to x

  $F(x,y) = \int (3x^2 - 2yx + 2)\,dx + f_1(y)$
- Evaluate integral

  $F(x,y) = x^3 - y\,x^2 + 2x + f_1(y)$
- Take derivative of $F(x,y)$ with respect to y

  $N(x,y) = \frac{\partial}{\partial y}F(x,y)$
- Compute derivative

  $-x^2 + 6y^2 + 3 = -x^2 + \frac{d}{dy}f_1(y)$
- Isolate for $\frac{d}{dy}f_1(y)$

  $\frac{d}{dy}f_1(y) = 6y^2 + 3$
- Solve for $f_1(y)$

  $f_1(y) = 2y^3 + 3y$
- Substitute $f_1(y)$ into equation for $F(x,y)$

  $F(x,y) = x^3 - y\,x^2 + 2y^3 + 2x + 3y$
- Substitute $F(x,y)$ into the solution of the ODE

  $x^3 - y\,x^2 + 2y^3 + 2x + 3y = c_1$
- Solve for y

  $$\left\{y = \frac{\left(-54x^3+54c_1-108x+6\sqrt{75x^6-162c_1x^3+378x^4+81c_1^2-324c_1x+162x^2+162}\right)^{\frac{1}{3}}}{6} - \frac{6(-}{\left(-54x^3+54c_1-108x+6\sqrt{75x^6-162c}\right.}\right.$$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
trying homogeneous types:
trying Chini
differential order: 1; looking for linear symmetries
trying exact
<- exact successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 372

```
dsolve(2+3*x^2-2*x*y(x)+(3-x^2+6*y(x)^2)*diff(y(x),x) = 0,y(x), singsol=all)
```

$$y(x) = \frac{\left(-54x^3-54c_1-108x+6\sqrt{75x^6+162c_1x^3+378x^4+81c_1^2+324c_1x+162x^2+162}\right)^{\frac{2}{3}}+6x^2-18}{6\left(-54x^3-54c_1-108x+6\sqrt{75x^6+162c_1x^3+378x^4+81c_1^2+324c_1x+162x^2+162}\right)^{\frac{1}{3}}}$$

$$y(x) = \frac{\left(-1-i\sqrt{3}\right)\left(-54x^3-54c_1-108x+6\sqrt{75x^6+162c_1x^3+378x^4+81c_1^2+324c_1x+162x^2+162}\right)^{\frac{1}{3}}}{12} + \frac{\left(x^2-3\right)\left(i\sqrt{3}-1\right)}{2\left(-54x^3-54c_1-108x+6\sqrt{75x^6+162c_1x^3+378x^4+81c_1^2+324c_1x+162x^2+162}\right)^{\frac{1}{3}}}$$

$$y(x) = \frac{\frac{\left(i\sqrt{3}-1\right)\left(-54x^3-54c_1-108x+6\sqrt{75x^6+162c_1x^3+378x^4+81c_1^2+324c_1x+162x^2+162}\right)^{\frac{2}{3}}}{6}+\left(-1-i\sqrt{3}\right)\left(x^2-3\right)}{2\left(-54x^3-54c_1-108x+6\sqrt{75x^6+162c_1x^3+378x^4+81c_1^2+324c_1x+162x^2+162}\right)^{\frac{1}{3}}}$$

✓ Solution by Mathematica

Time used: 8.724 (sec). Leaf size: 421

```
DSolve[2+3*x^2-2*x*y[x]+(3-x^2+6*y[x]^2)*y'[x] == 0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to -\frac{x^2-3}{\sqrt[3]{6}\sqrt[3]{9x^3+\sqrt{3}\sqrt{-2\left(x^2-3\right)^3+27\left(x^3+2x+c_1\right)^2}+18x+9c_1}} - \frac{\sqrt[3]{9x^3+\sqrt{3}\sqrt{-2\left(x^2-3\right)^3+27\left(x^3+2x+c_1\right)^2}+18x+9c_1}}{6^{2/3}}$$

$$y(x) \to \frac{\sqrt[3]{6}\left(1+i\sqrt{3}\right)\left(x^2-3\right)+\left(1-i\sqrt{3}\right)\left(9x^3+\sqrt{3}\sqrt{-2\left(x^2-3\right)^3+27\left(x^3+2x+c_1\right)^2}+18x+9c_1\right)^{2/3}}{2\ 6^{2/3}\sqrt[3]{9x^3+\sqrt{3}\sqrt{-2\left(x^2-3\right)^3+27\left(x^3+2x+c_1\right)^2}+18x+9c_1}}$$

$$y(x) \to \frac{\sqrt[3]{6}\left(1-i\sqrt{3}\right)\left(x^2-3\right)+\left(1+i\sqrt{3}\right)\left(9x^3+\sqrt{3}\sqrt{-2\left(x^2-3\right)^3+27\left(x^3+2x+c_1\right)^2}+18x+9c_1\right)^{2/3}}{2\ 6^{2/3}\sqrt[3]{9x^3+\sqrt{3}\sqrt{-2\left(x^2-3\right)^3+27\left(x^3+2x+c_1\right)^2}+18x+9c_1}}$$

5.4 problem 4

Internal problem ID [546]
Internal file name [`OUTPUT/546_Sunday_June_05_2022_01_44_05_AM_3993535/index.tex`]

Book: Elementary differential equations and boundary value problems, 10th ed., Boyce and DiPrima
Section: Section 2.6. Page 100
Problem number: 4.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"exact", "linear", "separable", "differentialType", "homogeneousTypeD2", "first_order_ode_lie_symmetry_lookup"**

Maple gives the following as the ode type

`[_separable]`

$$\boxed{2y + 2xy^2 + (2x + 2x^2y)\, y' = 0}$$

5.4.1 Solving as separable ode

In canonical form the ODE is

$$\begin{aligned} y' &= F(x,y) \\ &= f(x)g(y) \\ &= -\frac{y}{x} \end{aligned}$$

Where $f(x) = -\frac{1}{x}$ and $g(y) = y$. Integrating both sides gives

$$
\begin{aligned}
\frac{1}{y}\,dy &= -\frac{1}{x}\,dx \\
\int \frac{1}{y}\,dy &= \int -\frac{1}{x}\,dx \\
\ln(y) &= -\ln(x) + c_1 \\
y &= e^{-\ln(x)+c_1} \\
&= \frac{c_1}{x}
\end{aligned}
$$

Summary

The solution(s) found are the following

$$
y = \frac{c_1}{x} \tag{1}
$$

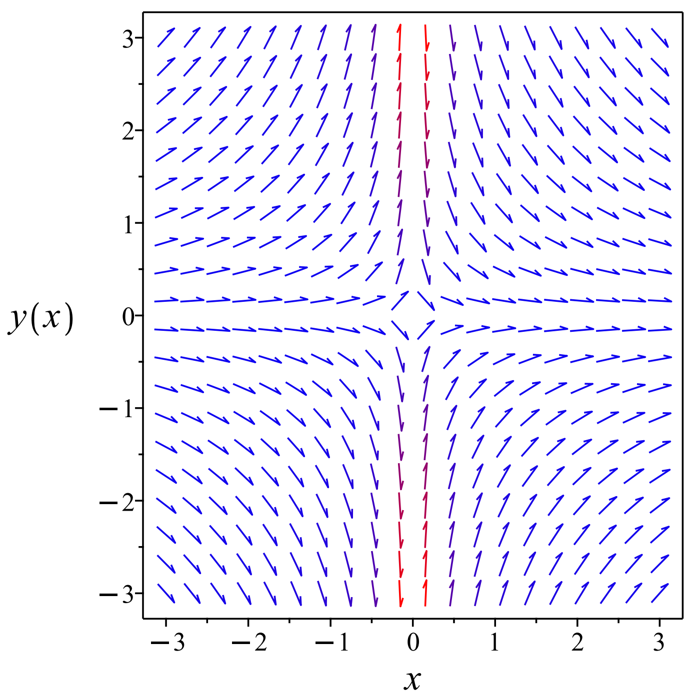


Figure 231: Slope field plot

Verification of solutions

$$
y = \frac{c_1}{x}
$$

Verified OK.

5.4.2 Solving as linear ode

Entering Linear first order ODE solver. In canonical form a linear first order is

$$y' + p(x)y = q(x)$$

Where here

$$\begin{aligned} p(x) &= \frac{1}{x} \\ q(x) &= 0 \end{aligned}$$

Hence the ode is

$$y' + \frac{y}{x} = 0$$

The integrating factor μ is

$$\begin{aligned} \mu &= e^{\int \frac{1}{x}dx} \\ &= x \end{aligned}$$

The ode becomes

$$\begin{aligned} \frac{d}{dx}\mu y &= 0 \\ \frac{d}{dx}(yx) &= 0 \end{aligned}$$

Integrating gives

$$yx = c_1$$

Dividing both sides by the integrating factor $\mu = x$ results in

$$y = \frac{c_1}{x}$$

Summary
The solution(s) found are the following

$$y = \frac{c_1}{x} \tag{1}$$

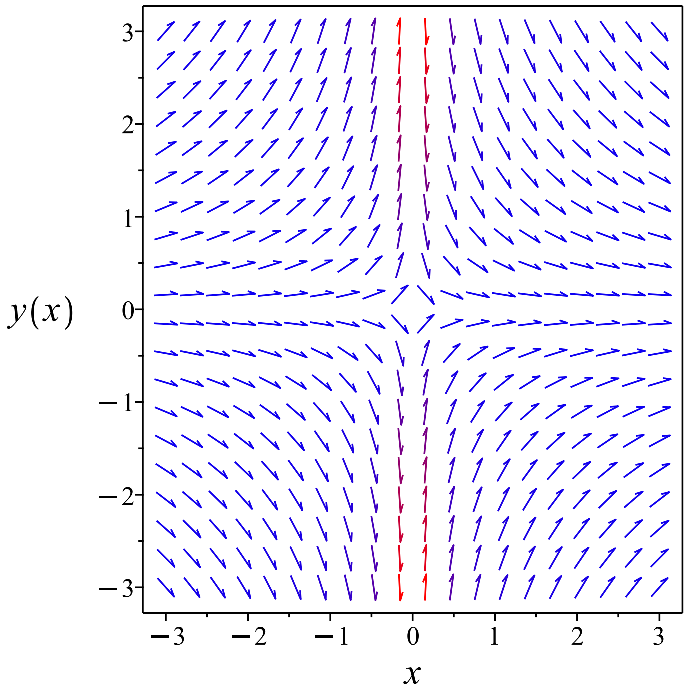


Figure 232: Slope field plot

Verification of solutions

$$y = \frac{c_1}{x}$$

Verified OK.

5.4.3 Solving as homogeneousTypeD2 ode

Using the change of variables $y = u(x)\,x$ on the above ode results in new ode in $u(x)$

$$2u(x)\,x + 2x^3u(x)^2 + \left(2x + 2x^3u(x)\right)\left(u'(x)\,x + u(x)\right) = 0$$

In canonical form the ODE is

$$\begin{aligned}
u' &= F(x,u) \\
&= f(x)g(u) \\
&= -\frac{2u}{x}
\end{aligned}$$

Where $f(x) = -\frac{2}{x}$ and $g(u) = u$. Integrating both sides gives

$$
\begin{aligned}
\frac{1}{u}\, du &= -\frac{2}{x}\, dx \\
\int \frac{1}{u}\, du &= \int -\frac{2}{x}\, dx \\
\ln(u) &= -2\ln(x) + c_2 \\
u &= \mathrm{e}^{-2\ln(x)+c_2} \\
&= \frac{c_2}{x^2}
\end{aligned}
$$

Therefore the solution y is

$$
\begin{aligned}
y &= ux \\
&= \frac{c_2}{x}
\end{aligned}
$$

Summary
The solution(s) found are the following

$$
y = \frac{c_2}{x} \tag{1}
$$

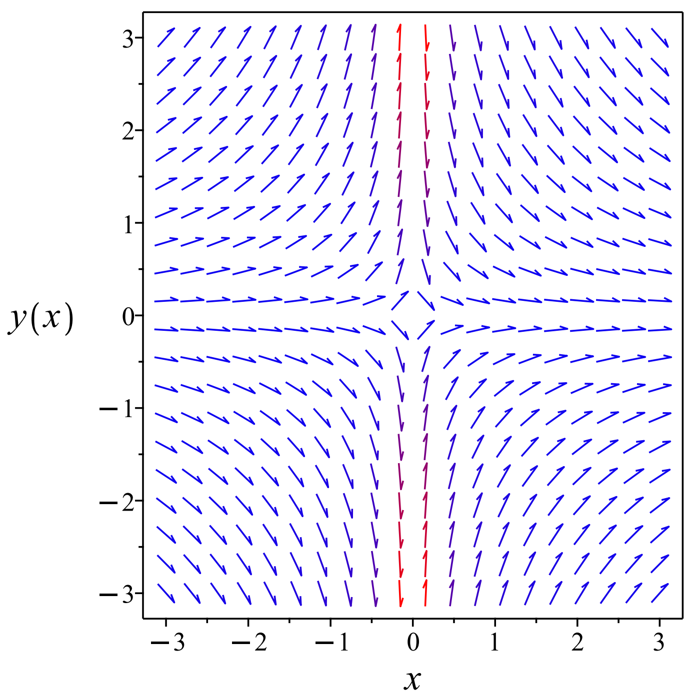


Figure 233: Slope field plot

Verification of solutions

$$y = \frac{c_2}{x}$$

Verified OK.

5.4.4 Solving as differentialType ode

Writing the ode as

$$y' = \frac{-2y - 2xy^2}{2x + 2x^2 y} \tag{1}$$

Which becomes

$$0 = (-x)\,dy + (-y)\,dx \tag{2}$$

But the RHS is complete differential because

$$(-x)\,dy + (-y)\,dx = d(-yx)$$

Hence (2) becomes

$$0 = d(-yx)$$

Integrating both sides gives gives these solutions

$$y = \frac{c_1}{x} + c_1$$

Summary
The solution(s) found are the following

$$y = \frac{c_1}{x} + c_1 \tag{1}$$

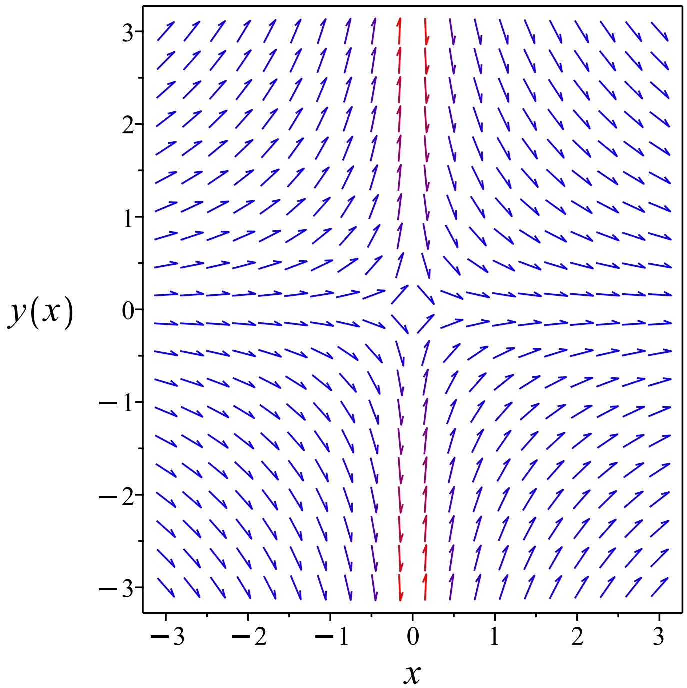


Figure 234: Slope field plot

Verification of solutions

$$y = \frac{c_1}{x} + c_1$$

Verified OK.

5.4.5 Solving as first order ode lie symmetry lookup ode

Writing the ode as

$$y' = -\frac{y}{x}$$
$$y' = \omega(x, y)$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x + \omega(\eta_y - \xi_x) - \omega^2 \xi_y - \omega_x \xi - \omega_y \eta = 0 \tag{A}$$

The type of this ode is known. It is of type `linear`. Therefore we do not need to solve the PDE (A), and can just use the lookup table shown below to find ξ, η

Table 238: Lie symmetry infinitesimal lookup table for known first order ODE's

ODE class	Form	ξ	η
linear ode	$y' = f(x)y(x) + g(x)$	0	$e^{\int f dx}$
separable ode	$y' = f(x)\, g(y)$	$\frac{1}{f}$	0
quadrature ode	$y' = f(x)$	0	1
quadrature ode	$y' = g(y)$	1	0
homogeneous ODEs of Class A	$y' = f\left(\frac{y}{x}\right)$	x	y
homogeneous ODEs of Class C	$y' = (a + bx + cy)^{\frac{n}{m}}$	1	$-\frac{b}{c}$
homogeneous class D	$y' = \frac{y}{x} + g(x)\, F\left(\frac{y}{x}\right)$	x^2	xy
First order special form ID 1	$y' = g(x)\, e^{h(x)+by} + f(x)$	$\frac{e^{-\int bf(x)dx - h(x)}}{g(x)}$	$\frac{f(x)e^{-\int bf(x)dx - h(x)}}{g(x)}$
polynomial type ode	$y' = \frac{a_1x+b_1y+c_1}{a_2x+b_2y+c_2}$	$\frac{a_1b_2x - a_2b_1x - b_1c_2 + b_2c_1}{a_1b_2 - a_2b_1}$	$\frac{a_1b_2y - a_2b_1y - a_1c_2 - a_2c_1}{a_1b_2 - a_2b_1}$
Bernoulli ode	$y' = f(x)\, y + g(x)\, y^n$	0	$e^{-\int (n-1)f(x)dx} y^n$
Reduced Riccati	$y' = f_1(x)\, y + f_2(x)\, y^2$	0	$e^{-\int f_1 dx}$

The above table shows that

$$\begin{aligned} \xi(x,y) &= 0 \\ \eta(x,y) &= \frac{1}{x} \end{aligned} \tag{A1}$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x,y) \to (R,S)$ where (R,S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y}\right) S(x,y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the

canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$R = x$$

S is found from

$$\begin{aligned} S &= \int \frac{1}{\eta} dy \\ &= \int \frac{1}{\frac{1}{x}} dy \end{aligned}$$

Which results in

$$S = yx$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x, y) S_y}{R_x + \omega(x, y) R_y} \tag{2}$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x, y)$ is the right hand side of the original ode given by

$$\omega(x, y) = -\frac{y}{x}$$

Evaluating all the partial derivatives gives

$$\begin{aligned} R_x &= 1 \\ R_y &= 0 \\ S_x &= y \\ S_y &= x \end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = 0 \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = 0$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by

integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$S(R) = c_1 \tag{4}$$

To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$yx = c_1$$

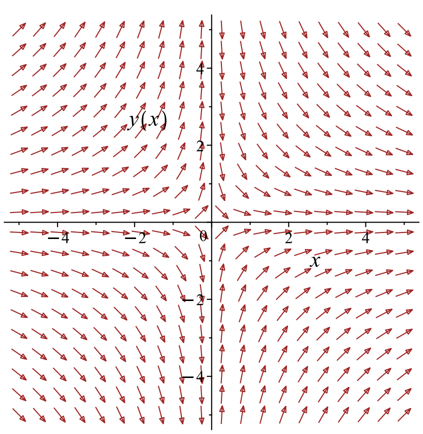
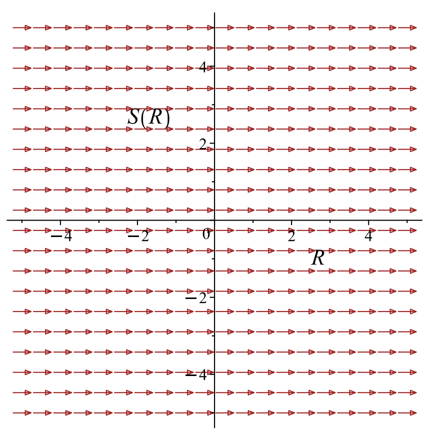
Which simplifies to

$$yx = c_1$$

Which gives

$$y = \frac{c_1}{x}$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in x, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dx} = -\frac{y}{x}$ 	$R = x$ $S = yx$	$\frac{dS}{dR} = 0$ 

Summary

The solution(s) found are the following

$$y = \frac{c_1}{x} \tag{1}$$

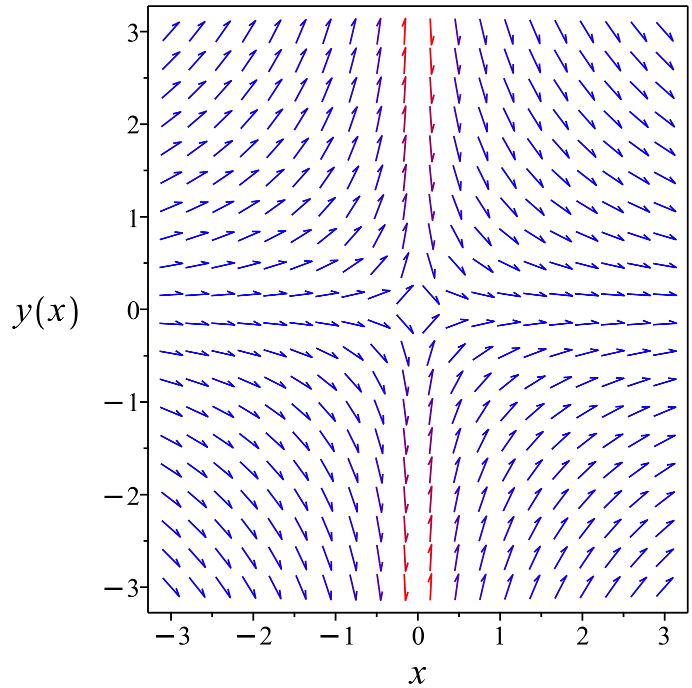


Figure 235: Slope field plot

Verification of solutions

$$y = \frac{c_1}{x}$$

Verified OK.

5.4.6 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x,y) + N(x,y)\frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x,y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y) = 0$$

Hence

$$\frac{\partial \phi}{\partial x} + \frac{\partial \phi}{\partial y}\frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$
\begin{aligned}
\frac{\partial \phi}{\partial x} &= M \\
\frac{\partial \phi}{\partial y} &= N
\end{aligned}
$$

But since $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ then for the above to be valid, we require that

$$
\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}
$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x, y)$ but at least we know now that we can do that since the condition $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$
M(x, y) \, \mathrm{d}x + N(x, y) \, \mathrm{d}y = 0 \tag{1A}
$$

Therefore

$$
\begin{aligned}
\left(-\frac{1}{y}\right) \mathrm{d}y &= \left(\frac{1}{x}\right) \mathrm{d}x \\
\left(-\frac{1}{x}\right) \mathrm{d}x + \left(-\frac{1}{y}\right) \mathrm{d}y &= 0 \qquad \text{(2A)}
\end{aligned}
$$

Comparing (1A) and (2A) shows that

$$
\begin{aligned}
M(x, y) &= -\frac{1}{x} \\
N(x, y) &= -\frac{1}{y}
\end{aligned}
$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$
\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}
$$

Using result found above gives

$$
\begin{aligned}
\frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}\left(-\frac{1}{x}\right) \\
&= 0
\end{aligned}
$$

And

$$\frac{\partial N}{\partial x} = \frac{\partial}{\partial x}\left(-\frac{1}{y}\right) \\ = 0$$

Since $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$, then the ODE is exact The following equations are now set up to solve for the function $\phi(x, y)$

$$\frac{\partial \phi}{\partial x} = M \tag{1}$$

$$\frac{\partial \phi}{\partial y} = N \tag{2}$$

Integrating (1) w.r.t. x gives

$$\int \frac{\partial \phi}{\partial x} \, dx = \int M \, dx$$

$$\int \frac{\partial \phi}{\partial x} \, dx = \int -\frac{1}{x} \, dx$$

$$\phi = -\ln(x) + f(y) \tag{3}$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$\frac{\partial \phi}{\partial y} = 0 + f'(y) \tag{4}$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = -\frac{1}{y}$. Therefore equation (4) becomes

$$-\frac{1}{y} = 0 + f'(y) \tag{5}$$

Solving equation (5) for $f'(y)$ gives

$$f'(y) = -\frac{1}{y}$$

Integrating the above w.r.t y gives

$$\int f'(y) \, dy = \int \left(-\frac{1}{y}\right) dy$$

$$f(y) = -\ln(y) + c_1$$

Where c_1 is constant of integration. Substituting result found above for $f(y)$ into equation (3) gives ϕ

$$\phi = -\ln(x) - \ln(y) + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = -\ln(x) - \ln(y)$$

The solution becomes

$$y = \frac{e^{-c_1}}{x}$$

Summary

The solution(s) found are the following

$$y = \frac{e^{-c_1}}{x} \tag{1}$$

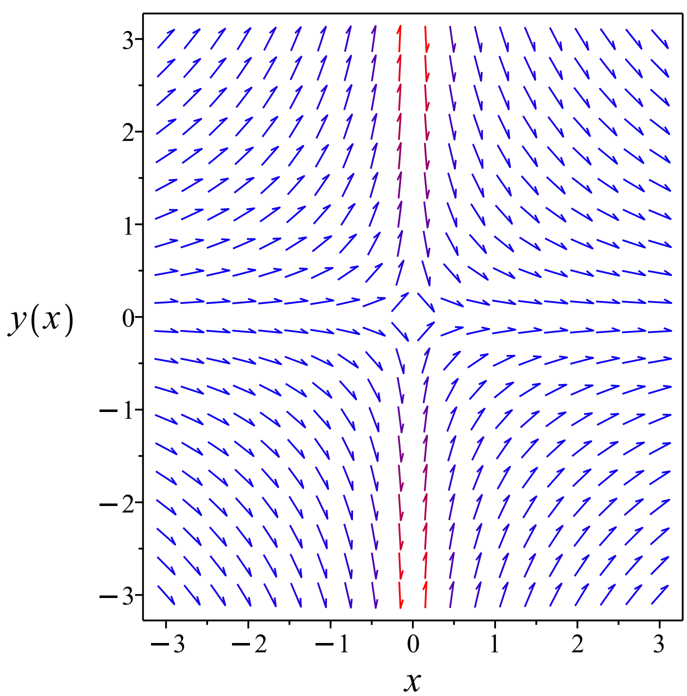


Figure 236: Slope field plot

Verification of solutions

$$y = \frac{e^{-c_1}}{x}$$

Verified OK.

5.4.7 Maple step by step solution

Let's solve

$2y + 2xy^2 + (2x + 2x^2y)\,y' = 0$

- Highest derivative means the order of the ODE is 1

  y'

- Integrate both sides with respect to x

  $\int (2y + 2xy^2 + (2x + 2x^2y)\,y')\,dx = \int 0dx + c_1$

- Evaluate integral

  $2x\left(\frac{xy^2}{2} + y\right) = c_1$

- Solve for y

  $\left\{y = \frac{-1-\sqrt{1+c_1}}{x}, y = \frac{-1+\sqrt{1+c_1}}{x}\right\}$

Maple trace

```
`Classification methods on request
Methods to be used are: [exact]
----------------------------
* Tackling ODE using method: exact
--- Trying classification methods ---
trying exact
<- exact successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 31

```
dsolve(2*y(x)+2*x*y(x)^2+(2*x+2*x^2*y(x))*diff(y(x),x) = 0,y(x), singsol=all)
```

$$y(x) = -\frac{1}{x}$$
$$y(x) = \frac{-1 - c_1}{x}$$
$$y(x) = \frac{c_1 - 1}{x}$$

✓ Solution by Mathematica

Time used: 0.023 (sec). Leaf size: 29

```
DSolve[2*y[x]+2*x*y[x]^2+(2*x+2*x^2*y[x])*y'[x] == 0,y[x],x,IncludeSingularSolutions -> True
```

$$y(x) \to -\frac{1}{x}$$
$$y(x) \to \frac{c_1}{x}$$
$$y(x) \to -\frac{1}{x}$$

5.5 problem 5

Internal problem ID [547]
Internal file name [`OUTPUT/547_Sunday_June_05_2022_01_44_06_AM_24958851/index.tex`]

Book: Elementary differential equations and boundary value problems, 10th ed., Boyce and DiPrima
Section: Section 2.6. Page 100
Problem number: 5.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"exact", "homogeneousTypeD2", "first_order_ode_lie_symmetry_calculated"**

Maple gives the following as the ode type

```
[[_homogeneous, `class A`], _rational, [_Abel, `2nd type`, `
   class A`]]
```

$$\boxed{y' - \frac{-ax - by}{bx + cy} = 0}$$

5.5.1 Solving as homogeneousTypeD2 ode

Using the change of variables $y = u(x)\,x$ on the above ode results in new ode in $u(x)$

$$u'(x)\,x + u(x) - \frac{-ax - bu(x)\,x}{bx + cu\,(x)\,x} = 0$$

In canonical form the ODE is

$$\begin{aligned} u' &= F(x,u) \\ &= f(x)g(u) \\ &= -\frac{c\,u^2 + 2bu + a}{x\,(cu + b)} \end{aligned}$$

Where $f(x) = -\frac{1}{x}$ and $g(u) = \frac{c\,u^2+2bu+a}{cu+b}$. Integrating both sides gives

$$\frac{1}{\frac{c\,u^2+2bu+a}{cu+b}}\,du = -\frac{1}{x}\,dx$$

$$\int \frac{1}{\frac{c\,u^2+2bu+a}{cu+b}}\,du = \int -\frac{1}{x}\,dx$$

$$\frac{\ln\left(c\,u^2+2bu+a\right)}{2} = -\ln\left(x\right)+c_2$$

Raising both side to exponential gives

$$\sqrt{c\,u^2+2bu+a} = \mathrm{e}^{-\ln(x)+c_2}$$

Which simplifies to

$$\sqrt{c\,u^2+2bu+a} = \frac{c_3}{x}$$

Which simplifies to

$$\sqrt{u\left(x\right)^2 c+2u\left(x\right) b+a} = \frac{c_3\mathrm{e}^{c_2}}{x}$$

The solution is

$$\sqrt{u\left(x\right)^2 c+2u\left(x\right) b+a} = \frac{c_3\mathrm{e}^{c_2}}{x}$$

Replacing $u(x)$ in the above solution by $\frac{y}{x}$ results in the solution for y in implicit form

$$\sqrt{\frac{y^2c}{x^2}+\frac{2yb}{x}+a} = \frac{c_3\mathrm{e}^{c_2}}{x}$$

$$\sqrt{\frac{cy^2+2byx+a\,x^2}{x^2}} = \frac{c_3\mathrm{e}^{c_2}}{x}$$

Summary

The solution(s) found are the following

$$\sqrt{\frac{cy^2+2byx+a\,x^2}{x^2}} = \frac{c_3\mathrm{e}^{c_2}}{x} \tag{1}$$

Verification of solutions

$$\sqrt{\frac{cy^2+2byx+a\,x^2}{x^2}} = \frac{c_3\mathrm{e}^{c_2}}{x}$$

Verified OK.

5.5.2 Solving as first order ode lie symmetry calculated ode

Writing the ode as

$$
\begin{aligned}
y' &= -\frac{ax+by}{bx+cy} \\
y' &= \omega(x,y)
\end{aligned}
$$

The condition of Lie symmetry is the linearized PDE given by

$$
\eta_x + \omega(\eta_y - \xi_x) - \omega^2 \xi_y - \omega_x \xi - \omega_y \eta = 0 \tag{A}
$$

The type of this ode is not in the lookup table. To determine ξ, η then (A) is solved using ansatz. Making bivariate polynomials of degree 1 to use as anstaz gives

$$
\xi = x a_2 + y a_3 + a_1 \tag{1E}
$$

$$
\eta = x b_2 + y b_3 + b_1 \tag{2E}
$$

Where the unknown coefficients are

$$
\{a_1, a_2, a_3, b_1, b_2, b_3\}
$$

Substituting equations (1E,2E) and ω into (A) gives

$$
\begin{aligned}
&b_2 - \frac{(ax+by)(b_3 - a_2)}{bx+cy} - \frac{(ax+by)^2 a_3}{(bx+cy)^2} \\
&- \left(-\frac{a}{bx+cy} + \frac{(ax+by) b}{(bx+cy)^2}\right)(x a_2 + y a_3 + a_1) \\
&- \left(-\frac{b}{bx+cy} + \frac{(ax+by) c}{(bx+cy)^2}\right)(x b_2 + y b_3 + b_1) = 0
\end{aligned} \tag{5E}
$$

Putting the above in normal form gives

$$
-\frac{a^2 x^2 a_3 - ab\, x^2 a_2 + ab\, x^2 b_3 + 2abxya_3 + ac\, x^2 b_2 - 2acxya_2 + 2acxyb_3 - ac\, y^2 a_3 - 2b^2 x^2 b_2 + 2b^2 y^2 a_3 - 2}{(bx+cy)^2}
$$

$$
= 0
$$

Setting the numerator to zero gives

$$
\begin{aligned}
&-a^2 x^2 a_3 + ab\, x^2 a_2 - ab\, x^2 b_3 - 2abxya_3 - ac\, x^2 b_2 + 2acxya_2 \\
&- 2acxyb_3 + ac\, y^2 a_3 + 2b^2 x^2 b_2 - 2b^2 y^2 a_3 + 2bcxyb_2 + bc\, y^2 a_2 \\
&- bc\, y^2 b_3 + c^2 y^2 b_2 - acxb_1 + acya_1 + b^2 x b_1 - b^2 y a_1 = 0
\end{aligned} \tag{6E}
$$

Looking at the above PDE shows the following are all the terms with $\{x, y\}$ in them.

$$\{x, y\}$$

The following substitution is now made to be able to collect on all terms with $\{x, y\}$ in them

$$\{x = v_1, y = v_2\}$$

The above PDE (6E) now becomes

$$-a^2a_3v_1^2 + aba_2v_1^2 - 2aba_3v_1v_2 - abb_3v_1^2 + 2aca_2v_1v_2 + aca_3v_2^2 - acb_2v_1^2 - 2acb_3v_1v_2 - 2b^2a_3v_2^2 + 2b^2b_2v_1^2 + bca_2v_2^2 + 2bcb_2v_1v_2 - bcb_3v_2^2 + c^2b_2v_2^2 + aca_1v_2 - acb_1v_1 - b^2a_1v_2 + b^2b_1v_1 = 0 \tag{7E}$$

Collecting the above on the terms v_i introduced, and these are

$$\{v_1, v_2\}$$

Equation (7E) now becomes

$$(-a^2a_3 + aba_2 - abb_3 - acb_2 + 2b^2b_2)\, v_1^2 + (-2aba_3 + 2aca_2 - 2acb_3 + 2bcb_2)\, v_1v_2 + (-acb_1 + b^2b_1)\, v_1 + (aca_3 - 2b^2a_3 + bca_2 - bcb_3 + c^2b_2)\, v_2^2 + (aca_1 - b^2a_1)\, v_2 = 0 \tag{8E}$$

Setting each coefficients in (8E) to zero gives the following equations to solve

$$\begin{aligned}
aca_1 - b^2a_1 &= 0\\
-acb_1 + b^2b_1 &= 0\\
-2aba_3 + 2aca_2 - 2acb_3 + 2bcb_2 &= 0\\
aca_3 - 2b^2a_3 + bca_2 - bcb_3 + c^2b_2 &= 0\\
-a^2a_3 + aba_2 - abb_3 - acb_2 + 2b^2b_2 &= 0
\end{aligned}$$

Solving the above equations for the unknowns gives

$$\begin{aligned}
a_1 &= 0\\
a_2 &= \frac{ab_3 - 2bb_2}{a}\\
a_3 &= -\frac{cb_2}{a}\\
b_1 &= 0\\
b_2 &= b_2\\
b_3 &= b_3
\end{aligned}$$

Substituting the above solution in the anstaz (1E,2E) (using 1 as arbitrary value for any unknown in the RHS) gives

$$\xi = x$$
$$\eta = y$$

Shifting is now applied to make $\xi = 0$ in order to simplify the rest of the computation

$$\begin{aligned}
\eta &= \eta - \omega(x,y)\,\xi \\
&= y - \left(-\frac{ax+by}{bx+cy}\right)(x) \\
&= \frac{a\,x^2+2bxy+c\,y^2}{bx+cy} \\
\xi &= 0
\end{aligned}$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x,y) \to (R,S)$ where (R,S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}$$

The above comes from the requirements that $\left(\xi\frac{\partial}{\partial x} + \eta\frac{\partial}{\partial y}\right)S(x,y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$R = x$$

S is found from

$$\begin{aligned}
S &= \int \frac{1}{\eta}dy \\
&= \int \frac{1}{\frac{a\,x^2+2bxy+c\,y^2}{bx+cy}}dy
\end{aligned}$$

Which results in

$$S = \frac{\ln\left(a\,x^2+2bxy+c\,y^2\right)}{2}$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x,y)S_y}{R_x + \omega(x,y)R_y} \tag{2}$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x, y)$ is the right hand side of the original ode given by

$$\omega(x, y) = -\frac{ax + by}{bx + cy}$$

Evaluating all the partial derivatives gives

$$R_x = 1$$
$$R_y = 0$$
$$S_x = \frac{ax + by}{a\,x^2 + 2bxy + c\,y^2}$$
$$S_y = \frac{bx + cy}{a\,x^2 + 2bxy + c\,y^2}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = 0 \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = 0$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$S(R) = c_1 \tag{4}$$

To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$\frac{\ln\left(cy^2 + 2byx + a\,x^2\right)}{2} = c_1$$

Which simplifies to

$$\frac{\ln\left(cy^2 + 2byx + a\,x^2\right)}{2} = c_1$$

Summary

The solution(s) found are the following

$$\frac{\ln\left(cy^2 + 2byx + a\,x^2\right)}{2} = c_1 \tag{1}$$

Verification of solutions

$$\frac{\ln(cy^2 + 2byx + a\,x^2)}{2} = c_1$$

Verified OK.

5.5.3 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x,y) + N(x,y)\frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x,y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y) = 0$$

Hence

$$\frac{\partial\phi}{\partial x} + \frac{\partial\phi}{\partial y}\frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$\begin{aligned}\frac{\partial\phi}{\partial x} &= M\\ \frac{\partial\phi}{\partial y} &= N\end{aligned}$$

But since $\frac{\partial^2\phi}{\partial x\partial y} = \frac{\partial^2\phi}{\partial y\partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x,y)$ but at least we know now that we can do that since the condition $\frac{\partial^2\phi}{\partial x\partial y} = \frac{\partial^2\phi}{\partial y\partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x,y)\,\mathrm{d}x + N(x,y)\,\mathrm{d}y = 0 \tag{1A}$$

Therefore

$$\begin{aligned}(bx + cy)\,\mathrm{d}y &= (-ax - by)\,\mathrm{d}x\\ (ax + by)\,\mathrm{d}x + (bx + cy)\,\mathrm{d}y &= 0 \end{aligned} \tag{2A}$$

Comparing (1A) and (2A) shows that

$$\begin{aligned}
M(x, y) &= ax + by \\
N(x, y) &= bx + cy
\end{aligned}$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Using result found above gives

$$\begin{aligned}
\frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}(ax + by) \\
&= b
\end{aligned}$$

And

$$\begin{aligned}
\frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}(bx + cy) \\
&= b
\end{aligned}$$

Since $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$, then the ODE is exact The following equations are now set up to solve for the function $\phi(x, y)$

$$\frac{\partial \phi}{\partial x} = M \tag{1}$$

$$\frac{\partial \phi}{\partial y} = N \tag{2}$$

Integrating (1) w.r.t. x gives

$$\begin{aligned}
\int \frac{\partial \phi}{\partial x} \, \mathrm{d}x &= \int M \, \mathrm{d}x \\
\int \frac{\partial \phi}{\partial x} \, \mathrm{d}x &= \int ax + by \, \mathrm{d}x \\
\phi &= \frac{1}{2} a\, x^2 + bxy + f(y) \qquad (3)
\end{aligned}$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$\frac{\partial \phi}{\partial y} = bx + f'(y) \tag{4}$$

But equation (2) says that $\frac{\partial\phi}{\partial y} = bx + cy$. Therefore equation (4) becomes

$$bx + cy = bx + f'(y) \tag{5}$$

Solving equation (5) for $f'(y)$ gives

$$f'(y) = cy$$

Integrating the above w.r.t y gives

$$\int f'(y)\,\mathrm{d}y = \int (cy)\,\mathrm{d}y$$

$$f(y) = \frac{c\,y^2}{2} + c_1$$

Where c_1 is constant of integration. Substituting result found above for $f(y)$ into equation (3) gives ϕ

$$\phi = \frac{1}{2}a\,x^2 + bxy + \frac{1}{2}c\,y^2 + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = \frac{1}{2}a\,x^2 + bxy + \frac{1}{2}c\,y^2$$

Summary

The solution(s) found are the following

$$\frac{a\,x^2}{2} + byx + \frac{cy^2}{2} = c_1 \tag{1}$$

Verification of solutions

$$\frac{a\,x^2}{2} + byx + \frac{cy^2}{2} = c_1$$

Verified OK.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
trying homogeneous types:
trying homogeneous D
<- homogeneous successful`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 77

```
dsolve(diff(y(x),x) = (-a*x-b*y(x))/(b*x+c*y(x)),y(x), singsol=all)
```

$$y(x) = \frac{-bxc_1 + \sqrt{-x^2 (ac - b^2) c_1^2 + c}}{cc_1}$$
$$y(x) = \frac{-bxc_1 - \sqrt{-x^2 (ac - b^2) c_1^2 + c}}{cc_1}$$

✓ Solution by Mathematica

Time used: 17.783 (sec). Leaf size: 139

```
DSolve[y'[x]== (-a*x-b*y[x])/(b*x+c*y[x]),y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to -\frac{bx + \sqrt{-acx^2 + b^2x^2 + ce^{2c_1}}}{c}$$
$$y(x) \to \frac{-bx + \sqrt{b^2x^2 + c (-ax^2 + e^{2c_1})}}{c}$$
$$y(x) \to -\frac{\sqrt{x^2 (b^2 - ac)} + bx}{c}$$
$$y(x) \to \frac{\sqrt{x^2 (b^2 - ac)} - bx}{c}$$

5.6 problem 6

Internal problem ID [548]
Internal file name [`OUTPUT/548_Sunday_June_05_2022_01_44_07_AM_46494958/index.tex`]

Book: Elementary differential equations and boundary value problems, 10th ed., Boyce and DiPrima
Section: Section 2.6. Page 100
Problem number: 6.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"homogeneousTypeD2", "first_order_ode_lie_symmetry_calculated"**

Maple gives the following as the ode type

```
[[_homogeneous, `class A`], _rational, [_Abel, `2nd type`, `
   class A`]]
```

$$\boxed{y' - \frac{-ax + by}{bx - cy} = 0}$$

5.6.1 Solving as homogeneousTypeD2 ode

Using the change of variables $y = u(x)\,x$ on the above ode results in new ode in $u(x)$

$$u'(x)\,x + u(x) - \frac{-ax + bu(x)\,x}{bx - cu\,(x)\,x} = 0$$

In canonical form the ODE is

$$\begin{aligned} u' &= F(x,u) \\ &= f(x)g(u) \\ &= -\frac{-c\,u^2 + a}{x\,(-cu + b)} \end{aligned}$$

Where $f(x) = -\frac{1}{x}$ and $g(u) = \frac{-c\,u^2+a}{-cu+b}$. Integrating both sides gives

$$\frac{1}{\frac{-c\,u^2+a}{-cu+b}}\,du = -\frac{1}{x}\,dx$$

$$\int \frac{1}{\frac{-c\,u^2+a}{-cu+b}}\,du = \int -\frac{1}{x}\,dx$$

$$\frac{\ln\left(c\,u^2-a\right)}{2}+\frac{b\,\operatorname{arctanh}\left(\frac{uc}{\sqrt{ac}}\right)}{\sqrt{ac}} = -\ln\left(x\right)+c_2$$

The solution is

$$\frac{\ln\left(u(x)^2\,c-a\right)}{2}+\frac{b\,\operatorname{arctanh}\left(\frac{u(x)c}{\sqrt{ac}}\right)}{\sqrt{ac}}+\ln\left(x\right)-c_2 = 0$$

Replacing $u(x)$ in the above solution by $\frac{y}{x}$ results in the solution for y in implicit form

$$\frac{\ln\left(\frac{y^2c}{x^2}-a\right)}{2}+\frac{b\,\operatorname{arctanh}\left(\frac{yc}{x\sqrt{ac}}\right)}{\sqrt{ac}}+\ln\left(x\right)-c_2 = 0$$

$$\frac{\ln\left(\frac{y^2c}{x^2}-a\right)}{2}+\frac{b\,\operatorname{arctanh}\left(\frac{yc}{x\sqrt{ac}}\right)}{\sqrt{ac}}+\ln\left(x\right)-c_2 = 0$$

<u>Summary</u>

The solution(s) found are the following

$$\frac{\ln\left(\frac{y^2c}{x^2}-a\right)}{2}+\frac{b\,\operatorname{arctanh}\left(\frac{yc}{x\sqrt{ac}}\right)}{\sqrt{ac}}+\ln\left(x\right)-c_2 = 0 \tag{1}$$

<u>Verification of solutions</u>

$$\frac{\ln\left(\frac{y^2c}{x^2}-a\right)}{2}+\frac{b\,\operatorname{arctanh}\left(\frac{yc}{x\sqrt{ac}}\right)}{\sqrt{ac}}+\ln\left(x\right)-c_2 = 0$$

Verified OK.

5.6.2 Solving as first order ode lie symmetry calculated ode

Writing the ode as

$$
\begin{aligned}
y' &= -\frac{-ax+by}{-bx+cy} \\
y' &= \omega(x,y)
\end{aligned}
$$

The condition of Lie symmetry is the linearized PDE given by

$$
\eta_x + \omega(\eta_y - \xi_x) - \omega^2 \xi_y - \omega_x \xi - \omega_y \eta = 0 \tag{A}
$$

The type of this ode is not in the lookup table. To determine ξ, η then (A) is solved using ansatz. Making bivariate polynomials of degree 1 to use as anstaz gives

$$
\xi = x a_2 + y a_3 + a_1 \tag{1E}
$$

$$
\eta = x b_2 + y b_3 + b_1 \tag{2E}
$$

Where the unknown coefficients are

$$
\{a_1, a_2, a_3, b_1, b_2, b_3\}
$$

Substituting equations (1E,2E) and ω into (A) gives

$$
\begin{aligned}
&b_2 - \frac{(-ax+by)(b_3-a_2)}{-bx+cy} - \frac{(-ax+by)^2 a_3}{(-bx+cy)^2} \\
&- \left(\frac{a}{-bx+cy} - \frac{(-ax+by)\, b}{(-bx+cy)^2}\right)(x a_2 + y a_3 + a_1) \\
&- \left(-\frac{b}{-bx+cy} + \frac{(-ax+by)\, c}{(-bx+cy)^2}\right)(x b_2 + y b_3 + b_1) = 0
\end{aligned} \tag{5E}
$$

Putting the above in normal form gives

$$
-\frac{a^2x^2a_3 - ab\,x^2a_2 + ab\,x^2b_3 - 2abxya_3 - ac\,x^2b_2 + 2acxya_2 - 2acxyb_3 + ac\,y^2a_3 + 2bcxyb_2 - bc\,y^2a_2 + b}{(bx-cy)^2}
$$

$= 0$

Setting the numerator to zero gives

$$
\begin{aligned}
&-a^2x^2a_3 + ab\,x^2a_2 - ab\,x^2b_3 + 2abxya_3 + ac\,x^2b_2 - 2acxya_2 + 2acxyb_3 - ac\,y^2a_3 \\
&- 2bcxyb_2 + bc\,y^2a_2 - bc\,y^2b_3 + c^2y^2b_2 + acxb_1 - acya_1 - b^2xb_1 + b^2ya_1 = 0
\end{aligned} \tag{6E}
$$

Looking at the above PDE shows the following are all the terms with $\{x, y\}$ in them.

$$\{x, y\}$$

The following substitution is now made to be able to collect on all terms with $\{x, y\}$ in them

$$\{x = v_1, y = v_2\}$$

The above PDE (6E) now becomes

$$\begin{aligned} -a^2a_3v_1^2 + aba_2v_1^2 + 2aba_3v_1v_2 - abb_3v_1^2 - 2aca_2v_1v_2 \\ - aca_3v_2^2 + acb_2v_1^2 + 2acb_3v_1v_2 + bca_2v_2^2 - 2bcb_2v_1v_2 \\ - bcb_3v_2^2 + c^2b_2v_2^2 - aca_1v_2 + acb_1v_1 + b^2a_1v_2 - b^2b_1v_1 = 0 \end{aligned} \tag{7E}$$

Collecting the above on the terms v_i introduced, and these are

$$\{v_1, v_2\}$$

Equation (7E) now becomes

$$\begin{aligned} (-a^2a_3 + aba_2 - abb_3 + acb_2)\, v_1^2 + (2aba_3 - 2aca_2 + 2acb_3 - 2bcb_2)\, v_1v_2 \\ + (acb_1 - b^2b_1)\, v_1 + (-aca_3 + bca_2 - bcb_3 + c^2b_2)\, v_2^2 + (-aca_1 + b^2a_1)\, v_2 = 0 \end{aligned} \tag{8E}$$

Setting each coefficients in (8E) to zero gives the following equations to solve

$$\begin{aligned} -aca_1 + b^2a_1 &= 0 \\ acb_1 - b^2b_1 &= 0 \\ -aca_3 + bca_2 - bcb_3 + c^2b_2 &= 0 \\ 2aba_3 - 2aca_2 + 2acb_3 - 2bcb_2 &= 0 \\ -a^2a_3 + aba_2 - abb_3 + acb_2 &= 0 \end{aligned}$$

Solving the above equations for the unknowns gives

$$\begin{aligned} a_1 &= 0 \\ a_2 &= b_3 \\ a_3 &= a_3 \\ b_1 &= 0 \\ b_2 &= \frac{a_3a}{c} \\ b_3 &= b_3 \end{aligned}$$

Substituting the above solution in the anstaz (1E,2E) (using 1 as arbitrary value for any unknown in the RHS) gives

$$\xi = x$$
$$\eta = y$$

Shifting is now applied to make $\xi = 0$ in order to simplify the rest of the computation

$$\begin{aligned}
\eta &= \eta - \omega(x, y)\,\xi \\
&= y - \left(-\frac{-ax + by}{-bx + cy}\right)(x) \\
&= \frac{a\,x^2 - c\,y^2}{bx - cy} \\
\xi &= 0
\end{aligned}$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x, y) \to (R, S)$ where (R, S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y}\right) S(x, y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$R = x$$

S is found from

$$\begin{aligned}
S &= \int \frac{1}{\eta} dy \\
&= \int \frac{1}{\frac{a\,x^2 - c\,y^2}{bx - cy}} dy
\end{aligned}$$

Which results in

$$S = \frac{\ln\left(-a\,x^2 + c\,y^2\right)}{2} + \frac{b \operatorname{arctanh}\left(\frac{yc}{x\sqrt{ac}}\right)}{\sqrt{ac}}$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x,y) S_y}{R_x + \omega(x,y) R_y} \tag{2}$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x,y)$ is the right hand side of the original ode given by

$$\omega(x,y) = -\frac{-ax + by}{-bx + cy}$$

Evaluating all the partial derivatives gives

$$\begin{aligned}
R_x &= 1 \\
R_y &= 0 \\
S_x &= \frac{ax - by}{a\,x^2 - c\,y^2} \\
S_y &= \frac{bx - cy}{a\,x^2 - c\,y^2}
\end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = 0 \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = 0$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$S(R) = c_1 \tag{4}$$

To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$\frac{\ln\left(cy^2 - a\,x^2\right)\sqrt{a}\,\sqrt{c} + 2b \operatorname{arctanh}\left(\frac{y\sqrt{c}}{x\sqrt{a}}\right)}{2\sqrt{a}\,\sqrt{c}} = c_1$$

Which simplifies to

$$\frac{\ln\left(cy^2-a\,x^2\right)\sqrt{a}\,\sqrt{c}+2b\,\operatorname{arctanh}\left(\frac{y\sqrt{c}}{x\sqrt{a}}\right)}{2\sqrt{a}\,\sqrt{c}}=c_1$$

Summary

The solution(s) found are the following

$$\frac{\ln\left(cy^2-a\,x^2\right)\sqrt{a}\,\sqrt{c}+2b\,\operatorname{arctanh}\left(\frac{y\sqrt{c}}{x\sqrt{a}}\right)}{2\sqrt{a}\,\sqrt{c}}=c_1 \tag{1}$$

Verification of solutions

$$\frac{\ln\left(cy^2-a\,x^2\right)\sqrt{a}\,\sqrt{c}+2b\,\operatorname{arctanh}\left(\frac{y\sqrt{c}}{x\sqrt{a}}\right)}{2\sqrt{a}\,\sqrt{c}}=c_1$$

Verified OK.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
trying homogeneous types:
trying homogeneous D
<- homogeneous successful`
```

✓ Solution by Maple

Time used: 0.422 (sec). Leaf size: 47

```
dsolve(diff(y(x),x) = (-a*x+b*y(x))/(b*x-c*y(x)),y(x), singsol=all)
```

$$y(x)=\operatorname{RootOf}\left(c\_Z^2-a-\mathrm{e}^{\operatorname{RootOf}\left(\mathrm{e}^{\_Z}\cosh\left(\frac{\sqrt{ac}\left(2c_1+\_Z+2\ln(x)\right)}{2b}\right)^2+a\right)}\right)x$$

✓ Solution by Mathematica

Time used: 0.072 (sec). Leaf size: 58

```
DSolve[y'[x] == (-a*x+b*y[x])/(b*x-c*y[x]),y[x],x,IncludeSingularSolutions -> True]
```

$$\text{Solve}\left[-\frac{b\,\text{arctanh}\left(\frac{\sqrt{c}y(x)}{\sqrt{a}x}\right)}{\sqrt{a}\sqrt{c}}-\frac{1}{2}\log\left(\frac{cy(x)^2}{x^2}-a\right)=\log(x)+c_1,y(x)\right]$$

5.7 problem 7

Internal problem ID [549]
Internal file name [OUTPUT/549_Sunday_June_05_2022_01_44_09_AM_4332274/index.tex]

Book: Elementary differential equations and boundary value problems, 10th ed., Boyce and DiPrima
Section: Section 2.6. Page 100
Problem number: 7.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"exact"**

Maple gives the following as the ode type

```
[_exact]
```

$$\boxed{e^{x} \sin \left(y\right)-2 \sin \left(x\right) y+\left(2 \cos \left(x\right)+e^{x} \cos \left(y\right)\right) y^{\prime}=0}$$

5.7.1 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x,y)+N(x,y)\frac{dy}{dx}=0 \tag{A}$$

We assume there exists a function $\phi(x,y)=c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y)=0$$

Hence

$$\frac{\partial \phi}{\partial x}+\frac{\partial \phi}{\partial y}\frac{dy}{dx}=0 \tag{B}$$

Comparing (A,B) shows that

$$\frac{\partial \phi}{\partial x}=M$$
$$\frac{\partial \phi}{\partial y}=N$$

But since $\frac{\partial^2\phi}{\partial x\partial y} = \frac{\partial^2\phi}{\partial y\partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x,y)$ but at least we know now that we can do that since the condition $\frac{\partial^2\phi}{\partial x\partial y} = \frac{\partial^2\phi}{\partial y\partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x,y)\,\mathrm{d}x + N(x,y)\,\mathrm{d}y = 0 \tag{1A}$$

Therefore

$$\begin{aligned} (2\cos(x) + \mathrm{e}^x\cos(y))\,\mathrm{d}y &= (-\mathrm{e}^x\sin(y) + 2\sin(x)\,y)\,\mathrm{d}x \\ (\mathrm{e}^x\sin(y) - 2\sin(x)\,y)\,\mathrm{d}x + (2\cos(x) + \mathrm{e}^x\cos(y))\,\mathrm{d}y &= 0 \end{aligned} \tag{2A}$$

Comparing (1A) and (2A) shows that

$$\begin{aligned} M(x,y) &= \mathrm{e}^x\sin(y) - 2\sin(x)\,y \\ N(x,y) &= 2\cos(x) + \mathrm{e}^x\cos(y) \end{aligned}$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Using result found above gives

$$\begin{aligned} \frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}(\mathrm{e}^x\sin(y) - 2\sin(x)\,y) \\ &= -2\sin(x) + \mathrm{e}^x\cos(y) \end{aligned}$$

And

$$\begin{aligned} \frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}(2\cos(x) + \mathrm{e}^x\cos(y)) \\ &= -2\sin(x) + \mathrm{e}^x\cos(y) \end{aligned}$$

Since $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$, then the ODE is exact The following equations are now set up to solve for the function $\phi(x,y)$

$$\frac{\partial\phi}{\partial x} = M \tag{1}$$

$$\frac{\partial\phi}{\partial y} = N \tag{2}$$

Integrating (1) w.r.t. x gives

$$\int \frac{\partial \phi}{\partial x} \mathrm{d}x = \int M \mathrm{d}x$$
$$\int \frac{\partial \phi}{\partial x} \mathrm{d}x = \int \mathrm{e}^{x} \sin(y) - 2\sin(x)\, y \,\mathrm{d}x$$
$$\phi = \mathrm{e}^{x} \sin(y) + 2\cos(x)\, y + f(y) \tag{3}$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$\frac{\partial \phi}{\partial y} = 2\cos(x) + \mathrm{e}^{x}\cos(y) + f'(y) \tag{4}$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = 2\cos(x) + \mathrm{e}^{x}\cos(y)$. Therefore equation (4) becomes

$$2\cos(x) + \mathrm{e}^{x}\cos(y) = 2\cos(x) + \mathrm{e}^{x}\cos(y) + f'(y) \tag{5}$$

Solving equation (5) for $f'(y)$ gives

$$f'(y) = 0$$

Therefore

$$f(y) = c_1$$

Where c_1 is constant of integration. Substituting this result for $f(y)$ into equation (3) gives ϕ

$$\phi = \mathrm{e}^{x}\sin(y) + 2\cos(x)\, y + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = \mathrm{e}^{x}\sin(y) + 2\cos(x)\, y$$

Summary

The solution(s) found are the following

$$\mathrm{e}^{x}\sin(y) + 2\cos(x)\, y = c_1 \tag{1}$$

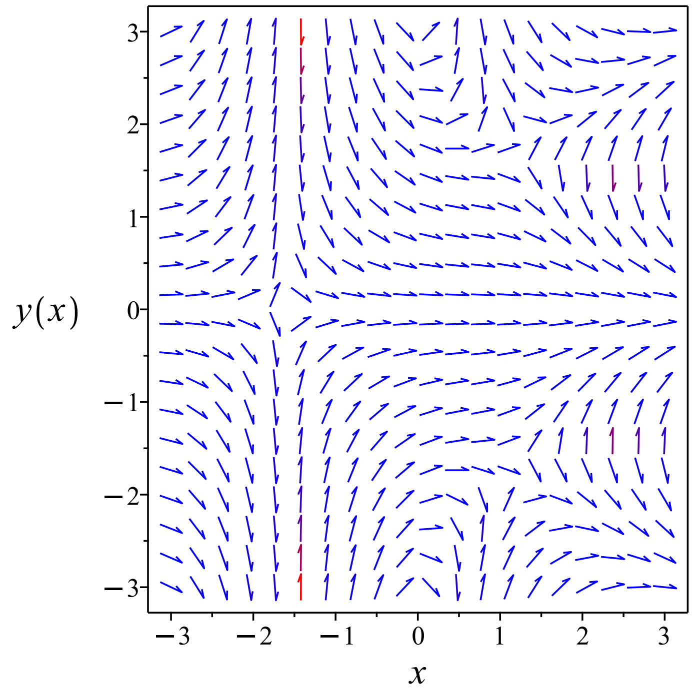


Figure 237: Slope field plot

Verification of solutions

$$e^{x} \sin (y)+2 \cos (x) y = c_1$$

Verified OK.

5.7.2 Maple step by step solution

Let's solve

$e^{x} \sin (y)-2 \sin (x) y+(2 \cos (x)+e^{x} \cos (y)) y' = 0$

- Highest derivative means the order of the ODE is 1

  y'

- □ Check if ODE is exact

  - ODE is exact if the lhs is the total derivative of a C^2 function

    $F'(x, y) = 0$

  - Compute derivative of lhs

    $F'(x, y)+\left(\frac{\partial}{\partial y}F(x, y)\right) y' = 0$

- Evaluate derivatives

  $-2\sin(x) + e^x \cos(y) = -2\sin(x) + e^x \cos(y)$

- Condition met, ODE is exact

- Exact ODE implies solution will be of this form

  $\left[F(x,y) = c_1, M(x,y) = F'(x,y), N(x,y) = \frac{\partial}{\partial y}F(x,y)\right]$

- Solve for $F(x,y)$ by integrating $M(x,y)$ with respect to x

  $F(x,y) = \int (e^x \sin(y) - 2\sin(x)\, y)\, dx + f_1(y)$

- Evaluate integral

  $F(x,y) = e^x \sin(y) + 2\cos(x)\, y + f_1(y)$

- Take derivative of $F(x,y)$ with respect to y

  $N(x,y) = \frac{\partial}{\partial y}F(x,y)$

- Compute derivative

  $2\cos(x) + e^x \cos(y) = e^x \cos(y) + 2\cos(x) + \frac{d}{dy}f_1(y)$

- Isolate for $\frac{d}{dy}f_1(y)$

  $\frac{d}{dy}f_1(y) = 0$

- Solve for $f_1(y)$

  $f_1(y) = 0$

- Substitute $f_1(y)$ into equation for $F(x,y)$

  $F(x,y) = e^x \sin(y) + 2\cos(x)\, y$

- Substitute $F(x,y)$ into the solution of the ODE

  $e^x \sin(y) + 2\cos(x)\, y = c_1$

- Solve for y

  $y = \mathit{RootOf}(-e^x \sin(\_Z) - 2\_Z\cos(x) + c_1)$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
trying homogeneous types:
trying Chini
differential order: 1; looking for linear symmetries
trying exact
<- exact successful`
```

✓ Solution by Maple

Time used: 0.015 (sec). Leaf size: 17

```
dsolve(exp(x)*sin(y(x))-2*sin(x)*y(x)+(2*cos(x)+exp(x)*cos(y(x)))*diff(y(x),x) = 0,y(x), sin
```

$$e^{x} \sin \left(y(x)\right)+2 \cos \left(x\right) y(x)+c_{1}=0$$

✓ Solution by Mathematica

Time used: 0.282 (sec). Leaf size: 20

```
DSolve[Exp[x]*Sin[y[x]]-2*Sin[x]*y[x]+(2*Cos[x]+Exp[x]*Cos[y[x]])*y'[x] == 0,y[x],x,IncludeS
```

$$\text{Solve}[e^{x} \sin(y(x))+2y(x)\cos(x)=c_1, y(x)]$$

5.8 problem 8

Internal problem ID [550]
Internal file name [OUTPUT/550_Sunday_June_05_2022_01_44_16_AM_39437276/index.tex]

Book: Elementary differential equations and boundary value problems, 10th ed., Boyce and DiPrima
Section: Section 2.6. Page 100
Problem number: 8.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"unknown"**

Maple gives the following as the ode type

```
[`x=_G(y,y')`]
```

Unable to solve or complete the solution.

$$\boxed{e^{x} \sin(y) + 3y - (3x - e^{x} \sin(y))\, y' = 0}$$

Unable to determine ODE type.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
trying homogeneous types:
trying Chini
differential order: 1; looking for linear symmetries
trying exact
Looking for potential symmetries
trying inverse_Riccati
trying an equivalence to an Abel ODE
differential order: 1; trying a linearization to 2nd order
--- trying a change of variables {x -> y(x), y(x) -> x}
differential order: 1; trying a linearization to 2nd order
trying 1st order ODE linearizable_by_differentiation
--- Trying Lie symmetry methods, 1st order ---
`, `-> Computing symmetries using: way = 3
`, `-> Computing symmetries using: way = 4
`, `-> Computing symmetries using: way = 5
trying symmetry patterns for 1st order ODEs
-> trying a symmetry pattern of the form [F(x)*G(y), 0]
-> trying a symmetry pattern of the form [0, F(x)*G(y)]
-> trying symmetry patterns of the forms [F(x),G(y)] and [G(y),F(x)]
`, `-> Computing symmetries using: way = HINT
   -> Calling odsolve with the ODE`, diff(y(x), x) = 0, y(x)`      *** Sublevel 2 ***
      Methods for first order ODEs:
      --- Trying classification methods ---
      trying a quadrature
      trying 1st order linear
      <- 1st order linear successful
   -> Calling odsolve with the ODE`, diff(y(x), x) = y(x)/x, y(x)`      *** Sublevel 2 ***
      Methods for first order ODEs:
      --- Trying classification methods ---
      trying a quadrature
      trying 1st order linear
      <- 1st order linear successful
   -> Calling odsolve with the ODE`, diff(y(x), x)-y(x)/x, y(x)`      *** Sublevel 2 ***
      Methods for first order ODEs:
      --- Trying classification methods ---
      trying a quadrature
      trying 1st order linear
```

✗ Solution by Maple

```
dsolve(exp(x)*sin(y(x))+3*y(x)-(3*x-exp(x)*sin(y(x)))*diff(y(x),x) = 0,y(x), singsol=all)
```

No solution found

✗ Solution by Mathematica

Time used: 0.0 (sec). Leaf size: 0

```
DSolve[Exp[x]*Sin[y[x]]+3*y[x]-(3*x-Exp[x]*Sin[y[x]])*y'[x] == 0,y[x],x,IncludeSingularSolut
```

Not solved

5.9 problem 9

Internal problem ID [551]
Internal file name [`OUTPUT/551_Sunday_June_05_2022_01_44_22_AM_93677906/index.tex`]

Book: Elementary differential equations and boundary value problems, 10th ed., Boyce and DiPrima
Section: Section 2.6. Page 100
Problem number: 9.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"exact"**

Maple gives the following as the ode type

`[_exact]`

$$\boxed{-2\,\mathrm{e}^{yx}\sin(2x)+\mathrm{e}^{yx}\cos(2x)\,y+\left(-3+\mathrm{e}^{yx}x\cos(2x)\right)y'=-2x}$$

5.9.1 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x,y)+N(x,y)\,\frac{dy}{dx}=0 \tag{A}$$

We assume there exists a function $\phi(x,y)=c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y)=0$$

Hence

$$\frac{\partial\phi}{\partial x}+\frac{\partial\phi}{\partial y}\frac{dy}{dx}=0 \tag{B}$$

Comparing (A,B) shows that

$$\begin{aligned}\frac{\partial\phi}{\partial x}&=M\\ \frac{\partial\phi}{\partial y}&=N\end{aligned}$$

But since $\frac{\partial^2\phi}{\partial x\partial y} = \frac{\partial^2\phi}{\partial y\partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x, y)$ but at least we know now that we can do that since the condition $\frac{\partial^2\phi}{\partial x\partial y} = \frac{\partial^2\phi}{\partial y\partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x, y)\,\mathrm{d}x + N(x, y)\,\mathrm{d}y = 0 \tag{1A}$$

Therefore

$$(-3 + \mathrm{e}^{yx} x \cos(2x))\,\mathrm{d}y = (-2x + 2\,\mathrm{e}^{yx} \sin(2x) - \mathrm{e}^{yx} \cos(2x)\, y)$$

$$(2x - 2\,\mathrm{e}^{yx} \sin(2x) + \mathrm{e}^{yx} \cos(2x)\, y)\,\mathrm{d}x + (-3 + \mathrm{e}^{yx} x \cos(2x))\,\mathrm{d}y = 0 \tag{2A}$$

Comparing (1A) and (2A) shows that

$$\begin{aligned}
M(x, y) &= 2x - 2\,\mathrm{e}^{yx} \sin(2x) + \mathrm{e}^{yx} \cos(2x)\, y \\
N(x, y) &= -3 + \mathrm{e}^{yx} x \cos(2x)
\end{aligned}$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Using result found above gives

$$\begin{aligned}
\frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}(2x - 2\,\mathrm{e}^{yx} \sin(2x) + \mathrm{e}^{yx} \cos(2x)\, y) \\
&= \mathrm{e}^{yx}(yx \cos(2x) + \cos(2x) - 2x \sin(2x))
\end{aligned}$$

And

$$\begin{aligned}
\frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}(-3 + \mathrm{e}^{yx} x \cos(2x)) \\
&= \mathrm{e}^{yx}(yx \cos(2x) + \cos(2x) - 2x \sin(2x))
\end{aligned}$$

Since $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$, then the ODE is <u>exact</u> The following equations are now set up to solve for the function $\phi(x, y)$

$$\frac{\partial \phi}{\partial x} = M \tag{1}$$

$$\frac{\partial \phi}{\partial y} = N \tag{2}$$

Integrating (1) w.r.t. x gives

$$
\begin{aligned}
\int \frac{\partial \phi}{\partial x} \mathrm{d}x &= \int M \,\mathrm{d}x \\
\int \frac{\partial \phi}{\partial x} \mathrm{d}x &= \int 2x - 2\,\mathrm{e}^{yx} \sin(2x) + \mathrm{e}^{yx} \cos(2x)\, y \,\mathrm{d}x \\
\phi &= \mathrm{e}^{yx} \cos(2x) + x^2 + f(y) \qquad (3)
\end{aligned}
$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$
\frac{\partial \phi}{\partial y} = \mathrm{e}^{yx} x \cos(2x) + f'(y) \qquad (4)
$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = -3 + \mathrm{e}^{yx} x \cos(2x)$. Therefore equation (4) becomes

$$
-3 + \mathrm{e}^{yx} x \cos(2x) = \mathrm{e}^{yx} x \cos(2x) + f'(y) \qquad (5)
$$

Solving equation (5) for $f'(y)$ gives

$$
f'(y) = -3
$$

Integrating the above w.r.t y gives

$$
\begin{aligned}
\int f'(y) \,\mathrm{d}y &= \int (-3) \,\mathrm{d}y \\
f(y) &= -3y + c_1
\end{aligned}
$$

Where c_1 is constant of integration. Substituting result found above for $f(y)$ into equation (3) gives ϕ

$$
\phi = \mathrm{e}^{yx} \cos(2x) + x^2 - 3y + c_1
$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$
c_1 = \mathrm{e}^{yx} \cos(2x) + x^2 - 3y
$$

The solution becomes

$$y = -\frac{-x^3 + c_1 x + 3\,\mathrm{LambertW}\left(-\frac{x\cos(2x)\mathrm{e}^{\frac{1}{3}x^3 - \frac{1}{3}c_1 x}}{3}\right)}{3x}$$

Summary

The solution(s) found are the following

$$y = -\frac{-x^3 + c_1 x + 3\,\mathrm{LambertW}\left(-\frac{x\cos(2x)\mathrm{e}^{\frac{1}{3}x^3 - \frac{1}{3}c_1 x}}{3}\right)}{3x} \tag{1}$$

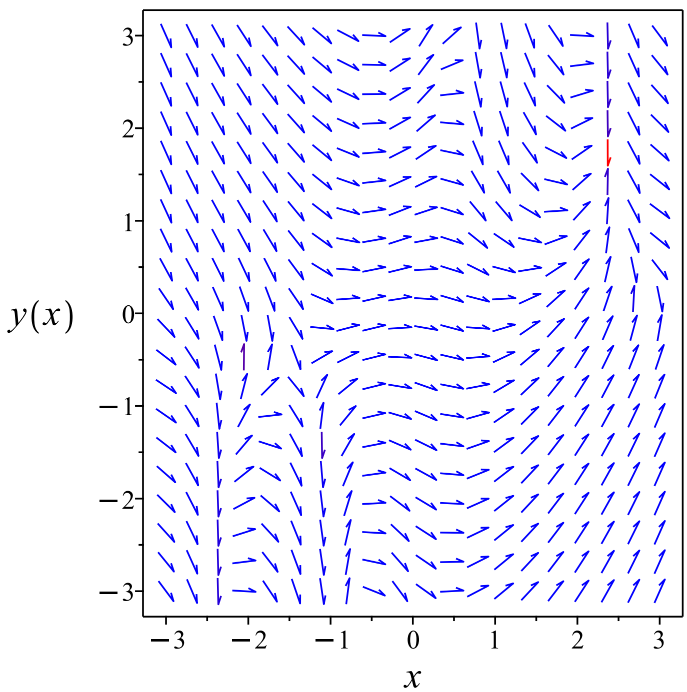


Figure 238: Slope field plot

Verification of solutions

$$y = -\frac{-x^3 + c_1 x + 3\,\mathrm{LambertW}\left(-\frac{x\cos(2x)\mathrm{e}^{\frac{1}{3}x^3 - \frac{1}{3}c_1 x}}{3}\right)}{3x}$$

Verified OK.

5.9.2 Maple step by step solution

Let's solve

$-2\,\mathrm{e}^{yx}\sin(2x)+\mathrm{e}^{yx}\cos(2x)\,y+(-3+\mathrm{e}^{yx}x\cos(2x))\,y'=-2x$

- Highest derivative means the order of the ODE is 1

  y'

□ Check if ODE is exact

  ◦ ODE is exact if the lhs is the total derivative of a C^2 function

    $F'(x,y)=0$

  ◦ Compute derivative of lhs

    $F'(x,y)+\left(\frac{\partial}{\partial y}F(x,y)\right)y'=0$

  ◦ Evaluate derivatives

    $\mathrm{e}^{yx}yx\cos(2x)+\mathrm{e}^{yx}\cos(2x)-2\,\mathrm{e}^{yx}x\sin(2x)=\mathrm{e}^{yx}yx\cos(2x)+\mathrm{e}^{yx}\cos(2x)-2\,\mathrm{e}^{yx}x\sin(2x)$

  ◦ Condition met, ODE is exact

- Exact ODE implies solution will be of this form

  $\left[F(x,y)=c_1, M(x,y)=F'(x,y), N(x,y)=\frac{\partial}{\partial y}F(x,y)\right]$

- Solve for $F(x,y)$ by integrating $M(x,y)$ with respect to x

  $F(x,y)=\int(2x-2\,\mathrm{e}^{yx}\sin(2x)+\mathrm{e}^{yx}\cos(2x)\,y)\,dx+f_1(y)$

- Evaluate integral

  $F(x,y)=y\left(\frac{y\,\mathrm{e}^{yx}\cos(2x)}{y^2+4}+\frac{2\,\mathrm{e}^{yx}\sin(2x)}{y^2+4}\right)+x^2+\frac{4\,\mathrm{e}^{yx}\cos(2x)}{y^2+4}-\frac{2y\,\mathrm{e}^{yx}\sin(2x)}{y^2+4}+f_1(y)$

- Take derivative of $F(x,y)$ with respect to y

  $N(x,y)=\frac{\partial}{\partial y}F(x,y)$

- Compute derivative

  $-3+\mathrm{e}^{yx}x\cos(2x)=\frac{y\,\mathrm{e}^{yx}\cos(2x)}{y^2+4}+y\left(\frac{\mathrm{e}^{yx}\cos(2x)}{y^2+4}-\frac{2y^2\mathrm{e}^{yx}\cos(2x)}{(y^2+4)^2}+\frac{y\,\mathrm{e}^{yx}x\cos(2x)}{y^2+4}-\frac{4\,\mathrm{e}^{yx}\sin(2x)y}{(y^2+4)^2}+\frac{2\,\mathrm{e}^{yx}x\sin}{y^2+}\right.$

- Isolate for $\frac{d}{dy}f_1(y)$

  $\frac{d}{dy}f_1(y)=-3+\mathrm{e}^{yx}x\cos(2x)-\frac{y\,\mathrm{e}^{yx}\cos(2x)}{y^2+4}-y\left(\frac{\mathrm{e}^{yx}\cos(2x)}{y^2+4}-\frac{2y^2\mathrm{e}^{yx}\cos(2x)}{(y^2+4)^2}+\frac{y\,\mathrm{e}^{yx}x\cos(2x)}{y^2+4}-\frac{4\,\mathrm{e}^{yx}\sin(2x)}{(y^2+4)^2}\right.$

- Solve for $f_1(y)$

  $f_1(y)=-3y$

- Substitute $f_1(y)$ into equation for $F(x,y)$

$$F(x,y)=y\left(\frac{y\,e^{yx}\cos(2x)}{y^2+4}+\frac{2\,e^{yx}\sin(2x)}{y^2+4}\right)+x^2+\frac{4\,e^{yx}\cos(2x)}{y^2+4}-\frac{2y\,e^{yx}\sin(2x)}{y^2+4}-3y$$

- Substitute $F(x,y)$ into the solution of the ODE

$$y\left(\frac{y\,e^{yx}\cos(2x)}{y^2+4}+\frac{2\,e^{yx}\sin(2x)}{y^2+4}\right)+x^2+\frac{4\,e^{yx}\cos(2x)}{y^2+4}-\frac{2y\,e^{yx}\sin(2x)}{y^2+4}-3y=c_1$$

- Solve for y

$$y=-\frac{-x^3+c_1x+3\mathit{LambertW}\left(-\frac{x\cos(2x)\mathrm{e}^{\frac{1}{3}x^3-\frac{1}{3}c_1x}}{3}\right)}{3x}$$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
trying homogeneous types:
trying Chini
differential order: 1; looking for linear symmetries
trying exact
<- exact successful`
```

✓ Solution by Maple

Time used: 0.031 (sec). Leaf size: 35

```
dsolve(2*x-2*exp(x*y(x))*sin(2*x)+exp(x*y(x))*cos(2*x)*y(x)+(-3+exp(x*y(x))*x*cos(2*x))*diff
```

$$y(x)=\frac{x^3+c_1x-3\,\mathrm{LambertW}\left(-\frac{x\cos(2x)\mathrm{e}^{\frac{x(x^2+c_1)}{3}}}{3}\right)}{3x}$$

✓ Solution by Mathematica

Time used: 5.197 (sec). Leaf size: 48

```
DSolve[2*x-2*Exp[x*y[x]]*Sin[2*x]+Exp[x*y[x]]*Cos[2*x]*y[x]+(-3+Exp[x*y[x]]*x*Cos[2*x])*y'[x
```

$$y(x) \to \frac{-3W\left(-\frac{1}{3}xe^{\frac{1}{3}x(x^2-c_1)}\cos(2x)\right)+x^3-c_1x}{3x}$$

5.10 problem 10

Internal problem ID [552]
Internal file name [`OUTPUT/552_Sunday_June_05_2022_01_44_25_AM_13151512/index.tex`]

Book: Elementary differential equations and boundary value problems, 10th ed., Boyce and DiPrima
Section: Section 2.6. Page 100
Problem number: 10.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"exact", "linear", "first_order_ode_lie_symmetry_lookup"**

Maple gives the following as the ode type

`[_linear]`

$$\boxed{\frac{y}{x}+(\ln (x)-2)\, y'=-6x}$$

5.10.1 Solving as linear ode

Entering Linear first order ODE solver. In canonical form a linear first order is

$$y'+p(x)y=q(x)$$

Where here

$$p(x)=\frac{1}{(\ln (x)-2)\, x}$$
$$q(x)=-\frac{6x}{\ln (x)-2}$$

Hence the ode is

$$y'+\frac{y}{(\ln (x)-2)\, x}=-\frac{6x}{\ln (x)-2}$$

The integrating factor μ is

$$
\begin{aligned}
\mu &= e^{\int \frac{1}{(\ln(x)-2)x}dx} \\
&= \ln(x) - 2
\end{aligned}
$$

The ode becomes

$$
\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}x}(\mu y) &= (\mu)\left(-\frac{6x}{\ln(x)-2}\right) \\
\frac{\mathrm{d}}{\mathrm{d}x}((\ln(x)-2)\, y) &= (\ln(x)-2)\left(-\frac{6x}{\ln(x)-2}\right) \\
\mathrm{d}((\ln(x)-2)\, y) &= (-6x)\, \mathrm{d}x
\end{aligned}
$$

Integrating gives

$$
\begin{aligned}
(\ln(x)-2)\, y &= \int -6x\, \mathrm{d}x \\
(\ln(x)-2)\, y &= -3x^2 + c_1
\end{aligned}
$$

Dividing both sides by the integrating factor $\mu = \ln(x) - 2$ results in

$$
y = -\frac{3x^2}{\ln(x)-2} + \frac{c_1}{\ln(x)-2}
$$

which simplifies to

$$
y = \frac{-3x^2 + c_1}{\ln(x)-2}
$$

Summary

The solution(s) found are the following

$$
y = \frac{-3x^2 + c_1}{\ln(x)-2} \tag{1}
$$

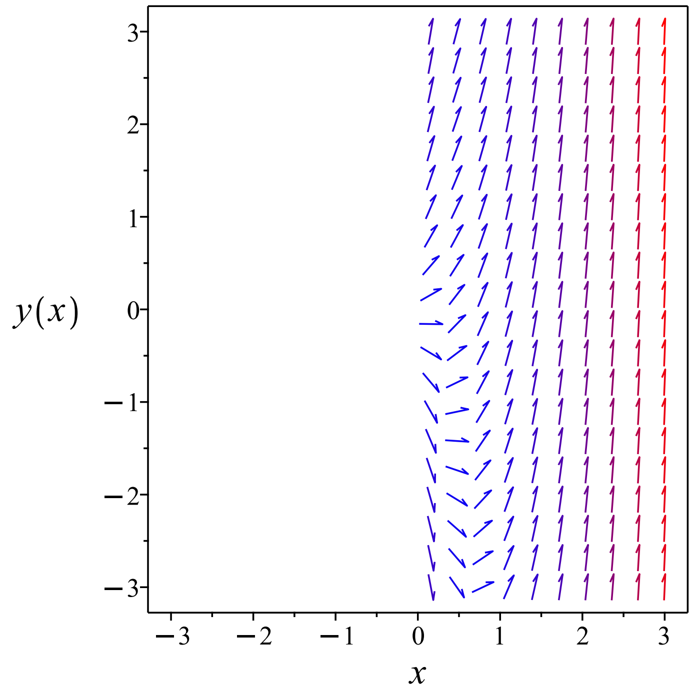


Figure 239: Slope field plot

Verification of solutions

$$y = \frac{-3x^2 + c_1}{\ln(x) - 2}$$

Verified OK.

5.10.2 Solving as first order ode lie symmetry lookup ode

Writing the ode as

$$y' = -\frac{6x^2 + y}{(\ln(x) - 2)x}$$
$$y' = \omega(x, y)$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x + \omega(\eta_y - \xi_x) - \omega^2\xi_y - \omega_x\xi - \omega_y\eta = 0 \tag{A}$$

The type of this ode is known. It is of type `linear`. Therefore we do not need to solve the PDE (A), and can just use the lookup table shown below to find ξ, η

Table 243: Lie symmetry infinitesimal lookup table for known first order ODE's

ODE class	Form	ξ	η
linear ode	$y' = f(x)y(x) + g(x)$	0	$e^{\int f dx}$
separable ode	$y' = f(x)\, g(y)$	$\frac{1}{f}$	0
quadrature ode	$y' = f(x)$	0	1
quadrature ode	$y' = g(y)$	1	0
homogeneous ODEs of Class A	$y' = f\left(\frac{y}{x}\right)$	x	y
homogeneous ODEs of Class C	$y' = (a + bx + cy)^{\frac{n}{m}}$	1	$-\frac{b}{c}$
homogeneous class D	$y' = \frac{y}{x} + g(x)\, F\left(\frac{y}{x}\right)$	x^2	xy
First order special form ID 1	$y' = g(x)\, e^{h(x)+by} + f(x)$	$\frac{e^{-\int bf(x)dx - h(x)}}{g(x)}$	$\frac{f(x)e^{-\int bf(x)dx - h(x)}}{g(x)}$
polynomial type ode	$y' = \frac{a_1x+b_1y+c_1}{a_2x+b_2y+c_2}$	$\frac{a_1b_2x - a_2b_1x - b_1c_2 + b_2c_1}{a_1b_2 - a_2b_1}$	$\frac{a_1b_2y - a_2b_1y - a_1c_2 - a_2c_1}{a_1b_2 - a_2b_1}$
Bernoulli ode	$y' = f(x)\, y + g(x)\, y^n$	0	$e^{-\int (n-1)f(x)dx} y^n$
Reduced Riccati	$y' = f_1(x)\, y + f_2(x)\, y^2$	0	$e^{-\int f_1 dx}$

The above table shows that

$$
\begin{aligned}
\xi(x, y) &= 0 \\
\eta(x, y) &= \frac{1}{\ln(x) - 2}
\end{aligned} \tag{A1}
$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x, y) \to (R, S)$ where (R, S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$
\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}
$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y}\right) S(x, y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the

canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$R = x$$

S is found from

$$\begin{aligned} S &= \int \frac{1}{\eta} dy \\ &= \int \frac{1}{\frac{1}{\ln(x)-2}} dy \end{aligned}$$

Which results in

$$S = (\ln (x) - 2)\, y$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x,y) S_y}{R_x + \omega(x,y) R_y} \tag{2}$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x,y)$ is the right hand side of the original ode given by

$$\omega(x,y) = -\frac{6x^2 + y}{(\ln (x) - 2)\, x}$$

Evaluating all the partial derivatives gives

$$\begin{aligned} R_x &= 1 \\ R_y &= 0 \\ S_x &= \frac{y}{x} \\ S_y &= \ln (x) - 2 \end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = -6x \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = -6R$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordinates R, S. Integrating the above gives

$$S(R) = -3R^2 + c_1 \tag{4}$$

To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$(\ln(x) - 2)\, y = -3x^2 + c_1$$

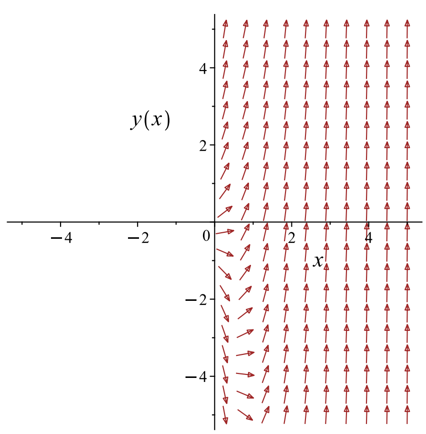
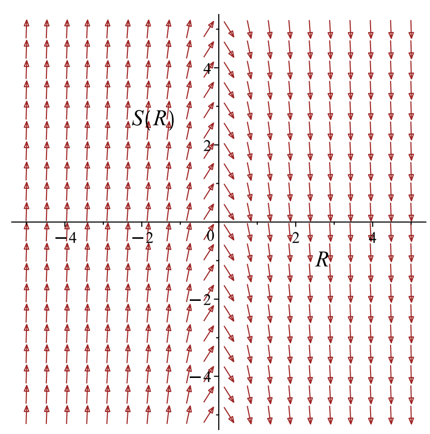
Which simplifies to

$$(\ln(x) - 2)\, y = -3x^2 + c_1$$

Which gives

$$y = \frac{-3x^2 + c_1}{\ln(x) - 2}$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in x, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dx} = -\frac{6x^2+y}{(\ln(x)-2)x}$ 	$R = x$ $S = (\ln(x) - 2)\, y$	$\frac{dS}{dR} = -6R$ 

Summary
The solution(s) found are the following

$$y = \frac{-3x^2 + c_1}{\ln(x) - 2} \tag{1}$$

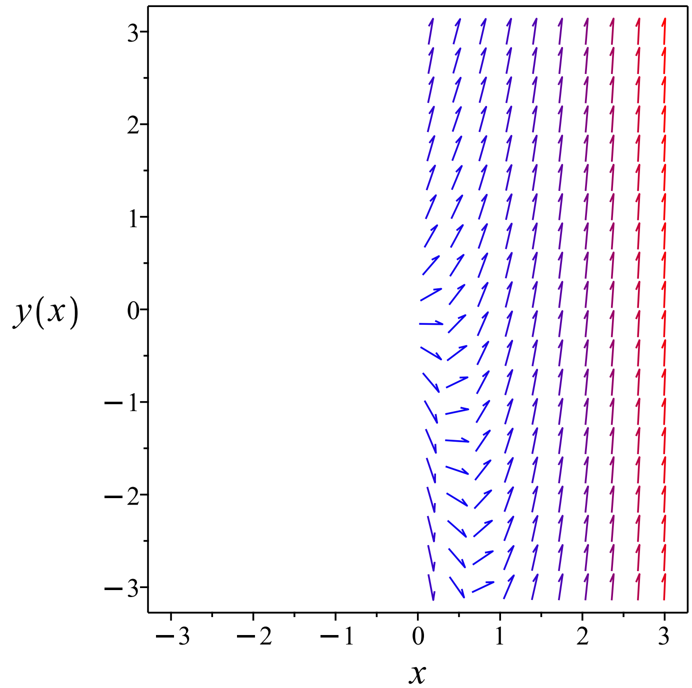


Figure 240: Slope field plot

Verification of solutions

$$y = \frac{-3x^2 + c_1}{\ln(x) - 2}$$

Verified OK.

5.10.3 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x,y) + N(x,y)\frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x,y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y) = 0$$

Hence

$$\frac{\partial \phi}{\partial x} + \frac{\partial \phi}{\partial y}\frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$\frac{\partial \phi}{\partial x} = M$$
$$\frac{\partial \phi}{\partial y} = N$$

But since $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x,y)$ but at least we know now that we can do that since the condition $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x,y)\,\mathrm{d}x + N(x,y)\,\mathrm{d}y = 0 \tag{1A}$$

Therefore

$$(\ln(x) - 2)\,\mathrm{d}y = \left(-\frac{y}{x} - 6x\right)\mathrm{d}x$$
$$\left(\frac{y}{x} + 6x\right)\mathrm{d}x + (\ln(x) - 2)\,\mathrm{d}y = 0 \tag{2A}$$

Comparing (1A) and (2A) shows that

$$M(x,y) = \frac{y}{x} + 6x$$
$$N(x,y) = \ln(x) - 2$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Using result found above gives

$$\frac{\partial M}{\partial y} = \frac{\partial}{\partial y}\left(\frac{y}{x} + 6x\right)$$
$$= \frac{1}{x}$$

And

$$\begin{aligned}\frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}(\ln(x) - 2)\\ &= \frac{1}{x}\end{aligned}$$

Since $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$, then the ODE is exact The following equations are now set up to solve for the function $\phi(x, y)$

$$\frac{\partial \phi}{\partial x} = M \tag{1}$$

$$\frac{\partial \phi}{\partial y} = N \tag{2}$$

Integrating (1) w.r.t. x gives

$$\int \frac{\partial \phi}{\partial x}\,\mathrm{d}x = \int M\,\mathrm{d}x$$

$$\int \frac{\partial \phi}{\partial x}\,\mathrm{d}x = \int \frac{y}{x} + 6x\,\mathrm{d}x$$

$$\phi = \ln(x)\,y + 3x^2 + f(y) \tag{3}$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$\frac{\partial \phi}{\partial y} = \ln(x) + f'(y) \tag{4}$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = \ln(x) - 2$. Therefore equation (4) becomes

$$\ln(x) - 2 = \ln(x) + f'(y) \tag{5}$$

Solving equation (5) for $f'(y)$ gives

$$f'(y) = -2$$

Integrating the above w.r.t y gives

$$\int f'(y)\,\mathrm{d}y = \int (-2)\,\mathrm{d}y$$

$$f(y) = -2y + c_1$$

Where c_1 is constant of integration. Substituting result found above for $f(y)$ into equation (3) gives ϕ

$$\phi = \ln(x)\, y + 3x^2 - 2y + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = \ln(x)\, y + 3x^2 - 2y$$

The solution becomes

$$y = \frac{-3x^2 + c_1}{\ln(x) - 2}$$

Summary
The solution(s) found are the following

$$y = \frac{-3x^2 + c_1}{\ln(x) - 2} \tag{1}$$

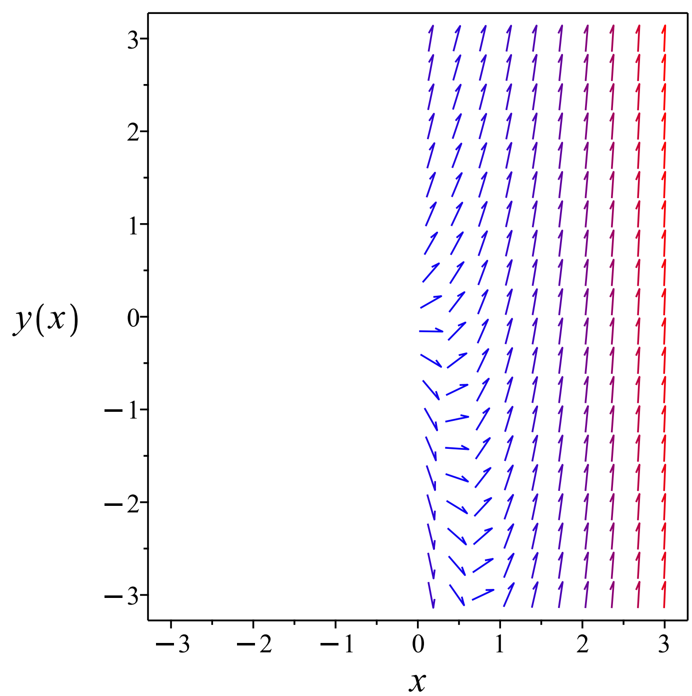


Figure 241: Slope field plot

Verification of solutions

$$y = \frac{-3x^2 + c_1}{\ln(x) - 2}$$

Verified OK.

5.10.4 Maple step by step solution

Let's solve

$\frac{y}{x} + (\ln(x) - 2)\, y' = -6x$

- Highest derivative means the order of the ODE is 1

  y'

- Isolate the derivative

  $y' = -\frac{y}{(\ln(x)-2)x} - \frac{6x}{\ln(x)-2}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE

  $y' + \frac{y}{(\ln(x)-2)x} = -\frac{6x}{\ln(x)-2}$

- The ODE is linear; multiply by an integrating factor $\mu(x)$

  $\mu(x)\left(y' + \frac{y}{(\ln(x)-2)x}\right) = -\frac{6\mu(x)x}{\ln(x)-2}$

- Assume the lhs of the ODE is the total derivative $\frac{d}{dx}(\mu(x)\, y)$

  $\mu(x)\left(y' + \frac{y}{(\ln(x)-2)x}\right) = \mu'(x)\, y + \mu(x)\, y'$

- Isolate $\mu'(x)$

  $\mu'(x) = \frac{\mu(x)}{(\ln(x)-2)x}$

- Solve to find the integrating factor

  $\mu(x) = \ln(x) - 2$

- Integrate both sides with respect to x

  $\int \left(\frac{d}{dx}(\mu(x)\, y)\right) dx = \int -\frac{6\mu(x)x}{\ln(x)-2} dx + c_1$

- Evaluate the integral on the lhs

  $\mu(x)\, y = \int -\frac{6\mu(x)x}{\ln(x)-2} dx + c_1$

- Solve for y

  $y = \frac{\int -\frac{6\mu(x)x}{\ln(x)-2} dx + c_1}{\mu(x)}$

- Substitute $\mu(x) = \ln(x) - 2$

  $y = \frac{\int -6x dx + c_1}{\ln(x) - 2}$

- Evaluate the integrals on the rhs

  $y = \frac{-3x^2 + c_1}{\ln(x) - 2}$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
<- 1st order linear successful`
```

✓ Solution by Maple

Time used: 0.219 (sec). Leaf size: 18

```
dsolve((y(x)/x+6*x)+(ln(x)-2)*diff(y(x),x) = 0,y(x), singsol=all)
```

$$y(x) = \frac{-3x^2 + c_1}{\ln(x) - 2}$$

✓ Solution by Mathematica

Time used: 0.079 (sec). Leaf size: 20

```
DSolve[(y[x]/x+6*x)+(Log[x]-2)*y'[x] == 0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \frac{-3x^2 + c_1}{\log(x) - 2}$$

5.11 problem 11

Internal problem ID [553]
Internal file name [`OUTPUT/553_Sunday_June_05_2022_01_44_26_AM_51220061/index.tex`]

Book: Elementary differential equations and boundary value problems, 10th ed., Boyce and DiPrima
Section: Section 2.6. Page 100
Problem number: 11.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"unknown"**

Maple gives the following as the ode type

```
[[_Abel, `2nd type`, `class B`]]
```

Unable to solve or complete the solution.

$$\boxed{yx + (\ln (x)\, y + yx)\, y' = -x \ln (x)}$$

Unable to determine ODE type.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
trying homogeneous types:
trying Chini
differential order: 1; looking for linear symmetries
trying exact
trying Abel
Looking for potential symmetries
Looking for potential symmetries
Looking for potential symmetries
trying inverse_Riccati
trying an equivalence to an Abel ODE
differential order: 1; trying a linearization to 2nd order
--- trying a change of variables {x -> y(x), y(x) -> x}
differential order: 1; trying a linearization to 2nd order
trying 1st order ODE linearizable_by_differentiation
--- Trying Lie symmetry methods, 1st order ---
`, `-> Computing symmetries using: way = 3
`, `-> Computing symmetries using: way = 4
`, `-> Computing symmetries using: way = 2
trying symmetry patterns for 1st order ODEs
-> trying a symmetry pattern of the form [F(x)*G(y), 0]
-> trying a symmetry pattern of the form [0, F(x)*G(y)]
-> trying symmetry patterns of the forms [F(x),G(y)] and [G(y),F(x)]
`, `-> Computing symmetries using: way = HINT
   -> Calling odsolve with the ODE`, diff(y(x), x)+y(x)*(ln(x)-1)/(x*(ln(x)+x)), y(x)`
      Methods for first order ODEs:
      --- Trying classification methods ---
      trying a quadrature
      trying 1st order linear
      <- 1st order linear successful
   -> Calling odsolve with the ODE`, diff(y(x), x)+y(x)*(ln(x)^2+x)/(x*(ln(x)+x)*ln(x)), y(x
      Methods for first order ODEs:
      --- Trying classification methods ---
      trying a quadrature
      trying 1st order linear
      <- 1st order linear successful
`, `-> Computing symmetries using: way = HINT
-> trying a symmetry pattern of the form [F(x),G(x)]
```

✗ Solution by Maple

```
dsolve((x*ln(x)+x*y(x))+(y(x)*ln(x)+x*y(x))*diff(y(x),x) = 0,y(x), singsol=all)
```

No solution found

✗ Solution by Mathematica

Time used: 0.0 (sec). Leaf size: 0

```
DSolve[(x*Log[x]+x*y[x])+(y[x]*Log[x]+x*y[x])*y'[x] == 0,y[x],x,IncludeSingularSolutions ->
```

Not solved

5.12 problem 12

Internal problem ID [554]
Internal file name [`OUTPUT/554_Sunday_June_05_2022_01_44_30_AM_73717968/index.tex`]

Book: Elementary differential equations and boundary value problems, 10th ed., Boyce and DiPrima
Section: Section 2.6. Page 100
Problem number: 12.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"exact", "separable", "differentialType", "homogeneousTypeD2", "first_order_ode_lie_symmetry_lookup"**

Maple gives the following as the ode type

`[_separable]`

$$\boxed{\frac{x}{(x^2+y^2)^{\frac{3}{2}}}+\frac{yy'}{(x^2+y^2)^{\frac{3}{2}}}=0}$$

5.12.1 Solving as separable ode

In canonical form the ODE is

$$\begin{aligned}y' &= F(x,y)\\ &= f(x)g(y)\\ &= -\frac{x}{y}\end{aligned}$$

Where $f(x) = -x$ and $g(y) = \frac{1}{y}$. Integrating both sides gives

$$\frac{1}{\frac{1}{y}}\,dy = -x\,dx$$

$$\int \frac{1}{\frac{1}{y}}\,dy = \int -x\,dx$$

$$\frac{y^2}{2} = -\frac{x^2}{2} + c_1$$

Which results in

$$y = \sqrt{-x^2 + 2c_1}$$

$$y = -\sqrt{-x^2 + 2c_1}$$

Summary

The solution(s) found are the following

$$y = \sqrt{-x^2 + 2c_1} \tag{1}$$

$$y = -\sqrt{-x^2 + 2c_1} \tag{2}$$

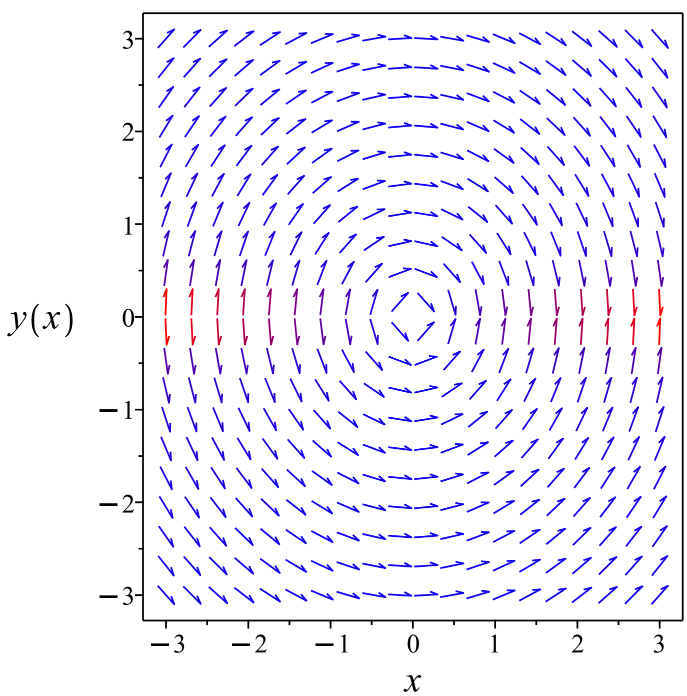


Figure 242: Slope field plot

Verification of solutions

$$y = \sqrt{-x^2 + 2c_1}$$

Verified OK.

$$y = -\sqrt{-x^2 + 2c_1}$$

Verified OK.

5.12.2 Solving as homogeneousTypeD2 ode

Using the change of variables $y = u(x)\,x$ on the above ode results in new ode in $u(x)$

$$\frac{x}{\left(x^2 + u\left(x\right)^2 x^2\right)^{\frac{3}{2}}} + \frac{u(x)\,x(u'(x)\,x + u(x))}{\left(x^2 + u\left(x\right)^2 x^2\right)^{\frac{3}{2}}} = 0$$

In canonical form the ODE is

$$\begin{aligned} u' &= F(x,u) \\ &= f(x)g(u) \\ &= -\frac{u^2+1}{ux} \end{aligned}$$

Where $f(x) = -\frac{1}{x}$ and $g(u) = \frac{u^2+1}{u}$. Integrating both sides gives

$$\begin{aligned} \frac{1}{\frac{u^2+1}{u}}\,du &= -\frac{1}{x}\,dx \\ \int \frac{1}{\frac{u^2+1}{u}}\,du &= \int -\frac{1}{x}\,dx \\ \frac{\ln\left(u^2+1\right)}{2} &= -\ln\left(x\right) + c_2 \end{aligned}$$

Raising both side to exponential gives

$$\sqrt{u^2+1} = \mathrm{e}^{-\ln(x)+c_2}$$

Which simplifies to

$$\sqrt{u^2+1} = \frac{c_3}{x}$$

Which simplifies to

$$\sqrt{u\left(x\right)^2 + 1} = \frac{c_3 \mathrm{e}^{c_2}}{x}$$

The solution is

$$\sqrt{u(x)^2+1} = \frac{c_3 e^{c_2}}{x}$$

Replacing $u(x)$ in the above solution by $\frac{y}{x}$ results in the solution for y in implicit form

$$\sqrt{\frac{y^2}{x^2}+1} = \frac{c_3 e^{c_2}}{x}$$
$$\sqrt{\frac{x^2+y^2}{x^2}} = \frac{c_3 e^{c_2}}{x}$$

Summary
The solution(s) found are the following

$$\sqrt{\frac{x^2+y^2}{x^2}} = \frac{c_3 e^{c_2}}{x} \tag{1}$$

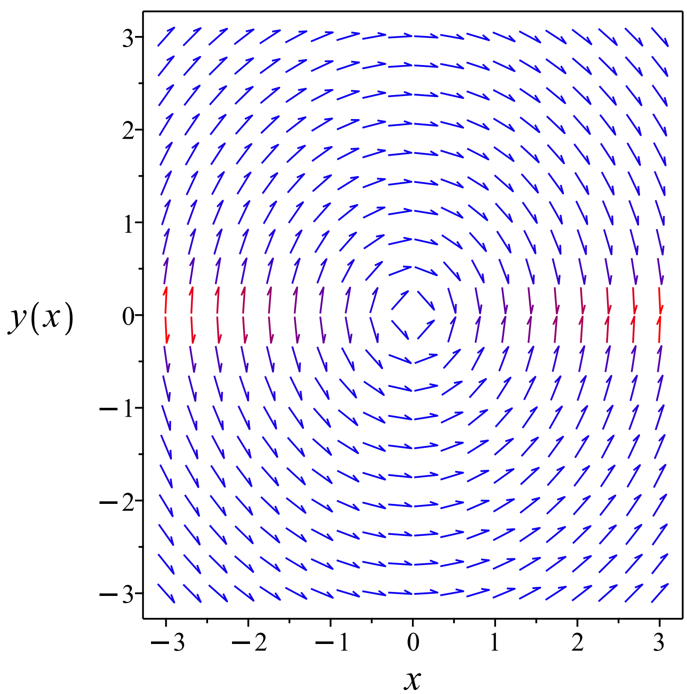


Figure 243: Slope field plot

Verification of solutions

$$\sqrt{\frac{x^2+y^2}{x^2}} = \frac{c_3 e^{c_2}}{x}$$

Verified OK.

5.12.3 Solving as differentialType ode

Writing the ode as

$$y' = -\frac{x}{y} \tag{1}$$

Which becomes

$$(y)\, dy = (-x)\, dx \tag{2}$$

But the RHS is complete differential because

$$(-x)\, dx = d\left(-\frac{x^2}{2}\right)$$

Hence (2) becomes

$$(y)\, dy = d\left(-\frac{x^2}{2}\right)$$

Integrating both sides gives gives these solutions

$$y = \sqrt{-x^2 + 2c_1} + c_1$$
$$y = -\sqrt{-x^2 + 2c_1} + c_1$$

<u>Summary</u>
The solution(s) found are the following

$$y = \sqrt{-x^2 + 2c_1} + c_1 \tag{1}$$
$$y = -\sqrt{-x^2 + 2c_1} + c_1 \tag{2}$$

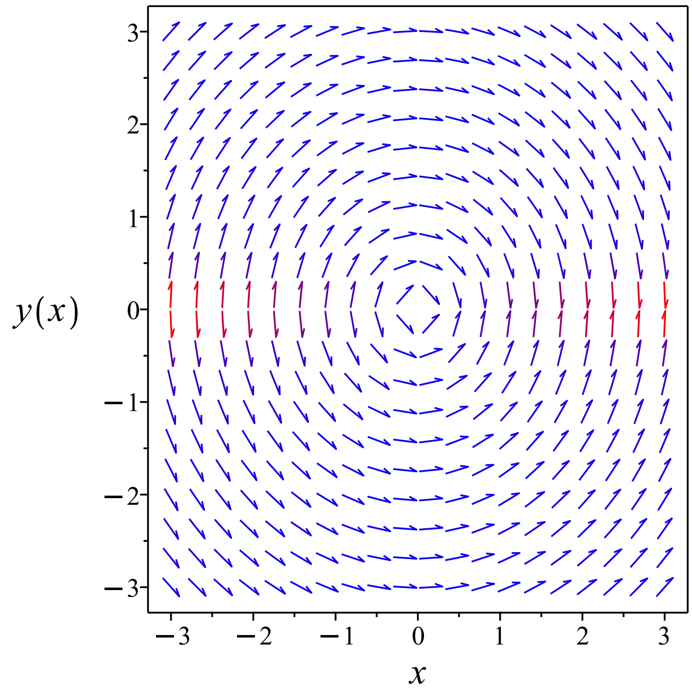


Figure 244: Slope field plot

Verification of solutions

$$y = \sqrt{-x^2 + 2c_1} + c_1$$

Verified OK.

$$y = -\sqrt{-x^2 + 2c_1} + c_1$$

Verified OK.

5.12.4 Solving as first order ode lie symmetry lookup ode

Writing the ode as

$$y' = -\frac{x}{y}$$
$$y' = \omega(x, y)$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x + \omega(\eta_y - \xi_x) - \omega^2 \xi_y - \omega_x \xi - \omega_y \eta = 0 \tag{A}$$

The type of this ode is known. It is of type `separable`. Therefore we do not need to solve the PDE (A), and can just use the lookup table shown below to find ξ, η

Table 246: Lie symmetry infinitesimal lookup table for known first order ODE's

ODE class	Form	ξ	η
linear ode	$y' = f(x)y(x) + g(x)$	0	$e^{\int f dx}$
separable ode	$y' = f(x)\,g(y)$	$\frac{1}{f}$	0
quadrature ode	$y' = f(x)$	0	1
quadrature ode	$y' = g(y)$	1	0
homogeneous ODEs of Class A	$y' = f\left(\frac{y}{x}\right)$	x	y
homogeneous ODEs of Class C	$y' = (a + bx + cy)^{\frac{n}{m}}$	1	$-\frac{b}{c}$
homogeneous class D	$y' = \frac{y}{x} + g(x)\,F\left(\frac{y}{x}\right)$	x^2	xy
First order special form ID 1	$y' = g(x)\,e^{h(x)+by} + f(x)$	$\frac{e^{-\int bf(x)dx-h(x)}}{g(x)}$	$\frac{f(x)e^{-\int bf(x)dx-h(x)}}{g(x)}$
polynomial type ode	$y' = \frac{a_1x+b_1y+c_1}{a_2x+b_2y+c_2}$	$\frac{a_1b_2x-a_2b_1x-b_1c_2+b_2c_1}{a_1b_2-a_2b_1}$	$\frac{a_1b_2y-a_2b_1y-a_1c_2-a_2c_1}{a_1b_2-a_2b_1}$
Bernoulli ode	$y' = f(x)\,y + g(x)\,y^n$	0	$e^{-\int(n-1)f(x)dx}y^n$
Reduced Riccati	$y' = f_1(x)\,y + f_2(x)\,y^2$	0	$e^{-\int f_1 dx}$

The above table shows that

$$
\begin{aligned}
\xi(x,y) &= -\frac{1}{x}\\
\eta(x,y) &= 0
\end{aligned}
\tag{A1}
$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x,y) \to (R,S)$ where (R,S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$
\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}
$$

The above comes from the requirements that $\left(\xi\frac{\partial}{\partial x} + \eta\frac{\partial}{\partial y}\right)S(x,y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the

canonical coordinates, where $S(R)$. Since $\eta = 0$ then in this special case

$$R = y$$

S is found from

$$\begin{aligned} S &= \int \frac{1}{\xi} dx \\ &= \int \frac{1}{-\frac{1}{x}} dx \end{aligned}$$

Which results in

$$S = -\frac{x^2}{2}$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x, y) S_y}{R_x + \omega(x, y) R_y} \tag{2}$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x, y)$ is the right hand side of the original ode given by

$$\omega(x, y) = -\frac{x}{y}$$

Evaluating all the partial derivatives gives

$$\begin{aligned} R_x &= 0 \\ R_y &= 1 \\ S_x &= -x \\ S_y &= 0 \end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = y \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = R$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by

integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$S(R) = \frac{R^2}{2} + c_1 \tag{4}$$

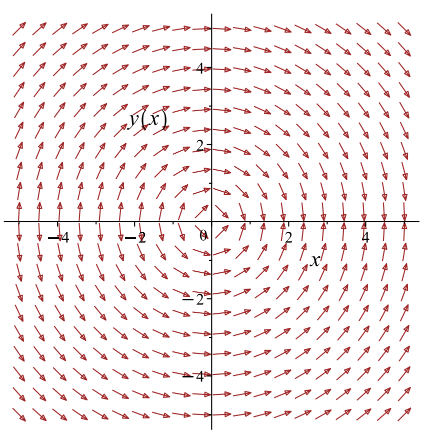
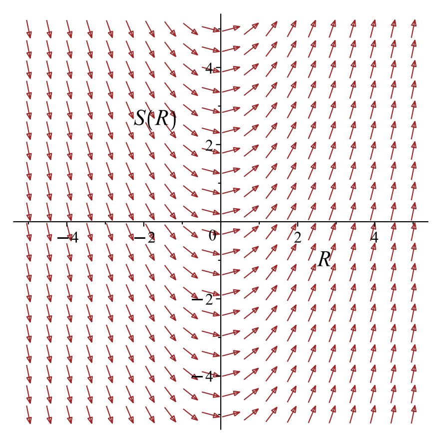
To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$-\frac{x^2}{2} = \frac{y^2}{2} + c_1$$

Which simplifies to

$$-\frac{x^2}{2} = \frac{y^2}{2} + c_1$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in x, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dx} = -\frac{x}{y}$ 	$R = y$ $S = -\frac{x^2}{2}$	$\frac{dS}{dR} = R$ 

Summary
The solution(s) found are the following

$$-\frac{x^2}{2} = \frac{y^2}{2} + c_1 \tag{1}$$

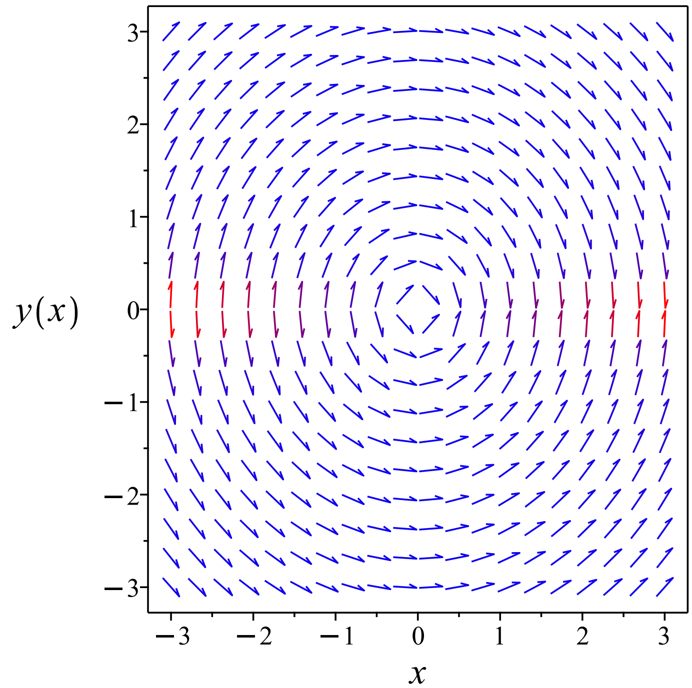


Figure 245: Slope field plot

Verification of solutions

$$-\frac{x^2}{2} = \frac{y^2}{2} + c_1$$

Verified OK.

5.12.5 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x, y) + N(x, y)\frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x, y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x, y) = 0$$

Hence

$$\frac{\partial \phi}{\partial x} + \frac{\partial \phi}{\partial y}\frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$\frac{\partial\phi}{\partial x} = M$$
$$\frac{\partial\phi}{\partial y} = N$$

But since $\frac{\partial^2\phi}{\partial x\partial y} = \frac{\partial^2\phi}{\partial y\partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x, y)$ but at least we know now that we can do that since the condition $\frac{\partial^2\phi}{\partial x\partial y} = \frac{\partial^2\phi}{\partial y\partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x, y)\,\mathrm{d}x + N(x, y)\,\mathrm{d}y = 0 \tag{1A}$$

Therefore

$$(-y)\,\mathrm{d}y = (x)\,\mathrm{d}x$$
$$(-x)\,\mathrm{d}x + (-y)\,\mathrm{d}y = 0 \tag{2A}$$

Comparing (1A) and (2A) shows that

$$M(x, y) = -x$$
$$N(x, y) = -y$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Using result found above gives

$$\frac{\partial M}{\partial y} = \frac{\partial}{\partial y}(-x)$$
$$= 0$$

And

$$\frac{\partial N}{\partial x} = \frac{\partial}{\partial x}(-y)$$
$$= 0$$

Since $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$, then the ODE is exact The following equations are now set up to solve for the function $\phi(x,y)$

$$\frac{\partial \phi}{\partial x} = M \tag{1}$$
$$\frac{\partial \phi}{\partial y} = N \tag{2}$$

Integrating (1) w.r.t. x gives

$$\int \frac{\partial \phi}{\partial x} \, \mathrm{d}x = \int M \, \mathrm{d}x$$
$$\int \frac{\partial \phi}{\partial x} \, \mathrm{d}x = \int -x \, \mathrm{d}x$$
$$\phi = -\frac{x^2}{2} + f(y) \tag{3}$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$\frac{\partial \phi}{\partial y} = 0 + f'(y) \tag{4}$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = -y$. Therefore equation (4) becomes

$$-y = 0 + f'(y) \tag{5}$$

Solving equation (5) for $f'(y)$ gives

$$f'(y) = -y$$

Integrating the above w.r.t y gives

$$\int f'(y) \, \mathrm{d}y = \int (-y) \, \mathrm{d}y$$
$$f(y) = -\frac{y^2}{2} + c_1$$

Where c_1 is constant of integration. Substituting result found above for $f(y)$ into equation (3) gives ϕ

$$\phi = -\frac{x^2}{2} - \frac{y^2}{2} + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = -\frac{x^2}{2} - \frac{y^2}{2}$$

Summary

The solution(s) found are the following

$$-\frac{x^2}{2} - \frac{y^2}{2} = c_1 \tag{1}$$

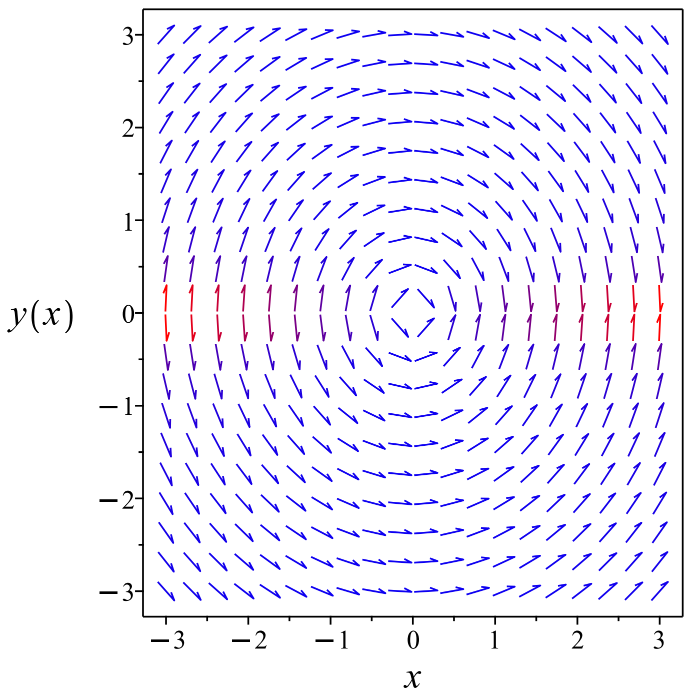


Figure 246: Slope field plot

Verification of solutions

$$-\frac{x^2}{2} - \frac{y^2}{2} = c_1$$

Verified OK.

5.12.6 Maple step by step solution

Let's solve

$\frac{x}{(x^2+y^2)^{\frac{3}{2}}}+\frac{yy'}{(x^2+y^2)^{\frac{3}{2}}}=0$

- Highest derivative means the order of the ODE is 1

  y'

- Integrate both sides with respect to x

  $\int \left(\frac{x}{(x^2+y^2)^{\frac{3}{2}}}+\frac{yy'}{(x^2+y^2)^{\frac{3}{2}}}\right)dx=\int 0dx+c_1$

- Evaluate integral

  $-\frac{1}{\sqrt{x^2+y^2}}=c_1$

- Solve for y

  $\left\{y=\frac{\sqrt{-c_1^2x^2+1}}{c_1}, y=-\frac{\sqrt{-c_1^2x^2+1}}{c_1}\right\}$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
<- Bernoulli successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 27

```
dsolve(x/(x^2+y(x)^2)^(3/2)+y(x)*diff(y(x),x)/(x^2+y(x)^2)^(3/2) = 0,y(x), singsol=all)
```

$$y(x)=\sqrt{-x^2+c_1}$$
$$y(x)=-\sqrt{-x^2+c_1}$$

✓ Solution by Mathematica

Time used: 0.09 (sec). Leaf size: 39

```
DSolve[x/(x^2+y[x]^2)^(3/2)+y[x]*y'[x]/(x^2+y[x]^2)^(3/2) == 0,y[x],x,IncludeSingularSolutio
```

$$y(x) \to -\sqrt{-x^2 + 2c_1}$$
$$y(x) \to \sqrt{-x^2 + 2c_1}$$

5.13 problem 13

Internal problem ID [555]
Internal file name [`OUTPUT/555_Sunday_June_05_2022_01_44_31_AM_94833341/index.tex`]

Book: Elementary differential equations and boundary value problems, 10th ed., Boyce and DiPrima
Section: Section 2.6. Page 100
Problem number: 13.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"exact", "differentialType", "homogeneousTypeD2", "first_order_ode_lie_symmetry_calculated"**

Maple gives the following as the ode type

```
[[_homogeneous, `class A`], _exact, _rational, [_Abel, `2nd
   type`, `class A`]]
```

$$\boxed{-y + (-x + 2y)\, y' = -2x}$$

With initial conditions

$$[y(1) = 3]$$

5.13.1 Existence and uniqueness analysis

This is non linear first order ODE. In canonical form it is written as

$$\begin{aligned} y' &= f(x,y) \\ &= \frac{-2x + y}{-x + 2y} \end{aligned}$$

The x domain of $f(x,y)$ when $y=3$ is

$$\{x<6\vee 6<x\}$$

And the point $x_0=1$ is inside this domain. The y domain of $f(x,y)$ when $x=1$ is

$$\left\{y<\frac{1}{2}\vee\frac{1}{2}<y\right\}$$

And the point $y_0=3$ is inside this domain. Now we will look at the continuity of

$$\begin{aligned}\frac{\partial f}{\partial y}&=\frac{\partial}{\partial y}\left(\frac{-2x+y}{-x+2y}\right)\\&=\frac{1}{-x+2y}-\frac{2(-2x+y)}{(-x+2y)^2}\end{aligned}$$

The x domain of $\frac{\partial f}{\partial y}$ when $y=3$ is

$$\{x<6\vee 6<x\}$$

And the point $x_0=1$ is inside this domain. The y domain of $\frac{\partial f}{\partial y}$ when $x=1$ is

$$\left\{y<\frac{1}{2}\vee\frac{1}{2}<y\right\}$$

And the point $y_0=3$ is inside this domain. Therefore solution exists and is unique.

5.13.2 Solving as homogeneousTypeD2 ode

Using the change of variables $y=u(x)\,x$ on the above ode results in new ode in $u(x)$

$$-u(x)\,x+(-x+2u(x)\,x)\,(u'(x)\,x+u(x))=-2x$$

In canonical form the ODE is

$$\begin{aligned}u'&=F(x,u)\\&=f(x)g(u)\\&=-\frac{2(u^2-u+1)}{x\,(2u-1)}\end{aligned}$$

Where $f(x) = -\frac{2}{x}$ and $g(u) = \frac{u^2-u+1}{2u-1}$. Integrating both sides gives

$$\frac{1}{\frac{u^2-u+1}{2u-1}}\,du = -\frac{2}{x}\,dx$$
$$\int \frac{1}{\frac{u^2-u+1}{2u-1}}\,du = \int -\frac{2}{x}\,dx$$
$$\ln\left(u^2-u+1\right) = -2\ln(x) + c_2$$

Raising both side to exponential gives

$$u^2 - u + 1 = \mathrm{e}^{-2\ln(x)+c_2}$$

Which simplifies to

$$u^2 - u + 1 = \frac{c_3}{x^2}$$

Which simplifies to

$$u(x)^2 - u(x) + 1 = \frac{c_3 \mathrm{e}^{c_2}}{x^2}$$

The solution is

$$u(x)^2 - u(x) + 1 = \frac{c_3 \mathrm{e}^{c_2}}{x^2}$$

Replacing $u(x)$ in the above solution by $\frac{y}{x}$ results in the solution for y in implicit form

$$\frac{y^2}{x^2} - \frac{y}{x} + 1 = \frac{c_3 \mathrm{e}^{c_2}}{x^2}$$
$$\frac{y^2}{x^2} - \frac{y}{x} + 1 = \frac{c_3 \mathrm{e}^{c_2}}{x^2}$$

Which simplifies to

$$y^2 - yx + x^2 = c_3 \mathrm{e}^{c_2}$$

Substituting initial conditions and solving for c_2 gives $c_2 = \ln\left(\frac{7}{c_3}\right)$. Hence the solution becomes

Summary

The solution(s) found are the following

$$y^2 - yx + x^2 = 7 \tag{1}$$

Verification of solutions

$$y^2 - yx + x^2 = 7$$

Verified OK.

5.13.3 Solving as differentialType ode

Writing the ode as

$$y' = \frac{-2x + y}{-x + 2y} \tag{1}$$

Which becomes

$$(-2y)\,dy = (-x)\,dy + (2x - y)\,dx \tag{2}$$

But the RHS is complete differential because

$$(-x)\,dy + (2x - y)\,dx = d\left(x^2 - yx\right)$$

Hence (2) becomes

$$(-2y)\,dy = d\left(x^2 - yx\right)$$

Integrating both sides gives gives these solutions

$$y = \frac{x}{2} + \frac{\sqrt{-3x^2 - 4c_1}}{2} + c_1$$
$$y = \frac{x}{2} - \frac{\sqrt{-3x^2 - 4c_1}}{2} + c_1$$

Initial conditions are used to solve for c_1. Substituting $x = 1$ and $y = 3$ in the above solution gives an equation to solve for the constant of integration.

$$3 = \frac{1}{2} - \frac{\sqrt{-3 - 4c_1}}{2} + c_1$$

Warning: Unable to solve for constant of integration. Initial conditions are used to solve for c_1. Substituting $x = 1$ and $y = 3$ in the above solution gives an equation to solve for the constant of integration.

$$3 = \frac{1}{2} + \frac{\sqrt{-3 - 4c_1}}{2} + c_1$$

$$c_1 = 2 - i\sqrt{3}$$

Substituting c_1 found above in the general solution gives

$$y = \frac{x}{2} + \frac{\sqrt{-3x^2 - 8 + 4i\sqrt{3}}}{2} + 2 - i\sqrt{3}$$

Summary

The solution(s) found are the following

$$y = \frac{x}{2} + \frac{\sqrt{-3x^2 - 8 + 4i\sqrt{3}}}{2} + 2 - i\sqrt{3} \tag{1}$$

Verification of solutions

$$y = \frac{x}{2} + \frac{\sqrt{-3x^2 - 8 + 4i\sqrt{3}}}{2} + 2 - i\sqrt{3}$$

Verified OK.

5.13.4 Solving as first order ode lie symmetry calculated ode

Writing the ode as

$$\begin{aligned} y' &= \frac{-2x + y}{-x + 2y} \\ y' &= \omega(x, y) \end{aligned}$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x + \omega(\eta_y - \xi_x) - \omega^2 \xi_y - \omega_x \xi - \omega_y \eta = 0 \tag{A}$$

The type of this ode is not in the lookup table. To determine ξ, η then (A) is solved using ansatz. Making bivariate polynomials of degree 1 to use as anstaz gives

$$\xi = x a_2 + y a_3 + a_1 \tag{1E}$$

$$\eta = x b_2 + y b_3 + b_1 \tag{2E}$$

Where the unknown coefficients are

$$\{a_1, a_2, a_3, b_1, b_2, b_3\}$$

Substituting equations (1E,2E) and ω into (A) gives

$$\begin{aligned} & b_2 + \frac{(-2x + y)(b_3 - a_2)}{-x + 2y} - \frac{(-2x + y)^2 a_3}{(-x + 2y)^2} \\ & - \left(-\frac{2}{-x + 2y} + \frac{-2x + y}{(-x + 2y)^2} \right) (x a_2 + y a_3 + a_1) \\ & - \left(\frac{1}{-x + 2y} - \frac{2(-2x + y)}{(-x + 2y)^2} \right) (x b_2 + y b_3 + b_1) = 0 \end{aligned} \tag{5E}$$

Putting the above in normal form gives

$$-\frac{2x^2a_2+4x^2a_3+2x^2b_2-2x^2b_3-8xya_2-4xya_3+4xyb_2+8xyb_3+2y^2a_2-2y^2a_3-4y^2b_2-2y^2b_3+3}{(x-2y)^2}$$
$$=0$$

Setting the numerator to zero gives

$$-2x^2a_2-4x^2a_3-2x^2b_2+2x^2b_3+8xya_2+4xya_3-4xyb_2 \\ -8xyb_3-2y^2a_2+2y^2a_3+4y^2b_2+2y^2b_3-3xb_1+3ya_1=0 \tag{6E}$$

Looking at the above PDE shows the following are all the terms with $\{x,y\}$ in them.

$$\{x,y\}$$

The following substitution is now made to be able to collect on all terms with $\{x,y\}$ in them

$$\{x=v_1,y=v_2\}$$

The above PDE (6E) now becomes

$$-2a_2v_1^2+8a_2v_1v_2-2a_2v_2^2-4a_3v_1^2+4a_3v_1v_2+2a_3v_2^2-2b_2v_1^2 \\ -4b_2v_1v_2+4b_2v_2^2+2b_3v_1^2-8b_3v_1v_2+2b_3v_2^2+3a_1v_2-3b_1v_1=0 \tag{7E}$$

Collecting the above on the terms v_i introduced, and these are

$$\{v_1,v_2\}$$

Equation (7E) now becomes

$$(-2a_2-4a_3-2b_2+2b_3)v_1^2+(8a_2+4a_3-4b_2-8b_3)v_1v_2 \\ -3b_1v_1+(-2a_2+2a_3+4b_2+2b_3)v_2^2+3a_1v_2=0 \tag{8E}$$

Setting each coefficients in (8E) to zero gives the following equations to solve

$$\begin{aligned}3a_1&=0\\-3b_1&=0\\-2a_2-4a_3-2b_2+2b_3&=0\\-2a_2+2a_3+4b_2+2b_3&=0\\8a_2+4a_3-4b_2-8b_3&=0\end{aligned}$$

Solving the above equations for the unknowns gives

$$
\begin{aligned}
a_1 &= 0 \\
a_2 &= b_2 + b_3 \\
a_3 &= -b_2 \\
b_1 &= 0 \\
b_2 &= b_2 \\
b_3 &= b_3
\end{aligned}
$$

Substituting the above solution in the anstaz (1E,2E) (using 1 as arbitrary value for any unknown in the RHS) gives

$$
\begin{aligned}
\xi &= x \\
\eta &= y
\end{aligned}
$$

Shifting is now applied to make $\xi = 0$ in order to simplify the rest of the computation

$$
\begin{aligned}
\eta &= \eta - \omega(x, y)\,\xi \\
&= y - \left(\frac{-2x + y}{-x + 2y}\right)(x) \\
&= \frac{-2x^2 + 2yx - 2y^2}{x - 2y} \\
\xi &= 0
\end{aligned}
$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x, y) \to (R, S)$ where (R, S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$
\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}
$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y}\right) S(x, y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$
R = x
$$

S is found from

$$
\begin{aligned}
S &= \int \frac{1}{\eta} dy \\
&= \int \frac{1}{\frac{-2x^2 + 2yx - 2y^2}{x - 2y}} dy
\end{aligned}
$$

Which results in

$$S = \frac{\ln(x^2 - yx + y^2)}{2}$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x,y)S_y}{R_x + \omega(x,y)R_y} \tag{2}$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x,y)$ is the right hand side of the original ode given by

$$\omega(x,y) = \frac{-2x + y}{-x + 2y}$$

Evaluating all the partial derivatives gives

$$\begin{aligned}
R_x &= 1 \\
R_y &= 0 \\
S_x &= \frac{2x - y}{2x^2 - 2yx + 2y^2} \\
S_y &= \frac{-x + 2y}{2x^2 - 2yx + 2y^2}
\end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = 0 \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = 0$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$S(R) = c_1 \tag{4}$$

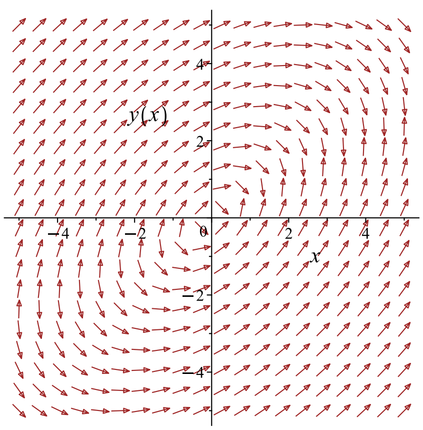
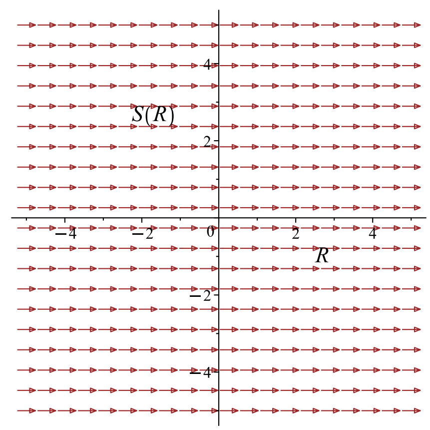
To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$\frac{\ln(y^2 - yx + x^2)}{2} = c_1$$

Which simplifies to

$$\frac{\ln \left(y^{2}-y x+x^{2}\right)}{2} = c_1$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in x, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dx} = \frac{-2x+y}{-x+2y}$ 	$R = x$ $S = \frac{\ln \left(x^{2}-y x+y^{2}\right)}{2}$	$\frac{dS}{dR} = 0$ 

Initial conditions are used to solve for c_1. Substituting $x = 1$ and $y = 3$ in the above solution gives an equation to solve for the constant of integration.

$$\frac{\ln \left(7\right)}{2} = c_1$$

$$c_1 = \frac{\ln \left(7\right)}{2}$$

Substituting c_1 found above in the general solution gives

$$\frac{\ln \left(x^{2}-y x+y^{2}\right)}{2} = \frac{\ln \left(7\right)}{2}$$

Summary

The solution(s) found are the following

$$\frac{\ln \left(y^{2}-y x+x^{2}\right)}{2} = \frac{\ln \left(7\right)}{2} \tag{1}$$

Verification of solutions

$$\frac{\ln\left(y^2 - yx + x^2\right)}{2} = \frac{\ln\left(7\right)}{2}$$

Verified OK.

5.13.5 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x,y) + N(x,y)\frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x,y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y) = 0$$

Hence

$$\frac{\partial \phi}{\partial x} + \frac{\partial \phi}{\partial y}\frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$\frac{\partial \phi}{\partial x} = M$$
$$\frac{\partial \phi}{\partial y} = N$$

But since $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x,y)$ but at least we know now that we can do that since the condition $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x,y)\,\mathrm{d}x + N(x,y)\,\mathrm{d}y = 0 \tag{1A}$$

Therefore

$$(-x + 2y)\,\mathrm{d}y = (-2x + y)\,\mathrm{d}x$$
$$(2x - y)\,\mathrm{d}x + (-x + 2y)\,\mathrm{d}y = 0 \tag{2A}$$

Comparing (1A) and (2A) shows that

$$
\begin{aligned}
M(x,y) &= 2x - y \\
N(x,y) &= -x + 2y
\end{aligned}
$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$
\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}
$$

Using result found above gives

$$
\begin{aligned}
\frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}(2x - y) \\
&= -1
\end{aligned}
$$

And

$$
\begin{aligned}
\frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}(-x + 2y) \\
&= -1
\end{aligned}
$$

Since $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$, then the ODE is exact The following equations are now set up to solve for the function $\phi(x,y)$

$$
\begin{aligned}
\frac{\partial \phi}{\partial x} &= M \qquad (1) \\
\frac{\partial \phi}{\partial y} &= N \qquad (2)
\end{aligned}
$$

Integrating (1) w.r.t. x gives

$$
\begin{aligned}
\int \frac{\partial \phi}{\partial x} \, \mathrm{d}x &= \int M \, \mathrm{d}x \\
\int \frac{\partial \phi}{\partial x} \, \mathrm{d}x &= \int 2x - y \, \mathrm{d}x \\
\phi &= x(x - y) + f(y) \qquad (3)
\end{aligned}
$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$
\frac{\partial \phi}{\partial y} = -x + f'(y) \qquad (4)
$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = -x + 2y$. Therefore equation (4) becomes

$$-x + 2y = -x + f'(y) \tag{5}$$

Solving equation (5) for $f'(y)$ gives

$$f'(y) = 2y$$

Integrating the above w.r.t y gives

$$\int f'(y)\, \mathrm{d}y = \int (2y)\, \mathrm{d}y$$
$$f(y) = y^2 + c_1$$

Where c_1 is constant of integration. Substituting result found above for $f(y)$ into equation (3) gives ϕ

$$\phi = x(x - y) + y^2 + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = x(x - y) + y^2$$

Initial conditions are used to solve for c_1. Substituting $x = 1$ and $y = 3$ in the above solution gives an equation to solve for the constant of integration.

$$7 = c_1$$

$$c_1 = 7$$

Substituting c_1 found above in the general solution gives

$$x(x - y) + y^2 = 7$$

Summary

The solution(s) found are the following

$$y^2 - yx + x^2 = 7 \tag{1}$$

Verification of solutions

$$y^2 - yx + x^2 = 7$$

Verified OK.

The solution

$$\frac{\ln (y^2 - yx + x^2)}{2} = \frac{\ln (7)}{2}$$

can be simplified to

$$\ln \left(y^2 - yx + x^2\right) = \ln (7)$$

5.13.6 Maple step by step solution

Let's solve

$[-y + (-x + 2y)\, y' = -2x, y(1) = 3]$

- Highest derivative means the order of the ODE is 1

  y'

- □ Check if ODE is exact
  - ODE is exact if the lhs is the total derivative of a C^2 function

    $F'(x, y) = 0$
  - Compute derivative of lhs

    $F'(x, y) + \left(\frac{\partial}{\partial y} F(x, y)\right) y' = 0$
  - Evaluate derivatives

    $-1 = -1$
  - Condition met, ODE is exact
- Exact ODE implies solution will be of this form

  $\left[F(x, y) = c_1, M(x, y) = F'(x, y)\, , N(x, y) = \frac{\partial}{\partial y} F(x, y)\right]$
- Solve for $F(x, y)$ by integrating $M(x, y)$ with respect to x

  $F(x, y) = \int (2x - y)\, dx + f_1(y)$
- Evaluate integral

  $F(x, y) = x^2 - yx + f_1(y)$
- Take derivative of $F(x, y)$ with respect to y

  $N(x, y) = \frac{\partial}{\partial y} F(x, y)$

- Compute derivative

  $-x+2y=-x+\frac{d}{dy}f_1(y)$
- Isolate for $\frac{d}{dy}f_1(y)$

  $\frac{d}{dy}f_1(y)=2y$
- Solve for $f_1(y)$

  $f_1(y)=y^2$
- Substitute $f_1(y)$ into equation for $F(x,y)$

  $F(x,y)=x^2-yx+y^2$
- Substitute $F(x,y)$ into the solution of the ODE

  $x^2-yx+y^2=c_1$
- Solve for y

  $\left\{y=\frac{x}{2}-\frac{\sqrt{-3x^2+4c_1}}{2}, y=\frac{x}{2}+\frac{\sqrt{-3x^2+4c_1}}{2}\right\}$
- Use initial condition $y(1)=3$

  $3=\frac{1}{2}-\frac{\sqrt{4c_1-3}}{2}$
- Solution does not satisfy initial condition
- Use initial condition $y(1)=3$

  $3=\frac{1}{2}+\frac{\sqrt{4c_1-3}}{2}$
- Solve for c_1

  $c_1=7$
- Substitute $c_1=7$ into general solution and simplify

  $y=\frac{x}{2}+\frac{\sqrt{-3x^2+28}}{2}$
- Solution to the IVP

  $y=\frac{x}{2}+\frac{\sqrt{-3x^2+28}}{2}$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
trying homogeneous types:
trying homogeneous D
<- homogeneous successful`
```

✓ Solution by Maple

Time used: 0.094 (sec). Leaf size: 19

```
dsolve([2*x-y(x)+(-x+2*y(x))*diff(y(x),x) = 0,y(1) = 3],y(x), singsol=all)
```

$$y(x) = \frac{x}{2} + \frac{\sqrt{-3x^2 + 28}}{2}$$

✓ Solution by Mathematica

Time used: 0.456 (sec). Leaf size: 22

```
DSolve[{2*x-y[x]+(-x+2*y[x])*y'[x] == 0,y[1]==3},y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \frac{1}{2}\left(\sqrt{28 - 3x^2} + x\right)$$

5.14 problem 14

Internal problem ID [556]
Internal file name [`OUTPUT/556_Sunday_June_05_2022_01_44_32_AM_62551946/index.tex`]

Book: Elementary differential equations and boundary value problems, 10th ed., Boyce and DiPrima
Section: Section 2.6. Page 100
Problem number: 14.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"exact", "differentialType"**

Maple gives the following as the ode type

```
[_exact, _rational, [_1st_order, `_with_symmetry_[F(x),G(x)]`],
    [_Abel, `2nd type`, `class A`]]
```

$$\boxed{y + (x - 4y)\, y' = -9x^2 + 1}$$

With initial conditions

$$[y(1) = 0]$$

5.14.1 Existence and uniqueness analysis

This is non linear first order ODE. In canonical form it is written as

$$\begin{aligned} y' &= f(x, y) \\ &= \frac{9x^2 + y - 1}{-x + 4y} \end{aligned}$$

The x domain of $f(x, y)$ when $y = 0$ is

$$\{x < 0 \lor 0 < x\}$$

And the point $x_0 = 1$ is inside this domain. The y domain of $f(x, y)$ when $x = 1$ is

$$\left\{y < \frac{1}{4} \vee \frac{1}{4} < y\right\}$$

And the point $y_0 = 0$ is inside this domain. Now we will look at the continuity of

$$\begin{aligned}\frac{\partial f}{\partial y} &= \frac{\partial}{\partial y}\left(\frac{9x^2 + y - 1}{-x + 4y}\right) \\ &= \frac{1}{-x + 4y} - \frac{4(9x^2 + y - 1)}{(-x + 4y)^2}\end{aligned}$$

The x domain of $\frac{\partial f}{\partial y}$ when $y = 0$ is

$$\{x < 0 \vee 0 < x\}$$

And the point $x_0 = 1$ is inside this domain. The y domain of $\frac{\partial f}{\partial y}$ when $x = 1$ is

$$\left\{y < \frac{1}{4} \vee \frac{1}{4} < y\right\}$$

And the point $y_0 = 0$ is inside this domain. Therefore solution exists and is unique.

5.14.2 Solving as differentialType ode

Writing the ode as

$$y' = \frac{1 - 9x^2 - y}{x - 4y} \tag{1}$$

Which becomes

$$(-4y)\, dy = (-x)\, dy + \left(-9x^2 - y + 1\right) dx \tag{2}$$

But the RHS is complete differential because

$$(-x)\, dy + \left(-9x^2 - y + 1\right) dx = d\left(-3x^3 - yx + x\right)$$

Hence (2) becomes

$$(-4y)\, dy = d\left(-3x^3 - yx + x\right)$$

Integrating both sides gives gives these solutions

$$y = \frac{x}{4} + \frac{\sqrt{24x^3 + x^2 - 8c_1 - 8x}}{4} + c_1$$
$$y = \frac{x}{4} - \frac{\sqrt{24x^3 + x^2 - 8c_1 - 8x}}{4} + c_1$$

Initial conditions are used to solve for c_1. Substituting $x = 1$ and $y = 0$ in the above solution gives an equation to solve for the constant of integration.

$$0 = \frac{1}{4} - \frac{\sqrt{17 - 8c_1}}{4} + c_1$$

$$c_1 = \frac{\sqrt{5}}{2} - \frac{1}{2}$$

Substituting c_1 found above in the general solution gives

$$y = \frac{x}{4} - \frac{\sqrt{24x^3 + x^2 - 4\sqrt{5} + 4 - 8x}}{4} + \frac{\sqrt{5}}{2} - \frac{1}{2}$$

Initial conditions are used to solve for c_1. Substituting $x = 1$ and $y = 0$ in the above solution gives an equation to solve for the constant of integration.

$$0 = \frac{1}{4} + \frac{\sqrt{17 - 8c_1}}{4} + c_1$$

$$c_1 = -\frac{1}{2} - \frac{\sqrt{5}}{2}$$

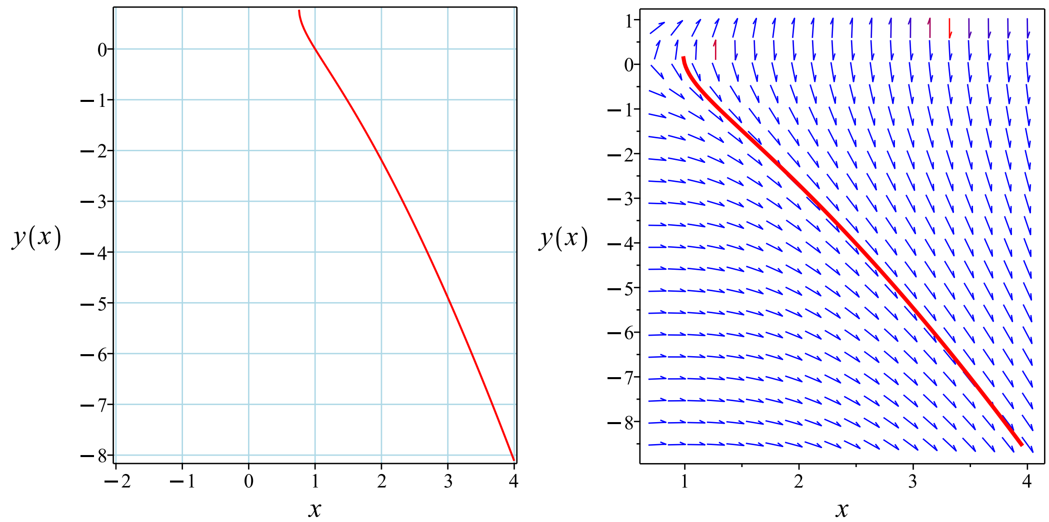
Substituting c_1 found above in the general solution gives

$$y = \frac{x}{4} + \frac{\sqrt{24x^3 + x^2 + 4 + 4\sqrt{5} - 8x}}{4} - \frac{1}{2} - \frac{\sqrt{5}}{2}$$

Summary

The solution(s) found are the following

$$y = \frac{x}{4} + \frac{\sqrt{24x^3 + x^2 + 4 + 4\sqrt{5} - 8x}}{4} - \frac{1}{2} - \frac{\sqrt{5}}{2} \tag{1}$$
$$y = \frac{x}{4} - \frac{\sqrt{24x^3 + x^2 - 4\sqrt{5} + 4 - 8x}}{4} + \frac{\sqrt{5}}{2} - \frac{1}{2} \tag{2}$$



(a) Solution plot

(b) Slope field plot

Verification of solutions

$$y = \frac{x}{4} + \frac{\sqrt{24x^3 + x^2 + 4 + 4\sqrt{5} - 8x}}{4} - \frac{1}{2} - \frac{\sqrt{5}}{2}$$

Verified OK.

$$y = \frac{x}{4} - \frac{\sqrt{24x^3 + x^2 - 4\sqrt{5} + 4 - 8x}}{4} + \frac{\sqrt{5}}{2} - \frac{1}{2}$$

Verified OK.

5.14.3 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x,y) + N(x,y)\frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x,y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y) = 0$$

Hence

$$\frac{\partial \phi}{\partial x} + \frac{\partial \phi}{\partial y}\frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$\frac{\partial \phi}{\partial x} = M$$
$$\frac{\partial \phi}{\partial y} = N$$

But since $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x, y)$ but at least we know now that we can do that since the condition $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x, y)\, \mathrm{d}x + N(x, y)\, \mathrm{d}y = 0 \tag{1A}$$

Therefore

$$(x - 4y)\, \mathrm{d}y = \left(-9x^2 - y + 1\right) \mathrm{d}x$$
$$\left(9x^2 + y - 1\right) \mathrm{d}x + (x - 4y)\, \mathrm{d}y = 0 \tag{2A}$$

Comparing (1A) and (2A) shows that

$$M(x, y) = 9x^2 + y - 1$$
$$N(x, y) = x - 4y$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Using result found above gives

$$\frac{\partial M}{\partial y} = \frac{\partial}{\partial y}\left(9x^2 + y - 1\right)$$
$$= 1$$

And

$$\frac{\partial N}{\partial x} = \frac{\partial}{\partial x}(x - 4y)$$
$$= 1$$

Since $\frac{\partial M}{\partial y}=\frac{\partial N}{\partial x}$, then the ODE is exact The following equations are now set up to solve for the function $\phi(x,y)$

$$\frac{\partial \phi}{\partial x}=M \tag{1}$$
$$\frac{\partial \phi}{\partial y}=N \tag{2}$$

Integrating (1) w.r.t. x gives

$$\int \frac{\partial \phi}{\partial x}\,\mathrm{d}x=\int M\,\mathrm{d}x$$
$$\int \frac{\partial \phi}{\partial x}\,\mathrm{d}x=\int 9x^2+y-1\,\mathrm{d}x$$
$$\phi=x\left(3x^2+y-1\right)+f(y) \tag{3}$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$\frac{\partial \phi}{\partial y}=x+f'(y) \tag{4}$$

But equation (2) says that $\frac{\partial \phi}{\partial y}=x-4y$. Therefore equation (4) becomes

$$x-4y=x+f'(y) \tag{5}$$

Solving equation (5) for $f'(y)$ gives

$$f'(y)=-4y$$

Integrating the above w.r.t y gives

$$\int f'(y)\,\mathrm{d}y=\int (-4y)\,\mathrm{d}y$$
$$f(y)=-2y^2+c_1$$

Where c_1 is constant of integration. Substituting result found above for $f(y)$ into equation (3) gives ϕ

$$\phi=x\left(3x^2+y-1\right)-2y^2+c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = x(3x^2 + y - 1) - 2y^2$$

Initial conditions are used to solve for c_1. Substituting $x = 1$ and $y = 0$ in the above solution gives an equation to solve for the constant of integration.

$$2 = c_1$$

$$c_1 = 2$$

Substituting c_1 found above in the general solution gives

$$x(3x^2 + y - 1) - 2y^2 = 2$$

Summary

The solution(s) found are the following

$$3x^3 + (y - 1)x - 2y^2 = 2 \tag{1}$$

Verification of solutions

$$3x^3 + (y - 1)x - 2y^2 = 2$$

Verified OK.

5.14.4 Maple step by step solution

Let's solve

$[y + (x - 4y)y' = -9x^2 + 1, y(1) = 0]$

- Highest derivative means the order of the ODE is 1

  y'

- □ Check if ODE is exact
  - ODE is exact if the lhs is the total derivative of a C^2 function

    $F'(x, y) = 0$
  - Compute derivative of lhs

    $F'(x, y) + \left(\frac{\partial}{\partial y}F(x, y)\right)y' = 0$

- Evaluate derivatives
  $1 = 1$
- Condition met, ODE is exact

- Exact ODE implies solution will be of this form
  $\left[F(x,y) = c_1, M(x,y) = F'(x,y), N(x,y) = \frac{\partial}{\partial y}F(x,y)\right]$
- Solve for $F(x,y)$ by integrating $M(x,y)$ with respect to x
  $F(x,y) = \int (9x^2 + y - 1)\, dx + f_1(y)$
- Evaluate integral
  $F(x,y) = 3x^3 + yx - x + f_1(y)$
- Take derivative of $F(x,y)$ with respect to y
  $N(x,y) = \frac{\partial}{\partial y}F(x,y)$
- Compute derivative
  $x - 4y = x + \frac{d}{dy}f_1(y)$
- Isolate for $\frac{d}{dy}f_1(y)$
  $\frac{d}{dy}f_1(y) = -4y$
- Solve for $f_1(y)$
  $f_1(y) = -2y^2$
- Substitute $f_1(y)$ into equation for $F(x,y)$
  $F(x,y) = 3x^3 + yx - 2y^2 - x$
- Substitute $F(x,y)$ into the solution of the ODE
  $3x^3 + yx - 2y^2 - x = c_1$
- Solve for y
  $\left\{y = \frac{x}{4} - \frac{\sqrt{24x^3+x^2-8c_1-8x}}{4}, y = \frac{x}{4} + \frac{\sqrt{24x^3+x^2-8c_1-8x}}{4}\right\}$
- Use initial condition $y(1) = 0$
  $0 = \frac{1}{4} - \frac{\sqrt{17-8c_1}}{4}$
- Solve for c_1
  $c_1 = 2$
- Substitute $c_1 = 2$ into general solution and simplify

$y = \frac{x}{4} - \frac{\sqrt{24x^3+x^2-8x-16}}{4}$

- Use initial condition $y(1) = 0$

  $0 = \frac{1}{4} + \frac{\sqrt{17-8c_1}}{4}$
- Solution does not satisfy initial condition
- Solution to the IVP

  $y = \frac{x}{4} - \frac{\sqrt{24x^3+x^2-8x-16}}{4}$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
trying homogeneous types:
trying Chini
differential order: 1; looking for linear symmetries
trying exact
<- exact successful`
```

✓ Solution by Maple
Time used: 0.047 (sec). Leaf size: 25

```
dsolve([-1+9*x^2+y(x)+(x-4*y(x))*diff(y(x),x) = 0,y(1) = 0],y(x), singsol=all)
```

$$y(x) = \frac{x}{4} - \frac{\sqrt{24x^3 + x^2 - 8x - 16}}{4}$$

✓ Solution by Mathematica
Time used: 0.148 (sec). Leaf size: 34

```
DSolve[{-1+9*x^2+y[x]+(x-4*y[x])*y'[x] == 0,y[1]==0},y[x],x,IncludeSingularSolutions -> True
```

$$y(x) \to \frac{1}{4}\left(x + i\sqrt{-24x^3 - x^2 + 8x + 16}\right)$$

5.15 problem 19

Internal problem ID [557]
Internal file name [`OUTPUT/557_Sunday_June_05_2022_01_44_33_AM_13501171/index.tex`]

Book: Elementary differential equations and boundary value problems, 10th ed., Boyce and DiPrima
Section: Section 2.6. Page 100
Problem number: 19.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"exact", "separable", "first_order_ode_lie_symmetry_lookup"**

Maple gives the following as the ode type

```
[_separable]
```

$$\boxed{y^3x^2 + x(1 + y^2)\, y' = 0}$$

5.15.1 Solving as separable ode

In canonical form the ODE is

$$\begin{aligned} y' &= F(x, y) \\ &= f(x)g(y) \\ &= -\frac{y^3x}{y^2 + 1} \end{aligned}$$

Where $f(x) = -x$ and $g(y) = \frac{y^3}{y^2+1}$. Integrating both sides gives

$$\frac{1}{\frac{y^3}{y^2+1}}\, dy = -x\, dx$$

$$\int \frac{1}{\frac{y^3}{y^2+1}}\, dy = \int -x\, dx$$

$$\ln(y) - \frac{1}{2y^2} = -\frac{x^2}{2} + c_1$$

Which results in

$$y = e^{\frac{\text{LambertW}\left(e^{x^2 - 2c_1}\right)}{2} - \frac{x^2}{2} + c_1}$$

Summary

The solution(s) found are the following

$$y = e^{\frac{\text{LambertW}\left(e^{x^2 - 2c_1}\right)}{2} - \frac{x^2}{2} + c_1} \tag{1}$$

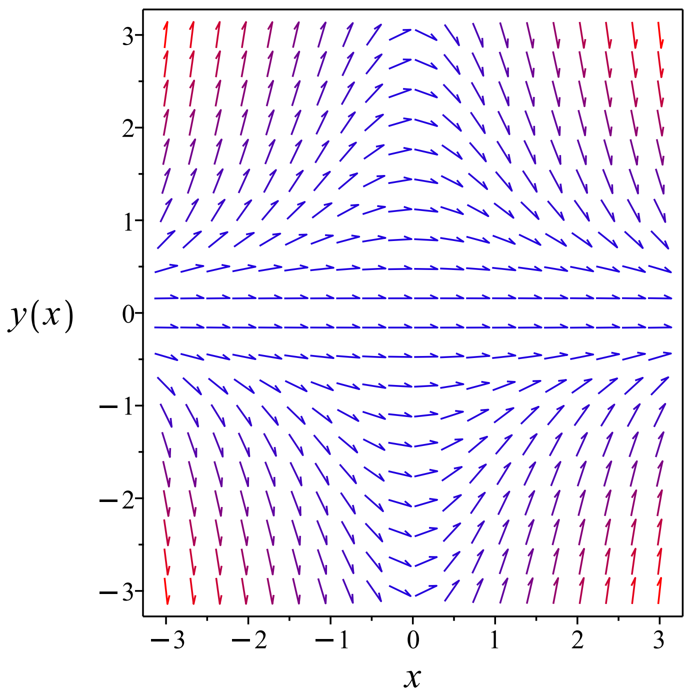


Figure 248: Slope field plot

Verification of solutions

$$y = e^{\frac{\text{LambertW}\left(e^{x^2 - 2c_1}\right)}{2} - \frac{x^2}{2} + c_1}$$

Verified OK.

5.15.2 Solving as first order ode lie symmetry lookup ode

Writing the ode as

$$
\begin{aligned}
y' &= -\frac{y^3 x}{y^2+1} \\
y' &= \omega(x,y)
\end{aligned}
$$

The condition of Lie symmetry is the linearized PDE given by

$$
\eta_x + \omega(\eta_y - \xi_x) - \omega^2 \xi_y - \omega_x \xi - \omega_y \eta = 0 \tag{A}
$$

The type of this ode is known. It is of type `separable`. Therefore we do not need to solve the PDE (A), and can just use the lookup table shown below to find ξ, η

Table 251: Lie symmetry infinitesimal lookup table for known first order ODE's

ODE class	Form	ξ	η
linear ode	$y' = f(x)y(x) + g(x)$	0	$e^{\int f dx}$
separable ode	$y' = f(x)\, g(y)$	$\frac{1}{f}$	0
quadrature ode	$y' = f(x)$	0	1
quadrature ode	$y' = g(y)$	1	0
homogeneous ODEs of Class A	$y' = f\left(\frac{y}{x}\right)$	x	y
homogeneous ODEs of Class C	$y' = (a + bx + cy)^{\frac{n}{m}}$	1	$-\frac{b}{c}$
homogeneous class D	$y' = \frac{y}{x} + g(x)\, F\left(\frac{y}{x}\right)$	x^2	xy
First order special form ID 1	$y' = g(x)\, e^{h(x)+by} + f(x)$	$\frac{e^{-\int bf(x)dx - h(x)}}{g(x)}$	$\frac{f(x)e^{-\int bf(x)dx - h(x)}}{g(x)}$
polynomial type ode	$y' = \frac{a_1x+b_1y+c_1}{a_2x+b_2y+c_2}$	$\frac{a_1b_2x - a_2b_1x - b_1c_2 + b_2c_1}{a_1b_2 - a_2b_1}$	$\frac{a_1b_2y - a_2b_1y - a_1c_2 - a_2c_1}{a_1b_2 - a_2b_1}$
Bernoulli ode	$y' = f(x)\, y + g(x)\, y^n$	0	$e^{-\int (n-1)f(x)dx} y^n$
Reduced Riccati	$y' = f_1(x)\, y + f_2(x)\, y^2$	0	$e^{-\int f_1 dx}$

The above table shows that

$$
\begin{aligned}
\xi(x,y) &= -\frac{1}{x} \\
\eta(x,y) &= 0
\end{aligned} \tag{A1}
$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x,y) \to (R,S)$ where (R,S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$
\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}
$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y}\right) S(x,y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the canonical coordinates, where $S(R)$. Since $\eta = 0$ then in this special case

$$
R = y
$$

S is found from

$$
\begin{aligned}
S &= \int \frac{1}{\xi} dx \\
&= \int \frac{1}{-\frac{1}{x}} dx
\end{aligned}
$$

Which results in

$$
S = -\frac{x^2}{2}
$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$
\frac{dS}{dR} = \frac{S_x + \omega(x,y) S_y}{R_x + \omega(x,y) R_y} \tag{2}
$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x,y)$ is the right hand side of the original ode given by

$$
\omega(x,y) = -\frac{y^3 x}{y^2 + 1}
$$

Evaluating all the partial derivatives gives

$$
\begin{aligned}
R_x &= 0 \\
R_y &= 1 \\
S_x &= -x \\
S_y &= 0
\end{aligned}
$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$
\frac{dS}{dR} = \frac{y^2+1}{y^3} \tag{2A}
$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$
\frac{dS}{dR} = \frac{R^2+1}{R^3}
$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$
S(R) = \ln(R) - \frac{1}{2R^2} + c_1 \tag{4}
$$

To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$
-\frac{x^2}{2} = \ln(y) - \frac{1}{2y^2} + c_1
$$

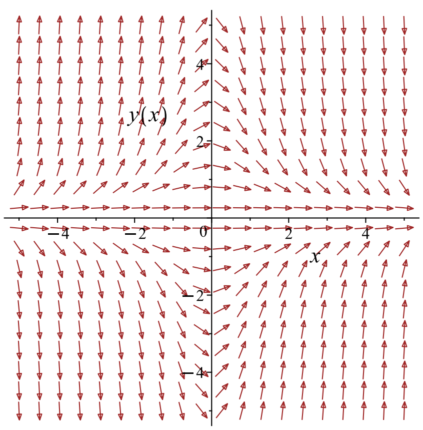
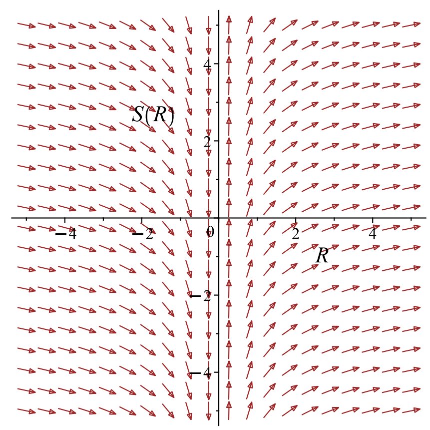
Which simplifies to

$$
-\frac{x^2}{2} = \ln(y) - \frac{1}{2y^2} + c_1
$$

Which gives

$$
y = \mathrm{e}^{\frac{\mathrm{LambertW}\left(\mathrm{e}^{x^2+2c_1}\right)}{2} - \frac{x^2}{2} - c_1}
$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in x, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dx} = -\frac{y^3 x}{y^2+1}$ 	$R = y$ $S = -\frac{x^2}{2}$	$\frac{dS}{dR} = \frac{R^2+1}{R^3}$ 

Summary

The solution(s) found are the following

$$y = e^{\frac{\mathrm{LambertW}\left(e^{x^2+2c_1}\right)}{2} - \frac{x^2}{2} - c_1} \tag{1}$$

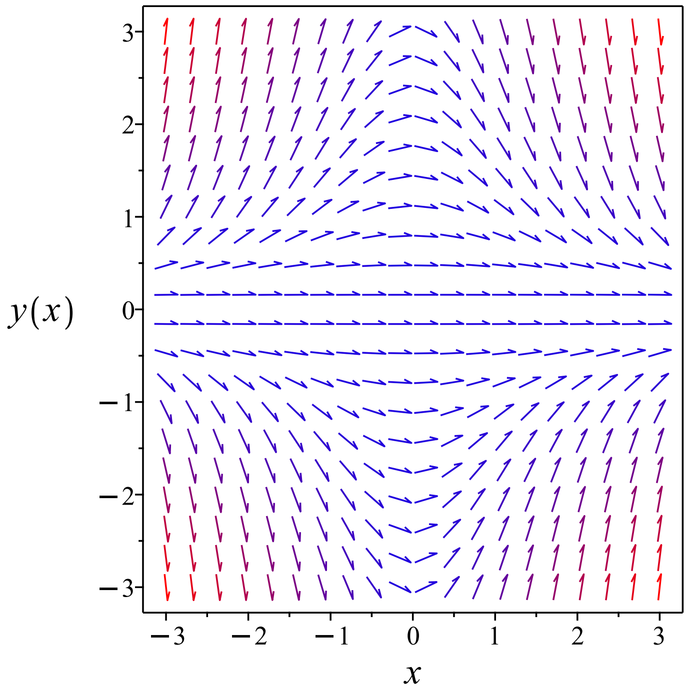


Figure 249: Slope field plot

Verification of solutions

$$y = e^{\frac{\text{LambertW}\left(e^{x^2+2c_1}\right)}{2}-\frac{x^2}{2}-c_1}$$

Verified OK.

5.15.3 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x,y)+N(x,y)\frac{dy}{dx}=0 \tag{A}$$

We assume there exists a function $\phi(x,y)=c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y)=0$$

Hence

$$\frac{\partial\phi}{\partial x}+\frac{\partial\phi}{\partial y}\frac{dy}{dx}=0 \tag{B}$$

Comparing (A,B) shows that

$$\frac{\partial\phi}{\partial x} = M$$
$$\frac{\partial\phi}{\partial y} = N$$

But since $\frac{\partial^2\phi}{\partial x\partial y} = \frac{\partial^2\phi}{\partial y\partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x, y)$ but at least we know now that we can do that since the condition $\frac{\partial^2\phi}{\partial x\partial y} = \frac{\partial^2\phi}{\partial y\partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x, y)\,\mathrm{d}x + N(x, y)\,\mathrm{d}y = 0 \tag{1A}$$

Therefore

$$\left(-\frac{y^2+1}{y^3}\right)\mathrm{d}y = (x)\,\mathrm{d}x$$
$$(-x)\,\mathrm{d}x + \left(-\frac{y^2+1}{y^3}\right)\mathrm{d}y = 0 \tag{2A}$$

Comparing (1A) and (2A) shows that

$$M(x, y) = -x$$
$$N(x, y) = -\frac{y^2+1}{y^3}$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Using result found above gives

$$\begin{aligned}\frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}(-x)\\ &= 0\end{aligned}$$

And

$$\frac{\partial N}{\partial x} = \frac{\partial}{\partial x}\left(-\frac{y^2+1}{y^3}\right)$$
$$= 0$$

Since $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$, then the ODE is exact The following equations are now set up to solve for the function $\phi(x,y)$

$$\frac{\partial \phi}{\partial x} = M \tag{1}$$
$$\frac{\partial \phi}{\partial y} = N \tag{2}$$

Integrating (1) w.r.t. x gives

$$\int \frac{\partial \phi}{\partial x} \,\mathrm{d}x = \int M \,\mathrm{d}x$$
$$\int \frac{\partial \phi}{\partial x} \,\mathrm{d}x = \int -x \,\mathrm{d}x$$
$$\phi = -\frac{x^2}{2} + f(y) \tag{3}$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$\frac{\partial \phi}{\partial y} = 0 + f'(y) \tag{4}$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = -\frac{y^2+1}{y^3}$. Therefore equation (4) becomes

$$-\frac{y^2+1}{y^3} = 0 + f'(y) \tag{5}$$

Solving equation (5) for $f'(y)$ gives

$$f'(y) = -\frac{y^2+1}{y^3}$$

Integrating the above w.r.t y gives

$$\int f'(y)\,\mathrm{d}y = \int \left(\frac{-y^2-1}{y^3}\right)\mathrm{d}y$$
$$f(y) = -\ln(y) + \frac{1}{2y^2} + c_1$$

Where c_1 is constant of integration. Substituting result found above for $f(y)$ into equation (3) gives ϕ

$$\phi = -\frac{x^2}{2} - \ln(y) + \frac{1}{2y^2} + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = -\frac{x^2}{2} - \ln(y) + \frac{1}{2y^2}$$

The solution becomes

$$y = \mathrm{e}^{\frac{\mathrm{LambertW}\left(\mathrm{e}^{x^2+2c_1}\right)}{2} - \frac{x^2}{2} - c_1}$$

Summary

The solution(s) found are the following

$$y = \mathrm{e}^{\frac{\mathrm{LambertW}\left(\mathrm{e}^{x^2+2c_1}\right)}{2} - \frac{x^2}{2} - c_1} \tag{1}$$

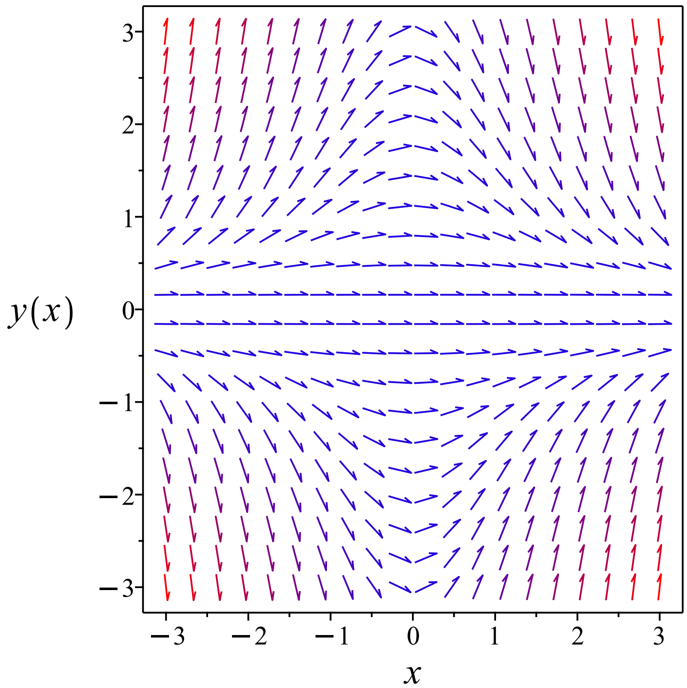


Figure 250: Slope field plot

Verification of solutions

$$y = e^{\frac{\operatorname{LambertW}\left(e^{x^2+2c_1}\right)}{2}-\frac{x^2}{2}-c_1}$$

Verified OK.

5.15.4 Maple step by step solution

Let's solve

$y^3x^2 + x(1+y^2)\,y' = 0$

- Highest derivative means the order of the ODE is 1

  y'

- Separate variables

  $\frac{y'(1+y^2)}{y^3} = -x$

- Integrate both sides with respect to x

  $\int \frac{y'(1+y^2)}{y^3}dx = \int -xdx + c_1$

- Evaluate integral

$$\ln (y)-\frac{1}{2y^2}=-\frac{x^2}{2}+c_1$$

- Solve for y

$$y=\mathrm{e}^{\frac{\mathit{LambertW}\left(\mathrm{e}^{x^2-2c_1}\right)}{2}-\frac{x^2}{2}+c_1}$$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
<- separable successful`
```

✓ Solution by Maple

Time used: 0.015 (sec). Leaf size: 37

```
dsolve(x^2*y(x)^3+x*(1+y(x)^2)*diff(y(x),x) = 0,y(x), singsol=all)
```

$$y(x)=\mathrm{e}^{-\frac{x^2}{2}-c_1}\sqrt{\frac{\mathrm{e}^{x^2+2c_1}}{\mathrm{LambertW}\left(\mathrm{e}^{x^2+2c_1}\right)}}$$

✓ Solution by Mathematica

Time used: 4.095 (sec). Leaf size: 46

```
DSolve[x^2*y[x]^3+x*(1+y[x]^2)*y'[x] == 0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x)\to -\frac{1}{\sqrt{W\left(e^{x^2-2c_1}\right)}}$$
$$y(x)\to \frac{1}{\sqrt{W\left(e^{x^2-2c_1}\right)}}$$
$$y(x)\to 0$$

5.16 problem 21

Internal problem ID [558]
Internal file name [`OUTPUT/558_Sunday_June_05_2022_01_44_35_AM_40301990/index.tex`]

Book: Elementary differential equations and boundary value problems, 10th ed., Boyce and DiPrima
Section: Section 2.6. Page 100
Problem number: 21.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"exactWithIntegrationFactor"**

Maple gives the following as the ode type

`[[_1st_order, `_with_symmetry_[F(x)*G(y),0]`]]`

$$\boxed{y + (2x - e^{y} y) y' = 0}$$

5.16.1 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x,y) + N(x,y)\frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x,y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y) = 0$$

Hence

$$\frac{\partial \phi}{\partial x} + \frac{\partial \phi}{\partial y}\frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$\frac{\partial \phi}{\partial x} = M$$
$$\frac{\partial \phi}{\partial y} = N$$

But since $\frac{\partial^2\phi}{\partial x\partial y} = \frac{\partial^2\phi}{\partial y\partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x,y)$ but at least we know now that we can do that since the condition $\frac{\partial^2\phi}{\partial x\partial y} = \frac{\partial^2\phi}{\partial y\partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x,y)\,\mathrm{d}x + N(x,y)\,\mathrm{d}y = 0 \tag{1A}$$

Therefore

$$\begin{aligned}(2x - \mathrm{e}^y y)\,\mathrm{d}y &= (-y)\,\mathrm{d}x\\ (y)\,\mathrm{d}x + (2x - \mathrm{e}^y y)\,\mathrm{d}y &= 0 \end{aligned} \tag{2A}$$

Comparing (1A) and (2A) shows that

$$\begin{aligned}M(x,y) &= y\\ N(x,y) &= 2x - \mathrm{e}^y y\end{aligned}$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Using result found above gives

$$\begin{aligned}\frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}(y)\\ &= 1\end{aligned}$$

And

$$\begin{aligned}\frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}(2x - \mathrm{e}^y y)\\ &= 2\end{aligned}$$

Since $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$, then the ODE is not exact. Since the ODE is not exact, we will try to find an integrating factor to make it exact. Let

$$\begin{aligned}A &= \frac{1}{N}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right)\\ &= \frac{1}{2x - \mathrm{e}^y y}((1) - (2))\\ &= \frac{1}{\mathrm{e}^y y - 2x}\end{aligned}$$

Since A depends on y, it can not be used to obtain an integrating factor. We will now try a second method to find an integrating factor. Let

$$\begin{aligned} B &= \frac{1}{M}\left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) \\ &= \frac{1}{y}((2) - (1)) \\ &= \frac{1}{y} \end{aligned}$$

Since B does not depend on x, it can be used to obtain an integrating factor. Let the integrating factor be μ. Then

$$\begin{aligned} \mu &= e^{\int B \, \mathrm{d}y} \\ &= e^{\int \frac{1}{y} \, \mathrm{d}y} \end{aligned}$$

The result of integrating gives

$$\begin{aligned} \mu &= e^{\ln(y)} \\ &= y \end{aligned}$$

M and N are now multiplied by this integrating factor, giving new M and new N which are called $\overline{M}$ and $\overline{N}$ so not to confuse them with the original M and N.

$$\begin{aligned} \overline{M} &= \mu M \\ &= y(y) \\ &= y^2 \end{aligned}$$

And

$$\begin{aligned} \overline{N} &= \mu N \\ &= y(2x - \mathrm{e}^y y) \\ &= -y^2 \mathrm{e}^y + 2yx \end{aligned}$$

So now a modified ODE is obtained from the original ODE which will be exact and can be solved using the standard method. The modified ODE is

$$\begin{aligned} \overline{M} + \overline{N} \frac{\mathrm{d}y}{\mathrm{d}x} &= 0 \\ (y^2) + (-y^2 \mathrm{e}^y + 2yx) \frac{\mathrm{d}y}{\mathrm{d}x} &= 0 \end{aligned}$$

The following equations are now set up to solve for the function $\phi(x, y)$

$$\frac{\partial \phi}{\partial x} = \overline{M} \tag{1}$$
$$\frac{\partial \phi}{\partial y} = \overline{N} \tag{2}$$

Integrating (1) w.r.t. x gives

$$\int \frac{\partial \phi}{\partial x} \, \mathrm{d}x = \int \overline{M} \, \mathrm{d}x$$
$$\int \frac{\partial \phi}{\partial x} \, \mathrm{d}x = \int y^2 \, \mathrm{d}x$$
$$\phi = x\,y^2 + f(y) \tag{3}$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$\frac{\partial \phi}{\partial y} = 2yx + f'(y) \tag{4}$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = -y^2 \mathrm{e}^y + 2yx$. Therefore equation (4) becomes

$$-y^2 \mathrm{e}^y + 2yx = 2yx + f'(y) \tag{5}$$

Solving equation (5) for $f'(y)$ gives

$$f'(y) = -y^2 \mathrm{e}^y$$

Integrating the above w.r.t y gives

$$\int f'(y) \, \mathrm{d}y = \int \left(-y^2 \mathrm{e}^y\right) \mathrm{d}y$$
$$f(y) = -\left(y^2 - 2y + 2\right) \mathrm{e}^y + c_1$$

Where c_1 is constant of integration. Substituting result found above for $f(y)$ into equation (3) gives ϕ

$$\phi = x\,y^2 - \left(y^2 - 2y + 2\right) \mathrm{e}^y + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = x\,y^2 - \left(y^2 - 2y + 2\right) \mathrm{e}^{y}$$

Summary

The solution(s) found are the following

$$xy^2 - \left(y^2 - 2y + 2\right) \mathrm{e}^{y} = c_1 \tag{1}$$

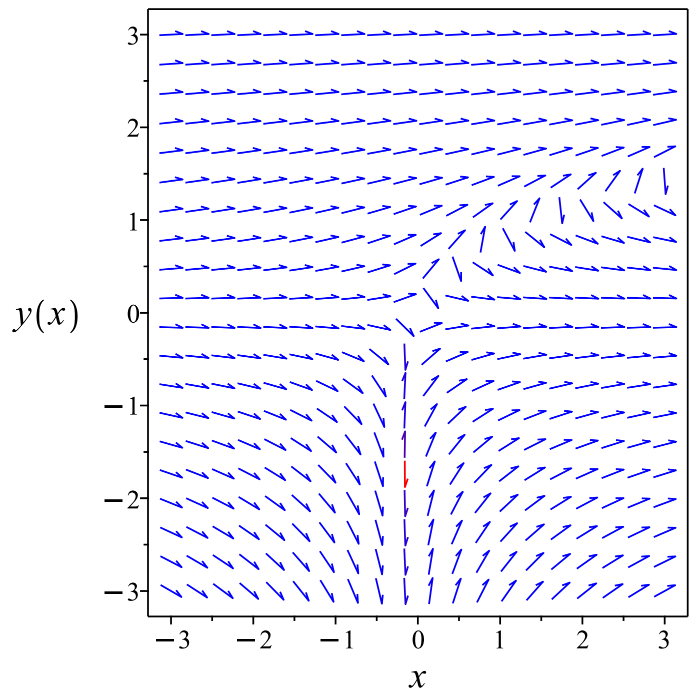


Figure 251: Slope field plot

Verification of solutions

$$xy^2 - \left(y^2 - 2y + 2\right) \mathrm{e}^{y} = c_1$$

Verified OK.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
<- 1st order linear successful
<- inverse linear successful`
```

✓ Solution by Maple

Time used: 0.031 (sec). Leaf size: 34

```
dsolve(y(x)+(2*x-exp(y(x))*y(x))*diff(y(x),x) = 0,y(x), singsol=all)
```

$$\frac{\left(-y(x)^2 + 2y(x) - 2\right) e^{y(x)} + x y(x)^2 - c_1}{y\left(x\right)^2} = 0$$

✓ Solution by Mathematica

Time used: 0.225 (sec). Leaf size: 32

```
DSolve[y[x]+(2*x-Exp[y[x]]*y[x])*y'[x] == 0,y[x],x,IncludeSingularSolutions -> True]
```

$$\text{Solve}\left[x = \frac{e^{y(x)}(y(x)^2 - 2y(x) + 2)}{y(x)^2} + \frac{c_1}{y(x)^2}, y(x)\right]$$

5.17 problem 22

Internal problem ID [559]
Internal file name [`OUTPUT/559_Sunday_June_05_2022_01_44_36_AM_63277223/index.tex`]

Book: Elementary differential equations and boundary value problems, 10th ed., Boyce and DiPrima
Section: Section 2.6. Page 100
Problem number: 22.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"exact", "separable", "first_order_ode_lie_symmetry_lookup"**

Maple gives the following as the ode type

`[_separable]`

$$\boxed{(2+x)\sin(y)+x\cos(y)\,y'=0}$$

5.17.1 Solving as separable ode

In canonical form the ODE is

$$\begin{aligned} y' &= F(x,y)\\ &= f(x)g(y)\\ &= -\frac{(2+x)\tan(y)}{x} \end{aligned}$$

Where $f(x)=-\frac{2+x}{x}$ and $g(y)=\tan(y)$. Integrating both sides gives

$$\begin{aligned} \frac{1}{\tan(y)}\,dy &= -\frac{2+x}{x}\,dx\\ \int \frac{1}{\tan(y)}\,dy &= \int -\frac{2+x}{x}\,dx\\ \ln(\sin(y)) &= -x-2\ln(x)+c_1 \end{aligned}$$

Raising both side to exponential gives

$$\sin(y) = e^{-x-2\ln(x)+c_1}$$

Which simplifies to

$$\sin(y) = c_2 e^{-x-2\ln(x)}$$

Summary

The solution(s) found are the following

$$y = \arcsin\left(\frac{c_2 e^{-x+c_1}}{x^2}\right) \tag{1}$$

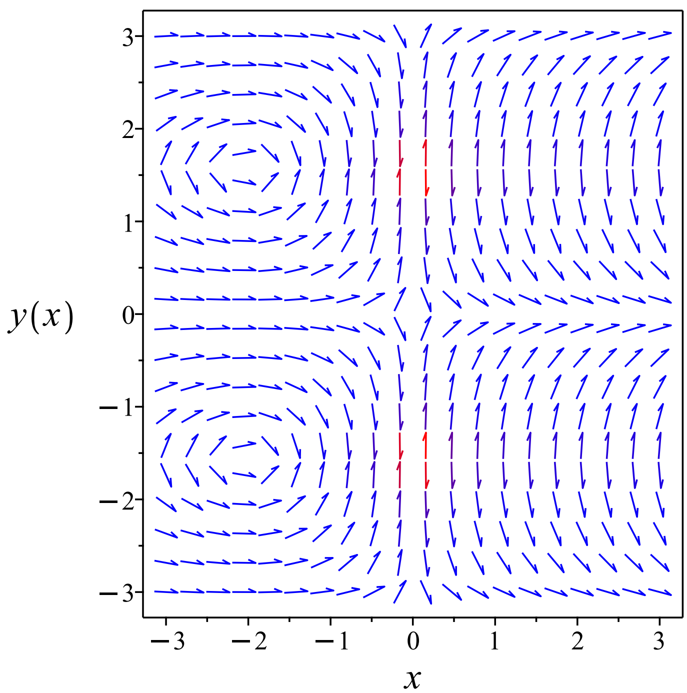


Figure 252: Slope field plot

Verification of solutions

$$y = \arcsin\left(\frac{c_2 e^{-x+c_1}}{x^2}\right)$$

Verified OK.

5.17.2 Solving as first order ode lie symmetry lookup ode

Writing the ode as

$$
\begin{aligned}
y' &= -\frac{(2+x)\sin(y)}{x\cos(y)} \\
y' &= \omega(x,y)
\end{aligned}
$$

The condition of Lie symmetry is the linearized PDE given by

$$
\eta_x + \omega(\eta_y - \xi_x) - \omega^2\xi_y - \omega_x\xi - \omega_y\eta = 0 \tag{A}
$$

The type of this ode is known. It is of type `separable`. Therefore we do not need to solve the PDE (A), and can just use the lookup table shown below to find ξ, η

Table 254: Lie symmetry infinitesimal lookup table for known first order ODE's

ODE class	Form	ξ	η
linear ode	$y' = f(x)y(x) + g(x)$	0	$e^{\int f dx}$
separable ode	$y' = f(x)\, g(y)$	$\frac{1}{f}$	0
quadrature ode	$y' = f(x)$	0	1
quadrature ode	$y' = g(y)$	1	0
homogeneous ODEs of Class A	$y' = f\left(\frac{y}{x}\right)$	x	y
homogeneous ODEs of Class C	$y' = (a + bx + cy)^{\frac{n}{m}}$	1	$-\frac{b}{c}$
homogeneous class D	$y' = \frac{y}{x} + g(x)\, F\left(\frac{y}{x}\right)$	x^2	xy
First order special form ID 1	$y' = g(x)\, e^{h(x)+by} + f(x)$	$\frac{e^{-\int bf(x)dx - h(x)}}{g(x)}$	$\frac{f(x)e^{-\int bf(x)dx - h(x)}}{g(x)}$
polynomial type ode	$y' = \frac{a_1x+b_1y+c_1}{a_2x+b_2y+c_2}$	$\frac{a_1b_2x - a_2b_1x - b_1c_2 + b_2c_1}{a_1b_2 - a_2b_1}$	$\frac{a_1b_2y - a_2b_1y - a_1c_2 - a_2c_1}{a_1b_2 - a_2b_1}$
Bernoulli ode	$y' = f(x)\, y + g(x)\, y^n$	0	$e^{-\int (n-1)f(x)dx}y^n$
Reduced Riccati	$y' = f_1(x)\, y + f_2(x)\, y^2$	0	$e^{-\int f_1 dx}$

The above table shows that

$$\xi(x, y) = -\frac{x}{2+x}$$
$$\eta(x, y) = 0 \tag{A1}$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x, y) \to (R, S)$ where (R, S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y}\right) S(x, y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the canonical coordinates, where $S(R)$. Since $\eta = 0$ then in this special case

$$R = y$$

S is found from

$$\begin{aligned} S &= \int \frac{1}{\xi} dx \\ &= \int \frac{1}{-\frac{x}{2+x}} dx \end{aligned}$$

Which results in

$$S = -x - 2\ln(x)$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x, y)S_y}{R_x + \omega(x, y)R_y} \tag{2}$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x, y)$ is the right hand side of the original ode given by

$$\omega(x, y) = -\frac{(2+x)\sin(y)}{x\cos(y)}$$

Evaluating all the partial derivatives gives

$$
\begin{aligned}
R_x &= 0 \\
R_y &= 1 \\
S_x &= \frac{-x-2}{x} \\
S_y &= 0
\end{aligned}
$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = \cot(y) \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = \cot(R)$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$S(R) = \ln(\sin(R)) + c_1 \tag{4}$$

To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$-x - 2\ln(x) = \ln(\sin(y)) + c_1$$

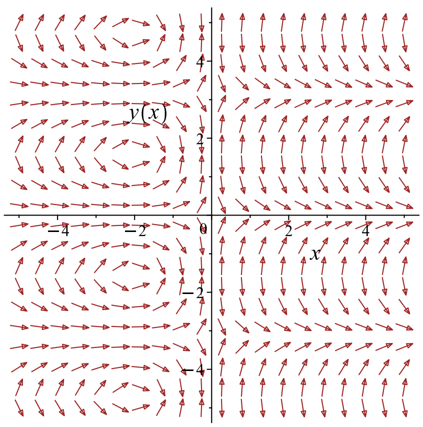
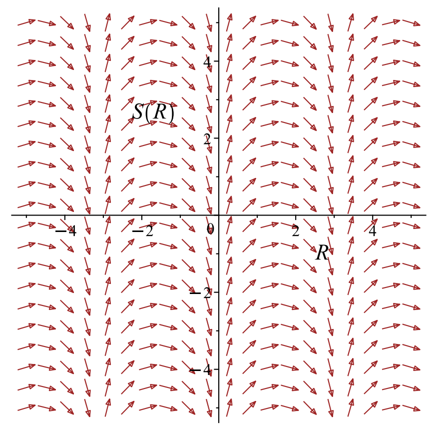
Which simplifies to

$$-x - 2\ln(x) = \ln(\sin(y)) + c_1$$

Which gives

$$y = \arcsin\left(\frac{e^{-x-c_1}}{x^2}\right)$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in x, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dx} = -\frac{(2+x)\sin(y)}{x\cos(y)}$ 	$R = y$ $S = -x - 2\ln(x)$	$\frac{dS}{dR} = \cot(R)$ 

Summary

The solution(s) found are the following

$$y = \arcsin\left(\frac{e^{-x-c_1}}{x^2}\right) \tag{1}$$

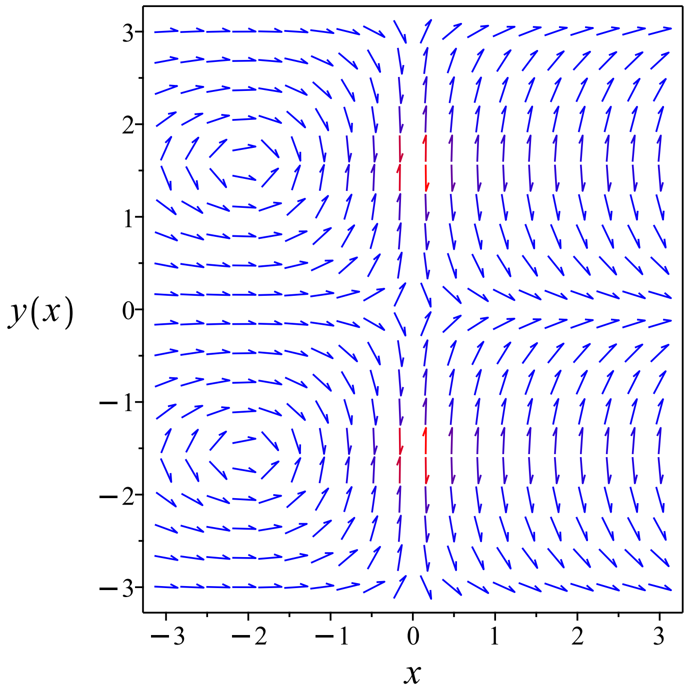


Figure 253: Slope field plot

Verification of solutions

$$y = \arcsin\left(\frac{e^{-x-c_1}}{x^2}\right)$$

Verified OK.

5.17.3 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x, y) + N(x, y)\frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x, y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x, y) = 0$$

Hence

$$\frac{\partial \phi}{\partial x} + \frac{\partial \phi}{\partial y}\frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$
\begin{aligned}
\frac{\partial \phi}{\partial x} &= M \\
\frac{\partial \phi}{\partial y} &= N
\end{aligned}
$$

But since $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ then for the above to be valid, we require that

$$
\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}
$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x, y)$ but at least we know now that we can do that since the condition $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$
M(x, y)\,\mathrm{d}x + N(x, y)\,\mathrm{d}y = 0 \tag{1A}
$$

Therefore

$$
\begin{aligned}
\left(-\frac{\cos(y)}{\sin(y)}\right) \mathrm{d}y &= \left(\frac{2 + x}{x}\right) \mathrm{d}x \\
\left(-\frac{2 + x}{x}\right) \mathrm{d}x + \left(-\frac{\cos(y)}{\sin(y)}\right) \mathrm{d}y &= 0 \qquad \text{(2A)}
\end{aligned}
$$

Comparing (1A) and (2A) shows that

$$
\begin{aligned}
M(x, y) &= -\frac{2 + x}{x} \\
N(x, y) &= -\frac{\cos(y)}{\sin(y)}
\end{aligned}
$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$
\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}
$$

Using result found above gives

$$
\begin{aligned}
\frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}\left(-\frac{2 + x}{x}\right) \\
&= 0
\end{aligned}
$$

And

$$
\begin{aligned}
\frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}\left(-\frac{\cos(y)}{\sin(y)}\right) \\
&= 0
\end{aligned}
$$

Since $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$, then the ODE is exact The following equations are now set up to solve for the function $\phi(x, y)$

$$
\frac{\partial \phi}{\partial x} = M \tag{1}
$$

$$
\frac{\partial \phi}{\partial y} = N \tag{2}
$$

Integrating (1) w.r.t. x gives

$$
\begin{aligned}
\int \frac{\partial \phi}{\partial x} \, \mathrm{d}x &= \int M \, \mathrm{d}x \\
\int \frac{\partial \phi}{\partial x} \, \mathrm{d}x &= \int -\frac{2+x}{x} \, \mathrm{d}x
\end{aligned}
$$

$$
\phi = -x - 2\ln(x) + f(y) \tag{3}
$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$
\frac{\partial \phi}{\partial y} = 0 + f'(y) \tag{4}
$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = -\frac{\cos(y)}{\sin(y)}$. Therefore equation (4) becomes

$$
-\frac{\cos(y)}{\sin(y)} = 0 + f'(y) \tag{5}
$$

Solving equation (5) for $f'(y)$ gives

$$
\begin{aligned}
f'(y) &= -\frac{\cos(y)}{\sin(y)} \\
&= -\cot(y)
\end{aligned}
$$

Integrating the above w.r.t y results in

$$\int f'(y)\,dy = \int (-\cot(y))\,dy$$
$$f(y) = -\ln(\sin(y)) + c_1$$

Where c_1 is constant of integration. Substituting result found above for $f(y)$ into equation (3) gives ϕ

$$\phi = -x - 2\ln(x) - \ln(\sin(y)) + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = -x - 2\ln(x) - \ln(\sin(y))$$

Summary
The solution(s) found are the following

$$-x - 2\ln(x) - \ln(\sin(y)) = c_1 \tag{1}$$

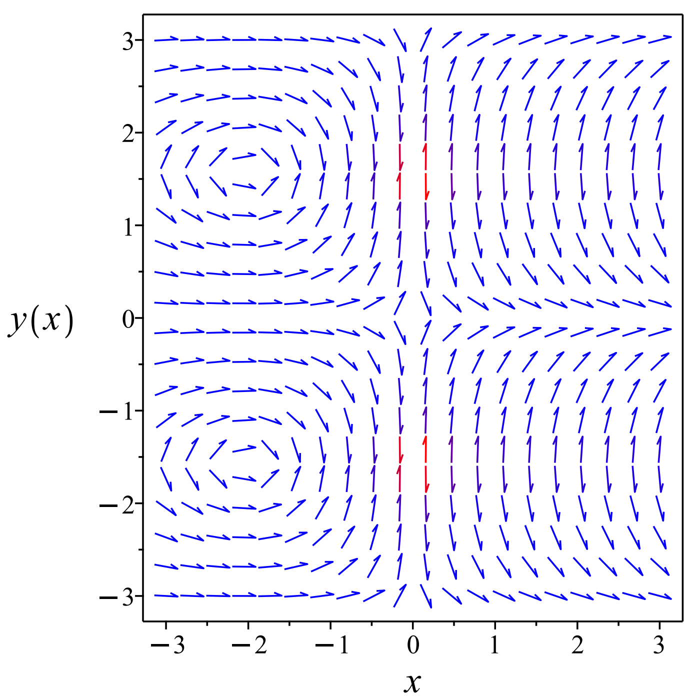


Figure 254: Slope field plot

Verification of solutions

$$-x-2\ln(x)-\ln(\sin(y))=c_1$$

Verified OK.

5.17.4 Maple step by step solution

Let's solve

$(2+x)\sin(y)+x\cos(y)\,y'=0$

- Highest derivative means the order of the ODE is 1

  y'

- Separate variables

  $\frac{y'\cos(y)}{\sin(y)}=-\frac{2+x}{x}$

- Integrate both sides with respect to x

  $\int\frac{y'\cos(y)}{\sin(y)}dx=\int-\frac{2+x}{x}dx+c_1$

- Evaluate integral

  $\ln(\sin(y))=-x-2\ln(x)+c_1$

- Solve for y

  $y=\arcsin\left(\frac{e^{-x+c_1}}{x^2}\right)$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
<- separable successful`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 16

```
dsolve((2+x)*sin(y(x))+x*cos(y(x))*diff(y(x),x) = 0,y(x), singsol=all)
```

$$y(x) = \arcsin\left(\frac{e^{-x}}{c_1 x^2}\right)$$

✓ Solution by Mathematica

Time used: 51.022 (sec). Leaf size: 23

```
DSolve[(2+x)*Sin[y[x]]+x*Cos[y[x]]*y'[x] == 0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \arcsin\left(\frac{e^{-x+c_1}}{x^2}\right)$$
$$y(x) \to 0$$

5.18 problem 25

Internal problem ID [560]
Internal file name [`OUTPUT/560_Sunday_June_05_2022_01_44_37_AM_81876546/index.tex`]

Book: Elementary differential equations and boundary value problems, 10th ed., Boyce and DiPrima
Section: Section 2.6. Page 100
Problem number: 25.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"homogeneousTypeD2", "exactWithIntegrationFactor"**

Maple gives the following as the ode type

```
[[_homogeneous, `class D`], _rational]
```

$$\boxed{2yx + 3x^2y + y^3 + (x^2 + y^2)\, y' = 0}$$

5.18.1 Solving as homogeneousTypeD2 ode

Using the change of variables $y = u(x)\, x$ on the above ode results in new ode in $u(x)$

$$2u(x)\, x^2 + 3x^3u(x) + u(x)^3\, x^3 + \left(x^2 + u(x)^2\, x^2\right)(u'(x)\, x + u(x)) = 0$$

In canonical form the ODE is

$$\begin{aligned} u' &= F(x,u) \\ &= f(x)g(u) \\ &= -\frac{u(u^2+3)\,(x+1)}{x\,(u^2+1)} \end{aligned}$$

Where $f(x) = -\frac{x+1}{x}$ and $g(u) = \frac{(u^2+3)u}{u^2+1}$. Integrating both sides gives

$$\frac{1}{\frac{(u^2+3)u}{u^2+1}}\, du = -\frac{x+1}{x}\, dx$$

$$
\int \frac{1}{\frac{(u^2+3)u}{u^2+1}} du = \int -\frac{x+1}{x} dx
$$

$$
\frac{\ln\left(\left(u^2+3\right)u\right)}{3} = -x - \ln\left(x\right) + c_2
$$

Raising both side to exponential gives

$$
\left(\left(u^2+3\right)u\right)^{\frac{1}{3}} = \mathrm{e}^{-x-\ln(x)+c_2}
$$

Which simplifies to

$$
\left(\left(u^2+3\right)u\right)^{\frac{1}{3}} = c_3 \mathrm{e}^{-x-\ln(x)}
$$

Which simplifies to

$$
\left(\left(u(x)^2+3\right)u(x)\right)^{\frac{1}{3}} = \frac{c_3 \mathrm{e}^{-x} \mathrm{e}^{c_2}}{x}
$$

The solution is

$$
\left(\left(u(x)^2+3\right)u(x)\right)^{\frac{1}{3}} = \frac{c_3 \mathrm{e}^{-x} \mathrm{e}^{c_2}}{x}
$$

Replacing $u(x)$ in the above solution by $\frac{y}{x}$ results in the solution for y in implicit form

$$
\left(\frac{\left(\frac{y^2}{x^2}+3\right)y}{x}\right)^{\frac{1}{3}} = \frac{c_3 \mathrm{e}^{-x} \mathrm{e}^{c_2}}{x}
$$

$$
\left(\frac{\left(y^2+3x^2\right)y}{x^3}\right)^{\frac{1}{3}} = \frac{c_3 \mathrm{e}^{-x+c_2}}{x}
$$

Summary

The solution(s) found are the following

$$
\left(\frac{\left(y^2+3x^2\right)y}{x^3}\right)^{\frac{1}{3}} = \frac{c_3 \mathrm{e}^{-x+c_2}}{x} \tag{1}
$$

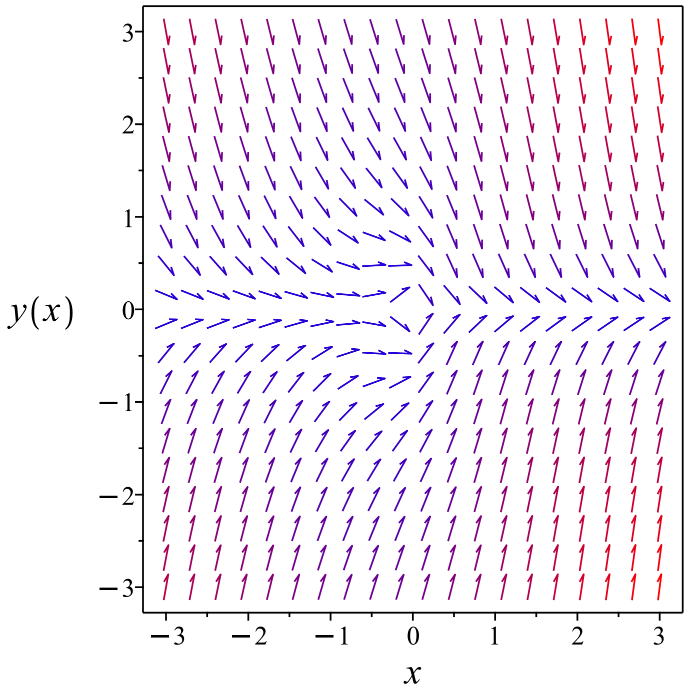


Figure 255: Slope field plot

Verification of solutions

$$\left(\frac{(y^2+3x^2)\,y}{x^3}\right)^{\frac{1}{3}} = \frac{c_3 \mathrm{e}^{-x+c_2}}{x}$$

Verified OK.

5.18.2 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x,y) + N(x,y)\,\frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x,y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y) = 0$$

Hence

$$\frac{\partial \phi}{\partial x} + \frac{\partial \phi}{\partial y}\frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$\frac{\partial \phi}{\partial x} = M$$
$$\frac{\partial \phi}{\partial y} = N$$

But since $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x, y)$ but at least we know now that we can do that since the condition $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x, y)\,\mathrm{d}x + N(x, y)\,\mathrm{d}y = 0 \tag{1A}$$

Therefore

$$\begin{aligned} (x^2 + y^2)\,\mathrm{d}y &= (-3y\,x^2 - y^3 - 2yx)\,\mathrm{d}x \\ (3y\,x^2 + y^3 + 2yx)\,\mathrm{d}x + (x^2 + y^2)\,\mathrm{d}y &= 0 \end{aligned} \tag{2A}$$

Comparing (1A) and (2A) shows that

$$\begin{aligned} M(x, y) &= 3y\,x^2 + y^3 + 2yx \\ N(x, y) &= x^2 + y^2 \end{aligned}$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Using result found above gives

$$\begin{aligned} \frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}(3y\,x^2 + y^3 + 2yx) \\ &= 3x^2 + 3y^2 + 2x \end{aligned}$$

And

$$\begin{aligned} \frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}(x^2 + y^2) \\ &= 2x \end{aligned}$$

Since $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$, then the ODE is not exact. Since the ODE is not exact, we will try to find an integrating factor to make it exact. Let

$$
\begin{aligned}
A &= \frac{1}{N}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right) \\
&= \frac{1}{x^2+y^2}\left((3x^2+3y^2+2x) - (2x)\right) \\
&= 3
\end{aligned}
$$

Since A does not depend on y, then it can be used to find an integrating factor. The integrating factor μ is

$$
\begin{aligned}
\mu &= e^{\int A \, \mathrm{d}x} \\
&= e^{\int 3 \, \mathrm{d}x}
\end{aligned}
$$

The result of integrating gives

$$
\begin{aligned}
\mu &= e^{3x} \\
&= \mathrm{e}^{3x}
\end{aligned}
$$

M and N are multiplied by this integrating factor, giving new M and new N which are called $\overline{M}$ and $\overline{N}$ for now so not to confuse them with the original M and N.

$$
\begin{aligned}
\overline{M} &= \mu M \\
&= \mathrm{e}^{3x}\left(3y\,x^2 + y^3 + 2yx\right) \\
&= y\left(3x^2 + y^2 + 2x\right)\mathrm{e}^{3x}
\end{aligned}
$$

And

$$
\begin{aligned}
\overline{N} &= \mu N \\
&= \mathrm{e}^{3x}\left(x^2 + y^2\right) \\
&= \left(x^2 + y^2\right)\mathrm{e}^{3x}
\end{aligned}
$$

Now a modified ODE is ontained from the original ODE, which is exact and can be solved. The modified ODE is

$$
\begin{aligned}
\overline{M} + \overline{N}\frac{\mathrm{d}y}{\mathrm{d}x} &= 0 \\
\left(y\left(3x^2 + y^2 + 2x\right)\mathrm{e}^{3x}\right) + \left(\left(x^2 + y^2\right)\mathrm{e}^{3x}\right)\frac{\mathrm{d}y}{\mathrm{d}x} &= 0
\end{aligned}
$$

The following equations are now set up to solve for the function $\phi(x,y)$

$$\frac{\partial\phi}{\partial x}=\overline{M} \tag{1}$$
$$\frac{\partial\phi}{\partial y}=\overline{N} \tag{2}$$

Integrating (1) w.r.t. x gives

$$\begin{aligned}\int\frac{\partial\phi}{\partial x}\,\mathrm{d}x&=\int\overline{M}\,\mathrm{d}x\\ \int\frac{\partial\phi}{\partial x}\,\mathrm{d}x&=\int y\left(3x^2+y^2+2x\right)\mathrm{e}^{3x}\,\mathrm{d}x\\ \phi&=\frac{(3x^2+y^2)\,y\,\mathrm{e}^{3x}}{3}+f(y)\end{aligned} \tag{3}$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$\begin{aligned}\frac{\partial\phi}{\partial y}&=\frac{2y^2\mathrm{e}^{3x}}{3}+\frac{(3x^2+y^2)\,\mathrm{e}^{3x}}{3}+f'(y) \\ &=\left(x^2+y^2\right)\mathrm{e}^{3x}+f'(y)\end{aligned} \tag{4}$$

But equation (2) says that $\frac{\partial\phi}{\partial y}=(x^2+y^2)\,\mathrm{e}^{3x}$. Therefore equation (4) becomes

$$\left(x^2+y^2\right)\mathrm{e}^{3x}=\left(x^2+y^2\right)\mathrm{e}^{3x}+f'(y) \tag{5}$$

Solving equation (5) for $f'(y)$ gives

$$f'(y)=0$$

Therefore

$$f(y)=c_1$$

Where c_1 is constant of integration. Substituting this result for $f(y)$ into equation (3) gives ϕ

$$\phi=\frac{(3x^2+y^2)\,y\,\mathrm{e}^{3x}}{3}+c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = \frac{(3x^2 + y^2)\, y\, e^{3x}}{3}$$

Summary

The solution(s) found are the following

$$\frac{(y^2 + 3x^2)\, y\, e^{3x}}{3} = c_1 \tag{1}$$

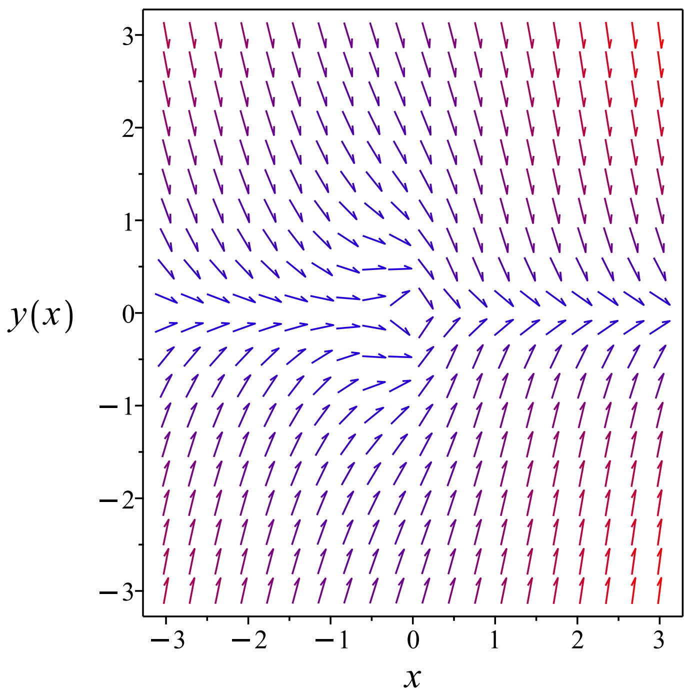


Figure 256: Slope field plot

Verification of solutions

$$\frac{(y^2 + 3x^2)\, y\, e^{3x}}{3} = c_1$$

Verified OK.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
trying homogeneous types:
trying homogeneous D
<- homogeneous successful`
```

✓ Solution by Maple

Time used: 0.015 (sec). Leaf size: 297

```
dsolve(2*x*y(x)+3*x^2*y(x)+y(x)^3+(x^2+y(x)^2)*diff(y(x),x) = 0,y(x), singsol=all)
```

$$y(x) = -\frac{\left(x^2 e^{6x} c_1^2 - \frac{2^{\frac{1}{3}}\left(\left(1+\sqrt{4x^6 e^{6x} c_1^2+1}\right) e^{6x} c_1^2\right)^{\frac{2}{3}}}{2}\right) 2^{\frac{1}{3}} e^{-3x}}{\left(\left(1+\sqrt{4x^6 e^{6x} c_1^2+1}\right) e^{6x} c_1^2\right)^{\frac{1}{3}} c_1}$$

$$y(x) = -\frac{e^{-3x} 2^{\frac{1}{3}}\left(2x^2\left(i\sqrt{3}-1\right) e^{6x} c_1^2 + 2^{\frac{1}{3}}\left(1+i\sqrt{3}\right)\left(\left(1+\sqrt{4x^6 e^{6x} c_1^2+1}\right) e^{6x} c_1^2\right)^{\frac{2}{3}}\right)}{4\left(\left(1+\sqrt{4x^6 e^{6x} c_1^2+1}\right) e^{6x} c_1^2\right)^{\frac{1}{3}} c_1}$$

$$y(x) = \frac{\left(2x^2\left(1+i\sqrt{3}\right) e^{6x} c_1^2 + 2^{\frac{1}{3}}\left(i\sqrt{3}-1\right)\left(\left(1+\sqrt{4x^6 e^{6x} c_1^2+1}\right) e^{6x} c_1^2\right)^{\frac{2}{3}}\right) e^{-3x} 2^{\frac{1}{3}}}{4\left(\left(1+\sqrt{4x^6 e^{6x} c_1^2+1}\right) e^{6x} c_1^2\right)^{\frac{1}{3}} c_1}$$

✓ Solution by Mathematica

Time used: 60.305 (sec). Leaf size: 383

```
DSolve[2*x*y[x]+3*x^2*y[x]+y[x]^3+(x^2+y[x]^2)*y'[x] == 0,y[x],x,IncludeSingularSolutions ->
```

$$y(x) \to \frac{e^{-3x}\left(-2e^{6x}x^2+\sqrt[3]{2}\left(\sqrt{4e^{18x}x^6+e^{6(2x+c_1)}}+e^{6x+3c_1}\right)^{2/3}\right)}{2^{2/3}\sqrt[3]{\sqrt{4e^{18x}x^6+e^{6(2x+c_1)}}+e^{6x+3c_1}}}$$

$$y(x) \to \frac{i\left(\sqrt{3}+i\right)e^{-3x}\sqrt[3]{\sqrt{4e^{18x}x^6+e^{6(2x+c_1)}}+e^{6x+3c_1}}}{2\sqrt[3]{2}} + \frac{\left(1+i\sqrt{3}\right)e^{3x}x^2}{2^{2/3}\sqrt[3]{\sqrt{4e^{18x}x^6+e^{6(2x+c_1)}}+e^{6x+3c_1}}}$$

$$y(x) \to \frac{\left(1-i\sqrt{3}\right)e^{3x}x^2}{2^{2/3}\sqrt[3]{\sqrt{4e^{18x}x^6+e^{6(2x+c_1)}}+e^{6x+3c_1}}} - \frac{\left(1+i\sqrt{3}\right)e^{-3x}\sqrt[3]{\sqrt{4e^{18x}x^6+e^{6(2x+c_1)}}+e^{6x+3c_1}}}{2\sqrt[3]{2}}$$

5.19 problem 26

Internal problem ID [561]
Internal file name [`OUTPUT/561_Sunday_June_05_2022_01_44_39_AM_71606817/index.tex`]

Book: Elementary differential equations and boundary value problems, 10th ed., Boyce and DiPrima
Section: Section 2.6. Page 100
Problem number: 26.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"linear", "exactWithIntegrationFactor", "first_order_ode_lie_symmetry_lookup"**

Maple gives the following as the ode type

```
[[_linear, `class A`]]
```

$$\boxed{y' - y = -1 + e^{2x}}$$

5.19.1 Solving as linear ode

Entering Linear first order ODE solver. In canonical form a linear first order is

$$y' + p(x)y = q(x)$$

Where here

$$\begin{aligned} p(x) &= -1 \\ q(x) &= -1 + e^{2x} \end{aligned}$$

Hence the ode is

$$y' - y = -1 + e^{2x}$$

The integrating factor μ is

$$
\begin{aligned}
\mu &= e^{\int (-1)dx} \\
&= e^{-x}
\end{aligned}
$$

The ode becomes

$$
\begin{aligned}
\frac{d}{dx}(\mu y) &= (\mu)\left(-1+e^{2x}\right) \\
\frac{d}{dx}\left(e^{-x} y\right) &= \left(e^{-x}\right)\left(-1+e^{2x}\right) \\
d\left(e^{-x} y\right) &= \left(e^{x}-e^{-x}\right)\, dx
\end{aligned}
$$

Integrating gives

$$
\begin{aligned}
e^{-x} y &= \int e^{x}-e^{-x}\, dx \\
e^{-x} y &= e^{x}+e^{-x}+c_1
\end{aligned}
$$

Dividing both sides by the integrating factor $\mu = e^{-x}$ results in

$$
y = e^{x}\left(e^{x}+e^{-x}\right)+c_1 e^{x}
$$

which simplifies to

$$
y = e^{2x}+1+c_1 e^{x}
$$

<u>Summary</u>
The solution(s) found are the following

$$
y = e^{2x}+1+c_1 e^{x} \tag{1}
$$

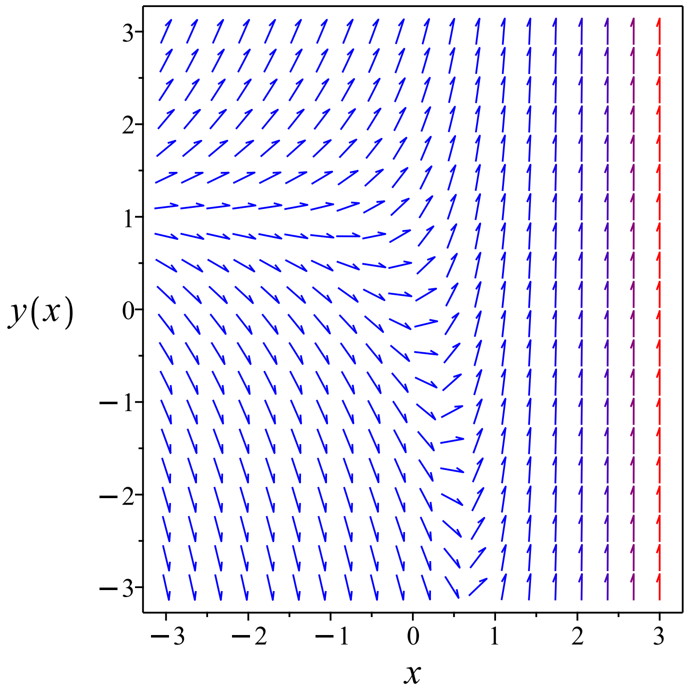


Figure 257: Slope field plot

Verification of solutions

$$y = e^{2x} + 1 + c_1 e^x$$

Verified OK.

5.19.2 Solving as first order ode lie symmetry lookup ode

Writing the ode as

$$\begin{aligned} y' &= -1 + e^{2x} + y \\ y' &= \omega(x, y) \end{aligned}$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x + \omega(\eta_y - \xi_x) - \omega^2 \xi_y - \omega_x \xi - \omega_y \eta = 0 \tag{A}$$

The type of this ode is known. It is of type `linear`. Therefore we do not need to solve the PDE (A), and can just use the lookup table shown below to find ξ, η

Table 257: Lie symmetry infinitesimal lookup table for known first order ODE's

ODE class	Form	ξ	η
linear ode	$y' = f(x)y(x) + g(x)$	0	$e^{\int f dx}$
separable ode	$y' = f(x)\, g(y)$	$\frac{1}{f}$	0
quadrature ode	$y' = f(x)$	0	1
quadrature ode	$y' = g(y)$	1	0
homogeneous ODEs of Class A	$y' = f\left(\frac{y}{x}\right)$	x	y
homogeneous ODEs of Class C	$y' = (a + bx + cy)^{\frac{n}{m}}$	1	$-\frac{b}{c}$
homogeneous class D	$y' = \frac{y}{x} + g(x)\, F\left(\frac{y}{x}\right)$	x^2	xy
First order special form ID 1	$y' = g(x)\, e^{h(x)+by} + f(x)$	$\frac{e^{-\int b f(x) dx - h(x)}}{g(x)}$	$\frac{f(x) e^{-\int b f(x) dx - h(x)}}{g(x)}$
polynomial type ode	$y' = \frac{a_1 x + b_1 y + c_1}{a_2 x + b_2 y + c_2}$	$\frac{a_1 b_2 x - a_2 b_1 x - b_1 c_2 + b_2 c_1}{a_1 b_2 - a_2 b_1}$	$\frac{a_1 b_2 y - a_2 b_1 y - a_1 c_2 - a_2 c_1}{a_1 b_2 - a_2 b_1}$
Bernoulli ode	$y' = f(x)\, y + g(x)\, y^n$	0	$e^{-\int (n-1) f(x) dx} y^n$
Reduced Riccati	$y' = f_1(x)\, y + f_2(x)\, y^2$	0	$e^{-\int f_1 dx}$

The above table shows that

$$
\begin{aligned}
\xi(x, y) &= 0 \\
\eta(x, y) &= e^x
\end{aligned} \tag{A1}
$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x, y) \to (R, S)$ where (R, S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$
\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}
$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y}\right) S(x, y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the

canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$R = x$$

S is found from

$$\begin{aligned} S &= \int \frac{1}{\eta} dy \\ &= \int \frac{1}{e^x} dy \end{aligned}$$

Which results in

$$S = e^{-x} y$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x, y) S_y}{R_x + \omega(x, y) R_y} \tag{2}$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x, y)$ is the right hand side of the original ode given by

$$\omega(x, y) = -1 + e^{2x} + y$$

Evaluating all the partial derivatives gives

$$\begin{aligned} R_x &= 1 \\ R_y &= 0 \\ S_x &= -e^{-x} y \\ S_y &= e^{-x} \end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = e^x - e^{-x} \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = e^R - e^{-R}$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by

integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$S(R) = e^{R} + e^{-R} + c_1 \tag{4}$$

To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$y\, e^{-x} = e^{x} + e^{-x} + c_1$$

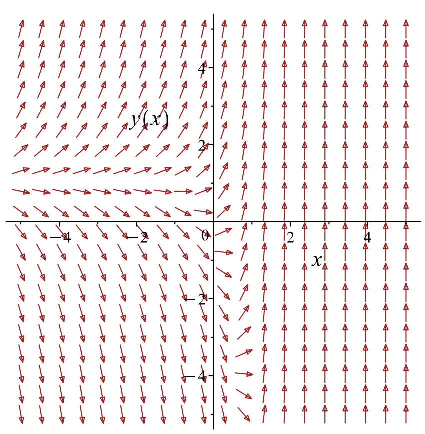
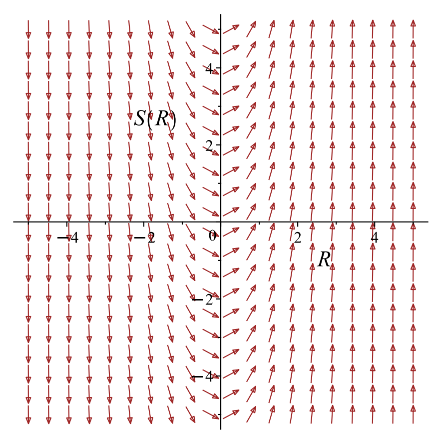
Which simplifies to

$$y\, e^{-x} = e^{x} + e^{-x} + c_1$$

Which gives

$$y = \left(e^{x} + e^{-x} + c_1\right) e^{x}$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in x, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dx} = -1 + e^{2x} + y$ 	$R = x$ $S = e^{-x} y$	$\frac{dS}{dR} = e^{R} - e^{-R}$ 

Summary

The solution(s) found are the following

$$y = \left(e^{x} + e^{-x} + c_1\right) e^{x} \tag{1}$$

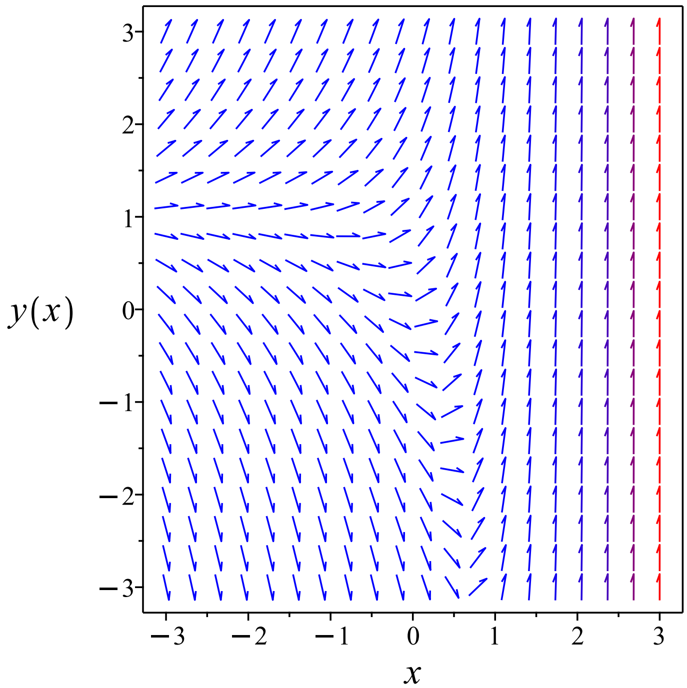


Figure 258: Slope field plot

Verification of solutions

$$y = \left(e^x + e^{-x} + c_1\right) e^x$$

Verified OK.

5.19.3 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x,y) + N(x,y)\frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x,y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y) = 0$$

Hence

$$\frac{\partial \phi}{\partial x} + \frac{\partial \phi}{\partial y}\frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$
\begin{aligned}
\frac{\partial \phi}{\partial x} &= M \\
\frac{\partial \phi}{\partial y} &= N
\end{aligned}
$$

But since $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ then for the above to be valid, we require that

$$
\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}
$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x, y)$ but at least we know now that we can do that since the condition $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$
M(x, y)\,\mathrm{d}x + N(x, y)\,\mathrm{d}y = 0 \tag{1A}
$$

Therefore

$$
\begin{aligned}
\mathrm{d}y &= \left(-1 + \mathrm{e}^{2x} + y\right) \mathrm{d}x \\
\left(1 - \mathrm{e}^{2x} - y\right) \mathrm{d}x + \mathrm{d}y &= 0 \qquad (2A)
\end{aligned}
$$

Comparing (1A) and (2A) shows that

$$
\begin{aligned}
M(x, y) &= 1 - \mathrm{e}^{2x} - y \\
N(x, y) &= 1
\end{aligned}
$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$
\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}
$$

Using result found above gives

$$
\begin{aligned}
\frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}\left(1 - \mathrm{e}^{2x} - y\right) \\
&= -1
\end{aligned}
$$

And

$$
\begin{aligned}
\frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}(1) \\
&= 0
\end{aligned}
$$

Since $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$, then the ODE is not exact. Since the ODE is not exact, we will try to find an integrating factor to make it exact. Let

$$\begin{aligned} A &= \frac{1}{N}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right) \\ &= 1((-1) - (0)) \\ &= -1 \end{aligned}$$

Since A does not depend on y, then it can be used to find an integrating factor. The integrating factor μ is

$$\begin{aligned} \mu &= e^{\int A \, \mathrm{d}x} \\ &= e^{\int -1 \, \mathrm{d}x} \end{aligned}$$

The result of integrating gives

$$\begin{aligned} \mu &= e^{-x} \\ &= \mathrm{e}^{-x} \end{aligned}$$

M and N are multiplied by this integrating factor, giving new M and new N which are called $\overline{M}$ and $\overline{N}$ for now so not to confuse them with the original M and N.

$$\begin{aligned} \overline{M} &= \mu M \\ &= \mathrm{e}^{-x}\left(1 - \mathrm{e}^{2x} - y\right) \\ &= \mathrm{e}^{-x} - \mathrm{e}^{x} - \mathrm{e}^{-x} y \end{aligned}$$

And

$$\begin{aligned} \overline{N} &= \mu N \\ &= \mathrm{e}^{-x}(1) \\ &= \mathrm{e}^{-x} \end{aligned}$$

Now a modified ODE is ontained from the original ODE, which is exact and can be solved. The modified ODE is

$$\begin{aligned} \overline{M} + \overline{N}\frac{\mathrm{d}y}{\mathrm{d}x} &= 0 \\ \left(\mathrm{e}^{-x} - \mathrm{e}^{x} - \mathrm{e}^{-x} y\right) + \left(\mathrm{e}^{-x}\right)\frac{\mathrm{d}y}{\mathrm{d}x} &= 0 \end{aligned}$$

The following equations are now set up to solve for the function $\phi(x, y)$

$$\frac{\partial \phi}{\partial x} = \overline{M} \tag{1}$$

$$\frac{\partial \phi}{\partial y} = \overline{N} \tag{2}$$

Integrating (1) w.r.t. x gives

$$\int \frac{\partial \phi}{\partial x}\,\mathrm{d}x = \int \overline{M}\,\mathrm{d}x$$
$$\int \frac{\partial \phi}{\partial x}\,\mathrm{d}x = \int \mathrm{e}^{-x} - \mathrm{e}^{x} - \mathrm{e}^{-x} y\,\mathrm{d}x$$
$$\phi = (y-1)\,\mathrm{e}^{-x} - \mathrm{e}^{x} + f(y) \tag{3}$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$\frac{\partial \phi}{\partial y} = \mathrm{e}^{-x} + f'(y) \tag{4}$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = \mathrm{e}^{-x}$. Therefore equation (4) becomes

$$\mathrm{e}^{-x} = \mathrm{e}^{-x} + f'(y) \tag{5}$$

Solving equation (5) for $f'(y)$ gives

$$f'(y) = 0$$

Therefore

$$f(y) = c_1$$

Where c_1 is constant of integration. Substituting this result for $f(y)$ into equation (3) gives ϕ

$$\phi = (y-1)\,\mathrm{e}^{-x} - \mathrm{e}^{x} + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = (y-1)\,\mathrm{e}^{-x} - \mathrm{e}^{x}$$

The solution becomes

$$y = \left(\mathrm{e}^{x} + \mathrm{e}^{-x} + c_1\right)\mathrm{e}^{x}$$

Summary

The solution(s) found are the following

$$y = \left(\mathrm{e}^{x} + \mathrm{e}^{-x} + c_1\right)\mathrm{e}^{x} \tag{1}$$

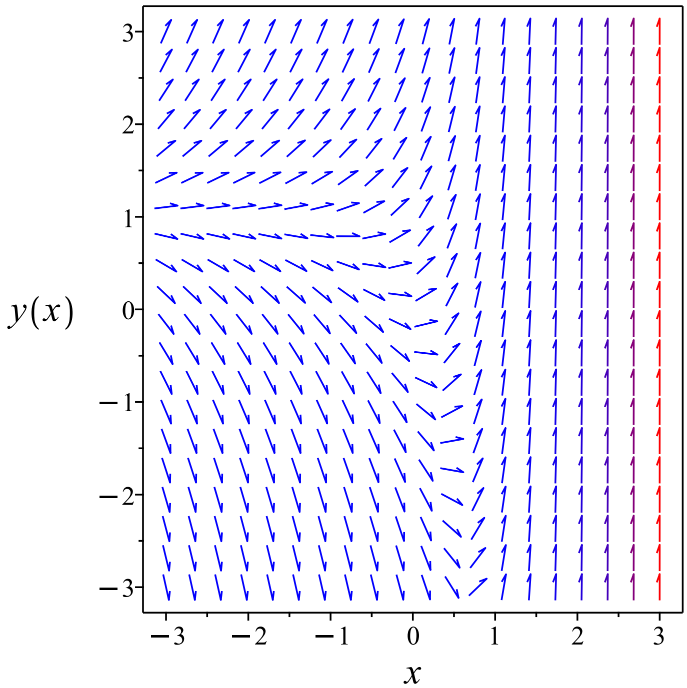


Figure 259: Slope field plot

Verification of solutions

$$y = \left(e^{x} + e^{-x} + c_1\right) e^{x}$$

Verified OK.

5.19.4 Maple step by step solution

Let's solve

$y' - y = -1 + e^{2x}$

- Highest derivative means the order of the ODE is 1

  y'

- Isolate the derivative

  $y' = -1 + e^{2x} + y$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE

  $y' - y = -1 + e^{2x}$

- The ODE is linear; multiply by an integrating factor $\mu(x)$

  $\mu(x)\left(y' - y\right) = \mu(x)\left(-1 + e^{2x}\right)$

- Assume the lhs of the ODE is the total derivative $\frac{d}{dx}(\mu(x)\, y)$

  $\mu(x)\,(y' - y) = \mu'(x)\, y + \mu(x)\, y'$
- Isolate $\mu'(x)$

  $\mu'(x) = -\mu(x)$
- Solve to find the integrating factor

  $\mu(x) = \mathrm{e}^{-x}$
- Integrate both sides with respect to x

  $\int \left(\frac{d}{dx}(\mu(x)\, y)\right) dx = \int \mu(x)\,(-1 + \mathrm{e}^{2x})\, dx + c_1$
- Evaluate the integral on the lhs

  $\mu(x)\, y = \int \mu(x)\,(-1 + \mathrm{e}^{2x})\, dx + c_1$
- Solve for y

  $y = \frac{\int \mu(x)(-1+\mathrm{e}^{2x})dx+c_1}{\mu(x)}$
- Substitute $\mu(x) = \mathrm{e}^{-x}$

  $y = \frac{\int \mathrm{e}^{-x}(-1+\mathrm{e}^{2x})dx+c_1}{\mathrm{e}^{-x}}$
- Evaluate the integrals on the rhs

  $y = \frac{\mathrm{e}^{x}+\frac{1}{\mathrm{e}^{x}}+c_1}{\mathrm{e}^{-x}}$
- Simplify

  $y = \mathrm{e}^{2x} + 1 + c_1\mathrm{e}^{x}$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
<- 1st order linear successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 14

```
dsolve(diff(y(x),x) = -1+exp(2*x)+y(x),y(x), singsol=all)
```

$$y(x) = e^{2x} + 1 + e^{x} c_1$$

✓ Solution by Mathematica

Time used: 0.047 (sec). Leaf size: 18

```
DSolve[y'[x] == -1+Exp[2*x]+y[x],y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to e^{2x} + c_1 e^{x} + 1$$

5.20 problem 27

Internal problem ID [562]
Internal file name [OUTPUT/562_Sunday_June_05_2022_01_44_40_AM_44632801/index.tex]

Book: Elementary differential equations and boundary value problems, 10th ed., Boyce and DiPrima
Section: Section 2.6. Page 100
Problem number: 27.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"exact", "differentialType"**

Maple gives the following as the ode type

```
[[_1st_order, `_with_symmetry_[F(x)*G(y),0]`]]
```

$$\boxed{\left(-\sin\left(y\right)+\frac{x}{y}\right)y'=-1}$$

5.20.1 Solving as differentialType ode

Writing the ode as

$$y'=-\frac{1}{-\sin\left(y\right)+\frac{x}{y}} \tag{1}$$

Which becomes

$$\left(y\sin\left(y\right)\right)dy=\left(x\right)dy+\left(y\right)dx \tag{2}$$

But the RHS is complete differential because

$$\left(x\right)dy+\left(y\right)dx=d(yx)$$

Hence (2) becomes

$$\left(y\sin\left(y\right)\right)dy=d(yx)$$

Integrating both sides gives gives the solution as

$$\sin(y) - y\cos(y) = yx + c_1$$

Summary

The solution(s) found are the following

$$\sin(y) - y\cos(y) = yx + c_1 \tag{1}$$

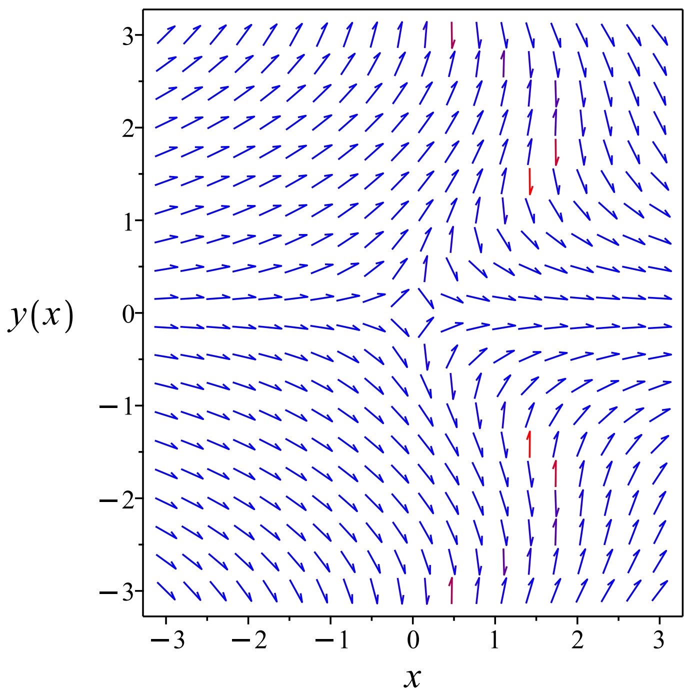


Figure 260: Slope field plot

Verification of solutions

$$\sin(y) - y\cos(y) = yx + c_1$$

Verified OK.

5.20.2 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x,y)+N(x,y)\frac{dy}{dx}=0 \tag{A}$$

We assume there exists a function $\phi(x,y)=c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y)=0$$

Hence

$$\frac{\partial\phi}{\partial x}+\frac{\partial\phi}{\partial y}\frac{dy}{dx}=0 \tag{B}$$

Comparing (A,B) shows that

$$\begin{aligned}\frac{\partial\phi}{\partial x}&=M\\ \frac{\partial\phi}{\partial y}&=N\end{aligned}$$

But since $\frac{\partial^2\phi}{\partial x\partial y}=\frac{\partial^2\phi}{\partial y\partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y}=\frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x,y)$ but at least we know now that we can do that since the condition $\frac{\partial^2\phi}{\partial x\partial y}=\frac{\partial^2\phi}{\partial y\partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x,y)\,\mathrm{d}x+N(x,y)\,\mathrm{d}y=0 \tag{1A}$$

Therefore

$$\begin{aligned}(y\sin(y)-x)\,\mathrm{d}y&=(y)\,\mathrm{d}x\\ (-y)\,\mathrm{d}x+(y\sin(y)-x)\,\mathrm{d}y&=0 \end{aligned}\tag{2A}$$

Comparing (1A) and (2A) shows that

$$\begin{aligned}M(x,y)&=-y\\ N(x,y)&=y\sin(y)-x\end{aligned}$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Using result found above gives

$$\begin{aligned}\frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}(-y) \\ &= -1\end{aligned}$$

And

$$\begin{aligned}\frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}(y \sin(y) - x) \\ &= -1\end{aligned}$$

Since $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$, then the ODE is exact The following equations are now set up to solve for the function $\phi(x, y)$

$$\frac{\partial \phi}{\partial x} = M \tag{1}$$

$$\frac{\partial \phi}{\partial y} = N \tag{2}$$

Integrating (1) w.r.t. x gives

$$\int \frac{\partial \phi}{\partial x} \, dx = \int M \, dx$$

$$\int \frac{\partial \phi}{\partial x} \, dx = \int -y \, dx$$

$$\phi = -yx + f(y) \tag{3}$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$\frac{\partial \phi}{\partial y} = -x + f'(y) \tag{4}$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = y \sin(y) - x$. Therefore equation (4) becomes

$$y \sin(y) - x = -x + f'(y) \tag{5}$$

Solving equation (5) for $f'(y)$ gives

$$f'(y) = y \sin(y)$$

Integrating the above w.r.t y gives

$$\int f'(y) \,\mathrm{d}y = \int (y \sin(y)) \,\mathrm{d}y$$
$$f(y) = \sin(y) - y \cos(y) + c_1$$

Where c_1 is constant of integration. Substituting result found above for $f(y)$ into equation (3) gives ϕ

$$\phi = -yx + \sin(y) - y \cos(y) + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = -yx + \sin(y) - y \cos(y)$$

Summary

The solution(s) found are the following

$$\sin(y) - y \cos(y) - yx = c_1 \tag{1}$$

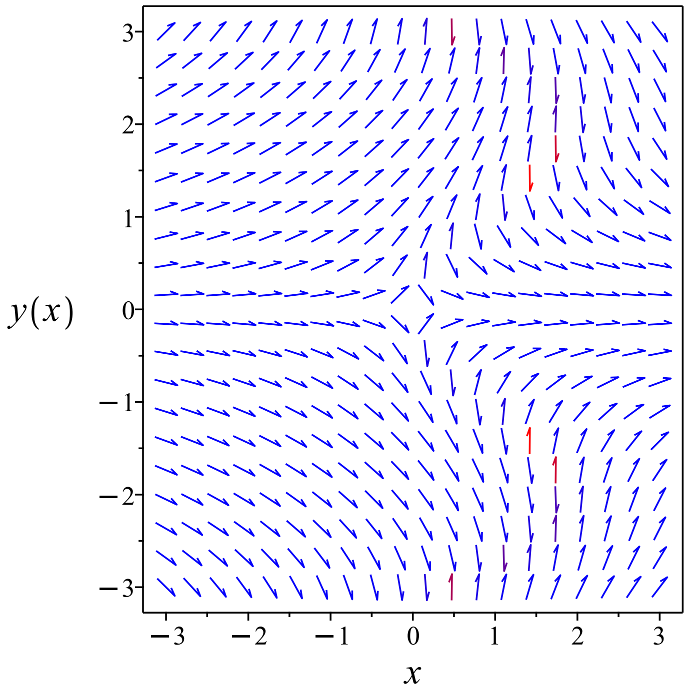


Figure 261: Slope field plot

Verification of solutions

$$\sin(y) - y\cos(y) - yx = c_1$$

Verified OK.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
<- 1st order linear successful
<- inverse linear successful`
```

✓ Solution by Maple

Time used: 0.078 (sec). Leaf size: 25

```
dsolve(1+(-sin(y(x))+x/y(x))*diff(y(x),x) = 0,y(x), singsol=all)
```

$$x + \frac{y(x)\cos(y(x)) - \sin(y(x)) - c_1}{y(x)} = 0$$

✓ Solution by Mathematica

Time used: 0.143 (sec). Leaf size: 29

```
DSolve[1+(-Sin[y[x]]+x/y[x])*y'[x] == 0,y[x],x,IncludeSingularSolutions -> True]
```

$$\text{Solve}\left[x = \frac{\sin(y(x)) - y(x)\cos(y(x))}{y(x)} + \frac{c_1}{y(x)}, y(x)\right]$$

5.21 problem 28

Internal problem ID [563]
Internal file name [OUTPUT/563_Sunday_June_05_2022_01_44_42_AM_83508478/index.tex]

Book: Elementary differential equations and boundary value problems, 10th ed., Boyce and DiPrima
Section: Section 2.6. Page 100
Problem number: 28.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"exactWithIntegrationFactor", "first_order_ode_lie_symmetry_calculated"**

Maple gives the following as the ode type

```
[[_1st_order, _with_exponential_symmetries]]
```

$$\boxed{y + \left(-\mathrm{e}^{-2y} + 2yx\right) y' = 0}$$

5.21.1 Solving as first order ode lie symmetry calculated ode

Writing the ode as

$$y' = -\frac{y}{-\mathrm{e}^{-2y} + 2yx}$$
$$y' = \omega(x, y)$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x + \omega(\eta_y - \xi_x) - \omega^2 \xi_y - \omega_x \xi - \omega_y \eta = 0 \tag{A}$$

The type of this ode is not in the lookup table. To determine ξ, η then (A) is solved using ansatz. Making bivariate polynomials of degree 2 to use as anstaz gives

$$\xi = x^2 a_4 + yxa_5 + y^2 a_6 + xa_2 + ya_3 + a_1 \tag{1E}$$
$$\eta = x^2 b_4 + yxb_5 + y^2 b_6 + xb_2 + yb_3 + b_1 \tag{2E}$$

Where the unknown coefficients are

$$\{a_1, a_2, a_3, a_4, a_5, a_6, b_1, b_2, b_3, b_4, b_5, b_6\}$$

Substituting equations (1E,2E) and ω into (A) gives

$$\begin{aligned} &2xb_4 + yb_5 + b_2 - \frac{y(-2xa_4 + xb_5 - ya_5 + 2yb_6 - a_2 + b_3)}{-\mathrm{e}^{-2y} + 2yx} \\ &- \frac{y^2(xa_5 + 2ya_6 + a_3)}{(-\mathrm{e}^{-2y} + 2yx)^2} - \frac{2y^2(x^2a_4 + yxa_5 + y^2a_6 + xa_2 + ya_3 + a_1)}{(-\mathrm{e}^{-2y} + 2yx)^2} \\ &- \left(-\frac{1}{-\mathrm{e}^{-2y} + 2yx} + \frac{y(2\,\mathrm{e}^{-2y} + 2x)}{(-\mathrm{e}^{-2y} + 2yx)^2}\right)\left(x^2b_4 \right. \\ &\left. + yxb_5 + y^2b_6 + xb_2 + yb_3 + b_1\right) = 0 \end{aligned} \tag{5E}$$

Putting the above in normal form gives

$$\frac{-10\,\mathrm{e}^{-2y}x^2yb_4 - 6\,\mathrm{e}^{-2y}x\,y^2b_5 - 2\,\mathrm{e}^{-2y}xya_4 - 6\,\mathrm{e}^{-2y}xyb_2 + \mathrm{e}^{-4y}yb_5 + 8x^3y^2b_4 + 4x^2y^3b_5 - 2y^4a_6 - 2y^3a_6 -}{}$$

$$= 0$$

Setting the numerator to zero gives

$$\begin{aligned} &-10\,\mathrm{e}^{-2y}x^2yb_4 - 6\,\mathrm{e}^{-2y}x\,y^2b_5 - 2\,\mathrm{e}^{-2y}xya_4 - 6\,\mathrm{e}^{-2y}xyb_2 + \mathrm{e}^{-4y}yb_5 \\ &+ 8x^3y^2b_4 + 4x^2y^3b_5 - 2y^4a_6 - 2y^3a_6 - 4x\,y^3b_6 + 2x^2y^2a_4 - 2x^2y^2b_5 \\ &- y^2a_3 + 4x^2y^2b_2 - 2x\,y^2b_3 - 2\,\mathrm{e}^{-2y}y^2b_3 - \mathrm{e}^{-2y}xb_2 - \mathrm{e}^{-2y}ya_2 \\ &- 2\,\mathrm{e}^{-2y}yb_1 - 2y^3a_3 - 2y^2a_1 - \mathrm{e}^{-2y}b_1 + 2\,\mathrm{e}^{-4y}xb_4 - x\,y^2a_5 \\ &- 2\,\mathrm{e}^{-2y}y^3b_6 - \mathrm{e}^{-2y}x^2b_4 - \mathrm{e}^{-2y}y^2a_5 + \mathrm{e}^{-2y}y^2b_6 + \mathrm{e}^{-4y}b_2 = 0 \end{aligned} \tag{6E}$$

Simplifying the above gives

$$\begin{aligned} &-10\,\mathrm{e}^{-2y}x^2yb_4 - 6\,\mathrm{e}^{-2y}x\,y^2b_5 - 2\,\mathrm{e}^{-2y}xya_4 - 6\,\mathrm{e}^{-2y}xyb_2 + \mathrm{e}^{-4y}yb_5 \\ &+ 8x^3y^2b_4 + 4x^2y^3b_5 - 2y^4a_6 - 2y^3a_6 - 4x\,y^3b_6 + 2x^2y^2a_4 - 2x^2y^2b_5 \\ &- y^2a_3 + 4x^2y^2b_2 - 2x\,y^2b_3 - 2\,\mathrm{e}^{-2y}y^2b_3 - \mathrm{e}^{-2y}xb_2 - \mathrm{e}^{-2y}ya_2 \\ &- 2\,\mathrm{e}^{-2y}yb_1 - 2y^3a_3 - 2y^2a_1 - \mathrm{e}^{-2y}b_1 + 2\,\mathrm{e}^{-4y}xb_4 - x\,y^2a_5 \\ &- 2\,\mathrm{e}^{-2y}y^3b_6 - \mathrm{e}^{-2y}x^2b_4 - \mathrm{e}^{-2y}y^2a_5 + \mathrm{e}^{-2y}y^2b_6 + \mathrm{e}^{-4y}b_2 = 0 \end{aligned} \tag{6E}$$

Looking at the above PDE shows the following are all the terms with $\{x, y\}$ in them.

$$\{x, y, \mathrm{e}^{-4y}, \mathrm{e}^{-2y}\}$$

The following substitution is now made to be able to collect on all terms with $\{x, y\}$ in them

$$\{x = v_1, y = v_2, \mathrm{e}^{-4y} = v_3, \mathrm{e}^{-2y} = v_4\}$$

The above PDE (6E) now becomes

$$\begin{aligned}
&8v_1^3v_2^2b_4 + 4v_1^2v_2^3b_5 + 2v_1^2v_2^2a_4 - 2v_2^4a_6 + 4v_1^2v_2^2b_2 - 10v_4v_1^2v_2b_4 - 2v_1^2v_2^2b_5 \\
&- 6v_4v_1v_2^2b_5 - 4v_1v_2^3b_6 - 2v_4v_2^3b_6 - 2v_2^3a_3 - 2v_4v_1v_2a_4 - v_1v_2^2a_5 - v_4v_2^2a_5 \\
&- 2v_2^3a_6 - 6v_4v_1v_2b_2 - 2v_1v_2^2b_3 - 2v_4v_2^2b_3 - v_4v_1^2b_4 + v_4v_2^2b_6 - 2v_2^2a_1 \\
&- v_4v_2a_2 - v_2^2a_3 - 2v_4v_2b_1 - v_4v_1b_2 + 2v_3v_1b_4 + v_3v_2b_5 - v_4b_1 + v_3b_2 = 0
\end{aligned} \tag{7E}$$

Collecting the above on the terms v_i introduced, and these are

$$\{v_1, v_2, v_3, v_4\}$$

Equation (7E) now becomes

$$\begin{aligned}
&8v_1^3v_2^2b_4 + 4v_1^2v_2^3b_5 + (2a_4 + 4b_2 - 2b_5)\, v_1^2v_2^2 - 10v_4v_1^2v_2b_4 - v_4v_1^2b_4 \\
&- 4v_1v_2^3b_6 - 6v_4v_1v_2^2b_5 + (-a_5 - 2b_3)\, v_1v_2^2 + (-2a_4 - 6b_2)\, v_1v_2v_4 + 2v_3v_1b_4 \\
&- v_4v_1b_2 - 2v_2^4a_6 - 2v_4v_2^3b_6 + (-2a_3 - 2a_6)\, v_2^3 + (-a_5 - 2b_3 + b_6)\, v_2^2v_4 \\
&+ (-2a_1 - a_3)\, v_2^2 + v_3v_2b_5 + (-a_2 - 2b_1)\, v_2v_4 + v_3b_2 - v_4b_1 = 0
\end{aligned} \tag{8E}$$

Setting each coefficients in (8E) to zero gives the following equations to solve

$$
\begin{aligned}
b_2 &= 0 \\
b_5 &= 0 \\
-2a_6 &= 0 \\
-b_1 &= 0 \\
-b_2 &= 0 \\
-10b_4 &= 0 \\
-b_4 &= 0 \\
2b_4 &= 0 \\
8b_4 &= 0 \\
-6b_5 &= 0 \\
4b_5 &= 0 \\
-4b_6 &= 0 \\
-2b_6 &= 0 \\
-2a_1 - a_3 &= 0 \\
-a_2 - 2b_1 &= 0 \\
-2a_3 - 2a_6 &= 0 \\
-2a_4 - 6b_2 &= 0 \\
-a_5 - 2b_3 &= 0 \\
2a_4 + 4b_2 - 2b_5 &= 0 \\
-a_5 - 2b_3 + b_6 &= 0
\end{aligned}
$$

Solving the above equations for the unknowns gives

$$
\begin{aligned}
a_1 &= 0 \\
a_2 &= 0 \\
a_3 &= 0 \\
a_4 &= 0 \\
a_5 &= -2b_3 \\
a_6 &= 0 \\
b_1 &= 0 \\
b_2 &= 0 \\
b_3 &= b_3 \\
b_4 &= 0 \\
b_5 &= 0 \\
b_6 &= 0
\end{aligned}
$$

Substituting the above solution in the anstaz (1E,2E) (using 1 as arbitrary value for any unknown in the RHS) gives

$$
\begin{aligned}
\xi &= -2yx \\
\eta &= y
\end{aligned}
$$

Shifting is now applied to make $\xi = 0$ in order to simplify the rest of the computation

$$
\begin{aligned}
\eta &= \eta - \omega(x,y)\,\xi \\
&= y - \left(-\frac{y}{-\mathrm{e}^{-2y} + 2yx}\right)(-2yx) \\
&= \frac{y\,\mathrm{e}^{-2y}}{-2yx + \mathrm{e}^{-2y}} \\
\xi &= 0
\end{aligned}
$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x,y) \to (R,S)$ where (R,S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$
\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}
$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial x}+\eta \frac{\partial}{\partial y}\right) S(x, y)=1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the canonical coordinates, where $S(R)$. Since $\xi=0$ then in this special case

$$R=x$$

S is found from

$$\begin{aligned} S &= \int \frac{1}{\eta} dy \\ &= \int \frac{1}{\frac{y\,\mathrm{e}^{-2y}}{-2yx+\mathrm{e}^{-2y}}} dy \end{aligned}$$

Which results in

$$S=\ln(y)-\mathrm{e}^{2y} x$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR}=\frac{S_x+\omega(x, y) S_y}{R_x+\omega(x, y) R_y} \tag{2}$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x, y)$ is the right hand side of the original ode given by

$$\omega(x, y)=-\frac{y}{-\mathrm{e}^{-2y}+2yx}$$

Evaluating all the partial derivatives gives

$$\begin{aligned} R_x &= 1 \\ R_y &= 0 \\ S_x &= -\mathrm{e}^{2y} \\ S_y &= \frac{-2\,\mathrm{e}^{2y} xy+1}{y} \end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR}=0 \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR}=0$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordinates R, S. Integrating the above gives

$$S(R) = c_1 \tag{4}$$

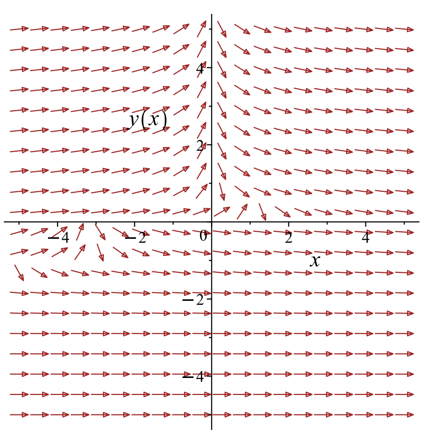
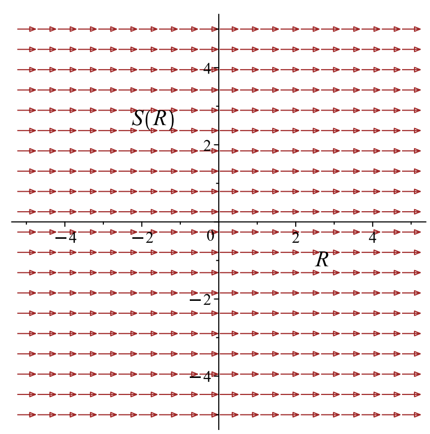
To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$\ln(y) - e^{2y}x = c_1$$

Which simplifies to

$$\ln(y) - e^{2y}x = c_1$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in x, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dx} = -\frac{y}{-e^{-2y}+2yx}$ 	$R = x$ $S = \ln(y) - e^{2y}x$	$\frac{dS}{dR} = 0$ 

Summary

The solution(s) found are the following

$$\ln(y) - e^{2y}x = c_1 \tag{1}$$

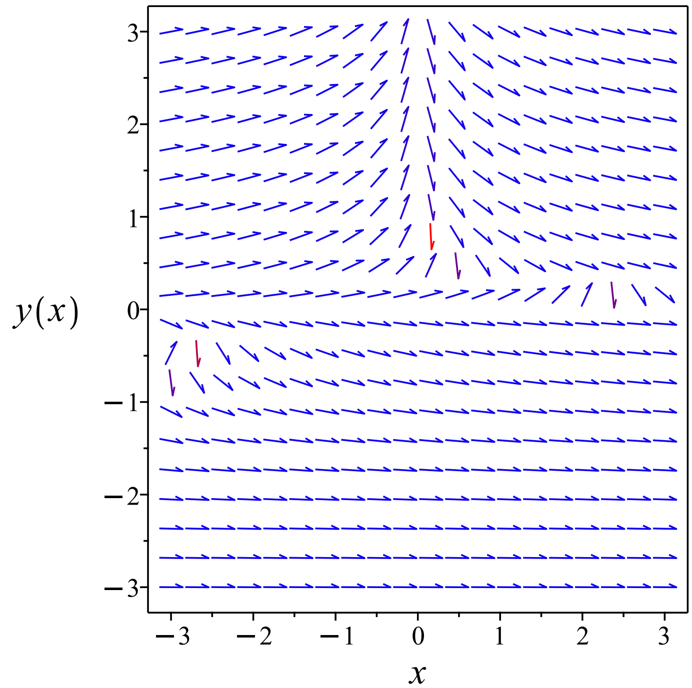


Figure 262: Slope field plot

Verification of solutions

$$\ln(y) - e^{2y}x = c_1$$

Verified OK.

5.21.2 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x,y) + N(x,y)\frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x,y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y) = 0$$

Hence

$$\frac{\partial\phi}{\partial x} + \frac{\partial\phi}{\partial y}\frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$\frac{\partial \phi}{\partial x} = M$$
$$\frac{\partial \phi}{\partial y} = N$$

But since $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x, y)$ but at least we know now that we can do that since the condition $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x, y) \, dx + N(x, y) \, dy = 0 \tag{1A}$$

Therefore

$$\begin{aligned} (-e^{-2y} + 2yx) \, dy &= (-y) \, dx \\ (y) \, dx + (-e^{-2y} + 2yx) \, dy &= 0 \end{aligned} \tag{2A}$$

Comparing (1A) and (2A) shows that

$$\begin{aligned} M(x, y) &= y \\ N(x, y) &= -e^{-2y} + 2yx \end{aligned}$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Using result found above gives

$$\begin{aligned} \frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}(y) \\ &= 1 \end{aligned}$$

And

$$\begin{aligned} \frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}(-e^{-2y} + 2yx) \\ &= 2y \end{aligned}$$

Since $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$, then the ODE is <u>not exact</u>. Since the ODE is not exact, we will try to find an integrating factor to make it exact. Let

$$
\begin{aligned}
A &= \frac{1}{N}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right) \\
&= \frac{1}{-\mathrm{e}^{-2y} + 2yx}((1) - (2y)) \\
&= \frac{-1 + 2y}{-2yx + \mathrm{e}^{-2y}}
\end{aligned}
$$

Since A depends on y, it can not be used to obtain an integrating factor. We will now try a second method to find an integrating factor. Let

$$
\begin{aligned}
B &= \frac{1}{M}\left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) \\
&= \frac{1}{y}((2y) - (1)) \\
&= \frac{-1 + 2y}{y}
\end{aligned}
$$

Since B does not depend on x, it can be used to obtain an integrating factor. Let the integrating factor be μ. Then

$$
\begin{aligned}
\mu &= e^{\int B \, \mathrm{d}y} \\
&= e^{\int \frac{-1+2y}{y} \, \mathrm{d}y}
\end{aligned}
$$

The result of integrating gives

$$
\begin{aligned}
\mu &= e^{2y - \ln(y)} \\
&= \frac{\mathrm{e}^{2y}}{y}
\end{aligned}
$$

M and N are now multiplied by this integrating factor, giving new M and new N which are called $\overline{M}$ and $\overline{N}$ so not to confuse them with the original M and N.

$$
\begin{aligned}
\overline{M} &= \mu M \\
&= \frac{\mathrm{e}^{2y}}{y}(y) \\
&= \mathrm{e}^{2y}
\end{aligned}
$$

And

$$
\begin{aligned}
\overline{N} &= \mu N \\
&= \frac{\mathrm{e}^{2y}}{y}\left(-\mathrm{e}^{-2y} + 2yx\right) \\
&= \frac{2\,\mathrm{e}^{2y} xy - 1}{y}
\end{aligned}
$$

So now a modified ODE is obtained from the original ODE which will be exact and can be solved using the standard method. The modified ODE is

$$\overline{M} + \overline{N}\frac{\mathrm{d}y}{\mathrm{d}x} = 0$$

$$(\mathrm{e}^{2y}) + \left(\frac{2\,\mathrm{e}^{2y} xy - 1}{y}\right)\frac{\mathrm{d}y}{\mathrm{d}x} = 0$$

The following equations are now set up to solve for the function $\phi(x, y)$

$$\frac{\partial \phi}{\partial x} = \overline{M} \tag{1}$$

$$\frac{\partial \phi}{\partial y} = \overline{N} \tag{2}$$

Integrating (1) w.r.t. x gives

$$\int \frac{\partial \phi}{\partial x} \,\mathrm{d}x = \int \overline{M} \,\mathrm{d}x$$

$$\int \frac{\partial \phi}{\partial x} \,\mathrm{d}x = \int \mathrm{e}^{2y} \,\mathrm{d}x$$

$$\phi = \mathrm{e}^{2y} x + f(y) \tag{3}$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$\frac{\partial \phi}{\partial y} = 2\,\mathrm{e}^{2y} x + f'(y) \tag{4}$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = \frac{2\,\mathrm{e}^{2y} xy - 1}{y}$. Therefore equation (4) becomes

$$\frac{2\,\mathrm{e}^{2y} xy - 1}{y} = 2\,\mathrm{e}^{2y} x + f'(y) \tag{5}$$

Solving equation (5) for $f'(y)$ gives

$$f'(y) = -\frac{1}{y}$$

Integrating the above w.r.t y gives

$$\int f'(y) \,\mathrm{d}y = \int \left(-\frac{1}{y}\right) \mathrm{d}y$$

$$f(y) = -\ln(y) + c_1$$

Where c_1 is constant of integration. Substituting result found above for $f(y)$ into equation (3) gives ϕ

$$\phi = e^{2y}x - \ln(y) + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = e^{2y}x - \ln(y)$$

Summary

The solution(s) found are the following

$$-\ln(y) + e^{2y}x = c_1 \tag{1}$$

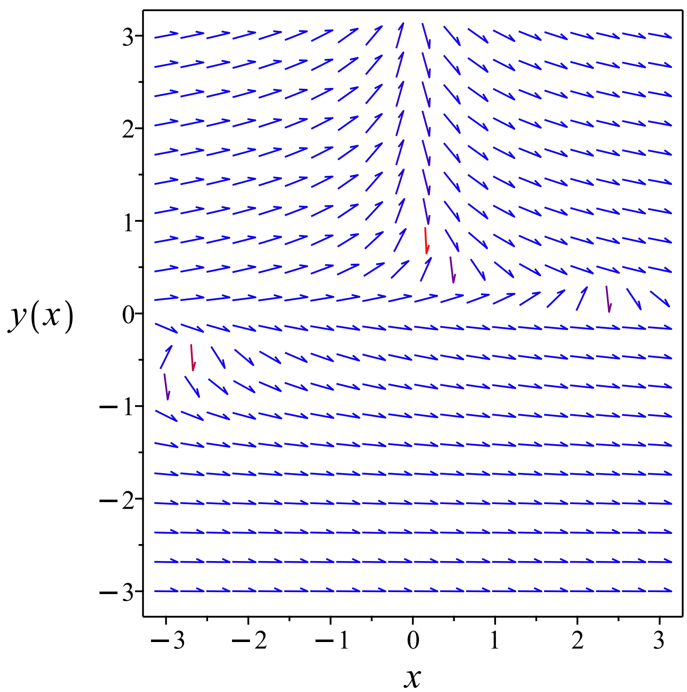


Figure 263: Slope field plot

Verification of solutions

$$-\ln(y) + e^{2y}x = c_1$$

Verified OK.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
<- 1st order linear successful
<- inverse linear successful`
```

✓ Solution by Maple

Time used: 0.031 (sec). Leaf size: 24

```
dsolve(y(x)+(-exp(-2*y(x))+2*x*y(x))*diff(y(x),x) = 0,y(x), singsol=all)
```

$$y(x) = \mathrm{e}^{\mathrm{RootOf}\left(c_1 \mathrm{e}^{-2\,\mathrm{e}^{\_Z}} + \_Z \mathrm{e}^{-2\,\mathrm{e}^{\_Z}} - x\right)}$$

✓ Solution by Mathematica

Time used: 0.243 (sec). Leaf size: 25

```
DSolve[y[x]+(-Exp[-2*y[x]]+2*x*y[x])*y'[x] == 0,y[x],x,IncludeSingularSolutions -> True]
```

$$\text{Solve}\left[x = e^{-2y(x)} \log(y(x)) + c_1 e^{-2y(x)}, y(x)\right]$$

5.22 problem 29

Internal problem ID [564]
Internal file name [`OUTPUT/564_Sunday_June_05_2022_01_44_43_AM_26805416/index.tex`]

Book: Elementary differential equations and boundary value problems, 10th ed., Boyce and DiPrima
Section: Section 2.6. Page 100
Problem number: 29.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"exactWithIntegrationFactor"**

Maple gives the following as the ode type

`[[_1st_order, `_with_symmetry_[F(x)*G(y),0]`]]`

$$\boxed{(e^{x} \cot(y) + 2 \csc(y) y) y' = -e^{x}}$$

5.22.1 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x,y) + N(x,y)\frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x,y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y) = 0$$

Hence

$$\frac{\partial\phi}{\partial x} + \frac{\partial\phi}{\partial y}\frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$\frac{\partial\phi}{\partial x} = M$$
$$\frac{\partial\phi}{\partial y} = N$$

But since $\frac{\partial^2\phi}{\partial x\partial y} = \frac{\partial^2\phi}{\partial y\partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x, y)$ but at least we know now that we can do that since the condition $\frac{\partial^2\phi}{\partial x\partial y} = \frac{\partial^2\phi}{\partial y\partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x, y)\,\mathrm{d}x + N(x, y)\,\mathrm{d}y = 0 \tag{1A}$$

Therefore

$$\begin{aligned}\left(\mathrm{e}^{x} \cot\left(y\right)+2 \csc\left(y\right) y\right)\mathrm{d}y &= \left(-\mathrm{e}^{x}\right)\mathrm{d}x\\ \left(\mathrm{e}^{x}\right)\mathrm{d}x + \left(\mathrm{e}^{x} \cot\left(y\right)+2 \csc\left(y\right) y\right)\mathrm{d}y &= 0 \qquad (2A)\end{aligned}$$

Comparing (1A) and (2A) shows that

$$\begin{aligned}M(x, y) &= \mathrm{e}^{x}\\ N(x, y) &= \mathrm{e}^{x} \cot\left(y\right)+2 \csc\left(y\right) y\end{aligned}$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Using result found above gives

$$\begin{aligned}\frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}\left(\mathrm{e}^{x}\right)\\ &= 0\end{aligned}$$

And

$$\begin{aligned}\frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}\left(\mathrm{e}^{x} \cot\left(y\right)+2 \csc\left(y\right) y\right)\\ &= \mathrm{e}^{x} \cot\left(y\right)\end{aligned}$$

Since $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$, then the ODE is not exact. Since the ODE is not exact, we will try to find an integrating factor to make it exact. Let

$$\begin{aligned}
A &= \frac{1}{N}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right) \\
&= \frac{\sin(y)}{e^x \cos(y) + 2y}((0) - (e^x \cot(y))) \\
&= -\frac{\cos(y)\, e^x}{e^x \cos(y) + 2y}
\end{aligned}$$

Since A depends on y, it can not be used to obtain an integrating factor. We will now try a second method to find an integrating factor. Let

$$\begin{aligned}
B &= \frac{1}{M}\left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) \\
&= e^{-x}((e^x \cot(y)) - (0)) \\
&= \cot(y)
\end{aligned}$$

Since B does not depend on x, it can be used to obtain an integrating factor. Let the integrating factor be μ. Then

$$\begin{aligned}
\mu &= e^{\int B \, \mathrm{d}y} \\
&= e^{\int \cot(y) \, \mathrm{d}y}
\end{aligned}$$

The result of integrating gives

$$\begin{aligned}
\mu &= e^{\ln(\sin(y))} \\
&= \sin(y)
\end{aligned}$$

M and N are now multiplied by this integrating factor, giving new M and new N which are called $\overline{M}$ and $\overline{N}$ so not to confuse them with the original M and N.

$$\begin{aligned}
\overline{M} &= \mu M \\
&= \sin(y)(e^x) \\
&= e^x \sin(y)
\end{aligned}$$

And

$$\begin{aligned}
\overline{N} &= \mu N \\
&= \sin(y)(e^x \cot(y) + 2\csc(y)\, y) \\
&= e^x \cos(y) + 2y
\end{aligned}$$

So now a modified ODE is obtained from the original ODE which will be exact and can be solved using the standard method. The modified ODE is

$$\overline{M} + \overline{N}\frac{\mathrm{d}y}{\mathrm{d}x} = 0$$
$$(\mathrm{e}^{x}\sin(y)) + (\mathrm{e}^{x}\cos(y) + 2y)\frac{\mathrm{d}y}{\mathrm{d}x} = 0$$

The following equations are now set up to solve for the function $\phi(x,y)$

$$\frac{\partial\phi}{\partial x} = \overline{M} \tag{1}$$
$$\frac{\partial\phi}{\partial y} = \overline{N} \tag{2}$$

Integrating (1) w.r.t. x gives

$$\int \frac{\partial\phi}{\partial x}\,\mathrm{d}x = \int \overline{M}\,\mathrm{d}x$$
$$\int \frac{\partial\phi}{\partial x}\,\mathrm{d}x = \int \mathrm{e}^{x}\sin(y)\,\mathrm{d}x$$
$$\phi = \mathrm{e}^{x}\sin(y) + f(y) \tag{3}$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$\frac{\partial\phi}{\partial y} = \mathrm{e}^{x}\cos(y) + f'(y) \tag{4}$$

But equation (2) says that $\frac{\partial\phi}{\partial y} = \mathrm{e}^{x}\cos(y) + 2y$. Therefore equation (4) becomes

$$\mathrm{e}^{x}\cos(y) + 2y = \mathrm{e}^{x}\cos(y) + f'(y) \tag{5}$$

Solving equation (5) for $f'(y)$ gives

$$f'(y) = 2y$$

Integrating the above w.r.t y gives

$$\int f'(y)\,\mathrm{d}y = \int (2y)\,\mathrm{d}y$$
$$f(y) = y^{2} + c_1$$

Where c_1 is constant of integration. Substituting result found above for $f(y)$ into equation (3) gives ϕ

$$\phi = e^x \sin(y) + y^2 + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = e^x \sin(y) + y^2$$

<u>Summary</u>
The solution(s) found are the following

$$e^x \sin(y) + y^2 = c_1 \tag{1}$$

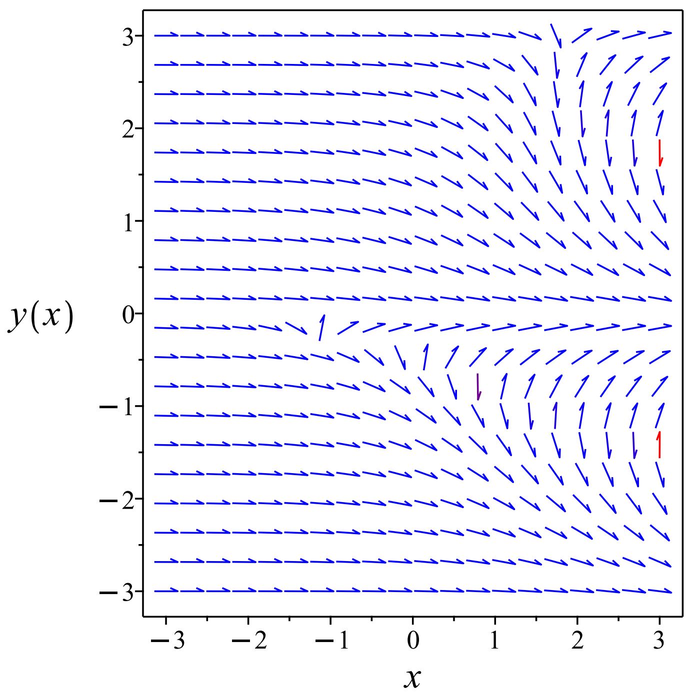


Figure 264: Slope field plot

<u>Verification of solutions</u>

$$e^x \sin(y) + y^2 = c_1$$

Verified OK.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
trying homogeneous types:
trying Chini
differential order: 1; looking for linear symmetries
trying exact
<- exact successful`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 15

```
dsolve(exp(x)+(exp(x)*cot(y(x))+2*csc(y(x))*y(x))*diff(y(x),x) = 0,y(x), singsol=all)
```

$$e^{x} \sin \left(y(x)\right)+y(x)^{2}+c_{1} = 0$$

✓ Solution by Mathematica

Time used: 0.312 (sec). Leaf size: 18

```
DSolve[Exp[x]+(Exp[x]*Cot[y[x]]+2*Csc[y[x]]*y[x])*y'[x] == 0,y[x],x,IncludeSingularSolutions
```

$$\text{Solve}\left[y(x)^2+e^x \sin(y(x))=c_1,y(x)\right]$$

5.23 problem 30

Internal problem ID [565]
Internal file name [`OUTPUT/565_Sunday_June_05_2022_01_44_50_AM_82120052/index.tex`]

Book: Elementary differential equations and boundary value problems, 10th ed., Boyce and DiPrima
Section: Section 2.6. Page 100
Problem number: 30.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"exact", "differentialType"**

Maple gives the following as the ode type

`[_rational]`

$$\boxed{\frac{4x^3}{y^2}+\frac{3}{y}+\left(\frac{3x}{y^2}+4y\right)y'=0}$$

5.23.1 Solving as differentialType ode

Writing the ode as

$$y'=\frac{-\frac{4x^3}{y^2}-\frac{3}{y}}{\frac{3x}{y^2}+4y} \tag{1}$$

Which becomes

$$\left(4y^3\right)dy=(-3x)\,dy+\left(-4x^3-3y\right)dx \tag{2}$$

But the RHS is complete differential because

$$(-3x)\,dy+\left(-4x^3-3y\right)dx=d\left(-x^4-3yx\right)$$

Hence (2) becomes

$$\left(4y^3\right)dy=d\left(-x^4-3yx\right)$$

Integrating both sides gives gives the solution as

$$y^4 = -x^4 - 3yx + c_1$$

Summary
The solution(s) found are the following

$$y^4 = -x^4 - 3yx + c_1 \tag{1}$$

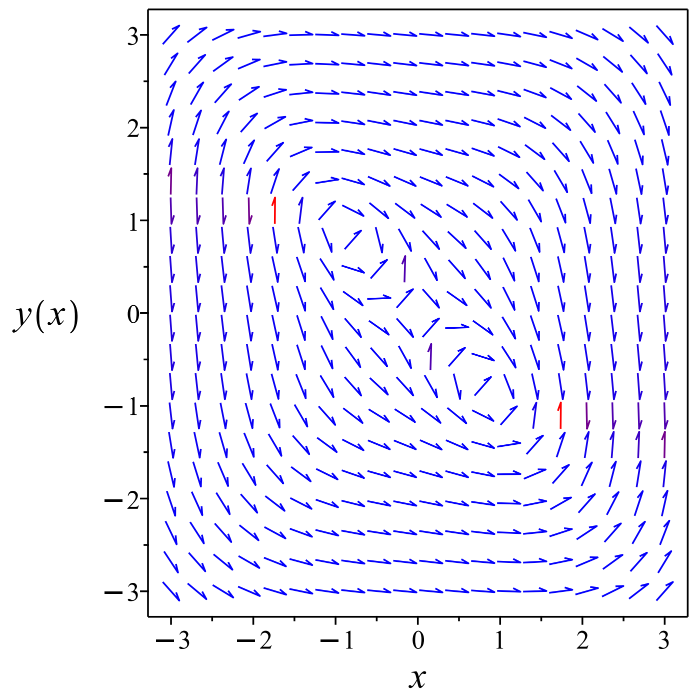


Figure 265: Slope field plot

Verification of solutions

$$y^4 = -x^4 - 3yx + c_1$$

Verified OK.

5.23.2 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x,y) + N(x,y)\frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x,y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y) = 0$$

Hence

$$\frac{\partial \phi}{\partial x} + \frac{\partial \phi}{\partial y}\frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$\begin{aligned}\frac{\partial \phi}{\partial x} &= M\\ \frac{\partial \phi}{\partial y} &= N\end{aligned}$$

But since $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x,y)$ but at least we know now that we can do that since the condition $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x,y)\,\mathrm{d}x + N(x,y)\,\mathrm{d}y = 0 \tag{1A}$$

Therefore

$$\begin{aligned}\left(4y^3 + 3x\right)\mathrm{d}y &= \left(-4x^3 - 3y\right)\mathrm{d}x\\ \left(4x^3 + 3y\right)\mathrm{d}x + \left(4y^3 + 3x\right)\mathrm{d}y &= 0 \end{aligned} \tag{2A}$$

Comparing (1A) and (2A) shows that

$$\begin{aligned}M(x,y) &= 4x^3 + 3y\\ N(x,y) &= 4y^3 + 3x\end{aligned}$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Using result found above gives

$$\begin{aligned}\frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}(4x^3 + 3y)\\ &= 3\end{aligned}$$

And

$$\begin{aligned}\frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}(4y^3 + 3x)\\ &= 3\end{aligned}$$

Since $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$, then the ODE is exact The following equations are now set up to solve for the function $\phi(x, y)$

$$\frac{\partial \phi}{\partial x} = M \tag{1}$$
$$\frac{\partial \phi}{\partial y} = N \tag{2}$$

Integrating (1) w.r.t. x gives

$$\begin{aligned}\int \frac{\partial \phi}{\partial x}\,\mathrm{d}x &= \int M\,\mathrm{d}x\\ \int \frac{\partial \phi}{\partial x}\,\mathrm{d}x &= \int 4x^3 + 3y\,\mathrm{d}x\\ \phi &= x(x^3 + 3y) + f(y)\end{aligned} \tag{3}$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$\frac{\partial \phi}{\partial y} = 3x + f'(y) \tag{4}$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = 4y^3 + 3x$. Therefore equation (4) becomes

$$4y^3 + 3x = 3x + f'(y) \tag{5}$$

Solving equation (5) for $f'(y)$ gives

$$f'(y) = 4y^3$$

Integrating the above w.r.t y gives

$$\int f'(y)\,\mathrm{d}y = \int (4y^3)\,\mathrm{d}y$$
$$f(y) = y^4 + c_1$$

Where c_1 is constant of integration. Substituting result found above for $f(y)$ into equation (3) gives ϕ

$$\phi = x(x^3 + 3y) + y^4 + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = x(x^3 + 3y) + y^4$$

Summary

The solution(s) found are the following

$$x(3y + x^3) + y^4 = c_1 \tag{1}$$

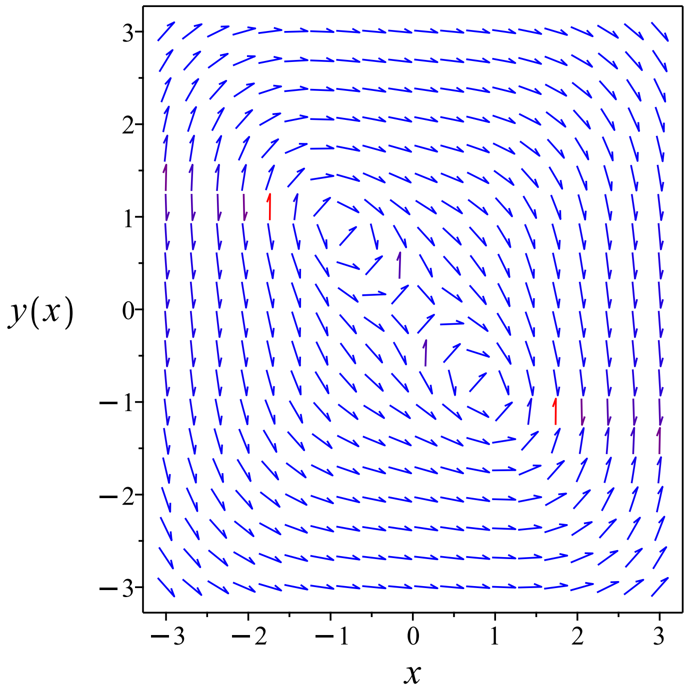


Figure 266: Slope field plot

Verification of solutions

$$x(3y + x^3) + y^4 = c_1$$

Verified OK.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
trying homogeneous types:
trying Chini
differential order: 1; looking for linear symmetries
trying exact
<- exact successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 17

```
dsolve(4*x^3/y(x)^2+3/y(x)+(3*x/y(x)^2+4*y(x))*diff(y(x),x) = 0,y(x), singsol=all)
```

$$x^4 + y(x)^4 + 3xy(x) + c_1 = 0$$

✓ Solution by Mathematica

Time used: 60.148 (sec). Leaf size: 1181

```
DSolve[4*x^3/y[x]^2+3/y[x]+(3*x/y[x]^2+4*y[x])*y'[x]== 0,y[x],x,IncludeSingularSolutions ->
```

$$y(x)\to -\frac{1}{2}\sqrt{\frac{6x}{\sqrt{\frac{4\sqrt[3]{2}(x^4-c_1)}{\sqrt[3]{243x^2+\sqrt{59049x^4-6912(x^4-c_1)^3}}}+\frac{\sqrt[3]{243x^2+\sqrt{59049x^4-6912(x^4-c_1)^3}}}{3\sqrt[3]{2}}}}-\frac{\quad}{\sqrt[3]{243}}}$$
$$-\frac{1}{2}\sqrt{\frac{4\sqrt[3]{2}\left(x^4-c_1\right)}{\sqrt[3]{243x^2+\sqrt{59049x^4-6912\left(x^4-c_1\right)^3}}}+\frac{\sqrt[3]{243x^2+\sqrt{59049x^4-6912\left(x^4-c_1\right)^3}}}{3\sqrt[3]{2}}}$$

$$y(x)\to \frac{1}{2}\sqrt{\frac{6x}{\sqrt{\frac{4\sqrt[3]{2}(x^4-c_1)}{\sqrt[3]{243x^2+\sqrt{59049x^4-6912(x^4-c_1)^3}}}+\frac{\sqrt[3]{243x^2+\sqrt{59049x^4-6912(x^4-c_1)^3}}}{3\sqrt[3]{2}}}}-\frac{\quad}{\sqrt[3]{243x^2}}}$$
$$-\frac{1}{2}\sqrt{\frac{4\sqrt[3]{2}\left(x^4-c_1\right)}{\sqrt[3]{243x^2+\sqrt{59049x^4-6912\left(x^4-c_1\right)^3}}}+\frac{\sqrt[3]{243x^2+\sqrt{59049x^4-6912\left(x^4-c_1\right)^3}}}{3\sqrt[3]{2}}}$$

$$y(x)\to \frac{1}{2}\sqrt{\frac{4\sqrt[3]{2}\left(x^4-c_1\right)}{\sqrt[3]{243x^2+\sqrt{59049x^4-6912\left(x^4-c_1\right)^3}}}+\frac{\sqrt[3]{243x^2+\sqrt{59049x^4-6912\left(x^4-c_1\right)^3}}}{3\sqrt[3]{2}}}$$
$$-\frac{1}{2}\sqrt{-\frac{6x}{\sqrt{\frac{4\sqrt[3]{2}(x^4-c_1)}{\sqrt[3]{243x^2+\sqrt{59049x^4-6912(x^4-c_1)^3}}}+\frac{\sqrt[3]{243x^2+\sqrt{59049x^4-6912(x^4-c_1)^3}}}{3\sqrt[3]{2}}}}-\frac{\quad}{\sqrt[3]{24}}}$$

$$y(x)\to \frac{1}{2}\sqrt{-\frac{6x}{\sqrt{\frac{4\sqrt[3]{2}(x^4-c_1)}{\sqrt[3]{243x^2+\sqrt{59049x^4-6912(x^4-c_1)^3}}}+\frac{\sqrt[3]{243x^2+\sqrt{59049x^4-6912(x^4-c_1)^3}}}{3\sqrt[3]{2}}}}-\frac{\quad}{\sqrt[3]{243}}}$$
$$+\frac{1}{2}\sqrt{\frac{4\sqrt[3]{2}\left(x^4-c_1\right)}{\sqrt[3]{243x^2+\sqrt{59049x^4-6912\left(x^4-c_1\right)^3}}}+\frac{\sqrt[3]{243x^2+\sqrt{59049x^4-6912\left(x^4-c_1\right)^3}}}{3\sqrt[3]{2}}}$$

5.24 problem 30

Internal problem ID [566]
Internal file name [`OUTPUT/566_Sunday_June_05_2022_01_44_51_AM_23911975/index.tex`]

Book: Elementary differential equations and boundary value problems, 10th ed., Boyce and DiPrima
Section: Section 2.6. Page 100
Problem number: 30.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"exact", "differentialType"**

Maple gives the following as the ode type

`[_rational]`

$$\boxed{\frac{6}{y}+\left(\frac{x^{2}}{y}+\frac{3y}{x}\right)y'=-3x}$$

5.24.1 Solving as differentialType ode

Writing the ode as

$$y' = \frac{-3x-\frac{6}{y}}{\frac{x^{2}}{y}+\frac{3y}{x}} \tag{1}$$

Which becomes

$$(3y^{2})\,dy = (-x^{3})\,dy + (-3x(yx+2))\,dx \tag{2}$$

But the RHS is complete differential because

$$(-x^{3})\,dy + (-3x(yx+2))\,dx = d\left(-y\,x^{3}-3x^{2}\right)$$

Hence (2) becomes

$$(3y^{2})\,dy = d\left(-y\,x^{3}-3x^{2}\right)$$

Integrating both sides gives gives these solutions

$$y = \frac{\left(-324x^2 + 108c_1 + 12\sqrt{12x^9 + 729x^4 - 486c_1x^2 + 81c_1^2}\right)^{\frac{1}{3}}}{6} - \frac{2x^3}{\left(-324x^2 + 108c_1 + 12\sqrt{12x^9 + 729x^4}\right.}$$

$$y = -\frac{\left(-324x^2 + 108c_1 + 12\sqrt{12x^9 + 729x^4 - 486c_1x^2 + 81c_1^2}\right)^{\frac{1}{3}}}{12} + \frac{x^3}{\left(-324x^2 + 108c_1 + 12\sqrt{12x^9 + 729\right.}$$

$$y = -\frac{\left(-324x^2 + 108c_1 + 12\sqrt{12x^9 + 729x^4 - 486c_1x^2 + 81c_1^2}\right)^{\frac{1}{3}}}{12} + \frac{x^3}{\left(-324x^2 + 108c_1 + 12\sqrt{12x^9 + 729\right.}$$

Summary
The solution(s) found are the following

$$y = \frac{\left(-324x^2+108c_1+12\sqrt{12x^9+729x^4-486c_1x^2+81c_1^2}\right)^{\frac{1}{3}}}{6} - \frac{2x^3}{\left(-324x^2+108c_1+12\sqrt{12x^9+729x^4-486c_1x^2+81c_1^2}\right)^{\frac{1}{3}}} + c_1 \tag{1}$$

$$y = -\frac{\left(-324x^2+108c_1+12\sqrt{12x^9+729x^4-486c_1x^2+81c_1^2}\right)^{\frac{1}{3}}}{12} + \frac{x^3}{\left(-324x^2+108c_1+12\sqrt{12x^9+729x^4-486c_1x^2+81c_1^2}\right)^{\frac{1}{3}}} + \frac{i\sqrt{3}\left(\frac{\left(-324x^2+108c_1+12\sqrt{12x^9+729x^4-486c_1x^2+81c_1^2}\right)^{\frac{1}{3}}}{6} + \frac{2x^3}{\left(-324x^2+108c_1+12\sqrt{12x^9+729x^4-486c_1x^2+81c_1^2}\right)^{\frac{1}{3}}}\right)}{2} + c_1 \tag{2}$$

$$y = -\frac{\left(-324x^2+108c_1+12\sqrt{12x^9+729x^4-486c_1x^2+81c_1^2}\right)^{\frac{1}{3}}}{12} + \frac{x^3}{\left(-324x^2+108c_1+12\sqrt{12x^9+729x^4-486c_1x^2+81c_1^2}\right)^{\frac{1}{3}}} - \frac{i\sqrt{3}\left(\frac{\left(-324x^2+108c_1+12\sqrt{12x^9+729x^4-486c_1x^2+81c_1^2}\right)^{\frac{1}{3}}}{6} + \frac{2x^3}{\left(-324x^2+108c_1+12\sqrt{12x^9+729x^4-486c_1x^2+81c_1^2}\right)^{\frac{1}{3}}}\right)}{2} + c_1 \tag{3}$$

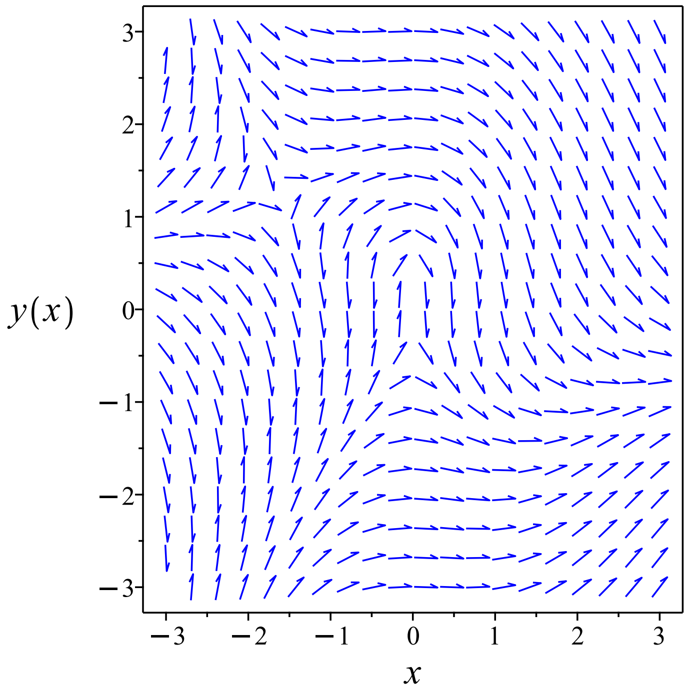


Figure 267: Slope field plot

Verification of solutions

$$y = \frac{\left(-324x^2+108c_1+12\sqrt{12x^9+729x^4-486c_1x^2+81c_1^2}\right)^{\frac{1}{3}}}{6} - \frac{2x^3}{\left(-324x^2+108c_1+12\sqrt{12x^9+729x^4-486c_1x^2+81c_1^2}\right)^{\frac{1}{3}}} + c_1$$

Verified OK.

$$y = -\frac{\left(-324x^2+108c_1+12\sqrt{12x^9+729x^4-486c_1x^2+81c_1^2}\right)^{\frac{1}{3}}}{12} + \frac{x^3}{\left(-324x^2+108c_1+12\sqrt{12x^9+729x^4-486c_1x^2+81c_1^2}\right)^{\frac{1}{3}}} + \frac{i\sqrt{3}\left(\frac{\left(-324x^2+108c_1+12\sqrt{12x^9+729x^4-486c_1x^2+81c_1^2}\right)^{\frac{1}{3}}}{6} + \frac{2x^3}{\left(-324x^2+108c_1+12\sqrt{12x^9+729x^4-486c_1x^2+81c_1^2}\right)^{\frac{1}{3}}}\right)}{2} + c_1$$

Verified OK.

$$y = -\frac{\left(-324x^2+108c_1+12\sqrt{12x^9+729x^4-486c_1x^2+81c_1^2}\right)^{\frac{1}{3}}}{12} + \frac{x^3}{\left(-324x^2+108c_1+12\sqrt{12x^9+729x^4-486c_1x^2+81c_1^2}\right)^{\frac{1}{3}}} - \frac{i\sqrt{3}\left(\frac{\left(-324x^2+108c_1+12\sqrt{12x^9+729x^4-486c_1x^2+81c_1^2}\right)^{\frac{1}{3}}}{6} + \frac{2x^3}{\left(-324x^2+108c_1+12\sqrt{12x^9+729x^4-486c_1x^2+81c_1^2}\right)^{\frac{1}{3}}}\right)}{2} + c_1$$

Verified OK.

5.24.2 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x,y)+N(x,y)\frac{dy}{dx}=0 \tag{A}$$

We assume there exists a function $\phi(x,y)=c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y)=0$$

Hence

$$\frac{\partial\phi}{\partial x}+\frac{\partial\phi}{\partial y}\frac{dy}{dx}=0 \tag{B}$$

Comparing (A,B) shows that

$$\begin{aligned}\frac{\partial\phi}{\partial x}&=M\\ \frac{\partial\phi}{\partial y}&=N\end{aligned}$$

But since $\frac{\partial^2\phi}{\partial x\partial y}=\frac{\partial^2\phi}{\partial y\partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y}=\frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x,y)$ but at least we know now that we can do that since the condition $\frac{\partial^2\phi}{\partial x\partial y}=\frac{\partial^2\phi}{\partial y\partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x,y)\,\mathrm{d}x+N(x,y)\,\mathrm{d}y=0 \tag{1A}$$

Therefore

$$\begin{aligned}\left(x^3+3y^2\right)\mathrm{d}y&=(-3x(yx+2))\,\mathrm{d}x\\ (3x(yx+2))\,\mathrm{d}x+\left(x^3+3y^2\right)\mathrm{d}y&=0 \end{aligned}\tag{2A}$$

Comparing (1A) and (2A) shows that

$$\begin{aligned}M(x,y)&=3x(yx+2)\\ N(x,y)&=x^3+3y^2\end{aligned}$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Using result found above gives

$$\begin{aligned}\frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}(3x(yx+2)) \\ &= 3x^2\end{aligned}$$

And

$$\begin{aligned}\frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}(x^3+3y^2) \\ &= 3x^2\end{aligned}$$

Since $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$, then the ODE is exact The following equations are now set up to solve for the function $\phi(x,y)$

$$\frac{\partial \phi}{\partial x} = M \tag{1}$$

$$\frac{\partial \phi}{\partial y} = N \tag{2}$$

Integrating (1) w.r.t. x gives

$$\int \frac{\partial \phi}{\partial x} \mathrm{d}x = \int M \mathrm{d}x$$

$$\int \frac{\partial \phi}{\partial x} \mathrm{d}x = \int 3x(yx+2) \mathrm{d}x$$

$$\phi = y x^3 + 3x^2 + f(y) \tag{3}$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$\frac{\partial \phi}{\partial y} = x^3 + f'(y) \tag{4}$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = x^3 + 3y^2$. Therefore equation (4) becomes

$$x^3 + 3y^2 = x^3 + f'(y) \tag{5}$$

Solving equation (5) for $f'(y)$ gives

$$f'(y) = 3y^2$$

Integrating the above w.r.t y gives

$$\int f'(y) \, \mathrm{d}y = \int (3y^2) \, \mathrm{d}y$$
$$f(y) = y^3 + c_1$$

Where c_1 is constant of integration. Substituting result found above for $f(y)$ into equation (3) gives ϕ

$$\phi = y\,x^3 + y^3 + 3x^2 + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = y\,x^3 + y^3 + 3x^2$$

Summary
The solution(s) found are the following

$$yx^3 + y^3 + 3x^2 = c_1 \tag{1}$$

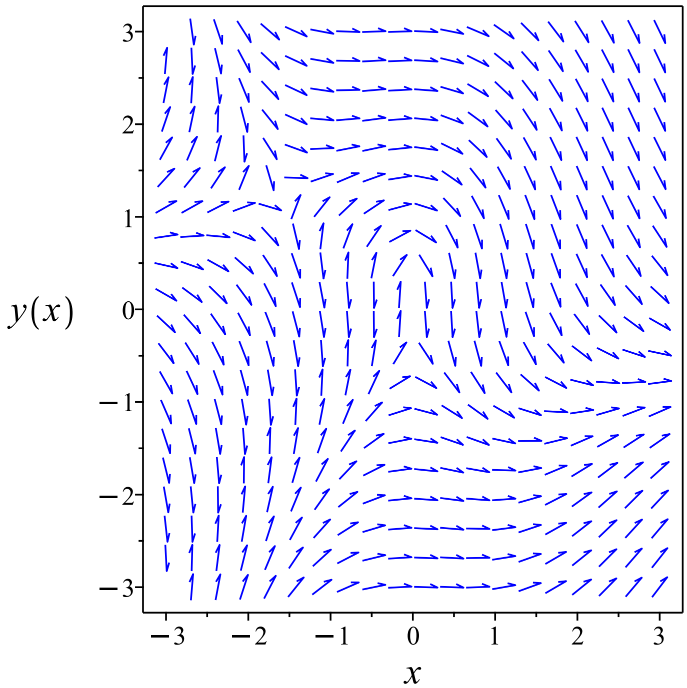


Figure 268: Slope field plot

Verification of solutions

$$
y x^{3}+y^{3}+3 x^{2}=c_{1}
$$

Verified OK.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
trying homogeneous types:
trying Chini
differential order: 1; looking for linear symmetries
trying exact
<- exact successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 326

```
dsolve(3*x+6/y(x)+(x^2/y(x)+3*y(x)/x)*diff(y(x),x) = 0,y(x), singsol=all)
```

$$y(x) = \frac{-12x^3 + \left(-324x^2 - 108c_1 + 12\sqrt{12x^9 + 729x^4 + 486c_1x^2 + 81c_1^2}\right)^{\frac{2}{3}}}{6\left(-324x^2 - 108c_1 + 12\sqrt{12x^9 + 729x^4 + 486c_1x^2 + 81c_1^2}\right)^{\frac{1}{3}}}$$

$$y(x) = -\frac{\left(1 + i\sqrt{3}\right)\left(-324x^2 - 108c_1 + 12\sqrt{12x^9 + 729x^4 + 486c_1x^2 + 81c_1^2}\right)^{\frac{1}{3}}}{12} - \frac{x^3\left(i\sqrt{3} - 1\right)}{\left(-324x^2 - 108c_1 + 12\sqrt{12x^9 + 729x^4 + 486c_1x^2 + 81c_1^2}\right)^{\frac{1}{3}}}$$

$$y(x) = \frac{12i\sqrt{3}\,x^3 + i\sqrt{3}\left(-324x^2 - 108c_1 + 12\sqrt{12x^9 + 729x^4 + 486c_1x^2 + 81c_1^2}\right)^{\frac{2}{3}} + 12x^3 - \left(-324x^2 - 108c_1\right.}{12\left(-324x^2 - 108c_1 + 12\sqrt{12x^9 + 729x^4 + 486c_1x^2 + 81c_1^2}\right)^{\frac{1}{3}}}$$

✓ Solution by Mathematica

Time used: 4.558 (sec). Leaf size: 331

```
DSolve[3*x+6/y[x]+(x^2/y[x]+3*y[x]/x)*y'[x] == 0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \frac{\sqrt[3]{-81x^2+\sqrt{108x^9+729\left(-3x^2+c_1\right)^2}+27c_1}}{3\sqrt[3]{2}} - \frac{\sqrt[3]{2}x^3}{\sqrt[3]{-81x^2+\sqrt{108x^9+729\left(-3x^2+c_1\right)^2}+27c_1}}$$

$$y(x) \to \frac{\left(-1+i\sqrt{3}\right)\sqrt[3]{-81x^2+\sqrt{108x^9+729\left(-3x^2+c_1\right)^2}+27c_1}}{6\sqrt[3]{2}} + \frac{\left(1+i\sqrt{3}\right)x^3}{2^{2/3}\sqrt[3]{-81x^2+\sqrt{108x^9+729\left(-3x^2+c_1\right)^2}+27c_1}}$$

$$y(x) \to \frac{\left(1-i\sqrt{3}\right)x^3}{2^{2/3}\sqrt[3]{-81x^2+\sqrt{108x^9+729\left(-3x^2+c_1\right)^2}+27c_1}} - \frac{\left(1+i\sqrt{3}\right)\sqrt[3]{-81x^2+\sqrt{108x^9+729\left(-3x^2+c_1\right)^2}+27c_1}}{6\sqrt[3]{2}}$$

5.25 problem 32

Internal problem ID [567]
Internal file name [`OUTPUT/567_Sunday_June_05_2022_01_44_53_AM_42477074/index.tex`]

Book: Elementary differential equations and boundary value problems, 10th ed., Boyce and DiPrima
Section: Section 2.6. Page 100
Problem number: 32.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"homogeneousTypeD2", "exactWithIntegrationFactor", "first_order_ode_lie_symmetry_calculated"**

Maple gives the following as the ode type

```
[[_homogeneous, `class A`], _rational, [_Abel, `2nd type`, `
   class B`]]
```

$$\boxed{3yx + y^2 + (x^2 + yx)\, y' = 0}$$

5.25.1 Solving as homogeneousTypeD2 ode

Using the change of variables $y = u(x)\, x$ on the above ode results in new ode in $u(x)$

$$3u(x)\, x^2 + u(x)^2\, x^2 + (x^2 + u(x)\, x^2)\, (u'(x)\, x + u(x)) = 0$$

In canonical form the ODE is

$$\begin{aligned} u' &= F(x, u) \\ &= f(x)g(u) \\ &= -\frac{2u(u+2)}{x\,(u+1)} \end{aligned}$$

Where $f(x) = -\frac{2}{x}$ and $g(u) = \frac{u(u+2)}{u+1}$. Integrating both sides gives

$$\frac{1}{\frac{u(u+2)}{u+1}}\, du = -\frac{2}{x}\, dx$$

$$\int \frac{1}{\frac{u(u+2)}{u+1}}\, du = \int -\frac{2}{x}\, dx$$

$$\frac{\ln\left(u(u+2)\right)}{2} = -2\ln\left(x\right) + c_2$$

Raising both side to exponential gives

$$\sqrt{u\left(u+2\right)} = \mathrm{e}^{-2\ln(x)+c_2}$$

Which simplifies to

$$\sqrt{u\left(u+2\right)} = \frac{c_3}{x^2}$$

Which simplifies to

$$\sqrt{u\left(x\right)\left(u\left(x\right)+2\right)} = \frac{c_3 \mathrm{e}^{c_2}}{x^2}$$

The solution is

$$\sqrt{u\left(x\right)\left(u\left(x\right)+2\right)} = \frac{c_3 \mathrm{e}^{c_2}}{x^2}$$

Replacing $u(x)$ in the above solution by $\frac{y}{x}$ results in the solution for y in implicit form

$$\sqrt{\frac{y\left(\frac{y}{x}+2\right)}{x}} = \frac{c_3 \mathrm{e}^{c_2}}{x^2}$$

$$\sqrt{\frac{y\left(2x+y\right)}{x^2}} = \frac{c_3 \mathrm{e}^{c_2}}{x^2}$$

Summary

The solution(s) found are the following

$$\sqrt{\frac{y\left(2x+y\right)}{x^2}} = \frac{c_3 \mathrm{e}^{c_2}}{x^2} \tag{1}$$

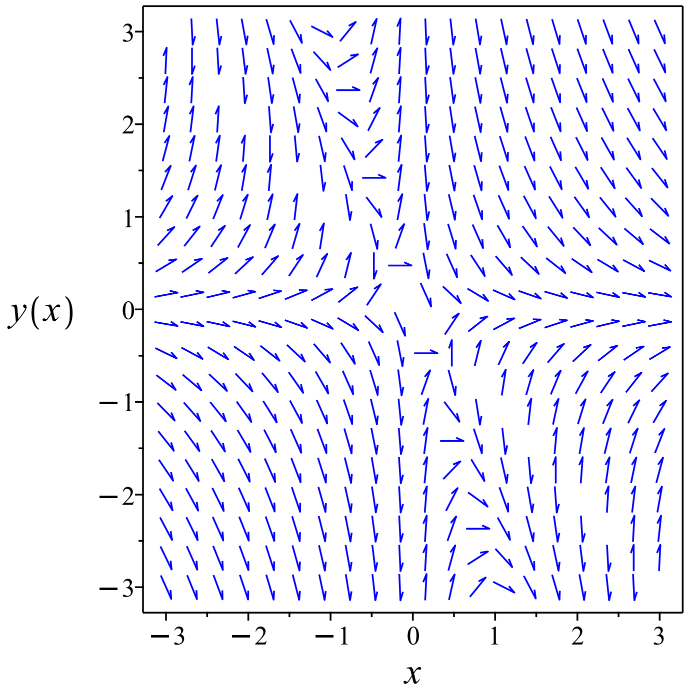


Figure 269: Slope field plot

Verification of solutions

$$\sqrt{\frac{y\left(2x+y\right)}{x^{2}}}=\frac{c_{3}\mathrm{e}^{c_{2}}}{x^{2}}$$

Verified OK.

5.25.2 Solving as first order ode lie symmetry calculated ode

Writing the ode as

$$y'=-\frac{y(3x+y)}{x\left(x+y\right)}$$
$$y'=\omega(x,y)$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x+\omega(\eta_y-\xi_x)-\omega^2\xi_y-\omega_x\xi-\omega_y\eta=0 \tag{A}$$

The type of this ode is not in the lookup table. To determine ξ,η then (A) is solved using ansatz. Making bivariate polynomials of degree 1 to use as anstaz gives

$$\xi=xa_2+ya_3+a_1 \tag{1E}$$
$$\eta=xb_2+yb_3+b_1 \tag{2E}$$

Where the unknown coefficients are

$$\{a_1, a_2, a_3, b_1, b_2, b_3\}$$

Substituting equations (1E,2E) and ω into (A) gives

$$\begin{aligned} &b_2 - \frac{y(3x+y)\,(b_3-a_2)}{x\,(x+y)} - \frac{y^2(3x+y)^2\,a_3}{x^2\,(x+y)^2} \\ &- \left(-\frac{3y}{x\,(x+y)} + \frac{y(3x+y)}{x^2\,(x+y)} + \frac{y(3x+y)}{x\,(x+y)^2}\right)(xa_2 + ya_3 + a_1) \\ &- \left(-\frac{3x+y}{x\,(x+y)} - \frac{y}{x\,(x+y)} + \frac{y(3x+y)}{x\,(x+y)^2}\right)(xb_2 + yb_3 + b_1) = 0 \end{aligned} \tag{5E}$$

Putting the above in normal form gives

$$\frac{4x^4b_2 + 4x^3yb_2 + 2x^2y^2a_2 - 12x^2y^2a_3 + 2x^2y^2b_2 - 2x^2y^2b_3 - 8x\,y^3a_3 - 2y^4a_3 + 3x^3b_1 - 3x^2ya_1 + 2x^2yb_1}{x^2\,(x+y)^2}$$
$$= 0$$

Setting the numerator to zero gives

$$\begin{aligned} &4x^4b_2 + 4x^3yb_2 + 2x^2y^2a_2 - 12x^2y^2a_3 + 2x^2y^2b_2 - 2x^2y^2b_3 - 8x\,y^3a_3 \\ &- 2y^4a_3 + 3x^3b_1 - 3x^2ya_1 + 2x^2yb_1 - 2x\,y^2a_1 + x\,y^2b_1 - y^3a_1 = 0 \end{aligned} \tag{6E}$$

Looking at the above PDE shows the following are all the terms with $\{x, y\}$ in them.

$$\{x, y\}$$

The following substitution is now made to be able to collect on all terms with $\{x, y\}$ in them

$$\{x = v_1, y = v_2\}$$

The above PDE (6E) now becomes

$$\begin{aligned} &2a_2v_1^2v_2^2 - 12a_3v_1^2v_2^2 - 8a_3v_1v_2^3 - 2a_3v_2^4 + 4b_2v_1^4 + 4b_2v_1^3v_2 + 2b_2v_1^2v_2^2 \\ &- 2b_3v_1^2v_2^2 - 3a_1v_1^2v_2 - 2a_1v_1v_2^2 - a_1v_2^3 + 3b_1v_1^3 + 2b_1v_1^2v_2 + b_1v_1v_2^2 = 0 \end{aligned} \tag{7E}$$

Collecting the above on the terms v_i introduced, and these are

$$\{v_1, v_2\}$$

Equation (7E) now becomes

$$4b_2v_1^4 + 4b_2v_1^3v_2 + 3b_1v_1^3 + (2a_2 - 12a_3 + 2b_2 - 2b_3)\, v_1^2v_2^2 \\ + (-3a_1 + 2b_1)\, v_1^2v_2 - 8a_3v_1v_2^3 + (-2a_1 + b_1)\, v_1v_2^2 - 2a_3v_2^4 - a_1v_2^3 = 0 \tag{8E}$$

Setting each coefficients in (8E) to zero gives the following equations to solve

$$\begin{aligned} -a_1 &= 0 \\ -8a_3 &= 0 \\ -2a_3 &= 0 \\ 3b_1 &= 0 \\ 4b_2 &= 0 \\ -3a_1 + 2b_1 &= 0 \\ -2a_1 + b_1 &= 0 \\ 2a_2 - 12a_3 + 2b_2 - 2b_3 &= 0 \end{aligned}$$

Solving the above equations for the unknowns gives

$$\begin{aligned} a_1 &= 0 \\ a_2 &= b_3 \\ a_3 &= 0 \\ b_1 &= 0 \\ b_2 &= 0 \\ b_3 &= b_3 \end{aligned}$$

Substituting the above solution in the anstaz (1E,2E) (using 1 as arbitrary value for any unknown in the RHS) gives

$$\begin{aligned} \xi &= x \\ \eta &= y \end{aligned}$$

Shifting is now applied to make $\xi = 0$ in order to simplify the rest of the computation

$$\begin{aligned}\eta &= \eta - \omega(x,y)\,\xi \\ &= y - \left(-\frac{y(3x+y)}{x\,(x+y)}\right)(x) \\ &= \frac{4yx+2y^2}{x+y} \\ \xi &= 0\end{aligned}$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x,y) \to (R,S)$ where (R,S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}$$

The above comes from the requirements that $\left(\xi\frac{\partial}{\partial x} + \eta\frac{\partial}{\partial y}\right)S(x,y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$R = x$$

S is found from

$$\begin{aligned}S &= \int \frac{1}{\eta} dy \\ &= \int \frac{1}{\frac{4yx+2y^2}{x+y}} dy\end{aligned}$$

Which results in

$$S = \frac{\ln\left(y(2x+y)\right)}{4}$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x,y)S_y}{R_x + \omega(x,y)R_y} \tag{2}$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x,y)$ is the right hand side of the original ode given by

$$\omega(x,y) = -\frac{y(3x+y)}{x\,(x+y)}$$

Evaluating all the partial derivatives gives

$$\begin{aligned}
R_x &= 1 \\
R_y &= 0 \\
S_x &= \frac{1}{4x + 2y} \\
S_y &= \frac{x + y}{2y\,(2x + y)}
\end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = -\frac{1}{2x} \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = -\frac{1}{2R}$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$S(R) = -\frac{\ln(R)}{2} + c_1 \tag{4}$$

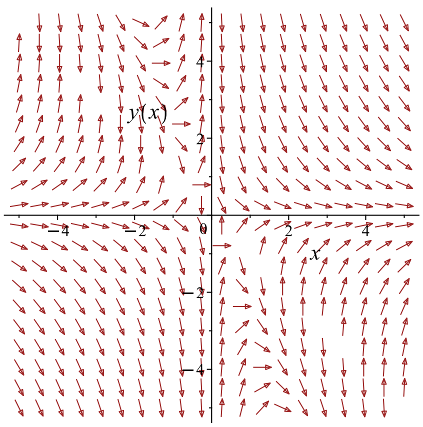
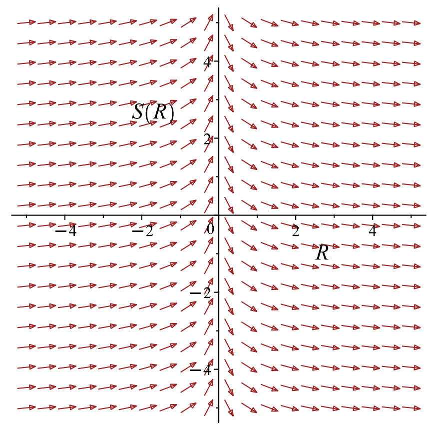
To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$\frac{\ln(y)}{4} + \frac{\ln(2x + y)}{4} = -\frac{\ln(x)}{2} + c_1$$

Which simplifies to

$$\frac{\ln(y)}{4} + \frac{\ln(2x + y)}{4} = -\frac{\ln(x)}{2} + c_1$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in x, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dx} = -\frac{y(3x+y)}{x(x+y)}$ 	$R = x$ $S = \frac{\ln(y)}{4} + \frac{\ln(2x+y)}{4}$	$\frac{dS}{dR} = -\frac{1}{2R}$ 

Summary

The solution(s) found are the following

$$\frac{\ln(y)}{4} + \frac{\ln(2x+y)}{4} = -\frac{\ln(x)}{2} + c_1 \tag{1}$$

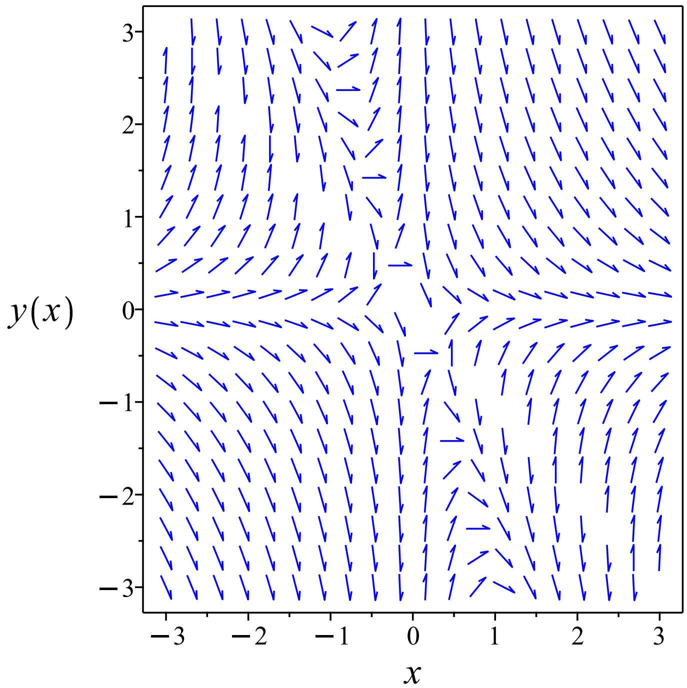


Figure 270: Slope field plot

Verification of solutions

$$\frac{\ln(y)}{4} + \frac{\ln(2x+y)}{4} = -\frac{\ln(x)}{2} + c_1$$

Verified OK.

5.25.3 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x,y) + N(x,y)\frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x,y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y) = 0$$

Hence

$$\frac{\partial \phi}{\partial x} + \frac{\partial \phi}{\partial y}\frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$\frac{\partial \phi}{\partial x} = M$$
$$\frac{\partial \phi}{\partial y} = N$$

But since $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x,y)$ but at least we know now that we can do that since the condition $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x,y)\,\mathrm{d}x + N(x,y)\,\mathrm{d}y = 0 \tag{1A}$$

Therefore

$$\begin{aligned}(x^2 + yx)\,\mathrm{d}y &= (-3yx - y^2)\,\mathrm{d}x\\ (3yx + y^2)\,\mathrm{d}x + (x^2 + yx)\,\mathrm{d}y &= 0 \qquad (2A)\end{aligned}$$

Comparing (1A) and (2A) shows that

$$M(x,y) = 3yx + y^2$$
$$N(x,y) = x^2 + yx$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Using result found above gives

$$\begin{aligned}\frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}(3yx + y^2)\\ &= 3x + 2y\end{aligned}$$

And

$$\begin{aligned}\frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}(x^2 + yx)\\ &= 2x + y\end{aligned}$$

Since $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$, then the ODE is not exact. Since the ODE is not exact, we will try to find an integrating factor to make it exact. Let

$$\begin{aligned}
A &= \frac{1}{N}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right) \\
&= \frac{1}{x\left(x+y\right)}\left(\left(3x+2y\right)-\left(2x+y\right)\right) \\
&= \frac{1}{x}
\end{aligned}$$

Since A does not depend on y, then it can be used to find an integrating factor. The integrating factor μ is

$$\begin{aligned}
\mu &= e^{\int A \, \mathrm{d}x} \\
&= e^{\int \frac{1}{x} \, \mathrm{d}x}
\end{aligned}$$

The result of integrating gives

$$\begin{aligned}
\mu &= e^{\ln(x)} \\
&= x
\end{aligned}$$

M and N are multiplied by this integrating factor, giving new M and new N which are called $\overline{M}$ and $\overline{N}$ for now so not to confuse them with the original M and N.

$$\begin{aligned}
\overline{M} &= \mu M \\
&= x\left(3yx+y^2\right) \\
&= y(3x+y)\,x
\end{aligned}$$

And

$$\begin{aligned}
\overline{N} &= \mu N \\
&= x\left(x^2+yx\right) \\
&= x^2(x+y)
\end{aligned}$$

Now a modified ODE is ontained from the original ODE, which is exact and can be solved. The modified ODE is

$$\begin{aligned}
\overline{M} + \overline{N}\frac{\mathrm{d}y}{\mathrm{d}x} &= 0 \\
\left(y(3x+y)\,x\right) + \left(x^2(x+y)\right)\frac{\mathrm{d}y}{\mathrm{d}x} &= 0
\end{aligned}$$

The following equations are now set up to solve for the function $\phi(x,y)$

$$\frac{\partial \phi}{\partial x} = \overline{M} \tag{1}$$
$$\frac{\partial \phi}{\partial y} = \overline{N} \tag{2}$$

Integrating (1) w.r.t. x gives

$$\int \frac{\partial \phi}{\partial x} \, \mathrm{d}x = \int \overline{M} \, \mathrm{d}x$$
$$\int \frac{\partial \phi}{\partial x} \, \mathrm{d}x = \int y(3x+y)\, x \, \mathrm{d}x$$
$$\phi = \frac{y\, x^2(2x+y)}{2} + f(y) \tag{3}$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$\frac{\partial \phi}{\partial y} = \frac{x^2(2x+y)}{2} + \frac{y\, x^2}{2} + f'(y) \tag{4}$$
$$= x^2(x+y) + f'(y)$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = x^2(x+y)$. Therefore equation (4) becomes

$$x^2(x+y) = x^2(x+y) + f'(y) \tag{5}$$

Solving equation (5) for $f'(y)$ gives

$$f'(y) = 0$$

Therefore

$$f(y) = c_1$$

Where c_1 is constant of integration. Substituting this result for $f(y)$ into equation (3) gives ϕ

$$\phi = \frac{y\, x^2(2x+y)}{2} + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = \frac{y\,x^2(2x + y)}{2}$$

Summary

The solution(s) found are the following

$$\frac{yx^2(2x + y)}{2} = c_1 \tag{1}$$

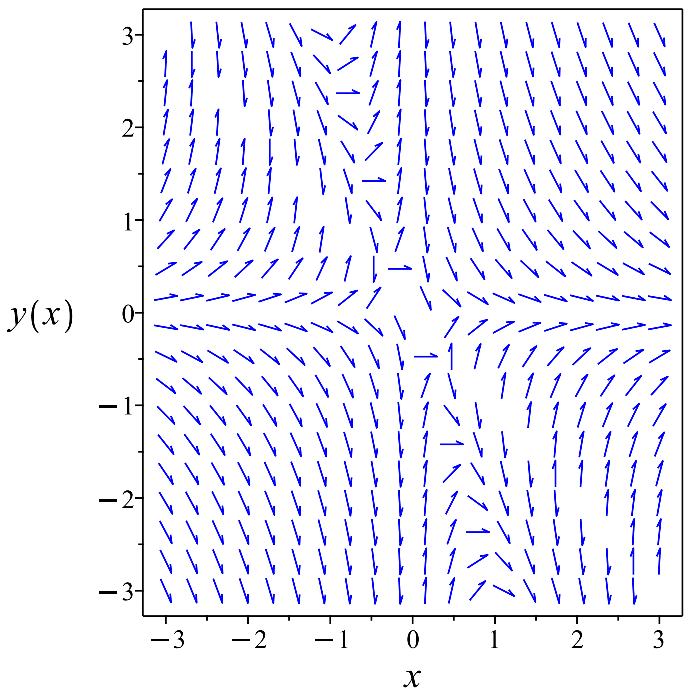


Figure 271: Slope field plot

Verification of solutions

$$\frac{yx^2(2x + y)}{2} = c_1$$

Verified OK.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
trying homogeneous types:
trying homogeneous D
<- homogeneous successful`
```

✓ Solution by Maple

Time used: 0.031 (sec). Leaf size: 59

```
dsolve(3*x*y(x)+y(x)^2+(x^2+x*y(x))*diff(y(x),x) = 0,y(x), singsol=all)
```

$$y(x) = \frac{-c_1x^2 - \sqrt{c_1^2x^4 + 1}}{c_1x}$$
$$y(x) = \frac{-c_1x^2 + \sqrt{c_1^2x^4 + 1}}{c_1x}$$

✓ Solution by Mathematica

Time used: 0.661 (sec). Leaf size: 93

```
DSolve[3*x*y[x]+y[x]^2+(x^2+x*y[x])*y'[x]== 0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to -\frac{x^2 + \sqrt{x^4 + e^{2c_1}}}{x}$$
$$y(x) \to -x + \frac{\sqrt{x^4 + e^{2c_1}}}{x}$$
$$y(x) \to -\frac{\sqrt{x^4} + x^2}{x}$$
$$y(x) \to \frac{\sqrt{x^4}}{x} - x$$

6 Miscellaneous problems, end of chapter 2. Page 133

6.1 problem 1

Internal problem ID [568]
Internal file name [`OUTPUT/568_Sunday_June_05_2022_01_44_54_AM_67792014/index.tex`]

Book: Elementary differential equations and boundary value problems, 10th ed., Boyce and DiPrima
Section: Miscellaneous problems, end of chapter 2. Page 133
Problem number: 1.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"linear", "exactWithIntegrationFactor", "first_order_ode_lie_symmetry_lookup"**

Maple gives the following as the ode type

```
[_linear]
```

$$\boxed{y' - \frac{x^3 - 2y}{x} = 0}$$

6.1.1 Solving as linear ode

Entering Linear first order ODE solver. In canonical form a linear first order is

$$y' + p(x)y = q(x)$$

Where here

$$p(x) = \frac{2}{x}$$
$$q(x) = x^2$$

Hence the ode is

$$y' + \frac{2y}{x} = x^2$$

The integrating factor μ is

$$
\begin{aligned}
\mu &= \mathrm{e}^{\int \frac{2}{x}dx} \\
&= x^2
\end{aligned}
$$

The ode becomes

$$
\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}x}(\mu y) &= (\mu)\left(x^2\right) \\
\frac{\mathrm{d}}{\mathrm{d}x}\left(y\, x^2\right) &= \left(x^2\right)\left(x^2\right) \\
\mathrm{d}\left(y\, x^2\right) &= x^4\, \mathrm{d}x
\end{aligned}
$$

Integrating gives

$$
\begin{aligned}
y\, x^2 &= \int x^4\, \mathrm{d}x \\
y\, x^2 &= \frac{x^5}{5} + c_1
\end{aligned}
$$

Dividing both sides by the integrating factor $\mu = x^2$ results in

$$
y = \frac{x^3}{5} + \frac{c_1}{x^2}
$$

Summary
The solution(s) found are the following

$$
y = \frac{x^3}{5} + \frac{c_1}{x^2} \tag{1}
$$

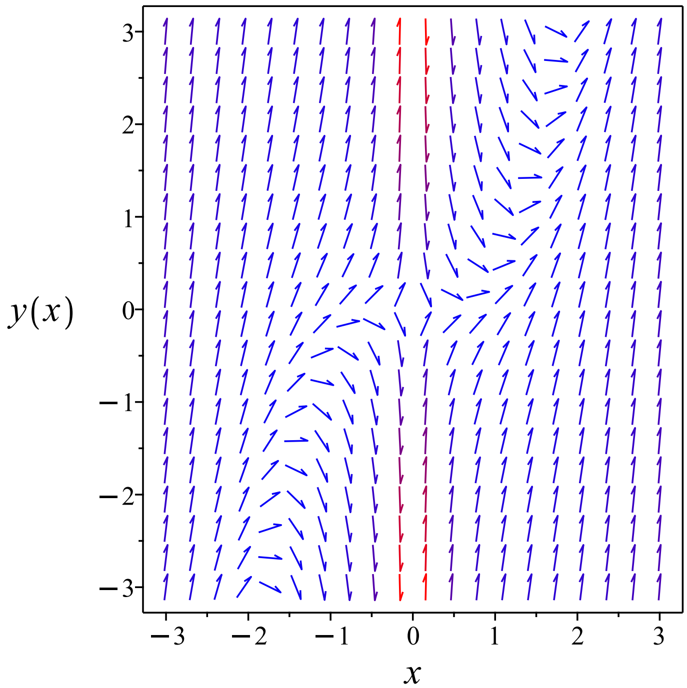


Figure 272: Slope field plot

Verification of solutions

$$y = \frac{x^3}{5} + \frac{c_1}{x^2}$$

Verified OK.

6.1.2 Solving as first order ode lie symmetry lookup ode

Writing the ode as

$$y' = -\frac{-x^3 + 2y}{x}$$
$$y' = \omega(x, y)$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x + \omega(\eta_y - \xi_x) - \omega^2 \xi_y - \omega_x \xi - \omega_y \eta = 0 \tag{A}$$

The type of this ode is known. It is of type `linear`. Therefore we do not need to solve the PDE (A), and can just use the lookup table shown below to find ξ, η

Table 260: Lie symmetry infinitesimal lookup table for known first order ODE's

ODE class	Form	ξ	η
linear ode	$y' = f(x)y(x) + g(x)$	0	$e^{\int f dx}$
separable ode	$y' = f(x)\, g(y)$	$\frac{1}{f}$	0
quadrature ode	$y' = f(x)$	0	1
quadrature ode	$y' = g(y)$	1	0
homogeneous ODEs of Class A	$y' = f\left(\frac{y}{x}\right)$	x	y
homogeneous ODEs of Class C	$y' = (a + bx + cy)^{\frac{n}{m}}$	1	$-\frac{b}{c}$
homogeneous class D	$y' = \frac{y}{x} + g(x)\, F\left(\frac{y}{x}\right)$	x^2	xy
First order special form ID 1	$y' = g(x)\, e^{h(x)+by} + f(x)$	$\frac{e^{-\int bf(x)dx - h(x)}}{g(x)}$	$\frac{f(x)e^{-\int bf(x)dx - h(x)}}{g(x)}$
polynomial type ode	$y' = \frac{a_1x+b_1y+c_1}{a_2x+b_2y+c_2}$	$\frac{a_1b_2x - a_2b_1x - b_1c_2 + b_2c_1}{a_1b_2 - a_2b_1}$	$\frac{a_1b_2y - a_2b_1y - a_1c_2 - a_2c_1}{a_1b_2 - a_2b_1}$
Bernoulli ode	$y' = f(x)\, y + g(x)\, y^n$	0	$e^{-\int (n-1)f(x)dx} y^n$
Reduced Riccati	$y' = f_1(x)\, y + f_2(x)\, y^2$	0	$e^{-\int f_1 dx}$

The above table shows that

$$
\begin{aligned}
\xi(x, y) &= 0 \\
\eta(x, y) &= \frac{1}{x^2}
\end{aligned} \tag{A1}
$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x, y) \to (R, S)$ where (R, S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$
\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}
$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y}\right) S(x, y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the

canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$R = x$$

S is found from

$$\begin{aligned} S &= \int \frac{1}{\eta} dy \\ &= \int \frac{1}{\frac{1}{x^2}} dy \end{aligned}$$

Which results in

$$S = y\,x^2$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x,y) S_y}{R_x + \omega(x,y) R_y} \tag{2}$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x,y)$ is the right hand side of the original ode given by

$$\omega(x,y) = -\frac{-x^3 + 2y}{x}$$

Evaluating all the partial derivatives gives

$$\begin{aligned} R_x &= 1 \\ R_y &= 0 \\ S_x &= 2yx \\ S_y &= x^2 \end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = x^4 \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = R^4$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by

integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$S(R) = \frac{R^5}{5} + c_1 \tag{4}$$

To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$x^2 y = \frac{x^5}{5} + c_1$$

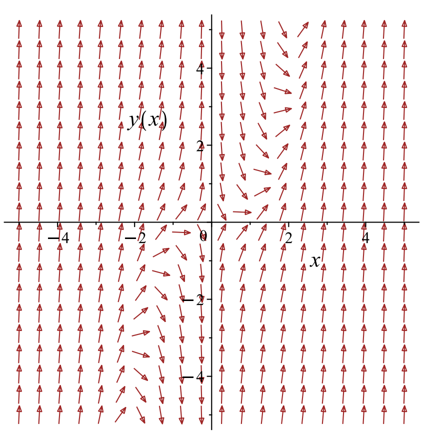
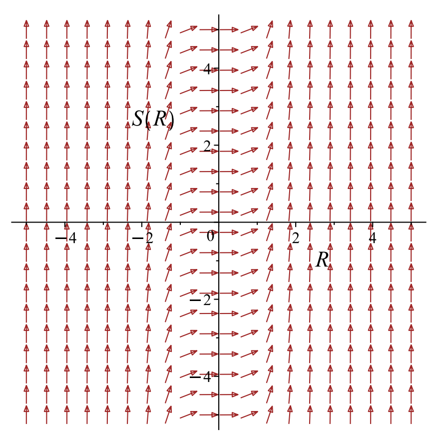
Which simplifies to

$$x^2 y = \frac{x^5}{5} + c_1$$

Which gives

$$y = \frac{x^5 + 5c_1}{5x^2}$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in x, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dx} = -\frac{-x^3+2y}{x}$ 	$R = x$ $S = y\,x^2$	$\frac{dS}{dR} = R^4$ 

Summary

The solution(s) found are the following

$$y = \frac{x^5 + 5c_1}{5x^2} \tag{1}$$

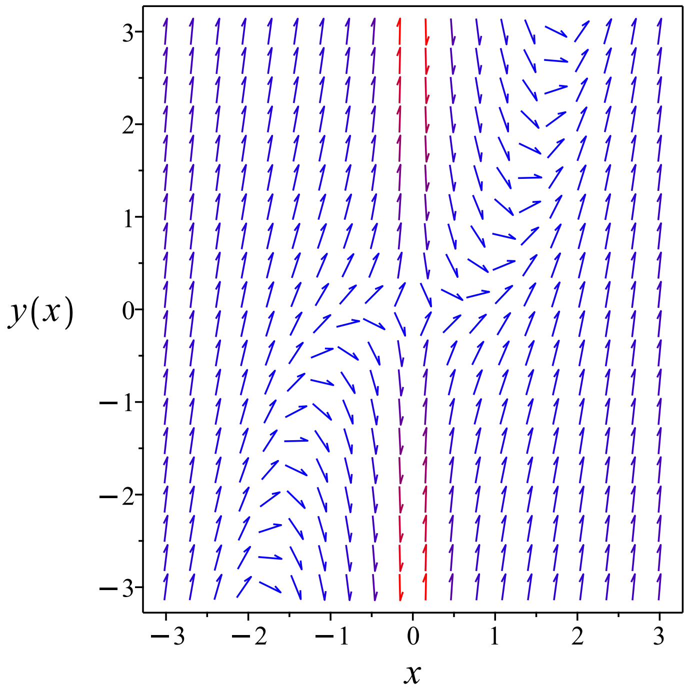


Figure 273: Slope field plot

Verification of solutions

$$y = \frac{x^5 + 5c_1}{5x^2}$$

Verified OK.

6.1.3 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x,y) + N(x,y)\frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x,y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y) = 0$$

Hence

$$\frac{\partial \phi}{\partial x} + \frac{\partial \phi}{\partial y}\frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$\frac{\partial \phi}{\partial x} = M$$
$$\frac{\partial \phi}{\partial y} = N$$

But since $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x, y)$ but at least we know now that we can do that since the condition $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x, y)\,\mathrm{d}x + N(x, y)\,\mathrm{d}y = 0 \tag{1A}$$

Therefore

$$\mathrm{d}y = \left(\frac{x^3 - 2y}{x}\right) \mathrm{d}x$$
$$\left(-\frac{x^3 - 2y}{x}\right) \mathrm{d}x + \mathrm{d}y = 0 \tag{2A}$$

Comparing (1A) and (2A) shows that

$$M(x, y) = -\frac{x^3 - 2y}{x}$$
$$N(x, y) = 1$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Using result found above gives

$$\frac{\partial M}{\partial y} = \frac{\partial}{\partial y}\left(-\frac{x^3 - 2y}{x}\right)$$
$$= \frac{2}{x}$$

And

$$
\begin{aligned}
\frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}(1) \\
&= 0
\end{aligned}
$$

Since $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$, then the ODE is <u>not exact</u>. Since the ODE is not exact, we will try to find an integrating factor to make it exact. Let

$$
\begin{aligned}
A &= \frac{1}{N}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right) \\
&= 1\left(\left(\frac{2}{x}\right) - (0)\right) \\
&= \frac{2}{x}
\end{aligned}
$$

Since A does not depend on y, then it can be used to find an integrating factor. The integrating factor μ is

$$
\begin{aligned}
\mu &= e^{\int A \, \mathrm{d}x} \\
&= e^{\int \frac{2}{x} \, \mathrm{d}x}
\end{aligned}
$$

The result of integrating gives

$$
\begin{aligned}
\mu &= e^{2\ln(x)} \\
&= x^2
\end{aligned}
$$

M and N are multiplied by this integrating factor, giving new M and new N which are called $\overline{M}$ and $\overline{N}$ for now so not to confuse them with the original M and N.

$$
\begin{aligned}
\overline{M} &= \mu M \\
&= x^2\left(-\frac{x^3 - 2y}{x}\right) \\
&= -x\left(x^3 - 2y\right)
\end{aligned}
$$

And

$$
\begin{aligned}
\overline{N} &= \mu N \\
&= x^2(1) \\
&= x^2
\end{aligned}
$$

Now a modified ODE is ontained from the original ODE, which is exact and can be solved. The modified ODE is

$$\overline{M} + \overline{N}\frac{\mathrm{d}y}{\mathrm{d}x} = 0$$
$$\left(-x\left(x^{3}-2y\right)\right) + \left(x^{2}\right)\frac{\mathrm{d}y}{\mathrm{d}x} = 0$$

The following equations are now set up to solve for the function $\phi(x, y)$

$$\frac{\partial \phi}{\partial x} = \overline{M} \tag{1}$$
$$\frac{\partial \phi}{\partial y} = \overline{N} \tag{2}$$

Integrating (1) w.r.t. x gives

$$\int \frac{\partial \phi}{\partial x} \mathrm{d}x = \int \overline{M} \mathrm{d}x$$
$$\int \frac{\partial \phi}{\partial x} \mathrm{d}x = \int -x\left(x^{3}-2y\right) \mathrm{d}x$$
$$\phi = -\frac{1}{5}x^{5} + y\,x^{2} + f(y) \tag{3}$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$\frac{\partial \phi}{\partial y} = x^{2} + f'(y) \tag{4}$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = x^{2}$. Therefore equation (4) becomes

$$x^{2} = x^{2} + f'(y) \tag{5}$$

Solving equation (5) for $f'(y)$ gives

$$f'(y) = 0$$

Therefore

$$f(y) = c_{1}$$

Where c_1 is constant of integration. Substituting this result for $f(y)$ into equation (3) gives ϕ

$$\phi = -\frac{1}{5}x^5 + y\,x^2 + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = -\frac{1}{5}x^5 + y\,x^2$$

The solution becomes

$$y = \frac{x^5 + 5c_1}{5x^2}$$

Summary

The solution(s) found are the following

$$y = \frac{x^5 + 5c_1}{5x^2} \tag{1}$$

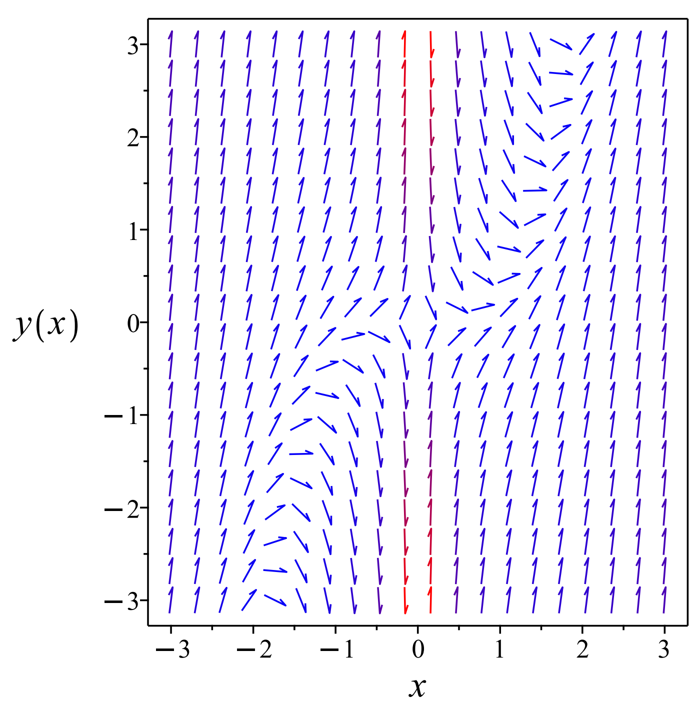


Figure 274: Slope field plot

Verification of solutions

$$y = \frac{x^5 + 5c_1}{5x^2}$$

Verified OK.

6.1.4 Maple step by step solution

Let's solve

$y' - \frac{x^3-2y}{x} = 0$

- Highest derivative means the order of the ODE is 1

  y'

- Isolate the derivative

  $y' = -\frac{2y}{x} + x^2$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE

  $y' + \frac{2y}{x} = x^2$

- The ODE is linear; multiply by an integrating factor $\mu(x)$

  $\mu(x)\left(y' + \frac{2y}{x}\right) = \mu(x)\, x^2$

- Assume the lhs of the ODE is the total derivative $\frac{d}{dx}(\mu(x)\, y)$

  $\mu(x)\left(y' + \frac{2y}{x}\right) = \mu'(x)\, y + \mu(x)\, y'$

- Isolate $\mu'(x)$

  $\mu'(x) = \frac{2\mu(x)}{x}$

- Solve to find the integrating factor

  $\mu(x) = x^2$

- Integrate both sides with respect to x

  $\int \left(\frac{d}{dx}(\mu(x)\, y)\right) dx = \int \mu(x)\, x^2 dx + c_1$

- Evaluate the integral on the lhs

  $\mu(x)\, y = \int \mu(x)\, x^2 dx + c_1$

- Solve for y

  $y = \frac{\int \mu(x)x^2 dx + c_1}{\mu(x)}$

- Substitute $\mu(x) = x^2$

$y = \frac{\int x^4 dx + c_1}{x^2}$

- Evaluate the integrals on the rhs

$y = \frac{\frac{x^5}{5} + c_1}{x^2}$

- Simplify

$y = \frac{x^5 + 5c_1}{5x^2}$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
<- 1st order linear successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 16

```
dsolve(diff(y(x),x) = (x^3-2*y(x))/x,y(x), singsol=all)
```

$$y(x) = \frac{x^5 + 5c_1}{5x^2}$$

✓ Solution by Mathematica

Time used: 0.024 (sec). Leaf size: 19

```
DSolve[y'[x]== (x^3-2*y[x])/x,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \frac{x^3}{5} + \frac{c_1}{x^2}$$

6.2 problem 2

Internal problem ID [569]
Internal file name [OUTPUT/569_Sunday_June_05_2022_01_44_55_AM_90905761/index.tex]

Book: Elementary differential equations and boundary value problems, 10th ed., Boyce and DiPrima
Section: Miscellaneous problems, end of chapter 2. Page 133
Problem number: 2.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"exact", "separable", "first_order_ode_lie_symmetry_lookup"**

Maple gives the following as the ode type

```
[_separable]
```

$$\boxed{y' - \frac{\cos(x) + 1}{2 - \sin(y)} = 0}$$

6.2.1 Solving as separable ode

In canonical form the ODE is

$$\begin{aligned} y' &= F(x,y) \\ &= f(x)g(y) \\ &= \frac{-\cos(x) - 1}{-2 + \sin(y)} \end{aligned}$$

Where $f(x) = -\cos(x) - 1$ and $g(y) = \frac{1}{-2+\sin(y)}$. Integrating both sides gives

$$\frac{1}{\frac{1}{-2+\sin(y)}}\,dy = -\cos(x) - 1\,dx$$

$$\int \frac{1}{\frac{1}{-2+\sin(y)}}\, dy = \int -\cos(x) - 1\, dx$$

$$-2y - \cos(y) = -x - \sin(x) + c_1$$

Which results in

$$y = \text{RootOf}\left(2\_Z + \cos(\_Z) - x - \sin(x) + c_1\right)$$

Summary

The solution(s) found are the following

$$y = \text{RootOf}\left(2\_Z + \cos(\_Z) - x - \sin(x) + c_1\right) \tag{1}$$

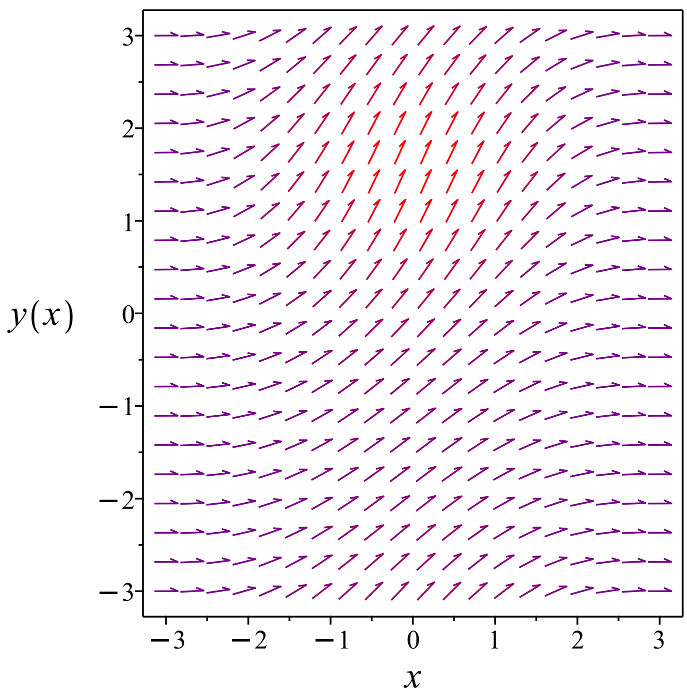


Figure 275: Slope field plot

Verification of solutions

$$y = \text{RootOf}\left(2\_Z + \cos(\_Z) - x - \sin(x) + c_1\right)$$

Verified OK.

6.2.2 Solving as first order ode lie symmetry lookup ode

Writing the ode as

$$
\begin{aligned}
y' &= -\frac{\cos(x)+1}{-2+\sin(y)} \\
y' &= \omega(x,y)
\end{aligned}
$$

The condition of Lie symmetry is the linearized PDE given by

$$
\eta_x + \omega(\eta_y - \xi_x) - \omega^2 \xi_y - \omega_x \xi - \omega_y \eta = 0 \tag{A}
$$

The type of this ode is known. It is of type `separable`. Therefore we do not need to solve the PDE (A), and can just use the lookup table shown below to find ξ, η

Table 263: Lie symmetry infinitesimal lookup table for known first order ODE's

ODE class	Form	ξ	η
linear ode	$y' = f(x)y(x) + g(x)$	0	$e^{\int f dx}$
separable ode	$y' = f(x)\,g(y)$	$\frac{1}{f}$	0
quadrature ode	$y' = f(x)$	0	1
quadrature ode	$y' = g(y)$	1	0
homogeneous ODEs of Class A	$y' = f\left(\frac{y}{x}\right)$	x	y
homogeneous ODEs of Class C	$y' = (a+bx+cy)^{\frac{n}{m}}$	1	$-\frac{b}{c}$
homogeneous class D	$y' = \frac{y}{x} + g(x)\,F\left(\frac{y}{x}\right)$	x^2	xy
First order special form ID 1	$y' = g(x)\,e^{h(x)+by} + f(x)$	$\frac{e^{-\int bf(x)dx-h(x)}}{g(x)}$	$\frac{f(x)e^{-\int bf(x)dx-h(x)}}{g(x)}$
polynomial type ode	$y' = \frac{a_1x+b_1y+c_1}{a_2x+b_2y+c_2}$	$\frac{a_1b_2x-a_2b_1x-b_1c_2+b_2c_1}{a_1b_2-a_2b_1}$	$\frac{a_1b_2y-a_2b_1y-a_1c_2-a_2c_1}{a_1b_2-a_2b_1}$
Bernoulli ode	$y' = f(x)\,y + g(x)\,y^n$	0	$e^{-\int (n-1)f(x)dx}y^n$
Reduced Riccati	$y' = f_1(x)\,y + f_2(x)\,y^2$	0	$e^{-\int f_1 dx}$

The above table shows that

$$
\begin{aligned}
\xi(x,y) &= \frac{1}{-\cos(x)-1} \\
\eta(x,y) &= 0
\end{aligned} \tag{A1}
$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x,y) \to (R,S)$ where (R,S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$
\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}
$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y}\right) S(x,y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the canonical coordinates, where $S(R)$. Since $\eta = 0$ then in this special case

$$
R = y
$$

S is found from

$$
\begin{aligned}
S &= \int \frac{1}{\xi} dx \\
&= \int \frac{1}{\frac{1}{-\cos(x)-1}} dx
\end{aligned}
$$

Which results in

$$
S = -x - \sin(x)
$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$
\frac{dS}{dR} = \frac{S_x + \omega(x,y) S_y}{R_x + \omega(x,y) R_y} \tag{2}
$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x,y)$ is the right hand side of the original ode given by

$$
\omega(x,y) = -\frac{\cos(x)+1}{-2+\sin(y)}
$$

Evaluating all the partial derivatives gives

$$
\begin{aligned}
R_x &= 0 \\
R_y &= 1 \\
S_x &= -\cos(x) - 1 \\
S_y &= 0
\end{aligned}
$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = -2 + \sin(y) \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = -2 + \sin(R)$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$S(R) = -\cos(R) - 2R + c_1 \tag{4}$$

To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$-x - \sin(x) = -\cos(y) - 2y + c_1$$

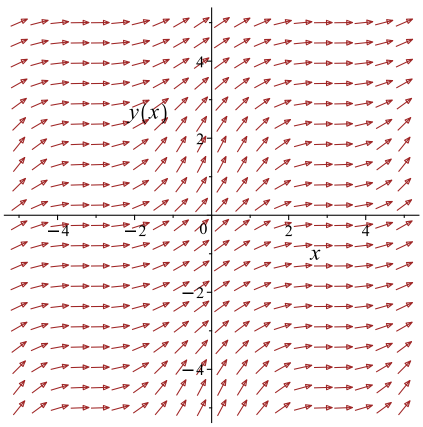
Which simplifies to

$$-x - \sin(x) = -\cos(y) - 2y + c_1$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

<table>
<tr><th>Original ode in x, y coordinates</th><th>Canonical coordinates transformation</th><th>ODE in canonical coordinates (R, S)</th></tr>
<tr><td>

$\frac{dy}{dx} = -\frac{\cos(x)+1}{-2+\sin(y)}$



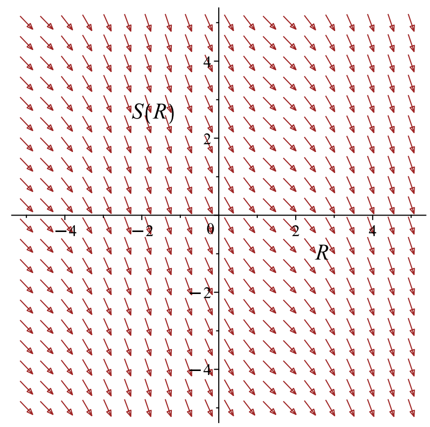
</td><td>

$R = y$

$S = -x - \sin(x)$

</td><td>

$\frac{dS}{dR} = -2 + \sin(R)$



</td></tr>
</table>

Summary

The solution(s) found are the following

$$-x - \sin(x) = -\cos(y) - 2y + c_1 \tag{1}$$

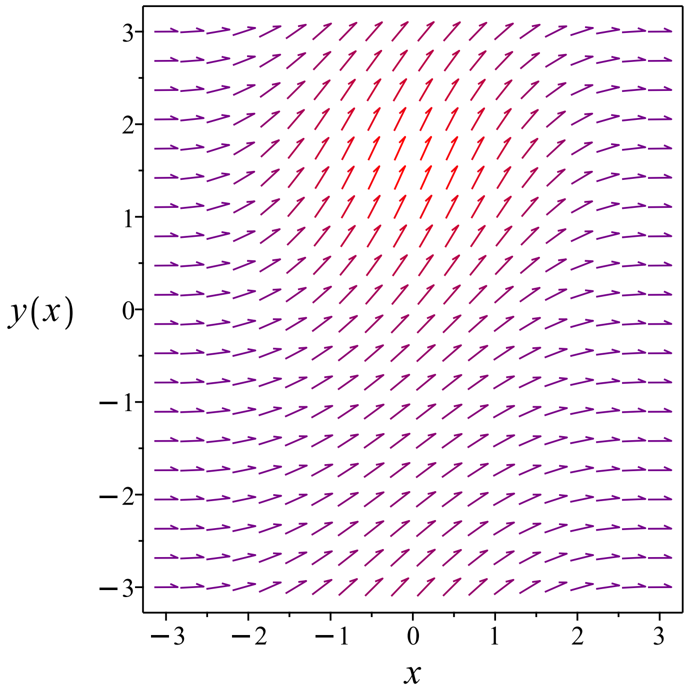


Figure 276: Slope field plot

Verification of solutions

$$-x - \sin(x) = -\cos(y) - 2y + c_1$$

Verified OK.

6.2.3 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x,y) + N(x,y)\frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x,y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y) = 0$$

Hence

$$\frac{\partial\phi}{\partial x} + \frac{\partial\phi}{\partial y}\frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$
\begin{aligned}
\frac{\partial \phi}{\partial x} &= M \\
\frac{\partial \phi}{\partial y} &= N
\end{aligned}
$$

But since $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ then for the above to be valid, we require that

$$
\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}
$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x, y)$ but at least we know now that we can do that since the condition $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$
M(x, y) \,\mathrm{d}x + N(x, y) \,\mathrm{d}y = 0 \tag{1A}
$$

Therefore

$$
\begin{aligned}
(2 - \sin(y)) \,\mathrm{d}y &= (\cos(x) + 1) \,\mathrm{d}x \\
(-\cos(x) - 1) \,\mathrm{d}x + (2 - \sin(y)) \,\mathrm{d}y &= 0 \qquad \text{(2A)}
\end{aligned}
$$

Comparing (1A) and (2A) shows that

$$
\begin{aligned}
M(x, y) &= -\cos(x) - 1 \\
N(x, y) &= 2 - \sin(y)
\end{aligned}
$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$
\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}
$$

Using result found above gives

$$
\begin{aligned}
\frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}(-\cos(x) - 1) \\
&= 0
\end{aligned}
$$

And

$$
\begin{aligned}
\frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}(2 - \sin(y)) \\
&= 0
\end{aligned}
$$

Since $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$, then the ODE is exact The following equations are now set up to solve for the function $\phi(x, y)$

$$\frac{\partial \phi}{\partial x} = M \tag{1}$$
$$\frac{\partial \phi}{\partial y} = N \tag{2}$$

Integrating (1) w.r.t. x gives

$$\int \frac{\partial \phi}{\partial x} \mathrm{d}x = \int M \mathrm{d}x$$
$$\int \frac{\partial \phi}{\partial x} \mathrm{d}x = \int -\cos(x) - 1 \mathrm{d}x$$
$$\phi = -x - \sin(x) + f(y) \tag{3}$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$\frac{\partial \phi}{\partial y} = 0 + f'(y) \tag{4}$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = 2 - \sin(y)$. Therefore equation (4) becomes

$$2 - \sin(y) = 0 + f'(y) \tag{5}$$

Solving equation (5) for $f'(y)$ gives

$$f'(y) = 2 - \sin(y)$$

Integrating the above w.r.t y gives

$$\int f'(y) \mathrm{d}y = \int (2 - \sin(y)) \mathrm{d}y$$
$$f(y) = 2y + \cos(y) + c_1$$

Where c_1 is constant of integration. Substituting result found above for $f(y)$ into equation (3) gives ϕ

$$\phi = -x - \sin(x) + 2y + \cos(y) + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = -x - \sin(x) + 2y + \cos(y)$$

Summary
The solution(s) found are the following

$$2y + \cos(y) - x - \sin(x) = c_1 \tag{1}$$

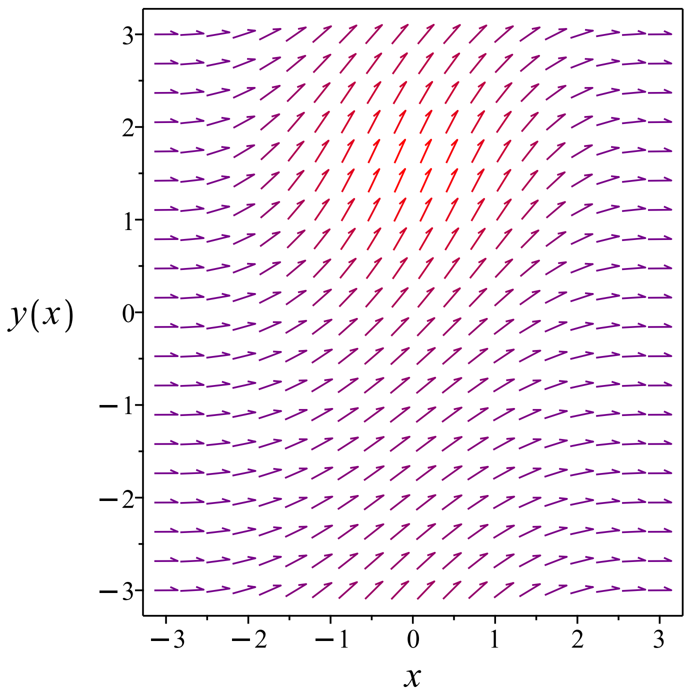


Figure 277: Slope field plot

Verification of solutions

$$2y + \cos(y) - x - \sin(x) = c_1$$

Verified OK.

6.2.4 Maple step by step solution

Let's solve

$y' - \frac{\cos(x)+1}{2-\sin(y)} = 0$

- Highest derivative means the order of the ODE is 1

  y'

- Separate variables

  $(2-\sin(y))\, y' = \cos(x) + 1$

- Integrate both sides with respect to x

  $\int (2-\sin(y))\, y' dx = \int (\cos(x) + 1)\, dx + c_1$

- Evaluate integral

  $2y + \cos(y) = x + \sin(x) + c_1$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
<- separable successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 17

```
dsolve(diff(y(x),x) = (1+cos(x))/(2-sin(y(x))),y(x), singsol=all)
```

$$x + \sin(x) - 2y(x) - \cos(y(x)) + c_1 = 0$$

✓ Solution by Mathematica

Time used: 0.372 (sec). Leaf size: 27

```
DSolve[y'[x] == (1+Cos[x])/(2-Sin[y[x]]),y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \text{InverseFunction}[-2\#1 - \cos(\#1)\&][-x - \sin(x) + c_1]$$

6.3 problem 3

Internal problem ID [570]
Internal file name [`OUTPUT/570_Sunday_June_05_2022_01_44_56_AM_17223756/index.tex`]

Book: Elementary differential equations and boundary value problems, 10th ed., Boyce and DiPrima
Section: Miscellaneous problems, end of chapter 2. Page 133
Problem number: 3.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"exact", "differentialType"**

Maple gives the following as the ode type

`[_rational]`

$$\boxed{y' - \frac{2x+y}{3-x+3y^2} = 0}$$

With initial conditions

$$[y(0) = 0]$$

6.3.1 Existence and uniqueness analysis

This is non linear first order ODE. In canonical form it is written as

$$\begin{aligned} y' &= f(x,y) \\ &= \frac{2x+y}{3y^2-x+3} \end{aligned}$$

The x domain of $f(x,y)$ when $y=0$ is

$$\{x < 3 \lor 3 < x\}$$

And the point $x_0 = 0$ is inside this domain. The y domain of $f(x,y)$ when $x=0$ is

$$\{-\infty < y < \infty\}$$

And the point $y_0 = 0$ is inside this domain. Now we will look at the continuity of

$$\begin{aligned}\frac{\partial f}{\partial y} &= \frac{\partial}{\partial y}\left(\frac{2x+y}{3y^2-x+3}\right) \\ &= \frac{1}{3y^2-x+3} - \frac{6(2x+y)\,y}{(3y^2-x+3)^2}\end{aligned}$$

The x domain of $\frac{\partial f}{\partial y}$ when $y = 0$ is

$$\{x < 3 \lor 3 < x\}$$

And the point $x_0 = 0$ is inside this domain. The y domain of $\frac{\partial f}{\partial y}$ when $x = 0$ is

$$\{-\infty < y < \infty\}$$

And the point $y_0 = 0$ is inside this domain. Therefore solution exists and is unique.

6.3.2 Solving as differentialType ode

Writing the ode as

$$y' = \frac{2x+y}{3-x+3y^2} \tag{1}$$

Which becomes

$$\left(-3y^2-3\right) dy = (-x)\, dy + (-2x-y)\, dx \tag{2}$$

But the RHS is complete differential because

$$(-x)\, dy + (-2x-y)\, dx = d\left(-x^2-yx\right)$$

Hence (2) becomes

$$\left(-3y^2-3\right) dy = d\left(-x^2-yx\right)$$

Integrating both sides gives gives these solutions

$$y = \frac{\left(108x^2 - 108c_1 + 12\sqrt{81x^4 - 162c_1x^2 - 12x^3 + 81c_1^2 + 108x^2 - 324x + 324}\right)^{\frac{1}{3}}}{6} - \frac{}{\left(108x^2 - 108c_1 + \right.}$$

$$y = -\frac{\left(108x^2 - 108c_1 + 12\sqrt{81x^4 - 162c_1x^2 - 12x^3 + 81c_1^2 + 108x^2 - 324x + 324}\right)^{\frac{1}{3}}}{12} + \frac{}{\left(108x^2 - 108c_1 \right.}$$

$$y = -\frac{\left(108x^2 - 108c_1 + 12\sqrt{81x^4 - 162c_1x^2 - 12x^3 + 81c_1^2 + 108x^2 - 324x + 324}\right)^{\frac{1}{3}}}{12} + \frac{}{\left(108x^2 - 108c_1 \right.}$$

Initial conditions are used to solve for c_1. Substituting $x = 0$ and $y = 0$ in the above solution gives an equation to solve for the constant of integration.

$$0 = \frac{-i\sqrt{3}\left(-4c_1 + 4\sqrt{c_1^2 + 4}\right)^{\frac{2}{3}} - 4i\sqrt{3} - \left(-4c_1 + 4\sqrt{c_1^2 + 4}\right)^{\frac{2}{3}} + 4c_1\left(-4c_1 + 4\sqrt{c_1^2 + 4}\right)^{\frac{1}{3}} + 4}{4\left(-4c_1 + 4\sqrt{c_1^2 + 4}\right)^{\frac{1}{3}}}$$

$$c_1 = \frac{i(2\sqrt{3} - 2i)\sqrt{3} + 4i\sqrt{3} + 2\sqrt{3} - 4 - 2i}{4\sqrt{2\sqrt{3} - 2i}}$$

Substituting c_1 found above in the general solution gives

$$y = \lim_{c_1 \to \frac{i(2\sqrt{3}-2i)\sqrt{3}+4i\sqrt{3}+2\sqrt{3}-4-2i}{4\sqrt{2\sqrt{3}-2i}}} \left(-\frac{\left(108x^2 - 108c_1 + 12\sqrt{81x^4 - 162c_1x^2 - 12x^3 + 81c_1^2 + 108x^2 - 324x \right.}}{12} \right.$$

Initial conditions are used to solve for c_1. Substituting $x = 0$ and $y = 0$ in the above solution gives an equation to solve for the constant of integration.

$$0 = \frac{i\sqrt{3}\left(-4c_1 + 4\sqrt{c_1^2 + 4}\right)^{\frac{2}{3}} + 4i\sqrt{3} - \left(-4c_1 + 4\sqrt{c_1^2 + 4}\right)^{\frac{2}{3}} + 4c_1\left(-4c_1 + 4\sqrt{c_1^2 + 4}\right)^{\frac{1}{3}} + 4}{4\left(-4c_1 + 4\sqrt{c_1^2 + 4}\right)^{\frac{1}{3}}}$$

$$c_1 = -\frac{i\left(2i+2\sqrt{3}\right)\sqrt{3}+4i\sqrt{3}+4-2i-2\sqrt{3}}{4\sqrt{2i+2\sqrt{3}}}$$

Substituting c_1 found above in the general solution gives

$$y = \text{Expression too large to display}$$

Initial conditions are used to solve for c_1. Substituting $x = 0$ and $y = 0$ in the above solution gives an equation to solve for the constant of integration.

$$0 = \frac{\left(-4c_1+4\sqrt{c_1^2+4}\right)^{\frac{2}{3}}+2c_1\left(-4c_1+4\sqrt{c_1^2+4}\right)^{\frac{1}{3}}-4}{2\left(-4c_1+4\sqrt{c_1^2+4}\right)^{\frac{1}{3}}}$$

$$c_1 = 0$$

Substituting c_1 found above in the general solution gives

$$y = \frac{\left(108x^2+12\sqrt{81x^4-12x^3+108x^2-324x+324}\right)^{\frac{2}{3}}+12x-36}{6\left(108x^2+12\sqrt{81x^4-12x^3+108x^2-324x+324}\right)^{\frac{1}{3}}}$$

Summary

The solution(s) found are the following

$$y = \frac{\left(108x^2+12\sqrt{81x^4-12x^3+108x^2-324x+324}\right)^{\frac{2}{3}}+12x-36}{6\left(108x^2+12\sqrt{81x^4-12x^3+108x^2-324x+324}\right)^{\frac{1}{3}}} \tag{1}$$

Expression too large to display (2)

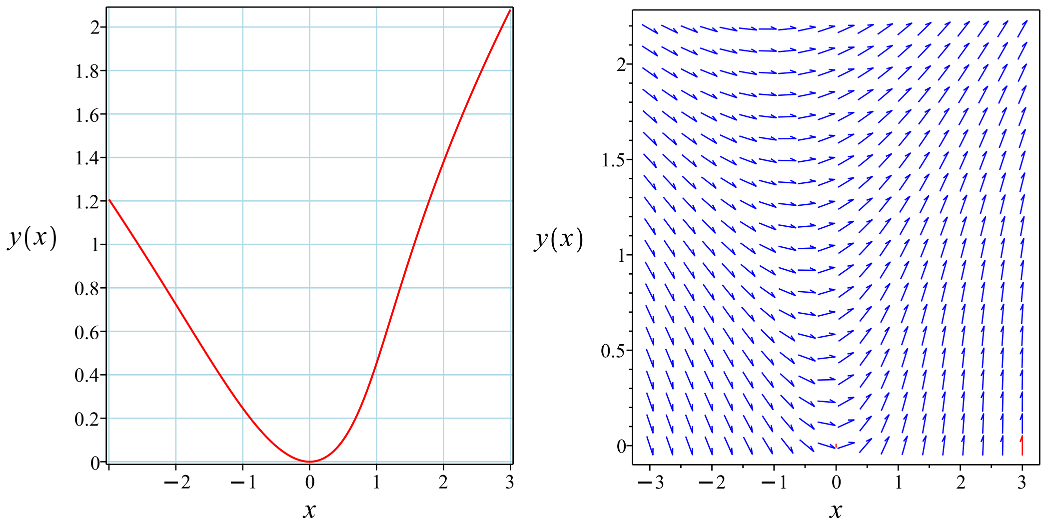
$$y \tag{3}$$

$$= \lim_{c_1 \to \frac{i\left(2\sqrt{3}-2i\right)\sqrt{3}+4i\sqrt{3}+2\sqrt{3}-4-2i}{4\sqrt{2\sqrt{3}-2i}}} \left(-\frac{\left(108x^2-108c_1+12\sqrt{81x^4-162c_1x^2-12x^3+81c_1^2+108x^2-324x+3\right.}{12} \right.$$

$$+ \frac{3-x}{\left(108x^2-108c_1+12\sqrt{81x^4-162c_1x^2-12x^3+81c_1^2+108x^2-324x+324}\right)^{\frac{1}{3}}}$$

$$- \frac{i\sqrt{3}\left(\frac{\left(108x^2-108c_1+12\sqrt{81x^4-162c_1x^2-12x^3+81c_1^2+108x^2-324x+324}\right)^{\frac{1}{3}}}{6} + \frac{-2x+6}{\left(108x^2-108c_1+12\sqrt{81x^4-162c_1x^2-12x^3+81c_1^2+1\right.}\right.}{2}$$

$$\left. + c_1 \right)$$



(a) Solution plot

(b) Slope field plot

Verification of solutions

$$y = \frac{\left(108x^2 + 12\sqrt{81x^4 - 12x^3 + 108x^2 - 324x + 324}\right)^{\frac{2}{3}} + 12x - 36}{6\left(108x^2 + 12\sqrt{81x^4 - 12x^3 + 108x^2 - 324x + 324}\right)^{\frac{1}{3}}}$$

Verified OK.

Expression too large to display

Warning, solution could not be verified

$$y = \lim_{c_1 \to \frac{i\left(2\sqrt{3}-2i\right)\sqrt{3}+4i\sqrt{3}+2\sqrt{3}-4-2i}{4\sqrt{2\sqrt{3}-2i}}} \left(-\frac{\left(108x^2 - 108c_1 + 12\sqrt{81x^4 - 162c_1x^2 - 12x^3 + 81c_1^2 + 108x^2 - 324x + 3\cdots}\right.}{12} \right.$$

$$+ \frac{3 - x}{\left(108x^2 - 108c_1 + 12\sqrt{81x^4 - 162c_1x^2 - 12x^3 + 81c_1^2 + 108x^2 - 324x + 324}\right)^{\frac{1}{3}}}$$

$$- \frac{i\sqrt{3}\left(\frac{\left(108x^2-108c_1+12\sqrt{81x^4-162c_1x^2-12x^3+81c_1^2+108x^2-324x+324}\right)^{\frac{1}{3}}}{6} + \frac{-2x+6}{\left(108x^2-108c_1+12\sqrt{81x^4-162c_1x^2-12x^3+81c_1^2+1\cdots}\right.}\right.}{2}$$

$$\left. + c_1 \right)$$

Warning, solution could not be verified

6.3.3 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x,y) + N(x,y)\frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x,y) = c$ where c is constant, that satisfies the

ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y) = 0$$

Hence

$$\frac{\partial \phi}{\partial x} + \frac{\partial \phi}{\partial y}\frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$\begin{aligned}\frac{\partial \phi}{\partial x} &= M \\ \frac{\partial \phi}{\partial y} &= N\end{aligned}$$

But since $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x,y)$ but at least we know now that we can do that since the condition $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x,y)\,\mathrm{d}x + N(x,y)\,\mathrm{d}y = 0 \tag{1A}$$

Therefore

$$\begin{aligned}\left(3y^2 - x + 3\right)\mathrm{d}y &= (2x + y)\,\mathrm{d}x \\ (-2x - y)\,\mathrm{d}x + \left(3y^2 - x + 3\right)\mathrm{d}y &= 0 \qquad \text{(2A)}\end{aligned}$$

Comparing (1A) and (2A) shows that

$$\begin{aligned}M(x,y) &= -2x - y \\ N(x,y) &= 3y^2 - x + 3\end{aligned}$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Using result found above gives

$$\begin{aligned}\frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}(-2x - y)\\ &= -1\end{aligned}$$

And

$$\begin{aligned}\frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}\left(3y^2 - x + 3\right)\\ &= -1\end{aligned}$$

Since $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$, then the ODE is exact The following equations are now set up to solve for the function $\phi(x, y)$

$$\frac{\partial \phi}{\partial x} = M \tag{1}$$

$$\frac{\partial \phi}{\partial y} = N \tag{2}$$

Integrating (1) w.r.t. x gives

$$\int \frac{\partial \phi}{\partial x} \, \mathrm{d}x = \int M \, \mathrm{d}x$$

$$\int \frac{\partial \phi}{\partial x} \, \mathrm{d}x = \int -2x - y \, \mathrm{d}x$$

$$\phi = -x(x + y) + f(y) \tag{3}$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$\frac{\partial \phi}{\partial y} = -x + f'(y) \tag{4}$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = 3y^2 - x + 3$. Therefore equation (4) becomes

$$3y^2 - x + 3 = -x + f'(y) \tag{5}$$

Solving equation (5) for $f'(y)$ gives

$$f'(y) = 3y^2 + 3$$

Integrating the above w.r.t y gives

$$\int f'(y)\,\mathrm{d}y = \int \left(3y^2+3\right)\mathrm{d}y$$
$$f(y) = y^3+3y+c_1$$

Where c_1 is constant of integration. Substituting result found above for $f(y)$ into equation (3) gives ϕ

$$\phi = -x(x+y)+y^3+3y+c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = -x(x+y)+y^3+3y$$

Initial conditions are used to solve for c_1. Substituting $x=0$ and $y=0$ in the above solution gives an equation to solve for the constant of integration.

$$0 = c_1$$

$$c_1 = 0$$

Substituting c_1 found above in the general solution gives

$$-x(x+y)+y^3+3y = 0$$

Summary

The solution(s) found are the following

$$y^3+(3-x)\,y-x^2 = 0 \tag{1}$$

Verification of solutions

$$y^3+(3-x)\,y-x^2 = 0$$

Verified OK.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
trying homogeneous types:
trying Chini
differential order: 1; looking for linear symmetries
trying exact
<- exact successful`
```

✓ Solution by Maple

Time used: 0.078 (sec). Leaf size: 75

```
dsolve([diff(y(x),x) = (2*x+y(x))/(3-x+3*y(x)^2),y(0) = 0],y(x), singsol=all)
```

$$y(x)=\frac{\left(108x^{2}+12\sqrt{81x^{4}-12x^{3}+108x^{2}-324x+324}\right)^{\frac{2}{3}}+12x-36}{6\left(108x^{2}+12\sqrt{81x^{4}-12x^{3}+108x^{2}-324x+324}\right)^{\frac{1}{3}}}$$

✓ Solution by Mathematica

Time used: 5.408 (sec). Leaf size: 114

```
DSolve[{y'[x] == (2*x+y[x])/(3-x+3*y[x]^2),y[0]==0},y[x],x,IncludeSingularSolutions -> True]
```

$$y(x)\to -\frac{\sqrt[3]{2}\left(\sqrt{3}\sqrt{27x^{4}-4x^{3}+36x^{2}-108x+108}-9x^{2}\right)^{2/3}+2\sqrt[3]{3}x-6\sqrt[3]{3}}{6^{2/3}\sqrt[3]{\sqrt{3}\sqrt{27x^{4}-4x^{3}+36x^{2}-108x+108}-9x^{2}}}$$

6.4 problem 4

Internal problem ID [571]
Internal file name [`OUTPUT/571_Sunday_June_05_2022_01_44_58_AM_16995004/index.tex`]

Book: Elementary differential equations and boundary value problems, 10th ed., Boyce and DiPrima
Section: Miscellaneous problems, end of chapter 2. Page 133
Problem number: 4.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"exact", "linear", "separable", "first_order_ode_lie_symmetry_lookup"**

Maple gives the following as the ode type

`[_separable]`

$$\boxed{y' - y + 2yx = -6x + 3}$$

6.4.1 Solving as separable ode

In canonical form the ODE is

$$\begin{aligned} y' &= F(x,y) \\ &= f(x)g(y) \\ &= (2x-1)(-y-3) \end{aligned}$$

Where $f(x) = 2x - 1$ and $g(y) = -y - 3$. Integrating both sides gives

$$\frac{1}{-y-3}\,dy = 2x - 1\,dx$$

$$\int \frac{1}{-y-3}\,dy = \int 2x - 1\,dx$$

$$-\ln(3+y) = x^2 + c_1 - x$$

Raising both side to exponential gives

$$\frac{1}{3+y} = \mathrm{e}^{x^2+c_1-x}$$

Which simplifies to

$$\frac{1}{3+y} = c_2 \mathrm{e}^{x^2-x}$$

Summary
The solution(s) found are the following

$$y = -\frac{\left(3c_2 \mathrm{e}^{x^2+c_1-x} - 1\right) \mathrm{e}^{-x^2-c_1+x}}{c_2} \tag{1}$$

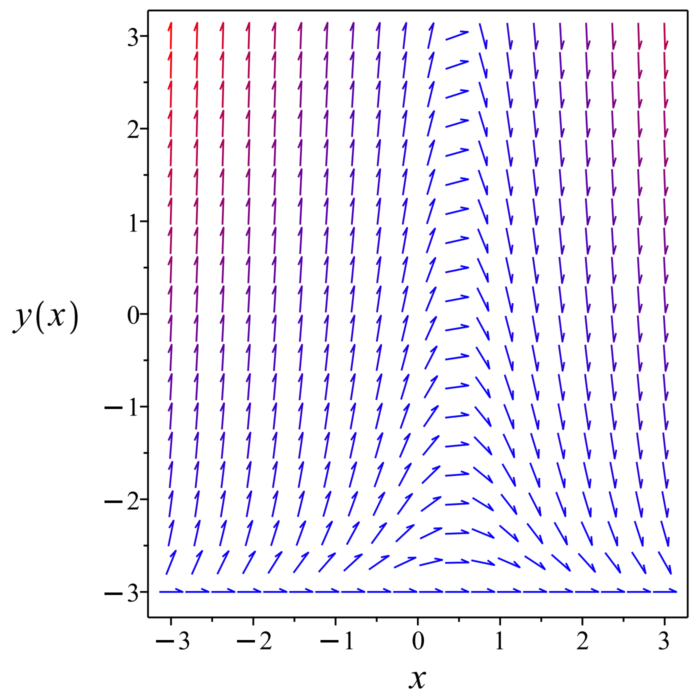


Figure 279: Slope field plot

Verification of solutions

$$y = -\frac{\left(3c_2 \mathrm{e}^{x^2+c_1-x} - 1\right) \mathrm{e}^{-x^2-c_1+x}}{c_2}$$

Verified OK.

6.4.2 Solving as linear ode

Entering Linear first order ODE solver. In canonical form a linear first order is

$$y' + p(x)y = q(x)$$

Where here

$$\begin{aligned}p(x) &= 2x - 1\\ q(x) &= -6x + 3\end{aligned}$$

Hence the ode is

$$y' + y(2x - 1) = -6x + 3$$

The integrating factor μ is

$$\begin{aligned}\mu &= \mathrm{e}^{\int (2x-1)dx}\\ &= \mathrm{e}^{x^2 - x}\end{aligned}$$

Which simplifies to

$$\mu = \mathrm{e}^{x(x-1)}$$

The ode becomes

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}x}(\mu y) &= (\mu)(-6x + 3)\\ \frac{\mathrm{d}}{\mathrm{d}x}\left(\mathrm{e}^{x(x-1)} y\right) &= \left(\mathrm{e}^{x(x-1)}\right)(-6x + 3)\\ \mathrm{d}\left(\mathrm{e}^{x(x-1)} y\right) &= \left((-6x + 3)\,\mathrm{e}^{x(x-1)}\right)\,\mathrm{d}x\end{aligned}$$

Integrating gives

$$\begin{aligned}\mathrm{e}^{x(x-1)} y &= \int (-6x + 3)\,\mathrm{e}^{x(x-1)}\,\mathrm{d}x\\ \mathrm{e}^{x(x-1)} y &= -3\,\mathrm{e}^{x(x-1)} + c_1\end{aligned}$$

Dividing both sides by the integrating factor $\mu = \mathrm{e}^{x(x-1)}$ results in

$$y = -3\,\mathrm{e}^{-x(x-1)}\mathrm{e}^{x(x-1)} + c_1 \mathrm{e}^{-x(x-1)}$$

which simplifies to

$$y = -3 + c_1 \mathrm{e}^{-x(x-1)}$$

Summary
The solution(s) found are the following

$$y = -3 + c_1 e^{-x(x-1)} \tag{1}$$

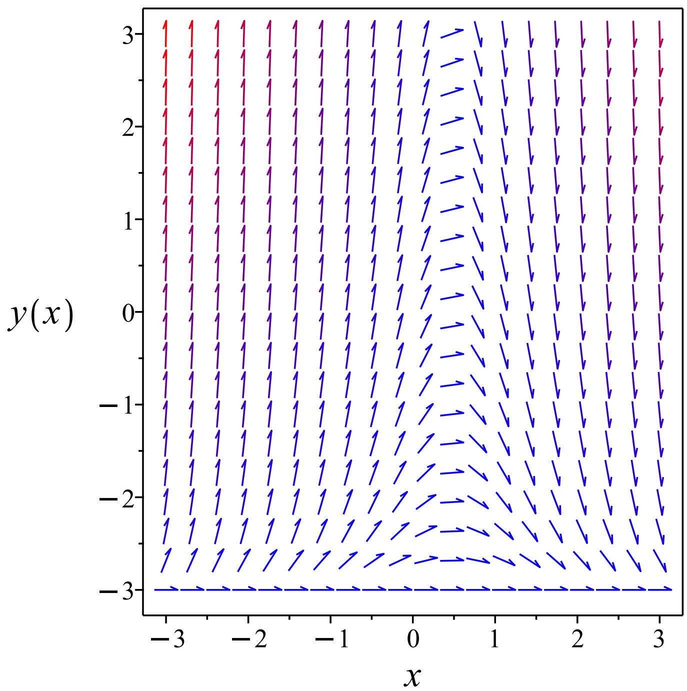


Figure 280: Slope field plot

Verification of solutions

$$y = -3 + c_1 e^{-x(x-1)}$$

Verified OK.

6.4.3 Solving as first order ode lie symmetry lookup ode

Writing the ode as

$$y' = -2yx - 6x + y + 3$$
$$y' = \omega(x, y)$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x + \omega(\eta_y - \xi_x) - \omega^2 \xi_y - \omega_x \xi - \omega_y \eta = 0 \tag{A}$$

The type of this ode is known. It is of type `linear`. Therefore we do not need to solve the PDE (A), and can just use the lookup table shown below to find ξ, η

Table 266: Lie symmetry infinitesimal lookup table for known first order ODE's

ODE class	Form	ξ	η
linear ode	$y' = f(x)y(x) + g(x)$	0	$e^{\int f dx}$
separable ode	$y' = f(x)\, g(y)$	$\frac{1}{f}$	0
quadrature ode	$y' = f(x)$	0	1
quadrature ode	$y' = g(y)$	1	0
homogeneous ODEs of Class A	$y' = f\left(\frac{y}{x}\right)$	x	y
homogeneous ODEs of Class C	$y' = (a + bx + cy)^{\frac{n}{m}}$	1	$-\frac{b}{c}$
homogeneous class D	$y' = \frac{y}{x} + g(x)\, F\left(\frac{y}{x}\right)$	x^2	xy
First order special form ID 1	$y' = g(x)\, e^{h(x)+by} + f(x)$	$\frac{e^{-\int bf(x)dx - h(x)}}{g(x)}$	$\frac{f(x)e^{-\int bf(x)dx - h(x)}}{g(x)}$
polynomial type ode	$y' = \frac{a_1x+b_1y+c_1}{a_2x+b_2y+c_2}$	$\frac{a_1b_2x - a_2b_1x - b_1c_2 + b_2c_1}{a_1b_2 - a_2b_1}$	$\frac{a_1b_2y - a_2b_1y - a_1c_2 - a_2c_1}{a_1b_2 - a_2b_1}$
Bernoulli ode	$y' = f(x)\, y + g(x)\, y^n$	0	$e^{-\int (n-1)f(x)dx} y^n$
Reduced Riccati	$y' = f_1(x)\, y + f_2(x)\, y^2$	0	$e^{-\int f_1 dx}$

The above table shows that

$$
\begin{aligned}
\xi(x, y) &= 0 \\
\eta(x, y) &= \mathrm{e}^{-x^2+x}
\end{aligned} \tag{A1}
$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x, y) \to (R, S)$ where (R, S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$
\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}
$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y}\right) S(x, y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the

canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$R = x$$

S is found from

$$\begin{aligned} S &= \int \frac{1}{\eta} dy \\ &= \int \frac{1}{\mathrm{e}^{-x^2+x}} dy \end{aligned}$$

Which results in

$$S = \mathrm{e}^{x^2-x} y$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x,y) S_y}{R_x + \omega(x,y) R_y} \tag{2}$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x,y)$ is the right hand side of the original ode given by

$$\omega(x,y) = -2yx - 6x + y + 3$$

Evaluating all the partial derivatives gives

$$\begin{aligned} R_x &= 1 \\ R_y &= 0 \\ S_x &= (2x-1)\,\mathrm{e}^{x(x-1)} y \\ S_y &= \mathrm{e}^{x(x-1)} \end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = (-6x+3)\,\mathrm{e}^{x(x-1)} \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = (-6R+3)\,\mathrm{e}^{R(R-1)}$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by

integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$S(R) = -3\,\mathrm{e}^{R(R-1)} + c_1 \tag{4}$$

To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$\mathrm{e}^{x(x-1)} y = -3\,\mathrm{e}^{x(x-1)} + c_1$$

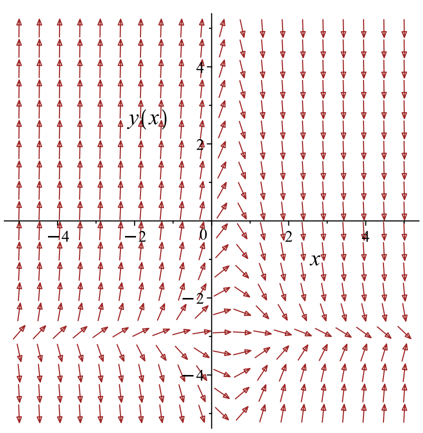
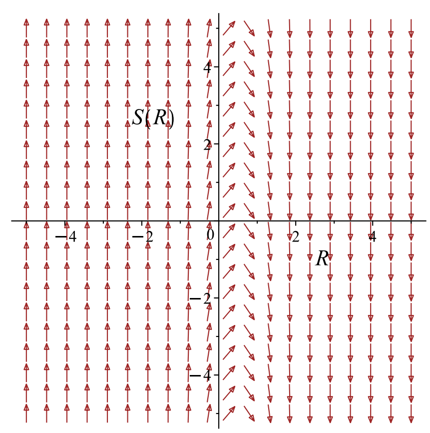
Which simplifies to

$$(3 + y)\,\mathrm{e}^{x(x-1)} - c_1 = 0$$

Which gives

$$y = -\left(3\,\mathrm{e}^{x(x-1)} - c_1\right) \mathrm{e}^{-x(x-1)}$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in x, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dx} = -2yx - 6x + y + 3$ 	$R = x$ $S = \mathrm{e}^{x(x-1)} y$	$\frac{dS}{dR} = (-6R + 3)\,\mathrm{e}^{R(R-1)}$ 

Summary
The solution(s) found are the following

$$y = -\left(3\,\mathrm{e}^{x(x-1)} - c_1\right) \mathrm{e}^{-x(x-1)} \tag{1}$$

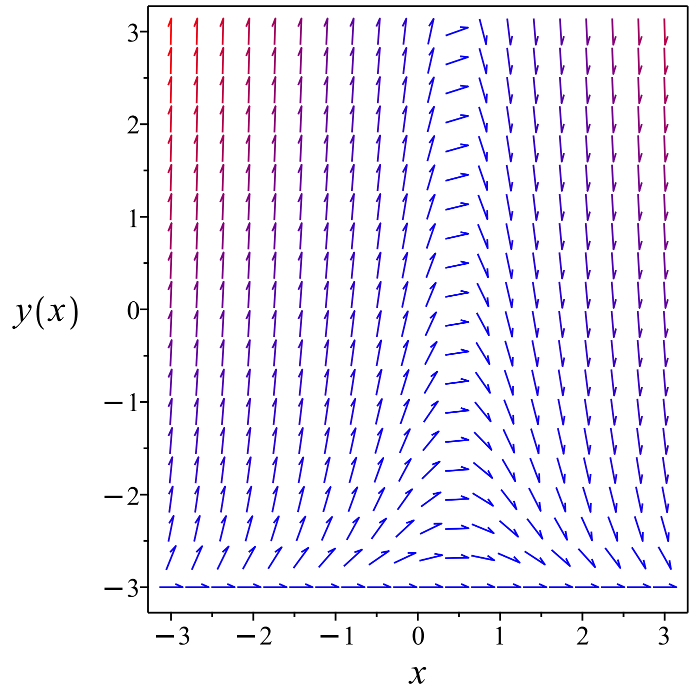


Figure 281: Slope field plot

Verification of solutions

$$y = -\left(3\,\mathrm{e}^{x(x-1)} - c_1\right)\mathrm{e}^{-x(x-1)}$$

Verified OK.

6.4.4 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x,y) + N(x,y)\,\frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x,y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y) = 0$$

Hence

$$\frac{\partial\phi}{\partial x} + \frac{\partial\phi}{\partial y}\frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$
\begin{aligned}
\frac{\partial \phi}{\partial x} &= M \\
\frac{\partial \phi}{\partial y} &= N
\end{aligned}
$$

But since $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ then for the above to be valid, we require that

$$
\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}
$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x, y)$ but at least we know now that we can do that since the condition $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$
M(x, y)\, \mathrm{d}x + N(x, y)\, \mathrm{d}y = 0 \tag{1A}
$$

Therefore

$$
\begin{aligned}
\left(\frac{1}{-y-3}\right) \mathrm{d}y &= (2x - 1)\, \mathrm{d}x \\
(1 - 2x)\, \mathrm{d}x + \left(\frac{1}{-y-3}\right) \mathrm{d}y &= 0 \qquad (2A)
\end{aligned}
$$

Comparing (1A) and (2A) shows that

$$
\begin{aligned}
M(x, y) &= 1 - 2x \\
N(x, y) &= \frac{1}{-y-3}
\end{aligned}
$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$
\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}
$$

Using result found above gives

$$
\begin{aligned}
\frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}(1 - 2x) \\
&= 0
\end{aligned}
$$

And

$$
\begin{aligned}
\frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}\left(\frac{1}{-y-3}\right) \\
&= 0
\end{aligned}
$$

Since $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$, then the ODE is exact The following equations are now set up to solve for the function $\phi(x, y)$

$$
\begin{aligned}
\frac{\partial \phi}{\partial x} &= M \qquad (1) \\
\frac{\partial \phi}{\partial y} &= N \qquad (2)
\end{aligned}
$$

Integrating (1) w.r.t. x gives

$$
\begin{aligned}
\int \frac{\partial \phi}{\partial x} \, \mathrm{d}x &= \int M \, \mathrm{d}x \\
\int \frac{\partial \phi}{\partial x} \, \mathrm{d}x &= \int 1 - 2x \, \mathrm{d}x \\
\phi &= -x^2 + x + f(y) \qquad (3)
\end{aligned}
$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$
\frac{\partial \phi}{\partial y} = 0 + f'(y) \qquad (4)
$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = \frac{1}{-y-3}$. Therefore equation (4) becomes

$$
\frac{1}{-y-3} = 0 + f'(y) \qquad (5)
$$

Solving equation (5) for $f'(y)$ gives

$$
f'(y) = -\frac{1}{3+y}
$$

Integrating the above w.r.t y gives

$$
\begin{aligned}
\int f'(y) \, \mathrm{d}y &= \int \left(-\frac{1}{3+y}\right) \mathrm{d}y \\
f(y) &= -\ln(3+y) + c_1
\end{aligned}
$$

Where c_1 is constant of integration. Substituting result found above for $f(y)$ into equation (3) gives ϕ

$$\phi = -x^2 + x - \ln(3+y) + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = -x^2 + x - \ln(3+y)$$

The solution becomes

$$y = e^{-x^2-c_1+x} - 3$$

Summary

The solution(s) found are the following

$$y = e^{-x^2-c_1+x} - 3 \tag{1}$$

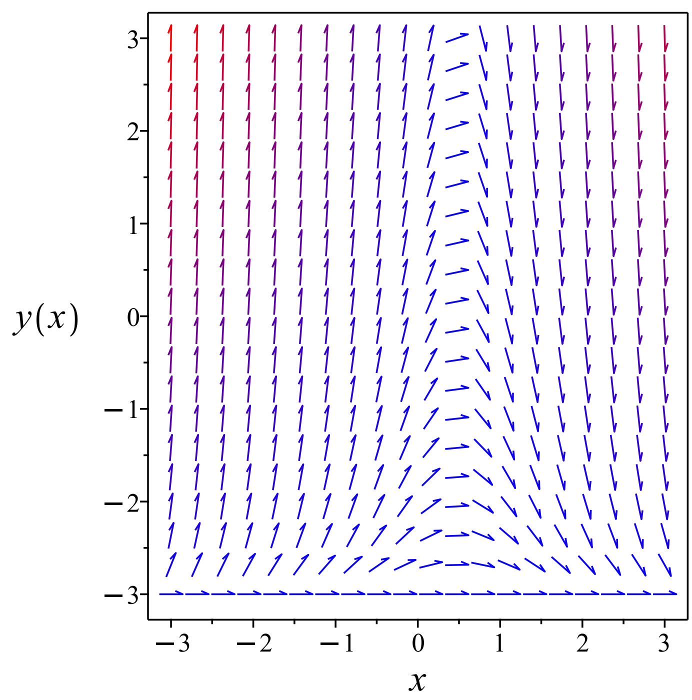


Figure 282: Slope field plot

Verification of solutions

$$y = e^{-x^2-c_1+x} - 3$$

Verified OK.

6.4.5 Maple step by step solution

Let's solve

$y' - y + 2yx = -6x + 3$

- Highest derivative means the order of the ODE is 1

  y'

- Separate variables

  $\frac{y'}{3+y} = 1 - 2x$

- Integrate both sides with respect to x

  $\int \frac{y'}{3+y}dx = \int (1 - 2x)\,dx + c_1$

- Evaluate integral

  $\ln(3 + y) = -x^2 + c_1 + x$

- Solve for y

  $y = \mathrm{e}^{-x^2+c_1+x} - 3$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
<- 1st order linear successful`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 15

```
dsolve(diff(y(x),x) = 3-6*x+y(x)-2*x*y(x),y(x), singsol=all)
```

$$y(x) = -3 + \mathrm{e}^{-x(x-1)}c_1$$

✓ Solution by Mathematica

Time used: 0.048 (sec). Leaf size: 24

```
DSolve[y'[x] == 3-6*x+y[x]-2*x*y[x],y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to -3 + c_1 e^{x-x^2}$$
$$y(x) \to -3$$

6.5 problem 5

Internal problem ID [572]
Internal file name [`OUTPUT/572_Sunday_June_05_2022_01_44_59_AM_84609071/index.tex`]

Book: Elementary differential equations and boundary value problems, 10th ed., Boyce and DiPrima
Section: Miscellaneous problems, end of chapter 2. Page 133
Problem number: 5.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"exact"**

Maple gives the following as the ode type

`[_rational, [_Abel, `2nd type`, `class B`]]`

$$\boxed{y' - \frac{-1 - 2yx - y^2}{x^2 + 2yx} = 0}$$

6.5.1 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x,y) + N(x,y)\frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x,y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y) = 0$$

Hence

$$\frac{\partial \phi}{\partial x} + \frac{\partial \phi}{\partial y}\frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$\frac{\partial \phi}{\partial x} = M$$
$$\frac{\partial \phi}{\partial y} = N$$

But since $\frac{\partial^2\phi}{\partial x\partial y} = \frac{\partial^2\phi}{\partial y\partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x, y)$ but at least we know now that we can do that since the condition $\frac{\partial^2\phi}{\partial x\partial y} = \frac{\partial^2\phi}{\partial y\partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x,y)\,\mathrm{d}x + N(x,y)\,\mathrm{d}y = 0 \tag{1A}$$

Therefore

$$\begin{aligned}(x(x+2y))\,\mathrm{d}y &= \left(-2yx - y^2 - 1\right)\mathrm{d}x\\ \left(2yx + y^2 + 1\right)\mathrm{d}x + (x(x+2y))\,\mathrm{d}y &= 0 \qquad (2\mathrm{A})\end{aligned}$$

Comparing (1A) and (2A) shows that

$$\begin{aligned}M(x,y) &= 2yx + y^2 + 1\\ N(x,y) &= x(x+2y)\end{aligned}$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Using result found above gives

$$\begin{aligned}\frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}\left(2yx + y^2 + 1\right)\\ &= 2x + 2y\end{aligned}$$

And

$$\begin{aligned}\frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}(x(x+2y))\\ &= 2x + 2y\end{aligned}$$

Since $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$, then the ODE is <u>exact</u> The following equations are now set up to solve for the function $\phi(x, y)$

$$\frac{\partial\phi}{\partial x} = M \tag{1}$$

$$\frac{\partial\phi}{\partial y} = N \tag{2}$$

Integrating (1) w.r.t. x gives

$$\int \frac{\partial \phi}{\partial x} \,\mathrm{d}x = \int M \,\mathrm{d}x$$
$$\int \frac{\partial \phi}{\partial x} \,\mathrm{d}x = \int 2yx + y^2 + 1 \,\mathrm{d}x$$
$$\phi = y\,x^2 + x\,y^2 + x + f(y) \tag{3}$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$\frac{\partial \phi}{\partial y} = x^2 + 2yx + f'(y) \tag{4}$$
$$= x(x + 2y) + f'(y)$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = x(x + 2y)$. Therefore equation (4) becomes

$$x(x + 2y) = x(x + 2y) + f'(y) \tag{5}$$

Solving equation (5) for $f'(y)$ gives

$$f'(y) = 0$$

Therefore

$$f(y) = c_1$$

Where c_1 is constant of integration. Substituting this result for $f(y)$ into equation (3) gives ϕ

$$\phi = y\,x^2 + x\,y^2 + x + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = y\,x^2 + x\,y^2 + x$$

Summary

The solution(s) found are the following

$$x^2y + xy^2 + x = c_1 \tag{1}$$

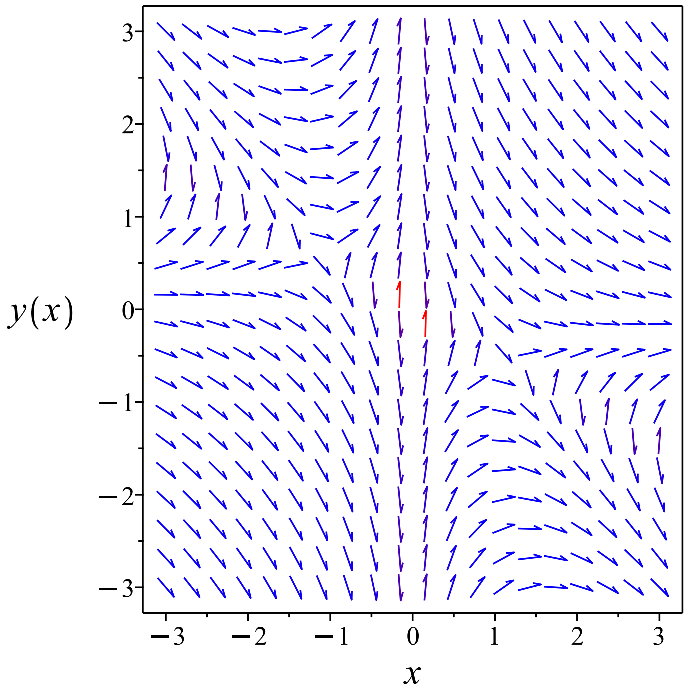


Figure 283: Slope field plot

Verification of solutions

$$x^2y + xy^2 + x = c_1$$

Verified OK.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
trying homogeneous types:
trying Chini
differential order: 1; looking for linear symmetries
trying exact
<- exact successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 59

```
dsolve(diff(y(x),x) = (-1-2*x*y(x)-y(x)^2)/(x^2+2*x*y(x)),y(x), singsol=all)
```

$$y(x) = \frac{-x^2 + \sqrt{x\,(x^3 - 4c_1 - 4x)}}{2x}$$
$$y(x) = \frac{-x^2 - \sqrt{x\,(x^3 - 4c_1 - 4x)}}{2x}$$

✓ Solution by Mathematica

Time used: 0.502 (sec). Leaf size: 67

```
DSolve[y'[x] == (-1-2*x*y[x]-y[x]^2)/(x^2+2*x*y[x]),y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to -\frac{x^2 + \sqrt{x\,(x^3 - 4x + 4c_1)}}{2x}$$
$$y(x) \to \frac{-x^2 + \sqrt{x\,(x^3 - 4x + 4c_1)}}{2x}$$

6.6 problem 6

Internal problem ID [573]
Internal file name [`OUTPUT/573_Sunday_June_05_2022_01_45_00_AM_94379850/index.tex`]

Book: Elementary differential equations and boundary value problems, 10th ed., Boyce and DiPrima
Section: Miscellaneous problems, end of chapter 2. Page 133
Problem number: 6.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"linear", "exactWithIntegrationFactor", "first_order_ode_lie_symmetry_lookup"**

Maple gives the following as the ode type

`[_linear]`

$$\boxed{yx + y'x + y = 1}$$

With initial conditions

$$[y(1) = 0]$$

6.6.1 Existence and uniqueness analysis

This is a linear ODE. In canonical form it is written as

$$y' + p(x)y = q(x)$$

Where here

$$p(x) = -\frac{-x-1}{x}$$
$$q(x) = \frac{1}{x}$$

Hence the ode is

$$y' - \frac{(-x-1)\,y}{x} = \frac{1}{x}$$

The domain of $p(x) = -\frac{-x-1}{x}$ is

$$\{x < 0 \lor 0 < x\}$$

And the point $x_0 = 1$ is inside this domain. The domain of $q(x) = \frac{1}{x}$ is

$$\{x < 0 \lor 0 < x\}$$

And the point $x_0 = 1$ is also inside this domain. Hence solution exists and is unique.

6.6.2 Solving as linear ode

Entering Linear first order ODE solver. The integrating factor μ is

$$\begin{aligned}\mu &= \mathrm{e}^{\int -\frac{-x-1}{x}dx} \\ &= \mathrm{e}^{x+\ln(x)}\end{aligned}$$

Which simplifies to

$$\mu = x\,\mathrm{e}^{x}$$

The ode becomes

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}x}(\mu y) &= (\mu)\left(\frac{1}{x}\right) \\ \frac{\mathrm{d}}{\mathrm{d}x}(x\,\mathrm{e}^{x} y) &= (x\,\mathrm{e}^{x})\left(\frac{1}{x}\right) \\ \mathrm{d}(x\,\mathrm{e}^{x} y) &= \mathrm{e}^{x}\,\mathrm{d}x\end{aligned}$$

Integrating gives

$$\begin{aligned}x\,\mathrm{e}^{x} y &= \int \mathrm{e}^{x}\,\mathrm{d}x \\ x\,\mathrm{e}^{x} y &= \mathrm{e}^{x} + c_1\end{aligned}$$

Dividing both sides by the integrating factor $\mu = x\,\mathrm{e}^{x}$ results in

$$y = \frac{\mathrm{e}^{-x}\mathrm{e}^{x}}{x} + \frac{c_1\mathrm{e}^{-x}}{x}$$

which simplifies to

$$y = \frac{c_1 e^{-x} + 1}{x}$$

Initial conditions are used to solve for c_1. Substituting $x = 1$ and $y = 0$ in the above solution gives an equation to solve for the constant of integration.

$$0 = e^{-1} c_1 + 1$$

$$c_1 = -e$$

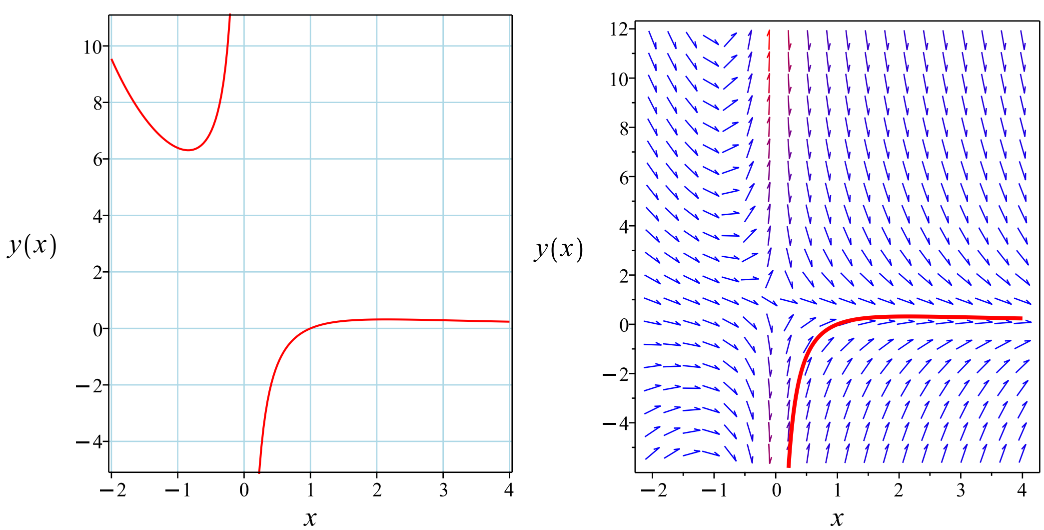
Substituting c_1 found above in the general solution gives

$$y = \frac{-e^{1-x} + 1}{x}$$

Summary

The solution(s) found are the following

$$y = \frac{-e^{1-x} + 1}{x} \tag{1}$$



(a) Solution plot

(b) Slope field plot

Verification of solutions

$$y = \frac{-e^{1-x} + 1}{x}$$

Verified OK.

6.6.3 Solving as first order ode lie symmetry lookup ode

Writing the ode as

$$
\begin{aligned}
y' &= -\frac{yx + y - 1}{x} \\
y' &= \omega(x, y)
\end{aligned}
$$

The condition of Lie symmetry is the linearized PDE given by

$$
\eta_x + \omega(\eta_y - \xi_x) - \omega^2 \xi_y - \omega_x \xi - \omega_y \eta = 0 \tag{A}
$$

The type of this ode is known. It is of type `linear`. Therefore we do not need to solve the PDE (A), and can just use the lookup table shown below to find ξ, η

Table 269: Lie symmetry infinitesimal lookup table for known first order ODE's

ODE class	Form	ξ	η
linear ode	$y' = f(x)y(x) + g(x)$	0	$e^{\int f dx}$
separable ode	$y' = f(x)\, g(y)$	$\frac{1}{f}$	0
quadrature ode	$y' = f(x)$	0	1
quadrature ode	$y' = g(y)$	1	0
homogeneous ODEs of Class A	$y' = f\left(\frac{y}{x}\right)$	x	y
homogeneous ODEs of Class C	$y' = (a + bx + cy)^{\frac{n}{m}}$	1	$-\frac{b}{c}$
homogeneous class D	$y' = \frac{y}{x} + g(x)\, F\left(\frac{y}{x}\right)$	x^2	xy
First order special form ID 1	$y' = g(x)\, e^{h(x)+by} + f(x)$	$\frac{e^{-\int bf(x)dx - h(x)}}{g(x)}$	$\frac{f(x)e^{-\int bf(x)dx - h(x)}}{g(x)}$
polynomial type ode	$y' = \frac{a_1 x + b_1 y + c_1}{a_2 x + b_2 y + c_2}$	$\frac{a_1 b_2 x - a_2 b_1 x - b_1 c_2 + b_2 c_1}{a_1 b_2 - a_2 b_1}$	$\frac{a_1 b_2 y - a_2 b_1 y - a_1 c_2 - a_2 c_1}{a_1 b_2 - a_2 b_1}$
Bernoulli ode	$y' = f(x)\, y + g(x)\, y^n$	0	$e^{-\int (n-1) f(x) dx} y^n$
Reduced Riccati	$y' = f_1(x)\, y + f_2(x)\, y^2$	0	$e^{-\int f_1 dx}$

The above table shows that

$$\begin{aligned} \xi(x,y) &= 0 \\ \eta(x,y) &= \mathrm{e}^{-x-\ln(x)} \end{aligned} \tag{A1}$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x,y) \to (R,S)$ where (R,S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y}\right) S(x,y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$R = x$$

S is found from

$$\begin{aligned} S &= \int \frac{1}{\eta} dy \\ &= \int \frac{1}{\mathrm{e}^{-x-\ln(x)}} dy \end{aligned}$$

Which results in

$$S = x\,\mathrm{e}^{x} y$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x,y) S_y}{R_x + \omega(x,y) R_y} \tag{2}$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x,y)$ is the right hand side of the original ode given by

$$\omega(x,y) = -\frac{yx + y - 1}{x}$$

Evaluating all the partial derivatives gives

$$\begin{aligned} R_x &= 1 \\ R_y &= 0 \\ S_x &= \mathrm{e}^{x} y(x+1) \\ S_y &= x\,\mathrm{e}^{x} \end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = \mathrm{e}^{x} \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = \mathrm{e}^{R}$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$S(R) = \mathrm{e}^{R} + c_1 \tag{4}$$

To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$y\,\mathrm{e}^{x} x = \mathrm{e}^{x} + c_1$$

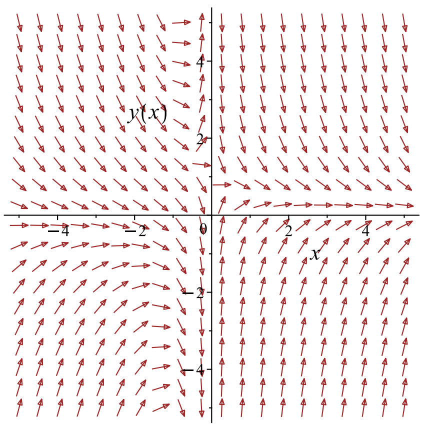
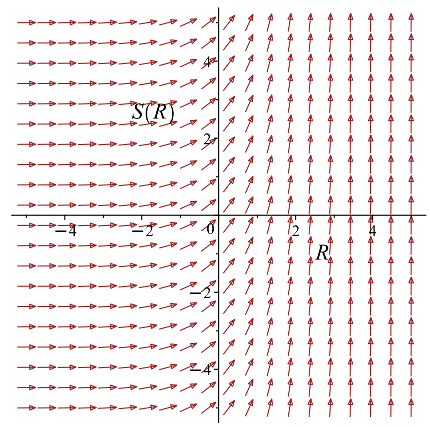
Which simplifies to

$$y\,\mathrm{e}^{x} x = \mathrm{e}^{x} + c_1$$

Which gives

$$y = \frac{(\mathrm{e}^{x} + c_1)\,\mathrm{e}^{-x}}{x}$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in x, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dx} = -\frac{yx+y-1}{x}$ 	$R = x$ $S = x\,\mathrm{e}^{x} y$	$\frac{dS}{dR} = \mathrm{e}^{R}$ 

Initial conditions are used to solve for c_1. Substituting $x = 1$ and $y = 0$ in the above solution gives an equation to solve for the constant of integration.

$$0 = \mathrm{e}^{-1} c_1 + 1$$

$$c_1 = -\mathrm{e}$$

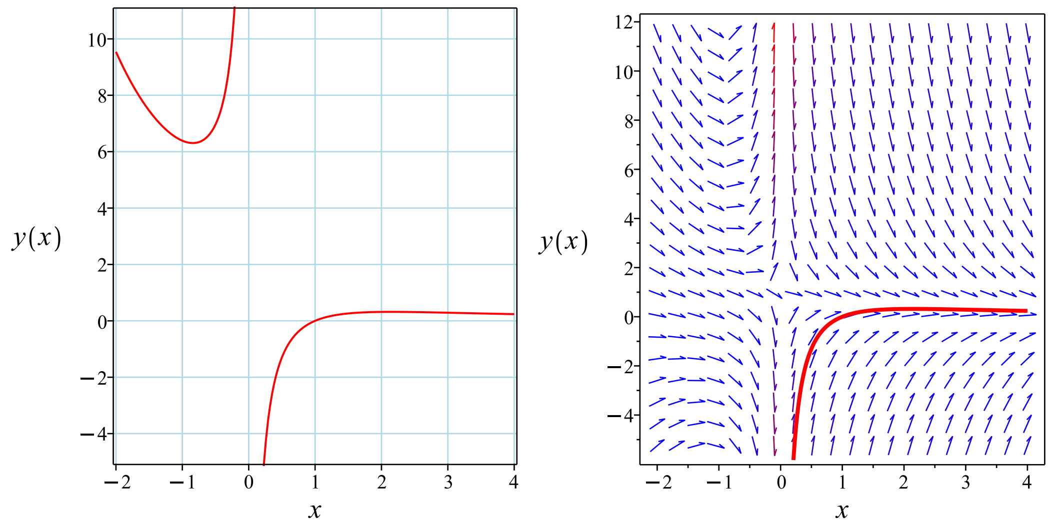
Substituting c_1 found above in the general solution gives

$$y = \frac{\mathrm{e}^{-x}\mathrm{e}^{x} - \mathrm{e}\,\mathrm{e}^{-x}}{x}$$

Summary

The solution(s) found are the following

$$y = \frac{\mathrm{e}^{-x}\mathrm{e}^{x} - \mathrm{e}\,\mathrm{e}^{-x}}{x} \tag{1}$$



(a) Solution plot

(b) Slope field plot

Verification of solutions

$$y = \frac{e^{-x} e^{x} - e\, e^{-x}}{x}$$

Verified OK.

6.6.4 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x, y) + N(x, y) \frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x, y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx} \phi(x, y) = 0$$

Hence

$$\frac{\partial \phi}{\partial x} + \frac{\partial \phi}{\partial y} \frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$\frac{\partial \phi}{\partial x} = M$$
$$\frac{\partial \phi}{\partial y} = N$$

But since $\frac{\partial^2\phi}{\partial x\partial y} = \frac{\partial^2\phi}{\partial y\partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x,y)$ but at least we know now that we can do that since the condition $\frac{\partial^2\phi}{\partial x\partial y} = \frac{\partial^2\phi}{\partial y\partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x,y)\,\mathrm{d}x + N(x,y)\,\mathrm{d}y = 0 \tag{1A}$$

Therefore

$$\begin{aligned}(x)\,\mathrm{d}y &= (-yx - y + 1)\,\mathrm{d}x\\ (yx + y - 1)\,\mathrm{d}x + (x)\,\mathrm{d}y &= 0 \qquad (2A)\end{aligned}$$

Comparing (1A) and (2A) shows that

$$\begin{aligned}M(x,y) &= yx + y - 1\\ N(x,y) &= x\end{aligned}$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Using result found above gives

$$\begin{aligned}\frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}(yx + y - 1)\\ &= x + 1\end{aligned}$$

And

$$\begin{aligned}\frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}(x)\\ &= 1\end{aligned}$$

Since $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$, then the ODE is <u>not exact</u>. Since the ODE is not exact, we will try to find an integrating factor to make it exact. Let

$$\begin{aligned}A &= \frac{1}{N}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right)\\ &= \frac{1}{x}((x+1) - (1))\\ &= 1\end{aligned}$$

Since A does not depend on y, then it can be used to find an integrating factor. The integrating factor μ is

$$\begin{aligned}\mu &= e^{\int A \, \mathrm{d}x} \\ &= e^{\int 1 \, \mathrm{d}x}\end{aligned}$$

The result of integrating gives

$$\begin{aligned}\mu &= e^{x} \\ &= \mathrm{e}^{x}\end{aligned}$$

M and N are multiplied by this integrating factor, giving new M and new N which are called $\overline{M}$ and $\overline{N}$ for now so not to confuse them with the original M and N.

$$\begin{aligned}\overline{M} &= \mu M \\ &= \mathrm{e}^{x}(yx + y - 1) \\ &= \mathrm{e}^{x}(yx + y - 1)\end{aligned}$$

And

$$\begin{aligned}\overline{N} &= \mu N \\ &= \mathrm{e}^{x}(x) \\ &= x\,\mathrm{e}^{x}\end{aligned}$$

Now a modified ODE is ontained from the original ODE, which is exact and can be solved. The modified ODE is

$$\begin{aligned}\overline{M} + \overline{N}\frac{\mathrm{d}y}{\mathrm{d}x} &= 0 \\ \left(\mathrm{e}^{x}(yx + y - 1)\right) + \left(x\,\mathrm{e}^{x}\right)\frac{\mathrm{d}y}{\mathrm{d}x} &= 0\end{aligned}$$

The following equations are now set up to solve for the function $\phi(x, y)$

$$\frac{\partial \phi}{\partial x} = \overline{M} \tag{1}$$

$$\frac{\partial \phi}{\partial y} = \overline{N} \tag{2}$$

Integrating (1) w.r.t. x gives

$$\begin{aligned}\int \frac{\partial \phi}{\partial x} \, \mathrm{d}x &= \int \overline{M} \, \mathrm{d}x \\ \int \frac{\partial \phi}{\partial x} \, \mathrm{d}x &= \int \mathrm{e}^{x}(yx + y - 1) \, \mathrm{d}x\end{aligned}$$

$$\phi = (yx - 1)\,\mathrm{e}^{x} + f(y) \tag{3}$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$\frac{\partial \phi}{\partial y} = x\,\mathrm{e}^{x} + f'(y) \tag{4}$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = x\,\mathrm{e}^{x}$. Therefore equation (4) becomes

$$x\,\mathrm{e}^{x} = x\,\mathrm{e}^{x} + f'(y) \tag{5}$$

Solving equation (5) for $f'(y)$ gives

$$f'(y) = 0$$

Therefore

$$f(y) = c_1$$

Where c_1 is constant of integration. Substituting this result for $f(y)$ into equation (3) gives ϕ

$$\phi = (yx - 1)\,\mathrm{e}^{x} + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = (yx - 1)\,\mathrm{e}^{x}$$

The solution becomes

$$y = \frac{(\mathrm{e}^{x} + c_1)\,\mathrm{e}^{-x}}{x}$$

Initial conditions are used to solve for c_1. Substituting $x = 1$ and $y = 0$ in the above solution gives an equation to solve for the constant of integration.

$$0 = \mathrm{e}^{-1}c_1 + 1$$

$$c_1 = -\mathrm{e}$$

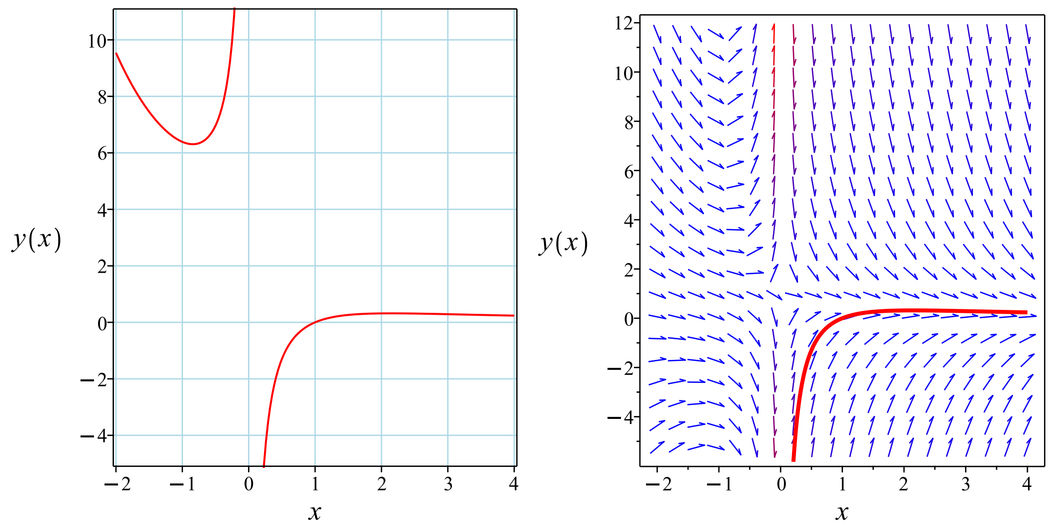
Substituting c_1 found above in the general solution gives

$$y = \frac{\mathrm{e}^{-x}\mathrm{e}^{x} - \mathrm{e}\,\mathrm{e}^{-x}}{x}$$

Summary

The solution(s) found are the following

$$y = \frac{e^{-x}e^{x} - e\,e^{-x}}{x} \tag{1}$$



(a) Solution plot

(b) Slope field plot

Verification of solutions

$$y = \frac{e^{-x}e^{x} - e\,e^{-x}}{x}$$

Verified OK.

6.6.5 Maple step by step solution

Let's solve

$[yx + y'x + y = 1, y(1) = 0]$

- Highest derivative means the order of the ODE is 1

  y'

- Isolate the derivative

  $y' = -\frac{(x+1)y}{x} + \frac{1}{x}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE

  $y' + \frac{(x+1)y}{x} = \frac{1}{x}$

- The ODE is linear; multiply by an integrating factor $\mu(x)$

  $\mu(x)\left(y'+\frac{(x+1)y}{x}\right)=\frac{\mu(x)}{x}$
- Assume the lhs of the ODE is the total derivative $\frac{d}{dx}(\mu(x)\,y)$

  $\mu(x)\left(y'+\frac{(x+1)y}{x}\right)=\mu'(x)\,y+\mu(x)\,y'$
- Isolate $\mu'(x)$

  $\mu'(x)=\frac{\mu(x)(x+1)}{x}$
- Solve to find the integrating factor

  $\mu(x)=x\,\mathrm{e}^{x}$
- Integrate both sides with respect to x

  $\int\left(\frac{d}{dx}(\mu(x)\,y)\right)dx=\int\frac{\mu(x)}{x}dx+c_1$
- Evaluate the integral on the lhs

  $\mu(x)\,y=\int\frac{\mu(x)}{x}dx+c_1$
- Solve for y

  $y=\frac{\int\frac{\mu(x)}{x}dx+c_1}{\mu(x)}$
- Substitute $\mu(x)=x\mathrm{e}^{x}$

  $y=\frac{\int\mathrm{e}^{x}dx+c_1}{x\,\mathrm{e}^{x}}$
- Evaluate the integrals on the rhs

  $y=\frac{\mathrm{e}^{x}+c_1}{\mathrm{e}^{x}x}$
- Simplify

  $y=\frac{c_1\mathrm{e}^{-x}+1}{x}$
- Use initial condition $y(1)=0$

  $0=\mathrm{e}^{-1}c_1+1$
- Solve for c_1

  $c_1=-\frac{1}{\mathrm{e}^{-1}}$
- Substitute $c_1=-\frac{1}{\mathrm{e}^{-1}}$ into general solution and simplify

  $y=\frac{-\mathrm{e}^{1-x}+1}{x}$
- Solution to the IVP

  $y=\frac{-\mathrm{e}^{1-x}+1}{x}$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
<- 1st order linear successful`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 18

```
dsolve([x*y(x)+x*diff(y(x),x) = 1-y(x),y(1) = 0],y(x), singsol=all)
```

$$y(x) = \frac{1 - \mathrm{e}^{1-x}}{x}$$

✓ Solution by Mathematica

Time used: 0.03 (sec). Leaf size: 20

```
DSolve[{x*y[x]+x*y'[x] == 1-y[x],y[1]==0},y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \frac{1 - e^{1-x}}{x}$$

6.7 problem 7

Internal problem ID [574]
Internal file name [`OUTPUT/574_Sunday_June_05_2022_01_45_01_AM_69403732/index.tex`]

Book: Elementary differential equations and boundary value problems, 10th ed., Boyce and DiPrima
Section: Miscellaneous problems, end of chapter 2. Page 133
Problem number: 7.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"exact", "separable", "differentialType", "first_order_ode_lie_symmetry_lookup"**

Maple gives the following as the ode type

`[_separable]`

$$\boxed{y' - \frac{4x^3 + 1}{y(2 + 3y)} = 0}$$

6.7.1 Solving as separable ode

In canonical form the ODE is

$$\begin{aligned} y' &= F(x, y) \\ &= f(x)g(y) \\ &= \frac{4x^3 + 1}{y(2 + 3y)} \end{aligned}$$

Where $f(x) = 4x^3 + 1$ and $g(y) = \frac{1}{y(2+3y)}$. Integrating both sides gives

$$\frac{1}{\frac{1}{y(2+3y)}}\, dy = 4x^3 + 1\, dx$$

$$\int \frac{1}{\frac{1}{y(2+3y)}}\,dy = \int 4x^3+1\,dx$$

$$y^3+y^2 = x^4+c_1+x$$

Which results in

$$y = \frac{\left(-8+108x^4+108c_1+108x+12\sqrt{81x^8+162c_1x^4+162x^5-12x^4+81c_1^2+162c_1x+81x^2-12c_1-\cdots}\right.}{6} + \frac{2}{3\left(-8+108x^4+108c_1+108x+12\sqrt{81x^8+162c_1x^4+162x^5-12x^4+81c_1^2+162c_1x+81x^2-12c\cdots}\right.} - \frac{1}{3}$$

$$y = -\frac{\left(-8+108x^4+108c_1+108x+12\sqrt{81x^8+162c_1x^4+162x^5-12x^4+81c_1^2+162c_1x+81x^2-12c_1\cdots}\right.}{12} - \frac{1}{3\left(-8+108x^4+108c_1+108x+12\sqrt{81x^8+162c_1x^4+162x^5-12x^4+81c_1^2+162c_1x+81x^2-12c\cdots}\right.} - \frac{1}{3} + \frac{i\sqrt{3}\left(\frac{\left(-8+108x^4+108c_1+108x+12\sqrt{81x^8+162c_1x^4+162x^5-12x^4+81c_1^2+162c_1x+81x^2-12c_1-12x}\right)^{\frac{1}{3}}}{6} - \frac{\cdots}{3\left(-8+108x^4+108c_1+10\cdots\right.}\right.}{2}$$

$$y = -\frac{\left(-8+108x^4+108c_1+108x+12\sqrt{81x^8+162c_1x^4+162x^5-12x^4+81c_1^2+162c_1x+81x^2-12c_1\cdots}\right.}{12} - \frac{1}{3\left(-8+108x^4+108c_1+108x+12\sqrt{81x^8+162c_1x^4+162x^5-12x^4+81c_1^2+162c_1x+81x^2-12c\cdots}\right.} - \frac{1}{3} - \frac{i\sqrt{3}\left(\frac{\left(-8+108x^4+108c_1+108x+12\sqrt{81x^8+162c_1x^4+162x^5-12x^4+81c_1^2+162c_1x+81x^2-12c_1-12x}\right)^{\frac{1}{3}}}{6} - \frac{\cdots}{3\left(-8+108x^4+108c_1+10\cdots\right.}\right.}{2}$$

Summary

The solution(s) found are the following

$$y = \frac{\left(-8+108x^4+108c_1+108x+12\sqrt{81x^8+162c_1x^4+162x^5-12x^4+81c_1^2+162c_1x+81x^2-12c_1-}\right.}{6} + \frac{2}{3\left(-8+108x^4+108c_1+108x+12\sqrt{81x^8+162c_1x^4+162x^5-12x^4+81c_1^2+162c_1x+81x^2-12c\right.} - \frac{1}{3} \tag{1}$$

$$y = -\frac{\left(-8+108x^4+108c_1+108x+12\sqrt{81x^8+162c_1x^4+162x^5-12x^4+81c_1^2+162c_1x+81x^2-12c_1}\right.}{12} - \frac{1}{3\left(-8+108x^4+108c_1+108x+12\sqrt{81x^8+162c_1x^4+162x^5-12x^4+81c_1^2+162c_1x+81x^2-12c\right.} - \frac{1}{3} + \frac{i\sqrt{3}\left(\frac{\left(-8+108x^4+108c_1+108x+12\sqrt{81x^8+162c_1x^4+162x^5-12x^4+81c_1^2+162c_1x+81x^2-12c_1-12x}\right)^{\frac{1}{3}}}{6} - \frac{}{3\left(-8+108x^4+108c_1+10\right.}\right.}{2} \tag{2}$$

$$y = -\frac{\left(-8+108x^4+108c_1+108x+12\sqrt{81x^8+162c_1x^4+162x^5-12x^4+81c_1^2+162c_1x+81x^2-12c_1}\right.}{12} - \frac{1}{3\left(-8+108x^4+108c_1+108x+12\sqrt{81x^8+162c_1x^4+162x^5-12x^4+81c_1^2+162c_1x+81x^2-12c\right.} - \frac{1}{3} - \frac{i\sqrt{3}\left(\frac{\left(-8+108x^4+108c_1+108x+12\sqrt{81x^8+162c_1x^4+162x^5-12x^4+81c_1^2+162c_1x+81x^2-12c_1-12x}\right)^{\frac{1}{3}}}{6} - \frac{}{3\left(-8+108x^4+108c_1+10\right.}\right.}{2} \tag{3}$$

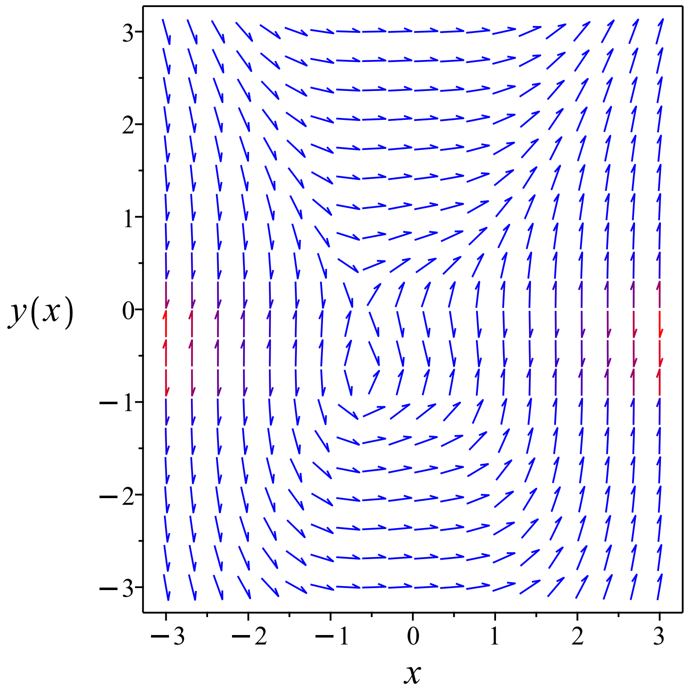


Figure 287: Slope field plot

Verification of solutions

$$y = \frac{\left(-8+108x^4+108c_1+108x+12\sqrt{81x^8+162c_1x^4+162x^5-12x^4+81c_1^2+162c_1x+81x^2-12c_1-}\right.}{6} + \frac{2}{3\left(-8+108x^4+108c_1+108x+12\sqrt{81x^8+162c_1x^4+162x^5-12x^4+81c_1^2+162c_1x+81x^2-12c}\right.} - \frac{1}{3}$$

Verified OK.

$$y = -\frac{\left(-8+108x^4+108c_1+108x+12\sqrt{81x^8+162c_1x^4+162x^5-12x^4+81c_1^2+162c_1x+81x^2-12c_1}\right.}{12} - \frac{1}{3\left(-8+108x^4+108c_1+108x+12\sqrt{81x^8+162c_1x^4+162x^5-12x^4+81c_1^2+162c_1x+81x^2-12c}\right.} - \frac{1}{3} + \frac{i\sqrt{3}\left(\frac{\left(-8+108x^4+108c_1+108x+12\sqrt{81x^8+162c_1x^4+162x^5-12x^4+81c_1^2+162c_1x+81x^2-12c_1-12x}\right)^{\frac{1}{3}}}{6} - \frac{}{3\left(-8+108x^4+108c_1+10\right.}\right.}{2}$$

Verified OK.

$$y = -\frac{\left(-8+108x^4+108c_1+108x+12\sqrt{81x^8+162c_1x^4+162x^5-12x^4+81c_1^2+162c_1x+81x^2-12c_1}\right.}{12} - \frac{1}{3\left(-8+108x^4+108c_1+108x+12\sqrt{81x^8+162c_1x^4+162x^5-12x^4+81c_1^2+162c_1x+81x^2-12c}\right.} - \frac{1}{3} - \frac{i\sqrt{3}\left(\frac{\left(-8+108x^4+108c_1+108x+12\sqrt{81x^8+162c_1x^4+162x^5-12x^4+81c_1^2+162c_1x+81x^2-12c_1-12x}\right)^{\frac{1}{3}}}{6} - \frac{}{3\left(-8+108x^4+108c_1+10\right.}\right.}{2}$$

Verified OK.

6.7.2 Solving as differentialType ode

Writing the ode as

$$y' = \frac{4x^3 + 1}{y\,(2 + 3y)} \tag{1}$$

Which becomes

$$\left(3y^2 + 2y\right) dy = \left(4x^3 + 1\right) dx \tag{2}$$

But the RHS is complete differential because

$$\left(4x^3 + 1\right) dx = d\left(x^4 + x\right)$$

Hence (2) becomes

$$\left(3y^2 + 2y\right) dy = d\left(x^4 + x\right)$$

Integrating both sides gives gives these solutions

$$y = \frac{\left(-8 + 108x^4 + 108c_1 + 108x + 12\sqrt{81x^8 + 162c_1x^4 + 162x^5 - 12x^4 + 81c_1^2 + 162c_1x + 81x^2 - 12c_1}\right.}{6}$$

$$y = -\frac{\left(-8 + 108x^4 + 108c_1 + 108x + 12\sqrt{81x^8 + 162c_1x^4 + 162x^5 - 12x^4 + 81c_1^2 + 162c_1x + 81x^2 - 12c}\right.}{12}$$

$$y = -\frac{\left(-8 + 108x^4 + 108c_1 + 108x + 12\sqrt{81x^8 + 162c_1x^4 + 162x^5 - 12x^4 + 81c_1^2 + 162c_1x + 81x^2 - 12c}\right.}{12}$$

Summary

The solution(s) found are the following

$$y = \frac{\left(-8 + 108x^4 + 108c_1 + 108x + 12\sqrt{81x^8 + 162c_1x^4 + 162x^5 - 12x^4 + 81c_1^2 + 162c_1x + 81x^2 - 12c_1 -}\right.}{6} + \frac{2}{3\left(-8 + 108x^4 + 108c_1 + 108x + 12\sqrt{81x^8 + 162c_1x^4 + 162x^5 - 12x^4 + 81c_1^2 + 162c_1x + 81x^2 - 12c}\right.} - \frac{1}{3} + c_1 \tag{1}$$

$$y = -\frac{\left(-8 + 108x^4 + 108c_1 + 108x + 12\sqrt{81x^8 + 162c_1x^4 + 162x^5 - 12x^4 + 81c_1^2 + 162c_1x + 81x^2 - 12c_1}\right.}{12} - \frac{1}{3\left(-8 + 108x^4 + 108c_1 + 108x + 12\sqrt{81x^8 + 162c_1x^4 + 162x^5 - 12x^4 + 81c_1^2 + 162c_1x + 81x^2 - 12c}\right.} - \frac{1}{3} + \frac{i\sqrt{3}\left(\frac{\left(-8+108x^4+108c_1+108x+12\sqrt{81x^8+162c_1x^4+162x^5-12x^4+81c_1^2+162c_1x+81x^2-12c_1-12x}\right)^{\frac{1}{3}}}{6} - \frac{}{3\left(-8+108x^4+108c_1+10\right.}\right.}{2} + c_1 \tag{2}$$

$$y = -\frac{\left(-8 + 108x^4 + 108c_1 + 108x + 12\sqrt{81x^8 + 162c_1x^4 + 162x^5 - 12x^4 + 81c_1^2 + 162c_1x + 81x^2 - 12c_1}\right.}{12} - \frac{1}{3\left(-8 + 108x^4 + 108c_1 + 108x + 12\sqrt{81x^8 + 162c_1x^4 + 162x^5 - 12x^4 + 81c_1^2 + 162c_1x + 81x^2 - 12c}\right.} - \frac{1}{3} - \frac{i\sqrt{3}\left(\frac{\left(-8+108x^4+108c_1+108x+12\sqrt{81x^8+162c_1x^4+162x^5-12x^4+81c_1^2+162c_1x+81x^2-12c_1-12x}\right)^{\frac{1}{3}}}{6} - \frac{}{3\left(-8+108x^4+108c_1+10\right.}\right.}{2} + c_1 \tag{3}$$

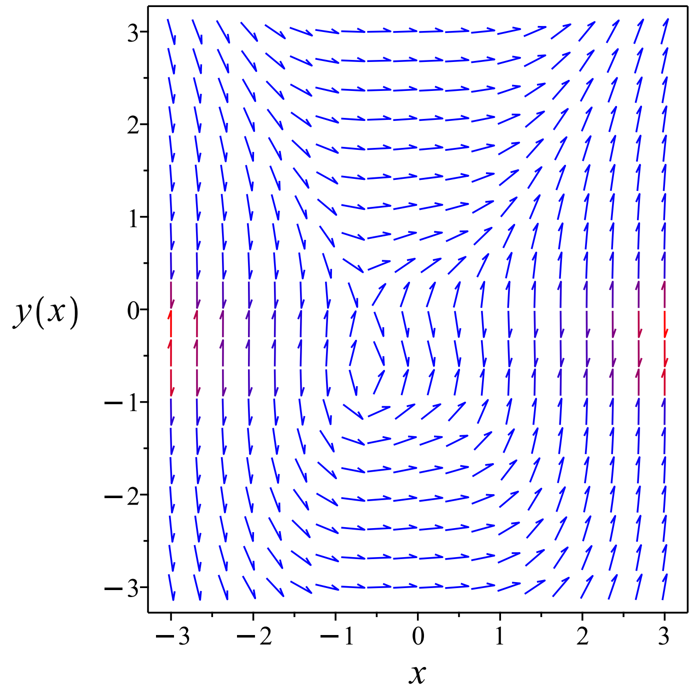


Figure 288: Slope field plot

Verification of solutions

$$y = \frac{\left(-8+108x^4+108c_1+108x+12\sqrt{81x^8+162c_1x^4+162x^5-12x^4+81c_1^2+162c_1x+81x^2-12c_1-}\right.}{6} + \frac{2}{3\left(-8+108x^4+108c_1+108x+12\sqrt{81x^8+162c_1x^4+162x^5-12x^4+81c_1^2+162c_1x+81x^2-12c}\right.} - \frac{1}{3} + c_1$$

Verified OK.

$$y = -\frac{\left(-8+108x^4+108c_1+108x+12\sqrt{81x^8+162c_1x^4+162x^5-12x^4+81c_1^2+162c_1x+81x^2-12c_1}\right.}{12} - \frac{1}{3\left(-8+108x^4+108c_1+108x+12\sqrt{81x^8+162c_1x^4+162x^5-12x^4+81c_1^2+162c_1x+81x^2-12c}\right.} - \frac{1}{3} + \frac{i\sqrt{3}\left(\frac{\left(-8+108x^4+108c_1+108x+12\sqrt{81x^8+162c_1x^4+162x^5-12x^4+81c_1^2+162c_1x+81x^2-12c_1-12x}\right)^{\frac{1}{3}}}{6} - \frac{}{3\left(-8+108x^4+108c_1+10\right.}\right.}{2} + c_1$$

Verified OK.

$$y = -\frac{\left(-8+108x^4+108c_1+108x+12\sqrt{81x^8+162c_1x^4+162x^5-12x^4+81c_1^2+162c_1x+81x^2-12c_1}\right.}{12} - \frac{1}{3\left(-8+108x^4+108c_1+108x+12\sqrt{81x^8+162c_1x^4+162x^5-12x^4+81c_1^2+162c_1x+81x^2-12c}\right.} - \frac{1}{3} - \frac{i\sqrt{3}\left(\frac{\left(-8+108x^4+108c_1+108x+12\sqrt{81x^8+162c_1x^4+162x^5-12x^4+81c_1^2+162c_1x+81x^2-12c_1-12x}\right)^{\frac{1}{3}}}{6} - \frac{}{3\left(-8+108x^4+108c_1+10\right.}\right.}{2} + c_1$$

Verified OK.

6.7.3 Solving as first order ode lie symmetry lookup ode

Writing the ode as

$$
\begin{aligned}
y' &= \frac{4x^3+1}{y(2+3y)} \\
y' &= \omega(x,y)
\end{aligned}
$$

The condition of Lie symmetry is the linearized PDE given by

$$
\eta_x + \omega(\eta_y - \xi_x) - \omega^2 \xi_y - \omega_x \xi - \omega_y \eta = 0 \tag{A}
$$

The type of this ode is known. It is of type `separable`. Therefore we do not need to solve the PDE (A), and can just use the lookup table shown below to find ξ, η

Table 272: Lie symmetry infinitesimal lookup table for known first order ODE's

ODE class	Form	ξ	η
linear ode	$y' = f(x)y(x) + g(x)$	0	$e^{\int f dx}$
separable ode	$y' = f(x)\, g(y)$	$\frac{1}{f}$	0
quadrature ode	$y' = f(x)$	0	1
quadrature ode	$y' = g(y)$	1	0
homogeneous ODEs of Class A	$y' = f\left(\frac{y}{x}\right)$	x	y
homogeneous ODEs of Class C	$y' = (a+bx+cy)^{\frac{n}{m}}$	1	$-\frac{b}{c}$
homogeneous class D	$y' = \frac{y}{x} + g(x)\, F\left(\frac{y}{x}\right)$	x^2	xy
First order special form ID 1	$y' = g(x)\, e^{h(x)+by} + f(x)$	$\frac{e^{-\int bf(x)dx - h(x)}}{g(x)}$	$\frac{f(x)e^{-\int bf(x)dx - h(x)}}{g(x)}$
polynomial type ode	$y' = \frac{a_1x+b_1y+c_1}{a_2x+b_2y+c_2}$	$\frac{a_1b_2x - a_2b_1x - b_1c_2 + b_2c_1}{a_1b_2 - a_2b_1}$	$\frac{a_1b_2y - a_2b_1y - a_1c_2 - a_2c_1}{a_1b_2 - a_2b_1}$
Bernoulli ode	$y' = f(x)\, y + g(x)\, y^n$	0	$e^{-\int (n-1)f(x)dx} y^n$
Reduced Riccati	$y' = f_1(x)\, y + f_2(x)\, y^2$	0	$e^{-\int f_1 dx}$

The above table shows that

$$
\begin{aligned}
\xi(x, y) &= \frac{1}{4x^3+1} \\
\eta(x, y) &= 0
\end{aligned} \tag{A1}
$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x, y) \to (R, S)$ where (R, S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$
\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}
$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y}\right) S(x, y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the canonical coordinates, where $S(R)$. Since $\eta = 0$ then in this special case

$$
R = y
$$

S is found from

$$
\begin{aligned}
S &= \int \frac{1}{\xi} dx \\
&= \int \frac{1}{\frac{1}{4x^3+1}} dx
\end{aligned}
$$

Which results in

$$
S = x^4 + x
$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$
\frac{dS}{dR} = \frac{S_x + \omega(x, y) S_y}{R_x + \omega(x, y) R_y} \tag{2}
$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x, y)$ is the right hand side of the original ode given by

$$
\omega(x, y) = \frac{4x^3+1}{y\,(2+3y)}
$$

Evaluating all the partial derivatives gives

$$
\begin{aligned}
R_x &= 0 \\
R_y &= 1 \\
S_x &= 4x^3 + 1 \\
S_y &= 0
\end{aligned}
$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$
\frac{dS}{dR} = 3y^2 + 2y \tag{2A}
$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$
\frac{dS}{dR} = 3R^2 + 2R
$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$
S(R) = R^3 + R^2 + c_1 \tag{4}
$$

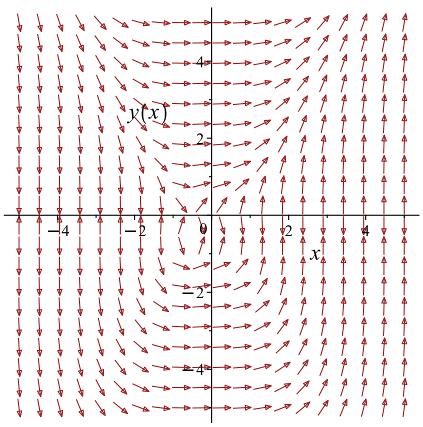
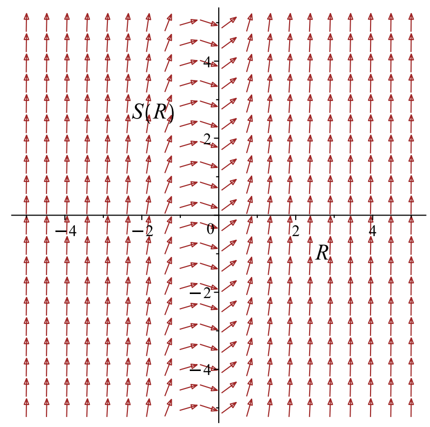
To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$
x^4 + x = y^3 + y^2 + c_1
$$

Which simplifies to

$$
x^4 + x = y^3 + y^2 + c_1
$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in x, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dx} = \frac{4x^3+1}{y(2+3y)}$ 	$R = y$ $S = x^4 + x$	$\frac{dS}{dR} = 3R^2 + 2R$ 

Summary

The solution(s) found are the following

$$x^4 + x = y^3 + y^2 + c_1 \tag{1}$$

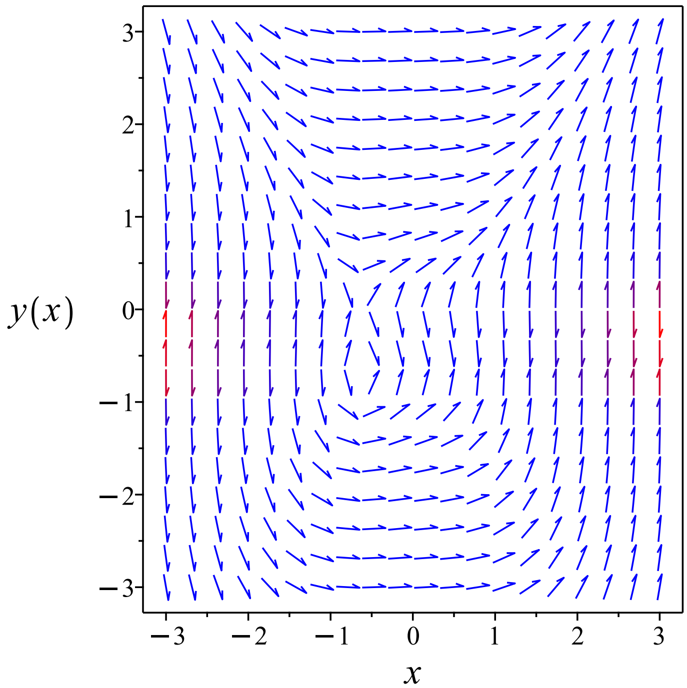


Figure 289: Slope field plot

Verification of solutions

$$x^4 + x = y^3 + y^2 + c_1$$

Verified OK.

6.7.4 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x, y) + N(x, y)\frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x, y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x, y) = 0$$

Hence

$$\frac{\partial \phi}{\partial x} + \frac{\partial \phi}{\partial y}\frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$\frac{\partial \phi}{\partial x} = M$$
$$\frac{\partial \phi}{\partial y} = N$$

But since $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x, y)$ but at least we know now that we can do that since the condition $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x, y)\,\mathrm{d}x + N(x, y)\,\mathrm{d}y = 0 \tag{1A}$$

Therefore

$$(y(2 + 3y))\,\mathrm{d}y = \left(4x^3 + 1\right)\mathrm{d}x$$
$$\left(-4x^3 - 1\right)\mathrm{d}x + (y(2 + 3y))\,\mathrm{d}y = 0 \tag{2A}$$

Comparing (1A) and (2A) shows that

$$M(x, y) = -4x^3 - 1$$
$$N(x, y) = y(2 + 3y)$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Using result found above gives

$$\begin{aligned}\frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}\left(-4x^3 - 1\right)\\ &= 0\end{aligned}$$

And

$$\begin{aligned}\frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}(y(2 + 3y))\\ &= 0\end{aligned}$$

Since $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$, then the ODE is exact The following equations are now set up to solve for the function $\phi(x, y)$

$$\frac{\partial \phi}{\partial x} = M \tag{1}$$

$$\frac{\partial \phi}{\partial y} = N \tag{2}$$

Integrating (1) w.r.t. x gives

$$\int \frac{\partial \phi}{\partial x} \, \mathrm{d}x = \int M \, \mathrm{d}x$$

$$\int \frac{\partial \phi}{\partial x} \, \mathrm{d}x = \int -4x^3 - 1 \, \mathrm{d}x$$

$$\phi = -x^4 - x + f(y) \tag{3}$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$\frac{\partial \phi}{\partial y} = 0 + f'(y) \tag{4}$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = y(2 + 3y)$. Therefore equation (4) becomes

$$y(2 + 3y) = 0 + f'(y) \tag{5}$$

Solving equation (5) for $f'(y)$ gives

$$f'(y) = y(2 + 3y)$$

Integrating the above w.r.t y gives

$$\int f'(y) \, \mathrm{d}y = \int (3y^2 + 2y) \, \mathrm{d}y$$

$$f(y) = y^3 + y^2 + c_1$$

Where c_1 is constant of integration. Substituting result found above for $f(y)$ into equation (3) gives ϕ

$$\phi = -x^4 + y^3 + y^2 - x + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = -x^4 + y^3 + y^2 - x$$

Summary
The solution(s) found are the following

$$y^3 + y^2 - x^4 - x = c_1 \tag{1}$$

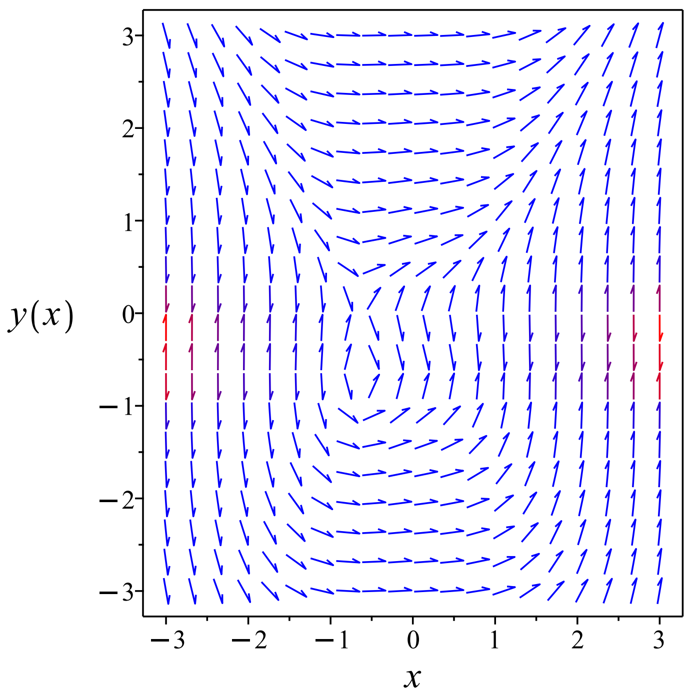


Figure 290: Slope field plot

Verification of solutions

$$y^3 + y^2 - x^4 - x = c_1$$

Verified OK.

6.7.5 Maple step by step solution

Let's solve

$y' - \frac{4x^3+1}{y(2+3y)} = 0$

- Highest derivative means the order of the ODE is 1

  y'

- Separate variables

  $y'y(2+3y) = 4x^3+1$

- Integrate both sides with respect to x

  $\int y'y(2+3y)\,dx = \int (4x^3+1)\,dx + c_1$

- Evaluate integral

  $y^3+y^2 = x^4+c_1+x$

- Solve for y

  $y = \frac{\left(-8+108x^4+108c_1+108x+12\sqrt{81x^8+162c_1x^4+162x^5-12x^4+81c_1^2+162c_1x+81x^2-12c_1-12x}\right)^{\frac{1}{3}}}{6} + \frac{}{3\left(-8+108x^4+108c_1+10\right.}$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
<- separable successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 382

```
dsolve(diff(y(x),x) = (4*x^3+1)/(y(x)*(2+3*y(x))),y(x), singsol=all)
```

$$y(x) = \frac{\left(-8+108x^4+108c_1+108x+12\sqrt{81}\sqrt{\left(x^4+c_1+x\right)\left(x^4+c_1+x-\frac{4}{27}\right)}\right)^{\frac{2}{3}} - 2\left(-8+108x^4+108c_1 \cdots\right.}{6\left(-8+108x^4+108c_1+108x+12\sqrt{81}\sqrt{\left(x^4+c_1+x\right)\left(x^4 \cdots\right.}\right.}$$

$$y(x) = -\frac{\left(1+i\sqrt{3}\right)\left(-8+108x^4+108c_1+108x+12\sqrt{81}\sqrt{\left(x^4+c_1+x\right)\left(x^4+c_1+x-\frac{4}{27}\right)}\right)^{\frac{2}{3}} - 4i\sqrt{3}+4\left(\cdots\right.}{12\left(-8+108x^4+108c_1+108x+12\sqrt{81}\sqrt{\left(x^4+ \cdots\right.}\right.}$$

$$y(x) = \frac{\left(i\sqrt{3}-1\right)\left(-8+108x^4+108c_1+108x+12\sqrt{81}\sqrt{\left(x^4+c_1+x\right)\left(x^4+c_1+x-\frac{4}{27}\right)}\right)^{\frac{2}{3}} - 4i\sqrt{3}-4\left(-\cdots\right.}{12\left(-8+108x^4+108c_1+108x+12\sqrt{81}\sqrt{\left(x^4+c_1 \cdots\right.}\right.}$$

✓ Solution by Mathematica

Time used: 4.502 (sec). Leaf size: 356

```
DSolve[y'[x]== (4*x^3+1)/(y[x]*(2+3*y[x])),y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \frac{1}{6}\left(2^{2/3}\sqrt[3]{27x^4+\sqrt{-4+(27x^4+27x-2+27c_1)^2}+27x-2+27c_1} + \frac{2\sqrt[3]{2}}{\sqrt[3]{27x^4+\sqrt{-4+(27x^4+27x-2+27c_1)^2}+27x-2+27c_1}} - 2\right)$$

$$y(x) \to \frac{1}{12}\left(i2^{2/3}\left(\sqrt{3}+i\right)\sqrt[3]{27x^4+\sqrt{-4+(27x^4+27x-2+27c_1)^2}+27x-2+27c_1} - \frac{2\sqrt[3]{2}(1+i\sqrt{3})}{\sqrt[3]{27x^4+\sqrt{-4+(27x^4+27x-2+27c_1)^2}+27x-2+27c_1}} - 4\right)$$

$$y(x) \to \frac{1}{12}\left(-2^{2/3}\left(1+i\sqrt{3}\right)\sqrt[3]{27x^4+\sqrt{-4+(27x^4+27x-2+27c_1)^2}+27x-2+27c_1} + \frac{2i\sqrt[3]{2}(\sqrt{3}+i)}{\sqrt[3]{27x^4+\sqrt{-4+(27x^4+27x-2+27c_1)^2}+27x-2+27c_1}} - 4\right)$$

6.8 problem 8

Internal problem ID [575]
Internal file name [`OUTPUT/575_Sunday_June_05_2022_01_45_02_AM_24783477/index.tex`]

Book: Elementary differential equations and boundary value problems, 10th ed., Boyce and DiPrima
Section: Miscellaneous problems, end of chapter 2. Page 133
Problem number: 8.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"exact", "linear", "first_order_ode_lie_symmetry_lookup"**

Maple gives the following as the ode type

`[_linear]`

$$\boxed{2y + y'x = \frac{\sin(x)}{x}}$$

With initial conditions

$$[y(2) = 1]$$

6.8.1 Existence and uniqueness analysis

This is a linear ODE. In canonical form it is written as

$$y' + p(x)y = q(x)$$

Where here

$$p(x) = \frac{2}{x}$$
$$q(x) = \frac{\sin(x)}{x^2}$$

Hence the ode is

$$y' + \frac{2y}{x} = \frac{\sin(x)}{x^2}$$

The domain of $p(x) = \frac{2}{x}$ is

$$\{x < 0 \lor 0 < x\}$$

And the point $x_0 = 2$ is inside this domain. The domain of $q(x) = \frac{\sin(x)}{x^2}$ is

$$\{x < 0 \lor 0 < x\}$$

And the point $x_0 = 2$ is also inside this domain. Hence solution exists and is unique.

6.8.2 Solving as linear ode

Entering Linear first order ODE solver. The integrating factor μ is

$$\begin{aligned}\mu &= e^{\int \frac{2}{x} dx} \\ &= x^2\end{aligned}$$

The ode becomes

$$\begin{aligned}\frac{d}{dx}(\mu y) &= (\mu)\left(\frac{\sin(x)}{x^2}\right) \\ \frac{d}{dx}\left(y\,x^2\right) &= \left(x^2\right)\left(\frac{\sin(x)}{x^2}\right) \\ d\left(y\,x^2\right) &= \sin(x)\,dx\end{aligned}$$

Integrating gives

$$\begin{aligned}y\,x^2 &= \int \sin(x)\,dx \\ y\,x^2 &= -\cos(x) + c_1\end{aligned}$$

Dividing both sides by the integrating factor $\mu = x^2$ results in

$$y = -\frac{\cos(x)}{x^2} + \frac{c_1}{x^2}$$

which simplifies to

$$y = \frac{-\cos(x) + c_1}{x^2}$$

Initial conditions are used to solve for c_1. Substituting $x = 2$ and $y = 1$ in the above solution gives an equation to solve for the constant of integration.

$$1 = -\frac{\cos(2)}{4} + \frac{c_1}{4}$$

$$c_1 = \cos(2) + 4$$

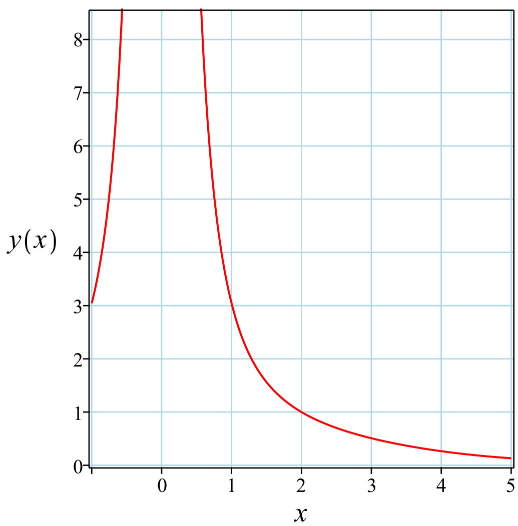
Substituting c_1 found above in the general solution gives

$$y = -\frac{\cos(x) - 4 - \cos(2)}{x^2}$$

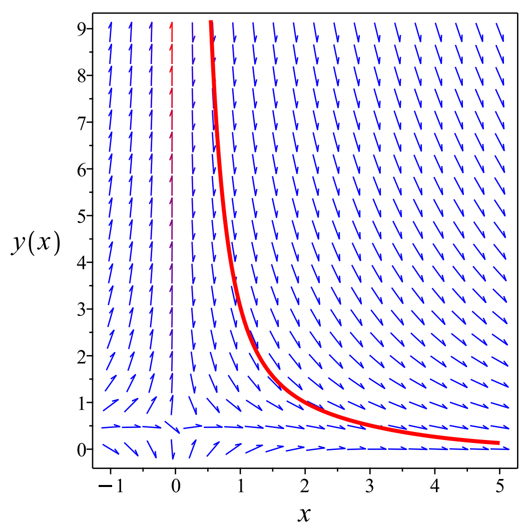
Summary

The solution(s) found are the following

$$y = -\frac{\cos(x) - 4 - \cos(2)}{x^2} \tag{1}$$



(a) Solution plot



(b) Slope field plot

Verification of solutions

$$y = -\frac{\cos(x) - 4 - \cos(2)}{x^2}$$

Verified OK.

6.8.3 Solving as first order ode lie symmetry lookup ode

Writing the ode as

$$y' = \frac{-2yx + \sin(x)}{x^2}$$
$$y' = \omega(x, y)$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x + \omega(\eta_y - \xi_x) - \omega^2\xi_y - \omega_x\xi - \omega_y\eta = 0 \tag{A}$$

The type of this ode is known. It is of type `linear`. Therefore we do not need to solve the PDE (A), and can just use the lookup table shown below to find ξ, η

Table 275: Lie symmetry infinitesimal lookup table for known first order ODE's

ODE class	Form	ξ	η
linear ode	$y' = f(x)y(x) + g(x)$	0	$e^{\int f dx}$
separable ode	$y' = f(x)\, g(y)$	$\frac{1}{f}$	0
quadrature ode	$y' = f(x)$	0	1
quadrature ode	$y' = g(y)$	1	0
homogeneous ODEs of Class A	$y' = f\left(\frac{y}{x}\right)$	x	y
homogeneous ODEs of Class C	$y' = (a + bx + cy)^{\frac{n}{m}}$	1	$-\frac{b}{c}$
homogeneous class D	$y' = \frac{y}{x} + g(x)\, F\left(\frac{y}{x}\right)$	x^2	xy
First order special form ID 1	$y' = g(x)\, e^{h(x)+by} + f(x)$	$\frac{e^{-\int bf(x)dx - h(x)}}{g(x)}$	$\frac{f(x)e^{-\int bf(x)dx - h(x)}}{g(x)}$
polynomial type ode	$y' = \frac{a_1x+b_1y+c_1}{a_2x+b_2y+c_2}$	$\frac{a_1b_2x-a_2b_1x-b_1c_2+b_2c_1}{a_1b_2-a_2b_1}$	$\frac{a_1b_2y-a_2b_1y-a_1c_2-a_2c_1}{a_1b_2-a_2b_1}$
Bernoulli ode	$y' = f(x)\, y + g(x)\, y^n$	0	$e^{-\int (n-1)f(x)dx}y^n$
Reduced Riccati	$y' = f_1(x)\, y + f_2(x)\, y^2$	0	$e^{-\int f_1 dx}$

The above table shows that

$$
\begin{aligned}
\xi(x,y) &= 0 \\
\eta(x,y) &= \frac{1}{x^2}
\end{aligned} \tag{A1}
$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x,y) \to (R,S)$ where (R,S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$
\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}
$$

The above comes from the requirements that $\left(\xi\frac{\partial}{\partial x} + \eta\frac{\partial}{\partial y}\right) S(x,y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$
R = x
$$

S is found from

$$
\begin{aligned}
S &= \int \frac{1}{\eta} dy \\
&= \int \frac{1}{\frac{1}{x^2}} dy
\end{aligned}
$$

Which results in

$$
S = y\,x^2
$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$
\frac{dS}{dR} = \frac{S_x + \omega(x,y)S_y}{R_x + \omega(x,y)R_y} \tag{2}
$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x,y)$ is the right hand side of the original ode given by

$$
\omega(x,y) = \frac{-2yx + \sin(x)}{x^2}
$$

Evaluating all the partial derivatives gives

$$
\begin{aligned}
R_x &= 1 \\
R_y &= 0 \\
S_x &= 2yx \\
S_y &= x^2
\end{aligned}
$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = \sin(x) \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = \sin(R)$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$S(R) = -\cos(R) + c_1 \tag{4}$$

To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$x^2 y = -\cos(x) + c_1$$

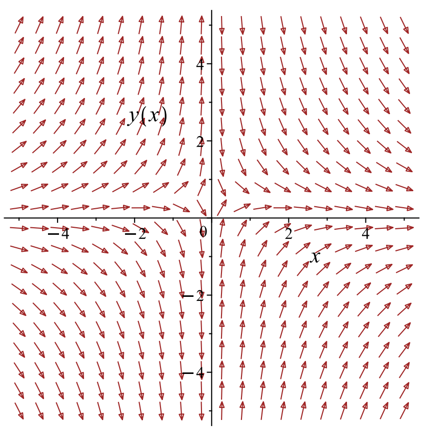
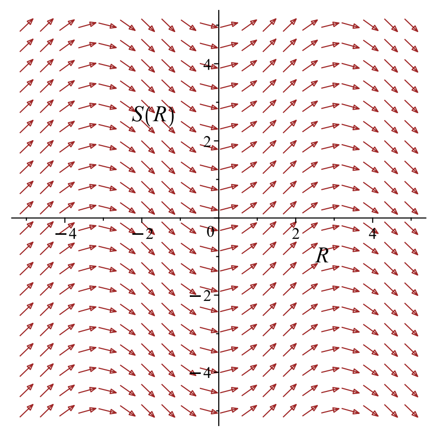
Which simplifies to

$$x^2 y = -\cos(x) + c_1$$

Which gives

$$y = -\frac{\cos(x) - c_1}{x^2}$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in x, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dx} = \frac{-2yx+\sin(x)}{x^2}$ 	$R = x$ $S = y\,x^2$	$\frac{dS}{dR} = \sin(R)$ 

Initial conditions are used to solve for c_1. Substituting $x = 2$ and $y = 1$ in the above solution gives an equation to solve for the constant of integration.

$$1 = -\frac{\cos(2)}{4} + \frac{c_1}{4}$$

$$c_1 = \cos(2) + 4$$

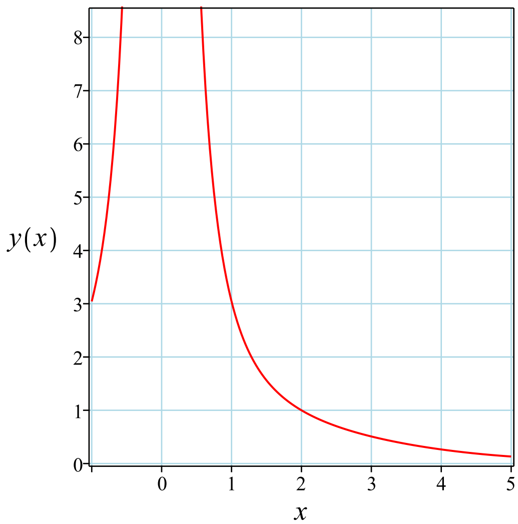
Substituting c_1 found above in the general solution gives

$$y = -\frac{\cos(x) - 4 - \cos(2)}{x^2}$$

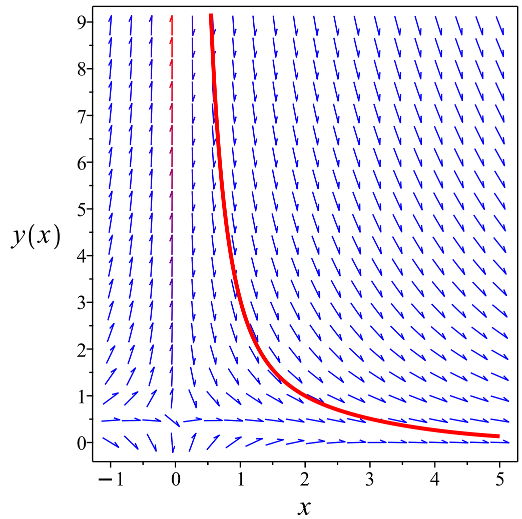
Summary

The solution(s) found are the following

$$y = -\frac{\cos(x) - 4 - \cos(2)}{x^2} \tag{1}$$



(a) Solution plot



(b) Slope field plot

Verification of solutions

$$y = -\frac{\cos(x) - 4 - \cos(2)}{x^2}$$

Verified OK.

6.8.4 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x, y) + N(x, y)\frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x, y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x, y) = 0$$

Hence

$$\frac{\partial \phi}{\partial x} + \frac{\partial \phi}{\partial y}\frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$\begin{aligned}\frac{\partial \phi}{\partial x} &= M \\ \frac{\partial \phi}{\partial y} &= N\end{aligned}$$

But since $\frac{\partial^2\phi}{\partial x\partial y} = \frac{\partial^2\phi}{\partial y\partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x, y)$ but at least we know now that we can do that since the condition $\frac{\partial^2\phi}{\partial x\partial y} = \frac{\partial^2\phi}{\partial y\partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x, y)\,\mathrm{d}x + N(x, y)\,\mathrm{d}y = 0 \tag{1A}$$

Therefore

$$\begin{aligned}(x^2)\,\mathrm{d}y &= (-2yx + \sin(x))\,\mathrm{d}x\\ (2yx - \sin(x))\,\mathrm{d}x + (x^2)\,\mathrm{d}y &= 0 \qquad (2A)\end{aligned}$$

Comparing (1A) and (2A) shows that

$$\begin{aligned}M(x, y) &= 2yx - \sin(x)\\ N(x, y) &= x^2\end{aligned}$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Using result found above gives

$$\begin{aligned}\frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}(2yx - \sin(x))\\ &= 2x\end{aligned}$$

And

$$\begin{aligned}\frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}(x^2)\\ &= 2x\end{aligned}$$

Since $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$, then the ODE is <u>exact</u> The following equations are now set up to solve for the function $\phi(x, y)$

$$\frac{\partial \phi}{\partial x} = M \tag{1}$$

$$\frac{\partial \phi}{\partial y} = N \tag{2}$$

Integrating (1) w.r.t. x gives

$$\int \frac{\partial \phi}{\partial x} \mathrm{d}x = \int M \mathrm{d}x$$
$$\int \frac{\partial \phi}{\partial x} \mathrm{d}x = \int 2yx - \sin(x) \mathrm{d}x$$
$$\phi = y\,x^2 + \cos(x) + f(y) \tag{3}$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$\frac{\partial \phi}{\partial y} = x^2 + f'(y) \tag{4}$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = x^2$. Therefore equation (4) becomes

$$x^2 = x^2 + f'(y) \tag{5}$$

Solving equation (5) for $f'(y)$ gives

$$f'(y) = 0$$

Therefore

$$f(y) = c_1$$

Where c_1 is constant of integration. Substituting this result for $f(y)$ into equation (3) gives ϕ

$$\phi = y\,x^2 + \cos(x) + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = y\,x^2 + \cos(x)$$

The solution becomes

$$y = -\frac{\cos(x) - c_1}{x^2}$$

Initial conditions are used to solve for c_1. Substituting $x = 2$ and $y = 1$ in the above solution gives an equation to solve for the constant of integration.

$$1 = -\frac{\cos(2)}{4} + \frac{c_1}{4}$$

$$c_1 = \cos(2) + 4$$

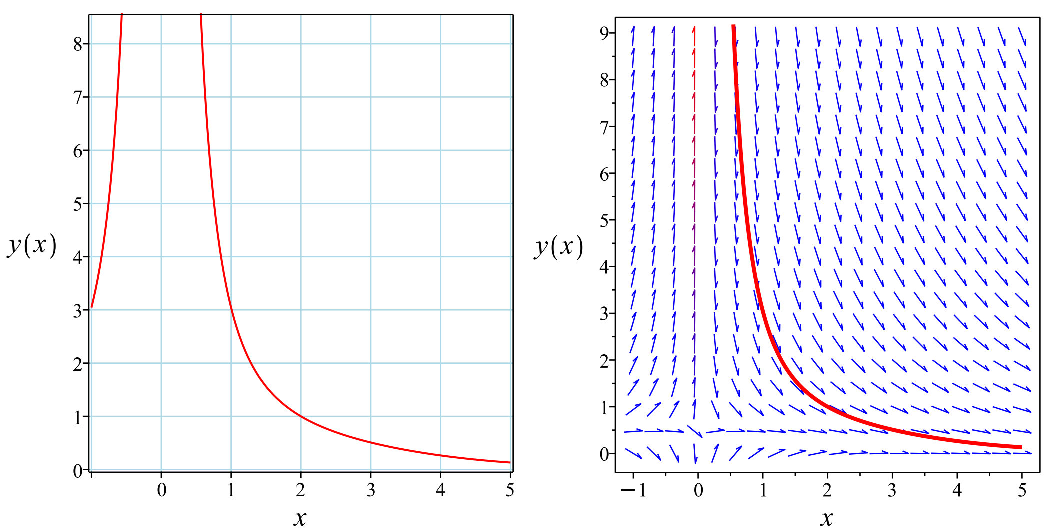
Substituting c_1 found above in the general solution gives

$$y = -\frac{\cos(x) - 4 - \cos(2)}{x^2}$$

Summary

The solution(s) found are the following

$$y = -\frac{\cos(x) - 4 - \cos(2)}{x^2} \tag{1}$$



(a) Solution plot (b) Slope field plot

Verification of solutions

$$y = -\frac{\cos(x) - 4 - \cos(2)}{x^2}$$

Verified OK.

6.8.5 Maple step by step solution

Let's solve

$\left[2y+y'x=\frac{\sin(x)}{x}, y(2)=1\right]$

- Highest derivative means the order of the ODE is 1

  y'

- Isolate the derivative

  $y'=-\frac{2y}{x}+\frac{\sin(x)}{x^2}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE

  $y'+\frac{2y}{x}=\frac{\sin(x)}{x^2}$

- The ODE is linear; multiply by an integrating factor $\mu(x)$

  $\mu(x)\left(y'+\frac{2y}{x}\right)=\frac{\mu(x)\sin(x)}{x^2}$

- Assume the lhs of the ODE is the total derivative $\frac{d}{dx}(\mu(x)\,y)$

  $\mu(x)\left(y'+\frac{2y}{x}\right)=\mu'(x)\,y+\mu(x)\,y'$

- Isolate $\mu'(x)$

  $\mu'(x)=\frac{2\mu(x)}{x}$

- Solve to find the integrating factor

  $\mu(x)=x^2$

- Integrate both sides with respect to x

  $\int\left(\frac{d}{dx}(\mu(x)\,y)\right)dx=\int\frac{\mu(x)\sin(x)}{x^2}dx+c_1$

- Evaluate the integral on the lhs

  $\mu(x)\,y=\int\frac{\mu(x)\sin(x)}{x^2}dx+c_1$

- Solve for y

  $y=\frac{\int\frac{\mu(x)\sin(x)}{x^2}dx+c_1}{\mu(x)}$

- Substitute $\mu(x)=x^2$

  $y=\frac{\int\sin(x)dx+c_1}{x^2}$

- Evaluate the integrals on the rhs

  $y=\frac{-\cos(x)+c_1}{x^2}$

- Use initial condition $y(2)=1$

$$1 = -\frac{\cos(2)}{4} + \frac{c_1}{4}$$

- Solve for c_1

$$c_1 = \cos(2) + 4$$

- Substitute $c_1 = \cos(2) + 4$ into general solution and simplify

$$y = \frac{-\cos(x) + 4 + \cos(2)}{x^2}$$

- Solution to the IVP

$$y = \frac{-\cos(x) + 4 + \cos(2)}{x^2}$$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
<- 1st order linear successful`
```

✓ Solution by Maple

Time used: 0.015 (sec). Leaf size: 16

```
dsolve([2*y(x)+x*diff(y(x),x) = sin(x)/x,y(2) = 1],y(x), singsol=all)
```

$$y(x) = \frac{-\cos(x) + 4 + \cos(2)}{x^2}$$

✓ Solution by Mathematica

Time used: 0.039 (sec). Leaf size: 17

```
DSolve[{2*y[x]+x*y'[x] == Sin[x]/x,y[2]==1},y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \frac{-\cos(x) + 4 + \cos(2)}{x^2}$$

6.9 problem 9

Internal problem ID [576]
Internal file name [`OUTPUT/576_Sunday_June_05_2022_01_45_03_AM_51375781/index.tex`]

Book: Elementary differential equations and boundary value problems, 10th ed., Boyce and DiPrima
Section: Miscellaneous problems, end of chapter 2. Page 133
Problem number: 9.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"exact", "differentialType"**

Maple gives the following as the ode type

```
[_rational, [_1st_order, `_with_symmetry_[F(x),G(x)]`], [_Abel,
    `2nd type`, `class A`]]
```

$$\boxed{y' - \frac{-1 - 2yx}{x^2 + 2y} = 0}$$

6.9.1 Solving as differentialType ode

Writing the ode as

$$y' = \frac{-1 - 2yx}{x^2 + 2y} \tag{1}$$

Which becomes

$$(2y)\, dy = \left(-x^2\right) dy + \left(-2yx - 1\right) dx \tag{2}$$

But the RHS is complete differential because

$$\left(-x^2\right) dy + \left(-2yx - 1\right) dx = d\left(-y\, x^2 - x\right)$$

Hence (2) becomes

$$(2y)\, dy = d\left(-y\, x^2 - x\right)$$

Integrating both sides gives gives these solutions

$$y = -\frac{x^2}{2} + \frac{\sqrt{x^4 + 4c_1 - 4x}}{2} + c_1$$
$$y = -\frac{x^2}{2} - \frac{\sqrt{x^4 + 4c_1 - 4x}}{2} + c_1$$

Summary

The solution(s) found are the following

$$y = -\frac{x^2}{2} + \frac{\sqrt{x^4 + 4c_1 - 4x}}{2} + c_1 \tag{1}$$
$$y = -\frac{x^2}{2} - \frac{\sqrt{x^4 + 4c_1 - 4x}}{2} + c_1 \tag{2}$$

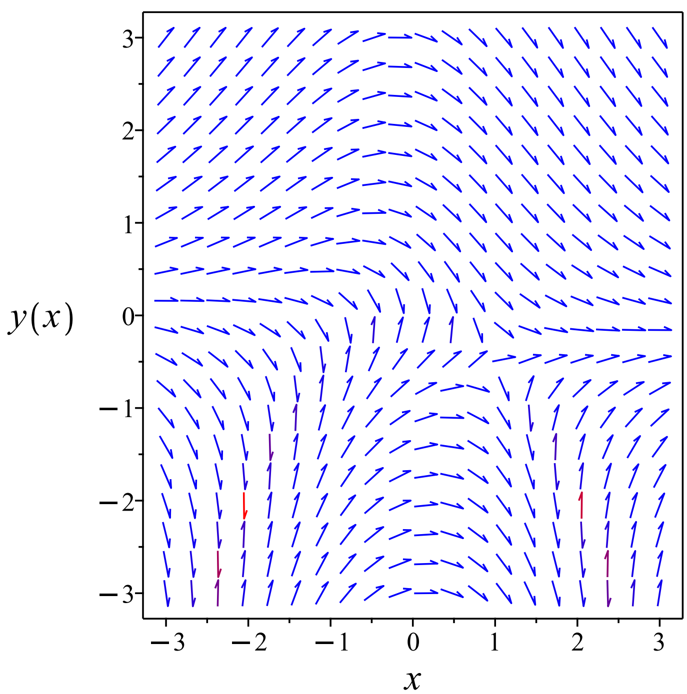


Figure 294: Slope field plot

Verification of solutions

$$y = -\frac{x^2}{2} + \frac{\sqrt{x^4 + 4c_1 - 4x}}{2} + c_1$$

Verified OK.

$$y = -\frac{x^2}{2} - \frac{\sqrt{x^4 + 4c_1 - 4x}}{2} + c_1$$

Verified OK.

6.9.2 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x,y) + N(x,y)\frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x,y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y) = 0$$

Hence

$$\frac{\partial \phi}{\partial x} + \frac{\partial \phi}{\partial y}\frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$\begin{aligned}\frac{\partial \phi}{\partial x} &= M\\ \frac{\partial \phi}{\partial y} &= N\end{aligned}$$

But since $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x,y)$ but at least we know now that we can do that since the condition $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x,y)\,\mathrm{d}x + N(x,y)\,\mathrm{d}y = 0 \tag{1A}$$

Therefore

$$
\begin{aligned}
\left(x^{2}+2y\right) \mathrm{d}y &= (-2yx-1) \mathrm{d}x \\
(2yx+1) \mathrm{d}x+\left(x^{2}+2y\right) \mathrm{d}y &= 0 \qquad (2A)
\end{aligned}
$$

Comparing (1A) and (2A) shows that

$$
\begin{aligned}
M(x,y) &= 2yx+1 \\
N(x,y) &= x^{2}+2y
\end{aligned}
$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$
\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}
$$

Using result found above gives

$$
\begin{aligned}
\frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}(2yx+1) \\
&= 2x
\end{aligned}
$$

And

$$
\begin{aligned}
\frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}\left(x^{2}+2y\right) \\
&= 2x
\end{aligned}
$$

Since $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$, then the ODE is exact The following equations are now set up to solve for the function $\phi(x,y)$

$$
\begin{aligned}
\frac{\partial \phi}{\partial x} &= M \qquad (1) \\
\frac{\partial \phi}{\partial y} &= N \qquad (2)
\end{aligned}
$$

Integrating (1) w.r.t. x gives

$$
\begin{aligned}
\int \frac{\partial \phi}{\partial x} \mathrm{d}x &= \int M \mathrm{d}x \\
\int \frac{\partial \phi}{\partial x} \mathrm{d}x &= \int 2yx+1 \mathrm{d}x \\
\phi &= y\,x^{2}+x+f(y) \qquad (3)
\end{aligned}
$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$\frac{\partial \phi}{\partial y} = x^2 + f'(y) \tag{4}$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = x^2 + 2y$. Therefore equation (4) becomes

$$x^2 + 2y = x^2 + f'(y) \tag{5}$$

Solving equation (5) for $f'(y)$ gives

$$f'(y) = 2y$$

Integrating the above w.r.t y gives

$$\int f'(y)\,\mathrm{d}y = \int (2y)\,\mathrm{d}y$$
$$f(y) = y^2 + c_1$$

Where c_1 is constant of integration. Substituting result found above for $f(y)$ into equation (3) gives ϕ

$$\phi = y\,x^2 + y^2 + x + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = y\,x^2 + y^2 + x$$

Summary

The solution(s) found are the following

$$x^2y + y^2 + x = c_1 \tag{1}$$

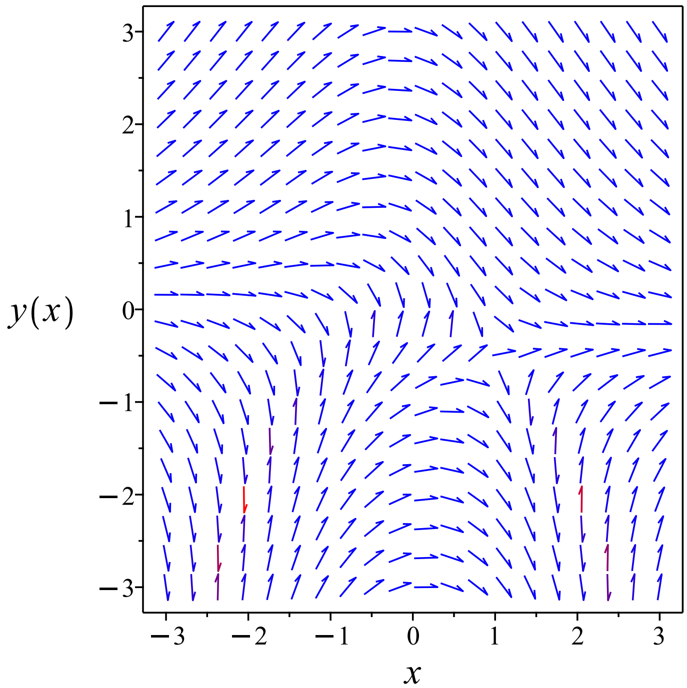


Figure 295: Slope field plot

Verification of solutions

$$x^2y + y^2 + x = c_1$$

Verified OK.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
trying homogeneous types:
trying Chini
differential order: 1; looking for linear symmetries
trying exact
<- exact successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 47

```
dsolve(diff(y(x),x) = (-1-2*x*y(x))/(x^2+2*y(x)),y(x), singsol=all)
```

$$y(x) = -\frac{x^2}{2} - \frac{\sqrt{x^4 - 4c_1 - 4x}}{2}$$
$$y(x) = -\frac{x^2}{2} + \frac{\sqrt{x^4 - 4c_1 - 4x}}{2}$$

✓ Solution by Mathematica

Time used: 0.15 (sec). Leaf size: 61

```
DSolve[y'[x]== (-1-2*x*y[x])/(x^2+2*y[x]),y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \frac{1}{2}\left(-x^2 - \sqrt{x^4 - 4x + 4c_1}\right)$$
$$y(x) \to \frac{1}{2}\left(-x^2 + \sqrt{x^4 - 4x + 4c_1}\right)$$

6.10 problem 10

Internal problem ID [577]
Internal file name [`OUTPUT/577_Sunday_June_05_2022_01_45_05_AM_92187482/index.tex`]

Book: Elementary differential equations and boundary value problems, 10th ed., Boyce and DiPrima
Section: Miscellaneous problems, end of chapter 2. Page 133
Problem number: 10.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"exact", "separable", "first_order_ode_lie_symmetry_lookup"**

Maple gives the following as the ode type

```
[_separable]
```

$$\boxed{\frac{yy'}{y-2} = -\frac{-x^2+x+1}{x^2}}$$

6.10.1 Solving as separable ode

In canonical form the ODE is

$$\begin{aligned}
y' &= F(x,y) \\
&= f(x)g(y) \\
&= \frac{(x^2-x-1)(y-2)}{y\,x^2}
\end{aligned}$$

Where $f(x) = \frac{x^2-x-1}{x^2}$ and $g(y) = \frac{y-2}{y}$. Integrating both sides gives

$$\frac{1}{\frac{y-2}{y}}\,dy = \frac{x^2-x-1}{x^2}\,dx$$

$$\int \frac{1}{\frac{y-2}{y}}\, dy = \int \frac{x^2 - x - 1}{x^2}\, dx$$

$$y + 2\ln(y-2) = x - \ln(x) + \frac{1}{x} + c_1$$

Which results in

$$y = \mathrm{e}^{-\frac{x\ln(x) + 2\,\mathrm{LambertW}\left(\frac{\mathrm{e}^{-\frac{-1 - c_1 x - x^2 + x\ln(x) + 2x}{2x}}}{2}\right)x - c_1 x - x^2 + 2x - 1}{2x}} + 2$$

Summary

The solution(s) found are the following

$$y = \mathrm{e}^{-\frac{x\ln(x) + 2\,\mathrm{LambertW}\left(\frac{\mathrm{e}^{-\frac{-1 - c_1 x - x^2 + x\ln(x) + 2x}{2x}}}{2}\right)x - c_1 x - x^2 + 2x - 1}{2x}} + 2 \tag{1}$$

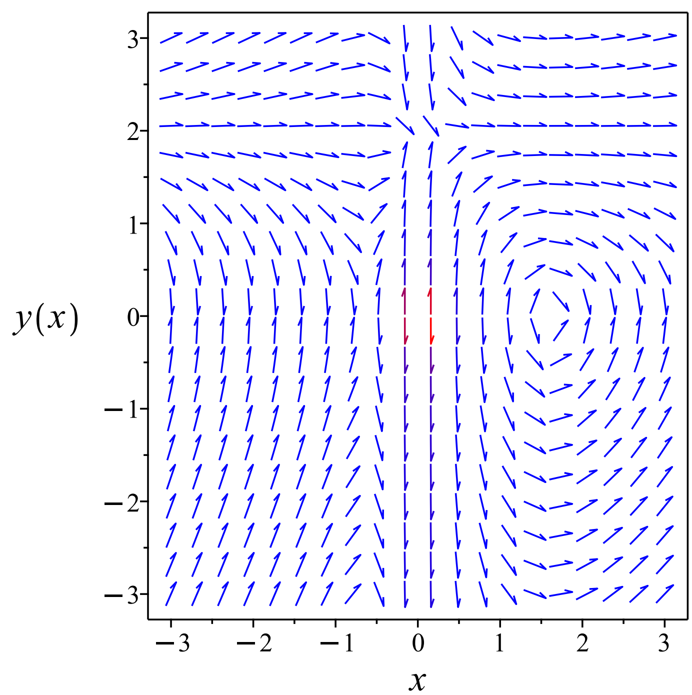


Figure 296: Slope field plot

Verification of solutions

$$
y = \mathrm{e}^{-\frac{x\ln(x)+2\,\mathrm{LambertW}\left(\frac{\mathrm{e}^{-\frac{-1-c_1x-x^2+x\ln(x)+2x}{2x}}}{2}\right)x-c_1x-x^2+2x-1}{2x}}+2
$$

Verified OK.

6.10.2 Solving as first order ode lie symmetry lookup ode

Writing the ode as

$$
\begin{aligned}
y' &= \frac{y\,x^2-2x^2-yx+2x-y+2}{y\,x^2} \\
y' &= \omega(x,y)
\end{aligned}
$$

The condition of Lie symmetry is the linearized PDE given by

$$
\eta_x+\omega(\eta_y-\xi_x)-\omega^2\xi_y-\omega_x\xi-\omega_y\eta=0 \tag{A}
$$

The type of this ode is known. It is of type `separable`. Therefore we do not need to solve the PDE (A), and can just use the lookup table shown below to find ξ,η

Table 278: Lie symmetry infinitesimal lookup table for known first order ODE's

ODE class	Form	ξ	η
linear ode	$y' = f(x)y(x) + g(x)$	0	$e^{\int f dx}$
separable ode	$y' = f(x)\,g(y)$	$\frac{1}{f}$	0
quadrature ode	$y' = f(x)$	0	1
quadrature ode	$y' = g(y)$	1	0
homogeneous ODEs of Class A	$y' = f\left(\frac{y}{x}\right)$	x	y
homogeneous ODEs of Class C	$y' = (a + bx + cy)^{\frac{n}{m}}$	1	$-\frac{b}{c}$
homogeneous class D	$y' = \frac{y}{x} + g(x)\,F\left(\frac{y}{x}\right)$	x^2	xy
First order special form ID 1	$y' = g(x)\,e^{h(x)+by} + f(x)$	$\frac{e^{-\int bf(x)dx-h(x)}}{g(x)}$	$\frac{f(x)e^{-\int bf(x)dx-h(x)}}{g(x)}$
polynomial type ode	$y' = \frac{a_1x+b_1y+c_1}{a_2x+b_2y+c_2}$	$\frac{a_1b_2x-a_2b_1x-b_1c_2+b_2c_1}{a_1b_2-a_2b_1}$	$\frac{a_1b_2y-a_2b_1y-a_1c_2-a_2c_1}{a_1b_2-a_2b_1}$
Bernoulli ode	$y' = f(x)\,y + g(x)\,y^n$	0	$e^{-\int(n-1)f(x)dx}y^n$
Reduced Riccati	$y' = f_1(x)\,y + f_2(x)\,y^2$	0	$e^{-\int f_1 dx}$

The above table shows that

$$\begin{aligned}\xi(x,y) &= \frac{x^2}{x^2 - x - 1}\\ \eta(x,y) &= 0 \end{aligned}\tag{A1}$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x,y) \to (R,S)$ where (R,S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}$$

The above comes from the requirements that $\left(\xi\frac{\partial}{\partial x} + \eta\frac{\partial}{\partial y}\right) S(x,y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the

canonical coordinates, where $S(R)$. Since $\eta = 0$ then in this special case

$$R = y$$

S is found from

$$\begin{aligned} S &= \int \frac{1}{\xi} dx \\ &= \int \frac{1}{\frac{x^2}{x^2-x-1}} dx \end{aligned}$$

Which results in

$$S = x - \ln(x) + \frac{1}{x}$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x,y) S_y}{R_x + \omega(x,y) R_y} \tag{2}$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x, y)$ is the right hand side of the original ode given by

$$\omega(x,y) = \frac{y\,x^2 - 2x^2 - yx + 2x - y + 2}{y\,x^2}$$

Evaluating all the partial derivatives gives

$$\begin{aligned} R_x &= 0 \\ R_y &= 1 \\ S_x &= \frac{x^2 - x - 1}{x^2} \\ S_y &= 0 \end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = \frac{y}{y-2} \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = \frac{R}{R-2}$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordinates R, S. Integrating the above gives

$$S(R) = R + 2\ln(R-2) + c_1 \tag{4}$$

To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$\frac{-x\ln(x) + x^2 + 1}{x} = y + 2\ln(y-2) + c_1$$

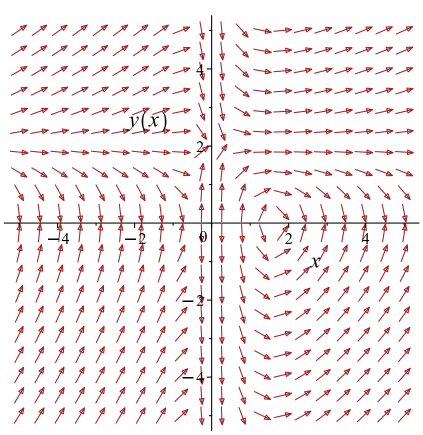
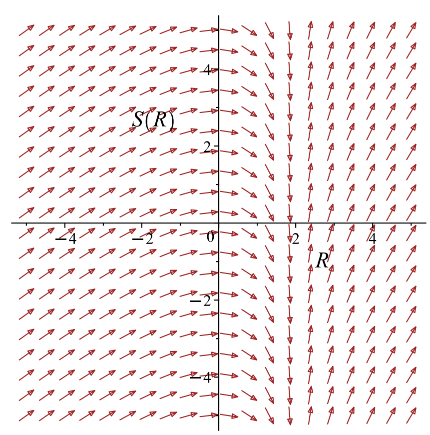
Which simplifies to

$$\frac{-x\ln(x) + x^2 + 1}{x} = y + 2\ln(y-2) + c_1$$

Which gives

$$y = 2\,\mathrm{LambertW}\left(\frac{\mathrm{e}^{-\frac{x\ln(x)+c_1x-x^2+2x-1}{2x}}}{2}\right) + 2$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in x, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dx} = \frac{y\,x^2-2x^2-yx+2x-y+2}{y\,x^2}$  	$R = y$ $S = \frac{-x\ln(x) + x^2 + 1}{x}$	$\frac{dS}{dR} = \frac{R}{R-2}$ 

Summary

The solution(s) found are the following

$$y = 2\,\mathrm{LambertW}\left(\frac{\mathrm{e}^{-\frac{x\ln(x)+c_1x-x^2+2x-1}{2x}}}{2}\right)+2 \tag{1}$$

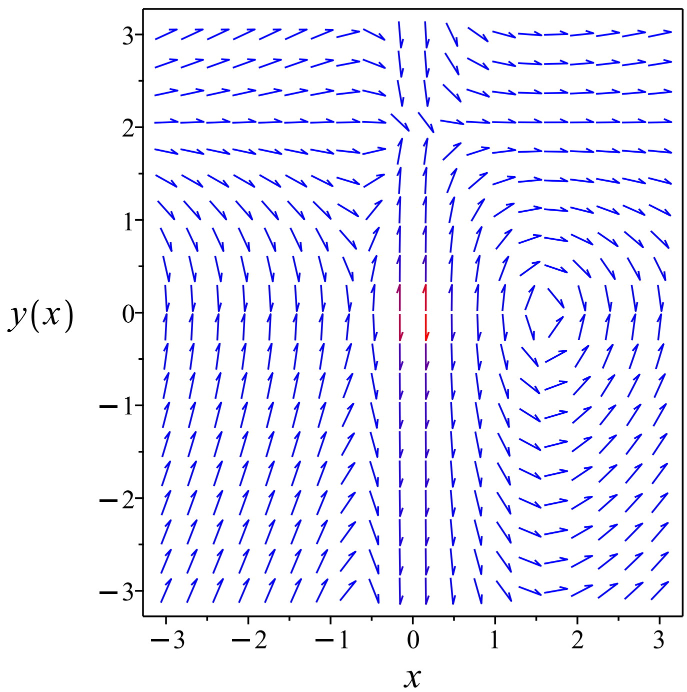


Figure 297: Slope field plot

Verification of solutions

$$y = 2\,\mathrm{LambertW}\left(\frac{\mathrm{e}^{-\frac{x\ln(x)+c_1x-x^2+2x-1}{2x}}}{2}\right)+2$$

Verified OK.

6.10.3 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x,y)+N(x,y)\frac{dy}{dx}=0 \tag{A}$$

We assume there exists a function $\phi(x,y)=c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y)=0$$

Hence

$$\frac{\partial\phi}{\partial x}+\frac{\partial\phi}{\partial y}\frac{dy}{dx}=0 \tag{B}$$

Comparing (A,B) shows that

$$\begin{aligned}\frac{\partial\phi}{\partial x}&=M\\ \frac{\partial\phi}{\partial y}&=N\end{aligned}$$

But since $\frac{\partial^2\phi}{\partial x\partial y}=\frac{\partial^2\phi}{\partial y\partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y}=\frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x,y)$ but at least we know now that we can do that since the condition $\frac{\partial^2\phi}{\partial x\partial y}=\frac{\partial^2\phi}{\partial y\partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x,y)\,\mathrm{d}x+N(x,y)\,\mathrm{d}y=0 \tag{1A}$$

Therefore

$$\begin{aligned}\left(\frac{y}{y-2}\right)\mathrm{d}y&=\left(\frac{x^2-x-1}{x^2}\right)\mathrm{d}x\\ \left(-\frac{x^2-x-1}{x^2}\right)\mathrm{d}x+\left(\frac{y}{y-2}\right)\mathrm{d}y&=0 \end{aligned} \tag{2A}$$

Comparing (1A) and (2A) shows that

$$\begin{aligned}M(x,y) &= -\frac{x^2-x-1}{x^2}\\ N(x,y) &= \frac{y}{y-2}\end{aligned}$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Using result found above gives

$$\begin{aligned}\frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}\left(-\frac{x^2-x-1}{x^2}\right)\\ &= 0\end{aligned}$$

And

$$\begin{aligned}\frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}\left(\frac{y}{y-2}\right)\\ &= 0\end{aligned}$$

Since $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$, then the ODE is exact The following equations are now set up to solve for the function $\phi(x,y)$

$$\frac{\partial \phi}{\partial x} = M \tag{1}$$

$$\frac{\partial \phi}{\partial y} = N \tag{2}$$

Integrating (1) w.r.t. x gives

$$\int \frac{\partial \phi}{\partial x}\,\mathrm{d}x = \int M\,\mathrm{d}x$$

$$\int \frac{\partial \phi}{\partial x}\,\mathrm{d}x = \int -\frac{x^2-x-1}{x^2}\,\mathrm{d}x$$

$$\phi = -x + \ln(x) - \frac{1}{x} + f(y) \tag{3}$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$\frac{\partial \phi}{\partial y} = 0 + f'(y) \tag{4}$$

But equation (2) says that $\frac{\partial\phi}{\partial y} = \frac{y}{y-2}$. Therefore equation (4) becomes

$$\frac{y}{y-2} = 0 + f'(y) \tag{5}$$

Solving equation (5) for $f'(y)$ gives

$$f'(y) = \frac{y}{y-2}$$

Integrating the above w.r.t y gives

$$\int f'(y)\,\mathrm{d}y = \int \left(\frac{y}{y-2}\right) \mathrm{d}y$$
$$f(y) = y + 2\ln(y-2) + c_1$$

Where c_1 is constant of integration. Substituting result found above for $f(y)$ into equation (3) gives ϕ

$$\phi = -x + \ln(x) - \frac{1}{x} + y + 2\ln(y-2) + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = -x + \ln(x) - \frac{1}{x} + y + 2\ln(y-2)$$

The solution becomes

$$y = \mathrm{e}^{-\frac{x\ln(x)+2\,\mathrm{LambertW}\left(\frac{\mathrm{e}^{-\frac{-1-c_1x-x^2+x\ln(x)+2x}{2x}}}{2}\right)x-c_1x-x^2+2x-1}{2x}} + 2$$

Summary

The solution(s) found are the following

$$y = \mathrm{e}^{-\frac{x\ln(x)+2\,\mathrm{LambertW}\left(\frac{\mathrm{e}^{-\frac{-1-c_1x-x^2+x\ln(x)+2x}{2x}}}{2}\right)x-c_1x-x^2+2x-1}{2x}} + 2 \tag{1}$$

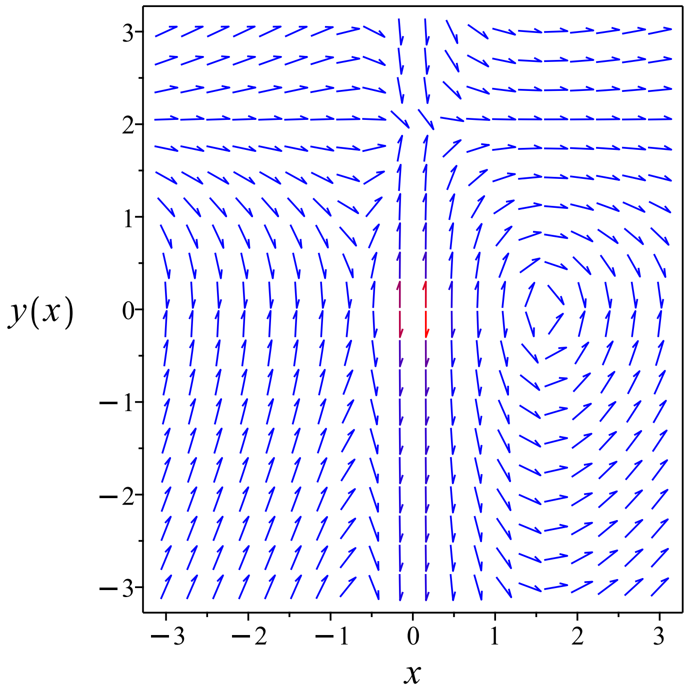


Figure 298: Slope field plot

Verification of solutions

$$y = \mathrm{e}^{-\frac{x \ln(x)+2 \,\mathrm{LambertW}\left(\frac{\mathrm{e}^{-\frac{-1-c_1 x-x^2+x \ln(x)+2x}{2x}}}{2}\right) x-c_1 x-x^2+2x-1}{2x}}+2$$

Verified OK.

6.10.4 Maple step by step solution

Let's solve

$\frac{yy'}{y-2} = -\frac{-x^2+x+1}{x^2}$

- Highest derivative means the order of the ODE is 1

  y'

- Integrate both sides with respect to x

  $\int \frac{yy'}{y-2}dx = \int -\frac{-x^2+x+1}{x^2}dx + c_1$

- Evaluate integral

  $y + 2\ln(y-2) = x - \ln(x) + \frac{1}{x} + c_1$

- Solve for y

$$y = \mathrm{e}^{-\frac{x\ln(x)+2\,LambertW\left(\frac{\mathrm{e}^{-\frac{-1-c_1x-x^2+x\ln(x)+2x}{2x}}}{2}\right)x-c_1x-x^2+2x-1}{2x}}+2$$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
<- separable successful`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 26

```
dsolve((-x^2+x+1)/x^2+y(x)*diff(y(x),x)/(-2+y(x)) = 0,y(x), singsol=all)
```

$$y(x) = 2\,\mathrm{LambertW}\left(\frac{c_1\mathrm{e}^{\frac{(x-1)^2}{2x}}}{2\sqrt{x}}\right)+2$$

✓ Solution by Mathematica

Time used: 60.036 (sec). Leaf size: 63

```
DSolve[(-x^2+x+1)/x^2+y[x]*y'[x]/(-2+y[x]) == 0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to 2\left(1+W\left(-\frac{1}{2}\sqrt{\frac{e^{x+\frac{1}{x}-2+c_1}}{x}}\right)\right)$$
$$y(x) \to 2\left(1+W\left(\frac{1}{2}\sqrt{\frac{e^{x+\frac{1}{x}-2+c_1}}{x}}\right)\right)$$

6.11 problem 11

Internal problem ID [578]
Internal file name [`OUTPUT/578_Sunday_June_05_2022_01_45_06_AM_15814802/index.tex`]

Book: Elementary differential equations and boundary value problems, 10th ed., Boyce and DiPrima
Section: Miscellaneous problems, end of chapter 2. Page 133
Problem number: 11.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"exact", "differentialType"**

Maple gives the following as the ode type

`[_exact]`

$$\boxed{y + (e^y + x)\, y' = -x^2}$$

6.11.1 Solving as differentialType ode

Writing the ode as

$$y' = \frac{-x^2 - y}{e^y + x} \tag{1}$$

Which becomes

$$(e^y)\, dy = (-x)\, dy + \left(-x^2 - y\right) dx \tag{2}$$

But the RHS is complete differential because

$$(-x)\, dy + \left(-x^2 - y\right) dx = d\left(-\frac{1}{3}x^3 - yx\right)$$

Hence (2) becomes

$$(e^y)\, dy = d\left(-\frac{1}{3}x^3 - yx\right)$$

Integrating both sides gives gives these solutions

$$y = -\operatorname{LambertW}\left(\frac{e^{\frac{-x^3+3c_1}{3x}}}{x}\right) + \frac{-x^3+3c_1}{3x} + c_1$$

Summary

The solution(s) found are the following

$$y = -\operatorname{LambertW}\left(\frac{e^{\frac{-x^3+3c_1}{3x}}}{x}\right) + \frac{-x^3+3c_1}{3x} + c_1 \tag{1}$$

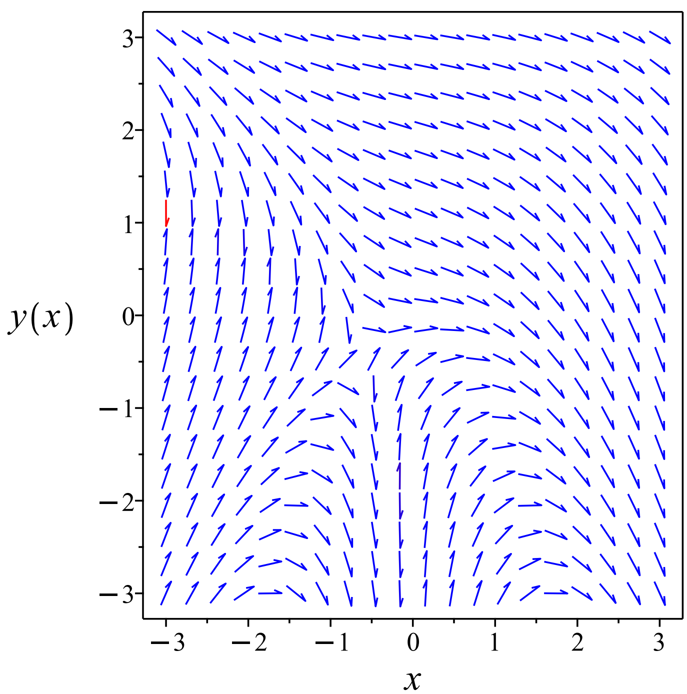


Figure 299: Slope field plot

Verification of solutions

$$y = -\operatorname{LambertW}\left(\frac{e^{\frac{-x^3+3c_1}{3x}}}{x}\right) + \frac{-x^3+3c_1}{3x} + c_1$$

Verified OK.

6.11.2 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x,y)+N(x,y)\frac{dy}{dx}=0 \tag{A}$$

We assume there exists a function $\phi(x,y)=c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y)=0$$

Hence

$$\frac{\partial\phi}{\partial x}+\frac{\partial\phi}{\partial y}\frac{dy}{dx}=0 \tag{B}$$

Comparing (A,B) shows that

$$\begin{aligned}\frac{\partial\phi}{\partial x}&=M\\ \frac{\partial\phi}{\partial y}&=N\end{aligned}$$

But since $\frac{\partial^2\phi}{\partial x\partial y}=\frac{\partial^2\phi}{\partial y\partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y}=\frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x,y)$ but at least we know now that we can do that since the condition $\frac{\partial^2\phi}{\partial x\partial y}=\frac{\partial^2\phi}{\partial y\partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x,y)\,\mathrm{d}x+N(x,y)\,\mathrm{d}y=0 \tag{1A}$$

Therefore

$$\begin{aligned}(\mathrm{e}^y+x)\,\mathrm{d}y&=\left(-x^2-y\right)\mathrm{d}x\\ \left(x^2+y\right)\mathrm{d}x+(\mathrm{e}^y+x)\,\mathrm{d}y&=0 \end{aligned}\tag{2A}$$

Comparing (1A) and (2A) shows that

$$\begin{aligned}M(x,y)&=x^2+y\\ N(x,y)&=\mathrm{e}^y+x\end{aligned}$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Using result found above gives

$$\begin{aligned}\frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}(x^2 + y)\\ &= 1\end{aligned}$$

And

$$\begin{aligned}\frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}(\mathrm{e}^y + x)\\ &= 1\end{aligned}$$

Since $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$, then the ODE is exact The following equations are now set up to solve for the function $\phi(x, y)$

$$\frac{\partial \phi}{\partial x} = M \tag{1}$$

$$\frac{\partial \phi}{\partial y} = N \tag{2}$$

Integrating (1) w.r.t. x gives

$$\int \frac{\partial \phi}{\partial x} \mathrm{d}x = \int M \,\mathrm{d}x$$

$$\int \frac{\partial \phi}{\partial x} \mathrm{d}x = \int x^2 + y \,\mathrm{d}x$$

$$\phi = \frac{1}{3}x^3 + yx + f(y) \tag{3}$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$\frac{\partial \phi}{\partial y} = x + f'(y) \tag{4}$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = \mathrm{e}^y + x$. Therefore equation (4) becomes

$$\mathrm{e}^y + x = x + f'(y) \tag{5}$$

Solving equation (5) for $f'(y)$ gives

$$f'(y) = e^y$$

Integrating the above w.r.t y gives

$$\int f'(y)\, dy = \int (e^y)\, dy$$
$$f(y) = e^y + c_1$$

Where c_1 is constant of integration. Substituting result found above for $f(y)$ into equation (3) gives ϕ

$$\phi = \frac{x^3}{3} + yx + e^y + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = \frac{x^3}{3} + yx + e^y$$

The solution becomes

$$y = -\operatorname{LambertW}\left(\frac{e^{\frac{-x^3+3c_1}{3x}}}{x}\right) + \frac{-x^3 + 3c_1}{3x}$$

<u>Summary</u>
The solution(s) found are the following

$$y = -\operatorname{LambertW}\left(\frac{e^{\frac{-x^3+3c_1}{3x}}}{x}\right) + \frac{-x^3 + 3c_1}{3x} \tag{1}$$

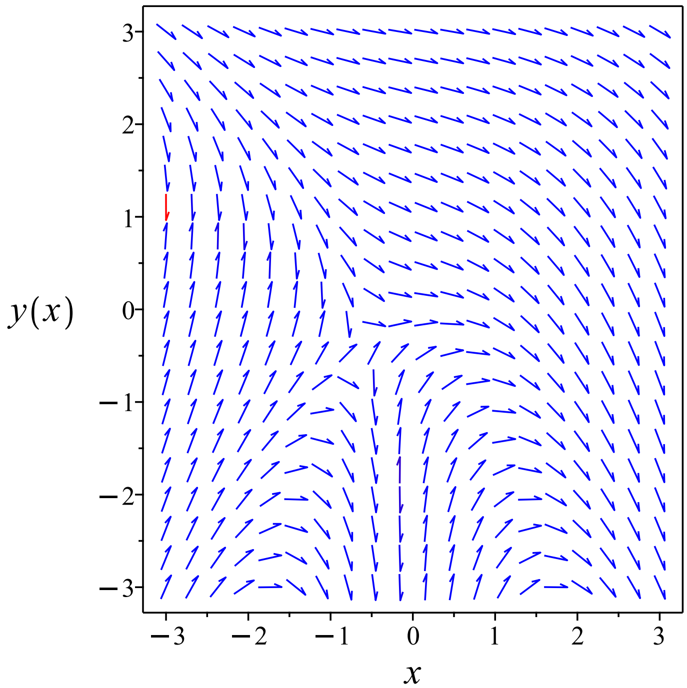


Figure 300: Slope field plot

Verification of solutions

$$y = -\operatorname{LambertW}\left(\frac{e^{\frac{-x^3+3c_1}{3x}}}{x}\right) + \frac{-x^3+3c_1}{3x}$$

Verified OK.

6.11.3 Maple step by step solution

Let's solve

$y + (e^y + x)\, y' = -x^2$

- Highest derivative means the order of the ODE is 1

  y'

□ Check if ODE is exact

  ◦ ODE is exact if the lhs is the total derivative of a C^2 function

    $F'(x, y) = 0$

  ◦ Compute derivative of lhs

$F'(x,y)+\left(\frac{\partial}{\partial y}F(x,y)\right)y'=0$

- Evaluate derivatives

  $1=1$

- Condition met, ODE is exact

- Exact ODE implies solution will be of this form

  $\left[F(x,y)=c_1, M(x,y)=F'(x,y), N(x,y)=\frac{\partial}{\partial y}F(x,y)\right]$

- Solve for $F(x,y)$ by integrating $M(x,y)$ with respect to x

  $F(x,y)=\int (x^2+y)\,dx+f_1(y)$

- Evaluate integral

  $F(x,y)=\frac{x^3}{3}+yx+f_1(y)$

- Take derivative of $F(x,y)$ with respect to y

  $N(x,y)=\frac{\partial}{\partial y}F(x,y)$

- Compute derivative

  $e^y+x=x+\frac{d}{dy}f_1(y)$

- Isolate for $\frac{d}{dy}f_1(y)$

  $\frac{d}{dy}f_1(y)=e^y$

- Solve for $f_1(y)$

  $f_1(y)=e^y$

- Substitute $f_1(y)$ into equation for $F(x,y)$

  $F(x,y)=\frac{x^3}{3}+yx+e^y$

- Substitute $F(x,y)$ into the solution of the ODE

  $\frac{x^3}{3}+yx+e^y=c_1$

- Solve for y

  $y=-LambertW\left(\frac{e^{\frac{-x^3+3c_1}{3x}}}{x}\right)+\frac{-x^3+3c_1}{3x}$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
trying homogeneous types:
trying Chini
differential order: 1; looking for linear symmetries
trying exact
<- exact successful`
```

✓ Solution by Maple

Time used: 0.015 (sec). Leaf size: 39

```
dsolve(x^2+y(x)+(exp(y(x))+x)*diff(y(x),x) = 0,y(x), singsol=all)
```

$$y(x) = \frac{-x^3 - 3x \operatorname{LambertW}\left(\frac{e^{-\frac{x^3+3c_1}{3x}}}{x}\right) - 3c_1}{3x}$$

✓ Solution by Mathematica

Time used: 3.877 (sec). Leaf size: 42

```
DSolve[x^2+y[x]+(Exp[y[x]]+x)*y'[x] == 0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to -W\left(\frac{e^{-\frac{x^2}{3}+\frac{c_1}{x}}}{x}\right) - \frac{x^2}{3} + \frac{c_1}{x}$$

6.12 problem 12

Internal problem ID [579]
Internal file name [`OUTPUT/579_Sunday_June_05_2022_01_45_07_AM_95196417/index.tex`]

Book: Elementary differential equations and boundary value problems, 10th ed., Boyce and DiPrima
Section: Miscellaneous problems, end of chapter 2. Page 133
Problem number: 12.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"linear", "exactWithIntegrationFactor", "first_order_ode_lie_symmetry_lookup"**

Maple gives the following as the ode type

```
[_linear]
```

$$\boxed{y' + y = \frac{1}{1 + e^{x}}}$$

6.12.1 Solving as linear ode

Entering Linear first order ODE solver. In canonical form a linear first order is

$$y' + p(x)y = q(x)$$

Where here

$$p(x) = 1$$
$$q(x) = \frac{1}{1 + e^{x}}$$

Hence the ode is

$$y' + y = \frac{1}{1 + e^{x}}$$

The integrating factor μ is

$$\mu = e^{\int 1 dx}$$
$$= e^x$$

The ode becomes

$$\frac{d}{dx}(\mu y) = (\mu)\left(\frac{1}{1+e^x}\right)$$
$$\frac{d}{dx}(e^x y) = (e^x)\left(\frac{1}{1+e^x}\right)$$
$$d(e^x y) = \left(\frac{e^x}{1+e^x}\right) dx$$

Integrating gives

$$e^x y = \int \frac{e^x}{1+e^x} dx$$
$$e^x y = \ln(1+e^x) + c_1$$

Dividing both sides by the integrating factor $\mu = e^x$ results in

$$y = e^{-x}\ln(1+e^x) + c_1 e^{-x}$$

which simplifies to

$$y = e^{-x}(\ln(1+e^x) + c_1)$$

Summary

The solution(s) found are the following

$$y = e^{-x}(\ln(1+e^x) + c_1) \tag{1}$$

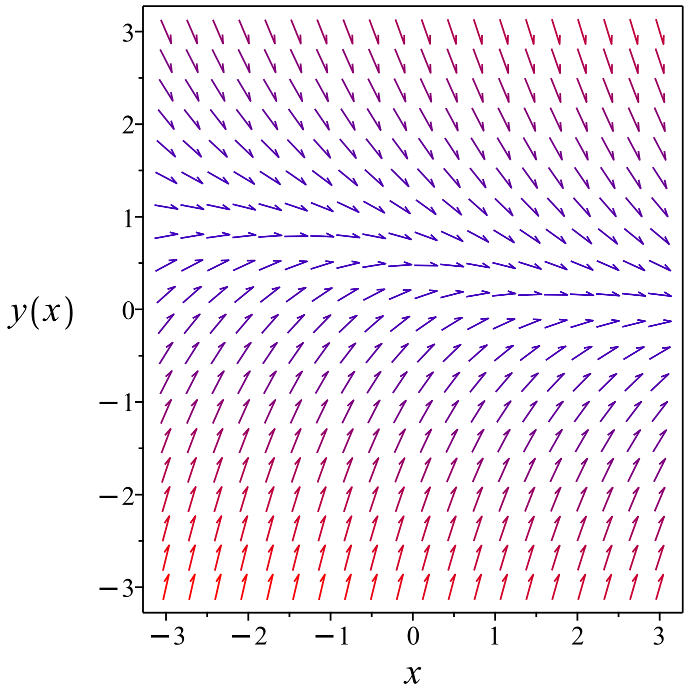


Figure 301: Slope field plot

Verification of solutions

$$y = e^{-x}(\ln(1 + e^x) + c_1)$$

Verified OK.

6.12.2 Solving as first order ode lie symmetry lookup ode

Writing the ode as

$$y' = -\frac{e^x y + y - 1}{1 + e^x}$$
$$y' = \omega(x, y)$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x + \omega(\eta_y - \xi_x) - \omega^2 \xi_y - \omega_x \xi - \omega_y \eta = 0 \tag{A}$$

The type of this ode is known. It is of type `linear`. Therefore we do not need to solve the PDE (A), and can just use the lookup table shown below to find ξ, η

Table 282: Lie symmetry infinitesimal lookup table for known first order ODE's

ODE class	Form	ξ	η
linear ode	$y' = f(x)y(x) + g(x)$	0	$e^{\int f dx}$
separable ode	$y' = f(x)\, g(y)$	$\frac{1}{f}$	0
quadrature ode	$y' = f(x)$	0	1
quadrature ode	$y' = g(y)$	1	0
homogeneous ODEs of Class A	$y' = f\left(\frac{y}{x}\right)$	x	y
homogeneous ODEs of Class C	$y' = (a + bx + cy)^{\frac{n}{m}}$	1	$-\frac{b}{c}$
homogeneous class D	$y' = \frac{y}{x} + g(x)\, F\left(\frac{y}{x}\right)$	x^2	xy
First order special form ID 1	$y' = g(x)\, e^{h(x)+by} + f(x)$	$\frac{e^{-\int bf(x)dx - h(x)}}{g(x)}$	$\frac{f(x)e^{-\int bf(x)dx - h(x)}}{g(x)}$
polynomial type ode	$y' = \frac{a_1x+b_1y+c_1}{a_2x+b_2y+c_2}$	$\frac{a_1b_2x - a_2b_1x - b_1c_2 + b_2c_1}{a_1b_2 - a_2b_1}$	$\frac{a_1b_2y - a_2b_1y - a_1c_2 - a_2c_1}{a_1b_2 - a_2b_1}$
Bernoulli ode	$y' = f(x)\, y + g(x)\, y^n$	0	$e^{-\int (n-1)f(x)dx} y^n$
Reduced Riccati	$y' = f_1(x)\, y + f_2(x)\, y^2$	0	$e^{-\int f_1 dx}$

The above table shows that

$$
\begin{aligned}
\xi(x, y) &= 0 \\
\eta(x, y) &= e^{-x}
\end{aligned} \tag{A1}
$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x, y) \to (R, S)$ where (R, S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$
\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}
$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y}\right) S(x, y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the

canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$R = x$$

S is found from

$$\begin{aligned} S &= \int \frac{1}{\eta} dy \\ &= \int \frac{1}{e^{-x}} dy \end{aligned}$$

Which results in

$$S = e^x y$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x, y)S_y}{R_x + \omega(x, y)R_y} \tag{2}$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x, y)$ is the right hand side of the original ode given by

$$\omega(x, y) = -\frac{e^x y + y - 1}{1 + e^x}$$

Evaluating all the partial derivatives gives

$$\begin{aligned} R_x &= 1 \\ R_y &= 0 \\ S_x &= e^x y \\ S_y &= e^x \end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = \frac{e^x}{1 + e^x} \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = \frac{e^R}{1 + e^R}$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by

integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$S(R) = \ln\left(1 + e^{R}\right) + c_1 \tag{4}$$

To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$e^{x} y = \ln\left(1 + e^{x}\right) + c_1$$

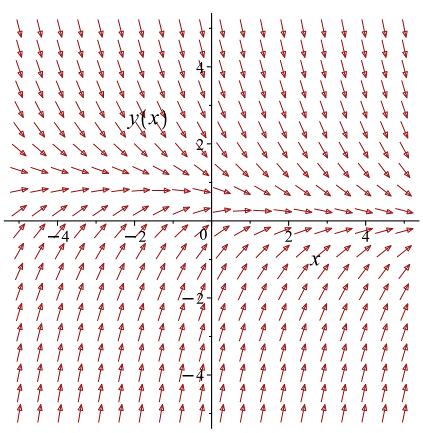
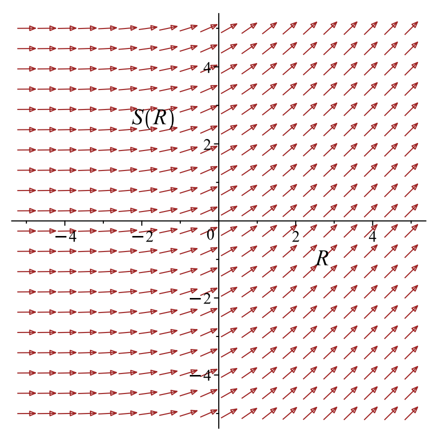
Which simplifies to

$$e^{x} y = \ln\left(1 + e^{x}\right) + c_1$$

Which gives

$$y = e^{-x}\left(\ln\left(1 + e^{x}\right) + c_1\right)$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in x, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dx} = -\frac{e^{x} y + y - 1}{1 + e^{x}}$ 	$R = x$ $S = e^{x} y$	$\frac{dS}{dR} = \frac{e^{R}}{1 + e^{R}}$ 

Summary

The solution(s) found are the following

$$y = e^{-x}\left(\ln\left(1 + e^{x}\right) + c_1\right) \tag{1}$$

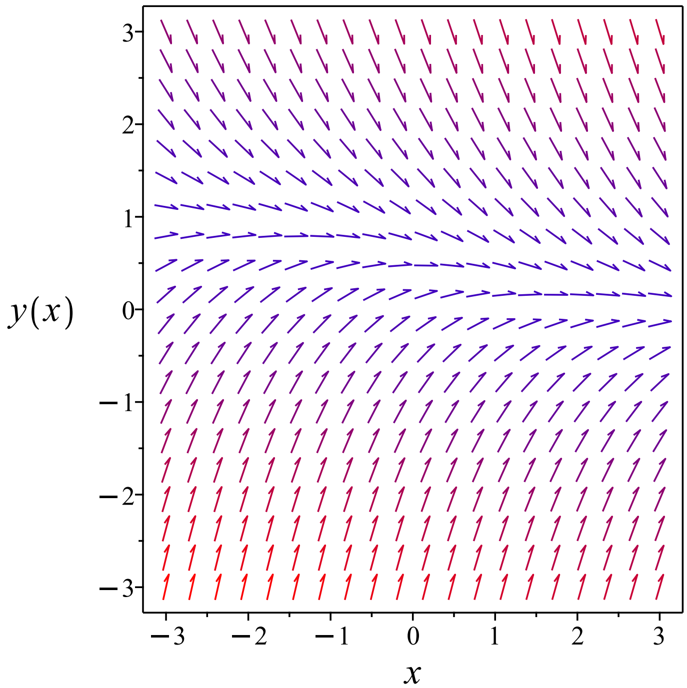


Figure 302: Slope field plot

Verification of solutions

$$y = e^{-x}(\ln(1 + e^x) + c_1)$$

Verified OK.

6.12.3 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x, y) + N(x, y)\frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x, y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x, y) = 0$$

Hence

$$\frac{\partial\phi}{\partial x} + \frac{\partial\phi}{\partial y}\frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$\frac{\partial \phi}{\partial x} = M$$
$$\frac{\partial \phi}{\partial y} = N$$

But since $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x,y)$ but at least we know now that we can do that since the condition $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x,y)\,\mathrm{d}x + N(x,y)\,\mathrm{d}y = 0 \tag{1A}$$

Therefore

$$\mathrm{d}y = \left(-y + \frac{1}{1+\mathrm{e}^x}\right)\mathrm{d}x$$
$$\left(y - \frac{1}{1+\mathrm{e}^x}\right)\mathrm{d}x + \mathrm{d}y = 0 \tag{2A}$$

Comparing (1A) and (2A) shows that

$$M(x,y) = y - \frac{1}{1+\mathrm{e}^x}$$
$$N(x,y) = 1$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Using result found above gives

$$\begin{aligned}\frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}\left(y - \frac{1}{1+\mathrm{e}^x}\right)\\ &= 1\end{aligned}$$

And

$$\begin{aligned}
\frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}(1) \\
&= 0
\end{aligned}$$

Since $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$, then the ODE is <u>not exact</u>. Since the ODE is not exact, we will try to find an integrating factor to make it exact. Let

$$\begin{aligned}
A &= \frac{1}{N}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right) \\
&= 1((1) - (0)) \\
&= 1
\end{aligned}$$

Since A does not depend on y, then it can be used to find an integrating factor. The integrating factor μ is

$$\begin{aligned}
\mu &= e^{\int A \,\mathrm{d}x} \\
&= e^{\int 1 \,\mathrm{d}x}
\end{aligned}$$

The result of integrating gives

$$\begin{aligned}
\mu &= e^{x} \\
&= \mathrm{e}^{x}
\end{aligned}$$

M and N are multiplied by this integrating factor, giving new M and new N which are called $\overline{M}$ and $\overline{N}$ for now so not to confuse them with the original M and N.

$$\begin{aligned}
\overline{M} &= \mu M \\
&= \mathrm{e}^{x}\left(y - \frac{1}{1 + \mathrm{e}^{x}}\right) \\
&= \frac{\mathrm{e}^{x}(\mathrm{e}^{x} y + y - 1)}{1 + \mathrm{e}^{x}}
\end{aligned}$$

And

$$\begin{aligned}
\overline{N} &= \mu N \\
&= \mathrm{e}^{x}(1) \\
&= \mathrm{e}^{x}
\end{aligned}$$

Now a modified ODE is ontained from the original ODE, which is exact and can be solved. The modified ODE is

$$\begin{aligned}
\overline{M} + \overline{N}\frac{\mathrm{d}y}{\mathrm{d}x} &= 0 \\
\left(\frac{\mathrm{e}^{x}(\mathrm{e}^{x} y + y - 1)}{1 + \mathrm{e}^{x}}\right) + (\mathrm{e}^{x})\frac{\mathrm{d}y}{\mathrm{d}x} &= 0
\end{aligned}$$

The following equations are now set up to solve for the function $\phi(x,y)$

$$\frac{\partial \phi}{\partial x} = \overline{M} \tag{1}$$

$$\frac{\partial \phi}{\partial y} = \overline{N} \tag{2}$$

Integrating (1) w.r.t. x gives

$$\int \frac{\partial \phi}{\partial x} \,\mathrm{d}x = \int \overline{M} \,\mathrm{d}x$$

$$\int \frac{\partial \phi}{\partial x} \,\mathrm{d}x = \int \frac{\mathrm{e}^x(\mathrm{e}^x y + y - 1)}{1 + \mathrm{e}^x} \,\mathrm{d}x$$

$$\phi = \mathrm{e}^x y - \ln(1 + \mathrm{e}^x) + f(y) \tag{3}$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$\frac{\partial \phi}{\partial y} = \mathrm{e}^x + f'(y) \tag{4}$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = \mathrm{e}^x$. Therefore equation (4) becomes

$$\mathrm{e}^x = \mathrm{e}^x + f'(y) \tag{5}$$

Solving equation (5) for $f'(y)$ gives

$$f'(y) = 0$$

Therefore

$$f(y) = c_1$$

Where c_1 is constant of integration. Substituting this result for $f(y)$ into equation (3) gives ϕ

$$\phi = \mathrm{e}^x y - \ln(1 + \mathrm{e}^x) + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = \mathrm{e}^x y - \ln(1 + \mathrm{e}^x)$$

The solution becomes

$$y = \mathrm{e}^{-x}(\ln(1+\mathrm{e}^{x}) + c_1)$$

Summary

The solution(s) found are the following

$$y = \mathrm{e}^{-x}(\ln(1+\mathrm{e}^{x}) + c_1) \tag{1}$$

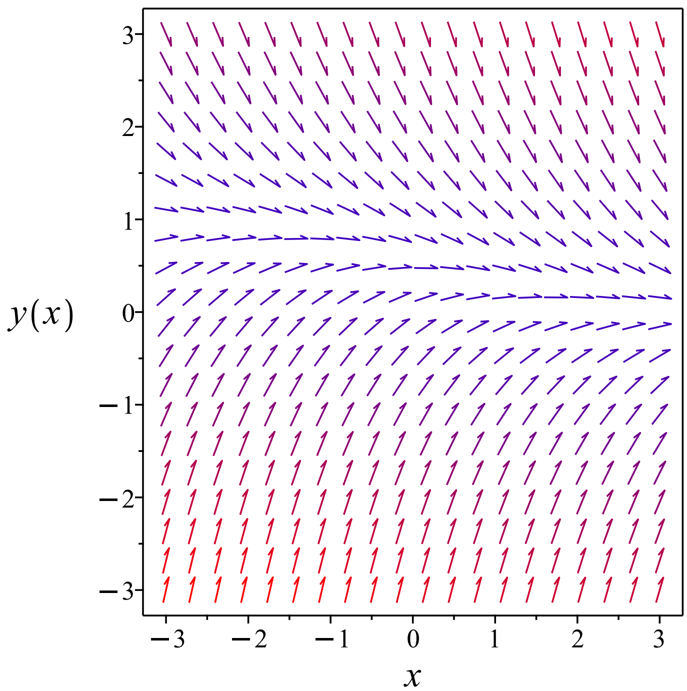


Figure 303: Slope field plot

Verification of solutions

$$y = \mathrm{e}^{-x}(\ln(1+\mathrm{e}^{x}) + c_1)$$

Verified OK.

6.12.4 Maple step by step solution

Let's solve

$y' + y = \frac{1}{1+e^x}$

- Highest derivative means the order of the ODE is 1

  y'

- Isolate the derivative

  $y' = -y + \frac{1}{1+e^x}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE

  $y' + y = \frac{1}{1+e^x}$

- The ODE is linear; multiply by an integrating factor $\mu(x)$

  $\mu(x)\left(y' + y\right) = \frac{\mu(x)}{1+e^x}$

- Assume the lhs of the ODE is the total derivative $\frac{d}{dx}(\mu(x)\, y)$

  $\mu(x)\left(y' + y\right) = \mu'(x)\, y + \mu(x)\, y'$

- Isolate $\mu'(x)$

  $\mu'(x) = \mu(x)$

- Solve to find the integrating factor

  $\mu(x) = e^x$

- Integrate both sides with respect to x

  $\int \left(\frac{d}{dx}(\mu(x)\, y)\right) dx = \int \frac{\mu(x)}{1+e^x} dx + c_1$

- Evaluate the integral on the lhs

  $\mu(x)\, y = \int \frac{\mu(x)}{1+e^x} dx + c_1$

- Solve for y

  $y = \frac{\int \frac{\mu(x)}{1+e^x} dx + c_1}{\mu(x)}$

- Substitute $\mu(x) = e^x$

  $y = \frac{\int \frac{e^x}{1+e^x} dx + c_1}{e^x}$

- Evaluate the integrals on the rhs

  $y = \frac{\ln(1+e^x) + c_1}{e^x}$

- Simplify

$$y = e^{-x}(\ln(1 + e^x) + c_1)$$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
<- 1st order linear successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 16

```
dsolve(y(x)+diff(y(x),x) = 1/(1+exp(x)),y(x), singsol=all)
```

$$y(x) = (\ln(1 + e^x) + c_1) e^{-x}$$

✓ Solution by Mathematica

Time used: 0.067 (sec). Leaf size: 20

```
DSolve[y[x]+y'[x] == 1/(1+Exp[x]),y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to e^{-x}(\log(e^x + 1) + c_1)$$

6.13 problem 13

Internal problem ID [580]
Internal file name [`OUTPUT/580_Sunday_June_05_2022_01_45_08_AM_16506274/index.tex`]

Book: Elementary differential equations and boundary value problems, 10th ed., Boyce and DiPrima
Section: Miscellaneous problems, end of chapter 2. Page 133
Problem number: 13.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"exact", "riccati", "separable", "first_order_ode_lie_symmetry_lookup"**

Maple gives the following as the ode type

`[_separable]`

$$\boxed{y' - y^2 - 2x y^2 = 1 + 2x}$$

6.13.1 Solving as separable ode

In canonical form the ODE is

$$\begin{aligned} y' &= F(x,y) \\ &= f(x)g(y) \\ &= (y^2+1)(1+2x) \end{aligned}$$

Where $f(x) = 1 + 2x$ and $g(y) = y^2 + 1$. Integrating both sides gives

$$\frac{1}{y^2+1}\,dy = 1 + 2x\,dx$$

$$\int \frac{1}{y^2+1}\,dy = \int 1 + 2x\,dx$$

$$\arctan(y) = x^2 + c_1 + x$$

Which results in

$$y = \tan\left(x^2 + c_1 + x\right)$$

Summary

The solution(s) found are the following

$$y = \tan\left(x^2 + c_1 + x\right) \tag{1}$$

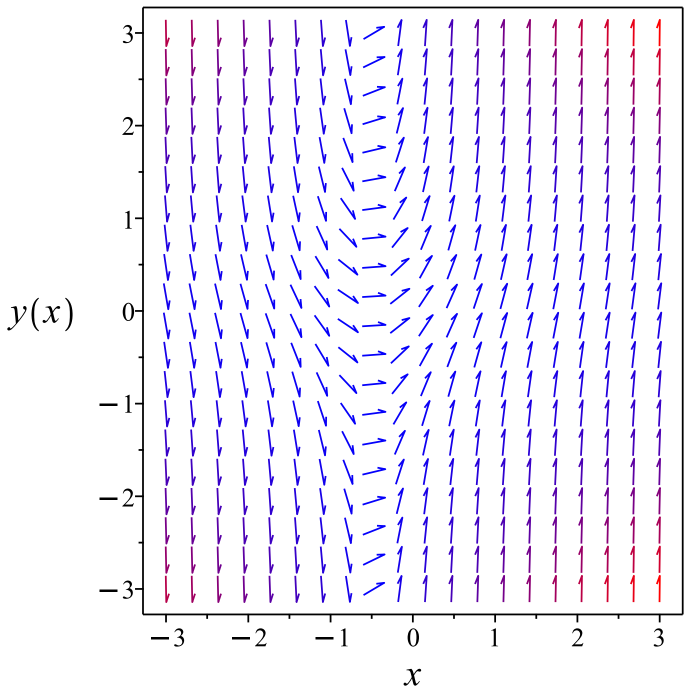


Figure 304: Slope field plot

Verification of solutions

$$y = \tan\left(x^2 + c_1 + x\right)$$

Verified OK.

6.13.2 Solving as first order ode lie symmetry lookup ode

Writing the ode as

$$
\begin{aligned}
y' &= 2x\,y^2 + y^2 + 2x + 1 \\
y' &= \omega(x, y)
\end{aligned}
$$

The condition of Lie symmetry is the linearized PDE given by

$$
\eta_x + \omega(\eta_y - \xi_x) - \omega^2 \xi_y - \omega_x \xi - \omega_y \eta = 0 \tag{A}
$$

The type of this ode is known. It is of type `separable`. Therefore we do not need to solve the PDE (A), and can just use the lookup table shown below to find ξ, η

Table 285: Lie symmetry infinitesimal lookup table for known first order ODE's

ODE class	Form	ξ	η
linear ode	$y' = f(x)y(x) + g(x)$	0	$e^{\int f dx}$
separable ode	$y' = f(x)\,g(y)$	$\frac{1}{f}$	0
quadrature ode	$y' = f(x)$	0	1
quadrature ode	$y' = g(y)$	1	0
homogeneous ODEs of Class A	$y' = f\left(\frac{y}{x}\right)$	x	y
homogeneous ODEs of Class C	$y' = (a + bx + cy)^{\frac{n}{m}}$	1	$-\frac{b}{c}$
homogeneous class D	$y' = \frac{y}{x} + g(x)\,F\left(\frac{y}{x}\right)$	x^2	xy
First order special form ID 1	$y' = g(x)\,e^{h(x)+by} + f(x)$	$\frac{e^{-\int bf(x)dx - h(x)}}{g(x)}$	$\frac{f(x)e^{-\int bf(x)dx - h(x)}}{g(x)}$
polynomial type ode	$y' = \frac{a_1x+b_1y+c_1}{a_2x+b_2y+c_2}$	$\frac{a_1b_2x - a_2b_1x - b_1c_2 + b_2c_1}{a_1b_2 - a_2b_1}$	$\frac{a_1b_2y - a_2b_1y - a_1c_2 - a_2c_1}{a_1b_2 - a_2b_1}$
Bernoulli ode	$y' = f(x)\,y + g(x)\,y^n$	0	$e^{-\int (n-1)f(x)dx} y^n$
Reduced Riccati	$y' = f_1(x)\,y + f_2(x)\,y^2$	0	$e^{-\int f_1 dx}$

The above table shows that

$$
\begin{aligned}
\xi(x,y) &= \frac{1}{1+2x} \\
\eta(x,y) &= 0
\end{aligned} \tag{A1}
$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x,y) \to (R,S)$ where (R,S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$
\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}
$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y}\right) S(x,y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the canonical coordinates, where $S(R)$. Since $\eta = 0$ then in this special case

$$
R = y
$$

S is found from

$$
\begin{aligned}
S &= \int \frac{1}{\xi} dx \\
&= \int \frac{1}{\frac{1}{1+2x}} dx
\end{aligned}
$$

Which results in

$$
S = x^2 + x
$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$
\frac{dS}{dR} = \frac{S_x + \omega(x,y) S_y}{R_x + \omega(x,y) R_y} \tag{2}
$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x,y)$ is the right hand side of the original ode given by

$$
\omega(x,y) = 2x\,y^2 + y^2 + 2x + 1
$$

Evaluating all the partial derivatives gives

$$
\begin{aligned}
R_x &= 0 \\
R_y &= 1 \\
S_x &= 1 + 2x \\
S_y &= 0
\end{aligned}
$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = \frac{1}{y^2 + 1} \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = \frac{1}{R^2 + 1}$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$S(R) = \arctan(R) + c_1 \tag{4}$$

To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$x^2 + x = \arctan(y) + c_1$$

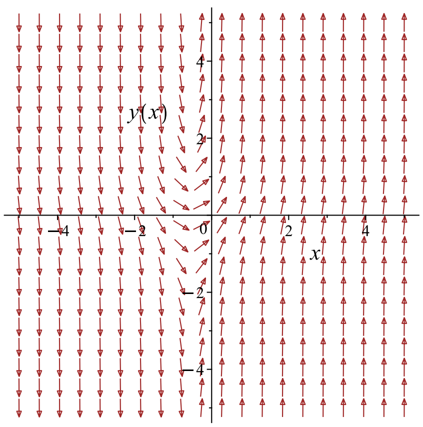
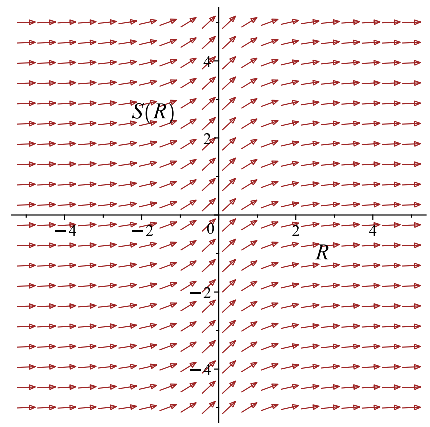
Which simplifies to

$$x^2 + x = \arctan(y) + c_1$$

Which gives

$$y = -\tan\left(-x^2 + c_1 - x\right)$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in x, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dx} = 2x\,y^2 + y^2 + 2x + 1$ 	$R = y$ $S = x^2 + x$	$\frac{dS}{dR} = \frac{1}{R^2+1}$ 

Summary

The solution(s) found are the following

$$y = -\tan\left(-x^2 + c_1 - x\right) \tag{1}$$

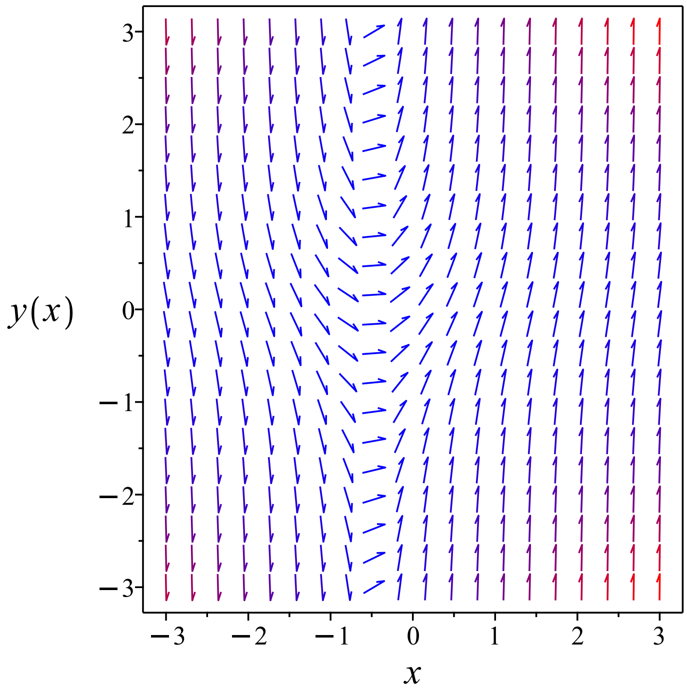


Figure 305: Slope field plot

Verification of solutions

$$y = -\tan\left(-x^2 + c_1 - x\right)$$

Verified OK.

6.13.3 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x,y) + N(x,y)\frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x,y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y) = 0$$

Hence

$$\frac{\partial \phi}{\partial x} + \frac{\partial \phi}{\partial y}\frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$\frac{\partial \phi}{\partial x} = M$$
$$\frac{\partial \phi}{\partial y} = N$$

But since $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x,y)$ but at least we know now that we can do that since the condition $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x,y)\,\mathrm{d}x + N(x,y)\,\mathrm{d}y = 0 \tag{1A}$$

Therefore

$$\left(\frac{1}{y^2+1}\right)\mathrm{d}y = (1+2x)\,\mathrm{d}x$$
$$(-1-2x)\,\mathrm{d}x + \left(\frac{1}{y^2+1}\right)\mathrm{d}y = 0 \tag{2A}$$

Comparing (1A) and (2A) shows that

$$M(x,y) = -1-2x$$
$$N(x,y) = \frac{1}{y^2+1}$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Using result found above gives

$$\begin{aligned}\frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}(-1-2x)\\ &= 0\end{aligned}$$

And

$$
\begin{aligned}
\frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}\left(\frac{1}{y^2+1}\right) \\
&= 0
\end{aligned}
$$

Since $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$, then the ODE is exact The following equations are now set up to solve for the function $\phi(x, y)$

$$
\begin{align}
\frac{\partial \phi}{\partial x} &= M \tag{1} \\
\frac{\partial \phi}{\partial y} &= N \tag{2}
\end{align}
$$

Integrating (1) w.r.t. x gives

$$
\begin{align}
\int \frac{\partial \phi}{\partial x} \mathrm{d}x &= \int M \mathrm{d}x \\
\int \frac{\partial \phi}{\partial x} \mathrm{d}x &= \int -1 - 2x \mathrm{d}x \\
\phi &= -x^2 - x + f(y) \tag{3}
\end{align}
$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$
\frac{\partial \phi}{\partial y} = 0 + f'(y) \tag{4}
$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = \frac{1}{y^2+1}$. Therefore equation (4) becomes

$$
\frac{1}{y^2+1} = 0 + f'(y) \tag{5}
$$

Solving equation (5) for $f'(y)$ gives

$$
f'(y) = \frac{1}{y^2+1}
$$

Integrating the above w.r.t y gives

$$
\begin{aligned}
\int f'(y) \mathrm{d}y &= \int \left(\frac{1}{y^2+1}\right) \mathrm{d}y \\
f(y) &= \arctan(y) + c_1
\end{aligned}
$$

Where c_1 is constant of integration. Substituting result found above for $f(y)$ into equation (3) gives ϕ

$$\phi = -x^2 - x + \arctan(y) + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = -x^2 - x + \arctan(y)$$

The solution becomes

$$y = \tan(x^2 + c_1 + x)$$

Summary

The solution(s) found are the following

$$y = \tan(x^2 + c_1 + x) \tag{1}$$

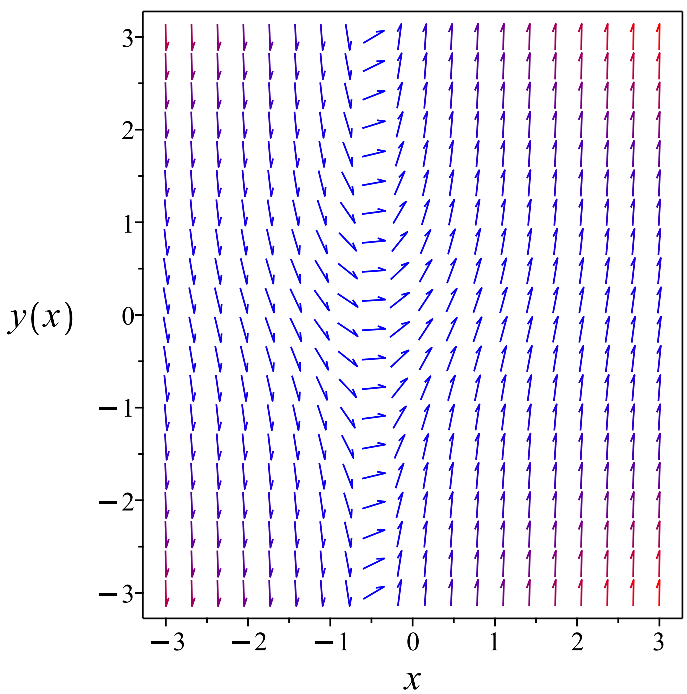


Figure 306: Slope field plot

Verification of solutions

$$y = \tan(x^2 + c_1 + x)$$

Verified OK.

6.13.4 Solving as riccati ode

In canonical form the ODE is

$$\begin{aligned} y' &= F(x,y) \\ &= 2x\,y^2 + y^2 + 2x + 1 \end{aligned}$$

This is a Riccati ODE. Comparing the ODE to solve

$$y' = 2x\,y^2 + y^2 + 2x + 1$$

With Riccati ODE standard form

$$y' = f_0(x) + f_1(x)y + f_2(x)y^2$$

Shows that $f_0(x) = 1 + 2x$, $f_1(x) = 0$ and $f_2(x) = 1 + 2x$. Let

$$\begin{aligned} y &= \frac{-u'}{f_2 u} \\ &= \frac{-u'}{(1+2x)\,u} \end{aligned} \tag{1}$$

Using the above substitution in the given ODE results (after some simplification)in a second order ODE to solve for $u(x)$ which is

$$f_2 u''(x) - (f_2' + f_1 f_2)\,u'(x) + f_2^2 f_0 u(x) = 0 \tag{2}$$

But

$$\begin{aligned} f_2' &= 2 \\ f_1 f_2 &= 0 \\ f_2^2 f_0 &= (1+2x)^3 \end{aligned}$$

Substituting the above terms back in equation (2) gives

$$(1+2x)\,u''(x) - 2u'(x) + (1+2x)^3\,u(x) = 0$$

Solving the above ODE (this ode solved using Maple, not this program), gives

$$u(x) = c_1 \sin\left(x^2 + x\right) + c_2 \cos\left(x^2 + x\right)$$

The above shows that

$$u'(x) = (1+2x)\left(c_1 \cos\left(x^2 + x\right) - c_2 \sin\left(x^2 + x\right)\right)$$

Using the above in (1) gives the solution

$$y = -\frac{c_1 \cos(x^2 + x) - c_2 \sin(x^2 + x)}{c_1 \sin(x^2 + x) + c_2 \cos(x^2 + x)}$$

Dividing both numerator and denominator by c_1 gives, after renaming the constant $\frac{c_2}{c_1} = c_3$ the following solution

$$y = \frac{-c_3 \cos(x^2 + x) + \sin(x^2 + x)}{c_3 \sin(x^2 + x) + \cos(x^2 + x)}$$

<u>Summary</u>
The solution(s) found are the following

$$y = \frac{-c_3 \cos(x^2 + x) + \sin(x^2 + x)}{c_3 \sin(x^2 + x) + \cos(x^2 + x)} \tag{1}$$

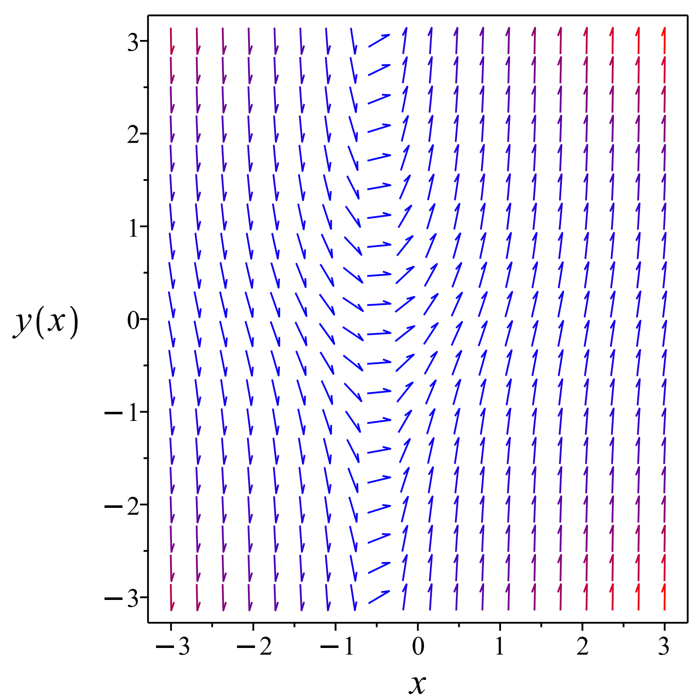


Figure 307: Slope field plot

<u>Verification of solutions</u>

$$y = \frac{-c_3 \cos(x^2 + x) + \sin(x^2 + x)}{c_3 \sin(x^2 + x) + \cos(x^2 + x)}$$

Verified OK.

6.13.5 Maple step by step solution

Let's solve

$y' - y^2 - 2xy^2 = 1 + 2x$

- Highest derivative means the order of the ODE is 1

  y'

- Separate variables

  $\frac{y'}{1+y^2} = 1 + 2x$

- Integrate both sides with respect to x

  $\int \frac{y'}{1+y^2}dx = \int (1 + 2x)\, dx + c_1$

- Evaluate integral

  $\arctan(y) = x^2 + c_1 + x$

- Solve for y

  $y = \tan(x^2 + c_1 + x)$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
<- separable successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 11

```
dsolve(diff(y(x),x) = 1+2*x+y(x)^2+2*x*y(x)^2,y(x), singsol=all)
```

$$y(x) = \tan\left(x^2 + c_1 + x\right)$$

✓ Solution by Mathematica

Time used: 0.178 (sec). Leaf size: 13

```
DSolve[y'[x] == 1+2*x+y[x]^2+2*x*y[x]^2,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \tan \left(x^2 + x + c_1\right)$$

6.14 problem 14

Internal problem ID [581]
Internal file name [`OUTPUT/581_Sunday_June_05_2022_01_45_09_AM_47757679/index.tex`]

Book: Elementary differential equations and boundary value problems, 10th ed., Boyce and DiPrima
Section: Miscellaneous problems, end of chapter 2. Page 133
Problem number: 14.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"exact", "differentialType", "homogeneousTypeD2", "first_order_ode_lie_symmetry_calculated"**

Maple gives the following as the ode type

```
[[_homogeneous, `class A`], _exact, _rational, [_Abel, `2nd
   type`, `class A`]]
```

$$\boxed{y + (x + 2y)\, y' = -x}$$

With initial conditions

$$[y(2) = 3]$$

6.14.1 Existence and uniqueness analysis

This is non linear first order ODE. In canonical form it is written as

$$\begin{aligned} y' &= f(x, y) \\ &= -\frac{x + y}{x + 2y} \end{aligned}$$

The x domain of $f(x,y)$ when $y=3$ is

$$\{x<-6\vee -6<x\}$$

And the point $x_0=2$ is inside this domain. The y domain of $f(x,y)$ when $x=2$ is

$$\{y<-1\vee -1<y\}$$

And the point $y_0=3$ is inside this domain. Now we will look at the continuity of

$$\begin{aligned}\frac{\partial f}{\partial y}&=\frac{\partial}{\partial y}\left(-\frac{x+y}{x+2y}\right)\\&=-\frac{1}{x+2y}+\frac{2x+2y}{(x+2y)^2}\end{aligned}$$

The x domain of $\frac{\partial f}{\partial y}$ when $y=3$ is

$$\{x<-6\vee -6<x\}$$

And the point $x_0=2$ is inside this domain. The y domain of $\frac{\partial f}{\partial y}$ when $x=2$ is

$$\{y<-1\vee -1<y\}$$

And the point $y_0=3$ is inside this domain. Therefore solution exists and is unique.

6.14.2 Solving as homogeneousTypeD2 ode

Using the change of variables $y=u(x)\,x$ on the above ode results in new ode in $u(x)$

$$u(x)\,x+(x+2u(x)\,x)\,(u'(x)\,x+u(x))=-x$$

In canonical form the ODE is

$$\begin{aligned}u'&=F(x,u)\\&=f(x)g(u)\\&=-\frac{2u^2+2u+1}{x\,(2u+1)}\end{aligned}$$

Where $f(x) = -\frac{1}{x}$ and $g(u) = \frac{2u^2+2u+1}{2u+1}$. Integrating both sides gives

$$\frac{1}{\frac{2u^2+2u+1}{2u+1}}\,du = -\frac{1}{x}\,dx$$

$$\int \frac{1}{\frac{2u^2+2u+1}{2u+1}}\,du = \int -\frac{1}{x}\,dx$$

$$\frac{\ln(2u^2+2u+1)}{2} = -\ln(x) + c_2$$

Raising both side to exponential gives

$$\sqrt{2u^2+2u+1} = \mathrm{e}^{-\ln(x)+c_2}$$

Which simplifies to

$$\sqrt{2u^2+2u+1} = \frac{c_3}{x}$$

Which simplifies to

$$\sqrt{2u(x)^2+2u(x)+1} = \frac{c_3\mathrm{e}^{c_2}}{x}$$

The solution is

$$\sqrt{2u(x)^2+2u(x)+1} = \frac{c_3\mathrm{e}^{c_2}}{x}$$

Replacing $u(x)$ in the above solution by $\frac{y}{x}$ results in the solution for y in implicit form

$$\sqrt{\frac{2y^2}{x^2}+\frac{2y}{x}+1} = \frac{c_3\mathrm{e}^{c_2}}{x}$$

$$\sqrt{\frac{2y^2+2yx+x^2}{x^2}} = \frac{c_3\mathrm{e}^{c_2}}{x}$$

Substituting initial conditions and solving for c_2 gives $c_2 = \frac{\ln\left(\frac{34}{c_3^2}\right)}{2}$. Hence the solution becomes Initial conditions are used to solve for c_3. Substituting $x = 2$ and $y = 3$ in the above solution gives an equation to solve for the constant of integration.

$$\frac{\sqrt{34}}{2} = \frac{c_3\sqrt{34}\,\sqrt{\frac{1}{c_3^2}}}{2}$$

This solution is valid for any c_3. Hence there are infinite number of solutions.

Summary
The solution(s) found are the following

$$\sqrt{\frac{2y^2+2yx+x^2}{x^2}} = \frac{c_3\sqrt{34}\,\sqrt{\frac{1}{c_3^2}}}{x} \tag{1}$$

Verification of solutions

$$\sqrt{\frac{2y^2+2yx+x^2}{x^2}} = \frac{c_3\sqrt{34}\,\sqrt{\frac{1}{c_3^2}}}{x}$$

Verified OK.

6.14.3 Solving as differentialType ode

Writing the ode as

$$y' = \frac{-x-y}{x+2y} \tag{1}$$

Which becomes

$$(2y)\,dy = (-x)\,dy + (-x-y)\,dx \tag{2}$$

But the RHS is complete differential because

$$(-x)\,dy + (-x-y)\,dx = d\left(-\frac{1}{2}x^2 - yx\right)$$

Hence (2) becomes

$$(2y)\,dy = d\left(-\frac{1}{2}x^2 - yx\right)$$

Integrating both sides gives gives these solutions

$$y = -\frac{x}{2} + \frac{\sqrt{-x^2+4c_1}}{2} + c_1$$
$$y = -\frac{x}{2} - \frac{\sqrt{-x^2+4c_1}}{2} + c_1$$

Initial conditions are used to solve for c_1. Substituting $x = 2$ and $y = 3$ in the above solution gives an equation to solve for the constant of integration.

$$3 = -1 - \sqrt{c_1 - 1} + c_1$$

$$
c_1 = \frac{9}{2} + \frac{\sqrt{13}}{2}
$$

Substituting c_1 found above in the general solution gives

$$
y = -\frac{x}{2} - \frac{\sqrt{-x^2 + 18 + 2\sqrt{13}}}{2} + \frac{9}{2} + \frac{\sqrt{13}}{2}
$$

Initial conditions are used to solve for c_1. Substituting $x = 2$ and $y = 3$ in the above solution gives an equation to solve for the constant of integration.

$$
3 = -1 + \sqrt{c_1 - 1} + c_1
$$

$$
c_1 = \frac{9}{2} - \frac{\sqrt{13}}{2}
$$

Substituting c_1 found above in the general solution gives

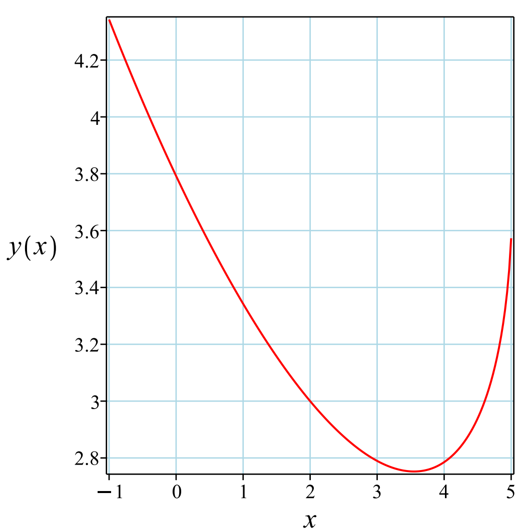
$$
y = -\frac{x}{2} + \frac{\sqrt{-x^2 + 18 - 2\sqrt{13}}}{2} + \frac{9}{2} - \frac{\sqrt{13}}{2}
$$

Summary

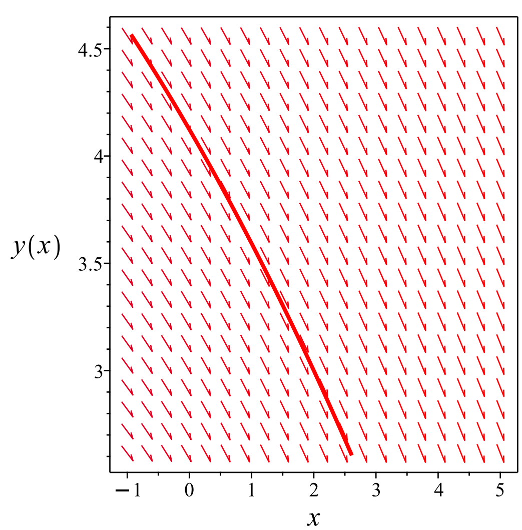
The solution(s) found are the following

$$
y = -\frac{x}{2} + \frac{\sqrt{-x^2 + 18 - 2\sqrt{13}}}{2} + \frac{9}{2} - \frac{\sqrt{13}}{2} \tag{1}
$$

$$
y = -\frac{x}{2} - \frac{\sqrt{-x^2 + 18 + 2\sqrt{13}}}{2} + \frac{9}{2} + \frac{\sqrt{13}}{2} \tag{2}
$$



(a) Solution plot



(b) Slope field plot

Verification of solutions

$$y = -\frac{x}{2}+\frac{\sqrt{-x^2+18-2\sqrt{13}}}{2}+\frac{9}{2}-\frac{\sqrt{13}}{2}$$

Verified OK. `{positive}`

$$y = -\frac{x}{2}-\frac{\sqrt{-x^2+18+2\sqrt{13}}}{2}+\frac{9}{2}+\frac{\sqrt{13}}{2}$$

Verified OK. `{positive}`

6.14.4 Solving as first order ode lie symmetry calculated ode

Writing the ode as

$$\begin{aligned} y' &= -\frac{x+y}{x+2y} \\ y' &= \omega(x,y) \end{aligned}$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x+\omega(\eta_y-\xi_x)-\omega^2\xi_y-\omega_x\xi-\omega_y\eta=0 \tag{A}$$

The type of this ode is not in the lookup table. To determine ξ,η then (A) is solved using ansatz. Making bivariate polynomials of degree 1 to use as anstaz gives

$$\xi = xa_2+ya_3+a_1 \tag{1E}$$

$$\eta = xb_2+yb_3+b_1 \tag{2E}$$

Where the unknown coefficients are

$$\{a_1,a_2,a_3,b_1,b_2,b_3\}$$

Substituting equations (1E,2E) and ω into (A) gives

$$b_2-\frac{(x+y)(b_3-a_2)}{x+2y}-\frac{(x+y)^2 a_3}{(x+2y)^2}-\left(-\frac{1}{x+2y}+\frac{x+y}{(x+2y)^2}\right)(xa_2+ya_3+a_1) \\ -\left(-\frac{1}{x+2y}+\frac{2x+2y}{(x+2y)^2}\right)(xb_2+yb_3+b_1)=0 \tag{5E}$$

Putting the above in normal form gives

$$\frac{x^2 a_2 - x^2 a_3 - x^2 b_3 + 4xya_2 - 2xya_3 + 4xyb_2 - 4xyb_3 + 2y^2 a_2 + 4y^2 b_2 - 2y^2 b_3 - xb_1 + ya_1}{(x+2y)^2} = 0$$

Setting the numerator to zero gives

$$x^2 a_2 - x^2 a_3 - x^2 b_3 + 4xya_2 - 2xya_3 + 4xyb_2 - 4xyb_3 + 2y^2 a_2 + 4y^2 b_2 - 2y^2 b_3 - xb_1 + ya_1 = 0 \tag{6E}$$

Looking at the above PDE shows the following are all the terms with $\{x, y\}$ in them.

$$\{x, y\}$$

The following substitution is now made to be able to collect on all terms with $\{x, y\}$ in them

$$\{x = v_1, y = v_2\}$$

The above PDE (6E) now becomes

$$a_2 v_1^2 + 4a_2 v_1 v_2 + 2a_2 v_2^2 - a_3 v_1^2 - 2a_3 v_1 v_2 + 4b_2 v_1 v_2 + 4b_2 v_2^2 - b_3 v_1^2 - 4b_3 v_1 v_2 - 2b_3 v_2^2 + a_1 v_2 - b_1 v_1 = 0 \tag{7E}$$

Collecting the above on the terms v_i introduced, and these are

$$\{v_1, v_2\}$$

Equation (7E) now becomes

$$(a_2 - a_3 - b_3) v_1^2 + (4a_2 - 2a_3 + 4b_2 - 4b_3) v_1 v_2 - b_1 v_1 + (2a_2 + 4b_2 - 2b_3) v_2^2 + a_1 v_2 = 0 \tag{8E}$$

Setting each coefficients in (8E) to zero gives the following equations to solve

$$\begin{aligned} a_1 &= 0 \\ -b_1 &= 0 \\ a_2 - a_3 - b_3 &= 0 \\ 2a_2 + 4b_2 - 2b_3 &= 0 \\ 4a_2 - 2a_3 + 4b_2 - 4b_3 &= 0 \end{aligned}$$

Solving the above equations for the unknowns gives

$$\begin{aligned}
a_1 &= 0 \\
a_2 &= -2b_2 + b_3 \\
a_3 &= -2b_2 \\
b_1 &= 0 \\
b_2 &= b_2 \\
b_3 &= b_3
\end{aligned}$$

Substituting the above solution in the anstaz (1E,2E) (using 1 as arbitrary value for any unknown in the RHS) gives

$$\begin{aligned}
\xi &= x \\
\eta &= y
\end{aligned}$$

Shifting is now applied to make $\xi = 0$ in order to simplify the rest of the computation

$$\begin{aligned}
\eta &= \eta - \omega(x, y)\,\xi \\
&= y - \left(-\frac{x+y}{x+2y}\right)(x) \\
&= \frac{x^2 + 2yx + 2y^2}{x + 2y} \\
\xi &= 0
\end{aligned}$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x, y) \to (R, S)$ where (R, S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y}\right) S(x, y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$R = x$$

S is found from

$$\begin{aligned}
S &= \int \frac{1}{\eta} dy \\
&= \int \frac{1}{\frac{x^2+2yx+2y^2}{x+2y}} dy
\end{aligned}$$

Which results in

$$S = \frac{\ln(x^2 + 2yx + 2y^2)}{2}$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x,y)S_y}{R_x + \omega(x,y)R_y} \tag{2}$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x,y)$ is the right hand side of the original ode given by

$$\omega(x,y) = -\frac{x+y}{x+2y}$$

Evaluating all the partial derivatives gives

$$\begin{aligned} R_x &= 1 \\ R_y &= 0 \\ S_x &= \frac{x+y}{x^2+2yx+2y^2} \\ S_y &= \frac{x+2y}{x^2+2yx+2y^2} \end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = 0 \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = 0$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$S(R) = c_1 \tag{4}$$

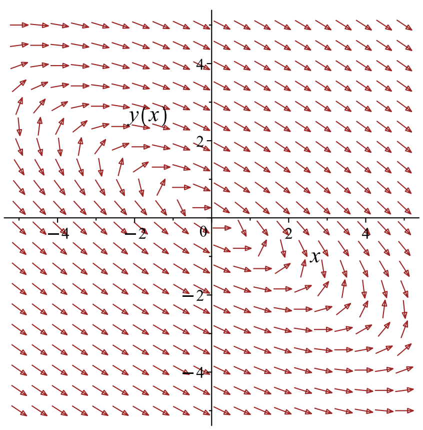
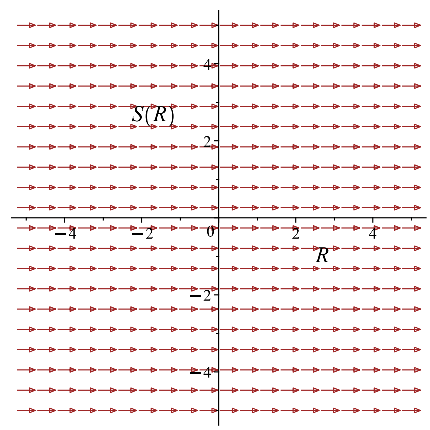
To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$\frac{\ln(2y^2 + 2yx + x^2)}{2} = c_1$$

Which simplifies to

$$\frac{\ln\left(2y^2+2yx+x^2\right)}{2}=c_1$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in x, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dx}=-\frac{x+y}{x+2y}$ 	$R=x$ $S=\frac{\ln\left(x^2+2yx+2y^2\right)}{2}$	$\frac{dS}{dR}=0$ 

Initial conditions are used to solve for c_1. Substituting $x=2$ and $y=3$ in the above solution gives an equation to solve for the constant of integration.

$$\frac{\ln(2)}{2}+\frac{\ln(17)}{2}=c_1$$

$$c_1=\frac{\ln(2)}{2}+\frac{\ln(17)}{2}$$

Substituting c_1 found above in the general solution gives

$$\frac{\ln\left(x^2+2yx+2y^2\right)}{2}=\frac{\ln(2)}{2}+\frac{\ln(17)}{2}$$

Summary

The solution(s) found are the following

$$\frac{\ln\left(2y^2+2yx+x^2\right)}{2}=\frac{\ln(2)}{2}+\frac{\ln(17)}{2} \tag{1}$$

Verification of solutions

$$\frac{\ln(2y^2+2yx+x^2)}{2} = \frac{\ln(2)}{2} + \frac{\ln(17)}{2}$$

Verified OK.

6.14.5 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x,y) + N(x,y)\frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x,y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y) = 0$$

Hence

$$\frac{\partial\phi}{\partial x} + \frac{\partial\phi}{\partial y}\frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$\frac{\partial\phi}{\partial x} = M$$
$$\frac{\partial\phi}{\partial y} = N$$

But since $\frac{\partial^2\phi}{\partial x\partial y} = \frac{\partial^2\phi}{\partial y\partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x,y)$ but at least we know now that we can do that since the condition $\frac{\partial^2\phi}{\partial x\partial y} = \frac{\partial^2\phi}{\partial y\partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x,y)\,\mathrm{d}x + N(x,y)\,\mathrm{d}y = 0 \tag{1A}$$

Therefore

$$(x+2y)\,\mathrm{d}y = (-x-y)\,\mathrm{d}x$$
$$(x+y)\,\mathrm{d}x + (x+2y)\,\mathrm{d}y = 0 \tag{2A}$$

Comparing (1A) and (2A) shows that

$$
\begin{aligned}
M(x,y) &= x+y \\
N(x,y) &= x+2y
\end{aligned}
$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$
\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}
$$

Using result found above gives

$$
\begin{aligned}
\frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}(x+y) \\
&= 1
\end{aligned}
$$

And

$$
\begin{aligned}
\frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}(x+2y) \\
&= 1
\end{aligned}
$$

Since $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$, then the ODE is exact The following equations are now set up to solve for the function $\phi(x,y)$

$$
\frac{\partial \phi}{\partial x} = M \tag{1}
$$

$$
\frac{\partial \phi}{\partial y} = N \tag{2}
$$

Integrating (1) w.r.t. x gives

$$
\begin{aligned}
\int \frac{\partial \phi}{\partial x} \,\mathrm{d}x &= \int M \,\mathrm{d}x \\
\int \frac{\partial \phi}{\partial x} \,\mathrm{d}x &= \int x+y \,\mathrm{d}x \\
\phi &= \frac{x(x+2y)}{2} + f(y) \qquad (3)
\end{aligned}
$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$
\frac{\partial \phi}{\partial y} = x + f'(y) \tag{4}
$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = x + 2y$. Therefore equation (4) becomes

$$x + 2y = x + f'(y) \tag{5}$$

Solving equation (5) for $f'(y)$ gives

$$f'(y) = 2y$$

Integrating the above w.r.t y gives

$$\int f'(y) \, dy = \int (2y) \, dy$$

$$f(y) = y^2 + c_1$$

Where c_1 is constant of integration. Substituting result found above for $f(y)$ into equation (3) gives ϕ

$$\phi = \frac{x(x + 2y)}{2} + y^2 + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = \frac{x(x + 2y)}{2} + y^2$$

Initial conditions are used to solve for c_1. Substituting $x = 2$ and $y = 3$ in the above solution gives an equation to solve for the constant of integration.

$$17 = c_1$$

$$c_1 = 17$$

Substituting c_1 found above in the general solution gives

$$\frac{x(x + 2y)}{2} + y^2 = 17$$

Summary
The solution(s) found are the following

$$\frac{x^2}{2} + yx + y^2 = 17 \tag{1}$$

Verification of solutions

$$\frac{x^2}{2}+yx+y^2=17$$

Verified OK.

6.14.6 Maple step by step solution

Let's solve

$[y+(x+2y)\,y'=-x, y(2)=3]$

- Highest derivative means the order of the ODE is 1

  y'

- □ Check if ODE is exact
  - ODE is exact if the lhs is the total derivative of a C^2 function

    $F'(x,y)=0$
  - Compute derivative of lhs

    $F'(x,y)+\left(\frac{\partial}{\partial y}F(x,y)\right)y'=0$
  - Evaluate derivatives

    $1=1$
  - Condition met, ODE is exact
- Exact ODE implies solution will be of this form

  $\left[F(x,y)=c_1, M(x,y)=F'(x,y), N(x,y)=\frac{\partial}{\partial y}F(x,y)\right]$
- Solve for $F(x,y)$ by integrating $M(x,y)$ with respect to x

  $F(x,y)=\int (x+y)\,dx+f_1(y)$
- Evaluate integral

  $F(x,y)=\frac{x^2}{2}+yx+f_1(y)$
- Take derivative of $F(x,y)$ with respect to y

  $N(x,y)=\frac{\partial}{\partial y}F(x,y)$
- Compute derivative

  $x+2y=x+\frac{d}{dy}f_1(y)$
- Isolate for $\frac{d}{dy}f_1(y)$

$\frac{d}{dy}f_1(y) = 2y$

- Solve for $f_1(y)$

  $f_1(y) = y^2$

- Substitute $f_1(y)$ into equation for $F(x, y)$

  $F(x, y) = \frac{1}{2}x^2 + yx + y^2$

- Substitute $F(x, y)$ into the solution of the ODE

  $\frac{1}{2}x^2 + yx + y^2 = c_1$

- Solve for y

  $\left\{y = -\frac{x}{2} - \frac{\sqrt{-x^2+4c_1}}{2}, y = -\frac{x}{2} + \frac{\sqrt{-x^2+4c_1}}{2}\right\}$

- Use initial condition $y(2) = 3$

  $3 = -1 - \frac{\sqrt{-4+4c_1}}{2}$

- Solution does not satisfy initial condition
- Use initial condition $y(2) = 3$

  $3 = -1 + \frac{\sqrt{-4+4c_1}}{2}$

- Solve for c_1

  $c_1 = 17$

- Substitute $c_1 = 17$ into general solution and simplify

  $y = -\frac{x}{2} + \frac{\sqrt{-x^2+68}}{2}$

- Solution to the IVP

  $y = -\frac{x}{2} + \frac{\sqrt{-x^2+68}}{2}$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
trying homogeneous types:
trying homogeneous D
<- homogeneous successful`
```

✓ Solution by Maple

Time used: 0.093 (sec). Leaf size: 19

```
dsolve([x+y(x)+(x+2*y(x))*diff(y(x),x) = 0,y(2) = 3],y(x), singsol=all)
```

$$y(x) = -\frac{x}{2} + \frac{\sqrt{-x^2 + 68}}{2}$$

✓ Solution by Mathematica

Time used: 0.458 (sec). Leaf size: 24

```
DSolve[{x+y[x]+(x+2*y[x])*y'[x] == 0,y[2]==3},y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \frac{1}{2}\left(\sqrt{68 - x^2} - x\right)$$

6.15 problem 15

Internal problem ID [582]
Internal file name [`OUTPUT/582_Sunday_June_05_2022_01_45_11_AM_3677461/index.tex`]

Book: Elementary differential equations and boundary value problems, 10th ed., Boyce and DiPrima
Section: Miscellaneous problems, end of chapter 2. Page 133
Problem number: 15.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"exact", "linear", "separable", "homogeneousTypeD2", "first_order_ode_lie_symmetry_lookup"**

Maple gives the following as the ode type

`[_separable]`

$$\boxed{(1+\mathrm{e}^{x})\, y' - y + \mathrm{e}^{x} y = 0}$$

6.15.1 Solving as separable ode

In canonical form the ODE is

$$\begin{aligned} y' &= F(x,y) \\ &= f(x)g(y) \\ &= -\frac{y(\mathrm{e}^{x}-1)}{1+\mathrm{e}^{x}} \end{aligned}$$

Where $f(x) = -\frac{e^x - 1}{1 + e^x}$ and $g(y) = y$. Integrating both sides gives

$$\begin{aligned}
\frac{1}{y}\, dy &= -\frac{e^x - 1}{1 + e^x}\, dx \\
\int \frac{1}{y}\, dy &= \int -\frac{e^x - 1}{1 + e^x}\, dx \\
\ln(y) &= \ln(e^x) - 2\ln(1 + e^x) + c_1 \\
y &= e^{\ln(e^x) - 2\ln(1+e^x) + c_1} \\
&= c_1 e^{\ln(e^x) - 2\ln(1+e^x)}
\end{aligned}$$

Which simplifies to

$$y = \frac{c_1 e^x}{(1 + e^x)^2}$$

Summary

The solution(s) found are the following

$$y = \frac{c_1 e^x}{(1 + e^x)^2} \tag{1}$$

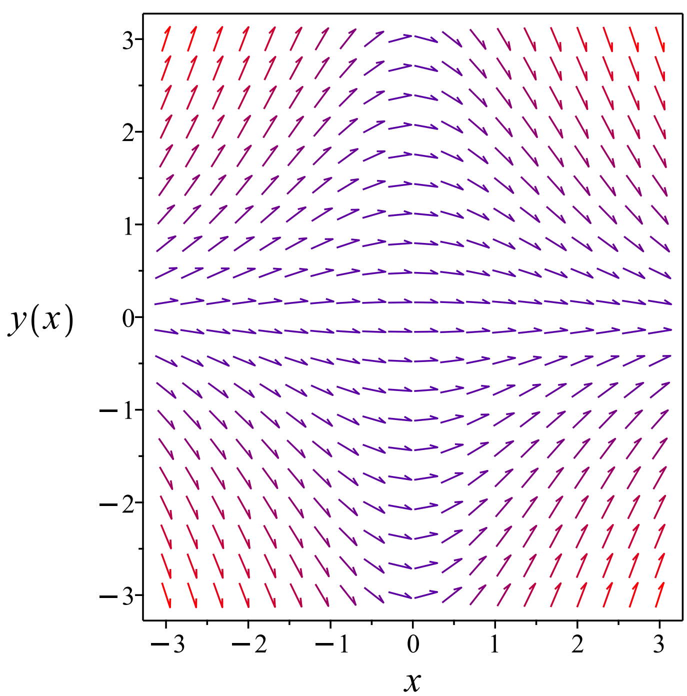


Figure 309: Slope field plot

Verification of solutions

$$y = \frac{c_1 \mathrm{e}^{x}}{(1+\mathrm{e}^{x})^{2}}$$

Verified OK.

6.15.2 Solving as linear ode

Entering Linear first order ODE solver. In canonical form a linear first order is

$$y' + p(x)y = q(x)$$

Where here

$$p(x) = -\frac{-\mathrm{e}^{x}+1}{1+\mathrm{e}^{x}}$$
$$q(x) = 0$$

Hence the ode is

$$y' - \frac{(-\mathrm{e}^{x}+1)\, y}{1+\mathrm{e}^{x}} = 0$$

The integrating factor μ is

$$\mu = \mathrm{e}^{\int -\frac{-\mathrm{e}^{x}+1}{1+\mathrm{e}^{x}}dx}$$
$$= \mathrm{e}^{-\ln(\mathrm{e}^{x})+2\ln(1+\mathrm{e}^{x})}$$

Which simplifies to

$$\mu = \mathrm{e}^{-x}(1+\mathrm{e}^{x})^{2}$$

The ode becomes

$$\frac{\mathrm{d}}{\mathrm{d}x}\mu y = 0$$
$$\frac{\mathrm{d}}{\mathrm{d}x}\left(\mathrm{e}^{-x}(1+\mathrm{e}^{x})^{2}\, y\right) = 0$$

Integrating gives

$$\mathrm{e}^{-x}(1+\mathrm{e}^{x})^{2}\, y = c_1$$

Dividing both sides by the integrating factor $\mu = \mathrm{e}^{-x}(1+\mathrm{e}^{x})^{2}$ results in

$$y = \frac{c_1 \mathrm{e}^{x}}{(1+\mathrm{e}^{x})^{2}}$$

Summary
The solution(s) found are the following

$$y = \frac{c_1 e^x}{(1 + e^x)^2} \tag{1}$$

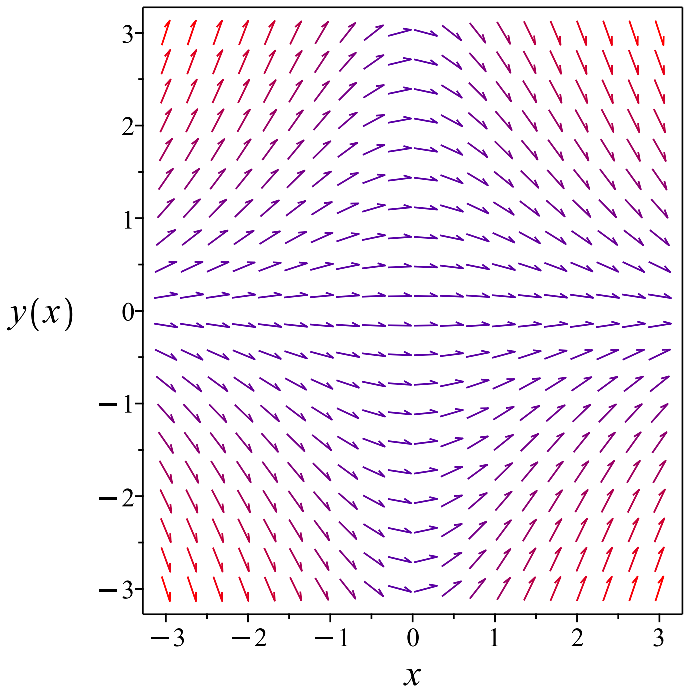


Figure 310: Slope field plot

Verification of solutions

$$y = \frac{c_1 e^x}{(1 + e^x)^2}$$

Verified OK.

6.15.3 Solving as homogeneousTypeD2 ode

Using the change of variables $y = u(x)\, x$ on the above ode results in new ode in $u(x)$

$$(1 + e^x)\,(u'(x)\, x + u(x)) - u(x)\, x + e^x u(x)\, x = 0$$

In canonical form the ODE is

$$
\begin{aligned}
u' &= F(x,u) \\
&= f(x)g(u) \\
&= -\frac{u(x\,\mathrm{e}^{x}+\mathrm{e}^{x}-x+1)}{(1+\mathrm{e}^{x})\,x}
\end{aligned}
$$

Where $f(x) = -\frac{x\,\mathrm{e}^{x}+\mathrm{e}^{x}-x+1}{x(1+\mathrm{e}^{x})}$ and $g(u) = u$. Integrating both sides gives

$$
\begin{aligned}
\frac{1}{u}\,du &= -\frac{x\,\mathrm{e}^{x}+\mathrm{e}^{x}-x+1}{x\,(1+\mathrm{e}^{x})}\,dx \\
\int \frac{1}{u}\,du &= \int -\frac{x\,\mathrm{e}^{x}+\mathrm{e}^{x}-x+1}{x\,(1+\mathrm{e}^{x})}\,dx \\
\ln(u) &= x-\ln(x)-2\ln(1+\mathrm{e}^{x})+c_{2} \\
u &= \mathrm{e}^{x-\ln(x)-2\ln(1+\mathrm{e}^{x})+c_{2}} \\
&= c_{2}\mathrm{e}^{x-\ln(x)-2\ln(1+\mathrm{e}^{x})}
\end{aligned}
$$

Which simplifies to

$$
u(x) = \frac{c_{2}\mathrm{e}^{x}}{x\,(1+\mathrm{e}^{x})^{2}}
$$

Therefore the solution y is

$$
\begin{aligned}
y &= ux \\
&= \frac{c_{2}\mathrm{e}^{x}}{(1+\mathrm{e}^{x})^{2}}
\end{aligned}
$$

Summary

The solution(s) found are the following

$$
y = \frac{c_{2}\mathrm{e}^{x}}{(1+\mathrm{e}^{x})^{2}} \tag{1}
$$

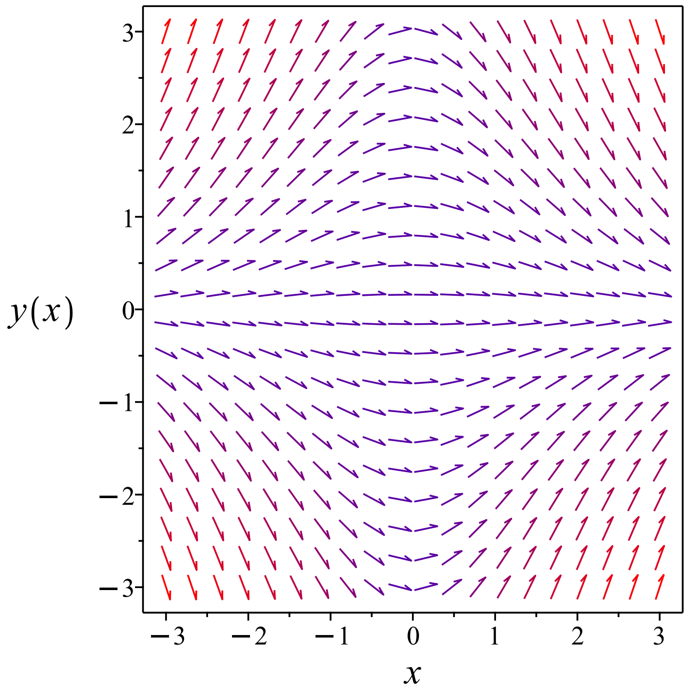


Figure 311: Slope field plot

Verification of solutions

$$y = \frac{c_2 e^x}{(1 + e^x)^2}$$

Verified OK.

6.15.4 Solving as first order ode lie symmetry lookup ode

Writing the ode as

$$y' = -\frac{y(e^x - 1)}{1 + e^x}$$
$$y' = \omega(x, y)$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x + \omega(\eta_y - \xi_x) - \omega^2 \xi_y - \omega_x \xi - \omega_y \eta = 0 \tag{A}$$

The type of this ode is known. It is of type `linear`. Therefore we do not need to solve the PDE (A), and can just use the lookup table shown below to find ξ, η

Table 289: Lie symmetry infinitesimal lookup table for known first order ODE's

ODE class	Form	ξ	η
linear ode	$y' = f(x)y(x) + g(x)$	0	$e^{\int f dx}$
separable ode	$y' = f(x)\, g(y)$	$\frac{1}{f}$	0
quadrature ode	$y' = f(x)$	0	1
quadrature ode	$y' = g(y)$	1	0
homogeneous ODEs of Class A	$y' = f\left(\frac{y}{x}\right)$	x	y
homogeneous ODEs of Class C	$y' = (a + bx + cy)^{\frac{n}{m}}$	1	$-\frac{b}{c}$
homogeneous class D	$y' = \frac{y}{x} + g(x)\, F\left(\frac{y}{x}\right)$	x^2	xy
First order special form ID 1	$y' = g(x)\, e^{h(x)+by} + f(x)$	$\frac{e^{-\int bf(x)dx-h(x)}}{g(x)}$	$\frac{f(x)e^{-\int bf(x)dx-h(x)}}{g(x)}$
polynomial type ode	$y' = \frac{a_1x+b_1y+c_1}{a_2x+b_2y+c_2}$	$\frac{a_1b_2x-a_2b_1x-b_1c_2+b_2c_1}{a_1b_2-a_2b_1}$	$\frac{a_1b_2y-a_2b_1y-a_1c_2-a_2c_1}{a_1b_2-a_2b_1}$
Bernoulli ode	$y' = f(x)\, y + g(x)\, y^n$	0	$e^{-\int (n-1)f(x)dx} y^n$
Reduced Riccati	$y' = f_1(x)\, y + f_2(x)\, y^2$	0	$e^{-\int f_1 dx}$

The above table shows that

$$\begin{aligned} \xi(x, y) &= 0 \\ \eta(x, y) &= \mathrm{e}^{\ln(\mathrm{e}^x)-2\ln(1+\mathrm{e}^x)} \end{aligned} \tag{A1}$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x, y) \to (R, S)$ where (R, S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y}\right) S(x, y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the

canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$R = x$$

S is found from

$$\begin{aligned} S &= \int \frac{1}{\eta} dy \\ &= \int \frac{1}{e^{\ln(e^x) - 2\ln(1+e^x)}} dy \end{aligned}$$

Which results in

$$S = e^{-x}(1 + e^x)^2 y$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x, y) S_y}{R_x + \omega(x, y) R_y} \tag{2}$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x, y)$ is the right hand side of the original ode given by

$$\omega(x, y) = -\frac{y(e^x - 1)}{1 + e^x}$$

Evaluating all the partial derivatives gives

$$\begin{aligned} R_x &= 1 \\ R_y &= 0 \\ S_x &= y(e^x - e^{-x}) \\ S_y &= e^{-x}(1 + e^x)^2 \end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = 0 \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = 0$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by

integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$S(R) = c_1 \tag{4}$$

To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$e^{-x}(1 + e^{x})^2 y = c_1$$

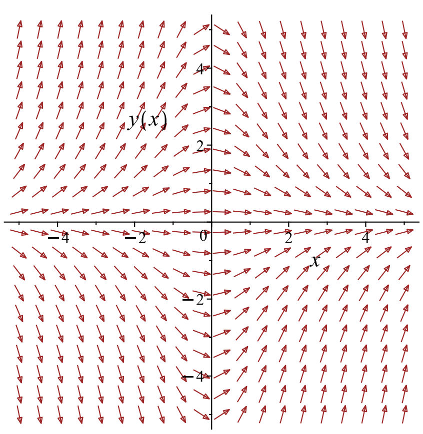
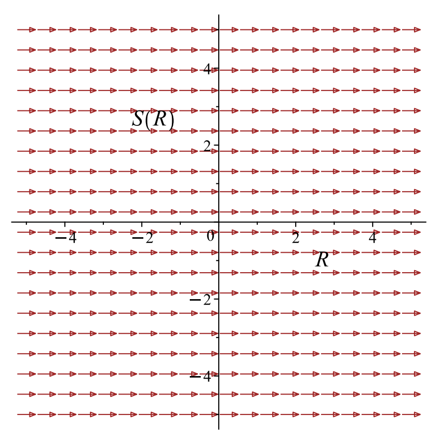
Which simplifies to

$$e^{-x}(1 + e^{x})^2 y = c_1$$

Which gives

$$y = \frac{c_1 e^{x}}{e^{2x} + 2e^{x} + 1}$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in x, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dx} = -\frac{y(e^{x}-1)}{1+e^{x}}$ 	$R = x$ $S = e^{-x}(1 + e^{x})^2 y$	$\frac{dS}{dR} = 0$ 

Summary

The solution(s) found are the following

$$y = \frac{c_1 e^{x}}{e^{2x} + 2e^{x} + 1} \tag{1}$$

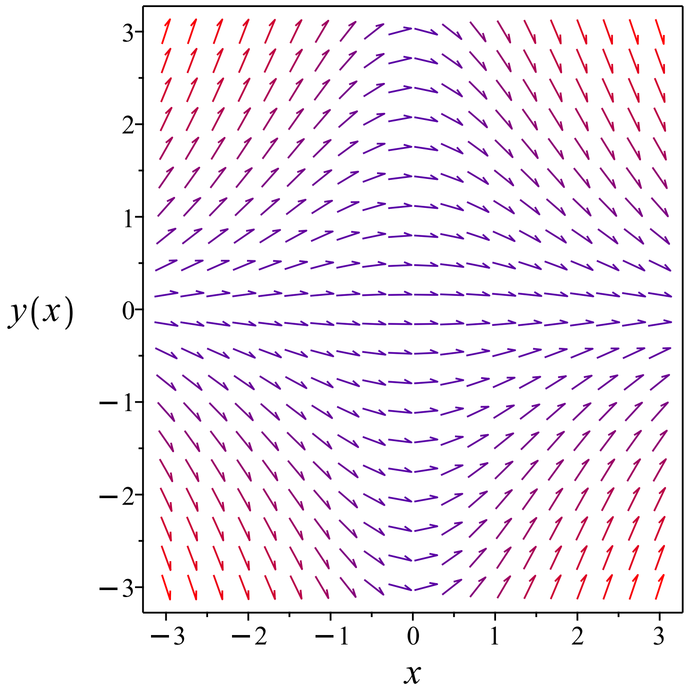


Figure 312: Slope field plot

Verification of solutions

$$y = \frac{c_1 e^x}{e^{2x} + 2 e^x + 1}$$

Verified OK.

6.15.5 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x,y) + N(x,y)\frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x,y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y) = 0$$

Hence

$$\frac{\partial \phi}{\partial x} + \frac{\partial \phi}{\partial y}\frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$
\begin{aligned}
\frac{\partial \phi}{\partial x} &= M \\
\frac{\partial \phi}{\partial y} &= N
\end{aligned}
$$

But since $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ then for the above to be valid, we require that

$$
\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}
$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x, y)$ but at least we know now that we can do that since the condition $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$
M(x, y)\, \mathrm{d}x + N(x, y)\, \mathrm{d}y = 0 \tag{1A}
$$

Therefore

$$
\begin{aligned}
\left(-\frac{1}{y}\right) \mathrm{d}y &= \left(\frac{\mathrm{e}^x - 1}{1 + \mathrm{e}^x}\right) \mathrm{d}x \\
\left(-\frac{\mathrm{e}^x - 1}{1 + \mathrm{e}^x}\right) \mathrm{d}x + \left(-\frac{1}{y}\right) \mathrm{d}y &= 0 \qquad \text{(2A)}
\end{aligned}
$$

Comparing (1A) and (2A) shows that

$$
\begin{aligned}
M(x, y) &= -\frac{\mathrm{e}^x - 1}{1 + \mathrm{e}^x} \\
N(x, y) &= -\frac{1}{y}
\end{aligned}
$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$
\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}
$$

Using result found above gives

$$
\begin{aligned}
\frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}\left(-\frac{\mathrm{e}^x - 1}{1 + \mathrm{e}^x}\right) \\
&= 0
\end{aligned}
$$

And

$$\frac{\partial N}{\partial x} = \frac{\partial}{\partial x}\left(-\frac{1}{y}\right)$$
$$= 0$$

Since $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$, then the ODE is exact The following equations are now set up to solve for the function $\phi(x, y)$

$$\frac{\partial \phi}{\partial x} = M \tag{1}$$
$$\frac{\partial \phi}{\partial y} = N \tag{2}$$

Integrating (1) w.r.t. x gives

$$\int \frac{\partial \phi}{\partial x} \, \mathrm{d}x = \int M \, \mathrm{d}x$$
$$\int \frac{\partial \phi}{\partial x} \, \mathrm{d}x = \int -\frac{\mathrm{e}^x - 1}{1 + \mathrm{e}^x} \, \mathrm{d}x$$
$$\phi = \ln(\mathrm{e}^x) - 2\ln(1 + \mathrm{e}^x) + f(y) \tag{3}$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$\frac{\partial \phi}{\partial y} = 0 + f'(y) \tag{4}$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = -\frac{1}{y}$. Therefore equation (4) becomes

$$-\frac{1}{y} = 0 + f'(y) \tag{5}$$

Solving equation (5) for $f'(y)$ gives

$$f'(y) = -\frac{1}{y}$$

Integrating the above w.r.t y gives

$$\int f'(y) \, \mathrm{d}y = \int \left(-\frac{1}{y}\right) \mathrm{d}y$$
$$f(y) = -\ln(y) + c_1$$

Where c_1 is constant of integration. Substituting result found above for $f(y)$ into equation (3) gives ϕ

$$\phi = \ln(e^x) - 2\ln(1 + e^x) - \ln(y) + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = \ln(e^x) - 2\ln(1 + e^x) - \ln(y)$$

The solution becomes

$$y = \frac{e^{x-c_1}}{(1 + e^x)^2}$$

Summary

The solution(s) found are the following

$$y = \frac{e^{x-c_1}}{(1 + e^x)^2} \tag{1}$$

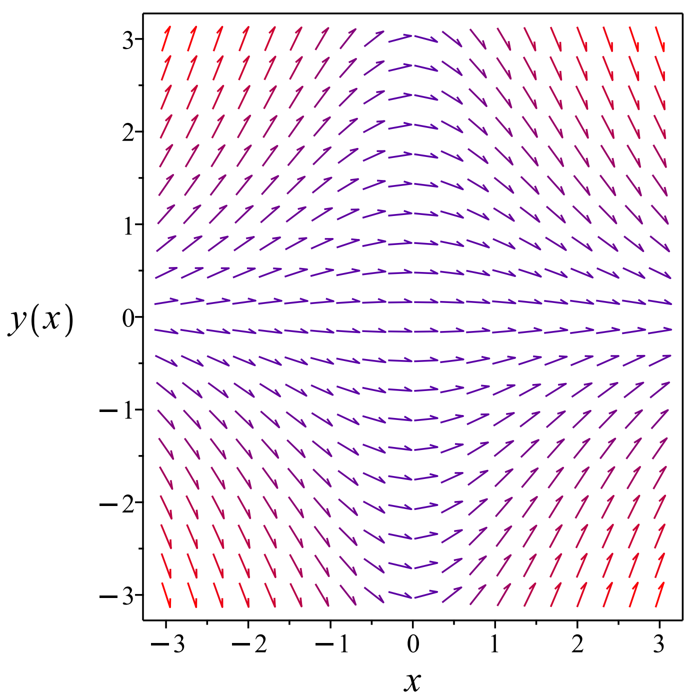


Figure 313: Slope field plot

Verification of solutions

$$y = \frac{e^{x-c_1}}{(1+e^x)^2}$$

Verified OK.

6.15.6 Maple step by step solution

Let's solve

$(1+e^x)\,y' - y + e^x y = 0$

- Highest derivative means the order of the ODE is 1

  y'

- Separate variables

  $\frac{y'}{y} = -\frac{e^x-1}{1+e^x}$

- Integrate both sides with respect to x

  $\int \frac{y'}{y}dx = \int -\frac{e^x-1}{1+e^x}dx + c_1$

- Evaluate integral

  $\ln(y) = \ln(e^x) - 2\ln(1+e^x) + c_1$

- Solve for y

  $y = \frac{e^{x+c_1}}{(1+e^x)^2}$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
<- 1st order linear successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 14

```
dsolve((1+exp(x))*diff(y(x),x) = y(x)-exp(x)*y(x),y(x), singsol=all)
```

$$y(x) = \frac{c_1 e^x}{(1 + e^x)^2}$$

✓ Solution by Mathematica

Time used: 0.077 (sec). Leaf size: 23

```
DSolve[(1+Exp[x])*y'[x]== y[x]-Exp[x]*y[x],y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \frac{c_1 e^x}{(e^x + 1)^2}$$
$$y(x) \to 0$$

6.16 problem 16

Internal problem ID [583]
Internal file name [`OUTPUT/583_Sunday_June_05_2022_01_45_12_AM_88473674/index.tex`]

Book: Elementary differential equations and boundary value problems, 10th ed., Boyce and DiPrima
Section: Miscellaneous problems, end of chapter 2. Page 133
Problem number: 16.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"exactWithIntegrationFactor"**

Maple gives the following as the ode type

`[NONE]`

$$\boxed{y' - \frac{-\mathrm{e}^{2y}\cos(x) + \cos(y)\,\mathrm{e}^{-x}}{2\,\mathrm{e}^{2y}\sin(x) - \sin(y)\,\mathrm{e}^{-x}} = 0}$$

6.16.1 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x,y) + N(x,y)\frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x,y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y) = 0$$

Hence

$$\frac{\partial \phi}{\partial x} + \frac{\partial \phi}{\partial y}\frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$\frac{\partial \phi}{\partial x} = M$$
$$\frac{\partial \phi}{\partial y} = N$$

But since $\frac{\partial^2\phi}{\partial x\partial y}=\frac{\partial^2\phi}{\partial y\partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y}=\frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x,y)$ but at least we know now that we can do that since the condition $\frac{\partial^2\phi}{\partial x\partial y}=\frac{\partial^2\phi}{\partial y\partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x,y)\,\mathrm{d}x+N(x,y)\,\mathrm{d}y=0 \tag{1A}$$

Therefore

$$\left(2\,\mathrm{e}^{2y}\sin(x)\,\mathrm{e}^{x}-\sin(y)\right)\mathrm{d}y=\left(-\mathrm{e}^{2y}\cos(x)\,\mathrm{e}^{x}+\cos(y)\right)\mathrm{d}x$$

$$\left(\mathrm{e}^{2y}\cos(x)\,\mathrm{e}^{x}-\cos(y)\right)\mathrm{d}x+\left(2\,\mathrm{e}^{2y}\sin(x)\,\mathrm{e}^{x}-\sin(y)\right)\mathrm{d}y=0 \tag{2A}$$

Comparing (1A) and (2A) shows that

$$\begin{aligned}M(x,y)&=\mathrm{e}^{2y}\cos(x)\,\mathrm{e}^{x}-\cos(y)\\N(x,y)&=2\,\mathrm{e}^{2y}\sin(x)\,\mathrm{e}^{x}-\sin(y)\end{aligned}$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y}=\frac{\partial N}{\partial x}$$

Using result found above gives

$$\begin{aligned}\frac{\partial M}{\partial y}&=\frac{\partial}{\partial y}\left(\mathrm{e}^{2y}\cos(x)\,\mathrm{e}^{x}-\cos(y)\right)\\&=2\cos(x)\,\mathrm{e}^{x+2y}+\sin(y)\end{aligned}$$

And

$$\begin{aligned}\frac{\partial N}{\partial x}&=\frac{\partial}{\partial x}\left(2\,\mathrm{e}^{2y}\sin(x)\,\mathrm{e}^{x}-\sin(y)\right)\\&=2(\cos(x)+\sin(x))\,\mathrm{e}^{x+2y}\end{aligned}$$

Since $\frac{\partial M}{\partial y}\neq\frac{\partial N}{\partial x}$, then the ODE is <u>not exact</u>. Since the ODE is not exact, we will try to find an integrating factor to make it exact. Let

$$\begin{aligned}A&=\frac{1}{N}\left(\frac{\partial M}{\partial y}-\frac{\partial N}{\partial x}\right)\\&=\frac{1}{2\sin(x)\,\mathrm{e}^{x+2y}-\sin(y)}\left(\left(2\,\mathrm{e}^{2y}\cos(x)\,\mathrm{e}^{x}+\sin(y)\right)-\left(2\,\mathrm{e}^{2y}\cos(x)\,\mathrm{e}^{x}+2\,\mathrm{e}^{2y}\sin(x)\,\mathrm{e}^{x}\right)\right)\\&=-1\end{aligned}$$

Since A does not depend on y, then it can be used to find an integrating factor. The integrating factor μ is

$$\begin{aligned}\mu &= e^{\int A\,\mathrm{d}x}\\ &= e^{\int -1\,\mathrm{d}x}\end{aligned}$$

The result of integrating gives

$$\begin{aligned}\mu &= e^{-x}\\ &= \mathrm{e}^{-x}\end{aligned}$$

M and N are multiplied by this integrating factor, giving new M and new N which are called $\overline{M}$ and $\overline{N}$ for now so not to confuse them with the original M and N.

$$\begin{aligned}\overline{M} &= \mu M\\ &= \mathrm{e}^{-x}\left(\mathrm{e}^{2y}\cos(x)\,\mathrm{e}^{x} - \cos(y)\right)\\ &= \left(\cos(x)\,\mathrm{e}^{x+2y} - \cos(y)\right)\mathrm{e}^{-x}\end{aligned}$$

And

$$\begin{aligned}\overline{N} &= \mu N\\ &= \mathrm{e}^{-x}\left(2\,\mathrm{e}^{2y}\sin(x)\,\mathrm{e}^{x} - \sin(y)\right)\\ &= \left(2\sin(x)\,\mathrm{e}^{x+2y} - \sin(y)\right)\mathrm{e}^{-x}\end{aligned}$$

Now a modified ODE is ontained from the original ODE, which is exact and can be solved. The modified ODE is

$$\begin{aligned}\overline{M} + \overline{N}\frac{\mathrm{d}y}{\mathrm{d}x} &= 0\\ \left(\left(\cos(x)\,\mathrm{e}^{x+2y} - \cos(y)\right)\mathrm{e}^{-x}\right) + \left(\left(2\sin(x)\,\mathrm{e}^{x+2y} - \sin(y)\right)\mathrm{e}^{-x}\right)\frac{\mathrm{d}y}{\mathrm{d}x} &= 0\end{aligned}$$

The following equations are now set up to solve for the function $\phi(x,y)$

$$\frac{\partial\phi}{\partial x} = \overline{M} \tag{1}$$

$$\frac{\partial\phi}{\partial y} = \overline{N} \tag{2}$$

Integrating (1) w.r.t. x gives

$$\begin{aligned}\int \frac{\partial\phi}{\partial x}\,\mathrm{d}x &= \int \overline{M}\,\mathrm{d}x\\ \int \frac{\partial\phi}{\partial x}\,\mathrm{d}x &= \int \left(\cos(x)\,\mathrm{e}^{x+2y} - \cos(y)\right)\mathrm{e}^{-x}\,\mathrm{d}x\end{aligned}$$

$$\phi = \mathrm{e}^{2y}\sin(x) + \cos(y)\,\mathrm{e}^{-x} + f(y) \tag{3}$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$\frac{\partial \phi}{\partial y} = 2\,\mathrm{e}^{2y} \sin(x) - \sin(y)\,\mathrm{e}^{-x} + f'(y) \tag{4}$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = (2\sin(x)\,\mathrm{e}^{x+2y} - \sin(y))\,\mathrm{e}^{-x}$. Therefore equation (4) becomes

$$\left(2\sin(x)\,\mathrm{e}^{x+2y} - \sin(y)\right)\mathrm{e}^{-x} = 2\,\mathrm{e}^{2y} \sin(x) - \sin(y)\,\mathrm{e}^{-x} + f'(y) \tag{5}$$

Solving equation (5) for $f'(y)$ gives

$$f'(y) = 0$$

Therefore

$$f(y) = c_1$$

Where c_1 is constant of integration. Substituting this result for $f(y)$ into equation (3) gives ϕ

$$\phi = \mathrm{e}^{2y} \sin(x) + \cos(y)\,\mathrm{e}^{-x} + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = \mathrm{e}^{2y} \sin(x) + \cos(y)\,\mathrm{e}^{-x}$$

Summary
The solution(s) found are the following

$$\mathrm{e}^{2y} \sin(x) + \cos(y)\,\mathrm{e}^{-x} = c_1 \tag{1}$$

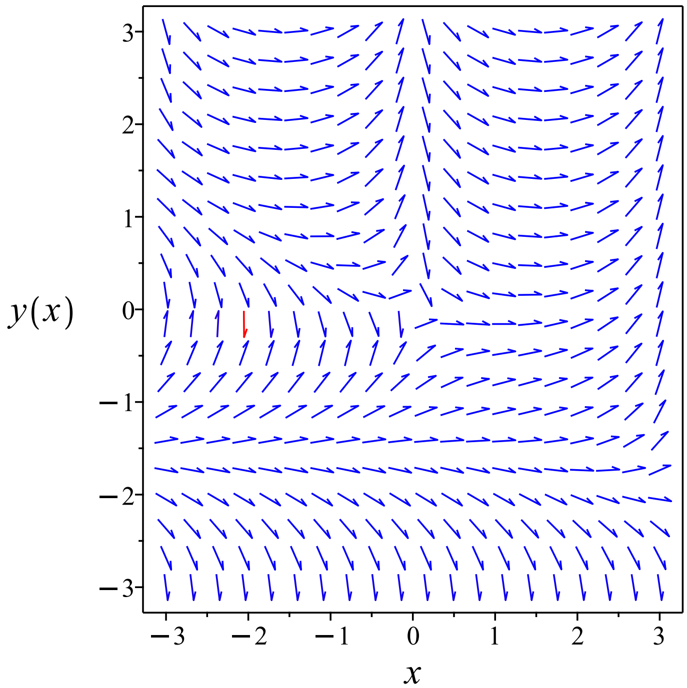


Figure 314: Slope field plot

Verification of solutions

$$e^{2y}\sin(x) + \cos(y)\,e^{-x} = c_1$$

Verified OK.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
trying homogeneous types:
trying Chini
differential order: 1; looking for linear symmetries
trying exact
<- exact successful`
```

✓ Solution by Maple

Time used: 0.047 (sec). Leaf size: 21

```
dsolve(diff(y(x),x) = (-exp(2*y(x))*cos(x)+cos(y(x))/exp(x))/(2*exp(2*y(x))*sin(x)-sin(y(x))
```

$$c_1 + \cos(y(x))\,e^{-x} + e^{2y(x)}\sin(x) = 0$$

✓ Solution by Mathematica

Time used: 0.473 (sec). Leaf size: 25

```
DSolve[y'[x] == (-Exp[2*y[x]]*Cos[x]+Cos[y[x]]/Exp[x])/(2*Exp[2*y[x]]*Sin[x]-Sin[y[x]]/Exp[x
```

$$\text{Solve}\left[e^{2y(x)}\sin(x) + e^{-x}\cos(y(x)) = c_1, y(x)\right]$$

6.17 problem 17

Internal problem ID [584]
Internal file name [`OUTPUT/584_Sunday_June_05_2022_01_45_16_AM_73508669/index.tex`]

Book: Elementary differential equations and boundary value problems, 10th ed., Boyce and DiPrima
Section: Miscellaneous problems, end of chapter 2. Page 133
Problem number: 17.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"linear", "exactWithIntegrationFactor", "first_order_ode_lie_symmetry_lookup"**

Maple gives the following as the ode type

```
[[_linear, `class A`]]
```

$$\boxed{y' - 3y = e^{2x}}$$

6.17.1 Solving as linear ode

Entering Linear first order ODE solver. In canonical form a linear first order is

$$y' + p(x)y = q(x)$$

Where here

$$p(x) = -3$$
$$q(x) = e^{2x}$$

Hence the ode is

$$y' - 3y = e^{2x}$$

The integrating factor μ is

$$
\begin{aligned}
\mu &= \mathrm{e}^{\int(-3)dx} \\
&= \mathrm{e}^{-3x}
\end{aligned}
$$

The ode becomes

$$
\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}x}(\mu y) &= (\mu)\left(\mathrm{e}^{2x}\right) \\
\frac{\mathrm{d}}{\mathrm{d}x}\left(\mathrm{e}^{-3x} y\right) &= \left(\mathrm{e}^{-3x}\right)\left(\mathrm{e}^{2x}\right) \\
\mathrm{d}\left(\mathrm{e}^{-3x} y\right) &= \mathrm{e}^{-x}\,\mathrm{d}x
\end{aligned}
$$

Integrating gives

$$
\begin{aligned}
\mathrm{e}^{-3x} y &= \int \mathrm{e}^{-x}\,\mathrm{d}x \\
\mathrm{e}^{-3x} y &= -\mathrm{e}^{-x} + c_1
\end{aligned}
$$

Dividing both sides by the integrating factor $\mu = \mathrm{e}^{-3x}$ results in

$$
y = -\mathrm{e}^{3x}\mathrm{e}^{-x} + c_1\mathrm{e}^{3x}
$$

which simplifies to

$$
y = -\mathrm{e}^{2x} + c_1\mathrm{e}^{3x}
$$

Summary

The solution(s) found are the following

$$
y = -\mathrm{e}^{2x} + c_1\mathrm{e}^{3x} \tag{1}
$$

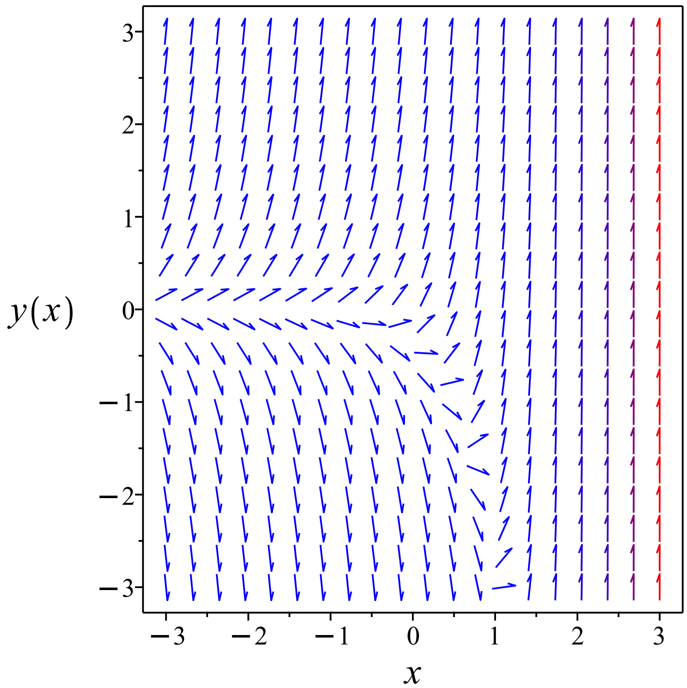


Figure 315: Slope field plot

Verification of solutions

$$y = -e^{2x} + c_1 e^{3x}$$

Verified OK.

6.17.2 Solving as first order ode lie symmetry lookup ode

Writing the ode as

$$y' = e^{2x} + 3y$$
$$y' = \omega(x, y)$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x + \omega(\eta_y - \xi_x) - \omega^2 \xi_y - \omega_x \xi - \omega_y \eta = 0 \tag{A}$$

The type of this ode is known. It is of type `linear`. Therefore we do not need to solve the PDE (A), and can just use the lookup table shown below to find ξ, η

Table 292: Lie symmetry infinitesimal lookup table for known first order ODE's

ODE class	Form	ξ	η
linear ode	$y' = f(x)y(x) + g(x)$	0	$e^{\int f dx}$
separable ode	$y' = f(x)\, g(y)$	$\frac{1}{f}$	0
quadrature ode	$y' = f(x)$	0	1
quadrature ode	$y' = g(y)$	1	0
homogeneous ODEs of Class A	$y' = f\left(\frac{y}{x}\right)$	x	y
homogeneous ODEs of Class C	$y' = (a + bx + cy)^{\frac{n}{m}}$	1	$-\frac{b}{c}$
homogeneous class D	$y' = \frac{y}{x} + g(x)\, F\left(\frac{y}{x}\right)$	x^2	xy
First order special form ID 1	$y' = g(x)\, e^{h(x)+by} + f(x)$	$\frac{e^{-\int bf(x)dx - h(x)}}{g(x)}$	$\frac{f(x)e^{-\int bf(x)dx - h(x)}}{g(x)}$
polynomial type ode	$y' = \frac{a_1x+b_1y+c_1}{a_2x+b_2y+c_2}$	$\frac{a_1b_2x - a_2b_1x - b_1c_2 + b_2c_1}{a_1b_2 - a_2b_1}$	$\frac{a_1b_2y - a_2b_1y - a_1c_2 - a_2c_1}{a_1b_2 - a_2b_1}$
Bernoulli ode	$y' = f(x)\, y + g(x)\, y^n$	0	$e^{-\int (n-1)f(x)dx} y^n$
Reduced Riccati	$y' = f_1(x)\, y + f_2(x)\, y^2$	0	$e^{-\int f_1 dx}$

The above table shows that

$$
\begin{aligned}
\xi(x, y) &= 0 \\
\eta(x, y) &= e^{3x}
\end{aligned}
\tag{A1}
$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x, y) \to (R, S)$ where (R, S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$
\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}
$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y}\right) S(x, y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the

canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$R = x$$

S is found from

$$\begin{aligned} S &= \int \frac{1}{\eta} dy \\ &= \int \frac{1}{e^{3x}} dy \end{aligned}$$

Which results in

$$S = e^{-3x} y$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x, y) S_y}{R_x + \omega(x, y) R_y} \tag{2}$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x, y)$ is the right hand side of the original ode given by

$$\omega(x, y) = e^{2x} + 3y$$

Evaluating all the partial derivatives gives

$$\begin{aligned} R_x &= 1 \\ R_y &= 0 \\ S_x &= -3 e^{-3x} y \\ S_y &= e^{-3x} \end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = e^{-x} \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = e^{-R}$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by

integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$S(R) = -e^{-R} + c_1 \tag{4}$$

To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$y\,e^{-3x} = -e^{-x} + c_1$$

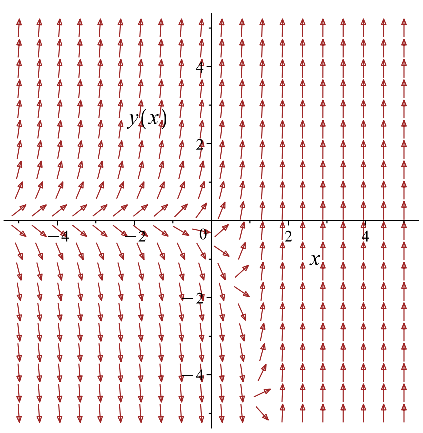
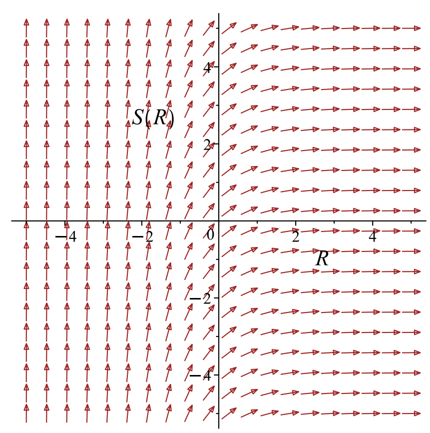
Which simplifies to

$$y\,e^{-3x} = -e^{-x} + c_1$$

Which gives

$$y = -\left(e^{-x} - c_1\right) e^{3x}$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in x, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dx} = e^{2x} + 3y$ 	$R = x$ $S = e^{-3x}y$	$\frac{dS}{dR} = e^{-R}$ 

Summary
The solution(s) found are the following

$$y = -\left(e^{-x} - c_1\right) e^{3x} \tag{1}$$

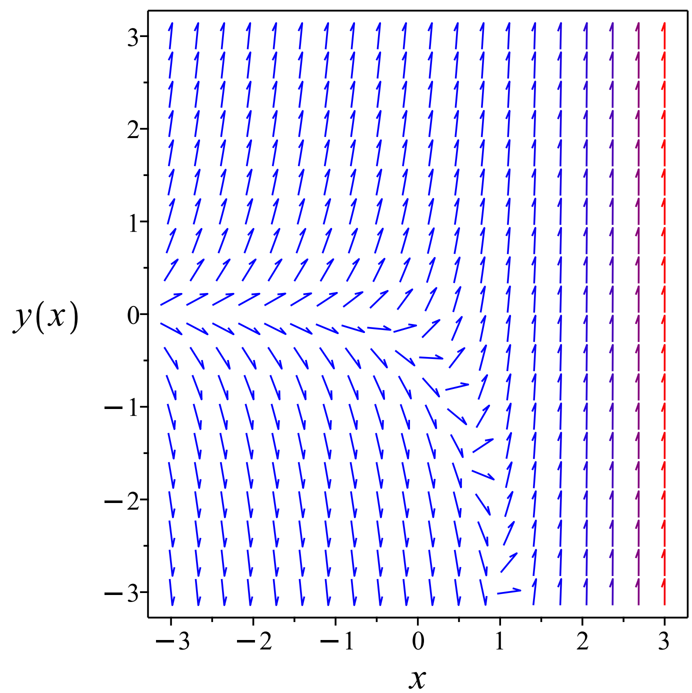


Figure 316: Slope field plot

Verification of solutions

$$y = -\left(\mathrm{e}^{-x} - c_1\right) \mathrm{e}^{3x}$$

Verified OK.

6.17.3 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x,y) + N(x,y)\frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x,y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y) = 0$$

Hence

$$\frac{\partial \phi}{\partial x} + \frac{\partial \phi}{\partial y}\frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$\frac{\partial \phi}{\partial x} = M$$
$$\frac{\partial \phi}{\partial y} = N$$

But since $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x, y)$ but at least we know now that we can do that since the condition $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x, y)\,\mathrm{d}x + N(x, y)\,\mathrm{d}y = 0 \tag{1A}$$

Therefore

$$\mathrm{d}y = \left(\mathrm{e}^{2x} + 3y\right) \mathrm{d}x$$
$$\left(-\mathrm{e}^{2x} - 3y\right) \mathrm{d}x + \mathrm{d}y = 0 \tag{2A}$$

Comparing (1A) and (2A) shows that

$$M(x, y) = -\mathrm{e}^{2x} - 3y$$
$$N(x, y) = 1$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Using result found above gives

$$\begin{aligned} \frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}\left(-\mathrm{e}^{2x} - 3y\right) \\ &= -3 \end{aligned}$$

And

$$\begin{aligned} \frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}(1) \\ &= 0 \end{aligned}$$

Since $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$, then the ODE is not exact. Since the ODE is not exact, we will try to find an integrating factor to make it exact. Let

$$
\begin{aligned}
A &= \frac{1}{N}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right) \\
&= 1((-3) - (0)) \\
&= -3
\end{aligned}
$$

Since A does not depend on y, then it can be used to find an integrating factor. The integrating factor μ is

$$
\begin{aligned}
\mu &= e^{\int A \, \mathrm{d}x} \\
&= e^{\int -3 \, \mathrm{d}x}
\end{aligned}
$$

The result of integrating gives

$$
\begin{aligned}
\mu &= e^{-3x} \\
&= \mathrm{e}^{-3x}
\end{aligned}
$$

M and N are multiplied by this integrating factor, giving new M and new N which are called $\overline{M}$ and $\overline{N}$ for now so not to confuse them with the original M and N.

$$
\begin{aligned}
\overline{M} &= \mu M \\
&= \mathrm{e}^{-3x}\left(-\mathrm{e}^{2x} - 3y\right) \\
&= \left(-\mathrm{e}^{2x} - 3y\right) \mathrm{e}^{-3x}
\end{aligned}
$$

And

$$
\begin{aligned}
\overline{N} &= \mu N \\
&= \mathrm{e}^{-3x}(1) \\
&= \mathrm{e}^{-3x}
\end{aligned}
$$

Now a modified ODE is ontained from the original ODE, which is exact and can be solved. The modified ODE is

$$
\begin{aligned}
\overline{M} + \overline{N}\frac{\mathrm{d}y}{\mathrm{d}x} &= 0 \\
\left(\left(-\mathrm{e}^{2x} - 3y\right) \mathrm{e}^{-3x}\right) + \left(\mathrm{e}^{-3x}\right) \frac{\mathrm{d}y}{\mathrm{d}x} &= 0
\end{aligned}
$$

The following equations are now set up to solve for the function $\phi(x, y)$

$$
\frac{\partial \phi}{\partial x} = \overline{M} \tag{1}
$$

$$
\frac{\partial \phi}{\partial y} = \overline{N} \tag{2}
$$

Integrating (1) w.r.t. x gives

$$\int \frac{\partial \phi}{\partial x} \, \mathrm{d}x = \int \overline{M} \, \mathrm{d}x$$

$$\int \frac{\partial \phi}{\partial x} \, \mathrm{d}x = \int \left(-\mathrm{e}^{2x} - 3y\right) \mathrm{e}^{-3x} \, \mathrm{d}x$$

$$\phi = \mathrm{e}^{-x} + \mathrm{e}^{-3x} y + f(y) \tag{3}$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$\frac{\partial \phi}{\partial y} = \mathrm{e}^{-3x} + f'(y) \tag{4}$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = \mathrm{e}^{-3x}$. Therefore equation (4) becomes

$$\mathrm{e}^{-3x} = \mathrm{e}^{-3x} + f'(y) \tag{5}$$

Solving equation (5) for $f'(y)$ gives

$$f'(y) = 0$$

Therefore

$$f(y) = c_1$$

Where c_1 is constant of integration. Substituting this result for $f(y)$ into equation (3) gives ϕ

$$\phi = \mathrm{e}^{-x} + \mathrm{e}^{-3x} y + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = \mathrm{e}^{-x} + \mathrm{e}^{-3x} y$$

The solution becomes

$$y = -\left(\mathrm{e}^{-x} - c_1\right) \mathrm{e}^{3x}$$

Summary

The solution(s) found are the following

$$y = -\left(\mathrm{e}^{-x} - c_1\right) \mathrm{e}^{3x} \tag{1}$$

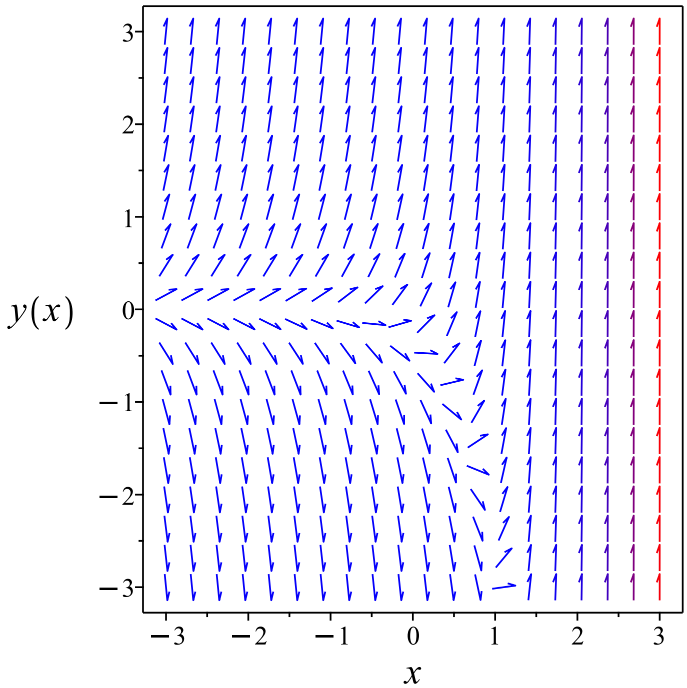


Figure 317: Slope field plot

Verification of solutions

$$y = -\left(e^{-x} - c_1\right) e^{3x}$$

Verified OK.

6.17.4 Maple step by step solution

Let's solve

$y' - 3y = e^{2x}$

- Highest derivative means the order of the ODE is 1

  y'

- Isolate the derivative

  $y' = e^{2x} + 3y$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE

  $y' - 3y = e^{2x}$

- The ODE is linear; multiply by an integrating factor $\mu(x)$

  $\mu(x)\left(y' - 3y\right) = \mu(x) e^{2x}$

- Assume the lhs of the ODE is the total derivative $\frac{d}{dx}(\mu(x)\,y)$

  $\mu(x)\,(y'-3y)=\mu'(x)\,y+\mu(x)\,y'$
- Isolate $\mu'(x)$

  $\mu'(x)=-3\mu(x)$
- Solve to find the integrating factor

  $\mu(x)=\mathrm{e}^{-3x}$
- Integrate both sides with respect to x

  $\int\left(\frac{d}{dx}(\mu(x)\,y)\right)dx=\int\mu(x)\,\mathrm{e}^{2x}dx+c_1$
- Evaluate the integral on the lhs

  $\mu(x)\,y=\int\mu(x)\,\mathrm{e}^{2x}dx+c_1$
- Solve for y

  $y=\frac{\int\mu(x)\mathrm{e}^{2x}dx+c_1}{\mu(x)}$
- Substitute $\mu(x)=\mathrm{e}^{-3x}$

  $y=\frac{\int\mathrm{e}^{2x}\mathrm{e}^{-3x}dx+c_1}{\mathrm{e}^{-3x}}$
- Evaluate the integrals on the rhs

  $y=\frac{-\mathrm{e}^{-x}+c_1}{\mathrm{e}^{-3x}}$
- Simplify

  $y=\mathrm{e}^{2x}(c_1\mathrm{e}^{x}-1)$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
<- 1st order linear successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 15

```
dsolve(diff(y(x),x) = exp(2*x)+3*y(x),y(x), singsol=all)
```

$$y(x) = (e^x c_1 - 1) e^{2x}$$

✓ Solution by Mathematica

Time used: 0.04 (sec). Leaf size: 19

```
DSolve[y'[x]== Exp[2*x]+3*y[x],y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \rightarrow e^{2x}(-1 + c_1 e^x)$$

6.18 problem 18

Internal problem ID [585]
Internal file name [`OUTPUT/585_Sunday_June_05_2022_01_45_17_AM_81100757/index.tex`]

Book: Elementary differential equations and boundary value problems, 10th ed., Boyce and DiPrima
Section: Miscellaneous problems, end of chapter 2. Page 133
Problem number: 18.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"linear", "exactWithIntegrationFactor", "first_order_ode_lie_symmetry_lookup"**

Maple gives the following as the ode type

```
[[_linear, `class A`]]
```

$$\boxed{2y + y' = \mathrm{e}^{-x^2-2x}}$$

6.18.1 Solving as linear ode

Entering Linear first order ODE solver. In canonical form a linear first order is

$$y' + p(x)y = q(x)$$

Where here

$$\begin{aligned} p(x) &= 2 \\ q(x) &= \mathrm{e}^{-x(2+x)} \end{aligned}$$

Hence the ode is

$$2y + y' = \mathrm{e}^{-x(2+x)}$$

The integrating factor μ is

$$\mu = e^{\int 2dx}$$
$$= e^{2x}$$

The ode becomes

$$\frac{d}{dx}(\mu y) = (\mu)\left(e^{-x(2+x)}\right)$$
$$\frac{d}{dx}\left(e^{2x}y\right) = \left(e^{2x}\right)\left(e^{-x(2+x)}\right)$$
$$d\left(e^{2x}y\right) = e^{-x^2}\, dx$$

Integrating gives

$$e^{2x}y = \int e^{-x^2}\, dx$$
$$e^{2x}y = \frac{\sqrt{\pi}\, \operatorname{erf}(x)}{2} + c_1$$

Dividing both sides by the integrating factor $\mu = e^{2x}$ results in

$$y = \frac{e^{-2x}\sqrt{\pi}\, \operatorname{erf}(x)}{2} + c_1 e^{-2x}$$

which simplifies to

$$y = e^{-2x}\left(\frac{\sqrt{\pi}\, \operatorname{erf}(x)}{2} + c_1\right)$$

Summary

The solution(s) found are the following

$$y = e^{-2x}\left(\frac{\sqrt{\pi}\, \operatorname{erf}(x)}{2} + c_1\right) \tag{1}$$

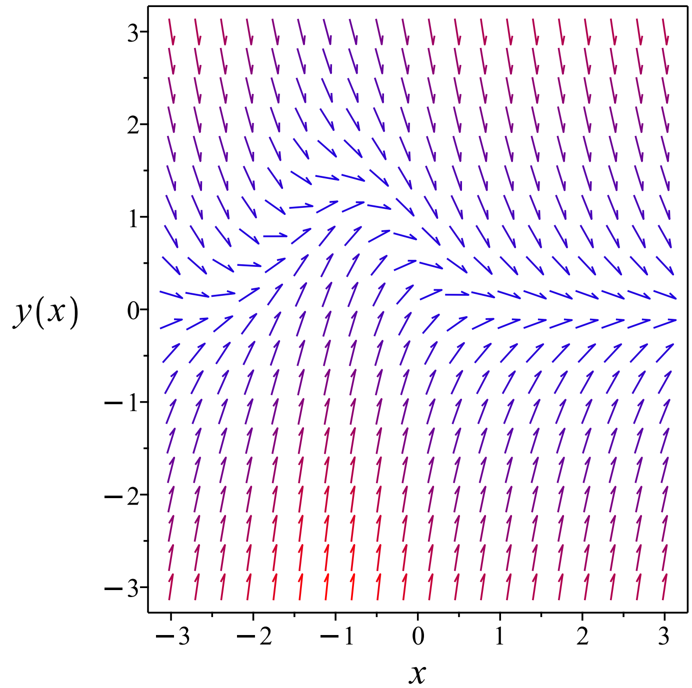


Figure 318: Slope field plot

Verification of solutions

$$y = e^{-2x}\left(\frac{\sqrt{\pi}\,\operatorname{erf}(x)}{2} + c_1\right)$$

Verified OK.

6.18.2 Solving as first order ode lie symmetry lookup ode

Writing the ode as

$$\begin{aligned} y' &= -2y + e^{-x^2-2x} \\ y' &= \omega(x, y) \end{aligned}$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x + \omega(\eta_y - \xi_x) - \omega^2 \xi_y - \omega_x \xi - \omega_y \eta = 0 \tag{A}$$

The type of this ode is known. It is of type `linear`. Therefore we do not need to solve the PDE (A), and can just use the lookup table shown below to find ξ, η

Table 295: Lie symmetry infinitesimal lookup table for known first order ODE's

ODE class	Form	ξ	η
linear ode	$y' = f(x)y(x) + g(x)$	0	$e^{\int f dx}$
separable ode	$y' = f(x)\,g(y)$	$\frac{1}{f}$	0
quadrature ode	$y' = f(x)$	0	1
quadrature ode	$y' = g(y)$	1	0
homogeneous ODEs of Class A	$y' = f\left(\frac{y}{x}\right)$	x	y
homogeneous ODEs of Class C	$y' = (a + bx + cy)^{\frac{n}{m}}$	1	$-\frac{b}{c}$
homogeneous class D	$y' = \frac{y}{x} + g(x)\,F\left(\frac{y}{x}\right)$	x^2	xy
First order special form ID 1	$y' = g(x)\,e^{h(x)+by} + f(x)$	$\frac{e^{-\int bf(x)dx - h(x)}}{g(x)}$	$\frac{f(x)e^{-\int bf(x)dx - h(x)}}{g(x)}$
polynomial type ode	$y' = \frac{a_1x+b_1y+c_1}{a_2x+b_2y+c_2}$	$\frac{a_1b_2x - a_2b_1x - b_1c_2 + b_2c_1}{a_1b_2 - a_2b_1}$	$\frac{a_1b_2y - a_2b_1y - a_1c_2 - a_2c_1}{a_1b_2 - a_2b_1}$
Bernoulli ode	$y' = f(x)\,y + g(x)\,y^n$	0	$e^{-\int (n-1)f(x)dx}y^n$
Reduced Riccati	$y' = f_1(x)\,y + f_2(x)\,y^2$	0	$e^{-\int f_1 dx}$

The above table shows that

$$
\begin{aligned}
\xi(x,y) &= 0 \\
\eta(x,y) &= e^{-2x}
\end{aligned} \tag{A1}
$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x,y) \to (R,S)$ where (R,S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$
\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}
$$

The above comes from the requirements that $\left(\xi\frac{\partial}{\partial x} + \eta\frac{\partial}{\partial y}\right)S(x,y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the

canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$R = x$$

S is found from

$$\begin{aligned} S &= \int \frac{1}{\eta} dy \\ &= \int \frac{1}{\mathrm{e}^{-2x}} dy \end{aligned}$$

Which results in

$$S = \mathrm{e}^{2x} y$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x,y) S_y}{R_x + \omega(x,y) R_y} \tag{2}$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x,y)$ is the right hand side of the original ode given by

$$\omega(x,y) = -2y + \mathrm{e}^{-x^2 - 2x}$$

Evaluating all the partial derivatives gives

$$\begin{aligned} R_x &= 1 \\ R_y &= 0 \\ S_x &= 2\,\mathrm{e}^{2x} y \\ S_y &= \mathrm{e}^{2x} \end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = \mathrm{e}^{-x^2} \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = \mathrm{e}^{-R^2}$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by

integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$S(R) = \frac{\sqrt{\pi}\ \mathrm{erf}\left(R\right)}{2} + c_1 \tag{4}$$

To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$y\,\mathrm{e}^{2x} = \frac{\sqrt{\pi}\ \mathrm{erf}\left(x\right)}{2} + c_1$$

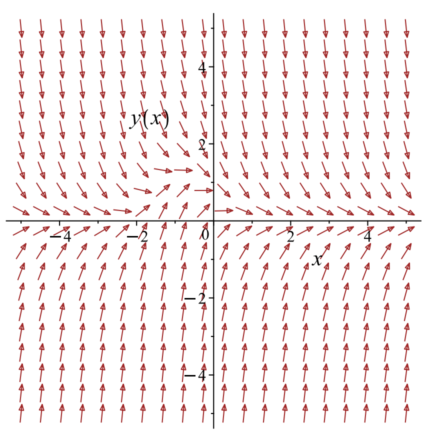
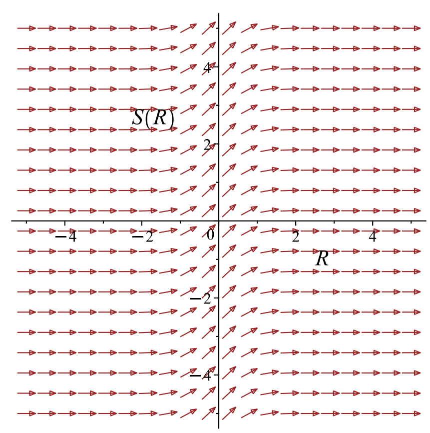
Which simplifies to

$$y\,\mathrm{e}^{2x} = \frac{\sqrt{\pi}\ \mathrm{erf}\left(x\right)}{2} + c_1$$

Which gives

$$y = \frac{\mathrm{e}^{-2x}\left(\sqrt{\pi}\ \mathrm{erf}\left(x\right) + 2c_1\right)}{2}$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in x, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dx} = -2y + \mathrm{e}^{-x^2-2x}$ 	$R = x$ $S = \mathrm{e}^{2x} y$	$\frac{dS}{dR} = \mathrm{e}^{-R^2}$ 

Summary

The solution(s) found are the following

$$y = \frac{\mathrm{e}^{-2x}\left(\sqrt{\pi}\ \mathrm{erf}\left(x\right) + 2c_1\right)}{2} \tag{1}$$

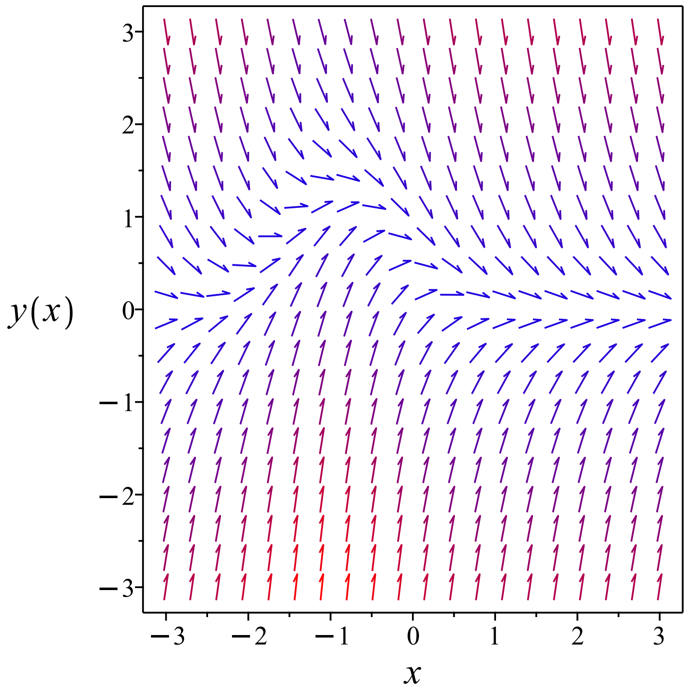


Figure 319: Slope field plot

Verification of solutions

$$y = \frac{e^{-2x}\left(\sqrt{\pi}\,\operatorname{erf}(x) + 2c_1\right)}{2}$$

Verified OK.

6.18.3 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x,y) + N(x,y)\frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x,y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y) = 0$$

Hence

$$\frac{\partial\phi}{\partial x} + \frac{\partial\phi}{\partial y}\frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$\frac{\partial \phi}{\partial x} = M$$
$$\frac{\partial \phi}{\partial y} = N$$

But since $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x, y)$ but at least we know now that we can do that since the condition $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x, y)\,\mathrm{d}x + N(x, y)\,\mathrm{d}y = 0 \tag{1A}$$

Therefore

$$\mathrm{d}y = \left(-2y + \mathrm{e}^{-x^2-2x}\right) \mathrm{d}x$$
$$\left(2y - \mathrm{e}^{-x^2-2x}\right) \mathrm{d}x + \mathrm{d}y = 0 \tag{2A}$$

Comparing (1A) and (2A) shows that

$$M(x, y) = 2y - \mathrm{e}^{-x^2-2x}$$
$$N(x, y) = 1$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Using result found above gives

$$\frac{\partial M}{\partial y} = \frac{\partial}{\partial y}\left(2y - \mathrm{e}^{-x^2-2x}\right)$$
$$= 2$$

And

$$\frac{\partial N}{\partial x} = \frac{\partial}{\partial x}(1)$$
$$= 0$$

Since $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$, then the ODE is not exact. Since the ODE is not exact, we will try to find an integrating factor to make it exact. Let

$$
\begin{aligned}
A &= \frac{1}{N}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right) \\
&= 1((2) - (0)) \\
&= 2
\end{aligned}
$$

Since A does not depend on y, then it can be used to find an integrating factor. The integrating factor μ is

$$
\begin{aligned}
\mu &= e^{\int A \,\mathrm{d}x} \\
&= e^{\int 2 \,\mathrm{d}x}
\end{aligned}
$$

The result of integrating gives

$$
\begin{aligned}
\mu &= e^{2x} \\
&= \mathrm{e}^{2x}
\end{aligned}
$$

M and N are multiplied by this integrating factor, giving new M and new N which are called $\overline{M}$ and $\overline{N}$ for now so not to confuse them with the original M and N.

$$
\begin{aligned}
\overline{M} &= \mu M \\
&= \mathrm{e}^{2x}\left(2y - \mathrm{e}^{-x^2-2x}\right) \\
&= \left(2y - \mathrm{e}^{-x(2+x)}\right)\mathrm{e}^{2x}
\end{aligned}
$$

And

$$
\begin{aligned}
\overline{N} &= \mu N \\
&= \mathrm{e}^{2x}(1) \\
&= \mathrm{e}^{2x}
\end{aligned}
$$

Now a modified ODE is ontained from the original ODE, which is exact and can be solved. The modified ODE is

$$
\begin{aligned}
\overline{M} + \overline{N}\frac{\mathrm{d}y}{\mathrm{d}x} &= 0 \\
\left(\left(2y - \mathrm{e}^{-x(2+x)}\right)\mathrm{e}^{2x}\right) + \left(\mathrm{e}^{2x}\right)\frac{\mathrm{d}y}{\mathrm{d}x} &= 0
\end{aligned}
$$

The following equations are now set up to solve for the function $\phi(x, y)$

$$
\frac{\partial \phi}{\partial x} = \overline{M} \tag{1}
$$

$$
\frac{\partial \phi}{\partial y} = \overline{N} \tag{2}
$$

Integrating (1) w.r.t. x gives

$$\int \frac{\partial \phi}{\partial x} \mathrm{d}x = \int \overline{M} \mathrm{d}x$$

$$\int \frac{\partial \phi}{\partial x} \mathrm{d}x = \int \left(2y - \mathrm{e}^{-x(2+x)}\right) \mathrm{e}^{2x} \mathrm{d}x$$

$$\phi = -\frac{\sqrt{\pi}\, \mathrm{erf}\,(x)}{2} + \mathrm{e}^{2x} y + f(y) \tag{3}$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$\frac{\partial \phi}{\partial y} = \mathrm{e}^{2x} + f'(y) \tag{4}$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = \mathrm{e}^{2x}$. Therefore equation (4) becomes

$$\mathrm{e}^{2x} = \mathrm{e}^{2x} + f'(y) \tag{5}$$

Solving equation (5) for $f'(y)$ gives

$$f'(y) = 0$$

Therefore

$$f(y) = c_1$$

Where c_1 is constant of integration. Substituting this result for $f(y)$ into equation (3) gives ϕ

$$\phi = -\frac{\sqrt{\pi}\, \mathrm{erf}\,(x)}{2} + \mathrm{e}^{2x} y + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = -\frac{\sqrt{\pi}\, \mathrm{erf}\,(x)}{2} + \mathrm{e}^{2x} y$$

The solution becomes

$$y = \frac{\mathrm{e}^{-2x}\left(\sqrt{\pi}\, \mathrm{erf}\,(x) + 2c_1\right)}{2}$$

Summary
The solution(s) found are the following

$$y = \frac{e^{-2x} \left(\sqrt{\pi}\, \operatorname{erf}(x) + 2c_1\right)}{2} \tag{1}$$

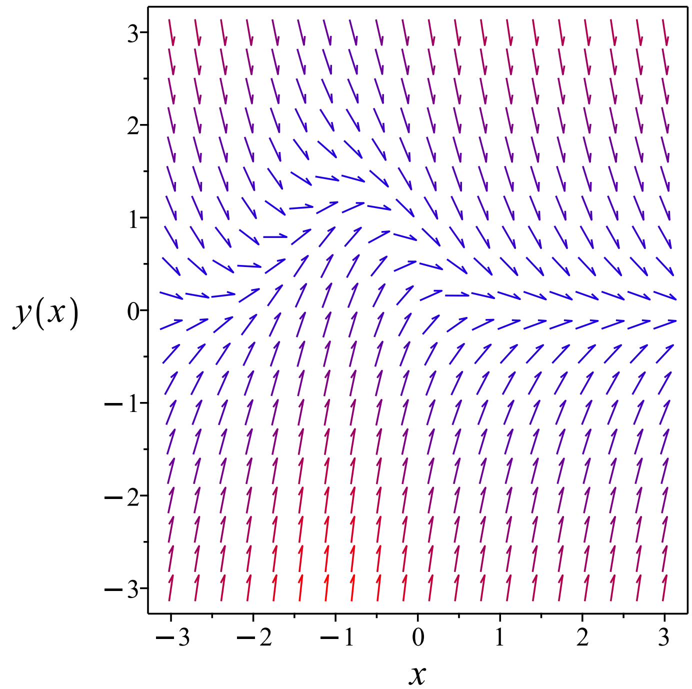


Figure 320: Slope field plot

Verification of solutions

$$y = \frac{e^{-2x} \left(\sqrt{\pi}\, \operatorname{erf}(x) + 2c_1\right)}{2}$$

Verified OK.

6.18.4 Maple step by step solution

Let's solve

$2y + y' = e^{-x^2-2x}$

- Highest derivative means the order of the ODE is 1

  y'

- Isolate the derivative

$$y' = -2y + e^{-x^2-2x}$$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE

$$2y + y' = e^{-x^2-2x}$$

- The ODE is linear; multiply by an integrating factor $\mu(x)$

$$\mu(x)\,(2y + y') = \mu(x)\,e^{-x^2-2x}$$

- Assume the lhs of the ODE is the total derivative $\frac{d}{dx}(\mu(x)\,y)$

$$\mu(x)\,(2y + y') = \mu'(x)\,y + \mu(x)\,y'$$

- Isolate $\mu'(x)$

$$\mu'(x) = 2\mu(x)$$

- Solve to find the integrating factor

$$\mu(x) = e^{2x}$$

- Integrate both sides with respect to x

$$\int \left(\frac{d}{dx}(\mu(x)\,y)\right) dx = \int \mu(x)\,e^{-x^2-2x} dx + c_1$$

- Evaluate the integral on the lhs

$$\mu(x)\,y = \int \mu(x)\,e^{-x^2-2x} dx + c_1$$

- Solve for y

$$y = \frac{\int \mu(x) e^{-x^2-2x} dx + c_1}{\mu(x)}$$

- Substitute $\mu(x) = e^{2x}$

$$y = \frac{\int e^{2x} e^{-x^2-2x} dx + c_1}{e^{2x}}$$

- Evaluate the integrals on the rhs

$$y = \frac{\frac{\sqrt{\pi}\,\mathrm{erf}(x)}{2} + c_1}{e^{2x}}$$

- Simplify

$$y = \frac{e^{-2x}\left(\sqrt{\pi}\,\mathrm{erf}(x) + 2c_1\right)}{2}$$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
<- 1st order linear successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 20

```
dsolve(2*y(x)+diff(y(x),x) = exp(-x^2-2*x),y(x), singsol=all)
```

$$y(x) = \frac{\left(\sqrt{\pi}\, \operatorname{erf}\left(x\right)+2c_1\right) \mathrm{e}^{-2x}}{2}$$

✓ Solution by Mathematica

Time used: 0.09 (sec). Leaf size: 27

```
DSolve[2*y[x]+y'[x] == Exp[-x^2-2*x],y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \frac{1}{2}e^{-2x}\left(\sqrt{\pi}\text{erf}(x)+2c_1\right)$$

6.19 problem 19

Internal problem ID [586]
Internal file name [`OUTPUT/586_Sunday_June_05_2022_01_45_18_AM_94339622/index.tex`]

Book: Elementary differential equations and boundary value problems, 10th ed., Boyce and DiPrima
Section: Miscellaneous problems, end of chapter 2. Page 133
Problem number: 19.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"exact"**

Maple gives the following as the ode type

`[_rational]`

$$\boxed{y' - \frac{3x^2 - 2y - y^3}{2x + 3x y^2} = 0}$$

6.19.1 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x,y) + N(x,y)\frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x,y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y) = 0$$

Hence

$$\frac{\partial \phi}{\partial x} + \frac{\partial \phi}{\partial y}\frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$\frac{\partial \phi}{\partial x} = M$$
$$\frac{\partial \phi}{\partial y} = N$$

But since $\frac{\partial^2\phi}{\partial x\partial y} = \frac{\partial^2\phi}{\partial y\partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x, y)$ but at least we know now that we can do that since the condition $\frac{\partial^2\phi}{\partial x\partial y} = \frac{\partial^2\phi}{\partial y\partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x, y)\,\mathrm{d}x + N(x, y)\,\mathrm{d}y = 0 \tag{1A}$$

Therefore

$$\begin{aligned}\left(x\left(3y^2+2\right)\right)\mathrm{d}y &= \left(-y^3+3x^2-2y\right)\mathrm{d}x\\ \left(y^3-3x^2+2y\right)\mathrm{d}x + \left(x\left(3y^2+2\right)\right)\mathrm{d}y &= 0 \qquad (2A)\end{aligned}$$

Comparing (1A) and (2A) shows that

$$\begin{aligned}M(x, y) &= y^3-3x^2+2y\\ N(x, y) &= x\left(3y^2+2\right)\end{aligned}$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Using result found above gives

$$\begin{aligned}\frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}\left(y^3-3x^2+2y\right)\\ &= 3y^2+2\end{aligned}$$

And

$$\begin{aligned}\frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}\left(x\left(3y^2+2\right)\right)\\ &= 3y^2+2\end{aligned}$$

Since $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$, then the ODE is exact The following equations are now set up to solve for the function $\phi(x, y)$

$$\frac{\partial\phi}{\partial x} = M \tag{1}$$

$$\frac{\partial\phi}{\partial y} = N \tag{2}$$

Integrating (1) w.r.t. x gives

$$\int \frac{\partial \phi}{\partial x} \mathrm{d}x = \int M \mathrm{d}x$$
$$\int \frac{\partial \phi}{\partial x} \mathrm{d}x = \int y^3 - 3x^2 + 2y \mathrm{d}x$$
$$\phi = -x(-y^3 + x^2 - 2y) + f(y) \tag{3}$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$\frac{\partial \phi}{\partial y} = -x(-3y^2 - 2) + f'(y) \tag{4}$$
$$= x(3y^2 + 2) + f'(y)$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = x(3y^2 + 2)$. Therefore equation (4) becomes

$$x(3y^2 + 2) = x(3y^2 + 2) + f'(y) \tag{5}$$

Solving equation (5) for $f'(y)$ gives

$$f'(y) = 0$$

Therefore

$$f(y) = c_1$$

Where c_1 is constant of integration. Substituting this result for $f(y)$ into equation (3) gives ϕ

$$\phi = -x(-y^3 + x^2 - 2y) + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = -x(-y^3 + x^2 - 2y)$$

Summary
The solution(s) found are the following

$$-x(-y^3 + x^2 - 2y) = c_1 \tag{1}$$

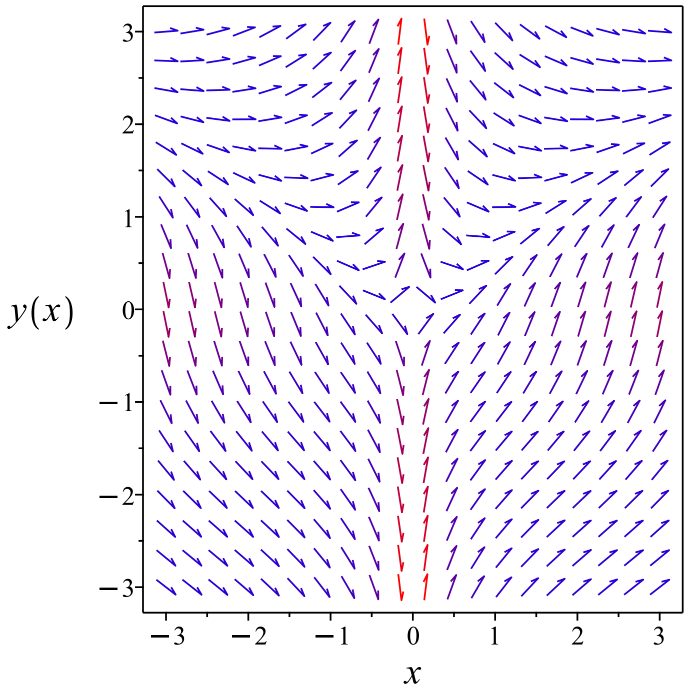


Figure 321: Slope field plot

Verification of solutions

$$-x\left(-y^{3}+x^{2}-2y\right)=c_{1}$$

Verified OK.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
trying homogeneous types:
trying Chini
differential order: 1; looking for linear symmetries
trying exact
<- exact successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 409

```
dsolve(diff(y(x),x) = (3*x^2-2*y(x)-y(x)^3)/(2*x+3*x*y(x)^2),y(x), singsol=all)
```

$$y(x) = -\frac{12^{\frac{1}{3}}\left(x^2 12^{\frac{1}{3}} - \frac{\left(\left(9x^3+\sqrt{3}\,\sqrt{27x^6-54c_1x^3+27c_1^2+32x^2}-9c_1\right)x^2\right)^{\frac{2}{3}}}{2}\right)}{3\left(\left(9x^3+\sqrt{3}\,\sqrt{27x^6-54c_1x^3+27c_1^2+32x^2}-9c_1\right)x^2\right)^{\frac{1}{3}}x}$$

$$y(x) = -\frac{2^{\frac{2}{3}}3^{\frac{1}{3}}\left(2i2^{\frac{2}{3}}3^{\frac{5}{6}}x^2 - 2x^2 2^{\frac{2}{3}}3^{\frac{1}{3}} + i\left(\left(9x^3+\sqrt{3}\,\sqrt{27x^6-54c_1x^3+27c_1^2+32x^2}-9c_1\right)x^2\right)^{\frac{2}{3}}\sqrt{3} + \left(\left(9x^3\right.\right.\right.}{12x\left(\left(9x^3+\sqrt{3}\,\sqrt{27x^6-54c_1x^3+27c_1^2+32x^2}-9c_1\right.\right.}$$

$$y(x) = \frac{2^{\frac{2}{3}}3^{\frac{1}{3}}\left(2\left(i3^{\frac{5}{6}}+3^{\frac{1}{3}}\right)x^2 2^{\frac{2}{3}} + \left(i\sqrt{3}-1\right)\left(\left(9x^3+\sqrt{3}\,\sqrt{27x^6-54c_1x^3+27c_1^2+32x^2}-9c_1\right)x^2\right)^{\frac{2}{3}}\right)}{12\left(\left(9x^3+\sqrt{3}\,\sqrt{27x^6-54c_1x^3+27c_1^2+32x^2}-9c_1\right)x^2\right)^{\frac{1}{3}}x}$$

✓ Solution by Mathematica

Time used: 32.075 (sec). Leaf size: 358

```
DSolve[y'[x] == (3*x^2-2*y[x]-y[x]^3)/(2*x+3*x*y[x]^2),y[x],x,IncludeSingularSolutions -> Tr
```

$$y(x) \to \frac{\sqrt[3]{27x^5 + 27c_1x^2 + \sqrt{864x^6 + 729x^4 (x^3 + c_1)^2}}}{3\sqrt[3]{2}x} - \frac{2\sqrt[3]{2}x}{\sqrt[3]{27x^5 + 27c_1x^2 + \sqrt{864x^6 + 729x^4 (x^3 + c_1)^2}}}$$

$$y(x) \to \frac{\sqrt[3]{2}(1 + i\sqrt{3})\, x}{\sqrt[3]{27x^5 + 27c_1x^2 + \sqrt{864x^6 + 729x^4 (x^3 + c_1)^2}}} - \frac{(1 - i\sqrt{3})\sqrt[3]{27x^5 + 27c_1x^2 + \sqrt{864x^6 + 729x^4 (x^3 + c_1)^2}}}{6\sqrt[3]{2}x}$$

$$y(x) \to \frac{\sqrt[3]{2}(1 - i\sqrt{3})\, x}{\sqrt[3]{27x^5 + 27c_1x^2 + \sqrt{864x^6 + 729x^4 (x^3 + c_1)^2}}} - \frac{(1 + i\sqrt{3})\sqrt[3]{27x^5 + 27c_1x^2 + \sqrt{864x^6 + 729x^4 (x^3 + c_1)^2}}}{6\sqrt[3]{2}x}$$

6.20 problem 20

Internal problem ID [587]
Internal file name [`OUTPUT/587_Sunday_June_05_2022_01_45_20_AM_44183904/index.tex`]

Book: Elementary differential equations and boundary value problems, 10th ed., Boyce and DiPrima
Section: Miscellaneous problems, end of chapter 2. Page 133
Problem number: 20.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"exact", "separable", "first order special form ID 1", "first_order_ode_lie_symmetry_lookup"**

Maple gives the following as the ode type

`[_separable]`

$$\boxed{y' - \mathrm{e}^{x+y} = 0}$$

6.20.1 Solving as separable ode

In canonical form the ODE is

$$\begin{aligned} y' &= F(x,y) \\ &= f(x)g(y) \\ &= \mathrm{e}^{x}\mathrm{e}^{y} \end{aligned}$$

Where $f(x) = \mathrm{e}^{x}$ and $g(y) = \mathrm{e}^{y}$. Integrating both sides gives

$$\begin{aligned} \frac{1}{\mathrm{e}^{y}}\, dy &= \mathrm{e}^{x}\, dx \\ \int \frac{1}{\mathrm{e}^{y}}\, dy &= \int \mathrm{e}^{x}\, dx \\ -\mathrm{e}^{-y} &= \mathrm{e}^{x} + c_1 \end{aligned}$$

Which results in

$$y = \ln\left(-\frac{1}{e^x + c_1}\right)$$

Summary

The solution(s) found are the following

$$y = \ln\left(-\frac{1}{e^x + c_1}\right) \tag{1}$$

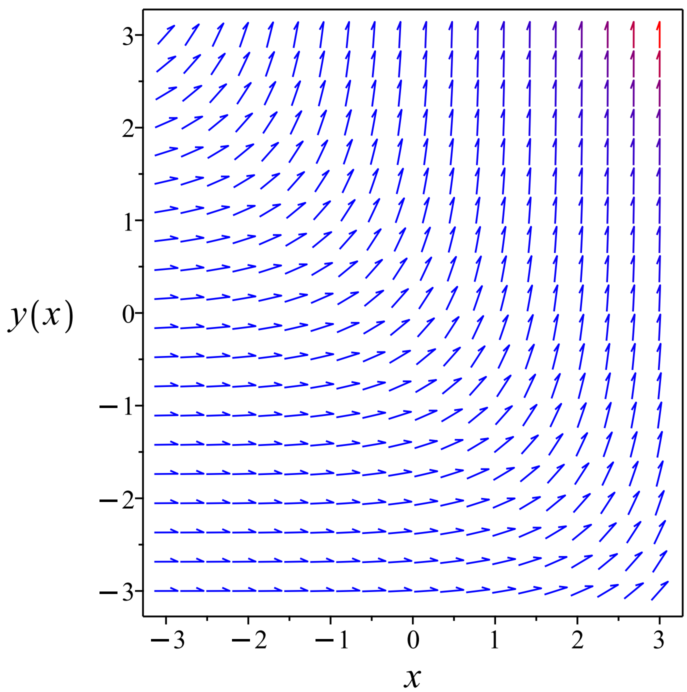


Figure 322: Slope field plot

Verification of solutions

$$y = \ln\left(-\frac{1}{e^x + c_1}\right)$$

Verified OK.

6.20.2 Solving as first order special form ID 1 ode

Writing the ode as

$$y' = e^{x+y} \tag{1}$$

And using the substitution $u = e^{-y}$ then

$$u' = -y' e^{-y}$$

The above shows that

$$\begin{aligned} y' &= -u'(x)\, e^{y} \\ &= -\frac{u'(x)}{u} \end{aligned}$$

Substituting this in (1) gives

$$-\frac{u'(x)}{u} = \frac{e^{x}}{u}$$

The above simplifies to

$$u'(x) = -e^{x} \tag{2}$$

Now ode (2) is solved for $u(x)$ Integrating both sides gives

$$\begin{aligned} u(x) &= \int -e^{x}\, dx \\ &= -e^{x} + c_1 \end{aligned}$$

Substituting the solution found for $u(x)$ in $u = e^{-y}$ gives

$$\begin{aligned} y &= -\ln(u(x)) \\ &= -\ln(-e^{x} + c_1) \\ &= -\ln(-e^{x} + c_1) \end{aligned}$$

Summary

The solution(s) found are the following

$$y = -\ln(-e^{x} + c_1) \tag{1}$$

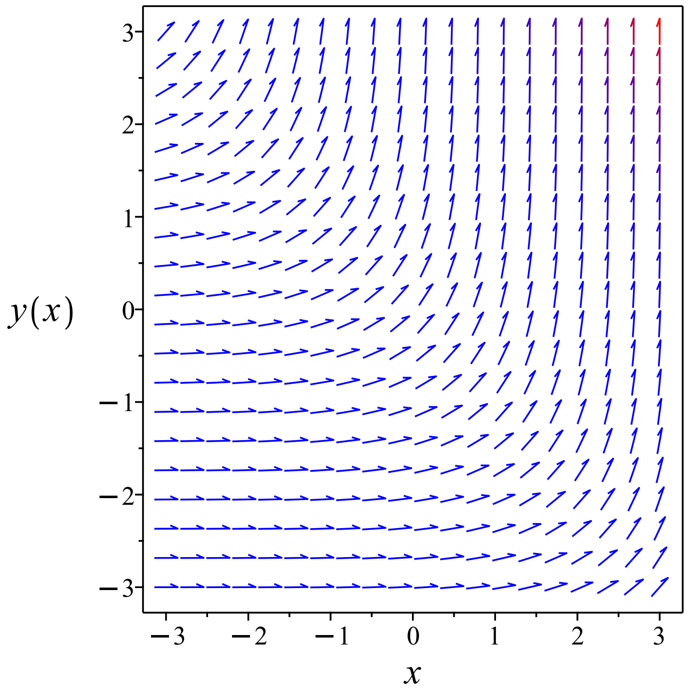


Figure 323: Slope field plot

Verification of solutions

$$y = -\ln\left(-e^{x} + c_1\right)$$

Verified OK.

6.20.3 Solving as first order ode lie symmetry lookup ode

Writing the ode as

$$y' = e^{x+y}$$
$$y' = \omega(x, y)$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x + \omega(\eta_y - \xi_x) - \omega^2 \xi_y - \omega_x \xi - \omega_y \eta = 0 \tag{A}$$

The type of this ode is known. It is of type `separable`. Therefore we do not need to solve the PDE (A), and can just use the lookup table shown below to find ξ, η

Table 298: Lie symmetry infinitesimal lookup table for known first order ODE's

ODE class	Form	ξ	η
linear ode	$y' = f(x)y(x) + g(x)$	0	$e^{\int f dx}$
separable ode	$y' = f(x)\, g(y)$	$\frac{1}{f}$	0
quadrature ode	$y' = f(x)$	0	1
quadrature ode	$y' = g(y)$	1	0
homogeneous ODEs of Class A	$y' = f\left(\frac{y}{x}\right)$	x	y
homogeneous ODEs of Class C	$y' = (a + bx + cy)^{\frac{n}{m}}$	1	$-\frac{b}{c}$
homogeneous class D	$y' = \frac{y}{x} + g(x)\, F\left(\frac{y}{x}\right)$	x^2	xy
First order special form ID 1	$y' = g(x)\, e^{h(x)+by} + f(x)$	$\frac{e^{-\int bf(x)dx-h(x)}}{g(x)}$	$\frac{f(x)e^{-\int bf(x)dx-h(x)}}{g(x)}$
polynomial type ode	$y' = \frac{a_1x+b_1y+c_1}{a_2x+b_2y+c_2}$	$\frac{a_1b_2x-a_2b_1x-b_1c_2+b_2c_1}{a_1b_2-a_2b_1}$	$\frac{a_1b_2y-a_2b_1y-a_1c_2-a_2c_1}{a_1b_2-a_2b_1}$
Bernoulli ode	$y' = f(x)\, y + g(x)\, y^n$	0	$e^{-\int (n-1)f(x)dx} y^n$
Reduced Riccati	$y' = f_1(x)\, y + f_2(x)\, y^2$	0	$e^{-\int f_1 dx}$

The above table shows that

$$\begin{aligned} \xi(x,y) &= \mathrm{e}^{-x} \\ \eta(x,y) &= 0 \end{aligned} \tag{A1}$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x, y) \to (R, S)$ where (R, S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y}\right) S(x,y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the

canonical coordinates, where $S(R)$. Since $\eta = 0$ then in this special case

$$R = y$$

S is found from

$$\begin{aligned} S &= \int \frac{1}{\xi} dx \\ &= \int \frac{1}{e^{-x}} dx \end{aligned}$$

Which results in

$$S = e^{x}$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x,y) S_y}{R_x + \omega(x,y) R_y} \tag{2}$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x,y)$ is the right hand side of the original ode given by

$$\omega(x,y) = e^{x+y}$$

Evaluating all the partial derivatives gives

$$\begin{aligned} R_x &= 0 \\ R_y &= 1 \\ S_x &= e^{x} \\ S_y &= 0 \end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = e^{-y} \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = e^{-R}$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by

integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$S(R) = -\mathrm{e}^{-R} + c_1 \tag{4}$$

To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$\mathrm{e}^{x} = -\mathrm{e}^{-y} + c_1$$

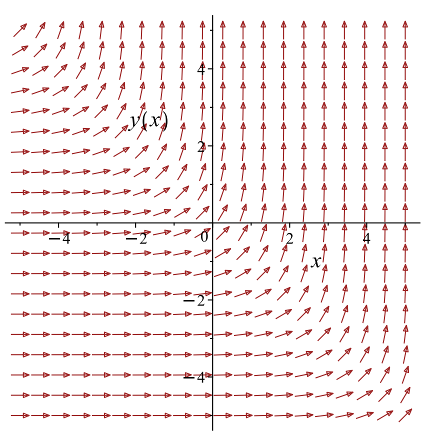
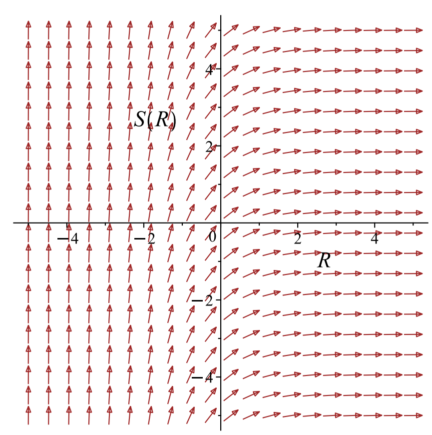
Which simplifies to

$$\mathrm{e}^{x} = -\mathrm{e}^{-y} + c_1$$

Which gives

$$y = -\ln\left(-\mathrm{e}^{x} + c_1\right)$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in x, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dx} = \mathrm{e}^{x+y}$ 	$R = y$ $S = \mathrm{e}^{x}$	$\frac{dS}{dR} = \mathrm{e}^{-R}$ 

Summary

The solution(s) found are the following

$$y = -\ln\left(-\mathrm{e}^{x} + c_1\right) \tag{1}$$

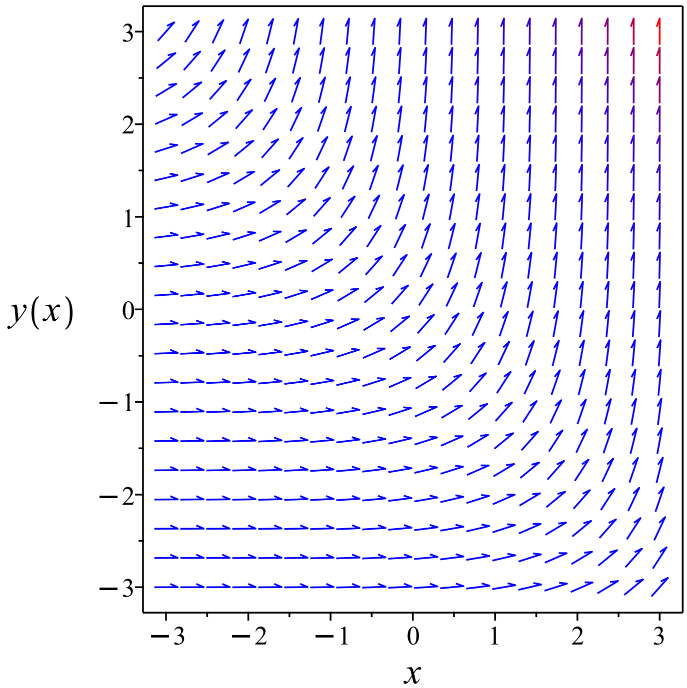


Figure 324: Slope field plot

Verification of solutions

$$y = -\ln\left(-\mathrm{e}^{x} + c_1\right)$$

Verified OK.

6.20.4 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x,y) + N(x,y)\frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x,y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y) = 0$$

Hence

$$\frac{\partial \phi}{\partial x} + \frac{\partial \phi}{\partial y}\frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$\frac{\partial \phi}{\partial x} = M$$
$$\frac{\partial \phi}{\partial y} = N$$

But since $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x, y)$ but at least we know now that we can do that since the condition $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x, y)\, \mathrm{d}x + N(x, y)\, \mathrm{d}y = 0 \tag{1A}$$

Therefore

$$\left(\mathrm{e}^{-y}\right) \mathrm{d}y = \left(\mathrm{e}^{x}\right) \mathrm{d}x$$
$$\left(-\mathrm{e}^{x}\right) \mathrm{d}x + \left(\mathrm{e}^{-y}\right) \mathrm{d}y = 0 \tag{2A}$$

Comparing (1A) and (2A) shows that

$$M(x, y) = -\mathrm{e}^{x}$$
$$N(x, y) = \mathrm{e}^{-y}$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Using result found above gives

$$\frac{\partial M}{\partial y} = \frac{\partial}{\partial y}\left(-\mathrm{e}^{x}\right)$$
$$= 0$$

And

$$\frac{\partial N}{\partial x} = \frac{\partial}{\partial x}\left(\mathrm{e}^{-y}\right)$$
$$= 0$$

Since $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$, then the ODE is <u>exact</u> The following equations are now set up to solve for the function $\phi(x, y)$

$$\frac{\partial \phi}{\partial x} = M \tag{1}$$

$$\frac{\partial \phi}{\partial y} = N \tag{2}$$

Integrating (1) w.r.t. x gives

$$\int \frac{\partial \phi}{\partial x} \,\mathrm{d}x = \int M \,\mathrm{d}x$$

$$\int \frac{\partial \phi}{\partial x} \,\mathrm{d}x = \int -\mathrm{e}^{x} \,\mathrm{d}x$$

$$\phi = -\mathrm{e}^{x} + f(y) \tag{3}$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$\frac{\partial \phi}{\partial y} = 0 + f'(y) \tag{4}$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = \mathrm{e}^{-y}$. Therefore equation (4) becomes

$$\mathrm{e}^{-y} = 0 + f'(y) \tag{5}$$

Solving equation (5) for $f'(y)$ gives

$$f'(y) = \mathrm{e}^{-y}$$

Integrating the above w.r.t y gives

$$\int f'(y) \,\mathrm{d}y = \int \left(\mathrm{e}^{-y}\right) \mathrm{d}y$$

$$f(y) = -\mathrm{e}^{-y} + c_1$$

Where c_1 is constant of integration. Substituting result found above for $f(y)$ into equation (3) gives ϕ

$$\phi = -\mathrm{e}^{x} - \mathrm{e}^{-y} + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = -e^{x} - e^{-y}$$

The solution becomes

$$y = -\ln\left(-e^{x} - c_1\right)$$

Summary

The solution(s) found are the following

$$y = -\ln\left(-e^{x} - c_1\right) \tag{1}$$

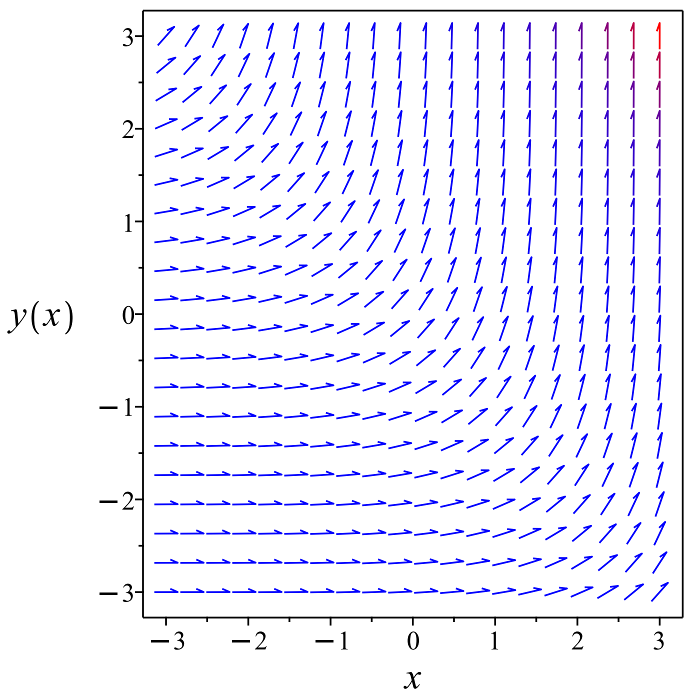


Figure 325: Slope field plot

Verification of solutions

$$y = -\ln\left(-e^{x} - c_1\right)$$

Verified OK.

6.20.5 Maple step by step solution

Let's solve

$y' - e^{x+y} = 0$

- Highest derivative means the order of the ODE is 1

  y'

- Separate variables

  $\frac{y'}{e^y} = e^x$

- Integrate both sides with respect to x

  $\int \frac{y'}{e^y} dx = \int e^x dx + c_1$

- Evaluate integral

  $-\frac{1}{e^y} = e^x + c_1$

- Solve for y

  $y = \ln\left(-\frac{1}{e^x + c_1}\right)$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
<- separable successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 13

```
dsolve(diff(y(x),x) = exp(x+y(x)),y(x), singsol=all)
```

$$y(x) = \ln\left(-\frac{1}{e^x + c_1}\right)$$

✓ Solution by Mathematica

Time used: 0.752 (sec). Leaf size: 18

```
DSolve[y'[x] == Exp[x+y[x]],y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to -\log\left(-e^x - c_1\right)$$

6.21 problem 21

Internal problem ID [588]
Internal file name [OUTPUT/588_Sunday_June_05_2022_01_45_20_AM_85449594/index.tex]

Book: Elementary differential equations and boundary value problems, 10th ed., Boyce and DiPrima
Section: Miscellaneous problems, end of chapter 2. Page 133
Problem number: 21.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"exact"**

Maple gives the following as the ode type

`[_rational]`

$$\boxed{\frac{-4+6yx+2y^2}{3x^2+4yx+3y^2}+y'=0}$$

6.21.1 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x,y)+N(x,y)\frac{dy}{dx}=0 \tag{A}$$

We assume there exists a function $\phi(x,y)=c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y)=0$$

Hence

$$\frac{\partial\phi}{\partial x}+\frac{\partial\phi}{\partial y}\frac{dy}{dx}=0 \tag{B}$$

Comparing (A,B) shows that

$$\frac{\partial\phi}{\partial x}=M$$
$$\frac{\partial\phi}{\partial y}=N$$

But since $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x, y)$ but at least we know now that we can do that since the condition $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x, y)\,\mathrm{d}x + N(x, y)\,\mathrm{d}y = 0 \tag{1A}$$

Therefore

$$\begin{aligned}(3x^2 + 4yx + 3y^2)\,\mathrm{d}y &= (-6yx - 2y^2 + 4)\,\mathrm{d}x \\ (6yx + 2y^2 - 4)\,\mathrm{d}x + (3x^2 + 4yx + 3y^2)\,\mathrm{d}y &= 0 \end{aligned} \tag{2A}$$

Comparing (1A) and (2A) shows that

$$\begin{aligned} M(x, y) &= 6yx + 2y^2 - 4 \\ N(x, y) &= 3x^2 + 4yx + 3y^2 \end{aligned}$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Using result found above gives

$$\begin{aligned} \frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}(6yx + 2y^2 - 4) \\ &= 6x + 4y \end{aligned}$$

And

$$\begin{aligned} \frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}(3x^2 + 4yx + 3y^2) \\ &= 6x + 4y \end{aligned}$$

Since $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$, then the ODE is exact The following equations are now set up to solve for the function $\phi(x, y)$

$$\frac{\partial \phi}{\partial x} = M \tag{1}$$

$$\frac{\partial \phi}{\partial y} = N \tag{2}$$

Integrating (1) w.r.t. x gives

$$\int \frac{\partial \phi}{\partial x} \, \mathrm{d}x = \int M \, \mathrm{d}x$$

$$\int \frac{\partial \phi}{\partial x} \, \mathrm{d}x = \int 6yx + 2y^2 - 4 \, \mathrm{d}x$$

$$\phi = 3y\,x^2 + 2x\,y^2 - 4x + f(y) \tag{3}$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$\frac{\partial \phi}{\partial y} = 3x^2 + 4yx + f'(y) \tag{4}$$

$$= x(3x + 4y) + f'(y)$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = 3x^2 + 4yx + 3y^2$. Therefore equation (4) becomes

$$3x^2 + 4yx + 3y^2 = x(3x + 4y) + f'(y) \tag{5}$$

Solving equation (5) for $f'(y)$ gives

$$f'(y) = 3y^2$$

Integrating the above w.r.t y gives

$$\int f'(y) \, \mathrm{d}y = \int (3y^2) \, \mathrm{d}y$$

$$f(y) = y^3 + c_1$$

Where c_1 is constant of integration. Substituting result found above for $f(y)$ into equation (3) gives ϕ

$$\phi = 3y\,x^2 + 2x\,y^2 + y^3 - 4x + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = 3y\,x^2 + 2x\,y^2 + y^3 - 4x$$

Summary

The solution(s) found are the following

$$3x^2y + 2xy^2 + y^3 - 4x = c_1 \tag{1}$$

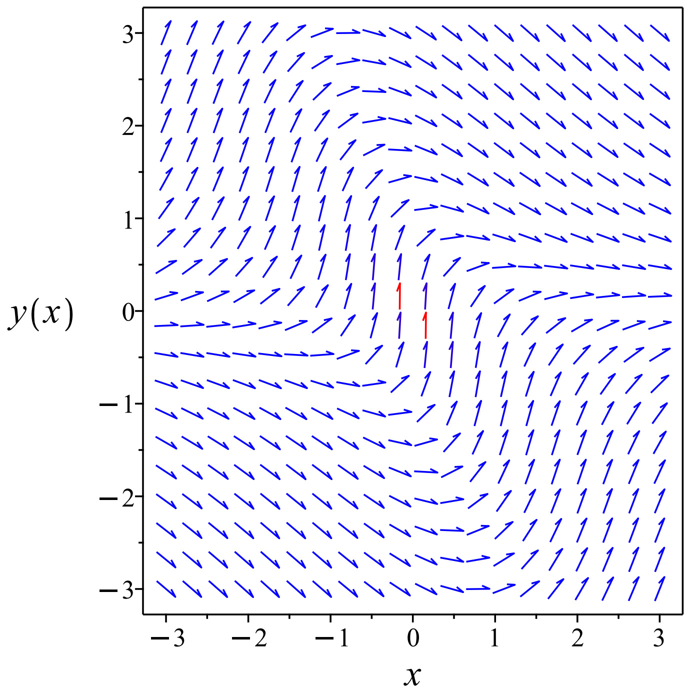


Figure 326: Slope field plot

Verification of solutions

$$3x^2y + 2xy^2 + y^3 - 4x = c_1$$

Verified OK.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
trying homogeneous types:
trying Chini
differential order: 1; looking for linear symmetries
trying exact
<- exact successful`
```

✓ Solution by Maple

Time used: 0.015 (sec). Leaf size: 517

```
dsolve((-4+6*x*y(x)+2*y(x)^2)/(3*x^2+4*x*y(x)+3*y(x)^2)+diff(y(x),x)=0,y(x), singsol=all)
```

$$y(x) = \frac{\left(152x^3 - 108c_1 + 432x + 12\sqrt{216x^6 - 228c_1x^3 + 912x^4 + 81c_1^2 - 648c_1x + 1296x^2}\right)^{\frac{1}{3}}}{6} - \frac{10x^2}{3\left(152x^3 - 108c_1 + 432x + 12\sqrt{216x^6 - 228c_1x^3 + 912x^4 + 81c_1^2 - 648c_1x + 1296x^2}\right)^{\frac{1}{3}}} - \frac{2x}{3}$$

$$y(x) = -\frac{\left(1+i\sqrt{3}\right)\left(152x^3 - 108c_1 + 432x + 12\sqrt{216x^6 - 228c_1x^3 + 912x^4 + 81c_1^2 - 648c_1x + 1296x^2}\right)^{\frac{1}{3}}}{12} - \frac{5x\left(i\sqrt{3}\,x - x + \frac{2\left(152x^3 - 108c_1 + 432x + 12\sqrt{216x^6 - 228c_1x^3 + 912x^4 + 81c_1^2 - 648c_1x + 1296x^2}\right)^{\frac{1}{3}}}{5}\right)}{3\left(152x^3 - 108c_1 + 432x + 12\sqrt{216x^6 - 228c_1x^3 + 912x^4 + 81c_1^2 - 648c_1x + 1296x^2}\right)^{\frac{1}{3}}}$$

$$y(x) = \frac{i\left(152x^3 - 108c_1 + 432x + 12\sqrt{216x^6 - 228c_1x^3 + 912x^4 + 81c_1^2 - 648c_1x + 1296x^2}\right)^{\frac{2}{3}}\sqrt{3} + 20i\sqrt{3}\,x^2 \cdots}{12 \cdots}$$

✓ Solution by Mathematica

Time used: 4.748 (sec). Leaf size: 383

```
DSolve[(-4+6*x*y[x]+2*y[x]^2)/(3*x^2+4*x*y[x]+3*y[x]^2)+y'[x]==0,y[x],x,IncludeSingularSolut
```

$$y(x) \to \frac{1}{6}\left(2^{2/3}\sqrt[3]{38x^3+\sqrt{500x^6+(38x^3+108x+27c_1)^2}+108x+27c_1} - \frac{10\sqrt[3]{2}x^2}{\sqrt[3]{38x^3+\sqrt{500x^6+(38x^3+108x+27c_1)^2}+108x+27c_1}} - 4x\right)$$

$$y(x) \to \frac{1}{12}\left(i2^{2/3}\left(\sqrt{3}+i\right)\sqrt[3]{38x^3+\sqrt{500x^6+(38x^3+108x+27c_1)^2}+108x+27c_1} + \frac{10\sqrt[3]{2}\left(1+i\sqrt{3}\right)x^2}{\sqrt[3]{38x^3+\sqrt{500x^6+(38x^3+108x+27c_1)^2}+108x+27c_1}} - 8x\right)$$

$$y(x) \to \frac{1}{12}\left(-2^{2/3}\left(1+i\sqrt{3}\right)\sqrt[3]{38x^3+\sqrt{500x^6+(38x^3+108x+27c_1)^2}+108x+27c_1} + \frac{10\sqrt[3]{2}\left(1-i\sqrt{3}\right)x^2}{\sqrt[3]{38x^3+\sqrt{500x^6+(38x^3+108x+27c_1)^2}+108x+27c_1}} - 8x\right)$$

6.22 problem 22

Internal problem ID [589]
Internal file name [`OUTPUT/589_Sunday_June_05_2022_01_45_22_AM_65389013/index.tex`]

Book: Elementary differential equations and boundary value problems, 10th ed., Boyce and DiPrima
Section: Miscellaneous problems, end of chapter 2. Page 133
Problem number: 22.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"exact", "separable", "differentialType", "first_order_ode_lie_symmetry_lookup"**

Maple gives the following as the ode type

`[_separable]`

$$\boxed{y' - \frac{x^2 - 1}{1 + y^2} = 0}$$

With initial conditions

$$[y(-1) = 1]$$

6.22.1 Existence and uniqueness analysis

This is non linear first order ODE. In canonical form it is written as

$$
\begin{aligned}
y' &= f(x, y) \\
&= \frac{x^2-1}{y^2+1}
\end{aligned}
$$

The x domain of $f(x, y)$ when $y = 1$ is

$$
\{-\infty < x < \infty\}
$$

And the point $x_0 = -1$ is inside this domain. The y domain of $f(x, y)$ when $x = -1$ is

$$
\{-\infty < y < \infty\}
$$

And the point $y_0 = 1$ is inside this domain. Now we will look at the continuity of

$$
\begin{aligned}
\frac{\partial f}{\partial y} &= \frac{\partial}{\partial y}\left(\frac{x^2-1}{y^2+1}\right) \\
&= -\frac{2(x^2-1)\,y}{\left(y^2+1\right)^2}
\end{aligned}
$$

The x domain of $\frac{\partial f}{\partial y}$ when $y = 1$ is

$$
\{-\infty < x < \infty\}
$$

And the point $x_0 = -1$ is inside this domain. The y domain of $\frac{\partial f}{\partial y}$ when $x = -1$ is

$$
\{-\infty < y < \infty\}
$$

And the point $y_0 = 1$ is inside this domain. Therefore solution exists and is unique.

6.22.2 Solving as separable ode

In canonical form the ODE is

$$
\begin{aligned}
y' &= F(x, y) \\
&= f(x)g(y) \\
&= \frac{x^2-1}{y^2+1}
\end{aligned}
$$

Where $f(x) = x^2 - 1$ and $g(y) = \frac{1}{y^2+1}$. Integrating both sides gives

$$\frac{1}{\frac{1}{y^2+1}}\,dy = x^2 - 1\,dx$$

$$\int \frac{1}{\frac{1}{y^2+1}}\,dy = \int x^2 - 1\,dx$$

$$\frac{1}{3}y^3 + y = \frac{1}{3}x^3 - x + c_1$$

Which results in

$$y = \frac{\left(4x^3 + 12c_1 - 12x + 4\sqrt{x^6 + 6c_1x^3 - 6x^4 + 9c_1^2 - 18c_1x + 9x^2 + 4}\right)^{\frac{1}{3}}}{2} - \frac{2}{\left(4x^3 + 12c_1 - 12x + 4\sqrt{x^6 + 6c_1x^3 - 6x^4 + 9c_1^2 - 18c_1x + 9x^2 + 4}\right)^{\frac{1}{3}}}$$

$$y = -\frac{\left(4x^3 + 12c_1 - 12x + 4\sqrt{x^6 + 6c_1x^3 - 6x^4 + 9c_1^2 - 18c_1x + 9x^2 + 4}\right)^{\frac{1}{3}}}{4} + \frac{1}{\left(4x^3 + 12c_1 - 12x + 4\sqrt{x^6 + 6c_1x^3 - 6x^4 + 9c_1^2 - 18c_1x + 9x^2 + 4}\right)^{\frac{1}{3}}} + \frac{i\sqrt{3}\left(\frac{\left(4x^3+12c_1-12x+4\sqrt{x^6+6c_1x^3-6x^4+9c_1^2-18c_1x+9x^2+4}\right)^{\frac{1}{3}}}{2} + \frac{2}{\left(4x^3+12c_1-12x+4\sqrt{x^6+6c_1x^3-6x^4+9c_1^2-18c_1x+9x^2+4}\right)^{\frac{1}{3}}}\right.}{2}$$

$$y = -\frac{\left(4x^3 + 12c_1 - 12x + 4\sqrt{x^6 + 6c_1x^3 - 6x^4 + 9c_1^2 - 18c_1x + 9x^2 + 4}\right)^{\frac{1}{3}}}{4} + \frac{1}{\left(4x^3 + 12c_1 - 12x + 4\sqrt{x^6 + 6c_1x^3 - 6x^4 + 9c_1^2 - 18c_1x + 9x^2 + 4}\right)^{\frac{1}{3}}} - \frac{i\sqrt{3}\left(\frac{\left(4x^3+12c_1-12x+4\sqrt{x^6+6c_1x^3-6x^4+9c_1^2-18c_1x+9x^2+4}\right)^{\frac{1}{3}}}{2} + \frac{2}{\left(4x^3+12c_1-12x+4\sqrt{x^6+6c_1x^3-6x^4+9c_1^2-18c_1x+9x^2+4}\right)^{\frac{1}{3}}}\right.}{2}$$

Initial conditions are used to solve for c_1. Substituting $x = -1$ and $y = 1$ in the above solution gives an equation to solve for the constant of integration.

$$1 = \frac{-i\sqrt{3}\left(8+12c_1+4\sqrt{9c_1^2+12c_1+8}\right)^{\frac{2}{3}} - 4i\sqrt{3} - \left(8+12c_1+4\sqrt{9c_1^2+12c_1+8}\right)^{\frac{2}{3}} + 4}{4\left(8+12c_1+4\sqrt{9c_1^2+12c_1+8}\right)^{\frac{1}{3}}}$$

Unable to solve for constant of integration.Initial conditions are used to solve for c_1. Substituting $x = -1$ and $y = 1$ in the above solution gives an equation to solve for the constant of integration.

$$1 = \frac{i\sqrt{3}\left(8+12c_1+4\sqrt{9c_1^2+12c_1+8}\right)^{\frac{2}{3}} - \left(8+12c_1+4\sqrt{9c_1^2+12c_1+8}\right)^{\frac{2}{3}} + 4i\sqrt{3} + 4}{4\left(8+12c_1+4\sqrt{9c_1^2+12c_1+8}\right)^{\frac{1}{3}}}$$

Unable to solve for constant of integration.Initial conditions are used to solve for c_1. Substituting $x = -1$ and $y = 1$ in the above solution gives an equation to solve for the constant of integration.

$$1 = \frac{\left(8+12c_1+4\sqrt{9c_1^2+12c_1+8}\right)^{\frac{2}{3}} - 4}{2\left(8+12c_1+4\sqrt{9c_1^2+12c_1+8}\right)^{\frac{1}{3}}}$$

$$c_1 = \frac{2}{3}$$

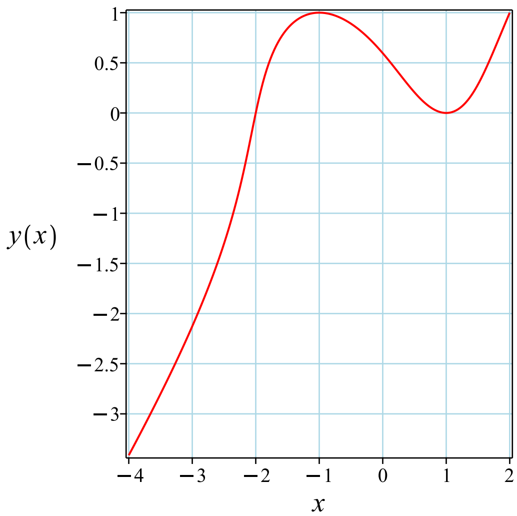
Substituting c_1 found above in the general solution gives

$$y = \frac{\left(4x^3+8-12x+4\sqrt{x^6-6x^4+4x^3+9x^2-12x+8}\right)^{\frac{2}{3}} - 4}{2\left(4x^3+8-12x+4\sqrt{x^6-6x^4+4x^3+9x^2-12x+8}\right)^{\frac{1}{3}}}$$

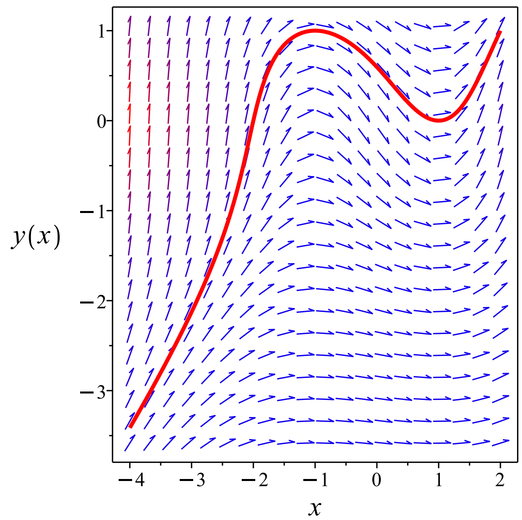
Summary

The solution(s) found are the following

$$y = \frac{\left(4x^3+8-12x+4\sqrt{x^6-6x^4+4x^3+9x^2-12x+8}\right)^{\frac{2}{3}} - 4}{2\left(4x^3+8-12x+4\sqrt{x^6-6x^4+4x^3+9x^2-12x+8}\right)^{\frac{1}{3}}} \tag{1}$$



(a) Solution plot



(b) Slope field plot

Verification of solutions

$$y = \frac{\left(4x^3 + 8 - 12x + 4\sqrt{x^6 - 6x^4 + 4x^3 + 9x^2 - 12x + 8}\right)^{\frac{2}{3}} - 4}{2\left(4x^3 + 8 - 12x + 4\sqrt{x^6 - 6x^4 + 4x^3 + 9x^2 - 12x + 8}\right)^{\frac{1}{3}}}$$

Verified OK.

6.22.3 Solving as differentialType ode

Writing the ode as

$$y' = \frac{x^2 - 1}{1 + y^2} \tag{1}$$

Which becomes

$$(y^2 + 1)\, dy = (x^2 - 1)\, dx \tag{2}$$

But the RHS is complete differential because

$$(x^2 - 1)\, dx = d\left(\frac{1}{3}x^3 - x\right)$$

Hence (2) becomes

$$(y^2 + 1)\, dy = d\left(\frac{1}{3}x^3 - x\right)$$

Integrating both sides gives gives these solutions

$$y = \frac{\left(4x^3 + 12c_1 - 12x + 4\sqrt{x^6 + 6c_1x^3 - 6x^4 + 9c_1^2 - 18c_1x + 9x^2 + 4}\right)^{\frac{1}{3}}}{2} - \frac{}{\left(4x^3 + 12c_1 - 12x + 4\sqrt{x^6}\right.}$$

$$y = -\frac{\left(4x^3 + 12c_1 - 12x + 4\sqrt{x^6 + 6c_1x^3 - 6x^4 + 9c_1^2 - 18c_1x + 9x^2 + 4}\right)^{\frac{1}{3}}}{4} + \frac{}{\left(4x^3 + 12c_1 - 12x + 4\sqrt{\right.}}$$

$$y = -\frac{\left(4x^3 + 12c_1 - 12x + 4\sqrt{x^6 + 6c_1x^3 - 6x^4 + 9c_1^2 - 18c_1x + 9x^2 + 4}\right)^{\frac{1}{3}}}{4} + \frac{}{\left(4x^3 + 12c_1 - 12x + 4\sqrt{\right.}}$$

Initial conditions are used to solve for c_1. Substituting $x = -1$ and $y = 1$ in the above solution gives an equation to solve for the constant of integration.

$$1 = \frac{-i\sqrt{3}\left(8 + 12c_1 + 4\sqrt{9c_1^2 + 12c_1 + 8}\right)^{\frac{2}{3}} - \left(8 + 12c_1 + 4\sqrt{9c_1^2 + 12c_1 + 8}\right)^{\frac{2}{3}} + 4c_1\left(8 + 12c_1 + 4\sqrt{9c_1^2}\right.}{4\left(8 + 12c_1 + 4\sqrt{9c_1^2 + 12c_1 + 8}\right)^{\frac{1}{3}}}$$

Warning: Unable to solve for constant of integration. Initial conditions are used to solve for c_1. Substituting $x = -1$ and $y = 1$ in the above solution gives an equation to solve for the constant of integration.

$$1 = \frac{i\sqrt{3}\left(8 + 12c_1 + 4\sqrt{9c_1^2 + 12c_1 + 8}\right)^{\frac{2}{3}} - \left(8 + 12c_1 + 4\sqrt{9c_1^2 + 12c_1 + 8}\right)^{\frac{2}{3}} + 4c_1\left(8 + 12c_1 + 4\sqrt{9c_1^2 -}\right.}{4\left(8 + 12c_1 + 4\sqrt{9c_1^2 + 12c_1 + 8}\right)^{\frac{1}{3}}}$$

Warning: Unable to solve for constant of integration. Initial conditions are used to solve for c_1. Substituting $x = -1$ and $y = 1$ in the above solution gives an equation to solve for the constant of integration.

$$1 = \frac{\left(8 + 12c_1 + 4\sqrt{9c_1^2 + 12c_1 + 8}\right)^{\frac{2}{3}} + 2c_1\left(8 + 12c_1 + 4\sqrt{9c_1^2 + 12c_1 + 8}\right)^{\frac{1}{3}} - 4}{2\left(8 + 12c_1 + 4\sqrt{9c_1^2 + 12c_1 + 8}\right)^{\frac{1}{3}}}$$

Warning: Unable to solve for constant of integration.

<u>Verification of solutions</u> N/A

6.22.4 Solving as first order ode lie symmetry lookup ode

Writing the ode as

$$y' = \frac{x^2 - 1}{y^2 + 1}$$
$$y' = \omega(x, y)$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x + \omega(\eta_y - \xi_x) - \omega^2 \xi_y - \omega_x \xi - \omega_y \eta = 0 \tag{A}$$

The type of this ode is known. It is of type `separable`. Therefore we do not need to solve the PDE (A), and can just use the lookup table shown below to find ξ, η

Table 301: Lie symmetry infinitesimal lookup table for known first order ODE's

ODE class	Form	ξ	η
linear ode	$y' = f(x)y(x) + g(x)$	0	$e^{\int f dx}$
separable ode	$y' = f(x)\, g(y)$	$\frac{1}{f}$	0
quadrature ode	$y' = f(x)$	0	1
quadrature ode	$y' = g(y)$	1	0
homogeneous ODEs of Class A	$y' = f\left(\frac{y}{x}\right)$	x	y
homogeneous ODEs of Class C	$y' = (a + bx + cy)^{\frac{n}{m}}$	1	$-\frac{b}{c}$
homogeneous class D	$y' = \frac{y}{x} + g(x)\, F\left(\frac{y}{x}\right)$	x^2	xy
First order special form ID 1	$y' = g(x)\, e^{h(x)+by} + f(x)$	$\frac{e^{-\int bf(x)dx - h(x)}}{g(x)}$	$\frac{f(x)e^{-\int bf(x)dx - h(x)}}{g(x)}$
polynomial type ode	$y' = \frac{a_1x + b_1y + c_1}{a_2x + b_2y + c_2}$	$\frac{a_1b_2x - a_2b_1x - b_1c_2 + b_2c_1}{a_1b_2 - a_2b_1}$	$\frac{a_1b_2y - a_2b_1y - a_1c_2 - a_2c_1}{a_1b_2 - a_2b_1}$
Bernoulli ode	$y' = f(x)\, y + g(x)\, y^n$	0	$e^{-\int (n-1)f(x)dx} y^n$
Reduced Riccati	$y' = f_1(x)\, y + f_2(x)\, y^2$	0	$e^{-\int f_1 dx}$

The above table shows that

$$
\begin{aligned}
\xi(x, y) &= \frac{1}{x^2 - 1} \\
\eta(x, y) &= 0
\end{aligned} \tag{A1}
$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x, y) \to (R, S)$ where (R, S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$
\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}
$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y}\right) S(x, y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the canonical coordinates, where $S(R)$. Since $\eta = 0$ then in this special case

$$
R = y
$$

S is found from

$$
\begin{aligned}
S &= \int \frac{1}{\xi} dx \\
&= \int \frac{1}{\frac{1}{x^2-1}} dx
\end{aligned}
$$

Which results in

$$
S = \frac{1}{3} x^3 - x
$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$
\frac{dS}{dR} = \frac{S_x + \omega(x, y) S_y}{R_x + \omega(x, y) R_y} \tag{2}
$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x, y)$ is the right hand side of the original ode given by

$$
\omega(x, y) = \frac{x^2 - 1}{y^2 + 1}
$$

Evaluating all the partial derivatives gives

$$
\begin{aligned}
R_x &= 0 \\
R_y &= 1 \\
S_x &= x^2 - 1 \\
S_y &= 0
\end{aligned}
$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = y^2 + 1 \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = R^2 + 1$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$S(R) = \frac{1}{3}R^3 + R + c_1 \tag{4}$$

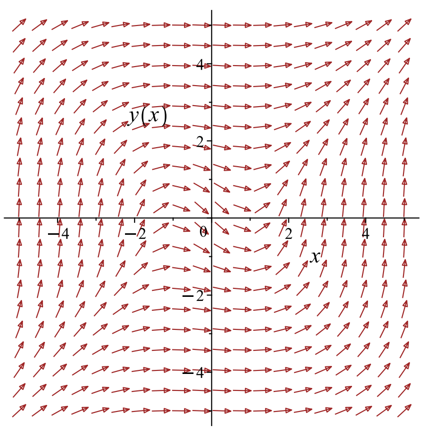
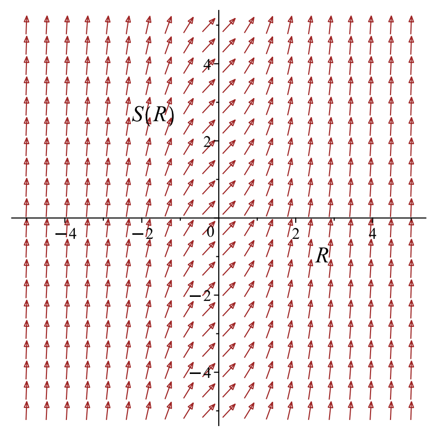
To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$\frac{1}{3}x^3 - x = \frac{y^3}{3} + y + c_1$$

Which simplifies to

$$\frac{1}{3}x^3 - x = \frac{y^3}{3} + y + c_1$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in x, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dx} = \frac{x^2-1}{y^2+1}$ 	$R = y$ $S = \frac{1}{3}x^3 - x$	$\frac{dS}{dR} = R^2 + 1$ 

Initial conditions are used to solve for c_1. Substituting $x = -1$ and $y = 1$ in the above solution gives an equation to solve for the constant of integration.

$$\frac{2}{3} = c_1 + \frac{4}{3}$$

$$c_1 = -\frac{2}{3}$$

Substituting c_1 found above in the general solution gives

$$\frac{1}{3}x^3 - x = \frac{1}{3}y^3 + y - \frac{2}{3}$$

Summary

The solution(s) found are the following

$$\frac{1}{3}x^3 - x = \frac{y^3}{3} + y - \frac{2}{3} \tag{1}$$

Verification of solutions

$$\frac{1}{3}x^3 - x = \frac{y^3}{3} + y - \frac{2}{3}$$

Verified OK.

6.22.5 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x,y)+N(x,y)\frac{dy}{dx}=0 \tag{A}$$

We assume there exists a function $\phi(x,y)=c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y)=0$$

Hence

$$\frac{\partial\phi}{\partial x}+\frac{\partial\phi}{\partial y}\frac{dy}{dx}=0 \tag{B}$$

Comparing (A,B) shows that

$$\begin{aligned}\frac{\partial\phi}{\partial x}&=M\\ \frac{\partial\phi}{\partial y}&=N\end{aligned}$$

But since $\frac{\partial^2\phi}{\partial x\partial y}=\frac{\partial^2\phi}{\partial y\partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y}=\frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x,y)$ but at least we know now that we can do that since the condition $\frac{\partial^2\phi}{\partial x\partial y}=\frac{\partial^2\phi}{\partial y\partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x,y)\,\mathrm{d}x+N(x,y)\,\mathrm{d}y=0 \tag{1A}$$

Therefore

$$\begin{aligned}\left(y^2+1\right)\mathrm{d}y&=\left(x^2-1\right)\mathrm{d}x\\ \left(-x^2+1\right)\mathrm{d}x+\left(y^2+1\right)\mathrm{d}y&=0 \end{aligned}\tag{2A}$$

Comparing (1A) and (2A) shows that

$$\begin{aligned}M(x,y)&=-x^2+1\\ N(x,y)&=y^2+1\end{aligned}$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Using result found above gives

$$\begin{aligned}\frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}(-x^2+1)\\ &= 0\end{aligned}$$

And

$$\begin{aligned}\frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}(y^2+1)\\ &= 0\end{aligned}$$

Since $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$, then the ODE is exact The following equations are now set up to solve for the function $\phi(x,y)$

$$\frac{\partial \phi}{\partial x} = M \tag{1}$$

$$\frac{\partial \phi}{\partial y} = N \tag{2}$$

Integrating (1) w.r.t. x gives

$$\begin{aligned}\int \frac{\partial \phi}{\partial x}\,\mathrm{d}x &= \int M\,\mathrm{d}x\\ \int \frac{\partial \phi}{\partial x}\,\mathrm{d}x &= \int -x^2+1\,\mathrm{d}x\\ \phi &= -\frac{1}{3}x^3 + x + f(y)\end{aligned} \tag{3}$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$\frac{\partial \phi}{\partial y} = 0 + f'(y) \tag{4}$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = y^2+1$. Therefore equation (4) becomes

$$y^2 + 1 = 0 + f'(y) \tag{5}$$

Solving equation (5) for $f'(y)$ gives

$$f'(y) = y^2 + 1$$

Integrating the above w.r.t y gives

$$\int f'(y)\, \mathrm{d}y = \int (y^2 + 1)\, \mathrm{d}y$$

$$f(y) = \frac{1}{3}y^3 + y + c_1$$

Where c_1 is constant of integration. Substituting result found above for $f(y)$ into equation (3) gives ϕ

$$\phi = -\frac{1}{3}x^3 + x + \frac{1}{3}y^3 + y + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = -\frac{1}{3}x^3 + x + \frac{1}{3}y^3 + y$$

Initial conditions are used to solve for c_1. Substituting $x = -1$ and $y = 1$ in the above solution gives an equation to solve for the constant of integration.

$$\frac{2}{3} = c_1$$

$$c_1 = \frac{2}{3}$$

Substituting c_1 found above in the general solution gives

$$-\frac{1}{3}x^3 + x + \frac{1}{3}y^3 + y = \frac{2}{3}$$

Summary

The solution(s) found are the following

$$-\frac{x^3}{3} + x + \frac{y^3}{3} + y = \frac{2}{3} \tag{1}$$

Verification of solutions

$$-\frac{x^3}{3} + x + \frac{y^3}{3} + y = \frac{2}{3}$$

Verified OK.

6.22.6 Maple step by step solution

Let's solve

$\left[y' - \frac{x^2-1}{1+y^2} = 0, y(-1) = 1\right]$

- Highest derivative means the order of the ODE is 1

  y'

- Separate variables

  $y'(1+y^2) = x^2 - 1$

- Integrate both sides with respect to x

  $\int y'(1+y^2)\,dx = \int (x^2-1)\,dx + c_1$

- Evaluate integral

  $\frac{y^3}{3} + y = \frac{1}{3}x^3 - x + c_1$

- Solve for y

  $y = \frac{\left(4x^3+12c_1-12x+4\sqrt{x^6+6c_1x^3-6x^4+9c_1^2-18c_1x+9x^2+4}\right)^{\frac{1}{3}}}{2} - \frac{2}{\left(4x^3+12c_1-12x+4\sqrt{x^6+6c_1x^3-6x^4+9c_1^2-18c_1x+9x^2+4}\right)^{\frac{1}{3}}}$

- Use initial condition $y(-1) = 1$

  $1 = \frac{\left(8+12c_1+4\sqrt{9c_1^2+12c_1+8}\right)^{\frac{1}{3}}}{2} - \frac{2}{\left(8+12c_1+4\sqrt{9c_1^2+12c_1+8}\right)^{\frac{1}{3}}}$

- Solve for c_1

  $c_1 = \frac{2}{3}$

- Substitute $c_1 = \frac{2}{3}$ into general solution and simplify

  $y = \frac{\left(4x^3+8-12x+4\sqrt{x^6-6x^4+4x^3+9x^2-12x+8}\right)^{\frac{2}{3}}-4}{2\left(4x^3+8-12x+4\sqrt{x^6-6x^4+4x^3+9x^2-12x+8}\right)^{\frac{1}{3}}}$

- Solution to the IVP

  $y = \frac{\left(4x^3+8-12x+4\sqrt{x^6-6x^4+4x^3+9x^2-12x+8}\right)^{\frac{2}{3}}-4}{2\left(4x^3+8-12x+4\sqrt{x^6-6x^4+4x^3+9x^2-12x+8}\right)^{\frac{1}{3}}}$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
<- separable successful`
```

✓ Solution by Maple

Time used: 0.11 (sec). Leaf size: 87

```
dsolve([diff(y(x),x) = (x^2-1)/(1+y(x)^2),y(-1) = 1],y(x), singsol=all)
```

$$y(x) = \frac{\left(8+4x^3-12x+4\sqrt{x^6-6x^4+4x^3+9x^2-12x+8}\right)^{\frac{2}{3}}-4}{2\left(8+4x^3-12x+4\sqrt{x^6-6x^4+4x^3+9x^2-12x+8}\right)^{\frac{1}{3}}}$$

✓ Solution by Mathematica

Time used: 2.95 (sec). Leaf size: 97

```
DSolve[{y'[x]== (x^2-1)/(1+y[x]^2),y[-1]==1},y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \frac{\sqrt[3]{2}\left(x^3+\sqrt{x^6-6x^4+4x^3+9x^2-12x+8}-3x+2\right)^{2/3}-2}{2^{2/3}\sqrt[3]{x^3+\sqrt{x^6-6x^4+4x^3+9x^2-12x+8}-3x+2}}$$

6.23 problem 23

Internal problem ID [590]
Internal file name [`OUTPUT/590_Sunday_June_05_2022_01_45_23_AM_42038206/index.tex`]

Book: Elementary differential equations and boundary value problems, 10th ed., Boyce and DiPrima
Section: Miscellaneous problems, end of chapter 2. Page 133
Problem number: 23.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"linear", "exactWithIntegrationFactor", "first_order_ode_lie_symmetry_lookup"**

Maple gives the following as the ode type

`[_linear]`

$$\boxed{(t+1)\, y+t y'={\mathrm e}^{2 t}}$$

6.23.1 Solving as linear ode

Entering Linear first order ODE solver. In canonical form a linear first order is

$$y'+p(t)y=q(t)$$

Where here

$$p(t)=-\frac{-t-1}{t}$$
$$q(t)=\frac{{\mathrm e}^{2 t}}{t}$$

Hence the ode is

$$y'-\frac{(-t-1)\, y}{t}=\frac{{\mathrm e}^{2 t}}{t}$$

The integrating factor μ is

$$\begin{aligned}
\mu &= \mathrm{e}^{\int -\frac{-t-1}{t}dt} \\
&= \mathrm{e}^{t+\ln(t)}
\end{aligned}$$

Which simplifies to

$$\mu = t\,\mathrm{e}^{t}$$

The ode becomes

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}(\mu y) &= (\mu)\left(\frac{\mathrm{e}^{2t}}{t}\right) \\
\frac{\mathrm{d}}{\mathrm{d}t}\left(t\,\mathrm{e}^{t} y\right) &= \left(t\,\mathrm{e}^{t}\right)\left(\frac{\mathrm{e}^{2t}}{t}\right) \\
\mathrm{d}\left(t\,\mathrm{e}^{t} y\right) &= \mathrm{e}^{3t}\,\mathrm{d}t
\end{aligned}$$

Integrating gives

$$\begin{aligned}
t\,\mathrm{e}^{t} y &= \int \mathrm{e}^{3t}\,\mathrm{d}t \\
t\,\mathrm{e}^{t} y &= \frac{\mathrm{e}^{3t}}{3} + c_1
\end{aligned}$$

Dividing both sides by the integrating factor $\mu = t\,\mathrm{e}^{t}$ results in

$$y = \frac{\mathrm{e}^{-t}\mathrm{e}^{3t}}{3t} + \frac{c_1\mathrm{e}^{-t}}{t}$$

which simplifies to

$$y = \frac{3c_1\mathrm{e}^{-t} + \mathrm{e}^{2t}}{3t}$$

Summary

The solution(s) found are the following

$$y = \frac{3c_1\mathrm{e}^{-t} + \mathrm{e}^{2t}}{3t} \tag{1}$$

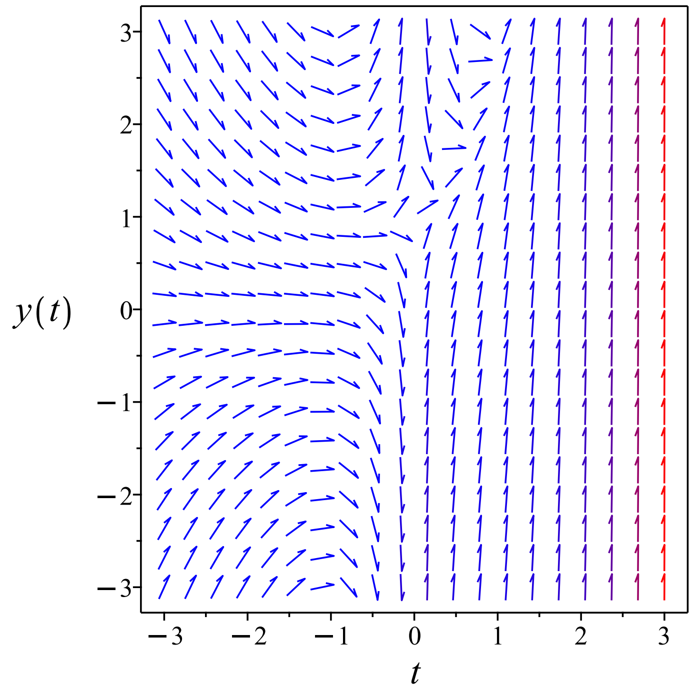


Figure 328: Slope field plot

Verification of solutions

$$y = \frac{3c_1 \mathrm{e}^{-t} + \mathrm{e}^{2t}}{3t}$$

Verified OK.

6.23.2 Solving as first order ode lie symmetry lookup ode

Writing the ode as

$$y' = \frac{-ty + \mathrm{e}^{2t} - y}{t}$$
$$y' = \omega(t, y)$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_t + \omega(\eta_y - \xi_t) - \omega^2 \xi_y - \omega_t \xi - \omega_y \eta = 0 \tag{A}$$

The type of this ode is known. It is of type `linear`. Therefore we do not need to solve the PDE (A), and can just use the lookup table shown below to find ξ, η

Table 304: Lie symmetry infinitesimal lookup table for known first order ODE's

ODE class	Form	ξ	η
linear ode	$y' = f(x)y(x) + g(x)$	0	$e^{\int f dx}$
separable ode	$y' = f(x)\, g(y)$	$\frac{1}{f}$	0
quadrature ode	$y' = f(x)$	0	1
quadrature ode	$y' = g(y)$	1	0
homogeneous ODEs of Class A	$y' = f\left(\frac{y}{x}\right)$	x	y
homogeneous ODEs of Class C	$y' = (a + bx + cy)^{\frac{n}{m}}$	1	$-\frac{b}{c}$
homogeneous class D	$y' = \frac{y}{x} + g(x)\, F\left(\frac{y}{x}\right)$	x^2	xy
First order special form ID 1	$y' = g(x)\, e^{h(x)+by} + f(x)$	$\frac{e^{-\int bf(x)dx - h(x)}}{g(x)}$	$\frac{f(x)e^{-\int bf(x)dx - h(x)}}{g(x)}$
polynomial type ode	$y' = \frac{a_1x+b_1y+c_1}{a_2x+b_2y+c_2}$	$\frac{a_1b_2x - a_2b_1x - b_1c_2 + b_2c_1}{a_1b_2 - a_2b_1}$	$\frac{a_1b_2y - a_2b_1y - a_1c_2 - a_2c_1}{a_1b_2 - a_2b_1}$
Bernoulli ode	$y' = f(x)\, y + g(x)\, y^n$	0	$e^{-\int (n-1)f(x)dx} y^n$
Reduced Riccati	$y' = f_1(x)\, y + f_2(x)\, y^2$	0	$e^{-\int f_1 dx}$

The above table shows that

$$\begin{aligned} \xi(t, y) &= 0 \\ \eta(t, y) &= \mathrm{e}^{-t - \ln(t)} \end{aligned} \tag{A1}$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(t, y) \to (R, S)$ where (R, S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$\frac{dt}{\xi} = \frac{dy}{\eta} = dS \tag{1}$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial t} + \eta \frac{\partial}{\partial y}\right) S(t, y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the

canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$R = t$$

S is found from

$$\begin{aligned}
S &= \int \frac{1}{\eta} dy \\
&= \int \frac{1}{\mathrm{e}^{-t-\ln(t)}} dy
\end{aligned}$$

Which results in

$$S = t\,\mathrm{e}^{t} y$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_t + \omega(t, y) S_y}{R_t + \omega(t, y) R_y} \tag{2}$$

Where in the above R_t, R_y, S_t, S_y are all partial derivatives and $\omega(t, y)$ is the right hand side of the original ode given by

$$\omega(t, y) = \frac{-ty + \mathrm{e}^{2t} - y}{t}$$

Evaluating all the partial derivatives gives

$$\begin{aligned}
R_t &= 1 \\
R_y &= 0 \\
S_t &= y\,\mathrm{e}^{t}(t + 1) \\
S_y &= t\,\mathrm{e}^{t}
\end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = \mathrm{e}^{3t} \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for t, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = \mathrm{e}^{3R}$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by

integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$S(R) = \frac{e^{3R}}{3} + c_1 \tag{4}$$

To complete the solution, we just need to transform (4) back to t, y coordinates. This results in

$$t\,e^{t}y = \frac{e^{3t}}{3} + c_1$$

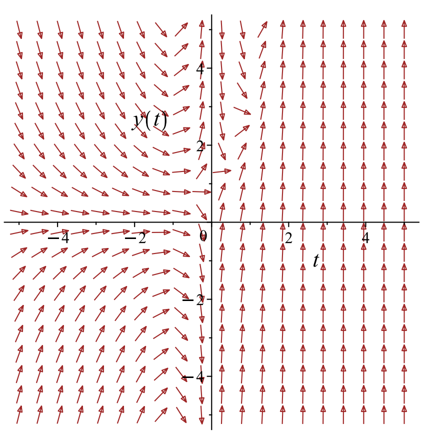
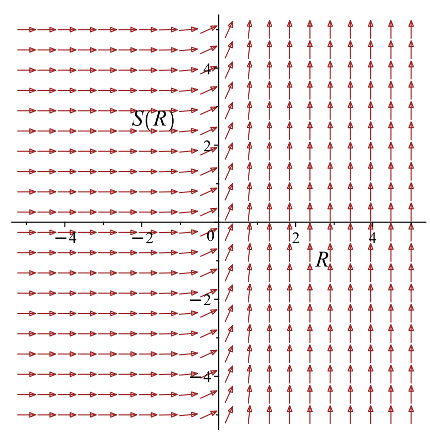
Which simplifies to

$$t\,e^{t}y = \frac{e^{3t}}{3} + c_1$$

Which gives

$$y = \frac{(e^{3t} + 3c_1)\,e^{-t}}{3t}$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in t, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dt} = \frac{-ty+e^{2t}-y}{t}$ 	$R = t$ $S = t\,e^{t}y$	$\frac{dS}{dR} = e^{3R}$ 

Summary

The solution(s) found are the following

$$y = \frac{(e^{3t} + 3c_1)\,e^{-t}}{3t} \tag{1}$$

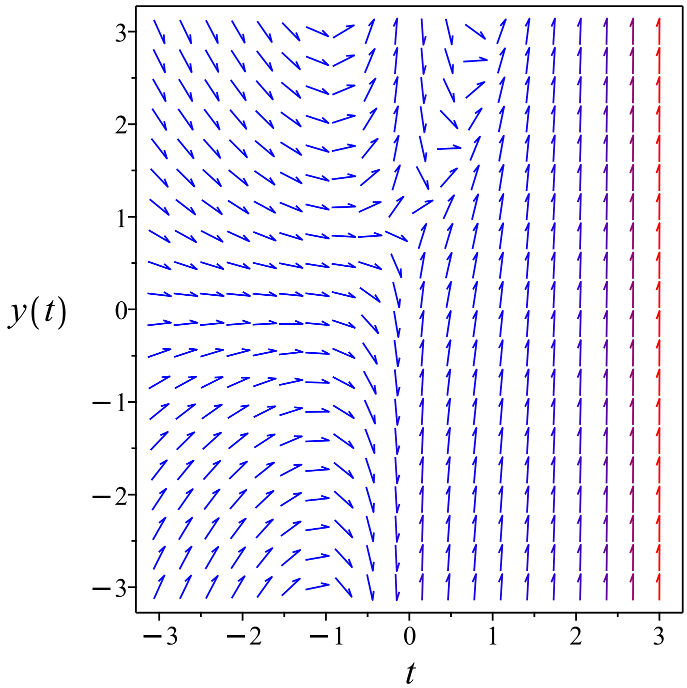


Figure 329: Slope field plot

Verification of solutions

$$y = \frac{\left(\mathrm{e}^{3t} + 3c_1\right)\mathrm{e}^{-t}}{3t}$$

Verified OK.

6.23.3 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x,y) + N(x,y)\frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x,y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y) = 0$$

Hence

$$\frac{\partial\phi}{\partial x} + \frac{\partial\phi}{\partial y}\frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$\frac{\partial \phi}{\partial x} = M$$
$$\frac{\partial \phi}{\partial y} = N$$

But since $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x, y)$ but at least we know now that we can do that since the condition $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(t, y)\, \mathrm{d}t + N(t, y)\, \mathrm{d}y = 0 \tag{1A}$$

Therefore

$$(t)\, \mathrm{d}y = \left(-(t+1)\, y + \mathrm{e}^{2t}\right) \mathrm{d}t$$
$$\left(-\mathrm{e}^{2t} + (t+1)\, y\right) \mathrm{d}t + (t)\, \mathrm{d}y = 0 \tag{2A}$$

Comparing (1A) and (2A) shows that

$$M(t, y) = -\mathrm{e}^{2t} + (t+1)\, y$$
$$N(t, y) = t$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial t}$$

Using result found above gives

$$\frac{\partial M}{\partial y} = \frac{\partial}{\partial y}\left(-\mathrm{e}^{2t} + (t+1)\, y\right)$$
$$= t + 1$$

And

$$\frac{\partial N}{\partial t} = \frac{\partial}{\partial t}(t)$$
$$= 1$$

Since $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial t}$, then the ODE is <u>not exact</u>. Since the ODE is not exact, we will try to find an integrating factor to make it exact. Let

$$
\begin{aligned}
A &= \frac{1}{N}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial t}\right) \\
&= \frac{1}{t}((t+1) - (1)) \\
&= 1
\end{aligned}
$$

Since A does not depend on y, then it can be used to find an integrating factor. The integrating factor μ is

$$
\begin{aligned}
\mu &= e^{\int A \, \mathrm{d}t} \\
&= e^{\int 1 \, \mathrm{d}t}
\end{aligned}
$$

The result of integrating gives

$$
\begin{aligned}
\mu &= e^{t} \\
&= \mathrm{e}^{t}
\end{aligned}
$$

M and N are multiplied by this integrating factor, giving new M and new N which are called $\overline{M}$ and $\overline{N}$ for now so not to confuse them with the original M and N.

$$
\begin{aligned}
\overline{M} &= \mu M \\
&= \mathrm{e}^{t}\left(-\mathrm{e}^{2t} + (t+1)\, y\right) \\
&= \left(-\mathrm{e}^{2t} + (t+1)\, y\right) \mathrm{e}^{t}
\end{aligned}
$$

And

$$
\begin{aligned}
\overline{N} &= \mu N \\
&= \mathrm{e}^{t}(t) \\
&= t\,\mathrm{e}^{t}
\end{aligned}
$$

Now a modified ODE is ontained from the original ODE, which is exact and can be solved. The modified ODE is

$$
\begin{aligned}
\overline{M} + \overline{N}\frac{\mathrm{d}y}{\mathrm{d}t} &= 0 \\
\left(\left(-\mathrm{e}^{2t} + (t+1)\, y\right) \mathrm{e}^{t}\right) + \left(t\,\mathrm{e}^{t}\right) \frac{\mathrm{d}y}{\mathrm{d}t} &= 0
\end{aligned}
$$

The following equations are now set up to solve for the function $\phi(t, y)$

$$
\frac{\partial \phi}{\partial t} = \overline{M} \tag{1}
$$

$$
\frac{\partial \phi}{\partial y} = \overline{N} \tag{2}
$$

Integrating (1) w.r.t. t gives

$$\int \frac{\partial \phi}{\partial t} \mathrm{d}t = \int \overline{M} \mathrm{d}t$$

$$\int \frac{\partial \phi}{\partial t} \mathrm{d}t = \int \left(-\mathrm{e}^{2t} + (t+1) y\right) \mathrm{e}^{t} \mathrm{d}t$$

$$\phi = t\,\mathrm{e}^{t} y - \frac{\mathrm{e}^{3t}}{3} + f(y) \tag{3}$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both t and y. Taking derivative of equation (3) w.r.t y gives

$$\frac{\partial \phi}{\partial y} = t\,\mathrm{e}^{t} + f'(y) \tag{4}$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = t\,\mathrm{e}^{t}$. Therefore equation (4) becomes

$$t\,\mathrm{e}^{t} = t\,\mathrm{e}^{t} + f'(y) \tag{5}$$

Solving equation (5) for $f'(y)$ gives

$$f'(y) = 0$$

Therefore

$$f(y) = c_1$$

Where c_1 is constant of integration. Substituting this result for $f(y)$ into equation (3) gives ϕ

$$\phi = t\,\mathrm{e}^{t} y - \frac{\mathrm{e}^{3t}}{3} + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = t\,\mathrm{e}^{t} y - \frac{\mathrm{e}^{3t}}{3}$$

The solution becomes

$$y = \frac{\left(\mathrm{e}^{3t} + 3c_1\right) \mathrm{e}^{-t}}{3t}$$

Summary

The solution(s) found are the following

$$y = \frac{(e^{3t} + 3c_1) e^{-t}}{3t} \tag{1}$$

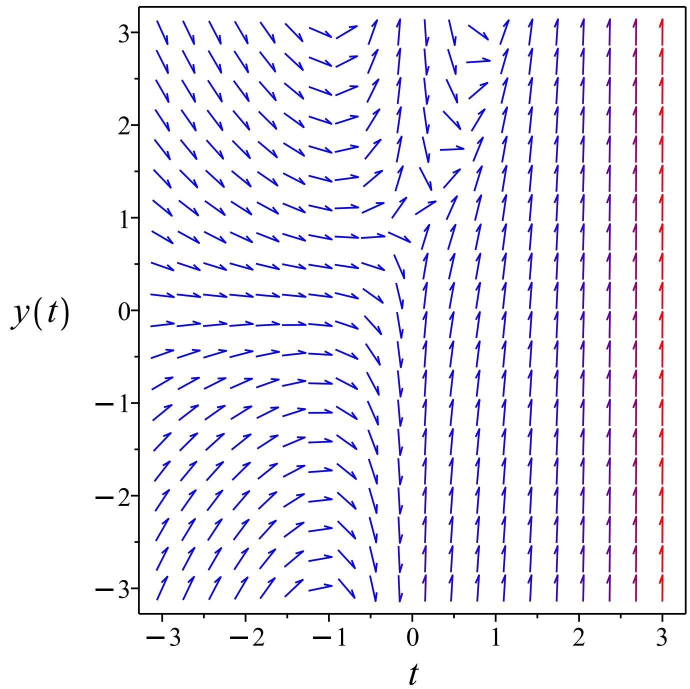


Figure 330: Slope field plot

Verification of solutions

$$y = \frac{(e^{3t} + 3c_1) e^{-t}}{3t}$$

Verified OK.

6.23.4 Maple step by step solution

Let's solve

$(t+1)\,y + ty' = e^{2t}$

- Highest derivative means the order of the ODE is 1

y'

- Isolate the derivative

$y' = -\frac{(t+1)y}{t} + \frac{e^{2t}}{t}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE

  $y' + \frac{(t+1)y}{t} = \frac{e^{2t}}{t}$

- The ODE is linear; multiply by an integrating factor $\mu(t)$

  $\mu(t)\left(y' + \frac{(t+1)y}{t}\right) = \frac{\mu(t)e^{2t}}{t}$

- Assume the lhs of the ODE is the total derivative $\frac{d}{dt}(\mu(t)\,y)$

  $\mu(t)\left(y' + \frac{(t+1)y}{t}\right) = \mu'(t)\,y + \mu(t)\,y'$

- Isolate $\mu'(t)$

  $\mu'(t) = \frac{\mu(t)(t+1)}{t}$

- Solve to find the integrating factor

  $\mu(t) = t\,e^{t}$

- Integrate both sides with respect to t

  $\int \left(\frac{d}{dt}(\mu(t)\,y)\right) dt = \int \frac{\mu(t)e^{2t}}{t} dt + c_1$

- Evaluate the integral on the lhs

  $\mu(t)\,y = \int \frac{\mu(t)e^{2t}}{t} dt + c_1$

- Solve for y

  $y = \frac{\int \frac{\mu(t)e^{2t}}{t} dt + c_1}{\mu(t)}$

- Substitute $\mu(t) = te^{t}$

  $y = \frac{\int e^{2t}e^{t} dt + c_1}{t\,e^{t}}$

- Evaluate the integrals on the rhs

  $y = \frac{\frac{e^{3t}}{3} + c_1}{t\,e^{t}}$

- Simplify

  $y = \frac{3c_1 e^{-t} + e^{2t}}{3t}$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
<- 1st order linear successful`
```

✓ Solution by Maple
Time used: 0.0 (sec). Leaf size: 21

```
dsolve((1+t)*y(t)+t*diff(y(t),t) = exp(2*t),y(t), singsol=all)
```

$$y(t) = \frac{e^{2t} + 3\,e^{-t}c_1}{3t}$$

✓ Solution by Mathematica
Time used: 0.053 (sec). Leaf size: 27

```
DSolve[(1+t)*y[t]+t*y'[t] == Exp[2*t],y[t],t,IncludeSingularSolutions -> True]
```

$$y(t) \to \frac{e^{2t} + 3c_1e^{-t}}{3t}$$

6.24 problem 24

Internal problem ID [591]
Internal file name [`OUTPUT/591_Sunday_June_05_2022_01_45_24_AM_63570909/index.tex`]

Book: Elementary differential equations and boundary value problems, 10th ed., Boyce and DiPrima
Section: Miscellaneous problems, end of chapter 2. Page 133
Problem number: 24.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"exact", "separable", "first_order_ode_lie_symmetry_lookup"**

Maple gives the following as the ode type

```
[_separable]
```

$$\boxed{2\cos(x)\sin(x)\sin(y)+\cos(y)\sin(x)^2 y'=0}$$

6.24.1 Solving as separable ode

In canonical form the ODE is

$$\begin{aligned} y' &= F(x,y) \\ &= f(x)g(y) \\ &= -\frac{2\cos(x)\tan(y)}{\sin(x)} \end{aligned}$$

Where $f(x)=-\frac{2\cos(x)}{\sin(x)}$ and $g(y)=\tan(y)$. Integrating both sides gives

$$\begin{aligned} \frac{1}{\tan(y)}\,dy &= -\frac{2\cos(x)}{\sin(x)}\,dx \\ \int \frac{1}{\tan(y)}\,dy &= \int -\frac{2\cos(x)}{\sin(x)}\,dx \\ \ln(\sin(y)) &= -2\ln(\sin(x))+c_1 \end{aligned}$$

Raising both side to exponential gives

$$\sin(y) = e^{-2\ln(\sin(x))+c_1}$$

Which simplifies to

$$\sin(y) = \frac{c_2}{\sin(x)^2}$$

Summary
The solution(s) found are the following

$$y = \arcsin\left(\frac{c_2 e^{c_1}}{\sin(x)^2}\right) \tag{1}$$

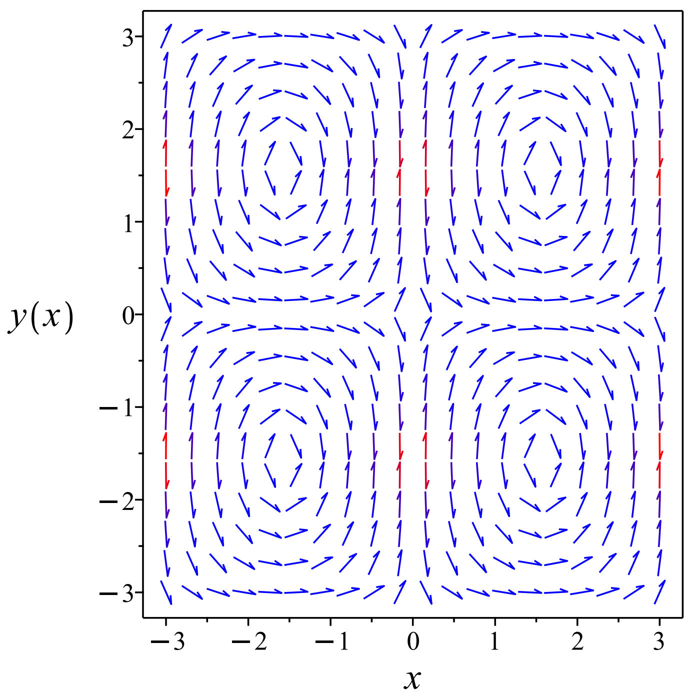


Figure 331: Slope field plot

Verification of solutions

$$y = \arcsin\left(\frac{c_2 e^{c_1}}{\sin(x)^2}\right)$$

Verified OK.

6.24.2 Solving as first order ode lie symmetry lookup ode

Writing the ode as

$$
\begin{aligned}
y' &= -\frac{2\cos(x)\sin(y)}{\sin(x)\cos(y)} \\
y' &= \omega(x,y)
\end{aligned}
$$

The condition of Lie symmetry is the linearized PDE given by

$$
\eta_x + \omega(\eta_y - \xi_x) - \omega^2 \xi_y - \omega_x \xi - \omega_y \eta = 0 \tag{A}
$$

The type of this ode is known. It is of type `separable`. Therefore we do not need to solve the PDE (A), and can just use the lookup table shown below to find ξ, η

Table 307: Lie symmetry infinitesimal lookup table for known first order ODE's

ODE class	Form	ξ	η
linear ode	$y' = f(x)y(x) + g(x)$	0	$e^{\int f dx}$
separable ode	$y' = f(x)\, g(y)$	$\frac{1}{f}$	0
quadrature ode	$y' = f(x)$	0	1
quadrature ode	$y' = g(y)$	1	0
homogeneous ODEs of Class A	$y' = f\left(\frac{y}{x}\right)$	x	y
homogeneous ODEs of Class C	$y' = (a + bx + cy)^{\frac{n}{m}}$	1	$-\frac{b}{c}$
homogeneous class D	$y' = \frac{y}{x} + g(x)\, F\left(\frac{y}{x}\right)$	x^2	xy
First order special form ID 1	$y' = g(x)\, e^{h(x)+by} + f(x)$	$\frac{e^{-\int bf(x)dx - h(x)}}{g(x)}$	$\frac{f(x)e^{-\int bf(x)dx - h(x)}}{g(x)}$
polynomial type ode	$y' = \frac{a_1x+b_1y+c_1}{a_2x+b_2y+c_2}$	$\frac{a_1b_2x-a_2b_1x-b_1c_2+b_2c_1}{a_1b_2-a_2b_1}$	$\frac{a_1b_2y-a_2b_1y-a_1c_2-a_2c_1}{a_1b_2-a_2b_1}$
Bernoulli ode	$y' = f(x)\, y + g(x)\, y^n$	0	$e^{-\int(n-1)f(x)dx} y^n$
Reduced Riccati	$y' = f_1(x)\, y + f_2(x)\, y^2$	0	$e^{-\int f_1 dx}$

The above table shows that

$$
\begin{aligned}
\xi(x,y) &= -\frac{\sin(x)}{2\cos(x)} \\
\eta(x,y) &= 0 \qquad \text{(A1)}
\end{aligned}
$$

The next step is to determine the canonical coordinates R,S. The canonical coordinates map $(x,y) \to (R,S)$ where (R,S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$
\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}
$$

The above comes from the requirements that $\left(\xi\frac{\partial}{\partial x} + \eta\frac{\partial}{\partial y}\right) S(x,y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the canonical coordinates, where $S(R)$. Since $\eta = 0$ then in this special case

$$
R = y
$$

S is found from

$$
\begin{aligned}
S &= \int \frac{1}{\xi} dx \\
&= \int \frac{1}{-\frac{\sin(x)}{2\cos(x)}} dx
\end{aligned}
$$

Which results in

$$
S = -2\ln(\sin(x))
$$

Now that R,S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$
\frac{dS}{dR} = \frac{S_x + \omega(x,y)S_y}{R_x + \omega(x,y)R_y} \tag{2}
$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x,y)$ is the right hand side of the original ode given by

$$
\omega(x,y) = -\frac{2\cos(x)\sin(y)}{\sin(x)\cos(y)}
$$

Evaluating all the partial derivatives gives

$$
\begin{aligned}
R_x &= 0 \\
R_y &= 1 \\
S_x &= -2 \cot (x) \\
S_y &= 0
\end{aligned}
$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = \cot (y) \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = \cot (R)$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$S(R) = \ln (\sin (R)) + c_1 \tag{4}$$

To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$-2 \ln (\sin (x)) = \ln (\sin (y)) + c_1$$

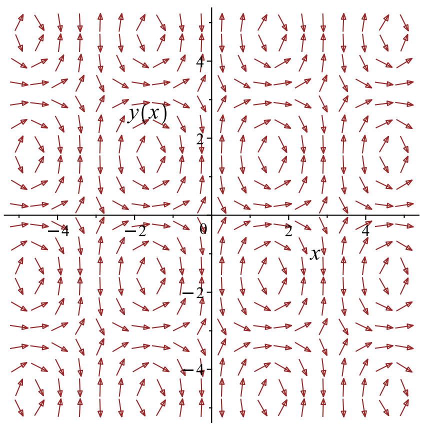
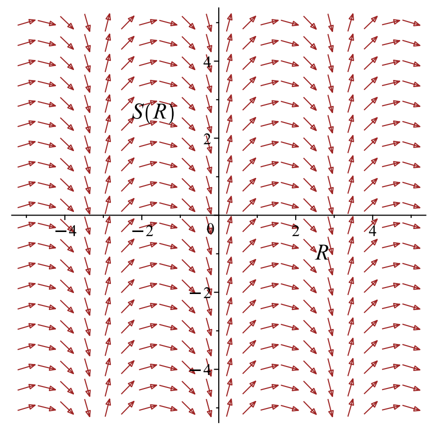
Which simplifies to

$$-2 \ln (\sin (x)) = \ln (\sin (y)) + c_1$$

Which gives

$$y = \arcsin \left(\frac{\mathrm{e}^{-c_1}}{\sin (x)^2}\right)$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

<table>
<tr><th>Original ode in x, y coordinates</th><th>Canonical coordinates transformation</th><th>ODE in canonical coordinates (R, S)</th></tr>
<tr><td>$\frac{dy}{dx} = -\frac{2\cos(x)\sin(y)}{\sin(x)\cos(y)}$
</td><td>$R = y$
$S = -2\ln(\sin(x))$</td><td>$\frac{dS}{dR} = \cot(R)$
</td></tr>
</table>

Summary

The solution(s) found are the following

$$y = \arcsin\left(\frac{e^{-c_1}}{\sin(x)^2}\right) \tag{1}$$

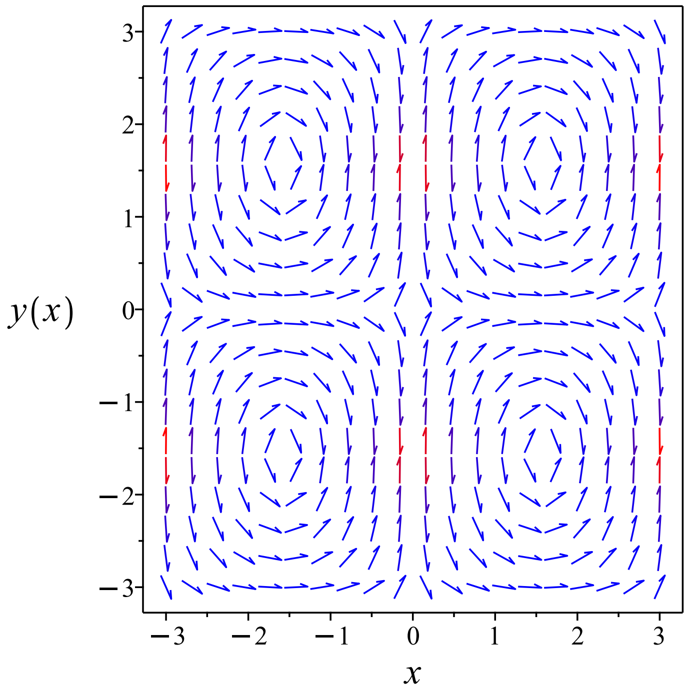


Figure 332: Slope field plot

Verification of solutions

$$y = \arcsin\left(\frac{e^{-c_1}}{\sin(x)^2}\right)$$

Verified OK.

6.24.3 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x,y) + N(x,y)\frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x,y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y) = 0$$

Hence

$$\frac{\partial \phi}{\partial x} + \frac{\partial \phi}{\partial y}\frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$\frac{\partial\phi}{\partial x} = M$$
$$\frac{\partial\phi}{\partial y} = N$$

But since $\frac{\partial^2\phi}{\partial x\partial y} = \frac{\partial^2\phi}{\partial y\partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x, y)$ but at least we know now that we can do that since the condition $\frac{\partial^2\phi}{\partial x\partial y} = \frac{\partial^2\phi}{\partial y\partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x, y)\,\mathrm{d}x + N(x, y)\,\mathrm{d}y = 0 \tag{1A}$$

Therefore

$$\left(-\frac{\cos(y)}{2\sin(y)}\right)\mathrm{d}y = \left(\frac{\cos(x)}{\sin(x)}\right)\mathrm{d}x$$
$$\left(-\frac{\cos(x)}{\sin(x)}\right)\mathrm{d}x + \left(-\frac{\cos(y)}{2\sin(y)}\right)\mathrm{d}y = 0 \tag{2A}$$

Comparing (1A) and (2A) shows that

$$M(x, y) = -\frac{\cos(x)}{\sin(x)}$$
$$N(x, y) = -\frac{\cos(y)}{2\sin(y)}$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Using result found above gives

$$\frac{\partial M}{\partial y} = \frac{\partial}{\partial y}\left(-\frac{\cos(x)}{\sin(x)}\right)$$
$$= 0$$

And

$$\begin{aligned}\frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}\left(-\frac{\cos(y)}{2\sin(y)}\right)\\&= 0\end{aligned}$$

Since $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$, then the ODE is exact The following equations are now set up to solve for the function $\phi(x, y)$

$$\frac{\partial \phi}{\partial x} = M \tag{1}$$

$$\frac{\partial \phi}{\partial y} = N \tag{2}$$

Integrating (1) w.r.t. x gives

$$\begin{aligned}\int \frac{\partial \phi}{\partial x}\,\mathrm{d}x &= \int M\,\mathrm{d}x\\\int \frac{\partial \phi}{\partial x}\,\mathrm{d}x &= \int -\frac{\cos(x)}{\sin(x)}\,\mathrm{d}x\\\phi &= -\ln(\sin(x)) + f(y)\end{aligned} \tag{3}$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$\frac{\partial \phi}{\partial y} = 0 + f'(y) \tag{4}$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = -\frac{\cos(y)}{2\sin(y)}$. Therefore equation (4) becomes

$$-\frac{\cos(y)}{2\sin(y)} = 0 + f'(y) \tag{5}$$

Solving equation (5) for $f'(y)$ gives

$$\begin{aligned}f'(y) &= -\frac{\cos(y)}{2\sin(y)}\\&= -\frac{\cot(y)}{2}\end{aligned}$$

Integrating the above w.r.t y results in

$$\int f'(y)\,\mathrm{d}y = \int \left(-\frac{\cot(y)}{2}\right) \mathrm{d}y$$

$$f(y) = -\frac{\ln(\sin(y))}{2} + c_1$$

Where c_1 is constant of integration. Substituting result found above for $f(y)$ into equation (3) gives ϕ

$$\phi = -\ln(\sin(x)) - \frac{\ln(\sin(y))}{2} + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = -\ln(\sin(x)) - \frac{\ln(\sin(y))}{2}$$

Summary

The solution(s) found are the following

$$-\ln(\sin(x)) - \frac{\ln(\sin(y))}{2} = c_1 \tag{1}$$

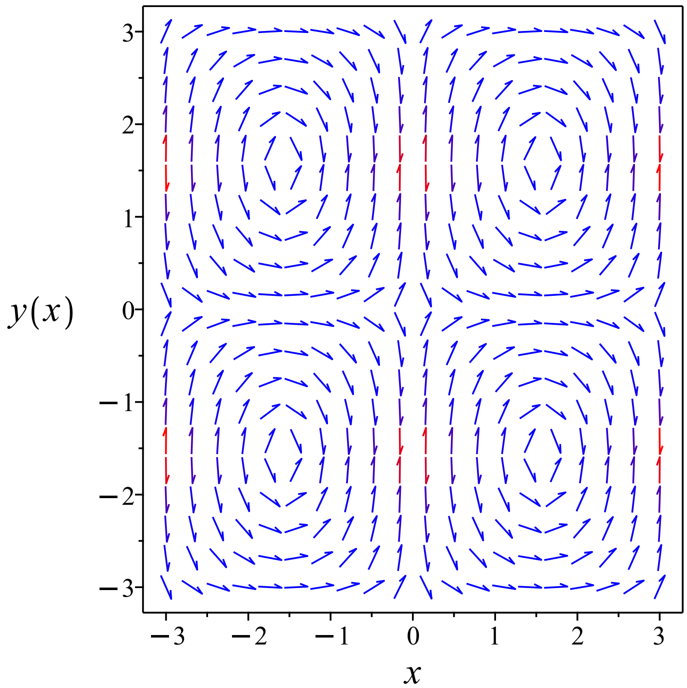


Figure 333: Slope field plot

Verification of solutions

$$-\ln(\sin(x)) - \frac{\ln(\sin(y))}{2} = c_1$$

Verified OK.

6.24.4 Maple step by step solution

Let's solve

$2\cos(x)\sin(x)\sin(y) + \cos(y)\sin(x)^2 y' = 0$

- Highest derivative means the order of the ODE is 1

  y'

- Integrate both sides with respect to x

  $\int \left(2\cos(x)\sin(x)\sin(y) + \cos(y)\sin(x)^2 y'\right) dx = \int 0 dx + c_1$

- Evaluate integral

  $\sin(x)^2 \sin(y) = c_1$

- Solve for y

$$y = \arcsin\left(\frac{c_1}{\sin(x)^2}\right)$$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
<- separable successful`
```

✓ Solution by Maple

Time used: 0.328 (sec). Leaf size: 18

```
dsolve(2*cos(x)*sin(x)*sin(y(x))+cos(y(x))*sin(x)^2*diff(y(x),x) = 0,y(x), singsol=all)
```

$$y(x) = -\arcsin\left(\frac{2c_1}{-1+\cos(2x)}\right)$$

✓ Solution by Mathematica

Time used: 5.176 (sec). Leaf size: 21

```
DSolve[2*Cos[x]*Sin[x]*Sin[y[x]]+Cos[y[x]]*Sin[x]^2*y'[x] == 0,y[x],x,IncludeSingularSolutio
```

$$y(x) \to \arcsin\left(\frac{1}{2}c_1\csc^2(x)\right)$$
$$y(x) \to 0$$

6.25 problem 25

Internal problem ID [592]
Internal file name [`OUTPUT/592_Sunday_June_05_2022_01_45_26_AM_93723432/index.tex`]

Book: Elementary differential equations and boundary value problems, 10th ed., Boyce and DiPrima
Section: Miscellaneous problems, end of chapter 2. Page 133
Problem number: 25.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"exact"**

Maple gives the following as the ode type

`[_exact, _rational]`

$$\boxed{\frac{2x}{y} - \frac{y}{x^2+y^2} + \left(-\frac{x^2}{y^2} + \frac{x}{x^2+y^2}\right) y' = 0}$$

6.25.1 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x,y) + N(x,y)\,\frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x,y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y) = 0$$

Hence

$$\frac{\partial \phi}{\partial x} + \frac{\partial \phi}{\partial y}\frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$\frac{\partial \phi}{\partial x} = M$$
$$\frac{\partial \phi}{\partial y} = N$$

But since $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x, y)$ but at least we know now that we can do that since the condition $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x, y)\,\mathrm{d}x + N(x, y)\,\mathrm{d}y = 0 \tag{1A}$$

Therefore

$$\left(-\frac{x^2}{y^2} + \frac{x}{x^2 + y^2}\right)\mathrm{d}y = \left(-\frac{2x}{y} + \frac{y}{x^2 + y^2}\right)\mathrm{d}x$$
$$\left(\frac{2x}{y} - \frac{y}{x^2 + y^2}\right)\mathrm{d}x + \left(-\frac{x^2}{y^2} + \frac{x}{x^2 + y^2}\right)\mathrm{d}y = 0 \tag{2A}$$

Comparing (1A) and (2A) shows that

$$M(x, y) = \frac{2x}{y} - \frac{y}{x^2 + y^2}$$
$$N(x, y) = -\frac{x^2}{y^2} + \frac{x}{x^2 + y^2}$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Using result found above gives

$$\frac{\partial M}{\partial y} = \frac{\partial}{\partial y}\left(\frac{2x}{y} - \frac{y}{x^2 + y^2}\right)$$
$$= -\frac{2x}{y^2} - \frac{1}{x^2 + y^2} + \frac{2y^2}{\left(x^2 + y^2\right)^2}$$

And

$$
\begin{aligned}
\frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}\left(-\frac{x^2}{y^2} + \frac{x}{x^2+y^2}\right) \\
&= -\frac{2x}{y^2} - \frac{2x^2}{\left(x^2+y^2\right)^2} + \frac{1}{x^2+y^2}
\end{aligned}
$$

Since $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$, then the ODE is exact The following equations are now set up to solve for the function $\phi(x,y)$

$$
\frac{\partial \phi}{\partial x} = M \tag{1}
$$

$$
\frac{\partial \phi}{\partial y} = N \tag{2}
$$

Integrating (1) w.r.t. x gives

$$
\begin{aligned}
\int \frac{\partial \phi}{\partial x}\,\mathrm{d}x &= \int M\,\mathrm{d}x \\
\int \frac{\partial \phi}{\partial x}\,\mathrm{d}x &= \int \frac{2x}{y} - \frac{y}{x^2+y^2}\,\mathrm{d}x
\end{aligned}
$$

$$
\phi = \frac{x^2}{y} - \arctan\left(\frac{x}{y}\right) + f(y) \tag{3}
$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$
\frac{\partial \phi}{\partial y} = -\frac{x^2}{y^2} + \frac{x}{y^2\left(\frac{x^2}{y^2}+1\right)} + f'(y) \tag{4}
$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = -\frac{x^2}{y^2} + \frac{x}{x^2+y^2}$. Therefore equation (4) becomes

$$
-\frac{x^2}{y^2} + \frac{x}{x^2+y^2} = -\frac{\left((x-1)\,y^2 + x^3\right)x}{y^2\left(x^2+y^2\right)} + f'(y) \tag{5}
$$

Solving equation (5) for $f'(y)$ gives

$$
f'(y) = 0
$$

Therefore

$$
f(y) = c_1
$$

Where c_1 is constant of integration. Substituting this result for $f(y)$ into equation (3) gives ϕ

$$\phi = \frac{x^2}{y} - \arctan\left(\frac{x}{y}\right) + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = \frac{x^2}{y} - \arctan\left(\frac{x}{y}\right)$$

Summary

The solution(s) found are the following

$$\frac{x^2}{y} - \arctan\left(\frac{x}{y}\right) = c_1 \tag{1}$$

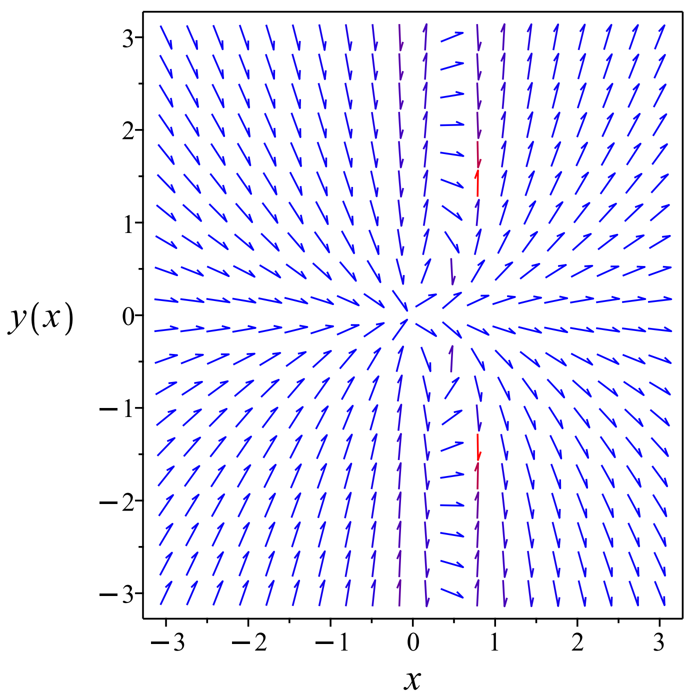


Figure 334: Slope field plot

Verification of solutions

$$\frac{x^2}{y} - \arctan\left(\frac{x}{y}\right) = c_1$$

Verified OK.

6.25.2 Maple step by step solution

Let's solve

$\frac{2x}{y}-\frac{y}{x^2+y^2}+\left(-\frac{x^2}{y^2}+\frac{x}{x^2+y^2}\right)y'=0$

- Highest derivative means the order of the ODE is 1

  y'

- □ Check if ODE is exact
  - ODE is exact if the lhs is the total derivative of a C^2 function

    $F'(x,y)=0$
  - Compute derivative of lhs

    $F'(x,y)+\left(\frac{\partial}{\partial y}F(x,y)\right)y'=0$
  - Evaluate derivatives

    $-\frac{2x}{y^2}-\frac{1}{x^2+y^2}+\frac{2y^2}{(x^2+y^2)^2}=-\frac{2x}{y^2}-\frac{2x^2}{(x^2+y^2)^2}+\frac{1}{x^2+y^2}$
  - Simplify

    $-\frac{2x}{y^2}-\frac{1}{x^2+y^2}+\frac{2y^2}{(x^2+y^2)^2}=-\frac{2x}{y^2}-\frac{2x^2}{(x^2+y^2)^2}+\frac{1}{x^2+y^2}$
  - Condition met, ODE is exact

- Exact ODE implies solution will be of this form

  $\left[F(x,y)=c_1, M(x,y)=F'(x,y), N(x,y)=\frac{\partial}{\partial y}F(x,y)\right]$

- Solve for $F(x,y)$ by integrating $M(x,y)$ with respect to x

  $F(x,y)=\int\left(\frac{2x}{y}-\frac{y}{x^2+y^2}\right)dx+f_1(y)$

- Evaluate integral

  $F(x,y)=\frac{x^2}{y}-\arctan\left(\frac{x}{y}\right)+f_1(y)$

- Take derivative of $F(x,y)$ with respect to y

  $N(x,y)=\frac{\partial}{\partial y}F(x,y)$

- Compute derivative

  $-\frac{x^2}{y^2}+\frac{x}{x^2+y^2}=-\frac{x^2}{y^2}+\frac{x}{y^2\left(\frac{x^2}{y^2}+1\right)}+\frac{d}{dy}f_1(y)$

- Isolate for $\frac{d}{dy}f_1(y)$

$$\frac{d}{dy}f_1(y) = \frac{x}{x^2+y^2} - \frac{x}{y^2\left(\frac{x^2}{y^2}+1\right)}$$

- Solve for $f_1(y)$

$$f_1(y) = 0$$

- Substitute $f_1(y)$ into equation for $F(x,y)$

$$F(x,y) = \frac{x^2}{y} - \arctan\left(\frac{x}{y}\right)$$

- Substitute $F(x,y)$ into the solution of the ODE

$$\frac{x^2}{y} - \arctan\left(\frac{x}{y}\right) = c_1$$

- Solve for y

$$y = \frac{x}{\tan\left(\mathit{RootOf}\left(\_Z - x\tan\left(\_Z\right) + c_1\right)\right)}$$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
trying homogeneous types:
trying Chini
differential order: 1; looking for linear symmetries
trying exact
<- exact successful`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 17

```
dsolve(2*x/y(x)-y(x)/(x^2+y(x)^2)+(-x^2/y(x)^2+x/(x^2+y(x)^2))*diff(y(x),x) = 0,y(x), singso
```

$$y(x) = \cot\left(\mathrm{RootOf}\left(-\_Z + x\tan\left(\_Z\right) + c_1\right)\right)x$$

✓ Solution by Mathematica

Time used: 0.255 (sec). Leaf size: 23

```
DSolve[2*x/y[x]-y[x]/(x^2+y[x]^2)+(-x^2/y[x]^2+x/(x^2+y[x]^2))*y'[x] == 0,y[x],x,IncludeSing
```

$$\text{Solve}\left[\arctan\left(\frac{x}{y(x)}\right)-\frac{x^2}{y(x)}=c_1,y(x)\right]$$

6.26 problem 26

Internal problem ID [593]
Internal file name [`OUTPUT/593_Sunday_June_05_2022_01_45_28_AM_41374823/index.tex`]

Book: Elementary differential equations and boundary value problems, 10th ed., Boyce and DiPrima
Section: Miscellaneous problems, end of chapter 2. Page 133
Problem number: 26.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"homogeneousTypeD", "homogeneousTypeD2", "first_order_ode_lie_symmetry_lookup"**

Maple gives the following as the ode type

```
[[_homogeneous, `class A`], _dAlembert]
```

$$\boxed{y'x - \mathrm{e}^{\frac{y}{x}} x - y = 0}$$

6.26.1 Solving as homogeneousTypeD ode

Writing the ode as

$$y' = \mathrm{e}^{\frac{y}{x}} + \frac{y}{x} \tag{A}$$

The given ode has the form

$$y' = \frac{y}{x} + g(x)\, f\left(b\frac{y}{x}\right)^{\frac{n}{m}} \tag{1}$$

Where b is scalar and $g(x)$ is function of x and n, m are integers. The solution is given in Kamke page 20. Using the substitution $y(x) = u(x)\, x$ then

$$\frac{dy}{dx} = \frac{du}{dx} x + u$$

Hence the given ode becomes

$$
\begin{aligned}
\frac{du}{dx}x + u &= u + g(x)\, f(bu)^{\frac{n}{m}} \\
u' &= \frac{1}{x} g(x)\, f(bu)^{\frac{n}{m}} \qquad (2)
\end{aligned}
$$

The above ode is always separable. This is easily solved for u assuming the integration can be resolved, and then the solution to the original ode becomes $y = ux$. Comapring the given ode (A) with the form (1) shows that

$$
\begin{aligned}
g(x) &= 1 \\
b &= 1 \\
f\left(\frac{bx}{y}\right) &= \mathrm{e}^{\frac{y}{x}}
\end{aligned}
$$

Substituting the above in (2) results in the $u(x)$ ode as

$$
u'(x) = \frac{\mathrm{e}^{u(x)}}{x}
$$

Which is now solved as separable In canonical form the ODE is

$$
\begin{aligned}
u' &= F(x,u) \\
&= f(x)g(u) \\
&= \frac{\mathrm{e}^{u}}{x}
\end{aligned}
$$

Where $f(x) = \frac{1}{x}$ and $g(u) = \mathrm{e}^{u}$. Integrating both sides gives

$$
\begin{aligned}
\frac{1}{\mathrm{e}^{u}}\, du &= \frac{1}{x}\, dx \\
\int \frac{1}{\mathrm{e}^{u}}\, du &= \int \frac{1}{x}\, dx \\
-\mathrm{e}^{-u} &= \ln(x) + c_1
\end{aligned}
$$

The solution is

$$
-\mathrm{e}^{-u(x)} - \ln(x) - c_1 = 0
$$

Therefore the solution is found using $y = ux$. Hence

$$
-\mathrm{e}^{-\frac{y}{x}} - \ln(x) - c_1 = 0
$$

Summary

The solution(s) found are the following

$$-\mathrm{e}^{-\frac{y}{x}}-\ln\left(x\right)-c_1=0 \tag{1}$$

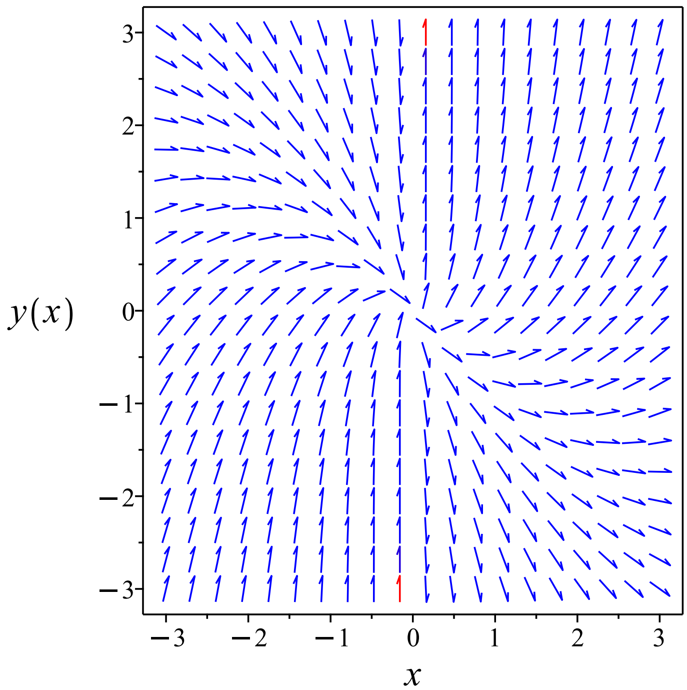


Figure 335: Slope field plot

Verification of solutions

$$-\mathrm{e}^{-\frac{y}{x}}-\ln\left(x\right)-c_1=0$$

Verified OK.

6.26.2 Solving as homogeneousTypeD2 ode

Using the change of variables $y=u(x)\,x$ on the above ode results in new ode in $u(x)$

$$(u'(x)\,x+u(x))\,x-\mathrm{e}^{u(x)}x-u(x)\,x=0$$

In canonical form the ODE is

$$\begin{aligned}u' &= F(x,u)\\ &= f(x)g(u)\\ &= \frac{\mathrm{e}^{u}}{x}\end{aligned}$$

Where $f(x) = \frac{1}{x}$ and $g(u) = e^u$. Integrating both sides gives

$$\frac{1}{e^u} du = \frac{1}{x} dx$$
$$\int \frac{1}{e^u} du = \int \frac{1}{x} dx$$
$$-e^{-u} = \ln(x) + c_2$$

The solution is

$$-e^{-u(x)} - \ln(x) - c_2 = 0$$

Replacing $u(x)$ in the above solution by $\frac{y}{x}$ results in the solution for y in implicit form

$$-e^{-\frac{y}{x}} - \ln(x) - c_2 = 0$$
$$-e^{-\frac{y}{x}} - \ln(x) - c_2 = 0$$

Summary
The solution(s) found are the following

$$-e^{-\frac{y}{x}} - \ln(x) - c_2 = 0 \tag{1}$$

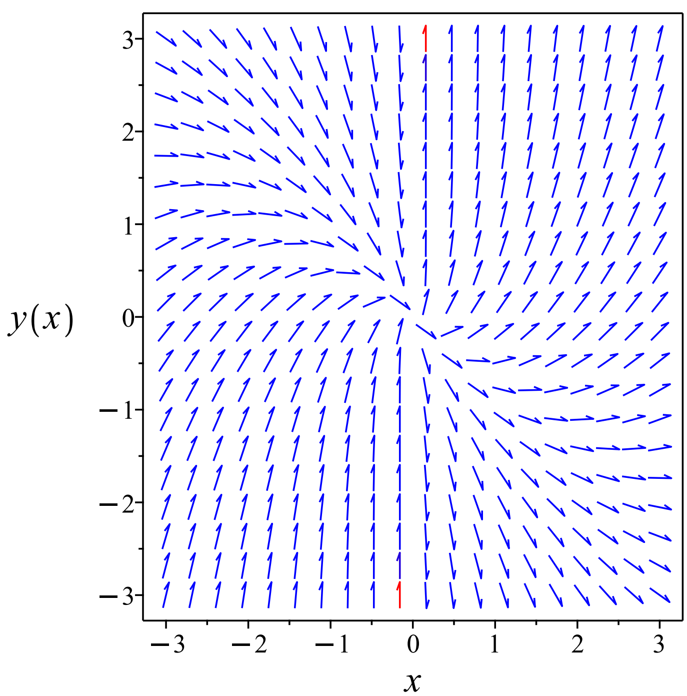


Figure 336: Slope field plot

Verification of solutions

$$-\mathrm{e}^{-\frac{y}{x}} - \ln(x) - c_2 = 0$$

Verified OK.

6.26.3 Solving as first order ode lie symmetry lookup ode

Writing the ode as

$$y' = \frac{\mathrm{e}^{\frac{y}{x}} x + y}{x}$$
$$y' = \omega(x, y)$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x + \omega(\eta_y - \xi_x) - \omega^2 \xi_y - \omega_x \xi - \omega_y \eta = 0 \tag{A}$$

The type of this ode is known. It is of type `homogeneous Type D`. Therefore we do not need to solve the PDE (A), and can just use the lookup table shown below to find ξ, η

Table 311: Lie symmetry infinitesimal lookup table for known first order ODE's

ODE class	Form	ξ	η
linear ode	$y' = f(x)y(x) + g(x)$	0	$e^{\int f dx}$
separable ode	$y' = f(x)\, g(y)$	$\frac{1}{f}$	0
quadrature ode	$y' = f(x)$	0	1
quadrature ode	$y' = g(y)$	1	0
homogeneous ODEs of Class A	$y' = f\left(\frac{y}{x}\right)$	x	y
homogeneous ODEs of Class C	$y' = (a + bx + cy)^{\frac{n}{m}}$	1	$-\frac{b}{c}$
homogeneous class D	$y' = \frac{y}{x} + g(x)\, F\left(\frac{y}{x}\right)$	x^2	xy
First order special form ID 1	$y' = g(x)\, e^{h(x)+by} + f(x)$	$\frac{e^{-\int bf(x)dx-h(x)}}{g(x)}$	$\frac{f(x)e^{-\int bf(x)dx-h(x)}}{g(x)}$
polynomial type ode	$y' = \frac{a_1x+b_1y+c_1}{a_2x+b_2y+c_2}$	$\frac{a_1b_2x-a_2b_1x-b_1c_2+b_2c_1}{a_1b_2-a_2b_1}$	$\frac{a_1b_2y-a_2b_1y-a_1c_2-a_2c_1}{a_1b_2-a_2b_1}$
Bernoulli ode	$y' = f(x)\, y + g(x)\, y^n$	0	$e^{-\int (n-1)f(x)dx} y^n$
Reduced Riccati	$y' = f_1(x)\, y + f_2(x)\, y^2$	0	$e^{-\int f_1 dx}$

The above table shows that

$$\begin{aligned} \xi(x,y) &= x^2 \\ \eta(x,y) &= yx \end{aligned} \tag{A1}$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x,y) \to (R,S)$ where (R,S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y}\right) S(x,y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the

canonical coordinates, where $S(R)$. Therefore

$$\frac{dy}{dx} = \frac{\eta}{\xi} = \frac{yx}{x^2} = \frac{y}{x}$$

This is easily solved to give

$$y = c_1 x$$

Where now the coordinate R is taken as the constant of integration. Hence

$$R = \frac{y}{x}$$

And S is found from

$$dS = \frac{dx}{\xi} = \frac{dx}{x^2}$$

Integrating gives

$$S = \int \frac{dx}{T} = -\frac{1}{x}$$

Where the constant of integration is set to zero as we just need one solution. Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x,y)S_y}{R_x + \omega(x,y)R_y} \tag{2}$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x,y)$ is the right hand side of the original ode given by

$$\omega(x,y) = \frac{e^{\frac{y}{x}} x + y}{x}$$

Evaluating all the partial derivatives gives

$$R_x = -\frac{y}{x^2}$$
$$R_y = \frac{1}{x}$$
$$S_x = \frac{1}{x^2}$$
$$S_y = 0$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = \frac{e^{-\frac{y}{x}}}{x} \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = -S(R)\,e^{-R}$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$S(R) = c_1 e^{e^{-R}} \tag{4}$$

To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$-\frac{1}{x} = c_1 e^{e^{-\frac{y}{x}}}$$

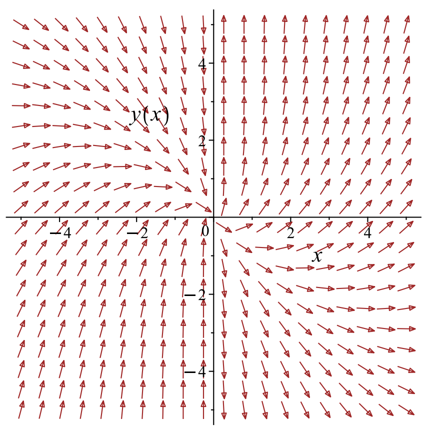
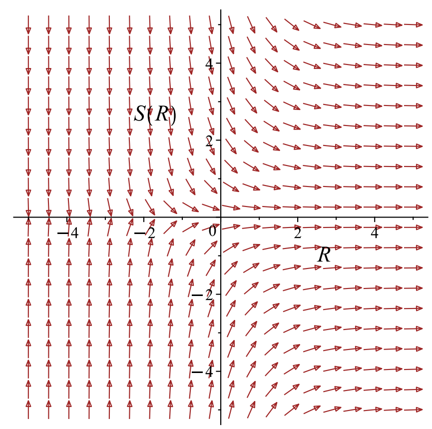
Which simplifies to

$$-\frac{1}{x} = c_1 e^{e^{-\frac{y}{x}}}$$

Which gives

$$y = -\ln\left(\ln\left(-\frac{1}{c_1 x}\right)\right) x$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in x, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dx} = \frac{e^{\frac{y}{x}} x+y}{x}$ 	$R = \frac{y}{x}$ $S = -\frac{1}{x}$	$\frac{dS}{dR} = -S(R)\, e^{-R}$ 

Summary

The solution(s) found are the following

$$y = -\ln\left(\ln\left(-\frac{1}{c_1 x}\right)\right) x \tag{1}$$

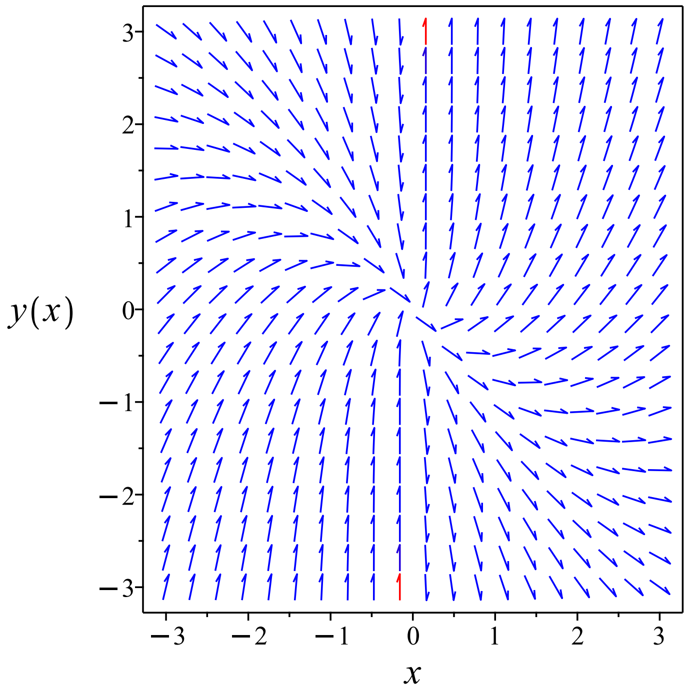


Figure 337: Slope field plot

Verification of solutions

$$y = -\ln\left(\ln\left(-\frac{1}{c_1 x}\right)\right)x$$

Verified OK.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
trying homogeneous types:
trying homogeneous D
<- homogeneous successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 15

```
dsolve(x*diff(y(x),x) = exp(y(x)/x)*x+y(x),y(x), singsol=all)
```

$$y(x) = \ln\left(-\frac{1}{\ln(x) + c_1}\right) x$$

✓ Solution by Mathematica

Time used: 0.316 (sec). Leaf size: 18

```
DSolve[x*y'[x] == Exp[y[x]/x]*x+y[x],y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to -x \log(-\log(x) - c_1)$$

6.27 problem 27

Internal problem ID [594]
Internal file name [`OUTPUT/594_Sunday_June_05_2022_01_45_29_AM_32116103/index.tex`]

Book: Elementary differential equations and boundary value problems, 10th ed., Boyce and DiPrima
Section: Miscellaneous problems, end of chapter 2. Page 133
Problem number: 27.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"exactWithIntegrationFactor"**

Maple gives the following as the ode type

```
[_rational, [_1st_order, `_with_symmetry_[F(x)*G(y),0]`]]
```

$$\boxed{y' - \frac{x}{x^2 + y + y^3} = 0}$$

6.27.1 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x,y) + N(x,y)\frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x,y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y) = 0$$

Hence

$$\frac{\partial\phi}{\partial x} + \frac{\partial\phi}{\partial y}\frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$\frac{\partial\phi}{\partial x} = M$$
$$\frac{\partial\phi}{\partial y} = N$$

But since $\frac{\partial^2\phi}{\partial x\partial y} = \frac{\partial^2\phi}{\partial y\partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x, y)$ but at least we know now that we can do that since the condition $\frac{\partial^2\phi}{\partial x\partial y} = \frac{\partial^2\phi}{\partial y\partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x, y)\,\mathrm{d}x + N(x, y)\,\mathrm{d}y = 0 \tag{1A}$$

Therefore

$$\begin{aligned}(y^3 + x^2 + y)\,\mathrm{d}y &= (x)\,\mathrm{d}x\\ (-x)\,\mathrm{d}x + (y^3 + x^2 + y)\,\mathrm{d}y &= 0 \end{aligned} \tag{2A}$$

Comparing (1A) and (2A) shows that

$$\begin{aligned}M(x, y) &= -x\\ N(x, y) &= y^3 + x^2 + y\end{aligned}$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Using result found above gives

$$\begin{aligned}\frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}(-x)\\ &= 0\end{aligned}$$

And

$$\begin{aligned}\frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}(y^3 + x^2 + y)\\ &= 2x\end{aligned}$$

Since $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$, then the ODE is not exact. Since the ODE is not exact, we will try to find an integrating factor to make it exact. Let

$$\begin{aligned}A &= \frac{1}{N}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right)\\ &= \frac{1}{y^3 + x^2 + y}((0) - (2x))\\ &= -\frac{2x}{y^3 + x^2 + y}\end{aligned}$$

Since A depends on y, it can not be used to obtain an integrating factor. We will now try a second method to find an integrating factor. Let

$$\begin{aligned} B &= \frac{1}{M}\left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) \\ &= -\frac{1}{x}((2x) - (0)) \\ &= -2 \end{aligned}$$

Since B does not depend on x, it can be used to obtain an integrating factor. Let the integrating factor be μ. Then

$$\begin{aligned} \mu &= e^{\int B \, \mathrm{d}y} \\ &= e^{\int -2 \, \mathrm{d}y} \end{aligned}$$

The result of integrating gives

$$\begin{aligned} \mu &= e^{-2y} \\ &= \mathrm{e}^{-2y} \end{aligned}$$

M and N are now multiplied by this integrating factor, giving new M and new N which are called $\overline{M}$ and $\overline{N}$ so not to confuse them with the original M and N.

$$\begin{aligned} \overline{M} &= \mu M \\ &= \mathrm{e}^{-2y}(-x) \\ &= -x\,\mathrm{e}^{-2y} \end{aligned}$$

And

$$\begin{aligned} \overline{N} &= \mu N \\ &= \mathrm{e}^{-2y}\left(y^3 + x^2 + y\right) \\ &= \left(y^3 + x^2 + y\right)\mathrm{e}^{-2y} \end{aligned}$$

So now a modified ODE is obtained from the original ODE which will be exact and can be solved using the standard method. The modified ODE is

$$\begin{aligned} \overline{M} + \overline{N}\frac{\mathrm{d}y}{\mathrm{d}x} &= 0 \\ \left(-x\,\mathrm{e}^{-2y}\right) + \left(\left(y^3 + x^2 + y\right)\mathrm{e}^{-2y}\right)\frac{\mathrm{d}y}{\mathrm{d}x} &= 0 \end{aligned}$$

The following equations are now set up to solve for the function $\phi(x, y)$

$$
\begin{aligned}
\frac{\partial \phi}{\partial x} &= \overline{M} \qquad (1)\\
\frac{\partial \phi}{\partial y} &= \overline{N} \qquad (2)
\end{aligned}
$$

Integrating (1) w.r.t. x gives

$$
\begin{aligned}
\int \frac{\partial \phi}{\partial x} \mathrm{d}x &= \int \overline{M} \,\mathrm{d}x \\
\int \frac{\partial \phi}{\partial x} \mathrm{d}x &= \int -x \,\mathrm{e}^{-2y} \,\mathrm{d}x \\
\phi &= -\frac{x^2 \mathrm{e}^{-2y}}{2} + f(y) \qquad (3)
\end{aligned}
$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$
\frac{\partial \phi}{\partial y} = x^2 \mathrm{e}^{-2y} + f'(y) \qquad (4)
$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = (y^3 + x^2 + y) \,\mathrm{e}^{-2y}$. Therefore equation (4) becomes

$$
(y^3 + x^2 + y) \,\mathrm{e}^{-2y} = x^2 \mathrm{e}^{-2y} + f'(y) \qquad (5)
$$

Solving equation (5) for $f'(y)$ gives

$$
\begin{aligned}
f'(y) &= \mathrm{e}^{-2y} y^3 + \mathrm{e}^{-2y} y \\
&= \mathrm{e}^{-2y} (y^3 + y)
\end{aligned}
$$

Integrating the above w.r.t y results in

$$
\begin{aligned}
\int f'(y) \,\mathrm{d}y &= \int \left(\mathrm{e}^{-2y} (y^3 + y)\right) \mathrm{d}y \\
f(y) &= -\frac{\mathrm{e}^{-2y} (4y^3 + 6y^2 + 10y + 5)}{8} + c_1
\end{aligned}
$$

Where c_1 is constant of integration. Substituting result found above for $f(y)$ into equation (3) gives ϕ

$$\phi = -\frac{x^2 e^{-2y}}{2} - \frac{e^{-2y}(4y^3 + 6y^2 + 10y + 5)}{8} + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = -\frac{x^2 e^{-2y}}{2} - \frac{e^{-2y}(4y^3 + 6y^2 + 10y + 5)}{8}$$

Summary

The solution(s) found are the following

$$-\frac{x^2 e^{-2y}}{2} - \frac{e^{-2y}(4y^3 + 6y^2 + 10y + 5)}{8} = c_1 \tag{1}$$

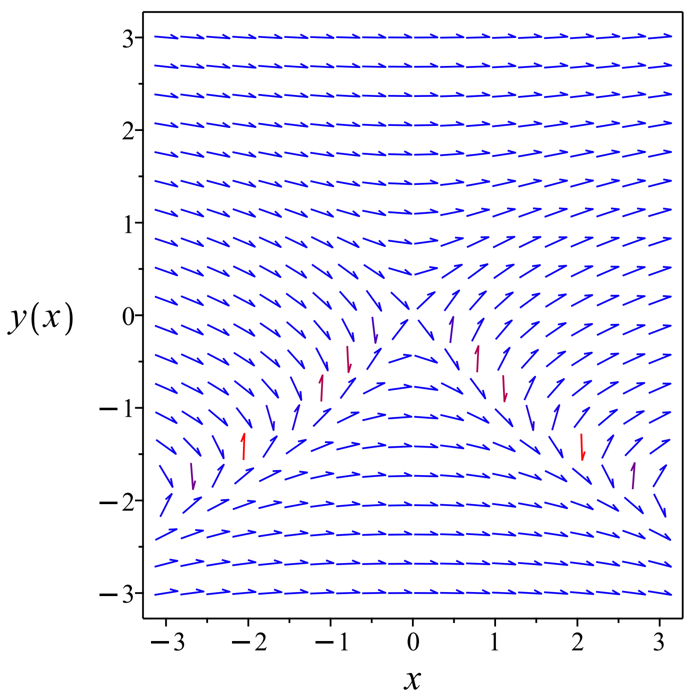


Figure 338: Slope field plot

Verification of solutions

$$-\frac{x^2 e^{-2y}}{2} - \frac{e^{-2y}(4y^3 + 6y^2 + 10y + 5)}{8} = c_1$$

Verified OK.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
trying homogeneous types:
trying Chini
differential order: 1; looking for linear symmetries
trying exact
<- exact successful`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 35

```
dsolve(diff(y(x),x) = x/(x^2+y(x)+y(x)^3),y(x), singsol=all)
```

$$\frac{\left(-4y(x)^3-4x^2-6y(x)^2-10y(x)-5\right)\mathrm{e}^{-2y(x)}}{4}+c_1=0$$

✓ Solution by Mathematica

Time used: 0.173 (sec). Leaf size: 48

```
DSolve[y'[x] == x/(x^2+y[x]+y[x]^3),y[x],x,IncludeSingularSolutions -> True]
```

$$\text{Solve}\left[-\frac{1}{2}x^2e^{-2y(x)}-\frac{1}{8}e^{-2y(x)}\left(4y(x)^3+6y(x)^2+10y(x)+5\right)=c_1,y(x)\right]$$

6.28 problem 28

Internal problem ID [595]
Internal file name [`OUTPUT/595_Sunday_June_05_2022_01_45_31_AM_56902748/index.tex`]

Book: Elementary differential equations and boundary value problems, 10th ed., Boyce and DiPrima
Section: Miscellaneous problems, end of chapter 2. Page 133
Problem number: 28.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"linear", "homogeneousTypeD2", "exactWithIntegrationFactor", "first_order_ode_lie_symmetry_lookup"**

Maple gives the following as the ode type

```
[_linear]
```

$$\boxed{2y + ty' = -3t}$$

6.28.1 Solving as linear ode

Entering Linear first order ODE solver. In canonical form a linear first order is

$$y' + p(t)y = q(t)$$

Where here

$$p(t) = \frac{2}{t}$$
$$q(t) = -3$$

Hence the ode is

$$y' + \frac{2y}{t} = -3$$

The integrating factor μ is

$$\mu = e^{\int \frac{2}{t} dt}$$
$$= t^2$$

The ode becomes

$$\frac{d}{dt}(\mu y) = (\mu)(-3)$$
$$\frac{d}{dt}(t^2 y) = (t^2)(-3)$$
$$d(t^2 y) = (-3t^2)\, dt$$

Integrating gives

$$t^2 y = \int -3t^2 \, dt$$
$$t^2 y = -t^3 + c_1$$

Dividing both sides by the integrating factor $\mu = t^2$ results in

$$y = -t + \frac{c_1}{t^2}$$

Summary

The solution(s) found are the following

$$y = -t + \frac{c_1}{t^2} \tag{1}$$

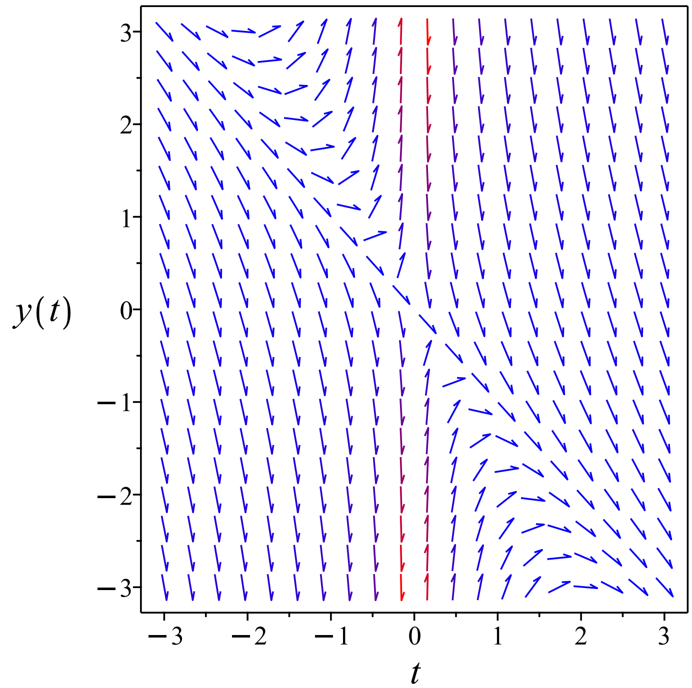


Figure 339: Slope field plot

Verification of solutions

$$y = -t + \frac{c_1}{t^2}$$

Verified OK.

6.28.2 Solving as homogeneousTypeD2 ode

Using the change of variables $y = u(t)\,t$ on the above ode results in new ode in $u(t)$

$$2u(t)\,t + t(u'(t)\,t + u(t)) = -3t$$

In canonical form the ODE is

$$\begin{aligned} u' &= F(t,u) \\ &= f(t)g(u) \\ &= \frac{-3u-3}{t} \end{aligned}$$

Where $f(t) = \frac{1}{t}$ and $g(u) = -3u - 3$. Integrating both sides gives

$$\frac{1}{-3u-3}\,du = \frac{1}{t}\,dt$$

$$
\int \frac{1}{-3u-3}du = \int \frac{1}{t}dt
$$

$$
-\frac{\ln (u+1)}{3} = \ln (t) + c_2
$$

Raising both side to exponential gives

$$
\frac{1}{(u+1)^{\frac{1}{3}}} = \mathrm{e}^{\ln(t)+c_2}
$$

Which simplifies to

$$
\frac{1}{(u+1)^{\frac{1}{3}}} = c_3 t
$$

Therefore the solution y is

$$
\begin{aligned}
y &= ut \\
&= -\frac{(c_3^3 \mathrm{e}^{3c_2} t^3 - 1)\, \mathrm{e}^{-3c_2}}{t^2 c_3^3}
\end{aligned}
$$

Summary

The solution(s) found are the following

$$
y = -\frac{(c_3^3 \mathrm{e}^{3c_2} t^3 - 1)\, \mathrm{e}^{-3c_2}}{t^2 c_3^3} \tag{1}
$$

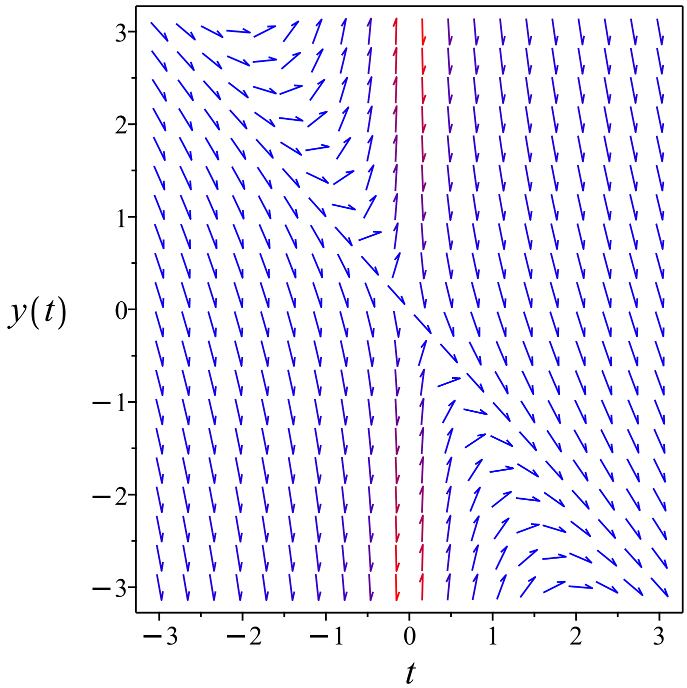


Figure 340: Slope field plot

Verification of solutions

$$y = -\frac{(c_3^3 \mathrm{e}^{3c_2} t^3 - 1)\,\mathrm{e}^{-3c_2}}{t^2 c_3^3}$$

Verified OK.

6.28.3 Solving as first order ode lie symmetry lookup ode

Writing the ode as

$$y' = -\frac{3t + 2y}{t}$$
$$y' = \omega(t, y)$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_t + \omega(\eta_y - \xi_t) - \omega^2 \xi_y - \omega_t \xi - \omega_y \eta = 0 \tag{A}$$

The type of this ode is known. It is of type `linear`. Therefore we do not need to solve the PDE (A), and can just use the lookup table shown below to find ξ, η

Table 313: Lie symmetry infinitesimal lookup table for known first order ODE's

ODE class	Form	ξ	η
linear ode	$y' = f(x)y(x) + g(x)$	0	$e^{\int f dx}$
separable ode	$y' = f(x)\, g(y)$	$\frac{1}{f}$	0
quadrature ode	$y' = f(x)$	0	1
quadrature ode	$y' = g(y)$	1	0
homogeneous ODEs of Class A	$y' = f\left(\frac{y}{x}\right)$	x	y
homogeneous ODEs of Class C	$y' = (a + bx + cy)^{\frac{n}{m}}$	1	$-\frac{b}{c}$
homogeneous class D	$y' = \frac{y}{x} + g(x)\, F\left(\frac{y}{x}\right)$	x^2	xy
First order special form ID 1	$y' = g(x)\, e^{h(x)+by} + f(x)$	$\frac{e^{-\int bf(x)dx - h(x)}}{g(x)}$	$\frac{f(x)e^{-\int bf(x)dx - h(x)}}{g(x)}$
polynomial type ode	$y' = \frac{a_1x+b_1y+c_1}{a_2x+b_2y+c_2}$	$\frac{a_1b_2x - a_2b_1x - b_1c_2 + b_2c_1}{a_1b_2 - a_2b_1}$	$\frac{a_1b_2y - a_2b_1y - a_1c_2 - a_2c_1}{a_1b_2 - a_2b_1}$
Bernoulli ode	$y' = f(x)\, y + g(x)\, y^n$	0	$e^{-\int (n-1)f(x)dx} y^n$
Reduced Riccati	$y' = f_1(x)\, y + f_2(x)\, y^2$	0	$e^{-\int f_1 dx}$

The above table shows that

$$
\begin{aligned}
\xi(t, y) &= 0 \\
\eta(t, y) &= \frac{1}{t^2}
\end{aligned} \tag{A1}
$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(t, y) \to (R, S)$ where (R, S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$
\frac{dt}{\xi} = \frac{dy}{\eta} = dS \tag{1}
$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial t} + \eta \frac{\partial}{\partial y}\right) S(t, y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the

canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$R = t$$

S is found from

$$\begin{aligned} S &= \int \frac{1}{\eta} dy \\ &= \int \frac{1}{\frac{1}{t^2}} dy \end{aligned}$$

Which results in

$$S = t^2 y$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_t + \omega(t, y) S_y}{R_t + \omega(t, y) R_y} \tag{2}$$

Where in the above R_t, R_y, S_t, S_y are all partial derivatives and $\omega(t, y)$ is the right hand side of the original ode given by

$$\omega(t, y) = -\frac{3t + 2y}{t}$$

Evaluating all the partial derivatives gives

$$\begin{aligned} R_t &= 1 \\ R_y &= 0 \\ S_t &= 2ty \\ S_y &= t^2 \end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = -3t^2 \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for t, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = -3R^2$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by

integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$S(R) = -R^3 + c_1 \tag{4}$$

To complete the solution, we just need to transform (4) back to t, y coordinates. This results in

$$yt^2 = -t^3 + c_1$$

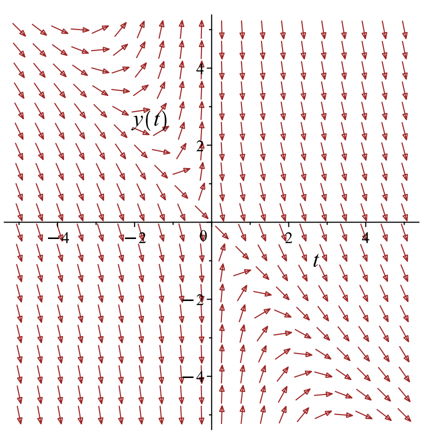
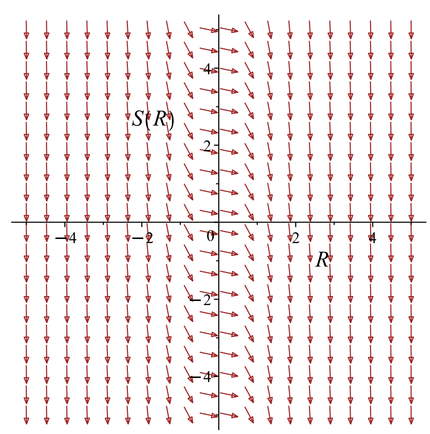
Which simplifies to

$$yt^2 = -t^3 + c_1$$

Which gives

$$y = \frac{-t^3 + c_1}{t^2}$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in t, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dt} = -\frac{3t+2y}{t}$ 	$R = t$ $S = t^2 y$	$\frac{dS}{dR} = -3R^2$ 

Summary

The solution(s) found are the following

$$y = \frac{-t^3 + c_1}{t^2} \tag{1}$$

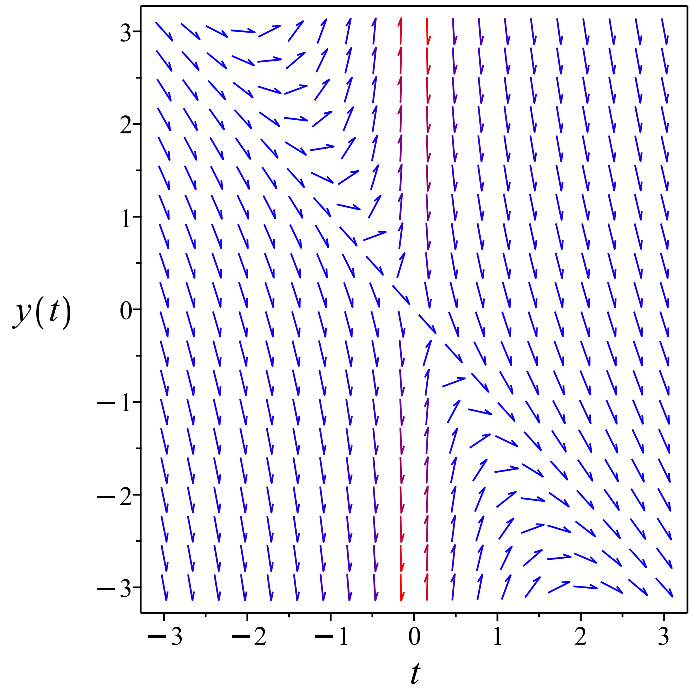


Figure 341: Slope field plot

<u>Verification of solutions</u>

$$y = \frac{-t^3 + c_1}{t^2}$$

Verified OK.

6.28.4 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x,y) + N(x,y)\frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x,y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y) = 0$$

Hence

$$\frac{\partial\phi}{\partial x} + \frac{\partial\phi}{\partial y}\frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$\frac{\partial \phi}{\partial x} = M$$
$$\frac{\partial \phi}{\partial y} = N$$

But since $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x, y)$ but at least we know now that we can do that since the condition $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(t, y)\,\mathrm{d}t + N(t, y)\,\mathrm{d}y = 0 \tag{1A}$$

Therefore

$$\begin{aligned}(t)\,\mathrm{d}y &= (-3t - 2y)\,\mathrm{d}t \\ (3t + 2y)\,\mathrm{d}t + (t)\,\mathrm{d}y &= 0 \end{aligned} \tag{2A}$$

Comparing (1A) and (2A) shows that

$$M(t, y) = 3t + 2y$$
$$N(t, y) = t$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial t}$$

Using result found above gives

$$\begin{aligned}\frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}(3t + 2y) \\ &= 2\end{aligned}$$

And

$$\begin{aligned}\frac{\partial N}{\partial t} &= \frac{\partial}{\partial t}(t) \\ &= 1\end{aligned}$$

Since $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial t}$, then the ODE is <u>not exact</u>. Since the ODE is not exact, we will try to find an integrating factor to make it exact. Let

$$\begin{aligned}
A &= \frac{1}{N}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial t}\right) \\
&= \frac{1}{t}((2) - (1)) \\
&= \frac{1}{t}
\end{aligned}$$

Since A does not depend on y, then it can be used to find an integrating factor. The integrating factor μ is

$$\begin{aligned}
\mu &= e^{\int A \, \mathrm{d}t} \\
&= e^{\int \frac{1}{t} \, \mathrm{d}t}
\end{aligned}$$

The result of integrating gives

$$\begin{aligned}
\mu &= e^{\ln(t)} \\
&= t
\end{aligned}$$

M and N are multiplied by this integrating factor, giving new M and new N which are called $\overline{M}$ and $\overline{N}$ for now so not to confuse them with the original M and N.

$$\begin{aligned}
\overline{M} &= \mu M \\
&= t(3t + 2y) \\
&= t(3t + 2y)
\end{aligned}$$

And

$$\begin{aligned}
\overline{N} &= \mu N \\
&= t(t) \\
&= t^2
\end{aligned}$$

Now a modified ODE is ontained from the original ODE, which is exact and can be solved. The modified ODE is

$$\begin{aligned}
\overline{M} + \overline{N}\frac{\mathrm{d}y}{\mathrm{d}t} &= 0 \\
(t(3t + 2y)) + \left(t^2\right)\frac{\mathrm{d}y}{\mathrm{d}t} &= 0
\end{aligned}$$

The following equations are now set up to solve for the function $\phi(t,y)$

$$\frac{\partial \phi}{\partial t} = \overline{M} \tag{1}$$

$$\frac{\partial \phi}{\partial y} = \overline{N} \tag{2}$$

Integrating (1) w.r.t. t gives

$$\int \frac{\partial \phi}{\partial t} \,\mathrm{d}t = \int \overline{M} \,\mathrm{d}t$$

$$\int \frac{\partial \phi}{\partial t} \,\mathrm{d}t = \int t(3t + 2y) \,\mathrm{d}t$$

$$\phi = t^2(t + y) + f(y) \tag{3}$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both t and y. Taking derivative of equation (3) w.r.t y gives

$$\frac{\partial \phi}{\partial y} = t^2 + f'(y) \tag{4}$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = t^2$. Therefore equation (4) becomes

$$t^2 = t^2 + f'(y) \tag{5}$$

Solving equation (5) for $f'(y)$ gives

$$f'(y) = 0$$

Therefore

$$f(y) = c_1$$

Where c_1 is constant of integration. Substituting this result for $f(y)$ into equation (3) gives ϕ

$$\phi = t^2(t + y) + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = t^2(t + y)$$

The solution becomes

$$y = \frac{-t^3 + c_1}{t^2}$$

Summary

The solution(s) found are the following

$$y = \frac{-t^3 + c_1}{t^2} \tag{1}$$

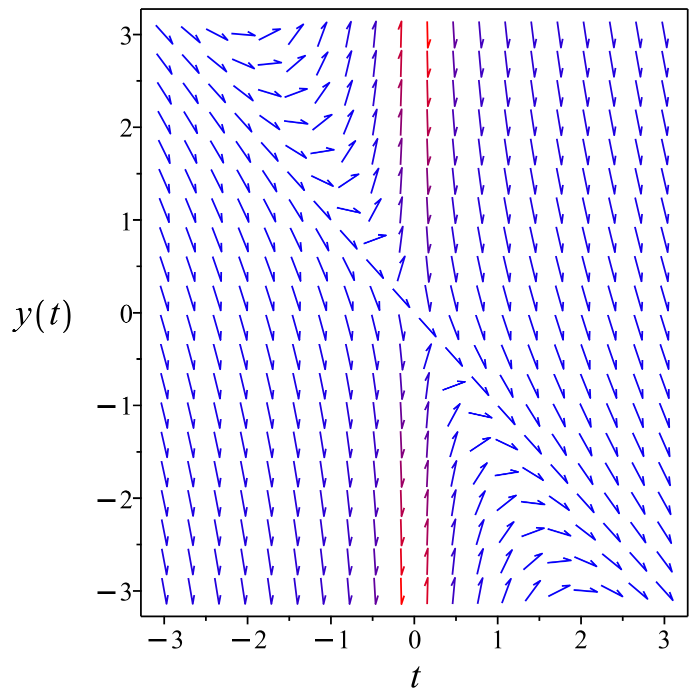


Figure 342: Slope field plot

Verification of solutions

$$y = \frac{-t^3 + c_1}{t^2}$$

Verified OK.

6.28.5 Maple step by step solution

Let's solve

$2y + ty' = -3t$

- Highest derivative means the order of the ODE is 1

  y'

- Isolate the derivative

  $y' = -3 - \frac{2y}{t}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE

  $y' + \frac{2y}{t} = -3$

- The ODE is linear; multiply by an integrating factor $\mu(t)$

  $\mu(t)\left(y' + \frac{2y}{t}\right) = -3\mu(t)$

- Assume the lhs of the ODE is the total derivative $\frac{d}{dt}(\mu(t)\, y)$

  $\mu(t)\left(y' + \frac{2y}{t}\right) = \mu'(t)\, y + \mu(t)\, y'$

- Isolate $\mu'(t)$

  $\mu'(t) = \frac{2\mu(t)}{t}$

- Solve to find the integrating factor

  $\mu(t) = t^2$

- Integrate both sides with respect to t

  $\int \left(\frac{d}{dt}(\mu(t)\, y)\right) dt = \int -3\mu(t)\, dt + c_1$

- Evaluate the integral on the lhs

  $\mu(t)\, y = \int -3\mu(t)\, dt + c_1$

- Solve for y

  $y = \frac{\int -3\mu(t)dt + c_1}{\mu(t)}$

- Substitute $\mu(t) = t^2$

  $y = \frac{\int -3t^2dt + c_1}{t^2}$

- Evaluate the integrals on the rhs

  $y = \frac{-t^3 + c_1}{t^2}$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
<- 1st order linear successful`
```

✓ Solution by Maple
Time used: 0.0 (sec). Leaf size: 13

```
dsolve(3*t+2*y(t) = -t*diff(y(t),t),y(t), singsol=all)
```

$$y(t) = -t + \frac{c_1}{t^2}$$

✓ Solution by Mathematica
Time used: 0.023 (sec). Leaf size: 15

```
DSolve[3*t+2*y[t] == -t*y'[t],y[t],t,IncludeSingularSolutions -> True]
```

$$y(t) \to -t + \frac{c_1}{t^2}$$

6.29 problem 29

Internal problem ID [596]
Internal file name [`OUTPUT/596_Sunday_June_05_2022_01_45_32_AM_62306633/index.tex`]

Book: Elementary differential equations and boundary value problems, 10th ed., Boyce and DiPrima
Section: Miscellaneous problems, end of chapter 2. Page 133
Problem number: 29.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"exactByInspection", "homogeneousTypeD2", "first_order_ode_lie_symmetry_calculated"**

Maple gives the following as the ode type

```
[[_homogeneous, `class A`], _rational, [_Abel, `2nd type`, `
   class A`]]
```

$$\boxed{y' - \frac{x + y}{x - y} = 0}$$

6.29.1 Solving as homogeneousTypeD2 ode

Using the change of variables $y = u(x)\,x$ on the above ode results in new ode in $u(x)$

$$u'(x)\,x + u(x) - \frac{x + u(x)\,x}{x - u\,(x)\,x} = 0$$

In canonical form the ODE is

$$\begin{aligned} u' &= F(x,u) \\ &= f(x)g(u) \\ &= -\frac{u^2 + 1}{x\,(u - 1)} \end{aligned}$$

Where $f(x) = -\frac{1}{x}$ and $g(u) = \frac{u^2+1}{u-1}$. Integrating both sides gives

$$\frac{1}{\frac{u^2+1}{u-1}} \, du = -\frac{1}{x} \, dx$$

$$\int \frac{1}{\frac{u^2+1}{u-1}} \, du = \int -\frac{1}{x} \, dx$$

$$\frac{\ln (u^2+1)}{2} - \arctan (u) = -\ln (x) + c_2$$

The solution is

$$\frac{\ln \left(u(x)^2+1\right)}{2} - \arctan (u(x)) + \ln (x) - c_2 = 0$$

Replacing $u(x)$ in the above solution by $\frac{y}{x}$ results in the solution for y in implicit form

$$\frac{\ln \left(\frac{y^2}{x^2}+1\right)}{2} - \arctan \left(\frac{y}{x}\right) + \ln (x) - c_2 = 0$$

$$\frac{\ln \left(\frac{y^2}{x^2}+1\right)}{2} - \arctan \left(\frac{y}{x}\right) + \ln (x) - c_2 = 0$$

Summary

The solution(s) found are the following

$$\frac{\ln \left(\frac{y^2}{x^2}+1\right)}{2} - \arctan \left(\frac{y}{x}\right) + \ln (x) - c_2 = 0 \tag{1}$$

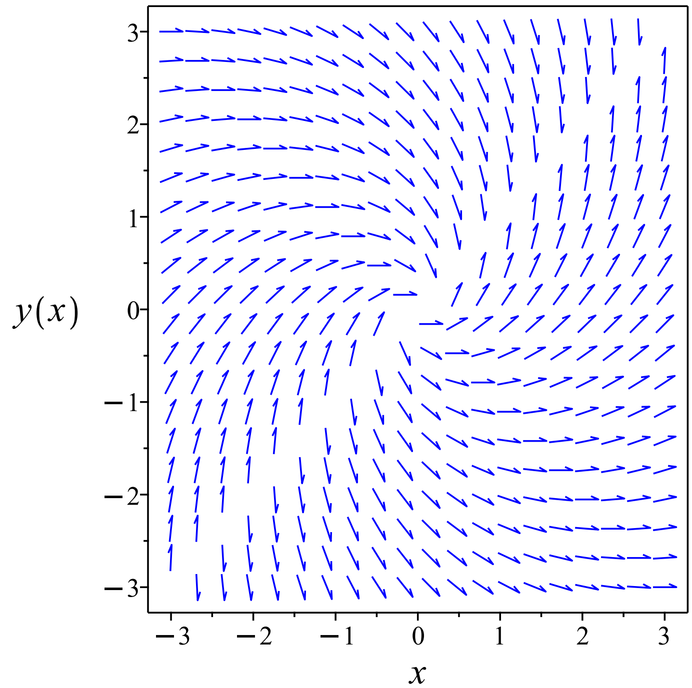


Figure 343: Slope field plot

Verification of solutions

$$\frac{\ln\left(\frac{y^2}{x^2}+1\right)}{2} - \arctan\left(\frac{y}{x}\right) + \ln(x) - c_2 = 0$$

Verified OK.

6.29.2 Solving as first order ode lie symmetry calculated ode

Writing the ode as

$$y' = -\frac{x+y}{-x+y}$$
$$y' = \omega(x,y)$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x + \omega(\eta_y - \xi_x) - \omega^2\xi_y - \omega_x\xi - \omega_y\eta = 0 \tag{A}$$

The type of this ode is not in the lookup table. To determine ξ, η then (A) is solved using ansatz. Making bivariate polynomials of degree 1 to use as anstaz gives

$$\xi = xa_2 + ya_3 + a_1 \tag{1E}$$
$$\eta = xb_2 + yb_3 + b_1 \tag{2E}$$

Where the unknown coefficients are

$$\{a_1, a_2, a_3, b_1, b_2, b_3\}$$

Substituting equations (1E,2E) and ω into (A) gives

$$b_2 - \frac{(x+y)(b_3-a_2)}{-x+y} - \frac{(x+y)^2 a_3}{(-x+y)^2} - \left(-\frac{1}{-x+y} - \frac{x+y}{(-x+y)^2}\right)(xa_2+ya_3+a_1) - \left(-\frac{1}{-x+y} + \frac{x+y}{(-x+y)^2}\right)(xb_2+yb_3+b_1) = 0 \tag{5E}$$

Putting the above in normal form gives

$$-\frac{x^2a_2 + x^2a_3 + x^2b_2 - x^2b_3 - 2xya_2 + 2xya_3 + 2xyb_2 + 2xyb_3 - y^2a_2 - y^2a_3 - y^2b_2 + y^2b_3 + 2xb_1 - 2ya}{(x-y)^2} = 0$$

Setting the numerator to zero gives

$$-x^2a_2 - x^2a_3 - x^2b_2 + x^2b_3 + 2xya_2 - 2xya_3 - 2xyb_2 - 2xyb_3 + y^2a_2 + y^2a_3 + y^2b_2 - y^2b_3 - 2xb_1 + 2ya_1 = 0 \tag{6E}$$

Looking at the above PDE shows the following are all the terms with $\{x, y\}$ in them.

$$\{x, y\}$$

The following substitution is now made to be able to collect on all terms with $\{x, y\}$ in them

$$\{x = v_1, y = v_2\}$$

The above PDE (6E) now becomes

$$-a_2v_1^2 + 2a_2v_1v_2 + a_2v_2^2 - a_3v_1^2 - 2a_3v_1v_2 + a_3v_2^2 - b_2v_1^2 - 2b_2v_1v_2 + b_2v_2^2 + b_3v_1^2 - 2b_3v_1v_2 - b_3v_2^2 + 2a_1v_2 - 2b_1v_1 = 0 \tag{7E}$$

Collecting the above on the terms v_i introduced, and these are

$$\{v_1, v_2\}$$

Equation (7E) now becomes

$$\begin{aligned}(-a_2 - a_3 - b_2 + b_3)\, v_1^2 + (2a_2 - 2a_3 - 2b_2 - 2b_3)\, v_1 v_2 \\ - 2b_1 v_1 + (a_2 + a_3 + b_2 - b_3)\, v_2^2 + 2a_1 v_2 = 0\end{aligned} \tag{8E}$$

Setting each coefficients in (8E) to zero gives the following equations to solve

$$\begin{aligned}2a_1 &= 0\\ -2b_1 &= 0\\ -a_2 - a_3 - b_2 + b_3 &= 0\\ a_2 + a_3 + b_2 - b_3 &= 0\\ 2a_2 - 2a_3 - 2b_2 - 2b_3 &= 0\end{aligned}$$

Solving the above equations for the unknowns gives

$$\begin{aligned}a_1 &= 0\\ a_2 &= b_3\\ a_3 &= -b_2\\ b_1 &= 0\\ b_2 &= b_2\\ b_3 &= b_3\end{aligned}$$

Substituting the above solution in the anstaz (1E,2E) (using 1 as arbitrary value for any unknown in the RHS) gives

$$\begin{aligned}\xi &= x\\ \eta &= y\end{aligned}$$

Shifting is now applied to make $\xi = 0$ in order to simplify the rest of the computation

$$\begin{aligned}\eta &= \eta - \omega(x, y)\, \xi\\ &= y - \left(-\frac{x+y}{-x+y}\right)(x)\\ &= \frac{-x^2 - y^2}{x - y}\\ \xi &= 0\end{aligned}$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x, y) \to (R, S)$ where (R, S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y}\right) S(x, y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$R = x$$

S is found from

$$\begin{aligned} S &= \int \frac{1}{\eta} dy \\ &= \int \frac{1}{\frac{-x^2-y^2}{x-y}} dy \end{aligned}$$

Which results in

$$S = \frac{\ln(x^2 + y^2)}{2} - \arctan\left(\frac{y}{x}\right)$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x, y) S_y}{R_x + \omega(x, y) R_y} \tag{2}$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x, y)$ is the right hand side of the original ode given by

$$\omega(x, y) = -\frac{x + y}{-x + y}$$

Evaluating all the partial derivatives gives

$$\begin{aligned} R_x &= 1 \\ R_y &= 0 \\ S_x &= \frac{x + y}{x^2 + y^2} \\ S_y &= \frac{-x + y}{x^2 + y^2} \end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = 0 \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = 0$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$S(R) = c_1 \tag{4}$$

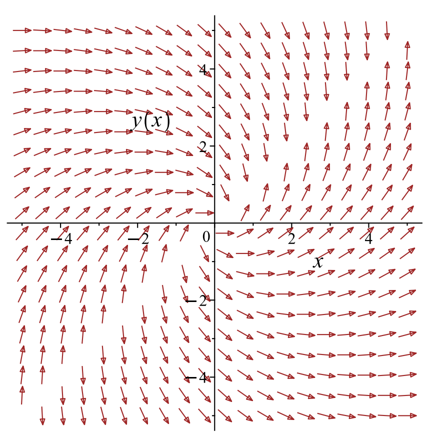
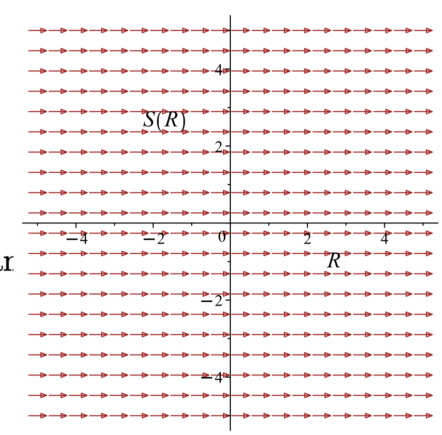
To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$\frac{\ln(x^2 + y^2)}{2} - \arctan\left(\frac{y}{x}\right) = c_1$$

Which simplifies to

$$\frac{\ln(x^2 + y^2)}{2} - \arctan\left(\frac{y}{x}\right) = c_1$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in x, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dx} = -\frac{x+y}{-x+y}$ 	$R = x$ $S = \frac{\ln(x^2+y^2)}{2} - \arctan$	$\frac{dS}{dR} = 0$ 

Summary
The solution(s) found are the following

$$\frac{\ln(x^2+y^2)}{2} - \arctan\left(\frac{y}{x}\right) = c_1 \tag{1}$$

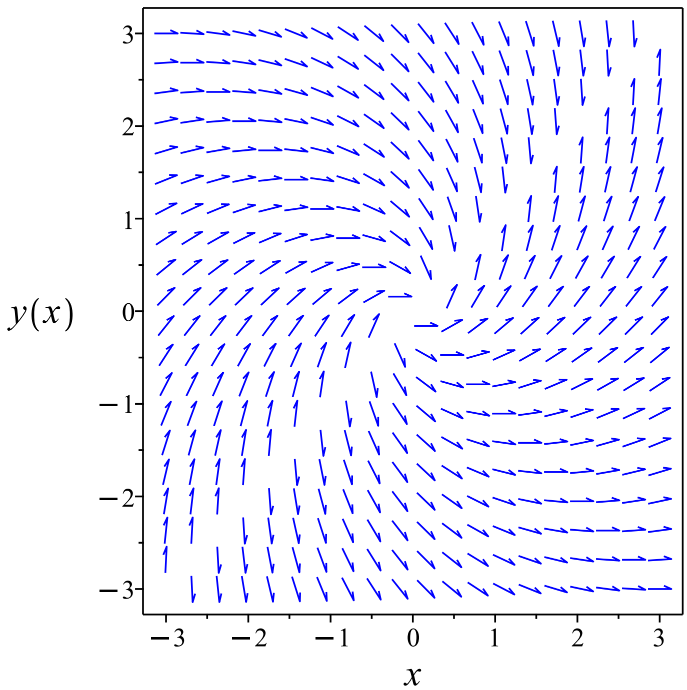


Figure 344: Slope field plot

Verification of solutions

$$\frac{\ln(x^2+y^2)}{2} - \arctan\left(\frac{y}{x}\right) = c_1$$

Verified OK.

6.29.3 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x,y) + N(x,y)\frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x,y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y) = 0$$

Hence

$$\frac{\partial\phi}{\partial x}+\frac{\partial\phi}{\partial y}\frac{dy}{dx}=0 \tag{B}$$

Comparing (A,B) shows that

$$\frac{\partial\phi}{\partial x}=M$$
$$\frac{\partial\phi}{\partial y}=N$$

But since $\frac{\partial^2\phi}{\partial x\partial y}=\frac{\partial^2\phi}{\partial y\partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y}=\frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x,y)$ but at least we know now that we can do that since the condition $\frac{\partial^2\phi}{\partial x\partial y}=\frac{\partial^2\phi}{\partial y\partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x,y)\,\mathrm{d}x+N(x,y)\,\mathrm{d}y=0 \tag{1A}$$

Therefore

$$(-x+y)\,\mathrm{d}y=(-x-y)\,\mathrm{d}x$$
$$(x+y)\,\mathrm{d}x+(-x+y)\,\mathrm{d}y=0 \tag{2A}$$

Comparing (1A) and (2A) shows that

$$M(x,y)=x+y$$
$$N(x,y)=-x+y$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y}=\frac{\partial N}{\partial x}$$

Using result found above gives

$$\begin{aligned}\frac{\partial M}{\partial y}&=\frac{\partial}{\partial y}(x+y)\\&=1\end{aligned}$$

And

$$\frac{\partial N}{\partial x} = \frac{\partial}{\partial x}(-x+y)$$
$$= -1$$

Since $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$, then the ODE is not exact. By inspection $\frac{1}{x^2+y^2}$ is an integrating factor. Therefore by multiplying $M = x + y$ and $N = -x + y$ by this integrating factor the ode becomes exact. The new M, N are

$$M = \frac{x+y}{x^2+y^2}$$
$$N = \frac{-x+y}{x^2+y^2}$$

To solve an ode of the form

$$M(x,y) + N(x,y)\frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x,y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y) = 0$$

Hence

$$\frac{\partial \phi}{\partial x} + \frac{\partial \phi}{\partial y}\frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$\frac{\partial \phi}{\partial x} = M$$
$$\frac{\partial \phi}{\partial y} = N$$

But since $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x,y)$ but at least we know now that we can do that since the condition $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x,y)\,\mathrm{d}x + N(x,y)\,\mathrm{d}y = 0 \tag{1A}$$

Therefore

$$\left(\frac{-x+y}{x^2+y^2}\right)\mathrm{d}y = \left(-\frac{x+y}{x^2+y^2}\right)\mathrm{d}x$$

$$\left(\frac{x+y}{x^2+y^2}\right)\mathrm{d}x + \left(\frac{-x+y}{x^2+y^2}\right)\mathrm{d}y = 0 \tag{2A}$$

Comparing (1A) and (2A) shows that

$$M(x,y) = \frac{x+y}{x^2+y^2}$$
$$N(x,y) = \frac{-x+y}{x^2+y^2}$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Using result found above gives

$$\begin{aligned}\frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}\left(\frac{x+y}{x^2+y^2}\right)\\ &= \frac{x^2-2yx-y^2}{(x^2+y^2)^2}\end{aligned}$$

And

$$\begin{aligned}\frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}\left(\frac{-x+y}{x^2+y^2}\right)\\ &= \frac{x^2-2yx-y^2}{(x^2+y^2)^2}\end{aligned}$$

Since $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$, then the ODE is exact The following equations are now set up to solve for the function $\phi(x,y)$

$$\frac{\partial \phi}{\partial x} = M \tag{1}$$
$$\frac{\partial \phi}{\partial y} = N \tag{2}$$

Integrating (1) w.r.t. x gives

$$\int \frac{\partial \phi}{\partial x} \, \mathrm{d}x = \int M \, \mathrm{d}x$$

$$\int \frac{\partial \phi}{\partial x} \, \mathrm{d}x = \int \frac{x + y}{x^2 + y^2} \, \mathrm{d}x$$

$$\phi = \frac{\ln(x^2 + y^2)}{2} + \arctan\left(\frac{x}{y}\right) + f(y) \tag{3}$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$\frac{\partial \phi}{\partial y} = \frac{y}{x^2 + y^2} - \frac{x}{y^2 \left(\frac{x^2}{y^2} + 1\right)} + f'(y) \tag{4}$$

$$= \frac{-x + y}{x^2 + y^2} + f'(y)$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = \frac{-x+y}{x^2+y^2}$. Therefore equation (4) becomes

$$\frac{-x + y}{x^2 + y^2} = \frac{-x + y}{x^2 + y^2} + f'(y) \tag{5}$$

Solving equation (5) for $f'(y)$ gives

$$f'(y) = 0$$

Therefore

$$f(y) = c_1$$

Where c_1 is constant of integration. Substituting this result for $f(y)$ into equation (3) gives ϕ

$$\phi = \frac{\ln(x^2 + y^2)}{2} + \arctan\left(\frac{x}{y}\right) + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = \frac{\ln(x^2 + y^2)}{2} + \arctan\left(\frac{x}{y}\right)$$

Summary

The solution(s) found are the following

$$\frac{\ln(x^2+y^2)}{2}+\arctan\left(\frac{x}{y}\right)=c_1 \tag{1}$$

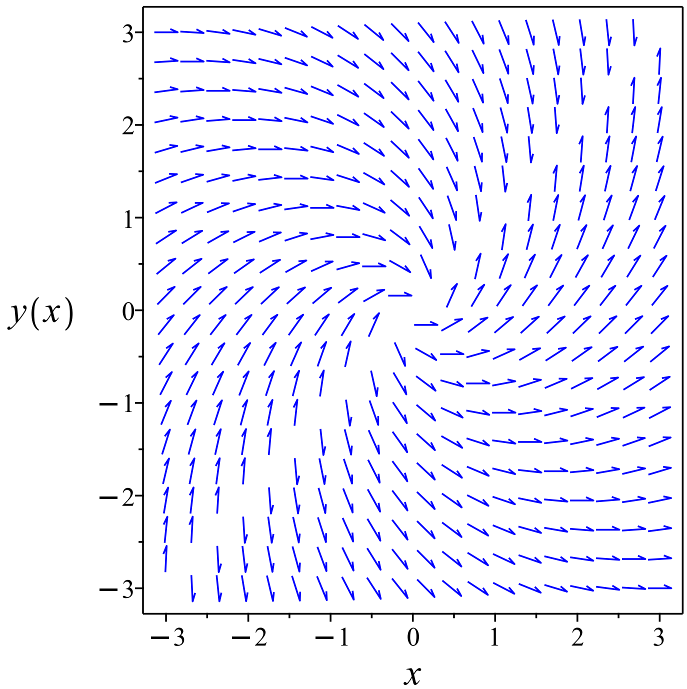


Figure 345: Slope field plot

Verification of solutions

$$\frac{\ln(x^2+y^2)}{2}+\arctan\left(\frac{x}{y}\right)=c_1$$

Verified OK.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
trying homogeneous types:
trying homogeneous D
<- homogeneous successful`
```

✓ Solution by Maple
Time used: 0.016 (sec). Leaf size: 24

```
dsolve(diff(y(x),x) = (x+y(x))/(x-y(x)),y(x), singsol=all)
```

$$y(x) = \tan\left(\text{RootOf}\left(-2\_Z + \ln\left(\sec\left(\_Z\right)^2\right) + 2\ln\left(x\right) + 2c_1\right)\right)x$$

✓ Solution by Mathematica
Time used: 0.033 (sec). Leaf size: 36

```
DSolve[y'[x] == (x+y[x])/(x-y[x]),y[x],x,IncludeSingularSolutions -> True]
```

$$\text{Solve}\left[\frac{1}{2}\log\left(\frac{y(x)^2}{x^2}+1\right) - \arctan\left(\frac{y(x)}{x}\right) = -\log(x) + c_1, y(x)\right]$$

6.30 problem 30

Internal problem ID [597]
Internal file name [`OUTPUT/597_Sunday_June_05_2022_01_45_33_AM_62087244/index.tex`]

Book: Elementary differential equations and boundary value problems, 10th ed., Boyce and DiPrima
Section: Miscellaneous problems, end of chapter 2. Page 133
Problem number: 30.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"homogeneousTypeD2", "exactWithIntegrationFactor", "first_order_ode_lie_symmetry_calculated"**

Maple gives the following as the ode type

```
[[_homogeneous, `class A`], _rational, [_Abel, `2nd type`, `
   class B`]]
```

$$\boxed{2yx + 3y^2 - \left(x^2 + 2yx\right) y' = 0}$$

6.30.1 Solving as homogeneousTypeD2 ode

Using the change of variables $y = u(x)\, x$ on the above ode results in new ode in $u(x)$

$$2u(x)\, x^2 + 3u(x)^2\, x^2 - \left(x^2 + 2u(x)\, x^2\right) \left(u'(x)\, x + u(x)\right) = 0$$

In canonical form the ODE is

$$\begin{aligned} u' &= F(x, u) \\ &= f(x) g(u) \\ &= \frac{u(u+1)}{(2u+1)\, x} \end{aligned}$$

Where $f(x) = \frac{1}{x}$ and $g(u) = \frac{u(u+1)}{2u+1}$. Integrating both sides gives

$$
\begin{aligned}
\frac{1}{\frac{u(u+1)}{2u+1}}\,du &= \frac{1}{x}\,dx \\
\int \frac{1}{\frac{u(u+1)}{2u+1}}\,du &= \int \frac{1}{x}\,dx \\
\ln\left(u(u+1)\right) &= \ln\left(x\right) + c_2
\end{aligned}
$$

Raising both side to exponential gives

$$
u(u+1) = \mathrm{e}^{\ln(x)+c_2}
$$

Which simplifies to

$$
u(u+1) = c_3 x
$$

Which simplifies to

$$
u(x)\left(u(x)+1\right) = c_3 \mathrm{e}^{c_2} x
$$

The solution is

$$
u(x)\left(u(x)+1\right) = c_3 \mathrm{e}^{c_2} x
$$

Replacing $u(x)$ in the above solution by $\frac{y}{x}$ results in the solution for y in implicit form

$$
\begin{aligned}
\frac{y\left(1+\frac{y}{x}\right)}{x} &= c_3 \mathrm{e}^{c_2} x \\
\frac{y(x+y)}{x^2} &= c_3 \mathrm{e}^{c_2} x
\end{aligned}
$$

<u>Summary</u>

The solution(s) found are the following

$$
\frac{y(x+y)}{x^2} = c_3 \mathrm{e}^{c_2} x \tag{1}
$$

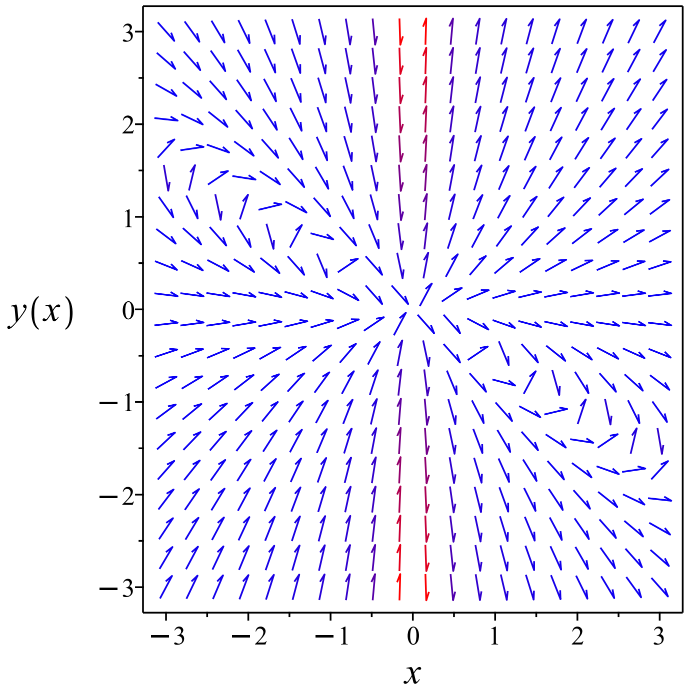


Figure 346: Slope field plot

Verification of solutions

$$\frac{y(x+y)}{x^2} = c_3 e^{c_2} x$$

Verified OK.

6.30.2 Solving as first order ode lie symmetry calculated ode

Writing the ode as

$$y' = \frac{y(2x+3y)}{x(x+2y)}$$
$$y' = \omega(x,y)$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x + \omega(\eta_y - \xi_x) - \omega^2 \xi_y - \omega_x \xi - \omega_y \eta = 0 \tag{A}$$

The type of this ode is not in the lookup table. To determine ξ, η then (A) is solved using ansatz. Making bivariate polynomials of degree 1 to use as anstaz gives

$$\xi = x a_2 + y a_3 + a_1 \tag{1E}$$
$$\eta = x b_2 + y b_3 + b_1 \tag{2E}$$

Where the unknown coefficients are

$$\{a_1, a_2, a_3, b_1, b_2, b_3\}$$

Substituting equations (1E,2E) and ω into (A) gives

$$\begin{aligned}
&b_2 + \frac{y(2x+3y)\,(b_3-a_2)}{x\,(x+2y)} - \frac{y^2(2x+3y)^2\,a_3}{x^2\,(x+2y)^2} \\
&- \left(\frac{2y}{x\,(x+2y)} - \frac{y(2x+3y)}{x^2\,(x+2y)} - \frac{y(2x+3y)}{x\,(x+2y)^2}\right)(xa_2+ya_3+a_1) \\
&- \left(\frac{2x+3y}{x\,(x+2y)} + \frac{3y}{x\,(x+2y)} - \frac{2y(2x+3y)}{x\,(x+2y)^2}\right)(xb_2+yb_3+b_1) = 0
\end{aligned} \tag{5E}$$

Putting the above in normal form gives

$$-\frac{x^4b_2 + 2x^3yb_2 + x^2y^2a_2 + 2x^2y^2a_3 + 2x^2y^2b_2 - x^2y^2b_3 + 6x\,y^3a_3 + 3y^4a_3 + 2x^3b_1 - 2x^2ya_1 + 6x^2yb_1 - }{x^2\,(x+2y)^2} = 0$$

Setting the numerator to zero gives

$$\begin{aligned}
&-x^4b_2 - 2x^3yb_2 - x^2y^2a_2 - 2x^2y^2a_3 - 2x^2y^2b_2 + x^2y^2b_3 - 6x\,y^3a_3 \\
&- 3y^4a_3 - 2x^3b_1 + 2x^2ya_1 - 6x^2yb_1 + 6x\,y^2a_1 - 6x\,y^2b_1 + 6y^3a_1 = 0
\end{aligned} \tag{6E}$$

Looking at the above PDE shows the following are all the terms with $\{x, y\}$ in them.

$$\{x, y\}$$

The following substitution is now made to be able to collect on all terms with $\{x, y\}$ in them

$$\{x = v_1, y = v_2\}$$

The above PDE (6E) now becomes

$$\begin{aligned}
&-a_2v_1^2v_2^2 - 2a_3v_1^2v_2^2 - 6a_3v_1v_2^3 - 3a_3v_2^4 - b_2v_1^4 - 2b_2v_1^3v_2 - 2b_2v_1^2v_2^2 \\
&+ b_3v_1^2v_2^2 + 2a_1v_1^2v_2 + 6a_1v_1v_2^2 + 6a_1v_2^3 - 2b_1v_1^3 - 6b_1v_1^2v_2 - 6b_1v_1v_2^2 = 0
\end{aligned} \tag{7E}$$

Collecting the above on the terms v_i introduced, and these are

$$\{v_1, v_2\}$$

Equation (7E) now becomes

$$\begin{aligned}
&-b_2 v_1^4 - 2b_2 v_1^3 v_2 - 2b_1 v_1^3 + (-a_2 - 2a_3 - 2b_2 + b_3) v_1^2 v_2^2 \\
&+ (2a_1 - 6b_1) v_1^2 v_2 - 6a_3 v_1 v_2^3 + (6a_1 - 6b_1) v_1 v_2^2 - 3a_3 v_2^4 + 6a_1 v_2^3 = 0
\end{aligned} \tag{8E}$$

Setting each coefficients in (8E) to zero gives the following equations to solve

$$\begin{aligned}
6a_1 &= 0 \\
-6a_3 &= 0 \\
-3a_3 &= 0 \\
-2b_1 &= 0 \\
-2b_2 &= 0 \\
-b_2 &= 0 \\
2a_1 - 6b_1 &= 0 \\
6a_1 - 6b_1 &= 0 \\
-a_2 - 2a_3 - 2b_2 + b_3 &= 0
\end{aligned}$$

Solving the above equations for the unknowns gives

$$\begin{aligned}
a_1 &= 0 \\
a_2 &= b_3 \\
a_3 &= 0 \\
b_1 &= 0 \\
b_2 &= 0 \\
b_3 &= b_3
\end{aligned}$$

Substituting the above solution in the anstaz (1E,2E) (using 1 as arbitrary value for any unknown in the RHS) gives

$$\begin{aligned}
\xi &= x \\
\eta &= y
\end{aligned}$$

Shifting is now applied to make $\xi = 0$ in order to simplify the rest of the computation

$$\begin{aligned}
\eta &= \eta - \omega(x,y)\,\xi \\
&= y - \left(\frac{y(2x+3y)}{x\,(x+2y)}\right)(x) \\
&= \frac{-yx-y^2}{x+2y} \\
\xi &= 0
\end{aligned}$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x,y) \to (R,S)$ where (R,S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y}\right) S(x,y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$R = x$$

S is found from

$$\begin{aligned}
S &= \int \frac{1}{\eta} dy \\
&= \int \frac{1}{\frac{-yx-y^2}{x+2y}} dy
\end{aligned}$$

Which results in

$$S = -\ln\left(y(x+y)\right)$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x,y)S_y}{R_x + \omega(x,y)R_y} \tag{2}$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x,y)$ is the right hand side of the original ode given by

$$\omega(x,y) = \frac{y(2x+3y)}{x\,(x+2y)}$$

Evaluating all the partial derivatives gives

$$
\begin{aligned}
R_x &= 1 \\
R_y &= 0 \\
S_x &= -\frac{1}{x+y} \\
S_y &= \frac{-2y-x}{y(x+y)}
\end{aligned}
$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$
\frac{dS}{dR} = -\frac{3}{x} \tag{2A}
$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$
\frac{dS}{dR} = -\frac{3}{R}
$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$
S(R) = -3\ln(R) + c_1 \tag{4}
$$

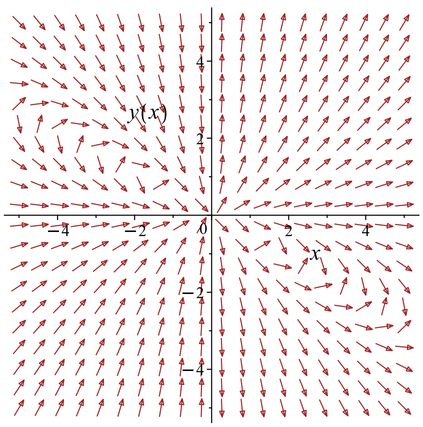
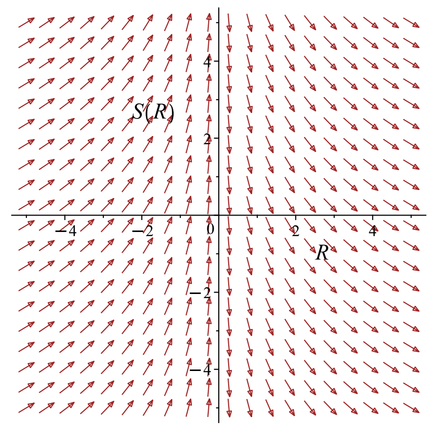
To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$
-\ln(y) - \ln(x+y) = -3\ln(x) + c_1
$$

Which simplifies to

$$
-\ln(y) - \ln(x+y) = -3\ln(x) + c_1
$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in x, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dx} = \frac{y(2x+3y)}{x(x+2y)}$ 	$R = x$ $S = -\ln(y) - \ln(x+y)$	$\frac{dS}{dR} = -\frac{3}{R}$ 

Summary

The solution(s) found are the following

$$-\ln(y) - \ln(x+y) = -3\ln(x) + c_1 \tag{1}$$

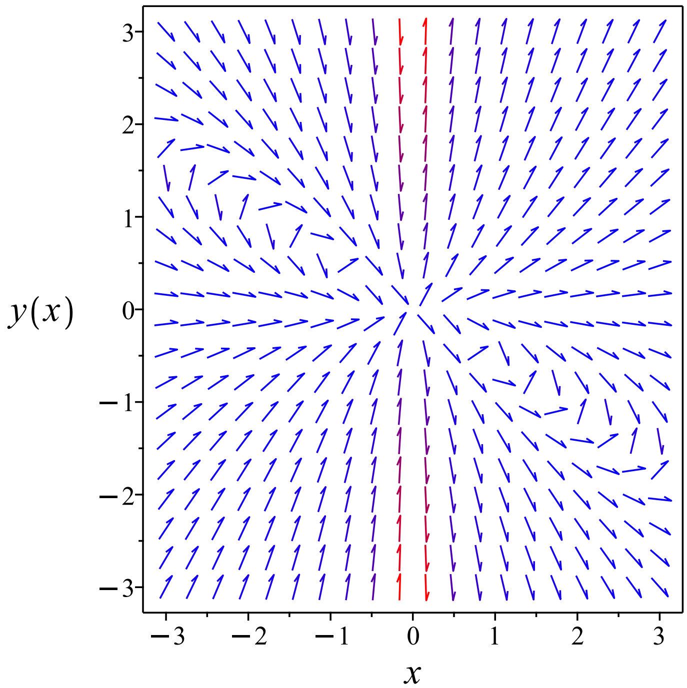


Figure 347: Slope field plot

Verification of solutions

$$-\ln(y) - \ln(x+y) = -3\ln(x) + c_1$$

Verified OK.

6.30.3 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x,y) + N(x,y)\frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x,y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y) = 0$$

Hence

$$\frac{\partial\phi}{\partial x} + \frac{\partial\phi}{\partial y}\frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$\frac{\partial \phi}{\partial x} = M$$
$$\frac{\partial \phi}{\partial y} = N$$

But since $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x, y)$ but at least we know now that we can do that since the condition $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x, y)\,\mathrm{d}x + N(x, y)\,\mathrm{d}y = 0 \tag{1A}$$

Therefore

$$\left(-x^2 - 2yx\right) \mathrm{d}y = \left(-2yx - 3y^2\right) \mathrm{d}x$$
$$\left(2yx + 3y^2\right) \mathrm{d}x + \left(-x^2 - 2yx\right) \mathrm{d}y = 0 \tag{2A}$$

Comparing (1A) and (2A) shows that

$$M(x, y) = 2yx + 3y^2$$
$$N(x, y) = -x^2 - 2yx$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Using result found above gives

$$\begin{aligned}\frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}\left(2yx + 3y^2\right) \\ &= 2x + 6y\end{aligned}$$

And

$$\begin{aligned}\frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}\left(-x^2 - 2yx\right) \\ &= -2x - 2y\end{aligned}$$

Since $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$, then the ODE is not exact. Since the ODE is not exact, we will try to find an integrating factor to make it exact. Let

$$\begin{aligned} A &= \frac{1}{N}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right) \\ &= -\frac{1}{x\left(x+2y\right)}\left(\left(2x+6y\right) - \left(-2x-2y\right)\right) \\ &= -\frac{4}{x} \end{aligned}$$

Since A does not depend on y, then it can be used to find an integrating factor. The integrating factor μ is

$$\begin{aligned} \mu &= e^{\int A \, \mathrm{d}x} \\ &= e^{\int -\frac{4}{x} \, \mathrm{d}x} \end{aligned}$$

The result of integrating gives

$$\begin{aligned} \mu &= e^{-4\ln(x)} \\ &= \frac{1}{x^4} \end{aligned}$$

M and N are multiplied by this integrating factor, giving new M and new N which are called $\overline{M}$ and $\overline{N}$ for now so not to confuse them with the original M and N.

$$\begin{aligned} \overline{M} &= \mu M \\ &= \frac{1}{x^4}\left(2yx+3y^2\right) \\ &= \frac{y(2x+3y)}{x^4} \end{aligned}$$

And

$$\begin{aligned} \overline{N} &= \mu N \\ &= \frac{1}{x^4}\left(-x^2-2yx\right) \\ &= \frac{-2y-x}{x^3} \end{aligned}$$

Now a modified ODE is ontained from the original ODE, which is exact and can be solved. The modified ODE is

$$\begin{aligned} \overline{M} + \overline{N}\frac{\mathrm{d}y}{\mathrm{d}x} &= 0 \\ \left(\frac{y(2x+3y)}{x^4}\right) + \left(\frac{-2y-x}{x^3}\right)\frac{\mathrm{d}y}{\mathrm{d}x} &= 0 \end{aligned}$$

The following equations are now set up to solve for the function $\phi(x, y)$

$$\frac{\partial \phi}{\partial x} = \overline{M} \tag{1}$$
$$\frac{\partial \phi}{\partial y} = \overline{N} \tag{2}$$

Integrating (1) w.r.t. x gives

$$\int \frac{\partial \phi}{\partial x} \, dx = \int \overline{M} \, dx$$
$$\int \frac{\partial \phi}{\partial x} \, dx = \int \frac{y(2x + 3y)}{x^4} \, dx$$
$$\phi = -\frac{y(x + y)}{x^3} + f(y) \tag{3}$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$\frac{\partial \phi}{\partial y} = -\frac{x + y}{x^3} - \frac{y}{x^3} + f'(y) \tag{4}$$
$$= \frac{-2y - x}{x^3} + f'(y)$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = \frac{-2y-x}{x^3}$. Therefore equation (4) becomes

$$\frac{-2y - x}{x^3} = \frac{-2y - x}{x^3} + f'(y) \tag{5}$$

Solving equation (5) for $f'(y)$ gives

$$f'(y) = 0$$

Therefore

$$f(y) = c_1$$

Where c_1 is constant of integration. Substituting this result for $f(y)$ into equation (3) gives ϕ

$$\phi = -\frac{y(x + y)}{x^3} + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = -\frac{y(x+y)}{x^3}$$

Summary
The solution(s) found are the following

$$-\frac{y(x+y)}{x^3} = c_1 \tag{1}$$

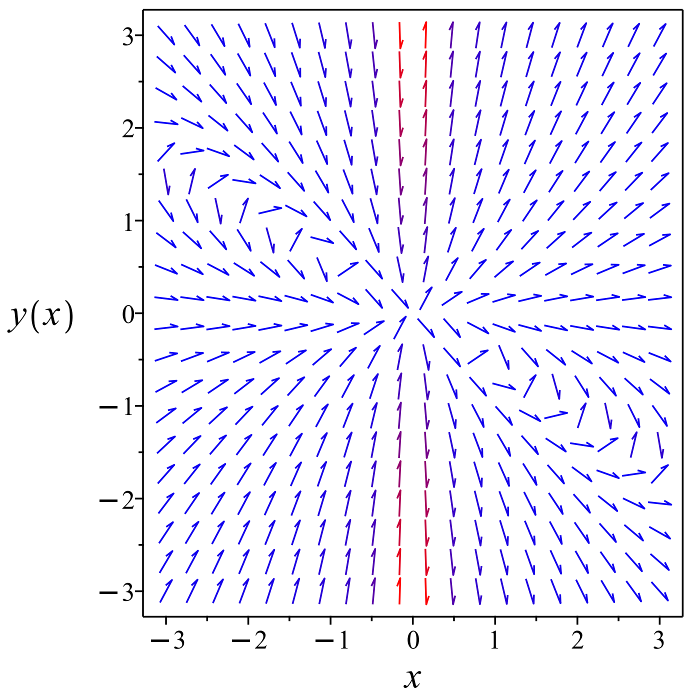


Figure 348: Slope field plot

Verification of solutions

$$-\frac{y(x+y)}{x^3} = c_1$$

Verified OK.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
trying homogeneous types:
trying homogeneous D
<- homogeneous successful`
```

✓ Solution by Maple

Time used: 0.015 (sec). Leaf size: 33

```
dsolve(2*x*y(x)+3*y(x)^2-(x^2+2*x*y(x))*diff(y(x),x) = 0,y(x), singsol=all)
```

$$y(x) = -\frac{\left(1+\sqrt{4c_1x+1}\right)x}{2}$$
$$y(x) = \frac{\left(-1+\sqrt{4c_1x+1}\right)x}{2}$$

✓ Solution by Mathematica

Time used: 0.408 (sec). Leaf size: 61

```
DSolve[2*x*y[x]+3*y[x]^2-(x^2+2*x*y[x])*y'[x] == 0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to -\frac{1}{2}x\left(1+\sqrt{1+4e^{c_1}x}\right)$$
$$y(x) \to \frac{1}{2}x\left(-1+\sqrt{1+4e^{c_1}x}\right)$$
$$y(x) \to 0$$
$$y(x) \to -x$$

6.31 problem 31

Internal problem ID [598]
Internal file name [`OUTPUT/598_Sunday_June_05_2022_01_45_35_AM_42029649/index.tex`]

Book: Elementary differential equations and boundary value problems, 10th ed., Boyce and DiPrima
Section: Miscellaneous problems, end of chapter 2. Page 133
Problem number: 31.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"exactWithIntegrationFactor", "first_order_ode_lie_symmetry_calculated"**

Maple gives the following as the ode type

```
[[_homogeneous, `class G`], _rational, [_Abel, `2nd type`, `
   class B`]]
```

$$\boxed{y' - \frac{-3x^2y - y^2}{2x^3 + 3yx} = 0}$$

With initial conditions

$$[y(1) = -2]$$

6.31.1 Solving as first order ode lie symmetry calculated ode

Writing the ode as

$$y' = -\frac{y(3x^2 + y)}{x\,(2x^2 + 3y)}$$
$$y' = \omega(x, y)$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x + \omega(\eta_y - \xi_x) - \omega^2\xi_y - \omega_x\xi - \omega_y\eta = 0 \tag{A}$$

The type of this ode is not in the lookup table. To determine ξ, η then (A) is solved using ansatz. Making bivariate polynomials of degree 1 to use as anstaz gives

$$\xi = xa_2 + ya_3 + a_1 \tag{1E}$$

$$\eta = xb_2 + yb_3 + b_1 \tag{2E}$$

Where the unknown coefficients are

$$\{a_1, a_2, a_3, b_1, b_2, b_3\}$$

Substituting equations (1E,2E) and ω into (A) gives

$$\begin{aligned} b_2 &- \frac{y(3x^2+y)(b_3-a_2)}{x(2x^2+3y)} - \frac{y^2(3x^2+y)^2 a_3}{x^2(2x^2+3y)^2} \\ &- \left(-\frac{6y}{2x^2+3y} + \frac{y(3x^2+y)}{x^2(2x^2+3y)} + \frac{4y(3x^2+y)}{(2x^2+3y)^2}\right)(xa_2+ya_3+a_1) \\ &- \left(-\frac{3x^2+y}{x(2x^2+3y)} - \frac{y}{x(2x^2+3y)} + \frac{3y(3x^2+y)}{x(2x^2+3y)^2}\right)(xb_2+yb_3+b_1) = 0 \end{aligned} \tag{5E}$$

Putting the above in normal form gives

$$\frac{10x^6b_2 - 15x^4y^2a_3 + 6x^5b_1 - 6x^4ya_1 + 16x^4yb_2 + 14x^3y^2a_2 - 7x^3y^2b_3 - 3x^2y^3a_3 + 4x^3yb_1 + 3x^2y^2a_1 + 1}{(2x^2+3y)^2 x^2}$$
$$= 0$$

Setting the numerator to zero gives

$$\begin{aligned} &10x^6b_2 - 15x^4y^2a_3 + 6x^5b_1 - 6x^4ya_1 + 16x^4yb_2 + 14x^3y^2a_2 - 7x^3y^2b_3 \\ &- 3x^2y^3a_3 + 4x^3yb_1 + 3x^2y^2a_1 + 12x^2y^2b_2 - 4y^4a_3 + 3x\,y^2b_1 - 3y^3a_1 = 0 \end{aligned} \tag{6E}$$

Looking at the above PDE shows the following are all the terms with $\{x, y\}$ in them.

$$\{x, y\}$$

The following substitution is now made to be able to collect on all terms with $\{x, y\}$ in them

$$\{x = v_1, y = v_2\}$$

The above PDE (6E) now becomes

$$
\begin{aligned}
-15a_3v_1^4v_2^2 + 10b_2v_1^6 - 6a_1v_1^4v_2 + 14a_2v_1^3v_2^2 - 3a_3v_1^2v_2^3 + 6b_1v_1^5 + 16b_2v_1^4v_2 \\
- 7b_3v_1^3v_2^2 + 3a_1v_1^2v_2^2 - 4a_3v_2^4 + 4b_1v_1^3v_2 + 12b_2v_1^2v_2^2 - 3a_1v_2^3 + 3b_1v_1v_2^2 = 0
\end{aligned} \tag{7E}
$$

Collecting the above on the terms v_i introduced, and these are

$$
\{v_1, v_2\}
$$

Equation (7E) now becomes

$$
\begin{aligned}
10b_2v_1^6 + 6b_1v_1^5 - 15a_3v_1^4v_2^2 + (-6a_1 + 16b_2)\, v_1^4v_2 + (14a_2 - 7b_3)\, v_1^3v_2^2 \\
+ 4b_1v_1^3v_2 - 3a_3v_1^2v_2^3 + (3a_1 + 12b_2)\, v_1^2v_2^2 + 3b_1v_1v_2^2 - 4a_3v_2^4 - 3a_1v_2^3 = 0
\end{aligned} \tag{8E}
$$

Setting each coefficients in (8E) to zero gives the following equations to solve

$$
\begin{aligned}
-3a_1 &= 0 \\
-15a_3 &= 0 \\
-4a_3 &= 0 \\
-3a_3 &= 0 \\
3b_1 &= 0 \\
4b_1 &= 0 \\
6b_1 &= 0 \\
10b_2 &= 0 \\
-6a_1 + 16b_2 &= 0 \\
3a_1 + 12b_2 &= 0 \\
14a_2 - 7b_3 &= 0
\end{aligned}
$$

Solving the above equations for the unknowns gives

$$
\begin{aligned}
a_1 &= 0 \\
a_2 &= a_2 \\
a_3 &= 0 \\
b_1 &= 0 \\
b_2 &= 0 \\
b_3 &= 2a_2
\end{aligned}
$$

Substituting the above solution in the anstaz (1E,2E) (using 1 as arbitrary value for any unknown in the RHS) gives

$$\xi = x$$
$$\eta = 2y$$

Shifting is now applied to make $\xi = 0$ in order to simplify the rest of the computation

$$\begin{aligned}\eta &= \eta - \omega(x, y)\,\xi \\ &= 2y - \left(-\frac{y(3x^2 + y)}{x\,(2x^2 + 3y)}\right)(x) \\ &= \frac{7y\,x^2 + 7y^2}{2x^2 + 3y} \\ \xi &= 0\end{aligned}$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x, y) \to (R, S)$ where (R, S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y}\right) S(x, y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$R = x$$

S is found from

$$\begin{aligned}S &= \int \frac{1}{\eta} dy \\ &= \int \frac{1}{\frac{7y\,x^2+7y^2}{2x^2+3y}} dy\end{aligned}$$

Which results in

$$S = \frac{\ln(x^2 + y)}{7} + \frac{2\ln(y)}{7}$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x, y)S_y}{R_x + \omega(x, y)R_y} \tag{2}$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x,y)$ is the right hand side of the original ode given by

$$\omega(x,y) = -\frac{y(3x^2+y)}{x\,(2x^2+3y)}$$

Evaluating all the partial derivatives gives

$$\begin{aligned}
R_x &= 1 \\
R_y &= 0 \\
S_x &= \frac{2x}{7x^2+7y} \\
S_y &= \frac{1}{7x^2+7y} + \frac{2}{7y}
\end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = -\frac{1}{7x} \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = -\frac{1}{7R}$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$S(R) = -\frac{\ln(R)}{7} + c_1 \tag{4}$$

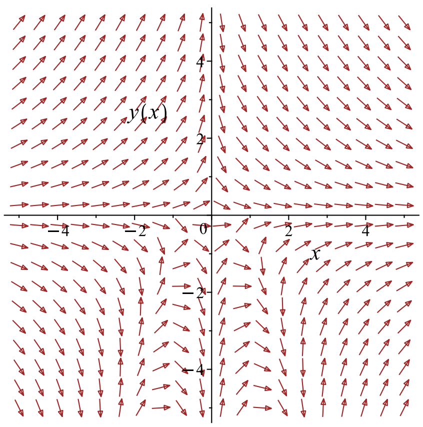
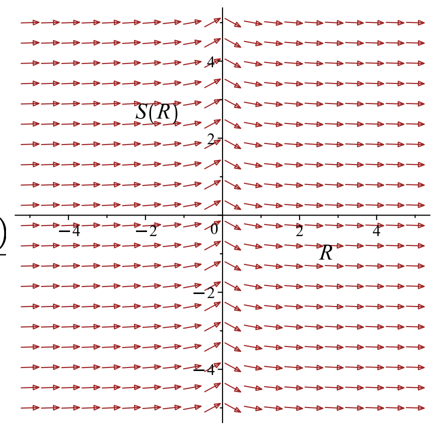
To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$\frac{\ln(x^2+y)}{7} + \frac{2\ln(y)}{7} = -\frac{\ln(x)}{7} + c_1$$

Which simplifies to

$$\frac{\ln(x^2+y)}{7} + \frac{2\ln(y)}{7} = -\frac{\ln(x)}{7} + c_1$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in x, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dx} = -\frac{y(3x^2+y)}{x(2x^2+3y)}$ 	$R = x$ $S = \frac{\ln(x^2+y)}{7} + \frac{2\ln(y)}{7}$	$\frac{dS}{dR} = -\frac{1}{7R}$ 

Initial conditions are used to solve for c_1. Substituting $x = 1$ and $y = -2$ in the above solution gives an equation to solve for the constant of integration.

$$\frac{3i\pi}{7} + \frac{2\ln(2)}{7} = c_1$$

$$c_1 = \frac{3i\pi}{7} + \frac{2\ln(2)}{7}$$

Substituting c_1 found above in the general solution gives

$$\frac{\ln(x^2+y)}{7} + \frac{2\ln(y)}{7} = -\frac{\ln(x)}{7} + \frac{3i\pi}{7} + \frac{2\ln(2)}{7}$$

Summary

The solution(s) found are the following

$$\frac{\ln(x^2+y)}{7} + \frac{2\ln(y)}{7} = -\frac{\ln(x)}{7} + \frac{3i\pi}{7} + \frac{2\ln(2)}{7} \tag{1}$$

Verification of solutions

$$\frac{\ln(x^2+y)}{7} + \frac{2\ln(y)}{7} = -\frac{\ln(x)}{7} + \frac{3i\pi}{7} + \frac{2\ln(2)}{7}$$

Verified OK.

6.31.2 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x,y)+N(x,y)\frac{dy}{dx}=0 \tag{A}$$

We assume there exists a function $\phi(x,y)=c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y)=0$$

Hence

$$\frac{\partial\phi}{\partial x}+\frac{\partial\phi}{\partial y}\frac{dy}{dx}=0 \tag{B}$$

Comparing (A,B) shows that

$$\begin{aligned}\frac{\partial\phi}{\partial x}&=M\\ \frac{\partial\phi}{\partial y}&=N\end{aligned}$$

But since $\frac{\partial^2\phi}{\partial x\partial y}=\frac{\partial^2\phi}{\partial y\partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y}=\frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x,y)$ but at least we know now that we can do that since the condition $\frac{\partial^2\phi}{\partial x\partial y}=\frac{\partial^2\phi}{\partial y\partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x,y)\,\mathrm{d}x+N(x,y)\,\mathrm{d}y=0 \tag{1A}$$

Therefore

$$\begin{aligned}\left(x\left(2x^2+3y\right)\right)\mathrm{d}y&=\left(-y\left(3x^2+y\right)\right)\mathrm{d}x\\ \left(y\left(3x^2+y\right)\right)\mathrm{d}x+\left(x\left(2x^2+3y\right)\right)\mathrm{d}y&=0\end{aligned} \tag{2A}$$

Comparing (1A) and (2A) shows that

$$\begin{aligned}M(x,y)&=y\left(3x^2+y\right)\\ N(x,y)&=x\left(2x^2+3y\right)\end{aligned}$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Using result found above gives

$$\begin{aligned}\frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}\big(y(3x^2 + y)\big)\\ &= 3x^2 + 2y\end{aligned}$$

And

$$\begin{aligned}\frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}\big(x(2x^2 + 3y)\big)\\ &= 6x^2 + 3y\end{aligned}$$

Since $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$, then the ODE is <u>not exact</u>. Since the ODE is not exact, we will try to find an integrating factor to make it exact. Let

$$\begin{aligned}A &= \frac{1}{N}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right)\\ &= \frac{1}{2x^3 + 3yx}\big((3x^2 + 2y) - (6x^2 + 3y)\big)\\ &= \frac{-3x^2 - y}{2x^3 + 3yx}\end{aligned}$$

Since A depends on y, it can not be used to obtain an integrating factor. We will now try a second method to find an integrating factor. Let

$$\begin{aligned}B &= \frac{1}{M}\left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right)\\ &= \frac{1}{3y\,x^2 + y^2}\big((6x^2 + 3y) - (3x^2 + 2y)\big)\\ &= \frac{1}{y}\end{aligned}$$

Since B does not depend on x, it can be used to obtain an integrating factor. Let the integrating factor be μ. Then

$$\begin{aligned}\mu &= e^{\int B \,\mathrm{d}y}\\ &= e^{\int \frac{1}{y} \,\mathrm{d}y}\end{aligned}$$

The result of integrating gives

$$
\begin{aligned}
\mu &= e^{\ln(y)} \\
&= y
\end{aligned}
$$

M and N are now multiplied by this integrating factor, giving new M and new N which are called $\overline{M}$ and $\overline{N}$ so not to confuse them with the original M and N.

$$
\begin{aligned}
\overline{M} &= \mu M \\
&= y\left(y\left(3x^2 + y\right)\right) \\
&= y^2\left(3x^2 + y\right)
\end{aligned}
$$

And

$$
\begin{aligned}
\overline{N} &= \mu N \\
&= y\left(x\left(2x^2 + 3y\right)\right) \\
&= yx\left(2x^2 + 3y\right)
\end{aligned}
$$

So now a modified ODE is obtained from the original ODE which will be exact and can be solved using the standard method. The modified ODE is

$$
\begin{aligned}
\overline{M} + \overline{N}\frac{\mathrm{d}y}{\mathrm{d}x} &= 0 \\
\left(y^2\left(3x^2 + y\right)\right) + \left(yx\left(2x^2 + 3y\right)\right)\frac{\mathrm{d}y}{\mathrm{d}x} &= 0
\end{aligned}
$$

The following equations are now set up to solve for the function $\phi(x, y)$

$$
\frac{\partial \phi}{\partial x} = \overline{M} \tag{1}
$$

$$
\frac{\partial \phi}{\partial y} = \overline{N} \tag{2}
$$

Integrating (1) w.r.t. x gives

$$
\begin{aligned}
\int \frac{\partial \phi}{\partial x}\,\mathrm{d}x &= \int \overline{M}\,\mathrm{d}x \\
\int \frac{\partial \phi}{\partial x}\,\mathrm{d}x &= \int y^2\left(3x^2 + y\right)\mathrm{d}x
\end{aligned}
$$

$$
\phi = y^2x\left(x^2 + y\right) + f(y) \tag{3}
$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$
\begin{aligned}
\frac{\partial \phi}{\partial y} &= 2yx(x^2+y) + x\,y^2 + f'(y) \\
&= yx(2x^2+3y) + f'(y)
\end{aligned} \tag{4}
$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = yx(2x^2+3y)$. Therefore equation (4) becomes

$$
yx(2x^2+3y) = yx(2x^2+3y) + f'(y) \tag{5}
$$

Solving equation (5) for $f'(y)$ gives

$$
f'(y) = 0
$$

Therefore

$$
f(y) = c_1
$$

Where c_1 is constant of integration. Substituting this result for $f(y)$ into equation (3) gives ϕ

$$
\phi = y^2x(x^2+y) + c_1
$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$
c_1 = y^2x(x^2+y)
$$

Initial conditions are used to solve for c_1. Substituting $x = 1$ and $y = -2$ in the above solution gives an equation to solve for the constant of integration.

$$
-4 = c_1
$$

$$
c_1 = -4
$$

Substituting c_1 found above in the general solution gives

$$
y^2x(x^2+y) = -4
$$

Summary

The solution(s) found are the following

$$
y^2x(x^2+y) = -4 \tag{1}
$$

Verification of solutions

$$y^2x(x^2+y) = -4$$

Verified OK.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
trying homogeneous types:
trying homogeneous G
<- homogeneous successful`
```

✓ Solution by Maple

Time used: 1.266 (sec). Leaf size: 111

```
dsolve([diff(y(x),x) = (-3*x^2*y(x)-y(x)^2)/(2*x^3+3*x*y(x)),y(1) = -2],y(x), singsol=all)
```

$$y(x) = \frac{\left(i\sqrt{3}-1\right)\left(-\left(x^7-6\sqrt{3}\,\sqrt{x^7+27}+54\right)x^2\right)^{\frac{2}{3}} - x^3\left(i\sqrt{3}\,x^3+x^3+2\left(-\left(x^7-6\sqrt{3}\,\sqrt{x^7+27}+54\right)x^2\right)^{\frac{1}{3}}\right.}{6\left(-\left(x^7-6\sqrt{3}\,\sqrt{x^7+27}+54\right)x^2\right)^{\frac{1}{3}}x}$$

✓ Solution by Mathematica

Time used: 40.923 (sec). Leaf size: 136

```
DSolve[{y'[x]== (-3*x^2*y[x]-y[x]^2)/(2*x^3+3*x*y[x]),y[1]==-2},y[x],x,IncludeSingularSoluti
```

$$y(x) \to \frac{i\left(\left(\sqrt{3}+i\right)x^3-\left(\sqrt{3}-i\right)x^3+\left(\sqrt{3}+i\right)\sqrt[3]{-x^9-54x^2+6\sqrt{3}\sqrt{x^4\left(x^7+27\right)}}-\frac{\left(\sqrt{3}-i\right)x^6}{\sqrt[3]{-x^9-54x^2+6\sqrt{3}}}\right.}{6x}$$

7 Chapter 3, Second order linear equations, 3.1 Homogeneous Equations with Constant Coefficients, page 144

7.1 problem 1

Internal problem ID [599]
Internal file name [`OUTPUT/599_Sunday_June_05_2022_01_45_39_AM_9710673/index.tex`]

Book: Elementary differential equations and boundary value problems, 10th ed., Boyce and DiPrima
Section: Chapter 3, Second order linear equations, 3.1 Homogeneous Equations with Constant Coefficients, page 144
Problem number: 1.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_linear_constant_coeff"**

Maple gives the following as the ode type

```
[[_2nd_order, _missing_x]]
```

$$\boxed{y'' + 2y' - 3y = 0}$$

7.1.1 Solving as second order linear constant coeff ode

This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = 0$$

Where in the above $A = 1, B = 2, C = -3$. Let the solution be $y = e^{\lambda x}$. Substituting this into the ODE gives

$$\lambda^2 e^{\lambda x} + 2\lambda e^{\lambda x} - 3 e^{\lambda x} = 0 \tag{1}$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda x}$ gives

$$\lambda^2 + 2\lambda - 3 = 0 \tag{2}$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form.Using the quadratic formula

$$\lambda_{1,2} = \frac{-B}{2A} \pm \frac{1}{2A}\sqrt{B^2 - 4AC}$$

Substituting $A = 1, B = 2, C = -3$ into the above gives

$$\lambda_{1,2} = \frac{-2}{(2)(1)} \pm \frac{1}{(2)(1)} \sqrt{2^2 - (4)(1)(-3)}$$
$$= -1 \pm 2$$

Hence

$$\lambda_1 = -1 + 2$$
$$\lambda_2 = -1 - 2$$

Which simplifies to

$$\lambda_1 = 1$$
$$\lambda_2 = -3$$

Since roots are real and distinct, then the solution is

$$y = c_1 e^{\lambda_1 x} + c_2 e^{\lambda_2 x}$$
$$y = c_1 e^{(1)x} + c_2 e^{(-3)x}$$

Or

$$y = c_1 \mathrm{e}^{x} + \mathrm{e}^{-3x} c_2$$

Summary

The solution(s) found are the following

$$y = c_1 \mathrm{e}^{x} + \mathrm{e}^{-3x} c_2 \tag{1}$$

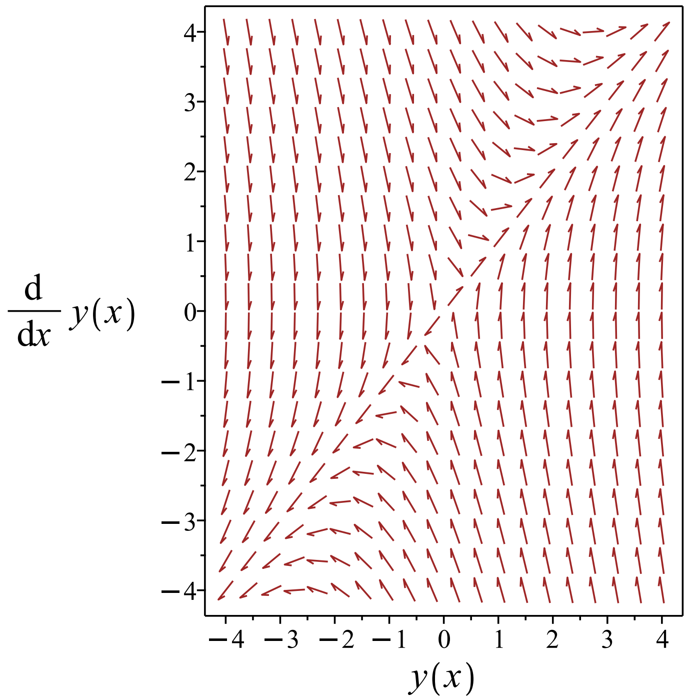


Figure 349: Slope field plot

Verification of solutions

$$y = c_1 e^{x} + e^{-3x} c_2$$

Verified OK.

7.1.2 Solving using Kovacic algorithm

Writing the ode as

$$y'' + 2y' - 3y = 0 \tag{1}$$

$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1 \\ B &= 2 \\ C &= -3 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\, dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{4}{1} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= 4 \\ t &= 1 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = 4z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 316: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 0 - 0 \\ &= 0 \end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Since $r = 4$ is not a function of x, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$z_1(x) = \mathrm{e}^{-2x}$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$\begin{aligned} y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx} \\ &= z_1 e^{-\int \frac{1}{2}\frac{2}{1}\,dx} \\ &= z_1 e^{-x} \\ &= z_1\left(\mathrm{e}^{-x}\right) \end{aligned}$$

Which simplifies to

$$y_1 = \mathrm{e}^{-3x}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx$$

Substituting gives

$$\begin{aligned} y_2 &= y_1 \int \frac{e^{\int -\frac{2}{1}\,dx}}{(y_1)^2}\,dx \\ &= y_1 \int \frac{e^{-2x}}{(y_1)^2}\,dx \\ &= y_1\left(\frac{\mathrm{e}^{4x}}{4}\right) \end{aligned}$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1 \left(\mathrm{e}^{-3x}\right) + c_2 \left(\mathrm{e}^{-3x} \left(\frac{\mathrm{e}^{4x}}{4}\right)\right)
\end{aligned}
$$

Summary

The solution(s) found are the following

$$
y = c_1 \mathrm{e}^{-3x} + \frac{c_2 \mathrm{e}^{x}}{4} \tag{1}
$$

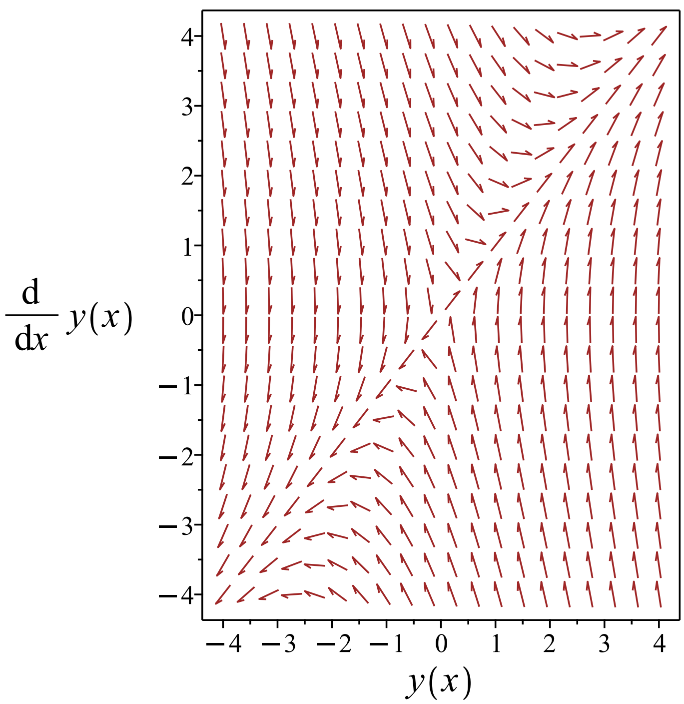


Figure 350: Slope field plot

Verification of solutions

$$
y = c_1 \mathrm{e}^{-3x} + \frac{c_2 \mathrm{e}^{x}}{4}
$$

Verified OK.

7.1.3 Maple step by step solution

Let's solve

$y'' + 2y' - 3y = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Characteristic polynomial of ODE

  $r^2 + 2r - 3 = 0$

- Factor the characteristic polynomial

  $(r + 3)(r - 1) = 0$

- Roots of the characteristic polynomial

  $r = (-3, 1)$

- 1st solution of the ODE

  $y_1(x) = \mathrm{e}^{-3x}$

- 2nd solution of the ODE

  $y_2(x) = \mathrm{e}^{x}$

- General solution of the ODE

  $y = c_1 y_1(x) + c_2 y_2(x)$

- Substitute in solutions

  $y = c_1 \mathrm{e}^{-3x} + c_2 \mathrm{e}^{x}$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
<- constant coefficients successful`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 17

```
dsolve(diff(y(x),x$2) +2*diff(y(x),x)-3*y(x) = 0,y(x), singsol=all)
```

$$y(x) = \left(c_1 e^{4x} + c_2\right) e^{-3x}$$

✓ Solution by Mathematica

Time used: 0.011 (sec). Leaf size: 20

```
DSolve[y''[x]+2*y'[x]-3*y[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to c_1 e^{-3x} + c_2 e^x$$

7.2 problem 2

Internal problem ID [600]
Internal file name [`OUTPUT/600_Sunday_June_05_2022_01_45_39_AM_98928217/index.tex`]

Book: Elementary differential equations and boundary value problems, 10th ed., Boyce and DiPrima
Section: Chapter 3, Second order linear equations, 3.1 Homogeneous Equations with Constant Coefficients, page 144
Problem number: 2.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_linear_constant_coeff"**

Maple gives the following as the ode type

```
[[_2nd_order, _missing_x]]
```

$$\boxed{y'' + 3y' + 2y = 0}$$

7.2.1 Solving as second order linear constant coeff ode

This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = 0$$

Where in the above $A = 1, B = 3, C = 2$. Let the solution be $y = e^{\lambda x}$. Substituting this into the ODE gives

$$\lambda^2 e^{\lambda x} + 3\lambda\, e^{\lambda x} + 2\, e^{\lambda x} = 0 \tag{1}$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda x}$ gives

$$\lambda^2 + 3\lambda + 2 = 0 \tag{2}$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form.Using the quadratic formula

$$\lambda_{1,2} = \frac{-B}{2A} \pm \frac{1}{2A}\sqrt{B^2 - 4AC}$$

Substituting $A = 1, B = 3, C = 2$ into the above gives

$$\begin{aligned}\lambda_{1,2} &= \frac{-3}{(2)(1)} \pm \frac{1}{(2)(1)} \sqrt{3^2 - (4)(1)(2)} \\ &= -\frac{3}{2} \pm \frac{1}{2}\end{aligned}$$

Hence

$$\begin{aligned}\lambda_1 &= -\frac{3}{2} + \frac{1}{2} \\ \lambda_2 &= -\frac{3}{2} - \frac{1}{2}\end{aligned}$$

Which simplifies to

$$\begin{aligned}\lambda_1 &= -1 \\ \lambda_2 &= -2\end{aligned}$$

Since roots are real and distinct, then the solution is

$$\begin{aligned}y &= c_1 e^{\lambda_1 x} + c_2 e^{\lambda_2 x} \\ y &= c_1 e^{(-1)x} + c_2 e^{(-2)x}\end{aligned}$$

Or

$$y = c_1 \mathrm{e}^{-x} + c_2 \mathrm{e}^{-2x}$$

Summary

The solution(s) found are the following

$$y = c_1 \mathrm{e}^{-x} + c_2 \mathrm{e}^{-2x} \tag{1}$$

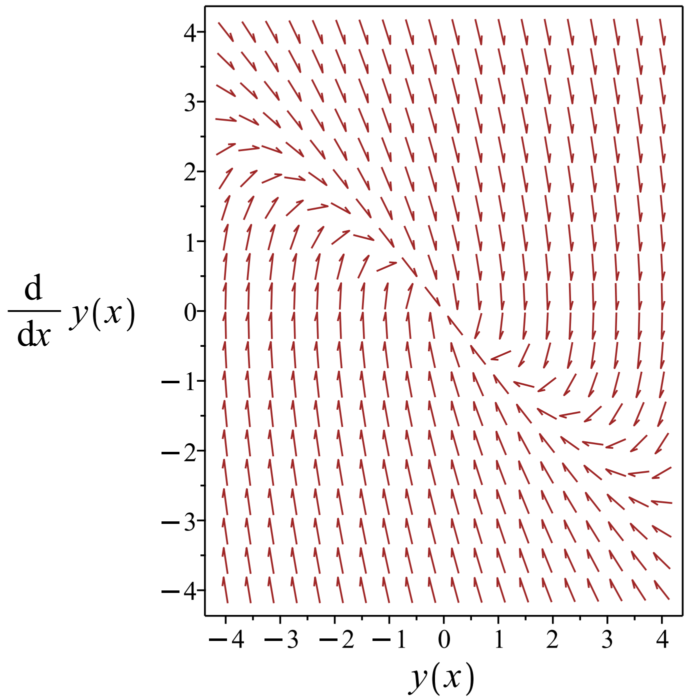


Figure 351: Slope field plot

Verification of solutions

$$y = c_1 e^{-x} + c_2 e^{-2x}$$

Verified OK.

7.2.2 Solving using Kovacic algorithm

Writing the ode as

$$y'' + 3y' + 2y = 0 \tag{1}$$

$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1 \\ B &= 3 \\ C &= 2 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = y e^{\int \frac{B}{2A} \, dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{1}{4} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= 1 \\ t &= 4 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = \frac{z(x)}{4} \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x) e^{-\int \frac{B}{2A} dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 318: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned}O(\infty) &= \deg(t) - \deg(s) \\ &= 0 - 0 \\ &= 0\end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Since $r = \frac{1}{4}$ is not a function of x, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$z_1(x) = \mathrm{e}^{-\frac{x}{2}}$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$\begin{aligned}y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx} \\ &= z_1 e^{-\int \frac{1}{2}\frac{3}{1}\,dx} \\ &= z_1 e^{-\frac{3x}{2}} \\ &= z_1\left(\mathrm{e}^{-\frac{3x}{2}}\right)\end{aligned}$$

Which simplifies to

$$y_1 = \mathrm{e}^{-2x}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx$$

Substituting gives

$$\begin{aligned}y_2 &= y_1 \int \frac{e^{\int -\frac{3}{1}\,dx}}{(y_1)^2}\,dx \\ &= y_1 \int \frac{e^{-3x}}{(y_1)^2}\,dx \\ &= y_1(\mathrm{e}^{x})\end{aligned}$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1\left(\mathrm{e}^{-2x}\right) + c_2\left(\mathrm{e}^{-2x}\left(\mathrm{e}^{x}\right)\right)
\end{aligned}
$$

Summary

The solution(s) found are the following

$$
y = c_1 \mathrm{e}^{-2x} + c_2 \mathrm{e}^{-x} \tag{1}
$$

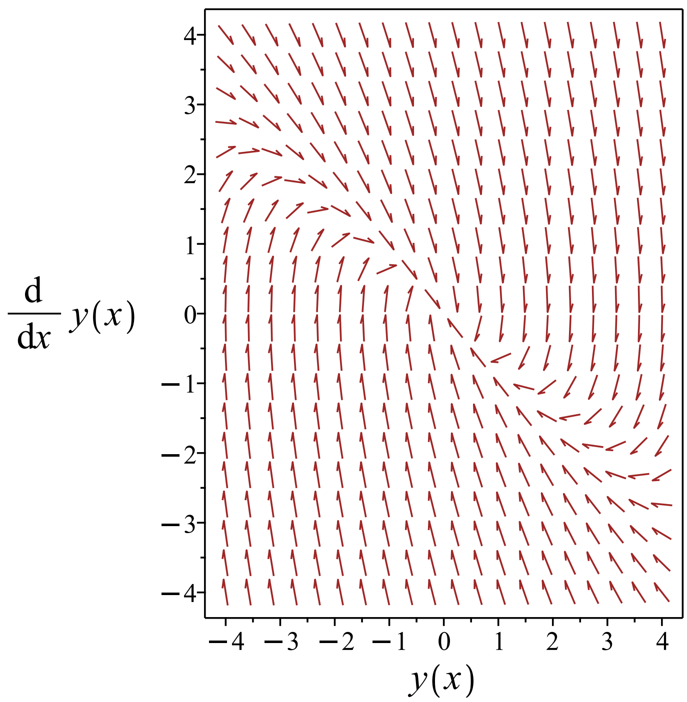


Figure 352: Slope field plot

Verification of solutions

$$
y = c_1 \mathrm{e}^{-2x} + c_2 \mathrm{e}^{-x}
$$

Verified OK.

7.2.3 Maple step by step solution

Let's solve

$y'' + 3y' + 2y = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Characteristic polynomial of ODE

  $r^2 + 3r + 2 = 0$

- Factor the characteristic polynomial

  $(r + 2)(r + 1) = 0$

- Roots of the characteristic polynomial

  $r = (-2, -1)$

- 1st solution of the ODE

  $y_1(x) = e^{-2x}$

- 2nd solution of the ODE

  $y_2(x) = e^{-x}$

- General solution of the ODE

  $y = c_1 y_1(x) + c_2 y_2(x)$

- Substitute in solutions

  $y = c_1 e^{-2x} + c_2 e^{-x}$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
<- constant coefficients successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 17

```
dsolve(diff(y(x),x$2) +3*diff(y(x),x)+2*y(x) = 0,y(x), singsol=all)
```

$$y(x) = \mathrm{e}^{-2x}c_1 + c_2\mathrm{e}^{-x}$$

✓ Solution by Mathematica

Time used: 0.013 (sec). Leaf size: 20

```
DSolve[y''[x]+3*y'[x]+2*y[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to e^{-2x}(c_2e^x + c_1)$$

7.3 problem 3

Internal problem ID [601]
Internal file name [`OUTPUT/601_Sunday_June_05_2022_01_45_40_AM_67843365/index.tex`]

Book: Elementary differential equations and boundary value problems, 10th ed., Boyce and DiPrima
Section: Chapter 3, Second order linear equations, 3.1 Homogeneous Equations with Constant Coefficients, page 144
Problem number: 3.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_linear_constant_coeff"**

Maple gives the following as the ode type

`[[_2nd_order, _missing_x]]`

$$\boxed{6y'' - y' - y = 0}$$

7.3.1 Solving as second order linear constant coeff ode

This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = 0$$

Where in the above $A = 6, B = -1, C = -1$. Let the solution be $y = e^{\lambda x}$. Substituting this into the ODE gives

$$6\lambda^2 \mathrm{e}^{\lambda x} - \lambda\, \mathrm{e}^{\lambda x} - \mathrm{e}^{\lambda x} = 0 \tag{1}$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda x}$ gives

$$6\lambda^2 - \lambda - 1 = 0 \tag{2}$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form.Using the quadratic formula

$$\lambda_{1,2} = \frac{-B}{2A} \pm \frac{1}{2A}\sqrt{B^2 - 4AC}$$

Substituting $A = 6, B = -1, C = -1$ into the above gives

$$\lambda_{1,2} = \frac{1}{(2)(6)} \pm \frac{1}{(2)(6)} \sqrt{-1^2 - (4)(6)(-1)}$$
$$= \frac{1}{12} \pm \frac{5}{12}$$

Hence

$$\lambda_1 = \frac{1}{12} + \frac{5}{12}$$
$$\lambda_2 = \frac{1}{12} - \frac{5}{12}$$

Which simplifies to

$$\lambda_1 = \frac{1}{2}$$
$$\lambda_2 = -\frac{1}{3}$$

Since roots are real and distinct, then the solution is

$$y = c_1 e^{\lambda_1 x} + c_2 e^{\lambda_2 x}$$
$$y = c_1 e^{\left(\frac{1}{2}\right)x} + c_2 e^{\left(-\frac{1}{3}\right)x}$$

Or

$$y = c_1 \mathrm{e}^{\frac{x}{2}} + c_2 \mathrm{e}^{-\frac{x}{3}}$$

Summary

The solution(s) found are the following

$$y = c_1 \mathrm{e}^{\frac{x}{2}} + c_2 \mathrm{e}^{-\frac{x}{3}} \tag{1}$$

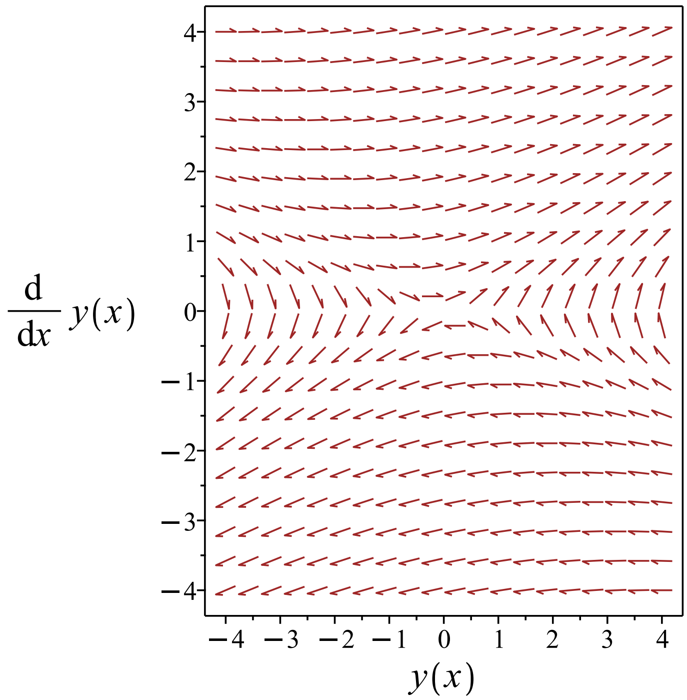


Figure 353: Slope field plot

Verification of solutions

$$y = c_1 e^{\frac{x}{2}} + c_2 e^{-\frac{x}{3}}$$

Verified OK.

7.3.2 Solving using Kovacic algorithm

Writing the ode as

$$6y'' - y' - y = 0 \tag{1}$$

$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 6 \\ B &= -1 \\ C &= -1 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\, dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{25}{144} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= 25 \\ t &= 144 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = \frac{25z(x)}{144} \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 320: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned}O(\infty) &= \deg(t) - \deg(s)\\ &= 0 - 0\\ &= 0\end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Since $r = \frac{25}{144}$ is not a function of x, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$z_1(x) = \mathrm{e}^{-\frac{5x}{12}}$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$\begin{aligned}y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx}\\ &= z_1 e^{-\int \frac{1}{2}\frac{-1}{6}\,dx}\\ &= z_1 e^{\frac{x}{12}}\\ &= z_1\left(\mathrm{e}^{\frac{x}{12}}\right)\end{aligned}$$

Which simplifies to

$$y_1 = \mathrm{e}^{-\frac{x}{3}}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx$$

Substituting gives

$$\begin{aligned}y_2 &= y_1 \int \frac{e^{\int -\frac{-1}{6}\,dx}}{(y_1)^2}\,dx\\ &= y_1 \int \frac{e^{\frac{x}{6}}}{(y_1)^2}\,dx\\ &= y_1\left(\frac{6\,\mathrm{e}^{\frac{5x}{6}}}{5}\right)\end{aligned}$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1\left(\mathrm{e}^{-\frac{x}{3}}\right) + c_2\left(\mathrm{e}^{-\frac{x}{3}}\left(\frac{6\,\mathrm{e}^{\frac{5x}{6}}}{5}\right)\right)
\end{aligned}
$$

Summary

The solution(s) found are the following

$$
y = c_1 \mathrm{e}^{-\frac{x}{3}} + \frac{6 c_2 \mathrm{e}^{\frac{x}{2}}}{5} \tag{1}
$$

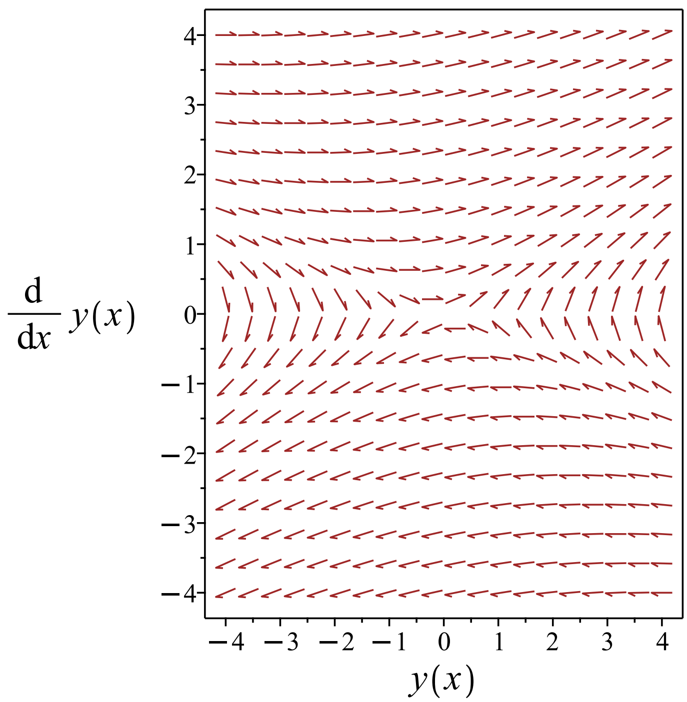


Figure 354: Slope field plot

Verification of solutions

$$
y = c_1 \mathrm{e}^{-\frac{x}{3}} + \frac{6 c_2 \mathrm{e}^{\frac{x}{2}}}{5}
$$

Verified OK.

7.3.3 Maple step by step solution

Let's solve

$6y'' - y' - y = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y'' = \frac{y'}{6} + \frac{y}{6}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y'' - \frac{y'}{6} - \frac{y}{6} = 0$

- Characteristic polynomial of ODE

  $r^2 - \frac{1}{6}r - \frac{1}{6} = 0$

- Factor the characteristic polynomial

  $\frac{(3r+1)(2r-1)}{6} = 0$

- Roots of the characteristic polynomial

  $r = \left(-\frac{1}{3}, \frac{1}{2}\right)$

- 1st solution of the ODE

  $y_1(x) = e^{-\frac{x}{3}}$

- 2nd solution of the ODE

  $y_2(x) = e^{\frac{x}{2}}$

- General solution of the ODE

  $y = c_1 y_1(x) + c_2 y_2(x)$

- Substitute in solutions

  $y = c_1 e^{-\frac{x}{3}} + c_2 e^{\frac{x}{2}}$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
<- constant coefficients successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 17

```
dsolve(6*diff(y(x),x$2) -diff(y(x),x)-y(x) = 0,y(x), singsol=all)
```

$$y(x) = \left(c_1 e^{\frac{5x}{6}} + c_2\right) e^{-\frac{x}{3}}$$

✓ Solution by Mathematica

Time used: 0.014 (sec). Leaf size: 26

```
DSolve[6*y''[x]-y'[x]-y[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to e^{-x/3}\left(c_2 e^{5x/6} + c_1\right)$$

7.4 problem 4

Internal problem ID [602]
Internal file name [`OUTPUT/602_Sunday_June_05_2022_01_45_41_AM_23423378/index.tex`]

Book: Elementary differential equations and boundary value problems, 10th ed., Boyce and DiPrima
Section: Chapter 3, Second order linear equations, 3.1 Homogeneous Equations with Constant Coefficients, page 144
Problem number: 4.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_linear_constant_coeff"**

Maple gives the following as the ode type

```
[[_2nd_order, _missing_x]]
```

$$\boxed{2y'' - 3y' + y = 0}$$

7.4.1 Solving as second order linear constant coeff ode

This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = 0$$

Where in the above $A = 2, B = -3, C = 1$. Let the solution be $y = e^{\lambda x}$. Substituting this into the ODE gives

$$2\lambda^2 e^{\lambda x} - 3\lambda\, e^{\lambda x} + e^{\lambda x} = 0 \tag{1}$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda x}$ gives

$$2\lambda^2 - 3\lambda + 1 = 0 \tag{2}$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form.Using the quadratic formula

$$\lambda_{1,2} = \frac{-B}{2A} \pm \frac{1}{2A}\sqrt{B^2 - 4AC}$$

Substituting $A = 2, B = -3, C = 1$ into the above gives

$$
\begin{aligned}
\lambda_{1,2} &= \frac{3}{(2)(2)} \pm \frac{1}{(2)(2)} \sqrt{-3^2 - (4)(2)(1)} \\
&= \frac{3}{4} \pm \frac{1}{4}
\end{aligned}
$$

Hence

$$
\begin{aligned}
\lambda_1 &= \frac{3}{4} + \frac{1}{4} \\
\lambda_2 &= \frac{3}{4} - \frac{1}{4}
\end{aligned}
$$

Which simplifies to

$$
\begin{aligned}
\lambda_1 &= 1 \\
\lambda_2 &= \frac{1}{2}
\end{aligned}
$$

Since roots are real and distinct, then the solution is

$$
\begin{aligned}
y &= c_1 e^{\lambda_1 x} + c_2 e^{\lambda_2 x} \\
y &= c_1 e^{(1)x} + c_2 e^{\left(\frac{1}{2}\right)x}
\end{aligned}
$$

Or

$$
y = c_1 e^x + c_2 e^{\frac{x}{2}}
$$

Summary

The solution(s) found are the following

$$
y = c_1 e^x + c_2 e^{\frac{x}{2}} \tag{1}
$$

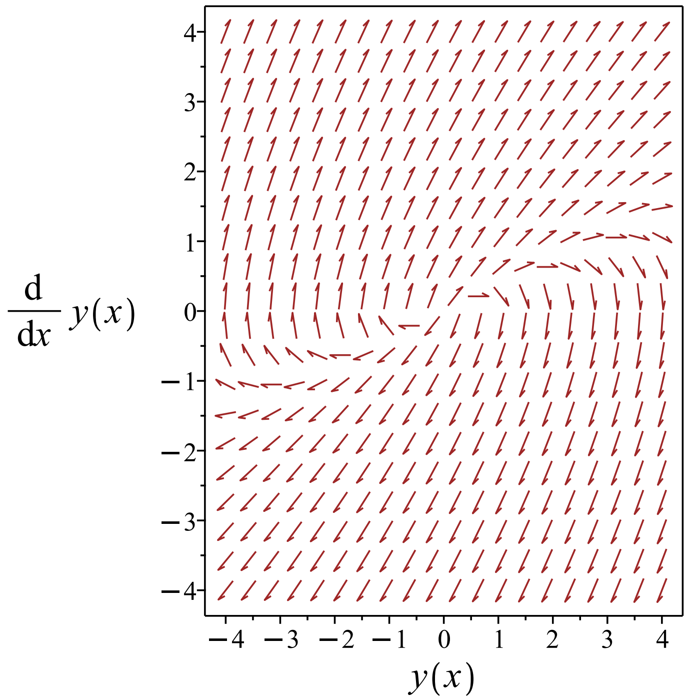


Figure 355: Slope field plot

Verification of solutions

$$y = c_1 e^{x} + c_2 e^{\frac{x}{2}}$$

Verified OK.

7.4.2 Solving using Kovacic algorithm

Writing the ode as

$$2y'' - 3y' + y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 2 \\ B &= -3 \\ C &= 1 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = y e^{\int \frac{B}{2A} \, dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{1}{16} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= 1 \\ t &= 16 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = \frac{z(x)}{16} \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 322: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 0 - 0 \\ &= 0 \end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Since $r = \frac{1}{16}$ is not a function of x, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$z_1(x) = \mathrm{e}^{-\frac{x}{4}}$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$\begin{aligned} y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx} \\ &= z_1 e^{-\int \frac{1}{2}\frac{-3}{2}\,dx} \\ &= z_1 e^{\frac{3x}{4}} \\ &= z_1 \left(\mathrm{e}^{\frac{3x}{4}}\right) \end{aligned}$$

Which simplifies to

$$y_1 = \mathrm{e}^{\frac{x}{2}}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx$$

Substituting gives

$$\begin{aligned} y_2 &= y_1 \int \frac{e^{\int -\frac{-3}{2}\,dx}}{(y_1)^2}\,dx \\ &= y_1 \int \frac{e^{\frac{3x}{2}}}{(y_1)^2}\,dx \\ &= y_1 \left(2\,\mathrm{e}^{\frac{x}{2}}\right) \end{aligned}$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1\left(\mathrm{e}^{\frac{x}{2}}\right) + c_2\left(\mathrm{e}^{\frac{x}{2}}\left(2\,\mathrm{e}^{\frac{x}{2}}\right)\right)
\end{aligned}
$$

Summary

The solution(s) found are the following

$$
y = c_1 \mathrm{e}^{\frac{x}{2}} + 2c_2 \mathrm{e}^{x} \tag{1}
$$

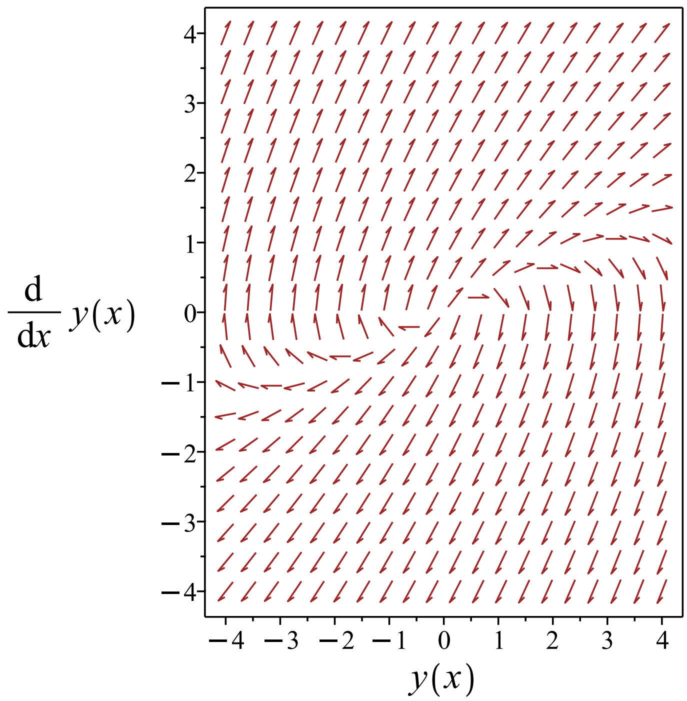


Figure 356: Slope field plot

Verification of solutions

$$
y = c_1 \mathrm{e}^{\frac{x}{2}} + 2c_2 \mathrm{e}^{x}
$$

Verified OK.

7.4.3 Maple step by step solution

Let's solve

$2y'' - 3y' + y = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y'' = \frac{3y'}{2} - \frac{y}{2}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y'' - \frac{3y'}{2} + \frac{y}{2} = 0$

- Characteristic polynomial of ODE

  $r^2 - \frac{3}{2}r + \frac{1}{2} = 0$

- Factor the characteristic polynomial

  $\frac{(2r-1)(r-1)}{2} = 0$

- Roots of the characteristic polynomial

  $r = \left(1, \frac{1}{2}\right)$

- 1st solution of the ODE

  $y_1(x) = \mathrm{e}^x$

- 2nd solution of the ODE

  $y_2(x) = \mathrm{e}^{\frac{x}{2}}$

- General solution of the ODE

  $y = c_1 y_1(x) + c_2 y_2(x)$

- Substitute in solutions

  $y = c_1 \mathrm{e}^x + c_2 \mathrm{e}^{\frac{x}{2}}$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
<- constant coefficients successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 15

```
dsolve(2*diff(y(x),x$2) -3*diff(y(x),x)+y(x) = 0,y(x), singsol=all)
```

$$y(x) = e^{x} c_1 + c_2 e^{\frac{x}{2}}$$

✓ Solution by Mathematica

Time used: 0.021 (sec). Leaf size: 35

```
DSolve[y''[x]-3*y'[x]+y[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to e^{-\frac{1}{2}\left(\sqrt{5}-3\right)x}\left(c_2 e^{\sqrt{5}x} + c_1\right)$$

7.5 problem 5

Internal problem ID [603]
Internal file name [`OUTPUT/603_Sunday_June_05_2022_01_45_42_AM_87737464/index.tex`]

Book: Elementary differential equations and boundary value problems, 10th ed., Boyce and DiPrima
Section: Chapter 3, Second order linear equations, 3.1 Homogeneous Equations with Constant Coefficients, page 144
Problem number: 5.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "exact linear second order ode", "second_order_integrable_as_is", "second_order_ode_missing_y", "second_order_linear_constant_coeff"**

Maple gives the following as the ode type

```
[[_2nd_order, _missing_x]]
```

$$\boxed{y'' + 5y' = 0}$$

7.5.1 Solving as second order linear constant coeff ode

This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = 0$$

Where in the above $A = 1, B = 5, C = 0$. Let the solution be $y = e^{\lambda x}$. Substituting this into the ODE gives

$$\lambda^2 e^{\lambda x} + 5\lambda\, e^{\lambda x} = 0 \tag{1}$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda x}$ gives

$$\lambda^2 + 5\lambda = 0 \tag{2}$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form.Using the quadratic formula

$$\lambda_{1,2} = \frac{-B}{2A} \pm \frac{1}{2A}\sqrt{B^2 - 4AC}$$

Substituting $A = 1, B = 5, C = 0$ into the above gives

$$\begin{aligned}\lambda_{1,2} &= \frac{-5}{(2)(1)} \pm \frac{1}{(2)(1)}\sqrt{5^2 - (4)(1)(0)} \\ &= -\frac{5}{2} \pm \frac{5}{2}\end{aligned}$$

Hence

$$\begin{aligned}\lambda_1 &= -\frac{5}{2} + \frac{5}{2} \\ \lambda_2 &= -\frac{5}{2} - \frac{5}{2}\end{aligned}$$

Which simplifies to

$$\begin{aligned}\lambda_1 &= 0 \\ \lambda_2 &= -5\end{aligned}$$

Since roots are real and distinct, then the solution is

$$\begin{aligned}y &= c_1 e^{\lambda_1 x} + c_2 e^{\lambda_2 x} \\ y &= c_1 e^{(0)x} + c_2 e^{(-5)x}\end{aligned}$$

Or

$$y = c_1 + c_2 \mathrm{e}^{-5x}$$

<u>Summary</u>
The solution(s) found are the following

$$y = c_1 + c_2 \mathrm{e}^{-5x} \tag{1}$$

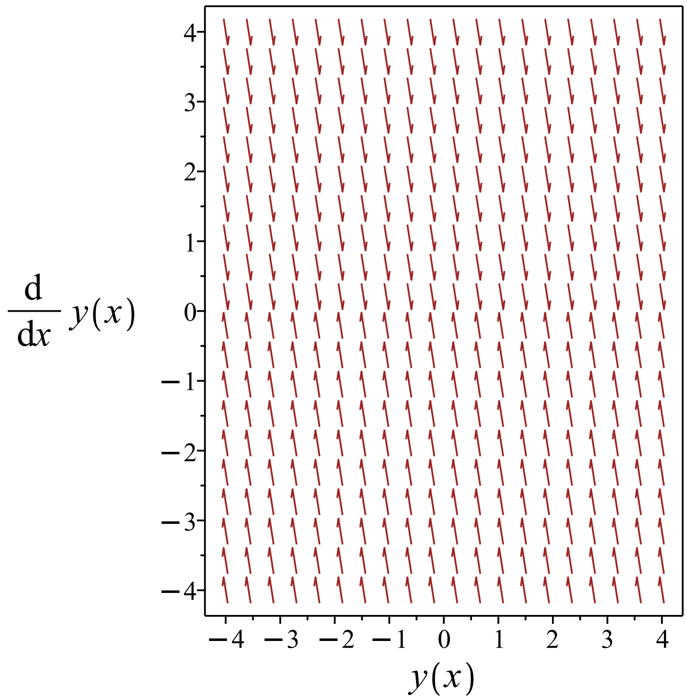


Figure 357: Slope field plot

Verification of solutions

$$y = c_1 + c_2 e^{-5x}$$

Verified OK.

7.5.2 Solving as second order integrable as is ode

Integrating both sides of the ODE w.r.t x gives

$$\int (y'' + 5y')\, dx = 0$$

$$5y + y' = c_1$$

Which is now solved for y. Integrating both sides gives

$$\int \frac{1}{-5y + c_1} dy = \int dx$$

$$-\frac{\ln(-5y + c_1)}{5} = x + c_2$$

Raising both side to exponential gives

$$\frac{1}{(-5y + c_1)^{\frac{1}{5}}} = e^{x + c_2}$$

Which simplifies to

$$\frac{1}{(-5y+c_1)^{\frac{1}{5}}} = c_3 e^{x}$$

Summary

The solution(s) found are the following

$$y = -\frac{e^{-5x}}{5c_3^5} + \frac{c_1}{5} \tag{1}$$

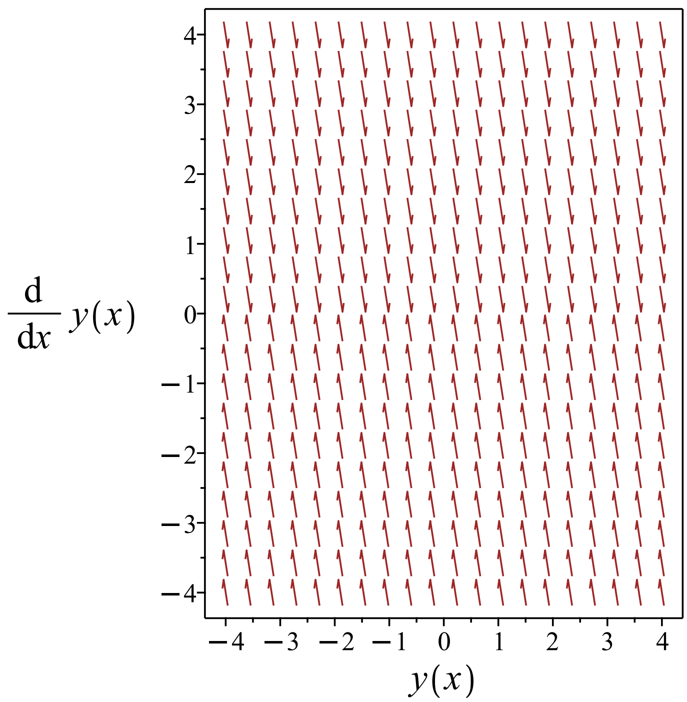


Figure 358: Slope field plot

Verification of solutions

$$y = -\frac{e^{-5x}}{5c_3^5} + \frac{c_1}{5}$$

Verified OK.

7.5.3 Solving as second order ode missing y ode

This is second order ode with missing dependent variable y. Let

$$p(x) = y'$$

Then

$$p'(x) = y''$$

Hence the ode becomes

$$p'(x) + 5p(x) = 0$$

Which is now solve for $p(x)$ as first order ode. Integrating both sides gives

$$\int -\frac{1}{5p}dp = \int dx$$
$$-\frac{\ln(p)}{5} = x + c_1$$

Raising both side to exponential gives

$$\frac{1}{p^{\frac{1}{5}}} = e^{x+c_1}$$

Which simplifies to

$$\frac{1}{p^{\frac{1}{5}}} = c_2 e^{x}$$

Since $p = y'$ then the new first order ode to solve is

$$y' = \frac{e^{-5x}}{c_2^5}$$

Integrating both sides gives

$$y = \int \frac{e^{-5x}}{c_2^5}\, dx$$
$$= -\frac{e^{-5x}}{5c_2^5} + c_3$$

Summary
The solution(s) found are the following

$$y = -\frac{e^{-5x}}{5c_2^5} + c_3 \tag{1}$$

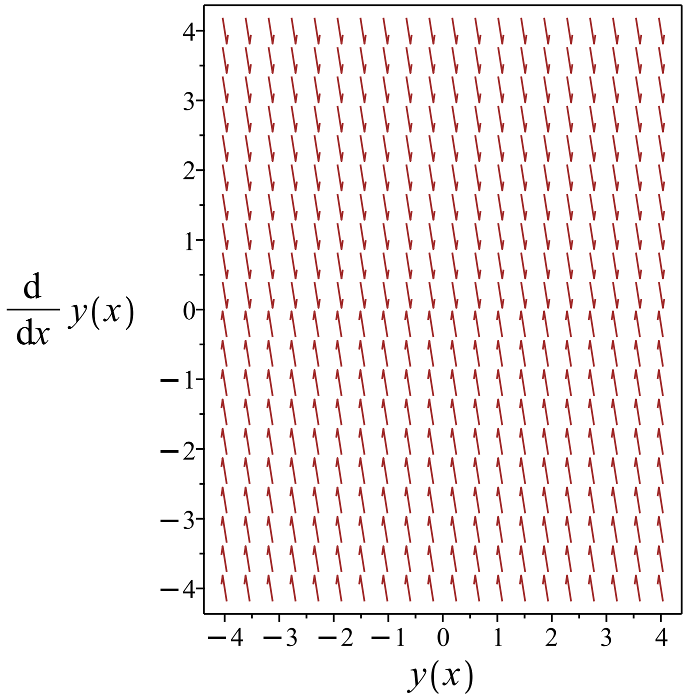


Figure 359: Slope field plot

Verification of solutions

$$y = -\frac{e^{-5x}}{5c_2^5} + c_3$$

Verified OK.

7.5.4 Solving as type second_order_integrable_as_is (not using ABC version)

Writing the ode as

$$y'' + 5y' = 0$$

Integrating both sides of the ODE w.r.t x gives

$$\int (y'' + 5y')\, dx = 0$$
$$5y + y' = c_1$$

Which is now solved for y. Integrating both sides gives

$$\int \frac{1}{-5y+c_1}dy = \int dx$$
$$-\frac{\ln(-5y+c_1)}{5} = x + c_2$$

Raising both side to exponential gives

$$\frac{1}{(-5y+c_1)^{\frac{1}{5}}} = \mathrm{e}^{x+c_2}$$

Which simplifies to

$$\frac{1}{(-5y+c_1)^{\frac{1}{5}}} = c_3\mathrm{e}^{x}$$

Summary

The solution(s) found are the following

$$y = -\frac{\mathrm{e}^{-5x}}{5c_3^5} + \frac{c_1}{5} \tag{1}$$

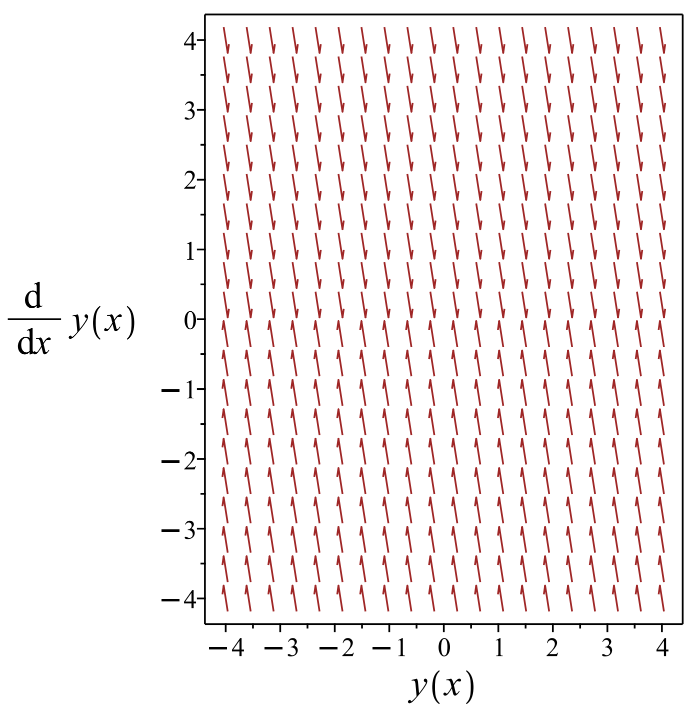


Figure 360: Slope field plot

Verification of solutions

$$y = -\frac{e^{-5x}}{5c_3^5} + \frac{c_1}{5}$$

Verified OK.

7.5.5 Solving using Kovacic algorithm

Writing the ode as

$$y'' + 5y' = 0 \tag{1}$$

$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1 \\ B &= 5 \\ C &= 0 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\, dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{25}{4} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= 25 \\ t &= 4 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = \frac{25z(x)}{4} \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 324: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 0 - 0 \\ &= 0 \end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Since $r = \frac{25}{4}$ is not a function of x, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$z_1(x) = \mathrm{e}^{-\frac{5x}{2}}$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{5}{1}\,dx} \\
&= z_1 e^{-\frac{5x}{2}} \\
&= z_1 \left(\mathrm{e}^{-\frac{5x}{2}}\right)
\end{aligned}$$

Which simplifies to

$$y_1 = \mathrm{e}^{-5x}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx$$

Substituting gives

$$\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{5}{1}\,dx}}{(y_1)^2}\,dx \\
&= y_1 \int \frac{e^{-5x}}{(y_1)^2}\,dx \\
&= y_1 \left(\frac{\mathrm{e}^{5x}}{5}\right)
\end{aligned}$$

Therefore the solution is

$$\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1\left(\mathrm{e}^{-5x}\right) + c_2\left(\mathrm{e}^{-5x}\left(\frac{\mathrm{e}^{5x}}{5}\right)\right)
\end{aligned}$$

Summary

The solution(s) found are the following

$$y = c_1 e^{-5x} + \frac{c_2}{5} \tag{1}$$

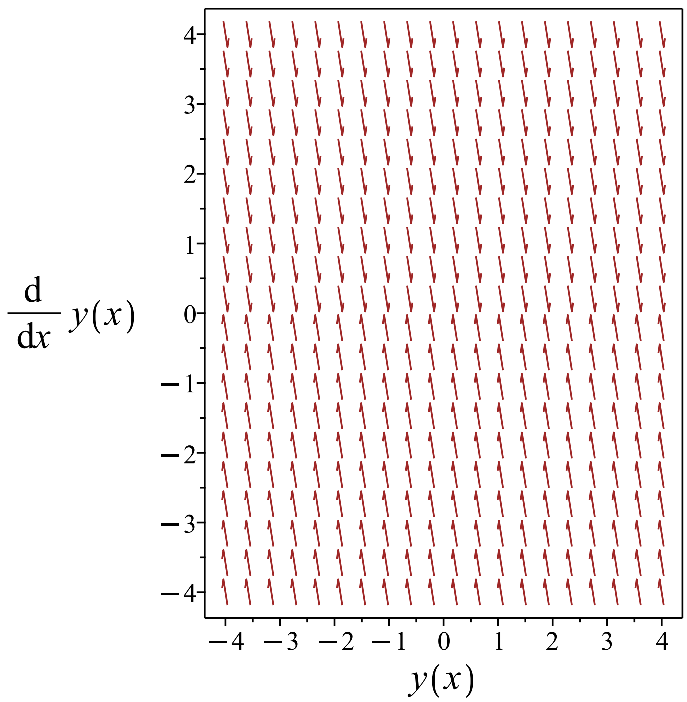


Figure 361: Slope field plot

Verification of solutions

$$y = c_1 e^{-5x} + \frac{c_2}{5}$$

Verified OK.

7.5.6 Solving as exact linear second order ode ode

An ode of the form

$$p(x) y'' + q(x) y' + r(x) y = s(x)$$

is exact if

$$p''(x) - q'(x) + r(x) = 0 \tag{1}$$

For the given ode we have

$$
\begin{aligned}
p(x) &= 1 \\
q(x) &= 5 \\
r(x) &= 0 \\
s(x) &= 0
\end{aligned}
$$

Hence

$$
\begin{aligned}
p''(x) &= 0 \\
q'(x) &= 0
\end{aligned}
$$

Therefore (1) becomes

$$
0 - (0) + (0) = 0
$$

Hence the ode is exact. Since we now know the ode is exact, it can be written as

$$
(p(x)\, y' + (q(x) - p'(x))\, y)' = s(x)
$$

Integrating gives

$$
p(x)\, y' + (q(x) - p'(x))\, y = \int s(x)\, dx
$$

Substituting the above values for p, q, r, s gives

$$
5y + y' = c_1
$$

We now have a first order ode to solve which is

$$
5y + y' = c_1
$$

Integrating both sides gives

$$
\begin{aligned}
\int \frac{1}{-5y + c_1} dy &= \int dx \\
-\frac{\ln(-5y + c_1)}{5} &= x + c_2
\end{aligned}
$$

Raising both side to exponential gives

$$
\frac{1}{(-5y + c_1)^{\frac{1}{5}}} = e^{x + c_2}
$$

Which simplifies to

$$\frac{1}{(-5y+c_1)^{\frac{1}{5}}} = c_3 e^{x}$$

Summary

The solution(s) found are the following

$$y = -\frac{e^{-5x}}{5c_3^5} + \frac{c_1}{5} \tag{1}$$

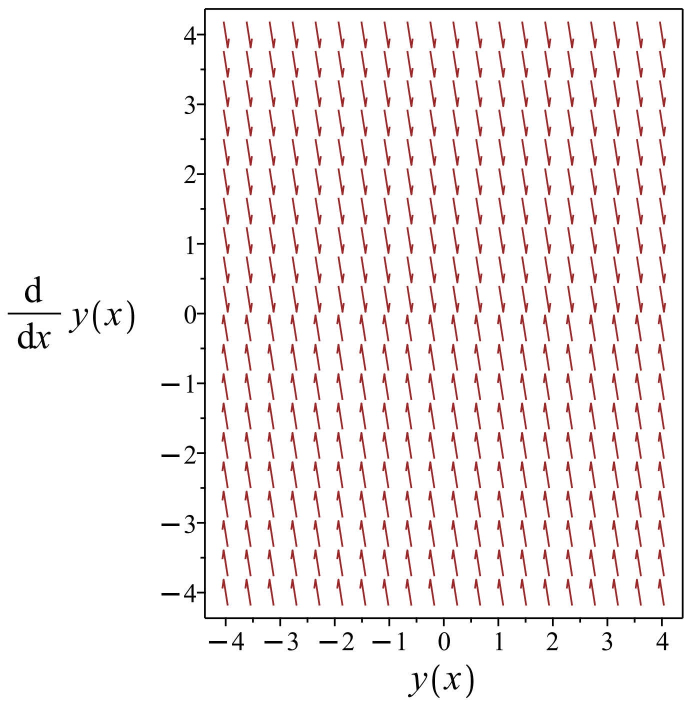


Figure 362: Slope field plot

Verification of solutions

$$y = -\frac{e^{-5x}}{5c_3^5} + \frac{c_1}{5}$$

Verified OK.

7.5.7 Maple step by step solution

Let's solve

$y'' + 5y' = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Characteristic polynomial of ODE

  $r^2 + 5r = 0$

- Factor the characteristic polynomial

  $r(r+5) = 0$

- Roots of the characteristic polynomial

  $r = (-5, 0)$

- 1st solution of the ODE

  $y_1(x) = \mathrm{e}^{-5x}$

- 2nd solution of the ODE

  $y_2(x) = 1$

- General solution of the ODE

  $y = c_1 y_1(x) + c_2 y_2(x)$

- Substitute in solutions

  $y = c_1 \mathrm{e}^{-5x} + c_2$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
<- constant coefficients successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 12

```
dsolve(diff(y(x),x$2) +5*diff(y(x),x) = 0,y(x), singsol=all)
```

$$y(x) = c_1 + c_2 e^{-5x}$$

✓ Solution by Mathematica

Time used: 0.01 (sec). Leaf size: 19

```
DSolve[y''[x]+5*y'[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to c_2 - \frac{1}{5} c_1 e^{-5x}$$

7.6 problem 6

Internal problem ID [604]
Internal file name [`OUTPUT/604_Sunday_June_05_2022_01_45_42_AM_9328719/index.tex`]

Book: Elementary differential equations and boundary value problems, 10th ed., Boyce and DiPrima
Section: Chapter 3, Second order linear equations, 3.1 Homogeneous Equations with Constant Coefficients, page 144
Problem number: 6.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_linear_constant_coeff", "second_order_ode_can_be_made_integrable"**

Maple gives the following as the ode type

```
[[_2nd_order, _missing_x]]
```

$$\boxed{4y'' - 9y = 0}$$

7.6.1 Solving as second order linear constant coeff ode

This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = 0$$

Where in the above $A = 4, B = 0, C = -9$. Let the solution be $y = e^{\lambda x}$. Substituting this into the ODE gives

$$4\lambda^2 e^{\lambda x} - 9 e^{\lambda x} = 0 \tag{1}$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda x}$ gives

$$4\lambda^2 - 9 = 0 \tag{2}$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form.Using the quadratic formula

$$\lambda_{1,2} = \frac{-B}{2A} \pm \frac{1}{2A}\sqrt{B^2 - 4AC}$$

Substituting $A = 4, B = 0, C = -9$ into the above gives

$$\begin{aligned}\lambda_{1,2} &= \frac{0}{(2)(4)} \pm \frac{1}{(2)(4)}\sqrt{0^2 - (4)(4)(-9)} \\ &= \pm\frac{3}{2}\end{aligned}$$

Hence

$$\begin{aligned}\lambda_1 &= +\frac{3}{2} \\ \lambda_2 &= -\frac{3}{2}\end{aligned}$$

Which simplifies to

$$\begin{aligned}\lambda_1 &= \frac{3}{2} \\ \lambda_2 &= -\frac{3}{2}\end{aligned}$$

Since roots are real and distinct, then the solution is

$$\begin{aligned}y &= c_1 e^{\lambda_1 x} + c_2 e^{\lambda_2 x} \\ y &= c_1 e^{\left(\frac{3}{2}\right)x} + c_2 e^{\left(-\frac{3}{2}\right)x}\end{aligned}$$

Or

$$y = c_1 \mathrm{e}^{\frac{3x}{2}} + c_2 \mathrm{e}^{-\frac{3x}{2}}$$

Summary

The solution(s) found are the following

$$y = c_1 \mathrm{e}^{\frac{3x}{2}} + c_2 \mathrm{e}^{-\frac{3x}{2}} \tag{1}$$

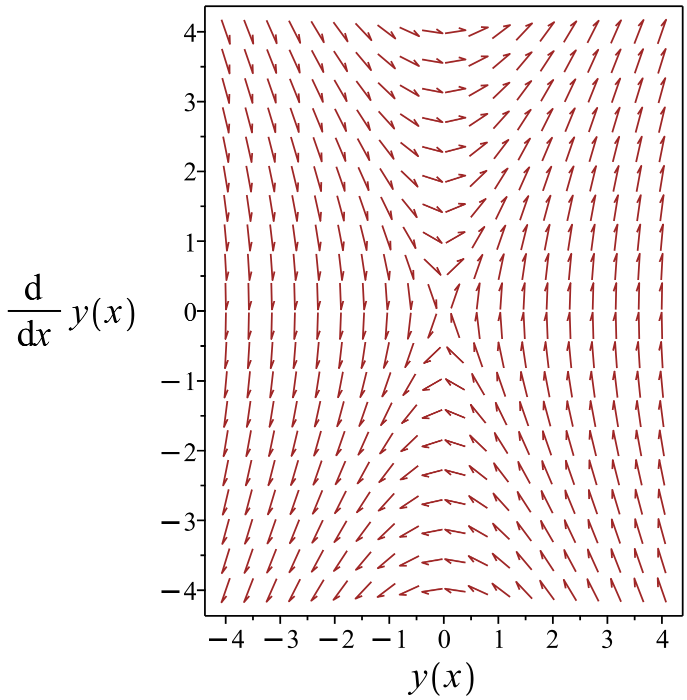


Figure 363: Slope field plot

Verification of solutions

$$y = c_1 e^{\frac{3x}{2}} + c_2 e^{-\frac{3x}{2}}$$

Verified OK.

7.6.2 Solving as second order ode can be made integrable ode

Multiplying the ode by y' gives

$$4y'y'' - 9yy' = 0$$

Integrating the above w.r.t x gives

$$\int (4y'y'' - 9yy')\,dx = 0$$

$$2y'^2 - \frac{9y^2}{2} = c_2$$

Which is now solved for y. Solving the given ode for y' results in 2 differential equations to solve. Each one of these will generate a solution. The equations generated are

$$y' = \frac{\sqrt{9y^2 + 2c_1}}{2} \tag{1}$$
$$y' = -\frac{\sqrt{9y^2 + 2c_1}}{2} \tag{2}$$

Now each one of the above ODE is solved.

Solving equation (1)

Integrating both sides gives

$$\int \frac{2}{\sqrt{9y^2 + 2c_1}} dy = \int dx$$
$$\frac{2 \ln\left(y\sqrt{9} + \sqrt{9y^2 + 2c_1}\right)\sqrt{9}}{9} = x + c_2$$

Raising both side to exponential gives

$$e^{\frac{2 \ln\left(y\sqrt{9} + \sqrt{9y^2 + 2c_1}\right)\sqrt{9}}{9}} = e^{x + c_2}$$

Which simplifies to

$$\left(3y + \sqrt{9y^2 + 2c_1}\right)^{\frac{2}{3}} = c_3 e^x$$

Solving equation (2)

Integrating both sides gives

$$\int -\frac{2}{\sqrt{9y^2 + 2c_1}} dy = \int dx$$
$$-\frac{2 \ln\left(y\sqrt{9} + \sqrt{9y^2 + 2c_1}\right)\sqrt{9}}{9} = x + c_4$$

Raising both side to exponential gives

$$e^{-\frac{2 \ln\left(y\sqrt{9} + \sqrt{9y^2 + 2c_1}\right)\sqrt{9}}{9}} = e^{x + c_4}$$

Which simplifies to

$$\frac{1}{\left(3y + \sqrt{9y^2 + 2c_1}\right)^{\frac{2}{3}}} = c_5 e^x$$

Summary
The solution(s) found are the following

$$y = \frac{(c_3^3 e^{3x} - 2c_1) e^{-x}}{6c_3 \sqrt{c_3 e^x}} \tag{1}$$

$$y = -\frac{(2c_1 c_5^3 e^{3x} - 1) e^{-x}}{6c_5 \sqrt{c_5 e^x}} \tag{2}$$

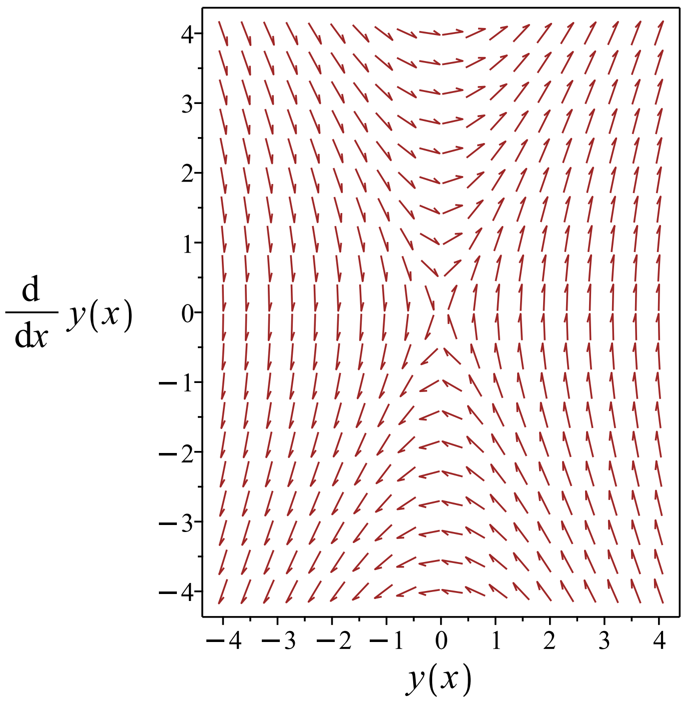


Figure 364: Slope field plot

Verification of solutions

$$y = \frac{(c_3^3 e^{3x} - 2c_1) e^{-x}}{6c_3 \sqrt{c_3 e^x}}$$

Verified OK.

$$y = -\frac{(2c_1 c_5^3 e^{3x} - 1) e^{-x}}{6c_5 \sqrt{c_5 e^x}}$$

Verified OK.

7.6.3 Solving using Kovacic algorithm

Writing the ode as

$$4y'' - 9y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 4 \\ B &= 0 \\ C &= -9 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{9}{4} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= 9 \\ t &= 4 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = \frac{9z(x)}{4} \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\,e^{-\int \frac{B}{2A}\,dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 326: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 0 - 0 \\ &= 0 \end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Since $r = \frac{9}{4}$ is not a function of x, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$z_1(x) = e^{-\frac{3x}{2}}$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$y_1 = z_1 e^{\int -\frac{1}{2} \frac{B}{A} \, dx}$$

Since $B = 0$ then the above reduces to

$$
\begin{aligned}
y_1 &= z_1 \\
&= e^{-\frac{3x}{2}}
\end{aligned}
$$

Which simplifies to

$$
y_1 = e^{-\frac{3x}{2}}
$$

The second solution y_2 to the original ode is found using reduction of order

$$
y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\, dx}}{y_1^2}\, dx
$$

Since $B = 0$ then the above becomes

$$
\begin{aligned}
y_2 &= y_1 \int \frac{1}{y_1^2}\, dx \\
&= e^{-\frac{3x}{2}} \int \frac{1}{e^{-3x}}\, dx \\
&= e^{-\frac{3x}{2}} \left(\frac{e^{3x}}{3} \right)
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1 \left(e^{-\frac{3x}{2}} \right) + c_2 \left(e^{-\frac{3x}{2}} \left(\frac{e^{3x}}{3} \right) \right)
\end{aligned}
$$

<u>Summary</u>
The solution(s) found are the following

$$
y = c_1 e^{-\frac{3x}{2}} + \frac{c_2 e^{\frac{3x}{2}}}{3} \tag{1}
$$

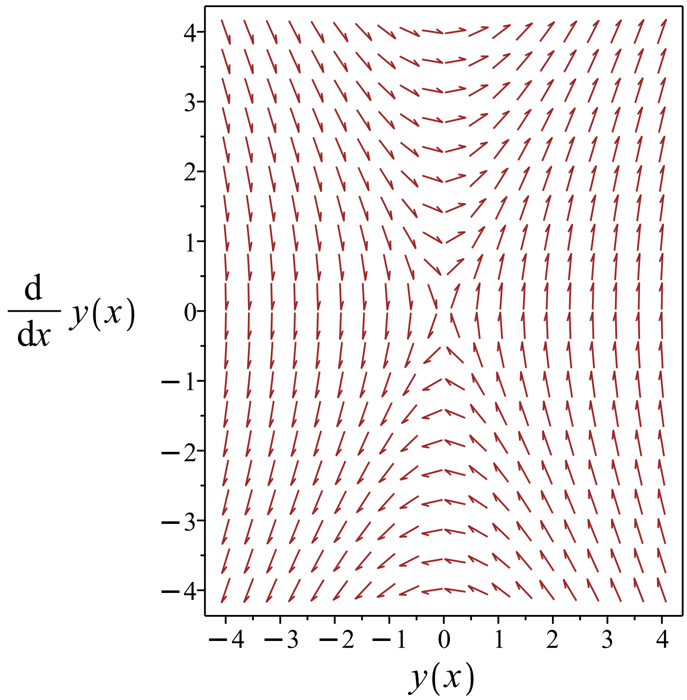


Figure 365: Slope field plot

Verification of solutions

$$y = c_1 e^{-\frac{3x}{2}} + \frac{c_2 e^{\frac{3x}{2}}}{3}$$

Verified OK.

7.6.4 Maple step by step solution

Let's solve

$4y'' - 9y = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y'' = \frac{9y}{4}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y'' - \frac{9y}{4} = 0$

- Characteristic polynomial of ODE

$r^2 - \frac{9}{4} = 0$

- Factor the characteristic polynomial

  $\frac{(2r-3)(2r+3)}{4} = 0$

- Roots of the characteristic polynomial

  $r = \left(-\frac{3}{2}, \frac{3}{2}\right)$

- 1st solution of the ODE

  $y_1(x) = e^{-\frac{3x}{2}}$

- 2nd solution of the ODE

  $y_2(x) = e^{\frac{3x}{2}}$

- General solution of the ODE

  $y = c_1 y_1(x) + c_2 y_2(x)$

- Substitute in solutions

  $y = c_1 e^{-\frac{3x}{2}} + c_2 e^{\frac{3x}{2}}$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
<- constant coefficients successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 17

```
dsolve(4*diff(y(x),x$2) -9*y(x) = 0,y(x), singsol=all)
```

$$y(x) = c_1 e^{-\frac{3x}{2}} + c_2 e^{\frac{3x}{2}}$$

✓ Solution by Mathematica

Time used: 0.013 (sec). Leaf size: 24

```
DSolve[4*y''[x]-9*y[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \rightarrow e^{-3x/2}\left(c_1 e^{3x} + c_2\right)$$

7.7 problem 7

Internal problem ID [605]
Internal file name [`OUTPUT/605_Sunday_June_05_2022_01_45_43_AM_24728919/index.tex`]

Book: Elementary differential equations and boundary value problems, 10th ed., Boyce and DiPrima
Section: Chapter 3, Second order linear equations, 3.1 Homogeneous Equations with Constant Coefficients, page 144
Problem number: 7.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_linear_constant_coeff"**

Maple gives the following as the ode type

```
[[_2nd_order, _missing_x]]
```

$$\boxed{y'' - 9y' + 9y = 0}$$

7.7.1 Solving as second order linear constant coeff ode

This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = 0$$

Where in the above $A = 1, B = -9, C = 9$. Let the solution be $y = e^{\lambda x}$. Substituting this into the ODE gives

$$\lambda^2 e^{\lambda x} - 9\lambda\, e^{\lambda x} + 9\, e^{\lambda x} = 0 \tag{1}$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda x}$ gives

$$\lambda^2 - 9\lambda + 9 = 0 \tag{2}$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form.Using the quadratic formula

$$\lambda_{1,2} = \frac{-B}{2A} \pm \frac{1}{2A}\sqrt{B^2 - 4AC}$$

Substituting $A = 1, B = -9, C = 9$ into the above gives

$$\begin{aligned}\lambda_{1,2} &= \frac{9}{(2)(1)} \pm \frac{1}{(2)(1)}\sqrt{-9^2 - (4)(1)(9)} \\ &= \frac{9}{2} \pm \frac{3\sqrt{5}}{2}\end{aligned}$$

Hence

$$\lambda_1 = \frac{9}{2} + \frac{3\sqrt{5}}{2}$$

$$\lambda_2 = \frac{9}{2} - \frac{3\sqrt{5}}{2}$$

Which simplifies to

$$\lambda_1 = \frac{9}{2} + \frac{3\sqrt{5}}{2}$$

$$\lambda_2 = \frac{9}{2} - \frac{3\sqrt{5}}{2}$$

Since roots are real and distinct, then the solution is

$$y = c_1 e^{\lambda_1 x} + c_2 e^{\lambda_2 x}$$

$$y = c_1 e^{\left(\frac{9}{2} + \frac{3\sqrt{5}}{2}\right)x} + c_2 e^{\left(\frac{9}{2} - \frac{3\sqrt{5}}{2}\right)x}$$

Or

$$y = c_1 \mathrm{e}^{\left(\frac{9}{2} + \frac{3\sqrt{5}}{2}\right)x} + c_2 \mathrm{e}^{\left(\frac{9}{2} - \frac{3\sqrt{5}}{2}\right)x}$$

Summary

The solution(s) found are the following

$$y = c_1 \mathrm{e}^{\left(\frac{9}{2} + \frac{3\sqrt{5}}{2}\right)x} + c_2 \mathrm{e}^{\left(\frac{9}{2} - \frac{3\sqrt{5}}{2}\right)x} \tag{1}$$

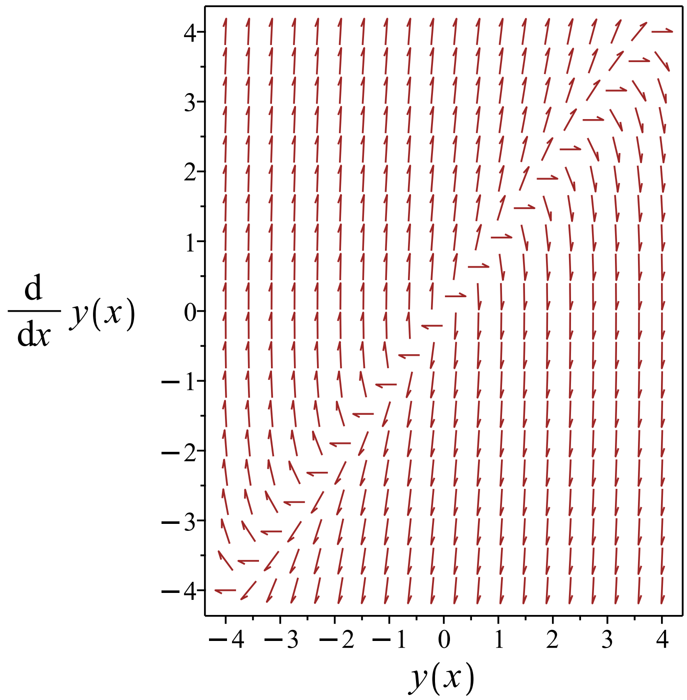


Figure 366: Slope field plot

Verification of solutions

$$y = c_1 e^{\left(\frac{9}{2}+\frac{3\sqrt{5}}{2}\right)x} + c_2 e^{\left(\frac{9}{2}-\frac{3\sqrt{5}}{2}\right)x}$$

Verified OK.

7.7.2 Solving using Kovacic algorithm

Writing the ode as

$$y'' - 9y' + 9y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1 \\ B &= -9 \\ C &= 9 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = y e^{\int \frac{B}{2A}\, dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{45}{4} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= 45 \\ t &= 4 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = \frac{45z(x)}{4} \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 328: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned}
O(\infty) &= \deg(t) - \deg(s) \\
&= 0 - 0 \\
&= 0
\end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Since $r = \frac{45}{4}$ is not a function of x, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$z_1(x) = \mathrm{e}^{-\frac{3x\sqrt{5}}{2}}$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{-9}{1}\,dx} \\
&= z_1 e^{\frac{9x}{2}} \\
&= z_1 \left(\mathrm{e}^{\frac{9x}{2}}\right)
\end{aligned}$$

Which simplifies to

$$y_1 = \mathrm{e}^{-\frac{3\left(\sqrt{5}-3\right)x}{2}}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx$$

Substituting gives

$$\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{-9}{1}\,dx}}{(y_1)^2}\,dx \\
&= y_1 \int \frac{e^{9x}}{(y_1)^2}\,dx \\
&= y_1 \left(\frac{\sqrt{5}\,\mathrm{e}^{3x\sqrt{5}}}{15}\right)
\end{aligned}$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1\left(\mathrm{e}^{-\frac{3\left(\sqrt{5}-3\right)x}{2}}\right) + c_2\left(\mathrm{e}^{-\frac{3\left(\sqrt{5}-3\right)x}{2}}\left(\frac{\sqrt{5}\,\mathrm{e}^{3x\sqrt{5}}}{15}\right)\right)
\end{aligned}
$$

Summary

The solution(s) found are the following

$$
y = c_1 \mathrm{e}^{-\frac{3\left(\sqrt{5}-3\right)x}{2}} + \frac{c_2\sqrt{5}\,\mathrm{e}^{\frac{3\left(3+\sqrt{5}\right)x}{2}}}{15} \tag{1}
$$

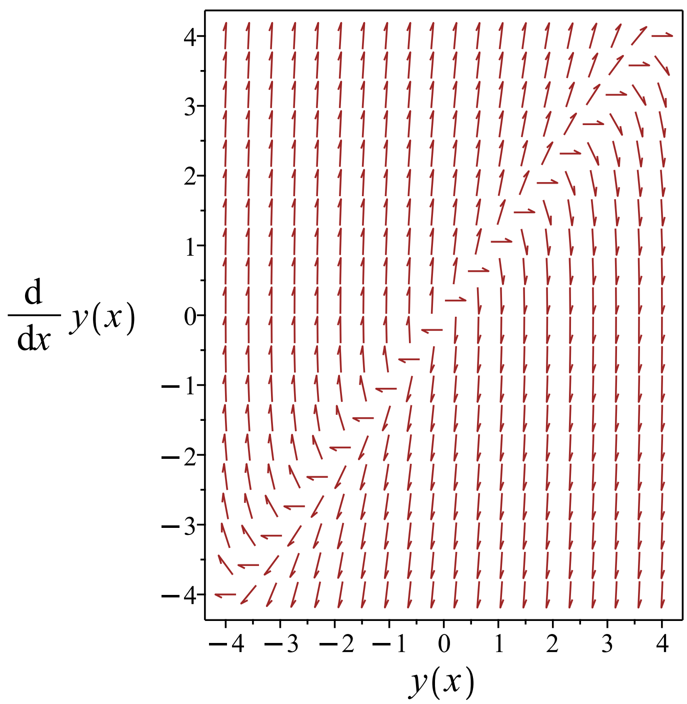


Figure 367: Slope field plot

Verification of solutions

$$
y = c_1 \mathrm{e}^{-\frac{3\left(\sqrt{5}-3\right)x}{2}} + \frac{c_2\sqrt{5}\,\mathrm{e}^{\frac{3\left(3+\sqrt{5}\right)x}{2}}}{15}
$$

Verified OK.

7.7.3 Maple step by step solution

Let's solve

$y'' - 9y' + 9y = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Characteristic polynomial of ODE

  $r^2 - 9r + 9 = 0$

- Use quadratic formula to solve for r

  $r = \frac{9 \pm \left(\sqrt{45}\right)}{2}$

- Roots of the characteristic polynomial

  $r = \left(\frac{9}{2} - \frac{3\sqrt{5}}{2}, \frac{9}{2} + \frac{3\sqrt{5}}{2}\right)$

- 1st solution of the ODE

  $y_1(x) = e^{\left(\frac{9}{2} - \frac{3\sqrt{5}}{2}\right)x}$

- 2nd solution of the ODE

  $y_2(x) = e^{\left(\frac{9}{2} + \frac{3\sqrt{5}}{2}\right)x}$

- General solution of the ODE

  $y = c_1 y_1(x) + c_2 y_2(x)$

- Substitute in solutions

  $y = c_1 e^{\left(\frac{9}{2} - \frac{3\sqrt{5}}{2}\right)x} + c_2 e^{\left(\frac{9}{2} + \frac{3\sqrt{5}}{2}\right)x}$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
<- constant coefficients successful`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 27

```
dsolve(diff(y(x),x$2) -9*diff(y(x),x)+9*y(x) = 0,y(x), singsol=all)
```

$$y(x) = c_1 e^{\frac{3(3+\sqrt{5})x}{2}} + c_2 e^{-\frac{3(\sqrt{5}-3)x}{2}}$$

✓ Solution by Mathematica

Time used: 0.022 (sec). Leaf size: 36

```
DSolve[y''[x]-9*y'[x]+9*y[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to e^{-\frac{3}{2}(\sqrt{5}-3)x}\left(c_2 e^{3\sqrt{5}x} + c_1\right)$$

7.8 problem 8

Internal problem ID [606]
Internal file name [`OUTPUT/606_Sunday_June_05_2022_01_45_44_AM_43508000/index.tex`]

Book: Elementary differential equations and boundary value problems, 10th ed., Boyce and DiPrima
Section: Chapter 3, Second order linear equations, 3.1 Homogeneous Equations with Constant Coefficients, page 144
Problem number: 8.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_linear_constant_coeff"**

Maple gives the following as the ode type

```
[[_2nd_order, _missing_x]]
```

$$\boxed{y'' - 2y' - 2y = 0}$$

7.8.1 Solving as second order linear constant coeff ode

This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = 0$$

Where in the above $A = 1, B = -2, C = -2$. Let the solution be $y = e^{\lambda x}$. Substituting this into the ODE gives

$$\lambda^2 \mathrm{e}^{\lambda x} - 2\lambda\, \mathrm{e}^{\lambda x} - 2\, \mathrm{e}^{\lambda x} = 0 \tag{1}$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda x}$ gives

$$\lambda^2 - 2\lambda - 2 = 0 \tag{2}$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form.Using the quadratic formula

$$\lambda_{1,2} = \frac{-B}{2A} \pm \frac{1}{2A}\sqrt{B^2 - 4AC}$$

Substituting $A = 1, B = -2, C = -2$ into the above gives

$$\begin{aligned}\lambda_{1,2} &= \frac{2}{(2)(1)} \pm \frac{1}{(2)(1)}\sqrt{-2^2 - (4)(1)(-2)} \\ &= 1 \pm \sqrt{3}\end{aligned}$$

Hence

$$\lambda_1 = 1 + \sqrt{3}$$
$$\lambda_2 = 1 - \sqrt{3}$$

Which simplifies to

$$\lambda_1 = 1 + \sqrt{3}$$
$$\lambda_2 = -\sqrt{3} + 1$$

Since roots are real and distinct, then the solution is

$$y = c_1 e^{\lambda_1 x} + c_2 e^{\lambda_2 x}$$
$$y = c_1 e^{\left(1+\sqrt{3}\right)x} + c_2 e^{\left(-\sqrt{3}+1\right)x}$$

Or

$$y = c_1 \mathrm{e}^{\left(1+\sqrt{3}\right)x} + c_2 \mathrm{e}^{\left(-\sqrt{3}+1\right)x}$$

Summary
The solution(s) found are the following

$$y = c_1 \mathrm{e}^{\left(1+\sqrt{3}\right)x} + c_2 \mathrm{e}^{\left(-\sqrt{3}+1\right)x} \tag{1}$$

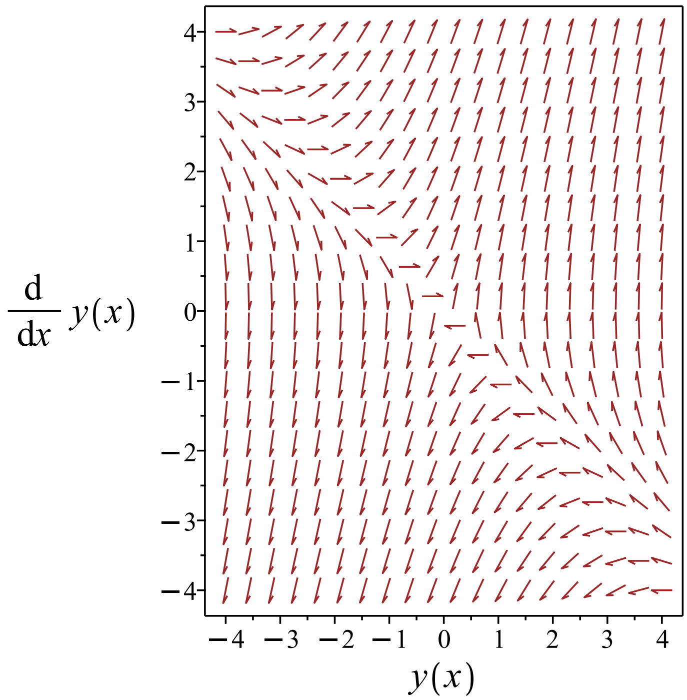


Figure 368: Slope field plot

Verification of solutions

$$y = c_1 e^{(1+\sqrt{3})x} + c_2 e^{(-\sqrt{3}+1)x}$$

Verified OK.

7.8.2 Solving using Kovacic algorithm

Writing the ode as

$$y'' - 2y' - 2y = 0 \tag{1}$$

$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1 \\ B &= -2 \\ C &= -2 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\, dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{3}{1} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= 3 \\ t &= 1 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = 3z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 330: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$
\begin{aligned}
O(\infty) &= \deg(t) - \deg(s) \\
&= 0 - 0 \\
&= 0
\end{aligned}
$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Therefore

$$
L = [1]
$$

Since $r = 3$ is not a function of x, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$
z_1(x) = \mathrm{e}^{-\sqrt{3}\,x}
$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$
\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{-2}{1}\,dx} \\
&= z_1 e^{x} \\
&= z_1 \left(\mathrm{e}^{x}\right)
\end{aligned}
$$

Which simplifies to

$$
y_1 = \mathrm{e}^{-\left(\sqrt{3}-1\right)x}
$$

The second solution y_2 to the original ode is found using reduction of order

$$
y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx
$$

Substituting gives

$$
\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{-2}{1}\,dx}}{\left(y_1\right)^2}\,dx \\
&= y_1 \int \frac{e^{2x}}{\left(y_1\right)^2}\,dx \\
&= y_1 \left(\frac{\sqrt{3}\,\mathrm{e}^{2\sqrt{3}\,x}}{6}\right)
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1 \left(\mathrm{e}^{-\left(\sqrt{3}-1\right)x}\right) + c_2 \left(\mathrm{e}^{-\left(\sqrt{3}-1\right)x} \left(\frac{\sqrt{3}\,\mathrm{e}^{2\sqrt{3}\,x}}{6}\right)\right)
\end{aligned}
$$

Summary

The solution(s) found are the following

$$
y = c_1 \mathrm{e}^{-\left(\sqrt{3}-1\right)x} + \frac{c_2 \sqrt{3}\,\mathrm{e}^{\left(1+\sqrt{3}\right)x}}{6} \tag{1}
$$

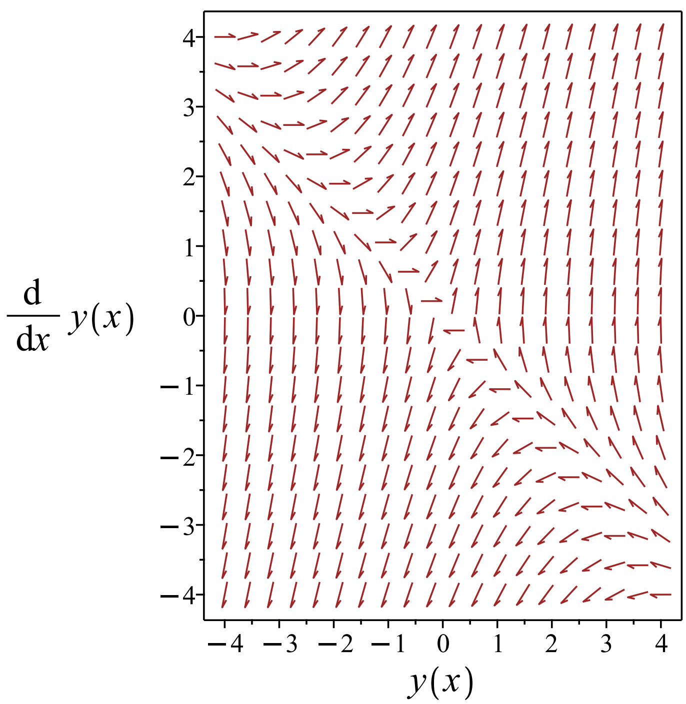


Figure 369: Slope field plot

Verification of solutions

$$
y = c_1 \mathrm{e}^{-\left(\sqrt{3}-1\right)x} + \frac{c_2 \sqrt{3}\,\mathrm{e}^{\left(1+\sqrt{3}\right)x}}{6}
$$

Verified OK.

7.8.3 Maple step by step solution

Let's solve

$y'' - 2y' - 2y = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Characteristic polynomial of ODE

  $r^2 - 2r - 2 = 0$

- Use quadratic formula to solve for r

  $r = \frac{2 \pm (\sqrt{12})}{2}$

- Roots of the characteristic polynomial

  $r = (1 + \sqrt{3}, -\sqrt{3} + 1)$

- 1st solution of the ODE

  $y_1(x) = e^{(1+\sqrt{3})x}$

- 2nd solution of the ODE

  $y_2(x) = e^{(-\sqrt{3}+1)x}$

- General solution of the ODE

  $y = c_1 y_1(x) + c_2 y_2(x)$

- Substitute in solutions

  $y = c_1 e^{(1+\sqrt{3})x} + c_2 e^{(-\sqrt{3}+1)x}$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
<- constant coefficients successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 26

```
dsolve(diff(y(x),x$2) -2*diff(y(x),x)-2*y(x) = 0,y(x), singsol=all)
```

$$y(x) = c_1 e^{\left(1+\sqrt{3}\right)x} + c_2 e^{-\left(\sqrt{3}-1\right)x}$$

✓ Solution by Mathematica

Time used: 0.019 (sec). Leaf size: 34

```
DSolve[y''[x]-2*y'[x]-2*y[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to e^{x-\sqrt{3}x}\left(c_2 e^{2\sqrt{3}x} + c_1\right)$$

7.9 problem 9

Internal problem ID [607]
Internal file name [OUTPUT/607_Sunday_June_05_2022_01_45_44_AM_60102257/index.tex]

Book: Elementary differential equations and boundary value problems, 10th ed., Boyce and DiPrima
Section: Chapter 3, Second order linear equations, 3.1 Homogeneous Equations with Constant Coefficients, page 144
Problem number: 9.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_linear_constant_coeff"**

Maple gives the following as the ode type

```
[[_2nd_order, _missing_x]]
```

$$\boxed{y'' + y' - 2y = 0}$$

With initial conditions

$$[y(0) = 0, y'(0) = 1]$$

7.9.1 Existence and uniqueness analysis

This is a linear ODE. In canonical form it is written as

$$y'' + p(x)y' + q(x)y = F$$

Where here

$$\begin{aligned} p(x) &= 1 \\ q(x) &= -2 \\ F &= 0 \end{aligned}$$

Hence the ode is

$$y'' + y' - 2y = 0$$

The domain of $p(x) = 1$ is

$$\{-\infty < x < \infty\}$$

And the point $x_0 = 0$ is inside this domain. The domain of $q(x) = -2$ is

$$\{-\infty < x < \infty\}$$

And the point $x_0 = 0$ is also inside this domain. Hence solution exists and is unique.

7.9.2 Solving as second order linear constant coeff ode

This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = 0$$

Where in the above $A = 1, B = 1, C = -2$. Let the solution be $y = e^{\lambda x}$. Substituting this into the ODE gives

$$\lambda^2 e^{\lambda x} + \lambda\, e^{\lambda x} - 2\, e^{\lambda x} = 0 \tag{1}$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda x}$ gives

$$\lambda^2 + \lambda - 2 = 0 \tag{2}$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form.Using the quadratic formula

$$\lambda_{1,2} = \frac{-B}{2A} \pm \frac{1}{2A}\sqrt{B^2 - 4AC}$$

Substituting $A = 1, B = 1, C = -2$ into the above gives

$$\begin{aligned}\lambda_{1,2} &= \frac{-1}{(2)\,(1)} \pm \frac{1}{(2)\,(1)}\sqrt{1^2 - (4)\,(1)\,(-2)}\\ &= -\frac{1}{2} \pm \frac{3}{2}\end{aligned}$$

Hence

$$\lambda_1 = -\frac{1}{2} + \frac{3}{2}$$
$$\lambda_2 = -\frac{1}{2} - \frac{3}{2}$$

Which simplifies to

$$\lambda_1 = 1$$
$$\lambda_2 = -2$$

Since roots are real and distinct, then the solution is

$$y = c_1 e^{\lambda_1 x} + c_2 e^{\lambda_2 x}$$
$$y = c_1 e^{(1)x} + c_2 e^{(-2)x}$$

Or

$$y = c_1 e^x + c_2 e^{-2x}$$

Initial conditions are used to solve for the constants of integration.

Looking at the above solution

$$y = c_1 e^x + c_2 e^{-2x} \tag{1}$$

Initial conditions are now substituted in the above solution. This will generate the required equations to solve for the integration constants. substituting $y = 0$ and $x = 0$ in the above gives

$$0 = c_1 + c_2 \tag{1A}$$

Taking derivative of the solution gives

$$y' = c_1 e^x - 2c_2 e^{-2x}$$

substituting $y' = 1$ and $x = 0$ in the above gives

$$1 = c_1 - 2c_2 \tag{2A}$$

Equations {1A,2A} are now solved for $\{c_1, c_2\}$. Solving for the constants gives

$$c_1 = \frac{1}{3}$$
$$c_2 = -\frac{1}{3}$$

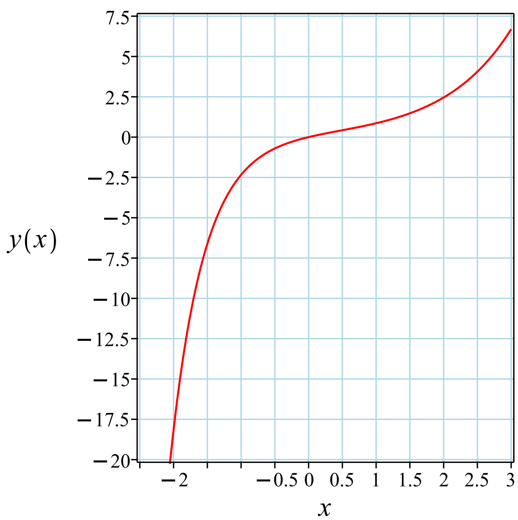
Substituting these values back in above solution results in

$$y = \frac{e^x}{3} - \frac{e^{-2x}}{3}$$

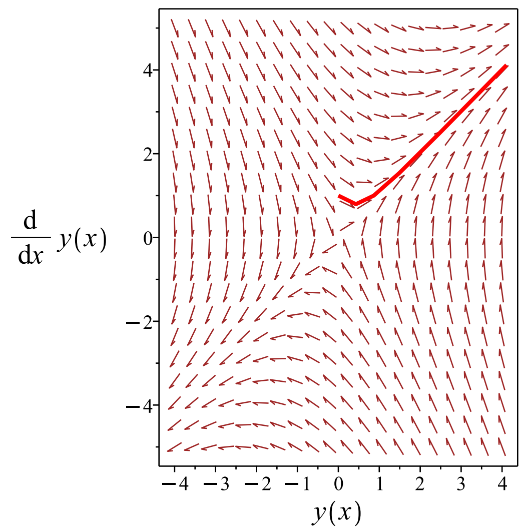
Summary

The solution(s) found are the following

$$y = \frac{e^{x}}{3} - \frac{e^{-2x}}{3} \tag{1}$$



(a) Solution plot



(b) Slope field plot

Verification of solutions

$$y = \frac{e^{x}}{3} - \frac{e^{-2x}}{3}$$

Verified OK.

7.9.3 Solving using Kovacic algorithm

Writing the ode as

$$y'' + y' - 2y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1 \\ B &= 1 \\ C &= -2 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{9}{4} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= 9 \\ t &= 4 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = \frac{9z(x)}{4} \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\,dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 332: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned}
O(\infty) &= \deg(t) - \deg(s) \\
&= 0 - 0 \\
&= 0
\end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Since $r = \frac{9}{4}$ is not a function of x, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$z_1(x) = \mathrm{e}^{-\frac{3x}{2}}$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\, dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{1}{1}\, dx} \\
&= z_1 e^{-\frac{x}{2}} \\
&= z_1 \left(\mathrm{e}^{-\frac{x}{2}}\right)
\end{aligned}$$

Which simplifies to

$$y_1 = \mathrm{e}^{-2x}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx$$

Substituting gives

$$\begin{aligned} y_2 &= y_1 \int \frac{e^{\int -\frac{1}{1}\,dx}}{(y_1)^2}\,dx \\ &= y_1 \int \frac{e^{-x}}{(y_1)^2}\,dx \\ &= y_1\left(\frac{\mathrm{e}^{3x}}{3}\right) \end{aligned}$$

Therefore the solution is

$$\begin{aligned} y &= c_1 y_1 + c_2 y_2 \\ &= c_1\left(\mathrm{e}^{-2x}\right) + c_2\left(\mathrm{e}^{-2x}\left(\frac{\mathrm{e}^{3x}}{3}\right)\right) \end{aligned}$$

Initial conditions are used to solve for the constants of integration.

Looking at the above solution

$$y = c_1\mathrm{e}^{-2x} + \frac{c_2\mathrm{e}^{x}}{3} \tag{1}$$

Initial conditions are now substituted in the above solution. This will generate the required equations to solve for the integration constants. substituting $y = 0$ and $x = 0$ in the above gives

$$0 = c_1 + \frac{c_2}{3} \tag{1A}$$

Taking derivative of the solution gives

$$y' = -2c_1\mathrm{e}^{-2x} + \frac{c_2\mathrm{e}^{x}}{3}$$

substituting $y' = 1$ and $x = 0$ in the above gives

$$1 = -2c_1 + \frac{c_2}{3} \tag{2A}$$

Equations {1A,2A} are now solved for $\{c_1, c_2\}$. Solving for the constants gives

$$c_1 = -\frac{1}{3}$$
$$c_2 = 1$$

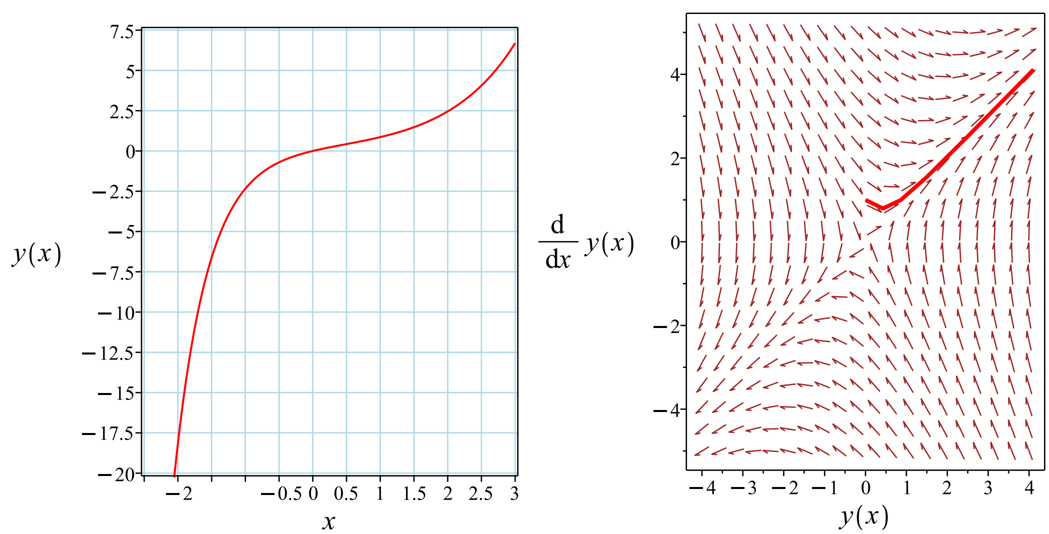
Substituting these values back in above solution results in

$$y = \frac{e^x}{3} - \frac{e^{-2x}}{3}$$

<u>Summary</u>

The solution(s) found are the following

$$y = \frac{e^x}{3} - \frac{e^{-2x}}{3} \tag{1}$$



(a) Solution plot (b) Slope field plot

<u>Verification of solutions</u>

$$y = \frac{e^x}{3} - \frac{e^{-2x}}{3}$$

Verified OK.

7.9.4 Maple step by step solution

Let's solve

$\left[y''+y'-2y=0, y(0)=0, y'\Big|_{\{x=0\}}=1\right]$

- Highest derivative means the order of the ODE is 2

  y''

- Characteristic polynomial of ODE

  $r^2+r-2=0$

- Factor the characteristic polynomial

  $(r+2)(r-1)=0$

- Roots of the characteristic polynomial

  $r=(-2,1)$

- 1st solution of the ODE

  $y_1(x)=\mathrm{e}^{-2x}$

- 2nd solution of the ODE

  $y_2(x)=\mathrm{e}^{x}$

- General solution of the ODE

  $y=c_1y_1(x)+c_2y_2(x)$

- Substitute in solutions

  $y=c_1\mathrm{e}^{-2x}+c_2\mathrm{e}^{x}$

□ Check validity of solution $y=c_1\mathrm{e}^{-2x}+c_2\mathrm{e}^{x}$

  ◦ Use initial condition $y(0)=0$

    $0=c_1+c_2$

  ◦ Compute derivative of the solution

    $y'=-2c_1\mathrm{e}^{-2x}+c_2\mathrm{e}^{x}$

  ◦ Use the initial condition $y'\Big|_{\{x=0\}}=1$

    $1=-2c_1+c_2$

  ◦ Solve for c_1 and c_2

    $\left\{c_1=-\frac{1}{3}, c_2=\frac{1}{3}\right\}$

- Substitute constant values into general solution and simplify

  $y = \frac{(e^{3x}-1)e^{-2x}}{3}$

- Solution to the IVP

  $y = \frac{(e^{3x}-1)e^{-2x}}{3}$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
<- constant coefficients successful`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 16

```
dsolve([diff(y(x),x$2) +diff(y(x),x)-2*y(x) = 0,y(0) = 0, D(y)(0) = 1],y(x), singsol=all)
```

$$y(x) = \frac{(e^{3x}-1)\, e^{-2x}}{3}$$

✓ Solution by Mathematica

Time used: 0.013 (sec). Leaf size: 21

```
DSolve[{y''[x]+y'[x]-2*y[x]==0,{y[0]==0,y'[0]==1}},y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \frac{1}{3}e^{-2x}\left(e^{3x}-1\right)$$

7.10 problem 10

Internal problem ID [608]
Internal file name [`OUTPUT/608_Sunday_June_05_2022_01_45_46_AM_30342769/index.tex`]

Book: Elementary differential equations and boundary value problems, 10th ed., Boyce and DiPrima
Section: Chapter 3, Second order linear equations, 3.1 Homogeneous Equations with Constant Coefficients, page 144
Problem number: 10.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_linear_constant_coeff"**

Maple gives the following as the ode type

```
[[_2nd_order, _missing_x]]
```

$$\boxed{y'' + 4y' + 3y = 0}$$

With initial conditions

$$[y(0) = 2, y'(0) = -1]$$

7.10.1 Existence and uniqueness analysis

This is a linear ODE. In canonical form it is written as

$$y'' + p(x)y' + q(x)y = F$$

Where here

$$\begin{aligned} p(x) &= 4 \\ q(x) &= 3 \\ F &= 0 \end{aligned}$$

Hence the ode is

$$y'' + 4y' + 3y = 0$$

The domain of $p(x) = 4$ is

$$\{-\infty < x < \infty\}$$

And the point $x_0 = 0$ is inside this domain. The domain of $q(x) = 3$ is

$$\{-\infty < x < \infty\}$$

And the point $x_0 = 0$ is also inside this domain. Hence solution exists and is unique.

7.10.2 Solving as second order linear constant coeff ode

This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = 0$$

Where in the above $A = 1, B = 4, C = 3$. Let the solution be $y = e^{\lambda x}$. Substituting this into the ODE gives

$$\lambda^2 e^{\lambda x} + 4\lambda\, e^{\lambda x} + 3\, e^{\lambda x} = 0 \tag{1}$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda x}$ gives

$$\lambda^2 + 4\lambda + 3 = 0 \tag{2}$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form.Using the quadratic formula

$$\lambda_{1,2} = \frac{-B}{2A} \pm \frac{1}{2A}\sqrt{B^2 - 4AC}$$

Substituting $A = 1, B = 4, C = 3$ into the above gives

$$\begin{aligned}\lambda_{1,2} &= \frac{-4}{(2)\,(1)} \pm \frac{1}{(2)\,(1)}\sqrt{4^2 - (4)\,(1)\,(3)}\\ &= -2 \pm 1\end{aligned}$$

Hence

$$\begin{aligned}\lambda_1 &= -2 + 1\\ \lambda_2 &= -2 - 1\end{aligned}$$

Which simplifies to

$$\lambda_1 = -1$$
$$\lambda_2 = -3$$

Since roots are real and distinct, then the solution is

$$y = c_1 e^{\lambda_1 x} + c_2 e^{\lambda_2 x}$$
$$y = c_1 e^{(-1)x} + c_2 e^{(-3)x}$$

Or

$$y = c_1 \mathrm{e}^{-x} + \mathrm{e}^{-3x} c_2$$

Initial conditions are used to solve for the constants of integration.

Looking at the above solution

$$y = c_1 \mathrm{e}^{-x} + \mathrm{e}^{-3x} c_2 \tag{1}$$

Initial conditions are now substituted in the above solution. This will generate the required equations to solve for the integration constants. substituting $y = 2$ and $x = 0$ in the above gives

$$2 = c_1 + c_2 \tag{1A}$$

Taking derivative of the solution gives

$$y' = -c_1 \mathrm{e}^{-x} - 3\,\mathrm{e}^{-3x} c_2$$

substituting $y' = -1$ and $x = 0$ in the above gives

$$-1 = -c_1 - 3c_2 \tag{2A}$$

Equations {1A,2A} are now solved for $\{c_1, c_2\}$. Solving for the constants gives

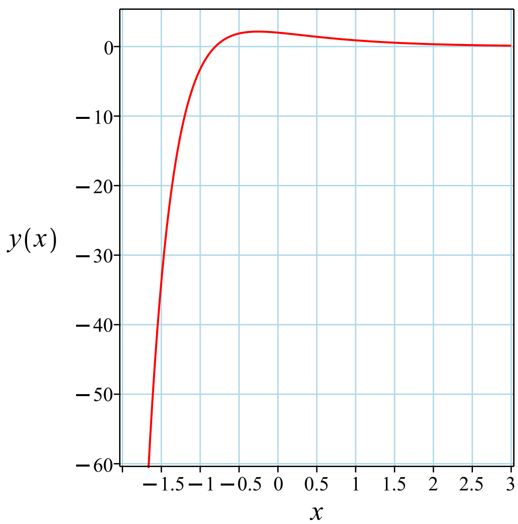
$$c_1 = \frac{5}{2}$$
$$c_2 = -\frac{1}{2}$$

Substituting these values back in above solution results in

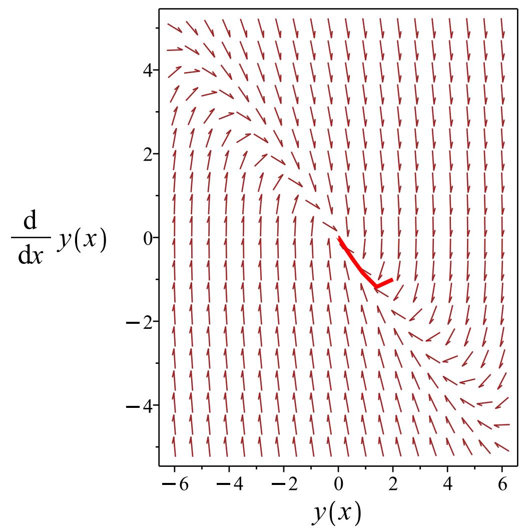
$$y = \frac{5\,\mathrm{e}^{-x}}{2} - \frac{\mathrm{e}^{-3x}}{2}$$

Summary
The solution(s) found are the following

$$y = \frac{5\,e^{-x}}{2} - \frac{e^{-3x}}{2} \tag{1}$$



(a) Solution plot



(b) Slope field plot

Verification of solutions

$$y = \frac{5\,e^{-x}}{2} - \frac{e^{-3x}}{2}$$

Verified OK.

7.10.3 Solving using Kovacic algorithm

Writing the ode as

$$y'' + 4y' + 3y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1 \\ B &= 4 \\ C &= 3 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{1}{1} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= 1 \\ t &= 1 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\,e^{-\int \frac{B}{2A}\,dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0,1,2,4,6,8,\cdots\}$	$\{\cdots,-6,-4,-2,0,2,3,4,5,6,\cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1,2\}$	$\{2,3,4,5,6,7,\cdots\}$

Table 334: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 0 - 0 \\ &= 0 \end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Since $r = 1$ is not a function of x, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$z_1(x) = \mathrm{e}^{-x}$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$\begin{aligned} y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx} \\ &= z_1 e^{-\int \frac{1}{2}\frac{4}{1}\,dx} \\ &= z_1 e^{-2x} \\ &= z_1 \left(\mathrm{e}^{-2x}\right) \end{aligned}$$

Which simplifies to

$$y_1 = \mathrm{e}^{-3x}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\, dx}}{y_1^2}\, dx$$

Substituting gives

$$\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{4}{1}\, dx}}{(y_1)^2}\, dx \\
&= y_1 \int \frac{e^{-4x}}{(y_1)^2}\, dx \\
&= y_1 \left(\frac{\mathrm{e}^{2x}}{2}\right)
\end{aligned}$$

Therefore the solution is

$$\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1 \left(\mathrm{e}^{-3x}\right) + c_2 \left(\mathrm{e}^{-3x} \left(\frac{\mathrm{e}^{2x}}{2}\right)\right)
\end{aligned}$$

Initial conditions are used to solve for the constants of integration.

Looking at the above solution

$$y = c_1 \mathrm{e}^{-3x} + \frac{c_2 \mathrm{e}^{-x}}{2} \tag{1}$$

Initial conditions are now substituted in the above solution. This will generate the required equations to solve for the integration constants. substituting $y = 2$ and $x = 0$ in the above gives

$$2 = c_1 + \frac{c_2}{2} \tag{1A}$$

Taking derivative of the solution gives

$$y' = -3c_1 \mathrm{e}^{-3x} - \frac{c_2 \mathrm{e}^{-x}}{2}$$

substituting $y' = -1$ and $x = 0$ in the above gives

$$-1 = -3c_1 - \frac{c_2}{2} \tag{2A}$$

Equations $\{1A, 2A\}$ are now solved for $\{c_1, c_2\}$. Solving for the constants gives

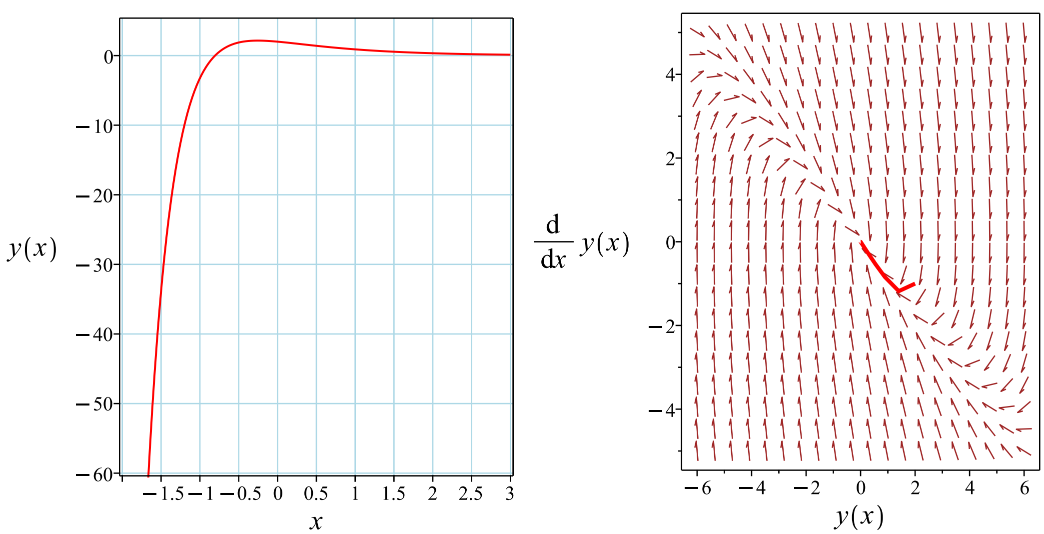
$$c_1 = -\frac{1}{2}$$
$$c_2 = 5$$

Substituting these values back in above solution results in

$$y = \frac{5\,e^{-x}}{2} - \frac{e^{-3x}}{2}$$

Summary
The solution(s) found are the following

$$y = \frac{5\,e^{-x}}{2} - \frac{e^{-3x}}{2} \tag{1}$$



(a) Solution plot (b) Slope field plot

Verification of solutions

$$y = \frac{5\,e^{-x}}{2} - \frac{e^{-3x}}{2}$$

Verified OK.

7.10.4 Maple step by step solution

Let's solve

$$\left[y''+4y'+3y=0, y(0)=2, y'\Big|_{\{x=0\}}=-1\right]$$

- Highest derivative means the order of the ODE is 2

  y''

- Characteristic polynomial of ODE

  $r^2+4r+3=0$

- Factor the characteristic polynomial

  $(r+3)(r+1)=0$

- Roots of the characteristic polynomial

  $r=(-3,-1)$

- 1st solution of the ODE

  $y_1(x)=e^{-3x}$

- 2nd solution of the ODE

  $y_2(x)=e^{-x}$

- General solution of the ODE

  $y=c_1y_1(x)+c_2y_2(x)$

- Substitute in solutions

  $y=c_1e^{-3x}+c_2e^{-x}$

□ Check validity of solution $y=c_1e^{-3x}+c_2e^{-x}$

  ◦ Use initial condition $y(0)=2$

    $2=c_1+c_2$

  ◦ Compute derivative of the solution

    $y'=-3c_1e^{-3x}-c_2e^{-x}$

  ◦ Use the initial condition $y'\Big|_{\{x=0\}}=-1$

    $-1=-3c_1-c_2$

  ◦ Solve for c_1 and c_2

    $\left\{c_1=-\frac{1}{2}, c_2=\frac{5}{2}\right\}$

- Substitute constant values into general solution and simplify

  $y = \frac{5\,e^{-x}}{2} - \frac{e^{-3x}}{2}$

- Solution to the IVP

  $y = \frac{5\,e^{-x}}{2} - \frac{e^{-3x}}{2}$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
<- constant coefficients successful`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 17

```
dsolve([diff(y(x),x$2) +4*diff(y(x),x)+3*y(x) = 0,y(0) = 2, D(y)(0) = -1],y(x), singsol=all)
```

$$y(x) = -\frac{e^{-3x}}{2} + \frac{5\,e^{-x}}{2}$$

✓ Solution by Mathematica

Time used: 0.013 (sec). Leaf size: 23

```
DSolve[{y''[x]+4*y'[x]+3*y[x]==0,{y[0]==2,y'[0]==-1}},y[x],x,IncludeSingularSolutions -> Tru
```

$$y(x) \to \frac{1}{2}e^{-3x}\left(5e^{2x} - 1\right)$$

7.11 problem 11

Internal problem ID [609]
Internal file name [`OUTPUT/609_Sunday_June_05_2022_01_45_47_AM_47206086/index.tex`]

Book: Elementary differential equations and boundary value problems, 10th ed., Boyce and DiPrima
Section: Chapter 3, Second order linear equations, 3.1 Homogeneous Equations with Constant Coefficients, page 144
Problem number: 11.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_linear_constant_coeff"**

Maple gives the following as the ode type

`[[_2nd_order, _missing_x]]`

$$\boxed{6y'' - 5y' + y = 0}$$

With initial conditions

$$[y(0) = 4, y'(0) = 0]$$

7.11.1 Existence and uniqueness analysis

This is a linear ODE. In canonical form it is written as

$$y'' + p(x)y' + q(x)y = F$$

Where here

$$\begin{aligned} p(x) &= -\frac{5}{6} \\ q(x) &= \frac{1}{6} \\ F &= 0 \end{aligned}$$

Hence the ode is

$$y'' - \frac{5y'}{6} + \frac{y}{6} = 0$$

The domain of $p(x) = -\frac{5}{6}$ is

$$\{-\infty < x < \infty\}$$

And the point $x_0 = 0$ is inside this domain. The domain of $q(x) = \frac{1}{6}$ is

$$\{-\infty < x < \infty\}$$

And the point $x_0 = 0$ is also inside this domain. Hence solution exists and is unique.

7.11.2 Solving as second order linear constant coeff ode

This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = 0$$

Where in the above $A = 6, B = -5, C = 1$. Let the solution be $y = e^{\lambda x}$. Substituting this into the ODE gives

$$6\lambda^2 e^{\lambda x} - 5\lambda\, e^{\lambda x} + e^{\lambda x} = 0 \tag{1}$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda x}$ gives

$$6\lambda^2 - 5\lambda + 1 = 0 \tag{2}$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form.Using the quadratic formula

$$\lambda_{1,2} = \frac{-B}{2A} \pm \frac{1}{2A}\sqrt{B^2 - 4AC}$$

Substituting $A = 6, B = -5, C = 1$ into the above gives

$$\begin{aligned}\lambda_{1,2} &= \frac{5}{(2)(6)} \pm \frac{1}{(2)(6)}\sqrt{-5^2 - (4)(6)(1)} \\ &= \frac{5}{12} \pm \frac{1}{12}\end{aligned}$$

Hence

$$\lambda_1 = \frac{5}{12} + \frac{1}{12}$$

$$\lambda_2 = \frac{5}{12} - \frac{1}{12}$$

Which simplifies to

$$\lambda_1 = \frac{1}{2}$$
$$\lambda_2 = \frac{1}{3}$$

Since roots are real and distinct, then the solution is

$$y = c_1 e^{\lambda_1 x} + c_2 e^{\lambda_2 x}$$
$$y = c_1 e^{\left(\frac{1}{2}\right)x} + c_2 e^{\left(\frac{1}{3}\right)x}$$

Or

$$y = c_1 e^{\frac{x}{2}} + c_2 e^{\frac{x}{3}}$$

Initial conditions are used to solve for the constants of integration.

Looking at the above solution

$$y = c_1 e^{\frac{x}{2}} + c_2 e^{\frac{x}{3}} \tag{1}$$

Initial conditions are now substituted in the above solution. This will generate the required equations to solve for the integration constants. substituting $y = 4$ and $x = 0$ in the above gives

$$4 = c_1 + c_2 \tag{1A}$$

Taking derivative of the solution gives

$$y' = \frac{c_1 e^{\frac{x}{2}}}{2} + \frac{c_2 e^{\frac{x}{3}}}{3}$$

substituting $y' = 0$ and $x = 0$ in the above gives

$$0 = \frac{c_1}{2} + \frac{c_2}{3} \tag{2A}$$

Equations {1A,2A} are now solved for $\{c_1, c_2\}$. Solving for the constants gives

$$c_1 = -8$$
$$c_2 = 12$$

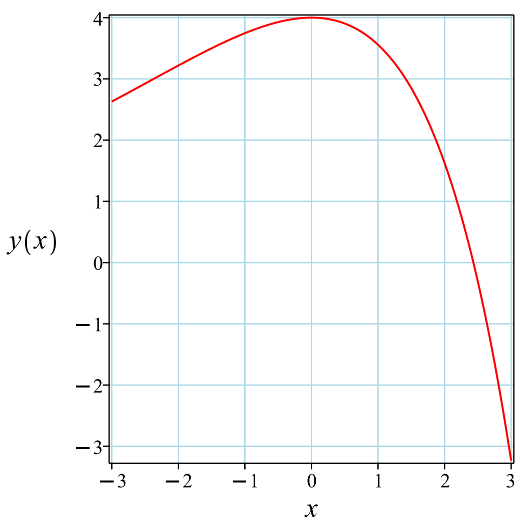
Substituting these values back in above solution results in

$$y = -8\,e^{\frac{x}{2}} + 12\,e^{\frac{x}{3}}$$

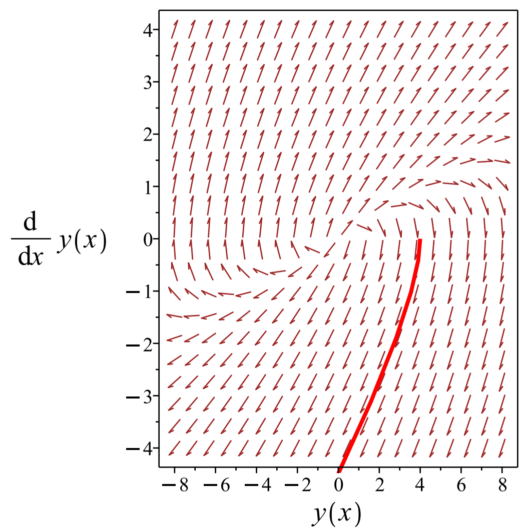
Summary

The solution(s) found are the following

$$y = -8\,e^{\frac{x}{2}} + 12\,e^{\frac{x}{3}} \tag{1}$$



(a) Solution plot



(b) Slope field plot

Verification of solutions

$$y = -8\,e^{\frac{x}{2}} + 12\,e^{\frac{x}{3}}$$

Verified OK.

7.11.3 Solving using Kovacic algorithm

Writing the ode as

$$6y'' - 5y' + y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 6 \\ B &= -5 \\ C &= 1 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = y e^{\int \frac{B}{2A}\, dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{1}{144} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= 1 \\ t &= 144 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = \frac{z(x)}{144} \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 336: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$
\begin{aligned}
O(\infty) &= \deg(t) - \deg(s) \\
&= 0 - 0 \\
&= 0
\end{aligned}
$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Therefore

$$
L = [1]
$$

Since $r = \frac{1}{144}$ is not a function of x, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$
z_1(x) = \mathrm{e}^{-\frac{x}{12}}
$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$
\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{-5}{6}\,dx} \\
&= z_1 e^{\frac{5x}{12}} \\
&= z_1 \left(\mathrm{e}^{\frac{5x}{12}}\right)
\end{aligned}
$$

Which simplifies to

$$
y_1 = \mathrm{e}^{\frac{x}{3}}
$$

The second solution y_2 to the original ode is found using reduction of order

$$
y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx
$$

Substituting gives

$$
\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{-5}{6}\,dx}}{(y_1)^2}\,dx \\
&= y_1 \int \frac{e^{\frac{5x}{6}}}{(y_1)^2}\,dx \\
&= y_1 \left(6\,\mathrm{e}^{\frac{x}{6}}\right)
\end{aligned}
$$

Therefore the solution is

$$\begin{aligned} y &= c_1 y_1 + c_2 y_2 \\ &= c_1\left(\mathrm{e}^{\frac{x}{3}}\right) + c_2\left(\mathrm{e}^{\frac{x}{3}}\left(6\,\mathrm{e}^{\frac{x}{6}}\right)\right) \end{aligned}$$

Initial conditions are used to solve for the constants of integration.

Looking at the above solution

$$y = c_1 \mathrm{e}^{\frac{x}{3}} + 6c_2 \mathrm{e}^{\frac{x}{2}} \tag{1}$$

Initial conditions are now substituted in the above solution. This will generate the required equations to solve for the integration constants. substituting $y = 4$ and $x = 0$ in the above gives

$$4 = c_1 + 6c_2 \tag{1A}$$

Taking derivative of the solution gives

$$y' = \frac{c_1 \mathrm{e}^{\frac{x}{3}}}{3} + 3c_2 \mathrm{e}^{\frac{x}{2}}$$

substituting $y' = 0$ and $x = 0$ in the above gives

$$0 = \frac{c_1}{3} + 3c_2 \tag{2A}$$

Equations {1A,2A} are now solved for $\{c_1, c_2\}$. Solving for the constants gives

$$\begin{aligned} c_1 &= 12 \\ c_2 &= -\frac{4}{3} \end{aligned}$$

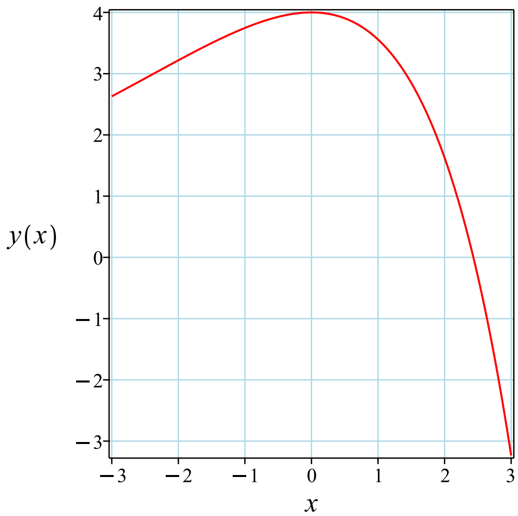
Substituting these values back in above solution results in

$$y = -8\,\mathrm{e}^{\frac{x}{2}} + 12\,\mathrm{e}^{\frac{x}{3}}$$

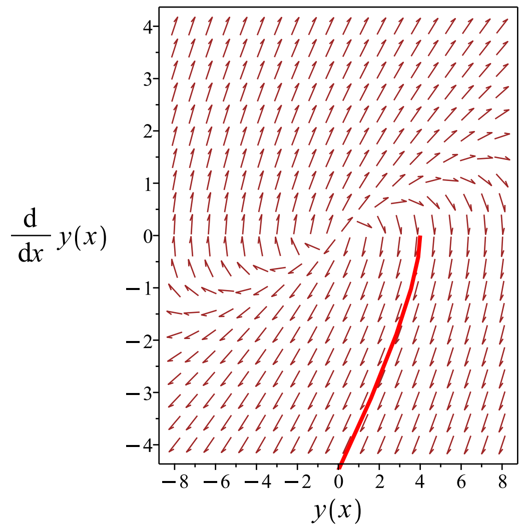
<u>Summary</u>

The solution(s) found are the following

$$y = -8\,\mathrm{e}^{\frac{x}{2}} + 12\,\mathrm{e}^{\frac{x}{3}} \tag{1}$$



(a) Solution plot



(b) Slope field plot

<u>Verification of solutions</u>

$$y = -8\,\mathrm{e}^{\frac{x}{2}} + 12\,\mathrm{e}^{\frac{x}{3}}$$

Verified OK.

7.11.4 Maple step by step solution

Let's solve

$\left[6y'' - 5y' + y = 0, y(0) = 4, y'\Big|_{\{x=0\}} = 0\right]$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y'' = \frac{5y'}{6} - \frac{y}{6}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y'' - \frac{5y'}{6} + \frac{y}{6} = 0$

- Characteristic polynomial of ODE

  $r^2 - \frac{5}{6}r + \frac{1}{6} = 0$

- Factor the characteristic polynomial

$\frac{(3r-1)(2r-1)}{6} = 0$

- Roots of the characteristic polynomial

  $r = \left(\frac{1}{2}, \frac{1}{3}\right)$

- 1st solution of the ODE

  $y_1(x) = \mathrm{e}^{\frac{x}{2}}$

- 2nd solution of the ODE

  $y_2(x) = \mathrm{e}^{\frac{x}{3}}$

- General solution of the ODE

  $y = c_1 y_1(x) + c_2 y_2(x)$

- Substitute in solutions

  $y = c_1 \mathrm{e}^{\frac{x}{2}} + c_2 \mathrm{e}^{\frac{x}{3}}$

□ Check validity of solution $y = c_1 \mathrm{e}^{\frac{x}{2}} + c_2 \mathrm{e}^{\frac{x}{3}}$

  ◦ Use initial condition $y(0) = 4$

    $4 = c_1 + c_2$

  ◦ Compute derivative of the solution

    $y' = \frac{c_1 \mathrm{e}^{\frac{x}{2}}}{2} + \frac{c_2 \mathrm{e}^{\frac{x}{3}}}{3}$

  ◦ Use the initial condition $y'\Big|_{\{x=0\}} = 0$

    $0 = \frac{c_1}{2} + \frac{c_2}{3}$

  ◦ Solve for c_1 and c_2

    $\{c_1 = -8, c_2 = 12\}$

  ◦ Substitute constant values into general solution and simplify

    $y = -8\,\mathrm{e}^{\frac{x}{2}} + 12\,\mathrm{e}^{\frac{x}{3}}$

- Solution to the IVP

  $y = -8\,\mathrm{e}^{\frac{x}{2}} + 12\,\mathrm{e}^{\frac{x}{3}}$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
<- constant coefficients successful`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 17

```
dsolve([6*diff(y(x),x$2) -5*diff(y(x),x)+y(x) = 0,y(0) = 4, D(y)(0) = 0],y(x), singsol=all)
```

$$y(x) = -8\,\mathrm{e}^{\frac{x}{2}} + 12\,\mathrm{e}^{\frac{x}{3}}$$

✓ Solution by Mathematica

Time used: 0.025 (sec). Leaf size: 48

```
DSolve[{6*y''[x]-5*y'[x]+2*y[x]==0,{y[0]==4,y'[0]==0}},y[x],x,IncludeSingularSolutions -> Tr
```

$$y(x) \to \frac{4}{23}e^{5x/12}\left(23\cos\left(\frac{\sqrt{23}x}{12}\right) - 5\sqrt{23}\sin\left(\frac{\sqrt{23}x}{12}\right)\right)$$

7.12 problem 12

Internal problem ID [610]
Internal file name [`OUTPUT/610_Sunday_June_05_2022_01_45_47_AM_93318101/index.tex`]

Book: Elementary differential equations and boundary value problems, 10th ed., Boyce and DiPrima
Section: Chapter 3, Second order linear equations, 3.1 Homogeneous Equations with Constant Coefficients, page 144
Problem number: 12.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "exact linear second order ode", "second_order_integrable_as_is", "second_order_ode_missing_y", "second_order_linear_constant_coeff"**

Maple gives the following as the ode type

```
[[_2nd_order, _missing_x]]
```

$$\boxed{y'' + 3y' = 0}$$

With initial conditions

$$[y(0) = -2, y'(0) = 3]$$

7.12.1 Existence and uniqueness analysis

This is a linear ODE. In canonical form it is written as

$$y'' + p(x)y' + q(x)y = F$$

Where here

$$\begin{aligned} p(x) &= 3 \\ q(x) &= 0 \\ F &= 0 \end{aligned}$$

Hence the ode is

$$y'' + 3y' = 0$$

The domain of $p(x) = 3$ is

$$\{-\infty < x < \infty\}$$

And the point $x_0 = 0$ is inside this domain. Hence solution exists and is unique.

7.12.2 Solving as second order linear constant coeff ode

This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = 0$$

Where in the above $A = 1, B = 3, C = 0$. Let the solution be $y = e^{\lambda x}$. Substituting this into the ODE gives

$$\lambda^2 \mathrm{e}^{\lambda x} + 3\lambda\, \mathrm{e}^{\lambda x} = 0 \tag{1}$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda x}$ gives

$$\lambda^2 + 3\lambda = 0 \tag{2}$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form.Using the quadratic formula

$$\lambda_{1,2} = \frac{-B}{2A} \pm \frac{1}{2A}\sqrt{B^2 - 4AC}$$

Substituting $A = 1, B = 3, C = 0$ into the above gives

$$\begin{aligned} \lambda_{1,2} &= \frac{-3}{(2)\,(1)} \pm \frac{1}{(2)\,(1)}\sqrt{3^2 - (4)\,(1)\,(0)} \\ &= -\frac{3}{2} \pm \frac{3}{2} \end{aligned}$$

Hence

$$\lambda_1 = -\frac{3}{2} + \frac{3}{2}$$
$$\lambda_2 = -\frac{3}{2} - \frac{3}{2}$$

Which simplifies to

$$\lambda_1 = 0$$
$$\lambda_2 = -3$$

Since roots are real and distinct, then the solution is

$$y = c_1 e^{\lambda_1 x} + c_2 e^{\lambda_2 x}$$
$$y = c_1 e^{(0)x} + c_2 e^{(-3)x}$$

Or

$$y = c_1 + \mathrm{e}^{-3x} c_2$$

Initial conditions are used to solve for the constants of integration.

Looking at the above solution

$$y = c_1 + \mathrm{e}^{-3x} c_2 \tag{1}$$

Initial conditions are now substituted in the above solution. This will generate the required equations to solve for the integration constants. substituting $y = -2$ and $x = 0$ in the above gives

$$-2 = c_1 + c_2 \tag{1A}$$

Taking derivative of the solution gives

$$y' = -3\,\mathrm{e}^{-3x} c_2$$

substituting $y' = 3$ and $x = 0$ in the above gives

$$3 = -3c_2 \tag{2A}$$

Equations {1A,2A} are now solved for $\{c_1, c_2\}$. Solving for the constants gives

$$c_1 = -1$$
$$c_2 = -1$$

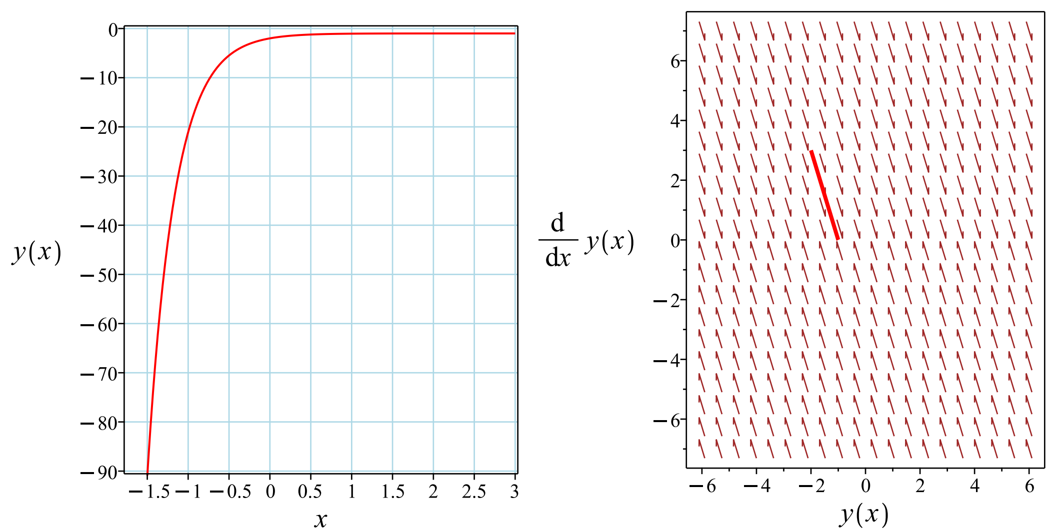
Substituting these values back in above solution results in

$$y = -1 - e^{-3x}$$

Summary

The solution(s) found are the following

$$y = -1 - e^{-3x} \tag{1}$$



(a) Solution plot

(b) Slope field plot

Verification of solutions

$$y = -1 - e^{-3x}$$

Verified OK.

7.12.3 Solving as second order integrable as is ode

Integrating both sides of the ODE w.r.t x gives

$$\int (y'' + 3y')\, dx = 0$$

$$3y + y' = c_1$$

Which is now solved for y. Integrating both sides gives

$$\int \frac{1}{-3y + c_1} dy = \int dx$$

$$-\frac{\ln(-3y + c_1)}{3} = x + c_2$$

Raising both side to exponential gives

$$\frac{1}{\left(-3y+c_1\right)^{\frac{1}{3}}} = \mathrm{e}^{x+c_2}$$

Which simplifies to

$$\frac{1}{\left(-3y+c_1\right)^{\frac{1}{3}}} = c_3\mathrm{e}^{x}$$

Initial conditions are used to solve for the constants of integration.

Looking at the above solution

$$y = -\frac{\mathrm{e}^{-3x}}{3c_3^3} + \frac{c_1}{3} \tag{1}$$

Initial conditions are now substituted in the above solution. This will generate the required equations to solve for the integration constants. substituting $y = -2$ and $x = 0$ in the above gives

$$-2 = \frac{c_1c_3^3 - 1}{3c_3^3} \tag{1A}$$

Taking derivative of the solution gives

$$y' = \frac{\mathrm{e}^{-3x}}{c_3^3}$$

substituting $y' = 3$ and $x = 0$ in the above gives

$$3 = \frac{1}{c_3^3} \tag{2A}$$

Equations {1A,2A} are now solved for $\{c_1, c_3\}$. Warning, unable to solve for constants of integrations.

Verification of solutions N/A

7.12.4 Solving as second order ode missing y ode

This is second order ode with missing dependent variable y. Let

$$p(x) = y'$$

Then

$$p'(x) = y''$$

Hence the ode becomes

$$p'(x) + 3p(x) = 0$$

Which is now solve for $p(x)$ as first order ode. Integrating both sides gives

$$\int -\frac{1}{3p} dp = \int dx$$
$$-\frac{\ln(p)}{3} = x + c_1$$

Raising both side to exponential gives

$$\frac{1}{p^{\frac{1}{3}}} = e^{x+c_1}$$

Which simplifies to

$$\frac{1}{p^{\frac{1}{3}}} = c_2 e^x$$

Initial conditions are used to solve for c_2. Substituting $x = 0$ and $p = 3$ in the above solution gives an equation to solve for the constant of integration.

$$3 = \frac{1}{c_2^3}$$

$$c_2 = \frac{3^{\frac{2}{3}}}{3}$$

Substituting c_2 found above in the general solution gives

$$p(x) = 3\,e^{-3x}$$

Since $p = y'$ then the new first order ode to solve is

$$y' = 3\,e^{-3x}$$

Integrating both sides gives

$$y = \int 3\,e^{-3x}\,\mathrm{d}x$$
$$= -e^{-3x} + c_3$$

Initial conditions are used to solve for c_3. Substituting $x = 0$ and $y = -2$ in the above solution gives an equation to solve for the constant of integration.

$$-2 = -1 + c_3$$

$$c_3 = -1$$

Substituting c_3 found above in the general solution gives

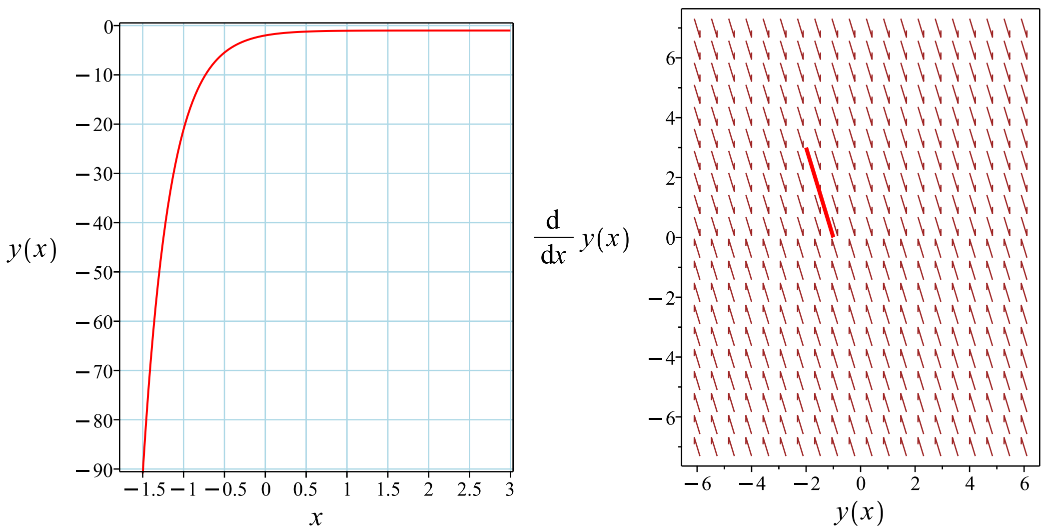
$$y = -1 - e^{-3x}$$

Initial conditions are used to solve for the constants of integration.

Summary

The solution(s) found are the following

$$y = -1 - e^{-3x} \tag{1}$$



(a) Solution plot (b) Slope field plot

Verification of solutions

$$y = -1 - e^{-3x}$$

Verified OK.

7.12.5 Solving as type second_order_integrable_as_is (not using ABC version)

Writing the ode as

$$y'' + 3y' = 0$$

Integrating both sides of the ODE w.r.t x gives

$$\int (y'' + 3y')\, dx = 0$$
$$3y + y' = c_1$$

Which is now solved for y. Integrating both sides gives

$$\int \frac{1}{-3y + c_1} dy = \int dx$$
$$-\frac{\ln(-3y + c_1)}{3} = x + c_2$$

Raising both side to exponential gives

$$\frac{1}{(-3y + c_1)^{\frac{1}{3}}} = \mathrm{e}^{x + c_2}$$

Which simplifies to

$$\frac{1}{(-3y + c_1)^{\frac{1}{3}}} = c_3 \mathrm{e}^{x}$$

Initial conditions are used to solve for the constants of integration.

Looking at the above solution

$$y = -\frac{\mathrm{e}^{-3x}}{3c_3^3} + \frac{c_1}{3} \tag{1}$$

Initial conditions are now substituted in the above solution. This will generate the required equations to solve for the integration constants. substituting $y = -2$ and $x = 0$ in the above gives

$$-2 = \frac{c_1 c_3^3 - 1}{3c_3^3} \tag{1A}$$

Taking derivative of the solution gives

$$y' = \frac{\mathrm{e}^{-3x}}{c_3^3}$$

substituting $y' = 3$ and $x = 0$ in the above gives

$$3 = \frac{1}{c_3^3} \tag{2A}$$

Equations {1A,2A} are now solved for $\{c_1, c_3\}$. Warning, unable to solve for constants of integrations.

7.12.6 Solving using Kovacic algorithm

Writing the ode as

$$y'' + 3y' = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1 \\ B &= 3 \\ C &= 0 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{9}{4} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= 9 \\ t &= 4 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = \frac{9z(x)}{4} \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\,dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 338: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$
\begin{aligned}
O(\infty) &= \deg(t) - \deg(s) \\
&= 0 - 0 \\
&= 0
\end{aligned}
$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Therefore

$$
L = [1]
$$

Since $r = \frac{9}{4}$ is not a function of x, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$
z_1(x) = \mathrm{e}^{-\frac{3x}{2}}
$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$
\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2} \frac{B}{A} \, dx} \\
&= z_1 e^{-\int \frac{1}{2} \frac{3}{1} \, dx}
\end{aligned}
$$

$$
\begin{aligned}
&= z_1 e^{-\frac{3x}{2}} \\
&= z_1 \left(\mathrm{e}^{-\frac{3x}{2}}\right)
\end{aligned}
$$

Which simplifies to

$$
y_1 = \mathrm{e}^{-3x}
$$

The second solution y_2 to the original ode is found using reduction of order

$$
y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\, dx}}{y_1^2}\, dx
$$

Substituting gives

$$
\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{3}{1}\, dx}}{(y_1)^2}\, dx \\
&= y_1 \int \frac{e^{-3x}}{(y_1)^2}\, dx \\
&= y_1 \left(\frac{\mathrm{e}^{3x}}{3}\right)
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1 \left(\mathrm{e}^{-3x}\right) + c_2 \left(\mathrm{e}^{-3x} \left(\frac{\mathrm{e}^{3x}}{3}\right)\right)
\end{aligned}
$$

Initial conditions are used to solve for the constants of integration.

Looking at the above solution

$$
y = c_1 \mathrm{e}^{-3x} + \frac{c_2}{3} \tag{1}
$$

Initial conditions are now substituted in the above solution. This will generate the required equations to solve for the integration constants. substituting $y = -2$ and $x = 0$ in the above gives

$$
-2 = c_1 + \frac{c_2}{3} \tag{1A}
$$

Taking derivative of the solution gives

$$y' = -3c_1 e^{-3x}$$

substituting $y' = 3$ and $x = 0$ in the above gives

$$3 = -3c_1 \tag{2A}$$

Equations {1A,2A} are now solved for $\{c_1, c_2\}$. Solving for the constants gives

$$c_1 = -1$$
$$c_2 = -3$$

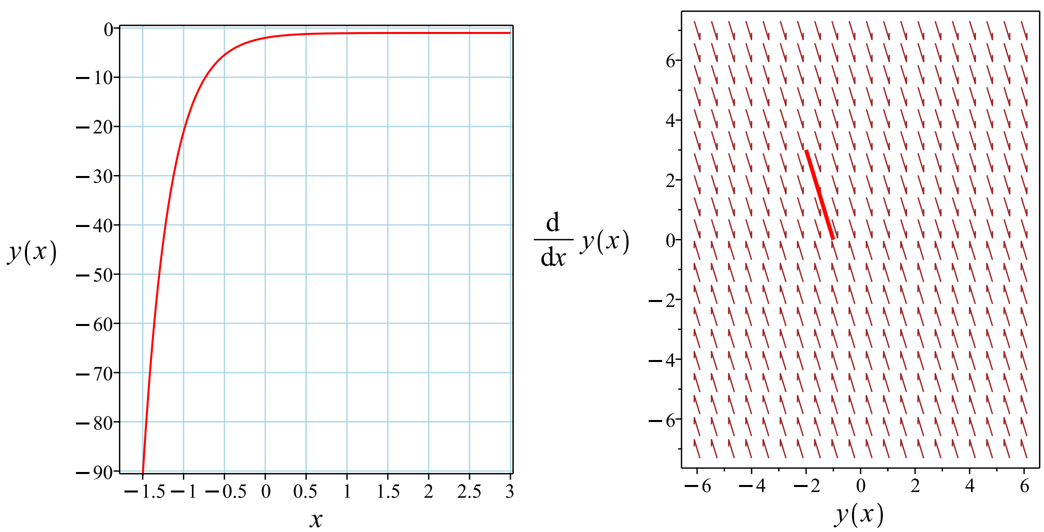
Substituting these values back in above solution results in

$$y = -1 - e^{-3x}$$

Summary

The solution(s) found are the following

$$y = -1 - e^{-3x} \tag{1}$$



(a) Solution plot

(b) Slope field plot

Verification of solutions

$$y = -1 - e^{-3x}$$

Verified OK.

7.12.7 Solving as exact linear second order ode ode

An ode of the form

$$p(x)\,y'' + q(x)\,y' + r(x)\,y = s(x)$$

is exact if

$$p''(x) - q'(x) + r(x) = 0 \tag{1}$$

For the given ode we have

$$\begin{aligned} p(x) &= 1 \\ q(x) &= 3 \\ r(x) &= 0 \\ s(x) &= 0 \end{aligned}$$

Hence

$$\begin{aligned} p''(x) &= 0 \\ q'(x) &= 0 \end{aligned}$$

Therefore (1) becomes

$$0 - (0) + (0) = 0$$

Hence the ode is exact. Since we now know the ode is exact, it can be written as

$$\left(p(x)\,y' + (q(x) - p'(x))\,y\right)' = s(x)$$

Integrating gives

$$p(x)\,y' + (q(x) - p'(x))\,y = \int s(x)\,dx$$

Substituting the above values for p, q, r, s gives

$$3y + y' = c_1$$

We now have a first order ode to solve which is

$$3y + y' = c_1$$

Integrating both sides gives

$$\int \frac{1}{-3y+c_1}dy = \int dx$$
$$-\frac{\ln(-3y+c_1)}{3} = x+c_2$$

Raising both side to exponential gives

$$\frac{1}{(-3y+c_1)^{\frac{1}{3}}} = \mathrm{e}^{x+c_2}$$

Which simplifies to

$$\frac{1}{(-3y+c_1)^{\frac{1}{3}}} = c_3\mathrm{e}^{x}$$

Initial conditions are used to solve for the constants of integration.

Looking at the above solution

$$y = -\frac{\mathrm{e}^{-3x}}{3c_3^3} + \frac{c_1}{3} \tag{1}$$

Initial conditions are now substituted in the above solution. This will generate the required equations to solve for the integration constants. substituting $y = -2$ and $x = 0$ in the above gives

$$-2 = \frac{c_1c_3^3 - 1}{3c_3^3} \tag{1A}$$

Taking derivative of the solution gives

$$y' = \frac{\mathrm{e}^{-3x}}{c_3^3}$$

substituting $y' = 3$ and $x = 0$ in the above gives

$$3 = \frac{1}{c_3^3} \tag{2A}$$

Equations {1A,2A} are now solved for $\{c_1, c_3\}$. Warning, unable to solve for constants of integrations.

<u>Verification of solutions</u> N/A

7.12.8 Maple step by step solution

Let's solve

$$\left[y'' + 3y' = 0, y(0) = -2, y'\Big|_{\{x=0\}} = 3\right]$$

- Highest derivative means the order of the ODE is 2

  y''

- Characteristic polynomial of ODE

  $r^2 + 3r = 0$

- Factor the characteristic polynomial

  $r(r+3) = 0$

- Roots of the characteristic polynomial

  $r = (-3, 0)$

- 1st solution of the ODE

  $y_1(x) = e^{-3x}$

- 2nd solution of the ODE

  $y_2(x) = 1$

- General solution of the ODE

  $y = c_1 y_1(x) + c_2 y_2(x)$

- Substitute in solutions

  $y = c_1 e^{-3x} + c_2$

□ Check validity of solution $y = c_1 e^{-3x} + c_2$

- ○ Use initial condition $y(0) = -2$

  $-2 = c_1 + c_2$

- ○ Compute derivative of the solution

  $y' = -3c_1 e^{-3x}$

- ○ Use the initial condition $y'\Big|_{\{x=0\}} = 3$

  $3 = -3c_1$

- ○ Solve for c_1 and c_2

  $\{c_1 = -1, c_2 = -1\}$

- ◦ Substitute constant values into general solution and simplify

  $$y = -1 - e^{-3x}$$

- Solution to the IVP

  $$y = -1 - e^{-3x}$$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
<- constant coefficients successful`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 12

```
dsolve([diff(y(x),x$2) +3*diff(y(x),x) = 0,y(0) = -2, D(y)(0) = 3],y(x), singsol=all)
```

$$y(x) = -1 - e^{-3x}$$

✓ Solution by Mathematica

Time used: 0.014 (sec). Leaf size: 14

```
DSolve[{y''[x]+3*y'[x]==0,{y[0]==-2,y'[0]==3}},y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to -e^{-3x} - 1$$

7.13 problem 13

Internal problem ID [611]
Internal file name [`OUTPUT/611_Sunday_June_05_2022_01_45_48_AM_49802737/index.tex`]

Book: Elementary differential equations and boundary value problems, 10th ed., Boyce and DiPrima
Section: Chapter 3, Second order linear equations, 3.1 Homogeneous Equations with Constant Coefficients, page 144
Problem number: 13.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_linear_constant_coeff"**

Maple gives the following as the ode type

`[[_2nd_order, _missing_x]]`

$$\boxed{y'' + 5y' + 3y = 0}$$

With initial conditions

$$[y(0) = 1, y'(0) = 0]$$

7.13.1 Existence and uniqueness analysis

This is a linear ODE. In canonical form it is written as

$$y'' + p(x)y' + q(x)y = F$$

Where here

$$\begin{aligned} p(x) &= 5 \\ q(x) &= 3 \\ F &= 0 \end{aligned}$$

Hence the ode is

$$y'' + 5y' + 3y = 0$$

The domain of $p(x) = 5$ is

$$\{-\infty < x < \infty\}$$

And the point $x_0 = 0$ is inside this domain. The domain of $q(x) = 3$ is

$$\{-\infty < x < \infty\}$$

And the point $x_0 = 0$ is also inside this domain. Hence solution exists and is unique.

7.13.2 Solving as second order linear constant coeff ode

This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = 0$$

Where in the above $A = 1, B = 5, C = 3$. Let the solution be $y = e^{\lambda x}$. Substituting this into the ODE gives

$$\lambda^2 e^{\lambda x} + 5\lambda\, e^{\lambda x} + 3\, e^{\lambda x} = 0 \tag{1}$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda x}$ gives

$$\lambda^2 + 5\lambda + 3 = 0 \tag{2}$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form.Using the quadratic formula

$$\lambda_{1,2} = \frac{-B}{2A} \pm \frac{1}{2A}\sqrt{B^2 - 4AC}$$

Substituting $A = 1, B = 5, C = 3$ into the above gives

$$\begin{aligned}\lambda_{1,2} &= \frac{-5}{(2)\,(1)} \pm \frac{1}{(2)\,(1)}\sqrt{5^2 - (4)\,(1)\,(3)} \\ &= -\frac{5}{2} \pm \frac{\sqrt{13}}{2}\end{aligned}$$

Hence

$$\lambda_1 = -\frac{5}{2} + \frac{\sqrt{13}}{2}$$

$$\lambda_2 = -\frac{5}{2} - \frac{\sqrt{13}}{2}$$

Which simplifies to

$$\lambda_1 = -\frac{5}{2}+\frac{\sqrt{13}}{2}$$
$$\lambda_2 = -\frac{5}{2}-\frac{\sqrt{13}}{2}$$

Since roots are real and distinct, then the solution is

$$y = c_1 e^{\lambda_1 x} + c_2 e^{\lambda_2 x}$$
$$y = c_1 e^{\left(-\frac{5}{2}+\frac{\sqrt{13}}{2}\right)x} + c_2 e^{\left(-\frac{5}{2}-\frac{\sqrt{13}}{2}\right)x}$$

Or

$$y = c_1 \mathrm{e}^{\left(-\frac{5}{2}+\frac{\sqrt{13}}{2}\right)x} + c_2 \mathrm{e}^{\left(-\frac{5}{2}-\frac{\sqrt{13}}{2}\right)x}$$

Initial conditions are used to solve for the constants of integration.

Looking at the above solution

$$y = c_1 \mathrm{e}^{\left(-\frac{5}{2}+\frac{\sqrt{13}}{2}\right)x} + c_2 \mathrm{e}^{\left(-\frac{5}{2}-\frac{\sqrt{13}}{2}\right)x} \tag{1}$$

Initial conditions are now substituted in the above solution. This will generate the required equations to solve for the integration constants. substituting $y = 1$ and $x = 0$ in the above gives

$$1 = c_1 + c_2 \tag{1A}$$

Taking derivative of the solution gives

$$y' = c_1 \left(-\frac{5}{2}+\frac{\sqrt{13}}{2}\right) \mathrm{e}^{\left(-\frac{5}{2}+\frac{\sqrt{13}}{2}\right)x} + c_2 \left(-\frac{5}{2}-\frac{\sqrt{13}}{2}\right) \mathrm{e}^{\left(-\frac{5}{2}-\frac{\sqrt{13}}{2}\right)x}$$

substituting $y' = 0$ and $x = 0$ in the above gives

$$0 = \frac{(c_1 - c_2)\sqrt{13}}{2} - \frac{5c_1}{2} - \frac{5c_2}{2} \tag{2A}$$

Equations {1A,2A} are now solved for $\{c_1, c_2\}$. Solving for the constants gives

$$c_1 = \frac{1}{2}+\frac{5\sqrt{13}}{26}$$
$$c_2 = \frac{1}{2}-\frac{5\sqrt{13}}{26}$$

Substituting these values back in above solution results in

$$y = \frac{e^{\frac{(-5+\sqrt{13})x}{2}}}{2} + \frac{5\,e^{\frac{(-5+\sqrt{13})x}{2}}\sqrt{13}}{26} + \frac{e^{-\frac{(5+\sqrt{13})x}{2}}}{2} - \frac{5\,e^{-\frac{(5+\sqrt{13})x}{2}}\sqrt{13}}{26}$$

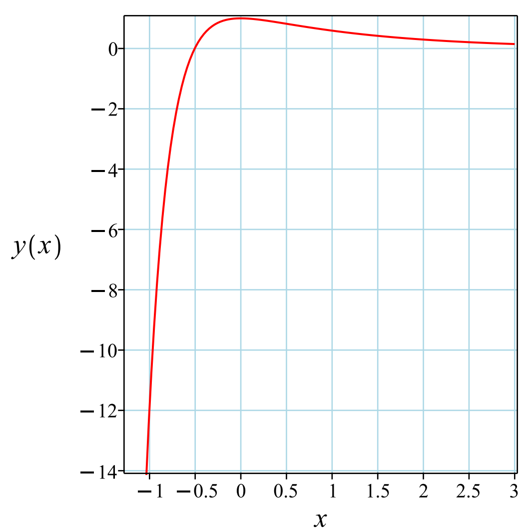
Which simplifies to

$$y = \frac{(13+5\sqrt{13})\,e^{\frac{(-5+\sqrt{13})x}{2}}}{26} + \frac{(13-5\sqrt{13})\,e^{-\frac{(5+\sqrt{13})x}{2}}}{26}$$

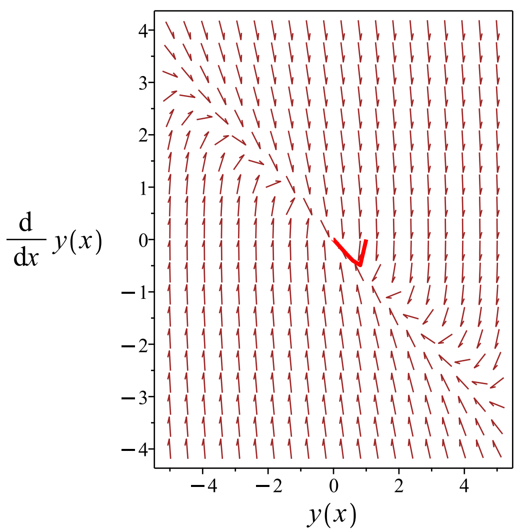
Summary

The solution(s) found are the following

$$y = \frac{(13+5\sqrt{13})\,e^{\frac{(-5+\sqrt{13})x}{2}}}{26} + \frac{(13-5\sqrt{13})\,e^{-\frac{(5+\sqrt{13})x}{2}}}{26} \tag{1}$$



(a) Solution plot



(b) Slope field plot

Verification of solutions

$$y = \frac{(13+5\sqrt{13})\,e^{\frac{(-5+\sqrt{13})x}{2}}}{26} + \frac{(13-5\sqrt{13})\,e^{-\frac{(5+\sqrt{13})x}{2}}}{26}$$

Verified OK.

7.13.3 Solving using Kovacic algorithm

Writing the ode as

$$y'' + 5y' + 3y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1 \\ B &= 5 \\ C &= 3 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{13}{4} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= 13 \\ t &= 4 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = \frac{13z(x)}{4} \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\,dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 340: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$
\begin{aligned}
O(\infty) &= \deg(t) - \deg(s) \\
&= 0 - 0 \\
&= 0
\end{aligned}
$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Therefore

$$
L = [1]
$$

Since $r = \frac{13}{4}$ is not a function of x, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$
z_1(x) = \mathrm{e}^{-\frac{x\sqrt{13}}{2}}
$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$
\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\, dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{5}{1}\, dx}
\end{aligned}
$$

$$
\begin{aligned}
&= z_1 e^{-\frac{5x}{2}} \\
&= z_1 \left(\mathrm{e}^{-\frac{5x}{2}}\right)
\end{aligned}
$$

Which simplifies to

$$
y_1 = \mathrm{e}^{-\frac{\left(5+\sqrt{13}\right)x}{2}}
$$

The second solution y_2 to the original ode is found using reduction of order

$$
y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\, dx}}{y_1^2}\, dx
$$

Substituting gives

$$
\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{5}{1}\, dx}}{(y_1)^2}\, dx \\
&= y_1 \int \frac{e^{-5x}}{(y_1)^2}\, dx \\
&= y_1 \left(\frac{\sqrt{13}\, \mathrm{e}^{x\sqrt{13}}}{13}\right)
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1 \left(\mathrm{e}^{-\frac{\left(5+\sqrt{13}\right)x}{2}}\right) + c_2 \left(\mathrm{e}^{-\frac{\left(5+\sqrt{13}\right)x}{2}} \left(\frac{\sqrt{13}\, \mathrm{e}^{x\sqrt{13}}}{13}\right)\right)
\end{aligned}
$$

Initial conditions are used to solve for the constants of integration.

Looking at the above solution

$$
y = c_1 \mathrm{e}^{-\frac{\left(5+\sqrt{13}\right)x}{2}} + \frac{c_2 \mathrm{e}^{\frac{\left(-5+\sqrt{13}\right)x}{2}} \sqrt{13}}{13} \tag{1}
$$

Initial conditions are now substituted in the above solution. This will generate the required equations to solve for the integration constants. substituting $y = 1$ and $x = 0$ in the above gives

$$
1 = c_1 + \frac{c_2\sqrt{13}}{13} \tag{1A}
$$

Taking derivative of the solution gives

$$y' = c_1\left(-\frac{5}{2}-\frac{\sqrt{13}}{2}\right)\mathrm{e}^{-\frac{\left(5+\sqrt{13}\right)x}{2}} + \frac{c_2\left(-\frac{5}{2}+\frac{\sqrt{13}}{2}\right)\mathrm{e}^{\frac{\left(-5+\sqrt{13}\right)x}{2}}\sqrt{13}}{13}$$

substituting $y' = 0$ and $x = 0$ in the above gives

$$0 = \frac{(-13c_1 - 5c_2)\sqrt{13}}{26} - \frac{5c_1}{2} + \frac{c_2}{2} \tag{2A}$$

Equations {1A,2A} are now solved for $\{c_1, c_2\}$. Solving for the constants gives

$$\begin{aligned} c_1 &= \frac{1}{2} - \frac{5\sqrt{13}}{26} \\ c_2 &= \frac{5}{2} + \frac{\sqrt{13}}{2} \end{aligned}$$

Substituting these values back in above solution results in

$$y = \frac{\mathrm{e}^{\frac{\left(-5+\sqrt{13}\right)x}{2}}}{2} + \frac{5\,\mathrm{e}^{\frac{\left(-5+\sqrt{13}\right)x}{2}}\sqrt{13}}{26} + \frac{\mathrm{e}^{-\frac{\left(5+\sqrt{13}\right)x}{2}}}{2} - \frac{5\,\mathrm{e}^{-\frac{\left(5+\sqrt{13}\right)x}{2}}\sqrt{13}}{26}$$

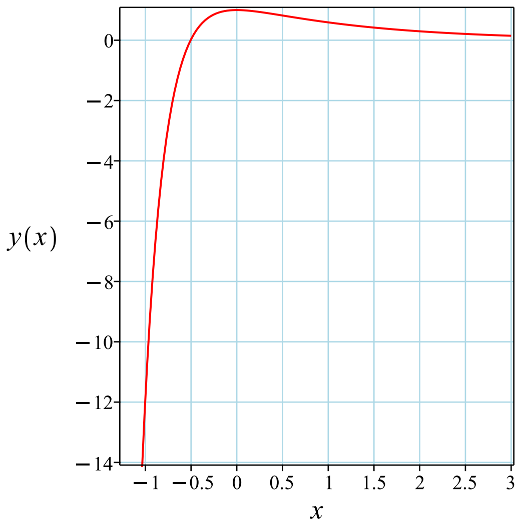
Which simplifies to

$$y = \frac{\left(13+5\sqrt{13}\right)\mathrm{e}^{\frac{\left(-5+\sqrt{13}\right)x}{2}}}{26} + \frac{\left(13-5\sqrt{13}\right)\mathrm{e}^{-\frac{\left(5+\sqrt{13}\right)x}{2}}}{26}$$

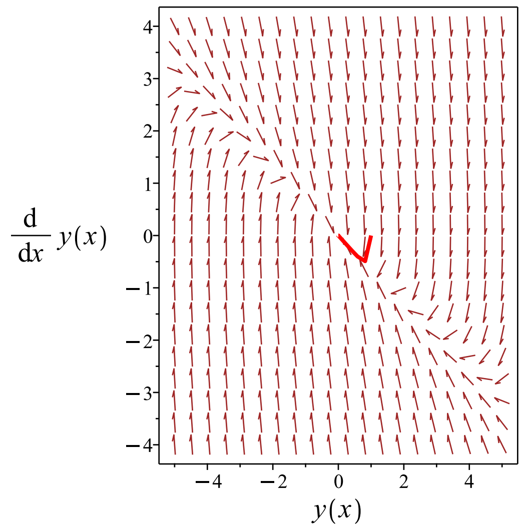
Summary

The solution(s) found are the following

$$y = \frac{\left(13+5\sqrt{13}\right)\mathrm{e}^{\frac{\left(-5+\sqrt{13}\right)x}{2}}}{26} + \frac{\left(13-5\sqrt{13}\right)\mathrm{e}^{-\frac{\left(5+\sqrt{13}\right)x}{2}}}{26} \tag{1}$$



(a) Solution plot



(b) Slope field plot

Verification of solutions

$$y = \frac{(13+5\sqrt{13})\,e^{\frac{(-5+\sqrt{13})x}{2}}}{26} + \frac{(13-5\sqrt{13})\,e^{-\frac{(5+\sqrt{13})x}{2}}}{26}$$

Verified OK.

7.13.4 Maple step by step solution

Let's solve

$\left[y''+5y'+3y=0, y(0)=1, y'\Big|_{\{x=0\}}=0\right]$

- Highest derivative means the order of the ODE is 2

  y''

- Characteristic polynomial of ODE

  $r^2+5r+3=0$

- Use quadratic formula to solve for r

  $r=\frac{(-5)\pm\left(\sqrt{13}\right)}{2}$

- Roots of the characteristic polynomial

  $r=\left(-\frac{5}{2}-\frac{\sqrt{13}}{2}, -\frac{5}{2}+\frac{\sqrt{13}}{2}\right)$

- 1st solution of the ODE

  $$y_1(x) = \mathrm{e}^{\left(-\frac{5}{2}-\frac{\sqrt{13}}{2}\right)x}$$

- 2nd solution of the ODE

  $$y_2(x) = \mathrm{e}^{\left(-\frac{5}{2}+\frac{\sqrt{13}}{2}\right)x}$$

- General solution of the ODE

  $$y = c_1 y_1(x) + c_2 y_2(x)$$

- Substitute in solutions

  $$y = c_1 \mathrm{e}^{\left(-\frac{5}{2}-\frac{\sqrt{13}}{2}\right)x} + c_2 \mathrm{e}^{\left(-\frac{5}{2}+\frac{\sqrt{13}}{2}\right)x}$$

□ Check validity of solution $y = c_1 \mathrm{e}^{\left(-\frac{5}{2}-\frac{\sqrt{13}}{2}\right)x} + c_2 \mathrm{e}^{\left(-\frac{5}{2}+\frac{\sqrt{13}}{2}\right)x}$

  - Use initial condition $y(0) = 1$

    $$1 = c_1 + c_2$$

  - Compute derivative of the solution

    $$y' = c_1 \left(-\frac{5}{2} - \frac{\sqrt{13}}{2}\right) \mathrm{e}^{\left(-\frac{5}{2}-\frac{\sqrt{13}}{2}\right)x} + c_2 \left(-\frac{5}{2} + \frac{\sqrt{13}}{2}\right) \mathrm{e}^{\left(-\frac{5}{2}+\frac{\sqrt{13}}{2}\right)x}$$

  - Use the initial condition $y'\Big|_{\{x=0\}} = 0$

    $$0 = c_1 \left(-\frac{5}{2} - \frac{\sqrt{13}}{2}\right) + \left(-\frac{5}{2} + \frac{\sqrt{13}}{2}\right) c_2$$

  - Solve for c_1 and c_2

    $$\left\{c_1 = \frac{1}{2} - \frac{5\sqrt{13}}{26}, c_2 = \frac{1}{2} + \frac{5\sqrt{13}}{26}\right\}$$

  - Substitute constant values into general solution and simplify

    $$y = \frac{\left(13+5\sqrt{13}\right)\mathrm{e}^{\frac{\left(-5+\sqrt{13}\right)x}{2}}}{26} + \frac{\left(13-5\sqrt{13}\right)\mathrm{e}^{-\frac{\left(5+\sqrt{13}\right)x}{2}}}{26}$$

- Solution to the IVP

  $$y = \frac{\left(13+5\sqrt{13}\right)\mathrm{e}^{\frac{\left(-5+\sqrt{13}\right)x}{2}}}{26} + \frac{\left(13-5\sqrt{13}\right)\mathrm{e}^{-\frac{\left(5+\sqrt{13}\right)x}{2}}}{26}$$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
<- constant coefficients successful`
```

✓ Solution by Maple

Time used: 0.047 (sec). Leaf size: 39

```
dsolve([diff(y(x),x$2) +5*diff(y(x),x)+3*y(x) = 0,y(0) = 1, D(y)(0) = 0],y(x), singsol=all)
```

$$y(x) = \frac{\left(13+5\sqrt{13}\right) \mathrm{e}^{\frac{\left(-5+\sqrt{13}\right)x}{2}}}{26}+\frac{\left(13-5\sqrt{13}\right) \mathrm{e}^{-\frac{\left(5+\sqrt{13}\right)x}{2}}}{26}$$

✓ Solution by Mathematica

Time used: 0.023 (sec). Leaf size: 51

```
DSolve[{y''[x]+5*y'[x]+3*y[x]==0,{y[0]==1,y'[0]==0}},y[x],x,IncludeSingularSolutions -> True
```

$$y(x) \to \frac{1}{26}e^{-\frac{1}{2}\left(5+\sqrt{13}\right)x}\left(\left(13+5\sqrt{13}\right)e^{\sqrt{13}x}+13-5\sqrt{13}\right)$$

7.14 problem 14

Internal problem ID [612]
Internal file name [`OUTPUT/612_Sunday_June_05_2022_01_45_50_AM_78937181/index.tex`]

Book: Elementary differential equations and boundary value problems, 10th ed., Boyce and DiPrima
Section: Chapter 3, Second order linear equations, 3.1 Homogeneous Equations with Constant Coefficients, page 144
Problem number: 14.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_linear_constant_coeff"**

Maple gives the following as the ode type

`[[_2nd_order, _missing_x]]`

$$\boxed{2y'' + y' - 4y = 0}$$

With initial conditions

$$[y(0) = 0, y'(0) = 1]$$

7.14.1 Existence and uniqueness analysis

This is a linear ODE. In canonical form it is written as

$$y'' + p(x)y' + q(x)y = F$$

Where here

$$\begin{aligned} p(x) &= \frac{1}{2} \\ q(x) &= -2 \\ F &= 0 \end{aligned}$$

Hence the ode is

$$y'' + \frac{y'}{2} - 2y = 0$$

The domain of $p(x) = \frac{1}{2}$ is

$$\{-\infty < x < \infty\}$$

And the point $x_0 = 0$ is inside this domain. The domain of $q(x) = -2$ is

$$\{-\infty < x < \infty\}$$

And the point $x_0 = 0$ is also inside this domain. Hence solution exists and is unique.

7.14.2 Solving as second order linear constant coeff ode

This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = 0$$

Where in the above $A = 2, B = 1, C = -4$. Let the solution be $y = e^{\lambda x}$. Substituting this into the ODE gives

$$2\lambda^2 e^{\lambda x} + \lambda\, e^{\lambda x} - 4\, e^{\lambda x} = 0 \tag{1}$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda x}$ gives

$$2\lambda^2 + \lambda - 4 = 0 \tag{2}$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form.Using the quadratic formula

$$\lambda_{1,2} = \frac{-B}{2A} \pm \frac{1}{2A}\sqrt{B^2 - 4AC}$$

Substituting $A = 2, B = 1, C = -4$ into the above gives

$$\begin{aligned}
\lambda_{1,2} &= \frac{-1}{(2)\,(2)} \pm \frac{1}{(2)\,(2)}\sqrt{1^2 - (4)\,(2)\,(-4)} \\
&= -\frac{1}{4} \pm \frac{\sqrt{33}}{4}
\end{aligned}$$

Hence

$$\lambda_1 = -\frac{1}{4} + \frac{\sqrt{33}}{4}$$

$$\lambda_2 = -\frac{1}{4} - \frac{\sqrt{33}}{4}$$

Which simplifies to

$$\lambda_1 = -\frac{1}{4} + \frac{\sqrt{33}}{4}$$
$$\lambda_2 = -\frac{1}{4} - \frac{\sqrt{33}}{4}$$

Since roots are real and distinct, then the solution is

$$y = c_1 e^{\lambda_1 x} + c_2 e^{\lambda_2 x}$$
$$y = c_1 e^{\left(-\frac{1}{4}+\frac{\sqrt{33}}{4}\right)x} + c_2 e^{\left(-\frac{1}{4}-\frac{\sqrt{33}}{4}\right)x}$$

Or

$$y = c_1 \mathrm{e}^{\left(-\frac{1}{4}+\frac{\sqrt{33}}{4}\right)x} + c_2 \mathrm{e}^{\left(-\frac{1}{4}-\frac{\sqrt{33}}{4}\right)x}$$

Initial conditions are used to solve for the constants of integration.

Looking at the above solution

$$y = c_1 \mathrm{e}^{\left(-\frac{1}{4}+\frac{\sqrt{33}}{4}\right)x} + c_2 \mathrm{e}^{\left(-\frac{1}{4}-\frac{\sqrt{33}}{4}\right)x} \tag{1}$$

Initial conditions are now substituted in the above solution. This will generate the required equations to solve for the integration constants. substituting $y = 0$ and $x = 0$ in the above gives

$$0 = c_1 + c_2 \tag{1A}$$

Taking derivative of the solution gives

$$y' = c_1 \left(-\frac{1}{4} + \frac{\sqrt{33}}{4}\right) \mathrm{e}^{\left(-\frac{1}{4}+\frac{\sqrt{33}}{4}\right)x} + c_2 \left(-\frac{1}{4} - \frac{\sqrt{33}}{4}\right) \mathrm{e}^{\left(-\frac{1}{4}-\frac{\sqrt{33}}{4}\right)x}$$

substituting $y' = 1$ and $x = 0$ in the above gives

$$1 = \frac{(c_1 - c_2)\sqrt{33}}{4} - \frac{c_1}{4} - \frac{c_2}{4} \tag{2A}$$

Equations {1A,2A} are now solved for $\{c_1, c_2\}$. Solving for the constants gives

$$c_1 = \frac{2\sqrt{33}}{33}$$
$$c_2 = -\frac{2\sqrt{33}}{33}$$

Substituting these values back in above solution results in

$$y = \frac{2\sqrt{33}\,\mathrm{e}^{\frac{\left(-1+\sqrt{33}\right)x}{4}}}{33} - \frac{2\sqrt{33}\,\mathrm{e}^{-\frac{\left(1+\sqrt{33}\right)x}{4}}}{33}$$

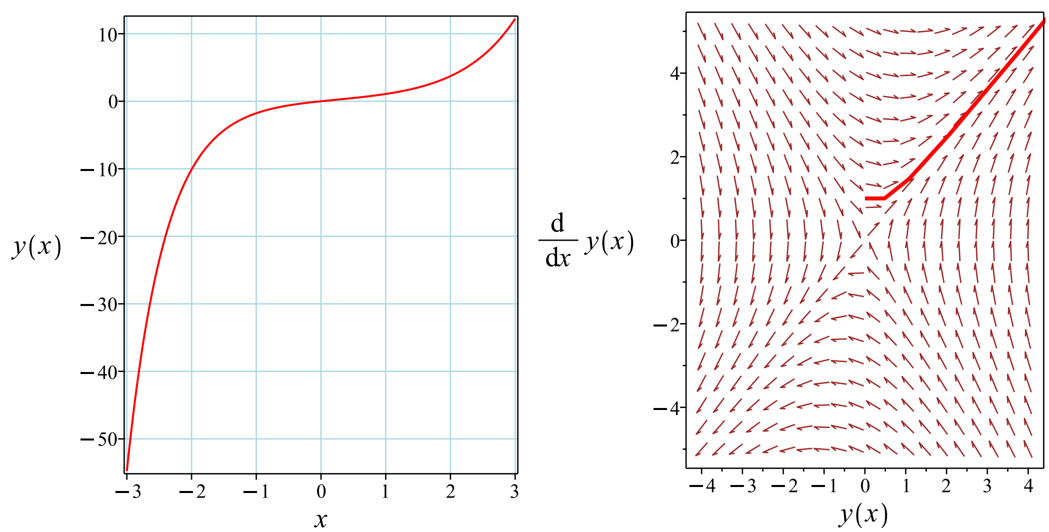
Which simplifies to

$$y = \frac{2\sqrt{33}\left(\mathrm{e}^{\frac{\left(-1+\sqrt{33}\right)x}{4}} - \mathrm{e}^{-\frac{\left(1+\sqrt{33}\right)x}{4}}\right)}{33}$$

Summary

The solution(s) found are the following

$$y = \frac{2\sqrt{33}\left(\mathrm{e}^{\frac{\left(-1+\sqrt{33}\right)x}{4}} - \mathrm{e}^{-\frac{\left(1+\sqrt{33}\right)x}{4}}\right)}{33} \tag{1}$$



(a) Solution plot

(b) Slope field plot

Verification of solutions

$$y = \frac{2\sqrt{33}\left(\mathrm{e}^{\frac{\left(-1+\sqrt{33}\right)x}{4}} - \mathrm{e}^{-\frac{\left(1+\sqrt{33}\right)x}{4}}\right)}{33}$$

Verified OK.

7.14.3 Solving using Kovacic algorithm

Writing the ode as

$$2y'' + y' - 4y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 2 \\ B &= 1 \\ C &= -4 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{33}{16} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= 33 \\ t &= 16 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = \frac{33z(x)}{16} \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\,dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 342: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$
\begin{aligned}
O(\infty) &= \deg(t) - \deg(s) \\
&= 0 - 0 \\
&= 0
\end{aligned}
$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Therefore

$$
L = [1]
$$

Since $r = \frac{33}{16}$ is not a function of x, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$
z_1(x) = \mathrm{e}^{-\frac{x\sqrt{33}}{4}}
$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$
\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{1}{2}\,dx}
\end{aligned}
$$

$$
\begin{aligned}
&= z_1 e^{-\frac{x}{4}} \\
&= z_1 \left(\mathrm{e}^{-\frac{x}{4}}\right)
\end{aligned}
$$

Which simplifies to

$$
y_1 = \mathrm{e}^{-\frac{\left(1+\sqrt{33}\right)x}{4}}
$$

The second solution y_2 to the original ode is found using reduction of order

$$
y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\, dx}}{y_1^2}\, dx
$$

Substituting gives

$$
\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{1}{2}\, dx}}{\left(y_1\right)^2}\, dx \\
&= y_1 \int \frac{e^{-\frac{x}{2}}}{\left(y_1\right)^2}\, dx \\
&= y_1 \left(\frac{2\sqrt{33}\,\mathrm{e}^{\frac{x\sqrt{33}}{2}}}{33}\right)
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1 \left(\mathrm{e}^{-\frac{\left(1+\sqrt{33}\right)x}{4}}\right) + c_2 \left(\mathrm{e}^{-\frac{\left(1+\sqrt{33}\right)x}{4}} \left(\frac{2\sqrt{33}\,\mathrm{e}^{\frac{x\sqrt{33}}{2}}}{33}\right)\right)
\end{aligned}
$$

Initial conditions are used to solve for the constants of integration.

Looking at the above solution

$$
y = c_1 \mathrm{e}^{-\frac{\left(1+\sqrt{33}\right)x}{4}} + \frac{2c_2\sqrt{33}\,\mathrm{e}^{\frac{\left(-1+\sqrt{33}\right)x}{4}}}{33} \tag{1}
$$

Initial conditions are now substituted in the above solution. This will generate the required equations to solve for the integration constants. substituting $y = 0$ and $x = 0$ in the above gives

$$
0 = c_1 + \frac{2c_2\sqrt{33}}{33} \tag{1A}
$$

Taking derivative of the solution gives

$$y' = c_1\left(-\frac{1}{4}-\frac{\sqrt{33}}{4}\right)\mathrm{e}^{-\frac{\left(1+\sqrt{33}\right)x}{4}}+\frac{2c_2\sqrt{33}\left(-\frac{1}{4}+\frac{\sqrt{33}}{4}\right)\mathrm{e}^{\frac{\left(-1+\sqrt{33}\right)x}{4}}}{33}$$

substituting $y' = 1$ and $x = 0$ in the above gives

$$1 = \frac{\left(-33c_1-2c_2\right)\sqrt{33}}{132}-\frac{c_1}{4}+\frac{c_2}{2} \tag{2A}$$

Equations {1A,2A} are now solved for $\{c_1, c_2\}$. Solving for the constants gives

$$c_1 = -\frac{2\sqrt{33}}{33}$$
$$c_2 = 1$$

Substituting these values back in above solution results in

$$y = \frac{2\sqrt{33}\,\mathrm{e}^{\frac{\left(-1+\sqrt{33}\right)x}{4}}}{33}-\frac{2\sqrt{33}\,\mathrm{e}^{-\frac{\left(1+\sqrt{33}\right)x}{4}}}{33}$$

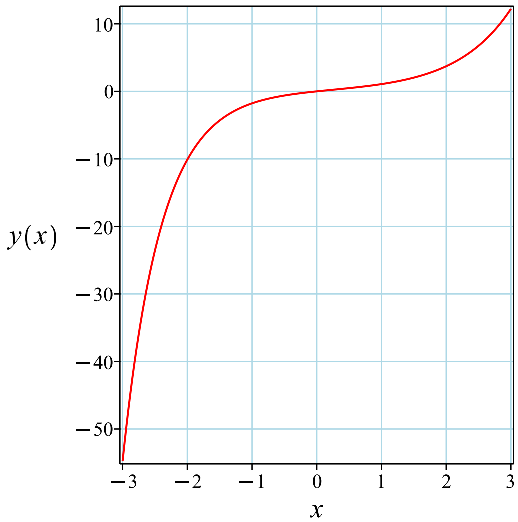
Which simplifies to

$$y = \frac{2\sqrt{33}\left(\mathrm{e}^{\frac{\left(-1+\sqrt{33}\right)x}{4}}-\mathrm{e}^{-\frac{\left(1+\sqrt{33}\right)x}{4}}\right)}{33}$$

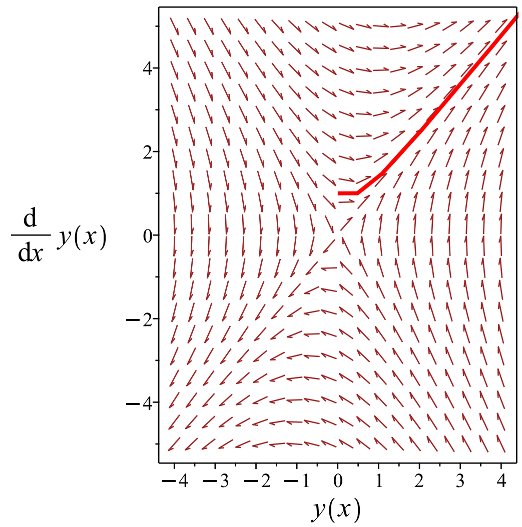
Summary

The solution(s) found are the following

$$y = \frac{2\sqrt{33}\left(\mathrm{e}^{\frac{\left(-1+\sqrt{33}\right)x}{4}}-\mathrm{e}^{-\frac{\left(1+\sqrt{33}\right)x}{4}}\right)}{33} \tag{1}$$



(a) Solution plot



(b) Slope field plot

Verification of solutions

$$y = \frac{2\sqrt{33}\left(e^{\frac{\left(-1+\sqrt{33}\right)x}{4}} - e^{-\frac{\left(1+\sqrt{33}\right)x}{4}}\right)}{33}$$

Verified OK.

7.14.4 Maple step by step solution

Let's solve

$$\left[2y'' + y' - 4y = 0, y(0) = 0, y'\Big|_{\{x=0\}} = 1\right]$$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y'' = -\frac{y'}{2} + 2y$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y'' + \frac{y'}{2} - 2y = 0$

- Characteristic polynomial of ODE

  $r^2 + \frac{1}{2}r - 2 = 0$

- Use quadratic formula to solve for r

  $r = \frac{\left(-\frac{1}{2}\right)\pm\left(\sqrt{\frac{33}{4}}\right)}{2}$
- Roots of the characteristic polynomial

  $r = \left(-\frac{1}{4}-\frac{\sqrt{33}}{4}, -\frac{1}{4}+\frac{\sqrt{33}}{4}\right)$
- 1st solution of the ODE

  $y_1(x) = \mathrm{e}^{\left(-\frac{1}{4}-\frac{\sqrt{33}}{4}\right)x}$
- 2nd solution of the ODE

  $y_2(x) = \mathrm{e}^{\left(-\frac{1}{4}+\frac{\sqrt{33}}{4}\right)x}$
- General solution of the ODE

  $y = c_1 y_1(x) + c_2 y_2(x)$
- Substitute in solutions

  $y = c_1 \mathrm{e}^{\left(-\frac{1}{4}-\frac{\sqrt{33}}{4}\right)x} + c_2 \mathrm{e}^{\left(-\frac{1}{4}+\frac{\sqrt{33}}{4}\right)x}$

□ Check validity of solution $y = c_1 \mathrm{e}^{\left(-\frac{1}{4}-\frac{\sqrt{33}}{4}\right)x} + c_2 \mathrm{e}^{\left(-\frac{1}{4}+\frac{\sqrt{33}}{4}\right)x}$

  - Use initial condition $y(0) = 0$

    $0 = c_1 + c_2$
  - Compute derivative of the solution

    $y' = c_1\left(-\frac{1}{4}-\frac{\sqrt{33}}{4}\right)\mathrm{e}^{\left(-\frac{1}{4}-\frac{\sqrt{33}}{4}\right)x} + c_2\left(-\frac{1}{4}+\frac{\sqrt{33}}{4}\right)\mathrm{e}^{\left(-\frac{1}{4}+\frac{\sqrt{33}}{4}\right)x}$
  - Use the initial condition $y'\Big|_{\{x=0\}} = 1$

    $1 = \left(-\frac{1}{4}-\frac{\sqrt{33}}{4}\right)c_1 + \left(-\frac{1}{4}+\frac{\sqrt{33}}{4}\right)c_2$
  - Solve for c_1 and c_2

    $\left\{c_1 = -\frac{2\sqrt{33}}{33}, c_2 = \frac{2\sqrt{33}}{33}\right\}$
  - Substitute constant values into general solution and simplify

    $y = \frac{2\sqrt{33}\left(\mathrm{e}^{\frac{\left(-1+\sqrt{33}\right)x}{4}} - \mathrm{e}^{-\frac{\left(1+\sqrt{33}\right)x}{4}}\right)}{33}$
- Solution to the IVP

$$y = \frac{2\sqrt{33}\left(\mathrm{e}^{\frac{\left(-1+\sqrt{33}\right)x}{4}} - \mathrm{e}^{-\frac{\left(1+\sqrt{33}\right)x}{4}}\right)}{33}$$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
<- constant coefficients successful`
```

✓ Solution by Maple

Time used: 0.047 (sec). Leaf size: 30

```
dsolve([2*diff(y(x),x$2) +diff(y(x),x)-4*y(x) = 0,y(0) = 0, D(y)(0) = 1],y(x), singsol=all)
```

$$y(x) = \frac{2\left(\mathrm{e}^{\frac{\left(-1+\sqrt{33}\right)x}{4}} - \mathrm{e}^{-\frac{\left(1+\sqrt{33}\right)x}{4}}\right)\sqrt{33}}{33}$$

✓ Solution by Mathematica

Time used: 0.027 (sec). Leaf size: 40

```
DSolve[{2*y''[x]+y'[x]-4*y[x]==0,{y[0]==0,y'[0]==1}},y[x],x,IncludeSingularSolutions -> True
```

$$y(x) \to \frac{2e^{-\frac{1}{4}\left(1+\sqrt{33}\right)x}\left(e^{\frac{\sqrt{33}x}{2}} - 1\right)}{\sqrt{33}}$$

7.15 problem 15

Internal problem ID [613]
Internal file name [`OUTPUT/613_Sunday_June_05_2022_01_45_51_AM_15862367/index.tex`]

Book: Elementary differential equations and boundary value problems, 10th ed., Boyce and DiPrima
Section: Chapter 3, Second order linear equations, 3.1 Homogeneous Equations with Constant Coefficients, page 144
Problem number: 15.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_linear_constant_coeff"**

Maple gives the following as the ode type

```
[[_2nd_order, _missing_x]]
```

$$\boxed{y'' + 8y' - 9y = 0}$$

With initial conditions

$$[y(1) = 1, y'(1) = 0]$$

7.15.1 Existence and uniqueness analysis

This is a linear ODE. In canonical form it is written as

$$y'' + p(x)y' + q(x)y = F$$

Where here

$$\begin{aligned} p(x) &= 8 \\ q(x) &= -9 \\ F &= 0 \end{aligned}$$

Hence the ode is

$$y'' + 8y' - 9y = 0$$

The domain of $p(x) = 8$ is

$$\{-\infty < x < \infty\}$$

And the point $x_0 = 1$ is inside this domain. The domain of $q(x) = -9$ is

$$\{-\infty < x < \infty\}$$

And the point $x_0 = 1$ is also inside this domain. Hence solution exists and is unique.

7.15.2 Solving as second order linear constant coeff ode

This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = 0$$

Where in the above $A = 1, B = 8, C = -9$. Let the solution be $y = e^{\lambda x}$. Substituting this into the ODE gives

$$\lambda^2 e^{\lambda x} + 8\lambda e^{\lambda x} - 9 e^{\lambda x} = 0 \tag{1}$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda x}$ gives

$$\lambda^2 + 8\lambda - 9 = 0 \tag{2}$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form.Using the quadratic formula

$$\lambda_{1,2} = \frac{-B}{2A} \pm \frac{1}{2A}\sqrt{B^2 - 4AC}$$

Substituting $A = 1, B = 8, C = -9$ into the above gives

$$\begin{aligned}\lambda_{1,2} &= \frac{-8}{(2)(1)} \pm \frac{1}{(2)(1)}\sqrt{8^2 - (4)(1)(-9)} \\ &= -4 \pm 5\end{aligned}$$

Hence

$$\lambda_1 = -4 + 5$$
$$\lambda_2 = -4 - 5$$

Which simplifies to

$$\lambda_1 = 1$$
$$\lambda_2 = -9$$

Since roots are real and distinct, then the solution is

$$y = c_1 e^{\lambda_1 x} + c_2 e^{\lambda_2 x}$$
$$y = c_1 e^{(1)x} + c_2 e^{(-9)x}$$

Or

$$y = c_1 \mathrm{e}^{x} + c_2 \mathrm{e}^{-9x}$$

Initial conditions are used to solve for the constants of integration.

Looking at the above solution

$$y = c_1 \mathrm{e}^{x} + c_2 \mathrm{e}^{-9x} \tag{1}$$

Initial conditions are now substituted in the above solution. This will generate the required equations to solve for the integration constants. substituting $y = 1$ and $x = 1$ in the above gives

$$1 = \left(\mathrm{e}^{10} c_1 + c_2\right) \mathrm{e}^{-9} \tag{1A}$$

Taking derivative of the solution gives

$$y' = c_1 \mathrm{e}^{x} - 9 c_2 \mathrm{e}^{-9x}$$

substituting $y' = 0$ and $x = 1$ in the above gives

$$0 = \left(\mathrm{e}^{10} c_1 - 9 c_2\right) \mathrm{e}^{-9} \tag{2A}$$

Equations $\{1A,2A\}$ are now solved for $\{c_1, c_2\}$. Solving for the constants gives

$$c_1 = \frac{9\,\mathrm{e}^{-1}}{10}$$
$$c_2 = \frac{\mathrm{e}^{9}}{10}$$

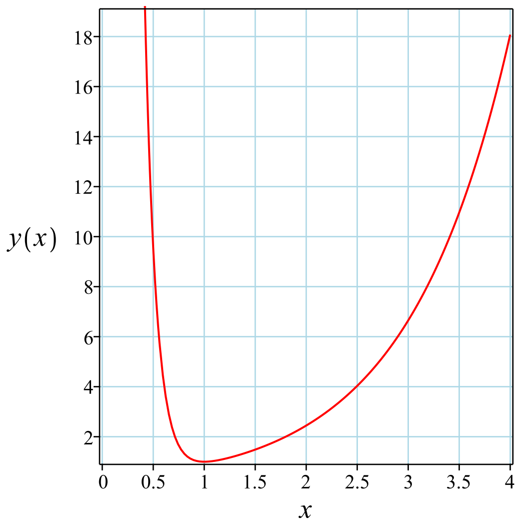
Substituting these values back in above solution results in

$$y = \frac{9\,\mathrm{e}^{x} \mathrm{e}^{-1}}{10} + \frac{\mathrm{e}^{-9x} \mathrm{e}^{9}}{10}$$

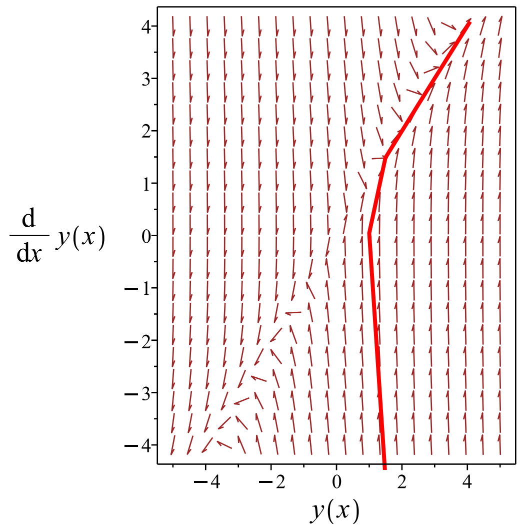
Summary

The solution(s) found are the following

$$y = \frac{9\,\mathrm{e}^{x}\mathrm{e}^{-1}}{10} + \frac{\mathrm{e}^{-9x}\mathrm{e}^{9}}{10} \tag{1}$$



(a) Solution plot



(b) Slope field plot

Verification of solutions

$$y = \frac{9\,\mathrm{e}^{x}\mathrm{e}^{-1}}{10} + \frac{\mathrm{e}^{-9x}\mathrm{e}^{9}}{10}$$

Verified OK.

7.15.3 Solving using Kovacic algorithm

Writing the ode as

$$y'' + 8y' - 9y = 0 \tag{1}$$

$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1 \\ B &= 8 \\ C &= -9 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{25}{1} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= 25 \\ t &= 1 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = 25z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\,e^{-\int \frac{B}{2A}\,dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 344: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$
\begin{aligned}
O(\infty) &= \deg(t) - \deg(s) \\
&= 0 - 0 \\
&= 0
\end{aligned}
$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Therefore

$$
L = [1]
$$

Since $r = 25$ is not a function of x, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$
z_1(x) = \mathrm{e}^{-5x}
$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$
\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\, dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{8}{1}\, dx} \\
&= z_1 e^{-4x} \\
&= z_1 \left(\mathrm{e}^{-4x}\right)
\end{aligned}
$$

Which simplifies to

$$y_1 = \mathrm{e}^{-9x}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\, dx}}{y_1^2}\, dx$$

Substituting gives

$$\begin{aligned} y_2 &= y_1 \int \frac{e^{\int -\frac{8}{1}\, dx}}{(y_1)^2}\, dx \\ &= y_1 \int \frac{e^{-8x}}{(y_1)^2}\, dx \\ &= y_1 \left(\frac{\mathrm{e}^{10x}}{10}\right) \end{aligned}$$

Therefore the solution is

$$\begin{aligned} y &= c_1 y_1 + c_2 y_2 \\ &= c_1 \left(\mathrm{e}^{-9x}\right) + c_2 \left(\mathrm{e}^{-9x} \left(\frac{\mathrm{e}^{10x}}{10}\right)\right) \end{aligned}$$

Initial conditions are used to solve for the constants of integration.

Looking at the above solution

$$y = c_1 \mathrm{e}^{-9x} + \frac{c_2 \mathrm{e}^{x}}{10} \tag{1}$$

Initial conditions are now substituted in the above solution. This will generate the required equations to solve for the integration constants. substituting $y = 1$ and $x = 1$ in the above gives

$$1 = \frac{\left(\mathrm{e}^{10} c_2 + 10 c_1\right) \mathrm{e}^{-9}}{10} \tag{1A}$$

Taking derivative of the solution gives

$$y' = -9 c_1 \mathrm{e}^{-9x} + \frac{c_2 \mathrm{e}^{x}}{10}$$

substituting $y' = 0$ and $x = 1$ in the above gives

$$0 = \frac{(e^{10}c_2 - 90c_1)\,e^{-9}}{10} \tag{2A}$$

Equations {1A,2A} are now solved for $\{c_1, c_2\}$. Solving for the constants gives

$$c_1 = \frac{e^9}{10}$$
$$c_2 = 9\,e^{-1}$$

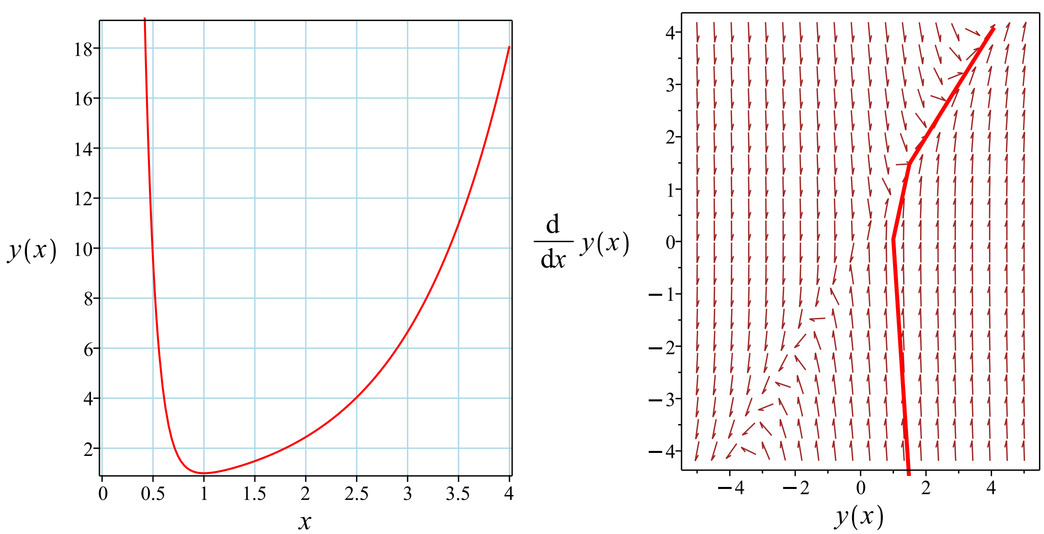
Substituting these values back in above solution results in

$$y = \frac{9\,e^x e^{-1}}{10} + \frac{e^{-9x} e^9}{10}$$

Summary

The solution(s) found are the following

$$y = \frac{9\,e^x e^{-1}}{10} + \frac{e^{-9x} e^9}{10} \tag{1}$$



(a) Solution plot (b) Slope field plot

Verification of solutions

$$y = \frac{9\,e^x e^{-1}}{10} + \frac{e^{-9x} e^9}{10}$$

Verified OK.

7.15.4 Maple step by step solution

Let's solve

$\left[y''+8y'-9y=0, y(1)=1, y'\Big|_{\{x=1\}}=0\right]$

- Highest derivative means the order of the ODE is 2

  y''
- Characteristic polynomial of ODE

  $r^2+8r-9=0$
- Factor the characteristic polynomial

  $(r+9)(r-1)=0$
- Roots of the characteristic polynomial

  $r=(-9,1)$
- 1st solution of the ODE

  $y_1(x)=e^{-9x}$
- 2nd solution of the ODE

  $y_2(x)=e^{x}$
- General solution of the ODE

  $y=c_1y_1(x)+c_2y_2(x)$
- Substitute in solutions

  $y=c_1e^{-9x}+c_2e^{x}$

□ Check validity of solution $y=c_1e^{-9x}+c_2e^{x}$
  - Use initial condition $y(1)=1$

    $1=c_1e^{-9}+ec_2$
  - Compute derivative of the solution

    $y'=-9c_1e^{-9x}+c_2e^{x}$
  - Use the initial condition $y'\Big|_{\{x=1\}}=0$

    $0=-9c_1e^{-9}+ec_2$
  - Solve for c_1 and c_2

    $\left\{c_1=\frac{1}{10\,e^{-9}}, c_2=\frac{9}{10\,e}\right\}$

- Substitute constant values into general solution and simplify

  $y = \frac{9\,e^{x-1}}{10} + \frac{e^{-9x+9}}{10}$

- Solution to the IVP

  $y = \frac{9\,e^{x-1}}{10} + \frac{e^{-9x+9}}{10}$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
<- constant coefficients successful`
```

✓ Solution by Maple

Time used: 0.031 (sec). Leaf size: 19

```
dsolve([diff(y(x),x$2) +8*diff(y(x),x)-9*y(x) = 0,y(1) = 1, D(y)(1) = 0],y(x), singsol=all)
```

$$y(x) = \frac{e^{9-9x}}{10} + \frac{9\,e^{x-1}}{10}$$

✓ Solution by Mathematica

Time used: 0.013 (sec). Leaf size: 26

```
DSolve[{y''[x]+8*y'[x]-9*y[x]==0,{y[1]==1,y'[1]==0}},y[x],x,IncludeSingularSolutions -> True
```

$$y(x) \to \frac{1}{10}e^{9-9x} + \frac{9e^{x-1}}{10}$$

7.16 problem 16

Internal problem ID [614]
Internal file name [`OUTPUT/614_Sunday_June_05_2022_01_45_52_AM_59244929/index.tex`]

Book: Elementary differential equations and boundary value problems, 10th ed., Boyce and DiPrima
Section: Chapter 3, Second order linear equations, 3.1 Homogeneous Equations with Constant Coefficients, page 144
Problem number: 16.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_linear_constant_coeff", "second_order_ode_can_be_made_integrable"**

Maple gives the following as the ode type

```
[[_2nd_order, _missing_x]]
```

$$\boxed{4y'' - y = 0}$$

With initial conditions

$$[y(-2) = 1, y'(-2) = -1]$$

7.16.1 Existence and uniqueness analysis

This is a linear ODE. In canonical form it is written as

$$y'' + p(x)y' + q(x)y = F$$

Where here

$$\begin{aligned} p(x) &= 0 \\ q(x) &= -\frac{1}{4} \\ F &= 0 \end{aligned}$$

Hence the ode is

$$y'' - \frac{y}{4} = 0$$

The domain of $p(x) = 0$ is

$$\{-\infty < x < \infty\}$$

And the point $x_0 = -2$ is inside this domain. The domain of $q(x) = -\frac{1}{4}$ is

$$\{-\infty < x < \infty\}$$

And the point $x_0 = -2$ is also inside this domain. Hence solution exists and is unique.

7.16.2 Solving as second order linear constant coeff ode

This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = 0$$

Where in the above $A = 4, B = 0, C = -1$. Let the solution be $y = e^{\lambda x}$. Substituting this into the ODE gives

$$4\lambda^2 \mathrm{e}^{\lambda x} - \mathrm{e}^{\lambda x} = 0 \tag{1}$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda x}$ gives

$$4\lambda^2 - 1 = 0 \tag{2}$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form.Using the quadratic formula

$$\lambda_{1,2} = \frac{-B}{2A} \pm \frac{1}{2A}\sqrt{B^2 - 4AC}$$

Substituting $A = 4, B = 0, C = -1$ into the above gives

$$\begin{aligned}
\lambda_{1,2} &= \frac{0}{(2)\,(4)} \pm \frac{1}{(2)\,(4)}\sqrt{0^2 - (4)\,(4)\,(-1)} \\
&= \pm\frac{1}{2}
\end{aligned}$$

Hence

$$\lambda_1 = +\frac{1}{2}$$

$$\lambda_2 = -\frac{1}{2}$$

Which simplifies to

$$\lambda_1 = \frac{1}{2}$$
$$\lambda_2 = -\frac{1}{2}$$

Since roots are real and distinct, then the solution is

$$y = c_1 e^{\lambda_1 x} + c_2 e^{\lambda_2 x}$$
$$y = c_1 e^{\left(\frac{1}{2}\right)x} + c_2 e^{\left(-\frac{1}{2}\right)x}$$

Or

$$y = c_1 \mathrm{e}^{\frac{x}{2}} + c_2 \mathrm{e}^{-\frac{x}{2}}$$

Initial conditions are used to solve for the constants of integration.

Looking at the above solution

$$y = c_1 \mathrm{e}^{\frac{x}{2}} + c_2 \mathrm{e}^{-\frac{x}{2}} \tag{1}$$

Initial conditions are now substituted in the above solution. This will generate the required equations to solve for the integration constants. substituting $y = 1$ and $x = -2$ in the above gives

$$1 = \mathrm{e}^{-1} c_1 + \mathrm{e} c_2 \tag{1A}$$

Taking derivative of the solution gives

$$y' = \frac{c_1 \mathrm{e}^{\frac{x}{2}}}{2} - \frac{c_2 \mathrm{e}^{-\frac{x}{2}}}{2}$$

substituting $y' = -1$ and $x = -2$ in the above gives

$$-1 = \frac{\mathrm{e}^{-1} c_1}{2} - \frac{\mathrm{e} c_2}{2} \tag{2A}$$

Equations {1A,2A} are now solved for $\{c_1, c_2\}$. Solving for the constants gives

$$c_1 = -\frac{\mathrm{e}}{2}$$
$$c_2 = \frac{3\,\mathrm{e}^{-1}}{2}$$

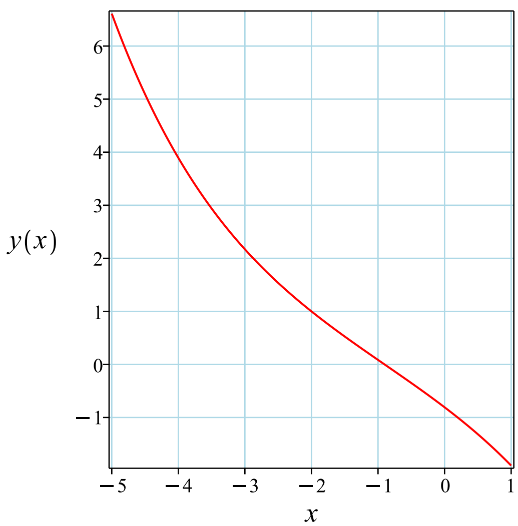
Substituting these values back in above solution results in

$$y = -\frac{e^{\frac{x}{2}} e}{2} + \frac{3 e^{-\frac{x}{2}} e^{-1}}{2}$$

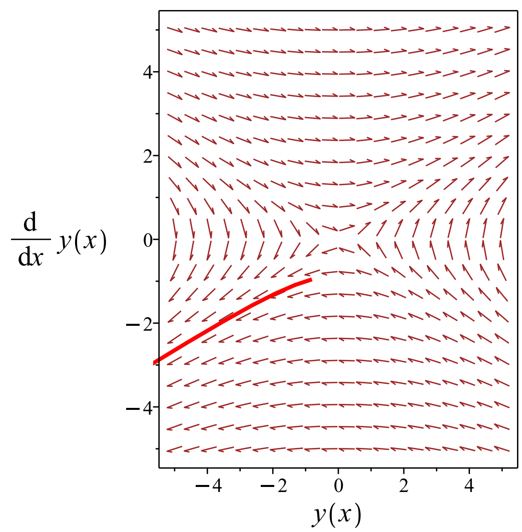
Summary

The solution(s) found are the following

$$y = -\frac{e^{\frac{x}{2}} e}{2} + \frac{3 e^{-\frac{x}{2}} e^{-1}}{2} \tag{1}$$



(a) Solution plot



(b) Slope field plot

Verification of solutions

$$y = -\frac{e^{\frac{x}{2}} e}{2} + \frac{3 e^{-\frac{x}{2}} e^{-1}}{2}$$

Verified OK.

7.16.3 Solving as second order ode can be made integrable ode

Multiplying the ode by y' gives

$$4y'y'' - yy' = 0$$

Integrating the above w.r.t x gives

$$\int (4y'y'' - yy')\, dx = 0$$

$$2y'^2 - \frac{y^2}{2} = c_2$$

Which is now solved for y. Solving the given ode for y' results in 2 differential equations to solve. Each one of these will generate a solution. The equations generated are

$$y' = \frac{\sqrt{y^2 + 2c_1}}{2} \tag{1}$$

$$y' = -\frac{\sqrt{y^2 + 2c_1}}{2} \tag{2}$$

Now each one of the above ODE is solved.

<u>Solving equation (1)</u>

Integrating both sides gives

$$\int \frac{2}{\sqrt{y^2 + 2c_1}} dy = \int dx$$

$$2 \ln\left(y + \sqrt{y^2 + 2c_1}\right) = x + c_2$$

Raising both side to exponential gives

$$\left(y + \sqrt{y^2 + 2c_1}\right)^2 = e^{x + c_2}$$

Which simplifies to

$$\left(y + \sqrt{y^2 + 2c_1}\right)^2 = c_3 e^x$$

<u>Solving equation (2)</u>

Integrating both sides gives

$$\int -\frac{2}{\sqrt{y^2 + 2c_1}} dy = \int dx$$

$$-2 \ln\left(y + \sqrt{y^2 + 2c_1}\right) = x + c_4$$

Raising both side to exponential gives

$$\frac{1}{\left(y + \sqrt{y^2 + 2c_1}\right)^2} = e^{x + c_4}$$

Which simplifies to

$$\frac{1}{\left(y + \sqrt{y^2 + 2c_1}\right)^2} = c_5 e^x$$

Initial conditions are used to solve for the constants of integration.

Looking at the First solution

$$y = \frac{c_3 e^x - 2c_1}{2\sqrt{c_3 e^x}} \tag{1}$$

Initial conditions are now substituted in the above solution. This will generate the required equations to solve for the integration constants. substituting $y = 1$ and $x = -2$ in the above gives

$$1 = -\frac{\left(e^2 c_1 - \frac{c_3}{2}\right) e^{-1}}{\sqrt{c_3}} \tag{1A}$$

Taking derivative of the solution gives

$$y' = \frac{c_3 e^x}{2\sqrt{c_3 e^x}} - \frac{(c_3 e^x - 2c_1) c_3 e^x}{4 (c_3 e^x)^{\frac{3}{2}}}$$

substituting $y' = -1$ and $x = -2$ in the above gives

$$-1 = \frac{2 e c_1 + c_3 e^{-1}}{4\sqrt{c_3}} \tag{2A}$$

Equations {1A,2A} are now solved for $\{c_1, c_3\}$. There is no solution for the constants of integrations. This solution is removed.

Looking at the Second solution

$$y = -\frac{2c_1 c_5 e^x - 1}{2\sqrt{c_5 e^x}} \tag{2}$$

Initial conditions are now substituted in the above solution. This will generate the required equations to solve for the integration constants. substituting $y = 1$ and $x = -2$ in the above gives

$$1 = \frac{-2c_1 c_5 e^{-1} + e}{2\sqrt{c_5}} \tag{1A}$$

Taking derivative of the solution gives

$$y' = -\frac{c_5 e^x c_1}{\sqrt{c_5 e^x}} + \frac{(2c_1 c_5 e^x - 1) c_5 e^x}{4 (c_5 e^x)^{\frac{3}{2}}}$$

substituting $y' = -1$ and $x = -2$ in the above gives

$$-1 = -\frac{c_1 c_5 e^{-1} + \frac{e}{2}}{2\sqrt{c_5}} \tag{2A}$$

Equations {1A,2A} are now solved for $\{c_1, c_5\}$. Solving for the constants gives

$$c_1 = \frac{3}{2}$$
$$c_5 = \frac{e^2}{9}$$

Substituting these values back in above solution results in

$$y = \frac{-e^{-1}e^{2+x} + 3\,e^{-1}}{2\sqrt{e^x}}$$

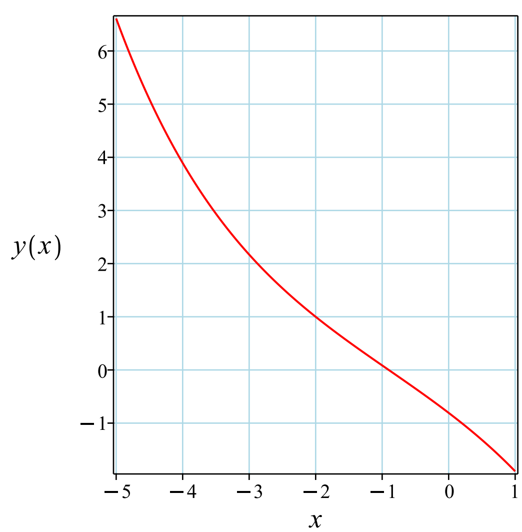
Which simplifies to

$$y = -\frac{(e^{2+x} - 3)\,e^{-1}}{2\sqrt{e^x}}$$

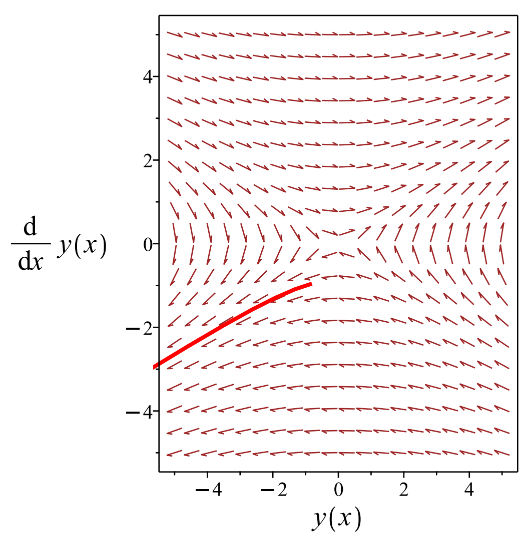
Summary

The solution(s) found are the following

$$y = -\frac{(e^{2+x} - 3)\,e^{-1}}{2\sqrt{e^x}} \tag{1}$$



(a) Solution plot



(b) Slope field plot

Verification of solutions

$$y = -\frac{(e^{2+x} - 3)\,e^{-1}}{2\sqrt{e^x}}$$

Verified OK.

7.16.4 Solving using Kovacic algorithm

Writing the ode as

$$4y'' - y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 4 \\ B &= 0 \\ C &= -1 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = y e^{\int \frac{B}{2A}\, dx}$$

Then (2) becomes

$$z''(x) = r z(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{1}{4} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= 1 \\ t &= 4 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = \frac{z(x)}{4} \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 346: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$
\begin{aligned}
O(\infty) &= \deg(t) - \deg(s) \\
&= 0 - 0 \\
&= 0
\end{aligned}
$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Therefore

$$
L = [1]
$$

Since $r = \frac{1}{4}$ is not a function of x, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$
z_1(x) = \mathrm{e}^{-\frac{x}{2}}
$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$
y_1 = z_1 e^{\int -\frac{1}{2} \frac{B}{A} \, dx}
$$

Since $B = 0$ then the above reduces to

$$
\begin{aligned}
y_1 &= z_1 \\
&= e^{-\frac{x}{2}}
\end{aligned}
$$

Which simplifies to

$$
y_1 = e^{-\frac{x}{2}}
$$

The second solution y_2 to the original ode is found using reduction of order

$$
y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\, dx}}{y_1^2}\, dx
$$

Since $B = 0$ then the above becomes

$$
\begin{aligned}
y_2 &= y_1 \int \frac{1}{y_1^2}\, dx \\
&= e^{-\frac{x}{2}} \int \frac{1}{e^{-x}}\, dx \\
&= e^{-\frac{x}{2}} (e^x)
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1 \left(e^{-\frac{x}{2}}\right) + c_2 \left(e^{-\frac{x}{2}} (e^x)\right)
\end{aligned}
$$

Initial conditions are used to solve for the constants of integration.

Looking at the above solution

$$
y = c_1 e^{-\frac{x}{2}} + c_2 e^{\frac{x}{2}} \tag{1}
$$

Initial conditions are now substituted in the above solution. This will generate the required equations to solve for the integration constants. substituting $y = 1$ and $x = -2$ in the above gives

$$
1 = e c_1 + c_2 e^{-1} \tag{1A}
$$

Taking derivative of the solution gives

$$y' = -\frac{c_1 \mathrm{e}^{-\frac{x}{2}}}{2} + \frac{c_2 \mathrm{e}^{\frac{x}{2}}}{2}$$

substituting $y' = -1$ and $x = -2$ in the above gives

$$-1 = -\frac{\mathrm{e} c_1}{2} + \frac{c_2 \mathrm{e}^{-1}}{2} \tag{2A}$$

Equations {1A,2A} are now solved for $\{c_1, c_2\}$. Solving for the constants gives

$$c_1 = \frac{3\,\mathrm{e}^{-1}}{2}$$
$$c_2 = -\frac{\mathrm{e}}{2}$$

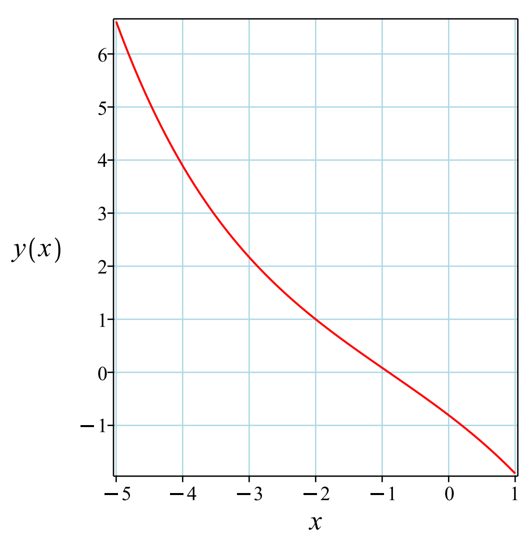
Substituting these values back in above solution results in

$$y = -\frac{\mathrm{e}^{\frac{x}{2}}\mathrm{e}}{2} + \frac{3\,\mathrm{e}^{-\frac{x}{2}}\mathrm{e}^{-1}}{2}$$

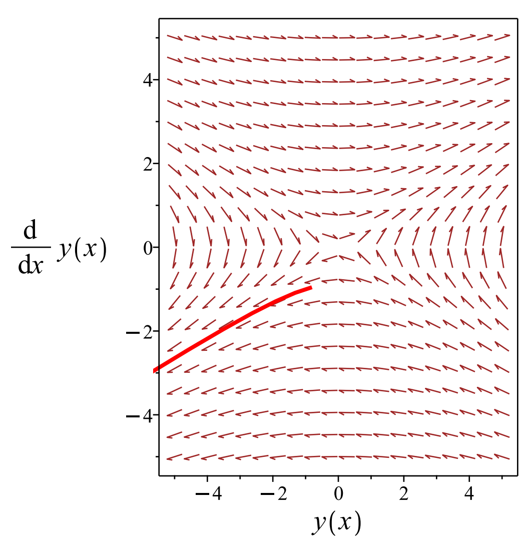
Summary

The solution(s) found are the following

$$y = -\frac{\mathrm{e}^{\frac{x}{2}}\mathrm{e}}{2} + \frac{3\,\mathrm{e}^{-\frac{x}{2}}\mathrm{e}^{-1}}{2} \tag{1}$$



(a) Solution plot



(b) Slope field plot

Verification of solutions

$$y = -\frac{e^{\frac{x}{2}} e}{2} + \frac{3 e^{-\frac{x}{2}} e^{-1}}{2}$$

Verified OK.

7.16.5 Maple step by step solution

Let's solve

$\left[4y'' - y = 0, y(-2) = 1, y'\Big|_{\{x=-2\}} = -1\right]$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y'' = \frac{y}{4}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y'' - \frac{y}{4} = 0$

- Characteristic polynomial of ODE

  $r^2 - \frac{1}{4} = 0$

- Factor the characteristic polynomial

  $\frac{(2r-1)(2r+1)}{4} = 0$

- Roots of the characteristic polynomial

  $r = \left(-\frac{1}{2}, \frac{1}{2}\right)$

- 1st solution of the ODE

  $y_1(x) = e^{-\frac{x}{2}}$

- 2nd solution of the ODE

  $y_2(x) = e^{\frac{x}{2}}$

- General solution of the ODE

  $y = c_1 y_1(x) + c_2 y_2(x)$

- Substitute in solutions

  $y = c_1 e^{-\frac{x}{2}} + c_2 e^{\frac{x}{2}}$

□ Check validity of solution $y = c_1 e^{-\frac{x}{2}} + c_2 e^{\frac{x}{2}}$

- Use initial condition $y(-2) = 1$

  $1 = \mathrm{e}c_1 + c_2\mathrm{e}^{-1}$

- Compute derivative of the solution

  $y' = -\frac{c_1\mathrm{e}^{-\frac{x}{2}}}{2} + \frac{c_2\mathrm{e}^{\frac{x}{2}}}{2}$

- Use the initial condition $y'\Big|_{\{x=-2\}} = -1$

  $-1 = -\frac{\mathrm{e}c_1}{2} + \frac{c_2\mathrm{e}^{-1}}{2}$

- Solve for c_1 and c_2

  $\left\{c_1 = \frac{3}{2\,\mathrm{e}}, c_2 = -\frac{1}{2\,\mathrm{e}^{-1}}\right\}$

- Substitute constant values into general solution and simplify

  $y = -\frac{\mathrm{e}^{\frac{x}{2}+1}}{2} + \frac{3\,\mathrm{e}^{-\frac{x}{2}-1}}{2}$

- Solution to the IVP

  $y = -\frac{\mathrm{e}^{\frac{x}{2}+1}}{2} + \frac{3\,\mathrm{e}^{-\frac{x}{2}-1}}{2}$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
<- constant coefficients successful`
```

✓ Solution by Maple

Time used: 0.031 (sec). Leaf size: 21

```
dsolve([4*diff(y(x),x$2) -y(x) = 0,y(-2) = 1, D(y)(-2) = -1],y(x), singsol=all)
```

$$y(x) = -\frac{\mathrm{e}^{1+\frac{x}{2}}}{2} + \frac{3\,\mathrm{e}^{-1-\frac{x}{2}}}{2}$$

✓ Solution by Mathematica

Time used: 0.012 (sec). Leaf size: 25

```
DSolve[{4*y''[x]-y[x]==0,{y[-2]==1,y'[-2]==-1}},y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to -\frac{1}{2}e^{-\frac{x}{2}-1}\left(e^{x+2}-3\right)$$

7.17 problem 19

Internal problem ID [615]
Internal file name [`OUTPUT/615_Sunday_June_05_2022_01_45_53_AM_95572567/index.tex`]

Book: Elementary differential equations and boundary value problems, 10th ed., Boyce and DiPrima
Section: Chapter 3, Second order linear equations, 3.1 Homogeneous Equations with Constant Coefficients, page 144
Problem number: 19.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_linear_constant_coeff", "second_order_ode_can_be_made_integrable"**

Maple gives the following as the ode type

```
[[_2nd_order, _missing_x]]
```

$$\boxed{y'' - y = 0}$$

With initial conditions

$$\left[y(0) = \frac{5}{4}, y'(0) = -\frac{3}{4}\right]$$

7.17.1 Existence and uniqueness analysis

This is a linear ODE. In canonical form it is written as

$$y'' + p(x)y' + q(x)y = F$$

Where here

$$p(x) = 0$$
$$q(x) = -1$$
$$F = 0$$

Hence the ode is

$$y'' - y = 0$$

The domain of $p(x) = 0$ is

$$\{-\infty < x < \infty\}$$

And the point $x_0 = 0$ is inside this domain. The domain of $q(x) = -1$ is

$$\{-\infty < x < \infty\}$$

And the point $x_0 = 0$ is also inside this domain. Hence solution exists and is unique.

7.17.2 Solving as second order linear constant coeff ode

This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = 0$$

Where in the above $A = 1, B = 0, C = -1$. Let the solution be $y = e^{\lambda x}$. Substituting this into the ODE gives

$$\lambda^2 \mathrm{e}^{\lambda x} - \mathrm{e}^{\lambda x} = 0 \tag{1}$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda x}$ gives

$$\lambda^2 - 1 = 0 \tag{2}$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form.Using the quadratic formula

$$\lambda_{1,2} = \frac{-B}{2A} \pm \frac{1}{2A}\sqrt{B^2 - 4AC}$$

Substituting $A = 1, B = 0, C = -1$ into the above gives

$$\begin{aligned}\lambda_{1,2} &= \frac{0}{(2)(1)} \pm \frac{1}{(2)(1)}\sqrt{0^2 - (4)(1)(-1)} \\ &= \pm 1\end{aligned}$$

Hence

$$\lambda_1 = +1$$
$$\lambda_2 = -1$$

Which simplifies to

$$\lambda_1 = 1$$
$$\lambda_2 = -1$$

Since roots are real and distinct, then the solution is

$$y = c_1 e^{\lambda_1 x} + c_2 e^{\lambda_2 x}$$
$$y = c_1 e^{(1)x} + c_2 e^{(-1)x}$$

Or

$$y = c_1 e^x + c_2 e^{-x}$$

Initial conditions are used to solve for the constants of integration.

Looking at the above solution

$$y = c_1 e^x + c_2 e^{-x} \tag{1}$$

Initial conditions are now substituted in the above solution. This will generate the required equations to solve for the integration constants. substituting $y = \frac{5}{4}$ and $x = 0$ in the above gives

$$\frac{5}{4} = c_1 + c_2 \tag{1A}$$

Taking derivative of the solution gives

$$y' = c_1 e^x - c_2 e^{-x}$$

substituting $y' = -\frac{3}{4}$ and $x = 0$ in the above gives

$$-\frac{3}{4} = c_1 - c_2 \tag{2A}$$

Equations {1A,2A} are now solved for $\{c_1, c_2\}$. Solving for the constants gives

$$c_1 = \frac{1}{4}$$
$$c_2 = 1$$

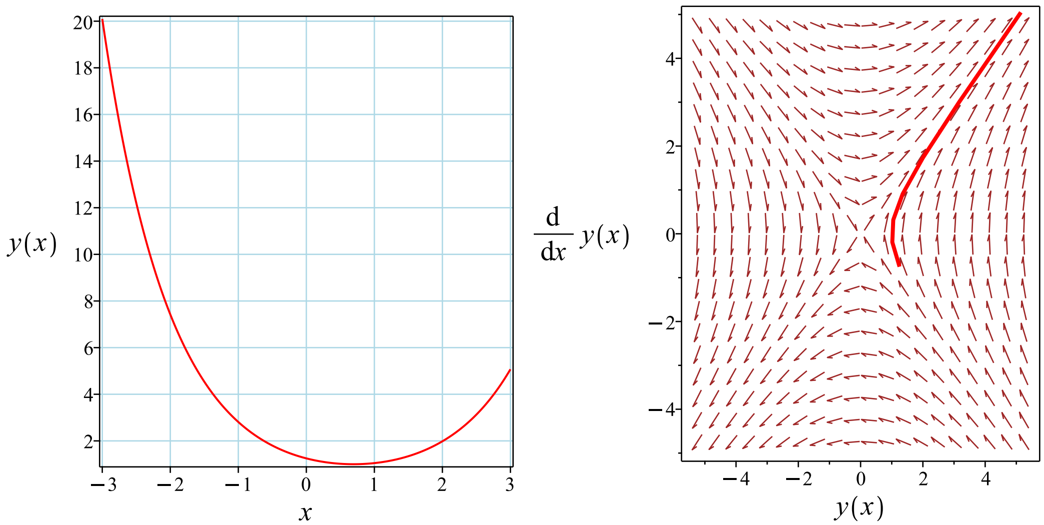
Substituting these values back in above solution results in

$$y = \frac{e^x}{4} + e^{-x}$$

Summary

The solution(s) found are the following

$$y = \frac{e^x}{4} + e^{-x} \tag{1}$$



(a) Solution plot (b) Slope field plot

Verification of solutions

$$y = \frac{e^x}{4} + e^{-x}$$

Verified OK.

7.17.3 Solving as second order ode can be made integrable ode

Multiplying the ode by y' gives

$$y'y'' - yy' = 0$$

Integrating the above w.r.t x gives

$$\int (y'y'' - yy') \, dx = 0$$
$$\frac{y'^2}{2} - \frac{y^2}{2} = c_2$$

Which is now solved for y. Solving the given ode for y' results in 2 differential equations to solve. Each one of these will generate a solution. The equations generated are

$$y' = \sqrt{y^2 + 2c_1} \tag{1}$$
$$y' = -\sqrt{y^2 + 2c_1} \tag{2}$$

Now each one of the above ODE is solved.

Solving equation (1)

Integrating both sides gives

$$\int \frac{1}{\sqrt{y^2 + 2c_1}} dy = \int dx$$
$$\ln\left(y + \sqrt{y^2 + 2c_1}\right) = x + c_2$$

Raising both side to exponential gives

$$y + \sqrt{y^2 + 2c_1} = e^{x+c_2}$$

Which simplifies to

$$y + \sqrt{y^2 + 2c_1} = c_3 e^x$$

Solving equation (2)

Integrating both sides gives

$$\int -\frac{1}{\sqrt{y^2 + 2c_1}} dy = \int dx$$
$$-\ln\left(y + \sqrt{y^2 + 2c_1}\right) = x + c_4$$

Raising both side to exponential gives

$$\frac{1}{y + \sqrt{y^2 + 2c_1}} = e^{x+c_4}$$

Which simplifies to

$$\frac{1}{y + \sqrt{y^2 + 2c_1}} = c_5 e^x$$

Initial conditions are used to solve for the constants of integration.

Looking at the First solution

$$y = \frac{\left(e^{2x} c_3^2 - 2c_1\right) e^{-x}}{2c_3} \tag{1}$$

Initial conditions are now substituted in the above solution. This will generate the required equations to solve for the integration constants. substituting $y = \frac{5}{4}$ and $x = 0$ in the above gives

$$\frac{5}{4} = \frac{c_3^2 - 2c_1}{2c_3} \tag{1A}$$

Taking derivative of the solution gives

$$y' = -\frac{\left(\mathrm{e}^{2x} c_3^2 - 2c_1\right) \mathrm{e}^{-x}}{2c_3} + c_3 \mathrm{e}^{x}$$

substituting $y' = -\frac{3}{4}$ and $x = 0$ in the above gives

$$-\frac{3}{4} = \frac{c_3^2 + 2c_1}{2c_3} \tag{2A}$$

Equations $\{1\text{A},2\text{A}\}$ are now solved for $\{c_1, c_3\}$. Solving for the constants gives

$$c_1 = -\frac{1}{2}$$
$$c_3 = \frac{1}{2}$$

Substituting these values back in above solution results in

$$y = \frac{\mathrm{e}^{-x}\left(4 + \mathrm{e}^{2x}\right)}{4}$$

Which simplifies to

$$y = \frac{\mathrm{e}^{x}}{4} + \mathrm{e}^{-x}$$

Looking at the Second solution

$$y = -\frac{\left(2c_1 c_5^2 \mathrm{e}^{2x} - 1\right) \mathrm{e}^{-x}}{2c_5} \tag{2}$$

Initial conditions are now substituted in the above solution. This will generate the required equations to solve for the integration constants. substituting $y = \frac{5}{4}$ and $x = 0$ in the above gives

$$\frac{5}{4} = \frac{-2c_1 c_5^2 + 1}{2c_5} \tag{1A}$$

Taking derivative of the solution gives

$$y' = -2c_1 c_5 \mathrm{e}^{x} + \frac{\left(2c_1 c_5^2 \mathrm{e}^{2x} - 1\right) \mathrm{e}^{-x}}{2c_5}$$

substituting $y' = -\frac{3}{4}$ and $x = 0$ in the above gives

$$-\frac{3}{4} = \frac{-2c_1c_5^2 - 1}{2c_5} \tag{2A}$$

Equations {1A,2A} are now solved for $\{c_1, c_5\}$. Solving for the constants gives

$$c_1 = -\frac{1}{2}$$
$$c_5 = \frac{1}{2}$$

Substituting these values back in above solution results in

$$y = \frac{e^{-x}(4 + e^{2x})}{4}$$

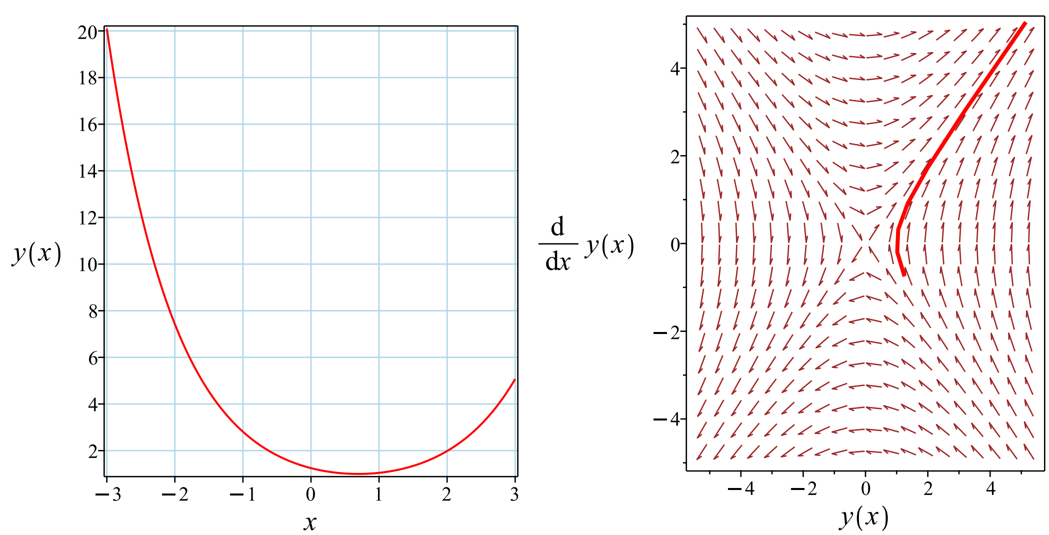
Which simplifies to

$$y = \frac{e^x}{4} + e^{-x}$$

Summary

The solution(s) found are the following

$$y = \frac{e^x}{4} + e^{-x} \tag{1}$$
$$y = \frac{e^x}{4} + e^{-x} \tag{2}$$



(a) Solution plot (b) Slope field plot

Verification of solutions

$$y = \frac{e^{x}}{4} + e^{-x}$$

Verified OK.

$$y = \frac{e^{x}}{4} + e^{-x}$$

Verified OK.

7.17.4 Solving using Kovacic algorithm

Writing the ode as

$$y'' - y = 0 \tag{1}$$

$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1 \\ B &= 0 \\ C &= -1 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = y e^{\int \frac{B}{2A}\, dx}$$

Then (2) becomes

$$z''(x) = r z(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{1}{1} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= 1 \\ t &= 1 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 348: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 0 - 0 \\ &= 0 \end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Since $r = 1$ is not a function of x, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$z_1(x) = \mathrm{e}^{-x}$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$y_1 = z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx}$$

Since $B = 0$ then the above reduces to

$$\begin{aligned} y_1 &= z_1 \\ &= \mathrm{e}^{-x} \end{aligned}$$

Which simplifies to

$$y_1 = \mathrm{e}^{-x}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx$$

Since $B = 0$ then the above becomes

$$\begin{aligned} y_2 &= y_1 \int \frac{1}{y_1^2}\,dx \\ &= \mathrm{e}^{-x} \int \frac{1}{\mathrm{e}^{-2x}}\,dx \\ &= \mathrm{e}^{-x}\left(\frac{\mathrm{e}^{2x}}{2}\right) \end{aligned}$$

Therefore the solution is

$$\begin{aligned} y &= c_1 y_1 + c_2 y_2 \\ &= c_1\left(\mathrm{e}^{-x}\right) + c_2\left(\mathrm{e}^{-x}\left(\frac{\mathrm{e}^{2x}}{2}\right)\right) \end{aligned}$$

Initial conditions are used to solve for the constants of integration.

Looking at the above solution

$$y = c_1 \mathrm{e}^{-x} + \frac{c_2 \mathrm{e}^{x}}{2} \tag{1}$$

Initial conditions are now substituted in the above solution. This will generate the required equations to solve for the integration constants. substituting $y = \frac{5}{4}$ and $x = 0$ in the above gives

$$\frac{5}{4} = c_1 + \frac{c_2}{2} \tag{1A}$$

Taking derivative of the solution gives

$$y' = -c_1 e^{-x} + \frac{c_2 e^{x}}{2}$$

substituting $y' = -\frac{3}{4}$ and $x = 0$ in the above gives

$$-\frac{3}{4} = -c_1 + \frac{c_2}{2} \tag{2A}$$

Equations {1A,2A} are now solved for $\{c_1, c_2\}$. Solving for the constants gives

$$c_1 = 1$$
$$c_2 = \frac{1}{2}$$

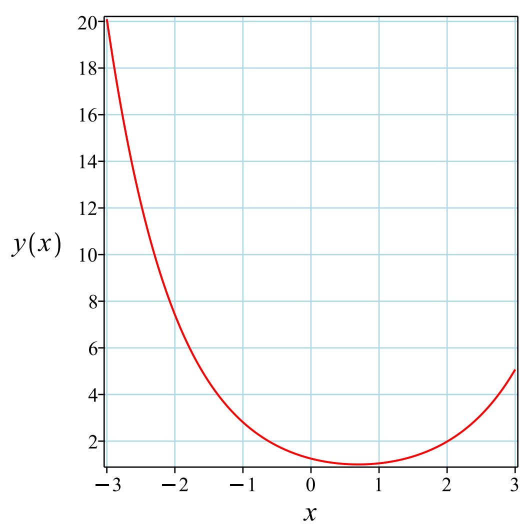
Substituting these values back in above solution results in

$$y = \frac{e^{x}}{4} + e^{-x}$$

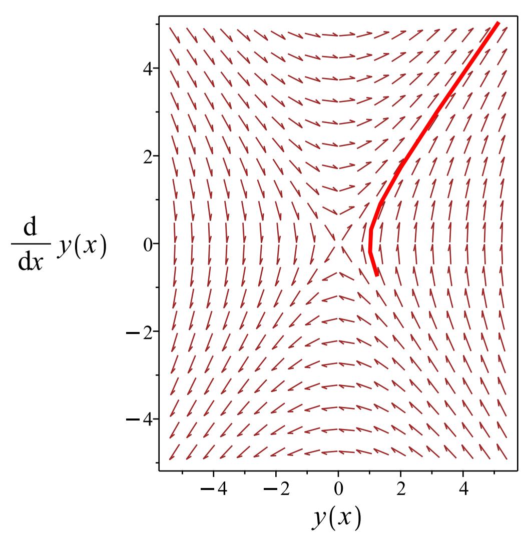
Summary

The solution(s) found are the following

$$y = \frac{e^{x}}{4} + e^{-x} \tag{1}$$



(a) Solution plot



(b) Slope field plot

Verification of solutions

$$y = \frac{e^{x}}{4} + e^{-x}$$

Verified OK.

7.17.5 Maple step by step solution

Let's solve

$\left[y'' - y = 0, y(0) = \frac{5}{4}, y'\Big|_{\{x=0\}} = -\frac{3}{4}\right]$

- Highest derivative means the order of the ODE is 2

  y''
- Characteristic polynomial of ODE

  $r^2 - 1 = 0$
- Factor the characteristic polynomial

  $(r-1)(r+1) = 0$
- Roots of the characteristic polynomial

  $r = (-1, 1)$
- 1st solution of the ODE

  $y_1(x) = e^{-x}$
- 2nd solution of the ODE

  $y_2(x) = e^{x}$
- General solution of the ODE

  $y = c_1 y_1(x) + c_2 y_2(x)$
- Substitute in solutions

  $y = c_1 e^{-x} + c_2 e^{x}$

☐ Check validity of solution $y = c_1 e^{-x} + c_2 e^{x}$

  ◦ Use initial condition $y(0) = \frac{5}{4}$

  $\frac{5}{4} = c_1 + c_2$

  ◦ Compute derivative of the solution

  $y' = -c_1 e^{-x} + c_2 e^{x}$

- Use the initial condition $y'\Big|_{\{x=0\}} = -\frac{3}{4}$

  $-\frac{3}{4} = -c_1 + c_2$
- Solve for c_1 and c_2

  $\left\{c_1 = 1, c_2 = \frac{1}{4}\right\}$
- Substitute constant values into general solution and simplify

  $y = \frac{e^x}{4} + e^{-x}$

- Solution to the IVP

  $y = \frac{e^x}{4} + e^{-x}$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
<- constant coefficients successful`
```

✓ Solution by Maple

Time used: 0.015 (sec). Leaf size: 13

```
dsolve([diff(y(x),x$2) -y(x) = 0,y(0) = 5/4, D(y)(0) = -3/4],y(x), singsol=all)
```

$$y(x) = \frac{e^x}{4} + e^{-x}$$

✓ Solution by Mathematica

Time used: 0.013 (sec). Leaf size: 18

```
DSolve[{y''[x]-y[x]==0,{y[0]==5/4,y'[0]==-3/4}},y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to e^{-x} + \frac{e^x}{4}$$

7.18 problem 20

Internal problem ID [616]
Internal file name [OUTPUT/616_Sunday_June_05_2022_01_45_54_AM_32300670/index.tex]

Book: Elementary differential equations and boundary value problems, 10th ed., Boyce and DiPrima
Section: Chapter 3, Second order linear equations, 3.1 Homogeneous Equations with Constant Coefficients, page 144
Problem number: 20.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_linear_constant_coeff"**

Maple gives the following as the ode type

```
[[_2nd_order, _missing_x]]
```

$$\boxed{2y'' - 3y' + y = 0}$$

With initial conditions

$$\left[y(0) = 2, y'(0) = \frac{1}{2}\right]$$

7.18.1 Existence and uniqueness analysis

This is a linear ODE. In canonical form it is written as

$$y'' + p(x)y' + q(x)y = F$$

Where here

$$p(x) = -\frac{3}{2}$$
$$q(x) = \frac{1}{2}$$
$$F = 0$$

Hence the ode is

$$y'' - \frac{3y'}{2} + \frac{y}{2} = 0$$

The domain of $p(x) = -\frac{3}{2}$ is

$$\{-\infty < x < \infty\}$$

And the point $x_0 = 0$ is inside this domain. The domain of $q(x) = \frac{1}{2}$ is

$$\{-\infty < x < \infty\}$$

And the point $x_0 = 0$ is also inside this domain. Hence solution exists and is unique.

7.18.2 Solving as second order linear constant coeff ode

This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = 0$$

Where in the above $A = 2, B = -3, C = 1$. Let the solution be $y = e^{\lambda x}$. Substituting this into the ODE gives

$$2\lambda^2 e^{\lambda x} - 3\lambda\, e^{\lambda x} + e^{\lambda x} = 0 \tag{1}$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda x}$ gives

$$2\lambda^2 - 3\lambda + 1 = 0 \tag{2}$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form.Using the quadratic formula

$$\lambda_{1,2} = \frac{-B}{2A} \pm \frac{1}{2A}\sqrt{B^2 - 4AC}$$

Substituting $A = 2, B = -3, C = 1$ into the above gives

$$\begin{aligned}\lambda_{1,2} &= \frac{3}{(2)\,(2)} \pm \frac{1}{(2)\,(2)}\sqrt{-3^2 - (4)\,(2)\,(1)}\\ &= \frac{3}{4} \pm \frac{1}{4}\end{aligned}$$

Hence

$$\lambda_1 = \frac{3}{4} + \frac{1}{4}$$

$$\lambda_2 = \frac{3}{4} - \frac{1}{4}$$

Which simplifies to

$$\lambda_1 = 1$$
$$\lambda_2 = \frac{1}{2}$$

Since roots are real and distinct, then the solution is

$$y = c_1 e^{\lambda_1 x} + c_2 e^{\lambda_2 x}$$
$$y = c_1 e^{(1)x} + c_2 e^{\left(\frac{1}{2}\right)x}$$

Or

$$y = c_1 e^x + c_2 e^{\frac{x}{2}}$$

Initial conditions are used to solve for the constants of integration.

Looking at the above solution

$$y = c_1 e^x + c_2 e^{\frac{x}{2}} \tag{1}$$

Initial conditions are now substituted in the above solution. This will generate the required equations to solve for the integration constants. substituting $y = 2$ and $x = 0$ in the above gives

$$2 = c_1 + c_2 \tag{1A}$$

Taking derivative of the solution gives

$$y' = c_1 e^x + \frac{c_2 e^{\frac{x}{2}}}{2}$$

substituting $y' = \frac{1}{2}$ and $x = 0$ in the above gives

$$\frac{1}{2} = c_1 + \frac{c_2}{2} \tag{2A}$$

Equations {1A,2A} are now solved for $\{c_1, c_2\}$. Solving for the constants gives

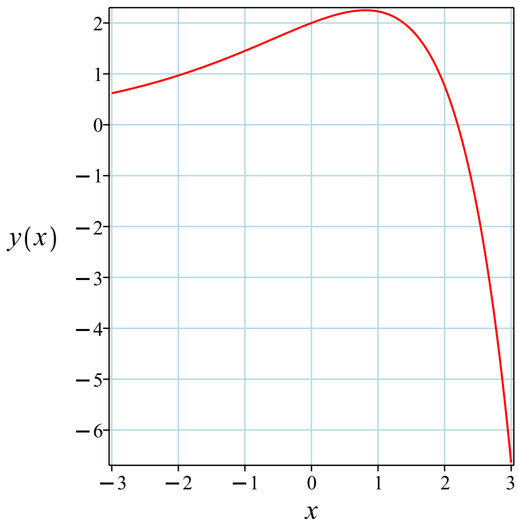
$$c_1 = -1$$
$$c_2 = 3$$

Substituting these values back in above solution results in

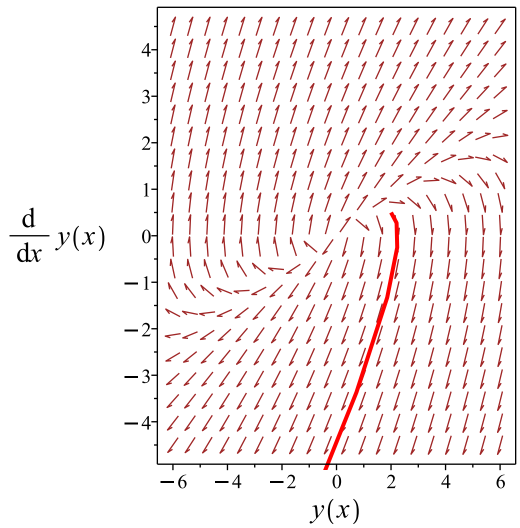
$$y = -e^x + 3\,e^{\frac{x}{2}}$$

Summary
The solution(s) found are the following

$$y = -e^{x} + 3\,e^{\frac{x}{2}} \tag{1}$$



(a) Solution plot



(b) Slope field plot

Verification of solutions

$$y = -e^{x} + 3\,e^{\frac{x}{2}}$$

Verified OK.

7.18.3 Solving using Kovacic algorithm

Writing the ode as

$$2y'' - 3y' + y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 2 \\ B &= -3 \\ C &= 1 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{1}{16} \tag{6}$$

Comparing the above to (5) shows that

$$s = 1$$
$$t = 16$$

Therefore eq. (4) becomes

$$z''(x) = \frac{z(x)}{16} \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 350: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$
\begin{aligned}
O(\infty) &= \deg(t) - \deg(s) \\
&= 0 - 0 \\
&= 0
\end{aligned}
$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Therefore

$$
L = [1]
$$

Since $r = \frac{1}{16}$ is not a function of x, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$
z_1(x) = \mathrm{e}^{-\frac{x}{4}}
$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$
\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{-3}{2}\,dx} \\
&= z_1 e^{\frac{3x}{4}} \\
&= z_1\left(\mathrm{e}^{\frac{3x}{4}}\right)
\end{aligned}
$$

Which simplifies to

$$
y_1 = \mathrm{e}^{\frac{x}{2}}
$$

The second solution y_2 to the original ode is found using reduction of order

$$
y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx
$$

Substituting gives

$$
\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{-3}{2}\,dx}}{(y_1)^2}\,dx \\
&= y_1 \int \frac{e^{\frac{3x}{2}}}{(y_1)^2}\,dx \\
&= y_1\left(2\,\mathrm{e}^{\frac{x}{2}}\right)
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1 \left(e^{\frac{x}{2}}\right) + c_2 \left(e^{\frac{x}{2}} \left(2\, e^{\frac{x}{2}}\right)\right)
\end{aligned}
$$

Initial conditions are used to solve for the constants of integration.

Looking at the above solution

$$
y = c_1 e^{\frac{x}{2}} + 2 c_2 e^{x} \tag{1}
$$

Initial conditions are now substituted in the above solution. This will generate the required equations to solve for the integration constants. substituting $y = 2$ and $x = 0$ in the above gives

$$
2 = c_1 + 2 c_2 \tag{1A}
$$

Taking derivative of the solution gives

$$
y' = \frac{c_1 e^{\frac{x}{2}}}{2} + 2 c_2 e^{x}
$$

substituting $y' = \frac{1}{2}$ and $x = 0$ in the above gives

$$
\frac{1}{2} = \frac{c_1}{2} + 2 c_2 \tag{2A}
$$

Equations {1A,2A} are now solved for $\{c_1, c_2\}$. Solving for the constants gives

$$
\begin{aligned}
c_1 &= 3 \\
c_2 &= -\frac{1}{2}
\end{aligned}
$$

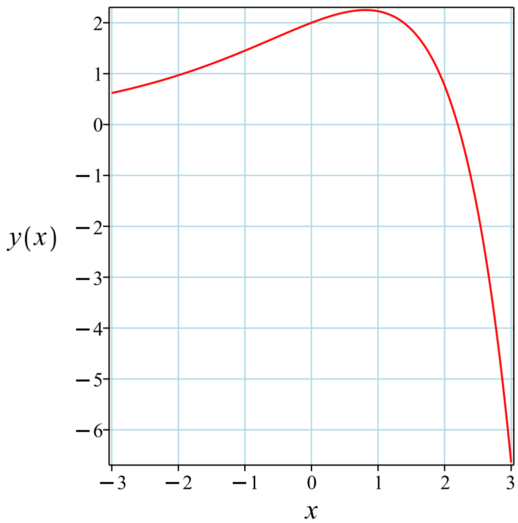
Substituting these values back in above solution results in

$$
y = -e^{x} + 3\, e^{\frac{x}{2}}
$$

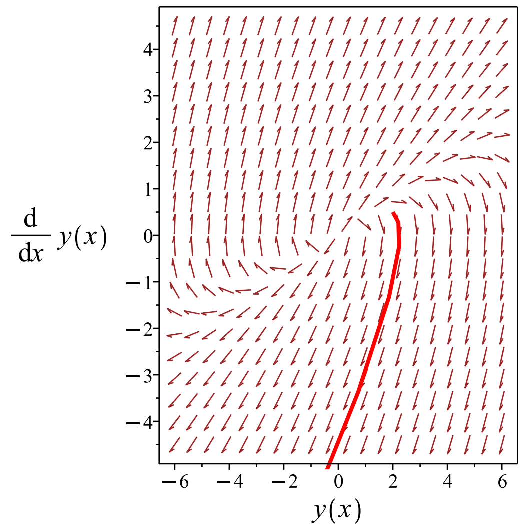
Summary

The solution(s) found are the following

$$
y = -e^{x} + 3\, e^{\frac{x}{2}} \tag{1}
$$



(a) Solution plot



(b) Slope field plot

<u>Verification of solutions</u>

$$y = -e^{x} + 3\,e^{\frac{x}{2}}$$

Verified OK.

7.18.4 Maple step by step solution

Let's solve

$$\left[2y'' - 3y' + y = 0, y(0) = 2, y'\Big|_{\{x=0\}} = \frac{1}{2}\right]$$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y'' = \frac{3y'}{2} - \frac{y}{2}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y'' - \frac{3y'}{2} + \frac{y}{2} = 0$

- Characteristic polynomial of ODE

  $r^2 - \frac{3}{2}r + \frac{1}{2} = 0$

- Factor the characteristic polynomial

$\frac{(2r-1)(r-1)}{2} = 0$

- Roots of the characteristic polynomial

  $r = \left(1, \frac{1}{2}\right)$

- 1st solution of the ODE

  $y_1(x) = e^x$

- 2nd solution of the ODE

  $y_2(x) = e^{\frac{x}{2}}$

- General solution of the ODE

  $y = c_1 y_1(x) + c_2 y_2(x)$

- Substitute in solutions

  $y = c_1 e^x + c_2 e^{\frac{x}{2}}$

☐ Check validity of solution $y = c_1 e^x + c_2 e^{\frac{x}{2}}$

  ◦ Use initial condition $y(0) = 2$

    $2 = c_1 + c_2$

  ◦ Compute derivative of the solution

    $y' = c_1 e^x + \frac{c_2 e^{\frac{x}{2}}}{2}$

  ◦ Use the initial condition $y'\Big|_{\{x=0\}} = \frac{1}{2}$

    $\frac{1}{2} = c_1 + \frac{c_2}{2}$

  ◦ Solve for c_1 and c_2

    $\{c_1 = -1, c_2 = 3\}$

  ◦ Substitute constant values into general solution and simplify

    $y = -e^x + 3\,e^{\frac{x}{2}}$

- Solution to the IVP

  $y = -e^x + 3\,e^{\frac{x}{2}}$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
<- constant coefficients successful`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 15

```
dsolve([2*diff(y(x),x$2) -3*diff(y(x),x)+y(x) = 0,y(0) = 2, D(y)(0) = 1/2],y(x), singsol=all
```

$$y(x) = -e^{x} + 3\,e^{\frac{x}{2}}$$

✓ Solution by Mathematica

Time used: 0.012 (sec). Leaf size: 20

```
DSolve[{2*y''[x]-3*y'[x]+y[x]==0,{y[0]==2,y'[0]==1/2}},y[x],x,IncludeSingularSolutions -> Tr
```

$$y(x) \to 3e^{x/2} - e^{x}$$

7.19 problem 21

Internal problem ID [617]
Internal file name [`OUTPUT/617_Sunday_June_05_2022_01_45_55_AM_95430660/index.tex`]

Book: Elementary differential equations and boundary value problems, 10th ed., Boyce and DiPrima
Section: Chapter 3, Second order linear equations, 3.1 Homogeneous Equations with Constant Coefficients, page 144
Problem number: 21.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_linear_constant_coeff"**

Maple gives the following as the ode type

`[[_2nd_order, _missing_x]]`

$$\boxed{y'' - y' - 2y = 0}$$

With initial conditions

$$[y(0) = \alpha, y'(0) = 2]$$

7.19.1 Existence and uniqueness analysis

This is a linear ODE. In canonical form it is written as

$$y'' + p(x)y' + q(x)y = F$$

Where here

$$p(x) = -1$$
$$q(x) = -2$$
$$F = 0$$

Hence the ode is

$$y'' - y' - 2y = 0$$

The domain of $p(x) = -1$ is

$$\{-\infty < x < \infty\}$$

And the point $x_0 = 0$ is inside this domain. The domain of $q(x) = -2$ is

$$\{-\infty < x < \infty\}$$

And the point $x_0 = 0$ is also inside this domain. Hence solution exists and is unique.

7.19.2 Solving as second order linear constant coeff ode

This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = 0$$

Where in the above $A = 1, B = -1, C = -2$. Let the solution be $y = e^{\lambda x}$. Substituting this into the ODE gives

$$\lambda^2 \mathrm{e}^{\lambda x} - \lambda\, \mathrm{e}^{\lambda x} - 2\, \mathrm{e}^{\lambda x} = 0 \tag{1}$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda x}$ gives

$$\lambda^2 - \lambda - 2 = 0 \tag{2}$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form.Using the quadratic formula

$$\lambda_{1,2} = \frac{-B}{2A} \pm \frac{1}{2A}\sqrt{B^2 - 4AC}$$

Substituting $A = 1, B = -1, C = -2$ into the above gives

$$\begin{aligned}\lambda_{1,2} &= \frac{1}{(2)\,(1)} \pm \frac{1}{(2)\,(1)}\sqrt{-1^2 - (4)\,(1)\,(-2)}\\ &= \frac{1}{2} \pm \frac{3}{2}\end{aligned}$$

Hence

$$\lambda_1 = \frac{1}{2} + \frac{3}{2}$$

$$\lambda_2 = \frac{1}{2} - \frac{3}{2}$$

Which simplifies to

$$\lambda_1 = 2$$
$$\lambda_2 = -1$$

Since roots are real and distinct, then the solution is

$$y = c_1 e^{\lambda_1 x} + c_2 e^{\lambda_2 x}$$
$$y = c_1 e^{(2)x} + c_2 e^{(-1)x}$$

Or

$$y = c_1 \mathrm{e}^{2x} + c_2 \mathrm{e}^{-x}$$

Initial conditions are used to solve for the constants of integration.

Looking at the above solution

$$y = c_1 \mathrm{e}^{2x} + c_2 \mathrm{e}^{-x} \tag{1}$$

Initial conditions are now substituted in the above solution. This will generate the required equations to solve for the integration constants. substituting $y = \alpha$ and $x = 0$ in the above gives

$$\alpha = c_1 + c_2 \tag{1A}$$

Taking derivative of the solution gives

$$y' = 2c_1 \mathrm{e}^{2x} - c_2 \mathrm{e}^{-x}$$

substituting $y' = 2$ and $x = 0$ in the above gives

$$2 = 2c_1 - c_2 \tag{2A}$$

Equations {1A,2A} are now solved for $\{c_1, c_2\}$. Solving for the constants gives

$$c_1 = \frac{\alpha}{3} + \frac{2}{3}$$
$$c_2 = -\frac{2}{3} + \frac{2\alpha}{3}$$

Substituting these values back in above solution results in

$$y = \frac{\mathrm{e}^{2x}\alpha}{3} + \frac{2\,\mathrm{e}^{2x}}{3} - \frac{2\,\mathrm{e}^{-x}}{3} + \frac{2\,\mathrm{e}^{-x}\alpha}{3}$$

Which simplifies to

$$y = \frac{(2\alpha - 2)\,\mathrm{e}^{-x}}{3} + \frac{\mathrm{e}^{2x}(\alpha + 2)}{3}$$

Summary

The solution(s) found are the following

$$y = \frac{(2\alpha - 2)\,\mathrm{e}^{-x}}{3} + \frac{\mathrm{e}^{2x}(\alpha + 2)}{3} \tag{1}$$

Verification of solutions

$$y = \frac{(2\alpha - 2)\,\mathrm{e}^{-x}}{3} + \frac{\mathrm{e}^{2x}(\alpha + 2)}{3}$$

Verified OK.

7.19.3 Solving using Kovacic algorithm

Writing the ode as

$$y'' - y' - 2y = 0 \tag{1}$$

$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1 \\ B &= -1 \\ C &= -2 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{9}{4} \tag{6}$$

Comparing the above to (5) shows that

$$s = 9$$
$$t = 4$$

Therefore eq. (4) becomes

$$z''(x) = \frac{9z(x)}{4} \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 352: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 0 - 0 \\ &= 0 \end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Since $r = \frac{9}{4}$ is not a function of x, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$z_1(x) = \mathrm{e}^{-\frac{3x}{2}}$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$\begin{aligned} y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx} \\ &= z_1 e^{-\int \frac{1}{2}\frac{-1}{1}\,dx} \\ &= z_1 e^{\frac{x}{2}} \\ &= z_1\left(\mathrm{e}^{\frac{x}{2}}\right) \end{aligned}$$

Which simplifies to

$$y_1 = \mathrm{e}^{-x}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx$$

Substituting gives

$$\begin{aligned} y_2 &= y_1 \int \frac{e^{\int -\frac{-1}{1}\,dx}}{(y_1)^2}\,dx \\ &= y_1 \int \frac{e^{x}}{(y_1)^2}\,dx \\ &= y_1\left(\frac{\mathrm{e}^{3x}}{3}\right) \end{aligned}$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1 \left(\mathrm{e}^{-x}\right) + c_2 \left(\mathrm{e}^{-x} \left(\frac{\mathrm{e}^{3x}}{3}\right)\right)
\end{aligned}
$$

Initial conditions are used to solve for the constants of integration.

Looking at the above solution

$$
y = c_1 \mathrm{e}^{-x} + \frac{c_2 \mathrm{e}^{2x}}{3} \tag{1}
$$

Initial conditions are now substituted in the above solution. This will generate the required equations to solve for the integration constants. substituting $y = \alpha$ and $x = 0$ in the above gives

$$
\alpha = c_1 + \frac{c_2}{3} \tag{1A}
$$

Taking derivative of the solution gives

$$
y' = -c_1 \mathrm{e}^{-x} + \frac{2 c_2 \mathrm{e}^{2x}}{3}
$$

substituting $y' = 2$ and $x = 0$ in the above gives

$$
2 = -c_1 + \frac{2 c_2}{3} \tag{2A}
$$

Equations {1A,2A} are now solved for $\{c_1, c_2\}$. Solving for the constants gives

$$
\begin{aligned}
c_1 &= -\frac{2}{3} + \frac{2\alpha}{3} \\
c_2 &= \alpha + 2
\end{aligned}
$$

Substituting these values back in above solution results in

$$
y = \frac{\mathrm{e}^{2x} \alpha}{3} + \frac{2\,\mathrm{e}^{2x}}{3} - \frac{2\,\mathrm{e}^{-x}}{3} + \frac{2\,\mathrm{e}^{-x} \alpha}{3}
$$

Which simplifies to

$$
y = \frac{(2\alpha - 2)\,\mathrm{e}^{-x}}{3} + \frac{\mathrm{e}^{2x} (\alpha + 2)}{3}
$$

Summary

The solution(s) found are the following

$$
y = \frac{(2\alpha - 2)\,\mathrm{e}^{-x}}{3} + \frac{\mathrm{e}^{2x} (\alpha + 2)}{3} \tag{1}
$$

Verification of solutions

$$y = \frac{(2\alpha - 2)\,\mathrm{e}^{-x}}{3} + \frac{\mathrm{e}^{2x}(\alpha + 2)}{3}$$

Verified OK.

7.19.4 Maple step by step solution

Let's solve

$\left[y'' - y' - 2y = 0, y(0) = \alpha, y'\Big|_{\{x=0\}} = 2\right]$

- Highest derivative means the order of the ODE is 2

  y''

- Characteristic polynomial of ODE

  $r^2 - r - 2 = 0$

- Factor the characteristic polynomial

  $(r + 1)(r - 2) = 0$

- Roots of the characteristic polynomial

  $r = (-1, 2)$

- 1st solution of the ODE

  $y_1(x) = \mathrm{e}^{-x}$

- 2nd solution of the ODE

  $y_2(x) = \mathrm{e}^{2x}$

- General solution of the ODE

  $y = c_1 y_1(x) + c_2 y_2(x)$

- Substitute in solutions

  $y = c_1 \mathrm{e}^{-x} + c_2 \mathrm{e}^{2x}$

□ Check validity of solution $y = c_1 \mathrm{e}^{-x} + c_2 \mathrm{e}^{2x}$

  ◦ Use initial condition $y(0) = \alpha$

    $\alpha = c_1 + c_2$

  ◦ Compute derivative of the solution

    $y' = -c_1 \mathrm{e}^{-x} + 2c_2 \mathrm{e}^{2x}$

- Use the initial condition $y'\Big|_{\{x=0\}} = 2$

  $2 = -c_1 + 2c_2$

- Solve for c_1 and c_2

  $\left\{c_1 = -\frac{2}{3} + \frac{2\alpha}{3}, c_2 = \frac{\alpha}{3} + \frac{2}{3}\right\}$

- Substitute constant values into general solution and simplify

  $y = \frac{(2\alpha-2)\,\mathrm{e}^{-x}}{3} + \frac{\mathrm{e}^{2x}(\alpha+2)}{3}$

- Solution to the IVP

  $y = \frac{(2\alpha-2)\,\mathrm{e}^{-x}}{3} + \frac{\mathrm{e}^{2x}(\alpha+2)}{3}$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
<- constant coefficients successful`
```

✓ Solution by Maple

Time used: 0.015 (sec). Leaf size: 25

```
dsolve([diff(y(x),x$2) -diff(y(x),x)-2*y(x) = 0,y(0) = alpha, D(y)(0) = 2],y(x), singsol=all
```

$$y(x) = \frac{(2\alpha-2)\,\mathrm{e}^{-x}}{3} + \frac{\mathrm{e}^{2x}(\alpha+2)}{3}$$

✓ Solution by Mathematica

Time used: 0.013 (sec). Leaf size: 29

```
DSolve[{y''[x]-y'[x]-2*y[x]==0,{y[0]==\[Alpha],y'[0]==2}},y[x],x,IncludeSingularSolutions ->
```

$$y(x) \to \frac{1}{3}e^{-x}\left(2(\alpha-1) + (\alpha+2)e^{3x}\right)$$

7.20 problem 22

Internal problem ID [618]
Internal file name [`OUTPUT/618_Sunday_June_05_2022_01_45_56_AM_92637636/index.tex`]

Book: Elementary differential equations and boundary value problems, 10th ed., Boyce and DiPrima
Section: Chapter 3, Second order linear equations, 3.1 Homogeneous Equations with Constant Coefficients, page 144
Problem number: 22.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_linear_constant_coeff", "second_order_ode_can_be_made_integrable"**

Maple gives the following as the ode type

```
[[_2nd_order, _missing_x]]
```

$$\boxed{4y'' - y = 0}$$

With initial conditions

$$[y(0) = 2, y'(0) = \beta]$$

7.20.1 Existence and uniqueness analysis

This is a linear ODE. In canonical form it is written as

$$y'' + p(x)y' + q(x)y = F$$

Where here

$$\begin{aligned} p(x) &= 0 \\ q(x) &= -\frac{1}{4} \\ F &= 0 \end{aligned}$$

Hence the ode is

$$y'' - \frac{y}{4} = 0$$

The domain of $p(x) = 0$ is

$$\{-\infty < x < \infty\}$$

And the point $x_0 = 0$ is inside this domain. The domain of $q(x) = -\frac{1}{4}$ is

$$\{-\infty < x < \infty\}$$

And the point $x_0 = 0$ is also inside this domain. Hence solution exists and is unique.

7.20.2 Solving as second order linear constant coeff ode

This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = 0$$

Where in the above $A = 4, B = 0, C = -1$. Let the solution be $y = e^{\lambda x}$. Substituting this into the ODE gives

$$4\lambda^2 \mathrm{e}^{\lambda x} - \mathrm{e}^{\lambda x} = 0 \tag{1}$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda x}$ gives

$$4\lambda^2 - 1 = 0 \tag{2}$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form.Using the quadratic formula

$$\lambda_{1,2} = \frac{-B}{2A} \pm \frac{1}{2A}\sqrt{B^2 - 4AC}$$

Substituting $A = 4, B = 0, C = -1$ into the above gives

$$\begin{aligned}\lambda_{1,2} &= \frac{0}{(2)(4)} \pm \frac{1}{(2)(4)}\sqrt{0^2 - (4)(4)(-1)} \\ &= \pm\frac{1}{2}\end{aligned}$$

Hence

$$\lambda_1 = +\frac{1}{2}$$

$$\lambda_2 = -\frac{1}{2}$$

Which simplifies to

$$\lambda_1 = \frac{1}{2}$$
$$\lambda_2 = -\frac{1}{2}$$

Since roots are real and distinct, then the solution is

$$y = c_1 e^{\lambda_1 x} + c_2 e^{\lambda_2 x}$$
$$y = c_1 e^{\left(\frac{1}{2}\right)x} + c_2 e^{\left(-\frac{1}{2}\right)x}$$

Or

$$y = c_1 \mathrm{e}^{\frac{x}{2}} + c_2 \mathrm{e}^{-\frac{x}{2}}$$

Initial conditions are used to solve for the constants of integration.

Looking at the above solution

$$y = c_1 \mathrm{e}^{\frac{x}{2}} + c_2 \mathrm{e}^{-\frac{x}{2}} \tag{1}$$

Initial conditions are now substituted in the above solution. This will generate the required equations to solve for the integration constants. substituting $y = 2$ and $x = 0$ in the above gives

$$2 = c_1 + c_2 \tag{1A}$$

Taking derivative of the solution gives

$$y' = \frac{c_1 \mathrm{e}^{\frac{x}{2}}}{2} - \frac{c_2 \mathrm{e}^{-\frac{x}{2}}}{2}$$

substituting $y' = \beta$ and $x = 0$ in the above gives

$$\beta = \frac{c_1}{2} - \frac{c_2}{2} \tag{2A}$$

Equations {1A,2A} are now solved for $\{c_1, c_2\}$. Solving for the constants gives

$$c_1 = 1 + \beta$$
$$c_2 = -\beta + 1$$

Substituting these values back in above solution results in

$$y = \mathrm{e}^{\frac{x}{2}}\beta - \mathrm{e}^{-\frac{x}{2}}\beta + \mathrm{e}^{\frac{x}{2}} + \mathrm{e}^{-\frac{x}{2}}$$

Which simplifies to

$$y = (1+\beta)\,e^{\frac{x}{2}} - (\beta - 1)\,e^{-\frac{x}{2}}$$

Summary

The solution(s) found are the following

$$y = (1+\beta)\,e^{\frac{x}{2}} - (\beta - 1)\,e^{-\frac{x}{2}} \tag{1}$$

Verification of solutions

$$y = (1+\beta)\,e^{\frac{x}{2}} - (\beta - 1)\,e^{-\frac{x}{2}}$$

Verified OK.

7.20.3 Solving as second order ode can be made integrable ode

Multiplying the ode by y' gives

$$4y'y'' - yy' = 0$$

Integrating the above w.r.t x gives

$$\int (4y'y'' - yy')\,dx = 0$$
$$2y'^2 - \frac{y^2}{2} = c_2$$

Which is now solved for y. Solving the given ode for y' results in 2 differential equations to solve. Each one of these will generate a solution. The equations generated are

$$y' = \frac{\sqrt{y^2 + 2c_1}}{2} \tag{1}$$
$$y' = -\frac{\sqrt{y^2 + 2c_1}}{2} \tag{2}$$

Now each one of the above ODE is solved.

Solving equation (1)

Integrating both sides gives

$$\int \frac{2}{\sqrt{y^2 + 2c_1}}dy = \int dx$$
$$2\ln\left(y + \sqrt{y^2 + 2c_1}\right) = x + c_2$$

Raising both side to exponential gives

$$\left(y+\sqrt{y^2+2c_1}\right)^2 = \mathrm{e}^{x+c_2}$$

Which simplifies to

$$\left(y+\sqrt{y^2+2c_1}\right)^2 = c_3\mathrm{e}^{x}$$

Solving equation (2)

Integrating both sides gives

$$\int -\frac{2}{\sqrt{y^2+2c_1}}dy = \int dx$$
$$-2\ln\left(y+\sqrt{y^2+2c_1}\right) = x + c_4$$

Raising both side to exponential gives

$$\frac{1}{\left(y+\sqrt{y^2+2c_1}\right)^2} = \mathrm{e}^{x+c_4}$$

Which simplifies to

$$\frac{1}{\left(y+\sqrt{y^2+2c_1}\right)^2} = c_5\mathrm{e}^{x}$$

Initial conditions are used to solve for the constants of integration.

Looking at the First solution

$$y = \frac{c_3\mathrm{e}^{x}-2c_1}{2\sqrt{c_3\mathrm{e}^{x}}} \tag{1}$$

Initial conditions are now substituted in the above solution. This will generate the required equations to solve for the integration constants. substituting $y=2$ and $x=0$ in the above gives

$$2 = \frac{c_3-2c_1}{2\sqrt{c_3}} \tag{1A}$$

Taking derivative of the solution gives

$$y' = \frac{c_3\mathrm{e}^{x}}{2\sqrt{c_3\mathrm{e}^{x}}} - \frac{\left(c_3\mathrm{e}^{x}-2c_1\right)c_3\mathrm{e}^{x}}{4\left(c_3\mathrm{e}^{x}\right)^{\frac{3}{2}}}$$

substituting $y' = \beta$ and $x = 0$ in the above gives

$$\beta = \frac{c_3 + 2c_1}{4\sqrt{c_3}} \tag{2A}$$

Equations {1A,2A} are now solved for $\{c_1, c_3\}$. Solving for the constants gives

$$c_1 = -4 \operatorname{csgn}(1+\beta)\,\beta + 2\beta^2 - 4 \operatorname{csgn}(1+\beta) + 4\beta + 2$$
$$c_3 = 4(1+\beta)^2$$

Substituting these values back in above solution results in

$$y = \frac{e^x\beta^2 + 2 \operatorname{csgn}(1+\beta)\,\beta + 2\,e^x\beta - \beta^2 + 2 \operatorname{csgn}(1+\beta) + e^x - 2\beta - 1}{\sqrt{e^x\beta^2 + 2\,e^x\beta + e^x}}$$

Which simplifies to

$$y = \frac{(1+\beta)\left(2 \operatorname{csgn}(1+\beta) + (e^x - 1)(1+\beta)\right)}{\sqrt{(1+\beta)^2 e^x}}$$

Looking at the Second solution

$$y = -\frac{2c_1c_5e^x - 1}{2\sqrt{c_5e^x}} \tag{2}$$

Initial conditions are now substituted in the above solution. This will generate the required equations to solve for the integration constants. substituting $y = 2$ and $x = 0$ in the above gives

$$2 = \frac{-2c_1c_5 + 1}{2\sqrt{c_5}} \tag{1A}$$

Taking derivative of the solution gives

$$y' = -\frac{c_5e^xc_1}{\sqrt{c_5e^x}} + \frac{(2c_1c_5e^x - 1)\,c_5e^x}{4\,(c_5e^x)^{\frac{3}{2}}}$$

substituting $y' = \beta$ and $x = 0$ in the above gives

$$\beta = \frac{-2c_1c_5 - 1}{4\sqrt{c_5}} \tag{2A}$$

Equations {1A,2A} are now solved for $\{c_1, c_5\}$. Solving for the constants gives

$$c_1 = 2\beta^2 - 2$$
$$c_5 = \frac{1}{4\,(\beta - 1)^2}$$

Substituting these values back in above solution results in

$$y = \frac{-e^{x}\beta - e^{x} + \beta - 1}{\sqrt{\frac{e^{x}}{\beta^{2}-2\beta+1}}\,\beta - \sqrt{\frac{e^{x}}{\beta^{2}-2\beta+1}}}$$

Which simplifies to

$$y = \frac{(-1-\beta)\,e^{x} + \beta - 1}{\sqrt{\frac{e^{x}}{(\beta-1)^{2}}}\,(\beta - 1)}$$

Summary

The solution(s) found are the following

$$y = \frac{(1+\beta)\,(2\,\operatorname{csgn}(1+\beta) + (e^{x}-1)\,(1+\beta))}{\sqrt{(1+\beta)^{2}\,e^{x}}} \tag{1}$$

$$y = \frac{(-1-\beta)\,e^{x} + \beta - 1}{\sqrt{\frac{e^{x}}{(\beta-1)^{2}}}\,(\beta - 1)} \tag{2}$$

Verification of solutions

$$y = \frac{(1+\beta)\,(2\,\operatorname{csgn}(1+\beta) + (e^{x}-1)\,(1+\beta))}{\sqrt{(1+\beta)^{2}\,e^{x}}}$$

Verified OK.

$$y = \frac{(-1-\beta)\,e^{x} + \beta - 1}{\sqrt{\frac{e^{x}}{(\beta-1)^{2}}}\,(\beta - 1)}$$

Verified OK.

7.20.4 Solving using Kovacic algorithm

Writing the ode as

$$4y'' - y = 0 \tag{1}$$

$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 4 \\ B &= 0 \\ C &= -1 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A} dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{1}{4} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= 1 \\ t &= 4 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = \frac{z(x)}{4} \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A} dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 354: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 0 - 0 \\ &= 0 \end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Since $r = \frac{1}{4}$ is not a function of x, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$z_1(x) = \mathrm{e}^{-\frac{x}{2}}$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$y_1 = z_1 e^{\int -\frac{1}{2} \frac{B}{A} \, dx}$$

Since $B = 0$ then the above reduces to

$$\begin{aligned} y_1 &= z_1 \\ &= \mathrm{e}^{-\frac{x}{2}} \end{aligned}$$

Which simplifies to

$$y_1 = \mathrm{e}^{-\frac{x}{2}}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\, dx}}{y_1^2}\, dx$$

Since $B = 0$ then the above becomes

$$\begin{aligned} y_2 &= y_1 \int \frac{1}{y_1^2}\, dx \\ &= \mathrm{e}^{-\frac{x}{2}} \int \frac{1}{\mathrm{e}^{-x}}\, dx \\ &= \mathrm{e}^{-\frac{x}{2}} (\mathrm{e}^{x}) \end{aligned}$$

Therefore the solution is

$$\begin{aligned} y &= c_1 y_1 + c_2 y_2 \\ &= c_1 \left(\mathrm{e}^{-\frac{x}{2}}\right) + c_2 \left(\mathrm{e}^{-\frac{x}{2}} (\mathrm{e}^{x})\right) \end{aligned}$$

Initial conditions are used to solve for the constants of integration.

Looking at the above solution

$$y = c_1 \mathrm{e}^{-\frac{x}{2}} + c_2 \mathrm{e}^{\frac{x}{2}} \tag{1}$$

Initial conditions are now substituted in the above solution. This will generate the required equations to solve for the integration constants. substituting $y = 2$ and $x = 0$ in the above gives

$$2 = c_1 + c_2 \tag{1A}$$

Taking derivative of the solution gives

$$y' = -\frac{c_1 \mathrm{e}^{-\frac{x}{2}}}{2} + \frac{c_2 \mathrm{e}^{\frac{x}{2}}}{2}$$

substituting $y' = \beta$ and $x = 0$ in the above gives

$$\beta = -\frac{c_1}{2} + \frac{c_2}{2} \tag{2A}$$

Equations {1A,2A} are now solved for $\{c_1, c_2\}$. Solving for the constants gives

$$c_1 = -\beta + 1$$
$$c_2 = 1 + \beta$$

Substituting these values back in above solution results in

$$y = e^{\frac{x}{2}}\beta - e^{-\frac{x}{2}}\beta + e^{\frac{x}{2}} + e^{-\frac{x}{2}}$$

Which simplifies to

$$y = (1+\beta)\, e^{\frac{x}{2}} - (\beta - 1)\, e^{-\frac{x}{2}}$$

Summary
The solution(s) found are the following

$$y = (1+\beta)\, e^{\frac{x}{2}} - (\beta - 1)\, e^{-\frac{x}{2}} \tag{1}$$

Verification of solutions

$$y = (1+\beta)\, e^{\frac{x}{2}} - (\beta - 1)\, e^{-\frac{x}{2}}$$

Verified OK.

7.20.5 Maple step by step solution

Let's solve

$\left[4y'' - y = 0, y(0) = 2, y'\Big|_{\{x=0\}} = \beta\right]$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y'' = \frac{y}{4}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y'' - \frac{y}{4} = 0$

- Characteristic polynomial of ODE

  $r^2 - \frac{1}{4} = 0$

- Factor the characteristic polynomial

  $\frac{(2r-1)(2r+1)}{4} = 0$

- Roots of the characteristic polynomial

  $r = \left(-\frac{1}{2}, \frac{1}{2}\right)$
- 1st solution of the ODE

  $y_1(x) = \mathrm{e}^{-\frac{x}{2}}$
- 2nd solution of the ODE

  $y_2(x) = \mathrm{e}^{\frac{x}{2}}$
- General solution of the ODE

  $y = c_1 y_1(x) + c_2 y_2(x)$
- Substitute in solutions

  $y = c_1 \mathrm{e}^{-\frac{x}{2}} + c_2 \mathrm{e}^{\frac{x}{2}}$

☐ Check validity of solution $y = c_1 \mathrm{e}^{-\frac{x}{2}} + c_2 \mathrm{e}^{\frac{x}{2}}$

  - Use initial condition $y(0) = 2$

    $2 = c_1 + c_2$
  - Compute derivative of the solution

    $y' = -\frac{c_1 \mathrm{e}^{-\frac{x}{2}}}{2} + \frac{c_2 \mathrm{e}^{\frac{x}{2}}}{2}$
  - Use the initial condition $y'\Big|_{\{x=0\}} = \beta$

    $\beta = -\frac{c_1}{2} + \frac{c_2}{2}$
  - Solve for c_1 and c_2

    $\{c_1 = -\beta + 1, c_2 = 1 + \beta\}$
  - Substitute constant values into general solution and simplify

    $y = (1 + \beta)\, \mathrm{e}^{\frac{x}{2}} - (\beta - 1)\, \mathrm{e}^{-\frac{x}{2}}$

- Solution to the IVP

  $y = (1 + \beta)\, \mathrm{e}^{\frac{x}{2}} - (\beta - 1)\, \mathrm{e}^{-\frac{x}{2}}$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
<- constant coefficients successful`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 22

```
dsolve([4*diff(y(x),x$2) -y(x) = 0,y(0) = 2, D(y)(0) = beta],y(x), singsol=all)
```

$$y(x) = (1 + \beta) e^{\frac{x}{2}} - (\beta - 1) e^{-\frac{x}{2}}$$

✓ Solution by Mathematica

Time used: 0.012 (sec). Leaf size: 25

```
DSolve[{4*y''[x]-y[x]==0,{y[0]==2,y'[0]==\[Beta]}},y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to e^{-x/2}(-\beta + (\beta + 1)e^{x} + 1)$$

7.21 problem 23

Internal problem ID [619]
Internal file name [`OUTPUT/619_Sunday_June_05_2022_01_45_56_AM_20917638/index.tex`]

Book: Elementary differential equations and boundary value problems, 10th ed., Boyce and DiPrima
Section: Chapter 3, Second order linear equations, 3.1 Homogeneous Equations with Constant Coefficients, page 144
Problem number: 23.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_linear_constant_coeff"**

Maple gives the following as the ode type

```
[[_2nd_order, _missing_x]]
```

$$\boxed{y'' - (2\alpha - 1)\, y' + \alpha(\alpha - 1)\, y = 0}$$

7.21.1 Solving as second order linear constant coeff ode

This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = 0$$

Where in the above $A = 1, B = -2\alpha + 1, C = \alpha^2 - \alpha$. Let the solution be $y = e^{\lambda x}$. Substituting this into the ODE gives

$$\lambda^2 e^{\lambda x} + (-2\alpha + 1)\, \lambda\, e^{\lambda x} + \left(\alpha^2 - \alpha\right) e^{\lambda x} = 0 \tag{1}$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda x}$ gives

$$\lambda^2 + (-2\alpha + 1)\, \lambda + \alpha^2 - \alpha = 0 \tag{2}$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form.Using the quadratic formula

$$\lambda_{1,2} = \frac{-B}{2A} \pm \frac{1}{2A}\sqrt{B^2 - 4AC}$$

Substituting $A = 1, B = -2\alpha + 1, C = \alpha^2 - \alpha$ into the above gives

$$\begin{aligned}\lambda_{1,2} &= \frac{2\alpha - 1}{(2)(1)} \pm \frac{1}{(2)(1)}\sqrt{-2\alpha + 1^2 - (4)(1)(\alpha^2 - \alpha)} \\ &= \alpha - \frac{1}{2} \pm \frac{1}{2}\end{aligned}$$

Hence

$$\lambda_1 = \alpha - \frac{1}{2} + \frac{1}{2}$$
$$\lambda_2 = \alpha - \frac{1}{2} - \frac{1}{2}$$

Which simplifies to

$$\lambda_1 = \alpha$$
$$\lambda_2 = \alpha - 1$$

Since roots are real and distinct, then the solution is

$$y = c_1 e^{\lambda_1 x} + c_2 e^{\lambda_2 x}$$
$$y = c_1 e^{(\alpha)x} + c_2 e^{(\alpha-1)x}$$

Or

$$y = c_1 e^{\alpha x} + c_2 e^{(\alpha-1)x}$$

Summary

The solution(s) found are the following

$$y = c_1 e^{\alpha x} + c_2 e^{(\alpha-1)x} \tag{1}$$

Verification of solutions

$$y = c_1 e^{\alpha x} + c_2 e^{(\alpha-1)x}$$

Verified OK.

7.21.2 Solving using Kovacic algorithm

Writing the ode as

$$y'' + (-2\alpha + 1)\, y' + (\alpha^2 - \alpha)\, y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1 \\ B &= -2\alpha + 1 \\ C &= \alpha^2 - \alpha \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = y e^{\int \frac{B}{2A}\, dx}$$

Then (2) becomes

$$z''(x) = r z(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{1}{4} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= 1 \\ t &= 4 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = \frac{z(x)}{4} \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 356: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 0 - 0 \\ &= 0 \end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Since $r = \frac{1}{4}$ is not a function of x, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$z_1(x) = e^{-\frac{x}{2}}$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$\begin{aligned} y_1 &= z_1 e^{\int -\frac{1}{2} \frac{B}{A} \, dx} \\ &= z_1 e^{-\int \frac{1}{2} \frac{-2\alpha+1}{1} \, dx} \end{aligned}$$

$$
\begin{aligned}
&= z_1 e^{\left(\alpha-\frac{1}{2}\right)x} \\
&= z_1 \left(\mathrm{e}^{\left(\alpha-\frac{1}{2}\right)x}\right)
\end{aligned}
$$

Which simplifies to

$$
y_1 = \mathrm{e}^{(\alpha-1)x}
$$

The second solution y_2 to the original ode is found using reduction of order

$$
y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\, dx}}{y_1^2}\, dx
$$

Substituting gives

$$
\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{-2\alpha+1}{1}\, dx}}{\left(y_1\right)^2}\, dx \\
&= y_1 \int \frac{e^{(2\alpha-1)x}}{\left(y_1\right)^2}\, dx \\
&= y_1(\mathrm{e}^x)
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1\left(\mathrm{e}^{(\alpha-1)x}\right) + c_2\left(\mathrm{e}^{(\alpha-1)x}(\mathrm{e}^x)\right)
\end{aligned}
$$

Summary

The solution(s) found are the following

$$
y = c_1 \mathrm{e}^{(\alpha-1)x} + c_2 \mathrm{e}^{\alpha x} \tag{1}
$$

Verification of solutions

$$
y = c_1 \mathrm{e}^{(\alpha-1)x} + c_2 \mathrm{e}^{\alpha x}
$$

Verified OK.

7.21.3 Maple step by step solution

Let's solve

$y'' + (-2\alpha + 1) y' + (\alpha^2 - \alpha) y = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Characteristic polynomial of ODE

  $r^2 + (-2\alpha + 1) r + \alpha^2 - \alpha = 0$

- Factor the characteristic polynomial

  $(\alpha - r)(\alpha - r - 1) = 0$

- Roots of the characteristic polynomial

  $r = (\alpha, \alpha - 1)$

- 1st solution of the ODE

  $y_1(x) = e^{\alpha x}$

- 2nd solution of the ODE

  $y_2(x) = e^{(\alpha - 1)x}$

- General solution of the ODE

  $y = c_1 y_1(x) + c_2 y_2(x)$

- Substitute in solutions

  $y = c_1 e^{\alpha x} + c_2 e^{(\alpha - 1)x}$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
<- constant coefficients successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 19

```
dsolve(diff(y(x),x$2) -(2*alpha-1)*diff(y(x),x)+alpha*(alpha-1)*y(x) = 0,y(x), singsol=all)
```

$$y(x) = c_1 e^{\alpha x} + c_2 e^{(\alpha-1)x}$$

✓ Solution by Mathematica

Time used: 0.015 (sec). Leaf size: 24

```
DSolve[y''[x]-(2*\[Alpha]-1)*y'[x]+\[Alpha]*(\[Alpha]-1)*y[x]==0,y[x],x,IncludeSingularSolut
```

$$y(x) \to c_1 e^{(\alpha-1)x} + c_2 e^{\alpha x}$$

7.22 problem 24

Internal problem ID [620]
Internal file name [`OUTPUT/620_Sunday_June_05_2022_01_45_57_AM_13631438/index.tex`]

Book: Elementary differential equations and boundary value problems, 10th ed., Boyce and DiPrima
Section: Chapter 3, Second order linear equations, 3.1 Homogeneous Equations with Constant Coefficients, page 144
Problem number: 24.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_linear_constant_coeff"**

Maple gives the following as the ode type

```
[[_2nd_order, _missing_x]]
```

$$\boxed{y'' + (3-\alpha)\,y' - 2(\alpha-1)\,y = 0}$$

7.22.1 Solving as second order linear constant coeff ode

This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = 0$$

Where in the above $A = 1, B = 3 - \alpha, C = -2\alpha + 2$. Let the solution be $y = e^{\lambda x}$. Substituting this into the ODE gives

$$\lambda^2 \mathrm{e}^{\lambda x} + (3-\alpha)\,\lambda\,\mathrm{e}^{\lambda x} + (-2\alpha+2)\,\mathrm{e}^{\lambda x} = 0 \tag{1}$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda x}$ gives

$$\lambda^2 + (3-\alpha)\,\lambda - 2\alpha + 2 = 0 \tag{2}$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form.Using the quadratic formula

$$\lambda_{1,2} = \frac{-B}{2A} \pm \frac{1}{2A}\sqrt{B^2 - 4AC}$$

Substituting $A = 1, B = 3 - \alpha, C = -2\alpha + 2$ into the above gives

$$\begin{aligned}\lambda_{1,2} &= \frac{\alpha - 3}{(2)(1)} \pm \frac{1}{(2)(1)}\sqrt{3 - \alpha^2 - (4)(1)(-2\alpha + 2)} \\ &= -\frac{3}{2} + \frac{\alpha}{2} \pm \frac{\sqrt{(\alpha + 1)^2}}{2}\end{aligned}$$

Hence

$$\begin{aligned}\lambda_1 &= -\frac{3}{2} + \frac{\alpha}{2} + \frac{\sqrt{(\alpha + 1)^2}}{2} \\ \lambda_2 &= -\frac{3}{2} + \frac{\alpha}{2} - \frac{\sqrt{(\alpha + 1)^2}}{2}\end{aligned}$$

Which simplifies to

$$\begin{aligned}\lambda_1 &= -\frac{3}{2} + \frac{\alpha}{2} + \frac{\sqrt{(\alpha + 1)^2}}{2} \\ \lambda_2 &= -\frac{3}{2} + \frac{\alpha}{2} - \frac{\sqrt{(\alpha + 1)^2}}{2}\end{aligned}$$

Since roots are real and distinct, then the solution is

$$\begin{aligned}y &= c_1 e^{\lambda_1 x} + c_2 e^{\lambda_2 x} \\ y &= c_1 e^{\left(-\frac{3}{2} + \frac{\alpha}{2} + \frac{\sqrt{(\alpha+1)^2}}{2}\right)x} + c_2 e^{\left(-\frac{3}{2} + \frac{\alpha}{2} - \frac{\sqrt{(\alpha+1)^2}}{2}\right)x}\end{aligned}$$

Or

$$y = c_1 \mathrm{e}^{\left(-\frac{3}{2} + \frac{\alpha}{2} + \frac{\sqrt{(\alpha+1)^2}}{2}\right)x} + c_2 \mathrm{e}^{\left(-\frac{3}{2} + \frac{\alpha}{2} - \frac{\sqrt{(\alpha+1)^2}}{2}\right)x}$$

Summary

The solution(s) found are the following

$$y = c_1 \mathrm{e}^{\left(-\frac{3}{2} + \frac{\alpha}{2} + \frac{\sqrt{(\alpha+1)^2}}{2}\right)x} + c_2 \mathrm{e}^{\left(-\frac{3}{2} + \frac{\alpha}{2} - \frac{\sqrt{(\alpha+1)^2}}{2}\right)x} \tag{1}$$

Verification of solutions

$$y = c_1 \mathrm{e}^{\left(-\frac{3}{2}+\frac{\alpha}{2}+\frac{\sqrt{(\alpha+1)^2}}{2}\right)x} + c_2 \mathrm{e}^{\left(-\frac{3}{2}+\frac{\alpha}{2}-\frac{\sqrt{(\alpha+1)^2}}{2}\right)x}$$

Verified OK.

7.22.2 Solving using Kovacic algorithm

Writing the ode as

$$y'' + (3-\alpha)\,y' + (-2\alpha+2)\,y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1 \\ B &= 3-\alpha \\ C &= -2\alpha+2 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = y e^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{\alpha^2 + 2\alpha + 1}{4} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= \alpha^2 + 2\alpha + 1 \\ t &= 4 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{1}{4}\alpha^2 + \frac{1}{2}\alpha + \frac{1}{4}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 358: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 0 - 0 \\ &= 0 \end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Since $r = \frac{1}{4}\alpha^2 + \frac{1}{2}\alpha + \frac{1}{4}$ is not a function of x, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$z_1(x) = \mathrm{e}^{\frac{\sqrt{(\alpha+1)^2}\, x}{2}}$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\, dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{3-\alpha}{1}\, dx} \\
&= z_1 e^{\left(-\frac{3}{2}+\frac{\alpha}{2}\right)x} \\
&= z_1 \left(\mathrm{e}^{\frac{(\alpha-3)x}{2}}\right)
\end{aligned}$$

Which simplifies to

$$y_1 = \mathrm{e}^{\frac{(\mathrm{csgn}(\alpha+1)\alpha+\mathrm{csgn}(\alpha+1)+\alpha-3)x}{2}}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\, dx}}{y_1^2}\, dx$$

Substituting gives

$$\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{3-\alpha}{1}\, dx}}{(y_1)^2}\, dx \\
&= y_1 \int \frac{e^{(\alpha-3)x}}{(y_1)^2}\, dx \\
&= y_1 \left(-\frac{\mathrm{csgn}\,(\alpha+1)\,\mathrm{e}^{-\mathrm{csgn}(\alpha+1)(\alpha+1)x}}{\alpha+1}\right)
\end{aligned}$$

Therefore the solution is

$$\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1 \left(\mathrm{e}^{\frac{(\mathrm{csgn}(\alpha+1)\alpha+\mathrm{csgn}(\alpha+1)+\alpha-3)x}{2}}\right) \\
&\quad + c_2 \left(\mathrm{e}^{\frac{(\mathrm{csgn}(\alpha+1)\alpha+\mathrm{csgn}(\alpha+1)+\alpha-3)x}{2}} \left(-\frac{\mathrm{csgn}\,(\alpha+1)\,\mathrm{e}^{-\mathrm{csgn}(\alpha+1)(\alpha+1)x}}{\alpha+1}\right)\right)
\end{aligned}$$

Simplifying the solution $y = c_1 e^{\frac{(\mathrm{csgn}(\alpha+1)\alpha+\mathrm{csgn}(\alpha+1)+\alpha-3)x}{2}} - \frac{c_2\,\mathrm{csgn}(\alpha+1)\mathrm{e}^{-\frac{(\mathrm{csgn}(\alpha+1)\alpha+\mathrm{csgn}(\alpha+1)-\alpha+3)x}{2}}}{\alpha+1}$ to $y = c_1 e^{\frac{(2\alpha-2)x}{2}} - \frac{c_2 e^{-2x}}{\alpha+1}$

Summary

The solution(s) found are the following

$$y = c_1 e^{\frac{(2\alpha-2)x}{2}} - \frac{c_2 e^{-2x}}{\alpha+1}$$

Verification of solutions

$$y = c_1 e^{\frac{(2\alpha-2)x}{2}} - \frac{c_2 e^{-2x}}{\alpha+1}$$

Verified OK.

7.22.3 Maple step by step solution

Let's solve

$y'' + (3-\alpha)\,y' + (-2\alpha+2)\,y = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Characteristic polynomial of ODE

  $r^2 + (3-\alpha)\,r - 2\alpha + 2 = 0$

- Factor the characteristic polynomial

  $-(r+2)\,(-r+\alpha-1) = 0$

- Roots of the characteristic polynomial

  $r = (-2, \alpha-1)$

- 1st solution of the ODE

  $y_1(x) = e^{-2x}$

- 2nd solution of the ODE

  $y_2(x) = e^{(\alpha-1)x}$

- General solution of the ODE

  $y = c_1 y_1(x) + c_2 y_2(x)$

- Substitute in solutions

  $y = c_1 e^{-2x} + c_2 e^{(\alpha-1)x}$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
<- constant coefficients successful`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 19

```
dsolve(diff(y(x),x$2) +(3-alpha)*diff(y(x),x)-2*(alpha-1)*y(x) = 0,y(x), singsol=all)
```

$$y(x) = \mathrm{e}^{-2x}c_1 + c_2\mathrm{e}^{(\alpha-1)x}$$

✓ Solution by Mathematica

Time used: 0.014 (sec). Leaf size: 24

```
DSolve[y''[x]+(3-\[Alpha])*y'[x]-2*(\[Alpha]-1)*y[x]==0,y[x],x,IncludeSingularSolutions -> T
```

$$y(x) \to e^{-2x}\left(c_1e^{\alpha x+x} + c_2\right)$$

7.23 problem 25

Internal problem ID [621]
Internal file name [`OUTPUT/621_Sunday_June_05_2022_01_45_58_AM_931008/index.tex`]

Book: Elementary differential equations and boundary value problems, 10th ed., Boyce and DiPrima
Section: Chapter 3, Second order linear equations, 3.1 Homogeneous Equations with Constant Coefficients, page 144
Problem number: 25.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_linear_constant_coeff"**

Maple gives the following as the ode type

`[[_2nd_order, _missing_x]]`

$$\boxed{2y'' + 3y' - 2y = 0}$$

With initial conditions

$$[y(0) = 1, y'(0) = -\beta]$$

7.23.1 Existence and uniqueness analysis

This is a linear ODE. In canonical form it is written as

$$y'' + p(x)y' + q(x)y = F$$

Where here

$$\begin{aligned} p(x) &= \frac{3}{2} \\ q(x) &= -1 \\ F &= 0 \end{aligned}$$

Hence the ode is

$$y'' + \frac{3y'}{2} - y = 0$$

The domain of $p(x) = \frac{3}{2}$ is

$$\{-\infty < x < \infty\}$$

And the point $x_0 = 0$ is inside this domain. The domain of $q(x) = -1$ is

$$\{-\infty < x < \infty\}$$

And the point $x_0 = 0$ is also inside this domain. Hence solution exists and is unique.

7.23.2 Solving as second order linear constant coeff ode

This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = 0$$

Where in the above $A = 2, B = 3, C = -2$. Let the solution be $y = e^{\lambda x}$. Substituting this into the ODE gives

$$2\lambda^2 e^{\lambda x} + 3\lambda\, e^{\lambda x} - 2\, e^{\lambda x} = 0 \tag{1}$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda x}$ gives

$$2\lambda^2 + 3\lambda - 2 = 0 \tag{2}$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form.Using the quadratic formula

$$\lambda_{1,2} = \frac{-B}{2A} \pm \frac{1}{2A}\sqrt{B^2 - 4AC}$$

Substituting $A = 2, B = 3, C = -2$ into the above gives

$$\begin{aligned}\lambda_{1,2} &= \frac{-3}{(2)\,(2)} \pm \frac{1}{(2)\,(2)}\sqrt{3^2 - (4)\,(2)\,(-2)}\\ &= -\frac{3}{4} \pm \frac{5}{4}\end{aligned}$$

Hence

$$\lambda_1 = -\frac{3}{4} + \frac{5}{4}$$

$$\lambda_2 = -\frac{3}{4} - \frac{5}{4}$$

Which simplifies to

$$\lambda_1 = \frac{1}{2}$$
$$\lambda_2 = -2$$

Since roots are real and distinct, then the solution is

$$y = c_1 e^{\lambda_1 x} + c_2 e^{\lambda_2 x}$$
$$y = c_1 e^{\left(\frac{1}{2}\right)x} + c_2 e^{(-2)x}$$

Or

$$y = c_1 \mathrm{e}^{\frac{x}{2}} + c_2 \mathrm{e}^{-2x}$$

Initial conditions are used to solve for the constants of integration.

Looking at the above solution

$$y = c_1 \mathrm{e}^{\frac{x}{2}} + c_2 \mathrm{e}^{-2x} \tag{1}$$

Initial conditions are now substituted in the above solution. This will generate the required equations to solve for the integration constants. substituting $y = 1$ and $x = 0$ in the above gives

$$1 = c_1 + c_2 \tag{1A}$$

Taking derivative of the solution gives

$$y' = \frac{c_1 \mathrm{e}^{\frac{x}{2}}}{2} - 2c_2 \mathrm{e}^{-2x}$$

substituting $y' = -\beta$ and $x = 0$ in the above gives

$$-\beta = \frac{c_1}{2} - 2c_2 \tag{2A}$$

Equations {1A,2A} are now solved for $\{c_1, c_2\}$. Solving for the constants gives

$$c_1 = \frac{4}{5} - \frac{2\beta}{5}$$
$$c_2 = \frac{2\beta}{5} + \frac{1}{5}$$

Substituting these values back in above solution results in

$$y = \frac{4\,\mathrm{e}^{\frac{x}{2}}}{5} - \frac{2\,\mathrm{e}^{\frac{x}{2}}\beta}{5} + \frac{2\,\mathrm{e}^{-2x}\beta}{5} + \frac{\mathrm{e}^{-2x}}{5}$$

Which simplifies to

$$y = -\frac{\left(2\,\mathrm{e}^{\frac{5x}{2}}\beta - 4\,\mathrm{e}^{\frac{5x}{2}} - 2\beta - 1\right)\mathrm{e}^{-2x}}{5}$$

<u>Summary</u>
The solution(s) found are the following

$$y = -\frac{\left(2\,\mathrm{e}^{\frac{5x}{2}}\beta - 4\,\mathrm{e}^{\frac{5x}{2}} - 2\beta - 1\right)\mathrm{e}^{-2x}}{5} \tag{1}$$

<u>Verification of solutions</u>

$$y = -\frac{\left(2\,\mathrm{e}^{\frac{5x}{2}}\beta - 4\,\mathrm{e}^{\frac{5x}{2}} - 2\beta - 1\right)\mathrm{e}^{-2x}}{5}$$

Verified OK.

7.23.3 Solving using Kovacic algorithm

Writing the ode as

$$2y'' + 3y' - 2y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 2 \\ B &= 3 \\ C &= -2 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{25}{16} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= 25 \\ t &= 16 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = \frac{25z(x)}{16} \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 360: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 0 - 0 \\ &= 0 \end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Since $r = \frac{25}{16}$ is not a function of x, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$z_1(x) = e^{-\frac{5x}{4}}$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$\begin{aligned} y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx} \\ &= z_1 e^{-\int \frac{1}{2}\frac{3}{2}\,dx} \\ &= z_1 e^{-\frac{3x}{4}} \\ &= z_1 \left(e^{-\frac{3x}{4}}\right) \end{aligned}$$

Which simplifies to

$$y_1 = e^{-2x}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx$$

Substituting gives

$$\begin{aligned} y_2 &= y_1 \int \frac{e^{\int -\frac{3}{2}\,dx}}{(y_1)^2}\,dx \\ &= y_1 \int \frac{e^{-\frac{3x}{2}}}{(y_1)^2}\,dx \\ &= y_1 \left(\frac{2\,e^{\frac{5x}{2}}}{5}\right) \end{aligned}$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1 \left(\mathrm{e}^{-2x}\right) + c_2 \left(\mathrm{e}^{-2x} \left(\frac{2\,\mathrm{e}^{\frac{5x}{2}}}{5}\right)\right)
\end{aligned}
$$

Initial conditions are used to solve for the constants of integration.

Looking at the above solution

$$
y = c_1 \mathrm{e}^{-2x} + \frac{2c_2 \mathrm{e}^{\frac{x}{2}}}{5} \tag{1}
$$

Initial conditions are now substituted in the above solution. This will generate the required equations to solve for the integration constants. substituting $y = 1$ and $x = 0$ in the above gives

$$
1 = c_1 + \frac{2c_2}{5} \tag{1A}
$$

Taking derivative of the solution gives

$$
y' = -2c_1 \mathrm{e}^{-2x} + \frac{c_2 \mathrm{e}^{\frac{x}{2}}}{5}
$$

substituting $y' = -\beta$ and $x = 0$ in the above gives

$$
-\beta = -2c_1 + \frac{c_2}{5} \tag{2A}
$$

Equations {1A,2A} are now solved for $\{c_1, c_2\}$. Solving for the constants gives

$$
\begin{aligned}
c_1 &= \frac{2\beta}{5} + \frac{1}{5} \\
c_2 &= 2 - \beta
\end{aligned}
$$

Substituting these values back in above solution results in

$$
y = \frac{4\,\mathrm{e}^{\frac{x}{2}}}{5} - \frac{2\,\mathrm{e}^{\frac{x}{2}}\beta}{5} + \frac{2\,\mathrm{e}^{-2x}\beta}{5} + \frac{\mathrm{e}^{-2x}}{5}
$$

Which simplifies to

$$
y = -\frac{\left(2\,\mathrm{e}^{\frac{5x}{2}}\beta - 4\,\mathrm{e}^{\frac{5x}{2}} - 2\beta - 1\right)\mathrm{e}^{-2x}}{5}
$$

Summary

The solution(s) found are the following

$$
y = -\frac{\left(2\,\mathrm{e}^{\frac{5x}{2}}\beta - 4\,\mathrm{e}^{\frac{5x}{2}} - 2\beta - 1\right)\mathrm{e}^{-2x}}{5} \tag{1}
$$

Verification of solutions

$$y = -\frac{\left(2\,\mathrm{e}^{\frac{5x}{2}}\beta - 4\,\mathrm{e}^{\frac{5x}{2}} - 2\beta - 1\right)\mathrm{e}^{-2x}}{5}$$

Verified OK.

7.23.4 Maple step by step solution

Let's solve

$\left[2y'' + 3y' - 2y = 0, y(0) = 1, y'\Big|_{\{x=0\}} = -\beta\right]$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y'' = -\frac{3y'}{2} + y$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y'' + \frac{3y'}{2} - y = 0$

- Characteristic polynomial of ODE

  $r^2 + \frac{3}{2}r - 1 = 0$

- Factor the characteristic polynomial

  $\frac{(r+2)(2r-1)}{2} = 0$

- Roots of the characteristic polynomial

  $r = \left(-2, \frac{1}{2}\right)$

- 1st solution of the ODE

  $y_1(x) = \mathrm{e}^{-2x}$

- 2nd solution of the ODE

  $y_2(x) = \mathrm{e}^{\frac{x}{2}}$

- General solution of the ODE

  $y = c_1 y_1(x) + c_2 y_2(x)$

- Substitute in solutions

  $y = c_1\mathrm{e}^{-2x} + c_2\mathrm{e}^{\frac{x}{2}}$

□ Check validity of solution $y = c_1\mathrm{e}^{-2x} + c_2\mathrm{e}^{\frac{x}{2}}$

- Use initial condition $y(0) = 1$

  $1 = c_1 + c_2$

- Compute derivative of the solution

  $y' = -2c_1 e^{-2x} + \frac{c_2 e^{\frac{x}{2}}}{2}$

- Use the initial condition $y'\big|_{\{x=0\}} = -\beta$

  $-\beta = -2c_1 + \frac{c_2}{2}$

- Solve for c_1 and c_2

  $\left\{c_1 = \frac{2\beta}{5} + \frac{1}{5}, c_2 = \frac{4}{5} - \frac{2\beta}{5}\right\}$

- Substitute constant values into general solution and simplify

  $y = -\frac{\left(2\,e^{\frac{5x}{2}}\beta - 4\,e^{\frac{5x}{2}} - 2\beta - 1\right)e^{-2x}}{5}$

- Solution to the IVP

  $y = -\frac{\left(2\,e^{\frac{5x}{2}}\beta - 4\,e^{\frac{5x}{2}} - 2\beta - 1\right)e^{-2x}}{5}$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
<- constant coefficients successful`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 28

```
dsolve([2*diff(y(x),x$2) +3*diff(y(x),x)-2*y(x) = 0,y(0) = 1, D(y)(0) = -beta],y(x), singsol
```

$$y(x) = -\frac{\left(2\,e^{\frac{5x}{2}}\beta - 4\,e^{\frac{5x}{2}} - 2\beta - 1\right)e^{-2x}}{5}$$

✓ Solution by Mathematica

Time used: 0.024 (sec). Leaf size: 67

```
DSolve[{y''[x]+3*y'[x]-2*y[x]==0,{y[0]==1,y'[0]==-\[Beta]}},y[x],x,IncludeSingularSolutions
```

$$y(x) \to \frac{1}{34} e^{-\frac{1}{2}\left(3+\sqrt{17}\right)x}\left(2\sqrt{17}\beta + \left(-2\sqrt{17}\beta + 3\sqrt{17} + 17\right)e^{\sqrt{17}x} - 3\sqrt{17} + 17\right)$$

7.24 problem 26

Internal problem ID [622]
Internal file name [`OUTPUT/622_Sunday_June_05_2022_01_45_59_AM_64820801/index.tex`]

Book: Elementary differential equations and boundary value problems, 10th ed., Boyce and DiPrima
Section: Chapter 3, Second order linear equations, 3.1 Homogeneous Equations with Constant Coefficients, page 144
Problem number: 26.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_linear_constant_coeff"**

Maple gives the following as the ode type

```
[[_2nd_order, _missing_x]]
```

$$\boxed{y'' + 5y' + 6y = 0}$$

With initial conditions

$$[y(0) = 2, y'(0) = \beta]$$

7.24.1 Existence and uniqueness analysis

This is a linear ODE. In canonical form it is written as

$$y'' + p(x)y' + q(x)y = F$$

Where here

$$\begin{aligned} p(x) &= 5 \\ q(x) &= 6 \\ F &= 0 \end{aligned}$$

Hence the ode is

$$y'' + 5y' + 6y = 0$$

The domain of $p(x) = 5$ is

$$\{-\infty < x < \infty\}$$

And the point $x_0 = 0$ is inside this domain. The domain of $q(x) = 6$ is

$$\{-\infty < x < \infty\}$$

And the point $x_0 = 0$ is also inside this domain. Hence solution exists and is unique.

7.24.2 Solving as second order linear constant coeff ode

This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = 0$$

Where in the above $A = 1, B = 5, C = 6$. Let the solution be $y = e^{\lambda x}$. Substituting this into the ODE gives

$$\lambda^2 e^{\lambda x} + 5\lambda\, e^{\lambda x} + 6\, e^{\lambda x} = 0 \tag{1}$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda x}$ gives

$$\lambda^2 + 5\lambda + 6 = 0 \tag{2}$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form.Using the quadratic formula

$$\lambda_{1,2} = \frac{-B}{2A} \pm \frac{1}{2A}\sqrt{B^2 - 4AC}$$

Substituting $A = 1, B = 5, C = 6$ into the above gives

$$\begin{aligned}\lambda_{1,2} &= \frac{-5}{(2)\,(1)} \pm \frac{1}{(2)\,(1)}\sqrt{5^2 - (4)\,(1)\,(6)}\\ &= -\frac{5}{2} \pm \frac{1}{2}\end{aligned}$$

Hence

$$\lambda_1 = -\frac{5}{2} + \frac{1}{2}$$

$$\lambda_2 = -\frac{5}{2} - \frac{1}{2}$$

Which simplifies to

$$\lambda_1 = -2$$
$$\lambda_2 = -3$$

Since roots are real and distinct, then the solution is

$$y = c_1 e^{\lambda_1 x} + c_2 e^{\lambda_2 x}$$
$$y = c_1 e^{(-2)x} + c_2 e^{(-3)x}$$

Or

$$y = c_1 \mathrm{e}^{-2x} + \mathrm{e}^{-3x} c_2$$

Initial conditions are used to solve for the constants of integration.

Looking at the above solution

$$y = c_1 \mathrm{e}^{-2x} + \mathrm{e}^{-3x} c_2 \tag{1}$$

Initial conditions are now substituted in the above solution. This will generate the required equations to solve for the integration constants. substituting $y = 2$ and $x = 0$ in the above gives

$$2 = c_1 + c_2 \tag{1A}$$

Taking derivative of the solution gives

$$y' = -2c_1 \mathrm{e}^{-2x} - 3\,\mathrm{e}^{-3x} c_2$$

substituting $y' = \beta$ and $x = 0$ in the above gives

$$\beta = -2c_1 - 3c_2 \tag{2A}$$

Equations {1A,2A} are now solved for $\{c_1, c_2\}$. Solving for the constants gives

$$c_1 = 6 + \beta$$
$$c_2 = -\beta - 4$$

Substituting these values back in above solution results in

$$y = \mathrm{e}^{-2x}\beta - \mathrm{e}^{-3x}\beta + 6\,\mathrm{e}^{-2x} - 4\,\mathrm{e}^{-3x}$$

Which simplifies to

$$y = (-\beta - 4)\,\mathrm{e}^{-3x} + (6+\beta)\,\mathrm{e}^{-2x}$$

Summary

The solution(s) found are the following

$$y = (-\beta - 4)\,\mathrm{e}^{-3x} + (6+\beta)\,\mathrm{e}^{-2x} \tag{1}$$

Verification of solutions

$$y = (-\beta - 4)\,\mathrm{e}^{-3x} + (6+\beta)\,\mathrm{e}^{-2x}$$

Verified OK.

7.24.3 Solving using Kovacic algorithm

Writing the ode as

$$y'' + 5y' + 6y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1 \\ B &= 5 \\ C &= 6 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = y e^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x) = r z(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{1}{4} \tag{6}$$

Comparing the above to (5) shows that

$$
\begin{aligned}
s &= 1 \\
t &= 4
\end{aligned}
$$

Therefore eq. (4) becomes

$$
z''(x) = \frac{z(x)}{4} \tag{7}
$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$
y = z(x)\, e^{-\int \frac{B}{2A}\, dx}
$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 362: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$
\begin{aligned}
O(\infty) &= \deg(t) - \deg(s) \\
&= 0 - 0 \\
&= 0
\end{aligned}
$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Therefore

$$
L = [1]
$$

Since $r = \frac{1}{4}$ is not a function of x, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$z_1(x) = \mathrm{e}^{-\frac{x}{2}}$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{5}{1}\,dx} \\
&= z_1 e^{-\frac{5x}{2}} \\
&= z_1 \left(\mathrm{e}^{-\frac{5x}{2}}\right)
\end{aligned}$$

Which simplifies to

$$y_1 = \mathrm{e}^{-3x}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx$$

Substituting gives

$$\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{5}{1}\,dx}}{(y_1)^2}\,dx \\
&= y_1 \int \frac{e^{-5x}}{(y_1)^2}\,dx \\
&= y_1(\mathrm{e}^{x})
\end{aligned}$$

Therefore the solution is

$$\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1\left(\mathrm{e}^{-3x}\right) + c_2\left(\mathrm{e}^{-3x}(\mathrm{e}^{x})\right)
\end{aligned}$$

Initial conditions are used to solve for the constants of integration.

Looking at the above solution

$$y = c_1 e^{-3x} + c_2 e^{-2x} \tag{1}$$

Initial conditions are now substituted in the above solution. This will generate the required equations to solve for the integration constants. substituting $y = 2$ and $x = 0$ in the above gives

$$2 = c_1 + c_2 \tag{1A}$$

Taking derivative of the solution gives

$$y' = -3c_1 e^{-3x} - 2c_2 e^{-2x}$$

substituting $y' = \beta$ and $x = 0$ in the above gives

$$\beta = -3c_1 - 2c_2 \tag{2A}$$

Equations {1A,2A} are now solved for $\{c_1, c_2\}$. Solving for the constants gives

$$\begin{aligned} c_1 &= -\beta - 4 \\ c_2 &= 6 + \beta \end{aligned}$$

Substituting these values back in above solution results in

$$y = e^{-2x}\beta - e^{-3x}\beta + 6\,e^{-2x} - 4\,e^{-3x}$$

Which simplifies to

$$y = (-\beta - 4)\,e^{-3x} + (6 + \beta)\,e^{-2x}$$

Summary

The solution(s) found are the following

$$y = (-\beta - 4)\,e^{-3x} + (6 + \beta)\,e^{-2x} \tag{1}$$

Verification of solutions

$$y = (-\beta - 4)\,e^{-3x} + (6 + \beta)\,e^{-2x}$$

Verified OK.

7.24.4 Maple step by step solution

Let's solve

$\left[y'' + 5y' + 6y = 0, y(0) = 2, y'\Big|_{\{x=0\}} = \beta\right]$

- Highest derivative means the order of the ODE is 2

  y''

- Characteristic polynomial of ODE

  $r^2 + 5r + 6 = 0$

- Factor the characteristic polynomial

  $(r + 3)(r + 2) = 0$

- Roots of the characteristic polynomial

  $r = (-3, -2)$

- 1st solution of the ODE

  $y_1(x) = e^{-3x}$

- 2nd solution of the ODE

  $y_2(x) = e^{-2x}$

- General solution of the ODE

  $y = c_1 y_1(x) + c_2 y_2(x)$

- Substitute in solutions

  $y = c_1 e^{-3x} + c_2 e^{-2x}$

□ Check validity of solution $y = c_1 e^{-3x} + c_2 e^{-2x}$

  - Use initial condition $y(0) = 2$

    $2 = c_1 + c_2$

  - Compute derivative of the solution

    $y' = -3c_1 e^{-3x} - 2c_2 e^{-2x}$

  - Use the initial condition $y'\Big|_{\{x=0\}} = \beta$

    $\beta = -3c_1 - 2c_2$

  - Solve for c_1 and c_2

    $\{c_1 = -\beta - 4, c_2 = 6 + \beta\}$

- Substitute constant values into general solution and simplify

  $y = (-\beta - 4)\,e^{-3x} + (6 + \beta)\,e^{-2x}$

- Solution to the IVP

  $y = (-\beta - 4)\,e^{-3x} + (6 + \beta)\,e^{-2x}$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
<- constant coefficients successful`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 23

```
dsolve([diff(y(x),x$2) +5*diff(y(x),x)+6*y(x) = 0,y(0) = 2, D(y)(0) = beta],y(x), singsol=al
```

$$y(x) = e^{-2x}(6 + \beta) + (-\beta - 4)\,e^{-3x}$$

✓ Solution by Mathematica

Time used: 0.013 (sec). Leaf size: 23

```
DSolve[{y''[x]+5*y'[x]+6*y[x]==0,{y[0]==2,y'[0]==\[Beta]}},y[x],x,IncludeSingularSolutions -
```

$$y(x) \to e^{-3x}(-\beta + (\beta + 6)e^{x} - 4)$$

8 Chapter 3, Second order linear equations, 3.3 Complex Roots of the Characteristic Equation , page 164

8.1 problem 7

Internal problem ID [623]
Internal file name [`OUTPUT/623_Sunday_June_05_2022_01_46_00_AM_10820559/index.tex`]

Book: Elementary differential equations and boundary value problems, 10th ed., Boyce and DiPrima
Section: Chapter 3, Second order linear equations, 3.3 Complex Roots of the Characteristic Equation , page 164
Problem number: 7.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_linear_constant_coeff"**

Maple gives the following as the ode type

```
[[_2nd_order, _missing_x]]
```

$$\boxed{y'' - 2y' + 2y = 0}$$

8.1.1 Solving as second order linear constant coeff ode

This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = 0$$

Where in the above $A = 1, B = -2, C = 2$. Let the solution be $y = e^{\lambda x}$. Substituting this into the ODE gives

$$\lambda^2 e^{\lambda x} - 2\lambda\, e^{\lambda x} + 2\, e^{\lambda x} = 0 \tag{1}$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda x}$ gives

$$\lambda^2 - 2\lambda + 2 = 0 \tag{2}$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form.Using the quadratic formula

$$\lambda_{1,2} = \frac{-B}{2A} \pm \frac{1}{2A}\sqrt{B^2 - 4AC}$$

Substituting $A = 1, B = -2, C = 2$ into the above gives

$$\lambda_{1,2} = \frac{2}{(2)(1)} \pm \frac{1}{(2)(1)}\sqrt{-2^2 - (4)(1)(2)}$$
$$= 1 \pm i$$

Hence

$$\lambda_1 = 1 + i$$
$$\lambda_2 = 1 - i$$

Which simplifies to

$$\lambda_1 = 1 + i$$
$$\lambda_2 = 1 - i$$

Since roots are complex conjugate of each others, then let the roots be

$$\lambda_{1,2} = \alpha \pm i\beta$$

Where $\alpha = 1$ and $\beta = 1$. Therefore the final solution, when using Euler relation, can be written as

$$y = e^{\alpha x}(c_1 \cos(\beta x) + c_2 \sin(\beta x))$$

Which becomes

$$y = e^{x}(c_1 \cos(x) + c_2 \sin(x))$$

Summary

The solution(s) found are the following

$$y = e^{x}(c_1 \cos(x) + c_2 \sin(x)) \tag{1}$$

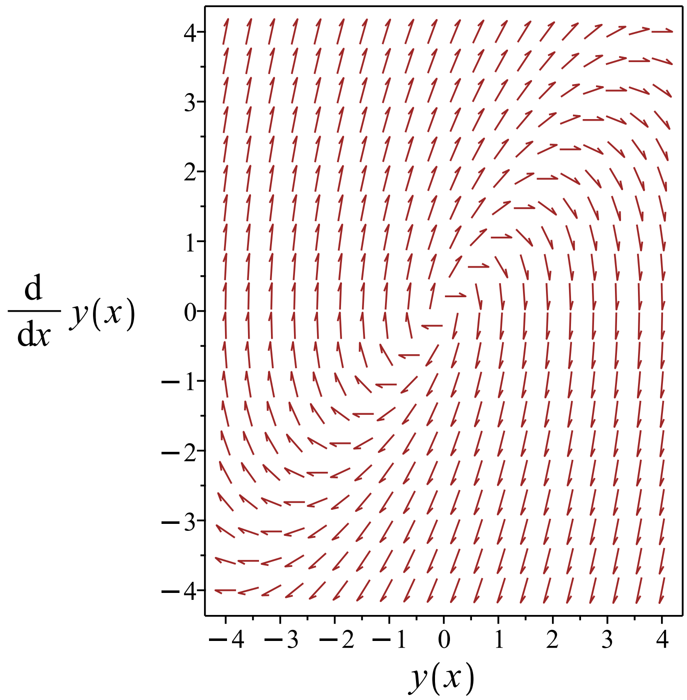


Figure 393: Slope field plot

Verification of solutions

$$y = e^x(c_1 \cos(x) + c_2 \sin(x))$$

Verified OK.

8.1.2 Solving using Kovacic algorithm

Writing the ode as

$$y'' - 2y' + 2y = 0 \tag{1}$$

$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1 \\ B &= -2 \\ C &= 2 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{-1}{1} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= -1 \\ t &= 1 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = -z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x) e^{-\int \frac{B}{2A} dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 364: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned}
O(\infty) &= \deg(t) - \deg(s) \\
&= 0 - 0 \\
&= 0
\end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Since $r = -1$ is not a function of x, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$z_1(x) = \cos(x)$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{-2}{1}\,dx} \\
&= z_1 e^{x} \\
&= z_1(\mathrm{e}^{x})
\end{aligned}$$

Which simplifies to

$$y_1 = \cos(x)\,\mathrm{e}^{x}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx$$

Substituting gives

$$\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{-2}{1}\,dx}}{(y_1)^2}\,dx \\
&= y_1 \int \frac{e^{2x}}{(y_1)^2}\,dx \\
&= y_1(\tan(x))
\end{aligned}$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1(\cos(x)\,\mathrm{e}^x) + c_2(\cos(x)\,\mathrm{e}^x(\tan(x)))
\end{aligned}
$$

Summary

The solution(s) found are the following

$$
y = c_1 \cos(x)\,\mathrm{e}^x + c_2 \sin(x)\,\mathrm{e}^x \tag{1}
$$

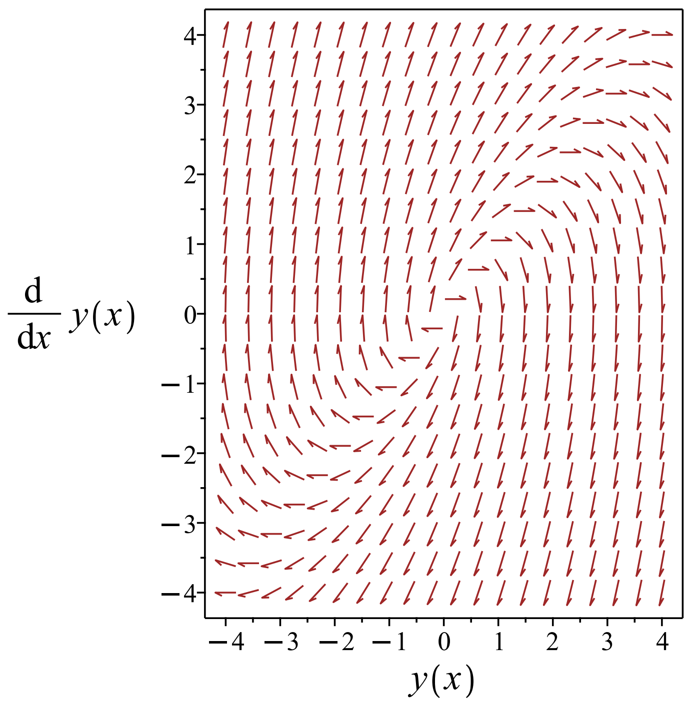


Figure 394: Slope field plot

Verification of solutions

$$
y = c_1 \cos(x)\,\mathrm{e}^x + c_2 \sin(x)\,\mathrm{e}^x
$$

Verified OK.

8.1.3 Maple step by step solution

Let's solve

$y'' - 2y' + 2y = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Characteristic polynomial of ODE

  $r^2 - 2r + 2 = 0$

- Use quadratic formula to solve for r

  $r = \frac{2 \pm (\sqrt{-4})}{2}$

- Roots of the characteristic polynomial

  $r = (1 - \mathrm{I}, 1 + \mathrm{I})$

- 1st solution of the ODE

  $y_1(x) = \cos(x)\,\mathrm{e}^x$

- 2nd solution of the ODE

  $y_2(x) = \sin(x)\,\mathrm{e}^x$

- General solution of the ODE

  $y = c_1 y_1(x) + c_2 y_2(x)$

- Substitute in solutions

  $y = c_1 \cos(x)\,\mathrm{e}^x + c_2 \sin(x)\,\mathrm{e}^x$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
<- constant coefficients successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 16

```
dsolve(diff(y(x),x$2) -2*diff(y(x),x)+2*y(x) = 0,y(x), singsol=all)
```

$$y(x) = e^{x}(c_1 \sin(x) + c_2 \cos(x))$$

✓ Solution by Mathematica

Time used: 0.013 (sec). Leaf size: 20

```
DSolve[y''[x]-2*y'[x]+2*y[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to e^{x}(c_2 \cos(x) + c_1 \sin(x))$$

8.2 problem 8

Internal problem ID [624]
Internal file name [`OUTPUT/624_Sunday_June_05_2022_01_46_01_AM_78108325/index.tex`]

Book: Elementary differential equations and boundary value problems, 10th ed., Boyce and DiPrima
Section: Chapter 3, Second order linear equations, 3.3 Complex Roots of the Characteristic Equation , page 164
Problem number: 8.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_linear_constant_coeff"**

Maple gives the following as the ode type

```
[[_2nd_order, _missing_x]]
```

$$\boxed{y'' - 2y' + 6y = 0}$$

8.2.1 Solving as second order linear constant coeff ode

This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = 0$$

Where in the above $A = 1, B = -2, C = 6$. Let the solution be $y = e^{\lambda x}$. Substituting this into the ODE gives

$$\lambda^2 e^{\lambda x} - 2\lambda e^{\lambda x} + 6 e^{\lambda x} = 0 \tag{1}$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda x}$ gives

$$\lambda^2 - 2\lambda + 6 = 0 \tag{2}$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form.Using the quadratic formula

$$\lambda_{1,2} = \frac{-B}{2A} \pm \frac{1}{2A}\sqrt{B^2 - 4AC}$$

Substituting $A = 1, B = -2, C = 6$ into the above gives

$$\begin{aligned}\lambda_{1,2} &= \frac{2}{(2)(1)} \pm \frac{1}{(2)(1)}\sqrt{-2^2 - (4)(1)(6)} \\ &= 1 \pm i\sqrt{5}\end{aligned}$$

Hence

$$\begin{aligned}\lambda_1 &= 1 + i\sqrt{5} \\ \lambda_2 &= 1 - i\sqrt{5}\end{aligned}$$

Which simplifies to

$$\begin{aligned}\lambda_1 &= 1 + i\sqrt{5} \\ \lambda_2 &= 1 - i\sqrt{5}\end{aligned}$$

Since roots are complex conjugate of each others, then let the roots be

$$\lambda_{1,2} = \alpha \pm i\beta$$

Where $\alpha = 1$ and $\beta = \sqrt{5}$. Therefore the final solution, when using Euler relation, can be written as

$$y = e^{\alpha x}(c_1 \cos(\beta x) + c_2 \sin(\beta x))$$

Which becomes

$$y = e^{x}\left(c_1 \cos\left(x\sqrt{5}\right) + c_2 \sin\left(x\sqrt{5}\right)\right)$$

Summary

The solution(s) found are the following

$$y = \mathrm{e}^{x}\left(c_1 \cos\left(x\sqrt{5}\right) + c_2 \sin\left(x\sqrt{5}\right)\right) \tag{1}$$

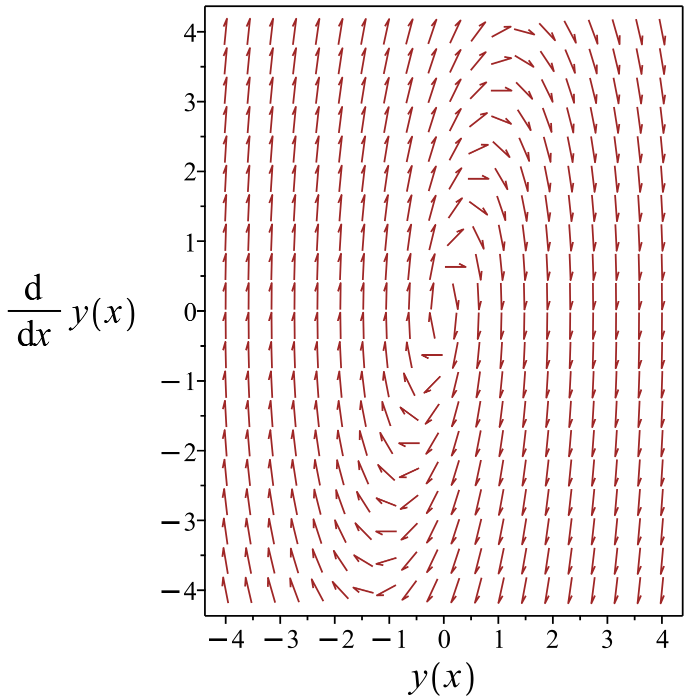


Figure 395: Slope field plot

Verification of solutions

$$y = e^x \left(c_1 \cos\left(x\sqrt{5}\right) + c_2 \sin\left(x\sqrt{5}\right)\right)$$

Verified OK.

8.2.2 Solving using Kovacic algorithm

Writing the ode as

$$y'' - 2y' + 6y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1 \\ B &= -2 \\ C &= 6 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{-5}{1} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= -5 \\ t &= 1 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = -5z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 366: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 0 - 0 \\ &= 0 \end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Since $r = -5$ is not a function of x, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$z_1(x) = \cos\left(x\sqrt{5}\right)$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$\begin{aligned} y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx} \\ &= z_1 e^{-\int \frac{1}{2}\frac{-2}{1}\,dx} \\ &= z_1 e^{x} \\ &= z_1\left(\mathrm{e}^{x}\right) \end{aligned}$$

Which simplifies to

$$y_1 = \mathrm{e}^{x}\cos\left(x\sqrt{5}\right)$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx$$

Substituting gives

$$\begin{aligned} y_2 &= y_1 \int \frac{e^{\int -\frac{-2}{1}\,dx}}{\left(y_1\right)^2}\,dx \\ &= y_1 \int \frac{e^{2x}}{\left(y_1\right)^2}\,dx \\ &= y_1\left(\frac{\sqrt{5}\,\tan\left(x\sqrt{5}\right)}{5}\right) \end{aligned}$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1 \left(\mathrm{e}^x \cos\left(x\sqrt{5}\right)\right) + c_2 \left(\mathrm{e}^x \cos\left(x\sqrt{5}\right) \left(\frac{\sqrt{5}\,\tan\left(x\sqrt{5}\right)}{5}\right)\right)
\end{aligned}
$$

<u>Summary</u>
The solution(s) found are the following

$$
y = c_1 \mathrm{e}^x \cos\left(x\sqrt{5}\right) + \frac{c_2 \sin\left(x\sqrt{5}\right) \mathrm{e}^x \sqrt{5}}{5} \tag{1}
$$

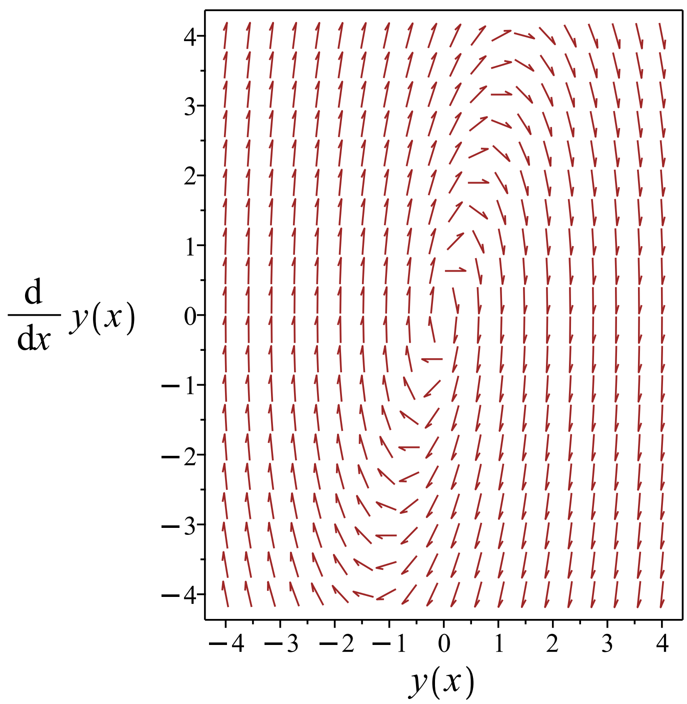


Figure 396: Slope field plot

<u>Verification of solutions</u>

$$
y = c_1 \mathrm{e}^x \cos\left(x\sqrt{5}\right) + \frac{c_2 \sin\left(x\sqrt{5}\right) \mathrm{e}^x \sqrt{5}}{5}
$$

Verified OK.

8.2.3 Maple step by step solution

Let's solve

$y'' - 2y' + 6y = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Characteristic polynomial of ODE

  $r^2 - 2r + 6 = 0$

- Use quadratic formula to solve for r

  $r = \frac{2 \pm (\sqrt{-20})}{2}$

- Roots of the characteristic polynomial

  $r = \left(1 - \mathrm{I}\sqrt{5}, 1 + \mathrm{I}\sqrt{5}\right)$

- 1st solution of the ODE

  $y_1(x) = \mathrm{e}^x \cos\left(x\sqrt{5}\right)$

- 2nd solution of the ODE

  $y_2(x) = \mathrm{e}^x \sin\left(x\sqrt{5}\right)$

- General solution of the ODE

  $y = c_1 y_1(x) + c_2 y_2(x)$

- Substitute in solutions

  $y = c_1 \mathrm{e}^x \cos\left(x\sqrt{5}\right) + c_2 \mathrm{e}^x \sin\left(x\sqrt{5}\right)$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
<- constant coefficients successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 24

```
dsolve(diff(y(x),x$2) -2*diff(y(x),x)+6*y(x) = 0,y(x), singsol=all)
```

$$y(x) = e^{x} \left(c_1 \sin \left(\sqrt{5}\, x\right) + c_2 \cos \left(\sqrt{5}\, x\right)\right)$$

✓ Solution by Mathematica

Time used: 0.02 (sec). Leaf size: 32

```
DSolve[y''[x]-2*y'[x]+6*y[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to e^{x} \left(c_2 \cos \left(\sqrt{5} x\right) + c_1 \sin \left(\sqrt{5} x\right)\right)$$

8.3 problem 9

Internal problem ID [625]
Internal file name [`OUTPUT/625_Sunday_June_05_2022_01_46_02_AM_10842904/index.tex`]

Book: Elementary differential equations and boundary value problems, 10th ed., Boyce and DiPrima
Section: Chapter 3, Second order linear equations, 3.3 Complex Roots of the Characteristic Equation , page 164
Problem number: 9.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_linear_constant_coeff"**

Maple gives the following as the ode type

`[[_2nd_order, _missing_x]]`

$$\boxed{y'' + 2y' - 8y = 0}$$

8.3.1 Solving as second order linear constant coeff ode

This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = 0$$

Where in the above $A = 1, B = 2, C = -8$. Let the solution be $y = e^{\lambda x}$. Substituting this into the ODE gives

$$\lambda^2 e^{\lambda x} + 2\lambda e^{\lambda x} - 8 e^{\lambda x} = 0 \tag{1}$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda x}$ gives

$$\lambda^2 + 2\lambda - 8 = 0 \tag{2}$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form.Using the quadratic formula

$$\lambda_{1,2} = \frac{-B}{2A} \pm \frac{1}{2A}\sqrt{B^2 - 4AC}$$

Substituting $A = 1, B = 2, C = -8$ into the above gives

$$\lambda_{1,2} = \frac{-2}{(2)(1)} \pm \frac{1}{(2)(1)} \sqrt{2^2 - (4)(1)(-8)}$$
$$= -1 \pm 3$$

Hence

$$\lambda_1 = -1 + 3$$
$$\lambda_2 = -1 - 3$$

Which simplifies to

$$\lambda_1 = 2$$
$$\lambda_2 = -4$$

Since roots are real and distinct, then the solution is

$$y = c_1 e^{\lambda_1 x} + c_2 e^{\lambda_2 x}$$
$$y = c_1 e^{(2)x} + c_2 e^{(-4)x}$$

Or

$$y = c_1 \mathrm{e}^{2x} + c_2 \mathrm{e}^{-4x}$$

Summary
The solution(s) found are the following

$$y = c_1 \mathrm{e}^{2x} + c_2 \mathrm{e}^{-4x} \tag{1}$$

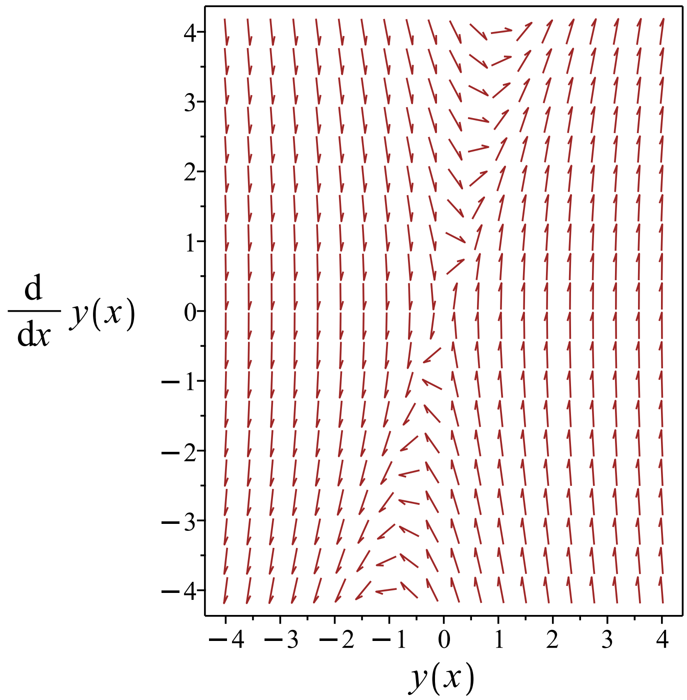


Figure 397: Slope field plot

Verification of solutions

$$y = c_1 e^{2x} + c_2 e^{-4x}$$

Verified OK.

8.3.2 Solving using Kovacic algorithm

Writing the ode as

$$y'' + 2y' - 8y = 0 \tag{1}$$

$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1 \\ B &= 2 \\ C &= -8 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = y e^{\int \frac{B}{2A} \, dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{9}{1} \tag{6}$$

Comparing the above to (5) shows that

$$s = 9$$
$$t = 1$$

Therefore eq. (4) becomes

$$z''(x) = 9z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1, 2\}$,$\{1, 3\}$,$\{2\}$,$\{3\}$,$\{3, 4\}$,$\{1, 2, 5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 368: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$
\begin{aligned}
O(\infty) &= \deg(t) - \deg(s) \\
&= 0 - 0 \\
&= 0
\end{aligned}
$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Therefore

$$
L = [1]
$$

Since $r = 9$ is not a function of x, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$
z_1(x) = \mathrm{e}^{-3x}
$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$
\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{2}{1}\,dx} \\
&= z_1 e^{-x} \\
&= z_1\left(\mathrm{e}^{-x}\right)
\end{aligned}
$$

Which simplifies to

$$
y_1 = \mathrm{e}^{-4x}
$$

The second solution y_2 to the original ode is found using reduction of order

$$
y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx
$$

Substituting gives

$$
\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{2}{1}\,dx}}{(y_1)^2}\,dx \\
&= y_1 \int \frac{e^{-2x}}{(y_1)^2}\,dx \\
&= y_1\left(\frac{\mathrm{e}^{6x}}{6}\right)
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1\left(\mathrm{e}^{-4x}\right) + c_2\left(\mathrm{e}^{-4x}\left(\frac{\mathrm{e}^{6x}}{6}\right)\right)
\end{aligned}
$$

Summary

The solution(s) found are the following

$$
y = c_1 \mathrm{e}^{-4x} + \frac{c_2 \mathrm{e}^{2x}}{6} \tag{1}
$$

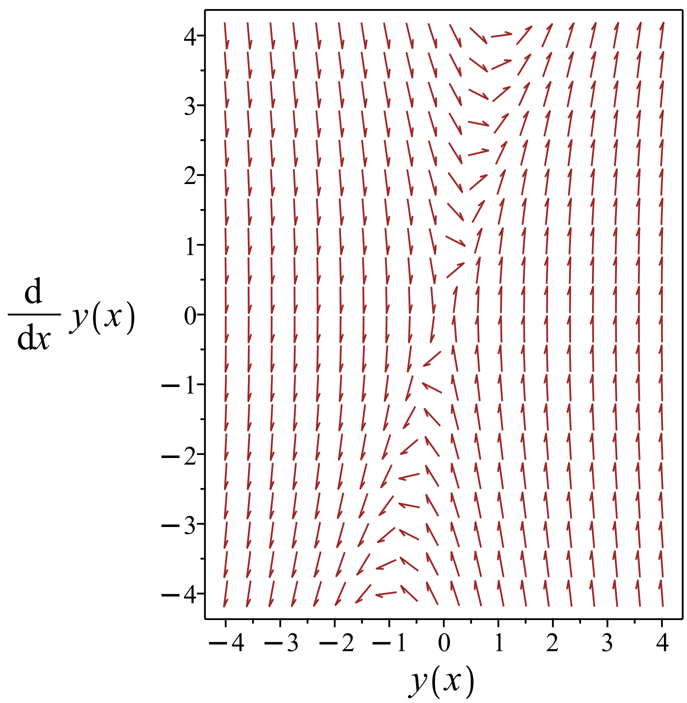


Figure 398: Slope field plot

Verification of solutions

$$
y = c_1 \mathrm{e}^{-4x} + \frac{c_2 \mathrm{e}^{2x}}{6}
$$

Verified OK.

8.3.3 Maple step by step solution

Let's solve

$y'' + 2y' - 8y = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Characteristic polynomial of ODE

  $r^2 + 2r - 8 = 0$

- Factor the characteristic polynomial

  $(r + 4)(r - 2) = 0$

- Roots of the characteristic polynomial

  $r = (-4, 2)$

- 1st solution of the ODE

  $y_1(x) = e^{-4x}$

- 2nd solution of the ODE

  $y_2(x) = e^{2x}$

- General solution of the ODE

  $y = c_1 y_1(x) + c_2 y_2(x)$

- Substitute in solutions

  $y = c_1 e^{-4x} + c_2 e^{2x}$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
<- constant coefficients successful`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 17

```
dsolve(diff(y(x),x$2) +2*diff(y(x),x)-8*y(x) = 0,y(x), singsol=all)
```

$$y(x) = \left(e^{6x}c_1 + c_2\right) e^{-4x}$$

✓ Solution by Mathematica

Time used: 0.012 (sec). Leaf size: 22

```
DSolve[y''[x]+2*y'[x]-8*y[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to e^{-4x}\left(c_2e^{6x} + c_1\right)$$

8.4 problem 10

Internal problem ID [626]
Internal file name [`OUTPUT/626_Sunday_June_05_2022_01_46_02_AM_81884660/index.tex`]

Book: Elementary differential equations and boundary value problems, 10th ed., Boyce and DiPrima
Section: Chapter 3, Second order linear equations, 3.3 Complex Roots of the Characteristic Equation , page 164
Problem number: 10.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_linear_constant_coeff"**

Maple gives the following as the ode type

```
[[_2nd_order, _missing_x]]
```

$$\boxed{y'' + 2y' + 2y = 0}$$

8.4.1 Solving as second order linear constant coeff ode

This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = 0$$

Where in the above $A = 1, B = 2, C = 2$. Let the solution be $y = e^{\lambda x}$. Substituting this into the ODE gives

$$\lambda^2 e^{\lambda x} + 2\lambda\, e^{\lambda x} + 2\, e^{\lambda x} = 0 \tag{1}$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda x}$ gives

$$\lambda^2 + 2\lambda + 2 = 0 \tag{2}$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form.Using the quadratic formula

$$\lambda_{1,2} = \frac{-B}{2A} \pm \frac{1}{2A}\sqrt{B^2 - 4AC}$$

Substituting $A = 1, B = 2, C = 2$ into the above gives

$$\begin{aligned}\lambda_{1,2} &= \frac{-2}{(2)(1)} \pm \frac{1}{(2)(1)}\sqrt{2^2 - (4)(1)(2)} \\ &= -1 \pm i\end{aligned}$$

Hence

$$\lambda_1 = -1 + i$$
$$\lambda_2 = -1 - i$$

Which simplifies to

$$\lambda_1 = -1 + i$$
$$\lambda_2 = -1 - i$$

Since roots are complex conjugate of each others, then let the roots be

$$\lambda_{1,2} = \alpha \pm i\beta$$

Where $\alpha = -1$ and $\beta = 1$. Therefore the final solution, when using Euler relation, can be written as

$$y = e^{\alpha x}(c_1 \cos(\beta x) + c_2 \sin(\beta x))$$

Which becomes

$$y = e^{-x}(c_1 \cos(x) + c_2 \sin(x))$$

Summary
The solution(s) found are the following

$$y = \mathrm{e}^{-x}(c_1 \cos(x) + c_2 \sin(x)) \tag{1}$$

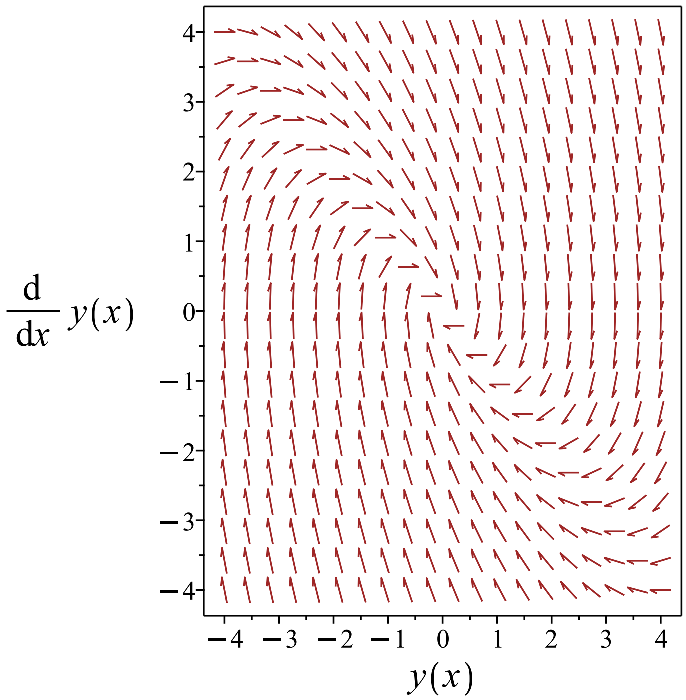


Figure 399: Slope field plot

Verification of solutions

$$y = e^{-x}(c_1 \cos(x) + c_2 \sin(x))$$

Verified OK.

8.4.2 Solving using Kovacic algorithm

Writing the ode as

$$y'' + 2y' + 2y = 0 \tag{1}$$

$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1 \\ B &= 2 \\ C &= 2 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = y e^{\int \frac{B}{2A}\, dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{-1}{1} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= -1 \\ t &= 1 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = -z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 370: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$
\begin{aligned}
O(\infty) &= \deg(t) - \deg(s) \\
&= 0 - 0 \\
&= 0
\end{aligned}
$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Therefore

$$
L = [1]
$$

Since $r = -1$ is not a function of x, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$
z_1(x) = \cos(x)
$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$
\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{2}{1}\,dx} \\
&= z_1 e^{-x} \\
&= z_1\left(\mathrm{e}^{-x}\right)
\end{aligned}
$$

Which simplifies to

$$
y_1 = \cos(x)\,\mathrm{e}^{-x}
$$

The second solution y_2 to the original ode is found using reduction of order

$$
y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx
$$

Substituting gives

$$
\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{2}{1}\,dx}}{(y_1)^2}\,dx \\
&= y_1 \int \frac{e^{-2x}}{(y_1)^2}\,dx \\
&= y_1(\tan(x))
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1\left(\cos(x)\,e^{-x}\right) + c_2\left(\cos(x)\,e^{-x}(\tan(x))\right)
\end{aligned}
$$

Summary

The solution(s) found are the following

$$
y = c_1 \cos(x)\,e^{-x} + c_2 \sin(x)\,e^{-x} \tag{1}
$$

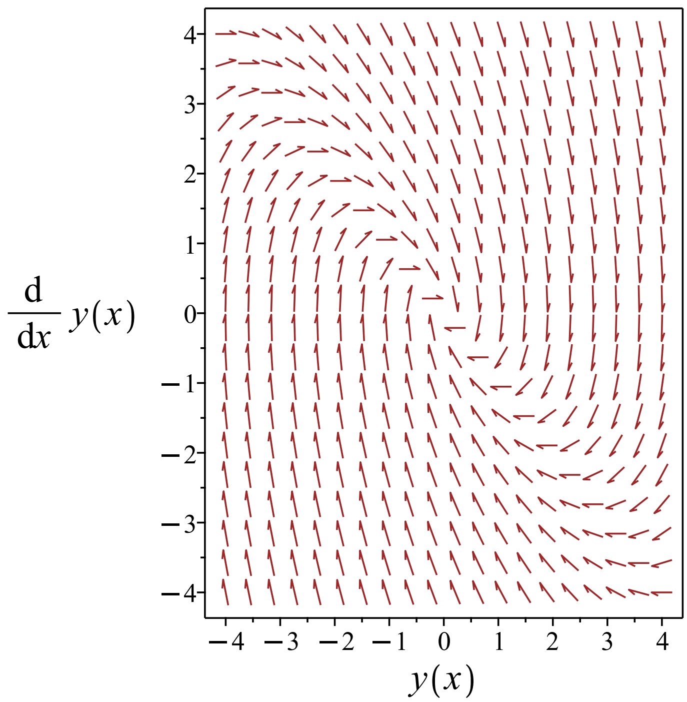


Figure 400: Slope field plot

Verification of solutions

$$
y = c_1 \cos(x)\,e^{-x} + c_2 \sin(x)\,e^{-x}
$$

Verified OK.

8.4.3 Maple step by step solution

Let's solve

$y'' + 2y' + 2y = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Characteristic polynomial of ODE

  $r^2 + 2r + 2 = 0$

- Use quadratic formula to solve for r

  $r = \frac{(-2)\pm(\sqrt{-4})}{2}$

- Roots of the characteristic polynomial

  $r = (-1 - \mathrm{I}, -1 + \mathrm{I})$

- 1st solution of the ODE

  $y_1(x) = \cos(x)\,\mathrm{e}^{-x}$

- 2nd solution of the ODE

  $y_2(x) = \sin(x)\,\mathrm{e}^{-x}$

- General solution of the ODE

  $y = c_1 y_1(x) + c_2 y_2(x)$

- Substitute in solutions

  $y = c_1 \cos(x)\,\mathrm{e}^{-x} + c_2 \sin(x)\,\mathrm{e}^{-x}$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
<- constant coefficients successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 18

```
dsolve(diff(y(x),x$2) +2*diff(y(x),x)+2*y(x) = 0,y(x), singsol=all)
```

$$y(x) = e^{-x}(c_1 \sin(x) + c_2 \cos(x))$$

✓ Solution by Mathematica

Time used: 0.015 (sec). Leaf size: 22

```
DSolve[y''[x]+2*y'[x]+2*y[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to e^{-x}(c_2 \cos(x) + c_1 \sin(x))$$

8.5 problem 11

Internal problem ID [627]
Internal file name [`OUTPUT/627_Sunday_June_05_2022_01_46_03_AM_30401613/index.tex`]

Book: Elementary differential equations and boundary value problems, 10th ed., Boyce and DiPrima
Section: Chapter 3, Second order linear equations, 3.3 Complex Roots of the Characteristic Equation , page 164
Problem number: 11.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_linear_constant_coeff"**

Maple gives the following as the ode type

```
[[_2nd_order, _missing_x]]
```

$$\boxed{y'' + 6y' + 13y = 0}$$

8.5.1 Solving as second order linear constant coeff ode

This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = 0$$

Where in the above $A = 1, B = 6, C = 13$. Let the solution be $y = e^{\lambda x}$. Substituting this into the ODE gives

$$\lambda^2 e^{\lambda x} + 6\lambda\, e^{\lambda x} + 13\, e^{\lambda x} = 0 \tag{1}$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda x}$ gives

$$\lambda^2 + 6\lambda + 13 = 0 \tag{2}$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form.Using the quadratic formula

$$\lambda_{1,2} = \frac{-B}{2A} \pm \frac{1}{2A}\sqrt{B^2 - 4AC}$$

Substituting $A = 1, B = 6, C = 13$ into the above gives

$$\begin{aligned}\lambda_{1,2} &= \frac{-6}{(2)(1)} \pm \frac{1}{(2)(1)}\sqrt{6^2 - (4)(1)(13)} \\ &= -3 \pm 2i\end{aligned}$$

Hence

$$\begin{aligned}\lambda_1 &= -3 + 2i \\ \lambda_2 &= -3 - 2i\end{aligned}$$

Which simplifies to

$$\begin{aligned}\lambda_1 &= -3 + 2i \\ \lambda_2 &= -3 - 2i\end{aligned}$$

Since roots are complex conjugate of each others, then let the roots be

$$\lambda_{1,2} = \alpha \pm i\beta$$

Where $\alpha = -3$ and $\beta = 2$. Therefore the final solution, when using Euler relation, can be written as

$$y = e^{\alpha x}(c_1 \cos(\beta x) + c_2 \sin(\beta x))$$

Which becomes

$$y = e^{-3x}(c_1 \cos(2x) + c_2 \sin(2x))$$

Summary

The solution(s) found are the following

$$y = \mathrm{e}^{-3x}(c_1 \cos(2x) + c_2 \sin(2x)) \tag{1}$$

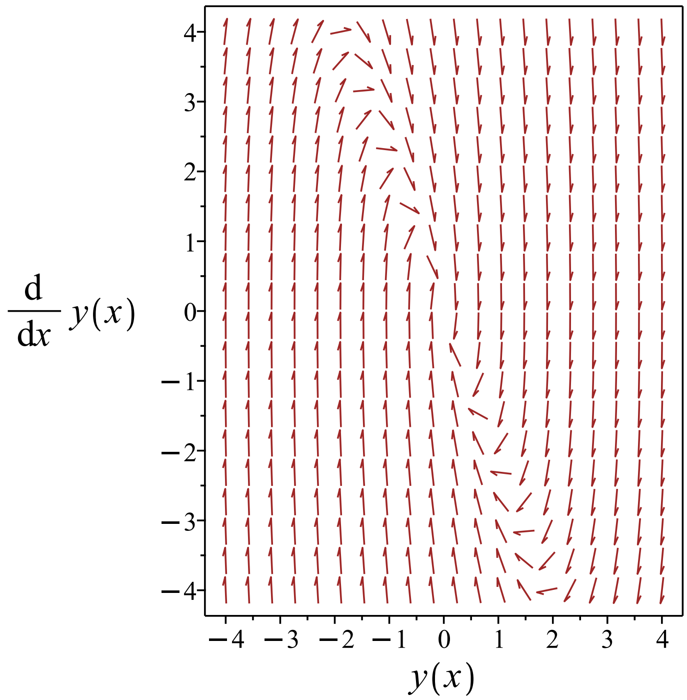


Figure 401: Slope field plot

Verification of solutions

$$y = e^{-3x}(c_1 \cos(2x) + c_2 \sin(2x))$$

Verified OK.

8.5.2 Solving using Kovacic algorithm

Writing the ode as

$$y'' + 6y' + 13y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1 \\ B &= 6 \\ C &= 13 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\, dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{-4}{1} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= -4 \\ t &= 1 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = -4z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 372: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 0 - 0 \\ &= 0 \end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Since $r = -4$ is not a function of x, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$z_1(x) = \cos(2x)$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$\begin{aligned} y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx} \\ &= z_1 e^{-\int \frac{1}{2}\frac{6}{1}\,dx} \\ &= z_1 e^{-3x} \\ &= z_1\left(\mathrm{e}^{-3x}\right) \end{aligned}$$

Which simplifies to

$$y_1 = \mathrm{e}^{-3x}\cos(2x)$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx$$

Substituting gives

$$\begin{aligned} y_2 &= y_1 \int \frac{e^{\int -\frac{6}{1}\,dx}}{(y_1)^2}\,dx \\ &= y_1 \int \frac{e^{-6x}}{(y_1)^2}\,dx \\ &= y_1\left(\frac{\tan(2x)}{2}\right) \end{aligned}$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1 \left(e^{-3x} \cos(2x)\right) + c_2 \left(e^{-3x} \cos(2x) \left(\frac{\tan(2x)}{2}\right)\right)
\end{aligned}
$$

<u>Summary</u>

The solution(s) found are the following

$$
y = c_1 e^{-3x} \cos(2x) + \frac{c_2 e^{-3x} \sin(2x)}{2} \tag{1}
$$

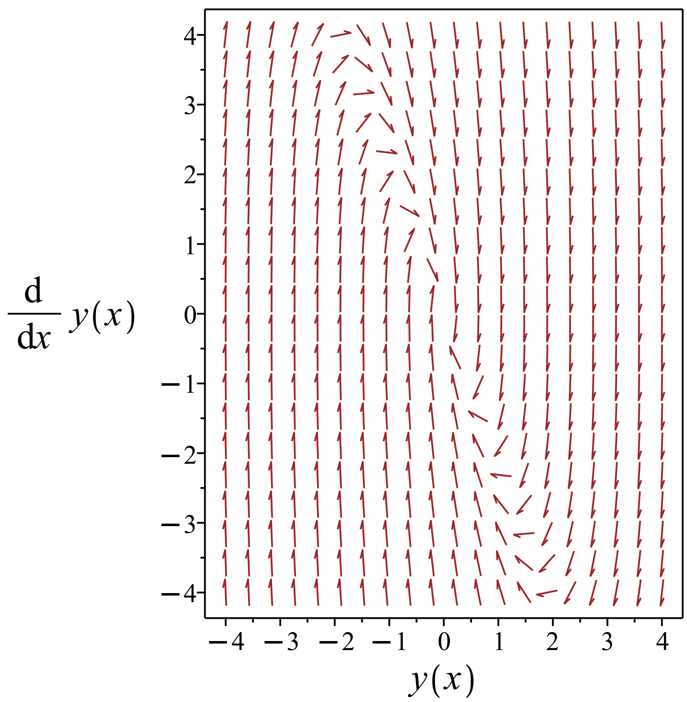


Figure 402: Slope field plot

<u>Verification of solutions</u>

$$
y = c_1 e^{-3x} \cos(2x) + \frac{c_2 e^{-3x} \sin(2x)}{2}
$$

Verified OK.

8.5.3 Maple step by step solution

Let's solve

$y'' + 6y' + 13y = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Characteristic polynomial of ODE

  $r^2 + 6r + 13 = 0$

- Use quadratic formula to solve for r

  $r = \frac{(-6)\pm(\sqrt{-16})}{2}$

- Roots of the characteristic polynomial

  $r = (-3 - 2\,\mathrm{I}, -3 + 2\,\mathrm{I})$

- 1st solution of the ODE

  $y_1(x) = \mathrm{e}^{-3x} \cos(2x)$

- 2nd solution of the ODE

  $y_2(x) = \mathrm{e}^{-3x} \sin(2x)$

- General solution of the ODE

  $y = c_1 y_1(x) + c_2 y_2(x)$

- Substitute in solutions

  $y = c_1 \mathrm{e}^{-3x} \cos(2x) + c_2 \mathrm{e}^{-3x} \sin(2x)$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
<- constant coefficients successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 22

```
dsolve(diff(y(x),x$2) +6*diff(y(x),x)+13*y(x) = 0,y(x), singsol=all)
```

$$y(x) = e^{-3x}(\sin(2x) c_1 + c_2 \cos(2x))$$

✓ Solution by Mathematica

Time used: 0.018 (sec). Leaf size: 26

```
DSolve[y''[x]+6*y'[x]+13*y[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to e^{-3x}(c_2 \cos(2x) + c_1 \sin(2x))$$

8.6 problem 12

Internal problem ID [628]
Internal file name [`OUTPUT/628_Sunday_June_05_2022_01_46_04_AM_31108206/index.tex`]

Book: Elementary differential equations and boundary value problems, 10th ed., Boyce and DiPrima
Section: Chapter 3, Second order linear equations, 3.3 Complex Roots of the Characteristic Equation , page 164
Problem number: 12.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_linear_constant_coeff", "second_order_ode_can_be_made_integrable"**

Maple gives the following as the ode type

```
[[_2nd_order, _missing_x]]
```

$$\boxed{4y'' + 9y = 0}$$

8.6.1 Solving as second order linear constant coeff ode

This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = 0$$

Where in the above $A = 4, B = 0, C = 9$. Let the solution be $y = e^{\lambda x}$. Substituting this into the ODE gives

$$4\lambda^2 e^{\lambda x} + 9\,e^{\lambda x} = 0 \tag{1}$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda x}$ gives

$$4\lambda^2 + 9 = 0 \tag{2}$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form.Using the quadratic formula

$$\lambda_{1,2} = \frac{-B}{2A} \pm \frac{1}{2A}\sqrt{B^2 - 4AC}$$

Substituting $A = 4, B = 0, C = 9$ into the above gives

$$\begin{aligned}
\lambda_{1,2} &= \frac{0}{(2)(4)} \pm \frac{1}{(2)(4)}\sqrt{0^2 - (4)(4)(9)} \\
&= \pm\frac{3i}{2}
\end{aligned}$$

Hence

$$\begin{aligned}
\lambda_1 &= +\frac{3i}{2} \\
\lambda_2 &= -\frac{3i}{2}
\end{aligned}$$

Which simplifies to

$$\begin{aligned}
\lambda_1 &= \frac{3i}{2} \\
\lambda_2 &= -\frac{3i}{2}
\end{aligned}$$

Since roots are complex conjugate of each others, then let the roots be

$$\lambda_{1,2} = \alpha \pm i\beta$$

Where $\alpha = 0$ and $\beta = \frac{3}{2}$. Therefore the final solution, when using Euler relation, can be written as

$$y = e^{\alpha x}(c_1 \cos(\beta x) + c_2 \sin(\beta x))$$

Which becomes

$$y = e^0\left(c_1 \cos\left(\frac{3x}{2}\right) + c_2 \sin\left(\frac{3x}{2}\right)\right)$$

Or

$$y = c_1 \cos\left(\frac{3x}{2}\right) + c_2 \sin\left(\frac{3x}{2}\right)$$

Summary
The solution(s) found are the following

$$y = c_1 \cos\left(\frac{3x}{2}\right) + c_2 \sin\left(\frac{3x}{2}\right) \tag{1}$$

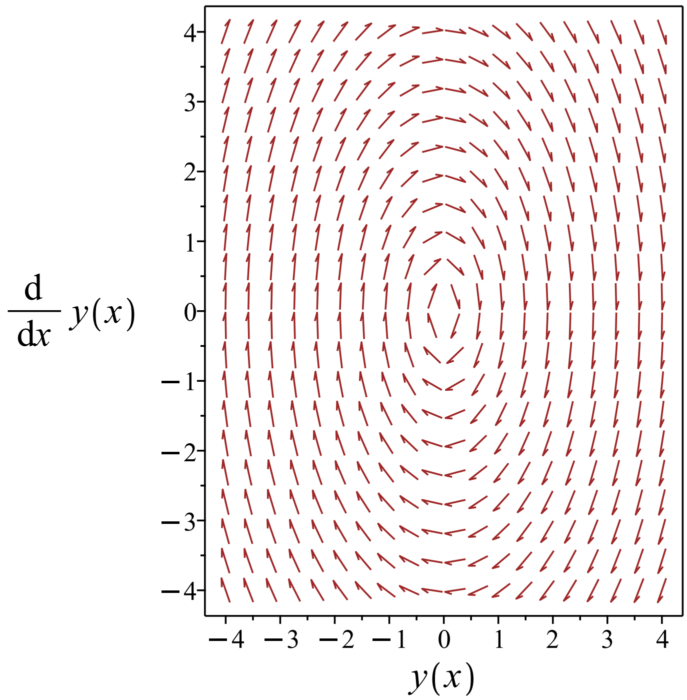


Figure 403: Slope field plot

Verification of solutions

$$y = c_1 \cos\left(\frac{3x}{2}\right) + c_2 \sin\left(\frac{3x}{2}\right)$$

Verified OK.

8.6.2 Solving as second order ode can be made integrable ode

Multiplying the ode by y' gives

$$4y'y'' + 9yy' = 0$$

Integrating the above w.r.t x gives

$$\int (4y'y'' + 9yy')\,dx = 0$$
$$2y'^2 + \frac{9y^2}{2} = c_2$$

Which is now solved for y. Solving the given ode for y' results in 2 differential equations to solve. Each one of these will generate a solution. The equations generated are

$$y' = \frac{\sqrt{-9y^2 + 2c_1}}{2} \tag{1}$$
$$y' = -\frac{\sqrt{-9y^2 + 2c_1}}{2} \tag{2}$$

Now each one of the above ODE is solved.

Solving equation (1)

Integrating both sides gives

$$\int \frac{2}{\sqrt{-9y^2 + 2c_1}} dy = \int dx$$
$$\frac{2\arctan\left(\frac{3y}{\sqrt{-9y^2+2c_1}}\right)}{3} = x + c_2$$

Solving equation (2)

Integrating both sides gives

$$\int -\frac{2}{\sqrt{-9y^2 + 2c_1}} dy = \int dx$$
$$-\frac{2\arctan\left(\frac{3y}{\sqrt{-9y^2+2c_1}}\right)}{3} = c_3 + x$$

Summary

The solution(s) found are the following

$$\frac{2\arctan\left(\frac{3y}{\sqrt{-9y^2+2c_1}}\right)}{3} = x + c_2 \tag{1}$$
$$-\frac{2\arctan\left(\frac{3y}{\sqrt{-9y^2+2c_1}}\right)}{3} = c_3 + x \tag{2}$$

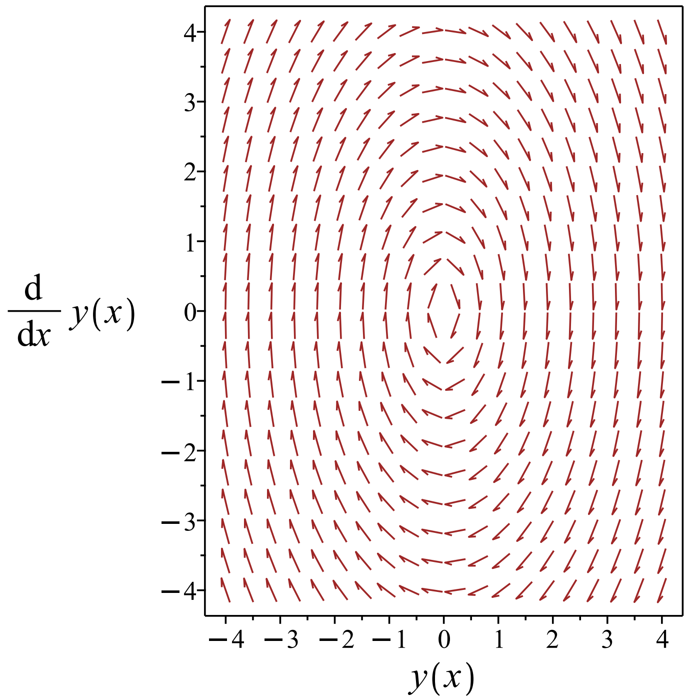


Figure 404: Slope field plot

Verification of solutions

$$\frac{2\arctan\left(\frac{3y}{\sqrt{-9y^2+2c_1}}\right)}{3} = x + c_2$$

Verified OK.

$$-\frac{2\arctan\left(\frac{3y}{\sqrt{-9y^2+2c_1}}\right)}{3} = c_3 + x$$

Verified OK.

8.6.3 Solving using Kovacic algorithm

Writing the ode as

$$4y'' + 9y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$
\begin{aligned}
A &= 4 \\
B &= 0 \\
C &= 9
\end{aligned} \tag{3}
$$

Applying the Liouville transformation on the dependent variable gives

$$
z(x) = ye^{\int \frac{B}{2A}\, dx}
$$

Then (2) becomes

$$
z''(x) = rz(x) \tag{4}
$$

Where r is given by

$$
\begin{aligned}
r &= \frac{s}{t} \\
&= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2}
\end{aligned} \tag{5}
$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$
r = \frac{-9}{4} \tag{6}
$$

Comparing the above to (5) shows that

$$
\begin{aligned}
s &= -9 \\
t &= 4
\end{aligned}
$$

Therefore eq. (4) becomes

$$
z''(x) = -\frac{9z(x)}{4} \tag{7}
$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$
y = z(x)\, e^{-\int \frac{B}{2A}\, dx}
$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 374: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 0 - 0 \\ &= 0 \end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Since $r = -\frac{9}{4}$ is not a function of x, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$z_1(x) = \cos\left(\frac{3x}{2}\right)$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$y_1 = z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx}$$

Since $B = 0$ then the above reduces to

$$\begin{aligned} y_1 &= z_1 \\ &= \cos\left(\frac{3x}{2}\right) \end{aligned}$$

Which simplifies to

$$y_1 = \cos\left(\frac{3x}{2}\right)$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\, dx}}{y_1^2}\, dx$$

Since $B = 0$ then the above becomes

$$\begin{aligned} y_2 &= y_1 \int \frac{1}{y_1^2}\, dx \\ &= \cos\left(\frac{3x}{2}\right) \int \frac{1}{\cos\left(\frac{3x}{2}\right)^2}\, dx \\ &= \cos\left(\frac{3x}{2}\right) \left(\frac{2\tan\left(\frac{3x}{2}\right)}{3}\right) \end{aligned}$$

Therefore the solution is

$$\begin{aligned} y &= c_1 y_1 + c_2 y_2 \\ &= c_1\left(\cos\left(\frac{3x}{2}\right)\right) + c_2\left(\cos\left(\frac{3x}{2}\right)\left(\frac{2\tan\left(\frac{3x}{2}\right)}{3}\right)\right) \end{aligned}$$

<u>Summary</u>
The solution(s) found are the following

$$y = c_1 \cos\left(\frac{3x}{2}\right) + \frac{2c_2 \sin\left(\frac{3x}{2}\right)}{3} \tag{1}$$

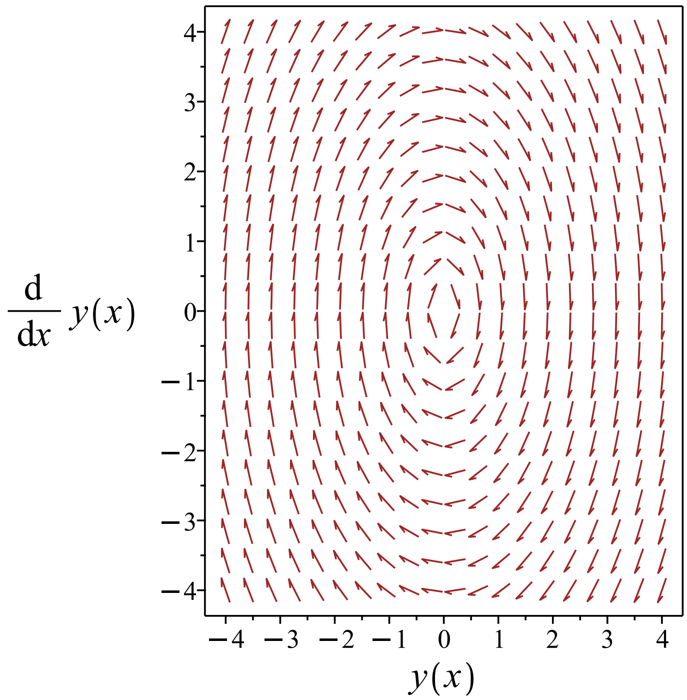


Figure 405: Slope field plot

Verification of solutions

$$y = c_1 \cos\left(\frac{3x}{2}\right) + \frac{2c_2 \sin\left(\frac{3x}{2}\right)}{3}$$

Verified OK.

8.6.4 Maple step by step solution

Let's solve

$4y'' + 9y = 0$

- Highest derivative means the order of the ODE is 2

y''

- Isolate 2nd derivative

$y'' = -\frac{9y}{4}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

$y'' + \frac{9y}{4} = 0$

- Characteristic polynomial of ODE

$r^2 + \frac{9}{4} = 0$

- Use quadratic formula to solve for r

  $r = \frac{0 \pm (\sqrt{-9})}{2}$

- Roots of the characteristic polynomial

  $r = \left(-\frac{3\,\mathrm{I}}{2}, \frac{3\,\mathrm{I}}{2}\right)$

- 1st solution of the ODE

  $y_1(x) = \cos\left(\frac{3x}{2}\right)$

- 2nd solution of the ODE

  $y_2(x) = \sin\left(\frac{3x}{2}\right)$

- General solution of the ODE

  $y = c_1 y_1(x) + c_2 y_2(x)$

- Substitute in solutions

  $y = c_1 \cos\left(\frac{3x}{2}\right) + c_2 \sin\left(\frac{3x}{2}\right)$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
<- constant coefficients successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 17

```
dsolve(4*diff(y(x),x$2) +9*y(x) = 0,y(x), singsol=all)
```

$$y(x) = c_1 \sin\left(\frac{3x}{2}\right) + c_2 \cos\left(\frac{3x}{2}\right)$$

✓ Solution by Mathematica

Time used: 0.013 (sec). Leaf size: 20

```
DSolve[y''[x]+9*y[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to c_1 \cos(3x) + c_2 \sin(3x)$$

8.7 problem 13

Internal problem ID [629]
Internal file name [`OUTPUT/629_Sunday_June_05_2022_01_46_05_AM_47762062/index.tex`]

Book: Elementary differential equations and boundary value problems, 10th ed., Boyce and DiPrima
Section: Chapter 3, Second order linear equations, 3.3 Complex Roots of the Characteristic Equation , page 164
Problem number: 13.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_linear_constant_coeff"**

Maple gives the following as the ode type

```
[[_2nd_order, _missing_x]]
```

$$\boxed{y'' + 2y' + \frac{5y}{4} = 0}$$

8.7.1 Solving as second order linear constant coeff ode

This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = 0$$

Where in the above $A = 1, B = 2, C = \frac{5}{4}$. Let the solution be $y = e^{\lambda x}$. Substituting this into the ODE gives

$$\lambda^2 e^{\lambda x} + 2\lambda\, e^{\lambda x} + \frac{5\, e^{\lambda x}}{4} = 0 \tag{1}$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda x}$ gives

$$\lambda^2 + 2\lambda + \frac{5}{4} = 0 \tag{2}$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form.Using the quadratic formula

$$\lambda_{1,2} = \frac{-B}{2A} \pm \frac{1}{2A}\sqrt{B^2 - 4AC}$$

Substituting $A = 1, B = 2, C = \frac{5}{4}$ into the above gives

$$\begin{aligned}\lambda_{1,2} &= \frac{-2}{(2)(1)} \pm \frac{1}{(2)(1)}\sqrt{2^2 - (4)(1)\left(\frac{5}{4}\right)} \\ &= -1 \pm \frac{i}{2}\end{aligned}$$

Hence

$$\begin{aligned}\lambda_1 &= -1 + \frac{i}{2} \\ \lambda_2 &= -1 - \frac{i}{2}\end{aligned}$$

Which simplifies to

$$\begin{aligned}\lambda_1 &= -1 + \frac{i}{2} \\ \lambda_2 &= -1 - \frac{i}{2}\end{aligned}$$

Since roots are complex conjugate of each others, then let the roots be

$$\lambda_{1,2} = \alpha \pm i\beta$$

Where $\alpha = -1$ and $\beta = \frac{1}{2}$. Therefore the final solution, when using Euler relation, can be written as

$$y = e^{\alpha x}(c_1 \cos(\beta x) + c_2 \sin(\beta x))$$

Which becomes

$$y = e^{-x}\left(c_1 \cos\left(\frac{x}{2}\right) + c_2 \sin\left(\frac{x}{2}\right)\right)$$

Summary

The solution(s) found are the following

$$y = \mathrm{e}^{-x}\left(c_1 \cos\left(\frac{x}{2}\right) + c_2 \sin\left(\frac{x}{2}\right)\right) \tag{1}$$

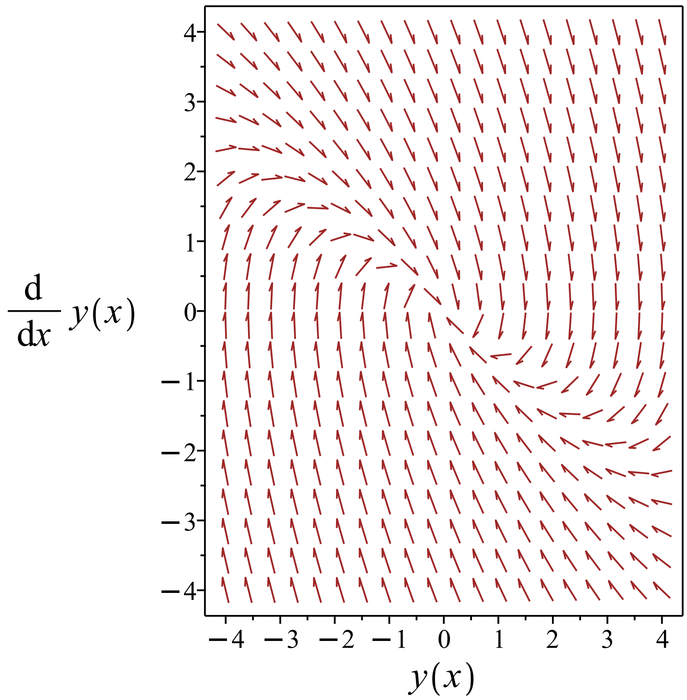


Figure 406: Slope field plot

Verification of solutions

$$y = \mathrm{e}^{-x}\left(c_1 \cos\left(\frac{x}{2}\right) + c_2 \sin\left(\frac{x}{2}\right)\right)$$

Verified OK.

8.7.2 Solving using Kovacic algorithm

Writing the ode as

$$y'' + 2y' + \frac{5y}{4} = 0 \tag{1}$$

$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1 \\ B &= 2 \\ C &= \frac{5}{4} \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = y e^{\int \frac{B}{2A}\, dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{-1}{4} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= -1 \\ t &= 4 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = -\frac{z(x)}{4} \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 376: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 0 - 0 \\ &= 0 \end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Since $r = -\frac{1}{4}$ is not a function of x, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$z_1(x) = \cos\left(\frac{x}{2}\right)$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$\begin{aligned} y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx} \\ &= z_1 e^{-\int \frac{1}{2}\frac{2}{1}\,dx} \\ &= z_1 e^{-x} \\ &= z_1\left(\mathrm{e}^{-x}\right) \end{aligned}$$

Which simplifies to

$$y_1 = \mathrm{e}^{-x}\cos\left(\frac{x}{2}\right)$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx$$

Substituting gives

$$\begin{aligned} y_2 &= y_1 \int \frac{e^{\int -\frac{2}{1}\,dx}}{(y_1)^2}\,dx \\ &= y_1 \int \frac{e^{-2x}}{(y_1)^2}\,dx \\ &= y_1\left(2\tan\left(\frac{x}{2}\right)\right) \end{aligned}$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1\left(e^{-x}\cos\left(\frac{x}{2}\right)\right) + c_2\left(e^{-x}\cos\left(\frac{x}{2}\right)\left(2\tan\left(\frac{x}{2}\right)\right)\right)
\end{aligned}
$$

<u>Summary</u>

The solution(s) found are the following

$$
y = c_1 e^{-x}\cos\left(\frac{x}{2}\right) + 2c_2 e^{-x}\sin\left(\frac{x}{2}\right) \tag{1}
$$

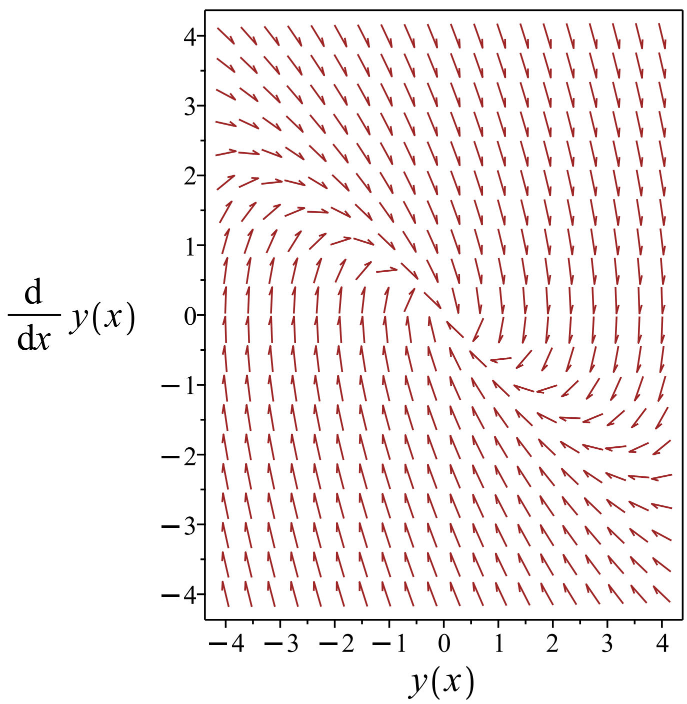


Figure 407: Slope field plot

<u>Verification of solutions</u>

$$
y = c_1 e^{-x}\cos\left(\frac{x}{2}\right) + 2c_2 e^{-x}\sin\left(\frac{x}{2}\right)
$$

Verified OK.

8.7.3 Maple step by step solution

Let's solve

$y'' + 2y' + \frac{5y}{4} = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Characteristic polynomial of ODE

  $r^2 + 2r + \frac{5}{4} = 0$

- Use quadratic formula to solve for r

  $r = \frac{(-2)\pm(\sqrt{-1})}{2}$

- Roots of the characteristic polynomial

  $r = \left(-1 - \frac{\mathrm{I}}{2}, -1 + \frac{\mathrm{I}}{2}\right)$

- 1st solution of the ODE

  $y_1(x) = \mathrm{e}^{-x} \cos\left(\frac{x}{2}\right)$

- 2nd solution of the ODE

  $y_2(x) = \mathrm{e}^{-x} \sin\left(\frac{x}{2}\right)$

- General solution of the ODE

  $y = c_1 y_1(x) + c_2 y_2(x)$

- Substitute in solutions

  $y = c_1 \mathrm{e}^{-x} \cos\left(\frac{x}{2}\right) + c_2 \mathrm{e}^{-x} \sin\left(\frac{x}{2}\right)$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
<- constant coefficients successful`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 22

```
dsolve(diff(y(x),x$2) +2*diff(y(x),x)+125/100*y(x) = 0,y(x), singsol=all)
```

$$y(x) = e^{-x}\left(c_1 \sin\left(\frac{x}{2}\right) + c_2 \cos\left(\frac{x}{2}\right)\right)$$

✓ Solution by Mathematica

Time used: 0.018 (sec). Leaf size: 30

```
DSolve[y''[x]+2*y'[x]+125/100*y[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to e^{-x}\left(c_2 \cos\left(\frac{x}{2}\right) + c_1 \sin\left(\frac{x}{2}\right)\right)$$

8.8 problem 14

Internal problem ID [630]
Internal file name [`OUTPUT/630_Sunday_June_05_2022_01_46_06_AM_17484274/index.tex`]

Book: Elementary differential equations and boundary value problems, 10th ed., Boyce and DiPrima
Section: Chapter 3, Second order linear equations, 3.3 Complex Roots of the Characteristic Equation , page 164
Problem number: 14.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_linear_constant_coeff"**

Maple gives the following as the ode type

```
[[_2nd_order, _missing_x]]
```

$$\boxed{9y'' + 9y' - 4y = 0}$$

8.8.1 Solving as second order linear constant coeff ode

This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = 0$$

Where in the above $A = 9, B = 9, C = -4$. Let the solution be $y = e^{\lambda x}$. Substituting this into the ODE gives

$$9\lambda^2 \mathrm{e}^{\lambda x} + 9\lambda\, \mathrm{e}^{\lambda x} - 4\, \mathrm{e}^{\lambda x} = 0 \tag{1}$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda x}$ gives

$$9\lambda^2 + 9\lambda - 4 = 0 \tag{2}$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form.Using the quadratic formula

$$\lambda_{1,2} = \frac{-B}{2A} \pm \frac{1}{2A}\sqrt{B^2 - 4AC}$$

Substituting $A = 9, B = 9, C = -4$ into the above gives

$$\begin{aligned}
\lambda_{1,2} &= \frac{-9}{(2)(9)} \pm \frac{1}{(2)(9)}\sqrt{9^2 - (4)(9)(-4)} \\
&= -\frac{1}{2} \pm \frac{5}{6}
\end{aligned}$$

Hence

$$\begin{aligned}
\lambda_1 &= -\frac{1}{2} + \frac{5}{6} \\
\lambda_2 &= -\frac{1}{2} - \frac{5}{6}
\end{aligned}$$

Which simplifies to

$$\begin{aligned}
\lambda_1 &= \frac{1}{3} \\
\lambda_2 &= -\frac{4}{3}
\end{aligned}$$

Since roots are real and distinct, then the solution is

$$\begin{aligned}
y &= c_1 e^{\lambda_1 x} + c_2 e^{\lambda_2 x} \\
y &= c_1 e^{\left(\frac{1}{3}\right)x} + c_2 e^{\left(-\frac{4}{3}\right)x}
\end{aligned}$$

Or

$$y = c_1 \mathrm{e}^{\frac{x}{3}} + c_2 \mathrm{e}^{-\frac{4x}{3}}$$

Summary

The solution(s) found are the following

$$y = c_1 \mathrm{e}^{\frac{x}{3}} + c_2 \mathrm{e}^{-\frac{4x}{3}} \tag{1}$$

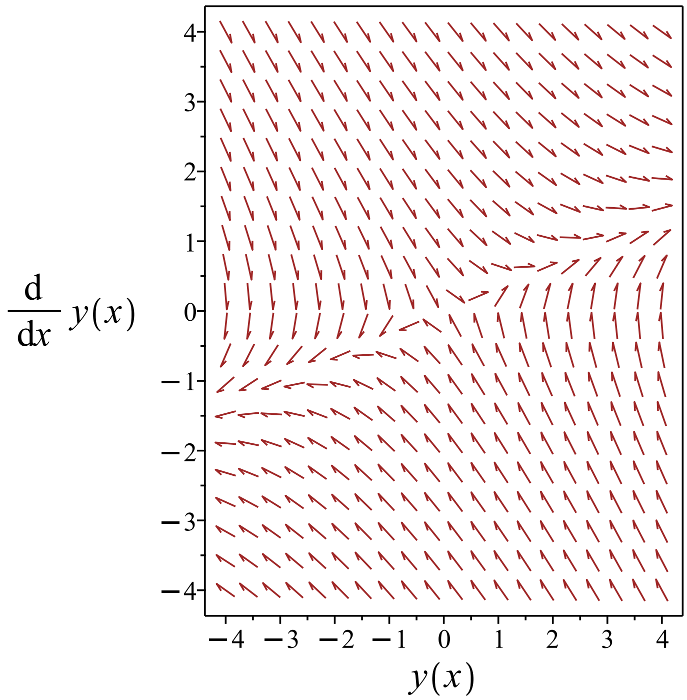


Figure 408: Slope field plot

Verification of solutions

$$y = c_1 e^{\frac{x}{3}} + c_2 e^{-\frac{4x}{3}}$$

Verified OK.

8.8.2 Solving using Kovacic algorithm

Writing the ode as

$$9y'' + 9y' - 4y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 9 \\ B &= 9 \\ C &= -4 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = y e^{\int \frac{B}{2A} \, dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{25}{36} \tag{6}$$

Comparing the above to (5) shows that

$$s = 25$$
$$t = 36$$

Therefore eq. (4) becomes

$$z''(x) = \frac{25z(x)}{36} \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 378: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$
\begin{aligned}
O(\infty) &= \deg(t) - \deg(s) \\
&= 0 - 0 \\
&= 0
\end{aligned}
$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Therefore

$$
L = [1]
$$

Since $r = \frac{25}{36}$ is not a function of x, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$
z_1(x) = \mathrm{e}^{-\frac{5x}{6}}
$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$
\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{9}{9}\,dx} \\
&= z_1 e^{-\frac{x}{2}} \\
&= z_1\left(\mathrm{e}^{-\frac{x}{2}}\right)
\end{aligned}
$$

Which simplifies to

$$
y_1 = \mathrm{e}^{-\frac{4x}{3}}
$$

The second solution y_2 to the original ode is found using reduction of order

$$
y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx
$$

Substituting gives

$$
\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{9}{9}\,dx}}{(y_1)^2}\,dx \\
&= y_1 \int \frac{e^{-x}}{(y_1)^2}\,dx \\
&= y_1\left(\frac{3\,\mathrm{e}^{\frac{5x}{3}}}{5}\right)
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1\left(\mathrm{e}^{-\frac{4x}{3}}\right) + c_2\left(\mathrm{e}^{-\frac{4x}{3}}\left(\frac{3\,\mathrm{e}^{\frac{5x}{3}}}{5}\right)\right)
\end{aligned}
$$

Summary

The solution(s) found are the following

$$
y = c_1 \mathrm{e}^{-\frac{4x}{3}} + \frac{3 c_2 \mathrm{e}^{\frac{x}{3}}}{5} \tag{1}
$$

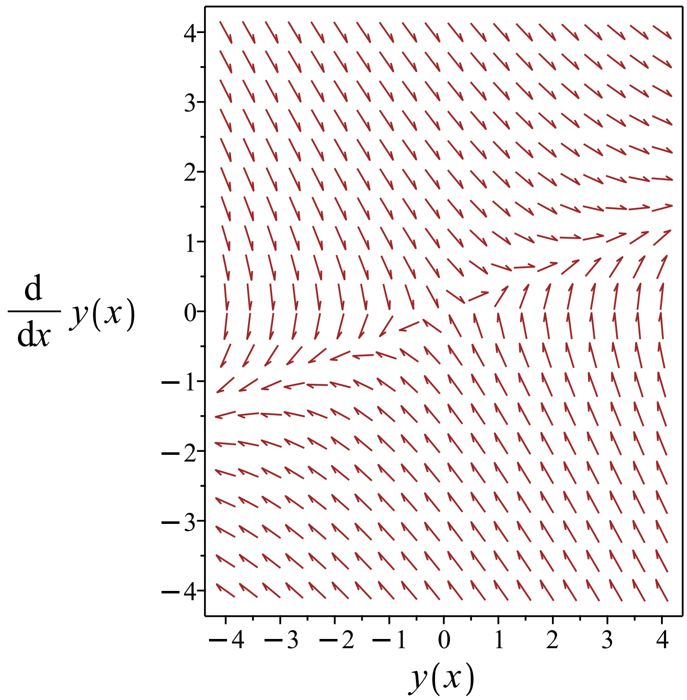


Figure 409: Slope field plot

Verification of solutions

$$
y = c_1 \mathrm{e}^{-\frac{4x}{3}} + \frac{3 c_2 \mathrm{e}^{\frac{x}{3}}}{5}
$$

Verified OK.

8.8.3 Maple step by step solution

Let's solve

$9y'' + 9y' - 4y = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y'' = -y' + \frac{4y}{9}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y'' + y' - \frac{4y}{9} = 0$

- Characteristic polynomial of ODE

  $r^2 + r - \frac{4}{9} = 0$

- Factor the characteristic polynomial

  $\frac{(3r+4)(3r-1)}{9} = 0$

- Roots of the characteristic polynomial

  $r = \left(-\frac{4}{3}, \frac{1}{3}\right)$

- 1st solution of the ODE

  $y_1(x) = e^{-\frac{4x}{3}}$

- 2nd solution of the ODE

  $y_2(x) = e^{\frac{x}{3}}$

- General solution of the ODE

  $y = c_1 y_1(x) + c_2 y_2(x)$

- Substitute in solutions

  $y = c_1 e^{-\frac{4x}{3}} + c_2 e^{\frac{x}{3}}$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
<- constant coefficients successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 17

```
dsolve(9*diff(y(x),x$2) +9*diff(y(x),x)-4*y(x) = 0,y(x), singsol=all)
```

$$y(x) = \left(c_2 e^{\frac{5x}{3}} + c_1\right) e^{-\frac{4x}{3}}$$

✓ Solution by Mathematica

Time used: 0.013 (sec). Leaf size: 26

```
DSolve[9*y''[x]+9*y'[x]-4*y[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to e^{-4x/3}\left(c_2 e^{5x/3} + c_1\right)$$

8.9 problem 15

Internal problem ID [631]
Internal file name [`OUTPUT/631_Sunday_June_05_2022_01_46_06_AM_45655324/index.tex`]

Book: Elementary differential equations and boundary value problems, 10th ed., Boyce and DiPrima
Section: Chapter 3, Second order linear equations, 3.3 Complex Roots of the Characteristic Equation , page 164
Problem number: 15.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_linear_constant_coeff"**

Maple gives the following as the ode type

```
[[_2nd_order, _missing_x]]
```

$$\boxed{y'' + y' + \frac{5y}{4} = 0}$$

8.9.1 Solving as second order linear constant coeff ode

This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = 0$$

Where in the above $A = 1, B = 1, C = \frac{5}{4}$. Let the solution be $y = e^{\lambda x}$. Substituting this into the ODE gives

$$\lambda^2 e^{\lambda x} + \lambda e^{\lambda x} + \frac{5 e^{\lambda x}}{4} = 0 \tag{1}$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda x}$ gives

$$\lambda^2 + \lambda + \frac{5}{4} = 0 \tag{2}$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form.Using the quadratic formula

$$\lambda_{1,2} = \frac{-B}{2A} \pm \frac{1}{2A}\sqrt{B^2 - 4AC}$$

Substituting $A = 1, B = 1, C = \frac{5}{4}$ into the above gives

$$\begin{aligned}\lambda_{1,2} &= \frac{-1}{(2)(1)} \pm \frac{1}{(2)(1)}\sqrt{1^2 - (4)(1)\left(\frac{5}{4}\right)}\\ &= -\frac{1}{2} \pm i\end{aligned}$$

Hence

$$\begin{aligned}\lambda_1 &= -\frac{1}{2} + i\\ \lambda_2 &= -\frac{1}{2} - i\end{aligned}$$

Which simplifies to

$$\begin{aligned}\lambda_1 &= -\frac{1}{2} + i\\ \lambda_2 &= -\frac{1}{2} - i\end{aligned}$$

Since roots are complex conjugate of each others, then let the roots be

$$\lambda_{1,2} = \alpha \pm i\beta$$

Where $\alpha = -\frac{1}{2}$ and $\beta = 1$. Therefore the final solution, when using Euler relation, can be written as

$$y = e^{\alpha x}(c_1 \cos(\beta x) + c_2 \sin(\beta x))$$

Which becomes

$$y = e^{-\frac{x}{2}}(c_1 \cos(x) + c_2 \sin(x))$$

Summary

The solution(s) found are the following

$$y = \mathrm{e}^{-\frac{x}{2}}(c_1 \cos(x) + c_2 \sin(x)) \tag{1}$$

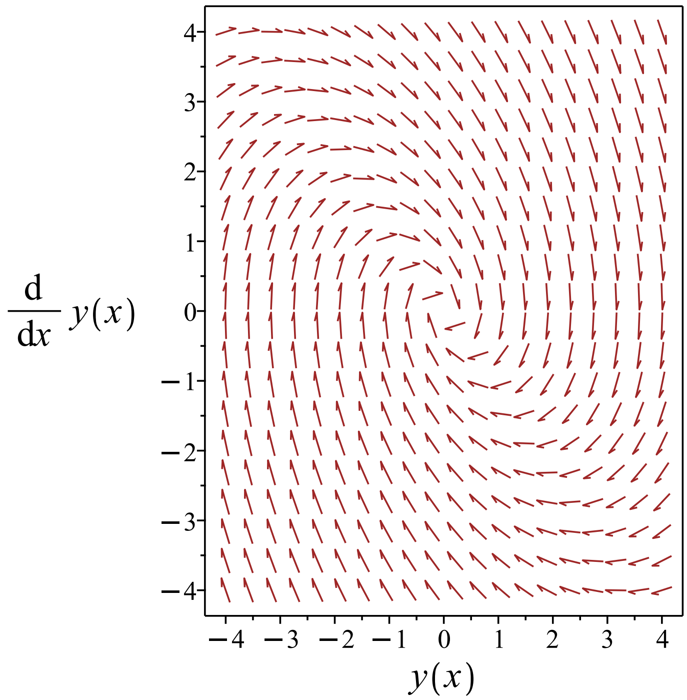


Figure 410: Slope field plot

Verification of solutions

$$y = e^{-\frac{x}{2}}(c_1 \cos(x) + c_2 \sin(x))$$

Verified OK.

8.9.2 Solving using Kovacic algorithm

Writing the ode as

$$y'' + y' + \frac{5y}{4} = 0 \tag{1}$$

$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1 \\ B &= 1 \\ C &= \frac{5}{4} \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\, dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{-1}{1} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= -1 \\ t &= 1 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = -z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 380: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 0 - 0 \\ &= 0 \end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Since $r = -1$ is not a function of x, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$z_1(x) = \cos(x)$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$\begin{aligned} y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx} \\ &= z_1 e^{-\int \frac{1}{2}\frac{1}{1}\,dx} \\ &= z_1 e^{-\frac{x}{2}} \\ &= z_1\left(\mathrm{e}^{-\frac{x}{2}}\right) \end{aligned}$$

Which simplifies to

$$y_1 = \mathrm{e}^{-\frac{x}{2}} \cos(x)$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx$$

Substituting gives

$$\begin{aligned} y_2 &= y_1 \int \frac{e^{\int -\frac{1}{1}\,dx}}{(y_1)^2}\,dx \\ &= y_1 \int \frac{e^{-x}}{(y_1)^2}\,dx \\ &= y_1(\tan(x)) \end{aligned}$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1\left(\mathrm{e}^{-\frac{x}{2}} \cos(x)\right) + c_2\left(\mathrm{e}^{-\frac{x}{2}} \cos(x) (\tan(x))\right)
\end{aligned}
$$

Summary
The solution(s) found are the following

$$
y = c_1 \mathrm{e}^{-\frac{x}{2}} \cos(x) + c_2 \mathrm{e}^{-\frac{x}{2}} \sin(x) \tag{1}
$$

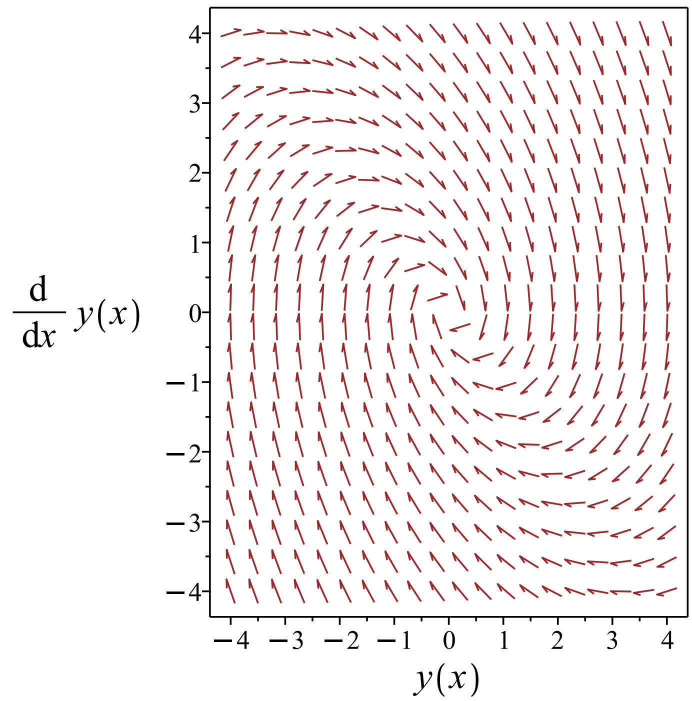


Figure 411: Slope field plot

Verification of solutions

$$
y = c_1 \mathrm{e}^{-\frac{x}{2}} \cos(x) + c_2 \mathrm{e}^{-\frac{x}{2}} \sin(x)
$$

Verified OK.

8.9.3 Maple step by step solution

Let's solve

$y'' + y' + \frac{5y}{4} = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Characteristic polynomial of ODE

  $r^2 + r + \frac{5}{4} = 0$

- Use quadratic formula to solve for r

  $r = \frac{(-1) \pm (\sqrt{-4})}{2}$

- Roots of the characteristic polynomial

  $r = \left(-\frac{1}{2} - \mathrm{I}, -\frac{1}{2} + \mathrm{I}\right)$

- 1st solution of the ODE

  $y_1(x) = \mathrm{e}^{-\frac{x}{2}} \cos(x)$

- 2nd solution of the ODE

  $y_2(x) = \mathrm{e}^{-\frac{x}{2}} \sin(x)$

- General solution of the ODE

  $y = c_1 y_1(x) + c_2 y_2(x)$

- Substitute in solutions

  $y = c_1 \mathrm{e}^{-\frac{x}{2}} \cos(x) + c_2 \mathrm{e}^{-\frac{x}{2}} \sin(x)$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
<- constant coefficients successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 18

```
dsolve(diff(y(x),x$2) +diff(y(x),x)+125/100*y(x) = 0,y(x), singsol=all)
```

$$y(x) = \mathrm{e}^{-\frac{x}{2}}(c_1 \sin(x) + c_2 \cos(x))$$

✓ Solution by Mathematica

Time used: 0.015 (sec). Leaf size: 24

```
DSolve[y''[x]+y'[x]+125/100*y[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to e^{-x/2}(c_2 \cos(x) + c_1 \sin(x))$$

8.10 problem 16

Internal problem ID [632]
Internal file name [`OUTPUT/632_Sunday_June_05_2022_01_46_07_AM_6524054/index.tex`]

Book: Elementary differential equations and boundary value problems, 10th ed., Boyce and DiPrima
Section: Chapter 3, Second order linear equations, 3.3 Complex Roots of the Characteristic Equation , page 164
Problem number: 16.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_linear_constant_coeff"**

Maple gives the following as the ode type

```
[[_2nd_order, _missing_x]]
```

$$\boxed{y'' + 4y' + \frac{25y}{4} = 0}$$

8.10.1 Solving as second order linear constant coeff ode

This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = 0$$

Where in the above $A = 1, B = 4, C = \frac{25}{4}$. Let the solution be $y = e^{\lambda x}$. Substituting this into the ODE gives

$$\lambda^2 e^{\lambda x} + 4\lambda\, e^{\lambda x} + \frac{25\, e^{\lambda x}}{4} = 0 \tag{1}$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda x}$ gives

$$\lambda^2 + 4\lambda + \frac{25}{4} = 0 \tag{2}$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form.Using the quadratic formula

$$\lambda_{1,2} = \frac{-B}{2A} \pm \frac{1}{2A}\sqrt{B^2 - 4AC}$$

Substituting $A = 1, B = 4, C = \frac{25}{4}$ into the above gives

$$\begin{aligned}\lambda_{1,2} &= \frac{-4}{(2)(1)} \pm \frac{1}{(2)(1)}\sqrt{4^2 - (4)(1)\left(\frac{25}{4}\right)} \\ &= -2 \pm \frac{3i}{2}\end{aligned}$$

Hence

$$\begin{aligned}\lambda_1 &= -2 + \frac{3i}{2} \\ \lambda_2 &= -2 - \frac{3i}{2}\end{aligned}$$

Which simplifies to

$$\begin{aligned}\lambda_1 &= -2 + \frac{3i}{2} \\ \lambda_2 &= -2 - \frac{3i}{2}\end{aligned}$$

Since roots are complex conjugate of each others, then let the roots be

$$\lambda_{1,2} = \alpha \pm i\beta$$

Where $\alpha = -2$ and $\beta = \frac{3}{2}$. Therefore the final solution, when using Euler relation, can be written as

$$y = e^{\alpha x}(c_1 \cos(\beta x) + c_2 \sin(\beta x))$$

Which becomes

$$y = e^{-2x}\left(c_1 \cos\left(\frac{3x}{2}\right) + c_2 \sin\left(\frac{3x}{2}\right)\right)$$

Summary

The solution(s) found are the following

$$y = \mathrm{e}^{-2x}\left(c_1 \cos\left(\frac{3x}{2}\right) + c_2 \sin\left(\frac{3x}{2}\right)\right) \tag{1}$$

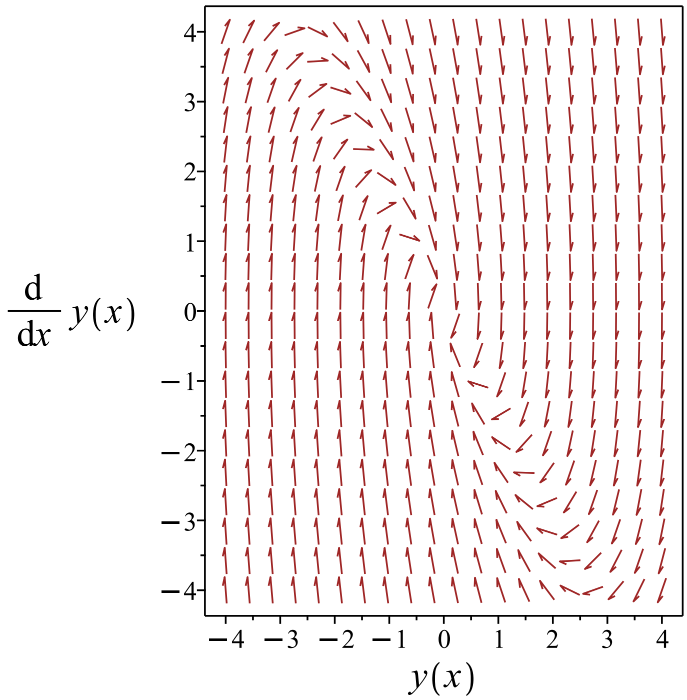


Figure 412: Slope field plot

Verification of solutions

$$y = e^{-2x}\left(c_1 \cos\left(\frac{3x}{2}\right) + c_2 \sin\left(\frac{3x}{2}\right)\right)$$

Verified OK.

8.10.2 Solving using Kovacic algorithm

Writing the ode as

$$y'' + 4y' + \frac{25y}{4} = 0 \tag{1}$$

$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1 \\ B &= 4 \\ C &= \frac{25}{4} \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\, dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{-9}{4} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= -9 \\ t &= 4 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = -\frac{9z(x)}{4} \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 382: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned}
O(\infty) &= \deg(t) - \deg(s) \\
&= 0 - 0 \\
&= 0
\end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Since $r = -\frac{9}{4}$ is not a function of x, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$z_1(x) = \cos\left(\frac{3x}{2}\right)$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\, dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{4}{1}\, dx} \\
&= z_1 e^{-2x} \\
&= z_1\left(\mathrm{e}^{-2x}\right)
\end{aligned}$$

Which simplifies to

$$y_1 = \mathrm{e}^{-2x} \cos\left(\frac{3x}{2}\right)$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\, dx}}{y_1^2}\, dx$$

Substituting gives

$$\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{4}{1}\, dx}}{(y_1)^2}\, dx \\
&= y_1 \int \frac{e^{-4x}}{(y_1)^2}\, dx \\
&= y_1 \left(\frac{2\tan\left(\frac{3x}{2}\right)}{3}\right)
\end{aligned}$$

Therefore the solution is

$$\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1 \left(\mathrm{e}^{-2x} \cos\left(\frac{3x}{2}\right)\right) + c_2 \left(\mathrm{e}^{-2x} \cos\left(\frac{3x}{2}\right) \left(\frac{2\tan\left(\frac{3x}{2}\right)}{3}\right)\right)
\end{aligned}$$

Summary

The solution(s) found are the following

$$y = c_1 \mathrm{e}^{-2x} \cos\left(\frac{3x}{2}\right) + \frac{2c_2 \mathrm{e}^{-2x} \sin\left(\frac{3x}{2}\right)}{3} \tag{1}$$

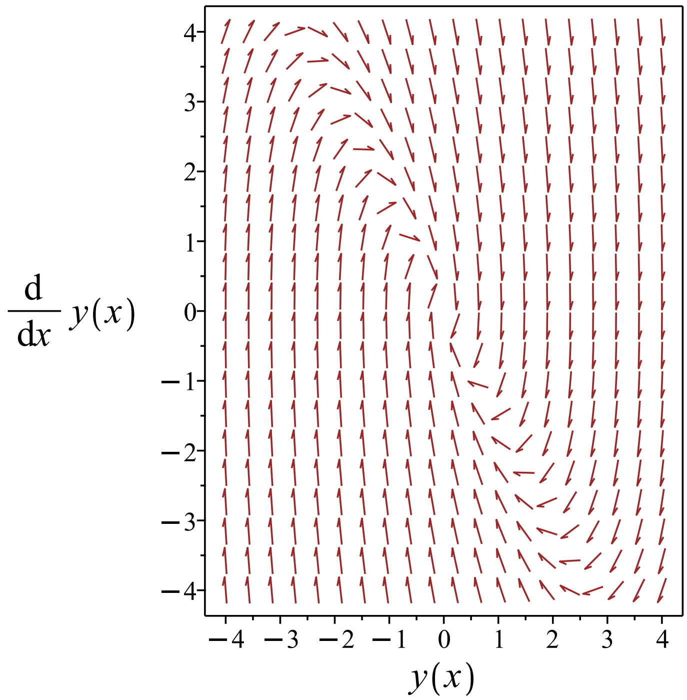


Figure 413: Slope field plot

Verification of solutions

$$y = c_1 \mathrm{e}^{-2x} \cos\left(\frac{3x}{2}\right) + \frac{2c_2 \mathrm{e}^{-2x} \sin\left(\frac{3x}{2}\right)}{3}$$

Verified OK.

8.10.3 Maple step by step solution

Let's solve

$y'' + 4y' + \frac{25y}{4} = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Characteristic polynomial of ODE

  $r^2 + 4r + \frac{25}{4} = 0$

- Use quadratic formula to solve for r

  $r = \frac{(-4)\pm(\sqrt{-9})}{2}$

- Roots of the characteristic polynomial

  $r = \left(-2 - \frac{3\,\mathrm{I}}{2}, -2 + \frac{3\,\mathrm{I}}{2}\right)$

- 1st solution of the ODE

  $y_1(x) = \mathrm{e}^{-2x} \cos\left(\frac{3x}{2}\right)$

- 2nd solution of the ODE

  $y_2(x) = \mathrm{e}^{-2x} \sin\left(\frac{3x}{2}\right)$

- General solution of the ODE

  $y = c_1 y_1(x) + c_2 y_2(x)$

- Substitute in solutions

  $y = c_1 \mathrm{e}^{-2x} \cos\left(\frac{3x}{2}\right) + c_2 \mathrm{e}^{-2x} \sin\left(\frac{3x}{2}\right)$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
<- constant coefficients successful`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 22

```
dsolve(diff(y(x),x$2)+ 4*diff(y(x),x)+625/100*y(x) = 0,y(x), singsol=all)
```

$$y(x) = \mathrm{e}^{-2x}\left(c_1 \sin\left(\frac{3x}{2}\right) + c_2 \cos\left(\frac{3x}{2}\right)\right)$$

✓ Solution by Mathematica

Time used: 0.018 (sec). Leaf size: 30

```
DSolve[y''[x]+4*y'[x]+625/100*y[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to e^{-2x}\left(c_2 \cos\left(\frac{3x}{2}\right) + c_1 \sin\left(\frac{3x}{2}\right)\right)$$

8.11 problem 17

Internal problem ID [633]
Internal file name [`OUTPUT/633_Sunday_June_05_2022_01_46_08_AM_45312836/index.tex`]

Book: Elementary differential equations and boundary value problems, 10th ed., Boyce and DiPrima
Section: Chapter 3, Second order linear equations, 3.3 Complex Roots of the Characteristic Equation , page 164
Problem number: 17.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_linear_constant_coeff", "second_order_ode_can_be_made_integrable"**

Maple gives the following as the ode type

```
[[_2nd_order, _missing_x]]
```

$$\boxed{y'' + 4y = 0}$$

With initial conditions

$$[y(0) = 0, y'(0) = 1]$$

8.11.1 Existence and uniqueness analysis

This is a linear ODE. In canonical form it is written as

$$y'' + p(x)y' + q(x)y = F$$

Where here

$$p(x) = 0$$
$$q(x) = 4$$
$$F = 0$$

Hence the ode is

$$y'' + 4y = 0$$

The domain of $p(x) = 0$ is

$$\{-\infty < x < \infty\}$$

And the point $x_0 = 0$ is inside this domain. The domain of $q(x) = 4$ is

$$\{-\infty < x < \infty\}$$

And the point $x_0 = 0$ is also inside this domain. Hence solution exists and is unique.

8.11.2 Solving as second order linear constant coeff ode

This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = 0$$

Where in the above $A = 1, B = 0, C = 4$. Let the solution be $y = e^{\lambda x}$. Substituting this into the ODE gives

$$\lambda^2 e^{\lambda x} + 4\,e^{\lambda x} = 0 \tag{1}$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda x}$ gives

$$\lambda^2 + 4 = 0 \tag{2}$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form.Using the quadratic formula

$$\lambda_{1,2} = \frac{-B}{2A} \pm \frac{1}{2A}\sqrt{B^2 - 4AC}$$

Substituting $A = 1, B = 0, C = 4$ into the above gives

$$\begin{aligned}
\lambda_{1,2} &= \frac{0}{(2)\,(1)} \pm \frac{1}{(2)\,(1)}\sqrt{0^2 - (4)\,(1)\,(4)} \\
&= \pm 2i
\end{aligned}$$

Hence

$$\begin{aligned}
\lambda_1 &= +2i \\
\lambda_2 &= -2i
\end{aligned}$$

Which simplifies to

$$\lambda_1 = 2i$$
$$\lambda_2 = -2i$$

Since roots are complex conjugate of each others, then let the roots be

$$\lambda_{1,2} = \alpha \pm i\beta$$

Where $\alpha = 0$ and $\beta = 2$. Therefore the final solution, when using Euler relation, can be written as

$$y = e^{\alpha x}(c_1 \cos(\beta x) + c_2 \sin(\beta x))$$

Which becomes

$$y = e^0(c_1 \cos(2x) + c_2 \sin(2x))$$

Or

$$y = c_1 \cos(2x) + c_2 \sin(2x)$$

Initial conditions are used to solve for the constants of integration.

Looking at the above solution

$$y = c_1 \cos(2x) + c_2 \sin(2x) \tag{1}$$

Initial conditions are now substituted in the above solution. This will generate the required equations to solve for the integration constants. substituting $y = 0$ and $x = 0$ in the above gives

$$0 = c_1 \tag{1A}$$

Taking derivative of the solution gives

$$y' = -2c_1 \sin(2x) + 2c_2 \cos(2x)$$

substituting $y' = 1$ and $x = 0$ in the above gives

$$1 = 2c_2 \tag{2A}$$

Equations {1A,2A} are now solved for $\{c_1, c_2\}$. Solving for the constants gives

$$c_1 = 0$$
$$c_2 = \frac{1}{2}$$

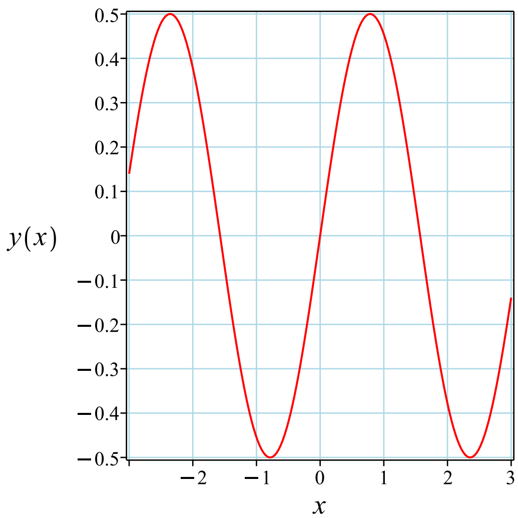
Substituting these values back in above solution results in

$$y = \frac{\sin(2x)}{2}$$

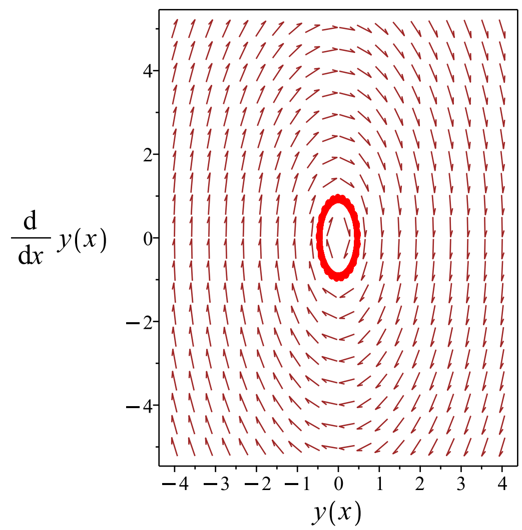
Summary

The solution(s) found are the following

$$y = \frac{\sin(2x)}{2} \tag{1}$$



(a) Solution plot



(b) Slope field plot

Verification of solutions

$$y = \frac{\sin(2x)}{2}$$

Verified OK.

8.11.3 Solving as second order ode can be made integrable ode

Multiplying the ode by y' gives

$$y'y'' + 4yy' = 0$$

Integrating the above w.r.t x gives

$$\int (y'y'' + 4yy')\, dx = 0$$
$$\frac{y'^2}{2} + 2y^2 = c_2$$

Which is now solved for y. Solving the given ode for y' results in 2 differential equations to solve. Each one of these will generate a solution. The equations generated are

$$y' = \sqrt{-4y^2 + 2c_1} \tag{1}$$
$$y' = -\sqrt{-4y^2 + 2c_1} \tag{2}$$

Now each one of the above ODE is solved.

Solving equation (1)

Integrating both sides gives

$$\int \frac{1}{\sqrt{-4y^2 + 2c_1}} dy = \int dx$$
$$\frac{\arctan\left(\frac{2y}{\sqrt{-4y^2+2c_1}}\right)}{2} = x + c_2$$

Solving equation (2)

Integrating both sides gives

$$\int -\frac{1}{\sqrt{-4y^2 + 2c_1}} dy = \int dx$$
$$-\frac{\arctan\left(\frac{2y}{\sqrt{-4y^2+2c_1}}\right)}{2} = c_3 + x$$

Initial conditions are used to solve for the constants of integration.

Looking at the First solution

$$\frac{\arctan\left(\frac{2y}{\sqrt{-4y^2+2c_1}}\right)}{2} = x + c_2 \tag{1}$$

Initial conditions are now substituted in the above solution. This will generate the required equations to solve for the integration constants. substituting $y = 0$ and $x = 0$ in the above gives

$$0 = c_2 \tag{1A}$$

Taking derivative of the solution gives

$$y' = \frac{\left(2\tan\left(2x+2c_2\right)^2 + 2\right)\sqrt{2}\,\sqrt{\frac{c_1}{\tan(2x+2c_2)^2+1}}}{2} - \frac{\tan\left(2x+2c_2\right)^2\sqrt{2}\,c_1\left(2\tan\left(2x+2c_2\right)^2+2\right)}{2\sqrt{\frac{c_1}{\tan(2x+2c_2)^2+1}}\,\left(\tan\left(2x+2c_2\right)^2+1\right)^2}$$

substituting $y' = 1$ and $x = 0$ in the above gives

$$1 = \frac{\cos(2c_2)^2 \sqrt{2}\, c_1}{\sqrt{\cos(2c_2)^2 c_1}} \tag{2A}$$

Equations {1A,2A} are now solved for $\{c_1, c_2\}$. Solving for the constants gives

$$c_1 = \frac{1}{2}$$
$$c_2 = 0$$

Substituting these values back in above solution results in

$$\frac{\arctan\left(\frac{2y}{\sqrt{-4y^2+1}}\right)}{2} = x$$

Looking at the Second solution

$$-\frac{\arctan\left(\frac{2y}{\sqrt{-4y^2+2c_1}}\right)}{2} = c_3 + x \tag{2}$$

Initial conditions are now substituted in the above solution. This will generate the required equations to solve for the integration constants. substituting $y = 0$ and $x = 0$ in the above gives

$$0 = c_3 \tag{1A}$$

Taking derivative of the solution gives

$$y' = -\frac{\left(2\tan(2c_3+2x)^2+2\right)\sqrt{2}\,\sqrt{\frac{c_1}{\tan(2c_3+2x)^2+1}}}{2} + \frac{\tan(2c_3+2x)^2\sqrt{2}\,c_1\left(2\tan(2c_3+2x)^2+2\right)}{2\sqrt{\frac{c_1}{\tan(2c_3+2x)^2+1}}\left(\tan(2c_3+2x)^2+1\right)^2}$$

substituting $y' = 1$ and $x = 0$ in the above gives

$$1 = -\frac{\cos(2c_3)^2 \sqrt{2}\, c_1}{\sqrt{\cos(2c_3)^2 c_1}} \tag{2A}$$

Equations {1A,2A} are now solved for $\{c_1, c_3\}$. Warning, unable to solve for constants of integrations.

Summary

The solution(s) found are the following

$$\frac{\arctan\left(\frac{2y}{\sqrt{-4y^2+1}}\right)}{2} = x \tag{1}$$

Verification of solutions

$$\frac{\arctan\left(\frac{2y}{\sqrt{-4y^2+1}}\right)}{2} = x$$

Verified OK.

8.11.4 Solving using Kovacic algorithm

Writing the ode as

$$y'' + 4y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1 \\ B &= 0 \\ C &= 4 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\, dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{-4}{1} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= -4 \\ t &= 1 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = -4z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 384: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 0 - 0 \\ &= 0 \end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Since $r = -4$ is not a function of x, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$z_1(x) = \cos(2x)$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$y_1 = z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx}$$

Since $B = 0$ then the above reduces to

$$\begin{aligned} y_1 &= z_1 \\ &= \cos(2x) \end{aligned}$$

Which simplifies to

$$y_1 = \cos(2x)$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx$$

Since $B = 0$ then the above becomes

$$\begin{aligned} y_2 &= y_1 \int \frac{1}{y_1^2}\,dx \\ &= \cos(2x) \int \frac{1}{\cos(2x)^2}\,dx \\ &= \cos(2x)\left(\frac{\tan(2x)}{2}\right) \end{aligned}$$

Therefore the solution is

$$\begin{aligned} y &= c_1 y_1 + c_2 y_2 \\ &= c_1(\cos(2x)) + c_2\left(\cos(2x)\left(\frac{\tan(2x)}{2}\right)\right) \end{aligned}$$

Initial conditions are used to solve for the constants of integration.

Looking at the above solution

$$y = c_1 \cos(2x) + \frac{c_2 \sin(2x)}{2} \tag{1}$$

Initial conditions are now substituted in the above solution. This will generate the required equations to solve for the integration constants. substituting $y = 0$ and $x = 0$ in the above gives

$$0 = c_1 \tag{1A}$$

Taking derivative of the solution gives

$$y' = -2c_1 \sin(2x) + c_2 \cos(2x)$$

substituting $y' = 1$ and $x = 0$ in the above gives

$$1 = c_2 \tag{2A}$$

Equations {1A,2A} are now solved for $\{c_1, c_2\}$. Solving for the constants gives

$$c_1 = 0$$
$$c_2 = 1$$

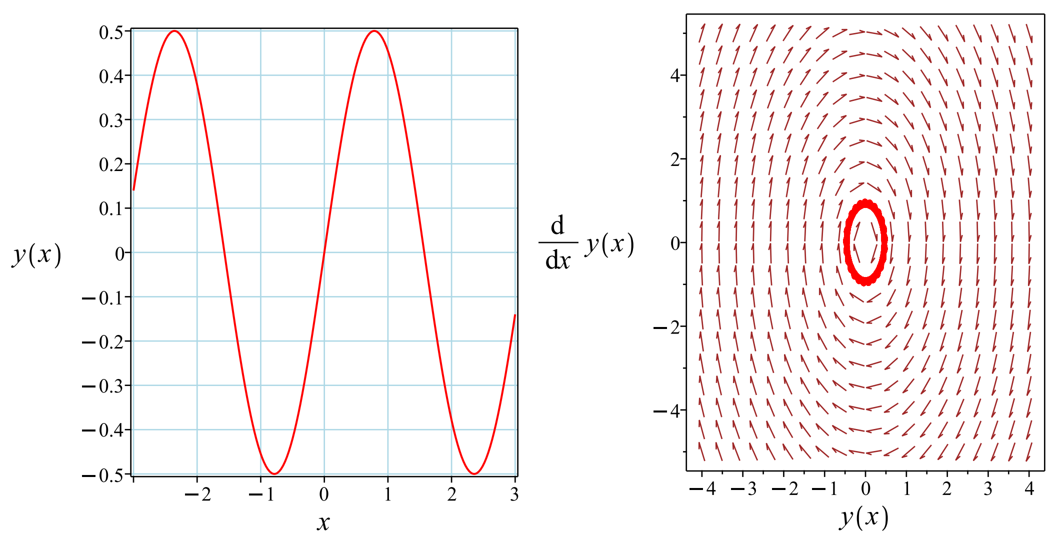
Substituting these values back in above solution results in

$$y = \frac{\sin(2x)}{2}$$

Summary

The solution(s) found are the following

$$y = \frac{\sin(2x)}{2} \tag{1}$$



(a) Solution plot

(b) Slope field plot

Verification of solutions

$$y = \frac{\sin(2x)}{2}$$

Verified OK.

8.11.5 Maple step by step solution

Let's solve

$\left[y'' + 4y = 0, y(0) = 0, y'\Big|_{\{x=0\}} = 1\right]$

- Highest derivative means the order of the ODE is 2

  y''

- Characteristic polynomial of ODE

  $r^2 + 4 = 0$

- Use quadratic formula to solve for r

  $r = \frac{0 \pm (\sqrt{-16})}{2}$

- Roots of the characteristic polynomial

  $r = (-2\,\mathrm{I}, 2\,\mathrm{I})$

- 1st solution of the ODE

  $y_1(x) = \cos(2x)$

- 2nd solution of the ODE

  $y_2(x) = \sin(2x)$

- General solution of the ODE

  $y = c_1 y_1(x) + c_2 y_2(x)$

- Substitute in solutions

  $y = c_1 \cos(2x) + c_2 \sin(2x)$

□ Check validity of solution $y = c_1 \cos(2x) + c_2 \sin(2x)$

  - Use initial condition $y(0) = 0$

    $0 = c_1$

  - Compute derivative of the solution

    $y' = -2c_1 \sin(2x) + 2c_2 \cos(2x)$

- Use the initial condition $y'\Big|_{\{x=0\}} = 1$

  $1 = 2c_2$

- Solve for c_1 and c_2

  $\{c_1 = 0, c_2 = \frac{1}{2}\}$

- Substitute constant values into general solution and simplify

  $y = \frac{\sin(2x)}{2}$

- Solution to the IVP

  $y = \frac{\sin(2x)}{2}$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
<- constant coefficients successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 10

```
dsolve([diff(y(x),x$2)+ 4*y(x) = 0,y(0) = 0, D(y)(0) = 1],y(x), singsol=all)
```

$$y(x) = \frac{\sin(2x)}{2}$$

✓ Solution by Mathematica

Time used: 0.012 (sec). Leaf size: 10

```
DSolve[{y''[x]+4*y[x]==0,{y[0]==0,y'[0]==1}},y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \sin(x)\cos(x)$$

8.12 problem 18

Internal problem ID [634]
Internal file name [`OUTPUT/634_Sunday_June_05_2022_01_46_09_AM_49749350/index.tex`]

Book: Elementary differential equations and boundary value problems, 10th ed., Boyce and DiPrima
Section: Chapter 3, Second order linear equations, 3.3 Complex Roots of the Characteristic Equation , page 164
Problem number: 18.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_linear_constant_coeff"**

Maple gives the following as the ode type

```
[[_2nd_order, _missing_x]]
```

$$\boxed{y'' + 4y' + 5y = 0}$$

With initial conditions

$$[y(0) = 1, y'(0) = 0]$$

8.12.1 Existence and uniqueness analysis

This is a linear ODE. In canonical form it is written as

$$y'' + p(x)y' + q(x)y = F$$

Where here

$$\begin{aligned} p(x) &= 4 \\ q(x) &= 5 \\ F &= 0 \end{aligned}$$

Hence the ode is

$$y'' + 4y' + 5y = 0$$

The domain of $p(x) = 4$ is

$$\{-\infty < x < \infty\}$$

And the point $x_0 = 0$ is inside this domain. The domain of $q(x) = 5$ is

$$\{-\infty < x < \infty\}$$

And the point $x_0 = 0$ is also inside this domain. Hence solution exists and is unique.

8.12.2 Solving as second order linear constant coeff ode

This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = 0$$

Where in the above $A = 1, B = 4, C = 5$. Let the solution be $y = e^{\lambda x}$. Substituting this into the ODE gives

$$\lambda^2 e^{\lambda x} + 4\lambda\, e^{\lambda x} + 5\, e^{\lambda x} = 0 \tag{1}$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda x}$ gives

$$\lambda^2 + 4\lambda + 5 = 0 \tag{2}$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form.Using the quadratic formula

$$\lambda_{1,2} = \frac{-B}{2A} \pm \frac{1}{2A}\sqrt{B^2 - 4AC}$$

Substituting $A = 1, B = 4, C = 5$ into the above gives

$$\begin{aligned}\lambda_{1,2} &= \frac{-4}{(2)\,(1)} \pm \frac{1}{(2)\,(1)}\sqrt{4^2 - (4)\,(1)\,(5)} \\ &= -2 \pm i\end{aligned}$$

Hence

$$\begin{aligned}\lambda_1 &= -2 + i \\ \lambda_2 &= -2 - i\end{aligned}$$

Which simplifies to

$$\lambda_1 = -2 + i$$
$$\lambda_2 = -2 - i$$

Since roots are complex conjugate of each others, then let the roots be

$$\lambda_{1,2} = \alpha \pm i\beta$$

Where $\alpha = -2$ and $\beta = 1$. Therefore the final solution, when using Euler relation, can be written as

$$y = e^{\alpha x}(c_1 \cos(\beta x) + c_2 \sin(\beta x))$$

Which becomes

$$y = e^{-2x}(c_1 \cos(x) + c_2 \sin(x))$$

Initial conditions are used to solve for the constants of integration.

Looking at the above solution

$$y = \mathrm{e}^{-2x}(c_1 \cos(x) + c_2 \sin(x)) \tag{1}$$

Initial conditions are now substituted in the above solution. This will generate the required equations to solve for the integration constants. substituting $y = 1$ and $x = 0$ in the above gives

$$1 = c_1 \tag{1A}$$

Taking derivative of the solution gives

$$y' = -2\,\mathrm{e}^{-2x}(c_1 \cos(x) + c_2 \sin(x)) + \mathrm{e}^{-2x}(-c_1 \sin(x) + c_2 \cos(x))$$

substituting $y' = 0$ and $x = 0$ in the above gives

$$0 = -2c_1 + c_2 \tag{2A}$$

Equations {1A,2A} are now solved for $\{c_1, c_2\}$. Solving for the constants gives

$$c_1 = 1$$
$$c_2 = 2$$

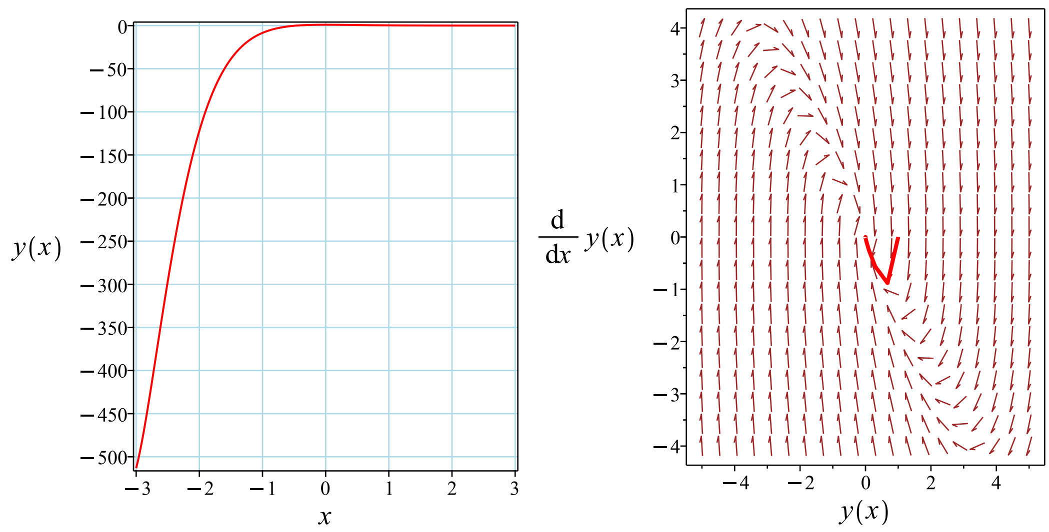
Substituting these values back in above solution results in

$$y = \mathrm{e}^{-2x}(\cos(x) + 2\sin(x))$$

Summary

The solution(s) found are the following

$$y = e^{-2x}(\cos(x) + 2\sin(x)) \tag{1}$$



(a) Solution plot

(b) Slope field plot

Verification of solutions

$$y = e^{-2x}(\cos(x) + 2\sin(x))$$

Verified OK.

8.12.3 Solving using Kovacic algorithm

Writing the ode as

$$y'' + 4y' + 5y = 0 \tag{1}$$

$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1 \\ B &= 4 \\ C &= 5 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{-1}{1} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= -1 \\ t &= 1 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = -z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 386: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned}
O(\infty) &= \deg(t) - \deg(s) \\
&= 0 - 0 \\
&= 0
\end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Since $r = -1$ is not a function of x, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$z_1(x) = \cos(x)$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{4}{1}\,dx} \\
&= z_1 e^{-2x} \\
&= z_1\left(\mathrm{e}^{-2x}\right)
\end{aligned}$$

Which simplifies to

$$y_1 = \mathrm{e}^{-2x}\cos(x)$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx$$

Substituting gives

$$\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{4}{1}\,dx}}{(y_1)^2}\,dx \\
&= y_1 \int \frac{e^{-4x}}{(y_1)^2}\,dx \\
&= y_1(\tan(x))
\end{aligned}$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1\left(\mathrm{e}^{-2x} \cos(x)\right) + c_2\left(\mathrm{e}^{-2x} \cos(x) (\tan(x))\right)
\end{aligned}
$$

Initial conditions are used to solve for the constants of integration.

Looking at the above solution

$$
y = c_1 \mathrm{e}^{-2x} \cos(x) + c_2 \mathrm{e}^{-2x} \sin(x) \tag{1}
$$

Initial conditions are now substituted in the above solution. This will generate the required equations to solve for the integration constants. substituting $y = 1$ and $x = 0$ in the above gives

$$
1 = c_1 \tag{1A}
$$

Taking derivative of the solution gives

$$
y' = -2c_1 \mathrm{e}^{-2x} \cos(x) - c_1 \mathrm{e}^{-2x} \sin(x) - 2c_2 \mathrm{e}^{-2x} \sin(x) + c_2 \mathrm{e}^{-2x} \cos(x)
$$

substituting $y' = 0$ and $x = 0$ in the above gives

$$
0 = -2c_1 + c_2 \tag{2A}
$$

Equations {1A,2A} are now solved for $\{c_1, c_2\}$. Solving for the constants gives

$$
\begin{aligned}
c_1 &= 1 \\
c_2 &= 2
\end{aligned}
$$

Substituting these values back in above solution results in

$$
y = \mathrm{e}^{-2x} \cos(x) + 2\,\mathrm{e}^{-2x} \sin(x)
$$

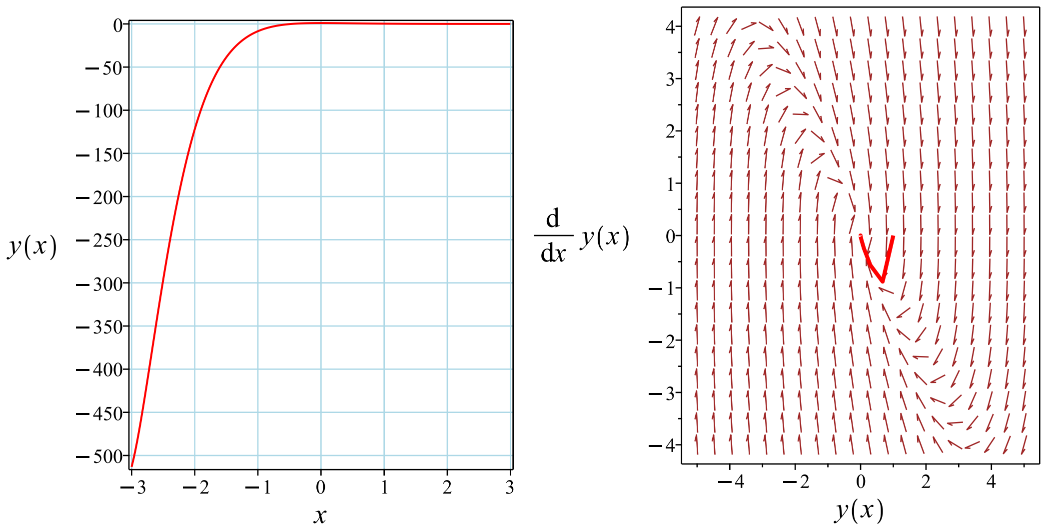
Which simplifies to

$$
y = \mathrm{e}^{-2x}(\cos(x) + 2\sin(x))
$$

Summary

The solution(s) found are the following

$$
y = \mathrm{e}^{-2x}(\cos(x) + 2\sin(x)) \tag{1}
$$



(a) Solution plot (b) Slope field plot

<u>Verification of solutions</u>

$$y = e^{-2x}(\cos(x) + 2\sin(x))$$

Verified OK.

8.12.4 Maple step by step solution

Let's solve

$$\left[y'' + 4y' + 5y = 0, y(0) = 1, y'\Big|_{\{x=0\}} = 0\right]$$

- Highest derivative means the order of the ODE is 2

  y''

- Characteristic polynomial of ODE

  $r^2 + 4r + 5 = 0$

- Use quadratic formula to solve for r

  $r = \frac{(-4)\pm(\sqrt{-4})}{2}$

- Roots of the characteristic polynomial

  $r = (-2 - \mathrm{I}, -2 + \mathrm{I})$

- 1st solution of the ODE

$y_1(x) = e^{-2x} \cos(x)$

- 2nd solution of the ODE

  $y_2(x) = e^{-2x} \sin(x)$

- General solution of the ODE

  $y = c_1 y_1(x) + c_2 y_2(x)$

- Substitute in solutions

  $y = c_1 e^{-2x} \cos(x) + c_2 e^{-2x} \sin(x)$

□ Check validity of solution $y = c_1 e^{-2x} \cos(x) + c_2 e^{-2x} \sin(x)$

  - Use initial condition $y(0) = 1$

    $1 = c_1$

  - Compute derivative of the solution

    $y' = -2c_1 e^{-2x} \cos(x) - c_1 e^{-2x} \sin(x) - 2c_2 e^{-2x} \sin(x) + c_2 e^{-2x} \cos(x)$

  - Use the initial condition $y'\Big|_{\{x=0\}} = 0$

    $0 = -2c_1 + c_2$

  - Solve for c_1 and c_2

    $\{c_1 = 1, c_2 = 2\}$

  - Substitute constant values into general solution and simplify

    $y = e^{-2x}(\cos(x) + 2\sin(x))$

- Solution to the IVP

  $y = e^{-2x}(\cos(x) + 2\sin(x))$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
<- constant coefficients successful`
```

✓ Solution by Maple

Time used: 0.015 (sec). Leaf size: 16

```
dsolve([diff(y(x),x$2)+ 4*diff(y(x),x)+5*y(x) = 0,y(0) = 1, D(y)(0) = 0],y(x), singsol=all)
```

$$y(x) = e^{-2x}(2\sin(x) + \cos(x))$$

✓ Solution by Mathematica

Time used: 0.015 (sec). Leaf size: 18

```
DSolve[{y''[x]+4*y'[x]+5*y[x]==0,{y[0]==1,y'[0]==0}},y[x],x,IncludeSingularSolutions -> True
```

$$y(x) \to e^{-2x}(2\sin(x) + \cos(x))$$

8.13 problem 19

Internal problem ID [635]
Internal file name [`OUTPUT/635_Sunday_June_05_2022_01_46_10_AM_46194284/index.tex`]

Book: Elementary differential equations and boundary value problems, 10th ed., Boyce and DiPrima
Section: Chapter 3, Second order linear equations, 3.3 Complex Roots of the Characteristic Equation , page 164
Problem number: 19.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_linear_constant_coeff"**

Maple gives the following as the ode type

```
[[_2nd_order, _missing_x]]
```

$$\boxed{y'' - 2y' + 5y = 0}$$

With initial conditions

$$\left[y\left(\frac{\pi}{2}\right) = 0, y'\left(\frac{\pi}{2}\right) = 2\right]$$

8.13.1 Existence and uniqueness analysis

This is a linear ODE. In canonical form it is written as

$$y'' + p(x)y' + q(x)y = F$$

Where here

$$\begin{aligned} p(x) &= -2 \\ q(x) &= 5 \\ F &= 0 \end{aligned}$$

Hence the ode is

$$y'' - 2y' + 5y = 0$$

The domain of $p(x) = -2$ is

$$\{-\infty < x < \infty\}$$

And the point $x_0 = \frac{\pi}{2}$ is inside this domain. The domain of $q(x) = 5$ is

$$\{-\infty < x < \infty\}$$

And the point $x_0 = \frac{\pi}{2}$ is also inside this domain. Hence solution exists and is unique.

8.13.2 Solving as second order linear constant coeff ode

This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = 0$$

Where in the above $A = 1, B = -2, C = 5$. Let the solution be $y = e^{\lambda x}$. Substituting this into the ODE gives

$$\lambda^2 \mathrm{e}^{\lambda x} - 2\lambda\, \mathrm{e}^{\lambda x} + 5\, \mathrm{e}^{\lambda x} = 0 \tag{1}$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda x}$ gives

$$\lambda^2 - 2\lambda + 5 = 0 \tag{2}$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form.Using the quadratic formula

$$\lambda_{1,2} = \frac{-B}{2A} \pm \frac{1}{2A}\sqrt{B^2 - 4AC}$$

Substituting $A = 1, B = -2, C = 5$ into the above gives

$$\begin{aligned}\lambda_{1,2} &= \frac{2}{(2)\,(1)} \pm \frac{1}{(2)\,(1)}\sqrt{-2^2 - (4)\,(1)\,(5)} \\ &= 1 \pm 2i\end{aligned}$$

Hence

$$\begin{aligned}\lambda_1 &= 1 + 2i \\ \lambda_2 &= 1 - 2i\end{aligned}$$

Which simplifies to

$$\lambda_1 = 1 + 2i$$
$$\lambda_2 = 1 - 2i$$

Since roots are complex conjugate of each others, then let the roots be

$$\lambda_{1,2} = \alpha \pm i\beta$$

Where $\alpha = 1$ and $\beta = 2$. Therefore the final solution, when using Euler relation, can be written as

$$y = e^{\alpha x}(c_1 \cos(\beta x) + c_2 \sin(\beta x))$$

Which becomes

$$y = e^{x}(c_1 \cos(2x) + c_2 \sin(2x))$$

Initial conditions are used to solve for the constants of integration.

Looking at the above solution

$$y = e^{x}(c_1 \cos(2x) + c_2 \sin(2x)) \tag{1}$$

Initial conditions are now substituted in the above solution. This will generate the required equations to solve for the integration constants. substituting $y = 0$ and $x = \frac{\pi}{2}$ in the above gives

$$0 = -c_1 e^{\frac{\pi}{2}} \tag{1A}$$

Taking derivative of the solution gives

$$y' = e^{x}(c_1 \cos(2x) + c_2 \sin(2x)) + e^{x}(-2c_1 \sin(2x) + 2c_2 \cos(2x))$$

substituting $y' = 2$ and $x = \frac{\pi}{2}$ in the above gives

$$2 = (-c_1 - 2c_2)\, e^{\frac{\pi}{2}} \tag{2A}$$

Equations {1A,2A} are now solved for $\{c_1, c_2\}$. Solving for the constants gives

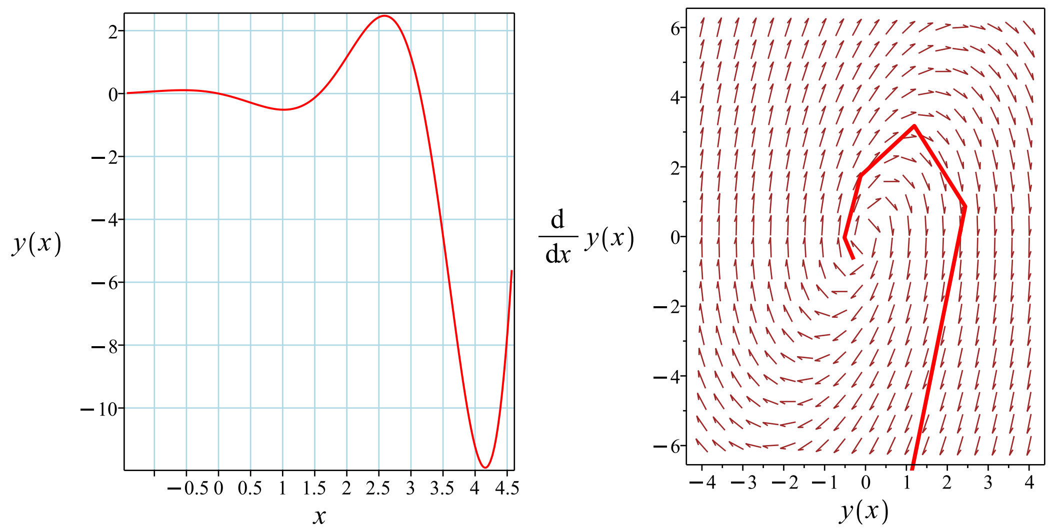
$$c_1 = 0$$
$$c_2 = -e^{-\frac{\pi}{2}}$$

Substituting these values back in above solution results in

$$y = -\sin(2x)\, e^{-\frac{\pi}{2}+x}$$

Summary
The solution(s) found are the following

$$y = -\sin(2x)\,e^{-\frac{\pi}{2}+x} \tag{1}$$



(a) Solution plot (b) Slope field plot

Verification of solutions

$$y = -\sin(2x)\,e^{-\frac{\pi}{2}+x}$$

Verified OK.

8.13.3 Solving using Kovacic algorithm

Writing the ode as

$$y'' - 2y' + 5y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1 \\ B &= -2 \\ C &= 5 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{-4}{1} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= -4 \\ t &= 1 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = -4z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 388: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 0 - 0 \\ &= 0 \end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Since $r = -4$ is not a function of x, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$z_1(x) = \cos(2x)$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$\begin{aligned} y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx} \\ &= z_1 e^{-\int \frac{1}{2}\frac{-2}{1}\,dx} \\ &= z_1 e^{x} \\ &= z_1(\mathrm{e}^{x}) \end{aligned}$$

Which simplifies to

$$y_1 = \mathrm{e}^{x}\cos(2x)$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx$$

Substituting gives

$$\begin{aligned} y_2 &= y_1 \int \frac{e^{\int -\frac{-2}{1}\,dx}}{(y_1)^2}\,dx \\ &= y_1 \int \frac{e^{2x}}{(y_1)^2}\,dx \\ &= y_1 \left(\frac{\tan(2x)}{2}\right) \end{aligned}$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1y_1 + c_2y_2 \\
&= c_1(\mathrm{e}^x \cos(2x)) + c_2\left(\mathrm{e}^x \cos(2x)\left(\frac{\tan(2x)}{2}\right)\right)
\end{aligned}
$$

Initial conditions are used to solve for the constants of integration.

Looking at the above solution

$$
y = c_1\mathrm{e}^x \cos(2x) + \frac{\mathrm{e}^x c_2 \sin(2x)}{2} \tag{1}
$$

Initial conditions are now substituted in the above solution. This will generate the required equations to solve for the integration constants. substituting $y = 0$ and $x = \frac{\pi}{2}$ in the above gives

$$
0 = -c_1\mathrm{e}^{\frac{\pi}{2}} \tag{1A}
$$

Taking derivative of the solution gives

$$
y' = c_1\mathrm{e}^x \cos(2x) - 2c_1\mathrm{e}^x \sin(2x) + \frac{\mathrm{e}^x c_2 \sin(2x)}{2} + \mathrm{e}^x c_2 \cos(2x)
$$

substituting $y' = 2$ and $x = \frac{\pi}{2}$ in the above gives

$$
2 = (-c_1 - c_2)\,\mathrm{e}^{\frac{\pi}{2}} \tag{2A}
$$

Equations {1A,2A} are now solved for $\{c_1, c_2\}$. Solving for the constants gives

$$
\begin{aligned}
c_1 &= 0 \\
c_2 &= -2\,\mathrm{e}^{-\frac{\pi}{2}}
\end{aligned}
$$

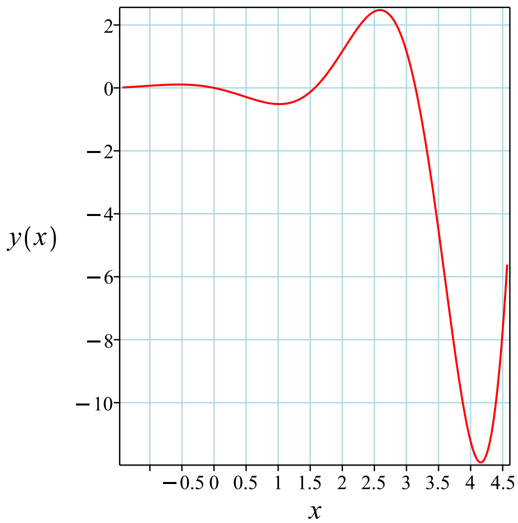
Substituting these values back in above solution results in

$$
y = -\sin(2x)\,\mathrm{e}^{-\frac{\pi}{2}+x}
$$

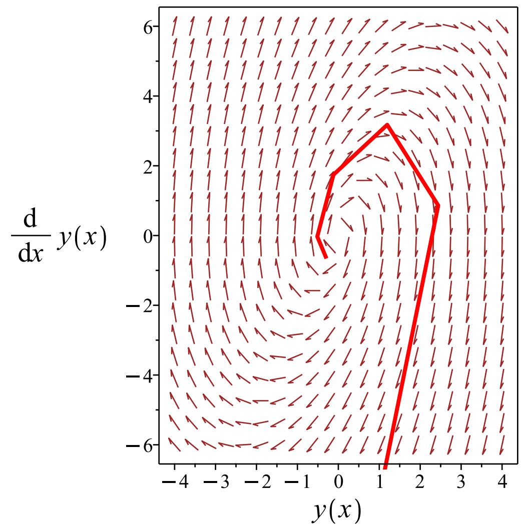
Summary

The solution(s) found are the following

$$
y = -\sin(2x)\,\mathrm{e}^{-\frac{\pi}{2}+x} \tag{1}
$$



(a) Solution plot



(b) Slope field plot

<u>Verification of solutions</u>

$$y = -\sin(2x)\,e^{-\frac{\pi}{2}+x}$$

Verified OK.

8.13.4 Maple step by step solution

Let's solve

$$\left[y'' - 2y' + 5y = 0, y\left(\frac{\pi}{2}\right) = 0, y'\Big|_{\left\{x=\frac{\pi}{2}\right\}} = 2\right]$$

- Highest derivative means the order of the ODE is 2

  y''
- Characteristic polynomial of ODE

  $r^2 - 2r + 5 = 0$
- Use quadratic formula to solve for r

  $r = \frac{2\pm(\sqrt{-16})}{2}$
- Roots of the characteristic polynomial

  $r = (1 - 2\,\mathrm{I}, 1 + 2\,\mathrm{I})$
- 1st solution of the ODE

$y_1(x) = e^x \cos(2x)$

- 2nd solution of the ODE

  $y_2(x) = e^x \sin(2x)$

- General solution of the ODE

  $y = c_1 y_1(x) + c_2 y_2(x)$

- Substitute in solutions

  $y = c_1 e^x \cos(2x) + e^x c_2 \sin(2x)$

□ Check validity of solution $y = c_1 e^x \cos(2x) + e^x c_2 \sin(2x)$

  - Use initial condition $y\left(\frac{\pi}{2}\right) = 0$

    $0 = -c_1 e^{\frac{\pi}{2}}$

  - Compute derivative of the solution

    $y' = c_1 e^x \cos(2x) - 2c_1 e^x \sin(2x) + e^x c_2 \sin(2x) + 2 e^x c_2 \cos(2x)$

  - Use the initial condition $y'\Big|_{\left\{x=\frac{\pi}{2}\right\}} = 2$

    $2 = -c_1 e^{\frac{\pi}{2}} - 2 e^{\frac{\pi}{2}} c_2$

  - Solve for c_1 and c_2

    $\left\{c_1 = 0, c_2 = -\frac{1}{e^{\frac{\pi}{2}}}\right\}$

  - Substitute constant values into general solution and simplify

    $y = -\sin(2x) e^{-\frac{\pi}{2}+x}$

- Solution to the IVP

  $y = -\sin(2x) e^{-\frac{\pi}{2}+x}$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
<- constant coefficients successful`
```

✓ Solution by Maple

Time used: 0.031 (sec). Leaf size: 16

```
dsolve([diff(y(x),x$2)- 2*diff(y(x),x)+5*y(x) = 0,y(1/2*Pi) = 0, D(y)(1/2*Pi) = 2],y(x), sin
```

$$y(x) = -\sin(2x)\,\mathrm{e}^{-\frac{\pi}{2}+x}$$

✓ Solution by Mathematica

Time used: 0.015 (sec). Leaf size: 20

```
DSolve[{y''[x]-2*y'[x]+5*y[x]==0,{y[Pi/2]==0,y'[Pi/2]==2}},y[x],x,IncludeSingularSolutions -
```

$$y(x) \to -e^{x-\frac{\pi}{2}}\sin(2x)$$

8.14 problem 20

Internal problem ID [636]
Internal file name [`OUTPUT/636_Sunday_June_05_2022_01_46_11_AM_56093143/index.tex`]

Book: Elementary differential equations and boundary value problems, 10th ed., Boyce and DiPrima
Section: Chapter 3, Second order linear equations, 3.3 Complex Roots of the Characteristic Equation , page 164
Problem number: 20.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_linear_constant_coeff", "second_order_ode_can_be_made_integrable"**

Maple gives the following as the ode type

```
[[_2nd_order, _missing_x]]
```

$$\boxed{y'' + y = 0}$$

With initial conditions

$$\left[y\left(\frac{\pi}{3}\right) = 2, y'\left(\frac{\pi}{3}\right) = -4\right]$$

8.14.1 Existence and uniqueness analysis

This is a linear ODE. In canonical form it is written as

$$y'' + p(x)y' + q(x)y = F$$

Where here

$$\begin{aligned} p(x) &= 0 \\ q(x) &= 1 \\ F &= 0 \end{aligned}$$

Hence the ode is

$$y'' + y = 0$$

The domain of $p(x) = 0$ is

$$\{-\infty < x < \infty\}$$

And the point $x_0 = \frac{\pi}{3}$ is inside this domain. The domain of $q(x) = 1$ is

$$\{-\infty < x < \infty\}$$

And the point $x_0 = \frac{\pi}{3}$ is also inside this domain. Hence solution exists and is unique.

8.14.2 Solving as second order linear constant coeff ode

This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = 0$$

Where in the above $A = 1, B = 0, C = 1$. Let the solution be $y = e^{\lambda x}$. Substituting this into the ODE gives

$$\lambda^2 \mathrm{e}^{\lambda x} + \mathrm{e}^{\lambda x} = 0 \tag{1}$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda x}$ gives

$$\lambda^2 + 1 = 0 \tag{2}$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form.Using the quadratic formula

$$\lambda_{1,2} = \frac{-B}{2A} \pm \frac{1}{2A}\sqrt{B^2 - 4AC}$$

Substituting $A = 1, B = 0, C = 1$ into the above gives

$$\begin{aligned}\lambda_{1,2} &= \frac{0}{(2)(1)} \pm \frac{1}{(2)(1)}\sqrt{0^2 - (4)(1)(1)} \\ &= \pm i\end{aligned}$$

Hence

$$\begin{aligned}\lambda_1 &= +i \\ \lambda_2 &= -i\end{aligned}$$

Which simplifies to

$$\lambda_1 = i$$
$$\lambda_2 = -i$$

Since roots are complex conjugate of each others, then let the roots be

$$\lambda_{1,2} = \alpha \pm i\beta$$

Where $\alpha = 0$ and $\beta = 1$. Therefore the final solution, when using Euler relation, can be written as

$$y = e^{\alpha x}(c_1 \cos(\beta x) + c_2 \sin(\beta x))$$

Which becomes

$$y = e^{0}(c_1 \cos(x) + c_2 \sin(x))$$

Or

$$y = c_1 \cos(x) + c_2 \sin(x)$$

Initial conditions are used to solve for the constants of integration.

Looking at the above solution

$$y = c_1 \cos(x) + c_2 \sin(x) \tag{1}$$

Initial conditions are now substituted in the above solution. This will generate the required equations to solve for the integration constants. substituting $y = 2$ and $x = \frac{\pi}{3}$ in the above gives

$$2 = \frac{c_1}{2} + \frac{\sqrt{3}\, c_2}{2} \tag{1A}$$

Taking derivative of the solution gives

$$y' = -c_1 \sin(x) + c_2 \cos(x)$$

substituting $y' = -4$ and $x = \frac{\pi}{3}$ in the above gives

$$-4 = -\frac{\sqrt{3}\, c_1}{2} + \frac{c_2}{2} \tag{2A}$$

Equations {1A,2A} are now solved for $\{c_1, c_2\}$. Solving for the constants gives

$$
\begin{aligned}
c_1 &= 2\sqrt{3}+1 \\
c_2 &= \sqrt{3}-2
\end{aligned}
$$

Substituting these values back in above solution results in

$$
y = 2\cos(x)\sqrt{3}+\sin(x)\sqrt{3}+\cos(x)-2\sin(x)
$$

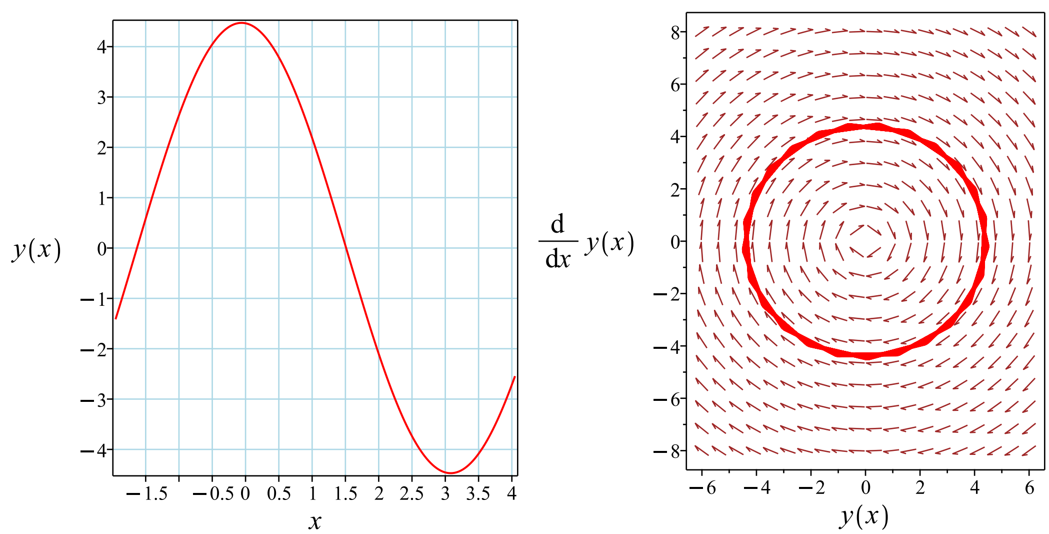
Which simplifies to

$$
y = (2\cos(x)+\sin(x))\sqrt{3}+\cos(x)-2\sin(x)
$$

Summary

The solution(s) found are the following

$$
y = (2\cos(x)+\sin(x))\sqrt{3}+\cos(x)-2\sin(x) \tag{1}
$$



(a) Solution plot

(b) Slope field plot

Verification of solutions

$$
y = (2\cos(x)+\sin(x))\sqrt{3}+\cos(x)-2\sin(x)
$$

Verified OK.

8.14.3 Solving as second order ode can be made integrable ode

Multiplying the ode by y' gives

$$y'y'' + yy' = 0$$

Integrating the above w.r.t x gives

$$\int (y'y'' + yy')\, dx = 0$$

$$\frac{y'^2}{2} + \frac{y^2}{2} = c_2$$

Which is now solved for y. Solving the given ode for y' results in 2 differential equations to solve. Each one of these will generate a solution. The equations generated are

$$y' = \sqrt{-y^2 + 2c_1} \tag{1}$$

$$y' = -\sqrt{-y^2 + 2c_1} \tag{2}$$

Now each one of the above ODE is solved.

Solving equation (1)

Integrating both sides gives

$$\int \frac{1}{\sqrt{-y^2 + 2c_1}} dy = \int dx$$

$$\arctan\left(\frac{y}{\sqrt{-y^2 + 2c_1}}\right) = x + c_2$$

Solving equation (2)

Integrating both sides gives

$$\int -\frac{1}{\sqrt{-y^2 + 2c_1}} dy = \int dx$$

$$-\arctan\left(\frac{y}{\sqrt{-y^2 + 2c_1}}\right) = c_3 + x$$

Initial conditions are used to solve for the constants of integration.

Looking at the First solution

$$\arctan\left(\frac{y}{\sqrt{-y^2 + 2c_1}}\right) = x + c_2 \tag{1}$$

Initial conditions are now substituted in the above solution. This will generate the required equations to solve for the integration constants. substituting $y = 2$ and $x = \frac{\pi}{3}$ in the above gives

$$\arctan\left(\frac{2}{\sqrt{-4+2c_1}}\right) = \frac{\pi}{3} + c_2 \tag{1A}$$

Taking derivative of the solution gives

$$y' = \left(\tan\left(x+c_2\right)^2+1\right)\sqrt{2}\,\sqrt{\frac{c_1}{\tan\left(x+c_2\right)^2+1}} - \frac{\tan\left(x+c_2\right)^2\sqrt{2}\,c_1}{\sqrt{\frac{c_1}{\tan(x+c_2)^2+1}}\,\left(\tan\left(x+c_2\right)^2+1\right)}$$

substituting $y' = -4$ and $x = \frac{\pi}{3}$ in the above gives

$$-4 = \frac{\cos\left(\frac{\pi}{3}+c_2\right)^2\sqrt{2}\,c_1}{\sqrt{\cos\left(\frac{\pi}{3}+c_2\right)^2 c_1}} \tag{2A}$$

Equations {1A,2A} are now solved for $\{c_1, c_2\}$. There is no solution for the constants of integrations. This solution is removed.

Looking at the Second solution

$$-\arctan\left(\frac{y}{\sqrt{-y^2+2c_1}}\right) = c_3 + x \tag{2}$$

Initial conditions are now substituted in the above solution. This will generate the required equations to solve for the integration constants. substituting $y = 2$ and $x = \frac{\pi}{3}$ in the above gives

$$-\arctan\left(\frac{2}{\sqrt{-4+2c_1}}\right) = c_3 + \frac{\pi}{3} \tag{1A}$$

Taking derivative of the solution gives

$$y' = -\left(\tan\left(c_3+x\right)^2+1\right)\sqrt{2}\,\sqrt{\frac{c_1}{\tan\left(c_3+x\right)^2+1}} + \frac{\tan\left(c_3+x\right)^2\sqrt{2}\,c_1}{\sqrt{\frac{c_1}{\tan(c_3+x)^2+1}}\,\left(\tan\left(c_3+x\right)^2+1\right)}$$

substituting $y' = -4$ and $x = \frac{\pi}{3}$ in the above gives

$$-4 = -\frac{\cos\left(c_3+\frac{\pi}{3}\right)^2\sqrt{2}\,c_1}{\sqrt{\cos\left(c_3+\frac{\pi}{3}\right)^2 c_1}} \tag{2A}$$

Equations {1A,2A} are now solved for $\{c_1, c_3\}$. Solving for the constants gives

$$
\begin{aligned}
c_1 &= 10 \\
c_3 &= -\arctan\left(\frac{1}{2}\right) - \frac{\pi}{3}
\end{aligned}
$$

Substituting these values back in above solution results in

$$
-\arctan\left(\frac{y}{\sqrt{-y^2+20}}\right) = -\arctan\left(\frac{1}{2}\right) - \frac{\pi}{3} + x
$$

Summary

The solution(s) found are the following

$$
-\arctan\left(\frac{y}{\sqrt{-y^2+20}}\right) = -\arctan\left(\frac{1}{2}\right) - \frac{\pi}{3} + x \tag{1}
$$

Verification of solutions

$$
-\arctan\left(\frac{y}{\sqrt{-y^2+20}}\right) = -\arctan\left(\frac{1}{2}\right) - \frac{\pi}{3} + x
$$

Verified OK.

8.14.4 Solving using Kovacic algorithm

Writing the ode as

$$
y'' + y = 0 \tag{1}
$$

$$
Ay'' + By' + Cy = 0 \tag{2}
$$

Comparing (1) and (2) shows that

$$
\begin{aligned}
A &= 1 \\
B &= 0 \\
C &= 1
\end{aligned} \tag{3}
$$

Applying the Liouville transformation on the dependent variable gives

$$
z(x) = ye^{\int \frac{B}{2A}\,dx}
$$

Then (2) becomes

$$
z''(x) = rz(x) \tag{4}
$$

Where r is given by

$$r = \frac{s}{t} = \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{-1}{1} \tag{6}$$

Comparing the above to (5) shows that

$$s = -1$$
$$t = 1$$

Therefore eq. (4) becomes

$$z''(x) = -z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 390: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 0 - 0 \\ &= 0 \end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Since $r = -1$ is not a function of x, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$z_1(x) = \cos(x)$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$y_1 = z_1 e^{\int -\frac{1}{2}\frac{B}{A}\, dx}$$

Since $B = 0$ then the above reduces to

$$\begin{aligned} y_1 &= z_1 \\ &= \cos(x) \end{aligned}$$

Which simplifies to

$$y_1 = \cos(x)$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\, dx}}{y_1^2}\, dx$$

Since $B = 0$ then the above becomes

$$\begin{aligned} y_2 &= y_1 \int \frac{1}{y_1^2}\, dx \\ &= \cos(x) \int \frac{1}{\cos(x)^2}\, dx \\ &= \cos(x)(\tan(x)) \end{aligned}$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1(\cos(x)) + c_2(\cos(x)(\tan(x)))
\end{aligned}
$$

Initial conditions are used to solve for the constants of integration.

Looking at the above solution

$$y = c_1 \cos(x) + c_2 \sin(x) \tag{1}$$

Initial conditions are now substituted in the above solution. This will generate the required equations to solve for the integration constants. substituting $y = 2$ and $x = \frac{\pi}{3}$ in the above gives

$$2 = \frac{c_1}{2} + \frac{\sqrt{3}\, c_2}{2} \tag{1A}$$

Taking derivative of the solution gives

$$y' = -c_1 \sin(x) + c_2 \cos(x)$$

substituting $y' = -4$ and $x = \frac{\pi}{3}$ in the above gives

$$-4 = -\frac{\sqrt{3}\, c_1}{2} + \frac{c_2}{2} \tag{2A}$$

Equations {1A,2A} are now solved for $\{c_1, c_2\}$. Solving for the constants gives

$$
\begin{aligned}
c_1 &= 2\sqrt{3} + 1 \\
c_2 &= \sqrt{3} - 2
\end{aligned}
$$

Substituting these values back in above solution results in

$$y = 2\cos(x)\sqrt{3} + \sin(x)\sqrt{3} + \cos(x) - 2\sin(x)$$

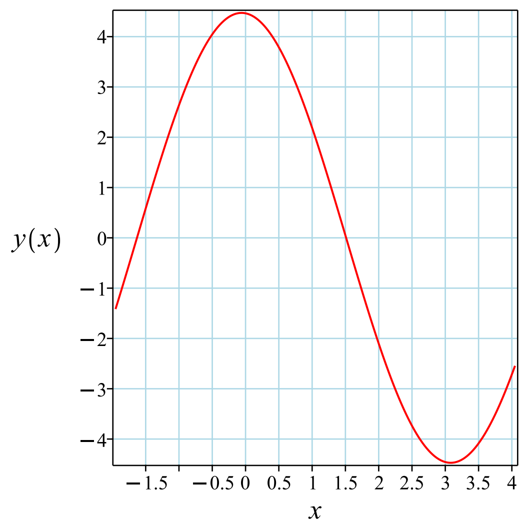
Which simplifies to

$$y = (2\cos(x) + \sin(x))\sqrt{3} + \cos(x) - 2\sin(x)$$

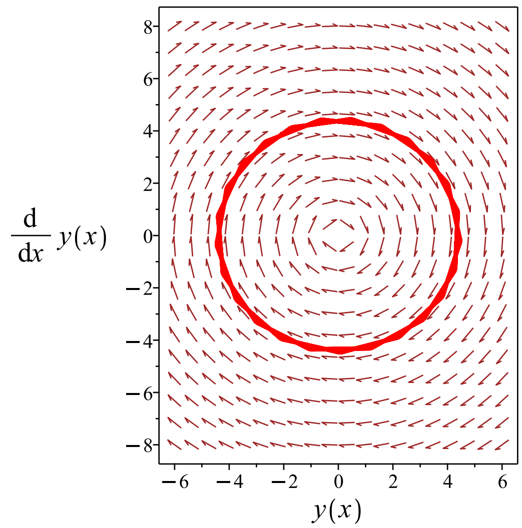
Summary

The solution(s) found are the following

$$y = (2\cos(x) + \sin(x))\sqrt{3} + \cos(x) - 2\sin(x) \tag{1}$$



(a) Solution plot



(b) Slope field plot

<u>Verification of solutions</u>

$$y = (2\cos(x) + \sin(x))\sqrt{3} + \cos(x) - 2\sin(x)$$

Verified OK.

8.14.5 Maple step by step solution

Let's solve

$$\left[y'' + y = 0, y\left(\frac{\pi}{3}\right) = 2, y'\Big|_{\{x=\frac{\pi}{3}\}} = -4\right]$$

- Highest derivative means the order of the ODE is 2

  y''
- Characteristic polynomial of ODE

  $r^2 + 1 = 0$
- Use quadratic formula to solve for r

  $r = \frac{0 \pm (\sqrt{-4})}{2}$
- Roots of the characteristic polynomial

  $r = (-\mathrm{I}, \mathrm{I})$
- 1st solution of the ODE

$y_1(x) = \cos(x)$

- 2nd solution of the ODE

  $y_2(x) = \sin(x)$

- General solution of the ODE

  $y = c_1 y_1(x) + c_2 y_2(x)$

- Substitute in solutions

  $y = c_1 \cos(x) + c_2 \sin(x)$

☐ Check validity of solution $y = c_1 \cos(x) + c_2 \sin(x)$

  - Use initial condition $y\left(\frac{\pi}{3}\right) = 2$

    $2 = \frac{c_1}{2} + \frac{\sqrt{3}\, c_2}{2}$

  - Compute derivative of the solution

    $y' = -c_1 \sin(x) + c_2 \cos(x)$

  - Use the initial condition $y'\Big|_{\left\{x=\frac{\pi}{3}\right\}} = -4$

    $-4 = -\frac{\sqrt{3}\, c_1}{2} + \frac{c_2}{2}$

  - Solve for c_1 and c_2

    $\left\{c_1 = 2\sqrt{3} + 1, c_2 = \sqrt{3} - 2\right\}$

  - Substitute constant values into general solution and simplify

    $y = (2\cos(x) + \sin(x))\sqrt{3} + \cos(x) - 2\sin(x)$

- Solution to the IVP

  $y = (2\cos(x) + \sin(x))\sqrt{3} + \cos(x) - 2\sin(x)$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
<- constant coefficients successful`
```

✓ Solution by Maple

Time used: 0.015 (sec). Leaf size: 23

```
dsolve([diff(y(x),x$2)+y(x) = 0,y(1/3*Pi) = 2, D(y)(1/3*Pi) = -4],y(x), singsol=all)
```

$$y(x) = (\sin(x) + 2\cos(x))\sqrt{3} + \cos(x) - 2\sin(x)$$

✓ Solution by Mathematica

Time used: 0.012 (sec). Leaf size: 28

```
DSolve[{y''[x]+y[x]==0,{y[Pi/3]==2,y'[Pi/3]==-4}},y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \left(\sqrt{3} - 2\right)\sin(x) + \left(1 + 2\sqrt{3}\right)\cos(x)$$

8.15 problem 21

Internal problem ID [637]
Internal file name [`OUTPUT/637_Sunday_June_05_2022_01_46_12_AM_12825249/index.tex`]

Book: Elementary differential equations and boundary value problems, 10th ed., Boyce and DiPrima
Section: Chapter 3, Second order linear equations, 3.3 Complex Roots of the Characteristic Equation , page 164
Problem number: 21.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_linear_constant_coeff"**

Maple gives the following as the ode type

`[[_2nd_order, _missing_x]]`

$$\boxed{y'' + y' + \frac{5y}{4} = 0}$$

With initial conditions

$$[y(0) = 3, y'(0) = 1]$$

8.15.1 Existence and uniqueness analysis

This is a linear ODE. In canonical form it is written as

$$y'' + p(x)y' + q(x)y = F$$

Where here

$$p(x) = 1$$
$$q(x) = \frac{5}{4}$$
$$F = 0$$

Hence the ode is

$$y'' + y' + \frac{5y}{4} = 0$$

The domain of $p(x) = 1$ is

$$\{-\infty < x < \infty\}$$

And the point $x_0 = 0$ is inside this domain. The domain of $q(x) = \frac{5}{4}$ is

$$\{-\infty < x < \infty\}$$

And the point $x_0 = 0$ is also inside this domain. Hence solution exists and is unique.

8.15.2 Solving as second order linear constant coeff ode

This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = 0$$

Where in the above $A = 1, B = 1, C = \frac{5}{4}$. Let the solution be $y = e^{\lambda x}$. Substituting this into the ODE gives

$$\lambda^2 e^{\lambda x} + \lambda e^{\lambda x} + \frac{5 e^{\lambda x}}{4} = 0 \tag{1}$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda x}$ gives

$$\lambda^2 + \lambda + \frac{5}{4} = 0 \tag{2}$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form.Using the quadratic formula

$$\lambda_{1,2} = \frac{-B}{2A} \pm \frac{1}{2A}\sqrt{B^2 - 4AC}$$

Substituting $A = 1, B = 1, C = \frac{5}{4}$ into the above gives

$$\begin{aligned}
\lambda_{1,2} &= \frac{-1}{(2)(1)} \pm \frac{1}{(2)(1)}\sqrt{1^2 - (4)(1)\left(\frac{5}{4}\right)} \\
&= -\frac{1}{2} \pm i
\end{aligned}$$

Hence

$$\lambda_1 = -\frac{1}{2} + i$$
$$\lambda_2 = -\frac{1}{2} - i$$

Which simplifies to

$$\lambda_1 = -\frac{1}{2} + i$$
$$\lambda_2 = -\frac{1}{2} - i$$

Since roots are complex conjugate of each others, then let the roots be

$$\lambda_{1,2} = \alpha \pm i\beta$$

Where $\alpha = -\frac{1}{2}$ and $\beta = 1$. Therefore the final solution, when using Euler relation, can be written as

$$y = e^{\alpha x}(c_1 \cos(\beta x) + c_2 \sin(\beta x))$$

Which becomes

$$y = e^{-\frac{x}{2}}(c_1 \cos(x) + c_2 \sin(x))$$

Initial conditions are used to solve for the constants of integration.

Looking at the above solution

$$y = \mathrm{e}^{-\frac{x}{2}}(c_1 \cos(x) + c_2 \sin(x)) \tag{1}$$

Initial conditions are now substituted in the above solution. This will generate the required equations to solve for the integration constants. substituting $y = 3$ and $x = 0$ in the above gives

$$3 = c_1 \tag{1A}$$

Taking derivative of the solution gives

$$y' = -\frac{\mathrm{e}^{-\frac{x}{2}}(c_1 \cos(x) + c_2 \sin(x))}{2} + \mathrm{e}^{-\frac{x}{2}}(-c_1 \sin(x) + c_2 \cos(x))$$

substituting $y' = 1$ and $x = 0$ in the above gives

$$1 = -\frac{c_1}{2} + c_2 \tag{2A}$$

Equations {1A,2A} are now solved for $\{c_1, c_2\}$. Solving for the constants gives

$$c_1 = 3$$
$$c_2 = \frac{5}{2}$$

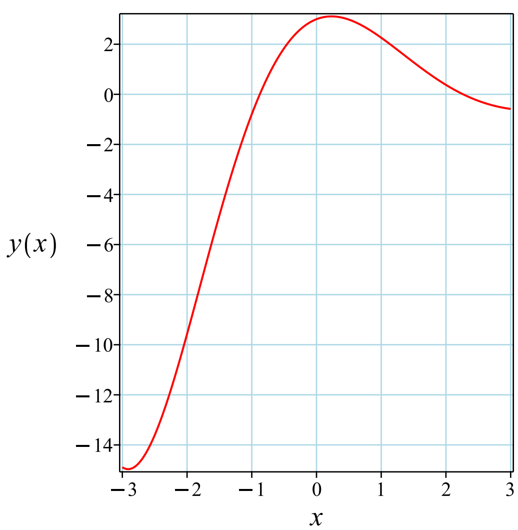
Substituting these values back in above solution results in

$$y = \frac{e^{-\frac{x}{2}}(6\cos(x) + 5\sin(x))}{2}$$

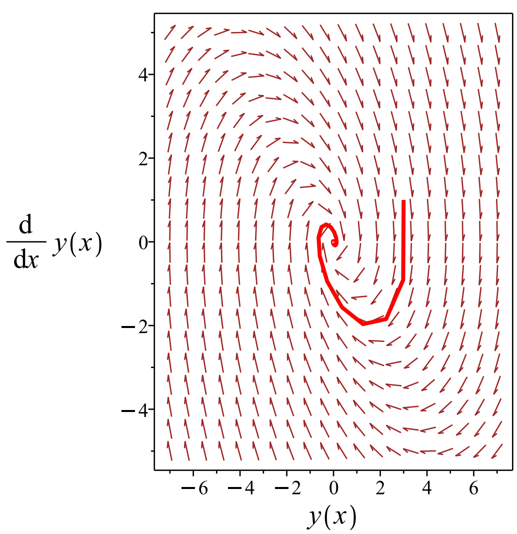
Summary

The solution(s) found are the following

$$y = \frac{e^{-\frac{x}{2}}(6\cos(x) + 5\sin(x))}{2} \tag{1}$$



(a) Solution plot



(b) Slope field plot

Verification of solutions

$$y = \frac{e^{-\frac{x}{2}}(6\cos(x) + 5\sin(x))}{2}$$

Verified OK.

8.15.3 Solving using Kovacic algorithm

Writing the ode as

$$y'' + y' + \frac{5y}{4} = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1 \\ B &= 1 \\ C &= \frac{5}{4} \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A} \, dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{-1}{1} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= -1 \\ t &= 1 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = -z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x) \, e^{-\int \frac{B}{2A} \, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 392: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned}
O(\infty) &= \deg(t) - \deg(s) \\
&= 0 - 0 \\
&= 0
\end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Since $r = -1$ is not a function of x, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$z_1(x) = \cos(x)$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{1}{1}\,dx}
\end{aligned}$$

$$
\begin{aligned}
&= z_1 e^{-\frac{x}{2}} \\
&= z_1 \left(\mathrm{e}^{-\frac{x}{2}}\right)
\end{aligned}
$$

Which simplifies to

$$
y_1 = \mathrm{e}^{-\frac{x}{2}} \cos(x)
$$

The second solution y_2 to the original ode is found using reduction of order

$$
y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\, dx}}{y_1^2} \, dx
$$

Substituting gives

$$
\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{1}{1}\, dx}}{(y_1)^2} \, dx \\
&= y_1 \int \frac{e^{-x}}{(y_1)^2} \, dx \\
&= y_1 (\tan(x))
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1 \left(\mathrm{e}^{-\frac{x}{2}} \cos(x)\right) + c_2 \left(\mathrm{e}^{-\frac{x}{2}} \cos(x) (\tan(x))\right)
\end{aligned}
$$

Initial conditions are used to solve for the constants of integration.

Looking at the above solution

$$
y = c_1 \mathrm{e}^{-\frac{x}{2}} \cos(x) + c_2 \mathrm{e}^{-\frac{x}{2}} \sin(x) \tag{1}
$$

Initial conditions are now substituted in the above solution. This will generate the required equations to solve for the integration constants. substituting $y = 3$ and $x = 0$ in the above gives

$$
3 = c_1 \tag{1A}
$$

Taking derivative of the solution gives

$$
y' = -\frac{c_1 \mathrm{e}^{-\frac{x}{2}} \cos(x)}{2} - c_1 \mathrm{e}^{-\frac{x}{2}} \sin(x) - \frac{c_2 \mathrm{e}^{-\frac{x}{2}} \sin(x)}{2} + c_2 \mathrm{e}^{-\frac{x}{2}} \cos(x)
$$

substituting $y' = 1$ and $x = 0$ in the above gives

$$1 = -\frac{c_1}{2} + c_2 \tag{2A}$$

Equations {1A,2A} are now solved for $\{c_1, c_2\}$. Solving for the constants gives

$$c_1 = 3$$
$$c_2 = \frac{5}{2}$$

Substituting these values back in above solution results in

$$y = \frac{5\,\mathrm{e}^{-\frac{x}{2}} \sin(x)}{2} + 3\,\mathrm{e}^{-\frac{x}{2}} \cos(x)$$

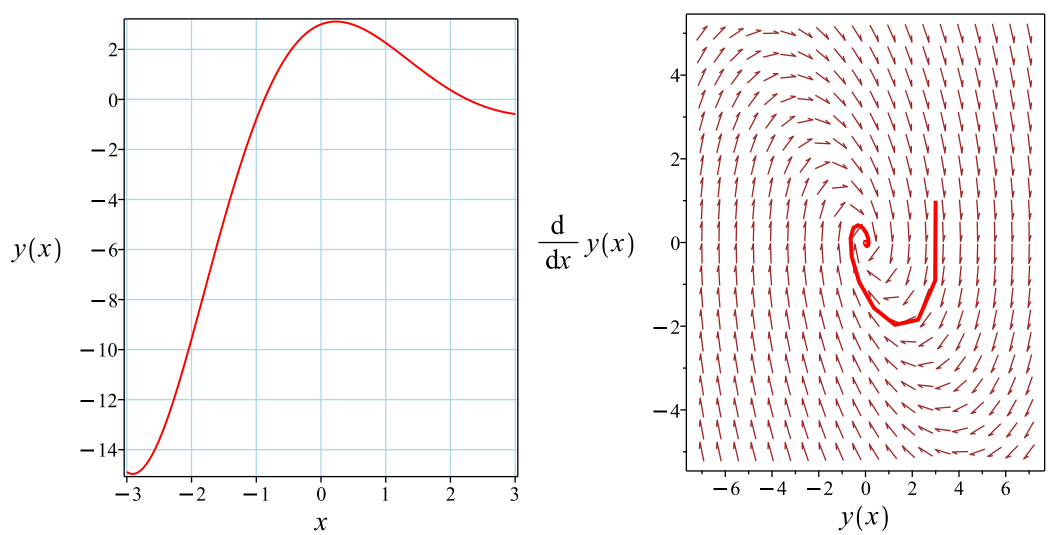
Which simplifies to

$$y = \frac{\mathrm{e}^{-\frac{x}{2}} (6\cos(x) + 5\sin(x))}{2}$$

<u>Summary</u>

The solution(s) found are the following

$$y = \frac{\mathrm{e}^{-\frac{x}{2}} (6\cos(x) + 5\sin(x))}{2} \tag{1}$$



(a) Solution plot

(b) Slope field plot

<u>Verification of solutions</u>

$$y = \frac{\mathrm{e}^{-\frac{x}{2}} (6\cos(x) + 5\sin(x))}{2}$$

Verified OK.

8.15.4 Maple step by step solution

Let's solve

$$\left[y''+y'+\frac{5y}{4}=0, y(0)=3, y'\Big|_{\{x=0\}}=1\right]$$

- Highest derivative means the order of the ODE is 2

  y''

- Characteristic polynomial of ODE

  $r^2+r+\frac{5}{4}=0$

- Use quadratic formula to solve for r

  $r=\frac{(-1)\pm(\sqrt{-4})}{2}$

- Roots of the characteristic polynomial

  $r=\left(-\frac{1}{2}-\mathrm{I}, -\frac{1}{2}+\mathrm{I}\right)$

- 1st solution of the ODE

  $y_1(x)=\mathrm{e}^{-\frac{x}{2}}\cos(x)$

- 2nd solution of the ODE

  $y_2(x)=\mathrm{e}^{-\frac{x}{2}}\sin(x)$

- General solution of the ODE

  $y=c_1y_1(x)+c_2y_2(x)$

- Substitute in solutions

  $y=c_1\mathrm{e}^{-\frac{x}{2}}\cos(x)+c_2\mathrm{e}^{-\frac{x}{2}}\sin(x)$

☐ Check validity of solution $y=c_1\mathrm{e}^{-\frac{x}{2}}\cos(x)+c_2\mathrm{e}^{-\frac{x}{2}}\sin(x)$

  - Use initial condition $y(0)=3$

    $3=c_1$

  - Compute derivative of the solution

    $y'=-\frac{c_1\mathrm{e}^{-\frac{x}{2}}\cos(x)}{2}-c_1\mathrm{e}^{-\frac{x}{2}}\sin(x)-\frac{c_2\mathrm{e}^{-\frac{x}{2}}\sin(x)}{2}+c_2\mathrm{e}^{-\frac{x}{2}}\cos(x)$

  - Use the initial condition $y'\Big|_{\{x=0\}}=1$

    $1=-\frac{c_1}{2}+c_2$

  - Solve for c_1 and c_2

$\left\{c_1 = 3, c_2 = \frac{5}{2}\right\}$

- Substitute constant values into general solution and simplify

$$y = \frac{e^{-\frac{x}{2}}(6\cos(x)+5\sin(x))}{2}$$

- Solution to the IVP

$$y = \frac{e^{-\frac{x}{2}}(6\cos(x)+5\sin(x))}{2}$$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
<- constant coefficients successful`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 19

```
dsolve([diff(y(x),x$2)+ diff(y(x),x)+125/100*y(x) = 0,y(0) = 3, D(y)(0) = 1],y(x), singsol=a
```

$$y(x) = \frac{e^{-\frac{x}{2}}(5\sin(x) + 6\cos(x))}{2}$$

✓ Solution by Mathematica

Time used: 0.016 (sec). Leaf size: 25

```
DSolve[{y''[x]+y'[x]+125/100*y[x]==0,{y[0]==3,y'[0]==1}},y[x],x,IncludeSingularSolutions ->
```

$$y(x) \to \frac{1}{2}e^{-x/2}(5\sin(x) + 6\cos(x))$$

8.16 problem 22

Internal problem ID [638]
Internal file name [`OUTPUT/638_Sunday_June_05_2022_01_46_13_AM_47679787/index.tex`]

Book: Elementary differential equations and boundary value problems, 10th ed., Boyce and DiPrima
Section: Chapter 3, Second order linear equations, 3.3 Complex Roots of the Characteristic Equation , page 164
Problem number: 22.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_linear_constant_coeff"**

Maple gives the following as the ode type

```
[[_2nd_order, _missing_x]]
```

$$\boxed{y'' + 2y' + 2y = 0}$$

With initial conditions

$$\left[y\left(\frac{\pi}{4}\right) = 2, y'\left(\frac{\pi}{4}\right) = -2\right]$$

8.16.1 Existence and uniqueness analysis

This is a linear ODE. In canonical form it is written as

$$y'' + p(x)y' + q(x)y = F$$

Where here

$$\begin{aligned} p(x) &= 2 \\ q(x) &= 2 \\ F &= 0 \end{aligned}$$

Hence the ode is

$$y'' + 2y' + 2y = 0$$

The domain of $p(x) = 2$ is

$$\{-\infty < x < \infty\}$$

And the point $x_0 = \frac{\pi}{4}$ is inside this domain. The domain of $q(x) = 2$ is

$$\{-\infty < x < \infty\}$$

And the point $x_0 = \frac{\pi}{4}$ is also inside this domain. Hence solution exists and is unique.

8.16.2 Solving as second order linear constant coeff ode

This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = 0$$

Where in the above $A = 1, B = 2, C = 2$. Let the solution be $y = e^{\lambda x}$. Substituting this into the ODE gives

$$\lambda^2 e^{\lambda x} + 2\lambda\, e^{\lambda x} + 2\, e^{\lambda x} = 0 \tag{1}$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda x}$ gives

$$\lambda^2 + 2\lambda + 2 = 0 \tag{2}$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form.Using the quadratic formula

$$\lambda_{1,2} = \frac{-B}{2A} \pm \frac{1}{2A}\sqrt{B^2 - 4AC}$$

Substituting $A = 1, B = 2, C = 2$ into the above gives

$$\begin{aligned}\lambda_{1,2} &= \frac{-2}{(2)\,(1)} \pm \frac{1}{(2)\,(1)}\sqrt{2^2 - (4)\,(1)\,(2)}\\ &= -1 \pm i\end{aligned}$$

Hence

$$\begin{aligned}\lambda_1 &= -1 + i\\ \lambda_2 &= -1 - i\end{aligned}$$

Which simplifies to

$$\lambda_1 = -1 + i$$
$$\lambda_2 = -1 - i$$

Since roots are complex conjugate of each others, then let the roots be

$$\lambda_{1,2} = \alpha \pm i\beta$$

Where $\alpha = -1$ and $\beta = 1$. Therefore the final solution, when using Euler relation, can be written as

$$y = e^{\alpha x}(c_1 \cos(\beta x) + c_2 \sin(\beta x))$$

Which becomes

$$y = e^{-x}(c_1 \cos(x) + c_2 \sin(x))$$

Initial conditions are used to solve for the constants of integration.

Looking at the above solution

$$y = \mathrm{e}^{-x}(c_1 \cos(x) + c_2 \sin(x)) \tag{1}$$

Initial conditions are now substituted in the above solution. This will generate the required equations to solve for the integration constants. substituting $y = 2$ and $x = \frac{\pi}{4}$ in the above gives

$$2 = \frac{(c_1 + c_2)\sqrt{2}\,\mathrm{e}^{-\frac{\pi}{4}}}{2} \tag{1A}$$

Taking derivative of the solution gives

$$y' = -\mathrm{e}^{-x}(c_1 \cos(x) + c_2 \sin(x)) + \mathrm{e}^{-x}(-c_1 \sin(x) + c_2 \cos(x))$$

substituting $y' = -2$ and $x = \frac{\pi}{4}$ in the above gives

$$-2 = -\sqrt{2}\,\mathrm{e}^{-\frac{\pi}{4}} c_1 \tag{2A}$$

Equations {1A,2A} are now solved for $\{c_1, c_2\}$. Solving for the constants gives

$$c_1 = \sqrt{2}\,\mathrm{e}^{\frac{\pi}{4}}$$
$$c_2 = \sqrt{2}\,\mathrm{e}^{\frac{\pi}{4}}$$

Substituting these values back in above solution results in

$$y = \sqrt{2}\, \cos(x)\, e^{\frac{\pi}{4}-x} + \sqrt{2}\, \sin(x)\, e^{\frac{\pi}{4}-x}$$

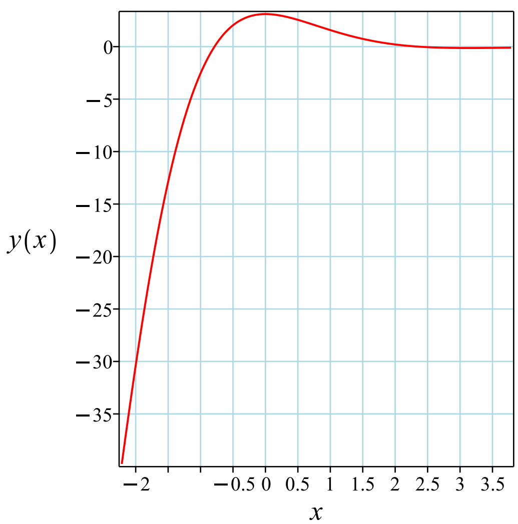
Which simplifies to

$$y = \sqrt{2}\, e^{\frac{\pi}{4}-x}(\cos(x) + \sin(x))$$

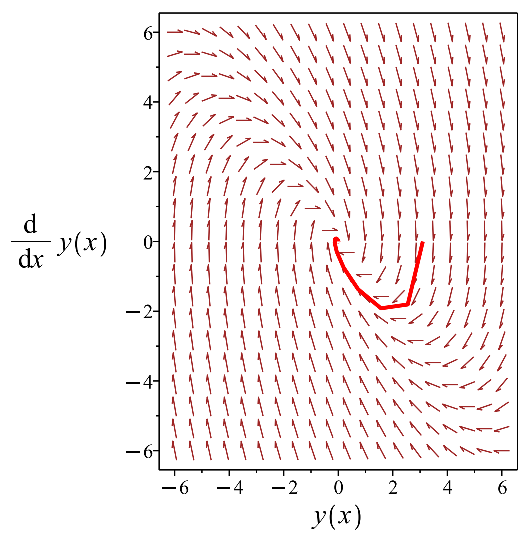
Summary

The solution(s) found are the following

$$y = \sqrt{2}\, e^{\frac{\pi}{4}-x}(\cos(x) + \sin(x)) \tag{1}$$



(a) Solution plot



(b) Slope field plot

Verification of solutions

$$y = \sqrt{2}\, e^{\frac{\pi}{4}-x}(\cos(x) + \sin(x))$$

Verified OK.

8.16.3 Solving using Kovacic algorithm

Writing the ode as

$$y'' + 2y' + 2y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1 \\ B &= 2 \\ C &= 2 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{-1}{1} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= -1 \\ t &= 1 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = -z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\,e^{-\int \frac{B}{2A}\,dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 394: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$
\begin{aligned}
O(\infty) &= \deg(t) - \deg(s) \\
&= 0 - 0 \\
&= 0
\end{aligned}
$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Therefore

$$
L = [1]
$$

Since $r = -1$ is not a function of x, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$
z_1(x) = \cos(x)
$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$
\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{2}{1}\,dx}
\end{aligned}
$$

$$
\begin{aligned}
&= z_1 e^{-x} \\
&= z_1 \left(\mathrm{e}^{-x}\right)
\end{aligned}
$$

Which simplifies to

$$
y_1 = \cos\left(x\right) \mathrm{e}^{-x}
$$

The second solution y_2 to the original ode is found using reduction of order

$$
y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\, dx}}{y_1^2}\, dx
$$

Substituting gives

$$
\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{2}{1}\, dx}}{\left(y_1\right)^2}\, dx \\
&= y_1 \int \frac{e^{-2x}}{\left(y_1\right)^2}\, dx \\
&= y_1 (\tan\left(x\right))
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1 \left(\cos\left(x\right) \mathrm{e}^{-x}\right) + c_2 \left(\cos\left(x\right) \mathrm{e}^{-x} (\tan\left(x\right))\right)
\end{aligned}
$$

Initial conditions are used to solve for the constants of integration.

Looking at the above solution

$$
y = c_1 \cos\left(x\right) \mathrm{e}^{-x} + c_2 \sin\left(x\right) \mathrm{e}^{-x} \tag{1}
$$

Initial conditions are now substituted in the above solution. This will generate the required equations to solve for the integration constants. substituting $y = 2$ and $x = \frac{\pi}{4}$ in the above gives

$$
2 = \frac{\left(c_1 + c_2\right) \sqrt{2}\, \mathrm{e}^{-\frac{\pi}{4}}}{2} \tag{1A}
$$

Taking derivative of the solution gives

$$
y' = -c_1 \sin\left(x\right) \mathrm{e}^{-x} - c_1 \cos\left(x\right) \mathrm{e}^{-x} + c_2 \cos\left(x\right) \mathrm{e}^{-x} - c_2 \sin\left(x\right) \mathrm{e}^{-x}
$$

substituting $y' = -2$ and $x = \frac{\pi}{4}$ in the above gives

$$-2 = -\sqrt{2}\,\mathrm{e}^{-\frac{\pi}{4}} c_1 \tag{2A}$$

Equations {1A,2A} are now solved for $\{c_1, c_2\}$. Solving for the constants gives

$$c_1 = \sqrt{2}\,\mathrm{e}^{\frac{\pi}{4}}$$
$$c_2 = \sqrt{2}\,\mathrm{e}^{\frac{\pi}{4}}$$

Substituting these values back in above solution results in

$$y = \sqrt{2}\,\cos(x)\,\mathrm{e}^{\frac{\pi}{4}-x} + \sqrt{2}\,\sin(x)\,\mathrm{e}^{\frac{\pi}{4}-x}$$

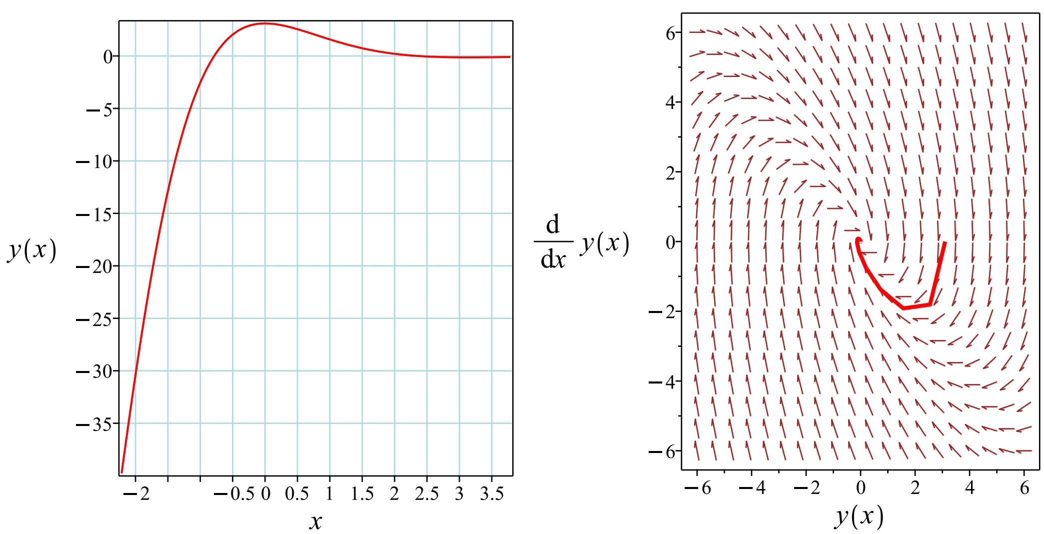
Which simplifies to

$$y = \sqrt{2}\,\mathrm{e}^{\frac{\pi}{4}-x}(\cos(x) + \sin(x))$$

Summary

The solution(s) found are the following

$$y = \sqrt{2}\,\mathrm{e}^{\frac{\pi}{4}-x}(\cos(x) + \sin(x)) \tag{1}$$



(a) Solution plot

(b) Slope field plot

Verification of solutions

$$y = \sqrt{2}\,\mathrm{e}^{\frac{\pi}{4}-x}(\cos(x) + \sin(x))$$

Verified OK.

8.16.4 Maple step by step solution

Let's solve

$$\left[y'' + 2y' + 2y = 0, y\left(\frac{\pi}{4}\right) = 2, y'\bigg|_{\left\{x=\frac{\pi}{4}\right\}} = -2\right]$$

- Highest derivative means the order of the ODE is 2

  y''

- Characteristic polynomial of ODE

  $r^2 + 2r + 2 = 0$

- Use quadratic formula to solve for r

  $r = \frac{(-2)\pm(\sqrt{-4})}{2}$

- Roots of the characteristic polynomial

  $r = (-1 - \mathrm{I}, -1 + \mathrm{I})$

- 1st solution of the ODE

  $y_1(x) = \cos(x)\,\mathrm{e}^{-x}$

- 2nd solution of the ODE

  $y_2(x) = \sin(x)\,\mathrm{e}^{-x}$

- General solution of the ODE

  $y = c_1 y_1(x) + c_2 y_2(x)$

- Substitute in solutions

  $y = c_1 \cos(x)\,\mathrm{e}^{-x} + c_2 \sin(x)\,\mathrm{e}^{-x}$

□ Check validity of solution $y = c_1 \cos(x)\,\mathrm{e}^{-x} + c_2 \sin(x)\,\mathrm{e}^{-x}$

- ◦ Use initial condition $y\left(\frac{\pi}{4}\right) = 2$

  $2 = \frac{\sqrt{2}\,\mathrm{e}^{-\frac{\pi}{4}} c_1}{2} + \frac{\sqrt{2}\,\mathrm{e}^{-\frac{\pi}{4}} c_2}{2}$

- ◦ Compute derivative of the solution

  $y' = -c_1 \sin(x)\,\mathrm{e}^{-x} - c_1 \cos(x)\,\mathrm{e}^{-x} + c_2 \cos(x)\,\mathrm{e}^{-x} - c_2 \sin(x)\,\mathrm{e}^{-x}$

- ◦ Use the initial condition $y'\Big|_{\left\{x=\frac{\pi}{4}\right\}} = -2$

  $-2 = -\sqrt{2}\,\mathrm{e}^{-\frac{\pi}{4}} c_1$

- ◦ Solve for c_1 and c_2

$$\left\{c_1 = \frac{\sqrt{2}}{e^{-\frac{\pi}{4}}}, c_2 = \frac{\sqrt{2}}{e^{-\frac{\pi}{4}}}\right\}$$

- ○ Substitute constant values into general solution and simplify

  $$y = \sqrt{2}\, e^{\frac{\pi}{4}-x}(\cos(x) + \sin(x))$$

- Solution to the IVP

  $$y = \sqrt{2}\, e^{\frac{\pi}{4}-x}(\cos(x) + \sin(x))$$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
<- constant coefficients successful`
```

✓ Solution by Maple

Time used: 0.047 (sec). Leaf size: 21

```
dsolve([diff(y(x),x$2)+ 2*diff(y(x),x)+2*y(x) = 0,y(1/4*Pi) = 2, D(y)(1/4*Pi) = -2],y(x), si
```

$$y(x) = \sqrt{2}\, e^{-x+\frac{\pi}{4}}(\sin(x) + \cos(x))$$

✓ Solution by Mathematica

Time used: 0.016 (sec). Leaf size: 27

```
DSolve[{y''[x]+2*y'[x]+2*y[x]==0,{y[Pi/4]==2,y'[Pi/4]==-2}},y[x],x,IncludeSingularSolutions
```

$$y(x) \to \sqrt{2} e^{\frac{\pi}{4}-x}(\sin(x) + \cos(x))$$

8.17 problem 23

Internal problem ID [639]
Internal file name [OUTPUT/639_Sunday_June_05_2022_01_46_15_AM_93629760/index.tex]

Book: Elementary differential equations and boundary value problems, 10th ed., Boyce and DiPrima
Section: Chapter 3, Second order linear equations, 3.3 Complex Roots of the Characteristic Equation , page 164
Problem number: 23.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_linear_constant_coeff"**

Maple gives the following as the ode type

`[[_2nd_order, _missing_x]]`

$$\boxed{u'' - u' + 2u = 0}$$

With initial conditions

$$[u(0) = 2, u'(0) = 0]$$

8.17.1 Existence and uniqueness analysis

This is a linear ODE. In canonical form it is written as

$$u'' + p(x)u' + q(x)u = F$$

Where here

$$p(x) = -1$$
$$q(x) = 2$$
$$F = 0$$

Hence the ode is

$$u'' - u' + 2u = 0$$

The domain of $p(x) = -1$ is

$$\{-\infty < x < \infty\}$$

And the point $x_0 = 0$ is inside this domain. The domain of $q(x) = 2$ is

$$\{-\infty < x < \infty\}$$

And the point $x_0 = 0$ is also inside this domain. Hence solution exists and is unique.

8.17.2 Solving as second order linear constant coeff ode

This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$Au''(x) + Bu'(x) + Cu(x) = 0$$

Where in the above $A = 1, B = -1, C = 2$. Let the solution be $u = e^{\lambda x}$. Substituting this into the ODE gives

$$\lambda^2 e^{\lambda x} - \lambda e^{\lambda x} + 2 e^{\lambda x} = 0 \tag{1}$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda x}$ gives

$$\lambda^2 - \lambda + 2 = 0 \tag{2}$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form.Using the quadratic formula

$$\lambda_{1,2} = \frac{-B}{2A} \pm \frac{1}{2A}\sqrt{B^2 - 4AC}$$

Substituting $A = 1, B = -1, C = 2$ into the above gives

$$\begin{aligned}
\lambda_{1,2} &= \frac{1}{(2)(1)} \pm \frac{1}{(2)(1)}\sqrt{-1^2 - (4)(1)(2)} \\
&= \frac{1}{2} \pm \frac{i\sqrt{7}}{2}
\end{aligned}$$

Hence

$$\begin{aligned}
\lambda_1 &= \frac{1}{2} + \frac{i\sqrt{7}}{2} \\
\lambda_2 &= \frac{1}{2} - \frac{i\sqrt{7}}{2}
\end{aligned}$$

Which simplifies to

$$\lambda_1 = \frac{1}{2} + \frac{i\sqrt{7}}{2}$$
$$\lambda_2 = \frac{1}{2} - \frac{i\sqrt{7}}{2}$$

Since roots are complex conjugate of each others, then let the roots be

$$\lambda_{1,2} = \alpha \pm i\beta$$

Where $\alpha = \frac{1}{2}$ and $\beta = \frac{\sqrt{7}}{2}$. Therefore the final solution, when using Euler relation, can be written as

$$u = e^{\alpha x}(c_1 \cos(\beta x) + c_2 \sin(\beta x))$$

Which becomes

$$u = e^{\frac{x}{2}}\left(c_1 \cos\left(\frac{\sqrt{7}\, x}{2}\right) + c_2 \sin\left(\frac{\sqrt{7}\, x}{2}\right)\right)$$

Initial conditions are used to solve for the constants of integration.

Looking at the above solution

$$u = \mathrm{e}^{\frac{x}{2}}\left(c_1 \cos\left(\frac{\sqrt{7}\, x}{2}\right) + c_2 \sin\left(\frac{\sqrt{7}\, x}{2}\right)\right) \tag{1}$$

Initial conditions are now substituted in the above solution. This will generate the required equations to solve for the integration constants. substituting $u = 2$ and $x = 0$ in the above gives

$$2 = c_1 \tag{1A}$$

Taking derivative of the solution gives

$$u' = \frac{\mathrm{e}^{\frac{x}{2}}\left(c_1 \cos\left(\frac{\sqrt{7}\, x}{2}\right) + c_2 \sin\left(\frac{\sqrt{7}\, x}{2}\right)\right)}{2} + \mathrm{e}^{\frac{x}{2}}\left(-\frac{c_1\sqrt{7}\, \sin\left(\frac{\sqrt{7}\, x}{2}\right)}{2} + \frac{c_2\sqrt{7}\, \cos\left(\frac{\sqrt{7}\, x}{2}\right)}{2}\right)$$

substituting $u' = 0$ and $x = 0$ in the above gives

$$0 = \frac{c_1}{2} + \frac{\sqrt{7}\, c_2}{2} \tag{2A}$$

Equations {1A,2A} are now solved for $\{c_1, c_2\}$. Solving for the constants gives

$$c_1 = 2$$
$$c_2 = -\frac{2\sqrt{7}}{7}$$

Substituting these values back in above solution results in

$$u = -\frac{2\sin\left(\frac{\sqrt{7}\,x}{2}\right)\mathrm{e}^{\frac{x}{2}}\sqrt{7}}{7} + 2\,\mathrm{e}^{\frac{x}{2}}\cos\left(\frac{\sqrt{7}\,x}{2}\right)$$

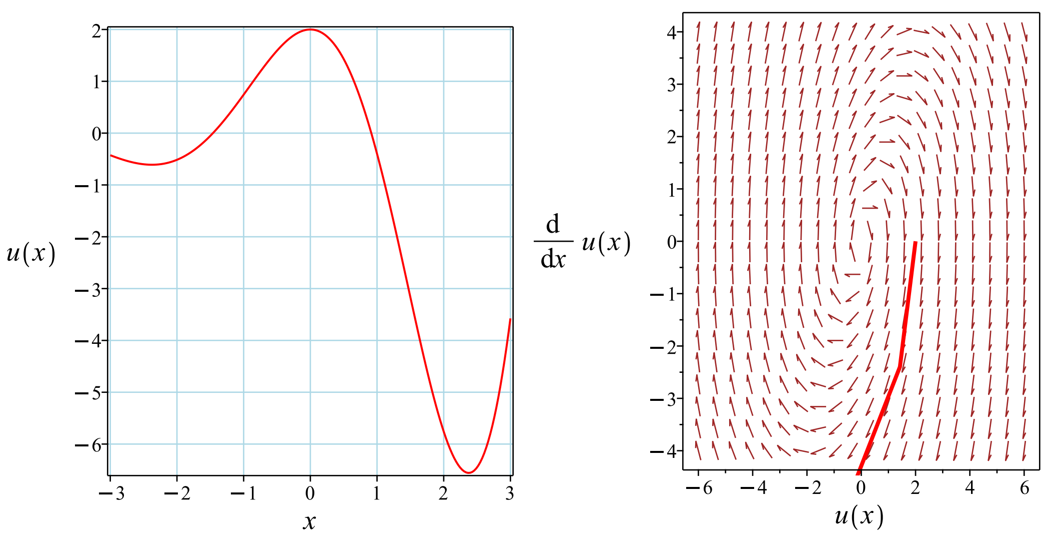
Which simplifies to

$$u = -\frac{2\left(\sqrt{7}\,\sin\left(\frac{\sqrt{7}\,x}{2}\right) - 7\cos\left(\frac{\sqrt{7}\,x}{2}\right)\right)\mathrm{e}^{\frac{x}{2}}}{7}$$

Summary

The solution(s) found are the following

$$u = -\frac{2\left(\sqrt{7}\,\sin\left(\frac{\sqrt{7}\,x}{2}\right) - 7\cos\left(\frac{\sqrt{7}\,x}{2}\right)\right)\mathrm{e}^{\frac{x}{2}}}{7} \tag{1}$$



(a) Solution plot (b) Slope field plot

Verification of solutions

$$u = -\frac{2\left(\sqrt{7}\,\sin\left(\frac{\sqrt{7}\,x}{2}\right) - 7\cos\left(\frac{\sqrt{7}\,x}{2}\right)\right)\mathrm{e}^{\frac{x}{2}}}{7}$$

Verified OK.

8.17.3 Solving using Kovacic algorithm

Writing the ode as

$$u'' - u' + 2u = 0 \tag{1}$$

$$Au'' + Bu' + Cu = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1 \\ B &= -1 \\ C &= 2 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ue^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{-7}{4} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= -7 \\ t &= 4 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = -\frac{7z(x)}{4} \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then u is found using the inverse transformation

$$u = z(x)\,e^{-\int \frac{B}{2A}\,dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 396: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 0 - 0 \\ &= 0 \end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Since $r = -\frac{7}{4}$ is not a function of x, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$z_1(x) = \cos\left(\frac{\sqrt{7}\, x}{2}\right)$$

Using the above, the solution for the original ode can now be found. The first solution

to the original ode in u is found from

$$
\begin{aligned}
u_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{-1}{1}\,dx} \\
&= z_1 e^{\frac{x}{2}} \\
&= z_1 \left(\mathrm{e}^{\frac{x}{2}}\right)
\end{aligned}
$$

Which simplifies to

$$
u_1 = \mathrm{e}^{\frac{x}{2}} \cos\left(\frac{\sqrt{7}\,x}{2}\right)
$$

The second solution u_2 to the original ode is found using reduction of order

$$
u_2 = u_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{u_1^2}\,dx
$$

Substituting gives

$$
\begin{aligned}
u_2 &= u_1 \int \frac{e^{\int -\frac{-1}{1}\,dx}}{\left(u_1\right)^2}\,dx \\
&= u_1 \int \frac{e^{x}}{\left(u_1\right)^2}\,dx \\
&= u_1 \left(\frac{2\sqrt{7}\,\tan\left(\frac{\sqrt{7}\,x}{2}\right)}{7}\right)
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
u &= c_1 u_1 + c_2 u_2 \\
&= c_1 \left(\mathrm{e}^{\frac{x}{2}} \cos\left(\frac{\sqrt{7}\,x}{2}\right)\right) + c_2 \left(\mathrm{e}^{\frac{x}{2}} \cos\left(\frac{\sqrt{7}\,x}{2}\right) \left(\frac{2\sqrt{7}\,\tan\left(\frac{\sqrt{7}\,x}{2}\right)}{7}\right)\right)
\end{aligned}
$$

Initial conditions are used to solve for the constants of integration.

Looking at the above solution

$$u = c_1 e^{\frac{x}{2}} \cos\left(\frac{\sqrt{7}\, x}{2}\right)+\frac{2 c_2 \sin\left(\frac{\sqrt{7}\, x}{2}\right) e^{\frac{x}{2}} \sqrt{7}}{7} \tag{1}$$

Initial conditions are now substituted in the above solution. This will generate the required equations to solve for the integration constants. substituting $u = 2$ and $x = 0$ in the above gives

$$2 = c_1 \tag{1A}$$

Taking derivative of the solution gives

$$u' = \frac{c_1 e^{\frac{x}{2}} \cos\left(\frac{\sqrt{7}\, x}{2}\right)}{2}-\frac{c_1 e^{\frac{x}{2}} \sqrt{7}\, \sin\left(\frac{\sqrt{7}\, x}{2}\right)}{2}+c_2 \cos\left(\frac{\sqrt{7}\, x}{2}\right) e^{\frac{x}{2}}+\frac{c_2 \sin\left(\frac{\sqrt{7}\, x}{2}\right) e^{\frac{x}{2}} \sqrt{7}}{7}$$

substituting $u' = 0$ and $x = 0$ in the above gives

$$0 = \frac{c_1}{2}+c_2 \tag{2A}$$

Equations {1A,2A} are now solved for $\{c_1, c_2\}$. Solving for the constants gives

$$c_1 = 2$$
$$c_2 = -1$$

Substituting these values back in above solution results in

$$u = -\frac{2 \sin\left(\frac{\sqrt{7}\, x}{2}\right) e^{\frac{x}{2}} \sqrt{7}}{7}+2 e^{\frac{x}{2}} \cos\left(\frac{\sqrt{7}\, x}{2}\right)$$

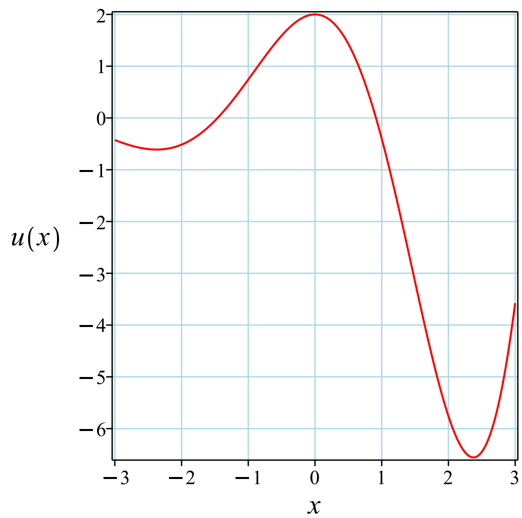
Which simplifies to

$$u = -\frac{2\left(\sqrt{7}\, \sin\left(\frac{\sqrt{7}\, x}{2}\right)-7 \cos\left(\frac{\sqrt{7}\, x}{2}\right)\right) e^{\frac{x}{2}}}{7}$$

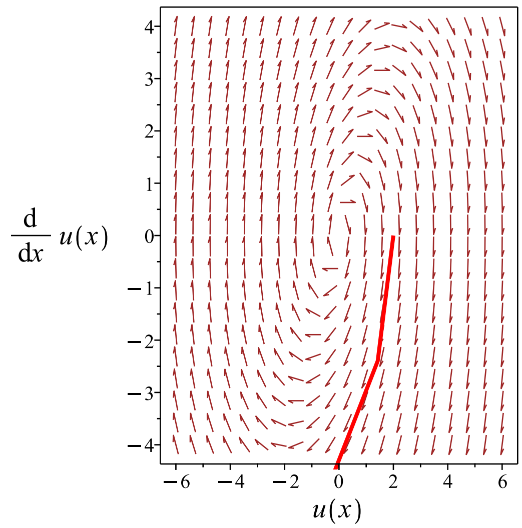
Summary

The solution(s) found are the following

$$u = -\frac{2\left(\sqrt{7}\, \sin\left(\frac{\sqrt{7}\, x}{2}\right)-7 \cos\left(\frac{\sqrt{7}\, x}{2}\right)\right) e^{\frac{x}{2}}}{7} \tag{1}$$



(a) Solution plot



(b) Slope field plot

<u>Verification of solutions</u>

$$u = -\frac{2\left(\sqrt{7}\,\sin\left(\frac{\sqrt{7}\,x}{2}\right) - 7\cos\left(\frac{\sqrt{7}\,x}{2}\right)\right)\mathrm{e}^{\frac{x}{2}}}{7}$$

Verified OK.

8.17.4 Maple step by step solution

Let's solve

$$\left[u'' - u' + 2u = 0, u(0) = 2, u'\Big|_{\{x=0\}} = 0\right]$$

- Highest derivative means the order of the ODE is 2

  u''

- Characteristic polynomial of ODE

  $r^2 - r + 2 = 0$

- Use quadratic formula to solve for r

  $r = \frac{1 \pm (\sqrt{-7})}{2}$

- Roots of the characteristic polynomial

  $r = \left(\frac{1}{2} - \frac{\mathrm{I}\sqrt{7}}{2}, \frac{1}{2} + \frac{\mathrm{I}\sqrt{7}}{2}\right)$

- 1st solution of the ODE

$$u_1(x) = e^{\frac{x}{2}} \cos\left(\frac{\sqrt{7}\,x}{2}\right)$$

- 2nd solution of the ODE

$$u_2(x) = e^{\frac{x}{2}} \sin\left(\frac{\sqrt{7}\,x}{2}\right)$$

- General solution of the ODE

$$u = c_1 u_1(x) + c_2 u_2(x)$$

- Substitute in solutions

$$u = c_1 e^{\frac{x}{2}} \cos\left(\frac{\sqrt{7}\,x}{2}\right) + \sin\left(\frac{\sqrt{7}\,x}{2}\right) e^{\frac{x}{2}} c_2$$

☐ Check validity of solution $u = c_1 e^{\frac{x}{2}} \cos\left(\frac{\sqrt{7}\,x}{2}\right) + \sin\left(\frac{\sqrt{7}\,x}{2}\right) e^{\frac{x}{2}} c_2$

- Use initial condition $u(0) = 2$

  $$2 = c_1$$

- Compute derivative of the solution

  $$u' = \frac{c_1 e^{\frac{x}{2}} \cos\left(\frac{\sqrt{7}\,x}{2}\right)}{2} - \frac{c_1 e^{\frac{x}{2}} \sqrt{7}\, \sin\left(\frac{\sqrt{7}\,x}{2}\right)}{2} + \frac{\sqrt{7}\, \cos\left(\frac{\sqrt{7}\,x}{2}\right) e^{\frac{x}{2}} c_2}{2} + \frac{\sin\left(\frac{\sqrt{7}\,x}{2}\right) e^{\frac{x}{2}} c_2}{2}$$

- Use the initial condition $u'\Big|_{\{x=0\}} = 0$

  $$0 = \frac{c_1}{2} + \frac{\sqrt{7}\,c_2}{2}$$

- Solve for c_1 and c_2

  $$\left\{c_1 = 2, c_2 = -\frac{2\sqrt{7}}{7}\right\}$$

- Substitute constant values into general solution and simplify

  $$u = -\frac{2\left(\sqrt{7}\, \sin\left(\frac{\sqrt{7}\,x}{2}\right) - 7\cos\left(\frac{\sqrt{7}\,x}{2}\right)\right) e^{\frac{x}{2}}}{7}$$

- Solution to the IVP

$$u = -\frac{2\left(\sqrt{7}\, \sin\left(\frac{\sqrt{7}\,x}{2}\right) - 7\cos\left(\frac{\sqrt{7}\,x}{2}\right)\right) e^{\frac{x}{2}}}{7}$$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
<- constant coefficients successful`
```

✓ Solution by Maple

Time used: 0.047 (sec). Leaf size: 31

```
dsolve([diff(u(x),x$2)- diff(u(x),x)+2*u(x) = 0,u(0) = 2, D(u)(0) = 0],u(x), singsol=all)
```

$$u(x) = -\frac{2\,\mathrm{e}^{\frac{x}{2}}\left(\sqrt{7}\,\sin\left(\frac{\sqrt{7}\,x}{2}\right)-7\cos\left(\frac{\sqrt{7}\,x}{2}\right)\right)}{7}$$

✓ Solution by Mathematica

Time used: 0.015 (sec). Leaf size: 19

```
DSolve[{u''[x]+4*u'[x]+5*u[x]==0,{u[0]==2,u'[0]==0}},u[x],x,IncludeSingularSolutions -> True
```

$$u(x) \to 2e^{-2x}(2\sin(x)+\cos(x))$$

8.18 problem 24

Internal problem ID [640]
Internal file name [`OUTPUT/640_Sunday_June_05_2022_01_46_16_AM_5872921/index.tex`]

Book: Elementary differential equations and boundary value problems, 10th ed., Boyce and DiPrima
Section: Chapter 3, Second order linear equations, 3.3 Complex Roots of the Characteristic Equation , page 164
Problem number: 24.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_linear_constant_coeff"**

Maple gives the following as the ode type

`[[_2nd_order, _missing_x]]`

$$\boxed{5u'' + 2u' + 7u = 0}$$

With initial conditions

$$[u(0) = 2, u'(0) = 1]$$

8.18.1 Existence and uniqueness analysis

This is a linear ODE. In canonical form it is written as

$$u'' + p(x)u' + q(x)u = F$$

Where here

$$\begin{aligned} p(x) &= \frac{2}{5} \\ q(x) &= \frac{7}{5} \\ F &= 0 \end{aligned}$$

Hence the ode is

$$u'' + \frac{2u'}{5} + \frac{7u}{5} = 0$$

The domain of $p(x) = \frac{2}{5}$ is

$$\{-\infty < x < \infty\}$$

And the point $x_0 = 0$ is inside this domain. The domain of $q(x) = \frac{7}{5}$ is

$$\{-\infty < x < \infty\}$$

And the point $x_0 = 0$ is also inside this domain. Hence solution exists and is unique.

8.18.2 Solving as second order linear constant coeff ode

This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$Au''(x) + Bu'(x) + Cu(x) = 0$$

Where in the above $A = 5, B = 2, C = 7$. Let the solution be $u = e^{\lambda x}$. Substituting this into the ODE gives

$$5\lambda^2 e^{\lambda x} + 2\lambda\, e^{\lambda x} + 7\, e^{\lambda x} = 0 \tag{1}$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda x}$ gives

$$5\lambda^2 + 2\lambda + 7 = 0 \tag{2}$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form.Using the quadratic formula

$$\lambda_{1,2} = \frac{-B}{2A} \pm \frac{1}{2A}\sqrt{B^2 - 4AC}$$

Substituting $A = 5, B = 2, C = 7$ into the above gives

$$\begin{aligned}
\lambda_{1,2} &= \frac{-2}{(2)\,(5)} \pm \frac{1}{(2)\,(5)}\sqrt{2^2 - (4)\,(5)\,(7)} \\
&= -\frac{1}{5} \pm \frac{i\sqrt{34}}{5}
\end{aligned}$$

Hence

$$\begin{aligned}
\lambda_1 &= -\frac{1}{5} + \frac{i\sqrt{34}}{5} \\
\lambda_2 &= -\frac{1}{5} - \frac{i\sqrt{34}}{5}
\end{aligned}$$

Which simplifies to

$$\lambda_1 = -\frac{1}{5} + \frac{i\sqrt{34}}{5}$$
$$\lambda_2 = -\frac{1}{5} - \frac{i\sqrt{34}}{5}$$

Since roots are complex conjugate of each others, then let the roots be

$$\lambda_{1,2} = \alpha \pm i\beta$$

Where $\alpha = -\frac{1}{5}$ and $\beta = \frac{\sqrt{34}}{5}$. Therefore the final solution, when using Euler relation, can be written as

$$u = e^{\alpha x}(c_1 \cos(\beta x) + c_2 \sin(\beta x))$$

Which becomes

$$u = e^{-\frac{x}{5}}\left(c_1 \cos\left(\frac{\sqrt{34}\,x}{5}\right) + c_2 \sin\left(\frac{\sqrt{34}\,x}{5}\right)\right)$$

Initial conditions are used to solve for the constants of integration.

Looking at the above solution

$$u = \mathrm{e}^{-\frac{x}{5}}\left(c_1 \cos\left(\frac{\sqrt{34}\,x}{5}\right) + c_2 \sin\left(\frac{\sqrt{34}\,x}{5}\right)\right) \tag{1}$$

Initial conditions are now substituted in the above solution. This will generate the required equations to solve for the integration constants. substituting $u = 2$ and $x = 0$ in the above gives

$$2 = c_1 \tag{1A}$$

Taking derivative of the solution gives

$$u' = -\frac{\mathrm{e}^{-\frac{x}{5}}\left(c_1 \cos\left(\frac{\sqrt{34}\,x}{5}\right) + c_2 \sin\left(\frac{\sqrt{34}\,x}{5}\right)\right)}{5} + \mathrm{e}^{-\frac{x}{5}}\left(-\frac{c_1\sqrt{34}\,\sin\left(\frac{\sqrt{34}\,x}{5}\right)}{5} + \frac{c_2\sqrt{34}\,\cos\left(\frac{\sqrt{34}\,x}{5}\right)}{5}\right)$$

substituting $u' = 1$ and $x = 0$ in the above gives

$$1 = -\frac{c_1}{5} + \frac{\sqrt{34}\,c_2}{5} \tag{2A}$$

Equations {1A,2A} are now solved for $\{c_1, c_2\}$. Solving for the constants gives

$$c_1 = 2$$
$$c_2 = \frac{7\sqrt{34}}{34}$$

Substituting these values back in above solution results in

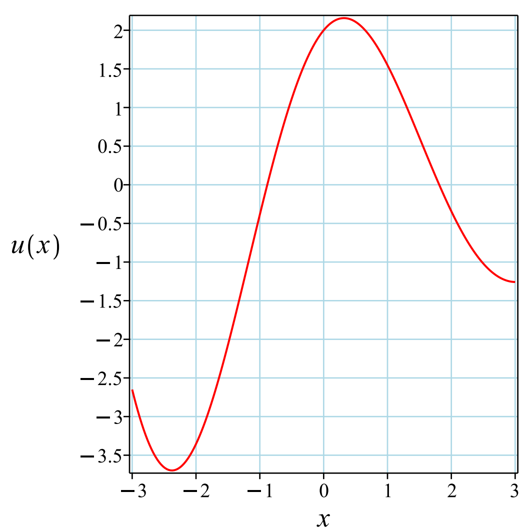
$$u = \frac{7\sin\left(\frac{\sqrt{34}\,x}{5}\right)\mathrm{e}^{-\frac{x}{5}}\sqrt{34}}{34} + 2\,\mathrm{e}^{-\frac{x}{5}}\cos\left(\frac{\sqrt{34}\,x}{5}\right)$$

Which simplifies to

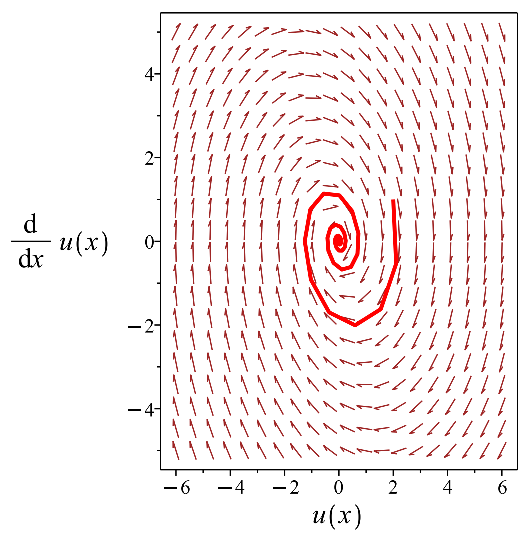
$$u = \frac{\left(7\sqrt{34}\,\sin\left(\frac{\sqrt{34}\,x}{5}\right) + 68\cos\left(\frac{\sqrt{34}\,x}{5}\right)\right)\mathrm{e}^{-\frac{x}{5}}}{34}$$

Summary
The solution(s) found are the following

$$u = \frac{\left(7\sqrt{34}\,\sin\left(\frac{\sqrt{34}\,x}{5}\right) + 68\cos\left(\frac{\sqrt{34}\,x}{5}\right)\right)\mathrm{e}^{-\frac{x}{5}}}{34} \tag{1}$$



(a) Solution plot



(b) Slope field plot

Verification of solutions

$$u = \frac{\left(7\sqrt{34}\,\sin\left(\frac{\sqrt{34}\,x}{5}\right) + 68\cos\left(\frac{\sqrt{34}\,x}{5}\right)\right)\mathrm{e}^{-\frac{x}{5}}}{34}$$

Verified OK.

8.18.3 Solving using Kovacic algorithm

Writing the ode as

$$5u'' + 2u' + 7u = 0 \tag{1}$$
$$Au'' + Bu' + Cu = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 5 \\ B &= 2 \\ C &= 7 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ue^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{-34}{25} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= -34 \\ t &= 25 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = -\frac{34z(x)}{25} \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then u is found using the inverse transformation

$$u = z(x)\, e^{-\int \frac{B}{2A}\,dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 398: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 0 - 0 \\ &= 0 \end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Since $r = -\frac{34}{25}$ is not a function of x, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$z_1(x) = \cos\left(\frac{\sqrt{34}\, x}{5}\right)$$

Using the above, the solution for the original ode can now be found. The first solution

to the original ode in u is found from

$$\begin{aligned}
u_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{2}{5}\,dx} \\
&= z_1 e^{-\frac{x}{5}} \\
&= z_1 \left(\mathrm{e}^{-\frac{x}{5}}\right)
\end{aligned}$$

Which simplifies to

$$u_1 = \mathrm{e}^{-\frac{x}{5}} \cos\left(\frac{\sqrt{34}\,x}{5}\right)$$

The second solution u_2 to the original ode is found using reduction of order

$$u_2 = u_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{u_1^2}\,dx$$

Substituting gives

$$\begin{aligned}
u_2 &= u_1 \int \frac{e^{\int -\frac{2}{5}\,dx}}{(u_1)^2}\,dx \\
&= u_1 \int \frac{e^{-\frac{2x}{5}}}{(u_1)^2}\,dx \\
&= u_1 \left(\frac{5\sqrt{34}\,\tan\left(\frac{\sqrt{34}\,x}{5}\right)}{34}\right)
\end{aligned}$$

Therefore the solution is

$$\begin{aligned}
u &= c_1 u_1 + c_2 u_2 \\
&= c_1 \left(\mathrm{e}^{-\frac{x}{5}} \cos\left(\frac{\sqrt{34}\,x}{5}\right)\right) + c_2 \left(\mathrm{e}^{-\frac{x}{5}} \cos\left(\frac{\sqrt{34}\,x}{5}\right) \left(\frac{5\sqrt{34}\,\tan\left(\frac{\sqrt{34}\,x}{5}\right)}{34}\right)\right)
\end{aligned}$$

Initial conditions are used to solve for the constants of integration.

Looking at the above solution

$$u = c_1 \mathrm{e}^{-\frac{x}{5}} \cos\left(\frac{\sqrt{34}\, x}{5}\right) + \frac{5 c_2 \sin\left(\frac{\sqrt{34}\, x}{5}\right) \mathrm{e}^{-\frac{x}{5}} \sqrt{34}}{34} \tag{1}$$

Initial conditions are now substituted in the above solution. This will generate the required equations to solve for the integration constants. substituting $u = 2$ and $x = 0$ in the above gives

$$2 = c_1 \tag{1A}$$

Taking derivative of the solution gives

$$u' = -\frac{c_1 \mathrm{e}^{-\frac{x}{5}} \cos\left(\frac{\sqrt{34}\, x}{5}\right)}{5} - \frac{c_1 \mathrm{e}^{-\frac{x}{5}} \sqrt{34}\, \sin\left(\frac{\sqrt{34}\, x}{5}\right)}{5} + c_2 \cos\left(\frac{\sqrt{34}\, x}{5}\right) \mathrm{e}^{-\frac{x}{5}} - \frac{c_2 \sin\left(\frac{\sqrt{34}\, x}{5}\right) \mathrm{e}^{-\frac{x}{5}} \sqrt{34}}{34}$$

substituting $u' = 1$ and $x = 0$ in the above gives

$$1 = -\frac{c_1}{5} + c_2 \tag{2A}$$

Equations {1A,2A} are now solved for $\{c_1, c_2\}$. Solving for the constants gives

$$c_1 = 2$$
$$c_2 = \frac{7}{5}$$

Substituting these values back in above solution results in

$$u = \frac{7 \sin\left(\frac{\sqrt{34}\, x}{5}\right) \mathrm{e}^{-\frac{x}{5}} \sqrt{34}}{34} + 2\, \mathrm{e}^{-\frac{x}{5}} \cos\left(\frac{\sqrt{34}\, x}{5}\right)$$

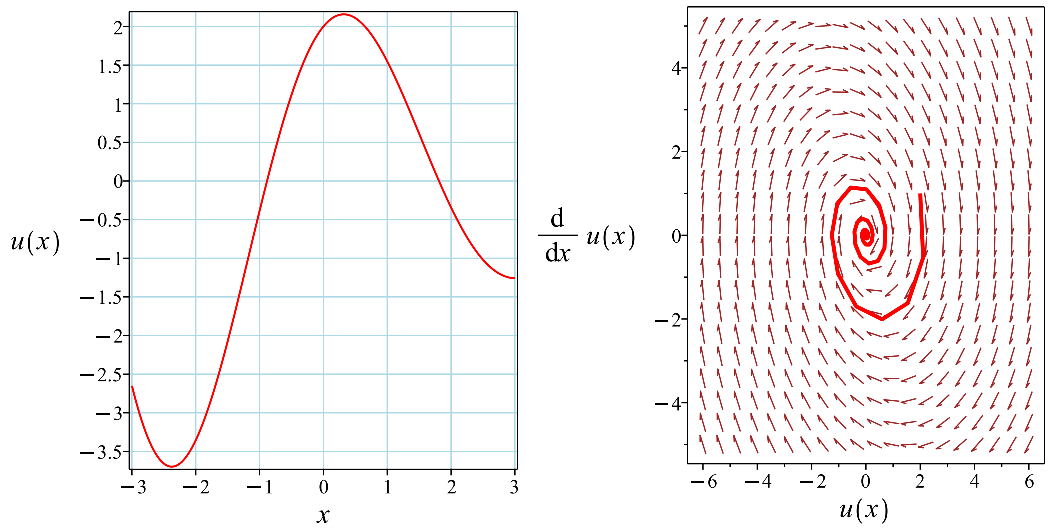
Which simplifies to

$$u = \frac{\left(7\sqrt{34}\, \sin\left(\frac{\sqrt{34}\, x}{5}\right) + 68 \cos\left(\frac{\sqrt{34}\, x}{5}\right)\right) \mathrm{e}^{-\frac{x}{5}}}{34}$$

Summary

The solution(s) found are the following

$$u = \frac{\left(7\sqrt{34}\, \sin\left(\frac{\sqrt{34}\, x}{5}\right) + 68 \cos\left(\frac{\sqrt{34}\, x}{5}\right)\right) \mathrm{e}^{-\frac{x}{5}}}{34} \tag{1}$$



(a) Solution plot (b) Slope field plot

Verification of solutions

$$u = \frac{\left(7\sqrt{34}\,\sin\left(\frac{\sqrt{34}\,x}{5}\right) + 68\cos\left(\frac{\sqrt{34}\,x}{5}\right)\right)e^{-\frac{x}{5}}}{34}$$

Verified OK.

8.18.4 Maple step by step solution

Let's solve

$\left[5u'' + 2u' + 7u = 0, u(0) = 2, u'\Big|_{\{x=0\}} = 1\right]$

- Highest derivative means the order of the ODE is 2

  u''

- Isolate 2nd derivative

  $u'' = -\frac{2u'}{5} - \frac{7u}{5}$

- Group terms with u on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $u'' + \frac{2u'}{5} + \frac{7u}{5} = 0$

- Characteristic polynomial of ODE

  $r^2 + \frac{2}{5}r + \frac{7}{5} = 0$

- Use quadratic formula to solve for r

  $r = \frac{\left(-\frac{2}{5}\right)\pm\left(\sqrt{-\frac{136}{25}}\right)}{2}$

- Roots of the characteristic polynomial

  $r = \left(-\frac{1}{5}-\frac{\mathrm{I}\sqrt{34}}{5}, -\frac{1}{5}+\frac{\mathrm{I}\sqrt{34}}{5}\right)$

- 1st solution of the ODE

  $u_1(x) = \mathrm{e}^{-\frac{x}{5}}\cos\left(\frac{\sqrt{34}\,x}{5}\right)$

- 2nd solution of the ODE

  $u_2(x) = \mathrm{e}^{-\frac{x}{5}}\sin\left(\frac{\sqrt{34}\,x}{5}\right)$

- General solution of the ODE

  $u = c_1u_1(x) + c_2u_2(x)$

- Substitute in solutions

  $u = c_1\mathrm{e}^{-\frac{x}{5}}\cos\left(\frac{\sqrt{34}\,x}{5}\right) + \sin\left(\frac{\sqrt{34}\,x}{5}\right)\mathrm{e}^{-\frac{x}{5}}c_2$

☐ Check validity of solution $u = c_1\mathrm{e}^{-\frac{x}{5}}\cos\left(\frac{\sqrt{34}\,x}{5}\right) + \sin\left(\frac{\sqrt{34}\,x}{5}\right)\mathrm{e}^{-\frac{x}{5}}c_2$

  - Use initial condition $u(0) = 2$

    $2 = c_1$

  - Compute derivative of the solution

    $u' = -\frac{c_1\mathrm{e}^{-\frac{x}{5}}\cos\left(\frac{\sqrt{34}\,x}{5}\right)}{5} - \frac{c_1\mathrm{e}^{-\frac{x}{5}}\sqrt{34}\sin\left(\frac{\sqrt{34}\,x}{5}\right)}{5} + \frac{\sqrt{34}\cos\left(\frac{\sqrt{34}\,x}{5}\right)\mathrm{e}^{-\frac{x}{5}}c_2}{5} - \frac{\sin\left(\frac{\sqrt{34}\,x}{5}\right)\mathrm{e}^{-\frac{x}{5}}c_2}{5}$

  - Use the initial condition $u'\Big|_{\{x=0\}} = 1$

    $1 = -\frac{c_1}{5} + \frac{\sqrt{34}\,c_2}{5}$

  - Solve for c_1 and c_2

    $\left\{c_1 = 2, c_2 = \frac{7\sqrt{34}}{34}\right\}$

  - Substitute constant values into general solution and simplify

    $u = \frac{\left(7\sqrt{34}\sin\left(\frac{\sqrt{34}\,x}{5}\right)+68\cos\left(\frac{\sqrt{34}\,x}{5}\right)\right)\mathrm{e}^{-\frac{x}{5}}}{34}$

- Solution to the IVP

  $u = \frac{\left(7\sqrt{34}\sin\left(\frac{\sqrt{34}\,x}{5}\right)+68\cos\left(\frac{\sqrt{34}\,x}{5}\right)\right)\mathrm{e}^{-\frac{x}{5}}}{34}$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
<- constant coefficients successful`
```

✓ Solution by Maple

Time used: 0.047 (sec). Leaf size: 32

```
dsolve([5*diff(u(x),x$2)+ 2*diff(u(x),x)+7*u(x) = 0,u(0) = 2, D(u)(0) = 1],u(x), singsol=all
```

$$u(x) = \frac{\mathrm{e}^{-\frac{x}{5}} \left(7\sqrt{34}\, \sin\left(\frac{\sqrt{34}\, x}{5}\right) + 68 \cos\left(\frac{\sqrt{34}\, x}{5}\right)\right)}{34}$$

✓ Solution by Mathematica

Time used: 0.025 (sec). Leaf size: 48

```
DSolve[{5*u''[x]+2*u'[x]+7*u[x]==0,{u[0]==2,u'[0]==1}},u[x],x,IncludeSingularSolutions -> Tr
```

$$u(x) \to \frac{1}{34} e^{-x/5} \left(7\sqrt{34} \sin\left(\frac{\sqrt{34}x}{5}\right) + 68 \cos\left(\frac{\sqrt{34}x}{5}\right)\right)$$

8.19 problem 25

Internal problem ID [641]
Internal file name [`OUTPUT/641_Sunday_June_05_2022_01_46_17_AM_9066687/index.tex`]

Book: Elementary differential equations and boundary value problems, 10th ed., Boyce and DiPrima
Section: Chapter 3, Second order linear equations, 3.3 Complex Roots of the Characteristic Equation , page 164
Problem number: 25.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_linear_constant_coeff"**

Maple gives the following as the ode type

```
[[_2nd_order, _missing_x]]
```

$$\boxed{y'' + 2y' + 6y = 0}$$

With initial conditions

$$[y(0) = 2, y'(0) = \alpha]$$

8.19.1 Existence and uniqueness analysis

This is a linear ODE. In canonical form it is written as

$$y'' + p(x)y' + q(x)y = F$$

Where here

$$\begin{aligned} p(x) &= 2 \\ q(x) &= 6 \\ F &= 0 \end{aligned}$$

Hence the ode is

$$y'' + 2y' + 6y = 0$$

The domain of $p(x) = 2$ is

$$\{-\infty < x < \infty\}$$

And the point $x_0 = 0$ is inside this domain. The domain of $q(x) = 6$ is

$$\{-\infty < x < \infty\}$$

And the point $x_0 = 0$ is also inside this domain. Hence solution exists and is unique.

8.19.2 Solving as second order linear constant coeff ode

This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = 0$$

Where in the above $A = 1, B = 2, C = 6$. Let the solution be $y = e^{\lambda x}$. Substituting this into the ODE gives

$$\lambda^2 e^{\lambda x} + 2\lambda\, e^{\lambda x} + 6\, e^{\lambda x} = 0 \tag{1}$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda x}$ gives

$$\lambda^2 + 2\lambda + 6 = 0 \tag{2}$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form.Using the quadratic formula

$$\lambda_{1,2} = \frac{-B}{2A} \pm \frac{1}{2A}\sqrt{B^2 - 4AC}$$

Substituting $A = 1, B = 2, C = 6$ into the above gives

$$\begin{aligned}
\lambda_{1,2} &= \frac{-2}{(2)\,(1)} \pm \frac{1}{(2)\,(1)}\sqrt{2^2 - (4)\,(1)\,(6)} \\
&= -1 \pm i\sqrt{5}
\end{aligned}$$

Hence

$$\begin{aligned}
\lambda_1 &= -1 + i\sqrt{5} \\
\lambda_2 &= -1 - i\sqrt{5}
\end{aligned}$$

Which simplifies to

$$\lambda_1 = -1 + i\sqrt{5}$$
$$\lambda_2 = -1 - i\sqrt{5}$$

Since roots are complex conjugate of each others, then let the roots be

$$\lambda_{1,2} = \alpha \pm i\beta$$

Where $\alpha = -1$ and $\beta = \sqrt{5}$. Therefore the final solution, when using Euler relation, can be written as

$$y = e^{\alpha x}(c_1 \cos(\beta x) + c_2 \sin(\beta x))$$

Which becomes

$$y = e^{-x}\left(c_1 \cos\left(x\sqrt{5}\right) + c_2 \sin\left(x\sqrt{5}\right)\right)$$

Initial conditions are used to solve for the constants of integration.

Looking at the above solution

$$y = \mathrm{e}^{-x}\left(c_1 \cos\left(x\sqrt{5}\right) + c_2 \sin\left(x\sqrt{5}\right)\right) \tag{1}$$

Initial conditions are now substituted in the above solution. This will generate the required equations to solve for the integration constants. substituting $y = 2$ and $x = 0$ in the above gives

$$2 = c_1 \tag{1A}$$

Taking derivative of the solution gives

$$y' = -\mathrm{e}^{-x}\left(c_1 \cos\left(x\sqrt{5}\right) + c_2 \sin\left(x\sqrt{5}\right)\right) + \mathrm{e}^{-x}\left(-c_1\sqrt{5}\, \sin\left(x\sqrt{5}\right) + c_2\sqrt{5}\, \cos\left(x\sqrt{5}\right)\right)$$

substituting $y' = \alpha$ and $x = 0$ in the above gives

$$\alpha = -c_1 + \sqrt{5}\, c_2 \tag{2A}$$

Equations {1A,2A} are now solved for $\{c_1, c_2\}$. Solving for the constants gives

$$c_1 = 2$$
$$c_2 = \frac{\sqrt{5}\,(\alpha + 2)}{5}$$

Substituting these values back in above solution results in

$$y = \frac{e^{-x} \sin\left(x\sqrt{5}\right) \sqrt{5}\,(\alpha + 2)}{5} + 2\,e^{-x} \cos\left(x\sqrt{5}\right)$$

Which simplifies to

$$y = \frac{\left(\sin\left(x\sqrt{5}\right) \sqrt{5}\,(\alpha + 2) + 10 \cos\left(x\sqrt{5}\right)\right) e^{-x}}{5}$$

Summary

The solution(s) found are the following

$$y = \frac{\left(\sin\left(x\sqrt{5}\right) \sqrt{5}\,(\alpha + 2) + 10 \cos\left(x\sqrt{5}\right)\right) e^{-x}}{5} \tag{1}$$

Verification of solutions

$$y = \frac{\left(\sin\left(x\sqrt{5}\right) \sqrt{5}\,(\alpha + 2) + 10 \cos\left(x\sqrt{5}\right)\right) e^{-x}}{5}$$

Verified OK.

8.19.3 Solving using Kovacic algorithm

Writing the ode as

$$y'' + 2y' + 6y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1 \\ B &= 2 \\ C &= 6 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$r = \frac{s}{t} \tag{5}$$
$$= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{-5}{1} \tag{6}$$

Comparing the above to (5) shows that

$$s = -5$$
$$t = 1$$

Therefore eq. (4) becomes

$$z''(x) = -5z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 400: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned}O(\infty) &= \deg(t) - \deg(s)\\ &= 0 - 0\\ &= 0\end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Since $r = -5$ is not a function of x, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$z_1(x) = \cos\left(x\sqrt{5}\right)$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$\begin{aligned}y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx}\\ &= z_1 e^{-\int \frac{1}{2}\frac{2}{1}\,dx}\\ &= z_1 e^{-x}\\ &= z_1\left(\mathrm{e}^{-x}\right)\end{aligned}$$

Which simplifies to

$$y_1 = \mathrm{e}^{-x}\cos\left(x\sqrt{5}\right)$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx$$

Substituting gives

$$\begin{aligned}y_2 &= y_1 \int \frac{e^{\int -\frac{2}{1}\,dx}}{(y_1)^2}\,dx\\ &= y_1 \int \frac{e^{-2x}}{(y_1)^2}\,dx\\ &= y_1\left(\frac{\sqrt{5}\,\tan\left(x\sqrt{5}\right)}{5}\right)\end{aligned}$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1\left(\mathrm{e}^{-x}\cos\left(x\sqrt{5}\right)\right) + c_2\left(\mathrm{e}^{-x}\cos\left(x\sqrt{5}\right)\left(\frac{\sqrt{5}\,\tan\left(x\sqrt{5}\right)}{5}\right)\right)
\end{aligned}
$$

Initial conditions are used to solve for the constants of integration.

Looking at the above solution

$$
y = c_1 \mathrm{e}^{-x}\cos\left(x\sqrt{5}\right) + \frac{c_2 \mathrm{e}^{-x}\sin\left(x\sqrt{5}\right)\sqrt{5}}{5} \tag{1}
$$

Initial conditions are now substituted in the above solution. This will generate the required equations to solve for the integration constants. substituting $y = 2$ and $x = 0$ in the above gives

$$
2 = c_1 \tag{1A}
$$

Taking derivative of the solution gives

$$
y' = -c_1 \mathrm{e}^{-x}\cos\left(x\sqrt{5}\right) - c_1 \mathrm{e}^{-x}\sqrt{5}\,\sin\left(x\sqrt{5}\right) - \frac{c_2 \mathrm{e}^{-x}\sin\left(x\sqrt{5}\right)\sqrt{5}}{5} + c_2 \mathrm{e}^{-x}\cos\left(x\sqrt{5}\right)
$$

substituting $y' = \alpha$ and $x = 0$ in the above gives

$$
\alpha = -c_1 + c_2 \tag{2A}
$$

Equations $\{1\mathrm{A}, 2\mathrm{A}\}$ are now solved for $\{c_1, c_2\}$. Solving for the constants gives

$$
\begin{aligned}
c_1 &= 2 \\
c_2 &= \alpha + 2
\end{aligned}
$$

Substituting these values back in above solution results in

$$
y = \frac{\mathrm{e}^{-x}\sin\left(x\sqrt{5}\right)\sqrt{5}\,(\alpha+2)}{5} + 2\,\mathrm{e}^{-x}\cos\left(x\sqrt{5}\right)
$$

Which simplifies to

$$
y = \frac{\left(\sin\left(x\sqrt{5}\right)\sqrt{5}\,(\alpha+2) + 10\cos\left(x\sqrt{5}\right)\right)\mathrm{e}^{-x}}{5}
$$

Summary

The solution(s) found are the following

$$y = \frac{\left(\sin\left(x\sqrt{5}\right)\sqrt{5}\,(\alpha+2)+10\cos\left(x\sqrt{5}\right)\right)\mathrm{e}^{-x}}{5} \tag{1}$$

Verification of solutions

$$y = \frac{\left(\sin\left(x\sqrt{5}\right)\sqrt{5}\,(\alpha+2)+10\cos\left(x\sqrt{5}\right)\right)\mathrm{e}^{-x}}{5}$$

Verified OK.

8.19.4 Maple step by step solution

Let's solve

$\left[y''+2y'+6y=0, y(0)=2, y'\Big|_{\{x=0\}}=\alpha\right]$

- Highest derivative means the order of the ODE is 2

  y''

- Characteristic polynomial of ODE

  $r^2+2r+6=0$

- Use quadratic formula to solve for r

  $r=\frac{(-2)\pm\left(\sqrt{-20}\right)}{2}$

- Roots of the characteristic polynomial

  $r=\left(-1-\mathrm{I}\sqrt{5}, -1+\mathrm{I}\sqrt{5}\right)$

- 1st solution of the ODE

  $y_1(x)=\mathrm{e}^{-x}\cos\left(x\sqrt{5}\right)$

- 2nd solution of the ODE

  $y_2(x)=\mathrm{e}^{-x}\sin\left(x\sqrt{5}\right)$

- General solution of the ODE

  $y=c_1y_1(x)+c_2y_2(x)$

- Substitute in solutions

  $y=c_1\mathrm{e}^{-x}\cos\left(x\sqrt{5}\right)+\mathrm{e}^{-x}\sin\left(x\sqrt{5}\right)c_2$

□ Check validity of solution $y=c_1\mathrm{e}^{-x}\cos\left(x\sqrt{5}\right)+\mathrm{e}^{-x}\sin\left(x\sqrt{5}\right)c_2$

- Use initial condition $y(0) = 2$

  $2 = c_1$

- Compute derivative of the solution

  $y' = -c_1 e^{-x} \cos\left(x\sqrt{5}\right) - c_1 e^{-x}\sqrt{5}\,\sin\left(x\sqrt{5}\right) - e^{-x}\sin\left(x\sqrt{5}\right) c_2 + e^{-x}\sqrt{5}\,\cos\left(x\sqrt{5}\right) c_2$

- Use the initial condition $y'\big|_{\{x=0\}} = \alpha$

  $\alpha = -c_1 + \sqrt{5}\,c_2$

- Solve for c_1 and c_2

  $\left\{c_1 = 2, c_2 = \frac{\sqrt{5}\,(\alpha+2)}{5}\right\}$

- Substitute constant values into general solution and simplify

  $y = \frac{\left(\sin\left(x\sqrt{5}\right)\sqrt{5}\,(\alpha+2)+10\cos\left(x\sqrt{5}\right)\right)e^{-x}}{5}$

- Solution to the IVP

  $y = \frac{\left(\sin\left(x\sqrt{5}\right)\sqrt{5}\,(\alpha+2)+10\cos\left(x\sqrt{5}\right)\right)e^{-x}}{5}$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
<- constant coefficients successful`
```

✓ Solution by Maple

Time used: 0.047 (sec). Leaf size: 32

```
dsolve([diff(y(x),x$2)+ 2*diff(y(x),x)+6*y(x) = 0,y(0) = 2, D(y)(0) = alpha],y(x), singsol=a
```

$$y(x) = \frac{e^{-x}\left(\sqrt{5}\,(\alpha+2)\sin\left(\sqrt{5}\,x\right) + 10\cos\left(\sqrt{5}\,x\right)\right)}{5}$$

✓ Solution by Mathematica

Time used: 0.021 (sec). Leaf size: 42

```
DSolve[{y''[x]+2*y'[x]+6*y[x]==0,{y[0]==2,y'[0]==\[Alpha]}},y[x],x,IncludeSingularSolutions
```

$$y(x) \to \frac{1}{5}e^{-x}\left(\sqrt{5}(\alpha+2)\sin\left(\sqrt{5}x\right)+10\cos\left(\sqrt{5}x\right)\right)$$

8.20 problem 26

Internal problem ID [642]
Internal file name [OUTPUT/642_Sunday_June_05_2022_01_46_18_AM_29576795/index.tex]

Book: Elementary differential equations and boundary value problems, 10th ed., Boyce and DiPrima
Section: Chapter 3, Second order linear equations, 3.3 Complex Roots of the Characteristic Equation , page 164
Problem number: 26.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_linear_constant_coeff"**

Maple gives the following as the ode type

```
[[_2nd_order, _missing_x]]
```

$$\boxed{y'' + 2ay' + (a^2 + 1)\, y = 0}$$

With initial conditions

$$[y(0) = 1, y'(0) = 0]$$

8.20.1 Existence and uniqueness analysis

This is a linear ODE. In canonical form it is written as

$$y'' + p(x)y' + q(x)y = F$$

Where here

$$\begin{aligned} p(x) &= 2a \\ q(x) &= a^2 + 1 \\ F &= 0 \end{aligned}$$

Hence the ode is

$$y'' + 2ay' + (a^2 + 1)\,y = 0$$

The domain of $p(x) = 2a$ is

$$\{-\infty < x < \infty\}$$

And the point $x_0 = 0$ is inside this domain. The domain of $q(x) = a^2 + 1$ is

$$\{-\infty < x < \infty\}$$

And the point $x_0 = 0$ is also inside this domain. Hence solution exists and is unique.

8.20.2 Solving as second order linear constant coeff ode

This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = 0$$

Where in the above $A = 1, B = 2a, C = a^2+1$. Let the solution be $y = e^{\lambda x}$. Substituting this into the ODE gives

$$\lambda^2 \mathrm{e}^{\lambda x} + 2a\lambda\,\mathrm{e}^{\lambda x} + (a^2 + 1)\,\mathrm{e}^{\lambda x} = 0 \tag{1}$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda x}$ gives

$$a^2 + 2a\lambda + \lambda^2 + 1 = 0 \tag{2}$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form.Using the quadratic formula

$$\lambda_{1,2} = \frac{-B}{2A} \pm \frac{1}{2A}\sqrt{B^2 - 4AC}$$

Substituting $A = 1, B = 2a, C = a^2 + 1$ into the above gives

$$\begin{aligned}\lambda_{1,2} &= \frac{-2a}{(2)\,(1)} \pm \frac{1}{(2)\,(1)}\sqrt{2a^2 - (4)\,(1)\,(a^2 + 1)}\\ &= -a \pm i\end{aligned}$$

Hence

$$\begin{aligned}\lambda_1 &= -a + i\\ \lambda_2 &= -a - i\end{aligned}$$

Which simplifies to

$$\lambda_1 = -a + i$$
$$\lambda_2 = -a - i$$

Since roots are complex conjugate of each others, then let the roots be

$$\lambda_{1,2} = \alpha \pm i\beta$$

Where $\alpha = -a$ and $\beta = 1$. Therefore the final solution, when using Euler relation, can be written as

$$y = e^{\alpha x}(c_1 \cos(\beta x) + c_2 \sin(\beta x))$$

Which becomes

$$y = e^{-ax}(c_1 \cos(x) + c_2 \sin(x))$$

Initial conditions are used to solve for the constants of integration.

Looking at the above solution

$$y = \mathrm{e}^{-ax}(c_1 \cos(x) + c_2 \sin(x)) \tag{1}$$

Initial conditions are now substituted in the above solution. This will generate the required equations to solve for the integration constants. substituting $y = 1$ and $x = 0$ in the above gives

$$1 = c_1 \tag{1A}$$

Taking derivative of the solution gives

$$y' = -a\,\mathrm{e}^{-ax}(c_1 \cos(x) + c_2 \sin(x)) + \mathrm{e}^{-ax}(-c_1 \sin(x) + c_2 \cos(x))$$

substituting $y' = 0$ and $x = 0$ in the above gives

$$0 = -ac_1 + c_2 \tag{2A}$$

Equations {1A,2A} are now solved for $\{c_1, c_2\}$. Solving for the constants gives

$$c_1 = 1$$
$$c_2 = a$$

Substituting these values back in above solution results in

$$y = \mathrm{e}^{-ax}(\sin(x)\,a + \cos(x))$$

Summary
The solution(s) found are the following

$$y = e^{-ax}(\sin(x)\, a + \cos(x)) \tag{1}$$

Verification of solutions

$$y = e^{-ax}(\sin(x)\, a + \cos(x))$$

Verified OK.

8.20.3 Solving using Kovacic algorithm

Writing the ode as

$$y'' + 2ay' + (a^2 + 1)\, y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1 \\ B &= 2a \\ C &= a^2 + 1 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = y e^{\int \frac{B}{2A}\, dx}$$

Then (2) becomes

$$z''(x) = r z(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{-1}{1} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= -1 \\ t &= 1 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = -z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 402: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 0 - 0 \\ &= 0 \end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Since $r = -1$ is not a function of x, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$z_1(x) = \cos(x)$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$
\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{2a}{1}\,dx} \\
&= z_1 e^{-ax} \\
&= z_1\left(\mathrm{e}^{-ax}\right)
\end{aligned}
$$

Which simplifies to

$$
y_1 = \mathrm{e}^{-ax}\cos(x)
$$

The second solution y_2 to the original ode is found using reduction of order

$$
y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx
$$

Substituting gives

$$
\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{2a}{1}\,dx}}{(y_1)^2}\,dx \\
&= y_1 \int \frac{e^{-2ax}}{(y_1)^2}\,dx \\
&= y_1(\tan(x))
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1\left(\mathrm{e}^{-ax}\cos(x)\right) + c_2\left(\mathrm{e}^{-ax}\cos(x)(\tan(x))\right)
\end{aligned}
$$

Initial conditions are used to solve for the constants of integration.

Looking at the above solution

$$
y = c_1 \mathrm{e}^{-ax}\cos(x) + c_2 \mathrm{e}^{-ax}\sin(x) \tag{1}
$$

Initial conditions are now substituted in the above solution. This will generate the required equations to solve for the integration constants. substituting $y = 1$ and $x = 0$ in the above gives

$$
1 = c_1 \tag{1A}
$$

Taking derivative of the solution gives

$$y' = -c_1 a\, \mathrm{e}^{-ax} \cos(x) - c_1 \mathrm{e}^{-ax} \sin(x) - c_2 a\, \mathrm{e}^{-ax} \sin(x) + c_2 \mathrm{e}^{-ax} \cos(x)$$

substituting $y' = 0$ and $x = 0$ in the above gives

$$0 = -a c_1 + c_2 \tag{2A}$$

Equations {1A,2A} are now solved for $\{c_1, c_2\}$. Solving for the constants gives

$$c_1 = 1$$
$$c_2 = a$$

Substituting these values back in above solution results in

$$y = \mathrm{e}^{-ax} \sin(x)\, a + \mathrm{e}^{-ax} \cos(x)$$

Which simplifies to

$$y = \mathrm{e}^{-ax} (\sin(x)\, a + \cos(x))$$

Summary

The solution(s) found are the following

$$y = \mathrm{e}^{-ax} (\sin(x)\, a + \cos(x)) \tag{1}$$

Verification of solutions

$$y = \mathrm{e}^{-ax} (\sin(x)\, a + \cos(x))$$

Verified OK.

8.20.4 Maple step by step solution

Let's solve

$$\left[y'' + 2ay' + (a^2 + 1)\, y = 0, y(0) = 1, y'\Big|_{\{x=0\}} = 0\right]$$

- Highest derivative means the order of the ODE is 2

  y''

- Characteristic polynomial of ODE

  $a^2 + 2ar + r^2 + 1 = 0$

- Use quadratic formula to solve for r

$r = \frac{(-2a)\pm(\sqrt{-4})}{2}$

- Roots of the characteristic polynomial

  $r = (-a - \mathrm{I}, -a + \mathrm{I})$

- 1st solution of the ODE

  $y_1(x) = \mathrm{e}^{-\Re(a)x} \cos\left(|\Im(a) + 1|x\right)$

- 2nd solution of the ODE

  $y_2(x) = \mathrm{e}^{-\Re(a)x} \sin\left(|\Im(a) + 1|x\right)$

- General solution of the ODE

  $y = c_1 y_1(x) + c_2 y_2(x)$

- Substitute in solutions

  $y = c_1 \mathrm{e}^{-\Re(a)x} \cos\left(|\Im(a) + 1|x\right) + c_2 \mathrm{e}^{-\Re(a)x} \sin\left(|\Im(a) + 1|x\right)$

□ Check validity of solution $y = c_1 \mathrm{e}^{-\Re(a)x} \cos\left(|\Im(a) + 1|x\right) + c_2 \mathrm{e}^{-\Re(a)x} \sin\left(|\Im(a) + 1|x\right)$

  - Use initial condition $y(0) = 1$

    $1 = c_1$

  - Compute derivative of the solution

    $y' = -c_1 \Re(a)\, \mathrm{e}^{-\Re(a)x} \cos\left(|\Im(a) + 1|x\right) - c_1 \mathrm{e}^{-\Re(a)x} |\Im(a) + 1| \sin\left(|\Im(a) + 1|x\right) - c_2 \Re(a)\, \mathrm{e}^{-\Re(a)x} \mathrm{si}$

  - Use the initial condition $y'\Big|_{\{x=0\}} = 0$

    $0 = -c_1 \Re(a) + c_2 |\Im(a) + 1|$

  - Solve for c_1 and c_2

    $\left\{c_1 = 1, c_2 = \frac{\Re(a)}{|\Im(a)+1|}\right\}$

  - Substitute constant values into general solution and simplify

    $y = \frac{\mathrm{e}^{-\Re(a)x}(\mathrm{signum}(\Im(a)+1) \sin(x(\Im(a)+1))\Re(a)+\cos(x(\Im(a)+1))|\Im(a)+1|)}{|\Im(a)+1|}$

- Solution to the IVP

  $y = \frac{\mathrm{e}^{-\Re(a)x}(\mathrm{signum}(\Im(a)+1) \sin(x(\Im(a)+1))\Re(a)+\cos(x(\Im(a)+1))|\Im(a)+1|)}{|\Im(a)+1|}$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
<- constant coefficients successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 17

```
dsolve([diff(y(x),x$2)+ 2*a*diff(y(x),x)+(a^2+1)*y(x) = 0,y(0) = 1, D(y)(0) = 0],y(x), sings
```

$$y(x) = e^{-ax}(a\sin(x) + \cos(x))$$

✓ Solution by Mathematica

Time used: 0.031 (sec). Leaf size: 94

```
DSolve[{y''[x]+2*a*y'[x]+(a^1+1)*y[x]==0,{y[0]==1,y'[0]==0}},y[x],x,IncludeSingularSolutions
```

$$y(x) \to \frac{e^{-\left(\left(\sqrt{a^2-a-1}+a\right)x\right)}\left(a\left(e^{2\sqrt{a^2-a-1}x}-1\right)+\sqrt{a^2-a-1}\left(e^{2\sqrt{a^2-a-1}x}+1\right)\right)}{2\sqrt{a^2-a-1}}$$

8.21 problem 35

Internal problem ID [643]
Internal file name [`OUTPUT/643_Sunday_June_05_2022_01_46_19_AM_92480596/index.tex`]

Book: Elementary differential equations and boundary value problems, 10th ed., Boyce and DiPrima
Section: Chapter 3, Second order linear equations, 3.3 Complex Roots of the Characteristic Equation , page 164
Problem number: 35.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_euler_ode", "second_order_change_of_variable_on_x_method_1", "second_order_change_of_variable_on_x_method_2", "second_order_change_of_variable_on_y_method_2"**

Maple gives the following as the ode type

```
[[_Emden, _Fowler], [_2nd_order, _linear, `_with_symmetry_[0,F(
   x)]`]]
```

$$\boxed{t^2y'' + ty' + y = 0}$$

8.21.1 Solving as second order euler ode ode

This is Euler second order ODE. Let the solution be $y = t^r$, then $y' = rt^{r-1}$ and $y'' = r(r-1)t^{r-2}$. Substituting these back into the given ODE gives

$$t^2(r(r-1))t^{r-2} + trt^{r-1} + t^r = 0$$

Simplifying gives

$$r(r-1)\,t^r + r\,t^r + t^r = 0$$

Since $t^r \neq 0$ then dividing throughout by t^r gives

$$r(r-1)+r+1=0$$

Or

$$r^2+1=0 \tag{1}$$

Equation (1) is the characteristic equation. Its roots determine the form of the general solution. Using the quadratic equation the roots are

$$r_1 = -i$$
$$r_2 = i$$

The roots are complex conjugate of each others. Let the roots be

$$r_1 = \alpha + i\beta$$
$$r_2 = \alpha - i\beta$$

Where in this case $\alpha = 0$ and $\beta = -1$. Hence the solution becomes

$$\begin{aligned} y &= c_1 t^{r_1} + c_2 t^{r_2} \\ &= c_1 t^{\alpha+i\beta} + c_2 t^{\alpha-i\beta} \\ &= t^{\alpha}\left(c_1 t^{i\beta} + c_2 t^{-i\beta}\right) \\ &= t^{\alpha}\left(c_1 e^{\ln\left(t^{i\beta}\right)} + c_2 e^{\ln\left(t^{-i\beta}\right)}\right) \\ &= t^{\alpha}\left(c_1 e^{i(\beta\ln t)} + c_2 e^{-i(\beta\ln t)}\right) \end{aligned}$$

Using the values for $\alpha = 0, \beta = -1$, the above becomes

$$y = t^0\left(c_1 e^{-i\ln(t)} + c_2 e^{i\ln(t)}\right)$$

Using Euler relation, the expression $c_1 e^{iA} + c_2 e^{-iA}$ is transformed to $c_1 \cos A + c_1 \sin A$ where the constants are free to change. Applying this to the above result gives

$$y = c_1 \cos(\ln(t)) + c_2 \sin(\ln(t))$$

Summary

The solution(s) found are the following

$$y = c_1 \cos(\ln(t)) + c_2 \sin(\ln(t)) \tag{1}$$

Verification of solutions

$$y = c_1 \cos(\ln(t)) + c_2 \sin(\ln(t))$$

Verified OK.

8.21.2 Solving as second order change of variable on x method 2 ode

In normal form the ode

$$t^2y'' + ty' + y = 0 \tag{1}$$

Becomes

$$y'' + p(t)\, y' + q(t)\, y = 0 \tag{2}$$

Where

$$\begin{aligned} p(t) &= \frac{1}{t} \\ q(t) &= \frac{1}{t^2} \end{aligned}$$

Applying change of variables $\tau = g(t)$ to (2) gives

$$\frac{d^2}{d\tau^2}y(\tau) + p_1\left(\frac{d}{d\tau}y(\tau)\right) + q_1 y(\tau) = 0 \tag{3}$$

Where τ is the new independent variable, and

$$p_1(\tau) = \frac{\tau''(t) + p(t)\,\tau'(t)}{\tau'(t)^2} \tag{4}$$

$$q_1(\tau) = \frac{q(t)}{\tau'(t)^2} \tag{5}$$

Let $p_1 = 0$. Eq (4) simplifies to

$$\tau''(t) + p(t)\,\tau'(t) = 0$$

This ode is solved resulting in

$$\begin{aligned} \tau &= \int \mathrm{e}^{-\left(\int p(t)dt\right)}dt \\ &= \int \mathrm{e}^{-\left(\int \frac{1}{t}dt\right)}dt \\ &= \int e^{-\ln(t)}\, dt \\ &= \int \frac{1}{t}dt \\ &= \ln(t) \end{aligned} \tag{6}$$

Using (6) to evaluate q_1 from (5) gives

$$\begin{aligned} q_1(\tau) &= \frac{q(t)}{\tau'(t)^2} \\ &= \frac{\frac{1}{t^2}}{\frac{1}{t^2}} \\ &= 1 \end{aligned} \tag{7}$$

Substituting the above in (3) and noting that now $p_1 = 0$ results in

$$\begin{aligned} \frac{d^2}{d\tau^2}y(\tau) + q_1 y(\tau) &= 0 \\ \frac{d^2}{d\tau^2}y(\tau) + y(\tau) &= 0 \end{aligned}$$

The above ode is now solved for $y(\tau)$.This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$Ay''(\tau) + By'(\tau) + Cy(\tau) = 0$$

Where in the above $A = 1, B = 0, C = 1$. Let the solution be $y(\tau) = e^{\lambda\tau}$. Substituting this into the ODE gives

$$\lambda^2 e^{\lambda\tau} + e^{\lambda\tau} = 0 \tag{1}$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda\tau}$ gives

$$\lambda^2 + 1 = 0 \tag{2}$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form.Using the quadratic formula

$$\lambda_{1,2} = \frac{-B}{2A} \pm \frac{1}{2A}\sqrt{B^2 - 4AC}$$

Substituting $A = 1, B = 0, C = 1$ into the above gives

$$\begin{aligned} \lambda_{1,2} &= \frac{0}{(2)(1)} \pm \frac{1}{(2)(1)}\sqrt{0^2 - (4)(1)(1)} \\ &= \pm i \end{aligned}$$

Hence

$$\begin{aligned} \lambda_1 &= +i \\ \lambda_2 &= -i \end{aligned}$$

Which simplifies to

$$\lambda_1 = i$$
$$\lambda_2 = -i$$

Since roots are complex conjugate of each others, then let the roots be

$$\lambda_{1,2} = \alpha \pm i\beta$$

Where $\alpha = 0$ and $\beta = 1$. Therefore the final solution, when using Euler relation, can be written as

$$y(\tau) = e^{\alpha\tau}(c_1 \cos(\beta\tau) + c_2 \sin(\beta\tau))$$

Which becomes

$$y(\tau) = e^0(c_1 \cos(\tau) + c_2 \sin(\tau))$$

Or

$$y(\tau) = c_1 \cos(\tau) + c_2 \sin(\tau)$$

The above solution is now transformed back to y using (6) which results in

$$y = c_1 \cos(\ln(t)) + c_2 \sin(\ln(t))$$

Summary

The solution(s) found are the following

$$y = c_1 \cos(\ln(t)) + c_2 \sin(\ln(t)) \tag{1}$$

Verification of solutions

$$y = c_1 \cos(\ln(t)) + c_2 \sin(\ln(t))$$

Verified OK.

8.21.3 Solving as second order change of variable on x method 1 ode

In normal form the ode

$$t^2y'' + ty' + y = 0 \tag{1}$$

Becomes

$$y'' + p(t)\,y' + q(t)\,y = 0 \tag{2}$$

Where

$$p(t) = \frac{1}{t}$$
$$q(t) = \frac{1}{t^2}$$

Applying change of variables $\tau = g(t)$ to (2) results

$$\frac{d^2}{d\tau^2}y(\tau) + p_1\left(\frac{d}{d\tau}y(\tau)\right) + q_1 y(\tau) = 0 \tag{3}$$

Where τ is the new independent variable, and

$$p_1(\tau) = \frac{\tau''(t) + p(t)\,\tau'(t)}{\tau'(t)^2} \tag{4}$$
$$q_1(\tau) = \frac{q(t)}{\tau'(t)^2} \tag{5}$$

Let $q_1 = c^2$ where c is some constant. Therefore from (5)

$$\begin{aligned} \tau' &= \frac{1}{c}\sqrt{q} \\ &= \frac{\sqrt{\frac{1}{t^2}}}{c} \\ \tau'' &= -\frac{1}{c\sqrt{\frac{1}{t^2}}\,t^3} \end{aligned} \tag{6}$$

Substituting the above into (4) results in

$$\begin{aligned} p_1(\tau) &= \frac{\tau''(t) + p(t)\,\tau'(t)}{\tau'(t)^2} \\ &= \frac{-\frac{1}{c\sqrt{\frac{1}{t^2}}\,t^3} + \frac{1}{t}\frac{\sqrt{\frac{1}{t^2}}}{c}}{\left(\frac{\sqrt{\frac{1}{t^2}}}{c}\right)^2} \\ &= 0 \end{aligned}$$

Therefore ode (3) now becomes

$$\begin{aligned} y(\tau)'' + p_1 y(\tau)' + q_1 y(\tau) &= 0 \\ \frac{d^2}{d\tau^2}y(\tau) + c^2 y(\tau) &= 0 \end{aligned} \tag{7}$$

The above ode is now solved for $y(\tau)$. Since the ode is now constant coefficients, it can be easily solved to give

$$y(\tau) = c_1 \cos(c\tau) + c_2 \sin(c\tau)$$

Now from (6)

$$\begin{aligned}\tau &= \int \frac{1}{c}\sqrt{q}\,dt \\ &= \frac{\int \sqrt{\frac{1}{t^2}}dt}{c} \\ &= \frac{\sqrt{\frac{1}{t^2}}\,t\ln(t)}{c}\end{aligned}$$

Substituting the above into the solution obtained gives

$$y = c_1 \cos(\ln(t)) + c_2 \sin(\ln(t))$$

Summary
The solution(s) found are the following

$$y = c_1 \cos(\ln(t)) + c_2 \sin(\ln(t)) \tag{1}$$

Verification of solutions

$$y = c_1 \cos(\ln(t)) + c_2 \sin(\ln(t))$$

Verified OK.

8.21.4 Solving as second order change of variable on y method 2 ode

In normal form the ode

$$t^2y'' + ty' + y = 0 \tag{1}$$

Becomes

$$y'' + p(t)\,y' + q(t)\,y = 0 \tag{2}$$

Where

$$\begin{aligned}p(t) &= \frac{1}{t} \\ q(t) &= \frac{1}{t^2}\end{aligned}$$

Applying change of variables on the depndent variable $y = v(t)\, t^n$ to (2) gives the following ode where the dependent variables is $v(t)$ and not y.

$$v''(t) + \left(\frac{2n}{t} + p\right) v'(t) + \left(\frac{n(n-1)}{t^2} + \frac{np}{t} + q\right) v(t) = 0 \tag{3}$$

Let the coefficient of $v(t)$ above be zero. Hence

$$\frac{n(n-1)}{t^2} + \frac{np}{t} + q = 0 \tag{4}$$

Substituting the earlier values found for $p(t)$ and $q(t)$ into (4) gives

$$\frac{n(n-1)}{t^2} + \frac{n}{t^2} + \frac{1}{t^2} = 0 \tag{5}$$

Solving (5) for n gives

$$n = i \tag{6}$$

Substituting this value in (3) gives

$$\begin{aligned} v''(t) + \left(\frac{2i}{t} + \frac{1}{t}\right) v'(t) &= 0 \\ v''(t) + \frac{(1+2i)\, v'(t)}{t} &= 0 \end{aligned} \tag{7}$$

Using the substitution

$$u(t) = v'(t)$$

Then (7) becomes

$$u'(t) + \frac{(1+2i)\, u(t)}{t} = 0 \tag{8}$$

The above is now solved for $u(t)$. In canonical form the ODE is

$$\begin{aligned} u' &= F(t, u) \\ &= f(t) g(u) \\ &= \frac{(-1-2i)\, u}{t} \end{aligned}$$

Where $f(t) = \frac{-1-2i}{t}$ and $g(u) = u$. Integrating both sides gives

$$\begin{aligned}
\frac{1}{u}\,du &= \frac{-1-2i}{t}\,dt \\
\int \frac{1}{u}\,du &= \int \frac{-1-2i}{t}\,dt \\
\ln(u) &= (-1-2i)\ln(t) + c_1 \\
u &= e^{(-1-2i)\ln(t)+c_1} \\
&= c_1 e^{(-1-2i)\ln(t)}
\end{aligned}$$

Which simplifies to

$$u(t) = \frac{c_1 t^{-2i}}{t}$$

Now that $u(t)$ is known, then

$$\begin{aligned}
v'(t) &= u(t) \\
v(t) &= \int u(t)\,dt + c_2 \\
&= \frac{i c_1 t^{-2i}}{2} + c_2
\end{aligned}$$

Hence

$$\begin{aligned}
y &= v(t)\,t^n \\
&= \left(\frac{i c_1 t^{-2i}}{2} + c_2\right) t^i \\
&= t^i c_2 + \frac{i t^{-i} c_1}{2}
\end{aligned}$$

Summary

The solution(s) found are the following

$$y = \left(\frac{i c_1 t^{-2i}}{2} + c_2\right) t^i \tag{1}$$

Verification of solutions

$$y = \left(\frac{i c_1 t^{-2i}}{2} + c_2\right) t^i$$

Verified OK.

8.21.5 Solving using Kovacic algorithm

Writing the ode as

$$t^2y'' + ty' + y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= t^2 \\ B &= t \\ C &= 1 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(t) = ye^{\int \frac{B}{2A}\, dt}$$

Then (2) becomes

$$z''(t) = rz(t) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{-5}{4t^2} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= -5 \\ t &= 4t^2 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(t) = \left(-\frac{5}{4t^2}\right) z(t) \tag{7}$$

Equation (7) is now solved. After finding $z(t)$ then y is found using the inverse transformation

$$y = z(t)\, e^{-\int \frac{B}{2A}\, dt}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 404: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 2 - 0 \\ &= 2 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 4t^2$. There is a pole at $t = 0$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 2 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Since pole order is not larger than 2 and the order at ∞ is 2 then the necessary conditions for case three are met. Therefore

$$L = [1, 2, 4, 6, 12]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = -\frac{5}{4t^2}$$

For the pole at $t = 0$ let b be the coefficient of $\frac{1}{t^2}$ in the partial fractions decomposition of r given above. Therefore $b = -\frac{5}{4}$. Hence

$$
\begin{aligned}
[\sqrt{r}]_c &= 0 \\
\alpha_c^{+} &= \frac{1}{2} + \sqrt{1+4b} = \frac{1}{2} + i \\
\alpha_c^{-} &= \frac{1}{2} - \sqrt{1+4b} = \frac{1}{2} - i
\end{aligned}
$$

Since the order of r at ∞ is 2 then $[\sqrt{r}]_\infty = 0$. Let b be the coefficient of $\frac{1}{t^2}$ in the Laurent series expansion of r at ∞. which can be found by dividing the leading coefficient of s by the leading coefficient of t from

$$
r = \frac{s}{t} = -\frac{5}{4t^2}
$$

Since the $\gcd(s,t) = 1$. This gives $b = -\frac{5}{4}$. Hence

$$
\begin{aligned}
[\sqrt{r}]_\infty &= 0 \\
\alpha_\infty^{+} &= \frac{1}{2} + \sqrt{1+4b} = \frac{1}{2} + i \\
\alpha_\infty^{-} &= \frac{1}{2} - \sqrt{1+4b} = \frac{1}{2} - i
\end{aligned}
$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$
r = -\frac{5}{4t^2}
$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^{+}	α_c^{-}
0	2	0	$\frac{1}{2} + i$	$\frac{1}{2} - i$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^{+}	α_∞^{-}
2	0	$\frac{1}{2} + i$	$\frac{1}{2} - i$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^{\pm}$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^{\pm}$ have also been found, the next step is to determine possible non negative integer d from these using

$$
d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}
$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^{\pm}$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω.

Trying $\alpha_{\infty}^{-} = \frac{1}{2} - i$ then

$$\begin{aligned} d &= \alpha_{\infty}^{-} - \left(\alpha_{c_1}^{-}\right) \\ &= \frac{1}{2} - i - \left(\frac{1}{2} - i\right) \\ &= 0 \end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c \in \Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{t - c} \right) + s(\infty)[\sqrt{r}]_{\infty}$$

The above gives

$$\begin{aligned} \omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^{-}}{t - c_1} \right) + (-)[\sqrt{r}]_{\infty} \\ &= \frac{\frac{1}{2} - i}{t} + (-)\,(0) \\ &= \frac{\frac{1}{2} - i}{t} \\ &= \frac{\frac{1}{2} - i}{t} \end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(t)$ of degree $d = 0$ to solve the ode. The polynomial $p(t)$ needs to satisfy the equation

$$p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right) p = 0 \tag{1A}$$

Let

$$p(t) = 1 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$\begin{aligned} (0) + 2\left(\frac{\frac{1}{2} - i}{t}\right)(0) + \left(\left(\frac{-\frac{1}{2} + i}{t^2}\right) + \left(\frac{\frac{1}{2} - i}{t}\right)^2 - \left(-\frac{5}{4t^2}\right) \right) &= 0 \\ 0 &= 0 \end{aligned}$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned} z_1(t) &= p e^{\int \omega \, dt} \\ &= \mathrm{e}^{\int \frac{\frac{1}{2} - i}{t} dt} \\ &= t^{\frac{1}{2} - i} \end{aligned}$$

The first solution to the original ode in y is found from

$$
\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dt} \\
&= z_1 e^{-\int \frac{1}{2}\frac{t}{t^2}\,dt} \\
&= z_1 e^{-\frac{\ln(t)}{2}} \\
&= z_1 \left(\frac{1}{\sqrt{t}}\right)
\end{aligned}
$$

Which simplifies to

$$
y_1 = t^{-i}
$$

The second solution y_2 to the original ode is found using reduction of order

$$
y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dt}}{y_1^2}\,dt
$$

Substituting gives

$$
\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{t}{t^2}\,dt}}{(y_1)^2}\,dt \\
&= y_1 \int \frac{e^{-\ln(t)}}{(y_1)^2}\,dt \\
&= y_1 \left(-\frac{i t^{2i}}{2}\right)
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1 \left(t^{-i}\right) + c_2 \left(t^{-i}\left(-\frac{i t^{2i}}{2}\right)\right)
\end{aligned}
$$

Summary
The solution(s) found are the following

$$
y = t^{-i} c_1 - \frac{i c_2 t^i}{2} \tag{1}
$$

Verification of solutions

$$
y = t^{-i} c_1 - \frac{i c_2 t^i}{2}
$$

Verified OK.

8.21.6 Maple step by step solution

Let's solve

$y''t^2 + ty' + y = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y'' = -\frac{y'}{t} - \frac{y}{t^2}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y'' + \frac{y'}{t} + \frac{y}{t^2} = 0$

- Multiply by denominators of the ODE

  $y''t^2 + ty' + y = 0$

- Make a change of variables

  $s = \ln(t)$

□ Substitute the change of variables back into the ODE

  - Calculate the 1st derivative of y with respect to t , using the chain rule

    $y' = \left(\frac{d}{ds}y(s)\right) s'(t)$

  - Compute derivative

    $y' = \frac{\frac{d}{ds}y(s)}{t}$

  - Calculate the 2nd derivative of y with respect to t , using the chain rule

    $y'' = \left(\frac{d^2}{ds^2}y(s)\right) s'(t)^2 + s''(t) \left(\frac{d}{ds}y(s)\right)$

  - Compute derivative

    $y'' = \frac{\frac{d^2}{ds^2}y(s)}{t^2} - \frac{\frac{d}{ds}y(s)}{t^2}$

  Substitute the change of variables back into the ODE

  $\left(\frac{\frac{d^2}{ds^2}y(s)}{t^2} - \frac{\frac{d}{ds}y(s)}{t^2}\right) t^2 + \frac{d}{ds}y(s) + y(s) = 0$

- Simplify

  $\frac{d^2}{ds^2}y(s) + y(s) = 0$

- Characteristic polynomial of ODE

  $r^2 + 1 = 0$

- Use quadratic formula to solve for r

  $r = \frac{0 \pm (\sqrt{-4})}{2}$
- Roots of the characteristic polynomial

  $r = (-\mathrm{I}, \mathrm{I})$
- 1st solution of the ODE

  $y_1(s) = \cos(s)$
- 2nd solution of the ODE

  $y_2(s) = \sin(s)$
- General solution of the ODE

  $y(s) = c_1 y_1(s) + c_2 y_2(s)$
- Substitute in solutions

  $y(s) = c_1 \cos(s) + c_2 \sin(s)$
- Change variables back using $s = \ln(t)$

  $y = c_1 \cos(\ln(t)) + c_2 \sin(\ln(t))$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
<- LODE of Euler type successful`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 15

```
dsolve(t^2*diff(y(t),t$2)+ t*diff(y(t),t)+y(t) = 0,y(t), singsol=all)
```

$$y(t) = c_1 \sin(\ln(t)) + c_2 \cos(\ln(t))$$

✓ Solution by Mathematica

Time used: 0.017 (sec). Leaf size: 18

```
DSolve[t^2*y''[t]+t*y'[t]+y[t]==0,y[t],t,IncludeSingularSolutions -> True]
```

$$y(t) \to c_1 \cos(\log(t)) + c_2 \sin(\log(t))$$

8.22 problem 36

Internal problem ID [644]
Internal file name [`OUTPUT/644_Sunday_June_05_2022_01_46_20_AM_35632985/index.tex`]

Book: Elementary differential equations and boundary value problems, 10th ed., Boyce and DiPrima
Section: Chapter 3, Second order linear equations, 3.3 Complex Roots of the Characteristic Equation , page 164
Problem number: 36.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_euler_ode", "exact linear second order ode", "second_order_integrable_as_is", "second_order_change_of_variable_on_x_method_1", "second_order_change_of_variable_on_x_method_2", "second_order_change_of_variable_on_y_method_1", "second_order_change_of_variable_on_y_method_2", "linear_second_order_ode_solved_by_an_integrating_factor"**

Maple gives the following as the ode type

```
[[_2nd_order, _exact, _linear, _homogeneous]]
```

$$\boxed{t^2y'' + 4ty' + 2y = 0}$$

8.22.1 Solving as second order euler ode ode

This is Euler second order ODE. Let the solution be $y = t^r$, then $y' = rt^{r-1}$ and $y'' = r(r-1)t^{r-2}$. Substituting these back into the given ODE gives

$$t^2(r(r-1))t^{r-2} + 4trt^{r-1} + 2t^r = 0$$

Simplifying gives

$$r(r-1)\,t^r + 4r\,t^r + 2t^r = 0$$

Since $t^r \neq 0$ then dividing throughout by t^r gives

$$r(r-1) + 4r + 2 = 0$$

Or

$$r^2 + 3r + 2 = 0 \tag{1}$$

Equation (1) is the characteristic equation. Its roots determine the form of the general solution. Using the quadratic equation the roots are

$$r_1 = -2$$
$$r_2 = -1$$

Since the roots are real and distinct, then the general solution is

$$y = c_1y_1 + c_2y_2$$

Where $y_1 = t^{r_1}$ and $y_2 = t^{r_2}$. Hence

$$y = \frac{c_1}{t^2} + \frac{c_2}{t}$$

Summary

The solution(s) found are the following

$$y = \frac{c_1}{t^2} + \frac{c_2}{t} \tag{1}$$

Verification of solutions

$$y = \frac{c_1}{t^2} + \frac{c_2}{t}$$

Verified OK.

8.22.2 Solving as linear second order ode solved by an integrating factor ode

The ode satisfies this form

$$y'' + p(t)\, y' + \frac{(p(t)^2 + p'(t))\, y}{2} = f(t)$$

Where $p(t) = \frac{4}{t}$. Therefore, there is an integrating factor given by

$$\begin{aligned} M(x) &= e^{\frac{1}{2}\int p\, dx} \\ &= e^{\int \frac{4}{t}\, dx} \\ &= t^2 \end{aligned}$$

Multiplying both sides of the ODE by the integrating factor $M(x)$ makes the left side of the ODE a complete differential

$$\begin{aligned} (M(x)y)'' &= 0 \\ (t^2y)'' &= 0 \end{aligned}$$

Integrating once gives

$$(t^2y)' = c_1$$

Integrating again gives

$$(t^2y) = c_1t + c_2$$

Hence the solution is

$$y = \frac{c_1t + c_2}{t^2}$$

Or

$$y = \frac{c_1}{t} + \frac{c_2}{t^2}$$

Summary

The solution(s) found are the following

$$y = \frac{c_1}{t} + \frac{c_2}{t^2} \tag{1}$$

Verification of solutions

$$y = \frac{c_1}{t} + \frac{c_2}{t^2}$$

Verified OK.

8.22.3 Solving as second order change of variable on x method 2 ode

In normal form the ode

$$t^2y'' + 4ty' + 2y = 0 \tag{1}$$

Becomes

$$y'' + p(t)\, y' + q(t)\, y = 0 \tag{2}$$

Where

$$\begin{aligned}p(t) &= \frac{4}{t}\\ q(t) &= \frac{2}{t^2}\end{aligned}$$

Applying change of variables $\tau = g(t)$ to (2) gives

$$\frac{d^2}{d\tau^2}y(\tau) + p_1\left(\frac{d}{d\tau}y(\tau)\right) + q_1 y(\tau) = 0 \tag{3}$$

Where τ is the new independent variable, and

$$p_1(\tau) = \frac{\tau''(t) + p(t)\,\tau'(t)}{\tau'(t)^2} \tag{4}$$

$$q_1(\tau) = \frac{q(t)}{\tau'(t)^2} \tag{5}$$

Let $p_1 = 0$. Eq (4) simplifies to

$$\tau''(t) + p(t)\,\tau'(t) = 0$$

This ode is solved resulting in

$$\begin{aligned}\tau &= \int e^{-\left(\int p(t)dt\right)}dt\\ &= \int e^{-\left(\int \frac{4}{t}dt\right)}dt\\ &= \int e^{-4\ln(t)}\, dt\\ &= \int \frac{1}{t^4}dt\\ &= -\frac{1}{3t^3}\end{aligned} \tag{6}$$

Using (6) to evaluate q_1 from (5) gives

$$\begin{aligned}
q_1(\tau) &= \frac{q(t)}{\tau'(t)^2} \\
&= \frac{\frac{2}{t^2}}{\frac{1}{t^8}} \\
&= 2t^6
\end{aligned} \tag{7}$$

Substituting the above in (3) and noting that now $p_1 = 0$ results in

$$\begin{aligned}
\frac{d^2}{d\tau^2}y(\tau) + q_1 y(\tau) &= 0 \\
\frac{d^2}{d\tau^2}y(\tau) + 2t^6 y(\tau) &= 0
\end{aligned}$$

But in terms of τ

$$2t^6 = \frac{2}{9\tau^2}$$

Hence the above ode becomes

$$\frac{d^2}{d\tau^2}y(\tau) + \frac{2y(\tau)}{9\tau^2} = 0$$

The above ode is now solved for $y(\tau)$. The ode can be written as

$$9\left(\frac{d^2}{d\tau^2}y(\tau)\right)\tau^2 + 2y(\tau) = 0$$

Which shows it is a Euler ODE. This is Euler second order ODE. Let the solution be $y(\tau) = \tau^r$, then $y' = r\tau^{r-1}$ and $y'' = r(r-1)\tau^{r-2}$. Substituting these back into the given ODE gives

$$9\tau^2(r(r-1))\tau^{r-2} + 0r\tau^{r-1} + 2\tau^r = 0$$

Simplifying gives

$$9r(r-1)\,\tau^r + 0\,\tau^r + 2\tau^r = 0$$

Since $\tau^r \neq 0$ then dividing throughout by τ^r gives

$$9r(r-1) + 0 + 2 = 0$$

Or

$$9r^2 - 9r + 2 = 0 \tag{1}$$

Equation (1) is the characteristic equation. Its roots determine the form of the general solution. Using the quadratic equation the roots are

$$r_1 = \frac{1}{3}$$
$$r_2 = \frac{2}{3}$$

Since the roots are real and distinct, then the general solution is

$$y(\tau) = c_1 y_1 + c_2 y_2$$

Where $y_1 = \tau^{r_1}$ and $y_2 = \tau^{r_2}$. Hence

$$y(\tau) = c_1 \tau^{\frac{1}{3}} + c_2 \tau^{\frac{2}{3}}$$

The above solution is now transformed back to y using (6) which results in

$$y = \frac{c_1 3^{\frac{2}{3}} \left(-\frac{1}{t^3}\right)^{\frac{1}{3}}}{3} + \frac{c_2 3^{\frac{1}{3}} \left(-\frac{1}{t^3}\right)^{\frac{2}{3}}}{3}$$

Summary

The solution(s) found are the following

$$y = \frac{c_1 3^{\frac{2}{3}} \left(-\frac{1}{t^3}\right)^{\frac{1}{3}}}{3} + \frac{c_2 3^{\frac{1}{3}} \left(-\frac{1}{t^3}\right)^{\frac{2}{3}}}{3} \tag{1}$$

Verification of solutions

$$y = \frac{c_1 3^{\frac{2}{3}} \left(-\frac{1}{t^3}\right)^{\frac{1}{3}}}{3} + \frac{c_2 3^{\frac{1}{3}} \left(-\frac{1}{t^3}\right)^{\frac{2}{3}}}{3}$$

Verified OK.

8.22.4 Solving as second order change of variable on x method 1 ode

In normal form the ode

$$t^2 y'' + 4ty' + 2y = 0 \tag{1}$$

Becomes

$$y'' + p(t)\, y' + q(t)\, y = 0 \tag{2}$$

Where

$$p(t) = \frac{4}{t}$$
$$q(t) = \frac{2}{t^2}$$

Applying change of variables $\tau = g(t)$ to (2) results

$$\frac{d^2}{d\tau^2}y(\tau) + p_1\left(\frac{d}{d\tau}y(\tau)\right) + q_1y(\tau) = 0 \tag{3}$$

Where τ is the new independent variable, and

$$p_1(\tau) = \frac{\tau''(t) + p(t)\,\tau'(t)}{\tau'(t)^2} \tag{4}$$

$$q_1(\tau) = \frac{q(t)}{\tau'(t)^2} \tag{5}$$

Let $q_1 = c^2$ where c is some constant. Therefore from (5)

$$\begin{aligned}\tau' &= \frac{1}{c}\sqrt{q}\\ &= \frac{\sqrt{2}\,\sqrt{\frac{1}{t^2}}}{c}\\ \tau'' &= -\frac{\sqrt{2}}{c\sqrt{\frac{1}{t^2}}\,t^3}\end{aligned} \tag{6}$$

Substituting the above into (4) results in

$$\begin{aligned}p_1(\tau) &= \frac{\tau''(t) + p(t)\,\tau'(t)}{\tau'(t)^2}\\ &= \frac{-\frac{\sqrt{2}}{c\sqrt{\frac{1}{t^2}}\,t^3} + \frac{4}{t}\frac{\sqrt{2}\,\sqrt{\frac{1}{t^2}}}{c}}{\left(\frac{\sqrt{2}\,\sqrt{\frac{1}{t^2}}}{c}\right)^2}\\ &= \frac{3c\sqrt{2}}{2}\end{aligned}$$

Therefore ode (3) now becomes

$$\begin{aligned}y(\tau)'' + p_1y(\tau)' + q_1y(\tau) &= 0\\ \frac{d^2}{d\tau^2}y(\tau) + \frac{3c\sqrt{2}\left(\frac{d}{d\tau}y(\tau)\right)}{2} + c^2y(\tau) &= 0\end{aligned} \tag{7}$$

The above ode is now solved for $y(\tau)$. Since the ode is now constant coefficients, it can be easily solved to give

$$y(\tau) = \mathrm{e}^{-\frac{3\sqrt{2}\,c\tau}{4}}\left(c_1\cosh\left(\frac{\sqrt{2}\,c\tau}{4}\right) + ic_2\sinh\left(\frac{\sqrt{2}\,c\tau}{4}\right)\right)$$

Now from (6)

$$
\begin{aligned}
\tau &= \int \frac{1}{c}\sqrt{q}\, dt \\
&= \frac{\int \sqrt{2}\, \sqrt{\frac{1}{t^2}} dt}{c} \\
&= \frac{\sqrt{2}\, \sqrt{\frac{1}{t^2}}\, t \ln(t)}{c}
\end{aligned}
$$

Substituting the above into the solution obtained gives

$$
y = \frac{c_1 \cosh\left(\frac{\ln(t)}{2}\right) + i c_2 \sinh\left(\frac{\ln(t)}{2}\right)}{t^{\frac{3}{2}}}
$$

Summary

The solution(s) found are the following

$$
y = \frac{c_1 \cosh\left(\frac{\ln(t)}{2}\right) + i c_2 \sinh\left(\frac{\ln(t)}{2}\right)}{t^{\frac{3}{2}}} \tag{1}
$$

Verification of solutions

$$
y = \frac{c_1 \cosh\left(\frac{\ln(t)}{2}\right) + i c_2 \sinh\left(\frac{\ln(t)}{2}\right)}{t^{\frac{3}{2}}}
$$

Verified OK.

8.22.5 Solving as second order change of variable on y method 1 ode

In normal form the given ode is written as

$$
y'' + p(t)\, y' + q(t)\, y = 0 \tag{2}
$$

Where

$$
\begin{aligned}
p(t) &= \frac{4}{t} \\
q(t) &= \frac{2}{t^2}
\end{aligned}
$$

Calculating the Liouville ode invariant Q given by

$$
\begin{aligned}
Q &= q - \frac{p'}{2} - \frac{p^2}{4} \\
&= \frac{2}{t^2} - \frac{\left(\frac{4}{t}\right)'}{2} - \frac{\left(\frac{4}{t}\right)^2}{4} \\
&= \frac{2}{t^2} - \frac{\left(-\frac{4}{t^2}\right)}{2} - \frac{\left(\frac{16}{t^2}\right)}{4} \\
&= \frac{2}{t^2} - \left(-\frac{2}{t^2}\right) - \frac{4}{t^2} \\
&= 0
\end{aligned}
$$

Since the Liouville ode invariant does not depend on the independent variable t then the transformation

$$
y = v(t)\, z(t) \tag{3}
$$

is used to change the original ode to a constant coefficients ode in v. In (3) the term $z(t)$ is given by

$$
\begin{aligned}
z(t) &= \mathrm{e}^{-\left(\int \frac{p(t)}{2} dt\right)} \\
&= e^{-\int \frac{\frac{4}{t}}{2}} \\
&= \frac{1}{t^2}
\end{aligned} \tag{5}
$$

Hence (3) becomes

$$
y = \frac{v(t)}{t^2} \tag{4}
$$

Applying this change of variable to the original ode results in

$$
v''(t) = 0
$$

Which is now solved for $v(t)$ Integrating twice gives the solution

$$
v(t) = c_1 t + c_2
$$

Now that $v(t)$ is known, then

$$
\begin{aligned}
y &= v(t)\, z(t) \\
&= (c_1 t + c_2)\,(z(t))
\end{aligned} \tag{7}
$$

But from (5)

$$z(t) = \frac{1}{t^2}$$

Hence (7) becomes

$$y = \frac{c_1 t + c_2}{t^2}$$

Summary

The solution(s) found are the following

$$y = \frac{c_1 t + c_2}{t^2} \tag{1}$$

Verification of solutions

$$y = \frac{c_1 t + c_2}{t^2}$$

Verified OK.

8.22.6 Solving as second order change of variable on y method 2 ode

In normal form the ode

$$t^2 y'' + 4ty' + 2y = 0 \tag{1}$$

Becomes

$$y'' + p(t)\, y' + q(t)\, y = 0 \tag{2}$$

Where

$$p(t) = \frac{4}{t}$$
$$q(t) = \frac{2}{t^2}$$

Applying change of variables on the depndent variable $y = v(t)\, t^n$ to (2) gives the following ode where the dependent variables is $v(t)$ and not y.

$$v''(t) + \left(\frac{2n}{t} + p\right) v'(t) + \left(\frac{n(n-1)}{t^2} + \frac{np}{t} + q\right) v(t) = 0 \tag{3}$$

Let the coefficient of $v(t)$ above be zero. Hence

$$\frac{n(n-1)}{t^2} + \frac{np}{t} + q = 0 \tag{4}$$

Substituting the earlier values found for $p(t)$ and $q(t)$ into (4) gives

$$\frac{n(n-1)}{t^2} + \frac{4n}{t^2} + \frac{2}{t^2} = 0 \tag{5}$$

Solving (5) for n gives

$$n = -1 \tag{6}$$

Substituting this value in (3) gives

$$\begin{aligned} v''(t) + \frac{2v'(t)}{t} &= 0 \\ v''(t) + \frac{2v'(t)}{t} &= 0 \end{aligned} \tag{7}$$

Using the substitution

$$u(t) = v'(t)$$

Then (7) becomes

$$u'(t) + \frac{2u(t)}{t} = 0 \tag{8}$$

The above is now solved for $u(t)$. In canonical form the ODE is

$$\begin{aligned} u' &= F(t, u) \\ &= f(t)g(u) \\ &= -\frac{2u}{t} \end{aligned}$$

Where $f(t) = -\frac{2}{t}$ and $g(u) = u$. Integrating both sides gives

$$\begin{aligned} \frac{1}{u}\, du &= -\frac{2}{t}\, dt \\ \int \frac{1}{u}\, du &= \int -\frac{2}{t}\, dt \\ \ln(u) &= -2\ln(t) + c_1 \\ u &= e^{-2\ln(t)+c_1} \\ &= \frac{c_1}{t^2} \end{aligned}$$

Now that $u(t)$ is known, then

$$
\begin{aligned}
v'(t) &= u(t) \\
v(t) &= \int u(t)\, dt + c_2 \\
&= -\frac{c_1}{t} + c_2
\end{aligned}
$$

Hence

$$
\begin{aligned}
y &= v(t)\, t^n \\
&= \frac{-\frac{c_1}{t} + c_2}{t} \\
&= \frac{c_2 t - c_1}{t^2}
\end{aligned}
$$

Summary

The solution(s) found are the following

$$
y = \frac{-\frac{c_1}{t} + c_2}{t} \tag{1}
$$

Verification of solutions

$$
y = \frac{-\frac{c_1}{t} + c_2}{t}
$$

Verified OK.

8.22.7 Solving as second order integrable as is ode

Integrating both sides of the ODE w.r.t t gives

$$
\begin{aligned}
\int \left(t^2 y'' + 4ty' + 2y\right) dt &= 0 \\
y' t^2 + 2yt &= c_1
\end{aligned}
$$

Which is now solved for y.

Entering Linear first order ODE solver. In canonical form a linear first order is

$$
y' + p(t)y = q(t)
$$

Where here

$$
\begin{aligned}
p(t) &= \frac{2}{t} \\
q(t) &= \frac{c_1}{t^2}
\end{aligned}
$$

Hence the ode is

$$y' + \frac{2y}{t} = \frac{c_1}{t^2}$$

The integrating factor μ is

$$\begin{aligned}\mu &= e^{\int \frac{2}{t}dt} \\ &= t^2\end{aligned}$$

The ode becomes

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}(\mu y) &= (\mu)\left(\frac{c_1}{t^2}\right) \\ \frac{\mathrm{d}}{\mathrm{d}t}(t^2 y) &= (t^2)\left(\frac{c_1}{t^2}\right) \\ \mathrm{d}(t^2 y) &= c_1\,\mathrm{d}t\end{aligned}$$

Integrating gives

$$\begin{aligned}t^2 y &= \int c_1\,\mathrm{d}t \\ t^2 y &= c_1 t + c_2\end{aligned}$$

Dividing both sides by the integrating factor $\mu = t^2$ results in

$$y = \frac{c_1}{t} + \frac{c_2}{t^2}$$

which simplifies to

$$y = \frac{c_1 t + c_2}{t^2}$$

Summary

The solution(s) found are the following

$$y = \frac{c_1 t + c_2}{t^2} \tag{1}$$

Verification of solutions

$$y = \frac{c_1 t + c_2}{t^2}$$

Verified OK.

8.22.8 Solving as type second_order_integrable_as_is (not using ABC version)

Writing the ode as

$$t^2y'' + 4ty' + 2y = 0$$

Integrating both sides of the ODE w.r.t t gives

$$\int (t^2y'' + 4ty' + 2y)\,dt = 0$$
$$y't^2 + 2yt = c_1$$

Which is now solved for y.

Entering Linear first order ODE solver. In canonical form a linear first order is

$$y' + p(t)y = q(t)$$

Where here

$$p(t) = \frac{2}{t}$$
$$q(t) = \frac{c_1}{t^2}$$

Hence the ode is

$$y' + \frac{2y}{t} = \frac{c_1}{t^2}$$

The integrating factor μ is

$$\mu = e^{\int \frac{2}{t}dt}$$
$$= t^2$$

The ode becomes

$$\frac{d}{dt}(\mu y) = (\mu)\left(\frac{c_1}{t^2}\right)$$
$$\frac{d}{dt}(t^2y) = (t^2)\left(\frac{c_1}{t^2}\right)$$
$$d(t^2y) = c_1\,dt$$

Integrating gives

$$t^2y = \int c_1\,dt$$
$$t^2y = c_1t + c_2$$

Dividing both sides by the integrating factor $\mu = t^2$ results in

$$y = \frac{c_1}{t} + \frac{c_2}{t^2}$$

which simplifies to

$$y = \frac{c_1 t + c_2}{t^2}$$

Summary

The solution(s) found are the following

$$y = \frac{c_1 t + c_2}{t^2} \tag{1}$$

Verification of solutions

$$y = \frac{c_1 t + c_2}{t^2}$$

Verified OK.

8.22.9 Solving using Kovacic algorithm

Writing the ode as

$$t^2 y'' + 4ty' + 2y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= t^2 \\ B &= 4t \\ C &= 2 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(t) = y e^{\int \frac{B}{2A} dt}$$

Then (2) becomes

$$z''(t) = r z(t) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{0}{1} \tag{6}$$

Comparing the above to (5) shows that

$$s = 0$$
$$t = 1$$

Therefore eq. (4) becomes

$$z''(t) = 0 \tag{7}$$

Equation (7) is now solved. After finding $z(t)$ then y is found using the inverse transformation

$$y = z(t)\, e^{-\int \frac{B}{2A}\, dt}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 406: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 0 - -\infty \\ &= \infty \end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is $infinity$ then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Since $r = 0$ is not a function of t, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$z_1(t) = 1$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dt} \\
&= z_1 e^{-\int \frac{1}{2}\frac{4t}{t^2}\,dt} \\
&= z_1 e^{-2\ln(t)} \\
&= z_1\left(\frac{1}{t^2}\right)
\end{aligned}$$

Which simplifies to

$$y_1 = \frac{1}{t^2}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dt}}{y_1^2}\,dt$$

Substituting gives

$$\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{4t}{t^2}\,dt}}{(y_1)^2}\,dt \\
&= y_1 \int \frac{e^{-4\ln(t)}}{(y_1)^2}\,dt \\
&= y_1(t)
\end{aligned}$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1 \left(\frac{1}{t^2}\right) + c_2 \left(\frac{1}{t^2}(t)\right)
\end{aligned}
$$

<u>Summary</u>
The solution(s) found are the following

$$
y = \frac{c_1}{t^2} + \frac{c_2}{t} \tag{1}
$$

<u>Verification of solutions</u>

$$
y = \frac{c_1}{t^2} + \frac{c_2}{t}
$$

Verified OK.

8.22.10 Solving as exact linear second order ode ode

An ode of the form

$$
p(t)\, y'' + q(t)\, y' + r(t)\, y = s(t)
$$

is exact if

$$
p''(t) - q'(t) + r(t) = 0 \tag{1}
$$

For the given ode we have

$$
\begin{aligned}
p(x) &= t^2 \\
q(x) &= 4t \\
r(x) &= 2 \\
s(x) &= 0
\end{aligned}
$$

Hence

$$
\begin{aligned}
p''(x) &= 2 \\
q'(x) &= 4
\end{aligned}
$$

Therefore (1) becomes

$$
2 - (4) + (2) = 0
$$

Hence the ode is exact. Since we now know the ode is exact, it can be written as

$$
\left(p(t)\, y' + \left(q(t) - p'(t)\right) y\right)' = s(x)
$$

Integrating gives

$$p(t)\, y' + (q(t) - p'(t))\, y = \int s(t)\ dt$$

Substituting the above values for p, q, r, s gives

$$y' t^2 + 2yt = c_1$$

We now have a first order ode to solve which is

$$y' t^2 + 2yt = c_1$$

Entering Linear first order ODE solver. In canonical form a linear first order is

$$y' + p(t)y = q(t)$$

Where here

$$\begin{aligned}
p(t) &= \frac{2}{t} \\
q(t) &= \frac{c_1}{t^2}
\end{aligned}$$

Hence the ode is

$$y' + \frac{2y}{t} = \frac{c_1}{t^2}$$

The integrating factor μ is

$$\begin{aligned}
\mu &= e^{\int \frac{2}{t} dt} \\
&= t^2
\end{aligned}$$

The ode becomes

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}(\mu y) &= (\mu)\left(\frac{c_1}{t^2}\right) \\
\frac{\mathrm{d}}{\mathrm{d}t}(t^2 y) &= (t^2)\left(\frac{c_1}{t^2}\right) \\
\mathrm{d}(t^2 y) &= c_1\, \mathrm{d}t
\end{aligned}$$

Integrating gives

$$\begin{aligned}
t^2 y &= \int c_1\, \mathrm{d}t \\
t^2 y &= c_1 t + c_2
\end{aligned}$$

Dividing both sides by the integrating factor $\mu = t^2$ results in

$$y = \frac{c_1}{t} + \frac{c_2}{t^2}$$

which simplifies to

$$y = \frac{c_1 t + c_2}{t^2}$$

Summary

The solution(s) found are the following

$$y = \frac{c_1 t + c_2}{t^2} \tag{1}$$

Verification of solutions

$$y = \frac{c_1 t + c_2}{t^2}$$

Verified OK.

8.22.11 Maple step by step solution

Let's solve

$y''t^2 + 4ty' + 2y = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y'' = -\frac{4y'}{t} - \frac{2y}{t^2}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y'' + \frac{4y'}{t} + \frac{2y}{t^2} = 0$

- Multiply by denominators of the ODE

  $y''t^2 + 4ty' + 2y = 0$

- Make a change of variables

  $s = \ln(t)$

□ Substitute the change of variables back into the ODE

  ◦ Calculate the 1st derivative of y with respect to t , using the chain rule

  $y' = \left(\frac{d}{ds} y(s)\right) s'(t)$

- Compute derivative

  $y' = \frac{\frac{d}{ds}y(s)}{t}$

- Calculate the 2nd derivative of y with respect to t , using the chain rule

  $y'' = \left(\frac{d^2}{ds^2}y(s)\right) s'(t)^2 + s''(t) \left(\frac{d}{ds}y(s)\right)$

- Compute derivative

  $y'' = \frac{\frac{d^2}{ds^2}y(s)}{t^2} - \frac{\frac{d}{ds}y(s)}{t^2}$

  Substitute the change of variables back into the ODE

  $\left(\frac{\frac{d^2}{ds^2}y(s)}{t^2} - \frac{\frac{d}{ds}y(s)}{t^2}\right) t^2 + 4\frac{d}{ds}y(s) + 2y(s) = 0$

- Simplify

  $\frac{d^2}{ds^2}y(s) + 3\frac{d}{ds}y(s) + 2y(s) = 0$

- Characteristic polynomial of ODE

  $r^2 + 3r + 2 = 0$

- Factor the characteristic polynomial

  $(r + 2)(r + 1) = 0$

- Roots of the characteristic polynomial

  $r = (-2, -1)$

- 1st solution of the ODE

  $y_1(s) = e^{-2s}$

- 2nd solution of the ODE

  $y_2(s) = e^{-s}$

- General solution of the ODE

  $y(s) = c_1y_1(s) + c_2y_2(s)$

- Substitute in solutions

  $y(s) = c_1e^{-2s} + c_2e^{-s}$

- Change variables back using $s = \ln(t)$

  $y = \frac{c_1}{t^2} + \frac{c_2}{t}$

- Simplify

  $y = \frac{c_1}{t^2} + \frac{c_2}{t}$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
<- LODE of Euler type successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 13

```
dsolve(t^2*diff(y(t),t$2)+ 4*t*diff(y(t),t)+2*y(t) = 0,y(t), singsol=all)
```

$$y(t) = \frac{c_2 t + c_1}{t^2}$$

✓ Solution by Mathematica

Time used: 0.016 (sec). Leaf size: 34

```
DSolve[t^2*y''[t]+4*t*y'[t]+y[t]==0,y[t],t,IncludeSingularSolutions -> True]
```

$$y(t) \to t^{-\frac{3}{2}-\frac{\sqrt{5}}{2}} \left(c_2 t^{\sqrt{5}} + c_1\right)$$

8.23 problem 37

Internal problem ID [645]
Internal file name [`OUTPUT/645_Sunday_June_05_2022_01_46_21_AM_42626299/index.tex`]

Book: Elementary differential equations and boundary value problems, 10th ed., Boyce and DiPrima
Section: Chapter 3, Second order linear equations, 3.3 Complex Roots of the Characteristic Equation , page 164
Problem number: 37.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_euler_ode", "second_order_change_of_variable_on_x_method_1", "second_order_change_of_variable_on_x_method_2", "second_order_change_of_variable_on_y_method_2"**

Maple gives the following as the ode type

```
[[_Emden, _Fowler]]
```

$$\boxed{t^2y'' + 3ty' + \frac{5y}{4} = 0}$$

The ode can be written as

$$4t^2y'' + 12ty' + 5y = 0$$

Which shows it is a Euler ODE.

8.23.1 Solving as second order euler ode ode

This is Euler second order ODE. Let the solution be $y = t^r$, then $y' = rt^{r-1}$ and $y'' = r(r-1)t^{r-2}$. Substituting these back into the given ODE gives

$$4t^2(r(r-1))t^{r-2} + 12trt^{r-1} + 5t^r = 0$$

Simplifying gives

$$4r(r-1)\,t^r + 12r\,t^r + 5t^r = 0$$

Since $t^r \neq 0$ then dividing throughout by t^r gives

$$4r(r-1) + 12r + 5 = 0$$

Or

$$4r^2 + 8r + 5 = 0 \tag{1}$$

Equation (1) is the characteristic equation. Its roots determine the form of the general solution. Using the quadratic equation the roots are

$$\begin{aligned} r_1 &= -1 - \frac{i}{2} \\ r_2 &= -1 + \frac{i}{2} \end{aligned}$$

The roots are complex conjugate of each others. Let the roots be

$$\begin{aligned} r_1 &= \alpha + i\beta \\ r_2 &= \alpha - i\beta \end{aligned}$$

Where in this case $\alpha = -1$ and $\beta = -\frac{1}{2}$. Hence the solution becomes

$$\begin{aligned} y &= c_1 t^{r_1} + c_2 t^{r_2} \\ &= c_1 t^{\alpha+i\beta} + c_2 t^{\alpha-i\beta} \\ &= t^\alpha \left(c_1 t^{i\beta} + c_2 t^{-i\beta}\right) \\ &= t^\alpha \left(c_1 e^{\ln\left(t^{i\beta}\right)} + c_2 e^{\ln\left(t^{-i\beta}\right)}\right) \\ &= t^\alpha \left(c_1 e^{i(\beta \ln t)} + c_2 e^{-i(\beta \ln t)}\right) \end{aligned}$$

Using the values for $\alpha = -1, \beta = -\frac{1}{2}$, the above becomes

$$y = t^{-1}\left(c_1 e^{-\frac{i\ln(t)}{2}} + c_2 e^{\frac{i\ln(t)}{2}}\right)$$

Using Euler relation, the expression $c_1 e^{iA} + c_2 e^{-iA}$ is transformed to $c_1 \cos A + c_1 \sin A$ where the constants are free to change. Applying this to the above result gives

$$y = \frac{1}{t}\left(c_1 \cos\left(\frac{\ln(t)}{2}\right) + c_2 \sin\left(\frac{\ln(t)}{2}\right)\right)$$

Summary

The solution(s) found are the following

$$y = \frac{c_1 \cos\left(\frac{\ln(t)}{2}\right) + c_2 \sin\left(\frac{\ln(t)}{2}\right)}{t} \tag{1}$$

Verification of solutions

$$y = \frac{c_1 \cos\left(\frac{\ln(t)}{2}\right) + c_2 \sin\left(\frac{\ln(t)}{2}\right)}{t}$$

Verified OK.

8.23.2 Solving as second order change of variable on x method 2 ode

In normal form the ode

$$4t^2 y'' + 12ty' + 5y = 0 \tag{1}$$

Becomes

$$y'' + p(t)\, y' + q(t)\, y = 0 \tag{2}$$

Where

$$\begin{aligned} p(t) &= \frac{3}{t} \\ q(t) &= \frac{5}{4t^2} \end{aligned}$$

Applying change of variables $\tau = g(t)$ to (2) gives

$$\frac{d^2}{d\tau^2} y(\tau) + p_1\left(\frac{d}{d\tau} y(\tau)\right) + q_1 y(\tau) = 0 \tag{3}$$

Where τ is the new independent variable, and

$$p_1(\tau) = \frac{\tau''(t) + p(t)\,\tau'(t)}{\tau'(t)^2} \tag{4}$$

$$q_1(\tau) = \frac{q(t)}{\tau'(t)^2} \tag{5}$$

Let $p_1 = 0$. Eq (4) simplifies to

$$\tau''(t) + p(t)\,\tau'(t) = 0$$

This ode is solved resulting in

$$\begin{aligned}
\tau &= \int e^{-\left(\int p(t)dt\right)} dt \\
&= \int e^{-\left(\int \frac{3}{t}dt\right)} dt \\
&= \int e^{-3\ln(t)}\, dt \\
&= \int \frac{1}{t^3} dt \\
&= -\frac{1}{2t^2}
\end{aligned} \tag{6}$$

Using (6) to evaluate q_1 from (5) gives

$$\begin{aligned}
q_1(\tau) &= \frac{q(t)}{\tau'(t)^2} \\
&= \frac{\frac{5}{4t^2}}{\frac{1}{t^6}} \\
&= \frac{5t^4}{4}
\end{aligned} \tag{7}$$

Substituting the above in (3) and noting that now $p_1 = 0$ results in

$$\begin{aligned}
\frac{d^2}{d\tau^2}y(\tau) + q_1 y(\tau) &= 0 \\
\frac{d^2}{d\tau^2}y(\tau) + \frac{5t^4 y(\tau)}{4} &= 0
\end{aligned}$$

But in terms of τ

$$\frac{5t^4}{4} = \frac{5}{16\tau^2}$$

Hence the above ode becomes

$$\frac{d^2}{d\tau^2}y(\tau) + \frac{5y(\tau)}{16\tau^2} = 0$$

The above ode is now solved for $y(\tau)$. The ode can be written as

$$16\left(\frac{d^2}{d\tau^2}y(\tau)\right)\tau^2 + 5y(\tau) = 0$$

Which shows it is a Euler ODE. This is Euler second order ODE. Let the solution be $y(\tau) = \tau^r$, then $y' = r\tau^{r-1}$ and $y'' = r(r-1)\tau^{r-2}$. Substituting these back into the given ODE gives

$$16\tau^2(r(r-1))\tau^{r-2} + 0r\tau^{r-1} + 5\tau^r = 0$$

Simplifying gives

$$16r(r-1)\,\tau^r + 0\,\tau^r + 5\tau^r = 0$$

Since $\tau^r \neq 0$ then dividing throughout by τ^r gives

$$16r(r-1) + 0 + 5 = 0$$

Or

$$16r^2 - 16r + 5 = 0 \tag{1}$$

Equation (1) is the characteristic equation. Its roots determine the form of the general solution. Using the quadratic equation the roots are

$$\begin{aligned} r_1 &= \frac{1}{2} - \frac{i}{4} \\ r_2 &= \frac{1}{2} + \frac{i}{4} \end{aligned}$$

The roots are complex conjugate of each others. Let the roots be

$$\begin{aligned} r_1 &= \alpha + i\beta \\ r_2 &= \alpha - i\beta \end{aligned}$$

Where in this case $\alpha = \frac{1}{2}$ and $\beta = -\frac{1}{4}$. Hence the solution becomes

$$\begin{aligned} y(\tau) &= c_1\tau^{r_1} + c_2\tau^{r_2} \\ &= c_1\tau^{\alpha+i\beta} + c_2\tau^{\alpha-i\beta} \\ &= \tau^{\alpha}\left(c_1\tau^{i\beta} + c_2\tau^{-i\beta}\right) \\ &= \tau^{\alpha}\left(c_1 e^{\ln(\tau^{i\beta})} + c_2 e^{\ln(\tau^{-i\beta})}\right) \\ &= \tau^{\alpha}\left(c_1 e^{i(\beta\ln\tau)} + c_2 e^{-i(\beta\ln\tau)}\right) \end{aligned}$$

Using the values for $\alpha = \frac{1}{2}, \beta = -\frac{1}{4}$, the above becomes

$$y(\tau) = \tau^{\frac{1}{2}}\left(c_1 e^{-\frac{i\ln(\tau)}{4}} + c_2 e^{\frac{i\ln(\tau)}{4}}\right)$$

Using Euler relation, the expression $c_1 e^{iA} + c_2 e^{-iA}$ is transformed to $c_1 \cos A + c_1 \sin A$ where the constants are free to change. Applying this to the above result gives

$$y(\tau) = \sqrt{\tau}\left(c_1\cos\left(\frac{\ln(\tau)}{4}\right) + c_2\sin\left(\frac{\ln(\tau)}{4}\right)\right)$$

The above solution is now transformed back to y using (6) which results in

$$y = \frac{\left(c_1 \cos\left(-\frac{\ln(2)}{4} + \frac{\ln\left(-\frac{1}{t^2}\right)}{4}\right) + c_2 \sin\left(-\frac{\ln(2)}{4} + \frac{\ln\left(-\frac{1}{t^2}\right)}{4}\right)\right)\sqrt{2}\,\sqrt{-\frac{1}{t^2}}}{2}$$

Summary
The solution(s) found are the following

$$y = \frac{\left(c_1 \cos\left(-\frac{\ln(2)}{4} + \frac{\ln\left(-\frac{1}{t^2}\right)}{4}\right) + c_2 \sin\left(-\frac{\ln(2)}{4} + \frac{\ln\left(-\frac{1}{t^2}\right)}{4}\right)\right)\sqrt{2}\,\sqrt{-\frac{1}{t^2}}}{2} \tag{1}$$

Verification of solutions

$$y = \frac{\left(c_1 \cos\left(-\frac{\ln(2)}{4} + \frac{\ln\left(-\frac{1}{t^2}\right)}{4}\right) + c_2 \sin\left(-\frac{\ln(2)}{4} + \frac{\ln\left(-\frac{1}{t^2}\right)}{4}\right)\right)\sqrt{2}\,\sqrt{-\frac{1}{t^2}}}{2}$$

Verified OK.

8.23.3 Solving as second order change of variable on x method 1 ode

In normal form the ode

$$4t^2y'' + 12ty' + 5y = 0 \tag{1}$$

Becomes

$$y'' + p(t)\,y' + q(t)\,y = 0 \tag{2}$$

Where

$$p(t) = \frac{3}{t}$$
$$q(t) = \frac{5}{4t^2}$$

Applying change of variables $\tau = g(t)$ to (2) results

$$\frac{d^2}{d\tau^2}y(\tau) + p_1\left(\frac{d}{d\tau}y(\tau)\right) + q_1 y(\tau) = 0 \tag{3}$$

Where τ is the new independent variable, and

$$p_1(\tau) = \frac{\tau''(t) + p(t)\,\tau'(t)}{\tau'(t)^2} \tag{4}$$

$$q_1(\tau) = \frac{q(t)}{\tau'(t)^2} \tag{5}$$

Let $q_1 = c^2$ where c is some constant. Therefore from (5)

$$\begin{aligned}
\tau' &= \frac{1}{c}\sqrt{q} \\
&= \frac{\sqrt{5}\,\sqrt{\frac{1}{t^2}}}{2c} \\
\tau'' &= -\frac{\sqrt{5}}{2c\sqrt{\frac{1}{t^2}}\,t^3}
\end{aligned} \tag{6}$$

Substituting the above into (4) results in

$$\begin{aligned}
p_1(\tau) &= \frac{\tau''(t) + p(t)\,\tau'(t)}{\tau'\,(t)^2} \\
&= \frac{-\frac{\sqrt{5}}{2c\sqrt{\frac{1}{t^2}}\,t^3} + \frac{3}{t}\frac{\sqrt{5}\,\sqrt{\frac{1}{t^2}}}{2c}}{\left(\frac{\sqrt{5}\,\sqrt{\frac{1}{t^2}}}{2c}\right)^2} \\
&= \frac{4c\sqrt{5}}{5}
\end{aligned}$$

Therefore ode (3) now becomes

$$\begin{aligned}
y(\tau)'' + p_1 y(\tau)' + q_1 y(\tau) &= 0 \\
\frac{d^2}{d\tau^2}y(\tau) + \frac{4c\sqrt{5}\left(\frac{d}{d\tau}y(\tau)\right)}{5} + c^2 y(\tau) &= 0
\end{aligned} \tag{7}$$

The above ode is now solved for $y(\tau)$. Since the ode is now constant coefficients, it can be easily solved to give

$$y(\tau) = \mathrm{e}^{-\frac{2\sqrt{5}\,c\tau}{5}}\left(c_1\cos\left(\frac{\sqrt{5}\,c\tau}{5}\right) + c_2\sin\left(\frac{\sqrt{5}\,c\tau}{5}\right)\right)$$

Now from (6)

$$\begin{aligned}
\tau &= \int \frac{1}{c}\sqrt{q}\,dt \\
&= \frac{\int \frac{\sqrt{5}\,\sqrt{\frac{1}{t^2}}}{2}dt}{c} \\
&= \frac{\sqrt{5}\,\sqrt{\frac{1}{t^2}}\,t\ln(t)}{2c}
\end{aligned}$$

Substituting the above into the solution obtained gives

$$y = \frac{c_1 \cos\left(\frac{\ln(t)}{2}\right) + c_2 \sin\left(\frac{\ln(t)}{2}\right)}{t}$$

Summary

The solution(s) found are the following

$$y = \frac{c_1 \cos\left(\frac{\ln(t)}{2}\right) + c_2 \sin\left(\frac{\ln(t)}{2}\right)}{t} \tag{1}$$

Verification of solutions

$$y = \frac{c_1 \cos\left(\frac{\ln(t)}{2}\right) + c_2 \sin\left(\frac{\ln(t)}{2}\right)}{t}$$

Verified OK.

8.23.4 Solving as second order change of variable on y method 2 ode

In normal form the ode

$$4t^2 y'' + 12ty' + 5y = 0 \tag{1}$$

Becomes

$$y'' + p(t)\, y' + q(t)\, y = 0 \tag{2}$$

Where

$$p(t) = \frac{3}{t}$$
$$q(t) = \frac{5}{4t^2}$$

Applying change of variables on the depndent variable $y = v(t)\, t^n$ to (2) gives the following ode where the dependent variables is $v(t)$ and not y.

$$v''(t) + \left(\frac{2n}{t} + p\right) v'(t) + \left(\frac{n(n-1)}{t^2} + \frac{np}{t} + q\right) v(t) = 0 \tag{3}$$

Let the coefficient of $v(t)$ above be zero. Hence

$$\frac{n(n-1)}{t^2} + \frac{np}{t} + q = 0 \tag{4}$$

Substituting the earlier values found for $p(t)$ and $q(t)$ into (4) gives

$$\frac{n(n-1)}{t^2} + \frac{3n}{t^2} + \frac{5}{4t^2} = 0 \tag{5}$$

Solving (5) for n gives

$$n = -1 + \frac{i}{2} \tag{6}$$

Substituting this value in (3) gives

$$\begin{aligned} v''(t) + \left(\frac{-2+i}{t} + \frac{3}{t}\right) v'(t) &= 0 \\ v''(t) + \frac{(1+i)\, v'(t)}{t} &= 0 \end{aligned} \tag{7}$$

Using the substitution

$$u(t) = v'(t)$$

Then (7) becomes

$$u'(t) + \frac{(1+i)\, u(t)}{t} = 0 \tag{8}$$

The above is now solved for $u(t)$. In canonical form the ODE is

$$\begin{aligned} u' &= F(t,u) \\ &= f(t)g(u) \\ &= \frac{(-1-i)\, u}{t} \end{aligned}$$

Where $f(t) = \frac{-1-i}{t}$ and $g(u) = u$. Integrating both sides gives

$$\begin{aligned} \frac{1}{u}\, du &= \frac{-1-i}{t}\, dt \\ \int \frac{1}{u}\, du &= \int \frac{-1-i}{t}\, dt \\ \ln(u) &= (-1-i)\ln(t) + c_1 \\ u &= e^{(-1-i)\ln(t) + c_1} \\ &= c_1 e^{(-1-i)\ln(t)} \end{aligned}$$

Which simplifies to

$$u(t) = \frac{c_1 t^{-i}}{t}$$

Now that $u(t)$ is known, then

$$\begin{aligned} v'(t) &= u(t) \\ v(t) &= \int u(t)\, dt + c_2 \\ &= i t^{-i} c_1 + c_2 \end{aligned}$$

Hence

$$\begin{aligned} y &= v(t)\, t^n \\ &= \left(i t^{-i} c_1 + c_2\right) t^{-1+\frac{i}{2}} \\ &= t^{-1-\frac{i}{2}} \left(i c_1 + t^i c_2\right) \end{aligned}$$

Summary

The solution(s) found are the following

$$y = \left(i t^{-i} c_1 + c_2\right) t^{-1+\frac{i}{2}} \tag{1}$$

Verification of solutions

$$y = \left(i t^{-i} c_1 + c_2\right) t^{-1+\frac{i}{2}}$$

Verified OK.

8.23.5 Solving using Kovacic algorithm

Writing the ode as

$$4t^2 y'' + 12 t y' + 5y = 0 \tag{1}$$
$$A y'' + B y' + C y = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 4t^2 \\ B &= 12t \\ C &= 5 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(t) = ye^{\int \frac{B}{2A}\,dt}$$

Then (2) becomes

$$z''(t) = rz(t) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{-1}{2t^2} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= -1 \\ t &= 2t^2 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(t) = \left(-\frac{1}{2t^2}\right) z(t) \tag{7}$$

Equation (7) is now solved. After finding $z(t)$ then y is found using the inverse transformation

$$y = z(t)\, e^{-\int \frac{B}{2A}\,dt}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 408: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$
\begin{aligned}
O(\infty) &= \deg(t) - \deg(s) \\
&= 2 - 0 \\
&= 2
\end{aligned}
$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 2t^2$. There is a pole at $t = 0$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 2 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Since pole order is not larger than 2 and the order at ∞ is 2 then the necessary conditions for case three are met. Therefore

$$
L = [1, 2, 4, 6, 12]
$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$
r = -\frac{1}{2t^2}
$$

For the pole at $t = 0$ let b be the coefficient of $\frac{1}{t^2}$ in the partial fractions decomposition

of r given above. Therefore $b = -\frac{1}{2}$. Hence

$$\begin{aligned}
[\sqrt{r}]_c &= 0 \\
\alpha_c^{+} &= \frac{1}{2} + \sqrt{1+4b} = \frac{1}{2} + \frac{i}{2} \\
\alpha_c^{-} &= \frac{1}{2} - \sqrt{1+4b} = \frac{1}{2} - \frac{i}{2}
\end{aligned}$$

Since the order of r at ∞ is 2 then $[\sqrt{r}]_\infty = 0$. Let b be the coefficient of $\frac{1}{t^2}$ in the Laurent series expansion of r at ∞. which can be found by dividing the leading coefficient of s by the leading coefficient of t from

$$r = \frac{s}{t} = -\frac{1}{2t^2}$$

Since the $\gcd(s,t) = 1$. This gives $b = -\frac{1}{2}$. Hence

$$\begin{aligned}
[\sqrt{r}]_\infty &= 0 \\
\alpha_\infty^{+} &= \frac{1}{2} + \sqrt{1+4b} = \frac{1}{2} + \frac{i}{2} \\
\alpha_\infty^{-} &= \frac{1}{2} - \sqrt{1+4b} = \frac{1}{2} - \frac{i}{2}
\end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = -\frac{1}{2t^2}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^{+}	α_c^{-}
0	2	0	$\frac{1}{2} + \frac{i}{2}$	$\frac{1}{2} - \frac{i}{2}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^{+}	α_∞^{-}
2	0	$\frac{1}{2} + \frac{i}{2}$	$\frac{1}{2} - \frac{i}{2}$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^{\pm}$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^{\pm}$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c \in \Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^{\pm}$. This is done by trial over all set of families $s = (s(c))_{c \in \Gamma \cup \infty}$ until such d is found to work in finding candidate ω.

Trying $\alpha_\infty^- = \frac{1}{2} - \frac{i}{2}$ then

$$
\begin{aligned}
d &= \alpha_\infty^- - (\alpha_{c_1}^-) \\
&= \frac{1}{2} - \frac{i}{2} - \left(\frac{1}{2} - \frac{i}{2}\right) \\
&= 0
\end{aligned}
$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$
\omega = \sum_{c \in \Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{t - c} \right) + s(\infty)[\sqrt{r}]_\infty
$$

The above gives

$$
\begin{aligned}
\omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^-}{t - c_1} \right) + (-)[\sqrt{r}]_\infty \\
&= \frac{\frac{1}{2} - \frac{i}{2}}{t} + (-)(0) \\
&= \frac{\frac{1}{2} - \frac{i}{2}}{t} \\
&= \frac{\frac{1}{2} - \frac{i}{2}}{t}
\end{aligned}
$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(t)$ of degree $d = 0$ to solve the ode. The polynomial $p(t)$ needs to satisfy the equation

$$
p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right) p = 0 \tag{1A}
$$

Let

$$
p(t) = 1 \tag{2A}
$$

Substituting the above in eq. (1A) gives

$$
\begin{aligned}
(0) + 2\left(\frac{\frac{1}{2} - \frac{i}{2}}{t}\right)(0) + \left(\left(\frac{-\frac{1}{2} + \frac{i}{2}}{t^2} \right) + \left(\frac{\frac{1}{2} - \frac{i}{2}}{t} \right)^2 - \left(-\frac{1}{2t^2} \right) \right) &= 0 \\
0 &= 0
\end{aligned}
$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z'' = rz$ is

$$
\begin{aligned}
z_1(t) &= p e^{\int \omega \, dt} \\
&= e^{\int \frac{\frac{1}{2} - \frac{i}{2}}{t} dt} \\
&= t^{\frac{1}{2} - \frac{i}{2}}
\end{aligned}
$$

The first solution to the original ode in y is found from

$$
\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dt} \\
&= z_1 e^{-\int \frac{1}{2}\frac{12t}{4t^2}\,dt} \\
&= z_1 e^{-\frac{3\ln(t)}{2}} \\
&= z_1\left(\frac{1}{t^{\frac{3}{2}}}\right)
\end{aligned}
$$

Which simplifies to

$$
y_1 = t^{-1-\frac{i}{2}}
$$

The second solution y_2 to the original ode is found using reduction of order

$$
y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dt}}{y_1^2}\,dt
$$

Substituting gives

$$
\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{12t}{4t^2}\,dt}}{(y_1)^2}\,dt \\
&= y_1 \int \frac{e^{-3\ln(t)}}{(y_1)^2}\,dt \\
&= y_1\left(-i t^{i}\right)
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1\left(t^{-1-\frac{i}{2}}\right) + c_2\left(t^{-1-\frac{i}{2}}\left(-i t^{i}\right)\right)
\end{aligned}
$$

Summary

The solution(s) found are the following

$$
y = c_1 t^{-1-\frac{i}{2}} - i c_2 t^{-1+\frac{i}{2}} \tag{1}
$$

Verification of solutions

$$
y = c_1 t^{-1-\frac{i}{2}} - i c_2 t^{-1+\frac{i}{2}}
$$

Verified OK.

8.23.6 Maple step by step solution

Let's solve

$4y''t^2+12ty'+5y=0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y''=-\frac{3y'}{t}-\frac{5y}{4t^2}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y''+\frac{3y'}{t}+\frac{5y}{4t^2}=0$

- Multiply by denominators of the ODE

  $4y''t^2+12ty'+5y=0$

- Make a change of variables

  $s=\ln(t)$

□ Substitute the change of variables back into the ODE

  ◦ Calculate the 1st derivative of y with respect to t , using the chain rule

  $y'=\left(\frac{d}{ds}y(s)\right)s'(t)$

  ◦ Compute derivative

  $y'=\frac{\frac{d}{ds}y(s)}{t}$

  ◦ Calculate the 2nd derivative of y with respect to t , using the chain rule

  $y''=\left(\frac{d^2}{ds^2}y(s)\right)s'(t)^2+s''(t)\left(\frac{d}{ds}y(s)\right)$

  ◦ Compute derivative

  $y''=\frac{\frac{d^2}{ds^2}y(s)}{t^2}-\frac{\frac{d}{ds}y(s)}{t^2}$

  Substitute the change of variables back into the ODE

  $4\left(\frac{\frac{d^2}{ds^2}y(s)}{t^2}-\frac{\frac{d}{ds}y(s)}{t^2}\right)t^2+12\frac{d}{ds}y(s)+5y(s)=0$

- Simplify

  $4\frac{d^2}{ds^2}y(s)+8\frac{d}{ds}y(s)+5y(s)=0$

- Isolate 2nd derivative

  $\frac{d^2}{ds^2}y(s)=-2\frac{d}{ds}y(s)-\frac{5y(s)}{4}$

- Group terms with $y(s)$ on the lhs of the ODE and the rest on the rhs of the ODE; ODE is lin

  $\frac{d^2}{ds^2}y(s) + 2\frac{d}{ds}y(s) + \frac{5y(s)}{4} = 0$

- Characteristic polynomial of ODE

  $r^2 + 2r + \frac{5}{4} = 0$

- Use quadratic formula to solve for r

  $r = \frac{(-2)\pm(\sqrt{-1})}{2}$

- Roots of the characteristic polynomial

  $r = \left(-1 - \frac{\mathrm{I}}{2}, -1 + \frac{\mathrm{I}}{2}\right)$

- 1st solution of the ODE

  $y_1(s) = \mathrm{e}^{-s}\cos\left(\frac{s}{2}\right)$

- 2nd solution of the ODE

  $y_2(s) = \mathrm{e}^{-s}\sin\left(\frac{s}{2}\right)$

- General solution of the ODE

  $y(s) = c_1 y_1(s) + c_2 y_2(s)$

- Substitute in solutions

  $y(s) = c_1\mathrm{e}^{-s}\cos\left(\frac{s}{2}\right) + c_2\mathrm{e}^{-s}\sin\left(\frac{s}{2}\right)$

- Change variables back using $s = \ln(t)$

  $y = \frac{c_1\cos\left(\frac{\ln(t)}{2}\right)}{t} + \frac{c_2\sin\left(\frac{\ln(t)}{2}\right)}{t}$

- Simplify

  $y = \frac{c_1\cos\left(\frac{\ln(t)}{2}\right)}{t} + \frac{c_2\sin\left(\frac{\ln(t)}{2}\right)}{t}$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
<- LODE of Euler type successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 23

```
dsolve(t^2*diff(y(t),t$2)+ 3*t*diff(y(t),t)+125/100*y(t) = 0,y(t), singsol=all)
```

$$y(t) = \frac{c_1 \sin\left(\frac{\ln(t)}{2}\right) + c_2 \cos\left(\frac{\ln(t)}{2}\right)}{t}$$

✓ Solution by Mathematica

Time used: 0.025 (sec). Leaf size: 30

```
DSolve[t^2*y''[t]+3*t*y'[t]+125/100*y[t]==0,y[t],t,IncludeSingularSolutions -> True]
```

$$y(t) \to \frac{c_2 \cos\left(\frac{\log(t)}{2}\right) + c_1 \sin\left(\frac{\log(t)}{2}\right)}{t}$$

8.24 problem 38

Internal problem ID [646]
Internal file name [`OUTPUT/646_Sunday_June_05_2022_01_46_22_AM_79982171/index.tex`]

Book: Elementary differential equations and boundary value problems, 10th ed., Boyce and DiPrima
Section: Chapter 3, Second order linear equations, 3.3 Complex Roots of the Characteristic Equation , page 164
Problem number: 38.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_euler_ode", "exact linear second order ode", "second_order_integrable_as_is", "second_order_change_of_variable_on_x_method_2", "second_order_change_of_variable_on_y_method_2"**

Maple gives the following as the ode type

```
[[_2nd_order, _exact, _linear, _homogeneous]]
```

$$\boxed{t^2y'' - 4ty' - 6y = 0}$$

8.24.1 Solving as second order euler ode ode

This is Euler second order ODE. Let the solution be $y = t^r$, then $y' = rt^{r-1}$ and $y'' = r(r-1)t^{r-2}$. Substituting these back into the given ODE gives

$$t^2(r(r-1))t^{r-2} - 4trt^{r-1} - 6t^r = 0$$

Simplifying gives

$$r(r-1)\,t^r - 4r\,t^r - 6t^r = 0$$

Since $t^r \neq 0$ then dividing throughout by t^r gives

$$r(r-1) - 4r - 6 = 0$$

Or

$$r^2 - 5r - 6 = 0 \tag{1}$$

Equation (1) is the characteristic equation. Its roots determine the form of the general solution. Using the quadratic equation the roots are

$$r_1 = -1$$
$$r_2 = 6$$

Since the roots are real and distinct, then the general solution is

$$y = c_1 y_1 + c_2 y_2$$

Where $y_1 = t^{r_1}$ and $y_2 = t^{r_2}$. Hence

$$y = \frac{c_1}{t} + c_2 t^6$$

Summary

The solution(s) found are the following

$$y = \frac{c_1}{t} + c_2 t^6 \tag{1}$$

Verification of solutions

$$y = \frac{c_1}{t} + c_2 t^6$$

Verified OK.

8.24.2 Solving as second order change of variable on x method 2 ode

In normal form the ode

$$t^2 y'' - 4ty' - 6y = 0 \tag{1}$$

Becomes

$$y'' + p(t)\, y' + q(t)\, y = 0 \tag{2}$$

Where

$$\begin{aligned} p(t) &= -\frac{4}{t} \\ q(t) &= -\frac{6}{t^2} \end{aligned}$$

Applying change of variables $\tau = g(t)$ to (2) gives

$$\frac{d^2}{d\tau^2} y(\tau) + p_1 \left(\frac{d}{d\tau} y(\tau)\right) + q_1 y(\tau) = 0 \tag{3}$$

Where τ is the new independent variable, and

$$p_1(\tau) = \frac{\tau''(t) + p(t)\, \tau'(t)}{\tau'(t)^2} \tag{4}$$

$$q_1(\tau) = \frac{q(t)}{\tau'(t)^2} \tag{5}$$

Let $p_1 = 0$. Eq (4) simplifies to

$$\tau''(t) + p(t)\, \tau'(t) = 0$$

This ode is solved resulting in

$$\begin{aligned} \tau &= \int e^{-\left(\int p(t)dt\right)} dt \\ &= \int e^{-\left(\int -\frac{4}{t} dt\right)} dt \\ &= \int e^{4\ln(t)}\, dt \\ &= \int t^4 dt \\ &= \frac{t^5}{5} \end{aligned} \tag{6}$$

Using (6) to evaluate q_1 from (5) gives

$$\begin{aligned} q_1(\tau) &= \frac{q(t)}{\tau'(t)^2} \\ &= \frac{-\frac{6}{t^2}}{t^8} \\ &= -\frac{6}{t^{10}} \end{aligned} \tag{7}$$

Substituting the above in (3) and noting that now $p_1 = 0$ results in

$$\begin{aligned} \frac{d^2}{d\tau^2}y(\tau) + q_1 y(\tau) &= 0 \\ \frac{d^2}{d\tau^2}y(\tau) - \frac{6y(\tau)}{t^{10}} &= 0 \end{aligned}$$

But in terms of τ

$$-\frac{6}{t^{10}} = -\frac{6}{25\tau^2}$$

Hence the above ode becomes

$$\frac{d^2}{d\tau^2}y(\tau) - \frac{6y(\tau)}{25\tau^2} = 0$$

The above ode is now solved for $y(\tau)$. The ode can be written as

$$25\left(\frac{d^2}{d\tau^2}y(\tau)\right)\tau^2 - 6y(\tau) = 0$$

Which shows it is a Euler ODE. This is Euler second order ODE. Let the solution be $y(\tau) = \tau^r$, then $y' = r\tau^{r-1}$ and $y'' = r(r-1)\tau^{r-2}$. Substituting these back into the given ODE gives

$$25\tau^2(r(r-1))\tau^{r-2} + 0r\tau^{r-1} - 6\tau^r = 0$$

Simplifying gives

$$25r(r-1)\tau^r + 0\,\tau^r - 6\tau^r = 0$$

Since $\tau^r \neq 0$ then dividing throughout by τ^r gives

$$25r(r-1) + 0 - 6 = 0$$

Or

$$25r^2 - 25r - 6 = 0 \tag{1}$$

Equation (1) is the characteristic equation. Its roots determine the form of the general solution. Using the quadratic equation the roots are

$$r_1 = -\frac{1}{5}$$
$$r_2 = \frac{6}{5}$$

Since the roots are real and distinct, then the general solution is

$$y(\tau) = c_1 y_1 + c_2 y_2$$

Where $y_1 = \tau^{r_1}$ and $y_2 = \tau^{r_2}$. Hence

$$y(\tau) = \frac{c_1}{\tau^{\frac{1}{5}}} + c_2 \tau^{\frac{6}{5}}$$

The above solution is now transformed back to y using (6) which results in

$$y = \frac{5^{\frac{1}{5}} \left(c_2 5^{\frac{3}{5}} t^5 (t^5)^{\frac{2}{5}} + 25 c_1\right)}{25 \left(t^5\right)^{\frac{1}{5}}}$$

Summary

The solution(s) found are the following

$$y = \frac{5^{\frac{1}{5}} \left(c_2 5^{\frac{3}{5}} t^5 (t^5)^{\frac{2}{5}} + 25 c_1\right)}{25 \left(t^5\right)^{\frac{1}{5}}} \tag{1}$$

Verification of solutions

$$y = \frac{5^{\frac{1}{5}} \left(c_2 5^{\frac{3}{5}} t^5 (t^5)^{\frac{2}{5}} + 25 c_1\right)}{25 \left(t^5\right)^{\frac{1}{5}}}$$

Verified OK.

8.24.3 Solving as second order change of variable on y method 2 ode

In normal form the ode

$$t^2 y'' - 4 t y' - 6 y = 0 \tag{1}$$

Becomes

$$y'' + p(t)\, y' + q(t)\, y = 0 \tag{2}$$

Where

$$
\begin{aligned}
p(t) &= -\frac{4}{t} \\
q(t) &= -\frac{6}{t^2}
\end{aligned}
$$

Applying change of variables on the depndent variable $y = v(t)\, t^n$ to (2) gives the following ode where the dependent variables is $v(t)$ and not y.

$$
v''(t) + \left(\frac{2n}{t} + p\right) v'(t) + \left(\frac{n(n-1)}{t^2} + \frac{np}{t} + q\right) v(t) = 0 \tag{3}
$$

Let the coefficient of $v(t)$ above be zero. Hence

$$
\frac{n(n-1)}{t^2} + \frac{np}{t} + q = 0 \tag{4}
$$

Substituting the earlier values found for $p(t)$ and $q(t)$ into (4) gives

$$
\frac{n(n-1)}{t^2} - \frac{4n}{t^2} - \frac{6}{t^2} = 0 \tag{5}
$$

Solving (5) for n gives

$$
n = 6 \tag{6}
$$

Substituting this value in (3) gives

$$
\begin{aligned}
v''(t) + \frac{8v'(t)}{t} &= 0 \\
v''(t) + \frac{8v'(t)}{t} &= 0 \qquad (7)
\end{aligned}
$$

Using the substitution

$$
u(t) = v'(t)
$$

Then (7) becomes

$$
u'(t) + \frac{8u(t)}{t} = 0 \tag{8}
$$

The above is now solved for $u(t)$. In canonical form the ODE is

$$\begin{aligned} u' &= F(t,u) \\ &= f(t)g(u) \\ &= -\frac{8u}{t} \end{aligned}$$

Where $f(t) = -\frac{8}{t}$ and $g(u) = u$. Integrating both sides gives

$$\begin{aligned} \frac{1}{u}\,du &= -\frac{8}{t}\,dt \\ \int \frac{1}{u}\,du &= \int -\frac{8}{t}\,dt \\ \ln(u) &= -8\ln(t) + c_1 \\ u &= e^{-8\ln(t)+c_1} \\ &= \frac{c_1}{t^8} \end{aligned}$$

Now that $u(t)$ is known, then

$$\begin{aligned} v'(t) &= u(t) \\ v(t) &= \int u(t)\,dt + c_2 \\ &= -\frac{c_1}{7t^7} + c_2 \end{aligned}$$

Hence

$$\begin{aligned} y &= v(t)\,t^n \\ &= \left(-\frac{c_1}{7t^7} + c_2\right)t^6 \\ &= \frac{7c_2t^7 - c_1}{7t} \end{aligned}$$

Summary
The solution(s) found are the following

$$y = \left(-\frac{c_1}{7t^7} + c_2\right)t^6 \tag{1}$$

Verification of solutions

$$y = \left(-\frac{c_1}{7t^7} + c_2\right)t^6$$

Verified OK.

8.24.4 Solving as second order integrable as is ode

Integrating both sides of the ODE w.r.t t gives

$$\int (t^2 y'' - 4ty' - 6y)\, dt = 0$$
$$y' t^2 - 6yt = c_1$$

Which is now solved for y.

Entering Linear first order ODE solver. In canonical form a linear first order is

$$y' + p(t)y = q(t)$$

Where here

$$p(t) = -\frac{6}{t}$$
$$q(t) = \frac{c_1}{t^2}$$

Hence the ode is

$$y' - \frac{6y}{t} = \frac{c_1}{t^2}$$

The integrating factor μ is

$$\mu = e^{\int -\frac{6}{t} dt}$$
$$= \frac{1}{t^6}$$

The ode becomes

$$\frac{d}{dt}(\mu y) = (\mu)\left(\frac{c_1}{t^2}\right)$$
$$\frac{d}{dt}\left(\frac{y}{t^6}\right) = \left(\frac{1}{t^6}\right)\left(\frac{c_1}{t^2}\right)$$
$$d\left(\frac{y}{t^6}\right) = \left(\frac{c_1}{t^8}\right) dt$$

Integrating gives

$$\frac{y}{t^6} = \int \frac{c_1}{t^8}\, dt$$
$$\frac{y}{t^6} = -\frac{c_1}{7t^7} + c_2$$

Dividing both sides by the integrating factor $\mu = \frac{1}{t^6}$ results in

$$y = -\frac{c_1}{7t} + c_2 t^6$$

Summary

The solution(s) found are the following

$$y = -\frac{c_1}{7t} + c_2 t^6 \tag{1}$$

Verification of solutions

$$y = -\frac{c_1}{7t} + c_2 t^6$$

Verified OK.

8.24.5 Solving as type second_order_integrable_as_is (not using ABC version)

Writing the ode as

$$t^2 y'' - 4ty' - 6y = 0$$

Integrating both sides of the ODE w.r.t t gives

$$\int \left(t^2 y'' - 4ty' - 6y\right) dt = 0$$
$$y' t^2 - 6yt = c_1$$

Which is now solved for y.

Entering Linear first order ODE solver. In canonical form a linear first order is

$$y' + p(t)y = q(t)$$

Where here

$$p(t) = -\frac{6}{t}$$
$$q(t) = \frac{c_1}{t^2}$$

Hence the ode is

$$y' - \frac{6y}{t} = \frac{c_1}{t^2}$$

The integrating factor μ is

$$\mu = e^{\int -\frac{6}{t}dt}$$
$$= \frac{1}{t^6}$$

The ode becomes

$$\frac{d}{dt}(\mu y) = (\mu)\left(\frac{c_1}{t^2}\right)$$
$$\frac{d}{dt}\left(\frac{y}{t^6}\right) = \left(\frac{1}{t^6}\right)\left(\frac{c_1}{t^2}\right)$$
$$d\left(\frac{y}{t^6}\right) = \left(\frac{c_1}{t^8}\right) dt$$

Integrating gives

$$\frac{y}{t^6} = \int \frac{c_1}{t^8} dt$$
$$\frac{y}{t^6} = -\frac{c_1}{7t^7} + c_2$$

Dividing both sides by the integrating factor $\mu = \frac{1}{t^6}$ results in

$$y = -\frac{c_1}{7t} + c_2 t^6$$

Summary
The solution(s) found are the following

$$y = -\frac{c_1}{7t} + c_2 t^6 \tag{1}$$

Verification of solutions

$$y = -\frac{c_1}{7t} + c_2 t^6$$

Verified OK.

8.24.6 Solving using Kovacic algorithm

Writing the ode as

$$t^2 y'' - 4ty' - 6y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= t^2 \\ B &= -4t \\ C &= -6 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(t) = y e^{\int \frac{B}{2A}\, dt}$$

Then (2) becomes

$$z''(t) = r z(t) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{12}{t^2} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= 12 \\ t &= t^2 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(t) = \left(\frac{12}{t^2}\right) z(t) \tag{7}$$

Equation (7) is now solved. After finding $z(t)$ then y is found using the inverse transformation

$$y = z(t)\, e^{-\int \frac{B}{2A}\, dt}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 410: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$
\begin{aligned}
O(\infty) &= \deg(t) - \deg(s) \\
&= 2 - 0 \\
&= 2
\end{aligned}
$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = t^2$. There is a pole at $t = 0$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 2 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Since pole order is not larger than 2 and the order at ∞ is 2 then the necessary conditions for case three are met. Therefore

$$
L = [1, 2, 4, 6, 12]
$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$
r = \frac{12}{t^2}
$$

For the pole at $t = 0$ let b be the coefficient of $\frac{1}{t^2}$ in the partial fractions decomposition

of r given above. Therefore $b = 12$. Hence

$$
\begin{aligned}
[\sqrt{r}]_c &= 0 \\
\alpha_c^{+} &= \frac{1}{2} + \sqrt{1+4b} = 4 \\
\alpha_c^{-} &= \frac{1}{2} - \sqrt{1+4b} = -3
\end{aligned}
$$

Since the order of r at ∞ is 2 then $[\sqrt{r}]_\infty = 0$. Let b be the coefficient of $\frac{1}{t^2}$ in the Laurent series expansion of r at ∞. which can be found by dividing the leading coefficient of s by the leading coefficient of t from

$$
r = \frac{s}{t} = \frac{12}{t^2}
$$

Since the $\gcd(s,t) = 1$. This gives $b = 12$. Hence

$$
\begin{aligned}
[\sqrt{r}]_\infty &= 0 \\
\alpha_\infty^{+} &= \frac{1}{2} + \sqrt{1+4b} = 4 \\
\alpha_\infty^{-} &= \frac{1}{2} - \sqrt{1+4b} = -3
\end{aligned}
$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$
r = \frac{12}{t^2}
$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^{+}	α_c^{-}
0	2	0	4	-3

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^{+}	α_∞^{-}
2	0	4	-3

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^{\pm}$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^{\pm}$ have also been found, the next step is to determine possible non negative integer d from these using

$$
d = \alpha_\infty^{s(\infty)} - \sum_{c \in \Gamma} \alpha_c^{s(c)}
$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^{\pm}$. This is done by trial over all set of families $s = (s(c))_{c \in \Gamma \cup \infty}$ until such d is found to work in finding candidate ω.

Trying $\alpha_\infty^- = -3$ then

$$\begin{aligned}
d &= \alpha_\infty^- - (\alpha_{c_1}^-) \\
&= -3 - (-3) \\
&= 0
\end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c \in \Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{t-c} \right) + s(\infty)[\sqrt{r}]_\infty$$

The above gives

$$\begin{aligned}
\omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^-}{t-c_1} \right) + (-)[\sqrt{r}]_\infty \\
&= -\frac{3}{t} + (-)(0) \\
&= -\frac{3}{t} \\
&= -\frac{3}{t}
\end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(t)$ of degree $d = 0$ to solve the ode. The polynomial $p(t)$ needs to satisfy the equation

$$p'' + 2\omega p' + (\omega' + \omega^2 - r)\, p = 0 \tag{1A}$$

Let

$$p(t) = 1 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$\begin{aligned}
(0) + 2\left(-\frac{3}{t}\right)(0) + \left(\left(\frac{3}{t^2}\right) + \left(-\frac{3}{t}\right)^2 - \left(\frac{12}{t^2}\right) \right) &= 0 \\
0 &= 0
\end{aligned}$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned}
z_1(t) &= p e^{\int \omega \, dt} \\
&= e^{\int -\frac{3}{t} dt} \\
&= \frac{1}{t^3}
\end{aligned}$$

The first solution to the original ode in y is found from

$$
\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dt} \\
&= z_1 e^{-\int \frac{1}{2}\frac{-4t}{t^2}\,dt} \\
&= z_1 e^{2\ln(t)} \\
&= z_1 \left(t^2\right)
\end{aligned}
$$

Which simplifies to

$$
y_1 = \frac{1}{t}
$$

The second solution y_2 to the original ode is found using reduction of order

$$
y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dt}}{y_1^2}\,dt
$$

Substituting gives

$$
\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{-4t}{t^2}\,dt}}{(y_1)^2}\,dt \\
&= y_1 \int \frac{e^{4\ln(t)}}{(y_1)^2}\,dt \\
&= y_1 \left(\frac{t^7}{7}\right)
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1 \left(\frac{1}{t}\right) + c_2 \left(\frac{1}{t}\left(\frac{t^7}{7}\right)\right)
\end{aligned}
$$

Summary
The solution(s) found are the following

$$
y = \frac{c_1}{t} + \frac{c_2 t^6}{7} \tag{1}
$$

Verification of solutions

$$
y = \frac{c_1}{t} + \frac{c_2 t^6}{7}
$$

Verified OK.

8.24.7 Solving as exact linear second order ode ode

An ode of the form

$$p(t)\,y'' + q(t)\,y' + r(t)\,y = s(t)$$

is exact if

$$p''(t) - q'(t) + r(t) = 0 \tag{1}$$

For the given ode we have

$$\begin{aligned} p(x) &= t^2 \\ q(x) &= -4t \\ r(x) &= -6 \\ s(x) &= 0 \end{aligned}$$

Hence

$$\begin{aligned} p''(x) &= 2 \\ q'(x) &= -4 \end{aligned}$$

Therefore (1) becomes

$$2 - (-4) + (-6) = 0$$

Hence the ode is exact. Since we now know the ode is exact, it can be written as

$$(p(t)\,y' + (q(t) - p'(t))\,y)' = s(x)$$

Integrating gives

$$p(t)\,y' + (q(t) - p'(t))\,y = \int s(t)\;dt$$

Substituting the above values for p, q, r, s gives

$$y't^2 - 6yt = c_1$$

We now have a first order ode to solve which is

$$y't^2 - 6yt = c_1$$

Entering Linear first order ODE solver. In canonical form a linear first order is

$$y' + p(t)y = q(t)$$

Where here

$$p(t) = -\frac{6}{t}$$
$$q(t) = \frac{c_1}{t^2}$$

Hence the ode is

$$y' - \frac{6y}{t} = \frac{c_1}{t^2}$$

The integrating factor μ is

$$\mu = e^{\int -\frac{6}{t}dt}$$
$$= \frac{1}{t^6}$$

The ode becomes

$$\frac{d}{dt}(\mu y) = (\mu)\left(\frac{c_1}{t^2}\right)$$
$$\frac{d}{dt}\left(\frac{y}{t^6}\right) = \left(\frac{1}{t^6}\right)\left(\frac{c_1}{t^2}\right)$$
$$d\left(\frac{y}{t^6}\right) = \left(\frac{c_1}{t^8}\right) dt$$

Integrating gives

$$\frac{y}{t^6} = \int \frac{c_1}{t^8} dt$$
$$\frac{y}{t^6} = -\frac{c_1}{7t^7} + c_2$$

Dividing both sides by the integrating factor $\mu = \frac{1}{t^6}$ results in

$$y = -\frac{c_1}{7t} + c_2 t^6$$

Summary

The solution(s) found are the following

$$y = -\frac{c_1}{7t} + c_2 t^6 \tag{1}$$

Verification of solutions

$$y = -\frac{c_1}{7t} + c_2 t^6$$

Verified OK.

8.24.8 Maple step by step solution

Let's solve

$y''t^2 - 4ty' - 6y = 0$

- Highest derivative means the order of the ODE is 2

y''

- Isolate 2nd derivative

$y'' = \frac{4y'}{t} + \frac{6y}{t^2}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

$y'' - \frac{4y'}{t} - \frac{6y}{t^2} = 0$

- Multiply by denominators of the ODE

$y''t^2 - 4ty' - 6y = 0$

- Make a change of variables

$s = \ln(t)$

□ Substitute the change of variables back into the ODE

  ◦ Calculate the 1st derivative of y with respect to t , using the chain rule

  $y' = \left(\frac{d}{ds}y(s)\right)s'(t)$

  ◦ Compute derivative

  $y' = \frac{\frac{d}{ds}y(s)}{t}$

  ◦ Calculate the 2nd derivative of y with respect to t , using the chain rule

  $y'' = \left(\frac{d^2}{ds^2}y(s)\right)s'(t)^2 + s''(t)\left(\frac{d}{ds}y(s)\right)$

  ◦ Compute derivative

  $y'' = \frac{\frac{d^2}{ds^2}y(s)}{t^2} - \frac{\frac{d}{ds}y(s)}{t^2}$

  Substitute the change of variables back into the ODE

  $\left(\frac{\frac{d^2}{ds^2}y(s)}{t^2} - \frac{\frac{d}{ds}y(s)}{t^2}\right)t^2 - 4\frac{d}{ds}y(s) - 6y(s) = 0$

- Simplify

$\frac{d^2}{ds^2}y(s) - 5\frac{d}{ds}y(s) - 6y(s) = 0$

- Characteristic polynomial of ODE

$r^2 - 5r - 6 = 0$

- Factor the characteristic polynomial

  $(r+1)(r-6)=0$
- Roots of the characteristic polynomial

  $r=(-1,6)$
- 1st solution of the ODE

  $y_1(s)=\mathrm{e}^{-s}$
- 2nd solution of the ODE

  $y_2(s)=\mathrm{e}^{6s}$
- General solution of the ODE

  $y(s)=c_1y_1(s)+c_2y_2(s)$
- Substitute in solutions

  $y(s)=c_1\mathrm{e}^{-s}+c_2\mathrm{e}^{6s}$
- Change variables back using $s=\ln(t)$

  $y=\frac{c_1}{t}+c_2t^6$
- Simplify

  $y=\frac{c_1}{t}+c_2t^6$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
<- LODE of Euler type successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 15

```
dsolve(t^2*diff(y(t),t$2)- 4*t*diff(y(t),t)-6*y(t) = 0,y(t), singsol=all)
```

$$y(t)=\frac{c_1t^7+c_2}{t}$$

✓ Solution by Mathematica

Time used: 0.01 (sec). Leaf size: 18

```
DSolve[t^2*y''[t]-4*t*y'[t]-6*y[t]==0,y[t],t,IncludeSingularSolutions -> True]
```

$$y(t) \to \frac{c_2 t^7 + c_1}{t}$$

8.25 problem 39

Internal problem ID [647]
Internal file name [`OUTPUT/647_Sunday_June_05_2022_01_46_22_AM_3994398/index.tex`]

Book: Elementary differential equations and boundary value problems, 10th ed., Boyce and DiPrima
Section: Chapter 3, Second order linear equations, 3.3 Complex Roots of the Characteristic Equation , page 164
Problem number: 39.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_euler_ode", "second_order_change_of_variable_on_x_method_1", "second_order_change_of_variable_on_x_method_2", "second_order_change_of_variable_on_y_method_1", "second_order_change_of_variable_on_y_method_2", "linear_second_order_ode_solved_by_an_integrating_factor"**

Maple gives the following as the ode type

```
[[_Emden, _Fowler], [_2nd_order, _linear, `_with_symmetry_[0,F(
   x)]`]]
```

$$\boxed{t^2y'' - 4ty' + 6y = 0}$$

8.25.1 Solving as second order euler ode ode

This is Euler second order ODE. Let the solution be $y = t^r$, then $y' = rt^{r-1}$ and $y'' = r(r-1)t^{r-2}$. Substituting these back into the given ODE gives

$$t^2(r(r-1))t^{r-2} - 4trt^{r-1} + 6t^r = 0$$

Simplifying gives

$$r(r-1)\,t^r - 4r\,t^r + 6t^r = 0$$

Since $t^r \neq 0$ then dividing throughout by t^r gives

$$r(r-1) - 4r + 6 = 0$$

Or

$$r^2 - 5r + 6 = 0 \tag{1}$$

Equation (1) is the characteristic equation. Its roots determine the form of the general solution. Using the quadratic equation the roots are

$$r_1 = 2$$
$$r_2 = 3$$

Since the roots are real and distinct, then the general solution is

$$y = c_1y_1 + c_2y_2$$

Where $y_1 = t^{r_1}$ and $y_2 = t^{r_2}$. Hence

$$y = c_2t^3 + c_1t^2$$

<u>Summary</u>
The solution(s) found are the following

$$y = c_2t^3 + c_1t^2 \tag{1}$$

<u>Verification of solutions</u>

$$y = c_2t^3 + c_1t^2$$

Verified OK.

8.25.2 Solving as linear second order ode solved by an integrating factor ode

The ode satisfies this form

$$y'' + p(t)\, y' + \frac{\left(p(t)^2 + p'(t)\right) y}{2} = f(t)$$

Where $p(t) = -\frac{4}{t}$. Therefore, there is an integrating factor given by

$$\begin{aligned}
M(x) &= e^{\frac{1}{2} \int p \, dx} \\
&= e^{\int -\frac{4}{t} \, dx} \\
&= \frac{1}{t^2}
\end{aligned}$$

Multiplying both sides of the ODE by the integrating factor $M(x)$ makes the left side of the ODE a complete differential

$$\begin{aligned}
(M(x)y)'' &= 0 \\
\left(\frac{y}{t^2}\right)'' &= 0
\end{aligned}$$

Integrating once gives

$$\left(\frac{y}{t^2}\right)' = c_1$$

Integrating again gives

$$\left(\frac{y}{t^2}\right) = c_1 t + c_2$$

Hence the solution is

$$y = \frac{c_1 t + c_2}{\frac{1}{t^2}}$$

Or

$$y = c_1 t^3 + c_2 t^2$$

Summary

The solution(s) found are the following

$$y = c_1 t^3 + c_2 t^2 \tag{1}$$

Verification of solutions

$$y = c_1 t^3 + c_2 t^2$$

Verified OK.

8.25.3 Solving as second order change of variable on x method 2 ode

In normal form the ode

$$t^2y'' - 4ty' + 6y = 0 \tag{1}$$

Becomes

$$y'' + p(t)\,y' + q(t)\,y = 0 \tag{2}$$

Where

$$\begin{aligned}
p(t) &= -\frac{4}{t}\\
q(t) &= \frac{6}{t^2}
\end{aligned}$$

Applying change of variables $\tau = g(t)$ to (2) gives

$$\frac{d^2}{d\tau^2}y(\tau) + p_1\left(\frac{d}{d\tau}y(\tau)\right) + q_1 y(\tau) = 0 \tag{3}$$

Where τ is the new independent variable, and

$$p_1(\tau) = \frac{\tau''(t) + p(t)\,\tau'(t)}{\tau'(t)^2} \tag{4}$$

$$q_1(\tau) = \frac{q(t)}{\tau'(t)^2} \tag{5}$$

Let $p_1 = 0$. Eq (4) simplifies to

$$\tau''(t) + p(t)\,\tau'(t) = 0$$

This ode is solved resulting in

$$\begin{aligned}
\tau &= \int \mathrm{e}^{-\left(\int p(t)dt\right)}\,dt\\
&= \int \mathrm{e}^{-\left(\int -\frac{4}{t}dt\right)}\,dt\\
&= \int e^{4\ln(t)}\,dt\\
&= \int t^4 dt\\
&= \frac{t^5}{5}
\end{aligned} \tag{6}$$

Using (6) to evaluate q_1 from (5) gives

$$\begin{aligned} q_1(\tau) &= \frac{q(t)}{\tau'(t)^2} \\ &= \frac{\frac{6}{t^2}}{t^8} \\ &= \frac{6}{t^{10}} \end{aligned} \tag{7}$$

Substituting the above in (3) and noting that now $p_1 = 0$ results in

$$\begin{aligned} \frac{d^2}{d\tau^2}y(\tau) + q_1 y(\tau) &= 0 \\ \frac{d^2}{d\tau^2}y(\tau) + \frac{6y(\tau)}{t^{10}} &= 0 \end{aligned}$$

But in terms of τ

$$\frac{6}{t^{10}} = \frac{6}{25\tau^2}$$

Hence the above ode becomes

$$\frac{d^2}{d\tau^2}y(\tau) + \frac{6y(\tau)}{25\tau^2} = 0$$

The above ode is now solved for $y(\tau)$. The ode can be written as

$$25\left(\frac{d^2}{d\tau^2}y(\tau)\right)\tau^2 + 6y(\tau) = 0$$

Which shows it is a Euler ODE. This is Euler second order ODE. Let the solution be $y(\tau) = \tau^r$, then $y' = r\tau^{r-1}$ and $y'' = r(r-1)\tau^{r-2}$. Substituting these back into the given ODE gives

$$25\tau^2(r(r-1))\tau^{r-2} + 0r\tau^{r-1} + 6\tau^r = 0$$

Simplifying gives

$$25r(r-1)\,\tau^r + 0\,\tau^r + 6\tau^r = 0$$

Since $\tau^r \neq 0$ then dividing throughout by τ^r gives

$$25r(r-1) + 0 + 6 = 0$$

Or

$$25r^2 - 25r + 6 = 0 \tag{1}$$

Equation (1) is the characteristic equation. Its roots determine the form of the general solution. Using the quadratic equation the roots are

$$r_1 = \frac{2}{5}$$
$$r_2 = \frac{3}{5}$$

Since the roots are real and distinct, then the general solution is

$$y(\tau) = c_1 y_1 + c_2 y_2$$

Where $y_1 = \tau^{r_1}$ and $y_2 = \tau^{r_2}$. Hence

$$y(\tau) = c_1 \tau^{\frac{2}{5}} + c_2 \tau^{\frac{3}{5}}$$

The above solution is now transformed back to y using (6) which results in

$$y = \frac{c_1 5^{\frac{3}{5}} (t^5)^{\frac{2}{5}}}{5} + \frac{c_2 5^{\frac{2}{5}} (t^5)^{\frac{3}{5}}}{5}$$

Summary

The solution(s) found are the following

$$y = \frac{c_1 5^{\frac{3}{5}} (t^5)^{\frac{2}{5}}}{5} + \frac{c_2 5^{\frac{2}{5}} (t^5)^{\frac{3}{5}}}{5} \tag{1}$$

Verification of solutions

$$y = \frac{c_1 5^{\frac{3}{5}} (t^5)^{\frac{2}{5}}}{5} + \frac{c_2 5^{\frac{2}{5}} (t^5)^{\frac{3}{5}}}{5}$$

Verified OK.

8.25.4 Solving as second order change of variable on x method 1 ode

In normal form the ode

$$t^2 y'' - 4ty' + 6y = 0 \tag{1}$$

Becomes

$$y'' + p(t)\, y' + q(t)\, y = 0 \tag{2}$$

Where

$$p(t) = -\frac{4}{t}$$
$$q(t) = \frac{6}{t^2}$$

Applying change of variables $\tau = g(t)$ to (2) results

$$\frac{d^2}{d\tau^2}y(\tau) + p_1\left(\frac{d}{d\tau}y(\tau)\right) + q_1 y(\tau) = 0 \tag{3}$$

Where τ is the new independent variable, and

$$p_1(\tau) = \frac{\tau''(t) + p(t)\,\tau'(t)}{\tau'(t)^2} \tag{4}$$

$$q_1(\tau) = \frac{q(t)}{\tau'(t)^2} \tag{5}$$

Let $q_1 = c^2$ where c is some constant. Therefore from (5)

$$\begin{aligned}\tau' &= \frac{1}{c}\sqrt{q} \\ &= \frac{\sqrt{6}\,\sqrt{\frac{1}{t^2}}}{c} \\ \tau'' &= -\frac{\sqrt{6}}{c\sqrt{\frac{1}{t^2}}\,t^3}\end{aligned} \tag{6}$$

Substituting the above into (4) results in

$$\begin{aligned}p_1(\tau) &= \frac{\tau''(t) + p(t)\,\tau'(t)}{\tau'(t)^2} \\ &= \frac{-\frac{\sqrt{6}}{c\sqrt{\frac{1}{t^2}}\,t^3} - \frac{4}{t}\frac{\sqrt{6}\,\sqrt{\frac{1}{t^2}}}{c}}{\left(\frac{\sqrt{6}\,\sqrt{\frac{1}{t^2}}}{c}\right)^2} \\ &= -\frac{5c\sqrt{6}}{6}\end{aligned}$$

Therefore ode (3) now becomes

$$\begin{aligned}y(\tau)'' + p_1 y(\tau)' + q_1 y(\tau) &= 0 \\ \frac{d^2}{d\tau^2}y(\tau) - \frac{5c\sqrt{6}\left(\frac{d}{d\tau}y(\tau)\right)}{6} + c^2 y(\tau) &= 0\end{aligned} \tag{7}$$

The above ode is now solved for $y(\tau)$. Since the ode is now constant coefficients, it can be easily solved to give

$$y(\tau) = \mathrm{e}^{\frac{5\sqrt{6}\,c\tau}{12}}\left(c_1 \cosh\left(\frac{\sqrt{6}\,c\tau}{12}\right) + i c_2 \sinh\left(\frac{\sqrt{6}\,c\tau}{12}\right)\right)$$

Now from (6)

$$\tau = \int \frac{1}{c}\sqrt{q}\,dt$$
$$= \frac{\int \sqrt{6}\,\sqrt{\frac{1}{t^2}}dt}{c}$$
$$= \frac{\sqrt{6}\,\sqrt{\frac{1}{t^2}}\,t\ln(t)}{c}$$

Substituting the above into the solution obtained gives

$$y = t^{\frac{5}{2}}\left(c_1\cosh\left(\frac{\ln(t)}{2}\right) + ic_2\sinh\left(\frac{\ln(t)}{2}\right)\right)$$

Summary

The solution(s) found are the following

$$y = t^{\frac{5}{2}}\left(c_1\cosh\left(\frac{\ln(t)}{2}\right) + ic_2\sinh\left(\frac{\ln(t)}{2}\right)\right) \tag{1}$$

Verification of solutions

$$y = t^{\frac{5}{2}}\left(c_1\cosh\left(\frac{\ln(t)}{2}\right) + ic_2\sinh\left(\frac{\ln(t)}{2}\right)\right)$$

Verified OK.

8.25.5 Solving as second order change of variable on y method 1 ode

In normal form the given ode is written as

$$y'' + p(t)\,y' + q(t)\,y = 0 \tag{2}$$

Where

$$p(t) = -\frac{4}{t}$$
$$q(t) = \frac{6}{t^2}$$

Calculating the Liouville ode invariant Q given by

$$
\begin{aligned}
Q &= q - \frac{p'}{2} - \frac{p^2}{4} \\
&= \frac{6}{t^2} - \frac{\left(-\frac{4}{t}\right)'}{2} - \frac{\left(-\frac{4}{t}\right)^2}{4} \\
&= \frac{6}{t^2} - \frac{\left(\frac{4}{t^2}\right)}{2} - \frac{\left(\frac{16}{t^2}\right)}{4} \\
&= \frac{6}{t^2} - \left(\frac{2}{t^2}\right) - \frac{4}{t^2} \\
&= 0
\end{aligned}
$$

Since the Liouville ode invariant does not depend on the independent variable t then the transformation

$$
y = v(t)\, z(t) \tag{3}
$$

is used to change the original ode to a constant coefficients ode in v. In (3) the term $z(t)$ is given by

$$
\begin{aligned}
z(t) &= \mathrm{e}^{-\left(\int \frac{p(t)}{2} dt\right)} \\
&= e^{-\int \frac{-\frac{4}{t}}{2}} \\
&= t^2
\end{aligned} \tag{5}
$$

Hence (3) becomes

$$
y = v(t)\, t^2 \tag{4}
$$

Applying this change of variable to the original ode results in

$$
t^4 v''(t) = 0
$$

Which is now solved for $v(t)$ Integrating twice gives the solution

$$
v(t) = c_1 t + c_2
$$

Now that $v(t)$ is known, then

$$
\begin{aligned}
y &= v(t)\, z(t) \\
&= (c_1 t + c_2)\,(z(t))
\end{aligned} \tag{7}
$$

But from (5)

$$
z(t) = t^2
$$

Hence (7) becomes

$$y = (c_1t + c_2)\, t^2$$

Summary
The solution(s) found are the following

$$y = (c_1t + c_2)\, t^2 \tag{1}$$

Verification of solutions

$$y = (c_1t + c_2)\, t^2$$

Verified OK.

8.25.6 Solving as second order change of variable on y method 2 ode

In normal form the ode

$$t^2y'' - 4ty' + 6y = 0 \tag{1}$$

Becomes

$$y'' + p(t)\, y' + q(t)\, y = 0 \tag{2}$$

Where

$$p(t) = -\frac{4}{t}$$
$$q(t) = \frac{6}{t^2}$$

Applying change of variables on the depndent variable $y = v(t)\, t^n$ to (2) gives the following ode where the dependent variables is $v(t)$ and not y.

$$v''(t) + \left(\frac{2n}{t} + p\right) v'(t) + \left(\frac{n(n-1)}{t^2} + \frac{np}{t} + q\right) v(t) = 0 \tag{3}$$

Let the coefficient of $v(t)$ above be zero. Hence

$$\frac{n(n-1)}{t^2} + \frac{np}{t} + q = 0 \tag{4}$$

Substituting the earlier values found for $p(t)$ and $q(t)$ into (4) gives

$$\frac{n(n-1)}{t^2} - \frac{4n}{t^2} + \frac{6}{t^2} = 0 \tag{5}$$

Solving (5) for n gives

$$n = 3 \tag{6}$$

Substituting this value in (3) gives

$$\begin{aligned} v''(t) + \frac{2v'(t)}{t} &= 0 \\ v''(t) + \frac{2v'(t)}{t} &= 0 \end{aligned} \tag{7}$$

Using the substitution

$$u(t) = v'(t)$$

Then (7) becomes

$$u'(t) + \frac{2u(t)}{t} = 0 \tag{8}$$

The above is now solved for $u(t)$. In canonical form the ODE is

$$\begin{aligned} u' &= F(t, u) \\ &= f(t)g(u) \\ &= -\frac{2u}{t} \end{aligned}$$

Where $f(t) = -\frac{2}{t}$ and $g(u) = u$. Integrating both sides gives

$$\begin{aligned} \frac{1}{u}\,du &= -\frac{2}{t}\,dt \\ \int \frac{1}{u}\,du &= \int -\frac{2}{t}\,dt \\ \ln(u) &= -2\ln(t) + c_1 \\ u &= \mathrm{e}^{-2\ln(t)+c_1} \\ &= \frac{c_1}{t^2} \end{aligned}$$

Now that $u(t)$ is known, then

$$\begin{aligned} v'(t) &= u(t) \\ v(t) &= \int u(t)\,dt + c_2 \\ &= -\frac{c_1}{t} + c_2 \end{aligned}$$

Hence

$$
\begin{aligned}
y &= v(t)\, t^n \\
&= \left(-\frac{c_1}{t} + c_2\right) t^3 \\
&= (c_2 t - c_1)\, t^2
\end{aligned}
$$

Summary

The solution(s) found are the following

$$y = \left(-\frac{c_1}{t} + c_2\right) t^3 \tag{1}$$

Verification of solutions

$$y = \left(-\frac{c_1}{t} + c_2\right) t^3$$

Verified OK.

8.25.7 Solving using Kovacic algorithm

Writing the ode as

$$t^2 y'' - 4ty' + 6y = 0 \tag{1}$$

$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$
\begin{aligned}
A &= t^2 \\
B &= -4t \\
C &= 6
\end{aligned}
\tag{3}
$$

Applying the Liouville transformation on the dependent variable gives

$$z(t) = y e^{\int \frac{B}{2A}\, dt}$$

Then (2) becomes

$$z''(t) = r z(t) \tag{4}$$

Where r is given by

$$
\begin{aligned}
r &= \frac{s}{t} \\
&= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2}
\end{aligned}
\tag{5}
$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{0}{1} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= 0 \\ t &= 1 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(t) = 0 \tag{7}$$

Equation (7) is now solved. After finding $z(t)$ then y is found using the inverse transformation

$$y = z(t)\, e^{-\int \frac{B}{2A}\, dt}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 412: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 0 - -\infty \\ &= \infty \end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is $infinity$ then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Since $r = 0$ is not a function of t, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$z_1(t) = 1$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$\begin{aligned} y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dt} \\ &= z_1 e^{-\int \frac{1}{2}\frac{-4t}{t^2}\,dt} \\ &= z_1 e^{2\ln(t)} \\ &= z_1(t^2) \end{aligned}$$

Which simplifies to

$$y_1 = t^2$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dt}}{y_1^2}\,dt$$

Substituting gives

$$\begin{aligned} y_2 &= y_1 \int \frac{e^{\int -\frac{-4t}{t^2}\,dt}}{(y_1)^2}\,dt \\ &= y_1 \int \frac{e^{4\ln(t)}}{(y_1)^2}\,dt \\ &= y_1(t) \end{aligned}$$

Therefore the solution is

$$\begin{aligned} y &= c_1 y_1 + c_2 y_2 \\ &= c_1(t^2) + c_2(t^2(t)) \end{aligned}$$

Summary
The solution(s) found are the following

$$y = c_2 t^3 + c_1 t^2 \tag{1}$$

Verification of solutions

$$y = c_2 t^3 + c_1 t^2$$

Verified OK.

8.25.8 Maple step by step solution

Let's solve

$y''t^2 - 4ty' + 6y = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y'' = \frac{4y'}{t} - \frac{6y}{t^2}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y'' - \frac{4y'}{t} + \frac{6y}{t^2} = 0$

- Multiply by denominators of the ODE

  $y''t^2 - 4ty' + 6y = 0$

- Make a change of variables

  $s = \ln(t)$

□ Substitute the change of variables back into the ODE

  - Calculate the 1st derivative of y with respect to t , using the chain rule

    $y' = \left(\frac{d}{ds}y(s)\right) s'(t)$

  - Compute derivative

    $y' = \frac{\frac{d}{ds}y(s)}{t}$

  - Calculate the 2nd derivative of y with respect to t , using the chain rule

    $y'' = \left(\frac{d^2}{ds^2}y(s)\right) s'(t)^2 + s''(t) \left(\frac{d}{ds}y(s)\right)$

  - Compute derivative

    $y'' = \frac{\frac{d^2}{ds^2}y(s)}{t^2} - \frac{\frac{d}{ds}y(s)}{t^2}$

Substitute the change of variables back into the ODE

$$\left(\frac{\frac{d^2}{ds^2}y(s)}{t^2} - \frac{\frac{d}{ds}y(s)}{t^2}\right)t^2 - 4\frac{d}{ds}y(s) + 6y(s) = 0$$

- Simplify

$$\frac{d^2}{ds^2}y(s) - 5\frac{d}{ds}y(s) + 6y(s) = 0$$

- Characteristic polynomial of ODE

$$r^2 - 5r + 6 = 0$$

- Factor the characteristic polynomial

$$(r-2)(r-3) = 0$$

- Roots of the characteristic polynomial

$$r = (2, 3)$$

- 1st solution of the ODE

$$y_1(s) = e^{2s}$$

- 2nd solution of the ODE

$$y_2(s) = e^{3s}$$

- General solution of the ODE

$$y(s) = c_1 y_1(s) + c_2 y_2(s)$$

- Substitute in solutions

$$y(s) = c_1 e^{2s} + c_2 e^{3s}$$

- Change variables back using $s = \ln(t)$

$$y = c_2 t^3 + c_1 t^2$$

- Simplify

$$y = t^2(c_2 t + c_1)$$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
<- LODE of Euler type successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 13

```
dsolve(t^2*diff(y(t),t$2)-4*t*diff(y(t),t)+6*y(t) = 0,y(t), singsol=all)
```

$$y(t) = t^2(c_2t + c_1)$$

✓ Solution by Mathematica

Time used: 0.01 (sec). Leaf size: 16

```
DSolve[t^2*y''[t]-4*t*y'[t]+6*y[t]==0,y[t],t,IncludeSingularSolutions -> True]
```

$$y(t) \to t^2(c_2t + c_1)$$

8.26 problem 40

Internal problem ID [648]
Internal file name [`OUTPUT/648_Sunday_June_05_2022_01_46_23_AM_19111792/index.tex`]

Book: Elementary differential equations and boundary value problems, 10th ed., Boyce and DiPrima
Section: Chapter 3, Second order linear equations, 3.3 Complex Roots of the Characteristic Equation , page 164
Problem number: 40.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_euler_ode", "second_order_change_of_variable_on_x_method_1", "second_order_change_of_variable_on_x_method_2", "second_order_change_of_variable_on_y_method_2"**

Maple gives the following as the ode type

```
[[_Emden, _Fowler]]
```

$$\boxed{t^2y'' - ty' + 5y = 0}$$

8.26.1 Solving as second order euler ode ode

This is Euler second order ODE. Let the solution be $y = t^r$, then $y' = rt^{r-1}$ and $y'' = r(r-1)t^{r-2}$. Substituting these back into the given ODE gives

$$t^2(r(r-1))t^{r-2} - trt^{r-1} + 5t^r = 0$$

Simplifying gives

$$r(r-1)\, t^r - r\, t^r + 5t^r = 0$$

Since $t^r \neq 0$ then dividing throughout by t^r gives

$$r(r-1) - r + 5 = 0$$

Or

$$r^2 - 2r + 5 = 0 \tag{1}$$

Equation (1) is the characteristic equation. Its roots determine the form of the general solution. Using the quadratic equation the roots are

$$\begin{aligned}r_1 &= 1 - 2i \\ r_2 &= 1 + 2i\end{aligned}$$

The roots are complex conjugate of each others. Let the roots be

$$\begin{aligned}r_1 &= \alpha + i\beta \\ r_2 &= \alpha - i\beta\end{aligned}$$

Where in this case $\alpha = 1$ and $\beta = -2$. Hence the solution becomes

$$\begin{aligned}y &= c_1 t^{r_1} + c_2 t^{r_2} \\ &= c_1 t^{\alpha + i\beta} + c_2 t^{\alpha - i\beta} \\ &= t^\alpha \left(c_1 t^{i\beta} + c_2 t^{-i\beta}\right) \\ &= t^\alpha \left(c_1 e^{\ln\left(t^{i\beta}\right)} + c_2 e^{\ln\left(t^{-i\beta}\right)}\right) \\ &= t^\alpha \left(c_1 e^{i(\beta \ln t)} + c_2 e^{-i(\beta \ln t)}\right)\end{aligned}$$

Using the values for $\alpha = 1, \beta = -2$, the above becomes

$$y = t^1 \left(c_1 e^{-2i \ln(t)} + c_2 e^{2i \ln(t)}\right)$$

Using Euler relation, the expression $c_1 e^{iA} + c_2 e^{-iA}$ is transformed to $c_1 \cos A + c_1 \sin A$ where the constants are free to change. Applying this to the above result gives

$$y = t(c_1 \cos(2\ln(t)) + c_2 \sin(2\ln(t)))$$

Summary

The solution(s) found are the following

$$y = t(c_1 \cos(2\ln(t)) + c_2 \sin(2\ln(t))) \tag{1}$$

Verification of solutions

$$y = t(c_1 \cos(2\ln(t)) + c_2 \sin(2\ln(t)))$$

Verified OK.

8.26.2 Solving as second order change of variable on x method 2 ode

In normal form the ode

$$t^2y'' - ty' + 5y = 0 \tag{1}$$

Becomes

$$y'' + p(t)\, y' + q(t)\, y = 0 \tag{2}$$

Where

$$\begin{aligned} p(t) &= -\frac{1}{t} \\ q(t) &= \frac{5}{t^2} \end{aligned}$$

Applying change of variables $\tau = g(t)$ to (2) gives

$$\frac{d^2}{d\tau^2}y(\tau) + p_1\left(\frac{d}{d\tau}y(\tau)\right) + q_1y(\tau) = 0 \tag{3}$$

Where τ is the new independent variable, and

$$p_1(\tau) = \frac{\tau''(t) + p(t)\,\tau'(t)}{\tau'(t)^2} \tag{4}$$

$$q_1(\tau) = \frac{q(t)}{\tau'(t)^2} \tag{5}$$

Let $p_1 = 0$. Eq (4) simplifies to

$$\tau''(t) + p(t)\,\tau'(t) = 0$$

This ode is solved resulting in

$$\begin{aligned} \tau &= \int \mathrm{e}^{-\left(\int p(t)dt\right)}dt \\ &= \int \mathrm{e}^{-\left(\int -\frac{1}{t}dt\right)}dt \\ &= \int e^{\ln(t)}\, dt \\ &= \int t dt \\ &= \frac{t^2}{2} \end{aligned} \tag{6}$$

Using (6) to evaluate q_1 from (5) gives

$$
\begin{aligned}
q_1(\tau) &= \frac{q(t)}{\tau'(t)^2} \\
&= \frac{\frac{5}{t^2}}{t^2} \\
&= \frac{5}{t^4}
\end{aligned} \tag{7}
$$

Substituting the above in (3) and noting that now $p_1 = 0$ results in

$$
\begin{aligned}
\frac{d^2}{d\tau^2} y(\tau) + q_1 y(\tau) &= 0 \\
\frac{d^2}{d\tau^2} y(\tau) + \frac{5y(\tau)}{t^4} &= 0
\end{aligned}
$$

But in terms of τ

$$
\frac{5}{t^4} = \frac{5}{4\tau^2}
$$

Hence the above ode becomes

$$
\frac{d^2}{d\tau^2} y(\tau) + \frac{5y(\tau)}{4\tau^2} = 0
$$

The above ode is now solved for $y(\tau)$. The ode can be written as

$$
4\left(\frac{d^2}{d\tau^2} y(\tau)\right)\tau^2 + 5y(\tau) = 0
$$

Which shows it is a Euler ODE. This is Euler second order ODE. Let the solution be $y(\tau) = \tau^r$, then $y' = r\tau^{r-1}$ and $y'' = r(r-1)\tau^{r-2}$. Substituting these back into the given ODE gives

$$
4\tau^2(r(r-1))\tau^{r-2} + 0r\tau^{r-1} + 5\tau^r = 0
$$

Simplifying gives

$$
4r(r-1)\,\tau^r + 0\,\tau^r + 5\tau^r = 0
$$

Since $\tau^r \neq 0$ then dividing throughout by τ^r gives

$$
4r(r-1) + 0 + 5 = 0
$$

Or

$$
4r^2 - 4r + 5 = 0 \tag{1}
$$

Equation (1) is the characteristic equation. Its roots determine the form of the general solution. Using the quadratic equation the roots are

$$
\begin{aligned}
r_1 &= \frac{1}{2} - i \\
r_2 &= \frac{1}{2} + i
\end{aligned}
$$

The roots are complex conjugate of each others. Let the roots be

$$
\begin{aligned}
r_1 &= \alpha + i\beta \\
r_2 &= \alpha - i\beta
\end{aligned}
$$

Where in this case $\alpha = \frac{1}{2}$ and $\beta = -1$. Hence the solution becomes

$$
\begin{aligned}
y(\tau) &= c_1 \tau^{r_1} + c_2 \tau^{r_2} \\
&= c_1 \tau^{\alpha + i\beta} + c_2 \tau^{\alpha - i\beta} \\
&= \tau^{\alpha} \left(c_1 \tau^{i\beta} + c_2 \tau^{-i\beta} \right) \\
&= \tau^{\alpha} \left(c_1 e^{\ln\left(\tau^{i\beta}\right)} + c_2 e^{\ln\left(\tau^{-i\beta}\right)} \right) \\
&= \tau^{\alpha} \left(c_1 e^{i(\beta \ln \tau)} + c_2 e^{-i(\beta \ln \tau)} \right)
\end{aligned}
$$

Using the values for $\alpha = \frac{1}{2}, \beta = -1$, the above becomes

$$
y(\tau) = \tau^{\frac{1}{2}} \left(c_1 e^{-i \ln(\tau)} + c_2 e^{i \ln(\tau)} \right)
$$

Using Euler relation, the expression $c_1 e^{iA} + c_2 e^{-iA}$ is transformed to $c_1 \cos A + c_1 \sin A$ where the constants are free to change. Applying this to the above result gives

$$
y(\tau) = \sqrt{\tau}(c_1 \cos(\ln(\tau)) + c_2 \sin(\ln(\tau)))
$$

The above solution is now transformed back to y using (6) which results in

$$
y = \frac{\sqrt{2}\, t(c_1 \cos(-\ln(2) + 2\ln(t)) + c_2 \sin(-\ln(2) + 2\ln(t)))}{2}
$$

Summary

The solution(s) found are the following

$$
y = \frac{\sqrt{2}\, t(c_1 \cos(-\ln(2) + 2\ln(t)) + c_2 \sin(-\ln(2) + 2\ln(t)))}{2} \tag{1}
$$

Verification of solutions

$$
y = \frac{\sqrt{2}\, t(c_1 \cos(-\ln(2) + 2\ln(t)) + c_2 \sin(-\ln(2) + 2\ln(t)))}{2}
$$

Verified OK.

8.26.3 Solving as second order change of variable on x method 1 ode

In normal form the ode

$$t^2y'' - ty' + 5y = 0 \tag{1}$$

Becomes

$$y'' + p(t)\, y' + q(t)\, y = 0 \tag{2}$$

Where

$$\begin{aligned} p(t) &= -\frac{1}{t} \\ q(t) &= \frac{5}{t^2} \end{aligned}$$

Applying change of variables $\tau = g(t)$ to (2) results

$$\frac{d^2}{d\tau^2}y(\tau) + p_1\left(\frac{d}{d\tau}y(\tau)\right) + q_1y(\tau) = 0 \tag{3}$$

Where τ is the new independent variable, and

$$p_1(\tau) = \frac{\tau''(t) + p(t)\, \tau'(t)}{\tau'(t)^2} \tag{4}$$

$$q_1(\tau) = \frac{q(t)}{\tau'(t)^2} \tag{5}$$

Let $q_1 = c^2$ where c is some constant. Therefore from (5)

$$\begin{aligned} \tau' &= \frac{1}{c}\sqrt{q} \\ &= \frac{\sqrt{5}\, \sqrt{\frac{1}{t^2}}}{c} \\ \tau'' &= -\frac{\sqrt{5}}{c\sqrt{\frac{1}{t^2}}\, t^3} \end{aligned} \tag{6}$$

Substituting the above into (4) results in

$$\begin{aligned} p_1(\tau) &= \frac{\tau''(t) + p(t)\, \tau'(t)}{\tau'(t)^2} \\ &= \frac{-\frac{\sqrt{5}}{c\sqrt{\frac{1}{t^2}}\, t^3} - \frac{1}{t}\frac{\sqrt{5}\sqrt{\frac{1}{t^2}}}{c}}{\left(\frac{\sqrt{5}\sqrt{\frac{1}{t^2}}}{c}\right)^2} \\ &= -\frac{2c\sqrt{5}}{5} \end{aligned}$$

Therefore ode (3) now becomes

$$y(\tau)'' + p_1 y(\tau)' + q_1 y(\tau) = 0$$

$$\frac{d^2}{d\tau^2}y(\tau) - \frac{2c\sqrt{5}\left(\frac{d}{d\tau}y(\tau)\right)}{5} + c^2 y(\tau) = 0 \tag{7}$$

The above ode is now solved for $y(\tau)$. Since the ode is now constant coefficients, it can be easily solved to give

$$y(\tau) = e^{\frac{\sqrt{5}\,c\tau}{5}}\left(c_1 \cos\left(\frac{2\sqrt{5}\,c\tau}{5}\right) + c_2 \sin\left(\frac{2\sqrt{5}\,c\tau}{5}\right)\right)$$

Now from (6)

$$\begin{aligned}\tau &= \int \frac{1}{c}\sqrt{q}\,dt \\ &= \frac{\int \sqrt{5}\,\sqrt{\frac{1}{t^2}}dt}{c} \\ &= \frac{\sqrt{5}\,\sqrt{\frac{1}{t^2}}\,t\ln(t)}{c}\end{aligned}$$

Substituting the above into the solution obtained gives

$$y = t(c_1 \cos(2\ln(t)) + c_2 \sin(2\ln(t)))$$

Summary

The solution(s) found are the following

$$y = t(c_1 \cos(2\ln(t)) + c_2 \sin(2\ln(t))) \tag{1}$$

Verification of solutions

$$y = t(c_1 \cos(2\ln(t)) + c_2 \sin(2\ln(t)))$$

Verified OK.

8.26.4 Solving as second order change of variable on y method 2 ode

In normal form the ode

$$t^2y'' - ty' + 5y = 0 \tag{1}$$

Becomes

$$y'' + p(t)\,y' + q(t)\,y = 0 \tag{2}$$

Where

$$
\begin{aligned}
p(t) &= -\frac{1}{t} \\
q(t) &= \frac{5}{t^2}
\end{aligned}
$$

Applying change of variables on the depndent variable $y = v(t)\, t^n$ to (2) gives the following ode where the dependent variables is $v(t)$ and not y.

$$
v''(t) + \left(\frac{2n}{t} + p\right) v'(t) + \left(\frac{n(n-1)}{t^2} + \frac{np}{t} + q\right) v(t) = 0 \tag{3}
$$

Let the coefficient of $v(t)$ above be zero. Hence

$$
\frac{n(n-1)}{t^2} + \frac{np}{t} + q = 0 \tag{4}
$$

Substituting the earlier values found for $p(t)$ and $q(t)$ into (4) gives

$$
\frac{n(n-1)}{t^2} - \frac{n}{t^2} + \frac{5}{t^2} = 0 \tag{5}
$$

Solving (5) for n gives

$$
n = 1 + 2i \tag{6}
$$

Substituting this value in (3) gives

$$
\begin{aligned}
v''(t) + \left(\frac{2+4i}{t} - \frac{1}{t}\right) v'(t) &= 0 \\
v''(t) + \frac{(1+4i)\, v'(t)}{t} &= 0
\end{aligned} \tag{7}
$$

Using the substitution

$$
u(t) = v'(t)
$$

Then (7) becomes

$$
u'(t) + \frac{(1+4i)\, u(t)}{t} = 0 \tag{8}
$$

The above is now solved for $u(t)$. In canonical form the ODE is

$$
\begin{aligned}
u' &= F(t,u) \\
&= f(t)g(u) \\
&= \frac{(-1-4i)\,u}{t}
\end{aligned}
$$

Where $f(t) = \frac{-1-4i}{t}$ and $g(u) = u$. Integrating both sides gives

$$
\begin{aligned}
\frac{1}{u}\,du &= \frac{-1-4i}{t}\,dt \\
\int \frac{1}{u}\,du &= \int \frac{-1-4i}{t}\,dt \\
\ln(u) &= (-1-4i)\ln(t) + c_1 \\
u &= \mathrm{e}^{(-1-4i)\ln(t)+c_1} \\
&= c_1 \mathrm{e}^{(-1-4i)\ln(t)}
\end{aligned}
$$

Which simplifies to

$$
u(t) = \frac{c_1 t^{-4i}}{t}
$$

Now that $u(t)$ is known, then

$$
\begin{aligned}
v'(t) &= u(t) \\
v(t) &= \int u(t)\,dt + c_2 \\
&= \frac{i c_1 t^{-4i}}{4} + c_2
\end{aligned}
$$

Hence

$$
\begin{aligned}
y &= v(t)\,t^n \\
&= \left(\frac{i c_1 t^{-4i}}{4} + c_2\right) t^{1+2i} \\
&= t^{1+2i} c_2 + \frac{i t^{1-2i} c_1}{4}
\end{aligned}
$$

<u>Summary</u>

The solution(s) found are the following

$$
y = \left(\frac{i c_1 t^{-4i}}{4} + c_2\right) t^{1+2i} \tag{1}
$$

Verification of solutions

$$y = \left(\frac{ic_1 t^{-4i}}{4} + c_2\right) t^{1+2i}$$

Verified OK.

8.26.5 Solving using Kovacic algorithm

Writing the ode as

$$t^2y'' - ty' + 5y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= t^2 \\ B &= -t \\ C &= 5 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(t) = ye^{\int \frac{B}{2A}\,dt}$$

Then (2) becomes

$$z''(t) = rz(t) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{-17}{4t^2} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= -17 \\ t &= 4t^2 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(t) = \left(-\frac{17}{4t^2}\right) z(t) \tag{7}$$

Equation (7) is now solved. After finding $z(t)$ then y is found using the inverse transformation

$$y = z(t)\, e^{-\int \frac{B}{2A}\, dt}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 414: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 2 - 0 \\ &= 2 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 4t^2$. There is a pole at $t = 0$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 2 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Since pole order is not larger than 2 and the order at ∞ is 2 then the necessary conditions for case three are met. Therefore

$$L = [1, 2, 4, 6, 12]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = -\frac{17}{4t^2}$$

For the pole at $t = 0$ let b be the coefficient of $\frac{1}{t^2}$ in the partial fractions decomposition of r given above. Therefore $b = -\frac{17}{4}$. Hence

$$\begin{aligned}
[\sqrt{r}]_c &= 0 \\
\alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{1}{2} + 2i \\
\alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = \frac{1}{2} - 2i
\end{aligned}$$

Since the order of r at ∞ is 2 then $[\sqrt{r}]_\infty = 0$. Let b be the coefficient of $\frac{1}{t^2}$ in the Laurent series expansion of r at ∞. which can be found by dividing the leading coefficient of s by the leading coefficient of t from

$$r = \frac{s}{t} = -\frac{17}{4t^2}$$

Since the $\gcd(s,t) = 1$. This gives $b = -\frac{17}{4}$. Hence

$$\begin{aligned}
[\sqrt{r}]_\infty &= 0 \\
\alpha_\infty^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{1}{2} + 2i \\
\alpha_\infty^- &= \frac{1}{2} - \sqrt{1+4b} = \frac{1}{2} - 2i
\end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = -\frac{17}{4t^2}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
0	2	0	$\frac{1}{2} + 2i$	$\frac{1}{2} - 2i$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
2	0	$\frac{1}{2} + 2i$	$\frac{1}{2} - 2i$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_{\infty}^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_{\infty}^{\pm}$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_{\infty}^{-} = \frac{1}{2} - 2i$ then

$$\begin{aligned}
d &= \alpha_{\infty}^{-} - \left(\alpha_{c_1}^{-}\right) \\
&= \frac{1}{2} - 2i - \left(\frac{1}{2} - 2i\right) \\
&= 0
\end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c\in\Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{t-c} \right) + s(\infty)[\sqrt{r}]_{\infty}$$

The above gives

$$\begin{aligned}
\omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^{-}}{t-c_1} \right) + (-)[\sqrt{r}]_{\infty} \\
&= \frac{\frac{1}{2} - 2i}{t} + (-)\,(0) \\
&= \frac{\frac{1}{2} - 2i}{t} \\
&= \frac{\frac{1}{2} - 2i}{t}
\end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(t)$ of degree $d = 0$ to solve the ode. The polynomial $p(t)$ needs to satisfy the equation

$$p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right) p = 0 \tag{1A}$$

Let

$$p(t) = 1 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$\begin{aligned}
(0) + 2\left(\frac{\frac{1}{2} - 2i}{t}\right)(0) + \left(\left(\frac{-\frac{1}{2} + 2i}{t^2}\right) + \left(\frac{\frac{1}{2} - 2i}{t}\right)^2 - \left(-\frac{17}{4t^2}\right) \right) &= 0 \\
0 &= 0
\end{aligned}$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned} z_1(t) &= p e^{\int \omega \, dt} \\ &= \mathrm{e}^{\int \frac{\frac{1}{2}-2i}{t} dt} \\ &= t^{\frac{1}{2}-2i} \end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned} y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A} \, dt} \\ &= z_1 e^{-\int \frac{1}{2}\frac{-t}{t^2} \, dt} \\ &= z_1 e^{\frac{\ln(t)}{2}} \\ &= z_1 \left(\sqrt{t}\right) \end{aligned}$$

Which simplifies to

$$y_1 = t^{1-2i}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A} \, dt}}{y_1^2} \, dt$$

Substituting gives

$$\begin{aligned} y_2 &= y_1 \int \frac{e^{\int -\frac{-t}{t^2} \, dt}}{(y_1)^2} \, dt \\ &= y_1 \int \frac{e^{\ln(t)}}{(y_1)^2} \, dt \\ &= y_1 \left(-\frac{i t^{4i}}{4}\right) \end{aligned}$$

Therefore the solution is

$$\begin{aligned} y &= c_1 y_1 + c_2 y_2 \\ &= c_1 \left(t^{1-2i}\right) + c_2 \left(t^{1-2i} \left(-\frac{i t^{4i}}{4}\right)\right) \end{aligned}$$

Summary
The solution(s) found are the following

$$y = t^{1-2i}c_1 - \frac{i c_2 t^{1+2i}}{4} \tag{1}$$

Verification of solutions

$$y = t^{1-2i}c_1 - \frac{i c_2 t^{1+2i}}{4}$$

Verified OK.

8.26.6 Maple step by step solution

Let's solve

$y''t^2 - ty' + 5y = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y'' = \frac{y'}{t} - \frac{5y}{t^2}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y'' - \frac{y'}{t} + \frac{5y}{t^2} = 0$

- Multiply by denominators of the ODE

  $y''t^2 - ty' + 5y = 0$

- Make a change of variables

  $s = \ln(t)$

- □ Substitute the change of variables back into the ODE
  - Calculate the 1st derivative of y with respect to t , using the chain rule

    $y' = \left(\frac{d}{ds}y(s)\right) s'(t)$

  - Compute derivative

    $y' = \frac{\frac{d}{ds}y(s)}{t}$

  - Calculate the 2nd derivative of y with respect to t , using the chain rule

    $y'' = \left(\frac{d^2}{ds^2}y(s)\right) s'(t)^2 + s''(t)\left(\frac{d}{ds}y(s)\right)$

  - Compute derivative

$y'' = \frac{\frac{d^2}{ds^2}y(s)}{t^2} - \frac{\frac{d}{ds}y(s)}{t^2}$

Substitute the change of variables back into the ODE

$\left(\frac{\frac{d^2}{ds^2}y(s)}{t^2} - \frac{\frac{d}{ds}y(s)}{t^2}\right)t^2 - \frac{d}{ds}y(s) + 5y(s) = 0$

- Simplify

  $\frac{d^2}{ds^2}y(s) - 2\frac{d}{ds}y(s) + 5y(s) = 0$

- Characteristic polynomial of ODE

  $r^2 - 2r + 5 = 0$

- Use quadratic formula to solve for r

  $r = \frac{2 \pm (\sqrt{-16})}{2}$

- Roots of the characteristic polynomial

  $r = (1 - 2\,\mathrm{I}, 1 + 2\,\mathrm{I})$

- 1st solution of the ODE

  $y_1(s) = \mathrm{e}^s \cos(2s)$

- 2nd solution of the ODE

  $y_2(s) = \mathrm{e}^s \sin(2s)$

- General solution of the ODE

  $y(s) = c_1 y_1(s) + c_2 y_2(s)$

- Substitute in solutions

  $y(s) = c_1 \mathrm{e}^s \cos(2s) + c_2 \mathrm{e}^s \sin(2s)$

- Change variables back using $s = \ln(t)$

  $y = c_1 t \cos(2\ln(t)) + c_2 t \sin(2\ln(t))$

- Simplify

  $y = t(c_1 \cos(2\ln(t)) + c_2 \sin(2\ln(t)))$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
<- LODE of Euler type successful`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 21

```
dsolve(t^2*diff(y(t),t$2)- t*diff(y(t),t)+5*y(t) = 0,y(t), singsol=all)
```

$$y(t) = t(c_1 \sin(2\ln(t)) + c_2 \cos(2\ln(t)))$$

✓ Solution by Mathematica

Time used: 0.023 (sec). Leaf size: 24

```
DSolve[t^2*y''[t]-t*y'[t]+5*y[t]==0,y[t],t,IncludeSingularSolutions -> True]
```

$$y(t) \to t(c_2 \cos(2\log(t)) + c_1 \sin(2\log(t)))$$

8.27 problem 41

Internal problem ID [649]
Internal file name [`OUTPUT/649_Sunday_June_05_2022_01_46_24_AM_12275308/index.tex`]

Book: Elementary differential equations and boundary value problems, 10th ed., Boyce and DiPrima
Section: Chapter 3, Second order linear equations, 3.3 Complex Roots of the Characteristic Equation , page 164
Problem number: 41.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_euler_ode", "second_order_change_of_variable_on_x_method_2", "second_order_change_of_variable_on_y_method_2", "second_order_ode_non_constant_coeff_transformation_on_B"**

Maple gives the following as the ode type

```
[[_Emden, _Fowler]]
```

$$\boxed{t^2y'' + 3ty' - 3y = 0}$$

8.27.1 Solving as second order euler ode ode

This is Euler second order ODE. Let the solution be $y = t^r$, then $y' = rt^{r-1}$ and $y'' = r(r-1)t^{r-2}$. Substituting these back into the given ODE gives

$$t^2(r(r-1))t^{r-2} + 3trt^{r-1} - 3t^r = 0$$

Simplifying gives

$$r(r-1)\,t^r + 3r\,t^r - 3t^r = 0$$

Since $t^r \neq 0$ then dividing throughout by t^r gives

$$r(r-1) + 3r - 3 = 0$$

Or

$$r^2 + 2r - 3 = 0 \tag{1}$$

Equation (1) is the characteristic equation. Its roots determine the form of the general solution. Using the quadratic equation the roots are

$$r_1 = -3$$
$$r_2 = 1$$

Since the roots are real and distinct, then the general solution is

$$y = c_1 y_1 + c_2 y_2$$

Where $y_1 = t^{r_1}$ and $y_2 = t^{r_2}$. Hence

$$y = \frac{c_1}{t^3} + c_2 t$$

Summary
The solution(s) found are the following

$$y = \frac{c_1}{t^3} + c_2 t \tag{1}$$

Verification of solutions

$$y = \frac{c_1}{t^3} + c_2 t$$

Verified OK.

8.27.2 Solving as second order change of variable on x method 2 ode

In normal form the ode

$$t^2 y'' + 3ty' - 3y = 0 \tag{1}$$

Becomes

$$y'' + p(t)\, y' + q(t)\, y = 0 \tag{2}$$

Where

$$p(t) = \frac{3}{t}$$
$$q(t) = -\frac{3}{t^2}$$

Applying change of variables $\tau = g(t)$ to (2) gives

$$\frac{d^2}{d\tau^2}y(\tau) + p_1\left(\frac{d}{d\tau}y(\tau)\right) + q_1 y(\tau) = 0 \tag{3}$$

Where τ is the new independent variable, and

$$p_1(\tau) = \frac{\tau''(t) + p(t)\,\tau'(t)}{\tau'(t)^2} \tag{4}$$

$$q_1(\tau) = \frac{q(t)}{\tau'(t)^2} \tag{5}$$

Let $p_1 = 0$. Eq (4) simplifies to

$$\tau''(t) + p(t)\,\tau'(t) = 0$$

This ode is solved resulting in

$$\begin{aligned}\tau &= \int \mathrm{e}^{-\left(\int p(t)dt\right)}dt \\ &= \int \mathrm{e}^{-\left(\int \frac{3}{t}dt\right)}dt \\ &= \int e^{-3\ln(t)}\,dt \\ &= \int \frac{1}{t^3}dt \\ &= -\frac{1}{2t^2}\end{aligned} \tag{6}$$

Using (6) to evaluate q_1 from (5) gives

$$\begin{aligned}q_1(\tau) &= \frac{q(t)}{\tau'(t)^2} \\ &= \frac{-\frac{3}{t^2}}{\frac{1}{t^6}} \\ &= -3t^4\end{aligned} \tag{7}$$

Substituting the above in (3) and noting that now $p_1 = 0$ results in

$$\frac{d^2}{d\tau^2}y(\tau) + q_1 y(\tau) = 0$$
$$\frac{d^2}{d\tau^2}y(\tau) - 3t^4 y(\tau) = 0$$

But in terms of τ

$$-3t^4 = -\frac{3}{4\tau^2}$$

Hence the above ode becomes

$$\frac{d^2}{d\tau^2}y(\tau) - \frac{3y(\tau)}{4\tau^2} = 0$$

The above ode is now solved for $y(\tau)$. The ode can be written as

$$4\left(\frac{d^2}{d\tau^2}y(\tau)\right)\tau^2 - 3y(\tau) = 0$$

Which shows it is a Euler ODE. This is Euler second order ODE. Let the solution be $y(\tau) = \tau^r$, then $y' = r\tau^{r-1}$ and $y'' = r(r-1)\tau^{r-2}$. Substituting these back into the given ODE gives

$$4\tau^2(r(r-1))\tau^{r-2} + 0r\tau^{r-1} - 3\tau^r = 0$$

Simplifying gives

$$4r(r-1)\,\tau^r + 0\,\tau^r - 3\tau^r = 0$$

Since $\tau^r \neq 0$ then dividing throughout by τ^r gives

$$4r(r-1) + 0 - 3 = 0$$

Or

$$4r^2 - 4r - 3 = 0 \tag{1}$$

Equation (1) is the characteristic equation. Its roots determine the form of the general solution. Using the quadratic equation the roots are

$$r_1 = -\frac{1}{2}$$
$$r_2 = \frac{3}{2}$$

Since the roots are real and distinct, then the general solution is

$$y(\tau) = c_1y_1 + c_2y_2$$

Where $y_1 = \tau^{r_1}$ and $y_2 = \tau^{r_2}$. Hence

$$y(\tau) = \frac{c_1}{\sqrt{\tau}} + c_2\tau^{\frac{3}{2}}$$

The above solution is now transformed back to y using (6) which results in

$$y = \frac{(4c_1t^4 + c_2)\sqrt{2}}{4t^4\sqrt{-\frac{1}{t^2}}}$$

Summary
The solution(s) found are the following

$$y = \frac{(4c_1t^4 + c_2)\sqrt{2}}{4t^4\sqrt{-\frac{1}{t^2}}} \tag{1}$$

Verification of solutions

$$y = \frac{(4c_1t^4 + c_2)\sqrt{2}}{4t^4\sqrt{-\frac{1}{t^2}}}$$

Verified OK.

8.27.3 Solving as second order change of variable on y method 2 ode

In normal form the ode

$$t^2y'' + 3ty' - 3y = 0 \tag{1}$$

Becomes

$$y'' + p(t)\,y' + q(t)\,y = 0 \tag{2}$$

Where

$$\begin{aligned} p(t) &= \frac{3}{t} \\ q(t) &= -\frac{3}{t^2} \end{aligned}$$

Applying change of variables on the depndent variable $y = v(t)\,t^n$ to (2) gives the following ode where the dependent variables is $v(t)$ and not y.

$$v''(t) + \left(\frac{2n}{t} + p\right) v'(t) + \left(\frac{n(n-1)}{t^2} + \frac{np}{t} + q\right) v(t) = 0 \tag{3}$$

Let the coefficient of $v(t)$ above be zero. Hence

$$\frac{n(n-1)}{t^2} + \frac{np}{t} + q = 0 \tag{4}$$

Substituting the earlier values found for $p(t)$ and $q(t)$ into (4) gives

$$\frac{n(n-1)}{t^2} + \frac{3n}{t^2} - \frac{3}{t^2} = 0 \tag{5}$$

Solving (5) for n gives

$$n = 1 \tag{6}$$

Substituting this value in (3) gives

$$\begin{aligned} v''(t) + \frac{5v'(t)}{t} &= 0 \\ v''(t) + \frac{5v'(t)}{t} &= 0 \end{aligned} \tag{7}$$

Using the substitution

$$u(t) = v'(t)$$

Then (7) becomes

$$u'(t) + \frac{5u(t)}{t} = 0 \tag{8}$$

The above is now solved for $u(t)$. In canonical form the ODE is

$$\begin{aligned} u' &= F(t, u) \\ &= f(t)g(u) \\ &= -\frac{5u}{t} \end{aligned}$$

Where $f(t) = -\frac{5}{t}$ and $g(u) = u$. Integrating both sides gives

$$\begin{aligned} \frac{1}{u}\, du &= -\frac{5}{t}\, dt \\ \int \frac{1}{u}\, du &= \int -\frac{5}{t}\, dt \\ \ln(u) &= -5\ln(t) + c_1 \\ u &= \mathrm{e}^{-5\ln(t)+c_1} \\ &= \frac{c_1}{t^5} \end{aligned}$$

Now that $u(t)$ is known, then

$$\begin{aligned} v'(t) &= u(t) \\ v(t) &= \int u(t)\, dt + c_2 \\ &= -\frac{c_1}{4t^4} + c_2 \end{aligned}$$

Hence

$$
\begin{aligned}
y &= v(t)\, t^n \\
&= \left(-\frac{c_1}{4t^4} + c_2\right) t \\
&= \left(-\frac{c_1}{4t^4} + c_2\right) t
\end{aligned}
$$

Summary
The solution(s) found are the following

$$
y = \left(-\frac{c_1}{4t^4} + c_2\right) t \tag{1}
$$

Verification of solutions

$$
y = \left(-\frac{c_1}{4t^4} + c_2\right) t
$$

Verified OK.

8.27.4 Solving as second order ode non constant coeff transformation on B ode

Given an ode of the form

$$
Ay'' + By' + Cy = F(t)
$$

This method reduces the order ode the ODE by one by applying the transformation

$$
y = Bv
$$

This results in

$$
\begin{aligned}
y' &= B'v + v'B \\
y'' &= B''v + B'v' + v''B + v'B' \\
&= v''B + 2v' + B' + B''v
\end{aligned}
$$

And now the original ode becomes

$$
\begin{aligned}
A(v''B + 2v'B' + B''v) + B(B'v + v'B) + CBv &= 0 \\
ABv'' + \left(2AB' + B^2\right) v' + (AB'' + BB' + CB)\, v &= 0 \qquad (1)
\end{aligned}
$$

If the term $AB'' + BB' + CB$ is zero, then this method works and can be used to solve

$$
ABv'' + \left(2AB' + B^2\right) v' = 0
$$

By Using $u = v'$ which reduces the order of the above ode to one. The new ode is

$$ABu' + \left(2AB' + B^2\right)u = 0$$

The above ode is first order ode which is solved for u. Now a new ode $v' = u$ is solved for v as first order ode. Then the final solution is obtain from $y = Bv$.

This method works only if the term $AB'' + BB' + CB$ is zero. The given ODE shows that

$$\begin{aligned}
A &= t^2 \\
B &= 3t \\
C &= -3 \\
F &= 0
\end{aligned}$$

The above shows that for this ode

$$\begin{aligned}
AB'' + BB' + CB &= \left(t^2\right)(0) + (3t)(3) + (-3)(3t) \\
&= 0
\end{aligned}$$

Hence the ode in v given in (1) now simplifies to

$$3t^3v'' + \left(15t^2\right)v' = 0$$

Now by applying $v' = u$ the above becomes

$$3t^2(u'(t)\,t + 5u(t)) = 0$$

Which is now solved for u. In canonical form the ODE is

$$\begin{aligned}
u' &= F(t, u) \\
&= f(t)g(u) \\
&= -\frac{5u}{t}
\end{aligned}$$

Where $f(t) = -\frac{5}{t}$ and $g(u) = u$. Integrating both sides gives

$$\begin{aligned}
\frac{1}{u}\,du &= -\frac{5}{t}\,dt \\
\int \frac{1}{u}\,du &= \int -\frac{5}{t}\,dt \\
\ln(u) &= -5\ln(t) + c_1 \\
u &= \mathrm{e}^{-5\ln(t)+c_1} \\
&= \frac{c_1}{t^5}
\end{aligned}$$

The ode for v now becomes

$$
\begin{aligned}
v' &= u \\
&= \frac{c_1}{t^5}
\end{aligned}
$$

Which is now solved for v. Integrating both sides gives

$$
\begin{aligned}
v(t) &= \int \frac{c_1}{t^5}\, \mathrm{d}t \\
&= -\frac{c_1}{4t^4} + c_2
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
y(t) &= Bv \\
&= (3t)\left(-\frac{c_1}{4t^4} + c_2\right) \\
&= \frac{3c_2t^4 - \frac{3c_1}{4}}{t^3}
\end{aligned}
$$

Summary
The solution(s) found are the following

$$
y = \frac{3c_2t^4 - \frac{3c_1}{4}}{t^3} \tag{1}
$$

Verification of solutions

$$
y = \frac{3c_2t^4 - \frac{3c_1}{4}}{t^3}
$$

Verified OK.

8.27.5 Solving using Kovacic algorithm

Writing the ode as

$$
t^2y'' + 3ty' - 3y = 0 \tag{1}
$$

$$
Ay'' + By' + Cy = 0 \tag{2}
$$

Comparing (1) and (2) shows that

$$
\begin{aligned}
A &= t^2 \\
B &= 3t \\
C &= -3
\end{aligned} \tag{3}
$$

Applying the Liouville transformation on the dependent variable gives

$$z(t) = ye^{\int \frac{B}{2A} dt}$$

Then (2) becomes

$$z''(t) = rz(t) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{15}{4t^2} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= 15 \\ t &= 4t^2 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(t) = \left(\frac{15}{4t^2}\right) z(t) \tag{7}$$

Equation (7) is now solved. After finding $z(t)$ then y is found using the inverse transformation

$$y = z(t)\, e^{-\int \frac{B}{2A} dt}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0,1,2,4,6,8,\cdots\}$	$\{\cdots,-6,-4,-2,0,2,3,4,5,6,\cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1,2\}$	$\{2,3,4,5,6,7,\cdots\}$

Table 416: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned}O(\infty) &= \deg(t) - \deg(s)\\ &= 2 - 0\\ &= 2\end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 4t^2$. There is a pole at $t = 0$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 2 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Since pole order is not larger than 2 and the order at ∞ is 2 then the necessary conditions for case three are met. Therefore

$$L = [1, 2, 4, 6, 12]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = \frac{15}{4t^2}$$

For the pole at $t = 0$ let b be the coefficient of $\frac{1}{t^2}$ in the partial fractions decomposition

of r given above. Therefore $b = \frac{15}{4}$. Hence

$$
\begin{aligned}
[\sqrt{r}]_c &= 0 \\
\alpha_c^{+} &= \frac{1}{2} + \sqrt{1+4b} = \frac{5}{2} \\
\alpha_c^{-} &= \frac{1}{2} - \sqrt{1+4b} = -\frac{3}{2}
\end{aligned}
$$

Since the order of r at ∞ is 2 then $[\sqrt{r}]_\infty = 0$. Let b be the coefficient of $\frac{1}{t^2}$ in the Laurent series expansion of r at ∞. which can be found by dividing the leading coefficient of s by the leading coefficient of t from

$$
r = \frac{s}{t} = \frac{15}{4t^2}
$$

Since the $\gcd(s,t) = 1$. This gives $b = \frac{15}{4}$. Hence

$$
\begin{aligned}
[\sqrt{r}]_\infty &= 0 \\
\alpha_\infty^{+} &= \frac{1}{2} + \sqrt{1+4b} = \frac{5}{2} \\
\alpha_\infty^{-} &= \frac{1}{2} - \sqrt{1+4b} = -\frac{3}{2}
\end{aligned}
$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$
r = \frac{15}{4t^2}
$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^{+}	α_c^{-}
0	2	0	$\frac{5}{2}$	$-\frac{3}{2}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^{+}	α_∞^{-}
2	0	$\frac{5}{2}$	$-\frac{3}{2}$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^{\pm}$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^{\pm}$ have also been found, the next step is to determine possible non negative integer d from these using

$$
d = \alpha_\infty^{s(\infty)} - \sum_{c \in \Gamma} \alpha_c^{s(c)}
$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^{\pm}$. This is done by trial over all set of families $s = (s(c))_{c \in \Gamma \cup \infty}$ until such d is found to work in finding candidate ω.

Trying $\alpha_\infty^- = -\frac{3}{2}$ then

$$
\begin{aligned}
d &= \alpha_\infty^- - \left(\alpha_{c_1}^-\right)\\
&= -\frac{3}{2} - \left(-\frac{3}{2}\right)\\
&= 0
\end{aligned}
$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$
\omega = \sum_{c\in\Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{t-c} \right) + s(\infty)[\sqrt{r}]_\infty
$$

The above gives

$$
\begin{aligned}
\omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^-}{t-c_1} \right) + (-)[\sqrt{r}]_\infty\\
&= -\frac{3}{2t} + (-)\,(0)\\
&= -\frac{3}{2t}\\
&= -\frac{3}{2t}
\end{aligned}
$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(t)$ of degree $d = 0$ to solve the ode. The polynomial $p(t)$ needs to satisfy the equation

$$
p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right) p = 0 \tag{1A}
$$

Let

$$
p(t) = 1 \tag{2A}
$$

Substituting the above in eq. (1A) gives

$$
\begin{aligned}
(0) + 2\left(-\frac{3}{2t}\right)(0) + \left(\left(\frac{3}{2t^2}\right) + \left(-\frac{3}{2t}\right)^2 - \left(\frac{15}{4t^2}\right) \right) &= 0\\
0 &= 0
\end{aligned}
$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z'' = rz$ is

$$
\begin{aligned}
z_1(t) &= p e^{\int \omega \, dt}\\
&= e^{\int -\frac{3}{2t} dt}\\
&= \frac{1}{t^{\frac{3}{2}}}
\end{aligned}
$$

The first solution to the original ode in y is found from

$$
\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dt} \\
&= z_1 e^{-\int \frac{1}{2}\frac{3t}{t^2}\,dt} \\
&= z_1 e^{-\frac{3\ln(t)}{2}} \\
&= z_1 \left(\frac{1}{t^{\frac{3}{2}}}\right)
\end{aligned}
$$

Which simplifies to

$$
y_1 = \frac{1}{t^3}
$$

The second solution y_2 to the original ode is found using reduction of order

$$
y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dt}}{y_1^2}\,dt
$$

Substituting gives

$$
\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{3t}{t^2}\,dt}}{(y_1)^2}\,dt \\
&= y_1 \int \frac{e^{-3\ln(t)}}{(y_1)^2}\,dt \\
&= y_1 \left(\frac{t^4}{4}\right)
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1 \left(\frac{1}{t^3}\right) + c_2 \left(\frac{1}{t^3}\left(\frac{t^4}{4}\right)\right)
\end{aligned}
$$

<u>Summary</u>
The solution(s) found are the following

$$
y = \frac{c_1}{t^3} + \frac{c_2 t}{4} \tag{1}
$$

Verification of solutions

$$y = \frac{c_1}{t^3} + \frac{c_2 t}{4}$$

Verified OK.

8.27.6 Maple step by step solution

Let's solve

$y''t^2 + 3ty' - 3y = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y'' = -\frac{3y'}{t} + \frac{3y}{t^2}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y'' + \frac{3y'}{t} - \frac{3y}{t^2} = 0$

- Multiply by denominators of the ODE

  $y''t^2 + 3ty' - 3y = 0$

- Make a change of variables

  $s = \ln(t)$

□ Substitute the change of variables back into the ODE

  - Calculate the 1st derivative of y with respect to t , using the chain rule

    $y' = \left(\frac{d}{ds}y(s)\right) s'(t)$

  - Compute derivative

    $y' = \frac{\frac{d}{ds}y(s)}{t}$

  - Calculate the 2nd derivative of y with respect to t , using the chain rule

    $y'' = \left(\frac{d^2}{ds^2}y(s)\right) s'(t)^2 + s''(t) \left(\frac{d}{ds}y(s)\right)$

  - Compute derivative

    $y'' = \frac{\frac{d^2}{ds^2}y(s)}{t^2} - \frac{\frac{d}{ds}y(s)}{t^2}$

  Substitute the change of variables back into the ODE

  $\left(\frac{\frac{d^2}{ds^2}y(s)}{t^2} - \frac{\frac{d}{ds}y(s)}{t^2}\right) t^2 + 3\frac{d}{ds}y(s) - 3y(s) = 0$

- Simplify

  $\frac{d^2}{ds^2}y(s) + 2\frac{d}{ds}y(s) - 3y(s) = 0$
- Characteristic polynomial of ODE

  $r^2 + 2r - 3 = 0$
- Factor the characteristic polynomial

  $(r+3)(r-1) = 0$
- Roots of the characteristic polynomial

  $r = (-3, 1)$
- 1st solution of the ODE

  $y_1(s) = e^{-3s}$
- 2nd solution of the ODE

  $y_2(s) = e^{s}$
- General solution of the ODE

  $y(s) = c_1 y_1(s) + c_2 y_2(s)$
- Substitute in solutions

  $y(s) = c_1 e^{-3s} + c_2 e^{s}$
- Change variables back using $s = \ln(t)$

  $y = \frac{c_1}{t^3} + c_2 t$
- Simplify

  $y = \frac{c_1}{t^3} + c_2 t$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
<- LODE of Euler type successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 15

```
dsolve(t^2*diff(y(t),t$2)+ 3*t*diff(y(t),t)-3*y(t) = 0,y(t), singsol=all)
```

$$y(t) = \frac{c_1 t^4 + c_2}{t^3}$$

✓ Solution by Mathematica

Time used: 0.011 (sec). Leaf size: 16

```
DSolve[t^2*y''[t]+3*t*y'[t]-3*y[t]==0,y[t],t,IncludeSingularSolutions -> True]
```

$$y(t) \to \frac{c_1}{t^3} + c_2 t$$

8.28 problem 42

Internal problem ID [650]
Internal file name [`OUTPUT/650_Sunday_June_05_2022_01_46_25_AM_47152662/index.tex`]

Book: Elementary differential equations and boundary value problems, 10th ed., Boyce and DiPrima
Section: Chapter 3, Second order linear equations, 3.3 Complex Roots of the Characteristic Equation , page 164
Problem number: 42.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_euler_ode", "second_order_change_of_variable_on_x_method_1", "second_order_change_of_variable_on_x_method_2", "second_order_change_of_variable_on_y_method_2"**

Maple gives the following as the ode type

```
[[_Emden, _Fowler]]
```

$$\boxed{t^2y'' + 7ty' + 10y = 0}$$

8.28.1 Solving as second order euler ode ode

This is Euler second order ODE. Let the solution be $y = t^r$, then $y' = rt^{r-1}$ and $y'' = r(r-1)t^{r-2}$. Substituting these back into the given ODE gives

$$t^2(r(r-1))t^{r-2} + 7trt^{r-1} + 10t^r = 0$$

Simplifying gives

$$r(r-1)\,t^r + 7r\,t^r + 10t^r = 0$$

Since $t^r \neq 0$ then dividing throughout by t^r gives

$$r(r-1) + 7r + 10 = 0$$

Or

$$r^2 + 6r + 10 = 0 \tag{1}$$

Equation (1) is the characteristic equation. Its roots determine the form of the general solution. Using the quadratic equation the roots are

$$\begin{aligned} r_1 &= -3 - i \\ r_2 &= -3 + i \end{aligned}$$

The roots are complex conjugate of each others. Let the roots be

$$\begin{aligned} r_1 &= \alpha + i\beta \\ r_2 &= \alpha - i\beta \end{aligned}$$

Where in this case $\alpha = -3$ and $\beta = -1$. Hence the solution becomes

$$\begin{aligned} y &= c_1 t^{r_1} + c_2 t^{r_2} \\ &= c_1 t^{\alpha + i\beta} + c_2 t^{\alpha - i\beta} \\ &= t^{\alpha}\left(c_1 t^{i\beta} + c_2 t^{-i\beta}\right) \\ &= t^{\alpha}\left(c_1 e^{\ln\left(t^{i\beta}\right)} + c_2 e^{\ln\left(t^{-i\beta}\right)}\right) \\ &= t^{\alpha}\left(c_1 e^{i(\beta \ln t)} + c_2 e^{-i(\beta \ln t)}\right) \end{aligned}$$

Using the values for $\alpha = -3, \beta = -1$, the above becomes

$$y = t^{-3}\left(c_1 e^{-i\ln(t)} + c_2 e^{i\ln(t)}\right)$$

Using Euler relation, the expression $c_1 e^{iA} + c_2 e^{-iA}$ is transformed to $c_1 \cos A + c_1 \sin A$ where the constants are free to change. Applying this to the above result gives

$$y = \frac{1}{t^3}\left(c_1 \cos\left(\ln(t)\right) + c_2 \sin\left(\ln(t)\right)\right)$$

Summary

The solution(s) found are the following

$$y = \frac{c_1 \cos\left(\ln(t)\right) + c_2 \sin\left(\ln(t)\right)}{t^3} \tag{1}$$

Verification of solutions

$$y = \frac{c_1 \cos\left(\ln(t)\right) + c_2 \sin\left(\ln(t)\right)}{t^3}$$

Verified OK.

8.28.2 Solving as second order change of variable on x method 2 ode

In normal form the ode

$$t^2y'' + 7ty' + 10y = 0 \tag{1}$$

Becomes

$$y'' + p(t)\, y' + q(t)\, y = 0 \tag{2}$$

Where

$$\begin{aligned} p(t) &= \frac{7}{t} \\ q(t) &= \frac{10}{t^2} \end{aligned}$$

Applying change of variables $\tau = g(t)$ to (2) gives

$$\frac{d^2}{d\tau^2}y(\tau) + p_1\left(\frac{d}{d\tau}y(\tau)\right) + q_1 y(\tau) = 0 \tag{3}$$

Where τ is the new independent variable, and

$$p_1(\tau) = \frac{\tau''(t) + p(t)\,\tau'(t)}{\tau'(t)^2} \tag{4}$$

$$q_1(\tau) = \frac{q(t)}{\tau'(t)^2} \tag{5}$$

Let $p_1 = 0$. Eq (4) simplifies to

$$\tau''(t) + p(t)\,\tau'(t) = 0$$

This ode is solved resulting in

$$\begin{aligned} \tau &= \int \mathrm{e}^{-\left(\int p(t)dt\right)}dt \\ &= \int \mathrm{e}^{-\left(\int \frac{7}{t}dt\right)}dt \\ &= \int e^{-7\ln(t)}\, dt \\ &= \int \frac{1}{t^7}dt \\ &= -\frac{1}{6t^6} \end{aligned} \tag{6}$$

Using (6) to evaluate q_1 from (5) gives

$$
\begin{aligned}
q_1(\tau) &= \frac{q(t)}{\tau'(t)^2} \\
&= \frac{\frac{10}{t^2}}{\frac{1}{t^{14}}} \\
&= 10t^{12}
\end{aligned} \tag{7}
$$

Substituting the above in (3) and noting that now $p_1 = 0$ results in

$$
\begin{aligned}
\frac{d^2}{d\tau^2}y(\tau) + q_1 y(\tau) = 0 \\
\frac{d^2}{d\tau^2}y(\tau) + 10t^{12}y(\tau) = 0
\end{aligned}
$$

But in terms of τ

$$
10t^{12} = \frac{5}{18\tau^2}
$$

Hence the above ode becomes

$$
\frac{d^2}{d\tau^2}y(\tau) + \frac{5y(\tau)}{18\tau^2} = 0
$$

The above ode is now solved for $y(\tau)$. The ode can be written as

$$
18\left(\frac{d^2}{d\tau^2}y(\tau)\right)\tau^2 + 5y(\tau) = 0
$$

Which shows it is a Euler ODE. This is Euler second order ODE. Let the solution be $y(\tau) = \tau^r$, then $y' = r\tau^{r-1}$ and $y'' = r(r-1)\tau^{r-2}$. Substituting these back into the given ODE gives

$$
18\tau^2(r(r-1))\tau^{r-2} + 0r\tau^{r-1} + 5\tau^r = 0
$$

Simplifying gives

$$
18r(r-1)\,\tau^r + 0\,\tau^r + 5\tau^r = 0
$$

Since $\tau^r \neq 0$ then dividing throughout by τ^r gives

$$
18r(r-1) + 0 + 5 = 0
$$

Or

$$
18r^2 - 18r + 5 = 0 \tag{1}
$$

Equation (1) is the characteristic equation. Its roots determine the form of the general solution. Using the quadratic equation the roots are

$$
\begin{aligned}
r_1 &= \frac{1}{2}-\frac{i}{6} \\
r_2 &= \frac{1}{2}+\frac{i}{6}
\end{aligned}
$$

The roots are complex conjugate of each others. Let the roots be

$$
\begin{aligned}
r_1 &= \alpha + i\beta \\
r_2 &= \alpha - i\beta
\end{aligned}
$$

Where in this case $\alpha = \frac{1}{2}$ and $\beta = -\frac{1}{6}$. Hence the solution becomes

$$
\begin{aligned}
y(\tau) &= c_1 \tau^{r_1} + c_2 \tau^{r_2} \\
&= c_1 \tau^{\alpha + i\beta} + c_2 \tau^{\alpha - i\beta} \\
&= \tau^{\alpha}\left(c_1 \tau^{i\beta} + c_2 \tau^{-i\beta}\right) \\
&= \tau^{\alpha}\left(c_1 e^{\ln\left(\tau^{i\beta}\right)} + c_2 e^{\ln\left(\tau^{-i\beta}\right)}\right) \\
&= \tau^{\alpha}\left(c_1 e^{i(\beta \ln \tau)} + c_2 e^{-i(\beta \ln \tau)}\right)
\end{aligned}
$$

Using the values for $\alpha = \frac{1}{2}, \beta = -\frac{1}{6}$, the above becomes

$$
y(\tau) = \tau^{\frac{1}{2}}\left(c_1 e^{-\frac{i \ln(\tau)}{6}} + c_2 e^{\frac{i \ln(\tau)}{6}}\right)
$$

Using Euler relation, the expression $c_1 e^{iA} + c_2 e^{-iA}$ is transformed to $c_1 \cos A + c_1 \sin A$ where the constants are free to change. Applying this to the above result gives

$$
y(\tau) = \sqrt{\tau}\left(c_1 \cos\left(\frac{\ln(\tau)}{6}\right) + c_2 \sin\left(\frac{\ln(\tau)}{6}\right)\right)
$$

The above solution is now transformed back to y using (6) which results in

$$
y = \frac{\left(c_1 \cos\left(-\frac{\ln(2)}{6} - \frac{\ln(3)}{6} + \frac{\ln\left(-\frac{1}{t^6}\right)}{6}\right) + c_2 \sin\left(-\frac{\ln(2)}{6} - \frac{\ln(3)}{6} + \frac{\ln\left(-\frac{1}{t^6}\right)}{6}\right)\right)\sqrt{6}\,\sqrt{-\frac{1}{t^6}}}{6}
$$

Summary

The solution(s) found are the following

$$
y = \frac{\left(c_1 \cos\left(-\frac{\ln(2)}{6} - \frac{\ln(3)}{6} + \frac{\ln\left(-\frac{1}{t^6}\right)}{6}\right) + c_2 \sin\left(-\frac{\ln(2)}{6} - \frac{\ln(3)}{6} + \frac{\ln\left(-\frac{1}{t^6}\right)}{6}\right)\right)\sqrt{6}\,\sqrt{-\frac{1}{t^6}}}{6} \tag{1}
$$

Verification of solutions

$$y = \frac{\left(c_1 \cos\left(-\frac{\ln(2)}{6}-\frac{\ln(3)}{6}+\frac{\ln\left(-\frac{1}{t^6}\right)}{6}\right)+c_2 \sin\left(-\frac{\ln(2)}{6}-\frac{\ln(3)}{6}+\frac{\ln\left(-\frac{1}{t^6}\right)}{6}\right)\right)\sqrt{6}\,\sqrt{-\frac{1}{t^6}}}{6}$$

Verified OK.

8.28.3 Solving as second order change of variable on x method 1 ode

In normal form the ode

$$t^2y''+7ty'+10y=0 \tag{1}$$

Becomes

$$y''+p(t)\,y'+q(t)\,y=0 \tag{2}$$

Where

$$\begin{aligned} p(t) &= \frac{7}{t} \\ q(t) &= \frac{10}{t^2} \end{aligned}$$

Applying change of variables $\tau = g(t)$ to (2) results

$$\frac{d^2}{d\tau^2}y(\tau)+p_1\left(\frac{d}{d\tau}y(\tau)\right)+q_1y(\tau)=0 \tag{3}$$

Where τ is the new independent variable, and

$$p_1(\tau) = \frac{\tau''(t)+p(t)\,\tau'(t)}{\tau'(t)^2} \tag{4}$$

$$q_1(\tau) = \frac{q(t)}{\tau'(t)^2} \tag{5}$$

Let $q_1 = c^2$ where c is some constant. Therefore from (5)

$$\begin{aligned} \tau' &= \frac{1}{c}\sqrt{q} \\ &= \frac{\sqrt{10}\,\sqrt{\frac{1}{t^2}}}{c} \end{aligned} \tag{6}$$

$$\tau'' = -\frac{\sqrt{10}}{c\sqrt{\frac{1}{t^2}}\,t^3}$$

Substituting the above into (4) results in

$$
\begin{aligned}
p_1(\tau) &= \frac{\tau''(t) + p(t)\,\tau'(t)}{\tau'(t)^2} \\
&= \frac{-\frac{\sqrt{10}}{c\sqrt{\frac{1}{t^2}}\,t^3} + \frac{7}{t}\frac{\sqrt{10}\,\sqrt{\frac{1}{t^2}}}{c}}{\left(\frac{\sqrt{10}\,\sqrt{\frac{1}{t^2}}}{c}\right)^2} \\
&= \frac{3c\sqrt{10}}{5}
\end{aligned}
$$

Therefore ode (3) now becomes

$$
\begin{gathered}
y(\tau)'' + p_1 y(\tau)' + q_1 y(\tau) = 0 \\
\frac{d^2}{d\tau^2}y(\tau) + \frac{3c\sqrt{10}\left(\frac{d}{d\tau}y(\tau)\right)}{5} + c^2 y(\tau) = 0 \qquad (7)
\end{gathered}
$$

The above ode is now solved for $y(\tau)$. Since the ode is now constant coefficients, it can be easily solved to give

$$
y(\tau) = \mathrm{e}^{-\frac{3\sqrt{10}\,c\tau}{10}}\left(c_1\cos\left(\frac{\sqrt{10}\,c\tau}{10}\right) + c_2\sin\left(\frac{\sqrt{10}\,c\tau}{10}\right)\right)
$$

Now from (6)

$$
\begin{aligned}
\tau &= \int \frac{1}{c}\sqrt{q}\,dt \\
&= \frac{\int \sqrt{10}\,\sqrt{\frac{1}{t^2}}dt}{c} \\
&= \frac{\sqrt{10}\,\sqrt{\frac{1}{t^2}}\,t\ln(t)}{c}
\end{aligned}
$$

Substituting the above into the solution obtained gives

$$
y = \frac{c_1\cos(\ln(t)) + c_2\sin(\ln(t))}{t^3}
$$

Summary

The solution(s) found are the following

$$
y = \frac{c_1\cos(\ln(t)) + c_2\sin(\ln(t))}{t^3} \qquad (1)
$$

Verification of solutions

$$y = \frac{c_1 \cos(\ln(t)) + c_2 \sin(\ln(t))}{t^3}$$

Verified OK.

8.28.4 Solving as second order change of variable on y method 2 ode

In normal form the ode

$$t^2 y'' + 7ty' + 10y = 0 \tag{1}$$

Becomes

$$y'' + p(t)\, y' + q(t)\, y = 0 \tag{2}$$

Where

$$p(t) = \frac{7}{t}$$
$$q(t) = \frac{10}{t^2}$$

Applying change of variables on the depndent variable $y = v(t)\, t^n$ to (2) gives the following ode where the dependent variables is $v(t)$ and not y.

$$v''(t) + \left(\frac{2n}{t} + p\right) v'(t) + \left(\frac{n(n-1)}{t^2} + \frac{np}{t} + q\right) v(t) = 0 \tag{3}$$

Let the coefficient of $v(t)$ above be zero. Hence

$$\frac{n(n-1)}{t^2} + \frac{np}{t} + q = 0 \tag{4}$$

Substituting the earlier values found for $p(t)$ and $q(t)$ into (4) gives

$$\frac{n(n-1)}{t^2} + \frac{7n}{t^2} + \frac{10}{t^2} = 0 \tag{5}$$

Solving (5) for n gives

$$n = -3 + i \tag{6}$$

Substituting this value in (3) gives

$$v''(t) + \left(\frac{-6+2i}{t} + \frac{7}{t}\right) v'(t) = 0$$
$$v''(t) + \frac{(1+2i)\, v'(t)}{t} = 0 \tag{7}$$

Using the substitution

$$u(t) = v'(t)$$

Then (7) becomes

$$u'(t) + \frac{(1 + 2i)\, u(t)}{t} = 0 \tag{8}$$

The above is now solved for $u(t)$. In canonical form the ODE is

$$\begin{aligned} u' &= F(t, u) \\ &= f(t)g(u) \\ &= \frac{(-1 - 2i)\, u}{t} \end{aligned}$$

Where $f(t) = \frac{-1-2i}{t}$ and $g(u) = u$. Integrating both sides gives

$$\begin{aligned} \frac{1}{u}\, du &= \frac{-1 - 2i}{t}\, dt \\ \int \frac{1}{u}\, du &= \int \frac{-1 - 2i}{t}\, dt \\ \ln(u) &= (-1 - 2i) \ln(t) + c_1 \\ u &= e^{(-1-2i)\ln(t)+c_1} \\ &= c_1 e^{(-1-2i)\ln(t)} \end{aligned}$$

Which simplifies to

$$u(t) = \frac{c_1 t^{-2i}}{t}$$

Now that $u(t)$ is known, then

$$\begin{aligned} v'(t) &= u(t) \\ v(t) &= \int u(t)\, dt + c_2 \\ &= \frac{i c_1 t^{-2i}}{2} + c_2 \end{aligned}$$

Hence

$$\begin{aligned} y &= v(t)\, t^n \\ &= \left(\frac{i c_1 t^{-2i}}{2} + c_2\right) t^{-3+i} \\ &= c_2 t^{-3+i} + \frac{i c_1 t^{-3-i}}{2} \end{aligned}$$

Summary
The solution(s) found are the following

$$y = \left(\frac{i c_1 t^{-2i}}{2} + c_2\right) t^{-3+i} \tag{1}$$

Verification of solutions

$$y = \left(\frac{i c_1 t^{-2i}}{2} + c_2\right) t^{-3+i}$$

Verified OK.

8.28.5 Solving using Kovacic algorithm

Writing the ode as

$$t^2 y'' + 7ty' + 10y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= t^2 \\ B &= 7t \\ C &= 10 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(t) = y e^{\int \frac{B}{2A} \, dt}$$

Then (2) becomes

$$z''(t) = r z(t) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{-5}{4t^2} \tag{6}$$

Comparing the above to (5) shows that

$$s = -5$$
$$t = 4t^2$$

Therefore eq. (4) becomes

$$z''(t) = \left(-\frac{5}{4t^2}\right) z(t) \tag{7}$$

Equation (7) is now solved. After finding $z(t)$ then y is found using the inverse transformation

$$y = z(t)\, e^{-\int \frac{B}{2A}\, dt}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 418: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 2 - 0 \\ &= 2 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 4t^2$. There is a pole at $t = 0$ of order 2. Since there is no odd order pole

larger than 2 and the order at ∞ is 2 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Since pole order is not larger than 2 and the order at ∞ is 2 then the necessary conditions for case three are met. Therefore

$$L = [1, 2, 4, 6, 12]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = -\frac{5}{4t^2}$$

For the pole at $t = 0$ let b be the coefficient of $\frac{1}{t^2}$ in the partial fractions decomposition of r given above. Therefore $b = -\frac{5}{4}$. Hence

$$\begin{aligned}
[\sqrt{r}]_c &= 0 \\
\alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{1}{2} + i \\
\alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = \frac{1}{2} - i
\end{aligned}$$

Since the order of r at ∞ is 2 then $[\sqrt{r}]_\infty = 0$. Let b be the coefficient of $\frac{1}{t^2}$ in the Laurent series expansion of r at ∞. which can be found by dividing the leading coefficient of s by the leading coefficient of t from

$$r = \frac{s}{t} = -\frac{5}{4t^2}$$

Since the $\gcd(s, t) = 1$. This gives $b = -\frac{5}{4}$. Hence

$$\begin{aligned}
[\sqrt{r}]_\infty &= 0 \\
\alpha_\infty^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{1}{2} + i \\
\alpha_\infty^- &= \frac{1}{2} - \sqrt{1+4b} = \frac{1}{2} - i
\end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = -\frac{5}{4t^2}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
0	2	0	$\frac{1}{2} + i$	$\frac{1}{2} - i$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
2	0	$\frac{1}{2}+i$	$\frac{1}{2}-i$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^- = \frac{1}{2} - i$ then

$$\begin{aligned} d &= \alpha_\infty^- - \left(\alpha_{c_1}^-\right) \\ &= \frac{1}{2} - i - \left(\frac{1}{2} - i\right) \\ &= 0 \end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c\in\Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{t-c} \right) + s(\infty)[\sqrt{r}]_\infty$$

The above gives

$$\begin{aligned} \omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^-}{t-c_1} \right) + (-)[\sqrt{r}]_\infty \\ &= \frac{\frac{1}{2}-i}{t} + (-)(0) \\ &= \frac{\frac{1}{2}-i}{t} \\ &= \frac{\frac{1}{2}-i}{t} \end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(t)$ of degree $d = 0$ to solve the ode. The polynomial $p(t)$ needs to satisfy the equation

$$p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right) p = 0 \tag{1A}$$

Let

$$p(t) = 1 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$(0) + 2\left(\frac{\frac{1}{2} - i}{t}\right)(0) + \left(\left(\frac{-\frac{1}{2} + i}{t^2}\right) + \left(\frac{\frac{1}{2} - i}{t}\right)^2 - \left(-\frac{5}{4t^2}\right)\right) = 0$$
$$0 = 0$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned} z_1(t) &= p e^{\int \omega \, dt} \\ &= \mathrm{e}^{\int \frac{\frac{1}{2} - i}{t} dt} \\ &= t^{\frac{1}{2} - i} \end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned} y_1 &= z_1 e^{\int -\frac{1}{2} \frac{B}{A} \, dt} \\ &= z_1 e^{-\int \frac{1}{2} \frac{7t}{t^2} \, dt} \\ &= z_1 e^{-\frac{7 \ln(t)}{2}} \\ &= z_1 \left(\frac{1}{t^{\frac{7}{2}}}\right) \end{aligned}$$

Which simplifies to

$$y_1 = t^{-3-i}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A} \, dt}}{y_1^2} \, dt$$

Substituting gives

$$\begin{aligned} y_2 &= y_1 \int \frac{e^{\int -\frac{7t}{t^2} \, dt}}{(y_1)^2} \, dt \\ &= y_1 \int \frac{e^{-7 \ln(t)}}{(y_1)^2} \, dt \\ &= y_1 \left(-\frac{i t^{2i}}{2}\right) \end{aligned}$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1 \left(t^{-3-i}\right) + c_2 \left(t^{-3-i} \left(-\frac{i t^{2i}}{2}\right)\right)
\end{aligned}
$$

Summary
The solution(s) found are the following

$$
y = c_1 t^{-3-i} - \frac{i c_2 t^{-3+i}}{2} \tag{1}
$$

Verification of solutions

$$
y = c_1 t^{-3-i} - \frac{i c_2 t^{-3+i}}{2}
$$

Verified OK.

8.28.6 Maple step by step solution

Let's solve

$y'' t^2 + 7 t y' + 10 y = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y'' = -\frac{7 y'}{t} - \frac{10 y}{t^2}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y'' + \frac{7 y'}{t} + \frac{10 y}{t^2} = 0$

- Multiply by denominators of the ODE

  $y'' t^2 + 7 t y' + 10 y = 0$

- Make a change of variables

  $s = \ln(t)$

□ Substitute the change of variables back into the ODE

  ◦ Calculate the 1st derivative of y with respect to t , using the chain rule

  $y' = \left(\frac{d}{ds} y(s)\right) s'(t)$

- Compute derivative

  $y' = \frac{\frac{d}{ds}y(s)}{t}$

- Calculate the 2nd derivative of y with respect to t , using the chain rule

  $y'' = \left(\frac{d^2}{ds^2}y(s)\right) s'(t)^2 + s''(t) \left(\frac{d}{ds}y(s)\right)$

- Compute derivative

  $y'' = \frac{\frac{d^2}{ds^2}y(s)}{t^2} - \frac{\frac{d}{ds}y(s)}{t^2}$

  Substitute the change of variables back into the ODE

  $\left(\frac{\frac{d^2}{ds^2}y(s)}{t^2} - \frac{\frac{d}{ds}y(s)}{t^2}\right) t^2 + 7\frac{d}{ds}y(s) + 10y(s) = 0$

- Simplify

  $\frac{d^2}{ds^2}y(s) + 6\frac{d}{ds}y(s) + 10y(s) = 0$

- Characteristic polynomial of ODE

  $r^2 + 6r + 10 = 0$

- Use quadratic formula to solve for r

  $r = \frac{(-6)\pm(\sqrt{-4})}{2}$

- Roots of the characteristic polynomial

  $r = (-3 - \mathrm{I}, -3 + \mathrm{I})$

- 1st solution of the ODE

  $y_1(s) = \mathrm{e}^{-3s} \cos(s)$

- 2nd solution of the ODE

  $y_2(s) = \mathrm{e}^{-3s} \sin(s)$

- General solution of the ODE

  $y(s) = c_1 y_1(s) + c_2 y_2(s)$

- Substitute in solutions

  $y(s) = c_1 \mathrm{e}^{-3s} \cos(s) + c_2 \mathrm{e}^{-3s} \sin(s)$

- Change variables back using $s = \ln(t)$

  $y = \frac{c_1 \cos(\ln(t))}{t^3} + \frac{c_2 \sin(\ln(t))}{t^3}$

- Simplify

  $y = \frac{c_1 \cos(\ln(t))}{t^3} + \frac{c_2 \sin(\ln(t))}{t^3}$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
<- LODE of Euler type successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 19

```
dsolve(t^2*diff(y(t),t$2)+ 7*t*diff(y(t),t)+10*y(t) = 0,y(t), singsol=all)
```

$$y(t) = \frac{c_1 \sin(\ln(t)) + c_2 \cos(\ln(t))}{t^3}$$

✓ Solution by Mathematica

Time used: 0.022 (sec). Leaf size: 22

```
DSolve[t^2*y''[t]+7*t*y'[t]+10*y[t]==0,y[t],t,IncludeSingularSolutions -> True]
```

$$y(t) \to \frac{c_2 \cos(\log(t)) + c_1 \sin(\log(t))}{t^3}$$

8.29 problem 44

Internal problem ID [651]
Internal file name [OUTPUT/651_Sunday_June_05_2022_01_46_25_AM_16115400/index.tex]

Book: Elementary differential equations and boundary value problems, 10th ed., Boyce and DiPrima
Section: Chapter 3, Second order linear equations, 3.3 Complex Roots of the Characteristic Equation , page 164
Problem number: 44.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second_order_change_of_variable_on_x_method_1", "second_order_change_of_variable_on_x_method_2"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries], [_2nd_order, _linear, `
   _with_symmetry_[0,F(x)]`]]
```

$$\boxed{y''+ty'+e^{-t^2}y=0}$$

8.29.1 Solving as second order change of variable on x method 2 ode

In normal form the ode

$$y''+ty'+e^{-t^2}y=0 \tag{1}$$

Becomes

$$y''+p(t)\,y'+q(t)\,y=0 \tag{2}$$

Where

$$p(t)=t$$
$$q(t)=e^{-t^2}$$

Applying change of variables $\tau = g(t)$ to (2) gives

$$\frac{d^2}{d\tau^2}y(\tau) + p_1\left(\frac{d}{d\tau}y(\tau)\right) + q_1 y(\tau) = 0 \tag{3}$$

Where τ is the new independent variable, and

$$p_1(\tau) = \frac{\tau''(t) + p(t)\,\tau'(t)}{\tau'(t)^2} \tag{4}$$

$$q_1(\tau) = \frac{q(t)}{\tau'(t)^2} \tag{5}$$

Let $p_1 = 0$. Eq (4) simplifies to

$$\tau''(t) + p(t)\,\tau'(t) = 0$$

This ode is solved resulting in

$$\begin{aligned}
\tau &= \int e^{-\left(\int p(t)dt\right)}dt \\
&= \int e^{-\left(\int t dt\right)}dt \\
&= \int e^{-\frac{t^2}{2}}dt \\
&= \int e^{-\frac{t^2}{2}}dt \\
&= \frac{\sqrt{\pi}\,\sqrt{2}\,\operatorname{erf}\left(\frac{\sqrt{2}\,t}{2}\right)}{2}
\end{aligned} \tag{6}$$

Using (6) to evaluate q_1 from (5) gives

$$\begin{aligned}
q_1(\tau) &= \frac{q(t)}{\tau'(t)^2} \\
&= \frac{e^{-t^2}}{e^{-t^2}} \\
&= 1
\end{aligned} \tag{7}$$

Substituting the above in (3) and noting that now $p_1 = 0$ results in

$$\frac{d^2}{d\tau^2}y(\tau) + q_1 y(\tau) = 0$$

$$\frac{d^2}{d\tau^2}y(\tau) + y(\tau) = 0$$

The above ode is now solved for $y(\tau)$.This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$Ay''(\tau) + By'(\tau) + Cy(\tau) = 0$$

Where in the above $A = 1, B = 0, C = 1$. Let the solution be $y(\tau) = e^{\lambda\tau}$. Substituting this into the ODE gives

$$\lambda^2 \mathrm{e}^{\lambda\tau} + \mathrm{e}^{\lambda\tau} = 0 \tag{1}$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda\tau}$ gives

$$\lambda^2 + 1 = 0 \tag{2}$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form.Using the quadratic formula

$$\lambda_{1,2} = \frac{-B}{2A} \pm \frac{1}{2A}\sqrt{B^2 - 4AC}$$

Substituting $A = 1, B = 0, C = 1$ into the above gives

$$\begin{aligned}\lambda_{1,2} &= \frac{0}{(2)\,(1)} \pm \frac{1}{(2)\,(1)}\sqrt{0^2 - (4)\,(1)\,(1)} \\ &= \pm i\end{aligned}$$

Hence

$$\begin{aligned}\lambda_1 &= +i \\ \lambda_2 &= -i\end{aligned}$$

Which simplifies to

$$\begin{aligned}\lambda_1 &= i \\ \lambda_2 &= -i\end{aligned}$$

Since roots are complex conjugate of each others, then let the roots be

$$\lambda_{1,2} = \alpha \pm i\beta$$

Where $\alpha = 0$ and $\beta = 1$. Therefore the final solution, when using Euler relation, can be written as

$$y(\tau) = e^{\alpha\tau}(c_1 \cos(\beta\tau) + c_2 \sin(\beta\tau))$$

Which becomes

$$y(\tau) = e^0(c_1 \cos(\tau) + c_2 \sin(\tau))$$

Or

$$y(\tau) = c_1 \cos(\tau) + c_2 \sin(\tau)$$

The above solution is now transformed back to y using (6) which results in

$$y = c_1 \cos\left(\frac{\sqrt{\pi}\,\sqrt{2}\,\operatorname{erf}\left(\frac{\sqrt{2}\,t}{2}\right)}{2}\right) + c_2 \sin\left(\frac{\sqrt{\pi}\,\sqrt{2}\,\operatorname{erf}\left(\frac{\sqrt{2}\,t}{2}\right)}{2}\right)$$

Summary

The solution(s) found are the following

$$y = c_1 \cos\left(\frac{\sqrt{\pi}\,\sqrt{2}\,\operatorname{erf}\left(\frac{\sqrt{2}\,t}{2}\right)}{2}\right) + c_2 \sin\left(\frac{\sqrt{\pi}\,\sqrt{2}\,\operatorname{erf}\left(\frac{\sqrt{2}\,t}{2}\right)}{2}\right) \tag{1}$$

Verification of solutions

$$y = c_1 \cos\left(\frac{\sqrt{\pi}\,\sqrt{2}\,\operatorname{erf}\left(\frac{\sqrt{2}\,t}{2}\right)}{2}\right) + c_2 \sin\left(\frac{\sqrt{\pi}\,\sqrt{2}\,\operatorname{erf}\left(\frac{\sqrt{2}\,t}{2}\right)}{2}\right)$$

Verified OK.

8.29.2 Solving as second order change of variable on x method 1 ode

In normal form the ode

$$y'' + ty' + e^{-t^2} y = 0 \tag{1}$$

Becomes

$$y'' + p(t)\, y' + q(t)\, y = 0 \tag{2}$$

Where

$$p(t) = t$$
$$q(t) = e^{-t^2}$$

Applying change of variables $\tau = g(t)$ to (2) results

$$\frac{d^2}{d\tau^2} y(\tau) + p_1 \left(\frac{d}{d\tau} y(\tau)\right) + q_1 y(\tau) = 0 \tag{3}$$

Where τ is the new independent variable, and

$$p_1(\tau) = \frac{\tau''(t) + p(t)\,\tau'(t)}{\tau'(t)^2} \tag{4}$$

$$q_1(\tau) = \frac{q(t)}{\tau'(t)^2} \tag{5}$$

Let $q_1 = c^2$ where c is some constant. Therefore from (5)

$$\begin{aligned} \tau' &= \frac{1}{c}\sqrt{q} \\ &= \frac{\sqrt{e^{-t^2}}}{c} \\ \tau'' &= -\frac{\sqrt{e^{-t^2}}\,t}{c} \end{aligned} \tag{6}$$

Substituting the above into (4) results in

$$\begin{aligned} p_1(\tau) &= \frac{\tau''(t) + p(t)\,\tau'(t)}{\tau'(t)^2} \\ &= \frac{-\frac{\sqrt{e^{-t^2}}\,t}{c} + t\frac{\sqrt{e^{-t^2}}}{c}}{\left(\frac{\sqrt{e^{-t^2}}}{c}\right)^2} \\ &= 0 \end{aligned}$$

Therefore ode (3) now becomes

$$\begin{aligned} y(\tau)'' + p_1 y(\tau)' + q_1 y(\tau) &= 0 \\ \frac{d^2}{d\tau^2}y(\tau) + c^2 y(\tau) &= 0 \end{aligned} \tag{7}$$

The above ode is now solved for $y(\tau)$. Since the ode is now constant coefficients, it can be easily solved to give

$$y(\tau) = c_1 \cos(c\tau) + c_2 \sin(c\tau)$$

Now from (6)

$$\begin{aligned} \tau &= \int \frac{1}{c}\sqrt{q}\,dt \\ &= \frac{\int \sqrt{e^{-t^2}}dt}{c} \\ &= \frac{\sqrt{e^{-t^2}}\,e^{\frac{t^2}{2}}\sqrt{2}\,\sqrt{\pi}\,\operatorname{erf}\left(\frac{\sqrt{2}\,t}{2}\right)}{2c} \end{aligned}$$

Substituting the above into the solution obtained gives

$$y = c_1 \cos\left(\frac{\sqrt{\pi}\,\sqrt{2}\,\mathrm{erf}\left(\frac{\sqrt{2}\,t}{2}\right)}{2}\right) + c_2 \sin\left(\frac{\sqrt{\pi}\,\sqrt{2}\,\mathrm{erf}\left(\frac{\sqrt{2}\,t}{2}\right)}{2}\right)$$

Summary

The solution(s) found are the following

$$y = c_1 \cos\left(\frac{\sqrt{\pi}\,\sqrt{2}\,\mathrm{erf}\left(\frac{\sqrt{2}\,t}{2}\right)}{2}\right) + c_2 \sin\left(\frac{\sqrt{\pi}\,\sqrt{2}\,\mathrm{erf}\left(\frac{\sqrt{2}\,t}{2}\right)}{2}\right) \tag{1}$$

Verification of solutions

$$y = c_1 \cos\left(\frac{\sqrt{\pi}\,\sqrt{2}\,\mathrm{erf}\left(\frac{\sqrt{2}\,t}{2}\right)}{2}\right) + c_2 \sin\left(\frac{\sqrt{\pi}\,\sqrt{2}\,\mathrm{erf}\left(\frac{\sqrt{2}\,t}{2}\right)}{2}\right)$$

Verified OK.

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a symmetry of the form [xi=0, eta=F(x)]
<- linear_1 successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 55

```
dsolve(diff(y(t),t$2)+ t*diff(y(t),t)+exp(-t^2)*y(t) = 0,y(t), singsol=all)
```

$$y(t) = c_1 \,\mathrm{csgn}\left(\mathrm{e}^{\frac{t^2}{2}}\right) \sin\left(\frac{\sqrt{2}\,\sqrt{\pi}\,\mathrm{erf}\left(\frac{t\sqrt{2}}{2}\right)}{2}\right) + c_2 \cos\left(\frac{\sqrt{2}\,\mathrm{csgn}\left(\mathrm{e}^{\frac{t^2}{2}}\right)\sqrt{\pi}\,\mathrm{erf}\left(\frac{t\sqrt{2}}{2}\right)}{2}\right)$$

✓ Solution by Mathematica

Time used: 0.059 (sec). Leaf size: 102

```
DSolve[y''[t]+t*y'[t]+exp(-t^2)*y[t]==0,y[t],t,IncludeSingularSolutions -> True]
```

$$y(t) \to e^{-\frac{1}{4}(\sqrt{4\exp+1}+1)t^2}\left(c_1 \operatorname{HermiteH}\left(-\frac{1}{2}-\frac{1}{2\sqrt{4\exp+1}}, \frac{\sqrt[4]{4\exp+1}t}{\sqrt{2}}\right) + c_2 \operatorname{Hypergeometric1F1}\left(\frac{1}{4}\left(1+\frac{1}{\sqrt{4\exp+1}}\right), \frac{1}{2}, \frac{1}{2}\sqrt{4\exp+1}t^2\right)\right)$$

8.30 problem 46

Internal problem ID [652]
Internal file name [`OUTPUT/652_Sunday_June_05_2022_01_46_26_AM_50356826/index.tex`]

Book: Elementary differential equations and boundary value problems, 10th ed., Boyce and DiPrima
Section: Chapter 3, Second order linear equations, 3.3 Complex Roots of the Characteristic Equation , page 164
Problem number: 46.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_change_of_variable_on_x_method_1", "second_order_change_of_variable_on_x_method_2"**

Maple gives the following as the ode type

`[[_2nd_order, _with_linear_symmetries]]`

$$\boxed{ty'' + (t^2 - 1) y' + yt^3 = 0}$$

8.30.1 Solving as second order change of variable on x method 2 ode

In normal form the ode

$$ty'' + (t^2 - 1) y' + yt^3 = 0 \tag{1}$$

Becomes

$$y'' + p(t) y' + q(t) y = 0 \tag{2}$$

Where

$$p(t) = \frac{t^2 - 1}{t}$$
$$q(t) = t^2$$

Applying change of variables $\tau = g(t)$ to (2) gives

$$\frac{d^2}{d\tau^2}y(\tau) + p_1\left(\frac{d}{d\tau}y(\tau)\right) + q_1 y(\tau) = 0 \tag{3}$$

Where τ is the new independent variable, and

$$p_1(\tau) = \frac{\tau''(t) + p(t)\,\tau'(t)}{\tau'(t)^2} \tag{4}$$

$$q_1(\tau) = \frac{q(t)}{\tau'(t)^2} \tag{5}$$

Let $p_1 = 0$. Eq (4) simplifies to

$$\tau''(t) + p(t)\,\tau'(t) = 0$$

This ode is solved resulting in

$$\begin{aligned}\tau &= \int e^{-\left(\int p(t)dt\right)}dt \\ &= \int e^{-\left(\int \frac{t^2-1}{t}dt\right)}dt \\ &= \int e^{-\frac{t^2}{2}+\ln(t)}\,dt \\ &= \int t\,e^{-\frac{t^2}{2}}dt \\ &= -e^{-\frac{t^2}{2}}\end{aligned} \tag{6}$$

Using (6) to evaluate q_1 from (5) gives

$$\begin{aligned}q_1(\tau) &= \frac{q(t)}{\tau'(t)^2} \\ &= \frac{t^2}{t^2 e^{-t^2}} \\ &= e^{t^2}\end{aligned} \tag{7}$$

Substituting the above in (3) and noting that now $p_1 = 0$ results in

$$\frac{d^2}{d\tau^2}y(\tau) + q_1 y(\tau) = 0$$
$$\frac{d^2}{d\tau^2}y(\tau) + e^{t^2}y(\tau) = 0$$

But in terms of τ

$$e^{t^2} = \frac{1}{\tau^2}$$

Hence the above ode becomes

$$\frac{d^2}{d\tau^2}y(\tau) + \frac{y(\tau)}{\tau^2} = 0$$

The above ode is now solved for $y(\tau)$. The ode can be written as

$$\left(\frac{d^2}{d\tau^2}y(\tau)\right)\tau^2 + y(\tau) = 0$$

Which shows it is a Euler ODE. This is Euler second order ODE. Let the solution be $y(\tau) = \tau^r$, then $y' = r\tau^{r-1}$ and $y'' = r(r-1)\tau^{r-2}$. Substituting these back into the given ODE gives

$$\tau^2(r(r-1))\tau^{r-2} + 0r\tau^{r-1} + \tau^r = 0$$

Simplifying gives

$$r(r-1)\,\tau^r + 0\,\tau^r + \tau^r = 0$$

Since $\tau^r \neq 0$ then dividing throughout by τ^r gives

$$r(r-1) + 0 + 1 = 0$$

Or

$$r^2 - r + 1 = 0 \tag{1}$$

Equation (1) is the characteristic equation. Its roots determine the form of the general solution. Using the quadratic equation the roots are

$$r_1 = \frac{1}{2} - \frac{i\sqrt{3}}{2}$$
$$r_2 = \frac{1}{2} + \frac{i\sqrt{3}}{2}$$

The roots are complex conjugate of each others. Let the roots be

$$r_1 = \alpha + i\beta$$
$$r_2 = \alpha - i\beta$$

Where in this case $\alpha = \frac{1}{2}$ and $\beta = -\frac{\sqrt{3}}{2}$. Hence the solution becomes

$$
\begin{aligned}
y(\tau) &= c_1 \tau^{r_1} + c_2 \tau^{r_2} \\
&= c_1 \tau^{\alpha + i\beta} + c_2 \tau^{\alpha - i\beta} \\
&= \tau^{\alpha} \left(c_1 \tau^{i\beta} + c_2 \tau^{-i\beta} \right) \\
&= \tau^{\alpha} \left(c_1 e^{\ln(\tau^{i\beta})} + c_2 e^{\ln(\tau^{-i\beta})} \right) \\
&= \tau^{\alpha} \left(c_1 e^{i(\beta \ln \tau)} + c_2 e^{-i(\beta \ln \tau)} \right)
\end{aligned}
$$

Using the values for $\alpha = \frac{1}{2}, \beta = -\frac{\sqrt{3}}{2}$, the above becomes

$$
y(\tau) = \tau^{\frac{1}{2}} \left(c_1 e^{-\frac{i\sqrt{3}\, \ln(\tau)}{2}} + c_2 e^{\frac{i\sqrt{3}\, \ln(\tau)}{2}} \right)
$$

Using Euler relation, the expression $c_1 e^{iA} + c_2 e^{-iA}$ is transformed to $c_1 \cos A + c_1 \sin A$ where the constants are free to change. Applying this to the above result gives

$$
y(\tau) = \sqrt{\tau} \left(c_1 \cos \left(\frac{\sqrt{3}\, \ln (\tau)}{2} \right) + c_2 \sin \left(\frac{\sqrt{3}\, \ln (\tau)}{2} \right) \right)
$$

The above solution is now transformed back to y using (6) which results in

$$
y = \sqrt{-\mathrm{e}^{-\frac{t^2}{2}}} \left(c_1 \cos \left(\frac{\sqrt{3}\, \ln \left(-\mathrm{e}^{-\frac{t^2}{2}} \right)}{2} \right) + c_2 \sin \left(\frac{\sqrt{3}\, \ln \left(-\mathrm{e}^{-\frac{t^2}{2}} \right)}{2} \right) \right)
$$

<u>Summary</u>

The solution(s) found are the following

$$
y = \sqrt{-\mathrm{e}^{-\frac{t^2}{2}}} \left(c_1 \cos \left(\frac{\sqrt{3}\, \ln \left(-\mathrm{e}^{-\frac{t^2}{2}} \right)}{2} \right) + c_2 \sin \left(\frac{\sqrt{3}\, \ln \left(-\mathrm{e}^{-\frac{t^2}{2}} \right)}{2} \right) \right) \tag{1}
$$

<u>Verification of solutions</u>

$$
y = \sqrt{-\mathrm{e}^{-\frac{t^2}{2}}} \left(c_1 \cos \left(\frac{\sqrt{3}\, \ln \left(-\mathrm{e}^{-\frac{t^2}{2}} \right)}{2} \right) + c_2 \sin \left(\frac{\sqrt{3}\, \ln \left(-\mathrm{e}^{-\frac{t^2}{2}} \right)}{2} \right) \right)
$$

Verified OK.

8.30.2 Solving as second order change of variable on x method 1 ode

In normal form the ode

$$ty'' + (t^2 - 1)\, y' + y t^3 = 0 \tag{1}$$

Becomes

$$y'' + p(t)\, y' + q(t)\, y = 0 \tag{2}$$

Where

$$\begin{aligned} p(t) &= \frac{t^2 - 1}{t} \\ q(t) &= t^2 \end{aligned}$$

Applying change of variables $\tau = g(t)$ to (2) results

$$\frac{d^2}{d\tau^2} y(\tau) + p_1 \left(\frac{d}{d\tau} y(\tau)\right) + q_1 y(\tau) = 0 \tag{3}$$

Where τ is the new independent variable, and

$$p_1(\tau) = \frac{\tau''(t) + p(t)\, \tau'(t)}{\tau'(t)^2} \tag{4}$$

$$q_1(\tau) = \frac{q(t)}{\tau'(t)^2} \tag{5}$$

Let $q_1 = c^2$ where c is some constant. Therefore from (5)

$$\begin{aligned} \tau' &= \frac{1}{c}\sqrt{q} \\ &= \frac{\sqrt{t^2}}{c} \\ \tau'' &= \frac{t}{c\sqrt{t^2}} \end{aligned} \tag{6}$$

Substituting the above into (4) results in

$$\begin{aligned} p_1(\tau) &= \frac{\tau''(t) + p(t)\, \tau'(t)}{\tau'(t)^2} \\ &= \frac{\frac{t}{c\sqrt{t^2}} + \frac{t^2-1}{t}\frac{\sqrt{t^2}}{c}}{\left(\frac{\sqrt{t^2}}{c}\right)^2} \\ &= c \end{aligned}$$

Therefore ode (3) now becomes

$$
\begin{aligned}
y(\tau)'' + p_1 y(\tau)' + q_1 y(\tau) &= 0 \\
\frac{d^2}{d\tau^2} y(\tau) + c\left(\frac{d}{d\tau} y(\tau)\right) + c^2 y(\tau) &= 0
\end{aligned} \tag{7}
$$

The above ode is now solved for $y(\tau)$. Since the ode is now constant coefficients, it can be easily solved to give

$$
y(\tau) = \mathrm{e}^{-\frac{c\tau}{2}} \left(c_1 \cos\left(\frac{c\sqrt{3}\,\tau}{2}\right) + c_2 \sin\left(\frac{c\sqrt{3}\,\tau}{2}\right)\right)
$$

Now from (6)

$$
\begin{aligned}
\tau &= \int \frac{1}{c}\sqrt{q}\, dt \\
&= \frac{\int \sqrt{t^2} dt}{c} \\
&= \frac{t\sqrt{t^2}}{2c}
\end{aligned}
$$

Substituting the above into the solution obtained gives

$$
y = \mathrm{e}^{-\frac{t^2}{4}} \left(c_1 \cos\left(\frac{\sqrt{3}\,t^2}{4}\right) + c_2 \sin\left(\frac{\sqrt{3}\,t^2}{4}\right)\right)
$$

Summary

The solution(s) found are the following

$$
y = \mathrm{e}^{-\frac{t^2}{4}} \left(c_1 \cos\left(\frac{\sqrt{3}\,t^2}{4}\right) + c_2 \sin\left(\frac{\sqrt{3}\,t^2}{4}\right)\right) \tag{1}
$$

Verification of solutions

$$
y = \mathrm{e}^{-\frac{t^2}{4}} \left(c_1 \cos\left(\frac{\sqrt{3}\,t^2}{4}\right) + c_2 \sin\left(\frac{\sqrt{3}\,t^2}{4}\right)\right)
$$

Verified OK.

8.30.3 Solving using Kovacic algorithm

Writing the ode as

$$ty'' + (t^2 - 1)\, y' + yt^3 = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= t \\ B &= t^2 - 1 \\ C &= t^3 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(t) = ye^{\int \frac{B}{2A}\, dt}$$

Then (2) becomes

$$z''(t) = rz(t) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{-3t^4 + 3}{4t^2} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= -3t^4 + 3 \\ t &= 4t^2 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(t) = \left(\frac{-3t^4 + 3}{4t^2}\right) z(t) \tag{7}$$

Equation (7) is now solved. After finding $z(t)$ then y is found using the inverse transformation

$$y = z(t)\, e^{-\int \frac{B}{2A}\, dt}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 420: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 2 - 4 \\ &= -2 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 4t^2$. There is a pole at $t = 0$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is -2 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Therefore

$$L = [1, 2]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = -\frac{3t^2}{4} + \frac{3}{4t^2}$$

For the pole at $t = 0$ let b be the coefficient of $\frac{1}{t^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{3}{4}$. Hence

$$
\begin{aligned}
[\sqrt{r}]_c &= 0 \\
\alpha_c^{+} &= \frac{1}{2} + \sqrt{1+4b} = \frac{3}{2} \\
\alpha_c^{-} &= \frac{1}{2} - \sqrt{1+4b} = -\frac{1}{2}
\end{aligned}
$$

Since the order of r at ∞ is $O_r(\infty) = -2$ then

$$
v = \frac{-O_r(\infty)}{2} = \frac{2}{2} = 1
$$

$[\sqrt{r}]_\infty$ is the sum of terms involving t^i for $0 \leq i \leq v$ in the Laurent series for $\sqrt{r}$ at ∞. Therefore

$$
\begin{aligned}
[\sqrt{r}]_\infty &= \sum_{i=0}^{v} a_i t^i \\
&= \sum_{i=0}^{1} a_i t^i
\end{aligned} \tag{8}
$$

Let a be the coefficient of $t^v = t^1$ in the above sum. The Laurent series of $\sqrt{r}$ at ∞ is

$$
\sqrt{r} \approx \frac{i\sqrt{3}\,t}{2} - \frac{i\sqrt{3}}{4t^3} - \frac{i\sqrt{3}}{16t^7} - \frac{i\sqrt{3}}{32t^{11}} - \frac{5i\sqrt{3}}{256t^{15}} - \frac{7i\sqrt{3}}{512t^{19}} - \frac{21i\sqrt{3}}{2048t^{23}} - \frac{33i\sqrt{3}}{4096t^{27}} + \dots \tag{9}
$$

Comparing Eq. (9) with Eq. (8) shows that

$$
a = \frac{i\sqrt{3}}{2}
$$

From Eq. (9) the sum up to $v = 1$ gives

$$
\begin{aligned}
[\sqrt{r}]_\infty &= \sum_{i=0}^{1} a_i t^i \\
&= \frac{i\sqrt{3}\,t}{2}
\end{aligned} \tag{10}
$$

Now we need to find b, where b be the coefficient of $t^{v-1} = t^0 = 1$ in r minus the coefficient of same term but in $\left([\sqrt{r}]_\infty\right)^2$ where $[\sqrt{r}]_\infty$ was found above in Eq (10). Hence

$$
\left([\sqrt{r}]_\infty\right)^2 = -\frac{3t^2}{4}
$$

This shows that the coefficient of 1 in the above is 0. Now we need to find the coefficient of 1 in r. How this is done depends on if $v = 0$ or not. Since $v = 1$ which is not zero, then starting $r = \frac{s}{t}$, we do long division and write this in the form

$$r = Q + \frac{R}{t}$$

Where Q is the quotient and R is the remainder. Then the coefficient of 1 in r will be the coefficient this term in the quotient. Doing long division gives

$$\begin{aligned}
r &= \frac{s}{t} \\
&= \frac{-3t^4 + 3}{4t^2} \\
&= Q + \frac{R}{4t^2} \\
&= \left(-\frac{3t^2}{4}\right) + \left(\frac{3}{4t^2}\right) \\
&= -\frac{3t^2}{4} + \frac{3}{4t^2}
\end{aligned}$$

We see that the coefficient of the term t in the quotient is 0. Now b can be found.

$$\begin{aligned}
b &= (0) - (0) \\
&= 0
\end{aligned}$$

Hence

$$\begin{aligned}
[\sqrt{r}]_\infty &= \frac{i\sqrt{3}\,t}{2} \\
\alpha_\infty^+ &= \frac{1}{2}\left(\frac{b}{a} - v\right) &= \frac{1}{2}\left(\frac{0}{\frac{i\sqrt{3}}{2}} - 1\right) &= -\frac{1}{2} \\
\alpha_\infty^- &= \frac{1}{2}\left(-\frac{b}{a} - v\right) &= \frac{1}{2}\left(-\frac{0}{\frac{i\sqrt{3}}{2}} - 1\right) &= -\frac{1}{2}
\end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = \frac{-3t^4 + 3}{4t^2}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
0	2	0	$\frac{3}{2}$	$-\frac{1}{2}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
-2	$\frac{i\sqrt{3}\,t}{2}$	$-\frac{1}{2}$	$-\frac{1}{2}$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^- = -\frac{1}{2}$ then

$$\begin{aligned} d &= \alpha_\infty^- - \left(\alpha_{c_1}^-\right) \\ &= -\frac{1}{2} - \left(-\frac{1}{2}\right) \\ &= 0 \end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c\in\Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{t-c} \right) + s(\infty)[\sqrt{r}]_\infty$$

The above gives

$$\begin{aligned} \omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^-}{t-c_1} \right) + (-)[\sqrt{r}]_\infty \\ &= -\frac{1}{2t} + (-)\left(\frac{i\sqrt{3}\,t}{2}\right) \\ &= -\frac{1}{2t} - \frac{i\sqrt{3}\,t}{2} \\ &= \frac{-i\sqrt{3}\,t^2 - 1}{2t} \end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(t)$ of degree $d = 0$ to solve the ode. The polynomial $p(t)$ needs to satisfy the equation

$$p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right) p = 0 \tag{1A}$$

Let

$$p(t) = 1 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$(0)+2\left(-\frac{1}{2t}-\frac{i\sqrt{3}\,t}{2}\right)(0)+\left(\left(\frac{1}{2t^2}-\frac{i\sqrt{3}}{2}\right)+\left(-\frac{1}{2t}-\frac{i\sqrt{3}\,t}{2}\right)^2-\left(\frac{-3t^4+3}{4t^2}\right)\right)=0$$
$$0=0$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z''=rz$ is

$$\begin{aligned}
z_1(t) &= pe^{\int \omega\, dt}\\
&= \mathrm{e}^{\int \left(-\frac{1}{2t}-\frac{i\sqrt{3}\,t}{2}\right)dt}\\
&= \frac{\mathrm{e}^{-\frac{i\sqrt{3}\,t^2}{4}}}{\sqrt{t}}
\end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\, dt}\\
&= z_1 e^{-\int \frac{1}{2}\frac{t^2-1}{t}\, dt}\\
&= z_1 e^{-\frac{t^2}{4}+\frac{\ln(t)}{2}}\\
&= z_1\left(\sqrt{t}\,\mathrm{e}^{-\frac{t^2}{4}}\right)
\end{aligned}$$

Which simplifies to

$$y_1=\mathrm{e}^{-\frac{t^2\left(1+i\sqrt{3}\right)}{4}}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2=y_1\int \frac{e^{\int -\frac{B}{A}\, dt}}{y_1^2}\, dt$$

Substituting gives

$$\begin{aligned}
y_2 &= y_1\int \frac{e^{\int -\frac{t^2-1}{t}\, dt}}{(y_1)^2}\, dt\\
&= y_1\int \frac{e^{-\frac{t^2}{2}+\ln(t)}}{(y_1)^2}\, dt\\
&= y_1\left(-\frac{i\sqrt{3}\,\mathrm{e}^{\frac{i\sqrt{3}\,t^2}{2}}}{3}\right)
\end{aligned}$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1\left(\mathrm{e}^{-\frac{t^2\left(1+i\sqrt{3}\right)}{4}}\right) + c_2\left(\mathrm{e}^{-\frac{t^2\left(1+i\sqrt{3}\right)}{4}}\left(-\frac{i\sqrt{3}\,\mathrm{e}^{\frac{i\sqrt{3}\,t^2}{2}}}{3}\right)\right)
\end{aligned}
$$

Summary

The solution(s) found are the following

$$
y = c_1 \mathrm{e}^{-\frac{t^2\left(1+i\sqrt{3}\right)}{4}} - \frac{i c_2 \sqrt{3}\,\mathrm{e}^{\frac{t^2\left(i\sqrt{3}-1\right)}{4}}}{3} \tag{1}
$$

Verification of solutions

$$
y = c_1 \mathrm{e}^{-\frac{t^2\left(1+i\sqrt{3}\right)}{4}} - \frac{i c_2 \sqrt{3}\,\mathrm{e}^{\frac{t^2\left(i\sqrt{3}-1\right)}{4}}}{3}
$$

Verified OK.

8.30.4 Maple step by step solution

Let's solve

$t y'' + (t^2 - 1)\, y' + y t^3 = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y'' = -\frac{(t^2-1)y'}{t} - y t^2$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y'' + \frac{(t^2-1)y'}{t} + y t^2 = 0$

□ Check to see if $t_0 = 0$ is a regular singular point

  ◦ Define functions

    $\left[P_2(t) = \frac{t^2-1}{t}, P_3(t) = t^2\right]$

  ◦ $t \cdot P_2(t)$ is analytic at $t = 0$

    $\left(t \cdot P_2(t)\right)\Big|_{t=0} = -1$

- $t^2 \cdot P_3(t)$ is analytic at $t = 0$

  $\left(t^2 \cdot P_3(t)\right)\bigg|_{t=0} = 0$

- $t = 0$is a regular singular point

  Check to see if $t_0 = 0$ is a regular singular point

  $t_0 = 0$

- Multiply by denominators

  $ty'' + (t^2 - 1)\, y' + yt^3 = 0$

- Assume series solution for y

  $y = \sum_{k=0}^{\infty} a_k t^{k+r}$

☐ Rewrite ODE with series expansions

- Convert $t^3 \cdot y$ to series expansion

  $t^3 \cdot y = \sum_{k=0}^{\infty} a_k t^{k+r+3}$

- Shift index using $k- >k - 3$

  $t^3 \cdot y = \sum_{k=3}^{\infty} a_{k-3} t^{k+r}$

- Convert $t^m \cdot y'$ to series expansion for $m = 0..2$

  $t^m \cdot y' = \sum_{k=0}^{\infty} a_k(k + r)\, t^{k+r-1+m}$

- Shift index using $k- >k + 1 - m$

  $t^m \cdot y' = \sum_{k=-1+m}^{\infty} a_{k+1-m}(k + 1 - m + r)\, t^{k+r}$

- Convert $t \cdot y''$ to series expansion

  $t \cdot y'' = \sum_{k=0}^{\infty} a_k(k + r)\,(k + r - 1)\, t^{k+r-1}$

- Shift index using $k- >k + 1$

  $t \cdot y'' = \sum_{k=-1}^{\infty} a_{k+1}(k + 1 + r)\,(k + r)\, t^{k+r}$

  Rewrite ODE with series expansions

  $a_0 r(-2 + r)\, t^{-1+r} + a_1(1 + r)\,(-1 + r)\, t^r + (a_2(2 + r)\, r + a_0 r)\, t^{1+r} + (a_3(3 + r)\,(1 + r) + a_1(1 +$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $r(-2+r)=0$
- Values of r that satisfy the indicial equation

  $r\in\{0,2\}$
- The coefficients of each power of t must be 0

  $[a_1(1+r)(-1+r)=0, a_2(2+r)r+a_0r=0, a_3(3+r)(1+r)+a_1(1+r)=0]$
- Solve for the dependent coefficient(s)

  $\left\{a_1=0, a_2=-\frac{a_0}{2+r}, a_3=0\right\}$
- Each term in the series must be 0, giving the recursion relation

  $a_{k+1}(k+1+r)(k+r-1)+a_{k-1}(k+r-1)+a_{k-3}=0$
- Shift index using $k->k+3$

  $a_{k+4}(k+4+r)(k+2+r)+a_{k+2}(k+2+r)+a_k=0$
- Recursion relation that defines series solution to ODE

  $a_{k+4}=-\frac{ka_{k+2}+ra_{k+2}+a_k+2a_{k+2}}{(k+4+r)(k+2+r)}$
- Recursion relation for $r=0$

  $a_{k+4}=-\frac{ka_{k+2}+a_k+2a_{k+2}}{(k+4)(k+2)}$
- Solution for $r=0$

  $\left[y=\sum_{k=0}^{\infty} a_k t^k, a_{k+4}=-\frac{ka_{k+2}+a_k+2a_{k+2}}{(k+4)(k+2)}, a_1=0, a_2=-\frac{a_0}{2}, a_3=0\right]$
- Recursion relation for $r=2$

  $a_{k+4}=-\frac{ka_{k+2}+a_k+4a_{k+2}}{(k+6)(k+4)}$
- Solution for $r=2$

  $\left[y=\sum_{k=0}^{\infty} a_k t^{k+2}, a_{k+4}=-\frac{ka_{k+2}+a_k+4a_{k+2}}{(k+6)(k+4)}, a_1=0, a_2=-\frac{a_0}{4}, a_3=0\right]$
- Combine solutions and rename parameters

  $\left[y=\left(\sum_{k=0}^{\infty} a_k t^k\right)+\left(\sum_{k=0}^{\infty} b_k t^{k+2}\right), a_{k+4}=-\frac{ka_{k+2}+a_k+2a_{k+2}}{(k+4)(k+2)}, a_1=0, a_2=-\frac{a_0}{2}, a_3=0, b_{k+4}=-\frac{kb_k\cdots}{(\cdots}\right.$

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Group is reducible or imprimitive
<- Kovacics algorithm successful`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 34

```
dsolve(t*diff(y(t),t$2)+ (t^2-1)*diff(y(t),t)+t^3*y(t) = 0,y(t), singsol=all)
```

$$y(t) = \mathrm{e}^{-\frac{t^2}{4}}\left(c_1\cos\left(\frac{t^2\sqrt{3}}{4}\right)+c_2\sin\left(\frac{t^2\sqrt{3}}{4}\right)\right)$$

✓ Solution by Mathematica

Time used: 0.03 (sec). Leaf size: 48

```
DSolve[t*y''[t]+(t^2-1)*y'[t]+t^3*y[t]==0,y[t],t,IncludeSingularSolutions -> True]
```

$$y(t) \to e^{-\frac{t^2}{4}}\left(c_2\cos\left(\frac{\sqrt{3}t^2}{4}\right)+c_1\sin\left(\frac{\sqrt{3}t^2}{4}\right)\right)$$

9 Chapter 3, Second order linear equations, 3.4 Repeated roots, reduction of order , page 172

9.1 problem 1

Internal problem ID [653]
Internal file name [`OUTPUT/653_Sunday_June_05_2022_01_46_28_AM_41212301/index.tex`]

Book: Elementary differential equations and boundary value problems, 10th ed., Boyce and DiPrima
Section: Chapter 3, Second order linear equations, 3.4 Repeated roots, reduction of order , page 172
Problem number: 1.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_linear_constant_coeff", "linear_second_order_ode_solved_by_an_integrating_factor"**

Maple gives the following as the ode type

```
[[_2nd_order, _missing_x]]
```

$$\boxed{y'' - 2y' + y = 0}$$

9.1.1 Solving as second order linear constant coeff ode

This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = 0$$

Where in the above $A = 1, B = -2, C = 1$. Let the solution be $y = e^{\lambda x}$. Substituting this into the ODE gives

$$\lambda^2 e^{\lambda x} - 2\lambda\, e^{\lambda x} + e^{\lambda x} = 0 \tag{1}$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda x}$ gives

$$\lambda^2 - 2\lambda + 1 = 0 \tag{2}$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form.Using the quadratic formula

$$\lambda_{1,2} = \frac{-B}{2A} \pm \frac{1}{2A}\sqrt{B^2 - 4AC}$$

Substituting $A = 1, B = -2, C = 1$ into the above gives

$$\begin{aligned}\lambda_{1,2} &= \frac{2}{(2)(1)} \pm \frac{1}{(2)(1)}\sqrt{(-2)^2 - (4)(1)(1)} \\ &= 1\end{aligned}$$

Hence this is the case of a double root $\lambda_{1,2} = -1$. Therefore the solution is

$$y = c_1 e^{x} + c_2 x\, e^{x} \tag{1}$$

Summary

The solution(s) found are the following

$$y = c_1 e^{x} + c_2 x\, e^{x} \tag{1}$$

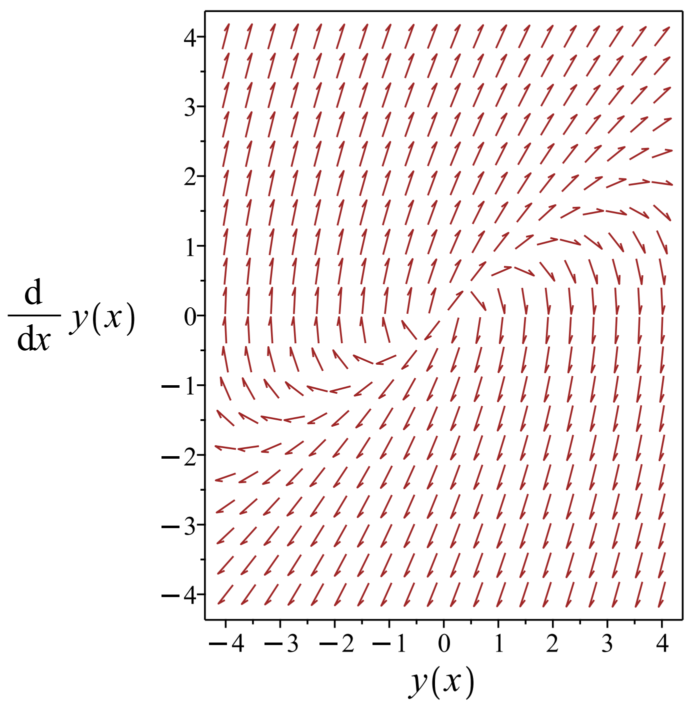


Figure 430: Slope field plot

Verification of solutions

$$y = c_1 e^{x} + c_2 x\, e^{x}$$

Verified OK.

9.1.2 Solving as linear second order ode solved by an integrating factor ode

The ode satisfies this form

$$y'' + p(x)\,y' + \frac{\left(p(x)^2 + p'(x)\right) y}{2} = f(x)$$

Where $p(x) = -2$. Therefore, there is an integrating factor given by

$$\begin{aligned} M(x) &= e^{\frac{1}{2}\int p\,dx} \\ &= e^{\int -2\,dx} \\ &= \mathrm{e}^{-x} \end{aligned}$$

Multiplying both sides of the ODE by the integrating factor $M(x)$ makes the left side of the ODE a complete differential

$$\begin{aligned} (M(x)y)'' &= 0 \\ \left(\mathrm{e}^{-x} y\right)'' &= 0 \end{aligned}$$

Integrating once gives

$$\left(\mathrm{e}^{-x} y\right)' = c_1$$

Integrating again gives

$$\left(\mathrm{e}^{-x} y\right) = c_1 x + c_2$$

Hence the solution is

$$y = \frac{c_1 x + c_2}{\mathrm{e}^{-x}}$$

Or

$$y = c_1 x\,\mathrm{e}^{x} + c_2 \mathrm{e}^{x}$$

<u>Summary</u>
The solution(s) found are the following

$$y = c_1 x\,\mathrm{e}^{x} + c_2 \mathrm{e}^{x} \tag{1}$$

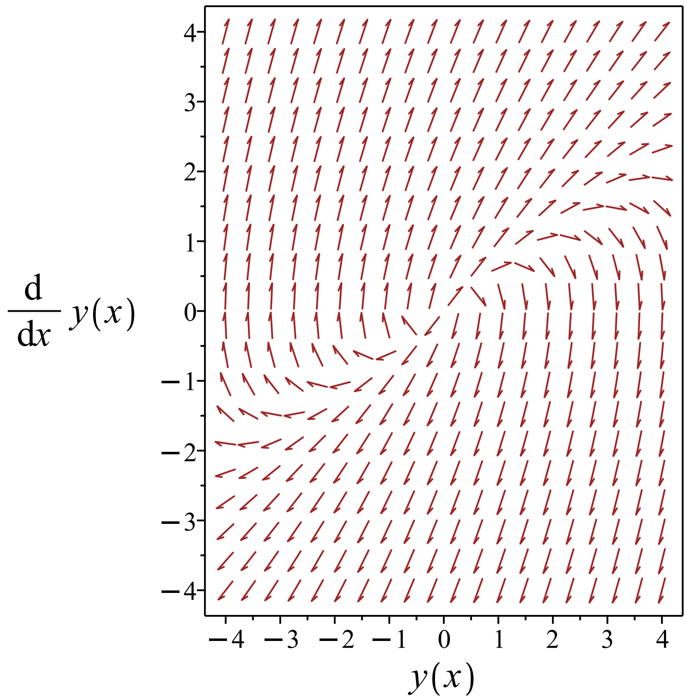


Figure 431: Slope field plot

Verification of solutions

$$y = c_1 x\, e^{x} + c_2 e^{x}$$

Verified OK.

9.1.3 Solving using Kovacic algorithm

Writing the ode as

$$y'' - 2y' + y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1 \\ B &= -2 \\ C &= 1 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = y e^{\int \frac{B}{2A}\, dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{0}{1} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= 0 \\ t &= 1 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = 0 \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 422: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$
\begin{aligned}
O(\infty) &= \deg(t) - \deg(s) \\
&= 0 - -\infty \\
&= \infty
\end{aligned}
$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is $infinity$ then the necessary conditions for case one are met. Therefore

$$
L = [1]
$$

Since $r = 0$ is not a function of x, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$
z_1(x) = 1
$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$
\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{-2}{1}\,dx} \\
&= z_1 e^{x} \\
&= z_1(\mathrm{e}^{x})
\end{aligned}
$$

Which simplifies to

$$
y_1 = \mathrm{e}^{x}
$$

The second solution y_2 to the original ode is found using reduction of order

$$
y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx
$$

Substituting gives

$$
\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{-2}{1}\,dx}}{(y_1)^2}\,dx \\
&= y_1 \int \frac{e^{2x}}{(y_1)^2}\,dx \\
&= y_1(x)
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1(\mathrm{e}^x) + c_2(\mathrm{e}^x(x))
\end{aligned}
$$

Summary
The solution(s) found are the following

$$
y = c_1 \mathrm{e}^x + c_2 x\,\mathrm{e}^x \tag{1}
$$

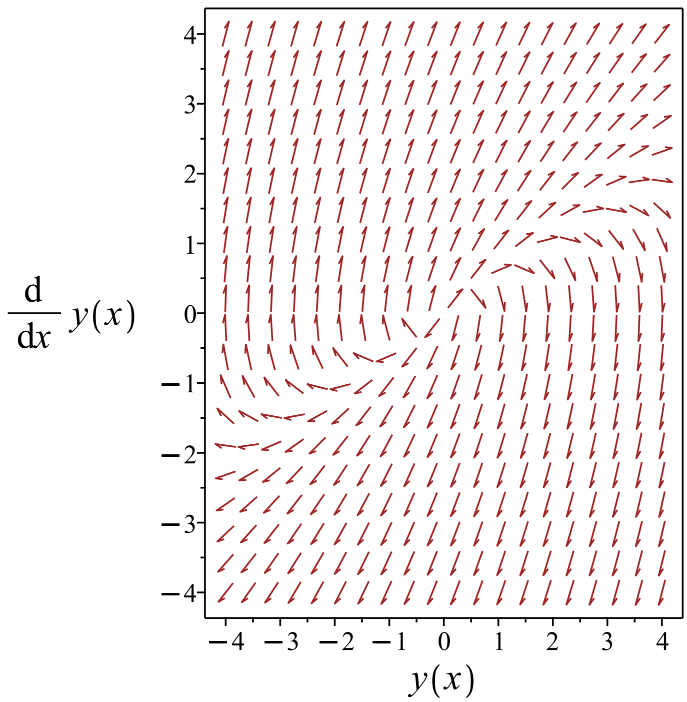


Figure 432: Slope field plot

Verification of solutions

$$
y = c_1 \mathrm{e}^x + c_2 x\,\mathrm{e}^x
$$

Verified OK.

9.1.4 Maple step by step solution

Let's solve

$y'' - 2y' + y = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Characteristic polynomial of ODE

  $r^2 - 2r + 1 = 0$

- Factor the characteristic polynomial

  $(r-1)^2 = 0$

- Root of the characteristic polynomial

  $r = 1$

- 1st solution of the ODE

  $y_1(x) = e^x$

- Repeated root, multiply $y_1(x)$ by x to ensure linear independence

  $y_2(x) = x\,e^x$

- General solution of the ODE

  $y = c_1 y_1(x) + c_2 y_2(x)$

- Substitute in solutions

  $y = c_1 e^x + c_2 x\,e^x$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
<- constant coefficients successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 12

```
dsolve(diff(y(x),x$2)-2*diff(y(x),x)+y(x) = 0,y(x), singsol=all)
```

$$y(x) = e^x(c_2x + c_1)$$

✓ Solution by Mathematica

Time used: 0.01 (sec). Leaf size: 16

```
DSolve[y''[x]-2*y'[x]+y[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to e^x(c_2x + c_1)$$

9.2 problem 2

Internal problem ID [654]
Internal file name [`OUTPUT/654_Sunday_June_05_2022_01_46_28_AM_19432812/index.tex`]

Book: Elementary differential equations and boundary value problems, 10th ed., Boyce and DiPrima
Section: Chapter 3, Second order linear equations, 3.4 Repeated roots, reduction of order , page 172
Problem number: 2.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_linear_constant_coeff", "linear_second_order_ode_solved_by_an_integrating_factor"**

Maple gives the following as the ode type

```
[[_2nd_order, _missing_x]]
```

$$\boxed{9y'' + 6y' + y = 0}$$

9.2.1 Solving as second order linear constant coeff ode

This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = 0$$

Where in the above $A = 9, B = 6, C = 1$. Let the solution be $y = e^{\lambda x}$. Substituting this into the ODE gives

$$9\lambda^2 e^{\lambda x} + 6\lambda\, e^{\lambda x} + e^{\lambda x} = 0 \tag{1}$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda x}$ gives

$$9\lambda^2 + 6\lambda + 1 = 0 \tag{2}$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form.Using the quadratic formula

$$\lambda_{1,2} = \frac{-B}{2A} \pm \frac{1}{2A}\sqrt{B^2 - 4AC}$$

Substituting $A = 9, B = 6, C = 1$ into the above gives

$$\begin{aligned}\lambda_{1,2} &= \frac{-6}{(2)\,(9)} \pm \frac{1}{(2)\,(9)}\sqrt{(6)^2 - (4)\,(9)\,(1)}\\ &= -\frac{1}{3}\end{aligned}$$

Hence this is the case of a double root $\lambda_{1,2} = \frac{1}{3}$. Therefore the solution is

$$y = c_1 e^{-\frac{x}{3}} + c_2 x\, e^{-\frac{x}{3}} \tag{1}$$

Summary

The solution(s) found are the following

$$y = c_1 e^{-\frac{x}{3}} + c_2 x\, e^{-\frac{x}{3}} \tag{1}$$

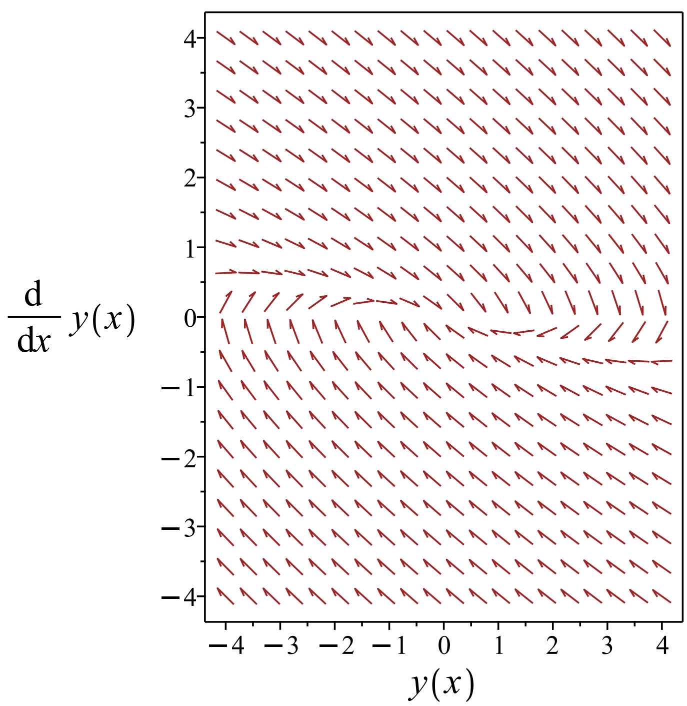


Figure 433: Slope field plot

Verification of solutions

$$y = c_1 e^{-\frac{x}{3}} + c_2 x\, e^{-\frac{x}{3}}$$

Verified OK.

9.2.2 Solving as linear second order ode solved by an integrating factor ode

The ode satisfies this form

$$y'' + p(x)\, y' + \frac{\left(p(x)^{2}+p'(x)\right) y}{2} = f(x)$$

Where $p(x) = \frac{2}{3}$. Therefore, there is an integrating factor given by

$$\begin{aligned}
M(x) &= e^{\frac{1}{2}\int p\,dx} \\
&= e^{\int \frac{2}{3}\,dx} \\
&= \mathrm{e}^{\frac{x}{3}}
\end{aligned}$$

Multiplying both sides of the ODE by the integrating factor $M(x)$ makes the left side of the ODE a complete differential

$$\begin{aligned}
(M(x)y)'' &= 0 \\
\left(\mathrm{e}^{\frac{x}{3}} y\right)'' &= 0
\end{aligned}$$

Integrating once gives

$$\left(\mathrm{e}^{\frac{x}{3}} y\right)' = c_1$$

Integrating again gives

$$\left(\mathrm{e}^{\frac{x}{3}} y\right) = c_1 x + c_2$$

Hence the solution is

$$y = \frac{c_1 x + c_2}{\mathrm{e}^{\frac{x}{3}}}$$

Or

$$y = c_1 x\,\mathrm{e}^{-\frac{x}{3}} + c_2 \mathrm{e}^{-\frac{x}{3}}$$

Summary

The solution(s) found are the following

$$y = c_1 x\,\mathrm{e}^{-\frac{x}{3}} + c_2 \mathrm{e}^{-\frac{x}{3}} \tag{1}$$

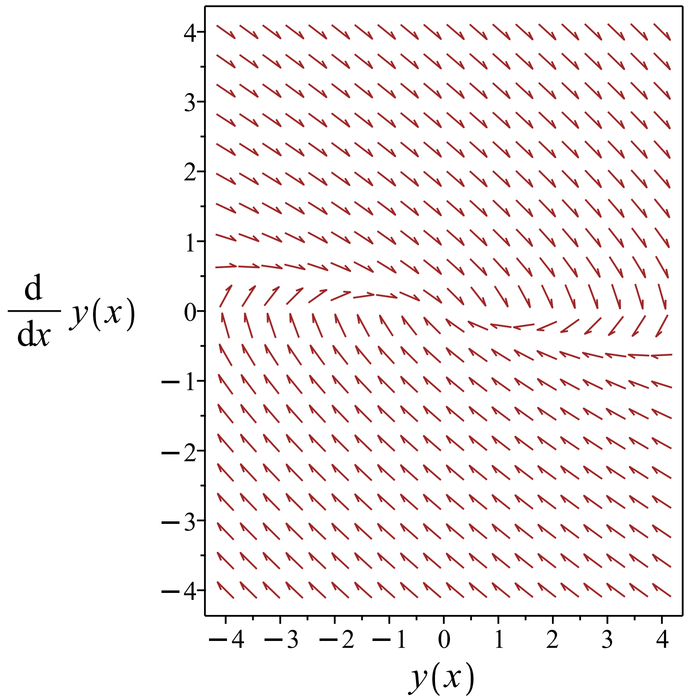


Figure 434: Slope field plot

Verification of solutions

$$y = c_1 x\,\mathrm{e}^{-\frac{x}{3}} + c_2 \mathrm{e}^{-\frac{x}{3}}$$

Verified OK.

9.2.3 Solving using Kovacic algorithm

Writing the ode as

$$9y'' + 6y' + y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 9 \\ B &= 6 \\ C &= 1 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = y e^{\int \frac{B}{2A}\, dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{0}{1} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= 0 \\ t &= 1 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = 0 \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x) e^{-\int \frac{B}{2A} dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 424: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned}O(\infty) &= \deg(t) - \deg(s)\\ &= 0 - -\infty\\ &= \infty\end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is $infinity$ then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Since $r = 0$ is not a function of x, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$z_1(x) = 1$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$\begin{aligned}y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx}\\ &= z_1 e^{-\int \frac{1}{2}\frac{6}{9}\,dx}\\ &= z_1 e^{-\frac{x}{3}}\\ &= z_1\left(\mathrm{e}^{-\frac{x}{3}}\right)\end{aligned}$$

Which simplifies to

$$y_1 = \mathrm{e}^{-\frac{x}{3}}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx$$

Substituting gives

$$\begin{aligned}y_2 &= y_1 \int \frac{e^{\int -\frac{6}{9}\,dx}}{(y_1)^2}\,dx\\ &= y_1 \int \frac{e^{-\frac{2x}{3}}}{(y_1)^2}\,dx\\ &= y_1(x)\end{aligned}$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1\left(\mathrm{e}^{-\frac{x}{3}}\right) + c_2\left(\mathrm{e}^{-\frac{x}{3}}(x)\right)
\end{aligned}
$$

Summary

The solution(s) found are the following

$$
y = c_1 \mathrm{e}^{-\frac{x}{3}} + c_2 x\,\mathrm{e}^{-\frac{x}{3}} \tag{1}
$$

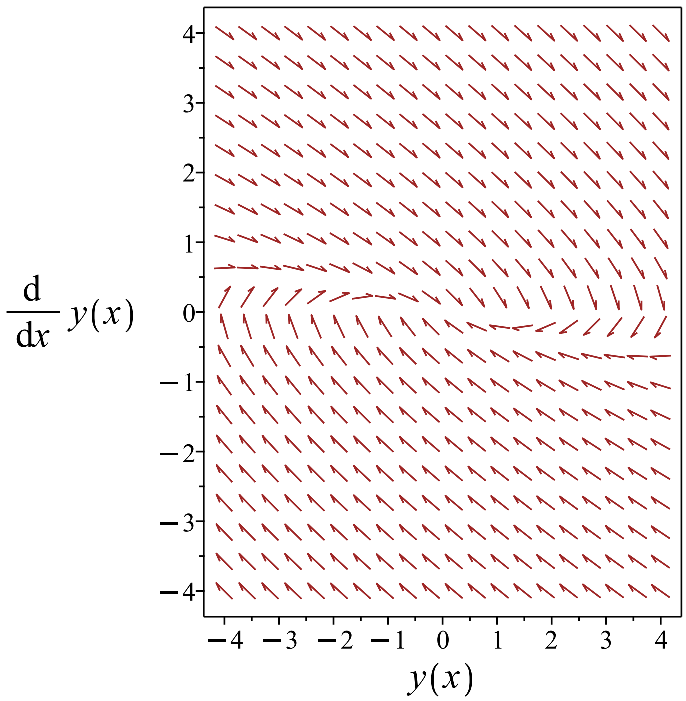


Figure 435: Slope field plot

Verification of solutions

$$
y = c_1 \mathrm{e}^{-\frac{x}{3}} + c_2 x\,\mathrm{e}^{-\frac{x}{3}}
$$

Verified OK.

9.2.4 Maple step by step solution

Let's solve

$9y'' + 6y' + y = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y'' = -\frac{2y'}{3} - \frac{y}{9}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y'' + \frac{2y'}{3} + \frac{y}{9} = 0$

- Characteristic polynomial of ODE

  $r^2 + \frac{2}{3}r + \frac{1}{9} = 0$

- Factor the characteristic polynomial

  $\frac{(3r+1)^2}{9} = 0$

- Root of the characteristic polynomial

  $r = -\frac{1}{3}$

- 1st solution of the ODE

  $y_1(x) = e^{-\frac{x}{3}}$

- Repeated root, multiply $y_1(x)$ by x to ensure linear independence

  $y_2(x) = x\, e^{-\frac{x}{3}}$

- General solution of the ODE

  $y = c_1 y_1(x) + c_2 y_2(x)$

- Substitute in solutions

  $y = c_1 e^{-\frac{x}{3}} + c_2 x\, e^{-\frac{x}{3}}$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
<- constant coefficients successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 14

```
dsolve(9*diff(y(x),x$2)+6*diff(y(x),x)+y(x) = 0,y(x), singsol=all)
```

$$y(x) = e^{-\frac{x}{3}}(c_2x + c_1)$$

✓ Solution by Mathematica

Time used: 0.013 (sec). Leaf size: 20

```
DSolve[9*y''[x]+6*y'[x]+y[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to e^{-x/3}(c_2x + c_1)$$

9.3 problem 3

Internal problem ID [655]
Internal file name [`OUTPUT/655_Sunday_June_05_2022_01_46_29_AM_67197608/index.tex`]

Book: Elementary differential equations and boundary value problems, 10th ed., Boyce and DiPrima
Section: Chapter 3, Second order linear equations, 3.4 Repeated roots, reduction of order , page 172
Problem number: 3.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_linear_constant_coeff"**

Maple gives the following as the ode type

```
[[_2nd_order, _missing_x]]
```

$$\boxed{4y'' - 4y' - 3y = 0}$$

9.3.1 Solving as second order linear constant coeff ode

This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = 0$$

Where in the above $A = 4, B = -4, C = -3$. Let the solution be $y = e^{\lambda x}$. Substituting this into the ODE gives

$$4\lambda^2 e^{\lambda x} - 4\lambda\, e^{\lambda x} - 3\, e^{\lambda x} = 0 \tag{1}$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda x}$ gives

$$4\lambda^2 - 4\lambda - 3 = 0 \tag{2}$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form.Using the quadratic formula

$$\lambda_{1,2} = \frac{-B}{2A} \pm \frac{1}{2A}\sqrt{B^2 - 4AC}$$

Substituting $A = 4, B = -4, C = -3$ into the above gives

$$\lambda_{1,2} = \frac{4}{(2)(4)} \pm \frac{1}{(2)(4)}\sqrt{-4^2 - (4)(4)(-3)}$$
$$= \frac{1}{2} \pm 1$$

Hence

$$\lambda_1 = \frac{1}{2} + 1$$
$$\lambda_2 = \frac{1}{2} - 1$$

Which simplifies to

$$\lambda_1 = \frac{3}{2}$$
$$\lambda_2 = -\frac{1}{2}$$

Since roots are real and distinct, then the solution is

$$y = c_1 e^{\lambda_1 x} + c_2 e^{\lambda_2 x}$$
$$y = c_1 e^{\left(\frac{3}{2}\right)x} + c_2 e^{\left(-\frac{1}{2}\right)x}$$

Or

$$y = c_1 \mathrm{e}^{\frac{3x}{2}} + c_2 \mathrm{e}^{-\frac{x}{2}}$$

Summary

The solution(s) found are the following

$$y = c_1 \mathrm{e}^{\frac{3x}{2}} + c_2 \mathrm{e}^{-\frac{x}{2}} \tag{1}$$

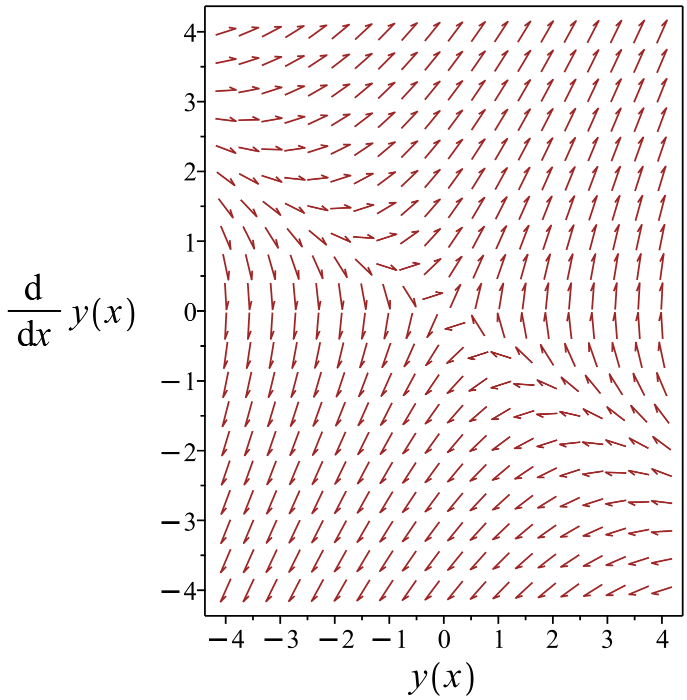


Figure 436: Slope field plot

Verification of solutions

$$y = c_1 e^{\frac{3x}{2}} + c_2 e^{-\frac{x}{2}}$$

Verified OK.

9.3.2 Solving using Kovacic algorithm

Writing the ode as

$$4y'' - 4y' - 3y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 4 \\ B &= -4 \\ C &= -3 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = y e^{\int \frac{B}{2A} \, dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{1}{1} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= 1 \\ t &= 1 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x) \, e^{-\int \frac{B}{2A} \, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 426: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$
\begin{aligned}
O(\infty) &= \deg(t) - \deg(s) \\
&= 0 - 0 \\
&= 0
\end{aligned}
$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Therefore

$$
L = [1]
$$

Since $r = 1$ is not a function of x, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$
z_1(x) = \mathrm{e}^{-x}
$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$
\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{-4}{4}\,dx} \\
&= z_1 e^{\frac{x}{2}} \\
&= z_1\left(\mathrm{e}^{\frac{x}{2}}\right)
\end{aligned}
$$

Which simplifies to

$$
y_1 = \mathrm{e}^{-\frac{x}{2}}
$$

The second solution y_2 to the original ode is found using reduction of order

$$
y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx
$$

Substituting gives

$$
\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{-4}{4}\,dx}}{(y_1)^2}\,dx \\
&= y_1 \int \frac{e^{x}}{(y_1)^2}\,dx \\
&= y_1\left(\frac{\mathrm{e}^{2x}}{2}\right)
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1\left(\mathrm{e}^{-\frac{x}{2}}\right) + c_2\left(\mathrm{e}^{-\frac{x}{2}}\left(\frac{\mathrm{e}^{2x}}{2}\right)\right)
\end{aligned}
$$

Summary

The solution(s) found are the following

$$
y = c_1 \mathrm{e}^{-\frac{x}{2}} + \frac{c_2 \mathrm{e}^{\frac{3x}{2}}}{2} \tag{1}
$$

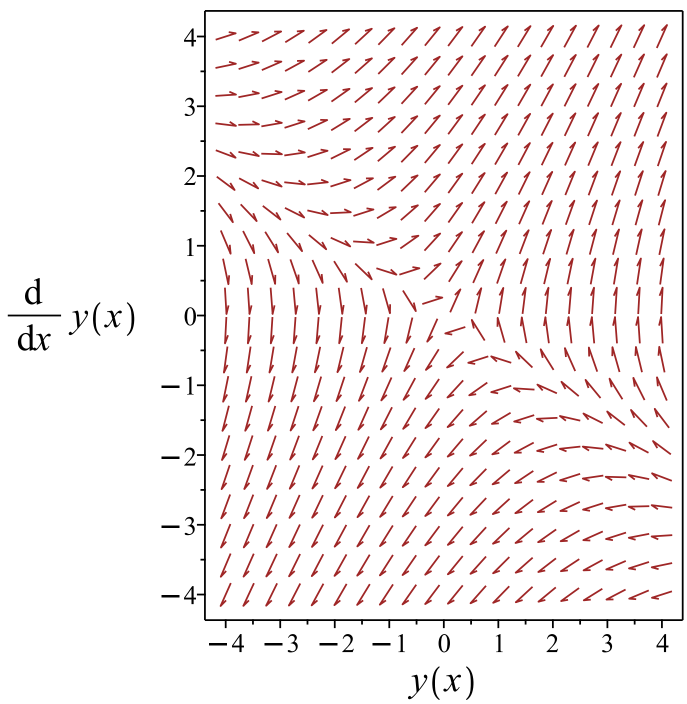


Figure 437: Slope field plot

Verification of solutions

$$
y = c_1 \mathrm{e}^{-\frac{x}{2}} + \frac{c_2 \mathrm{e}^{\frac{3x}{2}}}{2}
$$

Verified OK.

9.3.3 Maple step by step solution

Let's solve

$4y'' - 4y' - 3y = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y'' = y' + \frac{3y}{4}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y'' - y' - \frac{3y}{4} = 0$

- Characteristic polynomial of ODE

  $r^2 - r - \frac{3}{4} = 0$

- Factor the characteristic polynomial

  $\frac{(2r+1)(2r-3)}{4} = 0$

- Roots of the characteristic polynomial

  $r = \left(-\frac{1}{2}, \frac{3}{2}\right)$

- 1st solution of the ODE

  $y_1(x) = e^{-\frac{x}{2}}$

- 2nd solution of the ODE

  $y_2(x) = e^{\frac{3x}{2}}$

- General solution of the ODE

  $y = c_1 y_1(x) + c_2 y_2(x)$

- Substitute in solutions

  $y = c_1 e^{-\frac{x}{2}} + c_2 e^{\frac{3x}{2}}$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
<- constant coefficients successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 17

```
dsolve(4*diff(y(x),x$2)-4*diff(y(x),x)-3*y(x) = 0,y(x), singsol=all)
```

$$y(x) = c_1 e^{-\frac{x}{2}} + c_2 e^{\frac{3x}{2}}$$

✓ Solution by Mathematica

Time used: 0.013 (sec). Leaf size: 24

```
DSolve[4*y''[x]-4*y'[x]-3*y[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to e^{-x/2}\left(c_2 e^{2x} + c_1\right)$$

9.4 problem 4

Internal problem ID [656]
Internal file name [`OUTPUT/656_Sunday_June_05_2022_01_46_30_AM_9155155/index.tex`]

Book: Elementary differential equations and boundary value problems, 10th ed., Boyce and DiPrima
Section: Chapter 3, Second order linear equations, 3.4 Repeated roots, reduction of order , page 172
Problem number: 4.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_linear_constant_coeff", "linear_second_order_ode_solved_by_an_integrating_factor"**

Maple gives the following as the ode type

```
[[_2nd_order, _missing_x]]
```

$$\boxed{4y'' + 12y' + 9y = 0}$$

9.4.1 Solving as second order linear constant coeff ode

This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = 0$$

Where in the above $A = 4, B = 12, C = 9$. Let the solution be $y = e^{\lambda x}$. Substituting this into the ODE gives

$$4\lambda^2 e^{\lambda x} + 12\lambda\, e^{\lambda x} + 9\, e^{\lambda x} = 0 \tag{1}$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda x}$ gives

$$4\lambda^2 + 12\lambda + 9 = 0 \tag{2}$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form.Using the quadratic formula

$$\lambda_{1,2} = \frac{-B}{2A} \pm \frac{1}{2A}\sqrt{B^2 - 4AC}$$

Substituting $A = 4, B = 12, C = 9$ into the above gives

$$\begin{aligned}\lambda_{1,2} &= \frac{-12}{(2)(4)} \pm \frac{1}{(2)(4)}\sqrt{(12)^2 - (4)(4)(9)} \\ &= -\frac{3}{2}\end{aligned}$$

Hence this is the case of a double root $\lambda_{1,2} = \frac{3}{2}$. Therefore the solution is

$$y = c_1 \mathrm{e}^{-\frac{3x}{2}} + c_2 x\,\mathrm{e}^{-\frac{3x}{2}} \tag{1}$$

Summary

The solution(s) found are the following

$$y = c_1 \mathrm{e}^{-\frac{3x}{2}} + c_2 x\,\mathrm{e}^{-\frac{3x}{2}} \tag{1}$$

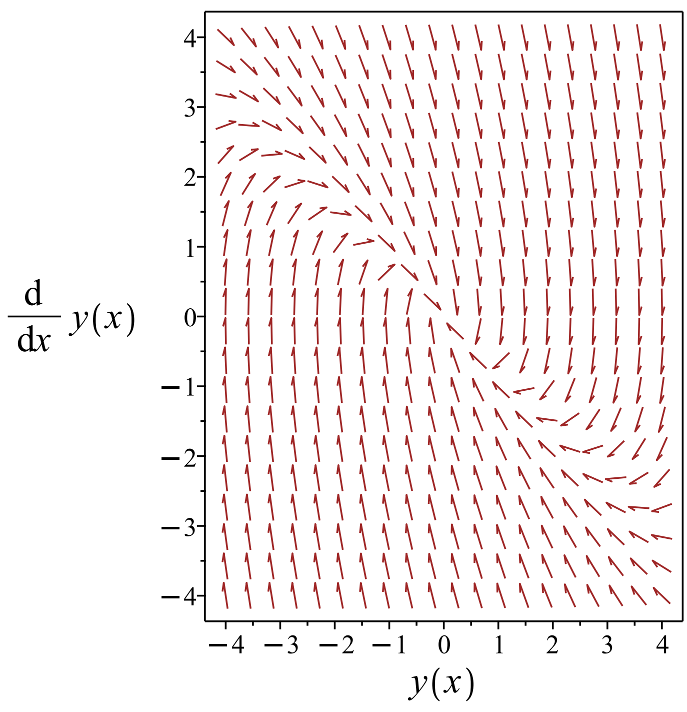


Figure 438: Slope field plot

Verification of solutions

$$y = c_1 \mathrm{e}^{-\frac{3x}{2}} + c_2 x\,\mathrm{e}^{-\frac{3x}{2}}$$

Verified OK.

9.4.2 Solving as linear second order ode solved by an integrating factor ode

The ode satisfies this form

$$y'' + p(x)\, y' + \frac{\left(p(x)^2 + p'(x)\right) y}{2} = f(x)$$

Where $p(x) = 3$. Therefore, there is an integrating factor given by

$$\begin{aligned} M(x) &= e^{\frac{1}{2}\int p\, dx} \\ &= e^{\int 3\, dx} \\ &= \mathrm{e}^{\frac{3x}{2}} \end{aligned}$$

Multiplying both sides of the ODE by the integrating factor $M(x)$ makes the left side of the ODE a complete differential

$$\begin{aligned} (M(x)y)'' &= 0 \\ \left(\mathrm{e}^{\frac{3x}{2}} y\right)'' &= 0 \end{aligned}$$

Integrating once gives

$$\left(\mathrm{e}^{\frac{3x}{2}} y\right)' = c_1$$

Integrating again gives

$$\left(\mathrm{e}^{\frac{3x}{2}} y\right) = c_1 x + c_2$$

Hence the solution is

$$y = \frac{c_1 x + c_2}{\mathrm{e}^{\frac{3x}{2}}}$$

Or

$$y = c_1 x\, \mathrm{e}^{-\frac{3x}{2}} + c_2 \mathrm{e}^{-\frac{3x}{2}}$$

Summary

The solution(s) found are the following

$$y = c_1 x\, \mathrm{e}^{-\frac{3x}{2}} + c_2 \mathrm{e}^{-\frac{3x}{2}} \tag{1}$$

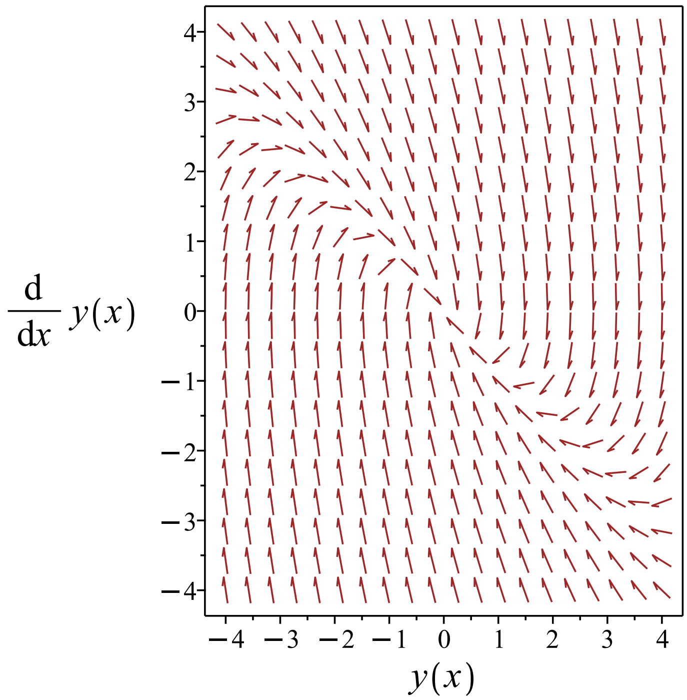


Figure 439: Slope field plot

Verification of solutions

$$y = c_1 x \,\mathrm{e}^{-\frac{3x}{2}} + c_2 \mathrm{e}^{-\frac{3x}{2}}$$

Verified OK.

9.4.3 Solving using Kovacic algorithm

Writing the ode as

$$4y'' + 12y' + 9y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 4 \\ B &= 12 \\ C &= 9 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = y e^{\int \frac{B}{2A} \, dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{0}{1} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= 0 \\ t &= 1 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = 0 \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x) e^{-\int \frac{B}{2A} dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 428: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$
\begin{aligned}
O(\infty) &= \deg(t) - \deg(s) \\
&= 0 - -\infty \\
&= \infty
\end{aligned}
$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is $infinity$ then the necessary conditions for case one are met. Therefore

$$
L = [1]
$$

Since $r = 0$ is not a function of x, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$
z_1(x) = 1
$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$
\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{12}{4}\,dx} \\
&= z_1 e^{-\frac{3x}{2}} \\
&= z_1 \left(\mathrm{e}^{-\frac{3x}{2}}\right)
\end{aligned}
$$

Which simplifies to

$$
y_1 = \mathrm{e}^{-\frac{3x}{2}}
$$

The second solution y_2 to the original ode is found using reduction of order

$$
y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx
$$

Substituting gives

$$
\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{12}{4}\,dx}}{(y_1)^2}\,dx \\
&= y_1 \int \frac{e^{-3x}}{(y_1)^2}\,dx \\
&= y_1(x)
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1\left(\mathrm{e}^{-\frac{3x}{2}}\right) + c_2\left(\mathrm{e}^{-\frac{3x}{2}}(x)\right)
\end{aligned}
$$

Summary

The solution(s) found are the following

$$
y = c_1 \mathrm{e}^{-\frac{3x}{2}} + c_2 x\,\mathrm{e}^{-\frac{3x}{2}} \tag{1}
$$

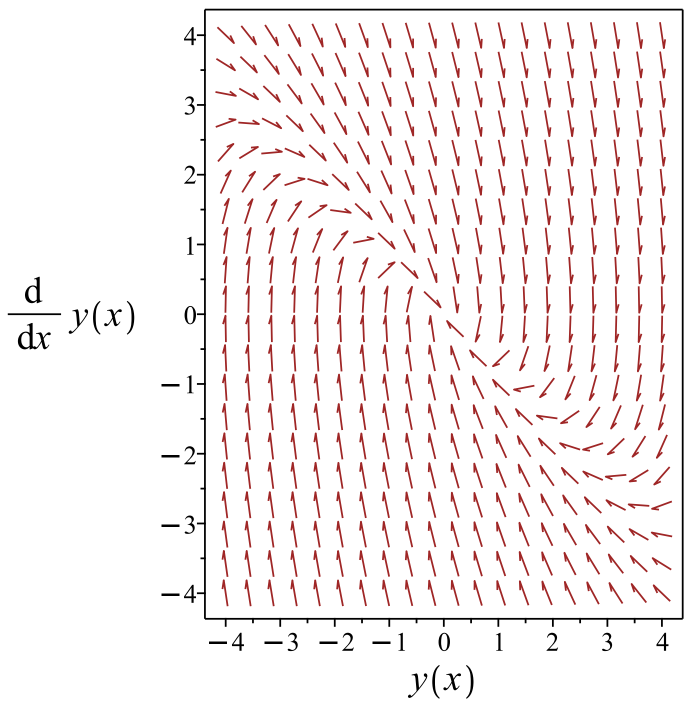


Figure 440: Slope field plot

Verification of solutions

$$
y = c_1 \mathrm{e}^{-\frac{3x}{2}} + c_2 x\,\mathrm{e}^{-\frac{3x}{2}}
$$

Verified OK.

9.4.4 Maple step by step solution

Let's solve

$4y'' + 12y' + 9y = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y'' = -3y' - \frac{9y}{4}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y'' + 3y' + \frac{9y}{4} = 0$

- Characteristic polynomial of ODE

  $r^2 + 3r + \frac{9}{4} = 0$

- Factor the characteristic polynomial

  $\frac{(2r+3)^2}{4} = 0$

- Root of the characteristic polynomial

  $r = -\frac{3}{2}$

- 1st solution of the ODE

  $y_1(x) = \mathrm{e}^{-\frac{3x}{2}}$

- Repeated root, multiply $y_1(x)$ by x to ensure linear independence

  $y_2(x) = x\,\mathrm{e}^{-\frac{3x}{2}}$

- General solution of the ODE

  $y = c_1 y_1(x) + c_2 y_2(x)$

- Substitute in solutions

  $y = c_1 \mathrm{e}^{-\frac{3x}{2}} + c_2 x\,\mathrm{e}^{-\frac{3x}{2}}$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
<- constant coefficients successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 14

```
dsolve(4*diff(y(x),x$2)+12*diff(y(x),x)+9*y(x) = 0,y(x), singsol=all)
```

$$y(x) = \mathrm{e}^{-\frac{3x}{2}}(c_2x + c_1)$$

✓ Solution by Mathematica

Time used: 0.014 (sec). Leaf size: 20

```
DSolve[4*y''[x]+12*y'[x]+9*y[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to e^{-3x/2}(c_2x + c_1)$$

9.5 problem 5

Internal problem ID [657]
Internal file name [`OUTPUT/657_Sunday_June_05_2022_01_46_31_AM_43935394/index.tex`]

Book: Elementary differential equations and boundary value problems, 10th ed., Boyce and DiPrima
Section: Chapter 3, Second order linear equations, 3.4 Repeated roots, reduction of order , page 172
Problem number: 5.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_linear_constant_coeff"**

Maple gives the following as the ode type

```
[[_2nd_order, _missing_x]]
```

$$\boxed{y'' - 2y' + 10y = 0}$$

9.5.1 Solving as second order linear constant coeff ode

This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = 0$$

Where in the above $A = 1, B = -2, C = 10$. Let the solution be $y = e^{\lambda x}$. Substituting this into the ODE gives

$$\lambda^2 \mathrm{e}^{\lambda x} - 2\lambda \, \mathrm{e}^{\lambda x} + 10 \, \mathrm{e}^{\lambda x} = 0 \tag{1}$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda x}$ gives

$$\lambda^2 - 2\lambda + 10 = 0 \tag{2}$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form.Using the quadratic formula

$$\lambda_{1,2} = \frac{-B}{2A} \pm \frac{1}{2A}\sqrt{B^2 - 4AC}$$

Substituting $A = 1, B = -2, C = 10$ into the above gives

$$\begin{aligned}\lambda_{1,2} &= \frac{2}{(2)(1)} \pm \frac{1}{(2)(1)}\sqrt{-2^2 - (4)(1)(10)} \\ &= 1 \pm 3i\end{aligned}$$

Hence

$$\lambda_1 = 1 + 3i$$
$$\lambda_2 = 1 - 3i$$

Which simplifies to

$$\lambda_1 = 1 + 3i$$
$$\lambda_2 = 1 - 3i$$

Since roots are complex conjugate of each others, then let the roots be

$$\lambda_{1,2} = \alpha \pm i\beta$$

Where $\alpha = 1$ and $\beta = 3$. Therefore the final solution, when using Euler relation, can be written as

$$y = e^{\alpha x}(c_1 \cos(\beta x) + c_2 \sin(\beta x))$$

Which becomes

$$y = e^{x}(c_1 \cos(3x) + c_2 \sin(3x))$$

Summary

The solution(s) found are the following

$$y = \mathrm{e}^{x}(c_1 \cos(3x) + c_2 \sin(3x)) \tag{1}$$

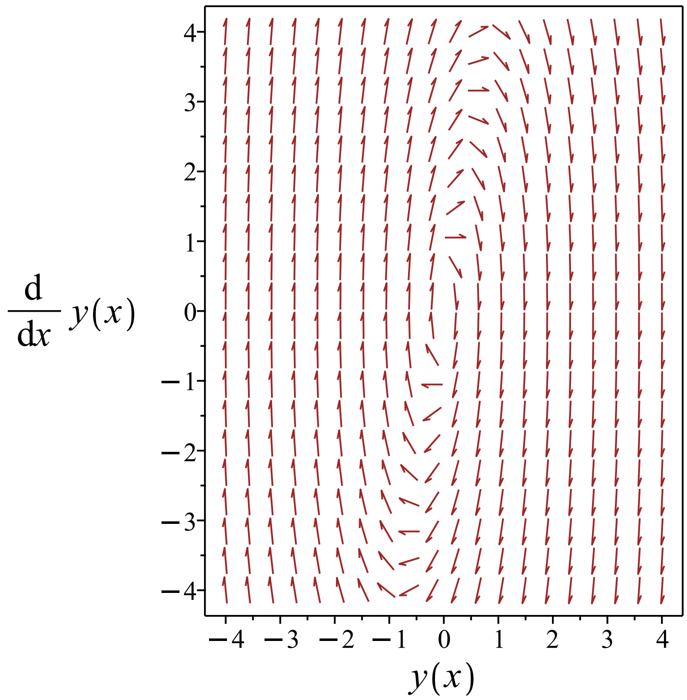


Figure 441: Slope field plot

Verification of solutions

$$y = e^{x}(c_1 \cos(3x) + c_2 \sin(3x))$$

Verified OK.

9.5.2 Solving using Kovacic algorithm

Writing the ode as

$$y'' - 2y' + 10y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1 \\ B &= -2 \\ C &= 10 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\, dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{-9}{1} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= -9 \\ t &= 1 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = -9z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 430: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned}O(\infty) &= \deg(t) - \deg(s) \\ &= 0 - 0 \\ &= 0\end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Since $r = -9$ is not a function of x, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$z_1(x) = \cos(3x)$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$\begin{aligned}y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx} \\ &= z_1 e^{-\int \frac{1}{2}\frac{-2}{1}\,dx} \\ &= z_1 e^{x} \\ &= z_1\left(e^{x}\right)\end{aligned}$$

Which simplifies to

$$y_1 = e^{x}\cos(3x)$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx$$

Substituting gives

$$\begin{aligned}y_2 &= y_1 \int \frac{e^{\int -\frac{-2}{1}\,dx}}{\left(y_1\right)^2}\,dx \\ &= y_1 \int \frac{e^{2x}}{\left(y_1\right)^2}\,dx \\ &= y_1\left(\frac{\tan(3x)}{3}\right)\end{aligned}$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1(e^x \cos(3x)) + c_2\left(e^x \cos(3x)\left(\frac{\tan(3x)}{3}\right)\right)
\end{aligned}
$$

Summary

The solution(s) found are the following

$$
y = c_1 e^x \cos(3x) + \frac{c_2 e^x \sin(3x)}{3} \tag{1}
$$

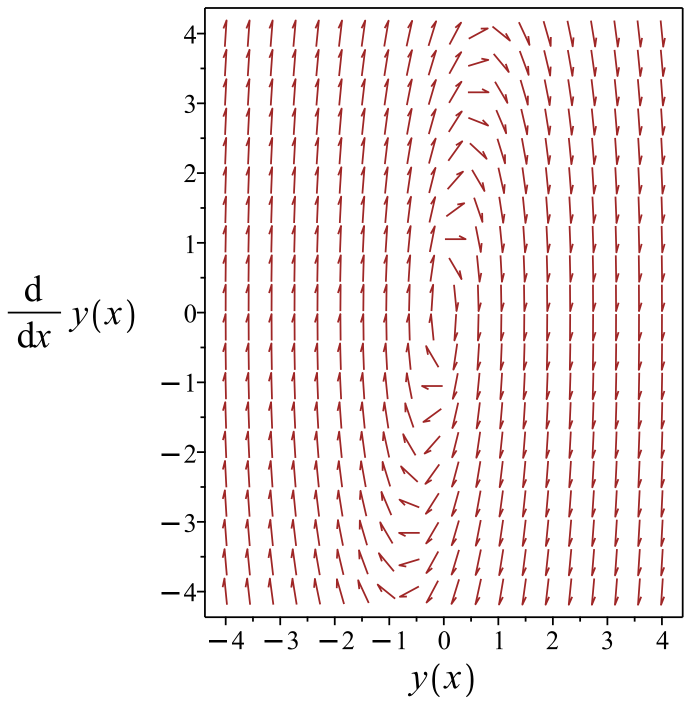


Figure 442: Slope field plot

Verification of solutions

$$
y = c_1 e^x \cos(3x) + \frac{c_2 e^x \sin(3x)}{3}
$$

Verified OK.

9.5.3 Maple step by step solution

Let's solve

$y'' - 2y' + 10y = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Characteristic polynomial of ODE

  $r^2 - 2r + 10 = 0$

- Use quadratic formula to solve for r

  $r = \frac{2 \pm (\sqrt{-36})}{2}$

- Roots of the characteristic polynomial

  $r = (1 - 3\,\mathrm{I}, 1 + 3\,\mathrm{I})$

- 1st solution of the ODE

  $y_1(x) = \mathrm{e}^x \cos(3x)$

- 2nd solution of the ODE

  $y_2(x) = \mathrm{e}^x \sin(3x)$

- General solution of the ODE

  $y = c_1 y_1(x) + c_2 y_2(x)$

- Substitute in solutions

  $y = c_1 \mathrm{e}^x \cos(3x) + c_2 \mathrm{e}^x \sin(3x)$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
<- constant coefficients successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 20

```
dsolve(diff(y(x),x$2)-2*diff(y(x),x)+10*y(x) = 0,y(x), singsol=all)
```

$$y(x) = e^{x}(c_1 \sin(3x) + c_2 \cos(3x))$$

✓ Solution by Mathematica

Time used: 0.015 (sec). Leaf size: 24

```
DSolve[y''[x]-2*y'[x]+10*y[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to e^{x}(c_2 \cos(3x) + c_1 \sin(3x))$$

9.6 problem 6

Internal problem ID [658]
Internal file name [`OUTPUT/658_Sunday_June_05_2022_01_46_32_AM_72033730/index.tex`]

Book: Elementary differential equations and boundary value problems, 10th ed., Boyce and DiPrima
Section: Chapter 3, Second order linear equations, 3.4 Repeated roots, reduction of order , page 172
Problem number: 6.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_linear_constant_coeff", "linear_second_order_ode_solved_by_an_integrating_factor"**

Maple gives the following as the ode type

```
[[_2nd_order, _missing_x]]
```

$$\boxed{y'' - 6y' + 9y = 0}$$

9.6.1 Solving as second order linear constant coeff ode

This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = 0$$

Where in the above $A = 1, B = -6, C = 9$. Let the solution be $y = e^{\lambda x}$. Substituting this into the ODE gives

$$\lambda^2 e^{\lambda x} - 6\lambda\, e^{\lambda x} + 9\, e^{\lambda x} = 0 \tag{1}$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda x}$ gives

$$\lambda^2 - 6\lambda + 9 = 0 \tag{2}$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form.Using the quadratic formula

$$\lambda_{1,2} = \frac{-B}{2A} \pm \frac{1}{2A}\sqrt{B^2 - 4AC}$$

Substituting $A = 1, B = -6, C = 9$ into the above gives

$$\begin{aligned}\lambda_{1,2} &= \frac{6}{(2)(1)} \pm \frac{1}{(2)(1)}\sqrt{(-6)^2 - (4)(1)(9)} \\ &= 3\end{aligned}$$

Hence this is the case of a double root $\lambda_{1,2} = -3$. Therefore the solution is

$$y = c_1 e^{3x} + c_2 x e^{3x} \tag{1}$$

Summary

The solution(s) found are the following

$$y = c_1 e^{3x} + c_2 x e^{3x} \tag{1}$$

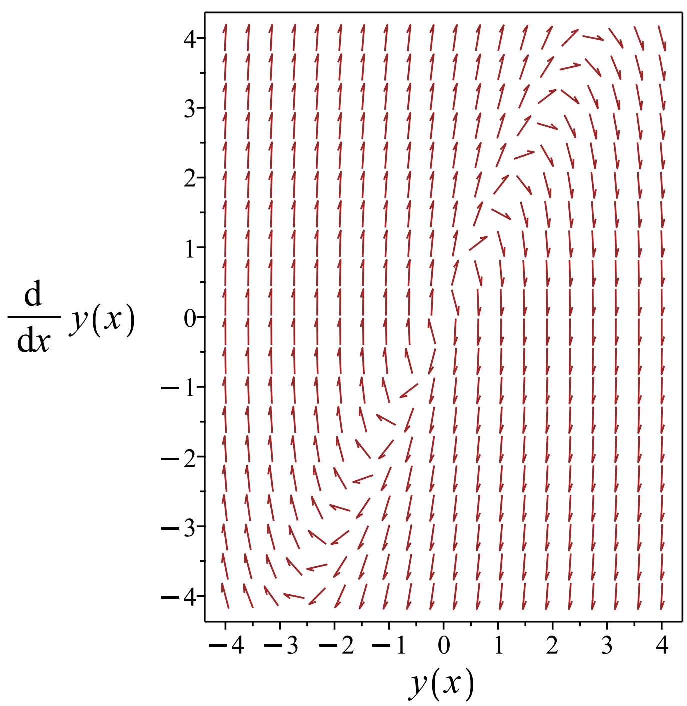


Figure 443: Slope field plot

Verification of solutions

$$y = c_1 e^{3x} + c_2 x e^{3x}$$

Verified OK.

9.6.2 Solving as linear second order ode solved by an integrating factor ode

The ode satisfies this form

$$y'' + p(x)\, y' + \frac{\left(p(x)^{2} + p'(x)\right) y}{2} = f(x)$$

Where $p(x) = -6$. Therefore, there is an integrating factor given by

$$\begin{aligned} M(x) &= e^{\frac{1}{2} \int p\, dx} \\ &= e^{\int -6\, dx} \\ &= \mathrm{e}^{-3x} \end{aligned}$$

Multiplying both sides of the ODE by the integrating factor $M(x)$ makes the left side of the ODE a complete differential

$$\begin{aligned} (M(x) y)'' &= 0 \\ \left(\mathrm{e}^{-3x} y\right)'' &= 0 \end{aligned}$$

Integrating once gives

$$\left(\mathrm{e}^{-3x} y\right)' = c_1$$

Integrating again gives

$$\left(\mathrm{e}^{-3x} y\right) = c_1 x + c_2$$

Hence the solution is

$$y = \frac{c_1 x + c_2}{\mathrm{e}^{-3x}}$$

Or

$$y = c_1 x\, \mathrm{e}^{3x} + c_2 \mathrm{e}^{3x}$$

Summary
The solution(s) found are the following

$$y = c_1 x\, \mathrm{e}^{3x} + c_2 \mathrm{e}^{3x} \tag{1}$$

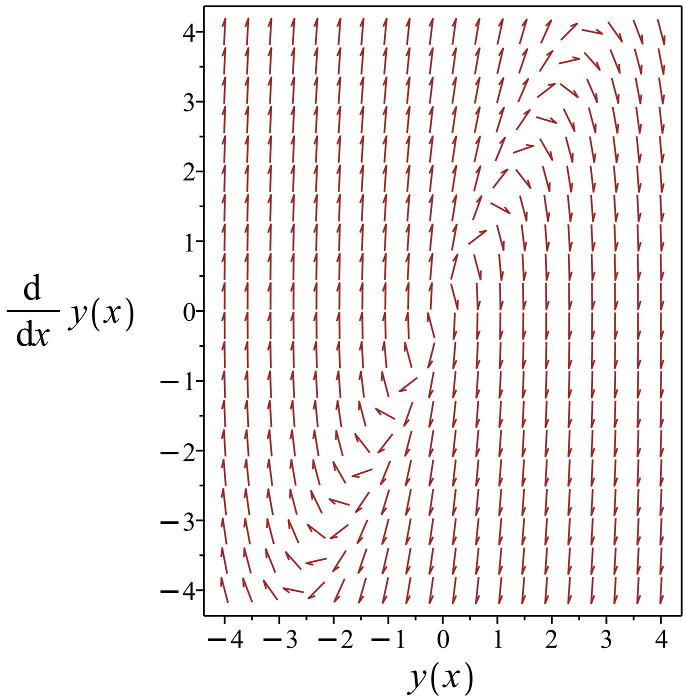


Figure 444: Slope field plot

Verification of solutions

$$y = c_1 x\, e^{3x} + c_2 e^{3x}$$

Verified OK.

9.6.3 Solving using Kovacic algorithm

Writing the ode as

$$y'' - 6y' + 9y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1 \\ B &= -6 \\ C &= 9 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = y e^{\int \frac{B}{2A}\, dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{0}{1} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= 0 \\ t &= 1 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = 0 \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 432: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$
\begin{aligned}
O(\infty) &= \deg(t) - \deg(s) \\
&= 0 - -\infty \\
&= \infty
\end{aligned}
$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is $infinity$ then the necessary conditions for case one are met. Therefore

$$
L = [1]
$$

Since $r = 0$ is not a function of x, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$
z_1(x) = 1
$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$
\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2} \frac{B}{A} \, dx} \\
&= z_1 e^{-\int \frac{1}{2} \frac{-6}{1} \, dx} \\
&= z_1 e^{3x} \\
&= z_1 \left(\mathrm{e}^{3x}\right)
\end{aligned}
$$

Which simplifies to

$$
y_1 = \mathrm{e}^{3x}
$$

The second solution y_2 to the original ode is found using reduction of order

$$
y_2 = y_1 \int \frac{e^{\int -\frac{B}{A} \, dx}}{y_1^2} \, dx
$$

Substituting gives

$$
\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{-6}{1} \, dx}}{(y_1)^2} \, dx \\
&= y_1 \int \frac{e^{6x}}{(y_1)^2} \, dx \\
&= y_1(x)
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1\left(\mathrm{e}^{3x}\right) + c_2\left(\mathrm{e}^{3x}(x)\right)
\end{aligned}
$$

Summary

The solution(s) found are the following

$$
y = c_1 \mathrm{e}^{3x} + c_2 x\,\mathrm{e}^{3x} \tag{1}
$$

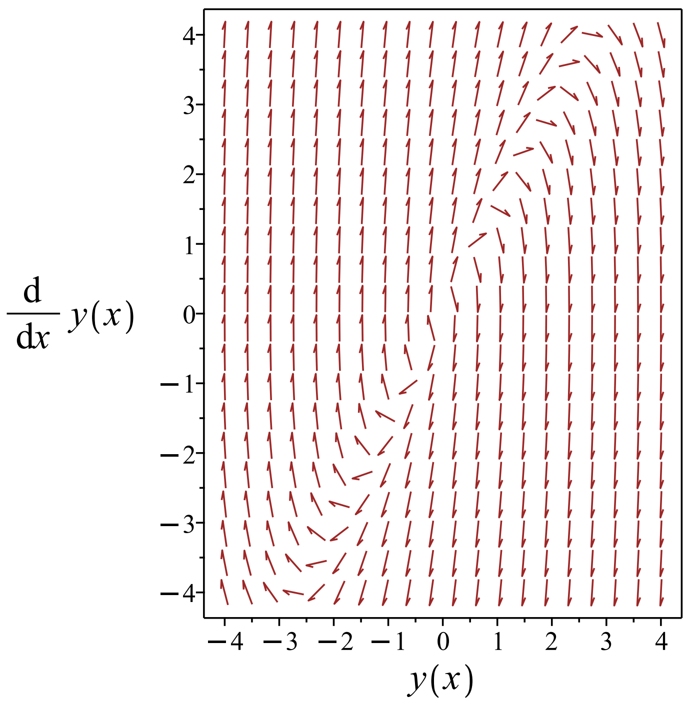


Figure 445: Slope field plot

Verification of solutions

$$
y = c_1 \mathrm{e}^{3x} + c_2 x\,\mathrm{e}^{3x}
$$

Verified OK.

9.6.4 Maple step by step solution

Let's solve

$y'' - 6y' + 9y = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Characteristic polynomial of ODE

  $r^2 - 6r + 9 = 0$

- Factor the characteristic polynomial

  $(r-3)^2 = 0$

- Root of the characteristic polynomial

  $r = 3$

- 1st solution of the ODE

  $y_1(x) = e^{3x}$

- Repeated root, multiply $y_1(x)$ by x to ensure linear independence

  $y_2(x) = x\, e^{3x}$

- General solution of the ODE

  $y = c_1 y_1(x) + c_2 y_2(x)$

- Substitute in solutions

  $y = c_1 e^{3x} + c_2 x\, e^{3x}$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
<- constant coefficients successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 14

```
dsolve(diff(y(x),x$2)-6*diff(y(x),x)+9*y(x) = 0,y(x), singsol=all)
```

$$y(x) = e^{3x}(c_2x + c_1)$$

✓ Solution by Mathematica

Time used: 0.013 (sec). Leaf size: 18

```
DSolve[y''[x]-6*y'[x]+9*y[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to e^{3x}(c_2x + c_1)$$

9.7 problem 7

Internal problem ID [659]
Internal file name [`OUTPUT/659_Sunday_June_05_2022_01_46_32_AM_89046047/index.tex`]

Book: Elementary differential equations and boundary value problems, 10th ed., Boyce and DiPrima
Section: Chapter 3, Second order linear equations, 3.4 Repeated roots, reduction of order , page 172
Problem number: 7.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_linear_constant_coeff"**

Maple gives the following as the ode type

```
[[_2nd_order, _missing_x]]
```

$$\boxed{4y'' + 17y' + 4y = 0}$$

9.7.1 Solving as second order linear constant coeff ode

This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = 0$$

Where in the above $A = 4, B = 17, C = 4$. Let the solution be $y = e^{\lambda x}$. Substituting this into the ODE gives

$$4\lambda^2 e^{\lambda x} + 17\lambda e^{\lambda x} + 4 e^{\lambda x} = 0 \tag{1}$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda x}$ gives

$$4\lambda^2 + 17\lambda + 4 = 0 \tag{2}$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form.Using the quadratic formula

$$\lambda_{1,2} = \frac{-B}{2A} \pm \frac{1}{2A}\sqrt{B^2 - 4AC}$$

Substituting $A = 4, B = 17, C = 4$ into the above gives

$$
\begin{aligned}
\lambda_{1,2} &= \frac{-17}{(2)(4)} \pm \frac{1}{(2)(4)} \sqrt{17^2 - (4)(4)(4)} \\
&= -\frac{17}{8} \pm \frac{15}{8}
\end{aligned}
$$

Hence

$$
\begin{aligned}
\lambda_1 &= -\frac{17}{8} + \frac{15}{8} \\
\lambda_2 &= -\frac{17}{8} - \frac{15}{8}
\end{aligned}
$$

Which simplifies to

$$
\begin{aligned}
\lambda_1 &= -\frac{1}{4} \\
\lambda_2 &= -4
\end{aligned}
$$

Since roots are real and distinct, then the solution is

$$
\begin{aligned}
y &= c_1 e^{\lambda_1 x} + c_2 e^{\lambda_2 x} \\
y &= c_1 e^{\left(-\frac{1}{4}\right)x} + c_2 e^{(-4)x}
\end{aligned}
$$

Or

$$
y = c_1 \mathrm{e}^{-\frac{x}{4}} + c_2 \mathrm{e}^{-4x}
$$

Summary
The solution(s) found are the following

$$
y = c_1 \mathrm{e}^{-\frac{x}{4}} + c_2 \mathrm{e}^{-4x} \tag{1}
$$

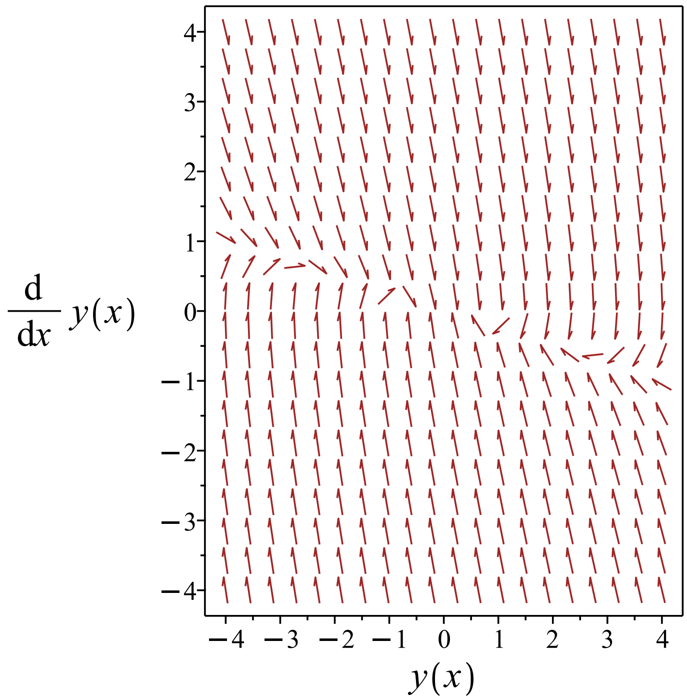


Figure 446: Slope field plot

Verification of solutions

$$y = c_1 e^{-\frac{x}{4}} + c_2 e^{-4x}$$

Verified OK.

9.7.2 Solving using Kovacic algorithm

Writing the ode as

$$4y'' + 17y' + 4y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 4 \\ B &= 17 \\ C &= 4 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{225}{64} \tag{6}$$

Comparing the above to (5) shows that

$$s = 225$$
$$t = 64$$

Therefore eq. (4) becomes

$$z''(x) = \frac{225z(x)}{64} \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 434: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$
\begin{aligned}
O(\infty) &= \deg(t) - \deg(s) \\
&= 0 - 0 \\
&= 0
\end{aligned}
$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Therefore

$$
L = [1]
$$

Since $r = \frac{225}{64}$ is not a function of x, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$
z_1(x) = \mathrm{e}^{-\frac{15x}{8}}
$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$
\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{17}{4}\,dx} \\
&= z_1 e^{-\frac{17x}{8}} \\
&= z_1 \left(\mathrm{e}^{-\frac{17x}{8}}\right)
\end{aligned}
$$

Which simplifies to

$$
y_1 = \mathrm{e}^{-4x}
$$

The second solution y_2 to the original ode is found using reduction of order

$$
y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx
$$

Substituting gives

$$
\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{17}{4}\,dx}}{(y_1)^2}\,dx \\
&= y_1 \int \frac{e^{-\frac{17x}{4}}}{(y_1)^2}\,dx \\
&= y_1 \left(\frac{4\,\mathrm{e}^{\frac{15x}{4}}}{15}\right)
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1 \left(\mathrm{e}^{-4x}\right) + c_2 \left(\mathrm{e}^{-4x} \left(\frac{4\,\mathrm{e}^{\frac{15x}{4}}}{15}\right)\right)
\end{aligned}
$$

Summary
The solution(s) found are the following

$$
y = c_1 \mathrm{e}^{-4x} + \frac{4c_2 \mathrm{e}^{-\frac{x}{4}}}{15} \tag{1}
$$

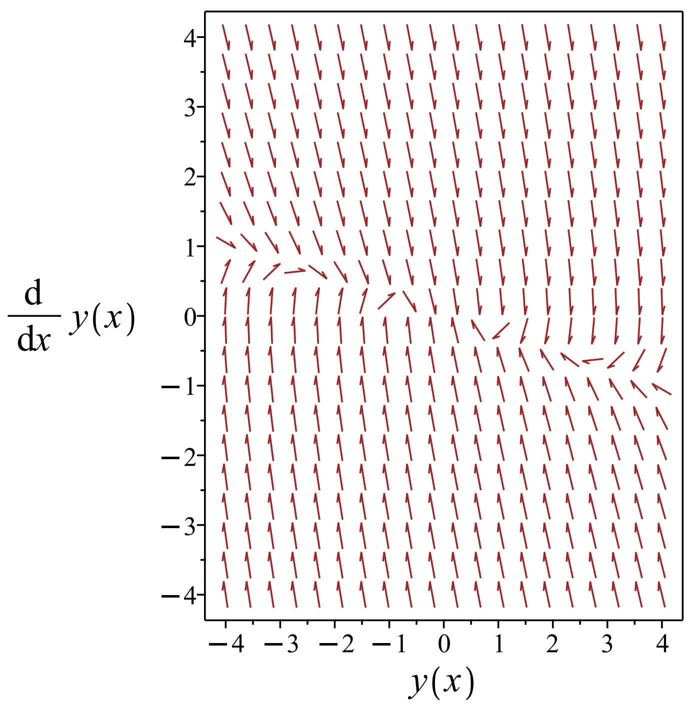


Figure 447: Slope field plot

Verification of solutions

$$
y = c_1 \mathrm{e}^{-4x} + \frac{4c_2 \mathrm{e}^{-\frac{x}{4}}}{15}
$$

Verified OK.

9.7.3 Maple step by step solution

Let's solve

$4y'' + 17y' + 4y = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y'' = -\frac{17y'}{4} - y$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y'' + \frac{17y'}{4} + y = 0$

- Characteristic polynomial of ODE

  $r^2 + \frac{17}{4}r + 1 = 0$

- Factor the characteristic polynomial

  $\frac{(r+4)(4r+1)}{4} = 0$

- Roots of the characteristic polynomial

  $r = \left(-4, -\frac{1}{4}\right)$

- 1st solution of the ODE

  $y_1(x) = \mathrm{e}^{-4x}$

- 2nd solution of the ODE

  $y_2(x) = \mathrm{e}^{-\frac{x}{4}}$

- General solution of the ODE

  $y = c_1 y_1(x) + c_2 y_2(x)$

- Substitute in solutions

  $y = c_1 \mathrm{e}^{-4x} + c_2 \mathrm{e}^{-\frac{x}{4}}$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
<- constant coefficients successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 17

```
dsolve(4*diff(y(x),x$2)+17*diff(y(x),x)+4*y(x) = 0,y(x), singsol=all)
```

$$y(x) = c_1 \mathrm{e}^{-4x} + c_2 \mathrm{e}^{-\frac{x}{4}}$$

✓ Solution by Mathematica

Time used: 0.012 (sec). Leaf size: 24

```
DSolve[4*y''[x]+17*y'[x]+4*y[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to e^{-4x} \left(c_1 e^{15x/4} + c_2\right)$$

9.8 problem 8

Internal problem ID [660]
Internal file name [`OUTPUT/660_Sunday_June_05_2022_01_46_33_AM_61322730/index.tex`]

Book: Elementary differential equations and boundary value problems, 10th ed., Boyce and DiPrima
Section: Chapter 3, Second order linear equations, 3.4 Repeated roots, reduction of order , page 172
Problem number: 8.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_linear_constant_coeff", "linear_second_order_ode_solved_by_an_integrating_factor"**

Maple gives the following as the ode type

```
[[_2nd_order, _missing_x]]
```

$$\boxed{16y'' + 24y' + 9y = 0}$$

9.8.1 Solving as second order linear constant coeff ode

This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = 0$$

Where in the above $A = 16, B = 24, C = 9$. Let the solution be $y = e^{\lambda x}$. Substituting this into the ODE gives

$$16\lambda^2 e^{\lambda x} + 24\lambda\, e^{\lambda x} + 9\, e^{\lambda x} = 0 \tag{1}$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda x}$ gives

$$16\lambda^2 + 24\lambda + 9 = 0 \tag{2}$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form.Using the quadratic formula

$$\lambda_{1,2} = \frac{-B}{2A} \pm \frac{1}{2A}\sqrt{B^2 - 4AC}$$

Substituting $A = 16, B = 24, C = 9$ into the above gives

$$\begin{aligned}\lambda_{1,2} &= \frac{-24}{(2)\,(16)} \pm \frac{1}{(2)\,(16)}\sqrt{(24)^2 - (4)\,(16)\,(9)} \\ &= -\frac{3}{4}\end{aligned}$$

Hence this is the case of a double root $\lambda_{1,2} = \frac{3}{4}$. Therefore the solution is

$$y = c_1 \mathrm{e}^{-\frac{3x}{4}} + c_2 x\, \mathrm{e}^{-\frac{3x}{4}} \tag{1}$$

Summary

The solution(s) found are the following

$$y = c_1 \mathrm{e}^{-\frac{3x}{4}} + c_2 x\, \mathrm{e}^{-\frac{3x}{4}} \tag{1}$$

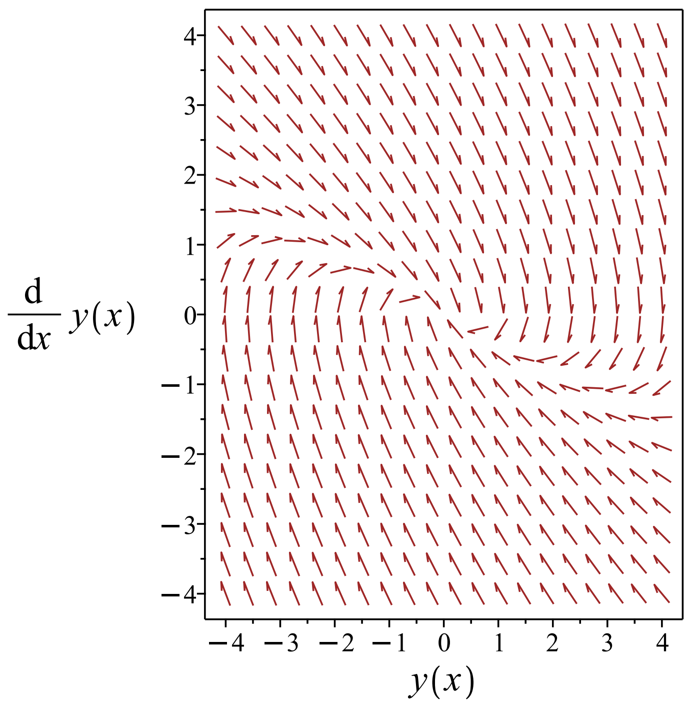


Figure 448: Slope field plot

Verification of solutions

$$y = c_1 \mathrm{e}^{-\frac{3x}{4}} + c_2 x\, \mathrm{e}^{-\frac{3x}{4}}$$

Verified OK.

9.8.2 Solving as linear second order ode solved by an integrating factor ode

The ode satisfies this form

$$y'' + p(x)\, y' + \frac{\left(p(x)^2 + p'(x)\right) y}{2} = f(x)$$

Where $p(x) = \frac{3}{2}$. Therefore, there is an integrating factor given by

$$\begin{aligned} M(x) &= e^{\frac{1}{2}\int p\, dx} \\ &= e^{\int \frac{3}{2}\, dx} \\ &= \mathrm{e}^{\frac{3x}{4}} \end{aligned}$$

Multiplying both sides of the ODE by the integrating factor $M(x)$ makes the left side of the ODE a complete differential

$$\begin{aligned} (M(x)y)'' &= 0 \\ \left(\mathrm{e}^{\frac{3x}{4}} y\right)'' &= 0 \end{aligned}$$

Integrating once gives

$$\left(\mathrm{e}^{\frac{3x}{4}} y\right)' = c_1$$

Integrating again gives

$$\left(\mathrm{e}^{\frac{3x}{4}} y\right) = c_1 x + c_2$$

Hence the solution is

$$y = \frac{c_1 x + c_2}{\mathrm{e}^{\frac{3x}{4}}}$$

Or

$$y = c_1 x\, \mathrm{e}^{-\frac{3x}{4}} + \mathrm{e}^{-\frac{3x}{4}} c_2$$

Summary

The solution(s) found are the following

$$y = c_1 x\, \mathrm{e}^{-\frac{3x}{4}} + \mathrm{e}^{-\frac{3x}{4}} c_2 \tag{1}$$

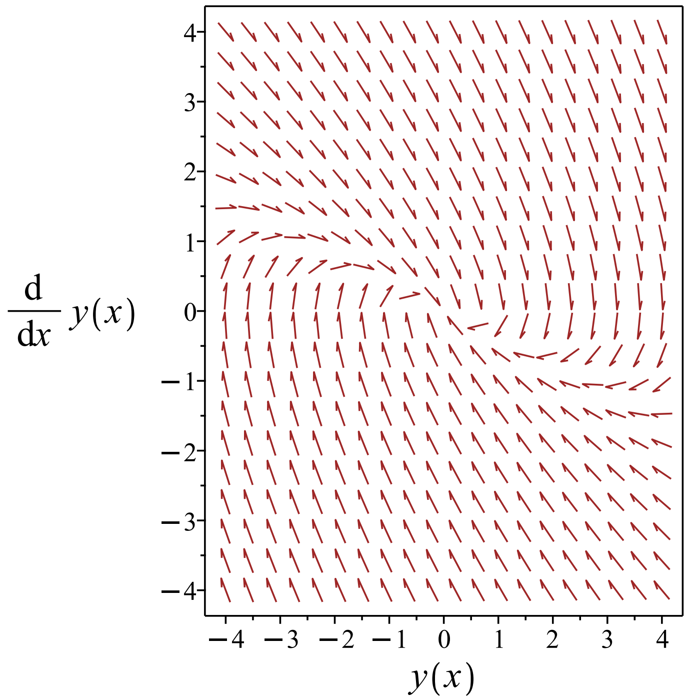


Figure 449: Slope field plot

Verification of solutions

$$y = c_1 x\,\mathrm{e}^{-\frac{3x}{4}} + \mathrm{e}^{-\frac{3x}{4}} c_2$$

Verified OK.

9.8.3 Solving using Kovacic algorithm

Writing the ode as

$$16y'' + 24y' + 9y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 16 \\ B &= 24 \\ C &= 9 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = y e^{\int \frac{B}{2A}\, dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{0}{1} \tag{6}$$

Comparing the above to (5) shows that

$$s = 0$$
$$t = 1$$

Therefore eq. (4) becomes

$$z''(x) = 0 \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 436: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 0 - -\infty \\ &= \infty \end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is $infinity$ then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Since $r = 0$ is not a function of x, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$z_1(x) = 1$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$\begin{aligned} y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx} \\ &= z_1 e^{-\int \frac{1}{2}\frac{24}{16}\,dx} \\ &= z_1 e^{-\frac{3x}{4}} \\ &= z_1\left(\mathrm{e}^{-\frac{3x}{4}}\right) \end{aligned}$$

Which simplifies to

$$y_1 = \mathrm{e}^{-\frac{3x}{4}}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx$$

Substituting gives

$$\begin{aligned} y_2 &= y_1 \int \frac{e^{\int -\frac{24}{16}\,dx}}{(y_1)^2}\,dx \\ &= y_1 \int \frac{e^{-\frac{3x}{2}}}{(y_1)^2}\,dx \\ &= y_1(x) \end{aligned}$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1\left(\mathrm{e}^{-\frac{3x}{4}}\right) + c_2\left(\mathrm{e}^{-\frac{3x}{4}}(x)\right)
\end{aligned}
$$

Summary
The solution(s) found are the following

$$
y = c_1\mathrm{e}^{-\frac{3x}{4}} + c_2 x\,\mathrm{e}^{-\frac{3x}{4}} \tag{1}
$$

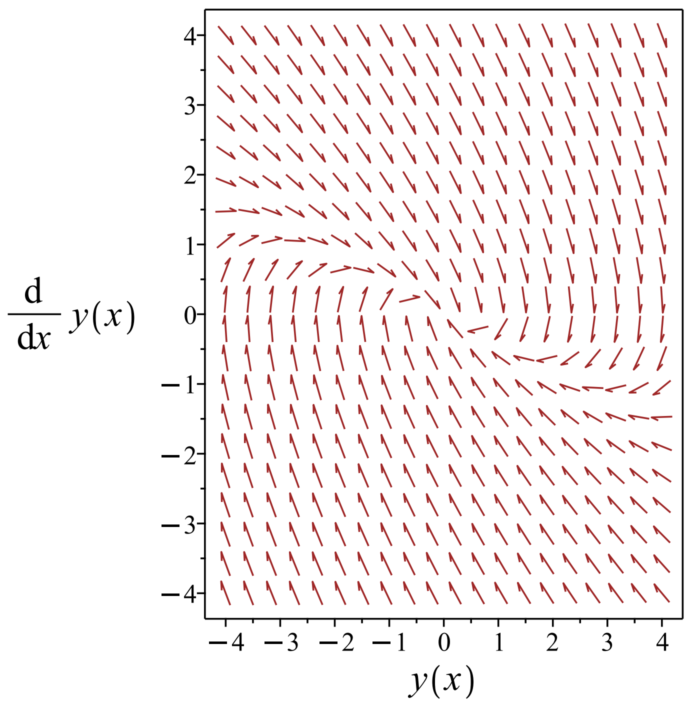


Figure 450: Slope field plot

Verification of solutions

$$
y = c_1\mathrm{e}^{-\frac{3x}{4}} + c_2 x\,\mathrm{e}^{-\frac{3x}{4}}
$$

Verified OK.

9.8.4 Maple step by step solution

Let's solve

$16y'' + 24y' + 9y = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y'' = -\frac{3y'}{2} - \frac{9y}{16}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y'' + \frac{3y'}{2} + \frac{9y}{16} = 0$

- Characteristic polynomial of ODE

  $r^2 + \frac{3}{2}r + \frac{9}{16} = 0$

- Factor the characteristic polynomial

  $\frac{(4r+3)^2}{16} = 0$

- Root of the characteristic polynomial

  $r = -\frac{3}{4}$

- 1st solution of the ODE

  $y_1(x) = \mathrm{e}^{-\frac{3x}{4}}$

- Repeated root, multiply $y_1(x)$ by x to ensure linear independence

  $y_2(x) = x\,\mathrm{e}^{-\frac{3x}{4}}$

- General solution of the ODE

  $y = c_1y_1(x) + c_2y_2(x)$

- Substitute in solutions

  $y = c_1\mathrm{e}^{-\frac{3x}{4}} + c_2x\,\mathrm{e}^{-\frac{3x}{4}}$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
<- constant coefficients successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 14

```
dsolve(16*diff(y(x),x$2)+24*diff(y(x),x)+9*y(x) = 0,y(x), singsol=all)
```

$$y(x) = \mathrm{e}^{-\frac{3x}{4}}(c_2x + c_1)$$

✓ Solution by Mathematica

Time used: 0.014 (sec). Leaf size: 20

```
DSolve[16*y''[x]+24*y'[x]+9*y[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to e^{-3x/4}(c_2x + c_1)$$

9.9 problem 9

Internal problem ID [661]
Internal file name [`OUTPUT/661_Sunday_June_05_2022_01_46_34_AM_88618893/index.tex`]

Book: Elementary differential equations and boundary value problems, 10th ed., Boyce and DiPrima
Section: Chapter 3, Second order linear equations, 3.4 Repeated roots, reduction of order , page 172
Problem number: 9.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_linear_constant_coeff", "linear_second_order_ode_solved_by_an_integrating_factor"**

Maple gives the following as the ode type

```
[[_2nd_order, _missing_x]]
```

$$\boxed{25y'' - 20y' + 4y = 0}$$

9.9.1 Solving as second order linear constant coeff ode

This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = 0$$

Where in the above $A = 25, B = -20, C = 4$. Let the solution be $y = e^{\lambda x}$. Substituting this into the ODE gives

$$25\lambda^2 e^{\lambda x} - 20\lambda e^{\lambda x} + 4 e^{\lambda x} = 0 \tag{1}$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda x}$ gives

$$25\lambda^2 - 20\lambda + 4 = 0 \tag{2}$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form.Using the quadratic formula

$$\lambda_{1,2} = \frac{-B}{2A} \pm \frac{1}{2A}\sqrt{B^2 - 4AC}$$

Substituting $A = 25, B = -20, C = 4$ into the above gives

$$\begin{aligned}\lambda_{1,2} &= \frac{20}{(2)(25)} \pm \frac{1}{(2)(25)}\sqrt{(-20)^2 - (4)(25)(4)}\\ &= \frac{2}{5}\end{aligned}$$

Hence this is the case of a double root $\lambda_{1,2} = -\frac{2}{5}$. Therefore the solution is

$$y = c_1 e^{\frac{2x}{5}} + c_2 x\, e^{\frac{2x}{5}} \tag{1}$$

Summary

The solution(s) found are the following

$$y = c_1 e^{\frac{2x}{5}} + c_2 x\, e^{\frac{2x}{5}} \tag{1}$$

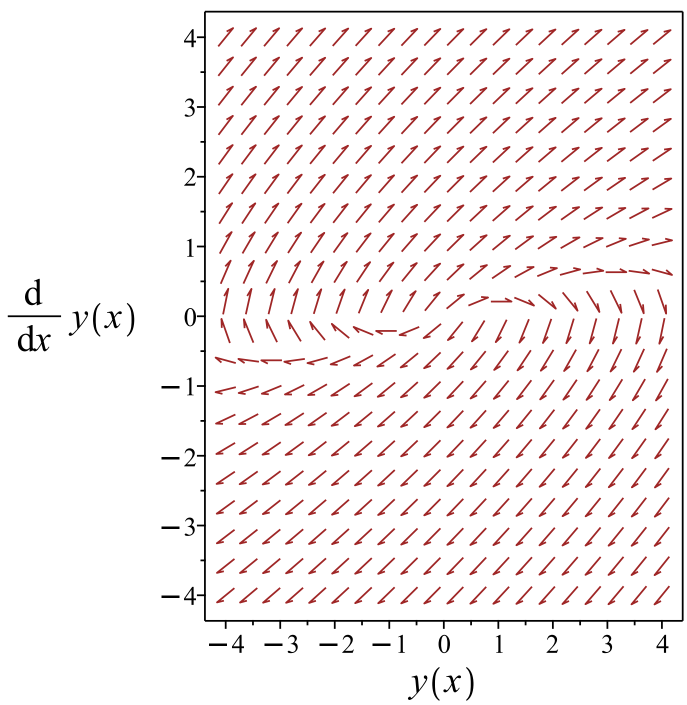


Figure 451: Slope field plot

Verification of solutions

$$y = c_1 e^{\frac{2x}{5}} + c_2 x\, e^{\frac{2x}{5}}$$

Verified OK.

9.9.2 Solving as linear second order ode solved by an integrating factor ode

The ode satisfies this form

$$y'' + p(x)\,y' + \frac{\left(p(x)^2 + p'(x)\right)y}{2} = f(x)$$

Where $p(x) = -\frac{4}{5}$. Therefore, there is an integrating factor given by

$$\begin{aligned} M(x) &= e^{\frac{1}{2}\int p\,dx} \\ &= e^{\int -\frac{4}{5}\,dx} \\ &= \mathrm{e}^{-\frac{2x}{5}} \end{aligned}$$

Multiplying both sides of the ODE by the integrating factor $M(x)$ makes the left side of the ODE a complete differential

$$\begin{aligned} (M(x)y)'' &= 0 \\ \left(\mathrm{e}^{-\frac{2x}{5}} y\right)'' &= 0 \end{aligned}$$

Integrating once gives

$$\left(\mathrm{e}^{-\frac{2x}{5}} y\right)' = c_1$$

Integrating again gives

$$\left(\mathrm{e}^{-\frac{2x}{5}} y\right) = c_1 x + c_2$$

Hence the solution is

$$y = \frac{c_1 x + c_2}{\mathrm{e}^{-\frac{2x}{5}}}$$

Or

$$y = c_1 x\,\mathrm{e}^{\frac{2x}{5}} + \mathrm{e}^{\frac{2x}{5}} c_2$$

Summary

The solution(s) found are the following

$$y = c_1 x\,\mathrm{e}^{\frac{2x}{5}} + \mathrm{e}^{\frac{2x}{5}} c_2 \tag{1}$$

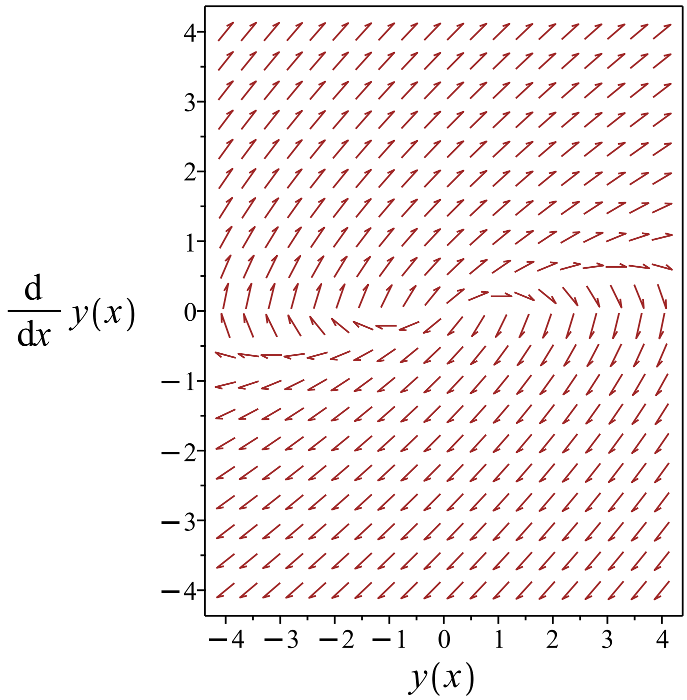


Figure 452: Slope field plot

Verification of solutions

$$y = c_1 x\,e^{\frac{2x}{5}} + e^{\frac{2x}{5}} c_2$$

Verified OK.

9.9.3 Solving using Kovacic algorithm

Writing the ode as

$$25y'' - 20y' + 4y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 25 \\ B &= -20 \\ C &= 4 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = y e^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{0}{1} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= 0 \\ t &= 1 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = 0 \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 438: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$
\begin{aligned}
O(\infty) &= \deg(t) - \deg(s) \\
&= 0 - -\infty \\
&= \infty
\end{aligned}
$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is $infinity$ then the necessary conditions for case one are met. Therefore

$$
L = [1]
$$

Since $r = 0$ is not a function of x, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$
z_1(x) = 1
$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$
\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{-20}{25}\,dx} \\
&= z_1 e^{\frac{2x}{5}} \\
&= z_1 \left(\mathrm{e}^{\frac{2x}{5}}\right)
\end{aligned}
$$

Which simplifies to

$$
y_1 = \mathrm{e}^{\frac{2x}{5}}
$$

The second solution y_2 to the original ode is found using reduction of order

$$
y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx
$$

Substituting gives

$$
\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{-20}{25}\,dx}}{(y_1)^2}\,dx \\
&= y_1 \int \frac{e^{\frac{4x}{5}}}{(y_1)^2}\,dx \\
&= y_1(x)
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1\left(\mathrm{e}^{\frac{2x}{5}}\right) + c_2\left(\mathrm{e}^{\frac{2x}{5}}(x)\right)
\end{aligned}
$$

Summary

The solution(s) found are the following

$$
y = c_1 \mathrm{e}^{\frac{2x}{5}} + c_2 x \,\mathrm{e}^{\frac{2x}{5}} \tag{1}
$$

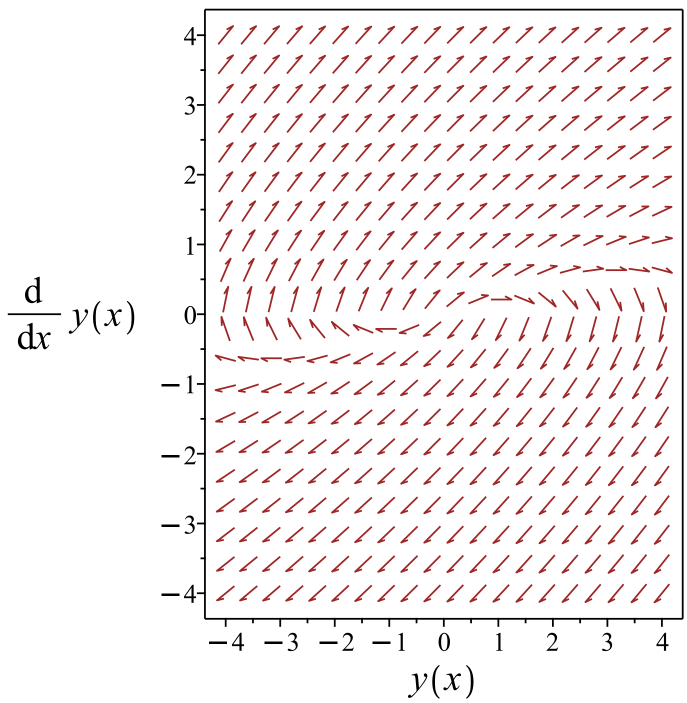


Figure 453: Slope field plot

Verification of solutions

$$
y = c_1 \mathrm{e}^{\frac{2x}{5}} + c_2 x \,\mathrm{e}^{\frac{2x}{5}}
$$

Verified OK.

9.9.4 Maple step by step solution

Let's solve

$25y'' - 20y' + 4y = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y'' = \frac{4y'}{5} - \frac{4y}{25}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y'' - \frac{4y'}{5} + \frac{4y}{25} = 0$

- Characteristic polynomial of ODE

  $r^2 - \frac{4}{5}r + \frac{4}{25} = 0$

- Factor the characteristic polynomial

  $\frac{(5r-2)^2}{25} = 0$

- Root of the characteristic polynomial

  $r = \frac{2}{5}$

- 1st solution of the ODE

  $y_1(x) = e^{\frac{2x}{5}}$

- Repeated root, multiply $y_1(x)$ by x to ensure linear independence

  $y_2(x) = x\,e^{\frac{2x}{5}}$

- General solution of the ODE

  $y = c_1 y_1(x) + c_2 y_2(x)$

- Substitute in solutions

  $y = c_1 e^{\frac{2x}{5}} + c_2 x\,e^{\frac{2x}{5}}$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
<- constant coefficients successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 14

```
dsolve(25*diff(y(x),x$2)-20*diff(y(x),x)+4*y(x) = 0,y(x), singsol=all)
```

$$y(x) = e^{\frac{2x}{5}}(c_2x + c_1)$$

✓ Solution by Mathematica

Time used: 0.015 (sec). Leaf size: 20

```
DSolve[25*y''[x]-20*y'[x]+4*y[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to e^{2x/5}(c_2x + c_1)$$

9.10 problem 10

Internal problem ID [662]
Internal file name [`OUTPUT/662_Sunday_June_05_2022_01_46_34_AM_57523948/index.tex`]

Book: Elementary differential equations and boundary value problems, 10th ed., Boyce and DiPrima
Section: Chapter 3, Second order linear equations, 3.4 Repeated roots, reduction of order , page 172
Problem number: 10.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_linear_constant_coeff"**

Maple gives the following as the ode type

```
[[_2nd_order, _missing_x]]
```

$$\boxed{2y'' + 2y' + y = 0}$$

9.10.1 Solving as second order linear constant coeff ode

This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = 0$$

Where in the above $A = 2, B = 2, C = 1$. Let the solution be $y = e^{\lambda x}$. Substituting this into the ODE gives

$$2\lambda^2 e^{\lambda x} + 2\lambda\, e^{\lambda x} + e^{\lambda x} = 0 \tag{1}$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda x}$ gives

$$2\lambda^2 + 2\lambda + 1 = 0 \tag{2}$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form.Using the quadratic formula

$$\lambda_{1,2} = \frac{-B}{2A} \pm \frac{1}{2A}\sqrt{B^2 - 4AC}$$

Substituting $A = 2, B = 2, C = 1$ into the above gives

$$\begin{aligned}\lambda_{1,2} &= \frac{-2}{(2)(2)} \pm \frac{1}{(2)(2)}\sqrt{2^2 - (4)(2)(1)} \\ &= -\frac{1}{2} \pm \frac{i}{2}\end{aligned}$$

Hence

$$\begin{aligned}\lambda_1 &= -\frac{1}{2} + \frac{i}{2} \\ \lambda_2 &= -\frac{1}{2} - \frac{i}{2}\end{aligned}$$

Which simplifies to

$$\begin{aligned}\lambda_1 &= -\frac{1}{2} + \frac{i}{2} \\ \lambda_2 &= -\frac{1}{2} - \frac{i}{2}\end{aligned}$$

Since roots are complex conjugate of each others, then let the roots be

$$\lambda_{1,2} = \alpha \pm i\beta$$

Where $\alpha = -\frac{1}{2}$ and $\beta = \frac{1}{2}$. Therefore the final solution, when using Euler relation, can be written as

$$y = e^{\alpha x}(c_1 \cos(\beta x) + c_2 \sin(\beta x))$$

Which becomes

$$y = e^{-\frac{x}{2}}\left(c_1 \cos\left(\frac{x}{2}\right) + c_2 \sin\left(\frac{x}{2}\right)\right)$$

Summary

The solution(s) found are the following

$$y = \mathrm{e}^{-\frac{x}{2}}\left(c_1 \cos\left(\frac{x}{2}\right) + c_2 \sin\left(\frac{x}{2}\right)\right) \tag{1}$$

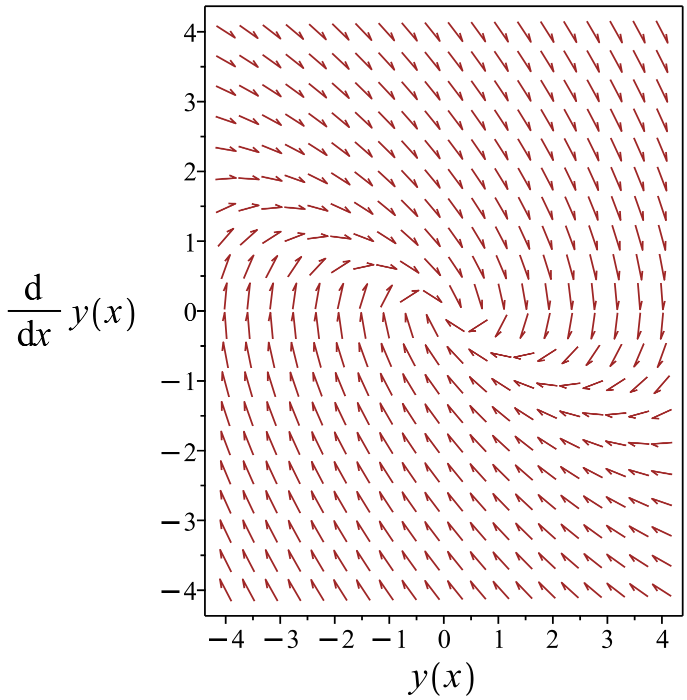


Figure 454: Slope field plot

Verification of solutions

$$y = e^{-\frac{x}{2}}\left(c_1 \cos\left(\frac{x}{2}\right) + c_2 \sin\left(\frac{x}{2}\right)\right)$$

Verified OK.

9.10.2 Solving using Kovacic algorithm

Writing the ode as

$$2y'' + 2y' + y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 2 \\ B &= 2 \\ C &= 1 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{-1}{4} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= -1 \\ t &= 4 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = -\frac{z(x)}{4} \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 440: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 0 - 0 \\ &= 0 \end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Since $r = -\frac{1}{4}$ is not a function of x, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$z_1(x) = \cos\left(\frac{x}{2}\right)$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$\begin{aligned} y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx} \\ &= z_1 e^{-\int \frac{1}{2}\frac{2}{2}\,dx} \\ &= z_1 e^{-\frac{x}{2}} \\ &= z_1\left(\mathrm{e}^{-\frac{x}{2}}\right) \end{aligned}$$

Which simplifies to

$$y_1 = \mathrm{e}^{-\frac{x}{2}} \cos\left(\frac{x}{2}\right)$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx$$

Substituting gives

$$\begin{aligned} y_2 &= y_1 \int \frac{e^{\int -\frac{2}{2}\,dx}}{(y_1)^2}\,dx \\ &= y_1 \int \frac{e^{-x}}{(y_1)^2}\,dx \\ &= y_1\left(2\tan\left(\frac{x}{2}\right)\right) \end{aligned}$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1 \left(\mathrm{e}^{-\frac{x}{2}} \cos\left(\frac{x}{2}\right)\right) + c_2 \left(\mathrm{e}^{-\frac{x}{2}} \cos\left(\frac{x}{2}\right) \left(2 \tan\left(\frac{x}{2}\right)\right)\right)
\end{aligned}
$$

Summary
The solution(s) found are the following

$$
y = c_1 \mathrm{e}^{-\frac{x}{2}} \cos\left(\frac{x}{2}\right) + 2 c_2 \mathrm{e}^{-\frac{x}{2}} \sin\left(\frac{x}{2}\right) \tag{1}
$$

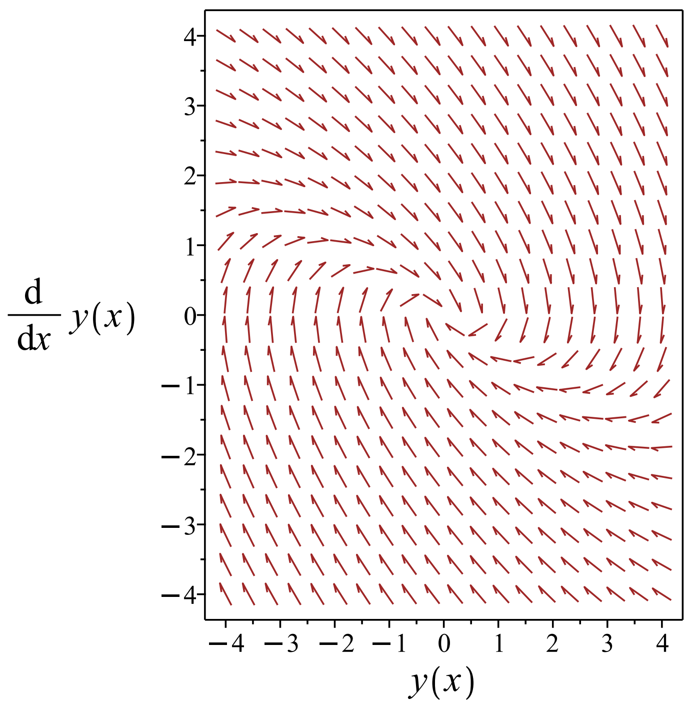


Figure 455: Slope field plot

Verification of solutions

$$
y = c_1 \mathrm{e}^{-\frac{x}{2}} \cos\left(\frac{x}{2}\right) + 2 c_2 \mathrm{e}^{-\frac{x}{2}} \sin\left(\frac{x}{2}\right)
$$

Verified OK.

9.10.3 Maple step by step solution

Let's solve

$2y'' + 2y' + y = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y'' = -y' - \frac{y}{2}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y'' + y' + \frac{y}{2} = 0$

- Characteristic polynomial of ODE

  $r^2 + r + \frac{1}{2} = 0$

- Use quadratic formula to solve for r

  $r = \frac{(-1)\pm(\sqrt{-1})}{2}$

- Roots of the characteristic polynomial

  $r = \left(-\frac{1}{2} - \frac{\mathrm{I}}{2}, -\frac{1}{2} + \frac{\mathrm{I}}{2}\right)$

- 1st solution of the ODE

  $y_1(x) = \mathrm{e}^{-\frac{x}{2}} \cos\left(\frac{x}{2}\right)$

- 2nd solution of the ODE

  $y_2(x) = \mathrm{e}^{-\frac{x}{2}} \sin\left(\frac{x}{2}\right)$

- General solution of the ODE

  $y = c_1 y_1(x) + c_2 y_2(x)$

- Substitute in solutions

  $y = c_1 \mathrm{e}^{-\frac{x}{2}} \cos\left(\frac{x}{2}\right) + c_2 \mathrm{e}^{-\frac{x}{2}} \sin\left(\frac{x}{2}\right)$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
<- constant coefficients successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 22

```
dsolve(2*diff(y(x),x$2)+2*diff(y(x),x)+y(x) = 0,y(x), singsol=all)
```

$$y(x) = e^{-\frac{x}{2}}\left(c_1 \sin\left(\frac{x}{2}\right) + c_2 \cos\left(\frac{x}{2}\right)\right)$$

✓ Solution by Mathematica

Time used: 0.017 (sec). Leaf size: 32

```
DSolve[2*y''[x]+2*y'[x]+y[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to e^{-x/2}\left(c_2 \cos\left(\frac{x}{2}\right) + c_1 \sin\left(\frac{x}{2}\right)\right)$$

9.11 problem 11

Internal problem ID [663]
Internal file name [`OUTPUT/663_Sunday_June_05_2022_01_46_35_AM_83642007/index.tex`]

Book: Elementary differential equations and boundary value problems, 10th ed., Boyce and DiPrima
Section: Chapter 3, Second order linear equations, 3.4 Repeated roots, reduction of order , page 172
Problem number: 11.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_linear_constant_coeff", "linear_second_order_ode_solved_by_an_integrating_factor"**

Maple gives the following as the ode type

```
[[_2nd_order, _missing_x]]
```

$$\boxed{9y'' - 12y' + 4y = 0}$$

With initial conditions

$$[y(0) = 2, y'(0) = -1]$$

9.11.1 Existence and uniqueness analysis

This is a linear ODE. In canonical form it is written as

$$y'' + p(t)y' + q(t)y = F$$

Where here

$$\begin{aligned} p(t) &= -\frac{4}{3} \\ q(t) &= \frac{4}{9} \\ F &= 0 \end{aligned}$$

Hence the ode is

$$y'' - \frac{4y'}{3} + \frac{4y}{9} = 0$$

The domain of $p(t) = -\frac{4}{3}$ is

$$\{-\infty < t < \infty\}$$

And the point $t_0 = 0$ is inside this domain. The domain of $q(t) = \frac{4}{9}$ is

$$\{-\infty < t < \infty\}$$

And the point $t_0 = 0$ is also inside this domain. Hence solution exists and is unique.

9.11.2 Solving as second order linear constant coeff ode

This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$Ay''(t) + By'(t) + Cy(t) = 0$$

Where in the above $A = 9, B = -12, C = 4$. Let the solution be $y = e^{\lambda t}$. Substituting this into the ODE gives

$$9\lambda^2 e^{\lambda t} - 12\lambda\, e^{\lambda t} + 4\, e^{\lambda t} = 0 \tag{1}$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda t}$ gives

$$9\lambda^2 - 12\lambda + 4 = 0 \tag{2}$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form.Using the quadratic formula

$$\lambda_{1,2} = \frac{-B}{2A} \pm \frac{1}{2A}\sqrt{B^2 - 4AC}$$

Substituting $A = 9, B = -12, C = 4$ into the above gives

$$\lambda_{1,2} = \frac{12}{(2)(9)} \pm \frac{1}{(2)(9)}\sqrt{(-12)^2 - (4)(9)(4)}$$
$$= \frac{2}{3}$$

Hence this is the case of a double root $\lambda_{1,2} = -\frac{2}{3}$. Therefore the solution is

$$y = c_1 e^{\frac{2t}{3}} + c_2 e^{\frac{2t}{3}} t \tag{1}$$

Initial conditions are used to solve for the constants of integration.

Looking at the above solution

$$y = c_1 e^{\frac{2t}{3}} + c_2 t\, e^{\frac{2t}{3}} \tag{1}$$

Initial conditions are now substituted in the above solution. This will generate the required equations to solve for the integration constants. substituting $y = 2$ and $t = 0$ in the above gives

$$2 = c_1 \tag{1A}$$

Taking derivative of the solution gives

$$y' = \frac{2c_1 e^{\frac{2t}{3}}}{3} + c_2 e^{\frac{2t}{3}} + \frac{2c_2 t\, e^{\frac{2t}{3}}}{3}$$

substituting $y' = -1$ and $t = 0$ in the above gives

$$-1 = \frac{2c_1}{3} + c_2 \tag{2A}$$

Equations {1A,2A} are now solved for $\{c_1, c_2\}$. Solving for the constants gives

$$c_1 = 2$$
$$c_2 = -\frac{7}{3}$$

Substituting these values back in above solution results in

$$y = -\frac{7\, e^{\frac{2t}{3}} t}{3} + 2\, e^{\frac{2t}{3}}$$

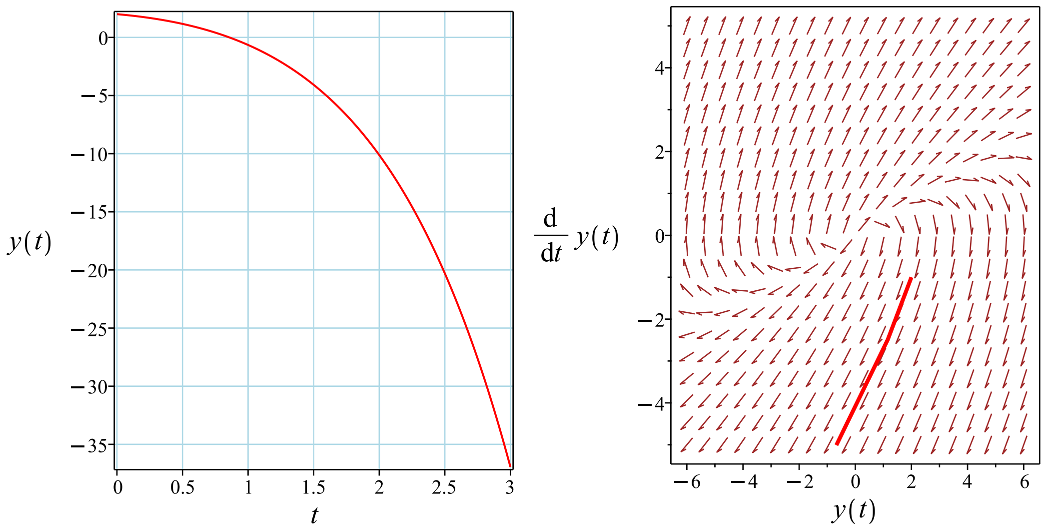
Which simplifies to

$$y = e^{\frac{2t}{3}} \left(2 - \frac{7t}{3}\right)$$

Summary

The solution(s) found are the following

$$y = e^{\frac{2t}{3}} \left(2 - \frac{7t}{3}\right) \tag{1}$$



(a) Solution plot

(b) Slope field plot

Verification of solutions

$$y = e^{\frac{2t}{3}} \left(2 - \frac{7t}{3}\right)$$

Verified OK.

9.11.3 Solving as linear second order ode solved by an integrating factor ode

The ode satisfies this form

$$y'' + p(t)\, y' + \frac{\left(p(t)^2 + p'(t)\right) y}{2} = f(t)$$

Where $p(t) = -\frac{4}{3}$. Therefore, there is an integrating factor given by

$$\begin{aligned} M(x) &= e^{\frac{1}{2} \int p\, dx} \\ &= e^{\int -\frac{4}{3}\, dx} \\ &= e^{-\frac{2t}{3}} \end{aligned}$$

Multiplying both sides of the ODE by the integrating factor $M(x)$ makes the left side of the ODE a complete differential

$$
\begin{aligned}
(M(x)y)'' &= 0 \\
\left(e^{-\frac{2t}{3}} y\right)'' &= 0
\end{aligned}
$$

Integrating once gives

$$
\left(e^{-\frac{2t}{3}} y\right)' = c_1
$$

Integrating again gives

$$
\left(e^{-\frac{2t}{3}} y\right) = c_1 t + c_2
$$

Hence the solution is

$$
y = \frac{c_1 t + c_2}{e^{-\frac{2t}{3}}}
$$

Or

$$
y = c_1 t\, e^{\frac{2t}{3}} + c_2 e^{\frac{2t}{3}}
$$

Initial conditions are used to solve for the constants of integration.

Looking at the above solution

$$
y = c_1 t\, e^{\frac{2t}{3}} + c_2 e^{\frac{2t}{3}} \tag{1}
$$

Initial conditions are now substituted in the above solution. This will generate the required equations to solve for the integration constants. substituting $y = 2$ and $t = 0$ in the above gives

$$
2 = c_2 \tag{1A}
$$

Taking derivative of the solution gives

$$
y' = c_1 e^{\frac{2t}{3}} + \frac{2c_1 t\, e^{\frac{2t}{3}}}{3} + \frac{2c_2 e^{\frac{2t}{3}}}{3}
$$

substituting $y' = -1$ and $t = 0$ in the above gives

$$
-1 = c_1 + \frac{2c_2}{3} \tag{2A}
$$

Equations {1A,2A} are now solved for $\{c_1, c_2\}$. Solving for the constants gives

$$
\begin{aligned}
c_1 &= -\frac{7}{3} \\
c_2 &= 2
\end{aligned}
$$

Substituting these values back in above solution results in

$$
y = -\frac{7\,\mathrm{e}^{\frac{2t}{3}} t}{3} + 2\,\mathrm{e}^{\frac{2t}{3}}
$$

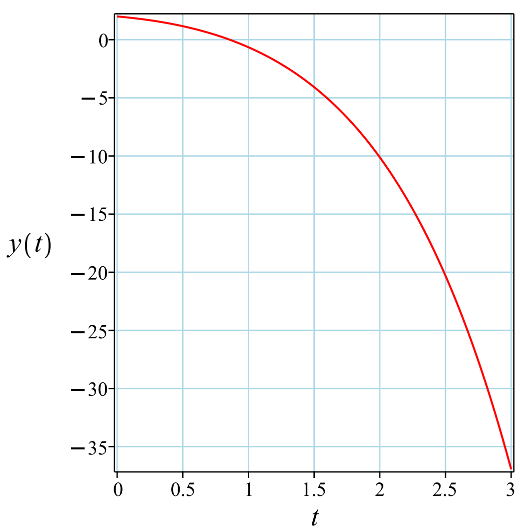
Which simplifies to

$$
y = \mathrm{e}^{\frac{2t}{3}} \left(2 - \frac{7t}{3}\right)
$$

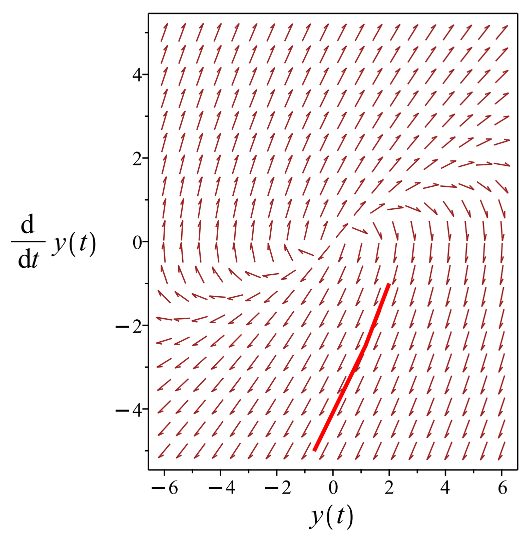
Summary

The solution(s) found are the following

$$
y = \mathrm{e}^{\frac{2t}{3}} \left(2 - \frac{7t}{3}\right) \tag{1}
$$



(a) Solution plot



(b) Slope field plot

Verification of solutions

$$
y = \mathrm{e}^{\frac{2t}{3}} \left(2 - \frac{7t}{3}\right)
$$

Verified OK.

9.11.4 Solving using Kovacic algorithm

Writing the ode as

$$9y'' - 12y' + 4y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 9 \\ B &= -12 \\ C &= 4 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(t) = ye^{\int \frac{B}{2A}\, dt}$$

Then (2) becomes

$$z''(t) = rz(t) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{0}{1} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= 0 \\ t &= 1 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(t) = 0 \tag{7}$$

Equation (7) is now solved. After finding $z(t)$ then y is found using the inverse transformation

$$y = z(t)\, e^{-\int \frac{B}{2A}\, dt}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 442: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$
\begin{aligned}
O(\infty) &= \deg(t) - \deg(s) \\
&= 0 - -\infty \\
&= \infty
\end{aligned}
$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is $infinity$ then the necessary conditions for case one are met. Therefore

$$
L = [1]
$$

Since $r = 0$ is not a function of t, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$
z_1(t) = 1
$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$
\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2} \frac{B}{A} \, dt} \\
&= z_1 e^{-\int \frac{1}{2} \frac{-12}{9} \, dt}
\end{aligned}
$$

$$
\begin{aligned}
&= z_1 e^{\frac{2t}{3}} \\
&= z_1 \left(\mathrm{e}^{\frac{2t}{3}}\right)
\end{aligned}
$$

Which simplifies to

$$
y_1 = \mathrm{e}^{\frac{2t}{3}}
$$

The second solution y_2 to the original ode is found using reduction of order

$$
y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\, dt}}{y_1^2}\, dt
$$

Substituting gives

$$
\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{-12}{9}\, dt}}{(y_1)^2}\, dt \\
&= y_1 \int \frac{e^{\frac{4t}{3}}}{(y_1)^2}\, dt \\
&= y_1(t)
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1 \left(\mathrm{e}^{\frac{2t}{3}}\right) + c_2 \left(\mathrm{e}^{\frac{2t}{3}} (t)\right)
\end{aligned}
$$

Initial conditions are used to solve for the constants of integration.

Looking at the above solution

$$
y = c_1 \mathrm{e}^{\frac{2t}{3}} + c_2 t\, \mathrm{e}^{\frac{2t}{3}} \tag{1}
$$

Initial conditions are now substituted in the above solution. This will generate the required equations to solve for the integration constants. substituting $y = 2$ and $t = 0$ in the above gives

$$
2 = c_1 \tag{1A}
$$

Taking derivative of the solution gives

$$
y' = \frac{2c_1 \mathrm{e}^{\frac{2t}{3}}}{3} + c_2 \mathrm{e}^{\frac{2t}{3}} + \frac{2c_2 t\, \mathrm{e}^{\frac{2t}{3}}}{3}
$$

substituting $y' = -1$ and $t = 0$ in the above gives

$$-1 = \frac{2c_1}{3} + c_2 \tag{2A}$$

Equations {1A,2A} are now solved for $\{c_1, c_2\}$. Solving for the constants gives

$$c_1 = 2$$
$$c_2 = -\frac{7}{3}$$

Substituting these values back in above solution results in

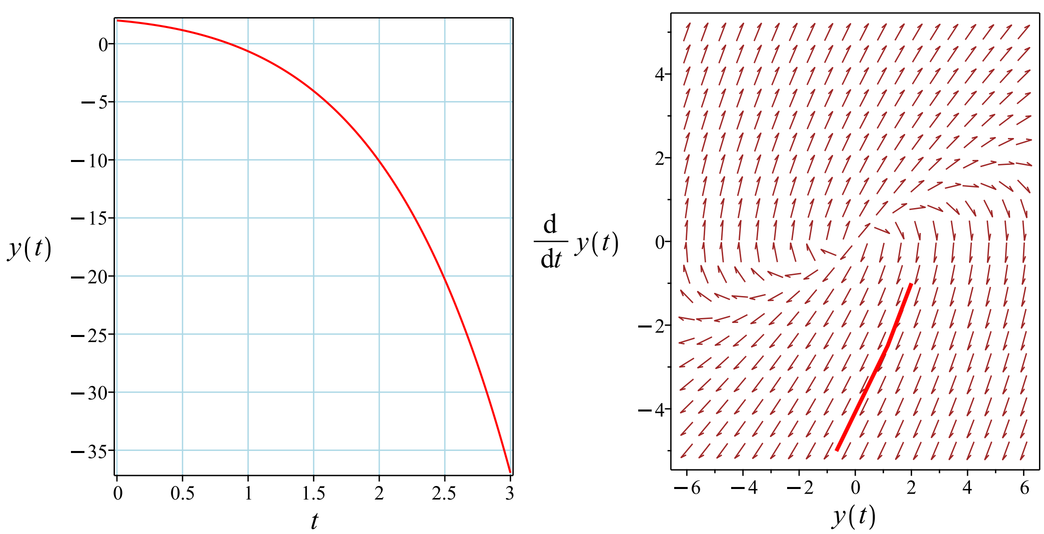
$$y = -\frac{7\,\mathrm{e}^{\frac{2t}{3}} t}{3} + 2\,\mathrm{e}^{\frac{2t}{3}}$$

Which simplifies to

$$y = \mathrm{e}^{\frac{2t}{3}} \left(2 - \frac{7t}{3}\right)$$

Summary
The solution(s) found are the following

$$y = \mathrm{e}^{\frac{2t}{3}} \left(2 - \frac{7t}{3}\right) \tag{1}$$



(a) Solution plot (b) Slope field plot

Verification of solutions

$$y = e^{\frac{2t}{3}} \left(2 - \frac{7t}{3}\right)$$

Verified OK.

9.11.5 Maple step by step solution

Let's solve

$\left[9y'' - 12y' + 4y = 0, y(0) = 2, y'\Big|_{\{t=0\}} = -1\right]$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y'' = \frac{4y'}{3} - \frac{4y}{9}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y'' - \frac{4y'}{3} + \frac{4y}{9} = 0$

- Characteristic polynomial of ODE

  $r^2 - \frac{4}{3}r + \frac{4}{9} = 0$

- Factor the characteristic polynomial

  $\frac{(3r-2)^2}{9} = 0$

- Root of the characteristic polynomial

  $r = \frac{2}{3}$

- 1st solution of the ODE

  $y_1(t) = e^{\frac{2t}{3}}$

- Repeated root, multiply $y_1(t)$ by t to ensure linear independence

  $y_2(t) = e^{\frac{2t}{3}} t$

- General solution of the ODE

  $y = c_1 y_1(t) + c_2 y_2(t)$

- Substitute in solutions

  $y = c_1 e^{\frac{2t}{3}} + c_2 t\, e^{\frac{2t}{3}}$

□ Check validity of solution $y = c_1 e^{\frac{2t}{3}} + c_2 t e^{\frac{2t}{3}}$

- Use initial condition $y(0) = 2$

  $2 = c_1$

- Compute derivative of the solution

  $y' = \frac{2c_1 e^{\frac{2t}{3}}}{3} + c_2 e^{\frac{2t}{3}} + \frac{2c_2 t\, e^{\frac{2t}{3}}}{3}$

- Use the initial condition $y'\Big|_{\{t=0\}} = -1$

  $-1 = \frac{2c_1}{3} + c_2$

- Solve for c_1 and c_2

  $\left\{c_1 = 2, c_2 = -\frac{7}{3}\right\}$

- Substitute constant values into general solution and simplify

  $y = e^{\frac{2t}{3}}\left(2 - \frac{7t}{3}\right)$

- Solution to the IVP

  $y = e^{\frac{2t}{3}}\left(2 - \frac{7t}{3}\right)$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
<- constant coefficients successful`
```

✓ Solution by Maple

Time used: 0.015 (sec). Leaf size: 14

```
dsolve([9*diff(y(t),t$2)-12*diff(y(t),t)+4*y(t) = 0,y(0) = 2, D(y)(0) = -1],y(t), singsol=al
```

$$y(t) = -\frac{e^{\frac{2t}{3}}(-6 + 7t)}{3}$$

✓ Solution by Mathematica

Time used: 0.015 (sec). Leaf size: 15

```
DSolve[{9*y''[t]-12*y'[t]+4*y[t]==0,{y[0]==0,y'[0]==-1}},y[t],t,IncludeSingularSolutions ->
```

$$y(t) \to -e^{2t/3}t$$

9.12 problem 12

Internal problem ID [664]
Internal file name [`OUTPUT/664_Sunday_June_05_2022_01_46_36_AM_49512898/index.tex`]

Book: Elementary differential equations and boundary value problems, 10th ed., Boyce and DiPrima
Section: Chapter 3, Second order linear equations, 3.4 Repeated roots, reduction of order , page 172
Problem number: 12.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_linear_constant_coeff", "linear_second_order_ode_solved_by_an_integrating_factor"**

Maple gives the following as the ode type

```
[[_2nd_order, _missing_x]]
```

$$\boxed{y'' - 6y' + 9y = 0}$$

With initial conditions

$$[y(0) = 0, y'(0) = 2]$$

9.12.1 Existence and uniqueness analysis

This is a linear ODE. In canonical form it is written as

$$y'' + p(t)y' + q(t)y = F$$

Where here

$$\begin{aligned} p(t) &= -6 \\ q(t) &= 9 \\ F &= 0 \end{aligned}$$

Hence the ode is

$$y'' - 6y' + 9y = 0$$

The domain of $p(t) = -6$ is

$$\{-\infty < t < \infty\}$$

And the point $t_0 = 0$ is inside this domain. The domain of $q(t) = 9$ is

$$\{-\infty < t < \infty\}$$

And the point $t_0 = 0$ is also inside this domain. Hence solution exists and is unique.

9.12.2 Solving as second order linear constant coeff ode

This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$Ay''(t) + By'(t) + Cy(t) = 0$$

Where in the above $A = 1, B = -6, C = 9$. Let the solution be $y = e^{\lambda t}$. Substituting this into the ODE gives

$$\lambda^2 e^{\lambda t} - 6\lambda\, e^{\lambda t} + 9\, e^{\lambda t} = 0 \tag{1}$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda t}$ gives

$$\lambda^2 - 6\lambda + 9 = 0 \tag{2}$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form.Using the quadratic formula

$$\lambda_{1,2} = \frac{-B}{2A} \pm \frac{1}{2A}\sqrt{B^2 - 4AC}$$

Substituting $A = 1, B = -6, C = 9$ into the above gives

$$\begin{aligned}\lambda_{1,2} &= \frac{6}{(2)(1)} \pm \frac{1}{(2)(1)}\sqrt{(-6)^2 - (4)(1)(9)} \\ &= 3\end{aligned}$$

Hence this is the case of a double root $\lambda_{1,2} = -3$. Therefore the solution is

$$y = c_1 e^{3t} + c_2 e^{3t} t \tag{1}$$

Initial conditions are used to solve for the constants of integration.

Looking at the above solution

$$y = c_1 e^{3t} + c_2 e^{3t} t \tag{1}$$

Initial conditions are now substituted in the above solution. This will generate the required equations to solve for the integration constants. substituting $y = 0$ and $t = 0$ in the above gives

$$0 = c_1 \tag{1A}$$

Taking derivative of the solution gives

$$y' = 3c_1 e^{3t} + 3c_2 e^{3t} t + c_2 e^{3t}$$

substituting $y' = 2$ and $t = 0$ in the above gives

$$2 = 3c_1 + c_2 \tag{2A}$$

Equations {1A,2A} are now solved for $\{c_1, c_2\}$. Solving for the constants gives

$$\begin{aligned}c_1 &= 0 \\ c_2 &= 2\end{aligned}$$

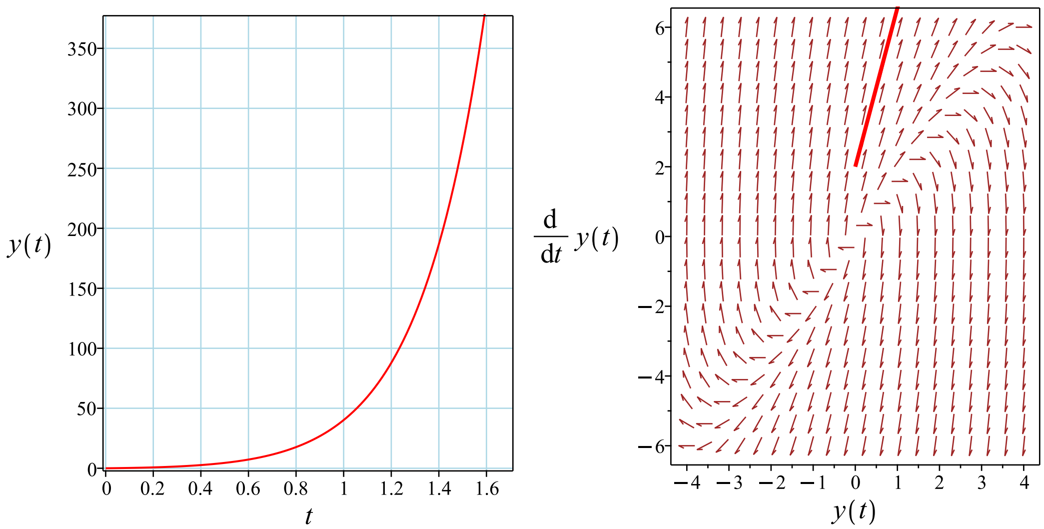
Substituting these values back in above solution results in

$$y = 2\,e^{3t} t$$

Summary

The solution(s) found are the following

$$y = 2\,e^{3t} t \tag{1}$$



(a) Solution plot

(b) Slope field plot

<u>Verification of solutions</u>

$$y = 2\,\mathrm{e}^{3t}t$$

Verified OK.

9.12.3 Solving as linear second order ode solved by an integrating factor ode

The ode satisfies this form

$$y'' + p(t)\,y' + \frac{\left(p(t)^2 + p'(t)\right)y}{2} = f(t)$$

Where $p(t) = -6$. Therefore, there is an integrating factor given by

$$\begin{aligned} M(x) &= e^{\frac{1}{2}\int p\,dx} \\ &= e^{\int -6\,dx} \\ &= \mathrm{e}^{-3t} \end{aligned}$$

Multiplying both sides of the ODE by the integrating factor $M(x)$ makes the left side of the ODE a complete differential

$$\begin{aligned} (M(x)y)'' &= 0 \\ \left(\mathrm{e}^{-3t}y\right)'' &= 0 \end{aligned}$$

Integrating once gives

$$(e^{-3t}y)' = c_1$$

Integrating again gives

$$(e^{-3t}y) = c_1 t + c_2$$

Hence the solution is

$$y = \frac{c_1 t + c_2}{e^{-3t}}$$

Or

$$y = c_1 t\, e^{3t} + c_2 e^{3t}$$

Initial conditions are used to solve for the constants of integration.

Looking at the above solution

$$y = c_1 t\, e^{3t} + c_2 e^{3t} \tag{1}$$

Initial conditions are now substituted in the above solution. This will generate the required equations to solve for the integration constants. substituting $y = 0$ and $t = 0$ in the above gives

$$0 = c_2 \tag{1A}$$

Taking derivative of the solution gives

$$y' = c_1 e^{3t} + 3c_1 t\, e^{3t} + 3c_2 e^{3t}$$

substituting $y' = 2$ and $t = 0$ in the above gives

$$2 = c_1 + 3c_2 \tag{2A}$$

Equations {1A,2A} are now solved for $\{c_1, c_2\}$. Solving for the constants gives

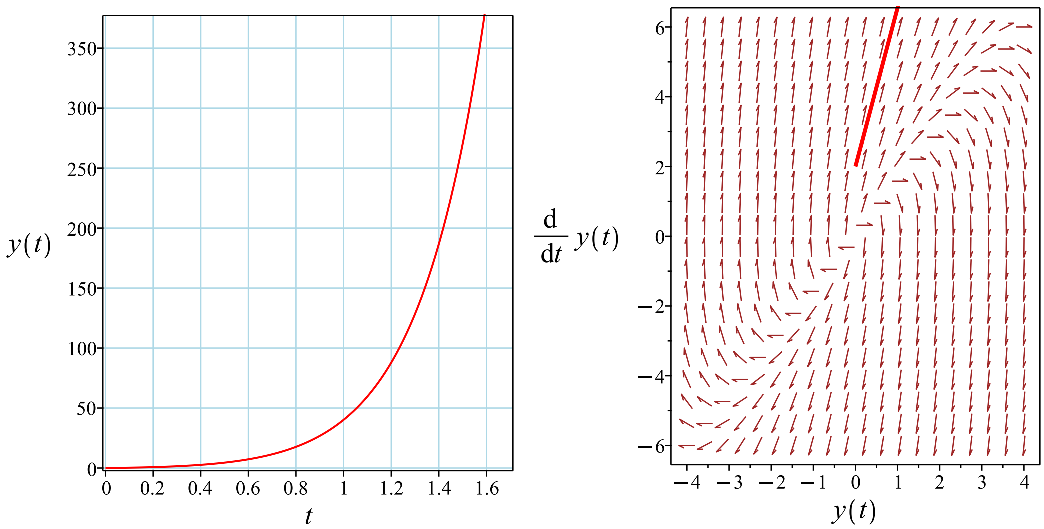
$$c_1 = 2$$
$$c_2 = 0$$

Substituting these values back in above solution results in

$$y = 2\, e^{3t} t$$

Summary
The solution(s) found are the following

$$y = 2\, e^{3t} t \tag{1}$$



(a) Solution plot (b) Slope field plot

Verification of solutions

$$y = 2\,\mathrm{e}^{3t}t$$

Verified OK.

9.12.4 Solving using Kovacic algorithm

Writing the ode as

$$y'' - 6y' + 9y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1 \\ B &= -6 \\ C &= 9 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(t) = ye^{\int \frac{B}{2A}\,dt}$$

Then (2) becomes

$$z''(t) = rz(t) \tag{4}$$

Where r is given by

$$r = \frac{s}{t} = \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{0}{1} \tag{6}$$

Comparing the above to (5) shows that

$$s = 0$$
$$t = 1$$

Therefore eq. (4) becomes

$$z''(t) = 0 \tag{7}$$

Equation (7) is now solved. After finding $z(t)$ then y is found using the inverse transformation

$$y = z(t)\, e^{-\int \frac{B}{2A}\, dt}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 444: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 0 - -\infty \\ &= \infty \end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is $infinity$ then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Since $r = 0$ is not a function of t, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$z_1(t) = 1$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$\begin{aligned} y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dt} \\ &= z_1 e^{-\int \frac{1}{2}\frac{-6}{1}\,dt} \\ &= z_1 e^{3t} \\ &= z_1 \left(e^{3t}\right) \end{aligned}$$

Which simplifies to

$$y_1 = e^{3t}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dt}}{y_1^2}\,dt$$

Substituting gives

$$\begin{aligned} y_2 &= y_1 \int \frac{e^{\int -\frac{-6}{1}\,dt}}{(y_1)^2}\,dt \\ &= y_1 \int \frac{e^{6t}}{(y_1)^2}\,dt \\ &= y_1(t) \end{aligned}$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1 \left(e^{3t}\right) + c_2 \left(e^{3t}(t)\right)
\end{aligned}
$$

Initial conditions are used to solve for the constants of integration.

Looking at the above solution

$$
y = c_1 e^{3t} + c_2 e^{3t} t \tag{1}
$$

Initial conditions are now substituted in the above solution. This will generate the required equations to solve for the integration constants. substituting $y = 0$ and $t = 0$ in the above gives

$$
0 = c_1 \tag{1A}
$$

Taking derivative of the solution gives

$$
y' = 3c_1 e^{3t} + 3c_2 e^{3t} t + c_2 e^{3t}
$$

substituting $y' = 2$ and $t = 0$ in the above gives

$$
2 = 3c_1 + c_2 \tag{2A}
$$

Equations {1A,2A} are now solved for $\{c_1, c_2\}$. Solving for the constants gives

$$
\begin{aligned}
c_1 &= 0 \\
c_2 &= 2
\end{aligned}
$$

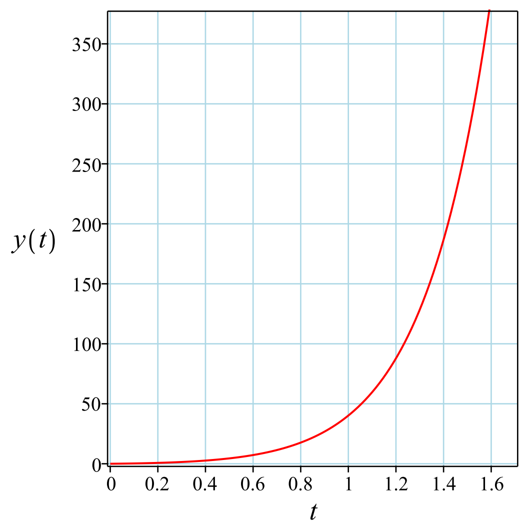
Substituting these values back in above solution results in

$$
y = 2 e^{3t} t
$$

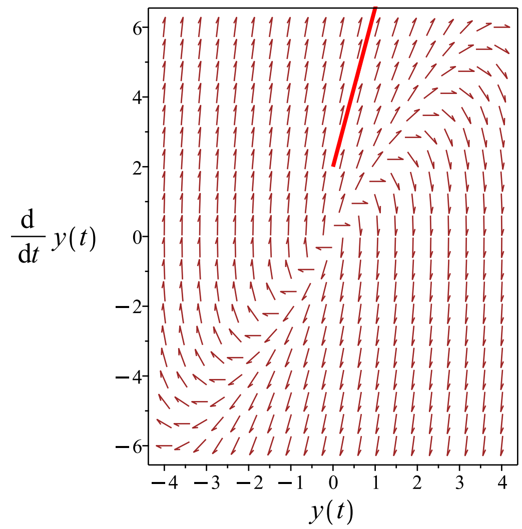
Summary

The solution(s) found are the following

$$
y = 2 e^{3t} t \tag{1}
$$



(a) Solution plot



(b) Slope field plot

<u>Verification of solutions</u>

$$y = 2\,\mathrm{e}^{3t} t$$

Verified OK.

9.12.5 Maple step by step solution

Let's solve

$$\left[y'' - 6y' + 9y = 0, y(0) = 0, y'\Big|_{\{t=0\}} = 2\right]$$

- Highest derivative means the order of the ODE is 2

  y''

- Characteristic polynomial of ODE

  $r^2 - 6r + 9 = 0$

- Factor the characteristic polynomial

  $(r-3)^2 = 0$

- Root of the characteristic polynomial

  $r = 3$

- 1st solution of the ODE

$y_1(t) = e^{3t}$

- Repeated root, multiply $y_1(t)$ by t to ensure linear independence

  $y_2(t) = e^{3t}t$

- General solution of the ODE

  $y = c_1 y_1(t) + c_2 y_2(t)$

- Substitute in solutions

  $y = c_1 e^{3t} + c_2 e^{3t}t$

- □ Check validity of solution $y = c_1 e^{3t} + c_2 e^{3t}t$
  - Use initial condition $y(0) = 0$

    $0 = c_1$
  - Compute derivative of the solution

    $y' = 3c_1 e^{3t} + 3c_2 e^{3t}t + c_2 e^{3t}$
  - Use the initial condition $y'\Big|_{\{t=0\}} = 2$

    $2 = 3c_1 + c_2$
  - Solve for c_1 and c_2

    $\{c_1 = 0, c_2 = 2\}$
  - Substitute constant values into general solution and simplify

    $y = 2\,e^{3t}t$

- Solution to the IVP

  $y = 2\,e^{3t}t$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
<- constant coefficients successful`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 11

```
dsolve([diff(y(t),t$2)-6*diff(y(t),t)+9*y(t) = 0,y(0) = 0, D(y)(0) = 2],y(t), singsol=all)
```

$$y(t) = 2\,\mathrm{e}^{3t}t$$

✓ Solution by Mathematica

Time used: 0.014 (sec). Leaf size: 13

```
DSolve[{y''[t]-6*y'[t]+9*y[t]==0,{y[0]==0,y'[0]==2}},y[t],t,IncludeSingularSolutions -> True
```

$$y(t) \to 2e^{3t}t$$

9.13 problem 13

Internal problem ID [665]
Internal file name [`OUTPUT/665_Sunday_June_05_2022_01_46_37_AM_75605177/index.tex`]

Book: Elementary differential equations and boundary value problems, 10th ed., Boyce and DiPrima
Section: Chapter 3, Second order linear equations, 3.4 Repeated roots, reduction of order , page 172
Problem number: 13.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_linear_constant_coeff"**

Maple gives the following as the ode type

`[[_2nd_order, _missing_x]]`

$$\boxed{9y'' + 6y' + 82y = 0}$$

With initial conditions

$$[y(0) = -1, y'(0) = 2]$$

9.13.1 Existence and uniqueness analysis

This is a linear ODE. In canonical form it is written as

$$y'' + p(t)y' + q(t)y = F$$

Where here

$$\begin{aligned} p(t) &= \frac{2}{3} \\ q(t) &= \frac{82}{9} \\ F &= 0 \end{aligned}$$

Hence the ode is

$$y'' + \frac{2y'}{3} + \frac{82y}{9} = 0$$

The domain of $p(t) = \frac{2}{3}$ is

$$\{-\infty < t < \infty\}$$

And the point $t_0 = 0$ is inside this domain. The domain of $q(t) = \frac{82}{9}$ is

$$\{-\infty < t < \infty\}$$

And the point $t_0 = 0$ is also inside this domain. Hence solution exists and is unique.

9.13.2 Solving as second order linear constant coeff ode

This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$Ay''(t) + By'(t) + Cy(t) = 0$$

Where in the above $A = 9, B = 6, C = 82$. Let the solution be $y = e^{\lambda t}$. Substituting this into the ODE gives

$$9\lambda^2 \mathrm{e}^{\lambda t} + 6\lambda\,\mathrm{e}^{\lambda t} + 82\,\mathrm{e}^{\lambda t} = 0 \tag{1}$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda t}$ gives

$$9\lambda^2 + 6\lambda + 82 = 0 \tag{2}$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form.Using the quadratic formula

$$\lambda_{1,2} = \frac{-B}{2A} \pm \frac{1}{2A}\sqrt{B^2 - 4AC}$$

Substituting $A = 9, B = 6, C = 82$ into the above gives

$$\begin{aligned}\lambda_{1,2} &= \frac{-6}{(2)\,(9)} \pm \frac{1}{(2)\,(9)}\sqrt{6^2 - (4)\,(9)\,(82)}\\ &= -\frac{1}{3} \pm 3i\end{aligned}$$

Hence

$$\begin{aligned}\lambda_1 &= -\frac{1}{3} + 3i\\ \lambda_2 &= -\frac{1}{3} - 3i\end{aligned}$$

Which simplifies to

$$\lambda_1 = -\frac{1}{3} + 3i$$
$$\lambda_2 = -\frac{1}{3} - 3i$$

Since roots are complex conjugate of each others, then let the roots be

$$\lambda_{1,2} = \alpha \pm i\beta$$

Where $\alpha = -\frac{1}{3}$ and $\beta = 3$. Therefore the final solution, when using Euler relation, can be written as

$$y = e^{\alpha t}(c_1 \cos(\beta t) + c_2 \sin(\beta t))$$

Which becomes

$$y = e^{-\frac{t}{3}}(c_1 \cos(3t) + c_2 \sin(3t))$$

Initial conditions are used to solve for the constants of integration.

Looking at the above solution

$$y = \mathrm{e}^{-\frac{t}{3}}(c_1 \cos(3t) + c_2 \sin(3t)) \tag{1}$$

Initial conditions are now substituted in the above solution. This will generate the required equations to solve for the integration constants. substituting $y = -1$ and $t = 0$ in the above gives

$$-1 = c_1 \tag{1A}$$

Taking derivative of the solution gives

$$y' = -\frac{\mathrm{e}^{-\frac{t}{3}}(c_1 \cos(3t) + c_2 \sin(3t))}{3} + \mathrm{e}^{-\frac{t}{3}}(-3c_1 \sin(3t) + 3c_2 \cos(3t))$$

substituting $y' = 2$ and $t = 0$ in the above gives

$$2 = -\frac{c_1}{3} + 3c_2 \tag{2A}$$

Equations {1A,2A} are now solved for $\{c_1, c_2\}$. Solving for the constants gives

$$c_1 = -1$$
$$c_2 = \frac{5}{9}$$

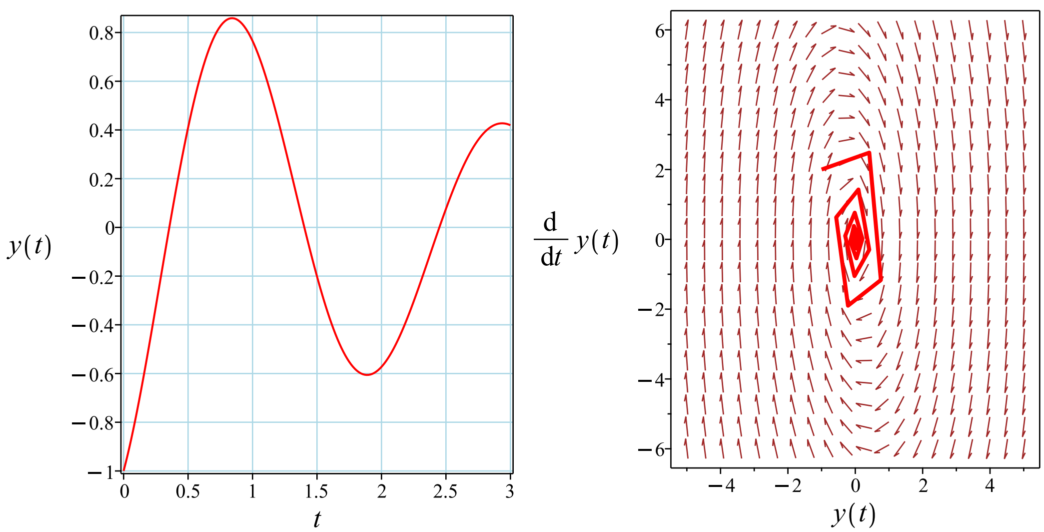
Substituting these values back in above solution results in

$$y = -\frac{e^{-\frac{t}{3}}(9\cos(3t) - 5\sin(3t))}{9}$$

Summary

The solution(s) found are the following

$$y = -\frac{e^{-\frac{t}{3}}(9\cos(3t) - 5\sin(3t))}{9} \tag{1}$$



(a) Solution plot

(b) Slope field plot

Verification of solutions

$$y = -\frac{e^{-\frac{t}{3}}(9\cos(3t) - 5\sin(3t))}{9}$$

Verified OK.

9.13.3 Solving using Kovacic algorithm

Writing the ode as

$$9y'' + 6y' + 82y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 9 \\ B &= 6 \\ C &= 82 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(t) = ye^{\int \frac{B}{2A}\,dt}$$

Then (2) becomes

$$z''(t) = rz(t) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{-9}{1} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= -9 \\ t &= 1 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(t) = -9z(t) \tag{7}$$

Equation (7) is now solved. After finding $z(t)$ then y is found using the inverse transformation

$$y = z(t)\, e^{-\int \frac{B}{2A}\,dt}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0,1,2,4,6,8,\cdots\}$	$\{\cdots,-6,-4,-2,0,2,3,4,5,6,\cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1,2\}$	$\{2,3,4,5,6,7,\cdots\}$

Table 446: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned}O(\infty) &= \deg(t) - \deg(s)\\ &= 0 - 0\\ &= 0\end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Since $r = -9$ is not a function of t, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$z_1(t) = \cos(3t)$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$\begin{aligned}y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dt}\\ &= z_1 e^{-\int \frac{1}{2}\frac{6}{9}\,dt}\\ &= z_1 e^{-\frac{t}{3}}\\ &= z_1\left(\mathrm{e}^{-\frac{t}{3}}\right)\end{aligned}$$

Which simplifies to

$$y_1 = \mathrm{e}^{-\frac{t}{3}} \cos(3t)$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\, dt}}{y_1^2}\, dt$$

Substituting gives

$$\begin{aligned} y_2 &= y_1 \int \frac{e^{\int -\frac{6}{9}\, dt}}{(y_1)^2}\, dt \\ &= y_1 \int \frac{e^{-\frac{2t}{3}}}{(y_1)^2}\, dt \\ &= y_1 \left(\frac{\tan(3t)}{3}\right) \end{aligned}$$

Therefore the solution is

$$\begin{aligned} y &= c_1 y_1 + c_2 y_2 \\ &= c_1 \left(\mathrm{e}^{-\frac{t}{3}} \cos(3t)\right) + c_2 \left(\mathrm{e}^{-\frac{t}{3}} \cos(3t) \left(\frac{\tan(3t)}{3}\right)\right) \end{aligned}$$

Initial conditions are used to solve for the constants of integration.

Looking at the above solution

$$y = c_1 \mathrm{e}^{-\frac{t}{3}} \cos(3t) + \frac{c_2 \mathrm{e}^{-\frac{t}{3}} \sin(3t)}{3} \tag{1}$$

Initial conditions are now substituted in the above solution. This will generate the required equations to solve for the integration constants. substituting $y = -1$ and $t = 0$ in the above gives

$$-1 = c_1 \tag{1A}$$

Taking derivative of the solution gives

$$y' = -\frac{c_1 \mathrm{e}^{-\frac{t}{3}} \cos(3t)}{3} - 3c_1 \mathrm{e}^{-\frac{t}{3}} \sin(3t) - \frac{c_2 \mathrm{e}^{-\frac{t}{3}} \sin(3t)}{9} + c_2 \mathrm{e}^{-\frac{t}{3}} \cos(3t)$$

substituting $y' = 2$ and $t = 0$ in the above gives

$$2 = -\frac{c_1}{3} + c_2 \tag{2A}$$

Equations {1A,2A} are now solved for $\{c_1, c_2\}$. Solving for the constants gives

$$c_1 = -1$$
$$c_2 = \frac{5}{3}$$

Substituting these values back in above solution results in

$$y = -\mathrm{e}^{-\frac{t}{3}} \cos(3t) + \frac{5\,\mathrm{e}^{-\frac{t}{3}} \sin(3t)}{9}$$

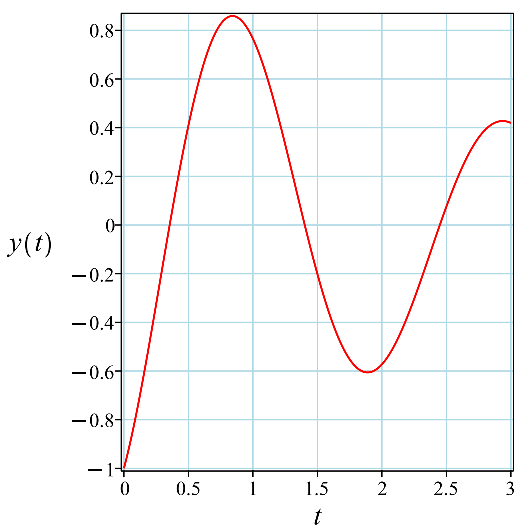
Which simplifies to

$$y = -\frac{\mathrm{e}^{-\frac{t}{3}} \left(9 \cos(3t) - 5 \sin(3t)\right)}{9}$$

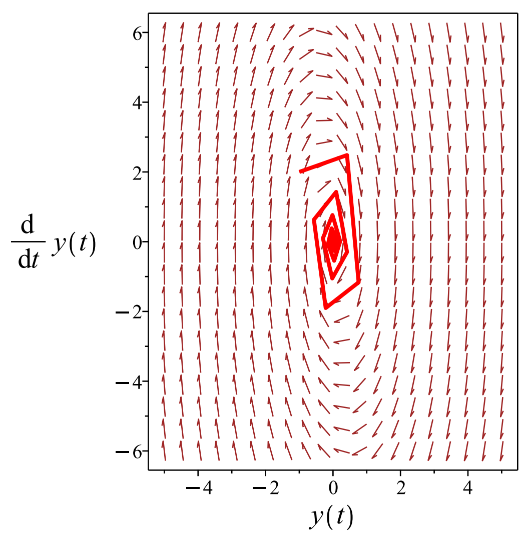
Summary

The solution(s) found are the following

$$y = -\frac{\mathrm{e}^{-\frac{t}{3}} \left(9 \cos(3t) - 5 \sin(3t)\right)}{9} \tag{1}$$



(a) Solution plot



(b) Slope field plot

Verification of solutions

$$y = -\frac{e^{-\frac{t}{3}} (9 \cos (3t) - 5 \sin (3t))}{9}$$

Verified OK.

9.13.4 Maple step by step solution

Let's solve

$\left[9y'' + 6y' + 82y = 0, y(0) = -1, y'\Big|_{\{t=0\}} = 2\right]$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y'' = -\frac{2y'}{3} - \frac{82y}{9}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y'' + \frac{2y'}{3} + \frac{82y}{9} = 0$

- Characteristic polynomial of ODE

  $r^2 + \frac{2}{3}r + \frac{82}{9} = 0$

- Use quadratic formula to solve for r

  $r = \frac{(-\frac{2}{3}) \pm (\sqrt{-36})}{2}$

- Roots of the characteristic polynomial

  $r = \left(-\frac{1}{3} - 3\,\mathrm{I}, -\frac{1}{3} + 3\,\mathrm{I}\right)$

- 1st solution of the ODE

  $y_1(t) = e^{-\frac{t}{3}} \cos (3t)$

- 2nd solution of the ODE

  $y_2(t) = e^{-\frac{t}{3}} \sin (3t)$

- General solution of the ODE

  $y = c_1 y_1(t) + c_2 y_2(t)$

- Substitute in solutions

  $y = c_1 e^{-\frac{t}{3}} \cos (3t) + c_2 e^{-\frac{t}{3}} \sin (3t)$

☐ Check validity of solution $y = c_1 e^{-\frac{t}{3}} \cos (3t) + c_2 e^{-\frac{t}{3}} \sin (3t)$

- Use initial condition $y(0) = -1$

  $-1 = c_1$
- Compute derivative of the solution

  $y' = -\frac{c_1 e^{-\frac{t}{3}} \cos(3t)}{3} - 3c_1 e^{-\frac{t}{3}} \sin(3t) - \frac{c_2 e^{-\frac{t}{3}} \sin(3t)}{3} + 3c_2 e^{-\frac{t}{3}} \cos(3t)$
- Use the initial condition $y'\Big|_{\{t=0\}} = 2$

  $2 = -\frac{c_1}{3} + 3c_2$
- Solve for c_1 and c_2

  $\left\{c_1 = -1, c_2 = \frac{5}{9}\right\}$
- Substitute constant values into general solution and simplify

  $y = -\frac{e^{-\frac{t}{3}}(9\cos(3t) - 5\sin(3t))}{9}$

- Solution to the IVP

  $y = -\frac{e^{-\frac{t}{3}}(9\cos(3t) - 5\sin(3t))}{9}$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
<- constant coefficients successful`
```

✓ Solution by Maple

Time used: 0.015 (sec). Leaf size: 23

```
dsolve([9*diff(y(t),t$2)+6*diff(y(t),t)+82*y(t) = 0,y(0) = -1, D(y)(0) = 2],y(t), singsol=al
```

$$y(t) = \frac{e^{-\frac{t}{3}}(5\sin(3t) - 9\cos(3t))}{9}$$

✓ Solution by Mathematica

Time used: 0.019 (sec). Leaf size: 29

```
DSolve[{9*y''[t]+6*y'[t]+82*y[t]==0,{y[0]==-1,y'[0]==2}},y[t],t,IncludeSingularSolutions ->
```

$$y(t) \to \frac{1}{9}e^{-t/3}(5\sin(3t) - 9\cos(3t))$$

9.14 problem 14

Internal problem ID [666]
Internal file name [`OUTPUT/666_Sunday_June_05_2022_01_46_38_AM_29132401/index.tex`]

Book: Elementary differential equations and boundary value problems, 10th ed., Boyce and DiPrima
Section: Chapter 3, Second order linear equations, 3.4 Repeated roots, reduction of order , page 172
Problem number: 14.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_linear_constant_coeff", "linear_second_order_ode_solved_by_an_integrating_factor"**

Maple gives the following as the ode type

```
[[_2nd_order, _missing_x]]
```

$$\boxed{y'' + 4y' + 4y = 0}$$

With initial conditions

$$[y(-1) = 2, y'(-1) = 1]$$

9.14.1 Existence and uniqueness analysis

This is a linear ODE. In canonical form it is written as

$$y'' + p(x)y' + q(x)y = F$$

Where here

$$\begin{aligned} p(x) &= 4 \\ q(x) &= 4 \\ F &= 0 \end{aligned}$$

Hence the ode is

$$y'' + 4y' + 4y = 0$$

The domain of $p(x) = 4$ is

$$\{-\infty < x < \infty\}$$

And the point $x_0 = -1$ is inside this domain. The domain of $q(x) = 4$ is

$$\{-\infty < x < \infty\}$$

And the point $x_0 = -1$ is also inside this domain. Hence solution exists and is unique.

9.14.2 Solving as second order linear constant coeff ode

This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = 0$$

Where in the above $A = 1, B = 4, C = 4$. Let the solution be $y = e^{\lambda x}$. Substituting this into the ODE gives

$$\lambda^2 e^{\lambda x} + 4\lambda\, e^{\lambda x} + 4\, e^{\lambda x} = 0 \tag{1}$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda x}$ gives

$$\lambda^2 + 4\lambda + 4 = 0 \tag{2}$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form.Using the quadratic formula

$$\lambda_{1,2} = \frac{-B}{2A} \pm \frac{1}{2A}\sqrt{B^2 - 4AC}$$

Substituting $A = 1, B = 4, C = 4$ into the above gives

$$
\begin{aligned}
\lambda_{1,2} &= \frac{-4}{(2)(1)} \pm \frac{1}{(2)(1)} \sqrt{(4)^2 - (4)(1)(4)} \\
&= -2
\end{aligned}
$$

Hence this is the case of a double root $\lambda_{1,2} = 2$. Therefore the solution is

$$
y = c_1 \mathrm{e}^{-2x} + c_2 \mathrm{e}^{-2x} x \tag{1}
$$

Initial conditions are used to solve for the constants of integration.

Looking at the above solution

$$
y = c_1 \mathrm{e}^{-2x} + x \,\mathrm{e}^{-2x} c_2 \tag{1}
$$

Initial conditions are now substituted in the above solution. This will generate the required equations to solve for the integration constants. substituting $y = 2$ and $x = -1$ in the above gives

$$
2 = \mathrm{e}^2 (c_1 - c_2) \tag{1A}
$$

Taking derivative of the solution gives

$$
y' = -2c_1 \mathrm{e}^{-2x} + c_2 \mathrm{e}^{-2x} - 2x \,\mathrm{e}^{-2x} c_2
$$

substituting $y' = 1$ and $x = -1$ in the above gives

$$
1 = \mathrm{e}^2 (-2c_1 + 3c_2) \tag{2A}
$$

Equations {1A,2A} are now solved for $\{c_1, c_2\}$. Solving for the constants gives

$$
\begin{aligned}
c_1 &= 7 \,\mathrm{e}^{-2} \\
c_2 &= 5 \,\mathrm{e}^{-2}
\end{aligned}
$$

Substituting these values back in above solution results in

$$
y = 5x \,\mathrm{e}^{-2x} \mathrm{e}^{-2} + 7 \,\mathrm{e}^{-2x} \mathrm{e}^{-2}
$$

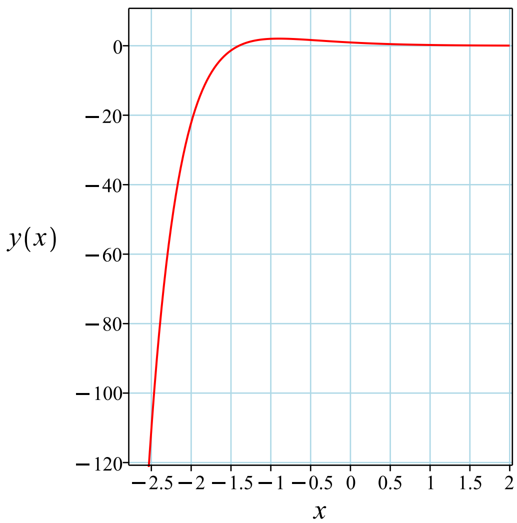
Which simplifies to

$$
y = \mathrm{e}^{-2x-2} (7 + 5x)
$$

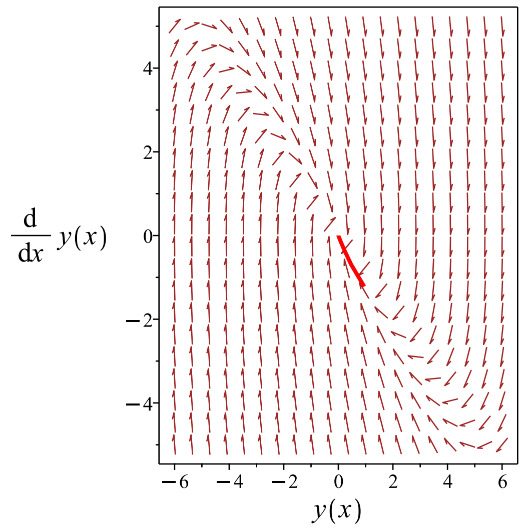
Summary

The solution(s) found are the following

$$
y = \mathrm{e}^{-2x-2} (7 + 5x) \tag{1}
$$



(a) Solution plot



(b) Slope field plot

<u>Verification of solutions</u>

$$y = e^{-2x-2}(7 + 5x)$$

Verified OK.

9.14.3 Solving as linear second order ode solved by an integrating factor ode

The ode satisfies this form

$$y'' + p(x)\,y' + \frac{\left(p(x)^2 + p'(x)\right)y}{2} = f(x)$$

Where $p(x) = 4$. Therefore, there is an integrating factor given by

$$\begin{aligned}M(x) &= e^{\frac{1}{2}\int p\,dx}\\ &= e^{\int 4\,dx}\\ &= e^{2x}\end{aligned}$$

Multiplying both sides of the ODE by the integrating factor $M(x)$ makes the left side of the ODE a complete differential

$$\begin{aligned}(M(x)y)'' &= 0\\ \left(e^{2x}y\right)'' &= 0\end{aligned}$$

Integrating once gives

$$\left(e^{2x}y\right)' = c_1$$

Integrating again gives

$$\left(e^{2x}y\right) = c_1x + c_2$$

Hence the solution is

$$y = \frac{c_1x + c_2}{e^{2x}}$$

Or

$$y = c_1x\,e^{-2x} + c_2e^{-2x}$$

Initial conditions are used to solve for the constants of integration.

Looking at the above solution

$$y = c_1x\,e^{-2x} + c_2e^{-2x} \tag{1}$$

Initial conditions are now substituted in the above solution. This will generate the required equations to solve for the integration constants. substituting $y = 2$ and $x = -1$ in the above gives

$$2 = (-c_1 + c_2)\,e^2 \tag{1A}$$

Taking derivative of the solution gives

$$y' = c_1e^{-2x} - 2c_1x\,e^{-2x} - 2c_2e^{-2x}$$

substituting $y' = 1$ and $x = -1$ in the above gives

$$1 = (3c_1 - 2c_2)\,e^2 \tag{2A}$$

Equations {1A,2A} are now solved for $\{c_1, c_2\}$. Solving for the constants gives

$$c_1 = 5\,e^{-2}$$
$$c_2 = 7\,e^{-2}$$

Substituting these values back in above solution results in

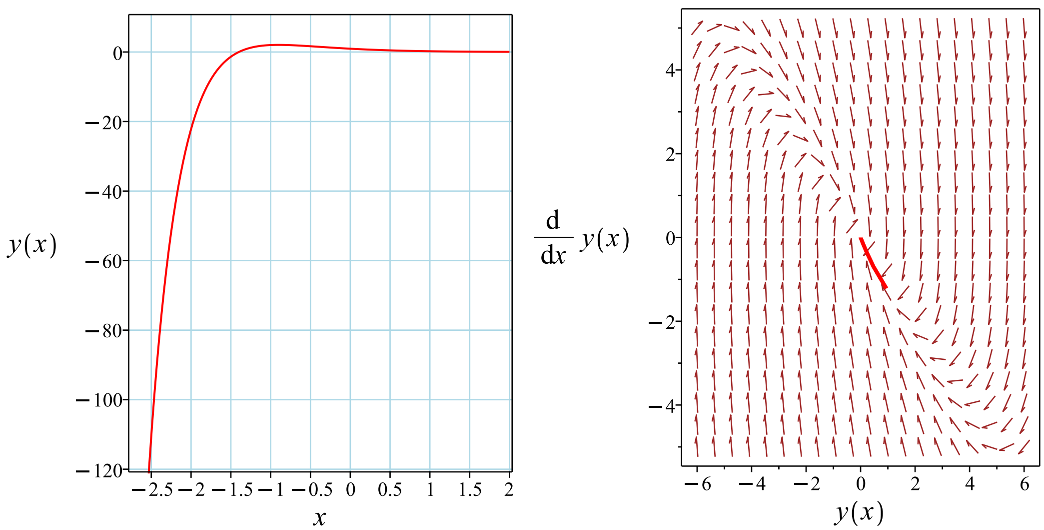
$$y = 5x\,e^{-2x-2} + 7\,e^{-2x}e^{-2}$$

Which simplifies to

$$y = e^{-2x-2}(7 + 5x)$$

Summary
The solution(s) found are the following

$$y = e^{-2x-2}(7+5x) \tag{1}$$



(a) Solution plot (b) Slope field plot

Verification of solutions

$$y = e^{-2x-2}(7+5x)$$

Verified OK.

9.14.4 Solving using Kovacic algorithm

Writing the ode as

$$y'' + 4y' + 4y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1 \\ B &= 4 \\ C &= 4 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = y e^{\int \frac{B}{2A}\, dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{0}{1} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= 0 \\ t &= 1 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = 0 \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x) e^{-\int \frac{B}{2A} dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 448: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned}
O(\infty) &= \deg(t) - \deg(s) \\
&= 0 - -\infty \\
&= \infty
\end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is $infinity$ then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Since $r = 0$ is not a function of x, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$z_1(x) = 1$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{4}{1}\,dx} \\
&= z_1 e^{-2x} \\
&= z_1 \left(\mathrm{e}^{-2x}\right)
\end{aligned}$$

Which simplifies to

$$y_1 = \mathrm{e}^{-2x}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx$$

Substituting gives

$$\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{4}{1}\,dx}}{(y_1)^2}\,dx \\
&= y_1 \int \frac{e^{-4x}}{(y_1)^2}\,dx \\
&= y_1(x)
\end{aligned}$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1\left(\mathrm{e}^{-2x}\right) + c_2\left(\mathrm{e}^{-2x}(x)\right)
\end{aligned}
$$

Initial conditions are used to solve for the constants of integration.

Looking at the above solution

$$
y = c_1 \mathrm{e}^{-2x} + x\,\mathrm{e}^{-2x} c_2 \tag{1}
$$

Initial conditions are now substituted in the above solution. This will generate the required equations to solve for the integration constants. substituting $y = 2$ and $x = -1$ in the above gives

$$
2 = \mathrm{e}^{2}(c_1 - c_2) \tag{1A}
$$

Taking derivative of the solution gives

$$
y' = -2c_1 \mathrm{e}^{-2x} + c_2 \mathrm{e}^{-2x} - 2x\,\mathrm{e}^{-2x} c_2
$$

substituting $y' = 1$ and $x = -1$ in the above gives

$$
1 = \mathrm{e}^{2}(-2c_1 + 3c_2) \tag{2A}
$$

Equations {1A,2A} are now solved for $\{c_1, c_2\}$. Solving for the constants gives

$$
\begin{aligned}
c_1 &= 7\,\mathrm{e}^{-2} \\
c_2 &= 5\,\mathrm{e}^{-2}
\end{aligned}
$$

Substituting these values back in above solution results in

$$
y = 5x\,\mathrm{e}^{-2x} \mathrm{e}^{-2} + 7\,\mathrm{e}^{-2x} \mathrm{e}^{-2}
$$

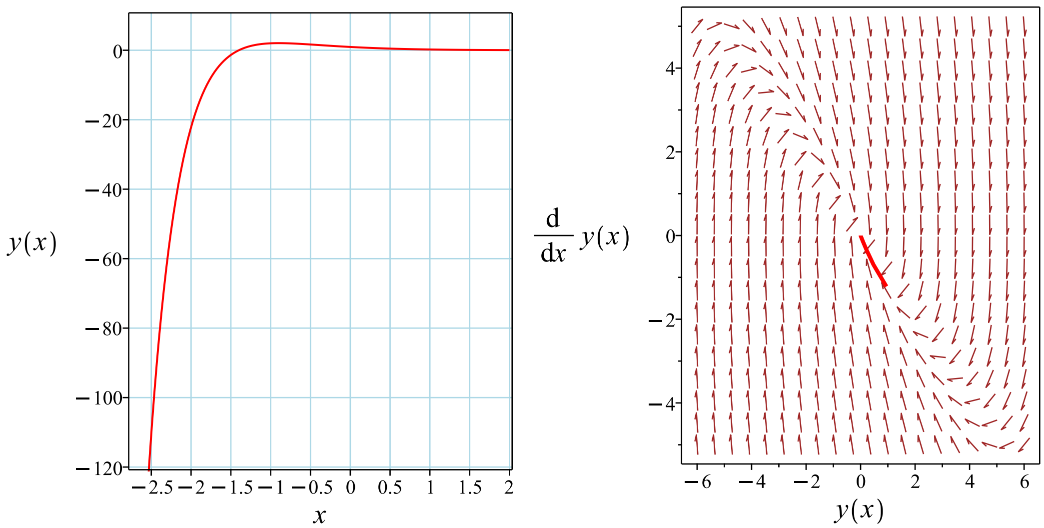
Which simplifies to

$$
y = \mathrm{e}^{-2x-2}(7 + 5x)
$$

Summary

The solution(s) found are the following

$$
y = \mathrm{e}^{-2x-2}(7 + 5x) \tag{1}
$$



(a) Solution plot

(b) Slope field plot

Verification of solutions

$$y = e^{-2x-2}(7 + 5x)$$

Verified OK.

9.14.5 Maple step by step solution

Let's solve

$\left[y'' + 4y' + 4y = 0, y(-1) = 2, y'\Big|_{\{x=-1\}} = 1\right]$

- Highest derivative means the order of the ODE is 2

  y''
- Characteristic polynomial of ODE

  $r^2 + 4r + 4 = 0$
- Factor the characteristic polynomial

  $(r + 2)^2 = 0$
- Root of the characteristic polynomial

  $r = -2$
- 1st solution of the ODE

$$y_1(x) = e^{-2x}$$

- Repeated root, multiply $y_1(x)$ by x to ensure linear independence

$$y_2(x) = e^{-2x}x$$

- General solution of the ODE

$$y = c_1y_1(x) + c_2y_2(x)$$

- Substitute in solutions

$$y = c_1e^{-2x} + x\,e^{-2x}c_2$$

□ Check validity of solution $y = c_1e^{-2x} + xe^{-2x}c_2$

  - Use initial condition $y(-1) = 2$

$$2 = e^2c_1 - e^2c_2$$

  - Compute derivative of the solution

$$y' = -2c_1e^{-2x} + c_2e^{-2x} - 2x\,e^{-2x}c_2$$

  - Use the initial condition $y'\Big|_{\{x=-1\}} = 1$

$$1 = -2\,e^2c_1 + 3\,e^2c_2$$

  - Solve for c_1 and c_2

$$\left\{c_1 = \frac{7}{e^2}, c_2 = \frac{5}{e^2}\right\}$$

  - Substitute constant values into general solution and simplify

$$y = e^{-2x-2}(7 + 5x)$$

- Solution to the IVP

$$y = e^{-2x-2}(7 + 5x)$$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
<- constant coefficients successful`
```

✓ Solution by Maple

Time used: 0.031 (sec). Leaf size: 16

```
dsolve([diff(y(x),x$2)+4*diff(y(x),x)+4*y(x) = 0,y(-1) = 2, D(y)(-1) = 1],y(x), singsol=all)
```

$$y(x) = \mathrm{e}^{-2x-2}(5x+7)$$

✓ Solution by Mathematica

Time used: 0.014 (sec). Leaf size: 18

```
DSolve[{y''[x]+4*y'[x]+4*y[x]==0,{y[-1]==2,y'[-1]==1}},y[x],x,IncludeSingularSolutions -> Tr
```

$$y(x) \to e^{-2(x+1)}(5x+7)$$

9.15 problem 15

Internal problem ID [667]
Internal file name [`OUTPUT/667_Sunday_June_05_2022_01_46_39_AM_14025215/index.tex`]

Book: Elementary differential equations and boundary value problems, 10th ed., Boyce and DiPrima
Section: Chapter 3, Second order linear equations, 3.4 Repeated roots, reduction of order , page 172
Problem number: 15.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_linear_constant_coeff", "linear_second_order_ode_solved_by_an_integrating_factor"**

Maple gives the following as the ode type

```
[[_2nd_order, _missing_x]]
```

$$\boxed{4y'' + 12y' + 9y = 0}$$

With initial conditions

$$[y(0) = 1, y'(0) = -4]$$

9.15.1 Existence and uniqueness analysis

This is a linear ODE. In canonical form it is written as

$$y'' + p(t)y' + q(t)y = F$$

Where here

$$\begin{aligned} p(t) &= 3 \\ q(t) &= \frac{9}{4} \\ F &= 0 \end{aligned}$$

Hence the ode is

$$y'' + 3y' + \frac{9y}{4} = 0$$

The domain of $p(t) = 3$ is

$$\{-\infty < t < \infty\}$$

And the point $t_0 = 0$ is inside this domain. The domain of $q(t) = \frac{9}{4}$ is

$$\{-\infty < t < \infty\}$$

And the point $t_0 = 0$ is also inside this domain. Hence solution exists and is unique.

9.15.2 Solving as second order linear constant coeff ode

This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$Ay''(t) + By'(t) + Cy(t) = 0$$

Where in the above $A = 4, B = 12, C = 9$. Let the solution be $y = e^{\lambda t}$. Substituting this into the ODE gives

$$4\lambda^2 e^{\lambda t} + 12\lambda\, e^{\lambda t} + 9\, e^{\lambda t} = 0 \tag{1}$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda t}$ gives

$$4\lambda^2 + 12\lambda + 9 = 0 \tag{2}$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form.Using the quadratic formula

$$\lambda_{1,2} = \frac{-B}{2A} \pm \frac{1}{2A}\sqrt{B^2 - 4AC}$$

Substituting $A = 4, B = 12, C = 9$ into the above gives

$$\begin{aligned}\lambda_{1,2} &= \frac{-12}{(2)(4)} \pm \frac{1}{(2)(4)}\sqrt{(12)^2 - (4)(4)(9)} \\ &= -\frac{3}{2}\end{aligned}$$

Hence this is the case of a double root $\lambda_{1,2} = \frac{3}{2}$. Therefore the solution is

$$y = c_1 e^{-\frac{3t}{2}} + c_2 e^{-\frac{3t}{2}} t \tag{1}$$

Initial conditions are used to solve for the constants of integration.

Looking at the above solution

$$y = c_1 e^{-\frac{3t}{2}} + c_2 t\, e^{-\frac{3t}{2}} \tag{1}$$

Initial conditions are now substituted in the above solution. This will generate the required equations to solve for the integration constants. substituting $y = 1$ and $t = 0$ in the above gives

$$1 = c_1 \tag{1A}$$

Taking derivative of the solution gives

$$y' = -\frac{3c_1 e^{-\frac{3t}{2}}}{2} + c_2 e^{-\frac{3t}{2}} - \frac{3c_2 t\, e^{-\frac{3t}{2}}}{2}$$

substituting $y' = -4$ and $t = 0$ in the above gives

$$-4 = -\frac{3c_1}{2} + c_2 \tag{2A}$$

Equations {1A,2A} are now solved for $\{c_1, c_2\}$. Solving for the constants gives

$$\begin{aligned}c_1 &= 1 \\ c_2 &= -\frac{5}{2}\end{aligned}$$

Substituting these values back in above solution results in

$$y = -\frac{5\, e^{-\frac{3t}{2}} t}{2} + e^{-\frac{3t}{2}}$$

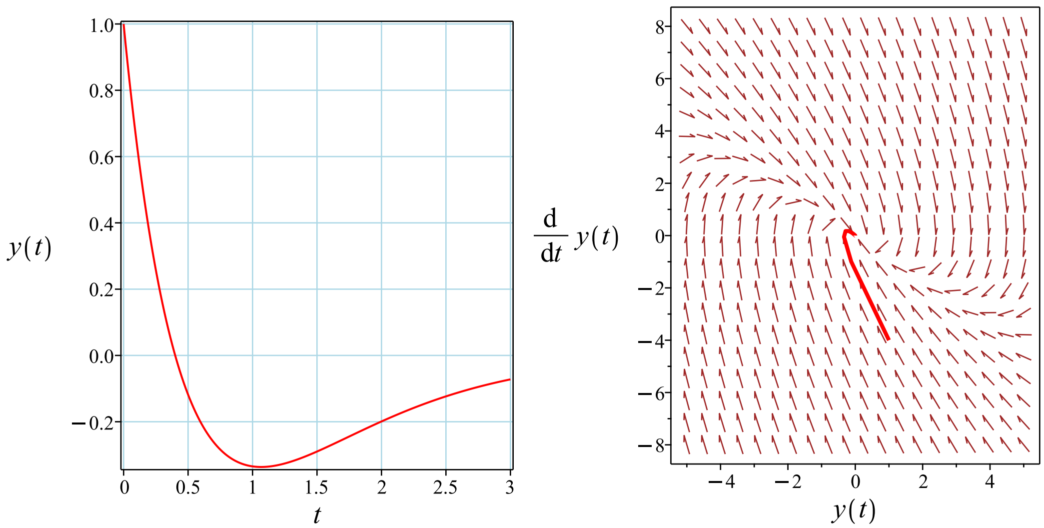
Which simplifies to

$$y = e^{-\frac{3t}{2}} \left(1 - \frac{5t}{2}\right)$$

Summary

The solution(s) found are the following

$$y = \mathrm{e}^{-\frac{3t}{2}} \left(1 - \frac{5t}{2}\right) \tag{1}$$



(a) Solution plot (b) Slope field plot

Verification of solutions

$$y = \mathrm{e}^{-\frac{3t}{2}} \left(1 - \frac{5t}{2}\right)$$

Verified OK.

9.15.3 Solving as linear second order ode solved by an integrating factor ode

The ode satisfies this form

$$y'' + p(t)\, y' + \frac{\left(p(t)^2 + p'(t)\right) y}{2} = f(t)$$

Where $p(t) = 3$. Therefore, there is an integrating factor given by

$$\begin{aligned} M(x) &= e^{\frac{1}{2} \int p\, dx} \\ &= e^{\int 3\, dx} \\ &= \mathrm{e}^{\frac{3t}{2}} \end{aligned}$$

Multiplying both sides of the ODE by the integrating factor $M(x)$ makes the left side of the ODE a complete differential

$$(M(x)y)'' = 0$$
$$\left(e^{\frac{3t}{2}} y\right)'' = 0$$

Integrating once gives

$$\left(e^{\frac{3t}{2}} y\right)' = c_1$$

Integrating again gives

$$\left(e^{\frac{3t}{2}} y\right) = c_1 t + c_2$$

Hence the solution is

$$y = \frac{c_1 t + c_2}{e^{\frac{3t}{2}}}$$

Or

$$y = c_1 t\, e^{-\frac{3t}{2}} + c_2 e^{-\frac{3t}{2}}$$

Initial conditions are used to solve for the constants of integration.

Looking at the above solution

$$y = c_1 t\, e^{-\frac{3t}{2}} + c_2 e^{-\frac{3t}{2}} \tag{1}$$

Initial conditions are now substituted in the above solution. This will generate the required equations to solve for the integration constants. substituting $y = 1$ and $t = 0$ in the above gives

$$1 = c_2 \tag{1A}$$

Taking derivative of the solution gives

$$y' = c_1 e^{-\frac{3t}{2}} - \frac{3c_1 t\, e^{-\frac{3t}{2}}}{2} - \frac{3c_2 e^{-\frac{3t}{2}}}{2}$$

substituting $y' = -4$ and $t = 0$ in the above gives

$$-4 = c_1 - \frac{3c_2}{2} \tag{2A}$$

Equations {1A,2A} are now solved for $\{c_1, c_2\}$. Solving for the constants gives

$$c_1 = -\frac{5}{2}$$
$$c_2 = 1$$

Substituting these values back in above solution results in

$$y = -\frac{5\,\mathrm{e}^{-\frac{3t}{2}} t}{2} + \mathrm{e}^{-\frac{3t}{2}}$$

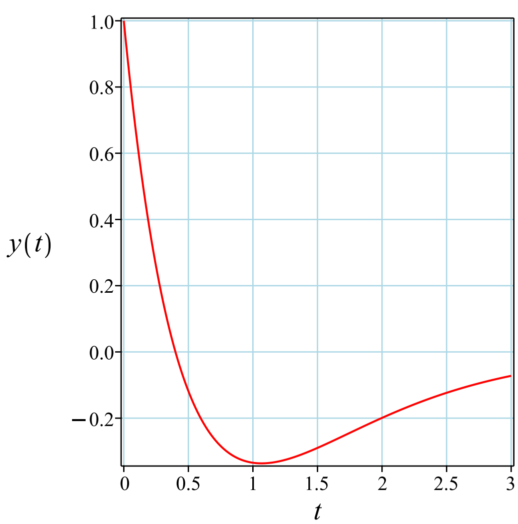
Which simplifies to

$$y = \mathrm{e}^{-\frac{3t}{2}}\left(1 - \frac{5t}{2}\right)$$

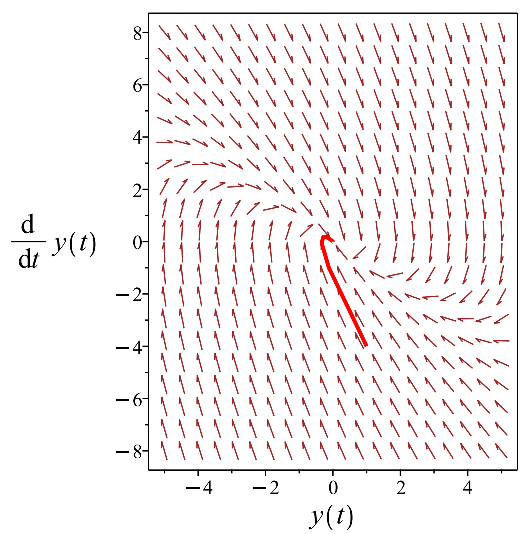
Summary

The solution(s) found are the following

$$y = \mathrm{e}^{-\frac{3t}{2}}\left(1 - \frac{5t}{2}\right) \tag{1}$$



(a) Solution plot



(b) Slope field plot

Verification of solutions

$$y = \mathrm{e}^{-\frac{3t}{2}}\left(1 - \frac{5t}{2}\right)$$

Verified OK.

9.15.4 Solving using Kovacic algorithm

Writing the ode as

$$
\begin{align*}
4y'' + 12y' + 9y &= 0 \tag{1} \\
Ay'' + By' + Cy &= 0 \tag{2}
\end{align*}
$$

Comparing (1) and (2) shows that

$$
\begin{align*}
A &= 4 \\
B &= 12 \tag{3} \\
C &= 9
\end{align*}
$$

Applying the Liouville transformation on the dependent variable gives

$$
z(t) = y e^{\int \frac{B}{2A} \, dt}
$$

Then (2) becomes

$$
z''(t) = r z(t) \tag{4}
$$

Where r is given by

$$
\begin{align*}
r &= \frac{s}{t} \tag{5} \\
&= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2}
\end{align*}
$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$
r = \frac{0}{1} \tag{6}
$$

Comparing the above to (5) shows that

$$
\begin{align*}
s &= 0 \\
t &= 1
\end{align*}
$$

Therefore eq. (4) becomes

$$
z''(t) = 0 \tag{7}
$$

Equation (7) is now solved. After finding $z(t)$ then y is found using the inverse transformation

$$
y = z(t) \, e^{-\int \frac{B}{2A} \, dt}
$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 450: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$
\begin{aligned}
O(\infty) &= \deg(t) - \deg(s) \\
&= 0 - -\infty \\
&= \infty
\end{aligned}
$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is $infinity$ then the necessary conditions for case one are met. Therefore

$$
L = [1]
$$

Since $r = 0$ is not a function of t, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$
z_1(t) = 1
$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$
\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dt} \\
&= z_1 e^{-\int \frac{1}{2}\frac{12}{4}\,dt}
\end{aligned}
$$

$$
\begin{aligned}
&= z_1 e^{-\frac{3t}{2}} \\
&= z_1 \left(\mathrm{e}^{-\frac{3t}{2}}\right)
\end{aligned}
$$

Which simplifies to

$$
y_1 = \mathrm{e}^{-\frac{3t}{2}}
$$

The second solution y_2 to the original ode is found using reduction of order

$$
y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\, dt}}{y_1^2}\, dt
$$

Substituting gives

$$
\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{12}{4}\, dt}}{(y_1)^2}\, dt \\
&= y_1 \int \frac{e^{-3t}}{(y_1)^2}\, dt \\
&= y_1(t)
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1 \left(\mathrm{e}^{-\frac{3t}{2}}\right) + c_2 \left(\mathrm{e}^{-\frac{3t}{2}}(t)\right)
\end{aligned}
$$

Initial conditions are used to solve for the constants of integration.

Looking at the above solution

$$
y = c_1 \mathrm{e}^{-\frac{3t}{2}} + c_2 t\, \mathrm{e}^{-\frac{3t}{2}} \tag{1}
$$

Initial conditions are now substituted in the above solution. This will generate the required equations to solve for the integration constants. substituting $y = 1$ and $t = 0$ in the above gives

$$
1 = c_1 \tag{1A}
$$

Taking derivative of the solution gives

$$
y' = -\frac{3c_1 \mathrm{e}^{-\frac{3t}{2}}}{2} + c_2 \mathrm{e}^{-\frac{3t}{2}} - \frac{3c_2 t\, \mathrm{e}^{-\frac{3t}{2}}}{2}
$$

substituting $y' = -4$ and $t = 0$ in the above gives

$$-4 = -\frac{3c_1}{2} + c_2 \tag{2A}$$

Equations {1A,2A} are now solved for $\{c_1, c_2\}$. Solving for the constants gives

$$\begin{aligned} c_1 &= 1 \\ c_2 &= -\frac{5}{2} \end{aligned}$$

Substituting these values back in above solution results in

$$y = -\frac{5\,\mathrm{e}^{-\frac{3t}{2}}t}{2} + \mathrm{e}^{-\frac{3t}{2}}$$

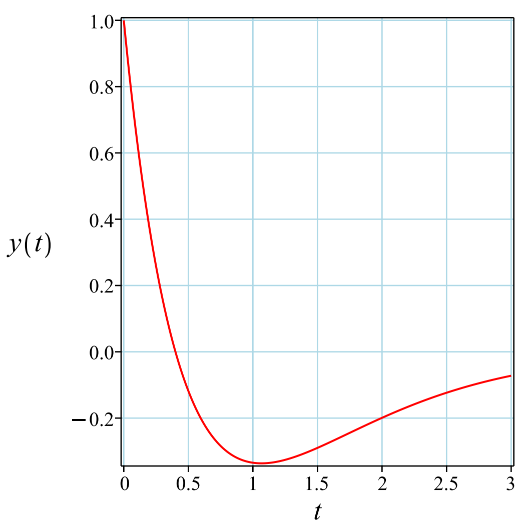
Which simplifies to

$$y = \mathrm{e}^{-\frac{3t}{2}}\left(1 - \frac{5t}{2}\right)$$

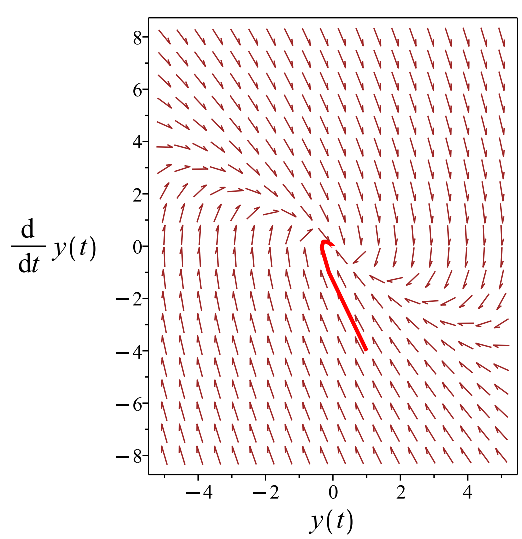
Summary

The solution(s) found are the following

$$y = \mathrm{e}^{-\frac{3t}{2}}\left(1 - \frac{5t}{2}\right) \tag{1}$$



(a) Solution plot



(b) Slope field plot

Verification of solutions

$$y = e^{-\frac{3t}{2}} \left(1 - \frac{5t}{2}\right)$$

Verified OK.

9.15.5 Maple step by step solution

Let's solve

$\left[4y'' + 12y' + 9y = 0, y(0) = 1, y'\Big|_{\{t=0\}} = -4\right]$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y'' = -3y' - \frac{9y}{4}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y'' + 3y' + \frac{9y}{4} = 0$

- Characteristic polynomial of ODE

  $r^2 + 3r + \frac{9}{4} = 0$

- Factor the characteristic polynomial

  $\frac{(2r+3)^2}{4} = 0$

- Root of the characteristic polynomial

  $r = -\frac{3}{2}$

- 1st solution of the ODE

  $y_1(t) = e^{-\frac{3t}{2}}$

- Repeated root, multiply $y_1(t)$ by t to ensure linear independence

  $y_2(t) = e^{-\frac{3t}{2}} t$

- General solution of the ODE

  $y = c_1 y_1(t) + c_2 y_2(t)$

- Substitute in solutions

  $y = c_1 e^{-\frac{3t}{2}} + c_2 t\, e^{-\frac{3t}{2}}$

□ Check validity of solution $y = c_1 e^{-\frac{3t}{2}} + c_2 t e^{-\frac{3t}{2}}$

- Use initial condition $y(0) = 1$

  $1 = c_1$
- Compute derivative of the solution

  $y' = -\frac{3c_1 \mathrm{e}^{-\frac{3t}{2}}}{2} + c_2 \mathrm{e}^{-\frac{3t}{2}} - \frac{3c_2 t\,\mathrm{e}^{-\frac{3t}{2}}}{2}$
- Use the initial condition $y'\Big|_{\{t=0\}} = -4$

  $-4 = -\frac{3c_1}{2} + c_2$
- Solve for c_1 and c_2

  $\left\{c_1 = 1, c_2 = -\frac{5}{2}\right\}$
- Substitute constant values into general solution and simplify

  $y = \mathrm{e}^{-\frac{3t}{2}} \left(1 - \frac{5t}{2}\right)$

• Solution to the IVP

  $y = \mathrm{e}^{-\frac{3t}{2}} \left(1 - \frac{5t}{2}\right)$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
<- constant coefficients successful`
```

✓ Solution by Maple

Time used: 0.015 (sec). Leaf size: 14

```
dsolve([4*diff(y(t),t$2)+12*diff(y(t),t)+9*y(t) = 0,y(0) = 1, D(y)(0) = -4],y(t), singsol=al
```

$$y(t) = -\frac{\mathrm{e}^{-\frac{3t}{2}}(-2 + 5t)}{2}$$

✓ Solution by Mathematica

Time used: 0.015 (sec). Leaf size: 21

```
DSolve[{4*y''[t]+12*y'[t]+9*y[t]==0,{y[0]==1,y'[0]==-4}},y[t],t,IncludeSingularSolutions ->
```

$$y(t) \to \frac{1}{2}e^{-3t/2}(2-5t)$$

9.16 problem 16

Internal problem ID [668]
Internal file name [`OUTPUT/668_Sunday_June_05_2022_01_46_40_AM_23845844/index.tex`]

Book: Elementary differential equations and boundary value problems, 10th ed., Boyce and DiPrima
Section: Chapter 3, Second order linear equations, 3.4 Repeated roots, reduction of order , page 172
Problem number: 16.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_linear_constant_coeff", "linear_second_order_ode_solved_by_an_integrating_factor"**

Maple gives the following as the ode type

```
[[_2nd_order, _missing_x]]
```

$$\boxed{y'' - y' + \frac{y}{4} = 0}$$

With initial conditions

$$[y(0) = 2, y'(0) = b]$$

9.16.1 Existence and uniqueness analysis

This is a linear ODE. In canonical form it is written as

$$y'' + p(t)y' + q(t)y = F$$

Where here

$$\begin{aligned} p(t) &= -1 \\ q(t) &= \frac{1}{4} \\ F &= 0 \end{aligned}$$

Hence the ode is

$$y'' - y' + \frac{y}{4} = 0$$

The domain of $p(t) = -1$ is

$$\{-\infty < t < \infty\}$$

And the point $t_0 = 0$ is inside this domain. The domain of $q(t) = \frac{1}{4}$ is

$$\{-\infty < t < \infty\}$$

And the point $t_0 = 0$ is also inside this domain. Hence solution exists and is unique.

9.16.2 Solving as second order linear constant coeff ode

This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$Ay''(t) + By'(t) + Cy(t) = 0$$

Where in the above $A = 1, B = -1, C = \frac{1}{4}$. Let the solution be $y = e^{\lambda t}$. Substituting this into the ODE gives

$$\lambda^2 e^{\lambda t} - \lambda e^{\lambda t} + \frac{e^{\lambda t}}{4} = 0 \tag{1}$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda t}$ gives

$$\lambda^2 - \lambda + \frac{1}{4} = 0 \tag{2}$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form.Using the quadratic formula

$$\lambda_{1,2} = \frac{-B}{2A} \pm \frac{1}{2A}\sqrt{B^2 - 4AC}$$

Substituting $A = 1, B = -1, C = \frac{1}{4}$ into the above gives

$$\begin{aligned}\lambda_{1,2} &= \frac{1}{(2)(1)} \pm \frac{1}{(2)(1)} \sqrt{(-1)^2 - (4)(1)\left(\frac{1}{4}\right)} \\ &= \frac{1}{2}\end{aligned}$$

Hence this is the case of a double root $\lambda_{1,2} = -\frac{1}{2}$. Therefore the solution is

$$y = c_1 e^{\frac{t}{2}} + c_2 t\, e^{\frac{t}{2}} \tag{1}$$

Initial conditions are used to solve for the constants of integration.

Looking at the above solution

$$y = c_1 e^{\frac{t}{2}} + c_2 t\, e^{\frac{t}{2}} \tag{1}$$

Initial conditions are now substituted in the above solution. This will generate the required equations to solve for the integration constants. substituting $y = 2$ and $t = 0$ in the above gives

$$2 = c_1 \tag{1A}$$

Taking derivative of the solution gives

$$y' = \frac{c_1 e^{\frac{t}{2}}}{2} + c_2 e^{\frac{t}{2}} + \frac{c_2 t\, e^{\frac{t}{2}}}{2}$$

substituting $y' = b$ and $t = 0$ in the above gives

$$b = \frac{c_1}{2} + c_2 \tag{2A}$$

Equations {1A,2A} are now solved for $\{c_1, c_2\}$. Solving for the constants gives

$$\begin{aligned}c_1 &= 2 \\ c_2 &= b - 1\end{aligned}$$

Substituting these values back in above solution results in

$$y = (b - 1)\, t\, e^{\frac{t}{2}} + 2\, e^{\frac{t}{2}}$$

Which simplifies to

$$y = (2 + t(b - 1))\, e^{\frac{t}{2}}$$

Summary

The solution(s) found are the following

$$y = (2 + t(b - 1))\,\mathrm{e}^{\frac{t}{2}} \tag{1}$$

Verification of solutions

$$y = (2 + t(b - 1))\,\mathrm{e}^{\frac{t}{2}}$$

Verified OK.

9.16.3 Solving as linear second order ode solved by an integrating factor ode

The ode satisfies this form

$$y'' + p(t)\,y' + \frac{\left(p(t)^2 + p'(t)\right)y}{2} = f(t)$$

Where $p(t) = -1$. Therefore, there is an integrating factor given by

$$\begin{aligned}
M(x) &= e^{\frac{1}{2}\int p\,dx} \\
&= e^{\int -1\,dx} \\
&= \mathrm{e}^{-\frac{t}{2}}
\end{aligned}$$

Multiplying both sides of the ODE by the integrating factor $M(x)$ makes the left side of the ODE a complete differential

$$\begin{aligned}
(M(x)y)'' &= 0 \\
\left(\mathrm{e}^{-\frac{t}{2}}y\right)'' &= 0
\end{aligned}$$

Integrating once gives

$$\left(\mathrm{e}^{-\frac{t}{2}}y\right)' = c_1$$

Integrating again gives

$$\left(\mathrm{e}^{-\frac{t}{2}}y\right) = c_1 t + c_2$$

Hence the solution is

$$y = \frac{c_1 t + c_2}{\mathrm{e}^{-\frac{t}{2}}}$$

Or

$$y = c_1 t\,\mathrm{e}^{\frac{t}{2}} + c_2 \mathrm{e}^{\frac{t}{2}}$$

Initial conditions are used to solve for the constants of integration.

Looking at the above solution

$$y = c_1 t \,e^{\frac{t}{2}} + c_2 e^{\frac{t}{2}} \tag{1}$$

Initial conditions are now substituted in the above solution. This will generate the required equations to solve for the integration constants. substituting $y = 2$ and $t = 0$ in the above gives

$$2 = c_2 \tag{1A}$$

Taking derivative of the solution gives

$$y' = c_1 e^{\frac{t}{2}} + \frac{c_1 t \,e^{\frac{t}{2}}}{2} + \frac{c_2 e^{\frac{t}{2}}}{2}$$

substituting $y' = b$ and $t = 0$ in the above gives

$$b = c_1 + \frac{c_2}{2} \tag{2A}$$

Equations {1A,2A} are now solved for $\{c_1, c_2\}$. Solving for the constants gives

$$c_1 = b - 1$$
$$c_2 = 2$$

Substituting these values back in above solution results in

$$y = t \,e^{\frac{t}{2}} b - t \,e^{\frac{t}{2}} + 2 \,e^{\frac{t}{2}}$$

Which simplifies to

$$y = (2 + t(b - 1)) \,e^{\frac{t}{2}}$$

Summary

The solution(s) found are the following

$$y = (2 + t(b - 1)) \,e^{\frac{t}{2}} \tag{1}$$

Verification of solutions

$$y = (2 + t(b - 1)) \,e^{\frac{t}{2}}$$

Verified OK.

9.16.4 Solving using Kovacic algorithm

Writing the ode as

$$y'' - y' + \frac{y}{4} = 0 \tag{1}$$

$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1 \\ B &= -1 \\ C &= \frac{1}{4} \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(t) = ye^{\int \frac{B}{2A} \, dt}$$

Then (2) becomes

$$z''(t) = rz(t) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{0}{1} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= 0 \\ t &= 1 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(t) = 0 \tag{7}$$

Equation (7) is now solved. After finding $z(t)$ then y is found using the inverse transformation

$$y = z(t) \, e^{-\int \frac{B}{2A} \, dt}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 452: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$
\begin{aligned}
O(\infty) &= \deg(t) - \deg(s) \\
&= 0 - -\infty \\
&= \infty
\end{aligned}
$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is $infinity$ then the necessary conditions for case one are met. Therefore

$$
L = [1]
$$

Since $r = 0$ is not a function of t, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$
z_1(t) = 1
$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$
\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2} \frac{B}{A} \, dt} \\
&= z_1 e^{-\int \frac{1}{2} \frac{-1}{1} \, dt}
\end{aligned}
$$

$$
\begin{aligned}
&= z_1 e^{\frac{t}{2}} \\
&= z_1 \left(\mathrm{e}^{\frac{t}{2}}\right)
\end{aligned}
$$

Which simplifies to

$$
y_1 = \mathrm{e}^{\frac{t}{2}}
$$

The second solution y_2 to the original ode is found using reduction of order

$$
y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\, dt}}{y_1^2}\, dt
$$

Substituting gives

$$
\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{-1}{1}\, dt}}{(y_1)^2}\, dt \\
&= y_1 \int \frac{e^t}{(y_1)^2}\, dt \\
&= y_1(t)
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1 \left(\mathrm{e}^{\frac{t}{2}}\right) + c_2 \left(\mathrm{e}^{\frac{t}{2}}(t)\right)
\end{aligned}
$$

Initial conditions are used to solve for the constants of integration.

Looking at the above solution

$$
y = c_1 \mathrm{e}^{\frac{t}{2}} + c_2 t\, \mathrm{e}^{\frac{t}{2}} \tag{1}
$$

Initial conditions are now substituted in the above solution. This will generate the required equations to solve for the integration constants. substituting $y = 2$ and $t = 0$ in the above gives

$$
2 = c_1 \tag{1A}
$$

Taking derivative of the solution gives

$$
y' = \frac{c_1 \mathrm{e}^{\frac{t}{2}}}{2} + c_2 \mathrm{e}^{\frac{t}{2}} + \frac{c_2 t\, \mathrm{e}^{\frac{t}{2}}}{2}
$$

substituting $y' = b$ and $t = 0$ in the above gives

$$b = \frac{c_1}{2} + c_2 \tag{2A}$$

Equations {1A,2A} are now solved for $\{c_1, c_2\}$. Solving for the constants gives

$$c_1 = 2$$
$$c_2 = b - 1$$

Substituting these values back in above solution results in

$$y = (b-1)\, t\, e^{\frac{t}{2}} + 2\, e^{\frac{t}{2}}$$

Which simplifies to

$$y = (2 + t(b-1))\, e^{\frac{t}{2}}$$

Summary

The solution(s) found are the following

$$y = (2 + t(b-1))\, e^{\frac{t}{2}} \tag{1}$$

Verification of solutions

$$y = (2 + t(b-1))\, e^{\frac{t}{2}}$$

Verified OK.

9.16.5 Maple step by step solution

Let's solve

$\left[y'' - y' + \frac{y}{4} = 0, y(0) = 2, y'\Big|_{\{t=0\}} = b\right]$

- Highest derivative means the order of the ODE is 2
  y''
- Characteristic polynomial of ODE
  $r^2 - r + \frac{1}{4} = 0$
- Factor the characteristic polynomial
  $\frac{(2r-1)^2}{4} = 0$
- Root of the characteristic polynomial

$r = \frac{1}{2}$

- 1st solution of the ODE

  $y_1(t) = e^{\frac{t}{2}}$

- Repeated root, multiply $y_1(t)$ by t to ensure linear independence

  $y_2(t) = t\, e^{\frac{t}{2}}$

- General solution of the ODE

  $y = c_1 y_1(t) + c_2 y_2(t)$

- Substitute in solutions

  $y = c_1 e^{\frac{t}{2}} + c_2 t\, e^{\frac{t}{2}}$

□ Check validity of solution $y = c_1 e^{\frac{t}{2}} + c_2 t e^{\frac{t}{2}}$

  ◦ Use initial condition $y(0) = 2$

    $2 = c_1$

  ◦ Compute derivative of the solution

    $y' = \frac{c_1 e^{\frac{t}{2}}}{2} + c_2 e^{\frac{t}{2}} + \frac{c_2 t\, e^{\frac{t}{2}}}{2}$

  ◦ Use the initial condition $y'\Big|_{\{t=0\}} = b$

    $b = \frac{c_1}{2} + c_2$

  ◦ Solve for c_1 and c_2

    $\{c_1 = 2, c_2 = b - 1\}$

  ◦ Substitute constant values into general solution and simplify

    $y = (2 + t(b-1))\, e^{\frac{t}{2}}$

- Solution to the IVP

  $y = (2 + t(b-1))\, e^{\frac{t}{2}}$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
<- constant coefficients successful`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 16

```
dsolve([diff(y(t),t$2)-diff(y(t),t)+25/100*y(t) = 0,y(0) = 2, D(y)(0) = b],y(t), singsol=all
```

$$y(t) = (2 + t(b - 1))\, \mathrm{e}^{\frac{t}{2}}$$

✓ Solution by Mathematica

Time used: 0.014 (sec). Leaf size: 20

```
DSolve[{y''[t]-y'[t]+25/100*y[t]==0,{y[0]==2,y'[0]==b}},y[t],t,IncludeSingularSolutions -> T
```

$$y(t) \to e^{t/2}((b - 1)t + 2)$$

9.17 problem 23

Internal problem ID [669]
Internal file name [`OUTPUT/669_Sunday_June_05_2022_01_46_41_AM_12322057/index.tex`]

Book: Elementary differential equations and boundary value problems, 10th ed., Boyce and DiPrima
Section: Chapter 3, Second order linear equations, 3.4 Repeated roots, reduction of order , page 172
Problem number: 23.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"reduction_of_order", "second_order_euler_ode", "second_order_change_of_variable_on_x_method_1", "second_order_change_of_variable_on_x_method_2", "second_order_change_of_variable_on_y_method_1", "second_order_change_of_variable_on_y_method_2", "linear_second_order_ode_solved_by_an_integrating_factor"**

Maple gives the following as the ode type

```
[[_Emden, _Fowler], [_2nd_order, _linear, `_with_symmetry_[0,F(
   x)]`]]
```

$$\boxed{t^2y'' - 4ty' + 6y = 0}$$

Given that one solution of the ode is

$$y_1 = t^2$$

Given one basis solution $y_1(t)$, then the second basis solution is given by

$$y_2(t) = y_1 \left(\int \frac{e^{-(\int p dt)}}{y_1^2} dt \right)$$

Where $p(x)$ is the coefficient of y' when the ode is written in the normal form

$$y'' + p(t)\, y' + q(t)\, y = f(t)$$

Looking at the ode to solve shows that

$$p(t) = -\frac{4}{t}$$

Therefore

$$y_2(t) = t^2\left(\int \frac{e^{-\left(\int -\frac{4}{t}dt\right)}}{t^4}dt\right)$$

$$y_2(t) = t^2 \int \frac{t^4}{t^4}, dt$$

$$y_2(t) = t^2\left(\int 1dt\right)$$

$$y_2(t) = t^3$$

Hence the solution is

$$\begin{aligned} y &= c_1y_1(t) + c_2y_2(t) \\ &= c_2t^3 + c_1t^2 \end{aligned}$$

Summary

The solution(s) found are the following

$$y = c_2t^3 + c_1t^2 \tag{1}$$

Verification of solutions

$$y = c_2t^3 + c_1t^2$$

Verified OK.

9.17.1 Maple step by step solution

Let's solve

$y''t^2 - 4ty' + 6y = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y'' = \frac{4y'}{t} - \frac{6y}{t^2}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y'' - \frac{4y'}{t} + \frac{6y}{t^2} = 0$

- Multiply by denominators of the ODE

  $y''t^2 - 4ty' + 6y = 0$

- Make a change of variables

  $s = \ln(t)$

☐ Substitute the change of variables back into the ODE

  - Calculate the 1st derivative of y with respect to t , using the chain rule

    $y' = \left(\frac{d}{ds}y(s)\right) s'(t)$

  - Compute derivative

    $y' = \frac{\frac{d}{ds}y(s)}{t}$

  - Calculate the 2nd derivative of y with respect to t , using the chain rule

    $y'' = \left(\frac{d^2}{ds^2}y(s)\right) s'(t)^2 + s''(t) \left(\frac{d}{ds}y(s)\right)$

  - Compute derivative

    $y'' = \frac{\frac{d^2}{ds^2}y(s)}{t^2} - \frac{\frac{d}{ds}y(s)}{t^2}$

  Substitute the change of variables back into the ODE

  $\left(\frac{\frac{d^2}{ds^2}y(s)}{t^2} - \frac{\frac{d}{ds}y(s)}{t^2}\right) t^2 - 4\frac{d}{ds}y(s) + 6y(s) = 0$

- Simplify

  $\frac{d^2}{ds^2}y(s) - 5\frac{d}{ds}y(s) + 6y(s) = 0$

- Characteristic polynomial of ODE

  $r^2 - 5r + 6 = 0$

- Factor the characteristic polynomial

  $(r-2)(r-3) = 0$

- Roots of the characteristic polynomial

  $r = (2, 3)$

- 1st solution of the ODE

  $y_1(s) = e^{2s}$

- 2nd solution of the ODE

$y_2(s) = e^{3s}$

- General solution of the ODE

  $y(s) = c_1 y_1(s) + c_2 y_2(s)$

- Substitute in solutions

  $y(s) = c_1 e^{2s} + c_2 e^{3s}$

- Change variables back using $s = \ln(t)$

  $y = c_2 t^3 + c_1 t^2$

- Simplify

  $y = t^2(c_2 t + c_1)$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
<- LODE of Euler type successful`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 13

```
dsolve([t^2*diff(y(t),t$2)-4*t*diff(y(t),t)+6*y(t)=0,t^2],singsol=all)
```

$$y(t) = t^2(c_2 t + c_1)$$

✓ Solution by Mathematica

Time used: 0.012 (sec). Leaf size: 16

```
DSolve[t^2*y''[t]-4*t*y'[t]+6*y[t]==0,y[t],t,IncludeSingularSolutions -> True]
```

$$y(t) \to t^2(c_2 t + c_1)$$

9.18 problem 24

Internal problem ID [670]
Internal file name [OUTPUT/670_Sunday_June_05_2022_01_46_42_AM_6399244/index.tex]

Book: Elementary differential equations and boundary value problems, 10th ed., Boyce and DiPrima
Section: Chapter 3, Second order linear equations, 3.4 Repeated roots, reduction of order , page 172
Problem number: 24.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"reduction_of_order", "second_order_euler_ode", "second_order_change_of_variable_on_x_method_2", "second_order_change_of_variable_on_y_method_2", "second_order_ode_non_constant_coeff_transformation_on_B"**

Maple gives the following as the ode type

```
[[_Emden, _Fowler]]
```

$$\boxed{t^2y'' + 2ty' - 2y = 0}$$

Given that one solution of the ode is

$$y_1 = t$$

Given one basis solution $y_1(t)$, then the second basis solution is given by

$$y_2(t) = y_1 \left(\int \frac{e^{-(\int p dt)}}{y_1^2} dt \right)$$

Where $p(x)$ is the coefficient of y' when the ode is written in the normal form

$$y'' + p(t)\, y' + q(t)\, y = f(t)$$

Looking at the ode to solve shows that

$$p(t) = \frac{2}{t}$$

Therefore

$$y_2(t) = t\left(\int \frac{e^{-\left(\int \frac{2}{t}dt\right)}}{t^2}dt\right)$$

$$y_2(t) = t\int \frac{\frac{1}{t^2}}{t^2}, dt$$

$$y_2(t) = t\left(\int \frac{1}{t^4}dt\right)$$

$$y_2(t) = -\frac{1}{3t^2}$$

Hence the solution is

$$\begin{aligned} y &= c_1 y_1(t) + c_2 y_2(t) \\ &= c_1 t - \frac{c_2}{3t^2} \end{aligned}$$

Summary

The solution(s) found are the following

$$y = c_1 t - \frac{c_2}{3t^2} \tag{1}$$

Verification of solutions

$$y = c_1 t - \frac{c_2}{3t^2}$$

Verified OK.

9.18.1 Maple step by step solution

Let's solve

$y''t^2 + 2ty' - 2y = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y'' = -\frac{2y'}{t} + \frac{2y}{t^2}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y'' + \frac{2y'}{t} - \frac{2y}{t^2} = 0$

- Multiply by denominators of the ODE

  $y'' t^2 + 2ty' - 2y = 0$
- Make a change of variables

  $s = \ln(t)$

□ Substitute the change of variables back into the ODE
  - Calculate the 1st derivative of y with respect to t , using the chain rule

    $y' = \left(\frac{d}{ds}y(s)\right) s'(t)$
  - Compute derivative

    $y' = \frac{\frac{d}{ds}y(s)}{t}$
  - Calculate the 2nd derivative of y with respect to t , using the chain rule

    $y'' = \left(\frac{d^2}{ds^2}y(s)\right) s'(t)^2 + s''(t)\left(\frac{d}{ds}y(s)\right)$
  - Compute derivative

    $y'' = \frac{\frac{d^2}{ds^2}y(s)}{t^2} - \frac{\frac{d}{ds}y(s)}{t^2}$

  Substitute the change of variables back into the ODE

  $\left(\frac{\frac{d^2}{ds^2}y(s)}{t^2} - \frac{\frac{d}{ds}y(s)}{t^2}\right) t^2 + 2\frac{d}{ds}y(s) - 2y(s) = 0$
- Simplify

  $\frac{d^2}{ds^2}y(s) + \frac{d}{ds}y(s) - 2y(s) = 0$
- Characteristic polynomial of ODE

  $r^2 + r - 2 = 0$
- Factor the characteristic polynomial

  $(r+2)(r-1) = 0$
- Roots of the characteristic polynomial

  $r = (-2, 1)$
- 1st solution of the ODE

  $y_1(s) = e^{-2s}$
- 2nd solution of the ODE

  $y_2(s) = e^{s}$
- General solution of the ODE

$$y(s) = c_1 y_1(s) + c_2 y_2(s)$$

- Substitute in solutions

$$y(s) = c_1 \mathrm{e}^{-2s} + c_2 \mathrm{e}^{s}$$

- Change variables back using $s = \ln(t)$

$$y = \frac{c_1}{t^2} + c_2 t$$

- Simplify

$$y = \frac{c_1}{t^2} + c_2 t$$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
<- LODE of Euler type successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 15

```
dsolve([t^2*diff(y(t),t$2)+2*t*diff(y(t),t)-2*y(t)=0,t],singsol=all)
```

$$y(t) = \frac{c_1 t^3 + c_2}{t^2}$$

✓ Solution by Mathematica

Time used: 0.01 (sec). Leaf size: 16

```
DSolve[t^2*y''[t]+2*t*y'[t]-2*y[t]==0,y[t],t,IncludeSingularSolutions -> True]
```

$$y(t) \to \frac{c_1}{t^2} + c_2 t$$

9.19 problem 25

Internal problem ID [671]
Internal file name [`OUTPUT/671_Sunday_June_05_2022_01_46_43_AM_97850935/index.tex`]

Book: Elementary differential equations and boundary value problems, 10th ed., Boyce and DiPrima
Section: Chapter 3, Second order linear equations, 3.4 Repeated roots, reduction of order , page 172
Problem number: 25.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"reduction_of_order", "second_order_euler_ode", "exact linear second order ode", "second_order_integrable_as_is", "second_order_change_of_variable_on_x_method_1", "second_order_change_of_variable_on_x_method_2", "second_order_change_of_variable_on_y_method_2"**

Maple gives the following as the ode type

```
[[_2nd_order, _exact, _linear, _homogeneous]]
```

$$\boxed{t^2y'' + 3ty' + y = 0}$$

Given that one solution of the ode is

$$y_1 = \frac{1}{t}$$

Given one basis solution $y_1(t)$, then the second basis solution is given by

$$y_2(t) = y_1 \left(\int \frac{e^{-(\int p dt)}}{y_1^2} dt \right)$$

Where $p(x)$ is the coefficient of y' when the ode is written in the normal form

$$y'' + p(t)\, y' + q(t)\, y = f(t)$$

Looking at the ode to solve shows that

$$p(t) = \frac{3}{t}$$

Therefore

$$y_2(t) = \frac{\int \mathrm{e}^{-\left(\int \frac{3}{t}dt\right)} t^2 dt}{t}$$

$$y_2(t) = \frac{1}{t} \int \frac{\frac{1}{t^3}}{\frac{1}{t^2}}, dt$$

$$y_2(t) = \frac{\int \frac{1}{t} dt}{t}$$

$$y_2(t) = \frac{\ln(t)}{t}$$

Hence the solution is

$$\begin{aligned} y &= c_1 y_1(t) + c_2 y_2(t) \\ &= \frac{c_1}{t} + \frac{c_2 \ln(t)}{t} \end{aligned}$$

Summary

The solution(s) found are the following

$$y = \frac{c_1}{t} + \frac{c_2 \ln(t)}{t} \tag{1}$$

Verification of solutions

$$y = \frac{c_1}{t} + \frac{c_2 \ln(t)}{t}$$

Verified OK.

9.19.1 Maple step by step solution

Let's solve

$y''t^2 + 3ty' + y = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y'' = -\frac{3y'}{t} - \frac{y}{t^2}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y'' + \frac{3y'}{t} + \frac{y}{t^2} = 0$

- Multiply by denominators of the ODE

  $y''t^2 + 3ty' + y = 0$

- Make a change of variables

  $s = \ln(t)$

□ Substitute the change of variables back into the ODE

  - Calculate the 1st derivative of y with respect to t , using the chain rule

    $y' = \left(\frac{d}{ds}y(s)\right) s'(t)$

  - Compute derivative

    $y' = \frac{\frac{d}{ds}y(s)}{t}$

  - Calculate the 2nd derivative of y with respect to t , using the chain rule

    $y'' = \left(\frac{d^2}{ds^2}y(s)\right) s'(t)^2 + s''(t) \left(\frac{d}{ds}y(s)\right)$

  - Compute derivative

    $y'' = \frac{\frac{d^2}{ds^2}y(s)}{t^2} - \frac{\frac{d}{ds}y(s)}{t^2}$

  Substitute the change of variables back into the ODE

  $\left(\frac{\frac{d^2}{ds^2}y(s)}{t^2} - \frac{\frac{d}{ds}y(s)}{t^2}\right) t^2 + 3\frac{d}{ds}y(s) + y(s) = 0$

- Simplify

  $\frac{d^2}{ds^2}y(s) + 2\frac{d}{ds}y(s) + y(s) = 0$

- Characteristic polynomial of ODE

  $r^2 + 2r + 1 = 0$

- Factor the characteristic polynomial

  $(r+1)^2 = 0$

- Root of the characteristic polynomial

  $r = -1$

- 1st solution of the ODE

  $y_1(s) = e^{-s}$

- Repeated root, multiply $y_1(s)$ by s to ensure linear independence

  $y_2(s) = s\,e^{-s}$

- General solution of the ODE

$$y(s) = c_1 y_1(s) + c_2 y_2(s)$$

- Substitute in solutions

$$y(s) = c_1 e^{-s} + c_2 s\, e^{-s}$$

- Change variables back using $s = \ln(t)$

$$y = \frac{c_1}{t} + \frac{c_2 \ln(t)}{t}$$

- Simplify

$$y = \frac{c_1}{t} + \frac{c_2 \ln(t)}{t}$$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
<- LODE of Euler type successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 14

```
dsolve([t^2*diff(y(t),t$2)+3*t*diff(y(t),t)+y(t)=0,1/t],singsol=all)
```

$$y(t) = \frac{c_2 \ln(t) + c_1}{t}$$

✓ Solution by Mathematica

Time used: 0.016 (sec). Leaf size: 17

```
DSolve[t^2*y''[t]+3*t*y'[t]+y[t]==0,y[t],t,IncludeSingularSolutions -> True]
```

$$y(t) \to \frac{c_2 \log(t) + c_1}{t}$$

9.20 problem 26

Internal problem ID [672]
Internal file name [`OUTPUT/672_Sunday_June_05_2022_01_46_43_AM_19696878/index.tex`]

Book: Elementary differential equations and boundary value problems, 10th ed., Boyce and DiPrima
Section: Chapter 3, Second order linear equations, 3.4 Repeated roots, reduction of order , page 172
Problem number: 26.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"reduction_of_order", "second_order_change_of_variable_on_y_method_1", "second_order_change_of_variable_on_y_method_2"**

Maple gives the following as the ode type

`[[_2nd_order, _with_linear_symmetries]]`

$$\boxed{t^2 y'' - t(2+t)\, y' + (2+t)\, y = 0}$$

Given that one solution of the ode is

$$y_1 = t$$

Given one basis solution $y_1(t)$, then the second basis solution is given by

$$y_2(t) = y_1 \left(\int \frac{\mathrm{e}^{-(\int p dt)}}{y_1^2} dt \right)$$

Where $p(x)$ is the coefficient of y' when the ode is written in the normal form

$$y'' + p(t)\, y' + q(t)\, y = f(t)$$

Looking at the ode to solve shows that

$$p(t) = \frac{-t^2 - 2t}{t^2}$$

Therefore

$$y_2(t) = t\left(\int \frac{e^{-\left(\int \frac{-t^2-2t}{t^2}dt\right)}}{t^2}dt\right)$$

$$y_2(t) = t\int \frac{e^{t+2\ln(t)}}{t^2}, dt$$

$$y_2(t) = t\left(\int e^t dt\right)$$

$$y_2(t) = t\,e^t$$

Hence the solution is

$$\begin{aligned} y &= c_1 y_1(t) + c_2 y_2(t) \\ &= c_1 t + c_2 t\,e^t \end{aligned}$$

Summary

The solution(s) found are the following

$$y = c_1 t + c_2 t\,e^t \tag{1}$$

Verification of solutions

$$y = c_1 t + c_2 t\,e^t$$

Verified OK.

9.20.1 Maple step by step solution

Let's solve

$y''t^2 + (-t^2 - 2t)\,y' + (2+t)\,y = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y'' = -\frac{(2+t)y}{t^2} + \frac{(2+t)y'}{t}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y'' - \frac{(2+t)y'}{t} + \frac{(2+t)y}{t^2} = 0$

□ Check to see if $t_0 = 0$ is a regular singular point

- Define functions

  $\left[P_2(t)=-\frac{2+t}{t}, P_3(t)=\frac{2+t}{t^2}\right]$

- $t\cdot P_2(t)$ is analytic at $t=0$

  $\left(t\cdot P_2(t)\right)\bigg|_{t=0}=-2$

- $t^2\cdot P_3(t)$ is analytic at $t=0$

  $\left(t^2\cdot P_3(t)\right)\bigg|_{t=0}=2$

- $t=0$is a regular singular point

  Check to see if $t_0=0$ is a regular singular point

  $t_0=0$

- Multiply by denominators

  $y''t^2-t(2+t)\,y'+(2+t)\,y=0$

- Assume series solution for y

  $y=\sum_{k=0}^{\infty} a_k t^{k+r}$

☐ Rewrite ODE with series expansions

- Convert $t^m\cdot y$ to series expansion for $m=0..1$

  $t^m\cdot y=\sum_{k=0}^{\infty} a_k t^{k+r+m}$

- Shift index using $k->k-m$

  $t^m\cdot y=\sum_{k=m}^{\infty} a_{k-m} t^{k+r}$

- Convert $t^m\cdot y'$ to series expansion for $m=1..2$

  $t^m\cdot y'=\sum_{k=0}^{\infty} a_k(k+r)\,t^{k+r-1+m}$

- Shift index using $k->k+1-m$

  $t^m\cdot y'=\sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\,t^{k+r}$

- Convert $t^2\cdot y''$ to series expansion

  $t^2\cdot y''=\sum_{k=0}^{\infty} a_k(k+r)\,(k+r-1)\,t^{k+r}$

  Rewrite ODE with series expansions

$$a_0(-1+r)(-2+r)t^r + \left(\sum_{k=1}^{\infty} \left(a_k(k+r-1)(k+r-2) - a_{k-1}(k+r-2)\right) t^{k+r}\right) = 0$$

- $a_0$cannot be 0 by assumption, giving the indicial equation
  $(-1+r)(-2+r) = 0$
- Values of r that satisfy the indicial equation
  $r \in \{1, 2\}$
- Each term in the series must be 0, giving the recursion relation
  $(k+r-2)(a_k(k+r-1) - a_{k-1}) = 0$
- Shift index using $k- > k+1$
  $(k+r-1)(a_{k+1}(k+r) - a_k) = 0$
- Recursion relation that defines series solution to ODE
  $a_{k+1} = \frac{a_k}{k+r}$
- Recursion relation for $r = 1$
  $a_{k+1} = \frac{a_k}{k+1}$
- Solution for $r = 1$
  $\left[y = \sum_{k=0}^{\infty} a_k t^{k+1}, a_{k+1} = \frac{a_k}{k+1}\right]$
- Recursion relation for $r = 2$
  $a_{k+1} = \frac{a_k}{k+2}$
- Solution for $r = 2$
  $\left[y = \sum_{k=0}^{\infty} a_k t^{k+2}, a_{k+1} = \frac{a_k}{k+2}\right]$
- Combine solutions and rename parameters
  $\left[y = \left(\sum_{k=0}^{\infty} a_k t^{k+1}\right) + \left(\sum_{k=0}^{\infty} b_k t^{k+2}\right), a_{k+1} = \frac{a_k}{k+1}, b_{k+1} = \frac{b_k}{k+2}\right]$

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Reducible group (found another exponential solution)
<- Kovacics algorithm successful`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 12

```
dsolve([t^2*diff(y(t),t$2)-t*(t+2)*diff(y(t),t)+(t+2)*y(t)=0,t],singsol=all)
```

$$y(t) = t(c_1 + c_2 e^t)$$

✓ Solution by Mathematica

Time used: 0.025 (sec). Leaf size: 16

```
DSolve[t^2*y''[t]-t*(t+2)*y'[t]+(t+2)*y[t]==0,y[t],t,IncludeSingularSolutions -> True]
```

$$y(t) \to t(c_2 e^t + c_1)$$

9.21 problem 27

Internal problem ID [673]
Internal file name [OUTPUT/673_Sunday_June_05_2022_01_46_44_AM_69159856/index.tex]

Book: Elementary differential equations and boundary value problems, 10th ed., Boyce and DiPrima
Section: Chapter 3, Second order linear equations, 3.4 Repeated roots, reduction of order , page 172
Problem number: 27.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"reduction_of_order", "second_order_bessel_ode", "second_order_change_of_variable_on_x_method_1", "second_order_change_of_variable_on_x_method_2"**

Maple gives the following as the ode type

```
[[_Emden, _Fowler], [_2nd_order, _linear, `_with_symmetry_[0,F(
   x)]`]]
```

$$\boxed{xy'' - y' + 4yx^3 = 0}$$

Given that one solution of the ode is

$$y_1 = \sin(x^2)$$

Given one basis solution $y_1(x)$, then the second basis solution is given by

$$y_2(x) = y_1 \left(\int \frac{e^{-(\int p dx)}}{y_1^2} dx \right)$$

Where $p(x)$ is the coefficient of y' when the ode is written in the normal form

$$y'' + p(x)\, y' + q(x)\, y = f(x)$$

Looking at the ode to solve shows that

$$p(x) = -\frac{1}{x}$$

Therefore

$$y_2(x) = \sin\left(x^2\right)\left(\int \frac{e^{-\left(\int -\frac{1}{x}dx\right)}}{\sin\left(x^2\right)^2}dx\right)$$

$$y_2(x) = \sin\left(x^2\right)\int \frac{x}{\sin\left(x^2\right)^2}, dx$$

$$y_2(x) = \sin\left(x^2\right)\left(\int \csc\left(x^2\right)^2 x dx\right)$$

$$y_2(x) = -\frac{\sin\left(x^2\right)\cot\left(x^2\right)}{2}$$

Hence the solution is

$$\begin{aligned} y &= c_1 y_1(x) + c_2 y_2(x) \\ &= \sin\left(x^2\right) c_1 - \frac{c_2 \sin\left(x^2\right)\cot\left(x^2\right)}{2} \end{aligned}$$

Summary

The solution(s) found are the following

$$y = \sin\left(x^2\right) c_1 - \frac{c_2 \sin\left(x^2\right)\cot\left(x^2\right)}{2} \tag{1}$$

Verification of solutions

$$y = \sin\left(x^2\right) c_1 - \frac{c_2 \sin\left(x^2\right)\cot\left(x^2\right)}{2}$$

Verified OK.

9.21.1 Maple step by step solution

Let's solve

$y''x - y' + 4yx^3 = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y'' = \frac{y'}{x} - 4x^2 y$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

$y'' - \frac{y'}{x} + 4x^2 y = 0$

- ☐ Check to see if $x_0 = 0$ is a regular singular point
  - ◦ Define functions

    $\left[P_2(x) = -\frac{1}{x}, P_3(x) = 4x^2\right]$
  - ◦ $x \cdot P_2(x)$ is analytic at $x = 0$

    $(x \cdot P_2(x))\Big|_{x=0} = -1$
  - ◦ $x^2 \cdot P_3(x)$ is analytic at $x = 0$

    $(x^2 \cdot P_3(x))\Big|_{x=0} = 0$
  - ◦ $x = 0$is a regular singular point

    Check to see if $x_0 = 0$ is a regular singular point

    $x_0 = 0$
- Multiply by denominators

  $y''x - y' + 4yx^3 = 0$
- Assume series solution for y

  $y = \sum_{k=0}^{\infty} a_k x^{k+r}$
- ☐ Rewrite ODE with series expansions
  - ◦ Convert $x^3 \cdot y$ to series expansion

    $x^3 \cdot y = \sum_{k=0}^{\infty} a_k x^{k+r+3}$
  - ◦ Shift index using $k- >k - 3$

    $x^3 \cdot y = \sum_{k=3}^{\infty} a_{k-3} x^{k+r}$
  - ◦ Convert y' to series expansion

    $y' = \sum_{k=0}^{\infty} a_k (k + r)\, x^{k+r-1}$
  - ◦ Shift index using $k- >k + 1$

    $y' = \sum_{k=-1}^{\infty} a_{k+1} (k + 1 + r)\, x^{k+r}$
  - ◦ Convert $x \cdot y''$ to series expansion

$$x \cdot y'' = \sum_{k=0}^{\infty} a_k (k+r)(k+r-1) x^{k+r-1}$$

- Shift index using $k->k+1$

$$x \cdot y'' = \sum_{k=-1}^{\infty} a_{k+1} (k+1+r)(k+r) x^{k+r}$$

Rewrite ODE with series expansions

$$a_0 r(-2+r) x^{-1+r} + a_1 (1+r)(-1+r) x^r + a_2 (2+r) r x^{1+r} + a_3 (3+r)(1+r) x^{2+r} + \left(\sum_{k=3}^{\infty} (a_k$$

- $a_0$cannot be 0 by assumption, giving the indicial equation

$r(-2+r) = 0$

- Values of r that satisfy the indicial equation

$r \in \{0, 2\}$

- The coefficients of each power of x must be 0

$[a_1(1+r)(-1+r) = 0, a_2(2+r) r = 0, a_3(3+r)(1+r) = 0]$

- Solve for the dependent coefficient(s)

$\{a_1 = 0, a_2 = 0, a_3 = 0\}$

- Each term in the series must be 0, giving the recursion relation

$a_{k+1}(k+1+r)(k+r-1) + 4a_{k-3} = 0$

- Shift index using $k->k+3$

$a_{k+4}(k+4+r)(k+2+r) + 4a_k = 0$

- Recursion relation that defines series solution to ODE

$a_{k+4} = -\frac{4a_k}{(k+4+r)(k+2+r)}$

- Recursion relation for $r = 0$

$a_{k+4} = -\frac{4a_k}{(k+4)(k+2)}$

- Solution for $r = 0$

$$\left[y = \sum_{k=0}^{\infty} a_k x^k, a_{k+4} = -\frac{4a_k}{(k+4)(k+2)}, a_1 = 0, a_2 = 0, a_3 = 0\right]$$

- Recursion relation for $r = 2$

$a_{k+4} = -\frac{4a_k}{(k+6)(k+4)}$

- Solution for $r = 2$

$$\left[y=\sum_{k=0}^{\infty} a_k x^{k+2}, a_{k+4}=-\frac{4a_k}{(k+6)(k+4)}, a_1=0, a_2=0, a_3=0\right]$$

- Combine solutions and rename parameters

$$\left[y=\left(\sum_{k=0}^{\infty} a_k x^{k}\right)+\left(\sum_{k=0}^{\infty} b_k x^{k+2}\right), a_{k+4}=-\frac{4a_k}{(k+4)(k+2)}, a_1=0, a_2=0, a_3=0, b_{k+4}=-\frac{4b_k}{(k+6)(k+4)}\right.$$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
<- linear_1 successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 17

```
dsolve([x*diff(y(x),x$2)-diff(y(x),x)+4*x^3*y(x)=0,sin(x^2)],singsol=all)
```

$$y(x)=c_1\sin\left(x^2\right)+c_2\cos\left(x^2\right)$$

✓ Solution by Mathematica

Time used: 0.019 (sec). Leaf size: 20

```
DSolve[x*y''[x]-y'[x]+4*x^3*y[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x)\to c_1\cos\left(x^2\right)+c_2\sin\left(x^2\right)$$

9.22 problem 28

Internal problem ID [674]
Internal file name [`OUTPUT/674_Sunday_June_05_2022_01_46_45_AM_64926509/index.tex`]

Book: Elementary differential equations and boundary value problems, 10th ed., Boyce and DiPrima
Section: Chapter 3, Second order linear equations, 3.4 Repeated roots, reduction of order , page 172
Problem number: 28.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"reduction_of_order", "second_order_change_of_variable_on_y_method_2", "second_order_ode_non_constant_coeff_transformation_on_B"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

$$\boxed{(x-1)\, y''-y' x+y=0}$$

Given that one solution of the ode is

$$y_1 = \mathrm{e}^{x}$$

Given one basis solution $y_1(x)$, then the second basis solution is given by

$$y_2(x) = y_1 \left(\int \frac{\mathrm{e}^{-\left(\int p dx\right)}}{y_1^2} dx\right)$$

Where $p(x)$ is the coefficient of y' when the ode is written in the normal form

$$y''+p(x)\, y'+q(x)\, y=f(x)$$

Looking at the ode to solve shows that

$$p(x) = -\frac{x}{x-1}$$

Therefore

$$y_2(x) = e^x \left(\int e^{-\left(\int -\frac{x}{x-1} dx\right)} e^{-2x} dx \right)$$

$$y_2(x) = e^x \int \frac{e^{x+\ln(x-1)}}{e^{2x}}, dx$$

$$y_2(x) = e^x \left(\int (x-1)\, e^{-x} dx \right)$$

$$y_2(x) = -e^x x\, e^{-x}$$

Hence the solution is

$$\begin{aligned} y &= c_1 y_1(x) + c_2 y_2(x) \\ &= c_1 e^x - c_2 e^x x\, e^{-x} \end{aligned}$$

Summary

The solution(s) found are the following

$$y = c_1 e^x - c_2 e^x x\, e^{-x} \tag{1}$$

Verification of solutions

$$y = c_1 e^x - c_2 e^x x\, e^{-x}$$

Verified OK.

9.22.1 Maple step by step solution

Let's solve

$(x-1)\, y'' - y'x + y = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y'' = -\frac{y}{x-1} + \frac{xy'}{x-1}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y'' - \frac{xy'}{x-1} + \frac{y}{x-1} = 0$

□ Check to see if $x_0 = 1$ is a regular singular point

  ◦ Define functions

$\left[P_2(x) = -\frac{x}{x-1}, P_3(x) = \frac{1}{x-1}\right]$

- $(x-1) \cdot P_2(x)$ is analytic at $x = 1$

  $$((x-1) \cdot P_2(x))\bigg|_{x=1} = -1$$

- $(x-1)^2 \cdot P_3(x)$ is analytic at $x = 1$

  $$\left((x-1)^2 \cdot P_3(x)\right)\bigg|_{x=1} = 0$$

- $x = 1$is a regular singular point

  Check to see if $x_0 = 1$ is a regular singular point

  $x_0 = 1$

- Multiply by denominators

  $(x-1)\, y'' - y'x + y = 0$

- Change variables using $x = u + 1$ so that the regular singular point is at $u = 0$

  $u\left(\frac{d^2}{du^2} y(u)\right) + (-u-1)\left(\frac{d}{du} y(u)\right) + y(u) = 0$

- Assume series solution for $y(u)$

  $$y(u) = \sum_{k=0}^{\infty} a_k u^{k+r}$$

☐ Rewrite ODE with series expansions

- Convert $u^m \cdot \left(\frac{d}{du} y(u)\right)$ to series expansion for $m = 0..1$

  $$u^m \cdot \left(\frac{d}{du} y(u)\right) = \sum_{k=0}^{\infty} a_k (k+r)\, u^{k+r-1+m}$$

- Shift index using $k- >k+1-m$

  $$u^m \cdot \left(\frac{d}{du} y(u)\right) = \sum_{k=-1+m}^{\infty} a_{k+1-m} (k+1-m+r)\, u^{k+r}$$

- Convert $u \cdot \left(\frac{d^2}{du^2} y(u)\right)$ to series expansion

  $$u \cdot \left(\frac{d^2}{du^2} y(u)\right) = \sum_{k=0}^{\infty} a_k (k+r)\,(k+r-1)\, u^{k+r-1}$$

- Shift index using $k- >k+1$

  $$u \cdot \left(\frac{d^2}{du^2} y(u)\right) = \sum_{k=-1}^{\infty} a_{k+1} (k+1+r)\,(k+r)\, u^{k+r}$$

  Rewrite ODE with series expansions

$$a_0 r(-2+r)\, u^{-1+r} + \left(\sum_{k=0}^{\infty} \left(a_{k+1}(k+1+r)\,(k+r-1) - a_k(k+r-1)\right) u^{k+r}\right) = 0$$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $r(-2+r) = 0$

- Values of r that satisfy the indicial equation

  $r \in \{0, 2\}$

- Each term in the series must be 0, giving the recursion relation

  $(k+r-1)\,(a_{k+1}(k+1+r) - a_k) = 0$

- Recursion relation that defines series solution to ODE

  $a_{k+1} = \frac{a_k}{k+1+r}$

- Recursion relation for $r = 0$

  $a_{k+1} = \frac{a_k}{k+1}$

- Solution for $r = 0$

  $\left[y(u) = \sum_{k=0}^{\infty} a_k u^k, a_{k+1} = \frac{a_k}{k+1}\right]$

- Revert the change of variables $u = x - 1$

  $\left[y = \sum_{k=0}^{\infty} a_k (x-1)^k, a_{k+1} = \frac{a_k}{k+1}\right]$

- Recursion relation for $r = 2$

  $a_{k+1} = \frac{a_k}{k+3}$

- Solution for $r = 2$

  $\left[y(u) = \sum_{k=0}^{\infty} a_k u^{k+2}, a_{k+1} = \frac{a_k}{k+3}\right]$

- Revert the change of variables $u = x - 1$

  $\left[y = \sum_{k=0}^{\infty} a_k (x-1)^{k+2}, a_{k+1} = \frac{a_k}{k+3}\right]$

- Combine solutions and rename parameters

  $\left[y = \left(\sum_{k=0}^{\infty} a_k (x-1)^k\right) + \left(\sum_{k=0}^{\infty} b_k (x-1)^{k+2}\right), a_{k+1} = \frac{a_k}{k+1}, b_{k+1} = \frac{b_k}{k+3}\right]$

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Reducible group (found another exponential solution)
<- Kovacics algorithm successful`
```

✓ Solution by Maple

Time used: 0.015 (sec). Leaf size: 12

```
dsolve([(x-1)*diff(y(x),x$2)-x*diff(y(x),x)+y(x)=0,exp(x)],singsol=all)
```

$$y(x) = c_1 x + e^x c_2$$

✓ Solution by Mathematica

Time used: 0.031 (sec). Leaf size: 17

```
DSolve[(x-1)*y''[x]-x*y'[x]+y[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to c_1 e^x - c_2 x$$

9.23 problem 29

Internal problem ID [675]
Internal file name [`OUTPUT/675_Sunday_June_05_2022_01_46_46_AM_68547472/index.tex`]

Book: Elementary differential equations and boundary value problems, 10th ed., Boyce and DiPrima
Section: Chapter 3, Second order linear equations, 3.4 Repeated roots, reduction of order , page 172
Problem number: 29.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"reduction_of_order", "second_order_bessel_ode"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

$$\boxed{x^2y'' - \left(x - \frac{3}{16}\right)y = 0}$$

Given that one solution of the ode is

$$y_1 = x^{\frac{1}{4}}e^{2\sqrt{x}}$$

Given one basis solution $y_1(x)$, then the second basis solution is given by

$$y_2(x) = y_1\left(\int \frac{e^{-\left(\int p dx\right)}}{y_1^2} dx\right)$$

Where $p(x)$ is the coefficient of y' when the ode is written in the normal form

$$y'' + p(x)\, y' + q(x)\, y = f(x)$$

Looking at the ode to solve shows that

$$p(x) = 0$$

Therefore

$$y_2(x) = x^{\frac{1}{4}} e^{2\sqrt{x}} \left(\int \frac{e^{-\left(\int 0 dx\right)} e^{-4\sqrt{x}}}{\sqrt{x}} dx\right)$$

$$y_2(x) = x^{\frac{1}{4}} e^{2\sqrt{x}} \int \frac{1}{\sqrt{x}\, e^{4\sqrt{x}}}, dx$$

$$y_2(x) = x^{\frac{1}{4}} e^{2\sqrt{x}} \left(\int \frac{e^{-4\sqrt{x}}}{\sqrt{x}} dx\right)$$

$$y_2(x) = -\frac{x^{\frac{1}{4}} e^{2\sqrt{x}} e^{-4\sqrt{x}}}{2}$$

Hence the solution is

$$\begin{aligned} y &= c_1 y_1(x) + c_2 y_2(x) \\ &= x^{\frac{1}{4}} e^{2\sqrt{x}} c_1 - \frac{c_2 x^{\frac{1}{4}} e^{2\sqrt{x}} e^{-4\sqrt{x}}}{2} \end{aligned}$$

Summary

The solution(s) found are the following

$$y = x^{\frac{1}{4}} e^{2\sqrt{x}} c_1 - \frac{c_2 x^{\frac{1}{4}} e^{2\sqrt{x}} e^{-4\sqrt{x}}}{2} \tag{1}$$

Verification of solutions

$$y = x^{\frac{1}{4}} e^{2\sqrt{x}} c_1 - \frac{c_2 x^{\frac{1}{4}} e^{2\sqrt{x}} e^{-4\sqrt{x}}}{2}$$

Verified OK.

9.23.1 Maple step by step solution

Let's solve

$x^2 y'' + \left(-x + \frac{3}{16}\right) y = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y'' = \frac{(16x-3)y}{16x^2}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

$y'' - \frac{(16x-3)y}{16x^2} = 0$

□ Check to see if $x_0 = 0$ is a regular singular point

- Define functions

  $\left[P_2(x) = 0, P_3(x) = -\frac{16x-3}{16x^2}\right]$

- $x \cdot P_2(x)$ is analytic at $x = 0$

  $\left(x \cdot P_2(x)\right)\Big|_{x=0} = 0$

- $x^2 \cdot P_3(x)$ is analytic at $x = 0$

  $\left(x^2 \cdot P_3(x)\right)\Big|_{x=0} = \frac{3}{16}$

- $x = 0$is a regular singular point

  Check to see if $x_0 = 0$ is a regular singular point

  $x_0 = 0$

- Multiply by denominators

  $16x^2y'' + (-16x + 3)\, y = 0$

- Assume series solution for y

  $y = \sum_{k=0}^{\infty} a_k x^{k+r}$

□ Rewrite ODE with series expansions

- Convert $x^m \cdot y$ to series expansion for $m = 0..1$

  $x^m \cdot y = \sum_{k=0}^{\infty} a_k x^{k+r+m}$

- Shift index using $k- > k - m$

  $x^m \cdot y = \sum_{k=m}^{\infty} a_{k-m} x^{k+r}$

- Convert $x^2 \cdot y''$ to series expansion

  $x^2 \cdot y'' = \sum_{k=0}^{\infty} a_k (k+r)\,(k+r-1)\, x^{k+r}$

  Rewrite ODE with series expansions

  $a_0(-1+4r)\,(-3+4r)\, x^r + \left(\sum_{k=1}^{\infty} \left(a_k(4k+4r-1)\,(4k+4r-3) - 16a_{k-1}\right) x^{k+r}\right) = 0$

- $a_0$cannot be 0 by assumption, giving the indicial equation

$$(-1+4r)(-3+4r)=0$$

- Values of r that satisfy the indicial equation

$$r \in \left\{\frac{1}{4}, \frac{3}{4}\right\}$$

- Each term in the series must be 0, giving the recursion relation

$$16\left(k+r-\frac{3}{4}\right)\left(k+r-\frac{1}{4}\right)a_k - 16a_{k-1} = 0$$

- Shift index using $k- >k+1$

$$16\left(k+\frac{1}{4}+r\right)\left(k+\frac{3}{4}+r\right)a_{k+1} - 16a_k = 0$$

- Recursion relation that defines series solution to ODE

$$a_{k+1} = \frac{16a_k}{(4k+1+4r)(4k+3+4r)}$$

- Recursion relation for $r=\frac{1}{4}$

$$a_{k+1} = \frac{16a_k}{(4k+2)(4k+4)}$$

- Solution for $r=\frac{1}{4}$

$$\left[y = \sum_{k=0}^{\infty} a_k x^{k+\frac{1}{4}}, a_{k+1} = \frac{16a_k}{(4k+2)(4k+4)}\right]$$

- Recursion relation for $r=\frac{3}{4}$

$$a_{k+1} = \frac{16a_k}{(4k+4)(4k+6)}$$

- Solution for $r=\frac{3}{4}$

$$\left[y = \sum_{k=0}^{\infty} a_k x^{k+\frac{3}{4}}, a_{k+1} = \frac{16a_k}{(4k+4)(4k+6)}\right]$$

- Combine solutions and rename parameters

$$\left[y = \left(\sum_{k=0}^{\infty} a_k x^{k+\frac{1}{4}}\right) + \left(\sum_{k=0}^{\infty} b_k x^{k+\frac{3}{4}}\right), a_{k+1} = \frac{16a_k}{(4k+2)(4k+4)}, b_{k+1} = \frac{16b_k}{(4k+4)(4k+6)}\right]$$

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Group is reducible or imprimitive
<- Kovacics algorithm successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 25

```
dsolve([x^2*diff(y(x),x$2)-(x-1875/10000)*y(x)=0,x^(1/4)*exp(2*sqrt(x))],singsol=all)
```

$$y(x) = x^{\frac{1}{4}}\left(c_1 \sinh\left(2\sqrt{x}\right) + c_2 \cosh\left(2\sqrt{x}\right)\right)$$

✓ Solution by Mathematica

Time used: 0.047 (sec). Leaf size: 41

```
DSolve[x^2*y''[x]-(x-1875/10000)*y[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \frac{1}{2}e^{-2\sqrt{x}}\sqrt[4]{x}\left(2c_1 e^{4\sqrt{x}} - c_2\right)$$

9.24 problem 30

Internal problem ID [676]
Internal file name [OUTPUT/676_Sunday_June_05_2022_01_46_46_AM_36615809/index.tex]

Book: Elementary differential equations and boundary value problems, 10th ed., Boyce and DiPrima
Section: Chapter 3, Second order linear equations, 3.4 Repeated roots, reduction of order , page 172
Problem number: 30.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"reduction_of_order", "second_order_bessel_ode", "second_order_change_of_variable_on_y_method_1"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

$$\boxed{x^2y'' + y'x + \left(x^2 - \frac{1}{4}\right)y = 0}$$

Given that one solution of the ode is

$$y_1 = \frac{\sin(x)}{\sqrt{x}}$$

Given one basis solution $y_1(x)$, then the second basis solution is given by

$$y_2(x) = y_1 \left(\int \frac{e^{-(\int p dx)}}{y_1^2} dx \right)$$

Where $p(x)$ is the coefficient of y' when the ode is written in the normal form

$$y'' + p(x)\, y' + q(x)\, y = f(x)$$

Looking at the ode to solve shows that

$$p(x) = \frac{1}{x}$$

Therefore

$$y_2(x) = \frac{\sin(x)\left(\int \frac{\mathrm{e}^{-\left(\int \frac{1}{x}dx\right)}x}{\sin(x)^2}dx\right)}{\sqrt{x}}$$

$$y_2(x) = \frac{\sin(x)}{\sqrt{x}}\int \frac{\frac{1}{x}}{\frac{\sin(x)^2}{x}}, dx$$

$$y_2(x) = \frac{\sin(x)\left(\int \csc(x)^2\, dx\right)}{\sqrt{x}}$$

$$y_2(x) = -\frac{\sin(x)\cot(x)}{\sqrt{x}}$$

Hence the solution is

$$\begin{aligned} y &= c_1 y_1(x) + c_2 y_2(x) \\ &= \frac{\sin(x)\, c_1}{\sqrt{x}} - \frac{c_2 \sin(x)\cot(x)}{\sqrt{x}} \end{aligned}$$

Summary

The solution(s) found are the following

$$y = \frac{\sin(x)\, c_1}{\sqrt{x}} - \frac{c_2 \sin(x)\cot(x)}{\sqrt{x}} \tag{1}$$

Verification of solutions

$$y = \frac{\sin(x)\, c_1}{\sqrt{x}} - \frac{c_2 \sin(x)\cot(x)}{\sqrt{x}}$$

Verified OK.

9.24.1 Maple step by step solution

Let's solve

$x^2 y'' + y'x + \left(x^2 - \frac{1}{4}\right) y = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y'' = -\frac{(4x^2-1)y}{4x^2} - \frac{y'}{x}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y''+\frac{y'}{x}+\frac{(4x^2-1)y}{4x^2}=0$

□ Check to see if $x_0=0$ is a regular singular point

  - Define functions

    $\left[P_2(x)=\frac{1}{x}, P_3(x)=\frac{4x^2-1}{4x^2}\right]$

  - $x\cdot P_2(x)$ is analytic at $x=0$

    $$\left(x\cdot P_2(x)\right)\bigg|_{x=0}=1$$

  - $x^2\cdot P_3(x)$ is analytic at $x=0$

    $$\left(x^2\cdot P_3(x)\right)\bigg|_{x=0}=-\frac{1}{4}$$

  - $x=0$is a regular singular point

    Check to see if $x_0=0$ is a regular singular point

    $x_0=0$

- Multiply by denominators

  $4x^2y''+4y'x+(4x^2-1)\,y=0$

- Assume series solution for y

  $$y=\sum_{k=0}^{\infty}a_kx^{k+r}$$

□ Rewrite ODE with series expansions

  - Convert $x^m\cdot y$ to series expansion for $m=0..2$

    $$x^m\cdot y=\sum_{k=0}^{\infty}a_kx^{k+r+m}$$

  - Shift index using $k->k-m$

    $$x^m\cdot y=\sum_{k=m}^{\infty}a_{k-m}x^{k+r}$$

  - Convert $x\cdot y'$ to series expansion

    $$x\cdot y'=\sum_{k=0}^{\infty}a_k(k+r)\,x^{k+r}$$

  - Convert $x^2\cdot y''$ to series expansion

    $$x^2\cdot y''=\sum_{k=0}^{\infty}a_k(k+r)\,(k+r-1)\,x^{k+r}$$

Rewrite ODE with series expansions

$$a_0(1+2r)(-1+2r)x^r + a_1(3+2r)(1+2r)x^{1+r} + \left(\sum_{k=2}^{\infty} (a_k(2k+2r+1)(2k+2r-1) + 4a_{k-}\right.$$

- $a_0$cannot be 0 by assumption, giving the indicial equation

$$(1+2r)(-1+2r) = 0$$

- Values of r that satisfy the indicial equation

$$r \in \left\{-\frac{1}{2}, \frac{1}{2}\right\}$$

- Each term must be 0

$$a_1(3+2r)(1+2r) = 0$$

- Solve for the dependent coefficient(s)

$$a_1 = 0$$

- Each term in the series must be 0, giving the recursion relation

$$a_k(4k^2 + 8kr + 4r^2 - 1) + 4a_{k-2} = 0$$

- Shift index using $k- > k+2$

$$a_{k+2}\left(4(k+2)^2 + 8(k+2)r + 4r^2 - 1\right) + 4a_k = 0$$

- Recursion relation that defines series solution to ODE

$$a_{k+2} = -\frac{4a_k}{4k^2+8kr+4r^2+16k+16r+15}$$

- Recursion relation for $r = -\frac{1}{2}$

$$a_{k+2} = -\frac{4a_k}{4k^2+12k+8}$$

- Solution for $r = -\frac{1}{2}$

$$\left[y = \sum_{k=0}^{\infty} a_k x^{k-\frac{1}{2}}, a_{k+2} = -\frac{4a_k}{4k^2+12k+8}, a_1 = 0\right]$$

- Recursion relation for $r = \frac{1}{2}$

$$a_{k+2} = -\frac{4a_k}{4k^2+20k+24}$$

- Solution for $r = \frac{1}{2}$

$$\left[y = \sum_{k=0}^{\infty} a_k x^{k+\frac{1}{2}}, a_{k+2} = -\frac{4a_k}{4k^2+20k+24}, a_1 = 0\right]$$

- Combine solutions and rename parameters

$$\left[y = \left(\sum_{k=0}^{\infty} a_k x^{k-\frac{1}{2}}\right) + \left(\sum_{k=0}^{\infty} b_k x^{k+\frac{1}{2}}\right), a_{k+2} = -\frac{4a_k}{4k^2+12k+8}, a_1 = 0, b_{k+2} = -\frac{4b_k}{4k^2+20k+24}, b_1 = 0\right]$$

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Group is reducible or imprimitive
<- Kovacics algorithm successful`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 17

```
dsolve([x^2*diff(y(x),x$2)+x*diff(y(x),x)+(x^2-25/100)*y(x)=0,x^(-1/2)*sin(x)],singsol=all)
```

$$y(x) = \frac{c_1 \sin(x) + c_2 \cos(x)}{\sqrt{x}}$$

✓ Solution by Mathematica

Time used: 0.042 (sec). Leaf size: 39

```
DSolve[x^2*y''[x]+x*y'[x]+(x^2-25/100)*y[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \frac{e^{-ix}(2c_1 - ic_2 e^{2ix})}{2\sqrt{x}}$$

9.25 problem 40

Internal problem ID [677]
Internal file name [`OUTPUT/677_Sunday_June_05_2022_01_46_47_AM_3427519/index.tex`]

Book: Elementary differential equations and boundary value problems, 10th ed., Boyce and DiPrima
Section: Chapter 3, Second order linear equations, 3.4 Repeated roots, reduction of order , page 172
Problem number: 40.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_euler_ode", "second_order_change_of_variable_on_x_method_1", "second_order_change_of_variable_on_x_method_2", "second_order_change_of_variable_on_y_method_2"**

Maple gives the following as the ode type

```
[[_Emden, _Fowler], [_2nd_order, _linear, `_with_symmetry_[0,F(
   x)]`]]
```

$$\boxed{t^2y'' - 3ty' + 4y = 0}$$

9.25.1 Solving as second order euler ode ode

This is Euler second order ODE. Let the solution be $y = t^r$, then $y' = rt^{r-1}$ and $y'' = r(r-1)t^{r-2}$. Substituting these back into the given ODE gives

$$t^2(r(r-1))t^{r-2} - 3trt^{r-1} + 4t^r = 0$$

Simplifying gives

$$r(r-1)\,t^r - 3r\,t^r + 4t^r = 0$$

Since $t^r \neq 0$ then dividing throughout by t^r gives

$$r(r-1) - 3r + 4 = 0$$

Or

$$r^2 - 4r + 4 = 0 \tag{1}$$

Equation (1) is the characteristic equation. Its roots determine the form of the general solution. Using the quadratic equation the roots are

$$\begin{aligned} r_1 &= 2 \\ r_2 &= 2 \end{aligned}$$

Since the roots are equal, then the general solution is

$$y = c_1 y_1 + c_2 y_2$$

Where $y_1 = t^r$ and $y_2 = t^r \ln(t)$. Hence

$$y = c_1 t^2 + c_2 t^2 \ln(t)$$

Summary

The solution(s) found are the following

$$y = c_1 t^2 + c_2 t^2 \ln(t) \tag{1}$$

Verification of solutions

$$y = c_1 t^2 + c_2 t^2 \ln(t)$$

Verified OK.

9.25.2 Solving as second order change of variable on x method 2 ode

In normal form the ode

$$t^2 y'' - 3ty' + 4y = 0 \tag{1}$$

Becomes

$$y'' + p(t)\, y' + q(t)\, y = 0 \tag{2}$$

Where

$$\begin{aligned} p(t) &= -\frac{3}{t} \\ q(t) &= \frac{4}{t^2} \end{aligned}$$

Applying change of variables $\tau = g(t)$ to (2) gives

$$\frac{d^2}{d\tau^2}y(\tau) + p_1\left(\frac{d}{d\tau}y(\tau)\right) + q_1 y(\tau) = 0 \tag{3}$$

Where τ is the new independent variable, and

$$p_1(\tau) = \frac{\tau''(t) + p(t)\,\tau'(t)}{\tau'(t)^2} \tag{4}$$

$$q_1(\tau) = \frac{q(t)}{\tau'(t)^2} \tag{5}$$

Let $p_1 = 0$. Eq (4) simplifies to

$$\tau''(t) + p(t)\,\tau'(t) = 0$$

This ode is solved resulting in

$$\begin{aligned}
\tau &= \int \mathrm{e}^{-\left(\int p(t)dt\right)} dt \\
&= \int \mathrm{e}^{-\left(\int -\frac{3}{t}dt\right)} dt \\
&= \int e^{3\ln(t)}\, dt \\
&= \int t^3 dt \\
&= \frac{t^4}{4}
\end{aligned} \tag{6}$$

Using (6) to evaluate q_1 from (5) gives

$$\begin{aligned}
q_1(\tau) &= \frac{q(t)}{\tau'(t)^2} \\
&= \frac{\frac{4}{t^2}}{t^6} \\
&= \frac{4}{t^8}
\end{aligned} \tag{7}$$

Substituting the above in (3) and noting that now $p_1 = 0$ results in

$$\frac{d^2}{d\tau^2}y(\tau) + q_1 y(\tau) = 0$$
$$\frac{d^2}{d\tau^2}y(\tau) + \frac{4y(\tau)}{t^8} = 0$$

But in terms of τ

$$\frac{4}{t^8} = \frac{1}{4\tau^2}$$

Hence the above ode becomes

$$\frac{d^2}{d\tau^2}y(\tau) + \frac{y(\tau)}{4\tau^2} = 0$$

The above ode is now solved for $y(\tau)$. The ode can be written as

$$4\left(\frac{d^2}{d\tau^2}y(\tau)\right)\tau^2 + y(\tau) = 0$$

Which shows it is a Euler ODE. This is Euler second order ODE. Let the solution be $y(\tau) = \tau^r$, then $y' = r\tau^{r-1}$ and $y'' = r(r-1)\tau^{r-2}$. Substituting these back into the given ODE gives

$$4\tau^2(r(r-1))\tau^{r-2} + 0r\tau^{r-1} + \tau^r = 0$$

Simplifying gives

$$4r(r-1)\,\tau^r + 0\,\tau^r + \tau^r = 0$$

Since $\tau^r \neq 0$ then dividing throughout by τ^r gives

$$4r(r-1) + 0 + 1 = 0$$

Or

$$4r^2 - 4r + 1 = 0 \tag{1}$$

Equation (1) is the characteristic equation. Its roots determine the form of the general solution. Using the quadratic equation the roots are

$$r_1 = \frac{1}{2}$$
$$r_2 = \frac{1}{2}$$

Since the roots are equal, then the general solution is

$$y(\tau) = c_1y_1 + c_2y_2$$

Where $y_1 = \tau^r$ and $y_2 = \tau^r \ln(\tau)$. Hence

$$y(\tau) = c_1\sqrt{\tau} + c_2\sqrt{\tau}\,\ln(\tau)$$

The above solution is now transformed back to y using (6) which results in

$$y = \frac{(-2c_2\ln(2) + c_2\ln(t^4) + c_1)\sqrt{t^4}}{2}$$

Summary

The solution(s) found are the following

$$y = \frac{\left(-2c_2 \ln \left(2\right)+c_2 \ln \left(t^4\right)+c_1\right) \sqrt{t^4}}{2} \tag{1}$$

Verification of solutions

$$y = \frac{\left(-2c_2 \ln \left(2\right)+c_2 \ln \left(t^4\right)+c_1\right) \sqrt{t^4}}{2}$$

Verified OK.

9.25.3 Solving as second order change of variable on x method 1 ode

In normal form the ode

$$t^2y''-3ty'+4y = 0 \tag{1}$$

Becomes

$$y''+p(t)\,y'+q(t)\,y = 0 \tag{2}$$

Where

$$\begin{aligned}
p(t) &= -\frac{3}{t}\\
q(t) &= \frac{4}{t^2}
\end{aligned}$$

Applying change of variables $\tau = g(t)$ to (2) results

$$\frac{d^2}{d\tau^2}y(\tau)+p_1\left(\frac{d}{d\tau}y(\tau)\right)+q_1y(\tau) = 0 \tag{3}$$

Where τ is the new independent variable, and

$$p_1(\tau) = \frac{\tau''(t)+p(t)\,\tau'(t)}{\tau'\left(t\right)^2} \tag{4}$$

$$q_1(\tau) = \frac{q(t)}{\tau'\left(t\right)^2} \tag{5}$$

Let $q_1 = c^2$ where c is some constant. Therefore from (5)

$$\begin{aligned}
\tau' &= \frac{1}{c}\sqrt{q}\\
&= \frac{2\sqrt{\frac{1}{t^2}}}{c} \qquad (6)\\
\tau'' &= -\frac{2}{c\sqrt{\frac{1}{t^2}}\,t^3}
\end{aligned}$$

Substituting the above into (4) results in

$$
\begin{aligned}
p_1(\tau) &= \frac{\tau''(t) + p(t)\,\tau'(t)}{\tau'(t)^2} \\
&= \frac{-\frac{2}{c\sqrt{\frac{1}{t^2}}\,t^3} - \frac{3}{t}\frac{2\sqrt{\frac{1}{t^2}}}{c}}{\left(\frac{2\sqrt{\frac{1}{t^2}}}{c}\right)^2} \\
&= -2c
\end{aligned}
$$

Therefore ode (3) now becomes

$$
\begin{aligned}
y(\tau)'' + p_1 y(\tau)' + q_1 y(\tau) &= 0 \\
\frac{d^2}{d\tau^2}y(\tau) - 2c\left(\frac{d}{d\tau}y(\tau)\right) + c^2 y(\tau) &= 0
\end{aligned} \tag{7}
$$

The above ode is now solved for $y(\tau)$. Since the ode is now constant coefficients, it can be easily solved to give

$$
y(\tau) = \mathrm{e}^{c\tau} c_1
$$

Now from (6)

$$
\begin{aligned}
\tau &= \int \frac{1}{c}\sqrt{q}\,dt \\
&= \frac{\int 2\sqrt{\frac{1}{t^2}}dt}{c} \\
&= \frac{2\sqrt{\frac{1}{t^2}}\,t\ln(t)}{c}
\end{aligned}
$$

Substituting the above into the solution obtained gives

$$
y = c_1 t^2
$$

Summary

The solution(s) found are the following

$$
y = c_1 t^2 \tag{1}
$$

Verification of solutions

$$
y = c_1 t^2
$$

Verified OK.

9.25.4 Solving as second order change of variable on y method 2 ode

In normal form the ode

$$t^2y'' - 3ty' + 4y = 0 \tag{1}$$

Becomes

$$y'' + p(t)\, y' + q(t)\, y = 0 \tag{2}$$

Where

$$\begin{aligned} p(t) &= -\frac{3}{t} \\ q(t) &= \frac{4}{t^2} \end{aligned}$$

Applying change of variables on the depndent variable $y = v(t)\, t^n$ to (2) gives the following ode where the dependent variables is $v(t)$ and not y.

$$v''(t) + \left(\frac{2n}{t} + p\right) v'(t) + \left(\frac{n(n-1)}{t^2} + \frac{np}{t} + q\right) v(t) = 0 \tag{3}$$

Let the coefficient of $v(t)$ above be zero. Hence

$$\frac{n(n-1)}{t^2} + \frac{np}{t} + q = 0 \tag{4}$$

Substituting the earlier values found for $p(t)$ and $q(t)$ into (4) gives

$$\frac{n(n-1)}{t^2} - \frac{3n}{t^2} + \frac{4}{t^2} = 0 \tag{5}$$

Solving (5) for n gives

$$n = 2 \tag{6}$$

Substituting this value in (3) gives

$$\begin{aligned} v''(t) + \frac{v'(t)}{t} &= 0 \\ v''(t) + \frac{v'(t)}{t} &= 0 \end{aligned} \tag{7}$$

Using the substitution

$$u(t) = v'(t)$$

Then (7) becomes

$$u'(t) + \frac{u(t)}{t} = 0 \tag{8}$$

The above is now solved for $u(t)$. In canonical form the ODE is

$$\begin{aligned} u' &= F(t, u) \\ &= f(t)g(u) \\ &= -\frac{u}{t} \end{aligned}$$

Where $f(t) = -\frac{1}{t}$ and $g(u) = u$. Integrating both sides gives

$$\begin{aligned} \frac{1}{u}\,du &= -\frac{1}{t}\,dt \\ \int \frac{1}{u}\,du &= \int -\frac{1}{t}\,dt \\ \ln(u) &= -\ln(t) + c_1 \\ u &= e^{-\ln(t)+c_1} \\ &= \frac{c_1}{t} \end{aligned}$$

Now that $u(t)$ is known, then

$$\begin{aligned} v'(t) &= u(t) \\ v(t) &= \int u(t)\,dt + c_2 \\ &= c_1 \ln(t) + c_2 \end{aligned}$$

Hence

$$\begin{aligned} y &= v(t)\,t^n \\ &= (c_1 \ln(t) + c_2)\,t^2 \\ &= (c_1 \ln(t) + c_2)\,t^2 \end{aligned}$$

Summary

The solution(s) found are the following

$$y = (c_1 \ln(t) + c_2)\,t^2 \tag{1}$$

Verification of solutions

$$y = (c_1 \ln(t) + c_2)\,t^2$$

Verified OK.

9.25.5 Solving using Kovacic algorithm

Writing the ode as

$$t^2y'' - 3ty' + 4y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= t^2 \\ B &= -3t \\ C &= 4 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(t) = ye^{\int \frac{B}{2A}\,dt}$$

Then (2) becomes

$$z''(t) = rz(t) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{-1}{4t^2} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= -1 \\ t &= 4t^2 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(t) = \left(-\frac{1}{4t^2}\right) z(t) \tag{7}$$

Equation (7) is now solved. After finding $z(t)$ then y is found using the inverse transformation

$$y = z(t)\, e^{-\int \frac{B}{2A}\,dt}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 462: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 2 - 0 \\ &= 2 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 4t^2$. There is a pole at $t = 0$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 2 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Since pole order is not larger than 2 and the order at ∞ is 2 then the necessary conditions for case three are met. Therefore

$$L = [1, 2, 4, 6, 12]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = -\frac{1}{4t^2}$$

For the pole at $t = 0$ let b be the coefficient of $\frac{1}{t^2}$ in the partial fractions decomposition of r given above. Therefore $b = -\frac{1}{4}$. Hence

$$\begin{aligned}
[\sqrt{r}]_c &= 0 \\
\alpha_c^{+} &= \frac{1}{2} + \sqrt{1+4b} = \frac{1}{2} \\
\alpha_c^{-} &= \frac{1}{2} - \sqrt{1+4b} = \frac{1}{2}
\end{aligned}$$

Since the order of r at ∞ is 2 then $[\sqrt{r}]_\infty = 0$. Let b be the coefficient of $\frac{1}{t^2}$ in the Laurent series expansion of r at ∞. which can be found by dividing the leading coefficient of s by the leading coefficient of t from

$$r = \frac{s}{t} = -\frac{1}{4t^2}$$

Since the $\gcd(s,t) = 1$. This gives $b = -\frac{1}{4}$. Hence

$$\begin{aligned}
[\sqrt{r}]_\infty &= 0 \\
\alpha_\infty^{+} &= \frac{1}{2} + \sqrt{1+4b} = \frac{1}{2} \\
\alpha_\infty^{-} &= \frac{1}{2} - \sqrt{1+4b} = \frac{1}{2}
\end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = -\frac{1}{4t^2}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^{+}	α_c^{-}
0	2	0	$\frac{1}{2}$	$\frac{1}{2}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^{+}	α_∞^{-}
2	0	$\frac{1}{2}$	$\frac{1}{2}$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^{\pm}$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^{\pm}$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^{\pm}$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω.

Trying $\alpha_\infty^- = \frac{1}{2}$ then

$$\begin{aligned} d &= \alpha_\infty^- - (\alpha_{c_1}^-) \\ &= \frac{1}{2} - \left(\frac{1}{2}\right) \\ &= 0 \end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c \in \Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{t - c} \right) + s(\infty)[\sqrt{r}]_\infty$$

The above gives

$$\begin{aligned} \omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^-}{t - c_1} \right) + (-)[\sqrt{r}]_\infty \\ &= \frac{1}{2t} + (-)(0) \\ &= \frac{1}{2t} \\ &= \frac{1}{2t} \end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(t)$ of degree $d = 0$ to solve the ode. The polynomial $p(t)$ needs to satisfy the equation

$$p'' + 2\omega p' + (\omega' + \omega^2 - r)\, p = 0 \tag{1A}$$

Let

$$p(t) = 1 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$\begin{aligned} (0) + 2\left(\frac{1}{2t}\right)(0) + \left(\left(-\frac{1}{2t^2}\right) + \left(\frac{1}{2t}\right)^2 - \left(-\frac{1}{4t^2}\right) \right) &= 0 \\ 0 &= 0 \end{aligned}$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned} z_1(t) &= p e^{\int \omega \, dt} \\ &= e^{\int \frac{1}{2t} dt} \\ &= \sqrt{t} \end{aligned}$$

The first solution to the original ode in y is found from

$$
\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dt} \\
&= z_1 e^{-\int \frac{1}{2}\frac{-3t}{t^2}\,dt} \\
&= z_1 e^{\frac{3\ln(t)}{2}} \\
&= z_1\left(t^{\frac{3}{2}}\right)
\end{aligned}
$$

Which simplifies to

$$
y_1 = t^2
$$

The second solution y_2 to the original ode is found using reduction of order

$$
y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dt}}{y_1^2}\,dt
$$

Substituting gives

$$
\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{-3t}{t^2}\,dt}}{(y_1)^2}\,dt \\
&= y_1 \int \frac{e^{3\ln(t)}}{(y_1)^2}\,dt \\
&= y_1(\ln(t))
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1\left(t^2\right) + c_2\left(t^2(\ln(t))\right)
\end{aligned}
$$

Summary
The solution(s) found are the following

$$
y = c_1 t^2 + c_2 t^2 \ln(t) \tag{1}
$$

Verification of solutions

$$
y = c_1 t^2 + c_2 t^2 \ln(t)
$$

Verified OK.

9.25.6 Maple step by step solution

Let's solve

$y''t^2 - 3ty' + 4y = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y'' = \frac{3y'}{t} - \frac{4y}{t^2}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y'' - \frac{3y'}{t} + \frac{4y}{t^2} = 0$

- Multiply by denominators of the ODE

  $y''t^2 - 3ty' + 4y = 0$

- Make a change of variables

  $s = \ln(t)$

□ Substitute the change of variables back into the ODE

- ○ Calculate the 1st derivative of y with respect to t , using the chain rule

  $y' = \left(\frac{d}{ds}y(s)\right) s'(t)$

- ○ Compute derivative

  $y' = \frac{\frac{d}{ds}y(s)}{t}$

- ○ Calculate the 2nd derivative of y with respect to t , using the chain rule

  $y'' = \left(\frac{d^2}{ds^2}y(s)\right) s'(t)^2 + s''(t)\left(\frac{d}{ds}y(s)\right)$

- ○ Compute derivative

  $y'' = \frac{\frac{d^2}{ds^2}y(s)}{t^2} - \frac{\frac{d}{ds}y(s)}{t^2}$

Substitute the change of variables back into the ODE

$\left(\frac{\frac{d^2}{ds^2}y(s)}{t^2} - \frac{\frac{d}{ds}y(s)}{t^2}\right) t^2 - 3\frac{d}{ds}y(s) + 4y(s) = 0$

- Simplify

  $\frac{d^2}{ds^2}y(s) - 4\frac{d}{ds}y(s) + 4y(s) = 0$

- Characteristic polynomial of ODE

  $r^2 - 4r + 4 = 0$

- Factor the characteristic polynomial

  $$(r-2)^2 = 0$$

- Root of the characteristic polynomial

  $$r = 2$$

- 1st solution of the ODE

  $$y_1(s) = e^{2s}$$

- Repeated root, multiply $y_1(s)$ by s to ensure linear independence

  $$y_2(s) = s\,e^{2s}$$

- General solution of the ODE

  $$y(s) = c_1 y_1(s) + c_2 y_2(s)$$

- Substitute in solutions

  $$y(s) = c_1 e^{2s} + c_2 s\,e^{2s}$$

- Change variables back using $s = \ln(t)$

  $$y = c_1 t^2 + c_2 t^2 \ln(t)$$

- Simplify

  $$y = t^2 (c_2 \ln(t) + c_1)$$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
<- LODE of Euler type successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 14

```
dsolve(t^2*diff(y(t),t$2)-3*t*diff(y(t),t)+4*y(t)=0,y(t), singsol=all)
```

$$y(t) = t^2 (c_2 \ln(t) + c_1)$$

✓ Solution by Mathematica

Time used: 0.016 (sec). Leaf size: 18

```
DSolve[t^2*y''[t]-3*t*y'[t]+4*y[t]==0,y[t],t,IncludeSingularSolutions -> True]
```

$$y(t) \to t^2(2c_2\log(t) + c_1)$$

9.26 problem 41

Internal problem ID [678]
Internal file name [`OUTPUT/678_Sunday_June_05_2022_01_46_48_AM_2297909/index.tex`]

Book: Elementary differential equations and boundary value problems, 10th ed., Boyce and DiPrima
Section: Chapter 3, Second order linear equations, 3.4 Repeated roots, reduction of order , page 172
Problem number: 41.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_euler_ode", "second_order_change_of_variable_on_x_method_1", "second_order_change_of_variable_on_x_method_2", "second_order_change_of_variable_on_y_method_2"**

Maple gives the following as the ode type

```
[[_Emden, _Fowler]]
```

$$\boxed{t^2y'' + 2ty' + \frac{y}{4} = 0}$$

The ode can be written as

$$4t^2y'' + 8ty' + y = 0$$

Which shows it is a Euler ODE.

9.26.1 Solving as second order euler ode ode

This is Euler second order ODE. Let the solution be $y = t^r$, then $y' = rt^{r-1}$ and $y'' = r(r-1)t^{r-2}$. Substituting these back into the given ODE gives

$$4t^2(r(r-1))t^{r-2} + 8trt^{r-1} + t^r = 0$$

Simplifying gives

$$4r(r-1)\,t^r + 8r\,t^r + t^r = 0$$

Since $t^r \neq 0$ then dividing throughout by t^r gives

$$4r(r-1) + 8r + 1 = 0$$

Or

$$4r^2 + 4r + 1 = 0 \tag{1}$$

Equation (1) is the characteristic equation. Its roots determine the form of the general solution. Using the quadratic equation the roots are

$$r_1 = -\frac{1}{2}$$
$$r_2 = -\frac{1}{2}$$

Since the roots are equal, then the general solution is

$$y = c_1 y_1 + c_2 y_2$$

Where $y_1 = t^r$ and $y_2 = t^r \ln(t)$. Hence

$$y = \frac{c_1}{\sqrt{t}} + \frac{c_2 \ln(t)}{\sqrt{t}}$$

Summary
The solution(s) found are the following

$$y = \frac{c_1}{\sqrt{t}} + \frac{c_2 \ln(t)}{\sqrt{t}} \tag{1}$$

Verification of solutions

$$y = \frac{c_1}{\sqrt{t}} + \frac{c_2 \ln(t)}{\sqrt{t}}$$

Verified OK.

9.26.2 Solving as second order change of variable on x method 2 ode

In normal form the ode

$$4t^2y'' + 8ty' + y = 0 \tag{1}$$

Becomes

$$y'' + p(t)\, y' + q(t)\, y = 0 \tag{2}$$

Where

$$\begin{aligned} p(t) &= \frac{2}{t} \\ q(t) &= \frac{1}{4t^2} \end{aligned}$$

Applying change of variables $\tau = g(t)$ to (2) gives

$$\frac{d^2}{d\tau^2}y(\tau) + p_1\left(\frac{d}{d\tau}y(\tau)\right) + q_1 y(\tau) = 0 \tag{3}$$

Where τ is the new independent variable, and

$$p_1(\tau) = \frac{\tau''(t) + p(t)\,\tau'(t)}{\tau'\,(t)^2} \tag{4}$$

$$q_1(\tau) = \frac{q(t)}{\tau'\,(t)^2} \tag{5}$$

Let $p_1 = 0$. Eq (4) simplifies to

$$\tau''(t) + p(t)\,\tau'(t) = 0$$

This ode is solved resulting in

$$\begin{aligned} \tau &= \int \mathrm{e}^{-\left(\int p(t)dt\right)}dt \\ &= \int \mathrm{e}^{-\left(\int \frac{2}{t}dt\right)}dt \\ &= \int e^{-2\ln(t)}\, dt \\ &= \int \frac{1}{t^2}dt \\ &= -\frac{1}{t} \end{aligned} \tag{6}$$

Using (6) to evaluate q_1 from (5) gives

$$\begin{aligned} q_1(\tau) &= \frac{q(t)}{\tau'(t)^2} \\ &= \frac{\frac{1}{4t^2}}{\frac{1}{t^4}} \\ &= \frac{t^2}{4} \end{aligned} \tag{7}$$

Substituting the above in (3) and noting that now $p_1 = 0$ results in

$$\begin{aligned} \frac{d^2}{d\tau^2}y(\tau) + q_1 y(\tau) &= 0 \\ \frac{d^2}{d\tau^2}y(\tau) + \frac{t^2 y(\tau)}{4} &= 0 \end{aligned}$$

But in terms of τ

$$\frac{t^2}{4} = \frac{1}{4\tau^2}$$

Hence the above ode becomes

$$\frac{d^2}{d\tau^2}y(\tau) + \frac{y(\tau)}{4\tau^2} = 0$$

The above ode is now solved for $y(\tau)$. The ode can be written as

$$4\left(\frac{d^2}{d\tau^2}y(\tau)\right)\tau^2 + y(\tau) = 0$$

Which shows it is a Euler ODE. This is Euler second order ODE. Let the solution be $y(\tau) = \tau^r$, then $y' = r\tau^{r-1}$ and $y'' = r(r-1)\tau^{r-2}$. Substituting these back into the given ODE gives

$$4\tau^2(r(r-1))\tau^{r-2} + 0r\tau^{r-1} + \tau^r = 0$$

Simplifying gives

$$4r(r-1)\,\tau^r + 0\,\tau^r + \tau^r = 0$$

Since $\tau^r \neq 0$ then dividing throughout by τ^r gives

$$4r(r-1) + 0 + 1 = 0$$

Or

$$4r^2 - 4r + 1 = 0 \tag{1}$$

Equation (1) is the characteristic equation. Its roots determine the form of the general solution. Using the quadratic equation the roots are

$$r_1 = \frac{1}{2}$$
$$r_2 = \frac{1}{2}$$

Since the roots are equal, then the general solution is

$$y(\tau) = c_1 y_1 + c_2 y_2$$

Where $y_1 = \tau^r$ and $y_2 = \tau^r \ln(\tau)$. Hence

$$y(\tau) = c_1 \sqrt{\tau} + c_2 \sqrt{\tau}\, \ln(\tau)$$

The above solution is now transformed back to y using (6) which results in

$$y = \left(\ln\left(-\frac{1}{t}\right) c_2 + c_1\right)\sqrt{-\frac{1}{t}}$$

Summary

The solution(s) found are the following

$$y = \left(\ln\left(-\frac{1}{t}\right) c_2 + c_1\right)\sqrt{-\frac{1}{t}} \tag{1}$$

Verification of solutions

$$y = \left(\ln\left(-\frac{1}{t}\right) c_2 + c_1\right)\sqrt{-\frac{1}{t}}$$

Verified OK.

9.26.3 Solving as second order change of variable on x method 1 ode

In normal form the ode

$$4t^2 y'' + 8ty' + y = 0 \tag{1}$$

Becomes

$$y'' + p(t)\, y' + q(t)\, y = 0 \tag{2}$$

Where

$$p(t) = \frac{2}{t}$$
$$q(t) = \frac{1}{4t^2}$$

Applying change of variables $\tau = g(t)$ to (2) results

$$\frac{d^2}{d\tau^2}y(\tau) + p_1\left(\frac{d}{d\tau}y(\tau)\right) + q_1 y(\tau) = 0 \tag{3}$$

Where τ is the new independent variable, and

$$p_1(\tau) = \frac{\tau''(t) + p(t)\,\tau'(t)}{\tau'(t)^2} \tag{4}$$

$$q_1(\tau) = \frac{q(t)}{\tau'(t)^2} \tag{5}$$

Let $q_1 = c^2$ where c is some constant. Therefore from (5)

$$\begin{aligned} \tau' &= \frac{1}{c}\sqrt{q} \\ &= \frac{\sqrt{\frac{1}{t^2}}}{2c} \\ \tau'' &= -\frac{1}{2c\sqrt{\frac{1}{t^2}}\,t^3} \end{aligned} \tag{6}$$

Substituting the above into (4) results in

$$\begin{aligned} p_1(\tau) &= \frac{\tau''(t) + p(t)\,\tau'(t)}{\tau'(t)^2} \\ &= \frac{-\frac{1}{2c\sqrt{\frac{1}{t^2}}\,t^3} + \frac{2}{t}\frac{\sqrt{\frac{1}{t^2}}}{2c}}{\left(\frac{\sqrt{\frac{1}{t^2}}}{2c}\right)^2} \\ &= 2c \end{aligned}$$

Therefore ode (3) now becomes

$$\begin{aligned} y(\tau)'' + p_1 y(\tau)' + q_1 y(\tau) &= 0 \\ \frac{d^2}{d\tau^2}y(\tau) + 2c\left(\frac{d}{d\tau}y(\tau)\right) + c^2 y(\tau) &= 0 \end{aligned} \tag{7}$$

The above ode is now solved for $y(\tau)$. Since the ode is now constant coefficients, it can be easily solved to give

$$y(\tau) = \mathrm{e}^{-c\tau} c_1$$

Now from (6)

$$\begin{aligned}
\tau &= \int \frac{1}{c}\sqrt{q}\,dt \\
&= \frac{\int \frac{\sqrt{\frac{1}{t^2}}}{2}dt}{c} \\
&= \frac{\sqrt{\frac{1}{t^2}}\,t\ln(t)}{2c}
\end{aligned}$$

Substituting the above into the solution obtained gives

$$y = \frac{c_1}{\sqrt{t}}$$

<u>Summary</u>

The solution(s) found are the following

$$y = \frac{c_1}{\sqrt{t}} \tag{1}$$

<u>Verification of solutions</u>

$$y = \frac{c_1}{\sqrt{t}}$$

Verified OK.

9.26.4 Solving as second order change of variable on y method 2 ode

In normal form the ode

$$4t^2y'' + 8ty' + y = 0 \tag{1}$$

Becomes

$$y'' + p(t)\,y' + q(t)\,y = 0 \tag{2}$$

Where

$$\begin{aligned}
p(t) &= \frac{2}{t} \\
q(t) &= \frac{1}{4t^2}
\end{aligned}$$

Applying change of variables on the depndent variable $y = v(t)\,t^n$ to (2) gives the following ode where the dependent variables is $v(t)$ and not y.

$$v''(t) + \left(\frac{2n}{t} + p\right)v'(t) + \left(\frac{n(n-1)}{t^2} + \frac{np}{t} + q\right)v(t) = 0 \tag{3}$$

Let the coefficient of $v(t)$ above be zero. Hence

$$\frac{n(n-1)}{t^2} + \frac{np}{t} + q = 0 \tag{4}$$

Substituting the earlier values found for $p(t)$ and $q(t)$ into (4) gives

$$\frac{n(n-1)}{t^2} + \frac{2n}{t^2} + \frac{1}{4t^2} = 0 \tag{5}$$

Solving (5) for n gives

$$n = -\frac{1}{2} \tag{6}$$

Substituting this value in (3) gives

$$\begin{aligned} v''(t) + \frac{v'(t)}{t} &= 0 \\ v''(t) + \frac{v'(t)}{t} &= 0 \end{aligned} \tag{7}$$

Using the substitution

$$u(t) = v'(t)$$

Then (7) becomes

$$u'(t) + \frac{u(t)}{t} = 0 \tag{8}$$

The above is now solved for $u(t)$. In canonical form the ODE is

$$\begin{aligned} u' &= F(t, u) \\ &= f(t)g(u) \\ &= -\frac{u}{t} \end{aligned}$$

Where $f(t) = -\frac{1}{t}$ and $g(u) = u$. Integrating both sides gives

$$\begin{aligned} \frac{1}{u}\, du &= -\frac{1}{t}\, dt \\ \int \frac{1}{u}\, du &= \int -\frac{1}{t}\, dt \\ \ln(u) &= -\ln(t) + c_1 \\ u &= e^{-\ln(t)+c_1} \\ &= \frac{c_1}{t} \end{aligned}$$

Now that $u(t)$ is known, then

$$
\begin{aligned}
v'(t) &= u(t) \\
v(t) &= \int u(t)\,dt + c_2 \\
&= c_1 \ln(t) + c_2
\end{aligned}
$$

Hence

$$
\begin{aligned}
y &= v(t)\,t^n \\
&= \frac{c_1 \ln(t) + c_2}{\sqrt{t}} \\
&= \frac{c_1 \ln(t) + c_2}{\sqrt{t}}
\end{aligned}
$$

Summary

The solution(s) found are the following

$$
y = \frac{c_1 \ln(t) + c_2}{\sqrt{t}} \tag{1}
$$

Verification of solutions

$$
y = \frac{c_1 \ln(t) + c_2}{\sqrt{t}}
$$

Verified OK.

9.26.5 Solving using Kovacic algorithm

Writing the ode as

$$
4t^2y'' + 8ty' + y = 0 \tag{1}
$$

$$
Ay'' + By' + Cy = 0 \tag{2}
$$

Comparing (1) and (2) shows that

$$
\begin{aligned}
A &= 4t^2 \\
B &= 8t \\
C &= 1
\end{aligned} \tag{3}
$$

Applying the Liouville transformation on the dependent variable gives

$$
z(t) = ye^{\int \frac{B}{2A}\,dt}
$$

Then (2) becomes

$$z''(t) = rz(t) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{-1}{4t^2} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= -1 \\ t &= 4t^2 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(t) = \left(-\frac{1}{4t^2}\right) z(t) \tag{7}$$

Equation (7) is now solved. After finding $z(t)$ then y is found using the inverse transformation

$$y = z(t)\, e^{-\int \frac{B}{2A}\, dt}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 464: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned}O(\infty) &= \deg(t) - \deg(s)\\ &= 2 - 0\\ &= 2\end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 4t^2$. There is a pole at $t = 0$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 2 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Since pole order is not larger than 2 and the order at ∞ is 2 then the necessary conditions for case three are met. Therefore

$$L = [1, 2, 4, 6, 12]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = -\frac{1}{4t^2}$$

For the pole at $t = 0$ let b be the coefficient of $\frac{1}{t^2}$ in the partial fractions decomposition of r given above. Therefore $b = -\frac{1}{4}$. Hence

$$\begin{aligned}[\sqrt{r}]_c &= 0\\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{1}{2}\\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = \frac{1}{2}\end{aligned}$$

Since the order of r at ∞ is 2 then $[\sqrt{r}]_\infty = 0$. Let b be the coefficient of $\frac{1}{t^2}$ in the Laurent series expansion of r at ∞. which can be found by dividing the leading coefficient of s by the leading coefficient of t from

$$r = \frac{s}{t} = -\frac{1}{4t^2}$$

Since the $\gcd(s, t) = 1$. This gives $b = -\frac{1}{4}$. Hence

$$\begin{aligned}[\sqrt{r}]_\infty &= 0\\ \alpha_\infty^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{1}{2}\\ \alpha_\infty^- &= \frac{1}{2} - \sqrt{1+4b} = \frac{1}{2}\end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = -\frac{1}{4t^2}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
0	2	0	$\frac{1}{2}$	$\frac{1}{2}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
2	0	$\frac{1}{2}$	$\frac{1}{2}$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^- = \frac{1}{2}$ then

$$\begin{aligned} d &= \alpha_\infty^- - \left(\alpha_{c_1}^-\right) \\ &= \frac{1}{2} - \left(\frac{1}{2}\right) \\ &= 0 \end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c\in\Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{t-c} \right) + s(\infty)[\sqrt{r}]_\infty$$

The above gives

$$\begin{aligned} \omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^-}{t-c_1} \right) + (-)[\sqrt{r}]_\infty \\ &= \frac{1}{2t} + (-)(0) \\ &= \frac{1}{2t} \\ &= \frac{1}{2t} \end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(t)$ of degree $d = 0$ to solve the ode. The polynomial $p(t)$ needs to satisfy the equation

$$p'' + 2\omega p' + (\omega' + \omega^2 - r)\, p = 0 \tag{1A}$$

Let

$$p(t) = 1 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$(0) + 2\left(\frac{1}{2t}\right)(0) + \left(\left(-\frac{1}{2t^2}\right) + \left(\frac{1}{2t}\right)^2 - \left(-\frac{1}{4t^2}\right)\right) = 0$$
$$0 = 0$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned} z_1(t) &= p e^{\int \omega\, dt} \\ &= \mathrm{e}^{\int \frac{1}{2t} dt} \\ &= \sqrt{t} \end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned} y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\, dt} \\ &= z_1 e^{-\int \frac{1}{2}\frac{8t}{4t^2}\, dt} \\ &= z_1 e^{-\ln(t)} \\ &= z_1 \left(\frac{1}{t}\right) \end{aligned}$$

Which simplifies to

$$y_1 = \frac{1}{\sqrt{t}}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\, dt}}{y_1^2}\, dt$$

Substituting gives

$$
\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{8t}{4t^2}\, dt}}{(y_1)^2}\, dt \\
&= y_1 \int \frac{e^{-2\ln(t)}}{(y_1)^2}\, dt \\
&= y_1(\ln(t))
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1 \left(\frac{1}{\sqrt{t}}\right) + c_2 \left(\frac{1}{\sqrt{t}}(\ln(t))\right)
\end{aligned}
$$

Summary

The solution(s) found are the following

$$
y = \frac{c_1}{\sqrt{t}} + \frac{c_2 \ln(t)}{\sqrt{t}} \tag{1}
$$

Verification of solutions

$$
y = \frac{c_1}{\sqrt{t}} + \frac{c_2 \ln(t)}{\sqrt{t}}
$$

Verified OK.

9.26.6 Maple step by step solution

Let's solve

$4y''t^2 + 8ty' + y = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y'' = -\frac{2y'}{t} - \frac{y}{4t^2}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y'' + \frac{2y'}{t} + \frac{y}{4t^2} = 0$

- Multiply by denominators of the ODE

$4y''t^2 + 8ty' + y = 0$

- Make a change of variables

  $s = \ln(t)$

☐ Substitute the change of variables back into the ODE

  - Calculate the 1st derivative of y with respect to t , using the chain rule

    $y' = \left(\frac{d}{ds}y(s)\right) s'(t)$

  - Compute derivative

    $y' = \frac{\frac{d}{ds}y(s)}{t}$

  - Calculate the 2nd derivative of y with respect to t , using the chain rule

    $y'' = \left(\frac{d^2}{ds^2}y(s)\right) s'(t)^2 + s''(t) \left(\frac{d}{ds}y(s)\right)$

  - Compute derivative

    $y'' = \frac{\frac{d^2}{ds^2}y(s)}{t^2} - \frac{\frac{d}{ds}y(s)}{t^2}$

  Substitute the change of variables back into the ODE

  $4\left(\frac{\frac{d^2}{ds^2}y(s)}{t^2} - \frac{\frac{d}{ds}y(s)}{t^2}\right) t^2 + 8\frac{d}{ds}y(s) + y(s) = 0$

- Simplify

  $4\frac{d^2}{ds^2}y(s) + 4\frac{d}{ds}y(s) + y(s) = 0$

- Isolate 2nd derivative

  $\frac{d^2}{ds^2}y(s) = -\frac{d}{ds}y(s) - \frac{y(s)}{4}$

- Group terms with $y(s)$ on the lhs of the ODE and the rest on the rhs of the ODE; ODE is lin

  $\frac{d^2}{ds^2}y(s) + \frac{d}{ds}y(s) + \frac{y(s)}{4} = 0$

- Characteristic polynomial of ODE

  $r^2 + r + \frac{1}{4} = 0$

- Factor the characteristic polynomial

  $\frac{(2r+1)^2}{4} = 0$

- Root of the characteristic polynomial

  $r = -\frac{1}{2}$

- 1st solution of the ODE

  $y_1(s) = e^{-\frac{s}{2}}$

- Repeated root, multiply $y_1(s)$ by s to ensure linear independence

  $y_2(s) = s\,\mathrm{e}^{-\frac{s}{2}}$
- General solution of the ODE

  $y(s) = c_1 y_1(s) + c_2 y_2(s)$
- Substitute in solutions

  $y(s) = c_1 \mathrm{e}^{-\frac{s}{2}} + c_2 s\,\mathrm{e}^{-\frac{s}{2}}$
- Change variables back using $s = \ln(t)$

  $y = \frac{c_1}{\sqrt{t}} + \frac{c_2 \ln(t)}{\sqrt{t}}$
- Simplify

  $y = \frac{c_1}{\sqrt{t}} + \frac{c_2 \ln(t)}{\sqrt{t}}$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
<- LODE of Euler type successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 14

```
dsolve(t^2*diff(y(t),t$2)+2*t*diff(y(t),t)+25/100*y(t)=0,y(t), singsol=all)
```

$$y(t) = \frac{c_2 \ln(t) + c_1}{\sqrt{t}}$$

✓ Solution by Mathematica

Time used: 0.019 (sec). Leaf size: 24

```
DSolve[t^2*y''[t]+2*t*y'[t]+25/100*y[t]==0,y[t],t,IncludeSingularSolutions -> True]
```

$$y(t) \to \frac{c_2 \log(t) + 2c_1}{2\sqrt{t}}$$

9.27 problem 42

Internal problem ID [679]
Internal file name [`OUTPUT/679_Sunday_June_05_2022_01_46_49_AM_62569225/index.tex`]

Book: Elementary differential equations and boundary value problems, 10th ed., Boyce and DiPrima
Section: Chapter 3, Second order linear equations, 3.4 Repeated roots, reduction of order , page 172
Problem number: 42.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_euler_ode", "second_order_change_of_variable_on_x_method_1", "second_order_change_of_variable_on_x_method_2", "second_order_change_of_variable_on_y_method_2", "second_order_ode_non_constant_coeff_transformation_on_B"**

Maple gives the following as the ode type

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[[_Emden, _Fowler]]
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$$\boxed{2t^2y'' - 5ty' + 5y = 0}$$

9.27.1 Solving as second order euler ode ode

This is Euler second order ODE. Let the solution be $y = t^r$, then $y' = rt^{r-1}$ and $y'' = r(r-1)t^{r-2}$. Substituting these back into the given ODE gives

$$2t^2(r(r-1))t^{r-2} - 5trt^{r-1} + 5t^r = 0$$

Simplifying gives

$$2r(r-1)\,t^r - 5r\,t^r + 5t^r = 0$$

Since $t^r \neq 0$ then dividing throughout by t^r gives

$$2r(r-1) - 5r + 5 = 0$$

Or

$$2r^2 - 7r + 5 = 0 \tag{1}$$

Equation (1) is the characteristic equation. Its roots determine the form of the general solution. Using the quadratic equation the roots are

$$r_1 = 1$$
$$r_2 = \frac{5}{2}$$

Since the roots are real and distinct, then the general solution is

$$y = c_1y_1 + c_2y_2$$

Where $y_1 = t^{r_1}$ and $y_2 = t^{r_2}$. Hence

$$y = c_1t + c_2t^{\frac{5}{2}}$$

Summary
The solution(s) found are the following

$$y = c_1t + c_2t^{\frac{5}{2}} \tag{1}$$

Verification of solutions

$$y = c_1t + c_2t^{\frac{5}{2}}$$

Verified OK.

9.27.2 Solving as second order change of variable on x method 2 ode

In normal form the ode

$$2t^2y'' - 5ty' + 5y = 0 \tag{1}$$

Becomes

$$y'' + p(t)\,y' + q(t)\,y = 0 \tag{2}$$

Where

$$\begin{aligned} p(t) &= -\frac{5}{2t} \\ q(t) &= \frac{5}{2t^2} \end{aligned}$$

Applying change of variables $\tau = g(t)$ to (2) gives

$$\frac{d^2}{d\tau^2}y(\tau) + p_1\left(\frac{d}{d\tau}y(\tau)\right) + q_1 y(\tau) = 0 \tag{3}$$

Where τ is the new independent variable, and

$$p_1(\tau) = \frac{\tau''(t) + p(t)\,\tau'(t)}{\tau'(t)^2} \tag{4}$$

$$q_1(\tau) = \frac{q(t)}{\tau'(t)^2} \tag{5}$$

Let $p_1 = 0$. Eq (4) simplifies to

$$\tau''(t) + p(t)\,\tau'(t) = 0$$

This ode is solved resulting in

$$\begin{aligned} \tau &= \int \mathrm{e}^{-\left(\int p(t)dt\right)}dt \\ &= \int \mathrm{e}^{-\left(\int -\frac{5}{2t}dt\right)}dt \\ &= \int e^{\frac{5\ln(t)}{2}}\,dt \\ &= \int t^{\frac{5}{2}}dt \\ &= \frac{2t^{\frac{7}{2}}}{7} \end{aligned} \tag{6}$$

Using (6) to evaluate q_1 from (5) gives

$$\begin{aligned} q_1(\tau) &= \frac{q(t)}{\tau'(t)^2} \\ &= \frac{\frac{5}{2t^2}}{t^5} \\ &= \frac{5}{2t^7} \end{aligned} \tag{7}$$

Substituting the above in (3) and noting that now $p_1 = 0$ results in

$$\begin{aligned} \frac{d^2}{d\tau^2}y(\tau) + q_1 y(\tau) &= 0 \\ \frac{d^2}{d\tau^2}y(\tau) + \frac{5y(\tau)}{2t^7} &= 0 \end{aligned}$$

But in terms of τ

$$\frac{5}{2t^7} = \frac{10}{49\tau^2}$$

Hence the above ode becomes

$$\frac{d^2}{d\tau^2}y(\tau) + \frac{10y(\tau)}{49\tau^2} = 0$$

The above ode is now solved for $y(\tau)$. The ode can be written as

$$49\left(\frac{d^2}{d\tau^2}y(\tau)\right)\tau^2 + 10y(\tau) = 0$$

Which shows it is a Euler ODE. This is Euler second order ODE. Let the solution be $y(\tau) = \tau^r$, then $y' = r\tau^{r-1}$ and $y'' = r(r-1)\tau^{r-2}$. Substituting these back into the given ODE gives

$$49\tau^2(r(r-1))\tau^{r-2} + 0r\tau^{r-1} + 10\tau^r = 0$$

Simplifying gives

$$49r(r-1)\,\tau^r + 0\,\tau^r + 10\tau^r = 0$$

Since $\tau^r \neq 0$ then dividing throughout by τ^r gives

$$49r(r-1) + 0 + 10 = 0$$

Or

$$49r^2 - 49r + 10 = 0 \tag{1}$$

Equation (1) is the characteristic equation. Its roots determine the form of the general solution. Using the quadratic equation the roots are

$$r_1 = \frac{2}{7}$$
$$r_2 = \frac{5}{7}$$

Since the roots are real and distinct, then the general solution is

$$y(\tau) = c_1 y_1 + c_2 y_2$$

Where $y_1 = \tau^{r_1}$ and $y_2 = \tau^{r_2}$. Hence

$$y(\tau) = c_1 \tau^{\frac{2}{7}} + c_2 \tau^{\frac{5}{7}}$$

The above solution is now transformed back to y using (6) which results in

$$y = \frac{c_1 2^{\frac{2}{7}} 7^{\frac{5}{7}} \left(t^{\frac{7}{2}}\right)^{\frac{2}{7}}}{7} + \frac{c_2 2^{\frac{5}{7}} 7^{\frac{2}{7}} \left(t^{\frac{7}{2}}\right)^{\frac{5}{7}}}{7}$$

Summary

The solution(s) found are the following

$$y = \frac{c_1 2^{\frac{2}{7}} 7^{\frac{5}{7}} \left(t^{\frac{7}{2}}\right)^{\frac{2}{7}}}{7} + \frac{c_2 2^{\frac{5}{7}} 7^{\frac{2}{7}} \left(t^{\frac{7}{2}}\right)^{\frac{5}{7}}}{7} \tag{1}$$

Verification of solutions

$$y = \frac{c_1 2^{\frac{2}{7}} 7^{\frac{5}{7}} \left(t^{\frac{7}{2}}\right)^{\frac{2}{7}}}{7} + \frac{c_2 2^{\frac{5}{7}} 7^{\frac{2}{7}} \left(t^{\frac{7}{2}}\right)^{\frac{5}{7}}}{7}$$

Verified OK.

9.27.3 Solving as second order change of variable on x method 1 ode

In normal form the ode

$$2t^2 y'' - 5ty' + 5y = 0 \tag{1}$$

Becomes

$$y'' + p(t)\, y' + q(t)\, y = 0 \tag{2}$$

Where

$$
\begin{aligned}
p(t) &= -\frac{5}{2t} \\
q(t) &= \frac{5}{2t^2}
\end{aligned}
$$

Applying change of variables $\tau = g(t)$ to (2) results

$$
\frac{d^2}{d\tau^2}y(\tau) + p_1\left(\frac{d}{d\tau}y(\tau)\right) + q_1y(\tau) = 0 \tag{3}
$$

Where τ is the new independent variable, and

$$
p_1(\tau) = \frac{\tau''(t) + p(t)\,\tau'(t)}{\tau'(t)^2} \tag{4}
$$

$$
q_1(\tau) = \frac{q(t)}{\tau'(t)^2} \tag{5}
$$

Let $q_1 = c^2$ where c is some constant. Therefore from (5)

$$
\begin{aligned}
\tau' &= \frac{1}{c}\sqrt{q} \\
&= \frac{\sqrt{10}\,\sqrt{\frac{1}{t^2}}}{2c} \\
\tau'' &= -\frac{\sqrt{10}}{2c\sqrt{\frac{1}{t^2}}\,t^3}
\end{aligned} \tag{6}
$$

Substituting the above into (4) results in

$$
\begin{aligned}
p_1(\tau) &= \frac{\tau''(t) + p(t)\,\tau'(t)}{\tau'(t)^2} \\
&= \frac{-\frac{\sqrt{10}}{2c\sqrt{\frac{1}{t^2}}\,t^3} - \frac{5}{2t}\frac{\sqrt{10}\,\sqrt{\frac{1}{t^2}}}{2c}}{\left(\frac{\sqrt{10}\,\sqrt{\frac{1}{t^2}}}{2c}\right)^2} \\
&= -\frac{7c\sqrt{10}}{10}
\end{aligned}
$$

Therefore ode (3) now becomes

$$
y(\tau)'' + p_1y(\tau)' + q_1y(\tau) = 0
$$

$$
\frac{d^2}{d\tau^2}y(\tau) - \frac{7c\sqrt{10}\left(\frac{d}{d\tau}y(\tau)\right)}{10} + c^2y(\tau) = 0 \tag{7}
$$

The above ode is now solved for $y(\tau)$. Since the ode is now constant coefficients, it can be easily solved to give

$$y(\tau) = e^{\frac{7\sqrt{10}\, c\tau}{20}} \left(c_1 \cosh\left(\frac{3\sqrt{10}\, c\tau}{20}\right) + i c_2 \sinh\left(\frac{3\sqrt{10}\, c\tau}{20}\right)\right)$$

Now from (6)

$$\begin{aligned}
\tau &= \int \frac{1}{c}\sqrt{q}\, dt \\
&= \frac{\int \frac{\sqrt{10}\sqrt{\frac{1}{t^2}}}{2} dt}{c} \\
&= \frac{\sqrt{10}\,\sqrt{\frac{1}{t^2}}\, t \ln(t)}{2c}
\end{aligned}$$

Substituting the above into the solution obtained gives

$$y = t^{\frac{7}{4}} \left(c_1 \cosh\left(\frac{3\ln(t)}{4}\right) + i c_2 \sinh\left(\frac{3\ln(t)}{4}\right)\right)$$

Summary

The solution(s) found are the following

$$y = t^{\frac{7}{4}} \left(c_1 \cosh\left(\frac{3\ln(t)}{4}\right) + i c_2 \sinh\left(\frac{3\ln(t)}{4}\right)\right) \tag{1}$$

Verification of solutions

$$y = t^{\frac{7}{4}} \left(c_1 \cosh\left(\frac{3\ln(t)}{4}\right) + i c_2 \sinh\left(\frac{3\ln(t)}{4}\right)\right)$$

Verified OK.

9.27.4 Solving as second order change of variable on y method 2 ode

In normal form the ode

$$2t^2y'' - 5ty' + 5y = 0 \tag{1}$$

Becomes

$$y'' + p(t)\, y' + q(t)\, y = 0 \tag{2}$$

Where

$$
\begin{aligned}
p(t) &= -\frac{5}{2t} \\
q(t) &= \frac{5}{2t^2}
\end{aligned}
$$

Applying change of variables on the depndent variable $y = v(t)\, t^n$ to (2) gives the following ode where the dependent variables is $v(t)$ and not y.

$$
v''(t) + \left(\frac{2n}{t} + p\right) v'(t) + \left(\frac{n(n-1)}{t^2} + \frac{np}{t} + q\right) v(t) = 0 \tag{3}
$$

Let the coefficient of $v(t)$ above be zero. Hence

$$
\frac{n(n-1)}{t^2} + \frac{np}{t} + q = 0 \tag{4}
$$

Substituting the earlier values found for $p(t)$ and $q(t)$ into (4) gives

$$
\frac{n(n-1)}{t^2} - \frac{5n}{2t^2} + \frac{5}{2t^2} = 0 \tag{5}
$$

Solving (5) for n gives

$$
n = \frac{5}{2} \tag{6}
$$

Substituting this value in (3) gives

$$
\begin{aligned}
v''(t) + \frac{5v'(t)}{2t} &= 0 \\
v''(t) + \frac{5v'(t)}{2t} &= 0
\end{aligned} \tag{7}
$$

Using the substitution

$$
u(t) = v'(t)
$$

Then (7) becomes

$$
u'(t) + \frac{5u(t)}{2t} = 0 \tag{8}
$$

The above is now solved for $u(t)$. In canonical form the ODE is

$$\begin{aligned} u' &= F(t,u) \\ &= f(t)g(u) \\ &= -\frac{5u}{2t} \end{aligned}$$

Where $f(t) = -\frac{5}{2t}$ and $g(u) = u$. Integrating both sides gives

$$\begin{aligned} \frac{1}{u}\,du &= -\frac{5}{2t}\,dt \\ \int \frac{1}{u}\,du &= \int -\frac{5}{2t}\,dt \\ \ln(u) &= -\frac{5\ln(t)}{2} + c_1 \\ u &= \mathrm{e}^{-\frac{5\ln(t)}{2}+c_1} \\ &= \frac{c_1}{t^{\frac{5}{2}}} \end{aligned}$$

Now that $u(t)$ is known, then

$$\begin{aligned} v'(t) &= u(t) \\ v(t) &= \int u(t)\,dt + c_2 \\ &= -\frac{2c_1}{3t^{\frac{3}{2}}} + c_2 \end{aligned}$$

Hence

$$\begin{aligned} y &= v(t)\,t^n \\ &= \left(-\frac{2c_1}{3t^{\frac{3}{2}}} + c_2\right) t^{\frac{5}{2}} \\ &= c_2 t^{\frac{5}{2}} - \frac{2c_1 t}{3} \end{aligned}$$

Summary

The solution(s) found are the following

$$y = \left(-\frac{2c_1}{3t^{\frac{3}{2}}} + c_2\right) t^{\frac{5}{2}} \tag{1}$$

Verification of solutions

$$y = \left(-\frac{2c_1}{3t^{\frac{3}{2}}} + c_2\right) t^{\frac{5}{2}}$$

Verified OK.

9.27.5 Solving as second order ode non constant coeff transformation on B ode

Given an ode of the form

$$Ay'' + By' + Cy = F(t)$$

This method reduces the order ode the ODE by one by applying the transformation

$$y = Bv$$

This results in

$$\begin{aligned}
y' &= B'v + v'B \\
y'' &= B''v + B'v' + v''B + v'B' \\
&= v''B + 2v' + B' + B''v
\end{aligned}$$

And now the original ode becomes

$$\begin{aligned}
A(v''B + 2v'B' + B''v) + B(B'v + v'B) + CBv &= 0 \\
ABv'' + \left(2AB' + B^2\right) v' + \left(AB'' + BB' + CB\right) v &= 0 \qquad (1)
\end{aligned}$$

If the term $AB'' + BB' + CB$ is zero, then this method works and can be used to solve

$$ABv'' + \left(2AB' + B^2\right) v' = 0$$

By Using $u = v'$ which reduces the order of the above ode to one. The new ode is

$$ABu' + \left(2AB' + B^2\right) u = 0$$

The above ode is first order ode which is solved for u. Now a new ode $v' = u$ is solved for v as first order ode. Then the final solution is obtain from $y = Bv$.

This method works only if the term $AB'' + BB' + CB$ is zero. The given ODE shows that

$$\begin{aligned}
A &= 2t^2 \\
B &= -5t \\
C &= 5 \\
F &= 0
\end{aligned}$$

The above shows that for this ode

$$\begin{aligned}
AB'' + BB' + CB &= \left(2t^2\right)(0) + (-5t)(-5) + (5)(-5t) \\
&= 0
\end{aligned}$$

Hence the ode in v given in (1) now simplifies to

$$-10t^3v'' + (5t^2)\, v' = 0$$

Now by applying $v' = u$ the above becomes

$$-10t^3u'(t) + 5t^2u(t) = 0$$

Which is now solved for u. In canonical form the ODE is

$$\begin{aligned}
u' &= F(t, u) \\
&= f(t)g(u) \\
&= \frac{u}{2t}
\end{aligned}$$

Where $f(t) = \frac{1}{2t}$ and $g(u) = u$. Integrating both sides gives

$$\begin{aligned}
\frac{1}{u}\, du &= \frac{1}{2t}\, dt \\
\int \frac{1}{u}\, du &= \int \frac{1}{2t}\, dt \\
\ln(u) &= \frac{\ln(t)}{2} + c_1 \\
u &= e^{\frac{\ln(t)}{2} + c_1} \\
&= c_1\sqrt{t}
\end{aligned}$$

The ode for v now becomes

$$\begin{aligned}
v' &= u \\
&= c_1\sqrt{t}
\end{aligned}$$

Which is now solved for v. Integrating both sides gives

$$\begin{aligned}
v(t) &= \int c_1\sqrt{t}\, \mathrm{d}t \\
&= \frac{2t^{\frac{3}{2}}c_1}{3} + c_2
\end{aligned}$$

Therefore the solution is

$$\begin{aligned}
y(t) &= Bv \\
&= (-5t)\left(\frac{2t^{\frac{3}{2}}c_1}{3} + c_2\right) \\
&= -\frac{5t\left(2t^{\frac{3}{2}}c_1 + 3c_2\right)}{3}
\end{aligned}$$

Summary
The solution(s) found are the following

$$y = -\frac{5t\left(2t^{\frac{3}{2}}c_1 + 3c_2\right)}{3} \tag{1}$$

Verification of solutions

$$y = -\frac{5t\left(2t^{\frac{3}{2}}c_1 + 3c_2\right)}{3}$$

Verified OK.

9.27.6 Solving using Kovacic algorithm

Writing the ode as

$$2t^2y'' - 5ty' + 5y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 2t^2 \\ B &= -5t \\ C &= 5 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(t) = ye^{\int \frac{B}{2A}\,dt}$$

Then (2) becomes

$$z''(t) = rz(t) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{5}{16t^2} \tag{6}$$

Comparing the above to (5) shows that

$$s = 5$$
$$t = 16t^2$$

Therefore eq. (4) becomes

$$z''(t) = \left(\frac{5}{16t^2}\right) z(t) \tag{7}$$

Equation (7) is now solved. After finding $z(t)$ then y is found using the inverse transformation

$$y = z(t)\, e^{-\int \frac{B}{2A}\, dt}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0,1,2,4,6,8,\cdots\}$	$\{\cdots,-6,-4,-2,0,2,3,4,5,6,\cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1,2\}$	$\{2,3,4,5,6,7,\cdots\}$

Table 466: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 2 - 0 \\ &= 2 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 16t^2$. There is a pole at $t = 0$ of order 2. Since there is no odd order pole

larger than 2 and the order at ∞ is 2 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Since pole order is not larger than 2 and the order at ∞ is 2 then the necessary conditions for case three are met. Therefore

$$L = [1, 2, 4, 6, 12]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = \frac{5}{16t^2}$$

For the pole at $t = 0$ let b be the coefficient of $\frac{1}{t^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{5}{16}$. Hence

$$\begin{aligned}
[\sqrt{r}]_c &= 0 \\
\alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{5}{4} \\
\alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = -\frac{1}{4}
\end{aligned}$$

Since the order of r at ∞ is 2 then $[\sqrt{r}]_\infty = 0$. Let b be the coefficient of $\frac{1}{t^2}$ in the Laurent series expansion of r at ∞. which can be found by dividing the leading coefficient of s by the leading coefficient of t from

$$r = \frac{s}{t} = \frac{5}{16t^2}$$

Since the $\gcd(s, t) = 1$. This gives $b = \frac{5}{16}$. Hence

$$\begin{aligned}
[\sqrt{r}]_\infty &= 0 \\
\alpha_\infty^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{5}{4} \\
\alpha_\infty^- &= \frac{1}{2} - \sqrt{1+4b} = -\frac{1}{4}
\end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = \frac{5}{16t^2}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
0	2	0	$\frac{5}{4}$	$-\frac{1}{4}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
2	0	$\frac{5}{4}$	$-\frac{1}{4}$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^- = -\frac{1}{4}$ then

$$\begin{aligned} d &= \alpha_\infty^- - \left(\alpha_{c_1}^-\right) \\ &= -\frac{1}{4} - \left(-\frac{1}{4}\right) \\ &= 0 \end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c\in\Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{t-c} \right) + s(\infty)[\sqrt{r}]_\infty$$

The above gives

$$\begin{aligned} \omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^-}{t-c_1} \right) + (-)[\sqrt{r}]_\infty \\ &= -\frac{1}{4t} + (-)\,(0) \\ &= -\frac{1}{4t} \\ &= -\frac{1}{4t} \end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(t)$ of degree $d = 0$ to solve the ode. The polynomial $p(t)$ needs to satisfy the equation

$$p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right) p = 0 \tag{1A}$$

Let

$$p(t) = 1 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$(0) + 2\left(-\frac{1}{4t}\right)(0) + \left(\left(\frac{1}{4t^2}\right) + \left(-\frac{1}{4t}\right)^2 - \left(\frac{5}{16t^2}\right)\right) = 0$$
$$0 = 0$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned} z_1(t) &= p e^{\int \omega \, dt} \\ &= \mathrm{e}^{\int -\frac{1}{4t} dt} \\ &= \frac{1}{t^{\frac{1}{4}}} \end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned} y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A} \, dt} \\ &= z_1 e^{-\int \frac{1}{2}\frac{-5t}{2t^2} \, dt} \\ &= z_1 e^{\frac{5\ln(t)}{4}} \\ &= z_1 \left(t^{\frac{5}{4}}\right) \end{aligned}$$

Which simplifies to

$$y_1 = t$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A} \, dt}}{y_1^2} \, dt$$

Substituting gives

$$\begin{aligned} y_2 &= y_1 \int \frac{e^{\int -\frac{-5t}{2t^2} \, dt}}{(y_1)^2} \, dt \\ &= y_1 \int \frac{e^{\frac{5\ln(t)}{2}}}{(y_1)^2} \, dt \\ &= y_1 \left(\frac{2t^{\frac{3}{2}}}{3}\right) \end{aligned}$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1(t) + c_2\left(t\left(\frac{2t^{\frac{3}{2}}}{3}\right)\right)
\end{aligned}
$$

Summary

The solution(s) found are the following

$$
y = c_1 t + \frac{2c_2 t^{\frac{5}{2}}}{3} \tag{1}
$$

Verification of solutions

$$
y = c_1 t + \frac{2c_2 t^{\frac{5}{2}}}{3}
$$

Verified OK.

9.27.7 Maple step by step solution

Let's solve

$2y''t^2 - 5ty' + 5y = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y'' = \frac{5y'}{2t} - \frac{5y}{2t^2}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y'' - \frac{5y'}{2t} + \frac{5y}{2t^2} = 0$

- Multiply by denominators of the ODE

  $2y''t^2 - 5ty' + 5y = 0$

- Make a change of variables

  $s = \ln(t)$

□ Substitute the change of variables back into the ODE

  ◦ Calculate the 1st derivative of y with respect to t , using the chain rule

  $y' = \left(\frac{d}{ds}y(s)\right) s'(t)$

- Compute derivative

  $y' = \frac{\frac{d}{ds}y(s)}{t}$

- Calculate the 2nd derivative of y with respect to t , using the chain rule

  $y'' = \left(\frac{d^2}{ds^2}y(s)\right)s'(t)^2 + s''(t)\left(\frac{d}{ds}y(s)\right)$

- Compute derivative

  $y'' = \frac{\frac{d^2}{ds^2}y(s)}{t^2} - \frac{\frac{d}{ds}y(s)}{t^2}$

  Substitute the change of variables back into the ODE

  $2\left(\frac{\frac{d^2}{ds^2}y(s)}{t^2} - \frac{\frac{d}{ds}y(s)}{t^2}\right)t^2 - 5\frac{d}{ds}y(s) + 5y(s) = 0$

- Simplify

  $2\frac{d^2}{ds^2}y(s) - 7\frac{d}{ds}y(s) + 5y(s) = 0$

- Isolate 2nd derivative

  $\frac{d^2}{ds^2}y(s) = \frac{7\frac{d}{ds}y(s)}{2} - \frac{5y(s)}{2}$

- Group terms with $y(s)$ on the lhs of the ODE and the rest on the rhs of the ODE; ODE is lin

  $\frac{d^2}{ds^2}y(s) - \frac{7\frac{d}{ds}y(s)}{2} + \frac{5y(s)}{2} = 0$

- Characteristic polynomial of ODE

  $r^2 - \frac{7}{2}r + \frac{5}{2} = 0$

- Factor the characteristic polynomial

  $\frac{(r-1)(2r-5)}{2} = 0$

- Roots of the characteristic polynomial

  $r = \left(1, \frac{5}{2}\right)$

- 1st solution of the ODE

  $y_1(s) = e^s$

- 2nd solution of the ODE

  $y_2(s) = e^{\frac{5s}{2}}$

- General solution of the ODE

  $y(s) = c_1 y_1(s) + c_2 y_2(s)$

- Substitute in solutions

  $y(s) = c_1 e^s + c_2 e^{\frac{5s}{2}}$

- Change variables back using $s = \ln(t)$

  $y = c_1 t + c_2 t^{\frac{5}{2}}$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
<- LODE of Euler type successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 13

```
dsolve(2*t^2*diff(y(t),t$2)-5*t*diff(y(t),t)+5*y(t)=0,y(t), singsol=all)
```

$$y(t) = c_1 t + c_2 t^{\frac{5}{2}}$$

✓ Solution by Mathematica

Time used: 0.011 (sec). Leaf size: 18

```
DSolve[2*t^2*y''[t]-5*t*y'[t]+5*y[t]==0,y[t],t,IncludeSingularSolutions -> True]
```

$$y(t) \to t\left(c_2 t^{3/2} + c_1\right)$$

9.28 problem 43

Internal problem ID [680]
Internal file name [`OUTPUT/680_Sunday_June_05_2022_01_46_49_AM_855305/index.tex`]

Book: Elementary differential equations and boundary value problems, 10th ed., Boyce and DiPrima
Section: Chapter 3, Second order linear equations, 3.4 Repeated roots, reduction of order , page 172
Problem number: 43.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_euler_ode", "exact linear second order ode", "second_order_integrable_as_is", "second_order_change_of_variable_on_x_method_1", "second_order_change_of_variable_on_x_method_2", "second_order_change_of_variable_on_y_method_2"**

Maple gives the following as the ode type

```
[[_2nd_order, _exact, _linear, _homogeneous]]
```

$$\boxed{t^2y'' + 3ty' + y = 0}$$

9.28.1 Solving as second order euler ode ode

This is Euler second order ODE. Let the solution be $y = t^r$, then $y' = rt^{r-1}$ and $y'' = r(r-1)t^{r-2}$. Substituting these back into the given ODE gives

$$t^2(r(r-1))t^{r-2} + 3trt^{r-1} + t^r = 0$$

Simplifying gives

$$r(r-1)\,t^r + 3r\,t^r + t^r = 0$$

Since $t^r \neq 0$ then dividing throughout by t^r gives

$$r(r-1) + 3r + 1 = 0$$

Or

$$r^2 + 2r + 1 = 0 \tag{1}$$

Equation (1) is the characteristic equation. Its roots determine the form of the general solution. Using the quadratic equation the roots are

$$\begin{aligned} r_1 &= -1 \\ r_2 &= -1 \end{aligned}$$

Since the roots are equal, then the general solution is

$$y = c_1 y_1 + c_2 y_2$$

Where $y_1 = t^r$ and $y_2 = t^r \ln(t)$. Hence

$$y = \frac{c_1}{t} + \frac{c_2 \ln(t)}{t}$$

Summary
The solution(s) found are the following

$$y = \frac{c_1}{t} + \frac{c_2 \ln(t)}{t} \tag{1}$$

Verification of solutions

$$y = \frac{c_1}{t} + \frac{c_2 \ln(t)}{t}$$

Verified OK.

9.28.2 Solving as second order change of variable on x method 2 ode

In normal form the ode

$$t^2y'' + 3ty' + y = 0 \tag{1}$$

Becomes

$$y'' + p(t)\, y' + q(t)\, y = 0 \tag{2}$$

Where

$$\begin{aligned} p(t) &= \frac{3}{t} \\ q(t) &= \frac{1}{t^2} \end{aligned}$$

Applying change of variables $\tau = g(t)$ to (2) gives

$$\frac{d^2}{d\tau^2}y(\tau) + p_1\left(\frac{d}{d\tau}y(\tau)\right) + q_1y(\tau) = 0 \tag{3}$$

Where τ is the new independent variable, and

$$p_1(\tau) = \frac{\tau''(t) + p(t)\, \tau'(t)}{\tau'(t)^2} \tag{4}$$

$$q_1(\tau) = \frac{q(t)}{\tau'(t)^2} \tag{5}$$

Let $p_1 = 0$. Eq (4) simplifies to

$$\tau''(t) + p(t)\, \tau'(t) = 0$$

This ode is solved resulting in

$$\begin{aligned} \tau &= \int \mathrm{e}^{-\left(\int p(t)dt\right)}dt \\ &= \int \mathrm{e}^{-\left(\int \frac{3}{t}dt\right)}dt \\ &= \int e^{-3\ln(t)}\, dt \\ &= \int \frac{1}{t^3}dt \\ &= -\frac{1}{2t^2} \end{aligned} \tag{6}$$

Using (6) to evaluate q_1 from (5) gives

$$\begin{aligned} q_1(\tau) &= \frac{q(t)}{\tau'(t)^2} \\ &= \frac{\frac{1}{t^2}}{\frac{1}{t^6}} \\ &= t^4 \end{aligned} \tag{7}$$

Substituting the above in (3) and noting that now $p_1 = 0$ results in

$$\begin{aligned} \frac{d^2}{d\tau^2}y(\tau) + q_1 y(\tau) &= 0 \\ \frac{d^2}{d\tau^2}y(\tau) + t^4 y(\tau) &= 0 \end{aligned}$$

But in terms of τ

$$t^4 = \frac{1}{4\tau^2}$$

Hence the above ode becomes

$$\frac{d^2}{d\tau^2}y(\tau) + \frac{y(\tau)}{4\tau^2} = 0$$

The above ode is now solved for $y(\tau)$. The ode can be written as

$$4\left(\frac{d^2}{d\tau^2}y(\tau)\right)\tau^2 + y(\tau) = 0$$

Which shows it is a Euler ODE. This is Euler second order ODE. Let the solution be $y(\tau) = \tau^r$, then $y' = r\tau^{r-1}$ and $y'' = r(r-1)\tau^{r-2}$. Substituting these back into the given ODE gives

$$4\tau^2(r(r-1))\tau^{r-2} + 0r\tau^{r-1} + \tau^r = 0$$

Simplifying gives

$$4r(r-1)\,\tau^r + 0\,\tau^r + \tau^r = 0$$

Since $\tau^r \neq 0$ then dividing throughout by τ^r gives

$$4r(r-1) + 0 + 1 = 0$$

Or

$$4r^2 - 4r + 1 = 0 \tag{1}$$

Equation (1) is the characteristic equation. Its roots determine the form of the general solution. Using the quadratic equation the roots are

$$r_1 = \frac{1}{2}$$
$$r_2 = \frac{1}{2}$$

Since the roots are equal, then the general solution is

$$y(\tau) = c_1 y_1 + c_2 y_2$$

Where $y_1 = \tau^r$ and $y_2 = \tau^r \ln(\tau)$. Hence

$$y(\tau) = c_1 \sqrt{\tau} + c_2 \sqrt{\tau}\, \ln(\tau)$$

The above solution is now transformed back to y using (6) which results in

$$y = \frac{\sqrt{2}\, \sqrt{-\frac{1}{t^2}}\, \left(c_1 - c_2 \ln(2) + c_2 \ln\left(-\frac{1}{t^2}\right)\right)}{2}$$

Summary
The solution(s) found are the following

$$y = \frac{\sqrt{2}\, \sqrt{-\frac{1}{t^2}}\, \left(c_1 - c_2 \ln(2) + c_2 \ln\left(-\frac{1}{t^2}\right)\right)}{2} \tag{1}$$

Verification of solutions

$$y = \frac{\sqrt{2}\, \sqrt{-\frac{1}{t^2}}\, \left(c_1 - c_2 \ln(2) + c_2 \ln\left(-\frac{1}{t^2}\right)\right)}{2}$$

Verified OK.

9.28.3 Solving as second order change of variable on x method 1 ode

In normal form the ode

$$t^2 y'' + 3ty' + y = 0 \tag{1}$$

Becomes

$$y'' + p(t)\, y' + q(t)\, y = 0 \tag{2}$$

Where

$$
\begin{aligned}
p(t) &= \frac{3}{t} \\
q(t) &= \frac{1}{t^2}
\end{aligned}
$$

Applying change of variables $\tau = g(t)$ to (2) results

$$
\frac{d^2}{d\tau^2}y(\tau) + p_1\left(\frac{d}{d\tau}y(\tau)\right) + q_1 y(\tau) = 0 \tag{3}
$$

Where τ is the new independent variable, and

$$
\begin{aligned}
p_1(\tau) &= \frac{\tau''(t) + p(t)\,\tau'(t)}{\tau'(t)^2} &(4)\\
q_1(\tau) &= \frac{q(t)}{\tau'(t)^2} &(5)
\end{aligned}
$$

Let $q_1 = c^2$ where c is some constant. Therefore from (5)

$$
\begin{aligned}
\tau' &= \frac{1}{c}\sqrt{q} \\
&= \frac{\sqrt{\frac{1}{t^2}}}{c} &(6)\\
\tau'' &= -\frac{1}{c\sqrt{\frac{1}{t^2}}\,t^3}
\end{aligned}
$$

Substituting the above into (4) results in

$$
\begin{aligned}
p_1(\tau) &= \frac{\tau''(t) + p(t)\,\tau'(t)}{\tau'(t)^2} \\
&= \frac{-\frac{1}{c\sqrt{\frac{1}{t^2}}\,t^3} + \frac{3}{t}\frac{\sqrt{\frac{1}{t^2}}}{c}}{\left(\frac{\sqrt{\frac{1}{t^2}}}{c}\right)^2} \\
&= 2c
\end{aligned}
$$

Therefore ode (3) now becomes

$$
\begin{aligned}
y(\tau)'' + p_1 y(\tau)' + q_1 y(\tau) &= 0 \\
\frac{d^2}{d\tau^2}y(\tau) + 2c\left(\frac{d}{d\tau}y(\tau)\right) + c^2 y(\tau) &= 0 &(7)
\end{aligned}
$$

The above ode is now solved for $y(\tau)$. Since the ode is now constant coefficients, it can be easily solved to give

$$y(\tau) = \mathrm{e}^{-c\tau} c_1$$

Now from (6)

$$\begin{aligned}
\tau &= \int \frac{1}{c}\sqrt{q}\, dt \\
&= \frac{\int \sqrt{\frac{1}{t^2}} dt}{c} \\
&= \frac{\sqrt{\frac{1}{t^2}}\, t \ln(t)}{c}
\end{aligned}$$

Substituting the above into the solution obtained gives

$$y = \frac{c_1}{t}$$

Summary

The solution(s) found are the following

$$y = \frac{c_1}{t} \tag{1}$$

Verification of solutions

$$y = \frac{c_1}{t}$$

Verified OK.

9.28.4 Solving as second order change of variable on y method 2 ode

In normal form the ode

$$t^2 y'' + 3ty' + y = 0 \tag{1}$$

Becomes

$$y'' + p(t)\, y' + q(t)\, y = 0 \tag{2}$$

Where

$$\begin{aligned}
p(t) &= \frac{3}{t} \\
q(t) &= \frac{1}{t^2}
\end{aligned}$$

Applying change of variables on the depndent variable $y = v(t)\, t^n$ to (2) gives the following ode where the dependent variables is $v(t)$ and not y.

$$v''(t) + \left(\frac{2n}{t} + p\right) v'(t) + \left(\frac{n(n-1)}{t^2} + \frac{np}{t} + q\right) v(t) = 0 \tag{3}$$

Let the coefficient of $v(t)$ above be zero. Hence

$$\frac{n(n-1)}{t^2} + \frac{np}{t} + q = 0 \tag{4}$$

Substituting the earlier values found for $p(t)$ and $q(t)$ into (4) gives

$$\frac{n(n-1)}{t^2} + \frac{3n}{t^2} + \frac{1}{t^2} = 0 \tag{5}$$

Solving (5) for n gives

$$n = -1 \tag{6}$$

Substituting this value in (3) gives

$$\begin{aligned} v''(t) + \frac{v'(t)}{t} &= 0 \\ v''(t) + \frac{v'(t)}{t} &= 0 \end{aligned} \tag{7}$$

Using the substitution

$$u(t) = v'(t)$$

Then (7) becomes

$$u'(t) + \frac{u(t)}{t} = 0 \tag{8}$$

The above is now solved for $u(t)$. In canonical form the ODE is

$$\begin{aligned} u' &= F(t, u) \\ &= f(t) g(u) \\ &= -\frac{u}{t} \end{aligned}$$

Where $f(t) = -\frac{1}{t}$ and $g(u) = u$. Integrating both sides gives

$$\begin{aligned}
\frac{1}{u}\, du &= -\frac{1}{t}\, dt \\
\int \frac{1}{u}\, du &= \int -\frac{1}{t}\, dt \\
\ln(u) &= -\ln(t) + c_1 \\
u &= \mathrm{e}^{-\ln(t)+c_1} \\
&= \frac{c_1}{t}
\end{aligned}$$

Now that $u(t)$ is known, then

$$\begin{aligned}
v'(t) &= u(t) \\
v(t) &= \int u(t)\, dt + c_2 \\
&= c_1 \ln(t) + c_2
\end{aligned}$$

Hence

$$\begin{aligned}
y &= v(t)\, t^n \\
&= \frac{c_1 \ln(t) + c_2}{t} \\
&= \frac{c_1 \ln(t) + c_2}{t}
\end{aligned}$$

Summary

The solution(s) found are the following

$$y = \frac{c_1 \ln(t) + c_2}{t} \tag{1}$$

Verification of solutions

$$y = \frac{c_1 \ln(t) + c_2}{t}$$

Verified OK.

9.28.5 Solving as second order integrable as is ode

Integrating both sides of the ODE w.r.t t gives

$$\int (t^2 y'' + 3ty' + y)\, dt = 0$$
$$y' t^2 + yt = c_1$$

Which is now solved for y.

Entering Linear first order ODE solver. In canonical form a linear first order is

$$y' + p(t)y = q(t)$$

Where here

$$p(t) = \frac{1}{t}$$
$$q(t) = \frac{c_1}{t^2}$$

Hence the ode is

$$y' + \frac{y}{t} = \frac{c_1}{t^2}$$

The integrating factor μ is

$$\mu = e^{\int \frac{1}{t} dt}$$
$$= t$$

The ode becomes

$$\frac{d}{dt}(\mu y) = (\mu)\left(\frac{c_1}{t^2}\right)$$
$$\frac{d}{dt}(ty) = (t)\left(\frac{c_1}{t^2}\right)$$
$$d(ty) = \left(\frac{c_1}{t}\right) dt$$

Integrating gives

$$ty = \int \frac{c_1}{t}\, dt$$
$$ty = c_1 \ln(t) + c_2$$

Dividing both sides by the integrating factor $\mu = t$ results in

$$y = \frac{c_1 \ln(t)}{t} + \frac{c_2}{t}$$

which simplifies to

$$y = \frac{c_1 \ln(t) + c_2}{t}$$

Summary

The solution(s) found are the following

$$y = \frac{c_1 \ln(t) + c_2}{t} \tag{1}$$

Verification of solutions

$$y = \frac{c_1 \ln(t) + c_2}{t}$$

Verified OK.

9.28.6 Solving as type second_order_integrable_as_is (not using ABC version)

Writing the ode as

$$t^2 y'' + 3ty' + y = 0$$

Integrating both sides of the ODE w.r.t t gives

$$\int (t^2 y'' + 3ty' + y)\, dt = 0$$
$$y' t^2 + yt = c_1$$

Which is now solved for y.

Entering Linear first order ODE solver. In canonical form a linear first order is

$$y' + p(t)y = q(t)$$

Where here

$$p(t) = \frac{1}{t}$$
$$q(t) = \frac{c_1}{t^2}$$

Hence the ode is

$$y' + \frac{y}{t} = \frac{c_1}{t^2}$$

The integrating factor μ is

$$
\begin{aligned}
\mu &= e^{\int \frac{1}{t}dt} \\
&= t
\end{aligned}
$$

The ode becomes

$$
\begin{aligned}
\frac{d}{dt}(\mu y) &= (\mu)\left(\frac{c_1}{t^2}\right) \\
\frac{d}{dt}(ty) &= (t)\left(\frac{c_1}{t^2}\right) \\
d(ty) &= \left(\frac{c_1}{t}\right) dt
\end{aligned}
$$

Integrating gives

$$
\begin{aligned}
ty &= \int \frac{c_1}{t}\, dt \\
ty &= c_1 \ln(t) + c_2
\end{aligned}
$$

Dividing both sides by the integrating factor $\mu = t$ results in

$$
y = \frac{c_1 \ln(t)}{t} + \frac{c_2}{t}
$$

which simplifies to

$$
y = \frac{c_1 \ln(t) + c_2}{t}
$$

Summary

The solution(s) found are the following

$$
y = \frac{c_1 \ln(t) + c_2}{t} \tag{1}
$$

Verification of solutions

$$
y = \frac{c_1 \ln(t) + c_2}{t}
$$

Verified OK.

9.28.7 Solving using Kovacic algorithm

Writing the ode as

$$t^2y'' + 3ty' + y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= t^2 \\ B &= 3t \\ C &= 1 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(t) = ye^{\int \frac{B}{2A}\, dt}$$

Then (2) becomes

$$z''(t) = rz(t) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{-1}{4t^2} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= -1 \\ t &= 4t^2 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(t) = \left(-\frac{1}{4t^2}\right) z(t) \tag{7}$$

Equation (7) is now solved. After finding $z(t)$ then y is found using the inverse transformation

$$y = z(t)\, e^{-\int \frac{B}{2A}\, dt}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 468: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 2 - 0 \\ &= 2 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 4t^2$. There is a pole at $t = 0$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 2 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Since pole order is not larger than 2 and the order at ∞ is 2 then the necessary conditions for case three are met. Therefore

$$L = [1, 2, 4, 6, 12]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = -\frac{1}{4t^2}$$

For the pole at $t = 0$ let b be the coefficient of $\frac{1}{t^2}$ in the partial fractions decomposition of r given above. Therefore $b = -\frac{1}{4}$. Hence

$$
\begin{aligned}
[\sqrt{r}]_c &= 0 \\
\alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{1}{2} \\
\alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = \frac{1}{2}
\end{aligned}
$$

Since the order of r at ∞ is 2 then $[\sqrt{r}]_\infty = 0$. Let b be the coefficient of $\frac{1}{t^2}$ in the Laurent series expansion of r at ∞. which can be found by dividing the leading coefficient of s by the leading coefficient of t from

$$
r = \frac{s}{t} = -\frac{1}{4t^2}
$$

Since the $\gcd(s, t) = 1$. This gives $b = -\frac{1}{4}$. Hence

$$
\begin{aligned}
[\sqrt{r}]_\infty &= 0 \\
\alpha_\infty^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{1}{2} \\
\alpha_\infty^- &= \frac{1}{2} - \sqrt{1+4b} = \frac{1}{2}
\end{aligned}
$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$
r = -\frac{1}{4t^2}
$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
0	2	0	$\frac{1}{2}$	$\frac{1}{2}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
2	0	$\frac{1}{2}$	$\frac{1}{2}$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$
d = \alpha_\infty^{s(\infty)} - \sum_{c \in \Gamma} \alpha_c^{s(c)}
$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c \in \Gamma \cup \infty}$ until such d is found to work in finding candidate ω.

Trying $\alpha_\infty^- = \frac{1}{2}$ then

$$\begin{aligned} d &= \alpha_\infty^- - (\alpha_{c_1}^-) \\ &= \frac{1}{2} - \left(\frac{1}{2}\right) \\ &= 0 \end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c\in\Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{t-c} \right) + s(\infty)[\sqrt{r}]_\infty$$

The above gives

$$\begin{aligned} \omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^-}{t-c_1} \right) + (-)[\sqrt{r}]_\infty \\ &= \frac{1}{2t} + (-)\,(0) \\ &= \frac{1}{2t} \\ &= \frac{1}{2t} \end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(t)$ of degree $d = 0$ to solve the ode. The polynomial $p(t)$ needs to satisfy the equation

$$p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right) p = 0 \tag{1A}$$

Let

$$p(t) = 1 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$\begin{aligned} (0) + 2\left(\frac{1}{2t}\right)(0) + \left(\left(-\frac{1}{2t^2}\right) + \left(\frac{1}{2t}\right)^2 - \left(-\frac{1}{4t^2}\right) \right) &= 0 \\ 0 &= 0 \end{aligned}$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned} z_1(t) &= p e^{\int \omega \, dt} \\ &= \mathrm{e}^{\int \frac{1}{2t} dt} \\ &= \sqrt{t} \end{aligned}$$

The first solution to the original ode in y is found from

$$
\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dt} \\
&= z_1 e^{-\int \frac{1}{2}\frac{3t}{t^2}\,dt} \\
&= z_1 e^{-\frac{3\ln(t)}{2}} \\
&= z_1 \left(\frac{1}{t^{\frac{3}{2}}}\right)
\end{aligned}
$$

Which simplifies to

$$
y_1 = \frac{1}{t}
$$

The second solution y_2 to the original ode is found using reduction of order

$$
y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dt}}{y_1^2}\,dt
$$

Substituting gives

$$
\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{3t}{t^2}\,dt}}{(y_1)^2}\,dt \\
&= y_1 \int \frac{e^{-3\ln(t)}}{(y_1)^2}\,dt \\
&= y_1(\ln(t))
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1\left(\frac{1}{t}\right) + c_2\left(\frac{1}{t}(\ln(t))\right)
\end{aligned}
$$

Summary

The solution(s) found are the following

$$
y = \frac{c_1}{t} + \frac{c_2 \ln(t)}{t} \tag{1}
$$

Verification of solutions

$$
y = \frac{c_1}{t} + \frac{c_2 \ln(t)}{t}
$$

Verified OK.

9.28.8 Solving as exact linear second order ode ode

An ode of the form

$$p(t)\,y'' + q(t)\,y' + r(t)\,y = s(t)$$

is exact if

$$p''(t) - q'(t) + r(t) = 0 \tag{1}$$

For the given ode we have

$$\begin{aligned}
p(x) &= t^2\\
q(x) &= 3t\\
r(x) &= 1\\
s(x) &= 0
\end{aligned}$$

Hence

$$\begin{aligned}
p''(x) &= 2\\
q'(x) &= 3
\end{aligned}$$

Therefore (1) becomes

$$2 - (3) + (1) = 0$$

Hence the ode is exact. Since we now know the ode is exact, it can be written as

$$(p(t)\,y' + (q(t) - p'(t))\,y)' = s(x)$$

Integrating gives

$$p(t)\,y' + (q(t) - p'(t))\,y = \int s(t)\ dt$$

Substituting the above values for p, q, r, s gives

$$y't^2 + yt = c_1$$

We now have a first order ode to solve which is

$$y't^2 + yt = c_1$$

Entering Linear first order ODE solver. In canonical form a linear first order is

$$y' + p(t)y = q(t)$$

Where here

$$p(t) = \frac{1}{t}$$
$$q(t) = \frac{c_1}{t^2}$$

Hence the ode is

$$y' + \frac{y}{t} = \frac{c_1}{t^2}$$

The integrating factor μ is

$$\mu = e^{\int \frac{1}{t}dt}$$
$$= t$$

The ode becomes

$$\frac{d}{dt}(\mu y) = (\mu)\left(\frac{c_1}{t^2}\right)$$
$$\frac{d}{dt}(ty) = (t)\left(\frac{c_1}{t^2}\right)$$
$$d(ty) = \left(\frac{c_1}{t}\right) dt$$

Integrating gives

$$ty = \int \frac{c_1}{t}\,dt$$
$$ty = c_1 \ln(t) + c_2$$

Dividing both sides by the integrating factor $\mu = t$ results in

$$y = \frac{c_1 \ln(t)}{t} + \frac{c_2}{t}$$

which simplifies to

$$y = \frac{c_1 \ln(t) + c_2}{t}$$

Summary

The solution(s) found are the following

$$y = \frac{c_1 \ln(t) + c_2}{t} \tag{1}$$

Verification of solutions

$$y = \frac{c_1 \ln(t) + c_2}{t}$$

Verified OK.

9.28.9 Maple step by step solution

Let's solve

$y''t^2+3ty'+y=0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y''=-\frac{3y'}{t}-\frac{y}{t^2}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y''+\frac{3y'}{t}+\frac{y}{t^2}=0$

- Multiply by denominators of the ODE

  $y''t^2+3ty'+y=0$

- Make a change of variables

  $s=\ln(t)$

□ Substitute the change of variables back into the ODE

  - Calculate the 1st derivative of y with respect to t , using the chain rule

    $y'=\left(\frac{d}{ds}y(s)\right)s'(t)$

  - Compute derivative

    $y'=\frac{\frac{d}{ds}y(s)}{t}$

  - Calculate the 2nd derivative of y with respect to t , using the chain rule

    $y''=\left(\frac{d^2}{ds^2}y(s)\right)s'(t)^2+s''(t)\left(\frac{d}{ds}y(s)\right)$

  - Compute derivative

    $y''=\frac{\frac{d^2}{ds^2}y(s)}{t^2}-\frac{\frac{d}{ds}y(s)}{t^2}$

  Substitute the change of variables back into the ODE

  $\left(\frac{\frac{d^2}{ds^2}y(s)}{t^2}-\frac{\frac{d}{ds}y(s)}{t^2}\right)t^2+3\frac{d}{ds}y(s)+y(s)=0$

- Simplify

  $\frac{d^2}{ds^2}y(s)+2\frac{d}{ds}y(s)+y(s)=0$

- Characteristic polynomial of ODE

  $r^2+2r+1=0$

- Factor the characteristic polynomial

  $(r+1)^2 = 0$
- Root of the characteristic polynomial

  $r = -1$
- 1st solution of the ODE

  $y_1(s) = e^{-s}$
- Repeated root, multiply $y_1(s)$ by s to ensure linear independence

  $y_2(s) = s\,e^{-s}$
- General solution of the ODE

  $y(s) = c_1 y_1(s) + c_2 y_2(s)$
- Substitute in solutions

  $y(s) = c_1 e^{-s} + c_2 s\,e^{-s}$
- Change variables back using $s = \ln(t)$

  $y = \frac{c_1}{t} + \frac{c_2 \ln(t)}{t}$
- Simplify

  $y = \frac{c_1}{t} + \frac{c_2 \ln(t)}{t}$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
<- LODE of Euler type successful`
```

✓ Solution by Maple

Time used: 0.015 (sec). Leaf size: 14

```
dsolve(t^2*diff(y(t),t$2)+3*t*diff(y(t),t)+y(t)=0,y(t), singsol=all)
```

$$y(t) = \frac{c_2 \ln(t) + c_1}{t}$$

✓ Solution by Mathematica

Time used: 0.015 (sec). Leaf size: 17

```
DSolve[t^2*y''[t]+3*t*y'[t]+y[t]==0,y[t],t,IncludeSingularSolutions -> True]
```

$$y(t) \to \frac{c_2 \log(t) + c_1}{t}$$

9.29 problem 44

Internal problem ID [681]
Internal file name [`OUTPUT/681_Sunday_June_05_2022_01_46_50_AM_23754413/index.tex`]

Book: Elementary differential equations and boundary value problems, 10th ed., Boyce and DiPrima
Section: Chapter 3, Second order linear equations, 3.4 Repeated roots, reduction of order , page 172
Problem number: 44.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_euler_ode", "second_order_change_of_variable_on_x_method_1", "second_order_change_of_variable_on_x_method_2", "second_order_change_of_variable_on_y_method_2"**

Maple gives the following as the ode type

```
[[_Emden, _Fowler]]
```

$$\boxed{4t^2y'' - 8ty' + 9y = 0}$$

9.29.1 Solving as second order euler ode ode

This is Euler second order ODE. Let the solution be $y = t^r$, then $y' = rt^{r-1}$ and $y'' = r(r-1)t^{r-2}$. Substituting these back into the given ODE gives

$$4t^2(r(r-1))t^{r-2} - 8trt^{r-1} + 9t^r = 0$$

Simplifying gives

$$4r(r-1)\,t^r - 8r\,t^r + 9t^r = 0$$

Since $t^r \neq 0$ then dividing throughout by t^r gives

$$4r(r-1) - 8r + 9 = 0$$

Or

$$4r^2 - 12r + 9 = 0 \tag{1}$$

Equation (1) is the characteristic equation. Its roots determine the form of the general solution. Using the quadratic equation the roots are

$$r_1 = \frac{3}{2}$$
$$r_2 = \frac{3}{2}$$

Since the roots are equal, then the general solution is

$$y = c_1 y_1 + c_2 y_2$$

Where $y_1 = t^r$ and $y_2 = t^r \ln(t)$. Hence

$$y = t^{\frac{3}{2}} c_1 + c_2 t^{\frac{3}{2}} \ln(t)$$

Summary
The solution(s) found are the following

$$y = t^{\frac{3}{2}} c_1 + c_2 t^{\frac{3}{2}} \ln(t) \tag{1}$$

Verification of solutions

$$y = t^{\frac{3}{2}} c_1 + c_2 t^{\frac{3}{2}} \ln(t)$$

Verified OK.

9.29.2 Solving as second order change of variable on x method 2 ode

In normal form the ode

$$4t^2 y'' - 8ty' + 9y = 0 \tag{1}$$

Becomes

$$y'' + p(t)\, y' + q(t)\, y = 0 \tag{2}$$

Where

$$p(t) = -\frac{2}{t}$$
$$q(t) = \frac{9}{4t^2}$$

Applying change of variables $\tau = g(t)$ to (2) gives

$$\frac{d^2}{d\tau^2}y(\tau) + p_1\left(\frac{d}{d\tau}y(\tau)\right) + q_1 y(\tau) = 0 \tag{3}$$

Where τ is the new independent variable, and

$$p_1(\tau) = \frac{\tau''(t) + p(t)\,\tau'(t)}{\tau'(t)^2} \tag{4}$$

$$q_1(\tau) = \frac{q(t)}{\tau'(t)^2} \tag{5}$$

Let $p_1 = 0$. Eq (4) simplifies to

$$\tau''(t) + p(t)\,\tau'(t) = 0$$

This ode is solved resulting in

$$\begin{aligned}
\tau &= \int \mathrm{e}^{-\left(\int p(t)dt\right)} dt \\
&= \int \mathrm{e}^{-\left(\int -\frac{2}{t}dt\right)} dt \\
&= \int e^{2\ln(t)}\, dt \\
&= \int t^2 dt \\
&= \frac{t^3}{3}
\end{aligned} \tag{6}$$

Using (6) to evaluate q_1 from (5) gives

$$\begin{aligned}
q_1(\tau) &= \frac{q(t)}{\tau'(t)^2} \\
&= \frac{\frac{9}{4t^2}}{t^4} \\
&= \frac{9}{4t^6}
\end{aligned} \tag{7}$$

Substituting the above in (3) and noting that now $p_1 = 0$ results in

$$\begin{aligned}
\frac{d^2}{d\tau^2}y(\tau) + q_1 y(\tau) &= 0 \\
\frac{d^2}{d\tau^2}y(\tau) + \frac{9y(\tau)}{4t^6} &= 0
\end{aligned}$$

But in terms of τ

$$\frac{9}{4t^6} = \frac{1}{4\tau^2}$$

Hence the above ode becomes

$$\frac{d^2}{d\tau^2}y(\tau) + \frac{y(\tau)}{4\tau^2} = 0$$

The above ode is now solved for $y(\tau)$. The ode can be written as

$$4\left(\frac{d^2}{d\tau^2}y(\tau)\right)\tau^2 + y(\tau) = 0$$

Which shows it is a Euler ODE. This is Euler second order ODE. Let the solution be $y(\tau) = \tau^r$, then $y' = r\tau^{r-1}$ and $y'' = r(r-1)\tau^{r-2}$. Substituting these back into the given ODE gives

$$4\tau^2(r(r-1))\tau^{r-2} + 0r\tau^{r-1} + \tau^r = 0$$

Simplifying gives

$$4r(r-1)\,\tau^r + 0\,\tau^r + \tau^r = 0$$

Since $\tau^r \neq 0$ then dividing throughout by τ^r gives

$$4r(r-1) + 0 + 1 = 0$$

Or

$$4r^2 - 4r + 1 = 0 \tag{1}$$

Equation (1) is the characteristic equation. Its roots determine the form of the general solution. Using the quadratic equation the roots are

$$r_1 = \frac{1}{2}$$
$$r_2 = \frac{1}{2}$$

Since the roots are equal, then the general solution is

$$y(\tau) = c_1y_1 + c_2y_2$$

Where $y_1 = \tau^r$ and $y_2 = \tau^r\ln(\tau)$. Hence

$$y(\tau) = c_1\sqrt{\tau} + c_2\sqrt{\tau}\,\ln(\tau)$$

The above solution is now transformed back to y using (6) which results in

$$y = \frac{\sqrt{3}\,\sqrt{t^3}\,(c_1 + c_2\ln(t^3) - c_2\ln(3))}{3}$$

Summary
The solution(s) found are the following

$$y = \frac{\sqrt{3}\,\sqrt{t^3}\,(c_1 + c_2 \ln(t^3) - c_2 \ln(3))}{3} \tag{1}$$

Verification of solutions

$$y = \frac{\sqrt{3}\,\sqrt{t^3}\,(c_1 + c_2 \ln(t^3) - c_2 \ln(3))}{3}$$

Verified OK.

9.29.3 Solving as second order change of variable on x method 1 ode

In normal form the ode

$$4t^2y'' - 8ty' + 9y = 0 \tag{1}$$

Becomes

$$y'' + p(t)\,y' + q(t)\,y = 0 \tag{2}$$

Where

$$p(t) = -\frac{2}{t}$$
$$q(t) = \frac{9}{4t^2}$$

Applying change of variables $\tau = g(t)$ to (2) results

$$\frac{d^2}{d\tau^2}y(\tau) + p_1\left(\frac{d}{d\tau}y(\tau)\right) + q_1 y(\tau) = 0 \tag{3}$$

Where τ is the new independent variable, and

$$p_1(\tau) = \frac{\tau''(t) + p(t)\,\tau'(t)}{\tau'(t)^2} \tag{4}$$

$$q_1(\tau) = \frac{q(t)}{\tau'(t)^2} \tag{5}$$

Let $q_1 = c^2$ where c is some constant. Therefore from (5)

$$\begin{aligned}\tau' &= \frac{1}{c}\sqrt{q} \\ &= \frac{3\sqrt{\frac{1}{t^2}}}{2c} \\ \tau'' &= -\frac{3}{2c\sqrt{\frac{1}{t^2}}\,t^3}\end{aligned} \tag{6}$$

Substituting the above into (4) results in

$$
\begin{aligned}
p_1(\tau) &= \frac{\tau''(t) + p(t)\,\tau'(t)}{\tau'(t)^2} \\
&= \frac{-\frac{3}{2c\sqrt{\frac{1}{t^2}}\,t^3} - \frac{2}{t}\,\frac{3\sqrt{\frac{1}{t^2}}}{2c}}{\left(\frac{3\sqrt{\frac{1}{t^2}}}{2c}\right)^2} \\
&= -2c
\end{aligned}
$$

Therefore ode (3) now becomes

$$
\begin{aligned}
y(\tau)'' + p_1 y(\tau)' + q_1 y(\tau) &= 0 \\
\frac{d^2}{d\tau^2} y(\tau) - 2c\left(\frac{d}{d\tau} y(\tau)\right) + c^2 y(\tau) &= 0 \qquad (7)
\end{aligned}
$$

The above ode is now solved for $y(\tau)$. Since the ode is now constant coefficients, it can be easily solved to give

$$
y(\tau) = \mathrm{e}^{c\tau} c_1
$$

Now from (6)

$$
\begin{aligned}
\tau &= \int \frac{1}{c}\sqrt{q}\,dt \\
&= \frac{\int \frac{3\sqrt{\frac{1}{t^2}}}{2} dt}{c} \\
&= \frac{3\sqrt{\frac{1}{t^2}}\,t\ln(t)}{2c}
\end{aligned}
$$

Substituting the above into the solution obtained gives

$$
y = t^{\frac{3}{2}} c_1
$$

Summary

The solution(s) found are the following

$$
y = t^{\frac{3}{2}} c_1 \qquad (1)
$$

Verification of solutions

$$
y = t^{\frac{3}{2}} c_1
$$

Verified OK.

9.29.4 Solving as second order change of variable on y method 2 ode

In normal form the ode

$$4t^2y'' - 8ty' + 9y = 0 \tag{1}$$

Becomes

$$y'' + p(t)\,y' + q(t)\,y = 0 \tag{2}$$

Where

$$\begin{aligned}p(t) &= -\frac{2}{t}\\ q(t) &= \frac{9}{4t^2}\end{aligned}$$

Applying change of variables on the depndent variable $y = v(t)\,t^n$ to (2) gives the following ode where the dependent variables is $v(t)$ and not y.

$$v''(t) + \left(\frac{2n}{t} + p\right) v'(t) + \left(\frac{n(n-1)}{t^2} + \frac{np}{t} + q\right) v(t) = 0 \tag{3}$$

Let the coefficient of $v(t)$ above be zero. Hence

$$\frac{n(n-1)}{t^2} + \frac{np}{t} + q = 0 \tag{4}$$

Substituting the earlier values found for $p(t)$ and $q(t)$ into (4) gives

$$\frac{n(n-1)}{t^2} - \frac{2n}{t^2} + \frac{9}{4t^2} = 0 \tag{5}$$

Solving (5) for n gives

$$n = \frac{3}{2} \tag{6}$$

Substituting this value in (3) gives

$$\begin{aligned}v''(t) + \frac{v'(t)}{t} &= 0\\ v''(t) + \frac{v'(t)}{t} &= 0\end{aligned} \tag{7}$$

Using the substitution

$$u(t) = v'(t)$$

Then (7) becomes

$$u'(t) + \frac{u(t)}{t} = 0 \tag{8}$$

The above is now solved for $u(t)$. In canonical form the ODE is

$$\begin{aligned} u' &= F(t,u) \\ &= f(t)g(u) \\ &= -\frac{u}{t} \end{aligned}$$

Where $f(t) = -\frac{1}{t}$ and $g(u) = u$. Integrating both sides gives

$$\begin{aligned} \frac{1}{u}\,du &= -\frac{1}{t}\,dt \\ \int \frac{1}{u}\,du &= \int -\frac{1}{t}\,dt \\ \ln(u) &= -\ln(t) + c_1 \\ u &= \mathrm{e}^{-\ln(t)+c_1} \\ &= \frac{c_1}{t} \end{aligned}$$

Now that $u(t)$ is known, then

$$\begin{aligned} v'(t) &= u(t) \\ v(t) &= \int u(t)\,dt + c_2 \\ &= c_1 \ln(t) + c_2 \end{aligned}$$

Hence

$$\begin{aligned} y &= v(t)\,t^n \\ &= (c_1 \ln(t) + c_2)\,t^{\frac{3}{2}} \\ &= (c_1 \ln(t) + c_2)\,t^{\frac{3}{2}} \end{aligned}$$

Summary

The solution(s) found are the following

$$y = (c_1 \ln(t) + c_2)\,t^{\frac{3}{2}} \tag{1}$$

Verification of solutions

$$y = (c_1 \ln(t) + c_2)\,t^{\frac{3}{2}}$$

Verified OK.

9.29.5 Solving using Kovacic algorithm

Writing the ode as

$$4t^2y'' - 8ty' + 9y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 4t^2 \\ B &= -8t \\ C &= 9 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(t) = ye^{\int \frac{B}{2A}\,dt}$$

Then (2) becomes

$$z''(t) = rz(t) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{-1}{4t^2} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= -1 \\ t &= 4t^2 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(t) = \left(-\frac{1}{4t^2}\right) z(t) \tag{7}$$

Equation (7) is now solved. After finding $z(t)$ then y is found using the inverse transformation

$$y = z(t)\, e^{-\int \frac{B}{2A}\,dt}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 470: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$
\begin{aligned}
O(\infty) &= \deg(t) - \deg(s) \\
&= 2 - 0 \\
&= 2
\end{aligned}
$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 4t^2$. There is a pole at $t = 0$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 2 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Since pole order is not larger than 2 and the order at ∞ is 2 then the necessary conditions for case three are met. Therefore

$$
L = [1, 2, 4, 6, 12]
$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$
r = -\frac{1}{4t^2}
$$

For the pole at $t = 0$ let b be the coefficient of $\frac{1}{t^2}$ in the partial fractions decomposition of r given above. Therefore $b = -\frac{1}{4}$. Hence

$$\begin{aligned}
[\sqrt{r}]_c &= 0 \\
\alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{1}{2} \\
\alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = \frac{1}{2}
\end{aligned}$$

Since the order of r at ∞ is 2 then $[\sqrt{r}]_\infty = 0$. Let b be the coefficient of $\frac{1}{t^2}$ in the Laurent series expansion of r at ∞. which can be found by dividing the leading coefficient of s by the leading coefficient of t from

$$r = \frac{s}{t} = -\frac{1}{4t^2}$$

Since the $\gcd(s,t) = 1$. This gives $b = -\frac{1}{4}$. Hence

$$\begin{aligned}
[\sqrt{r}]_\infty &= 0 \\
\alpha_\infty^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{1}{2} \\
\alpha_\infty^- &= \frac{1}{2} - \sqrt{1+4b} = \frac{1}{2}
\end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = -\frac{1}{4t^2}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
0	2	0	$\frac{1}{2}$	$\frac{1}{2}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
2	0	$\frac{1}{2}$	$\frac{1}{2}$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω.

Trying $\alpha_{\infty}^{-} = \frac{1}{2}$ then

$$
\begin{aligned}
d &= \alpha_{\infty}^{-} - \left(\alpha_{c_1}^{-}\right) \\
&= \frac{1}{2} - \left(\frac{1}{2}\right) \\
&= 0
\end{aligned}
$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$
\omega = \sum_{c \in \Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{t - c} \right) + s(\infty)[\sqrt{r}]_{\infty}
$$

The above gives

$$
\begin{aligned}
\omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^{-}}{t - c_1} \right) + (-)[\sqrt{r}]_{\infty} \\
&= \frac{1}{2t} + (-)\,(0) \\
&= \frac{1}{2t} \\
&= \frac{1}{2t}
\end{aligned}
$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(t)$ of degree $d = 0$ to solve the ode. The polynomial $p(t)$ needs to satisfy the equation

$$
p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right) p = 0 \tag{1A}
$$

Let

$$
p(t) = 1 \tag{2A}
$$

Substituting the above in eq. (1A) gives

$$
\begin{aligned}
(0) + 2\left(\frac{1}{2t}\right)(0) + \left(\left(-\frac{1}{2t^2}\right) + \left(\frac{1}{2t}\right)^2 - \left(-\frac{1}{4t^2}\right)\right) &= 0 \\
0 &= 0
\end{aligned}
$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z'' = rz$ is

$$
\begin{aligned}
z_1(t) &= p e^{\int \omega \, dt} \\
&= \mathrm{e}^{\int \frac{1}{2t} dt} \\
&= \sqrt{t}
\end{aligned}
$$

The first solution to the original ode in y is found from

$$
\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dt} \\
&= z_1 e^{-\int \frac{1}{2}\frac{-8t}{4t^2}\,dt} \\
&= z_1 e^{\ln(t)} \\
&= z_1(t)
\end{aligned}
$$

Which simplifies to

$$
y_1 = t^{\frac{3}{2}}
$$

The second solution y_2 to the original ode is found using reduction of order

$$
y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dt}}{y_1^2}\,dt
$$

Substituting gives

$$
\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{-8t}{4t^2}\,dt}}{(y_1)^2}\,dt \\
&= y_1 \int \frac{e^{2\ln(t)}}{(y_1)^2}\,dt \\
&= y_1(\ln(t))
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1\left(t^{\frac{3}{2}}\right) + c_2\left(t^{\frac{3}{2}}(\ln(t))\right)
\end{aligned}
$$

Summary

The solution(s) found are the following

$$
y = t^{\frac{3}{2}} c_1 + c_2 t^{\frac{3}{2}} \ln(t) \tag{1}
$$

Verification of solutions

$$
y = t^{\frac{3}{2}} c_1 + c_2 t^{\frac{3}{2}} \ln(t)
$$

Verified OK.

9.29.6 Maple step by step solution

Let's solve

$4y''t^2 - 8ty' + 9y = 0$

- Highest derivative means the order of the ODE is 2

y''

- Isolate 2nd derivative

$y'' = \frac{2y'}{t} - \frac{9y}{4t^2}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

$y'' - \frac{2y'}{t} + \frac{9y}{4t^2} = 0$

- Multiply by denominators of the ODE

$4y''t^2 - 8ty' + 9y = 0$

- Make a change of variables

$s = \ln(t)$

□ Substitute the change of variables back into the ODE

- ○ Calculate the 1st derivative of y with respect to t , using the chain rule

$y' = \left(\frac{d}{ds}y(s)\right)s'(t)$

- ○ Compute derivative

$y' = \frac{\frac{d}{ds}y(s)}{t}$

- ○ Calculate the 2nd derivative of y with respect to t , using the chain rule

$y'' = \left(\frac{d^2}{ds^2}y(s)\right)s'(t)^2 + s''(t)\left(\frac{d}{ds}y(s)\right)$

- ○ Compute derivative

$y'' = \frac{\frac{d^2}{ds^2}y(s)}{t^2} - \frac{\frac{d}{ds}y(s)}{t^2}$

Substitute the change of variables back into the ODE

$4\left(\frac{\frac{d^2}{ds^2}y(s)}{t^2} - \frac{\frac{d}{ds}y(s)}{t^2}\right)t^2 - 8\frac{d}{ds}y(s) + 9y(s) = 0$

- Simplify

$4\frac{d^2}{ds^2}y(s) - 12\frac{d}{ds}y(s) + 9y(s) = 0$

- Isolate 2nd derivative

$\frac{d^2}{ds^2}y(s) = 3\frac{d}{ds}y(s) - \frac{9y(s)}{4}$

- Group terms with $y(s)$ on the lhs of the ODE and the rest on the rhs of the ODE; ODE is lin
  $\frac{d^2}{ds^2}y(s) - 3\frac{d}{ds}y(s) + \frac{9y(s)}{4} = 0$
- Characteristic polynomial of ODE
  $r^2 - 3r + \frac{9}{4} = 0$
- Factor the characteristic polynomial
  $\frac{(2r-3)^2}{4} = 0$
- Root of the characteristic polynomial
  $r = \frac{3}{2}$
- 1st solution of the ODE
  $y_1(s) = e^{\frac{3s}{2}}$
- Repeated root, multiply $y_1(s)$ by s to ensure linear independence
  $y_2(s) = s\,e^{\frac{3s}{2}}$
- General solution of the ODE
  $y(s) = c_1 y_1(s) + c_2 y_2(s)$
- Substitute in solutions
  $y(s) = c_1 e^{\frac{3s}{2}} + c_2 s\,e^{\frac{3s}{2}}$
- Change variables back using $s = \ln(t)$
  $y = t^{\frac{3}{2}} c_1 + c_2 t^{\frac{3}{2}} \ln(t)$
- Simplify
  $y = (c_2 \ln(t) + c_1)\, t^{\frac{3}{2}}$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
<- LODE of Euler type successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 14

```
dsolve(4*t^2*diff(y(t),t$2)-8*t*diff(y(t),t)+9*y(t)=0,y(t), singsol=all)
```

$$y(t) = (c_2 \ln(t) + c_1)\, t^{\frac{3}{2}}$$

✓ Solution by Mathematica

Time used: 0.016 (sec). Leaf size: 25

```
DSolve[4*t^2*y''[t]-8*t*y'[t]+9*y[t]==0,y[t],t,IncludeSingularSolutions -> True]
```

$$y(t) \to \frac{1}{2} t^{3/2} (3c_2 \log(t) + 2c_1)$$

9.30 problem 45

Internal problem ID [682]
Internal file name [`OUTPUT/682_Sunday_June_05_2022_01_46_51_AM_41135738/index.tex`]

Book: Elementary differential equations and boundary value problems, 10th ed., Boyce and DiPrima
Section: Chapter 3, Second order linear equations, 3.4 Repeated roots, reduction of order , page 172
Problem number: 45.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_euler_ode", "second_order_change_of_variable_on_x_method_1", "second_order_change_of_variable_on_x_method_2", "second_order_change_of_variable_on_y_method_2"**

Maple gives the following as the ode type

```
[[_Emden, _Fowler]]
```

$$\boxed{t^2y'' + 5ty' + 13y = 0}$$

9.30.1 Solving as second order euler ode ode

This is Euler second order ODE. Let the solution be $y = t^r$, then $y' = rt^{r-1}$ and $y'' = r(r-1)t^{r-2}$. Substituting these back into the given ODE gives

$$t^2(r(r-1))t^{r-2} + 5trt^{r-1} + 13t^r = 0$$

Simplifying gives

$$r(r-1)\,t^r + 5r\,t^r + 13t^r = 0$$

Since $t^r \neq 0$ then dividing throughout by t^r gives

$$r(r-1) + 5r + 13 = 0$$

Or

$$r^2 + 4r + 13 = 0 \tag{1}$$

Equation (1) is the characteristic equation. Its roots determine the form of the general solution. Using the quadratic equation the roots are

$$\begin{aligned} r_1 &= -2 - 3i \\ r_2 &= -2 + 3i \end{aligned}$$

The roots are complex conjugate of each others. Let the roots be

$$\begin{aligned} r_1 &= \alpha + i\beta \\ r_2 &= \alpha - i\beta \end{aligned}$$

Where in this case $\alpha = -2$ and $\beta = -3$. Hence the solution becomes

$$\begin{aligned} y &= c_1 t^{r_1} + c_2 t^{r_2} \\ &= c_1 t^{\alpha+i\beta} + c_2 t^{\alpha-i\beta} \\ &= t^{\alpha}\left(c_1 t^{i\beta} + c_2 t^{-i\beta}\right) \\ &= t^{\alpha}\left(c_1 e^{\ln\left(t^{i\beta}\right)} + c_2 e^{\ln\left(t^{-i\beta}\right)}\right) \\ &= t^{\alpha}\left(c_1 e^{i(\beta \ln t)} + c_2 e^{-i(\beta \ln t)}\right) \end{aligned}$$

Using the values for $\alpha = -2, \beta = -3$, the above becomes

$$y = t^{-2}\left(c_1 e^{-3i\ln(t)} + c_2 e^{3i\ln(t)}\right)$$

Using Euler relation, the expression $c_1 e^{iA} + c_2 e^{-iA}$ is transformed to $c_1 \cos A + c_1 \sin A$ where the constants are free to change. Applying this to the above result gives

$$y = \frac{1}{t^2}(c_1 \cos(3\ln(t)) + c_2 \sin(3\ln(t)))$$

Summary

The solution(s) found are the following

$$y = \frac{c_1 \cos(3\ln(t)) + c_2 \sin(3\ln(t))}{t^2} \tag{1}$$

Verification of solutions

$$y = \frac{c_1 \cos(3\ln(t)) + c_2 \sin(3\ln(t))}{t^2}$$

Verified OK.

9.30.2 Solving as second order change of variable on x method 2 ode

In normal form the ode

$$t^2y'' + 5ty' + 13y = 0 \tag{1}$$

Becomes

$$y'' + p(t)\, y' + q(t)\, y = 0 \tag{2}$$

Where

$$\begin{aligned} p(t) &= \frac{5}{t} \\ q(t) &= \frac{13}{t^2} \end{aligned}$$

Applying change of variables $\tau = g(t)$ to (2) gives

$$\frac{d^2}{d\tau^2}y(\tau) + p_1\left(\frac{d}{d\tau}y(\tau)\right) + q_1y(\tau) = 0 \tag{3}$$

Where τ is the new independent variable, and

$$p_1(\tau) = \frac{\tau''(t) + p(t)\, \tau'(t)}{\tau'\left(t\right)^2} \tag{4}$$

$$q_1(\tau) = \frac{q(t)}{\tau'\left(t\right)^2} \tag{5}$$

Let $p_1 = 0$. Eq (4) simplifies to

$$\tau''(t) + p(t)\, \tau'(t) = 0$$

This ode is solved resulting in

$$\begin{aligned} \tau &= \int \mathrm{e}^{-\left(\int p(t)dt\right)}dt \\ &= \int \mathrm{e}^{-\left(\int \frac{5}{t}dt\right)}dt \\ &= \int e^{-5\ln(t)}\, dt \\ &= \int \frac{1}{t^5}dt \\ &= -\frac{1}{4t^4} \end{aligned} \tag{6}$$

Using (6) to evaluate q_1 from (5) gives

$$\begin{aligned} q_1(\tau) &= \frac{q(t)}{\tau'(t)^2} \\ &= \frac{\frac{13}{t^2}}{\frac{1}{t^{10}}} \\ &= 13t^8 \end{aligned} \tag{7}$$

Substituting the above in (3) and noting that now $p_1 = 0$ results in

$$\begin{aligned} \frac{d^2}{d\tau^2}y(\tau) + q_1 y(\tau) &= 0 \\ \frac{d^2}{d\tau^2}y(\tau) + 13t^8 y(\tau) &= 0 \end{aligned}$$

But in terms of τ

$$13t^8 = \frac{13}{16\tau^2}$$

Hence the above ode becomes

$$\frac{d^2}{d\tau^2}y(\tau) + \frac{13y(\tau)}{16\tau^2} = 0$$

The above ode is now solved for $y(\tau)$. The ode can be written as

$$16\left(\frac{d^2}{d\tau^2}y(\tau)\right)\tau^2 + 13y(\tau) = 0$$

Which shows it is a Euler ODE. This is Euler second order ODE. Let the solution be $y(\tau) = \tau^r$, then $y' = r\tau^{r-1}$ and $y'' = r(r-1)\tau^{r-2}$. Substituting these back into the given ODE gives

$$16\tau^2(r(r-1))\tau^{r-2} + 0r\tau^{r-1} + 13\tau^r = 0$$

Simplifying gives

$$16r(r-1)\,\tau^r + 0\,\tau^r + 13\tau^r = 0$$

Since $\tau^r \neq 0$ then dividing throughout by τ^r gives

$$16r(r-1) + 0 + 13 = 0$$

Or

$$16r^2 - 16r + 13 = 0 \tag{1}$$

Equation (1) is the characteristic equation. Its roots determine the form of the general solution. Using the quadratic equation the roots are

$$r_1 = \frac{1}{2} - \frac{3i}{4}$$
$$r_2 = \frac{1}{2} + \frac{3i}{4}$$

The roots are complex conjugate of each others. Let the roots be

$$r_1 = \alpha + i\beta$$
$$r_2 = \alpha - i\beta$$

Where in this case $\alpha = \frac{1}{2}$ and $\beta = -\frac{3}{4}$. Hence the solution becomes

$$\begin{aligned} y(\tau) &= c_1 \tau^{r_1} + c_2 \tau^{r_2} \\ &= c_1 \tau^{\alpha + i\beta} + c_2 \tau^{\alpha - i\beta} \\ &= \tau^{\alpha} \left(c_1 \tau^{i\beta} + c_2 \tau^{-i\beta}\right) \\ &= \tau^{\alpha} \left(c_1 e^{\ln(\tau^{i\beta})} + c_2 e^{\ln(\tau^{-i\beta})}\right) \\ &= \tau^{\alpha} \left(c_1 e^{i(\beta \ln \tau)} + c_2 e^{-i(\beta \ln \tau)}\right) \end{aligned}$$

Using the values for $\alpha = \frac{1}{2}, \beta = -\frac{3}{4}$, the above becomes

$$y(\tau) = \tau^{\frac{1}{2}} \left(c_1 e^{-\frac{3i \ln(\tau)}{4}} + c_2 e^{\frac{3i \ln(\tau)}{4}}\right)$$

Using Euler relation, the expression $c_1 e^{iA} + c_2 e^{-iA}$ is transformed to $c_1 \cos A + c_1 \sin A$ where the constants are free to change. Applying this to the above result gives

$$y(\tau) = \sqrt{\tau}\left(c_1 \cos\left(\frac{3\ln(\tau)}{4}\right) + c_2 \sin\left(\frac{3\ln(\tau)}{4}\right)\right)$$

The above solution is now transformed back to y using (6) which results in

$$y = \frac{\sqrt{-\frac{1}{t^4}}\left(c_1 \cos\left(-\frac{3\ln(2)}{2} + \frac{3\ln\left(-\frac{1}{t^4}\right)}{4}\right) + c_2 \sin\left(-\frac{3\ln(2)}{2} + \frac{3\ln\left(-\frac{1}{t^4}\right)}{4}\right)\right)}{2}$$

Summary

The solution(s) found are the following

$$y = \frac{\sqrt{-\frac{1}{t^4}}\left(c_1 \cos\left(-\frac{3\ln(2)}{2} + \frac{3\ln\left(-\frac{1}{t^4}\right)}{4}\right) + c_2 \sin\left(-\frac{3\ln(2)}{2} + \frac{3\ln\left(-\frac{1}{t^4}\right)}{4}\right)\right)}{2} \tag{1}$$

Verification of solutions

$$y = \frac{\sqrt{-\frac{1}{t^4}}\left(c_1 \cos\left(-\frac{3\ln(2)}{2}+\frac{3\ln\left(-\frac{1}{t^4}\right)}{4}\right)+c_2\sin\left(-\frac{3\ln(2)}{2}+\frac{3\ln\left(-\frac{1}{t^4}\right)}{4}\right)\right)}{2}$$

Verified OK.

9.30.3 Solving as second order change of variable on x method 1 ode

In normal form the ode

$$t^2y''+5ty'+13y=0 \tag{1}$$

Becomes

$$y''+p(t)\,y'+q(t)\,y=0 \tag{2}$$

Where

$$\begin{aligned} p(t) &= \frac{5}{t}\\ q(t) &= \frac{13}{t^2} \end{aligned}$$

Applying change of variables $\tau = g(t)$ to (2) results

$$\frac{d^2}{d\tau^2}y(\tau)+p_1\left(\frac{d}{d\tau}y(\tau)\right)+q_1y(\tau)=0 \tag{3}$$

Where τ is the new independent variable, and

$$p_1(\tau) = \frac{\tau''(t)+p(t)\,\tau'(t)}{\tau'(t)^2} \tag{4}$$

$$q_1(\tau) = \frac{q(t)}{\tau'(t)^2} \tag{5}$$

Let $q_1 = c^2$ where c is some constant. Therefore from (5)

$$\begin{aligned} \tau' &= \frac{1}{c}\sqrt{q}\\ &= \frac{\sqrt{13}\,\sqrt{\frac{1}{t^2}}}{c} \\ \tau'' &= -\frac{\sqrt{13}}{c\sqrt{\frac{1}{t^2}}\,t^3} \end{aligned} \tag{6}$$

Substituting the above into (4) results in

$$
\begin{aligned}
p_1(\tau) &= \frac{\tau''(t) + p(t)\,\tau'(t)}{\tau'(t)^2} \\
&= \frac{-\frac{\sqrt{13}}{c\sqrt{\frac{1}{t^2}}\,t^3} + \frac{5}{t}\frac{\sqrt{13}\,\sqrt{\frac{1}{t^2}}}{c}}{\left(\frac{\sqrt{13}\,\sqrt{\frac{1}{t^2}}}{c}\right)^2} \\
&= \frac{4c\sqrt{13}}{13}
\end{aligned}
$$

Therefore ode (3) now becomes

$$
\begin{aligned}
y(\tau)'' + p_1 y(\tau)' + q_1 y(\tau) = 0 \\
\frac{d^2}{d\tau^2} y(\tau) + \frac{4c\sqrt{13}\,\left(\frac{d}{d\tau} y(\tau)\right)}{13} + c^2 y(\tau) = 0
\end{aligned} \tag{7}
$$

The above ode is now solved for $y(\tau)$. Since the ode is now constant coefficients, it can be easily solved to give

$$
y(\tau) = \mathrm{e}^{-\frac{2\sqrt{13}\,c\tau}{13}} \left(c_1 \cos\left(\frac{3\sqrt{13}\,c\tau}{13}\right) + c_2 \sin\left(\frac{3\sqrt{13}\,c\tau}{13}\right)\right)
$$

Now from (6)

$$
\begin{aligned}
\tau &= \int \frac{1}{c}\sqrt{q}\,dt \\
&= \frac{\int \sqrt{13}\,\sqrt{\frac{1}{t^2}}dt}{c} \\
&= \frac{\sqrt{13}\,\sqrt{\frac{1}{t^2}}\,t\ln(t)}{c}
\end{aligned}
$$

Substituting the above into the solution obtained gives

$$
y = \frac{c_1 \cos(3\ln(t)) + c_2 \sin(3\ln(t))}{t^2}
$$

Summary

The solution(s) found are the following

$$
y = \frac{c_1 \cos(3\ln(t)) + c_2 \sin(3\ln(t))}{t^2} \tag{1}
$$

Verification of solutions

$$y = \frac{c_1 \cos(3 \ln(t)) + c_2 \sin(3 \ln(t))}{t^2}$$

Verified OK.

9.30.4 Solving as second order change of variable on y method 2 ode

In normal form the ode

$$t^2 y'' + 5ty' + 13y = 0 \tag{1}$$

Becomes

$$y'' + p(t)\, y' + q(t)\, y = 0 \tag{2}$$

Where

$$p(t) = \frac{5}{t}$$
$$q(t) = \frac{13}{t^2}$$

Applying change of variables on the depndent variable $y = v(t)\, t^n$ to (2) gives the following ode where the dependent variables is $v(t)$ and not y.

$$v''(t) + \left(\frac{2n}{t} + p\right) v'(t) + \left(\frac{n(n-1)}{t^2} + \frac{np}{t} + q\right) v(t) = 0 \tag{3}$$

Let the coefficient of $v(t)$ above be zero. Hence

$$\frac{n(n-1)}{t^2} + \frac{np}{t} + q = 0 \tag{4}$$

Substituting the earlier values found for $p(t)$ and $q(t)$ into (4) gives

$$\frac{n(n-1)}{t^2} + \frac{5n}{t^2} + \frac{13}{t^2} = 0 \tag{5}$$

Solving (5) for n gives

$$n = -2 + 3i \tag{6}$$

Substituting this value in (3) gives

$$v''(t) + \left(\frac{-4 + 6i}{t} + \frac{5}{t}\right) v'(t) = 0$$
$$v''(t) + \frac{(1 + 6i)\, v'(t)}{t} = 0 \tag{7}$$

Using the substitution

$$u(t) = v'(t)$$

Then (7) becomes

$$u'(t) + \frac{(1 + 6i)\, u(t)}{t} = 0 \tag{8}$$

The above is now solved for $u(t)$. In canonical form the ODE is

$$\begin{aligned} u' &= F(t, u) \\ &= f(t)g(u) \\ &= \frac{(-1 - 6i)\, u}{t} \end{aligned}$$

Where $f(t) = \frac{-1-6i}{t}$ and $g(u) = u$. Integrating both sides gives

$$\begin{aligned} \frac{1}{u}\, du &= \frac{-1 - 6i}{t}\, dt \\ \int \frac{1}{u}\, du &= \int \frac{-1 - 6i}{t}\, dt \\ \ln(u) &= (-1 - 6i)\ln(t) + c_1 \\ u &= \mathrm{e}^{(-1-6i)\ln(t)+c_1} \\ &= c_1 \mathrm{e}^{(-1-6i)\ln(t)} \end{aligned}$$

Which simplifies to

$$u(t) = \frac{c_1 t^{-6i}}{t}$$

Now that $u(t)$ is known, then

$$\begin{aligned} v'(t) &= u(t) \\ v(t) &= \int u(t)\, dt + c_2 \\ &= \frac{i c_1 t^{-6i}}{6} + c_2 \end{aligned}$$

Hence

$$\begin{aligned} y &= v(t)\, t^n \\ &= \left(\frac{i c_1 t^{-6i}}{6} + c_2\right) t^{-2+3i} \\ &= c_2 t^{-2+3i} + \frac{i c_1 t^{-2-3i}}{6} \end{aligned}$$

Summary
The solution(s) found are the following

$$y = \left(\frac{i c_1 t^{-6i}}{6} + c_2\right) t^{-2+3i} \tag{1}$$

Verification of solutions

$$y = \left(\frac{i c_1 t^{-6i}}{6} + c_2\right) t^{-2+3i}$$

Verified OK.

9.30.5 Solving using Kovacic algorithm

Writing the ode as

$$t^2 y'' + 5ty' + 13y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= t^2 \\ B &= 5t \\ C &= 13 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(t) = y e^{\int \frac{B}{2A}\, dt}$$

Then (2) becomes

$$z''(t) = r z(t) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{-37}{4t^2} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= -37 \\ t &= 4t^2 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(t) = \left(-\frac{37}{4t^2}\right) z(t) \tag{7}$$

Equation (7) is now solved. After finding $z(t)$ then y is found using the inverse transformation

$$y = z(t)\, e^{-\int \frac{B}{2A}\, dt}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 472: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 2 - 0 \\ &= 2 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 4t^2$. There is a pole at $t = 0$ of order 2. Since there is no odd order pole

larger than 2 and the order at ∞ is 2 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Since pole order is not larger than 2 and the order at ∞ is 2 then the necessary conditions for case three are met. Therefore

$$L = [1, 2, 4, 6, 12]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = -\frac{37}{4t^2}$$

For the pole at $t = 0$ let b be the coefficient of $\frac{1}{t^2}$ in the partial fractions decomposition of r given above. Therefore $b = -\frac{37}{4}$. Hence

$$\begin{aligned}
[\sqrt{r}]_c &= 0 \\
\alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{1}{2} + 3i \\
\alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = \frac{1}{2} - 3i
\end{aligned}$$

Since the order of r at ∞ is 2 then $[\sqrt{r}]_\infty = 0$. Let b be the coefficient of $\frac{1}{t^2}$ in the Laurent series expansion of r at ∞. which can be found by dividing the leading coefficient of s by the leading coefficient of t from

$$r = \frac{s}{t} = -\frac{37}{4t^2}$$

Since the $\gcd(s, t) = 1$. This gives $b = -\frac{37}{4}$. Hence

$$\begin{aligned}
[\sqrt{r}]_\infty &= 0 \\
\alpha_\infty^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{1}{2} + 3i \\
\alpha_\infty^- &= \frac{1}{2} - \sqrt{1+4b} = \frac{1}{2} - 3i
\end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = -\frac{37}{4t^2}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
0	2	0	$\frac{1}{2} + 3i$	$\frac{1}{2} - 3i$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
2	0	$\frac{1}{2}+3i$	$\frac{1}{2}-3i$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^- = \frac{1}{2} - 3i$ then

$$\begin{aligned}
d &= \alpha_\infty^- - (\alpha_{c_1}^-) \\
&= \frac{1}{2} - 3i - \left(\frac{1}{2} - 3i\right) \\
&= 0
\end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c\in\Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{t-c} \right) + s(\infty)[\sqrt{r}]_\infty$$

The above gives

$$\begin{aligned}
\omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^-}{t-c_1} \right) + (-)[\sqrt{r}]_\infty \\
&= \frac{\frac{1}{2} - 3i}{t} + (-)(0) \\
&= \frac{\frac{1}{2} - 3i}{t} \\
&= \frac{\frac{1}{2} - 3i}{t}
\end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(t)$ of degree $d = 0$ to solve the ode. The polynomial $p(t)$ needs to satisfy the equation

$$p'' + 2\omega p' + (\omega' + \omega^2 - r)\, p = 0 \tag{1A}$$

Let

$$p(t) = 1 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$(0)+2\left(\frac{\frac{1}{2}-3i}{t}\right)(0)+\left(\left(\frac{-\frac{1}{2}+3i}{t^2}\right)+\left(\frac{\frac{1}{2}-3i}{t}\right)^2-\left(-\frac{37}{4t^2}\right)\right)=0$$
$$0=0$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z''=rz$ is

$$\begin{aligned}
z_1(t) &= p e^{\int \omega \, dt} \\
&= \mathrm{e}^{\int \frac{\frac{1}{2}-3i}{t} dt} \\
&= t^{\frac{1}{2}-3i}
\end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A} \, dt} \\
&= z_1 e^{-\int \frac{1}{2}\frac{5t}{t^2} \, dt} \\
&= z_1 e^{-\frac{5\ln(t)}{2}} \\
&= z_1 \left(\frac{1}{t^{\frac{5}{2}}}\right)
\end{aligned}$$

Which simplifies to

$$y_1 = t^{-2-3i}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A} \, dt}}{y_1^2} \, dt$$

Substituting gives

$$\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{5t}{t^2} \, dt}}{(y_1)^2} \, dt \\
&= y_1 \int \frac{e^{-5\ln(t)}}{(y_1)^2} \, dt \\
&= y_1 \left(-\frac{i t^{6i}}{6}\right)
\end{aligned}$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1 \left(t^{-2-3i}\right) + c_2 \left(t^{-2-3i} \left(-\frac{i t^{6i}}{6}\right)\right)
\end{aligned}
$$

Summary

The solution(s) found are the following

$$
y = c_1 t^{-2-3i} - \frac{i c_2 t^{-2+3i}}{6} \tag{1}
$$

Verification of solutions

$$
y = c_1 t^{-2-3i} - \frac{i c_2 t^{-2+3i}}{6}
$$

Verified OK.

9.30.6 Maple step by step solution

Let's solve

$y'' t^2 + 5 t y' + 13 y = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y'' = -\frac{5y'}{t} - \frac{13y}{t^2}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y'' + \frac{5y'}{t} + \frac{13y}{t^2} = 0$

- Multiply by denominators of the ODE

  $y'' t^2 + 5 t y' + 13 y = 0$

- Make a change of variables

  $s = \ln(t)$

□ Substitute the change of variables back into the ODE

  - Calculate the 1st derivative of y with respect to t , using the chain rule

    $y' = \left(\frac{d}{ds} y(s)\right) s'(t)$

- Compute derivative

  $y' = \frac{\frac{d}{ds}y(s)}{t}$

- Calculate the 2nd derivative of y with respect to t , using the chain rule

  $y'' = \left(\frac{d^2}{ds^2}y(s)\right) s'(t)^2 + s''(t) \left(\frac{d}{ds}y(s)\right)$

- Compute derivative

  $y'' = \frac{\frac{d^2}{ds^2}y(s)}{t^2} - \frac{\frac{d}{ds}y(s)}{t^2}$

  Substitute the change of variables back into the ODE

  $\left(\frac{\frac{d^2}{ds^2}y(s)}{t^2} - \frac{\frac{d}{ds}y(s)}{t^2}\right) t^2 + 5\frac{d}{ds}y(s) + 13y(s) = 0$

- Simplify

  $\frac{d^2}{ds^2}y(s) + 4\frac{d}{ds}y(s) + 13y(s) = 0$

- Characteristic polynomial of ODE

  $r^2 + 4r + 13 = 0$

- Use quadratic formula to solve for r

  $r = \frac{(-4)\pm(\sqrt{-36})}{2}$

- Roots of the characteristic polynomial

  $r = (-2 - 3\,\mathrm{I}, -2 + 3\,\mathrm{I})$

- 1st solution of the ODE

  $y_1(s) = \mathrm{e}^{-2s} \cos(3s)$

- 2nd solution of the ODE

  $y_2(s) = \mathrm{e}^{-2s} \sin(3s)$

- General solution of the ODE

  $y(s) = c_1 y_1(s) + c_2 y_2(s)$

- Substitute in solutions

  $y(s) = c_1 \mathrm{e}^{-2s} \cos(3s) + c_2 \mathrm{e}^{-2s} \sin(3s)$

- Change variables back using $s = \ln(t)$

  $y = \frac{c_1 \cos(3\ln(t))}{t^2} + \frac{c_2 \sin(3\ln(t))}{t^2}$

- Simplify

  $y = \frac{c_1 \cos(3\ln(t))}{t^2} + \frac{c_2 \sin(3\ln(t))}{t^2}$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
<- LODE of Euler type successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 23

```
dsolve(t^2*diff(y(t),t$2)+5*t*diff(y(t),t)+13*y(t)=0,y(t), singsol=all)
```

$$y(t) = \frac{c_1 \sin(3\ln(t)) + c_2 \cos(3\ln(t))}{t^2}$$

✓ Solution by Mathematica

Time used: 0.024 (sec). Leaf size: 26

```
DSolve[t^2*y''[t]+5*t*y'[t]+13*y[t]==0,y[t],t,IncludeSingularSolutions -> True]
```

$$y(t) \to \frac{c_2 \cos(3\log(t)) + c_1 \sin(3\log(t))}{t^2}$$

10 Chapter 3, Second order linear equations, section 3.6, Variation of Parameters. page 190

10.1 problem 1

Internal problem ID [683]
Internal file name [OUTPUT/683_Sunday_June_05_2022_01_46_52_AM_10580708/index.tex]

Book: Elementary differential equations and boundary value problems, 10th ed., Boyce and DiPrima
Section: Chapter 3, Second order linear equations, section 3.6, Variation of Parameters. page 190
Problem number: 1.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_linear_constant_coeff"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

$$\boxed{y'' - 5y' + 6y = 2\,\mathrm{e}^{t}}$$

10.1.1 Solving as second order linear constant coeff ode

This is second order non-homogeneous ODE. In standard form the ODE is

$$Ay''(t) + By'(t) + Cy(t) = f(t)$$

Where $A = 1, B = -5, C = 6, f(t) = 2\,\mathrm{e}^{t}$. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(t) + By'(t) + Cy(t) = 0$, and y_p is a particular solution to the non-homogeneous ODE $Ay''(t) + By'(t) + Cy(t) = f(t)$. y_h is the solution to

$$y'' - 5y' + 6y = 0$$

This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$Ay''(t) + By'(t) + Cy(t) = 0$$

Where in the above $A = 1, B = -5, C = 6$. Let the solution be $y = e^{\lambda t}$. Substituting this into the ODE gives

$$\lambda^2 e^{\lambda t} - 5\lambda\, e^{\lambda t} + 6\, e^{\lambda t} = 0 \tag{1}$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda t}$ gives

$$\lambda^2 - 5\lambda + 6 = 0 \tag{2}$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form.Using the quadratic formula

$$\lambda_{1,2} = \frac{-B}{2A} \pm \frac{1}{2A}\sqrt{B^2 - 4AC}$$

Substituting $A = 1, B = -5, C = 6$ into the above gives

$$\begin{aligned}\lambda_{1,2} &= \frac{5}{(2)(1)} \pm \frac{1}{(2)(1)}\sqrt{-5^2 - (4)(1)(6)}\\ &= \frac{5}{2} \pm \frac{1}{2}\end{aligned}$$

Hence

$$\lambda_1 = \frac{5}{2} + \frac{1}{2}$$
$$\lambda_2 = \frac{5}{2} - \frac{1}{2}$$

Which simplifies to

$$\lambda_1 = 3$$
$$\lambda_2 = 2$$

Since roots are real and distinct, then the solution is

$$y = c_1 e^{\lambda_1 t} + c_2 e^{\lambda_2 t}$$
$$y = c_1 e^{(3)t} + c_2 e^{(2)t}$$

Or

$$y = c_1 e^{3t} + c_2 e^{2t}$$

Therefore the homogeneous solution y_h is

$$y_h = c_1 e^{3t} + c_2 e^{2t}$$

The particular solution is now found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$2\,\mathrm{e}^{t}$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$[\{\mathrm{e}^{t}\}]$$

While the set of the basis functions for the homogeneous solution found earlier is

$$\{\mathrm{e}^{2t}, \mathrm{e}^{3t}\}$$

Since there is no duplication between the basis function in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis in the UC_set.

$$y_p = A_1 \mathrm{e}^{t}$$

The unknowns $\{A_1\}$ are found by substituting the above trial solution y_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$2A_1 \mathrm{e}^{t} = 2\,\mathrm{e}^{t}$$

Solving for the unknowns by comparing coefficients results in

$$[A_1 = 1]$$

Substituting the above back in the above trial solution y_p, gives the particular solution

$$y_p = \mathrm{e}^{t}$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \left(c_1 \mathrm{e}^{3t} + c_2 \mathrm{e}^{2t}\right) + \left(\mathrm{e}^{t}\right) \end{aligned}$$

Summary

The solution(s) found are the following

$$y = c_1 \mathrm{e}^{3t} + c_2 \mathrm{e}^{2t} + \mathrm{e}^{t} \tag{1}$$

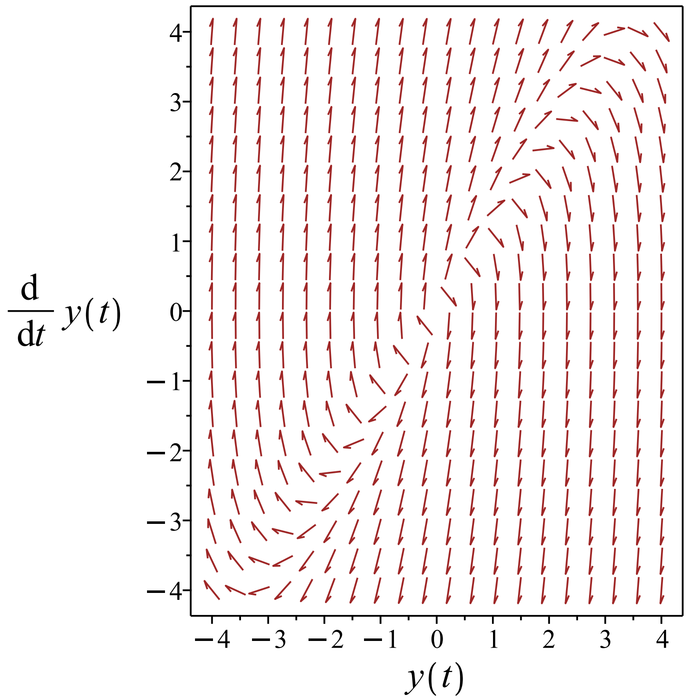


Figure 470: Slope field plot

Verification of solutions

$$y = c_1 e^{3t} + c_2 e^{2t} + e^t$$

Verified OK.

10.1.2 Solving using Kovacic algorithm

Writing the ode as

$$y'' - 5y' + 6y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1 \\ B &= -5 \\ C &= 6 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(t) = ye^{\int \frac{B}{2A}\, dt}$$

Then (2) becomes

$$z''(t) = rz(t) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{1}{4} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= 1 \\ t &= 4 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(t) = \frac{z(t)}{4} \tag{7}$$

Equation (7) is now solved. After finding $z(t)$ then y is found using the inverse transformation

$$y = z(t)\, e^{-\int \frac{B}{2A}\, dt}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 474: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned}O(\infty) &= \deg(t) - \deg(s)\\ &= 0 - 0\\ &= 0\end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Since $r = \frac{1}{4}$ is not a function of t, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$z_1(t) = \mathrm{e}^{-\frac{t}{2}}$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$\begin{aligned}y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dt}\\ &= z_1 e^{-\int \frac{1}{2}\frac{-5}{1}\,dt}\\ &= z_1 e^{\frac{5t}{2}}\\ &= z_1\left(\mathrm{e}^{\frac{5t}{2}}\right)\end{aligned}$$

Which simplifies to

$$y_1 = \mathrm{e}^{2t}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dt}}{y_1^2}\,dt$$

Substituting gives

$$\begin{aligned}y_2 &= y_1 \int \frac{e^{\int -\frac{-5}{1}\,dt}}{(y_1)^2}\,dt\\ &= y_1 \int \frac{e^{5t}}{(y_1)^2}\,dt\\ &= y_1\left(\mathrm{e}^{t}\right)\end{aligned}$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1 \left(\mathrm{e}^{2t}\right) + c_2 \left(\mathrm{e}^{2t} \left(\mathrm{e}^{t}\right)\right)
\end{aligned}
$$

This is second order nonhomogeneous ODE. Let the solution be

$$
y = y_h + y_p
$$

Where y_h is the solution to the homogeneous ODE $Ay''(t) + By'(t) + Cy(t) = 0$, and y_p is a particular solution to the nonhomogeneous ODE $Ay''(t) + By'(t) + Cy(t) = f(t)$. y_h is the solution to

$$
y'' - 5y' + 6y = 0
$$

The homogeneous solution is found using the Kovacic algorithm which results in

$$
y_h = c_1 \mathrm{e}^{2t} + c_2 \mathrm{e}^{3t}
$$

The particular solution is now found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$
2\,\mathrm{e}^{t}
$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$
[\{\mathrm{e}^{t}\}]
$$

While the set of the basis functions for the homogeneous solution found earlier is

$$
\{\mathrm{e}^{2t}, \mathrm{e}^{3t}\}
$$

Since there is no duplication between the basis function in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis in the UC_set.

$$
y_p = A_1 \mathrm{e}^{t}
$$

The unknowns $\{A_1\}$ are found by substituting the above trial solution y_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$
2A_1 \mathrm{e}^{t} = 2\,\mathrm{e}^{t}
$$

Solving for the unknowns by comparing coefficients results in

$$[A_1 = 1]$$

Substituting the above back in the above trial solution y_p, gives the particular solution

$$y_p = e^t$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= (c_1 e^{2t} + c_2 e^{3t}) + (e^t) \end{aligned}$$

<u>Summary</u>
The solution(s) found are the following

$$y = c_1 e^{2t} + c_2 e^{3t} + e^t \tag{1}$$

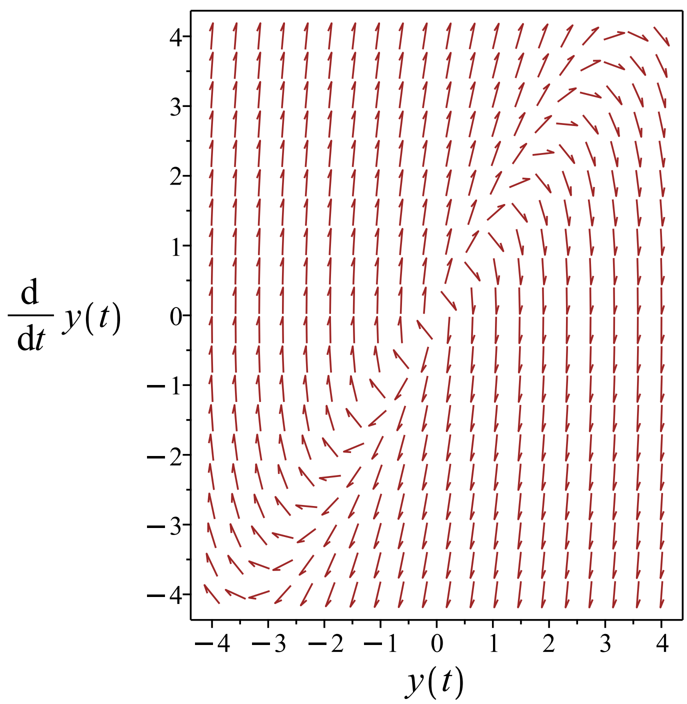


Figure 471: Slope field plot

<u>Verification of solutions</u>

$$y = c_1 e^{2t} + c_2 e^{3t} + e^t$$

Verified OK.

10.1.3 Maple step by step solution

Let's solve

$y'' - 5y' + 6y = 2\,\mathrm{e}^{t}$

- Highest derivative means the order of the ODE is 2

  y''

- Characteristic polynomial of homogeneous ODE

  $r^2 - 5r + 6 = 0$

- Factor the characteristic polynomial

  $(r-2)\,(r-3) = 0$

- Roots of the characteristic polynomial

  $r = (2, 3)$

- 1st solution of the homogeneous ODE

  $y_1(t) = \mathrm{e}^{2t}$

- 2nd solution of the homogeneous ODE

  $y_2(t) = \mathrm{e}^{3t}$

- General solution of the ODE

  $y = c_1 y_1(t) + c_2 y_2(t) + y_p(t)$

- Substitute in solutions of the homogeneous ODE

  $y = c_1 \mathrm{e}^{2t} + c_2 \mathrm{e}^{3t} + y_p(t)$

□ Find a particular solution $y_p(t)$ of the ODE

- ○ Use variation of parameters to find y_p here $f(t)$ is the forcing function

  $\left[y_p(t) = -y_1(t)\left(\int \frac{y_2(t) f(t)}{W(y_1(t), y_2(t))} dt\right) + y_2(t)\left(\int \frac{y_1(t) f(t)}{W(y_1(t), y_2(t))} dt\right), f(t) = 2\,\mathrm{e}^{t}\right]$

- ○ Wronskian of solutions of the homogeneous equation

  $W(y_1(t), y_2(t)) = \begin{bmatrix} \mathrm{e}^{2t} & \mathrm{e}^{3t} \\ 2\,\mathrm{e}^{2t} & 3\,\mathrm{e}^{3t} \end{bmatrix}$

- ○ Compute Wronskian

  $W(y_1(t), y_2(t)) = \mathrm{e}^{5t}$

- ○ Substitute functions into equation for $y_p(t)$

  $y_p(t) = -2\,\mathrm{e}^{2t}\left(\int \mathrm{e}^{-t} dt\right) + 2\,\mathrm{e}^{3t}\left(\int \mathrm{e}^{-2t} dt\right)$

- Compute integrals

  $y_p(t) = e^t$

- Substitute particular solution into general solution to ODE

  $y = c_1 e^{2t} + c_2 e^{3t} + e^t$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 2; linear nonhomogeneous with symmetry [0,1]
trying a double symmetry of the form [xi=0, eta=F(x)]
-> Try solving first the homogeneous part of the ODE
   checking if the LODE has constant coefficients
   <- constant coefficients successful
<- solving first the homogeneous part of the ODE successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 19

```
dsolve(diff(y(t),t$2)-5*diff(y(t),t)+6*y(t) = 2*exp(t),y(t), singsol=all)
```

$$y(t) = c_2 e^{2t} + c_1 e^{3t} + e^t$$

✓ Solution by Mathematica

Time used: 0.015 (sec). Leaf size: 25

```
DSolve[y''[t]-5*y'[t]+6*y[t] == 2*Exp[t],y[t],t,IncludeSingularSolutions -> True]
```

$$y(t) \to e^t (c_1 e^t + c_2 e^{2t} + 1)$$

10.2 problem 2

Internal problem ID [684]
Internal file name [`OUTPUT/684_Sunday_June_05_2022_01_46_53_AM_70729508/index.tex`]

Book: Elementary differential equations and boundary value problems, 10th ed., Boyce and DiPrima
Section: Chapter 3, Second order linear equations, section 3.6, Variation of Parameters. page 190
Problem number: 2.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_linear_constant_coeff"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

$$\boxed{y'' - y' - 2y = 2\,\mathrm{e}^{-t}}$$

10.2.1 Solving as second order linear constant coeff ode

This is second order non-homogeneous ODE. In standard form the ODE is

$$Ay''(t) + By'(t) + Cy(t) = f(t)$$

Where $A = 1, B = -1, C = -2, f(t) = 2\,\mathrm{e}^{-t}$. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(t) + By'(t) + Cy(t) = 0$, and y_p is a particular solution to the non-homogeneous ODE $Ay''(t) + By'(t) + Cy(t) = f(t)$. y_h is the solution to

$$y'' - y' - 2y = 0$$

This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$Ay''(t) + By'(t) + Cy(t) = 0$$

Where in the above $A = 1, B = -1, C = -2$. Let the solution be $y = e^{\lambda t}$. Substituting this into the ODE gives

$$\lambda^2 e^{\lambda t} - \lambda e^{\lambda t} - 2 e^{\lambda t} = 0 \tag{1}$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda t}$ gives

$$\lambda^2 - \lambda - 2 = 0 \tag{2}$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form.Using the quadratic formula

$$\lambda_{1,2} = \frac{-B}{2A} \pm \frac{1}{2A}\sqrt{B^2 - 4AC}$$

Substituting $A = 1, B = -1, C = -2$ into the above gives

$$\begin{aligned}\lambda_{1,2} &= \frac{1}{(2)(1)} \pm \frac{1}{(2)(1)}\sqrt{-1^2 - (4)(1)(-2)}\\ &= \frac{1}{2} \pm \frac{3}{2}\end{aligned}$$

Hence

$$\begin{aligned}\lambda_1 &= \frac{1}{2} + \frac{3}{2}\\ \lambda_2 &= \frac{1}{2} - \frac{3}{2}\end{aligned}$$

Which simplifies to

$$\begin{aligned}\lambda_1 &= 2\\ \lambda_2 &= -1\end{aligned}$$

Since roots are real and distinct, then the solution is

$$\begin{aligned}y &= c_1 e^{\lambda_1 t} + c_2 e^{\lambda_2 t}\\ y &= c_1 e^{(2)t} + c_2 e^{(-1)t}\end{aligned}$$

Or

$$y = c_1 e^{2t} + c_2 e^{-t}$$

Therefore the homogeneous solution y_h is

$$y_h = c_1 e^{2t} + c_2 e^{-t}$$

The particular solution is now found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$2\,\mathrm{e}^{-t}$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$[\{\mathrm{e}^{-t}\}]$$

While the set of the basis functions for the homogeneous solution found earlier is

$$\{\mathrm{e}^{-t}, \mathrm{e}^{2t}\}$$

Since e^{-t} is duplicated in the UC_set, then this basis is multiplied by extra t. The UC_set becomes

$$[\{t\,\mathrm{e}^{-t}\}]$$

Since there was duplication between the basis functions in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis function in the above updated UC_set.

$$y_p = A_1 t\,\mathrm{e}^{-t}$$

The unknowns $\{A_1\}$ are found by substituting the above trial solution y_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$-3A_1\mathrm{e}^{-t} = 2\,\mathrm{e}^{-t}$$

Solving for the unknowns by comparing coefficients results in

$$\left[A_1 = -\frac{2}{3}\right]$$

Substituting the above back in the above trial solution y_p, gives the particular solution

$$y_p = -\frac{2t\,\mathrm{e}^{-t}}{3}$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \left(c_1\mathrm{e}^{2t} + c_2\mathrm{e}^{-t}\right) + \left(-\frac{2t\,\mathrm{e}^{-t}}{3}\right) \end{aligned}$$

Summary
The solution(s) found are the following

$$y = c_1 e^{2t} + c_2 e^{-t} - \frac{2t\,e^{-t}}{3} \tag{1}$$

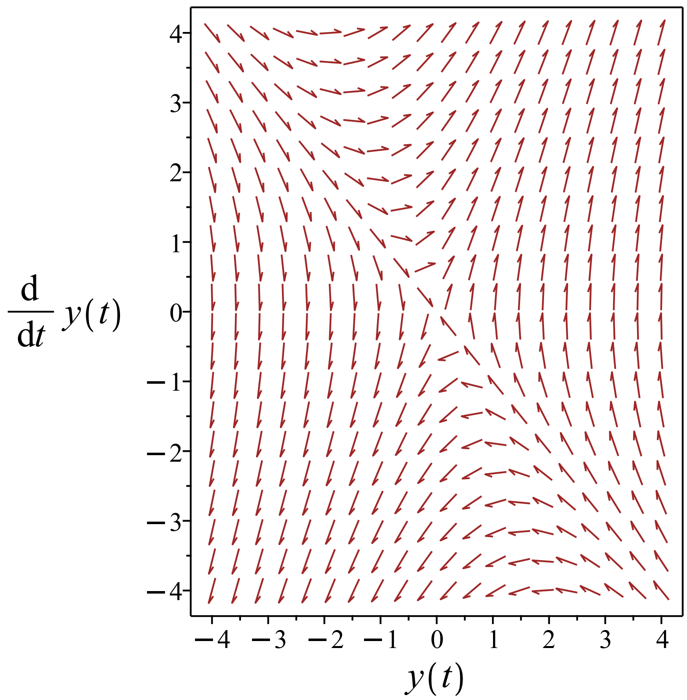


Figure 472: Slope field plot

Verification of solutions

$$y = c_1 e^{2t} + c_2 e^{-t} - \frac{2t\,e^{-t}}{3}$$

Verified OK.

10.2.2 Solving using Kovacic algorithm

Writing the ode as

$$y'' - y' - 2y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1 \\ B &= -1 \\ C &= -2 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(t) = ye^{\int \frac{B}{2A}\,dt}$$

Then (2) becomes

$$z''(t) = rz(t) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{9}{4} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= 9 \\ t &= 4 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(t) = \frac{9z(t)}{4} \tag{7}$$

Equation (7) is now solved. After finding $z(t)$ then y is found using the inverse transformation

$$y = z(t)\, e^{-\int \frac{B}{2A}\,dt}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0,1,2,4,6,8,\cdots\}$	$\{\cdots,-6,-4,-2,0,2,3,4,5,6,\cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1,2\}$	$\{2,3,4,5,6,7,\cdots\}$

Table 476: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 0 - 0 \\ &= 0 \end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Since $r = \frac{9}{4}$ is not a function of t, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$z_1(t) = \mathrm{e}^{-\frac{3t}{2}}$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$\begin{aligned} y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dt} \\ &= z_1 e^{-\int \frac{1}{2}\frac{-1}{1}\,dt} \\ &= z_1 e^{\frac{t}{2}} \\ &= z_1\left(\mathrm{e}^{\frac{t}{2}}\right) \end{aligned}$$

Which simplifies to

$$y_1 = e^{-t}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\, dt}}{y_1^2}\, dt$$

Substituting gives

$$\begin{aligned} y_2 &= y_1 \int \frac{e^{\int -\frac{-1}{1}\, dt}}{(y_1)^2}\, dt \\ &= y_1 \int \frac{e^t}{(y_1)^2}\, dt \\ &= y_1 \left(\frac{e^{3t}}{3}\right) \end{aligned}$$

Therefore the solution is

$$\begin{aligned} y &= c_1 y_1 + c_2 y_2 \\ &= c_1 \left(e^{-t}\right) + c_2 \left(e^{-t} \left(\frac{e^{3t}}{3}\right)\right) \end{aligned}$$

This is second order nonhomogeneous ODE. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(t) + By'(t) + Cy(t) = 0$, and y_p is a particular solution to the nonhomogeneous ODE $Ay''(t) + By'(t) + Cy(t) = f(t)$. y_h is the solution to

$$y'' - y' - 2y = 0$$

The homogeneous solution is found using the Kovacic algorithm which results in

$$y_h = c_1 e^{-t} + \frac{c_2 e^{2t}}{3}$$

The particular solution is now found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$2\,\mathrm{e}^{-t}$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$[\{\mathrm{e}^{-t}\}]$$

While the set of the basis functions for the homogeneous solution found earlier is

$$\left\{\frac{\mathrm{e}^{2t}}{3}, \mathrm{e}^{-t}\right\}$$

Since e^{-t} is duplicated in the UC_set, then this basis is multiplied by extra t. The UC_set becomes

$$[\{t\,\mathrm{e}^{-t}\}]$$

Since there was duplication between the basis functions in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis function in the above updated UC_set.

$$y_p = A_1 t\,\mathrm{e}^{-t}$$

The unknowns $\{A_1\}$ are found by substituting the above trial solution y_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$-3A_1\mathrm{e}^{-t} = 2\,\mathrm{e}^{-t}$$

Solving for the unknowns by comparing coefficients results in

$$\left[A_1 = -\frac{2}{3}\right]$$

Substituting the above back in the above trial solution y_p, gives the particular solution

$$y_p = -\frac{2t\,\mathrm{e}^{-t}}{3}$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \left(c_1\mathrm{e}^{-t} + \frac{c_2\mathrm{e}^{2t}}{3}\right) + \left(-\frac{2t\,\mathrm{e}^{-t}}{3}\right) \end{aligned}$$

Summary
The solution(s) found are the following

$$y = c_1 e^{-t} + \frac{c_2 e^{2t}}{3} - \frac{2t\, e^{-t}}{3} \tag{1}$$

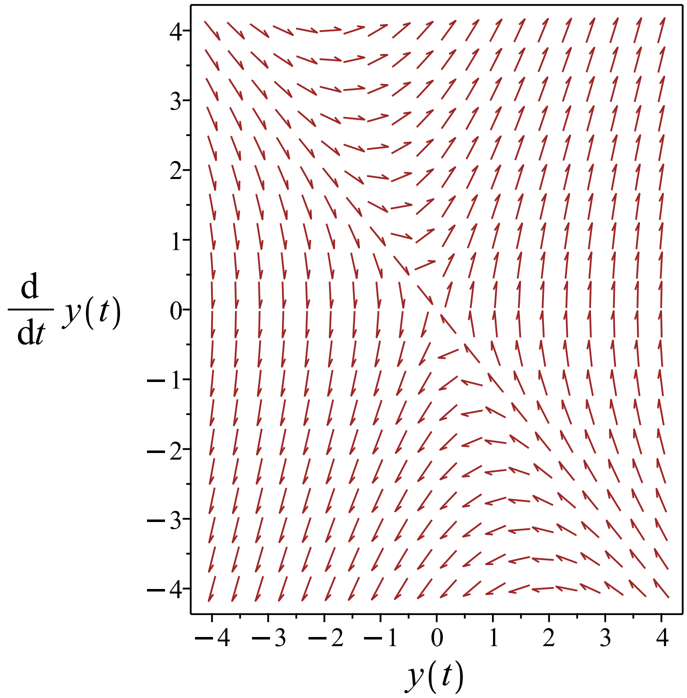


Figure 473: Slope field plot

Verification of solutions

$$y = c_1 e^{-t} + \frac{c_2 e^{2t}}{3} - \frac{2t\, e^{-t}}{3}$$

Verified OK.

10.2.3 Maple step by step solution

Let's solve

$y'' - y' - 2y = 2\, e^{-t}$

- Highest derivative means the order of the ODE is 2

  y''

- Characteristic polynomial of homogeneous ODE

$r^2 - r - 2 = 0$

- Factor the characteristic polynomial

  $(r+1)(r-2) = 0$

- Roots of the characteristic polynomial

  $r = (-1, 2)$

- 1st solution of the homogeneous ODE

  $y_1(t) = e^{-t}$

- 2nd solution of the homogeneous ODE

  $y_2(t) = e^{2t}$

- General solution of the ODE

  $y = c_1 y_1(t) + c_2 y_2(t) + y_p(t)$

- Substitute in solutions of the homogeneous ODE

  $y = c_1 e^{-t} + c_2 e^{2t} + y_p(t)$

☐ Find a particular solution $y_p(t)$ of the ODE

  ○ Use variation of parameters to find y_p here $f(t)$ is the forcing function

  $\left[y_p(t) = -y_1(t)\left(\int \frac{y_2(t)f(t)}{W(y_1(t),y_2(t))}dt\right) + y_2(t)\left(\int \frac{y_1(t)f(t)}{W(y_1(t),y_2(t))}dt\right), f(t) = 2\,e^{-t}\right]$

  ○ Wronskian of solutions of the homogeneous equation

  $$W(y_1(t), y_2(t)) = \begin{bmatrix} e^{-t} & e^{2t} \\ -e^{-t} & 2\,e^{2t} \end{bmatrix}$$

  ○ Compute Wronskian

  $W(y_1(t), y_2(t)) = 3\,e^{t}$

  ○ Substitute functions into equation for $y_p(t)$

  $y_p(t) = -\frac{2\,e^{-t}\left(\int 1 dt\right)}{3} + \frac{2\,e^{2t}\left(\int e^{-3t} dt\right)}{3}$

  ○ Compute integrals

  $y_p(t) = -\frac{2(1+3t)e^{-t}}{9}$

- Substitute particular solution into general solution to ODE

  $y = c_1 e^{-t} + c_2 e^{2t} - \frac{2(1+3t)e^{-t}}{9}$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 2; linear nonhomogeneous with symmetry [0,1]
trying a double symmetry of the form [xi=0, eta=F(x)]
-> Try solving first the homogeneous part of the ODE
   checking if the LODE has constant coefficients
   <- constant coefficients successful
<- solving first the homogeneous part of the ODE successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 24

```
dsolve(diff(y(t),t$2)-diff(y(t),t)-2*y(t) = 2*exp(-t),y(t), singsol=all)
```

$$y(t) = \frac{(-2t + 3c_1)\,\mathrm{e}^{-t}}{3} + c_2\mathrm{e}^{2t}$$

✓ Solution by Mathematica

Time used: 0.035 (sec). Leaf size: 32

```
DSolve[y''[t]-y'[t]-2*y[t] == 2*Exp[-t],y[t],t,IncludeSingularSolutions -> True]
```

$$y(t) \to \frac{1}{9}e^{-t}\left(-6t + 9c_2e^{3t} - 2 + 9c_1\right)$$

10.3 problem 3

Internal problem ID [685]
Internal file name [`OUTPUT/685_Sunday_June_05_2022_01_46_54_AM_20733644/index.tex`]

Book: Elementary differential equations and boundary value problems, 10th ed., Boyce and DiPrima
Section: Chapter 3, Second order linear equations, section 3.6, Variation of Parameters. page 190
Problem number: 3.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_linear_constant_coeff", "linear_second_order_ode_solved_by_an_integrating_factor"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

$$\boxed{y'' + 2y' + y = 3\,\mathrm{e}^{-t}}$$

10.3.1 Solving as second order linear constant coeff ode

This is second order non-homogeneous ODE. In standard form the ODE is

$$Ay''(t) + By'(t) + Cy(t) = f(t)$$

Where $A = 1, B = 2, C = 1, f(t) = 3\,\mathrm{e}^{-t}$. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(t) + By'(t) + Cy(t) = 0$, and y_p is a particular solution to the non-homogeneous ODE $Ay''(t) + By'(t) + Cy(t) = f(t)$. y_h is the solution to

$$y'' + 2y' + y = 0$$

This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$Ay''(t) + By'(t) + Cy(t) = 0$$

Where in the above $A = 1, B = 2, C = 1$. Let the solution be $y = e^{\lambda t}$. Substituting this into the ODE gives

$$\lambda^2 e^{\lambda t} + 2\lambda\, e^{\lambda t} + e^{\lambda t} = 0 \tag{1}$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda t}$ gives

$$\lambda^2 + 2\lambda + 1 = 0 \tag{2}$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form.Using the quadratic formula

$$\lambda_{1,2} = \frac{-B}{2A} \pm \frac{1}{2A}\sqrt{B^2 - 4AC}$$

Substituting $A = 1, B = 2, C = 1$ into the above gives

$$\begin{aligned}\lambda_{1,2} &= \frac{-2}{(2)\,(1)} \pm \frac{1}{(2)\,(1)}\sqrt{(2)^2 - (4)\,(1)\,(1)}\\ &= -1\end{aligned}$$

Hence this is the case of a double root $\lambda_{1,2} = 1$. Therefore the solution is

$$y = c_1 e^{-t} + c_2 t\, e^{-t} \tag{1}$$

Therefore the homogeneous solution y_h is

$$y_h = c_1 e^{-t} + c_2 e^{-t} t$$

The particular solution is now found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$3\, e^{-t}$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$[\{e^{-t}\}]$$

While the set of the basis functions for the homogeneous solution found earlier is

$$\{t\, e^{-t}, e^{-t}\}$$

Since e^{-t} is duplicated in the UC_set, then this basis is multiplied by extra t. The UC_set becomes

$$[\{t\,e^{-t}\}]$$

Since $t\,e^{-t}$ is duplicated in the UC_set, then this basis is multiplied by extra t. The UC_set becomes

$$[\{t^2e^{-t}\}]$$

Since there was duplication between the basis functions in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis function in the above updated UC_set.

$$y_p = A_1t^2e^{-t}$$

The unknowns $\{A_1\}$ are found by substituting the above trial solution y_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$2A_1e^{-t} = 3\,e^{-t}$$

Solving for the unknowns by comparing coefficients results in

$$\left[A_1 = \frac{3}{2}\right]$$

Substituting the above back in the above trial solution y_p, gives the particular solution

$$y_p = \frac{3t^2e^{-t}}{2}$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \left(c_1e^{-t} + c_2e^{-t}t\right) + \left(\frac{3t^2e^{-t}}{2}\right) \end{aligned}$$

Which simplifies to

$$y = (c_2t + c_1)\,e^{-t} + \frac{3t^2e^{-t}}{2}$$

Summary
The solution(s) found are the following

$$y = (c_2 t + c_1) e^{-t} + \frac{3t^2 e^{-t}}{2} \tag{1}$$

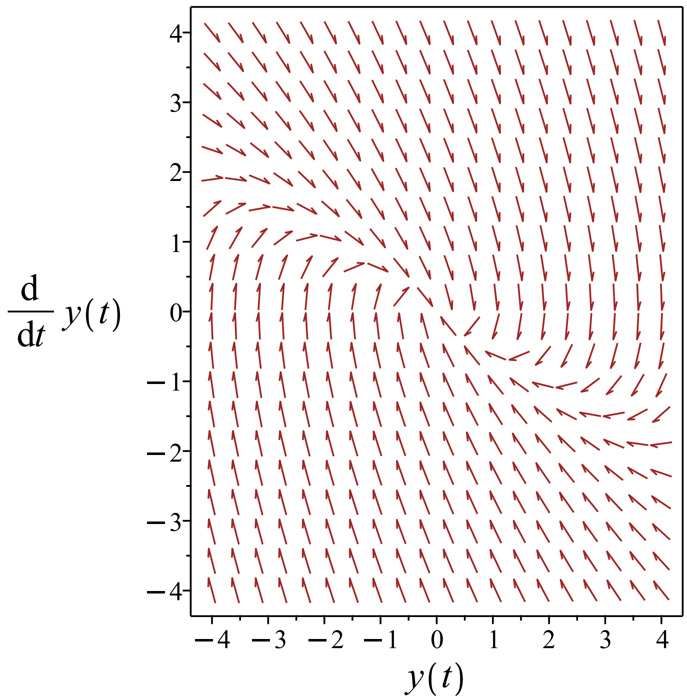


Figure 474: Slope field plot

Verification of solutions

$$y = (c_2 t + c_1) e^{-t} + \frac{3t^2 e^{-t}}{2}$$

Verified OK.

10.3.2 Solving as linear second order ode solved by an integrating factor ode

The ode satisfies this form

$$y'' + p(t) y' + \frac{(p(t)^2 + p'(t)) y}{2} = f(t)$$

Where $p(t) = 2$. Therefore, there is an integrating factor given by

$$\begin{aligned} M(x) &= e^{\frac{1}{2}\int p\,dx} \\ &= e^{\int 2\,dx} \\ &= \mathrm{e}^{t} \end{aligned}$$

Multiplying both sides of the ODE by the integrating factor $M(x)$ makes the left side of the ODE a complete differential

$$\begin{aligned} (M(x)y)'' &= 3\,\mathrm{e}^{-t}\mathrm{e}^{t} \\ \left(y\,\mathrm{e}^{t}\right)'' &= 3\,\mathrm{e}^{-t}\mathrm{e}^{t} \end{aligned}$$

Integrating once gives

$$\left(y\,\mathrm{e}^{t}\right)' = 3t + c_1$$

Integrating again gives

$$\left(y\,\mathrm{e}^{t}\right) = \frac{t(3t + 2c_1)}{2} + c_2$$

Hence the solution is

$$y = \frac{\frac{t(3t+2c_1)}{2} + c_2}{\mathrm{e}^{t}}$$

Or

$$y = c_1 t\,\mathrm{e}^{-t} + \frac{3t^2\mathrm{e}^{-t}}{2} + c_2\mathrm{e}^{-t}$$

Summary

The solution(s) found are the following

$$y = c_1 t\,\mathrm{e}^{-t} + \frac{3t^2\mathrm{e}^{-t}}{2} + c_2\mathrm{e}^{-t} \tag{1}$$

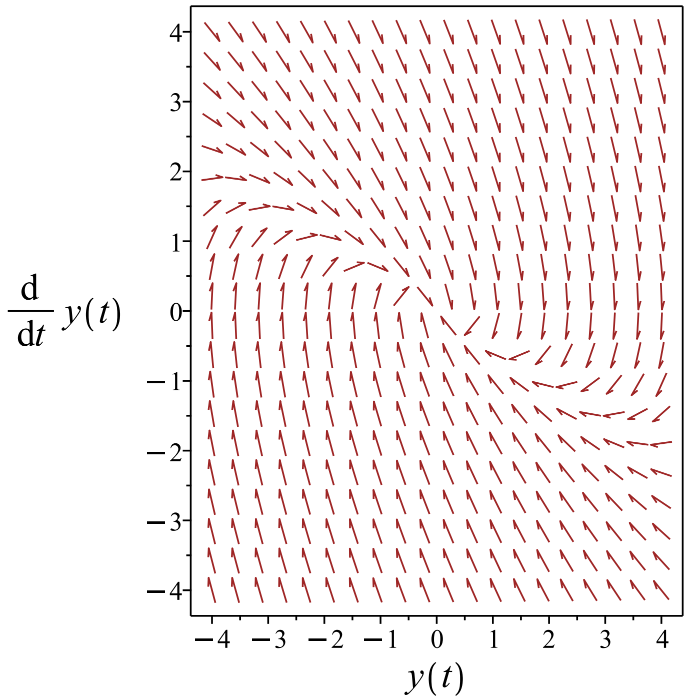


Figure 475: Slope field plot

Verification of solutions

$$y = c_1 t\,e^{-t} + \frac{3t^2 e^{-t}}{2} + c_2 e^{-t}$$

Verified OK.

10.3.3 Solving using Kovacic algorithm

Writing the ode as

$$y'' + 2y' + y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1 \\ B &= 2 \\ C &= 1 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(t) = y e^{\int \frac{B}{2A}\,dt}$$

Then (2) becomes

$$z''(t) = rz(t) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{0}{1} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= 0 \\ t &= 1 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(t) = 0 \tag{7}$$

Equation (7) is now solved. After finding $z(t)$ then y is found using the inverse transformation

$$y = z(t)\, e^{-\int \frac{B}{2A}\, dt}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 478: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$
\begin{aligned}
O(\infty) &= \deg(t) - \deg(s) \\
&= 0 - -\infty \\
&= \infty
\end{aligned}
$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is $infinity$ then the necessary conditions for case one are met. Therefore

$$
L = [1]
$$

Since $r = 0$ is not a function of t, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$
z_1(t) = 1
$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$
\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dt} \\
&= z_1 e^{-\int \frac{1}{2}\frac{2}{1}\,dt} \\
&= z_1 e^{-t} \\
&= z_1 \left(\mathrm{e}^{-t}\right)
\end{aligned}
$$

Which simplifies to

$$
y_1 = \mathrm{e}^{-t}
$$

The second solution y_2 to the original ode is found using reduction of order

$$
y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dt}}{y_1^2}\,dt
$$

Substituting gives

$$
\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{2}{1}\,dt}}{(y_1)^2}\,dt \\
&= y_1 \int \frac{e^{-2t}}{(y_1)^2}\,dt \\
&= y_1(t)
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1 \left(\mathrm{e}^{-t}\right) + c_2 \left(\mathrm{e}^{-t}(t)\right)
\end{aligned}
$$

This is second order nonhomogeneous ODE. Let the solution be

$$
y = y_h + y_p
$$

Where y_h is the solution to the homogeneous ODE $Ay''(t) + By'(t) + Cy(t) = 0$, and y_p is a particular solution to the nonhomogeneous ODE $Ay''(t) + By'(t) + Cy(t) = f(t)$. y_h is the solution to

$$
y'' + 2y' + y = 0
$$

The homogeneous solution is found using the Kovacic algorithm which results in

$$
y_h = c_1 \mathrm{e}^{-t} + c_2 \mathrm{e}^{-t} t
$$

The particular solution is now found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$
3\,\mathrm{e}^{-t}
$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$
[\{\mathrm{e}^{-t}\}]
$$

While the set of the basis functions for the homogeneous solution found earlier is

$$
\{t\,\mathrm{e}^{-t}, \mathrm{e}^{-t}\}
$$

Since e^{-t} is duplicated in the UC_set, then this basis is multiplied by extra t. The UC_set becomes

$$
[\{t\,\mathrm{e}^{-t}\}]
$$

Since $t\,\mathrm{e}^{-t}$ is duplicated in the UC_set, then this basis is multiplied by extra t. The UC_set becomes

$$
[\{t^2 \mathrm{e}^{-t}\}]
$$

Since there was duplication between the basis functions in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis function in the above updated UC_set.

$$y_p = A_1 t^2 \mathrm{e}^{-t}$$

The unknowns $\{A_1\}$ are found by substituting the above trial solution y_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$2A_1 \mathrm{e}^{-t} = 3\,\mathrm{e}^{-t}$$

Solving for the unknowns by comparing coefficients results in

$$\left[A_1 = \frac{3}{2}\right]$$

Substituting the above back in the above trial solution y_p, gives the particular solution

$$y_p = \frac{3t^2 \mathrm{e}^{-t}}{2}$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \left(c_1 \mathrm{e}^{-t} + c_2 \mathrm{e}^{-t} t\right) + \left(\frac{3t^2 \mathrm{e}^{-t}}{2}\right) \end{aligned}$$

Which simplifies to

$$y = (c_2 t + c_1)\,\mathrm{e}^{-t} + \frac{3t^2 \mathrm{e}^{-t}}{2}$$

Summary

The solution(s) found are the following

$$y = (c_2 t + c_1)\,\mathrm{e}^{-t} + \frac{3t^2 \mathrm{e}^{-t}}{2} \tag{1}$$

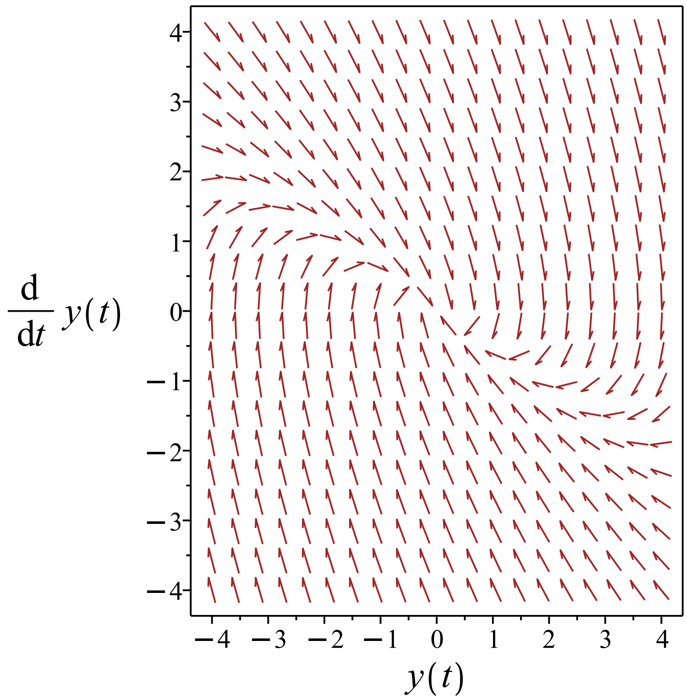


Figure 476: Slope field plot

Verification of solutions

$$y = (c_2 t + c_1) e^{-t} + \frac{3t^2 e^{-t}}{2}$$

Verified OK.

10.3.4 Maple step by step solution

Let's solve

$y'' + 2y' + y = 3 e^{-t}$

- Highest derivative means the order of the ODE is 2

  y''

- Characteristic polynomial of homogeneous ODE

  $r^2 + 2r + 1 = 0$

- Factor the characteristic polynomial

  $(r+1)^2 = 0$

- Root of the characteristic polynomial

$r = -1$

- 1st solution of the homogeneous ODE

  $y_1(t) = \mathrm{e}^{-t}$

- Repeated root, multiply $y_1(t)$ by t to ensure linear independence

  $y_2(t) = t\,\mathrm{e}^{-t}$

- General solution of the ODE

  $y = c_1 y_1(t) + c_2 y_2(t) + y_p(t)$

- Substitute in solutions of the homogeneous ODE

  $y = c_1 \mathrm{e}^{-t} + c_2 \mathrm{e}^{-t} t + y_p(t)$

☐ Find a particular solution $y_p(t)$ of the ODE

  - Use variation of parameters to find y_p here $f(t)$ is the forcing function

    $\left[y_p(t) = -y_1(t)\left(\int \frac{y_2(t) f(t)}{W(y_1(t), y_2(t))} dt\right) + y_2(t)\left(\int \frac{y_1(t) f(t)}{W(y_1(t), y_2(t))} dt\right), f(t) = 3\,\mathrm{e}^{-t}\right]$

  - Wronskian of solutions of the homogeneous equation

    $W(y_1(t), y_2(t)) = \begin{bmatrix} \mathrm{e}^{-t} & t\,\mathrm{e}^{-t} \\ -\mathrm{e}^{-t} & -t\,\mathrm{e}^{-t} + \mathrm{e}^{-t} \end{bmatrix}$

  - Compute Wronskian

    $W(y_1(t), y_2(t)) = \mathrm{e}^{-2t}$

  - Substitute functions into equation for $y_p(t)$

    $y_p(t) = -3\,\mathrm{e}^{-t}\left(\int t dt - \left(\int 1 dt\right) t\right)$

  - Compute integrals

    $y_p(t) = \frac{3t^2 \mathrm{e}^{-t}}{2}$

- Substitute particular solution into general solution to ODE

  $y = c_2 \mathrm{e}^{-t} t + c_1 \mathrm{e}^{-t} + \frac{3t^2 \mathrm{e}^{-t}}{2}$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 2; linear nonhomogeneous with symmetry [0,1]
trying a double symmetry of the form [xi=0, eta=F(x)]
-> Try solving first the homogeneous part of the ODE
   checking if the LODE has constant coefficients
   <- constant coefficients successful
<- solving first the homogeneous part of the ODE successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 19

```
dsolve(diff(y(t),t$2)+2*diff(y(t),t)+y(t) = 3*exp(-t),y(t), singsol=all)
```

$$y(t) = e^{-t}\left(c_2 + c_1 t + \frac{3}{2}t^2\right)$$

✓ Solution by Mathematica

Time used: 0.024 (sec). Leaf size: 29

```
DSolve[y''[t]+2*y'[t]+y[t] == 3*Exp[-t],y[t],t,IncludeSingularSolutions -> True]
```

$$y(t) \to \frac{1}{2}e^{-t}\left(3t^2 + 2c_2 t + 2c_1\right)$$

10.4 problem 4

Internal problem ID [686]
Internal file name [`OUTPUT/686_Sunday_June_05_2022_01_46_55_AM_1679039/index.tex`]

Book: Elementary differential equations and boundary value problems, 10th ed., Boyce and DiPrima
Section: Chapter 3, Second order linear equations, section 3.6, Variation of Parameters. page 190
Problem number: 4.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_linear_constant_coeff", "linear_second_order_ode_solved_by_an_integrating_factor"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

$$\boxed{4y'' - 4y' + y = 16\,\mathrm{e}^{\frac{t}{2}}}$$

10.4.1 Solving as second order linear constant coeff ode

This is second order non-homogeneous ODE. In standard form the ODE is

$$Ay''(t) + By'(t) + Cy(t) = f(t)$$

Where $A = 4, B = -4, C = 1, f(t) = 16\,\mathrm{e}^{\frac{t}{2}}$. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(t) + By'(t) + Cy(t) = 0$, and y_p is a particular solution to the non-homogeneous ODE $Ay''(t) + By'(t) + Cy(t) = f(t)$. y_h is the solution to

$$4y'' - 4y' + y = 0$$

This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$Ay''(t) + By'(t) + Cy(t) = 0$$

Where in the above $A = 4, B = -4, C = 1$. Let the solution be $y = e^{\lambda t}$. Substituting this into the ODE gives

$$4\lambda^2 \mathrm{e}^{\lambda t} - 4\lambda\, \mathrm{e}^{\lambda t} + \mathrm{e}^{\lambda t} = 0 \tag{1}$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda t}$ gives

$$4\lambda^2 - 4\lambda + 1 = 0 \tag{2}$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form.Using the quadratic formula

$$\lambda_{1,2} = \frac{-B}{2A} \pm \frac{1}{2A}\sqrt{B^2 - 4AC}$$

Substituting $A = 4, B = -4, C = 1$ into the above gives

$$\begin{aligned}\lambda_{1,2} &= \frac{4}{(2)\,(4)} \pm \frac{1}{(2)\,(4)}\sqrt{(-4)^2 - (4)\,(4)\,(1)}\\ &= \frac{1}{2}\end{aligned}$$

Hence this is the case of a double root $\lambda_{1,2} = -\frac{1}{2}$. Therefore the solution is

$$y = c_1 \mathrm{e}^{\frac{t}{2}} + c_2 t\, \mathrm{e}^{\frac{t}{2}} \tag{1}$$

Therefore the homogeneous solution y_h is

$$y_h = c_1 \mathrm{e}^{\frac{t}{2}} + c_2 t\, \mathrm{e}^{\frac{t}{2}}$$

The particular solution is now found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$16\, \mathrm{e}^{\frac{t}{2}}$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$\left[\left\{\mathrm{e}^{\frac{t}{2}}\right\}\right]$$

While the set of the basis functions for the homogeneous solution found earlier is

$$\left\{t\, \mathrm{e}^{\frac{t}{2}}, \mathrm{e}^{\frac{t}{2}}\right\}$$

Since $e^{\frac{t}{2}}$ is duplicated in the UC_set, then this basis is multiplied by extra t. The UC_set becomes

$$\left[\left\{t\,e^{\frac{t}{2}}\right\}\right]$$

Since $t\,e^{\frac{t}{2}}$ is duplicated in the UC_set, then this basis is multiplied by extra t. The UC_set becomes

$$\left[\left\{t^2 e^{\frac{t}{2}}\right\}\right]$$

Since there was duplication between the basis functions in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis function in the above updated UC_set.

$$y_p = A_1 t^2 e^{\frac{t}{2}}$$

The unknowns $\{A_1\}$ are found by substituting the above trial solution y_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$8A_1 e^{\frac{t}{2}} = 16\,e^{\frac{t}{2}}$$

Solving for the unknowns by comparing coefficients results in

$$[A_1 = 2]$$

Substituting the above back in the above trial solution y_p, gives the particular solution

$$y_p = 2t^2 e^{\frac{t}{2}}$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \left(c_1 e^{\frac{t}{2}} + c_2 t\,e^{\frac{t}{2}}\right) + \left(2t^2 e^{\frac{t}{2}}\right) \end{aligned}$$

Which simplifies to

$$y = e^{\frac{t}{2}}(c_2 t + c_1) + 2t^2 e^{\frac{t}{2}}$$

Summary

The solution(s) found are the following

$$y = e^{\frac{t}{2}}(c_2 t + c_1) + 2t^2 e^{\frac{t}{2}} \tag{1}$$

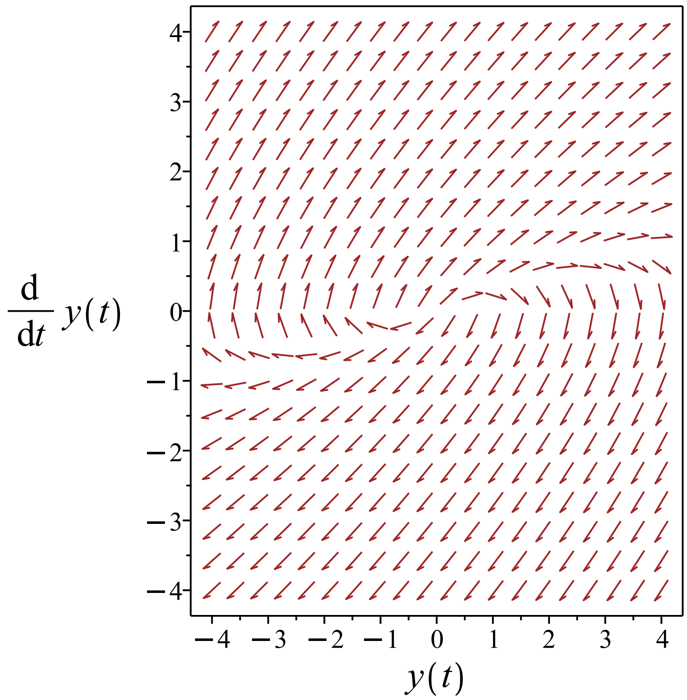


Figure 477: Slope field plot

Verification of solutions

$$y = e^{\frac{t}{2}}(c_2 t + c_1) + 2t^2 e^{\frac{t}{2}}$$

Verified OK.

10.4.2 Solving as linear second order ode solved by an integrating factor ode

The ode satisfies this form

$$y'' + p(t)\, y' + \frac{\left(p(t)^2 + p'(t)\right) y}{2} = f(t)$$

Where $p(t) = -1$. Therefore, there is an integrating factor given by

$$\begin{aligned} M(x) &= e^{\frac{1}{2}\int p\, dx} \\ &= e^{\int -1\, dx} \\ &= e^{-\frac{t}{2}} \end{aligned}$$

Multiplying both sides of the ODE by the integrating factor $M(x)$ makes the left side of the ODE a complete differential

$$
\begin{aligned}
(M(x)y)'' &= 4\,\mathrm{e}^{-\frac{t}{2}}\mathrm{e}^{\frac{t}{2}} \\
\left(\mathrm{e}^{-\frac{t}{2}}y\right)'' &= 4\,\mathrm{e}^{-\frac{t}{2}}\mathrm{e}^{\frac{t}{2}}
\end{aligned}
$$

Integrating once gives

$$
\left(\mathrm{e}^{-\frac{t}{2}}y\right)' = 4t + c_1
$$

Integrating again gives

$$
\left(\mathrm{e}^{-\frac{t}{2}}y\right) = t(c_1 + 2t) + c_2
$$

Hence the solution is

$$
y = \frac{t(c_1 + 2t) + c_2}{\mathrm{e}^{-\frac{t}{2}}}
$$

Or

$$
y = c_1 t\,\mathrm{e}^{\frac{t}{2}} + 2t^2\mathrm{e}^{\frac{t}{2}} + c_2\mathrm{e}^{\frac{t}{2}}
$$

Summary

The solution(s) found are the following

$$
y = c_1 t\,\mathrm{e}^{\frac{t}{2}} + 2t^2\mathrm{e}^{\frac{t}{2}} + c_2\mathrm{e}^{\frac{t}{2}} \tag{1}
$$

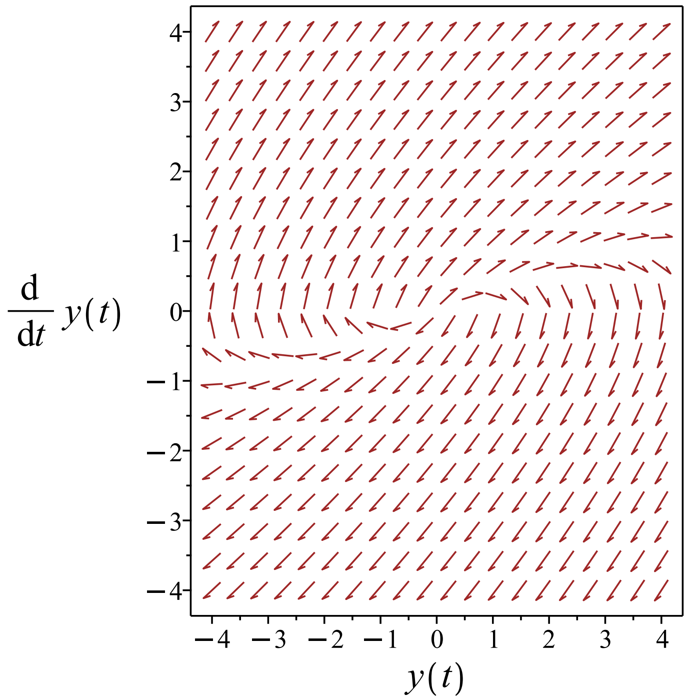


Figure 478: Slope field plot

Verification of solutions

$$y = c_1 t\, e^{\frac{t}{2}} + 2t^2 e^{\frac{t}{2}} + c_2 e^{\frac{t}{2}}$$

Verified OK.

10.4.3 Solving using Kovacic algorithm

Writing the ode as

$$4y'' - 4y' + y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 4 \\ B &= -4 \\ C &= 1 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(t) = y e^{\int \frac{B}{2A}\, dt}$$

Then (2) becomes

$$z''(t) = rz(t) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{0}{1} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= 0 \\ t &= 1 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(t) = 0 \tag{7}$$

Equation (7) is now solved. After finding $z(t)$ then y is found using the inverse transformation

$$y = z(t)\, e^{-\int \frac{B}{2A}\, dt}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 480: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 0 - -\infty \\ &= \infty \end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is $infinity$ then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Since $r = 0$ is not a function of t, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$z_1(t) = 1$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$\begin{aligned} y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dt} \\ &= z_1 e^{-\int \frac{1}{2}\frac{-4}{4}\,dt} \\ &= z_1 e^{\frac{t}{2}} \\ &= z_1 \left(\mathrm{e}^{\frac{t}{2}}\right) \end{aligned}$$

Which simplifies to

$$y_1 = \mathrm{e}^{\frac{t}{2}}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dt}}{y_1^2}\,dt$$

Substituting gives

$$\begin{aligned} y_2 &= y_1 \int \frac{e^{\int -\frac{-4}{4}\,dt}}{(y_1)^2}\,dt \\ &= y_1 \int \frac{e^t}{(y_1)^2}\,dt \\ &= y_1(t) \end{aligned}$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1 \left(\mathrm{e}^{\frac{t}{2}}\right) + c_2 \left(\mathrm{e}^{\frac{t}{2}} (t)\right)
\end{aligned}
$$

This is second order nonhomogeneous ODE. Let the solution be

$$
y = y_h + y_p
$$

Where y_h is the solution to the homogeneous ODE $Ay''(t) + By'(t) + Cy(t) = 0$, and y_p is a particular solution to the nonhomogeneous ODE $Ay''(t) + By'(t) + Cy(t) = f(t)$. y_h is the solution to

$$
4y'' - 4y' + y = 0
$$

The homogeneous solution is found using the Kovacic algorithm which results in

$$
y_h = c_1 \mathrm{e}^{\frac{t}{2}} + c_2 t\, \mathrm{e}^{\frac{t}{2}}
$$

The particular solution is now found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$
16\, \mathrm{e}^{\frac{t}{2}}
$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$
\left[\left\{\mathrm{e}^{\frac{t}{2}}\right\}\right]
$$

While the set of the basis functions for the homogeneous solution found earlier is

$$
\left\{t\, \mathrm{e}^{\frac{t}{2}}, \mathrm{e}^{\frac{t}{2}}\right\}
$$

Since $\mathrm{e}^{\frac{t}{2}}$ is duplicated in the UC_set, then this basis is multiplied by extra t. The UC_set becomes

$$
\left[\left\{t\, \mathrm{e}^{\frac{t}{2}}\right\}\right]
$$

Since $t\, \mathrm{e}^{\frac{t}{2}}$ is duplicated in the UC_set, then this basis is multiplied by extra t. The UC_set becomes

$$
\left[\left\{t^2 \mathrm{e}^{\frac{t}{2}}\right\}\right]
$$

Since there was duplication between the basis functions in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis function in the above updated UC_set.

$$y_p = A_1 t^2 \mathrm{e}^{\frac{t}{2}}$$

The unknowns $\{A_1\}$ are found by substituting the above trial solution y_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$8A_1 \mathrm{e}^{\frac{t}{2}} = 16\,\mathrm{e}^{\frac{t}{2}}$$

Solving for the unknowns by comparing coefficients results in

$$[A_1 = 2]$$

Substituting the above back in the above trial solution y_p, gives the particular solution

$$y_p = 2t^2 \mathrm{e}^{\frac{t}{2}}$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \left(c_1 \mathrm{e}^{\frac{t}{2}} + c_2 t\,\mathrm{e}^{\frac{t}{2}}\right) + \left(2t^2 \mathrm{e}^{\frac{t}{2}}\right) \end{aligned}$$

Which simplifies to

$$y = \mathrm{e}^{\frac{t}{2}}(c_2 t + c_1) + 2t^2 \mathrm{e}^{\frac{t}{2}}$$

Summary

The solution(s) found are the following

$$y = \mathrm{e}^{\frac{t}{2}}(c_2 t + c_1) + 2t^2 \mathrm{e}^{\frac{t}{2}} \tag{1}$$

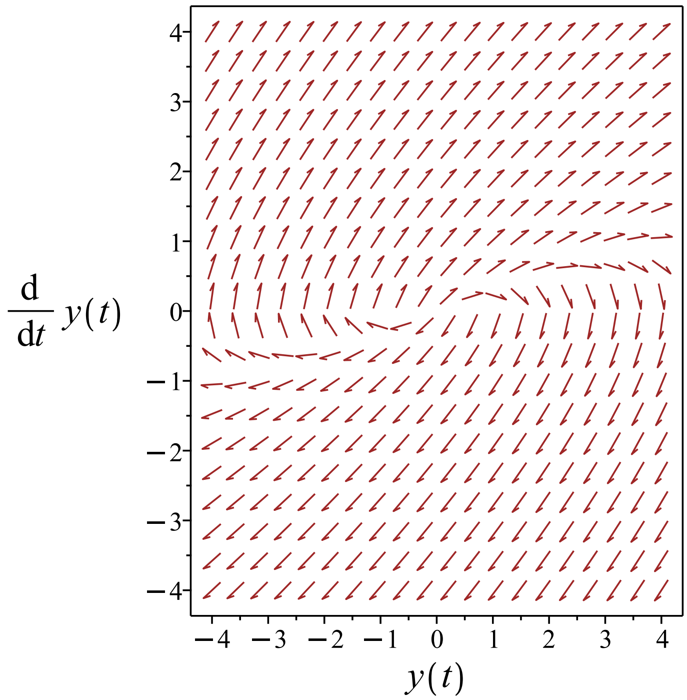


Figure 479: Slope field plot

Verification of solutions

$$y = e^{\frac{t}{2}}(c_2 t + c_1) + 2t^2 e^{\frac{t}{2}}$$

Verified OK.

10.4.4 Maple step by step solution

Let's solve

$4y'' - 4y' + y = 16\,e^{\frac{t}{2}}$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y'' = y' - \frac{y}{4} + 4\,e^{\frac{t}{2}}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y'' - y' + \frac{y}{4} = 4\,e^{\frac{t}{2}}$

- Characteristic polynomial of homogeneous ODE

$r^2 - r + \frac{1}{4} = 0$

- Factor the characteristic polynomial

  $\frac{(2r-1)^2}{4} = 0$

- Root of the characteristic polynomial

  $r = \frac{1}{2}$

- 1st solution of the homogeneous ODE

  $y_1(t) = e^{\frac{t}{2}}$

- Repeated root, multiply $y_1(t)$ by t to ensure linear independence

  $y_2(t) = t\, e^{\frac{t}{2}}$

- General solution of the ODE

  $y = c_1 y_1(t) + c_2 y_2(t) + y_p(t)$

- Substitute in solutions of the homogeneous ODE

  $y = c_1 e^{\frac{t}{2}} + c_2 t\, e^{\frac{t}{2}} + y_p(t)$

□ Find a particular solution $y_p(t)$ of the ODE

  ◦ Use variation of parameters to find y_p here $f(t)$ is the forcing function

    $\left[y_p(t) = -y_1(t)\left(\int \frac{y_2(t)f(t)}{W(y_1(t),y_2(t))}dt\right) + y_2(t)\left(\int \frac{y_1(t)f(t)}{W(y_1(t),y_2(t))}dt\right), f(t) = 4\, e^{\frac{t}{2}}\right]$

  ◦ Wronskian of solutions of the homogeneous equation

    $W(y_1(t), y_2(t)) = \begin{bmatrix} e^{\frac{t}{2}} & t\, e^{\frac{t}{2}} \\ \frac{e^{\frac{t}{2}}}{2} & e^{\frac{t}{2}} + \frac{t\, e^{\frac{t}{2}}}{2} \end{bmatrix}$

  ◦ Compute Wronskian

    $W(y_1(t), y_2(t)) = e^t$

  ◦ Substitute functions into equation for $y_p(t)$

    $y_p(t) = -4\, e^{\frac{t}{2}}\left(\int t dt - \left(\int 1 dt\right) t\right)$

  ◦ Compute integrals

    $y_p(t) = 2t^2 e^{\frac{t}{2}}$

- Substitute particular solution into general solution to ODE

  $y = c_2 t\, e^{\frac{t}{2}} + 2t^2 e^{\frac{t}{2}} + c_1 e^{\frac{t}{2}}$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 2; linear nonhomogeneous with symmetry [0,1]
trying a double symmetry of the form [xi=0, eta=F(x)]
-> Try solving first the homogeneous part of the ODE
   checking if the LODE has constant coefficients
   <- constant coefficients successful
<- solving first the homogeneous part of the ODE successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 19

```
dsolve(4*diff(y(t),t$2)-4*diff(y(t),t)+y(t) = 16*exp(t/2),y(t), singsol=all)
```

$$y(t) = e^{\frac{t}{2}} (c_1 t + 2t^2 + c_2)$$

✓ Solution by Mathematica

Time used: 0.021 (sec). Leaf size: 25

```
DSolve[4*y''[t]-4*y'[t]+y[t]== 16*Exp[t/2],y[t],t,IncludeSingularSolutions -> True]
```

$$y(t) \to e^{t/2} (2t^2 + c_2 t + c_1)$$

10.5 problem 5

Internal problem ID [687]
Internal file name [`OUTPUT/687_Sunday_June_05_2022_01_46_56_AM_81877378/index.tex`]

Book: Elementary differential equations and boundary value problems, 10th ed., Boyce and DiPrima
Section: Chapter 3, Second order linear equations, section 3.6, Variation of Parameters. page 190
Problem number: 5.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_linear_constant_coeff"**

Maple gives the following as the ode type

```
[[_2nd_order, _linear, _nonhomogeneous]]
```

$$\boxed{y'' + y = \tan(t)}$$

10.5.1 Solving as second order linear constant coeff ode

This is second order non-homogeneous ODE. In standard form the ODE is

$$Ay''(t) + By'(t) + Cy(t) = f(t)$$

Where $A = 1, B = 0, C = 1, f(t) = \tan(t)$. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(t) + By'(t) + Cy(t) = 0$, and y_p is a particular solution to the non-homogeneous ODE $Ay''(t) + By'(t) + Cy(t) = f(t)$. y_h is the solution to

$$y'' + y = 0$$

This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$Ay''(t) + By'(t) + Cy(t) = 0$$

Where in the above $A = 1, B = 0, C = 1$. Let the solution be $y = e^{\lambda t}$. Substituting this into the ODE gives

$$\lambda^2 \mathrm{e}^{\lambda t} + \mathrm{e}^{\lambda t} = 0 \tag{1}$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda t}$ gives

$$\lambda^2 + 1 = 0 \tag{2}$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form.Using the quadratic formula

$$\lambda_{1,2} = \frac{-B}{2A} \pm \frac{1}{2A}\sqrt{B^2 - 4AC}$$

Substituting $A = 1, B = 0, C = 1$ into the above gives

$$\begin{aligned}\lambda_{1,2} &= \frac{0}{(2)(1)} \pm \frac{1}{(2)(1)}\sqrt{0^2 - (4)(1)(1)} \\ &= \pm i\end{aligned}$$

Hence

$$\begin{aligned}\lambda_1 &= +i \\ \lambda_2 &= -i\end{aligned}$$

Which simplifies to

$$\begin{aligned}\lambda_1 &= i \\ \lambda_2 &= -i\end{aligned}$$

Since roots are complex conjugate of each others, then let the roots be

$$\lambda_{1,2} = \alpha \pm i\beta$$

Where $\alpha = 0$ and $\beta = 1$. Therefore the final solution, when using Euler relation, can be written as

$$y = e^{\alpha t}(c_1 \cos(\beta t) + c_2 \sin(\beta t))$$

Which becomes

$$y = e^0(c_1 \cos(t) + c_2 \sin(t))$$

Or

$$y = c_1 \cos(t) + c_2 \sin(t)$$

Therefore the homogeneous solution y_h is

$$y_h = c_1 \cos(t) + c_2 \sin(t)$$

The particular solution y_p can be found using either the method of undetermined coefficients, or the method of variation of parameters. The method of variation of parameters will be used as it is more general and can be used when the coefficients of the ODE depend on t as well. Let

$$y_p(t) = u_1 y_1 + u_2 y_2 \tag{1}$$

Where u_1, u_2 to be determined, and y_1, y_2 are the two basis solutions (the two linearly independent solutions of the homogeneous ODE) found earlier when solving the homogeneous ODE as

$$\begin{aligned} y_1 &= \cos(t) \\ y_2 &= \sin(t) \end{aligned}$$

In the Variation of parameters u_1, u_2 are found using

$$u_1 = -\int \frac{y_2 f(t)}{aW(t)} \tag{2}$$

$$u_2 = \int \frac{y_1 f(t)}{aW(t)} \tag{3}$$

Where $W(t)$ is the Wronskian and a is the coefficient in front of y'' in the given ODE. The Wronskian is given by $W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$. Hence

$$W = \begin{vmatrix} \cos(t) & \sin(t) \\ \frac{d}{dt}(\cos(t)) & \frac{d}{dt}(\sin(t)) \end{vmatrix}$$

Which gives

$$W = \begin{vmatrix} \cos(t) & \sin(t) \\ -\sin(t) & \cos(t) \end{vmatrix}$$

Therefore

$$W = (\cos(t))(\cos(t)) - (\sin(t))(-\sin(t))$$

Which simplifies to

$$W = \cos(t)^2 + \sin(t)^2$$

Which simplifies to

$$W = 1$$

Therefore Eq. (2) becomes

$$u_1 = -\int \frac{\sin(t)\tan(t)}{1}\, dt$$

Which simplifies to

$$u_1 = -\int \sin(t)\tan(t)\, dt$$

Hence

$$u_1 = \sin(t) - \ln(\sec(t) + \tan(t))$$

And Eq. (3) becomes

$$u_2 = \int \frac{\cos(t)\tan(t)}{1}\, dt$$

Which simplifies to

$$u_2 = \int \sin(t)\, dt$$

Hence

$$u_2 = -\cos(t)$$

Therefore the particular solution, from equation (1) is

$$y_p(t) = (\sin(t) - \ln(\sec(t) + \tan(t)))\cos(t) - \sin(t)\cos(t)$$

Which simplifies to

$$y_p(t) = -\cos(t)\ln(\sec(t) + \tan(t))$$

Therefore the general solution is

$$
\begin{aligned}
y &= y_h + y_p \\
&= (c_1 \cos(t) + c_2 \sin(t)) + (-\cos(t) \ln(\sec(t) + \tan(t)))
\end{aligned}
$$

Summary

The solution(s) found are the following

$$
y = c_1 \cos(t) + c_2 \sin(t) - \cos(t) \ln(\sec(t) + \tan(t)) \tag{1}
$$

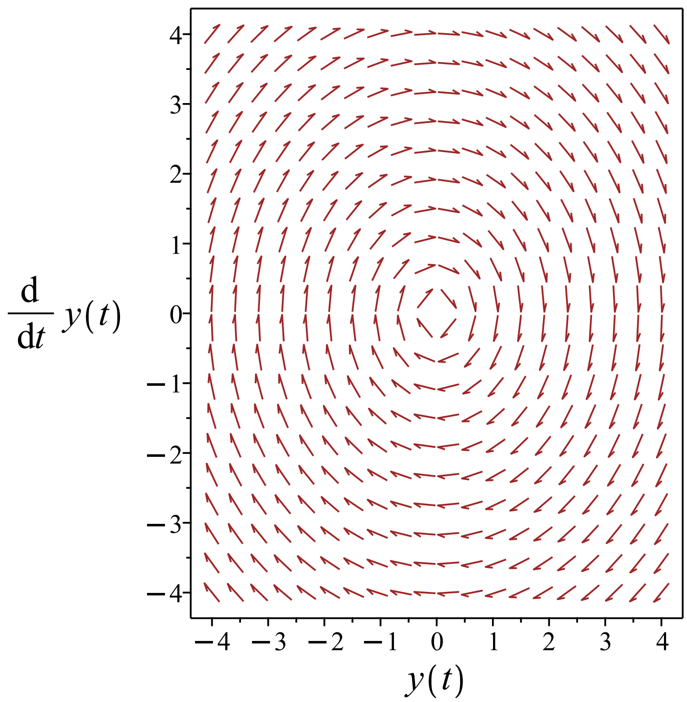


Figure 480: Slope field plot

Verification of solutions

$$
y = c_1 \cos(t) + c_2 \sin(t) - \cos(t) \ln(\sec(t) + \tan(t))
$$

Verified OK.

10.5.2 Solving using Kovacic algorithm

Writing the ode as

$$y'' + y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1 \\ B &= 0 \\ C &= 1 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(t) = ye^{\int \frac{B}{2A}\,dt}$$

Then (2) becomes

$$z''(t) = rz(t) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{-1}{1} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= -1 \\ t &= 1 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(t) = -z(t) \tag{7}$$

Equation (7) is now solved. After finding $z(t)$ then y is found using the inverse transformation

$$y = z(t)\,e^{-\int \frac{B}{2A}\,dt}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 482: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 0 - 0 \\ &= 0 \end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Since $r = -1$ is not a function of t, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$z_1(t) = \cos(t)$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$y_1 = z_1 e^{\int -\frac{1}{2} \frac{B}{A} \, dt}$$

Since $B=0$ then the above reduces to

$$
\begin{aligned}
y_1 &= z_1 \\
&= \cos(t)
\end{aligned}
$$

Which simplifies to

$$
y_1 = \cos(t)
$$

The second solution y_2 to the original ode is found using reduction of order

$$
y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\, dt}}{y_1^2}\, dt
$$

Since $B=0$ then the above becomes

$$
\begin{aligned}
y_2 &= y_1 \int \frac{1}{y_1^2}\, dt \\
&= \cos(t) \int \frac{1}{\cos(t)^2}\, dt \\
&= \cos(t)\,(\tan(t))
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1(\cos(t)) + c_2(\cos(t)\,(\tan(t)))
\end{aligned}
$$

This is second order nonhomogeneous ODE. Let the solution be

$$
y = y_h + y_p
$$

Where y_h is the solution to the homogeneous ODE $Ay''(t)+By'(t)+Cy(t)=0$, and y_p is a particular solution to the nonhomogeneous ODE $Ay''(t)+By'(t)+Cy(t)=f(t)$. y_h is the solution to

$$
y'' + y = 0
$$

The homogeneous solution is found using the Kovacic algorithm which results in

$$
y_h = c_1 \cos(t) + c_2 \sin(t)
$$

The particular solution y_p can be found using either the method of undetermined coefficients, or the method of variation of parameters. The method of variation of parameters will be used as it is more general and can be used when the coefficients of the ODE depend on t as well. Let

$$y_p(t) = u_1 y_1 + u_2 y_2 \tag{1}$$

Where u_1, u_2 to be determined, and y_1, y_2 are the two basis solutions (the two linearly independent solutions of the homogeneous ODE) found earlier when solving the homogeneous ODE as

$$y_1 = \cos(t)$$
$$y_2 = \sin(t)$$

In the Variation of parameters u_1, u_2 are found using

$$u_1 = -\int \frac{y_2 f(t)}{a W(t)} \tag{2}$$

$$u_2 = \int \frac{y_1 f(t)}{a W(t)} \tag{3}$$

Where $W(t)$ is the Wronskian and a is the coefficient in front of y'' in the given ODE. The Wronskian is given by $W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$. Hence

$$W = \begin{vmatrix} \cos(t) & \sin(t) \\ \frac{d}{dt}(\cos(t)) & \frac{d}{dt}(\sin(t)) \end{vmatrix}$$

Which gives

$$W = \begin{vmatrix} \cos(t) & \sin(t) \\ -\sin(t) & \cos(t) \end{vmatrix}$$

Therefore

$$W = (\cos(t))(\cos(t)) - (\sin(t))(-\sin(t))$$

Which simplifies to

$$W = \cos(t)^2 + \sin(t)^2$$

Which simplifies to

$$W = 1$$

Therefore Eq. (2) becomes

$$u_1 = -\int \frac{\sin(t)\tan(t)}{1}\,dt$$

Which simplifies to

$$u_1 = -\int \sin(t)\tan(t)\,dt$$

Hence

$$u_1 = \sin(t) - \ln(\sec(t) + \tan(t))$$

And Eq. (3) becomes

$$u_2 = \int \frac{\cos(t)\tan(t)}{1}\,dt$$

Which simplifies to

$$u_2 = \int \sin(t)\,dt$$

Hence

$$u_2 = -\cos(t)$$

Therefore the particular solution, from equation (1) is

$$y_p(t) = (\sin(t) - \ln(\sec(t) + \tan(t)))\cos(t) - \sin(t)\cos(t)$$

Which simplifies to

$$y_p(t) = -\cos(t)\ln(\sec(t) + \tan(t))$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= (c_1\cos(t) + c_2\sin(t)) + (-\cos(t)\ln(\sec(t) + \tan(t))) \end{aligned}$$

Summary
The solution(s) found are the following

$$y = c_1 \cos(t) + c_2 \sin(t) - \cos(t) \ln(\sec(t) + \tan(t)) \tag{1}$$

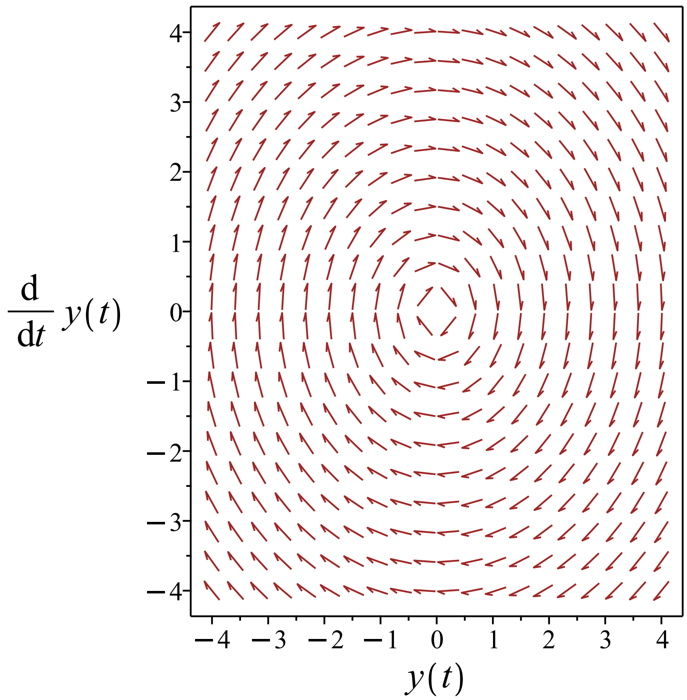


Figure 481: Slope field plot

Verification of solutions

$$y = c_1 \cos(t) + c_2 \sin(t) - \cos(t) \ln(\sec(t) + \tan(t))$$

Verified OK.

10.5.3 Maple step by step solution

Let's solve

$y'' + y = \tan(t)$

- Highest derivative means the order of the ODE is 2

  y''

- Characteristic polynomial of homogeneous ODE

  $r^2 + 1 = 0$

- Use quadratic formula to solve for r

  $r = \frac{0 \pm (\sqrt{-4})}{2}$

- Roots of the characteristic polynomial

  $r = (-\mathrm{I}, \mathrm{I})$

- 1st solution of the homogeneous ODE

  $y_1(t) = \cos(t)$

- 2nd solution of the homogeneous ODE

  $y_2(t) = \sin(t)$

- General solution of the ODE

  $y = c_1 y_1(t) + c_2 y_2(t) + y_p(t)$

- Substitute in solutions of the homogeneous ODE

  $y = c_1 \cos(t) + c_2 \sin(t) + y_p(t)$

☐ Find a particular solution $y_p(t)$ of the ODE

  ◦ Use variation of parameters to find y_p here $f(t)$ is the forcing function

  $\left[y_p(t) = -y_1(t) \left(\int \frac{y_2(t) f(t)}{W(y_1(t), y_2(t))} dt\right) + y_2(t) \left(\int \frac{y_1(t) f(t)}{W(y_1(t), y_2(t))} dt\right), f(t) = \tan(t)\right]$

  ◦ Wronskian of solutions of the homogeneous equation

  $W(y_1(t), y_2(t)) = \begin{bmatrix} \cos(t) & \sin(t) \\ -\sin(t) & \cos(t) \end{bmatrix}$

  ◦ Compute Wronskian

  $W(y_1(t), y_2(t)) = 1$

  ◦ Substitute functions into equation for $y_p(t)$

  $y_p(t) = -\cos(t) \left(\int \sin(t) \tan(t) \, dt\right) + \sin(t) \left(\int \sin(t) \, dt\right)$

  ◦ Compute integrals

  $y_p(t) = -\cos(t) \ln(\sec(t) + \tan(t))$

- Substitute particular solution into general solution to ODE

  $y = c_1 \cos(t) + c_2 \sin(t) - \cos(t) \ln(\sec(t) + \tan(t))$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 2; linear nonhomogeneous with symmetry [0,1]
trying a double symmetry of the form [xi=0, eta=F(x)]
-> Try solving first the homogeneous part of the ODE
   checking if the LODE has constant coefficients
   <- constant coefficients successful
<- solving first the homogeneous part of the ODE successful`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 23

```
dsolve(diff(y(t),t$2)+y(t) = tan(t),y(t), singsol=all)
```

$$y(t) = c_2 \sin(t) + \cos(t) c_1 - \cos(t) \ln(\sec(t) + \tan(t))$$

✓ Solution by Mathematica

Time used: 0.04 (sec). Leaf size: 23

```
DSolve[y''[t]+y[t] == Tan[t],y[t],t,IncludeSingularSolutions -> True]
```

$$y(t) \to \cos(t)(-\operatorname{arctanh}(\sin(t))) + c_1 \cos(t) + c_2 \sin(t)$$

10.6 problem 6

Internal problem ID [688]
Internal file name [`OUTPUT/688_Sunday_June_05_2022_01_46_58_AM_10201183/index.tex`]

Book: Elementary differential equations and boundary value problems, 10th ed., Boyce and DiPrima
Section: Chapter 3, Second order linear equations, section 3.6, Variation of Parameters. page 190
Problem number: 6.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_linear_constant_coeff"**

Maple gives the following as the ode type

```
[[_2nd_order, _linear, _nonhomogeneous]]
```

$$\boxed{y'' + 9y = 9\sec(3t)^2}$$

10.6.1 Solving as second order linear constant coeff ode

This is second order non-homogeneous ODE. In standard form the ODE is

$$Ay''(t) + By'(t) + Cy(t) = f(t)$$

Where $A = 1, B = 0, C = 9, f(t) = 9\sec(3t)^2$. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(t) + By'(t) + Cy(t) = 0$, and y_p is a particular solution to the non-homogeneous ODE $Ay''(t) + By'(t) + Cy(t) = f(t)$. y_h is the solution to

$$y'' + 9y = 0$$

This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$Ay''(t) + By'(t) + Cy(t) = 0$$

Where in the above $A = 1, B = 0, C = 9$. Let the solution be $y = e^{\lambda t}$. Substituting this into the ODE gives

$$\lambda^2 \mathrm{e}^{\lambda t} + 9\,\mathrm{e}^{\lambda t} = 0 \tag{1}$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda t}$ gives

$$\lambda^2 + 9 = 0 \tag{2}$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form.Using the quadratic formula

$$\lambda_{1,2} = \frac{-B}{2A} \pm \frac{1}{2A}\sqrt{B^2 - 4AC}$$

Substituting $A = 1, B = 0, C = 9$ into the above gives

$$\begin{aligned}\lambda_{1,2} &= \frac{0}{(2)(1)} \pm \frac{1}{(2)(1)}\sqrt{0^2 - (4)(1)(9)} \\ &= \pm 3i\end{aligned}$$

Hence

$$\begin{aligned}\lambda_1 &= +3i \\ \lambda_2 &= -3i\end{aligned}$$

Which simplifies to

$$\begin{aligned}\lambda_1 &= 3i \\ \lambda_2 &= -3i\end{aligned}$$

Since roots are complex conjugate of each others, then let the roots be

$$\lambda_{1,2} = \alpha \pm i\beta$$

Where $\alpha = 0$ and $\beta = 3$. Therefore the final solution, when using Euler relation, can be written as

$$y = e^{\alpha t}(c_1 \cos(\beta t) + c_2 \sin(\beta t))$$

Which becomes

$$y = e^0(c_1 \cos(3t) + c_2 \sin(3t))$$

Or

$$y = c_1 \cos(3t) + c_2 \sin(3t)$$

Therefore the homogeneous solution y_h is

$$y_h = c_1 \cos(3t) + c_2 \sin(3t)$$

The particular solution y_p can be found using either the method of undetermined coefficients, or the method of variation of parameters. The method of variation of parameters will be used as it is more general and can be used when the coefficients of the ODE depend on t as well. Let

$$y_p(t) = u_1 y_1 + u_2 y_2 \tag{1}$$

Where u_1, u_2 to be determined, and y_1, y_2 are the two basis solutions (the two linearly independent solutions of the homogeneous ODE) found earlier when solving the homogeneous ODE as

$$y_1 = \cos(3t)$$
$$y_2 = \sin(3t)$$

In the Variation of parameters u_1, u_2 are found using

$$u_1 = -\int \frac{y_2 f(t)}{aW(t)} \tag{2}$$

$$u_2 = \int \frac{y_1 f(t)}{aW(t)} \tag{3}$$

Where $W(t)$ is the Wronskian and a is the coefficient in front of y'' in the given ODE. The Wronskian is given by $W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$. Hence

$$W = \begin{vmatrix} \cos(3t) & \sin(3t) \\ \frac{d}{dt}(\cos(3t)) & \frac{d}{dt}(\sin(3t)) \end{vmatrix}$$

Which gives

$$W = \begin{vmatrix} \cos(3t) & \sin(3t) \\ -3\sin(3t) & 3\cos(3t) \end{vmatrix}$$

Therefore

$$W = (\cos(3t))(3\cos(3t)) - (\sin(3t))(-3\sin(3t))$$

Which simplifies to

$$W = 3\cos(3t)^2 + 3\sin(3t)^2$$

Which simplifies to

$$W = 3$$

Therefore Eq. (2) becomes

$$u_1 = -\int \frac{9\sin(3t)\sec(3t)^2}{3}\,dt$$

Which simplifies to

$$u_1 = -\int 3\sec(3t)\tan(3t)\,dt$$

Hence

$$u_1 = -\sec(3t)$$

And Eq. (3) becomes

$$u_2 = \int \frac{9\cos(3t)\sec(3t)^2}{3}\,dt$$

Which simplifies to

$$u_2 = \int 3\sec(3t)\,dt$$

Hence

$$u_2 = \ln(\sec(3t) + \tan(3t))$$

Therefore the particular solution, from equation (1) is

$$y_p(t) = -\sec(3t)\cos(3t) + \ln(\sec(3t) + \tan(3t))\sin(3t)$$

Which simplifies to

$$y_p(t) = -1 + \ln(\sec(3t) + \tan(3t))\sin(3t)$$

Therefore the general solution is

$$
\begin{aligned}
y &= y_h + y_p \\
&= (c_1 \cos(3t) + c_2 \sin(3t)) + (-1 + \ln(\sec(3t) + \tan(3t)) \sin(3t))
\end{aligned}
$$

Summary

The solution(s) found are the following

$$
y = c_1 \cos(3t) + c_2 \sin(3t) - 1 + \ln(\sec(3t) + \tan(3t)) \sin(3t) \tag{1}
$$

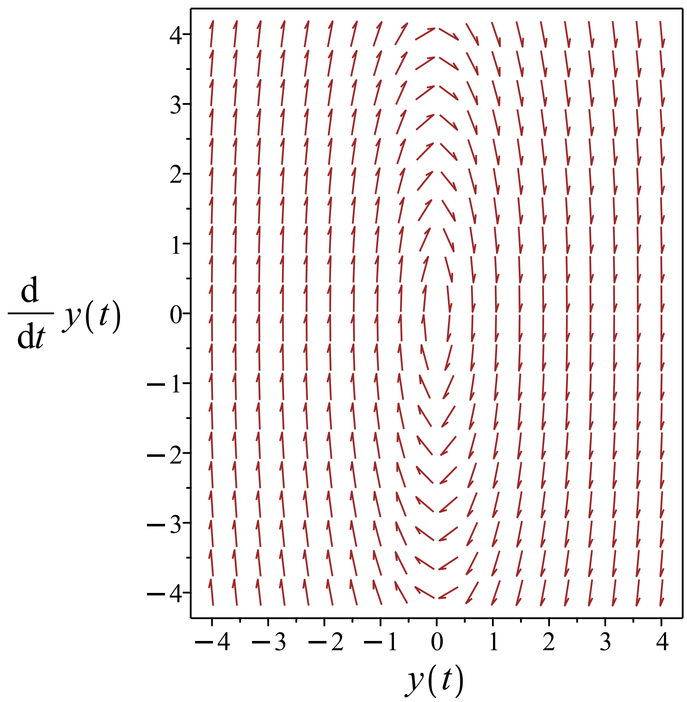


Figure 482: Slope field plot

Verification of solutions

$$
y = c_1 \cos(3t) + c_2 \sin(3t) - 1 + \ln(\sec(3t) + \tan(3t)) \sin(3t)
$$

Verified OK.

10.6.2 Solving using Kovacic algorithm

Writing the ode as

$$y'' + 9y = 0 \tag{1}$$

$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1 \\ B &= 0 \\ C &= 9 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(t) = ye^{\int \frac{B}{2A}\, dt}$$

Then (2) becomes

$$z''(t) = rz(t) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{-9}{1} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= -9 \\ t &= 1 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(t) = -9z(t) \tag{7}$$

Equation (7) is now solved. After finding $z(t)$ then y is found using the inverse transformation

$$y = z(t)\, e^{-\int \frac{B}{2A}\, dt}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 484: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 0 - 0 \\ &= 0 \end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Since $r = -9$ is not a function of t, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$z_1(t) = \cos(3t)$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$y_1 = z_1 e^{\int -\frac{1}{2} \frac{B}{A} \, dt}$$

Since $B = 0$ then the above reduces to

$$\begin{aligned} y_1 &= z_1 \\ &= \cos(3t) \end{aligned}$$

Which simplifies to

$$y_1 = \cos(3t)$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dt}}{y_1^2}\,dt$$

Since $B = 0$ then the above becomes

$$\begin{aligned} y_2 &= y_1 \int \frac{1}{y_1^2}\,dt \\ &= \cos(3t) \int \frac{1}{\cos(3t)^2}\,dt \\ &= \cos(3t)\left(\frac{\tan(3t)}{3}\right) \end{aligned}$$

Therefore the solution is

$$\begin{aligned} y &= c_1 y_1 + c_2 y_2 \\ &= c_1(\cos(3t)) + c_2\left(\cos(3t)\left(\frac{\tan(3t)}{3}\right)\right) \end{aligned}$$

This is second order nonhomogeneous ODE. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(t) + By'(t) + Cy(t) = 0$, and y_p is a particular solution to the nonhomogeneous ODE $Ay''(t) + By'(t) + Cy(t) = f(t)$. y_h is the solution to

$$y'' + 9y = 0$$

The homogeneous solution is found using the Kovacic algorithm which results in

$$y_h = c_1 \cos(3t) + \frac{c_2 \sin(3t)}{3}$$

The particular solution y_p can be found using either the method of undetermined coefficients, or the method of variation of parameters. The method of variation of parameters will be used as it is more general and can be used when the coefficients of the ODE depend on t as well. Let

$$y_p(t) = u_1 y_1 + u_2 y_2 \tag{1}$$

Where u_1, u_2 to be determined, and y_1, y_2 are the two basis solutions (the two linearly independent solutions of the homogeneous ODE) found earlier when solving the homogeneous ODE as

$$y_1 = \cos(3t)$$
$$y_2 = \frac{\sin(3t)}{3}$$

In the Variation of parameters u_1, u_2 are found using

$$u_1 = -\int \frac{y_2 f(t)}{aW(t)} \tag{2}$$

$$u_2 = \int \frac{y_1 f(t)}{aW(t)} \tag{3}$$

Where $W(t)$ is the Wronskian and a is the coefficient in front of y'' in the given ODE. The Wronskian is given by $W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$. Hence

$$W = \begin{vmatrix} \cos(3t) & \frac{\sin(3t)}{3} \\ \frac{d}{dt}(\cos(3t)) & \frac{d}{dt}\left(\frac{\sin(3t)}{3}\right) \end{vmatrix}$$

Which gives

$$W = \begin{vmatrix} \cos(3t) & \frac{\sin(3t)}{3} \\ -3\sin(3t) & \cos(3t) \end{vmatrix}$$

Therefore

$$W = (\cos(3t))(\cos(3t)) - \left(\frac{\sin(3t)}{3}\right)(-3\sin(3t))$$

Which simplifies to

$$W = \cos(3t)^2 + \sin(3t)^2$$

Which simplifies to

$$W = 1$$

Therefore Eq. (2) becomes

$$u_1 = -\int \frac{3\sin(3t)\sec(3t)^2}{1}\,dt$$

Which simplifies to

$$u_1 = -\int 3\sec(3t)\tan(3t)\,dt$$

Hence

$$u_1 = -\sec(3t)$$

And Eq. (3) becomes

$$u_2 = \int \frac{9\cos(3t)\sec(3t)^2}{1}\,dt$$

Which simplifies to

$$u_2 = \int 9\sec(3t)\,dt$$

Hence

$$u_2 = 3\ln(\sec(3t) + \tan(3t))$$

Therefore the particular solution, from equation (1) is

$$y_p(t) = -\sec(3t)\cos(3t) + \ln(\sec(3t) + \tan(3t))\sin(3t)$$

Which simplifies to

$$y_p(t) = -1 + \ln(\sec(3t) + \tan(3t))\sin(3t)$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \left(c_1\cos(3t) + \frac{c_2\sin(3t)}{3}\right) + (-1 + \ln(\sec(3t) + \tan(3t))\sin(3t)) \end{aligned}$$

Summary

The solution(s) found are the following

$$y = c_1\cos(3t) + \frac{c_2\sin(3t)}{3} - 1 + \ln(\sec(3t) + \tan(3t))\sin(3t) \tag{1}$$

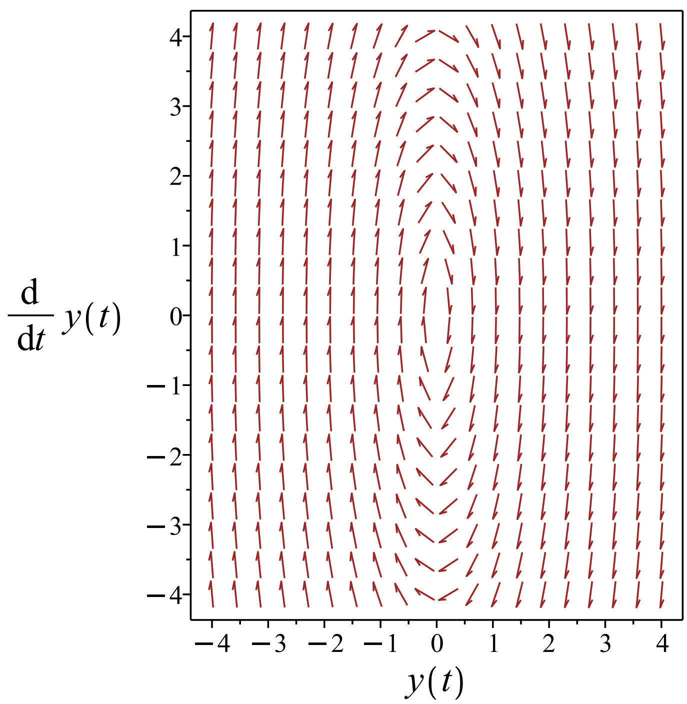


Figure 483: Slope field plot

Verification of solutions

$$y = c_1\cos(3t) + \frac{c_2\sin(3t)}{3} - 1 + \ln(\sec(3t) + \tan(3t))\sin(3t)$$

Verified OK.

10.6.3 Maple step by step solution

Let's solve

$y'' + 9y = 9 \sec(3t)^2$

- Highest derivative means the order of the ODE is 2

  y''

- Characteristic polynomial of homogeneous ODE

  $r^2 + 9 = 0$

- Use quadratic formula to solve for r

  $r = \frac{0 \pm (\sqrt{-36})}{2}$

- Roots of the characteristic polynomial

  $r = (-3\,\mathrm{I}, 3\,\mathrm{I})$

- 1st solution of the homogeneous ODE

  $y_1(t) = \cos(3t)$

- 2nd solution of the homogeneous ODE

  $y_2(t) = \sin(3t)$

- General solution of the ODE

  $y = c_1 y_1(t) + c_2 y_2(t) + y_p(t)$

- Substitute in solutions of the homogeneous ODE

  $y = c_1 \cos(3t) + c_2 \sin(3t) + y_p(t)$

□ Find a particular solution $y_p(t)$ of the ODE

- Use variation of parameters to find y_p here $f(t)$ is the forcing function

  $\left[y_p(t) = -y_1(t) \left(\int \frac{y_2(t) f(t)}{W(y_1(t), y_2(t))} dt\right) + y_2(t) \left(\int \frac{y_1(t) f(t)}{W(y_1(t), y_2(t))} dt\right), f(t) = 9 \sec(3t)^2\right]$

- Wronskian of solutions of the homogeneous equation

  $W(y_1(t), y_2(t)) = \begin{bmatrix} \cos(3t) & \sin(3t) \\ -3\sin(3t) & 3\cos(3t) \end{bmatrix}$

- Compute Wronskian

  $W(y_1(t), y_2(t)) = 3$

- Substitute functions into equation for $y_p(t)$

$$y_p(t) = -3\cos(3t)\left(\int \sec(3t)\tan(3t)\,dt\right) + 3\sin(3t)\left(\int \sec(3t)\,dt\right)$$

- Compute integrals

$$y_p(t) = -1 + \ln(\sec(3t) + \tan(3t))\sin(3t)$$

- Substitute particular solution into general solution to ODE

$$y = c_1\cos(3t) + c_2\sin(3t) - 1 + \ln(\sec(3t) + \tan(3t))\sin(3t)$$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 2; linear nonhomogeneous with symmetry [0,1]
trying a double symmetry of the form [xi=0, eta=F(x)]
-> Try solving first the homogeneous part of the ODE
   checking if the LODE has constant coefficients
   <- constant coefficients successful
<- solving first the homogeneous part of the ODE successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 33

```
dsolve(diff(y(t),t$2)+9*y(t) = 9*sec(3*t)^2,y(t), singsol=all)
```

$$y(t) = c_2\sin(3t) + c_1\cos(3t) + \ln(\sec(3t) + \tan(3t))\sin(3t) - 1$$

✓ Solution by Mathematica

Time used: 0.117 (sec). Leaf size: 31

```
DSolve[y''[t]+9*y[t] == 9*Sec[3*t]^2,y[t],t,IncludeSingularSolutions -> True]
```

$$y(t) \to c_1\cos(3t) + \sin(3t)\coth^{-1}(\sin(3t)) + c_2\sin(3t) - 1$$

10.7 problem 7

Internal problem ID [689]
Internal file name [`OUTPUT/689_Sunday_June_05_2022_01_46_59_AM_21493213/index.tex`]

Book: Elementary differential equations and boundary value problems, 10th ed., Boyce and DiPrima
Section: Chapter 3, Second order linear equations, section 3.6, Variation of Parameters. page 190
Problem number: 7.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_linear_constant_coeff", "linear_second_order_ode_solved_by_an_integrating_factor"**

Maple gives the following as the ode type

```
[[_2nd_order, _linear, _nonhomogeneous]]
```

$$\boxed{y'' + 4y' + 4y = \frac{\mathrm{e}^{-2t}}{t^2}}$$

10.7.1 Solving as second order linear constant coeff ode

This is second order non-homogeneous ODE. In standard form the ODE is

$$Ay''(t) + By'(t) + Cy(t) = f(t)$$

Where $A = 1, B = 4, C = 4, f(t) = \frac{\mathrm{e}^{-2t}}{t^2}$. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(t) + By'(t) + Cy(t) = 0$, and y_p is a particular solution to the non-homogeneous ODE $Ay''(t) + By'(t) + Cy(t) = f(t)$. y_h is the solution to

$$y'' + 4y' + 4y = 0$$

This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$Ay''(t) + By'(t) + Cy(t) = 0$$

Where in the above $A = 1, B = 4, C = 4$. Let the solution be $y = e^{\lambda t}$. Substituting this into the ODE gives

$$\lambda^2 e^{\lambda t} + 4\lambda e^{\lambda t} + 4 e^{\lambda t} = 0 \tag{1}$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda t}$ gives

$$\lambda^2 + 4\lambda + 4 = 0 \tag{2}$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form.Using the quadratic formula

$$\lambda_{1,2} = \frac{-B}{2A} \pm \frac{1}{2A}\sqrt{B^2 - 4AC}$$

Substituting $A = 1, B = 4, C = 4$ into the above gives

$$\begin{aligned}\lambda_{1,2} &= \frac{-4}{(2)(1)} \pm \frac{1}{(2)(1)}\sqrt{(4)^2 - (4)(1)(4)}\\ &= -2\end{aligned}$$

Hence this is the case of a double root $\lambda_{1,2} = 2$. Therefore the solution is

$$y = c_1 e^{-2t} + c_2 t\, e^{-2t} \tag{1}$$

Therefore the homogeneous solution y_h is

$$y_h = c_1 e^{-2t} + c_2 t\, e^{-2t}$$

The particular solution y_p can be found using either the method of undetermined coefficients, or the method of variation of parameters. The method of variation of parameters will be used as it is more general and can be used when the coefficients of the ODE depend on t as well. Let

$$y_p(t) = u_1 y_1 + u_2 y_2 \tag{1}$$

Where u_1, u_2 to be determined, and y_1, y_2 are the two basis solutions (the two linearly independent solutions of the homogeneous ODE) found earlier when solving the homogeneous ODE as

$$\begin{aligned}y_1 &= e^{-2t}\\ y_2 &= t\, e^{-2t}\end{aligned}$$

In the Variation of parameters u_1, u_2 are found using

$$u_1 = -\int \frac{y_2 f(t)}{aW(t)} \tag{2}$$

$$u_2 = \int \frac{y_1 f(t)}{aW(t)} \tag{3}$$

Where $W(t)$ is the Wronskian and a is the coefficient in front of y'' in the given ODE. The Wronskian is given by $W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$. Hence

$$W = \begin{vmatrix} e^{-2t} & t\,e^{-2t} \\ \frac{d}{dt}(e^{-2t}) & \frac{d}{dt}(t\,e^{-2t}) \end{vmatrix}$$

Which gives

$$W = \begin{vmatrix} e^{-2t} & t\,e^{-2t} \\ -2\,e^{-2t} & e^{-2t} - 2t\,e^{-2t} \end{vmatrix}$$

Therefore

$$W = (e^{-2t})\left(e^{-2t} - 2t\,e^{-2t}\right) - (t\,e^{-2t})\left(-2\,e^{-2t}\right)$$

Which simplifies to

$$W = e^{-4t}$$

Which simplifies to

$$W = e^{-4t}$$

Therefore Eq. (2) becomes

$$u_1 = -\int \frac{\frac{e^{-4t}}{t}}{e^{-4t}}\,dt$$

Which simplifies to

$$u_1 = -\int \frac{1}{t} dt$$

Hence

$$u_1 = -\ln(t)$$

And Eq. (3) becomes

$$u_2 = \int \frac{\frac{e^{-4t}}{t^2}}{e^{-4t}} \, dt$$

Which simplifies to

$$u_2 = \int \frac{1}{t^2} dt$$

Hence

$$u_2 = -\frac{1}{t}$$

Therefore the particular solution, from equation (1) is

$$y_p(t) = -\ln(t)\, e^{-2t} - e^{-2t}$$

Which simplifies to

$$y_p(t) = e^{-2t}(-1 - \ln(t))$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \left(c_1 e^{-2t} + c_2 t\, e^{-2t}\right) + \left(e^{-2t}(-1 - \ln(t))\right) \end{aligned}$$

Which simplifies to

$$y = e^{-2t}(c_2 t + c_1) + e^{-2t}(-1 - \ln(t))$$

Summary
The solution(s) found are the following

$$y = e^{-2t}(c_2 t + c_1) + e^{-2t}(-1 - \ln(t)) \tag{1}$$

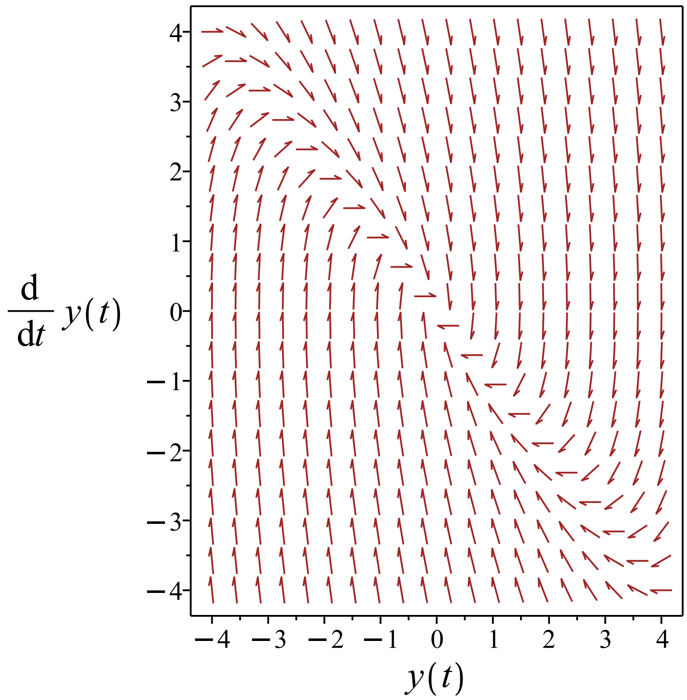


Figure 484: Slope field plot

Verification of solutions

$$y = e^{-2t}(c_2 t + c_1) + e^{-2t}(-1 - \ln(t))$$

Verified OK.

10.7.2 Solving as linear second order ode solved by an integrating factor ode

The ode satisfies this form

$$y'' + p(t)\, y' + \frac{(p(t)^2 + p'(t))\, y}{2} = f(t)$$

Where $p(t) = 4$. Therefore, there is an integrating factor given by

$$\begin{aligned} M(x) &= e^{\frac{1}{2}\int p\, dx} \\ &= e^{\int 4\, dx} \\ &= e^{2t} \end{aligned}$$

Multiplying both sides of the ODE by the integrating factor $M(x)$ makes the left side of the ODE a complete differential

$$(M(x)y)'' = \frac{e^{2t}e^{-2t}}{t^2}$$

$$\left(y\,e^{2t}\right)'' = \frac{e^{2t}e^{-2t}}{t^2}$$

Integrating once gives

$$\left(y\,e^{2t}\right)' = -\frac{1}{t} + c_1$$

Integrating again gives

$$\left(y\,e^{2t}\right) = c_1 t - \ln(t) + c_2$$

Hence the solution is

$$y = \frac{c_1 t - \ln(t) + c_2}{e^{2t}}$$

Or

$$y = c_1 t\,e^{-2t} + e^{-2t}c_2 - \ln(t)\,e^{-2t}$$

Summary

The solution(s) found are the following

$$y = c_1 t\,e^{-2t} + e^{-2t}c_2 - \ln(t)\,e^{-2t} \tag{1}$$

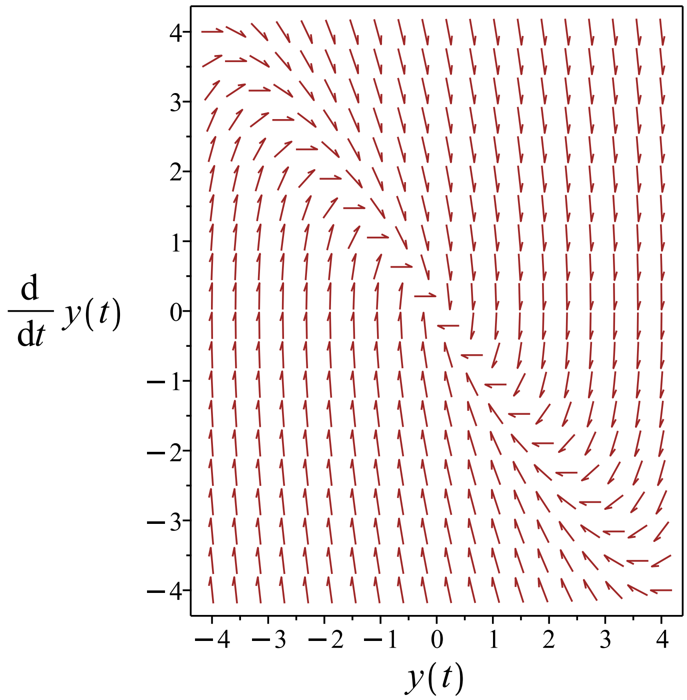


Figure 485: Slope field plot

Verification of solutions

$$y = c_1 t\,\mathrm{e}^{-2t} + \mathrm{e}^{-2t} c_2 - \ln(t)\,\mathrm{e}^{-2t}$$

Verified OK.

10.7.3 Solving using Kovacic algorithm

Writing the ode as

$$y'' + 4y' + 4y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1 \\ B &= 4 \\ C &= 4 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(t) = y e^{\int \frac{B}{2A}\,dt}$$

Then (2) becomes

$$z''(t) = rz(t) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{0}{1} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= 0 \\ t &= 1 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(t) = 0 \tag{7}$$

Equation (7) is now solved. After finding $z(t)$ then y is found using the inverse transformation

$$y = z(t)\, e^{-\int \frac{B}{2A}\, dt}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 486: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 0 - -\infty \\ &= \infty \end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is $infinity$ then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Since $r = 0$ is not a function of t, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$z_1(t) = 1$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$\begin{aligned} y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dt} \\ &= z_1 e^{-\int \frac{1}{2}\frac{4}{1}\,dt} \\ &= z_1 e^{-2t} \\ &= z_1 \left(\mathrm{e}^{-2t}\right) \end{aligned}$$

Which simplifies to

$$y_1 = \mathrm{e}^{-2t}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dt}}{y_1^2}\,dt$$

Substituting gives

$$\begin{aligned} y_2 &= y_1 \int \frac{e^{\int -\frac{4}{1}\,dt}}{(y_1)^2}\,dt \\ &= y_1 \int \frac{e^{-4t}}{(y_1)^2}\,dt \\ &= y_1(t) \end{aligned}$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1y_1 + c_2y_2 \\
&= c_1\left(\mathrm{e}^{-2t}\right) + c_2\left(\mathrm{e}^{-2t}(t)\right)
\end{aligned}
$$

This is second order nonhomogeneous ODE. Let the solution be

$$
y = y_h + y_p
$$

Where y_h is the solution to the homogeneous ODE $Ay''(t) + By'(t) + Cy(t) = 0$, and y_p is a particular solution to the nonhomogeneous ODE $Ay''(t) + By'(t) + Cy(t) = f(t)$. y_h is the solution to

$$
y'' + 4y' + 4y = 0
$$

The homogeneous solution is found using the Kovacic algorithm which results in

$$
y_h = c_1\mathrm{e}^{-2t} + c_2t\,\mathrm{e}^{-2t}
$$

The particular solution y_p can be found using either the method of undetermined coefficients, or the method of variation of parameters. The method of variation of parameters will be used as it is more general and can be used when the coefficients of the ODE depend on t as well. Let

$$
y_p(t) = u_1y_1 + u_2y_2 \tag{1}
$$

Where u_1, u_2 to be determined, and y_1, y_2 are the two basis solutions (the two linearly independent solutions of the homogeneous ODE) found earlier when solving the homogeneous ODE as

$$
\begin{aligned}
y_1 &= \mathrm{e}^{-2t} \\
y_2 &= t\,\mathrm{e}^{-2t}
\end{aligned}
$$

In the Variation of parameters u_1, u_2 are found using

$$
u_1 = -\int \frac{y_2f(t)}{aW(t)} \tag{2}
$$

$$
u_2 = \int \frac{y_1f(t)}{aW(t)} \tag{3}
$$

Where $W(t)$ is the Wronskian and a is the coefficient in front of y'' in the given ODE.

The Wronskian is given by $W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$. Hence

$$W = \begin{vmatrix} e^{-2t} & t\,e^{-2t} \\ \frac{d}{dt}(e^{-2t}) & \frac{d}{dt}(t\,e^{-2t}) \end{vmatrix}$$

Which gives

$$W = \begin{vmatrix} e^{-2t} & t\,e^{-2t} \\ -2\,e^{-2t} & e^{-2t} - 2t\,e^{-2t} \end{vmatrix}$$

Therefore

$$W = \left(e^{-2t}\right)\left(e^{-2t} - 2t\,e^{-2t}\right) - \left(t\,e^{-2t}\right)\left(-2\,e^{-2t}\right)$$

Which simplifies to

$$W = e^{-4t}$$

Which simplifies to

$$W = e^{-4t}$$

Therefore Eq. (2) becomes

$$u_1 = -\int \frac{\frac{e^{-4t}}{t}}{e^{-4t}}\,dt$$

Which simplifies to

$$u_1 = -\int \frac{1}{t}dt$$

Hence

$$u_1 = -\ln(t)$$

And Eq. (3) becomes

$$u_2 = \int \frac{\frac{e^{-4t}}{t^2}}{e^{-4t}}\,dt$$

Which simplifies to

$$u_2 = \int \frac{1}{t^2}dt$$

Hence

$$u_2 = -\frac{1}{t}$$

Therefore the particular solution, from equation (1) is

$$y_p(t) = -\ln(t)\,e^{-2t} - e^{-2t}$$

Which simplifies to

$$y_p(t) = e^{-2t}(-1 - \ln(t))$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \left(c_1 e^{-2t} + c_2 t\,e^{-2t}\right) + \left(e^{-2t}(-1 - \ln(t))\right) \end{aligned}$$

Which simplifies to

$$y = e^{-2t}(c_2 t + c_1) + e^{-2t}(-1 - \ln(t))$$

Summary

The solution(s) found are the following

$$y = e^{-2t}(c_2 t + c_1) + e^{-2t}(-1 - \ln(t)) \tag{1}$$

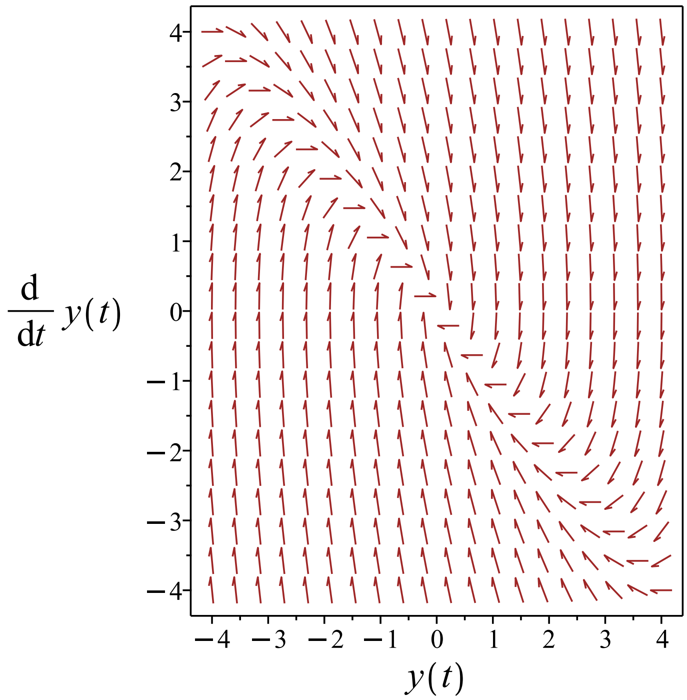


Figure 486: Slope field plot

Verification of solutions

$$y = e^{-2t}(c_2 t + c_1) + e^{-2t}(-1 - \ln(t))$$

Verified OK.

10.7.4 Maple step by step solution

Let's solve

$y'' + 4y' + 4y = \frac{e^{-2t}}{t^2}$

- Highest derivative means the order of the ODE is 2

  y''

- Characteristic polynomial of homogeneous ODE

  $r^2 + 4r + 4 = 0$

- Factor the characteristic polynomial

  $(r + 2)^2 = 0$

- Root of the characteristic polynomial

$r=-2$

- 1st solution of the homogeneous ODE

  $y_1(t)=\mathrm{e}^{-2t}$

- Repeated root, multiply $y_1(t)$ by t to ensure linear independence

  $y_2(t)=t\,\mathrm{e}^{-2t}$

- General solution of the ODE

  $y=c_1y_1(t)+c_2y_2(t)+y_p(t)$

- Substitute in solutions of the homogeneous ODE

  $y=c_1\mathrm{e}^{-2t}+c_2t\,\mathrm{e}^{-2t}+y_p(t)$

☐ Find a particular solution $y_p(t)$ of the ODE

  - Use variation of parameters to find y_p here $f(t)$ is the forcing function

    $\left[y_p(t)=-y_1(t)\left(\int \frac{y_2(t)f(t)}{W(y_1(t),y_2(t))}dt\right)+y_2(t)\left(\int \frac{y_1(t)f(t)}{W(y_1(t),y_2(t))}dt\right), f(t)=\frac{\mathrm{e}^{-2t}}{t^2}\right]$

  - Wronskian of solutions of the homogeneous equation

    $W(y_1(t),y_2(t))=\left[\begin{array}{cc}\mathrm{e}^{-2t} & t\,\mathrm{e}^{-2t} \\ -2\,\mathrm{e}^{-2t} & \mathrm{e}^{-2t}-2t\,\mathrm{e}^{-2t}\end{array}\right]$

  - Compute Wronskian

    $W(y_1(t),y_2(t))=\mathrm{e}^{-4t}$

  - Substitute functions into equation for $y_p(t)$

    $y_p(t)=\mathrm{e}^{-2t}\left(-\left(\int \frac{1}{t}dt\right)+\left(\int \frac{1}{t^2}dt\right)t\right)$

  - Compute integrals

    $y_p(t)=-\mathrm{e}^{-2t}(1+\ln(t))$

- Substitute particular solution into general solution to ODE

  $y=c_1\mathrm{e}^{-2t}+c_2t\,\mathrm{e}^{-2t}-\mathrm{e}^{-2t}(1+\ln(t))$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 2; linear nonhomogeneous with symmetry [0,1]
trying a double symmetry of the form [xi=0, eta=F(x)]
-> Try solving first the homogeneous part of the ODE
   checking if the LODE has constant coefficients
   <- constant coefficients successful
<- solving first the homogeneous part of the ODE successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 19

```
dsolve(diff(y(t),t$2)+4*diff(y(t),t)+4*y(t) = t^(-2)*exp(-2*t),y(t), singsol=all)
```

$$y(t) = e^{-2t}(-1 + c_1 t - \ln(t) + c_2)$$

✓ Solution by Mathematica

Time used: 0.028 (sec). Leaf size: 23

```
DSolve[y''[t]+4*y'[t]+4*y[t] == t^(-2)*Exp[-2*t],y[t],t,IncludeSingularSolutions -> True]
```

$$y(t) \to e^{-2t}(-\log(t) + c_2 t - 1 + c_1)$$

10.8 problem 8

Internal problem ID [690]
Internal file name [`OUTPUT/690_Sunday_June_05_2022_01_47_00_AM_95123719/index.tex`]

Book: Elementary differential equations and boundary value problems, 10th ed., Boyce and DiPrima
Section: Chapter 3, Second order linear equations, section 3.6, Variation of Parameters. page 190
Problem number: 8.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_linear_constant_coeff"**

Maple gives the following as the ode type

```
[[_2nd_order, _linear, _nonhomogeneous]]
```

$$\boxed{y'' + 4y = 3\csc(2t)}$$

10.8.1 Solving as second order linear constant coeff ode

This is second order non-homogeneous ODE. In standard form the ODE is

$$Ay''(t) + By'(t) + Cy(t) = f(t)$$

Where $A = 1, B = 0, C = 4, f(t) = 3\csc(2t)$. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(t) + By'(t) + Cy(t) = 0$, and y_p is a particular solution to the non-homogeneous ODE $Ay''(t) + By'(t) + Cy(t) = f(t)$. y_h is the solution to

$$y'' + 4y = 0$$

This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$Ay''(t) + By'(t) + Cy(t) = 0$$

Where in the above $A = 1, B = 0, C = 4$. Let the solution be $y = e^{\lambda t}$. Substituting this into the ODE gives

$$\lambda^2 e^{\lambda t} + 4 e^{\lambda t} = 0 \tag{1}$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda t}$ gives

$$\lambda^2 + 4 = 0 \tag{2}$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form.Using the quadratic formula

$$\lambda_{1,2} = \frac{-B}{2A} \pm \frac{1}{2A}\sqrt{B^2 - 4AC}$$

Substituting $A = 1, B = 0, C = 4$ into the above gives

$$\begin{aligned}\lambda_{1,2} &= \frac{0}{(2)(1)} \pm \frac{1}{(2)(1)}\sqrt{0^2 - (4)(1)(4)}\\ &= \pm 2i\end{aligned}$$

Hence

$$\begin{aligned}\lambda_1 &= +2i\\ \lambda_2 &= -2i\end{aligned}$$

Which simplifies to

$$\begin{aligned}\lambda_1 &= 2i\\ \lambda_2 &= -2i\end{aligned}$$

Since roots are complex conjugate of each others, then let the roots be

$$\lambda_{1,2} = \alpha \pm i\beta$$

Where $\alpha = 0$ and $\beta = 2$. Therefore the final solution, when using Euler relation, can be written as

$$y = e^{\alpha t}(c_1 \cos(\beta t) + c_2 \sin(\beta t))$$

Which becomes

$$y = e^0(c_1 \cos(2t) + c_2 \sin(2t))$$

Or

$$y = c_1 \cos(2t) + c_2 \sin(2t)$$

Therefore the homogeneous solution y_h is

$$y_h = c_1 \cos(2t) + c_2 \sin(2t)$$

The particular solution y_p can be found using either the method of undetermined coefficients, or the method of variation of parameters. The method of variation of parameters will be used as it is more general and can be used when the coefficients of the ODE depend on t as well. Let

$$y_p(t) = u_1 y_1 + u_2 y_2 \tag{1}$$

Where u_1, u_2 to be determined, and y_1, y_2 are the two basis solutions (the two linearly independent solutions of the homogeneous ODE) found earlier when solving the homogeneous ODE as

$$y_1 = \cos(2t)$$
$$y_2 = \sin(2t)$$

In the Variation of parameters u_1, u_2 are found using

$$u_1 = -\int \frac{y_2 f(t)}{a W(t)} \tag{2}$$

$$u_2 = \int \frac{y_1 f(t)}{a W(t)} \tag{3}$$

Where $W(t)$ is the Wronskian and a is the coefficient in front of y'' in the given ODE. The Wronskian is given by $W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$. Hence

$$W = \begin{vmatrix} \cos(2t) & \sin(2t) \\ \frac{d}{dt}(\cos(2t)) & \frac{d}{dt}(\sin(2t)) \end{vmatrix}$$

Which gives

$$W = \begin{vmatrix} \cos(2t) & \sin(2t) \\ -2\sin(2t) & 2\cos(2t) \end{vmatrix}$$

Therefore

$$W = (\cos(2t))(2\cos(2t)) - (\sin(2t))(-2\sin(2t))$$

Which simplifies to

$$W = 2\cos(2t)^2 + 2\sin(2t)^2$$

Which simplifies to

$$W = 2$$

Therefore Eq. (2) becomes

$$u_1 = -\int \frac{3\sin(2t)\csc(2t)}{2}dt$$

Which simplifies to

$$u_1 = -\int \frac{3}{2}dt$$

Hence

$$u_1 = -\frac{3t}{2}$$

And Eq. (3) becomes

$$u_2 = \int \frac{3\cos(2t)\csc(2t)}{2}dt$$

Which simplifies to

$$u_2 = \int \frac{3\cot(2t)}{2}dt$$

Hence

$$u_2 = -\frac{3\ln\left(\cot(2t)^2 + 1\right)}{8}$$

Which simplifies to

$$u_1 = -\frac{3t}{2}$$
$$u_2 = -\frac{3\ln\left(\csc(2t)^2\right)}{8}$$

Therefore the particular solution, from equation (1) is

$$y_p(t) = -\frac{3\cos(2t)\, t}{2} - \frac{3\ln\left(\csc(2t)^2\right)\sin(2t)}{8}$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= (c_1\cos(2t) + c_2\sin(2t)) + \left(-\frac{3\cos(2t)\, t}{2} - \frac{3\ln\left(\csc(2t)^2\right)\sin(2t)}{8}\right) \end{aligned}$$

Summary
The solution(s) found are the following

$$y = c_1\cos(2t) + c_2\sin(2t) - \frac{3\cos(2t)\, t}{2} - \frac{3\ln\left(\csc(2t)^2\right)\sin(2t)}{8} \tag{1}$$

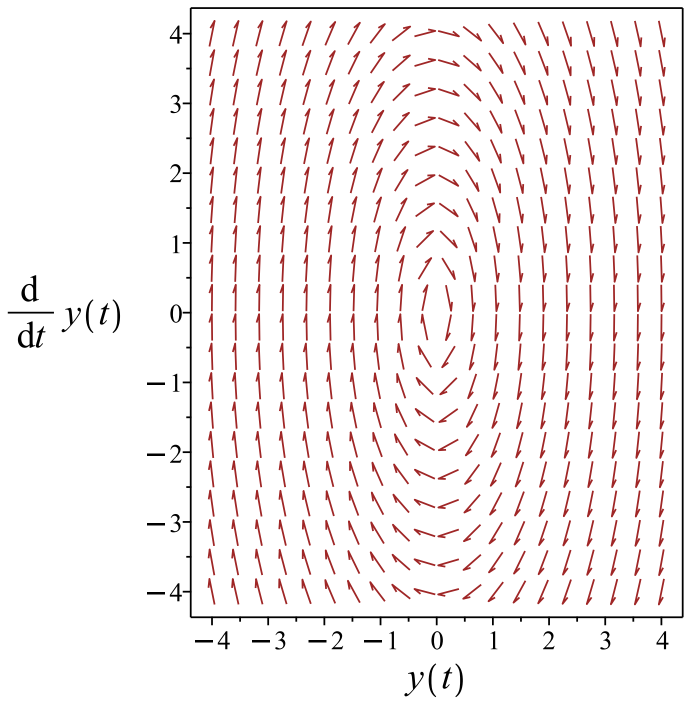


Figure 487: Slope field plot

Verification of solutions

$$y = c_1\cos(2t) + c_2\sin(2t) - \frac{3\cos(2t)\, t}{2} - \frac{3\ln\left(\csc(2t)^2\right)\sin(2t)}{8}$$

Verified OK.

10.8.2 Solving using Kovacic algorithm

Writing the ode as

$$y'' + 4y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1 \\ B &= 0 \\ C &= 4 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(t) = ye^{\int \frac{B}{2A}\, dt}$$

Then (2) becomes

$$z''(t) = rz(t) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{-4}{1} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= -4 \\ t &= 1 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(t) = -4z(t) \tag{7}$$

Equation (7) is now solved. After finding $z(t)$ then y is found using the inverse transformation

$$y = z(t)\, e^{-\int \frac{B}{2A}\, dt}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0,1,2,4,6,8,\cdots\}$	$\{\cdots,-6,-4,-2,0,2,3,4,5,6,\cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1,2\}$	$\{2,3,4,5,6,7,\cdots\}$

Table 488: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$
\begin{aligned}
O(\infty) &= \deg(t) - \deg(s) \\
&= 0 - 0 \\
&= 0
\end{aligned}
$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Therefore

$$
L = [1]
$$

Since $r = -4$ is not a function of t, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$
z_1(t) = \cos(2t)
$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$
y_1 = z_1 e^{\int -\frac{1}{2}\frac{B}{A}\, dt}
$$

Since $B = 0$ then the above reduces to

$$
\begin{aligned}
y_1 &= z_1 \\
&= \cos(2t)
\end{aligned}
$$

Which simplifies to

$$
y_1 = \cos(2t)
$$

The second solution y_2 to the original ode is found using reduction of order

$$
y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dt}}{y_1^2}\,dt
$$

Since $B = 0$ then the above becomes

$$
\begin{aligned}
y_2 &= y_1 \int \frac{1}{y_1^2}\,dt \\
&= \cos(2t) \int \frac{1}{\cos(2t)^2}\,dt \\
&= \cos(2t) \left(\frac{\tan(2t)}{2}\right)
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1(\cos(2t)) + c_2\left(\cos(2t)\left(\frac{\tan(2t)}{2}\right)\right)
\end{aligned}
$$

This is second order nonhomogeneous ODE. Let the solution be

$$
y = y_h + y_p
$$

Where y_h is the solution to the homogeneous ODE $Ay''(t) + By'(t) + Cy(t) = 0$, and y_p is a particular solution to the nonhomogeneous ODE $Ay''(t) + By'(t) + Cy(t) = f(t)$. y_h is the solution to

$$
y'' + 4y = 0
$$

The homogeneous solution is found using the Kovacic algorithm which results in

$$y_h = c_1 \cos(2t) + \frac{c_2 \sin(2t)}{2}$$

The particular solution y_p can be found using either the method of undetermined coefficients, or the method of variation of parameters. The method of variation of parameters will be used as it is more general and can be used when the coefficients of the ODE depend on t as well. Let

$$y_p(t) = u_1 y_1 + u_2 y_2 \tag{1}$$

Where u_1, u_2 to be determined, and y_1, y_2 are the two basis solutions (the two linearly independent solutions of the homogeneous ODE) found earlier when solving the homogeneous ODE as

$$y_1 = \cos(2t)$$
$$y_2 = \frac{\sin(2t)}{2}$$

In the Variation of parameters u_1, u_2 are found using

$$u_1 = -\int \frac{y_2 f(t)}{aW(t)} \tag{2}$$
$$u_2 = \int \frac{y_1 f(t)}{aW(t)} \tag{3}$$

Where $W(t)$ is the Wronskian and a is the coefficient in front of y'' in the given ODE. The Wronskian is given by $W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$. Hence

$$W = \begin{vmatrix} \cos(2t) & \frac{\sin(2t)}{2} \\ \frac{d}{dt}(\cos(2t)) & \frac{d}{dt}\left(\frac{\sin(2t)}{2}\right) \end{vmatrix}$$

Which gives

$$W = \begin{vmatrix} \cos(2t) & \frac{\sin(2t)}{2} \\ -2\sin(2t) & \cos(2t) \end{vmatrix}$$

Therefore

$$W = (\cos(2t))(\cos(2t)) - \left(\frac{\sin(2t)}{2}\right)(-2\sin(2t))$$

Which simplifies to

$$W = \cos(2t)^2 + \sin(2t)^2$$

Which simplifies to

$$W = 1$$

Therefore Eq. (2) becomes

$$u_1 = -\int \frac{\frac{3\sin(2t)\csc(2t)}{2}}{1}\,dt$$

Which simplifies to

$$u_1 = -\int \frac{3}{2}dt$$

Hence

$$u_1 = -\frac{3t}{2}$$

And Eq. (3) becomes

$$u_2 = \int \frac{3\cos(2t)\csc(2t)}{1}\,dt$$

Which simplifies to

$$u_2 = \int 3\cot(2t)\,dt$$

Hence

$$u_2 = -\frac{3\ln\left(\cot(2t)^2 + 1\right)}{4}$$

Which simplifies to

$$u_1 = -\frac{3t}{2}$$

$$u_2 = -\frac{3\ln\left(\csc(2t)^2\right)}{4}$$

Therefore the particular solution, from equation (1) is

$$y_p(t) = -\frac{3\cos(2t)\,t}{2} - \frac{3\ln\left(\csc(2t)^2\right)\sin(2t)}{8}$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \left(c_1\cos(2t) + \frac{c_2\sin(2t)}{2}\right) + \left(-\frac{3\cos(2t)\,t}{2} - \frac{3\ln\left(\csc(2t)^2\right)\sin(2t)}{8}\right) \end{aligned}$$

Summary
The solution(s) found are the following

$$y = c_1\cos(2t) + \frac{c_2\sin(2t)}{2} - \frac{3\cos(2t)\,t}{2} - \frac{3\ln\left(\csc(2t)^2\right)\sin(2t)}{8} \tag{1}$$

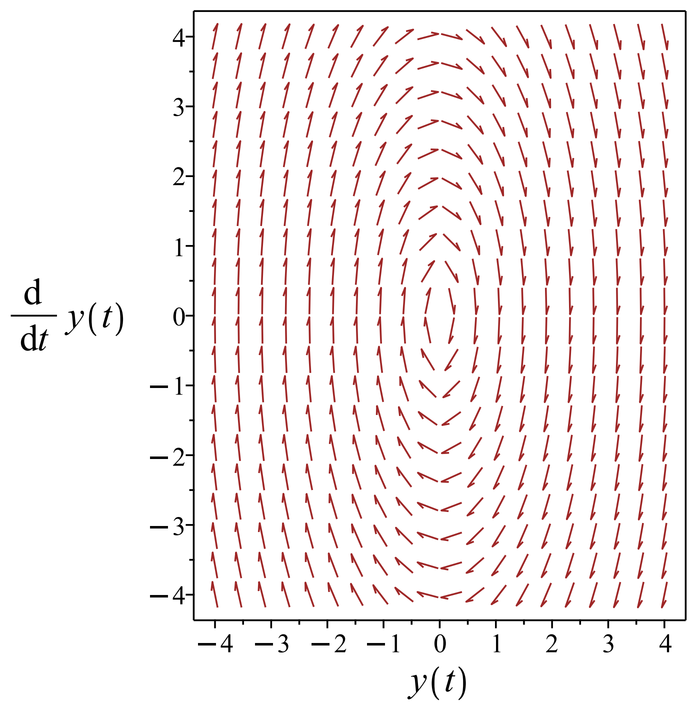


Figure 488: Slope field plot

Verification of solutions

$$y = c_1\cos(2t) + \frac{c_2\sin(2t)}{2} - \frac{3\cos(2t)\,t}{2} - \frac{3\ln\left(\csc(2t)^2\right)\sin(2t)}{8}$$

Verified OK.

10.8.3 Maple step by step solution

Let's solve

$y'' + 4y = 3\csc(2t)$

- Highest derivative means the order of the ODE is 2

  y''

- Characteristic polynomial of homogeneous ODE

  $r^2 + 4 = 0$

- Use quadratic formula to solve for r

  $r = \frac{0 \pm (\sqrt{-16})}{2}$

- Roots of the characteristic polynomial

  $r = (-2\,\mathrm{I}, 2\,\mathrm{I})$

- 1st solution of the homogeneous ODE

  $y_1(t) = \cos(2t)$

- 2nd solution of the homogeneous ODE

  $y_2(t) = \sin(2t)$

- General solution of the ODE

  $y = c_1 y_1(t) + c_2 y_2(t) + y_p(t)$

- Substitute in solutions of the homogeneous ODE

  $y = c_1 \cos(2t) + c_2 \sin(2t) + y_p(t)$

□ Find a particular solution $y_p(t)$ of the ODE

  ◦ Use variation of parameters to find y_p here $f(t)$ is the forcing function

  $\left[y_p(t) = -y_1(t)\left(\int \frac{y_2(t) f(t)}{W(y_1(t), y_2(t))} dt\right) + y_2(t)\left(\int \frac{y_1(t) f(t)}{W(y_1(t), y_2(t))} dt\right), f(t) = 3\csc(2t)\right]$

  ◦ Wronskian of solutions of the homogeneous equation

  $W(y_1(t), y_2(t)) = \begin{bmatrix} \cos(2t) & \sin(2t) \\ -2\sin(2t) & 2\cos(2t) \end{bmatrix}$

  ◦ Compute Wronskian

  $W(y_1(t), y_2(t)) = 2$

  ◦ Substitute functions into equation for $y_p(t)$

$$y_p(t) = -\frac{3\cos(2t)\left(\int 1dt\right)}{2} + \frac{3\sin(2t)\left(\int \cot(2t)dt\right)}{2}$$

- Compute integrals

$$y_p(t) = -\frac{3\cos(2t)t}{2} - \frac{3\ln\left(\csc(2t)^2\right)\sin(2t)}{8}$$

- Substitute particular solution into general solution to ODE

$$y = c_1\cos(2t) + c_2\sin(2t) - \frac{3\cos(2t)t}{2} - \frac{3\ln\left(\csc(2t)^2\right)\sin(2t)}{8}$$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 2; linear nonhomogeneous with symmetry [0,1]
trying a double symmetry of the form [xi=0, eta=F(x)]
-> Try solving first the homogeneous part of the ODE
   checking if the LODE has constant coefficients
   <- constant coefficients successful
<- solving first the homogeneous part of the ODE successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 35

```
dsolve(diff(y(t),t$2)+4*y(t) = 3*csc(2*t),y(t), singsol=all)
```

$$y(t) = -\frac{3\ln(\csc(2t))\sin(2t)}{4} + \frac{(-6t+4c_1)\cos(2t)}{4} + c_2\sin(2t)$$

✓ Solution by Mathematica

Time used: 0.036 (sec). Leaf size: 39

```
DSolve[y''[t]+4*y[t] ==3*Csc[2*t],y[t],t,IncludeSingularSolutions -> True]
```

$$y(t) \to \left(-\frac{3t}{2} + c_1\right)\cos(2t) + \frac{1}{4}\sin(2t)(3\log(\sin(2t)) + 4c_2)$$

10.9 problem 9

Internal problem ID [691]
Internal file name [`OUTPUT/691_Sunday_June_05_2022_01_47_02_AM_48256359/index.tex`]

Book: Elementary differential equations and boundary value problems, 10th ed., Boyce and DiPrima
Section: Chapter 3, Second order linear equations, section 3.6, Variation of Parameters. page 190
Problem number: 9.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_linear_constant_coeff"**

Maple gives the following as the ode type

```
[[_2nd_order, _linear, _nonhomogeneous]]
```

$$\boxed{y'' + y = 2\sec\left(\frac{t}{2}\right)}$$

10.9.1 Solving as second order linear constant coeff ode

This is second order non-homogeneous ODE. In standard form the ODE is

$$Ay''(t) + By'(t) + Cy(t) = f(t)$$

Where $A = 1, B = 0, C = 1, f(t) = 2\sec\left(\frac{t}{2}\right)$. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(t) + By'(t) + Cy(t) = 0$, and y_p is a particular solution to the non-homogeneous ODE $Ay''(t) + By'(t) + Cy(t) = f(t)$. y_h is the solution to

$$y'' + y = 0$$

This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$Ay''(t) + By'(t) + Cy(t) = 0$$

Where in the above $A = 1, B = 0, C = 1$. Let the solution be $y = e^{\lambda t}$. Substituting this into the ODE gives

$$\lambda^2 e^{\lambda t} + e^{\lambda t} = 0 \tag{1}$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda t}$ gives

$$\lambda^2 + 1 = 0 \tag{2}$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form.Using the quadratic formula

$$\lambda_{1,2} = \frac{-B}{2A} \pm \frac{1}{2A}\sqrt{B^2 - 4AC}$$

Substituting $A = 1, B = 0, C = 1$ into the above gives

$$\begin{aligned}\lambda_{1,2} &= \frac{0}{(2)(1)} \pm \frac{1}{(2)(1)}\sqrt{0^2 - (4)(1)(1)} \\ &= \pm i\end{aligned}$$

Hence

$$\begin{aligned}\lambda_1 &= +i \\ \lambda_2 &= -i\end{aligned}$$

Which simplifies to

$$\begin{aligned}\lambda_1 &= i \\ \lambda_2 &= -i\end{aligned}$$

Since roots are complex conjugate of each others, then let the roots be

$$\lambda_{1,2} = \alpha \pm i\beta$$

Where $\alpha = 0$ and $\beta = 1$. Therefore the final solution, when using Euler relation, can be written as

$$y = e^{\alpha t}(c_1 \cos(\beta t) + c_2 \sin(\beta t))$$

Which becomes

$$y = e^0(c_1 \cos(t) + c_2 \sin(t))$$

Or

$$y = c_1 \cos(t) + c_2 \sin(t)$$

Therefore the homogeneous solution y_h is

$$y_h = c_1 \cos(t) + c_2 \sin(t)$$

The particular solution y_p can be found using either the method of undetermined coefficients, or the method of variation of parameters. The method of variation of parameters will be used as it is more general and can be used when the coefficients of the ODE depend on t as well. Let

$$y_p(t) = u_1 y_1 + u_2 y_2 \tag{1}$$

Where u_1, u_2 to be determined, and y_1, y_2 are the two basis solutions (the two linearly independent solutions of the homogeneous ODE) found earlier when solving the homogeneous ODE as

$$\begin{aligned} y_1 &= \cos(t) \\ y_2 &= \sin(t) \end{aligned}$$

In the Variation of parameters u_1, u_2 are found using

$$u_1 = -\int \frac{y_2 f(t)}{aW(t)} \tag{2}$$

$$u_2 = \int \frac{y_1 f(t)}{aW(t)} \tag{3}$$

Where $W(t)$ is the Wronskian and a is the coefficient in front of y'' in the given ODE. The Wronskian is given by $W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$. Hence

$$W = \begin{vmatrix} \cos(t) & \sin(t) \\ \frac{d}{dt}(\cos(t)) & \frac{d}{dt}(\sin(t)) \end{vmatrix}$$

Which gives

$$W = \begin{vmatrix} \cos(t) & \sin(t) \\ -\sin(t) & \cos(t) \end{vmatrix}$$

Therefore

$$W = (\cos(t))(\cos(t)) - (\sin(t))(-\sin(t))$$

Which simplifies to

$$W = \cos(t)^2 + \sin(t)^2$$

Which simplifies to

$$W = 1$$

Therefore Eq. (2) becomes

$$u_1 = -\int \frac{2\sin(t)\sec\left(\frac{t}{2}\right)}{1}\,dt$$

Which simplifies to

$$u_1 = -\int 4\sin\left(\frac{t}{2}\right)dt$$

Hence

$$u_1 = 8\cos\left(\frac{t}{2}\right)$$

And Eq. (3) becomes

$$u_2 = \int \frac{2\cos(t)\sec\left(\frac{t}{2}\right)}{1}\,dt$$

Which simplifies to

$$u_2 = \int 2\cos(t)\sec\left(\frac{t}{2}\right)dt$$

Hence

$$u_2 = -4\ln\left(\sec\left(\frac{t}{2}\right) + \tan\left(\frac{t}{2}\right)\right) + 8\sin\left(\frac{t}{2}\right)$$

Therefore the particular solution, from equation (1) is

$$y_p(t) = 8\cos\left(\frac{t}{2}\right)\cos(t) + \left(-4\ln\left(\sec\left(\frac{t}{2}\right) + \tan\left(\frac{t}{2}\right)\right) + 8\sin\left(\frac{t}{2}\right)\right)\sin(t)$$

Which simplifies to

$$y_p(t) = \cos\left(\frac{t}{2}\right)\left(-8\ln\left(\sec\left(\frac{t}{2}\right)+\tan\left(\frac{t}{2}\right)\right)\sin\left(\frac{t}{2}\right)+8\right)$$

Therefore the general solution is

$$y = y_h + y_p$$

$$= (c_1\cos(t) + c_2\sin(t)) + \left(\cos\left(\frac{t}{2}\right)\left(-8\ln\left(\sec\left(\frac{t}{2}\right)+\tan\left(\frac{t}{2}\right)\right)\sin\left(\frac{t}{2}\right)+8\right)\right)$$

Summary

The solution(s) found are the following

$$y = c_1\cos(t) + c_2\sin(t) + \cos\left(\frac{t}{2}\right)\left(-8\ln\left(\sec\left(\frac{t}{2}\right)+\tan\left(\frac{t}{2}\right)\right)\sin\left(\frac{t}{2}\right)+8\right) \quad (1)$$

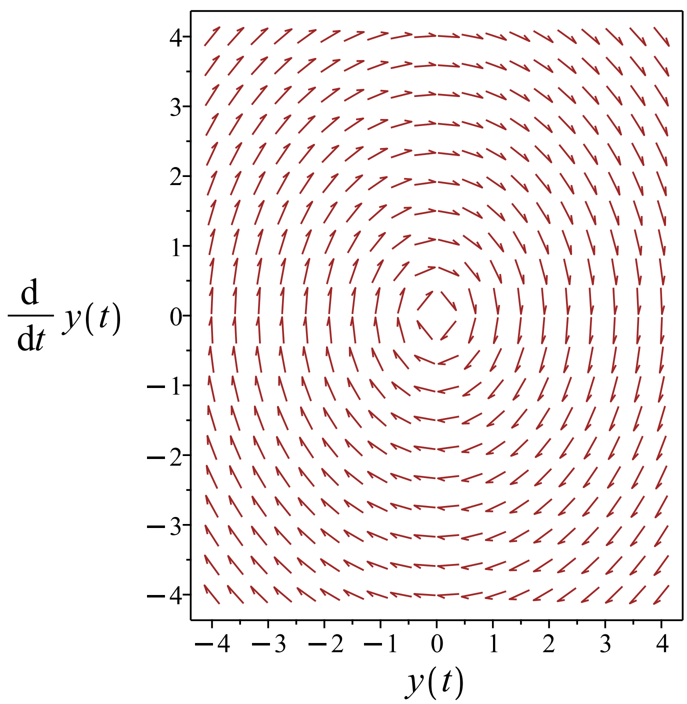


Figure 489: Slope field plot

Verification of solutions

$$y = c_1\cos(t) + c_2\sin(t) + \cos\left(\frac{t}{2}\right)\left(-8\ln\left(\sec\left(\frac{t}{2}\right)+\tan\left(\frac{t}{2}\right)\right)\sin\left(\frac{t}{2}\right)+8\right)$$

Verified OK.

10.9.2 Solving using Kovacic algorithm

Writing the ode as

$$y'' + y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1 \\ B &= 0 \\ C &= 1 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(t) = ye^{\int \frac{B}{2A}\, dt}$$

Then (2) becomes

$$z''(t) = rz(t) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{-1}{1} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= -1 \\ t &= 1 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(t) = -z(t) \tag{7}$$

Equation (7) is now solved. After finding $z(t)$ then y is found using the inverse transformation

$$y = z(t)\, e^{-\int \frac{B}{2A}\, dt}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 490: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned}
O(\infty) &= \deg(t) - \deg(s) \\
&= 0 - 0 \\
&= 0
\end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Since $r = -1$ is not a function of t, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$z_1(t) = \cos(t)$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$y_1 = z_1 e^{\int -\frac{1}{2} \frac{B}{A} \, dt}$$

Since $B=0$ then the above reduces to

$$\begin{aligned} y_1 &= z_1 \\ &= \cos(t) \end{aligned}$$

Which simplifies to

$$y_1 = \cos(t)$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\, dt}}{y_1^2}\, dt$$

Since $B=0$ then the above becomes

$$\begin{aligned} y_2 &= y_1 \int \frac{1}{y_1^2}\, dt \\ &= \cos(t) \int \frac{1}{\cos(t)^2}\, dt \\ &= \cos(t)(\tan(t)) \end{aligned}$$

Therefore the solution is

$$\begin{aligned} y &= c_1 y_1 + c_2 y_2 \\ &= c_1(\cos(t)) + c_2(\cos(t)(\tan(t))) \end{aligned}$$

This is second order nonhomogeneous ODE. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(t)+By'(t)+Cy(t)=0$, and y_p is a particular solution to the nonhomogeneous ODE $Ay''(t)+By'(t)+Cy(t)=f(t)$. y_h is the solution to

$$y'' + y = 0$$

The homogeneous solution is found using the Kovacic algorithm which results in

$$y_h = c_1 \cos(t) + c_2 \sin(t)$$

The particular solution y_p can be found using either the method of undetermined coefficients, or the method of variation of parameters. The method of variation of parameters will be used as it is more general and can be used when the coefficients of the ODE depend on t as well. Let

$$y_p(t) = u_1 y_1 + u_2 y_2 \tag{1}$$

Where u_1, u_2 to be determined, and y_1, y_2 are the two basis solutions (the two linearly independent solutions of the homogeneous ODE) found earlier when solving the homogeneous ODE as

$$y_1 = \cos(t)$$
$$y_2 = \sin(t)$$

In the Variation of parameters u_1, u_2 are found using

$$u_1 = -\int \frac{y_2 f(t)}{a W(t)} \tag{2}$$

$$u_2 = \int \frac{y_1 f(t)}{a W(t)} \tag{3}$$

Where $W(t)$ is the Wronskian and a is the coefficient in front of y'' in the given ODE. The Wronskian is given by $W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$. Hence

$$W = \begin{vmatrix} \cos(t) & \sin(t) \\ \frac{d}{dt}(\cos(t)) & \frac{d}{dt}(\sin(t)) \end{vmatrix}$$

Which gives

$$W = \begin{vmatrix} \cos(t) & \sin(t) \\ -\sin(t) & \cos(t) \end{vmatrix}$$

Therefore

$$W = (\cos(t))(\cos(t)) - (\sin(t))(-\sin(t))$$

Which simplifies to

$$W = \cos(t)^2 + \sin(t)^2$$

Which simplifies to

$$W = 1$$

Therefore Eq. (2) becomes

$$u_1 = -\int \frac{2\sin(t)\sec\left(\frac{t}{2}\right)}{1}\,dt$$

Which simplifies to

$$u_1 = -\int 4\sin\left(\frac{t}{2}\right)dt$$

Hence

$$u_1 = 8\cos\left(\frac{t}{2}\right)$$

And Eq. (3) becomes

$$u_2 = \int \frac{2\cos(t)\sec\left(\frac{t}{2}\right)}{1}\,dt$$

Which simplifies to

$$u_2 = \int 2\cos(t)\sec\left(\frac{t}{2}\right)dt$$

Hence

$$u_2 = -4\ln\left(\sec\left(\frac{t}{2}\right) + \tan\left(\frac{t}{2}\right)\right) + 8\sin\left(\frac{t}{2}\right)$$

Therefore the particular solution, from equation (1) is

$$y_p(t) = 8\cos\left(\frac{t}{2}\right)\cos(t) + \left(-4\ln\left(\sec\left(\frac{t}{2}\right) + \tan\left(\frac{t}{2}\right)\right) + 8\sin\left(\frac{t}{2}\right)\right)\sin(t)$$

Which simplifies to

$$y_p(t) = \cos\left(\frac{t}{2}\right)\left(-8\ln\left(\sec\left(\frac{t}{2}\right) + \tan\left(\frac{t}{2}\right)\right)\sin\left(\frac{t}{2}\right) + 8\right)$$

Therefore the general solution is

$$
\begin{aligned}
y &= y_h + y_p \\
&= (c_1 \cos(t) + c_2 \sin(t)) + \left(\cos\left(\frac{t}{2}\right)\left(-8\ln\left(\sec\left(\frac{t}{2}\right) + \tan\left(\frac{t}{2}\right)\right)\sin\left(\frac{t}{2}\right) + 8\right)\right)
\end{aligned}
$$

Summary
The solution(s) found are the following

$$
y = c_1 \cos(t) + c_2 \sin(t) + \cos\left(\frac{t}{2}\right)\left(-8\ln\left(\sec\left(\frac{t}{2}\right) + \tan\left(\frac{t}{2}\right)\right)\sin\left(\frac{t}{2}\right) + 8\right) \tag{1}
$$

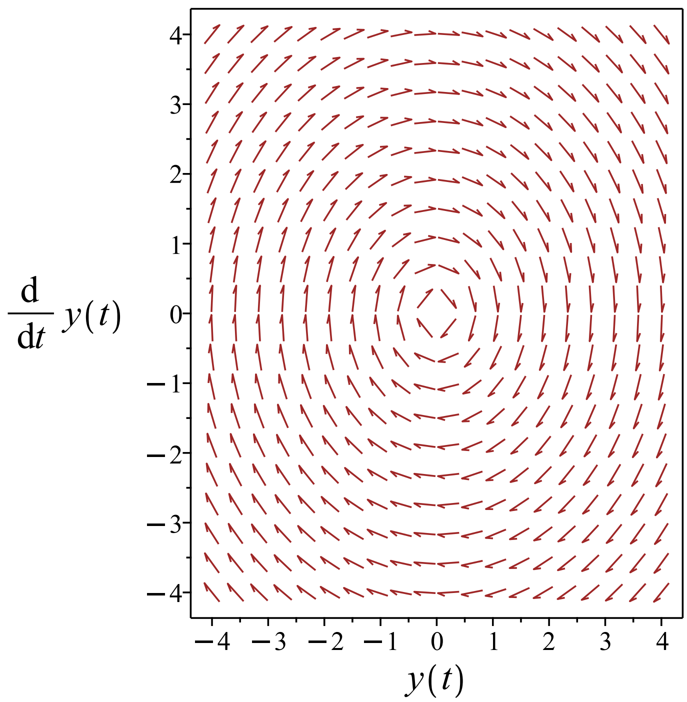


Figure 490: Slope field plot

Verification of solutions

$$
y = c_1 \cos(t) + c_2 \sin(t) + \cos\left(\frac{t}{2}\right)\left(-8\ln\left(\sec\left(\frac{t}{2}\right) + \tan\left(\frac{t}{2}\right)\right)\sin\left(\frac{t}{2}\right) + 8\right)
$$

Verified OK.

10.9.3 Maple step by step solution

Let's solve

$y'' + y = 2\sec\left(\frac{t}{2}\right)$

- Highest derivative means the order of the ODE is 2

  y''

- Characteristic polynomial of homogeneous ODE

  $r^2 + 1 = 0$

- Use quadratic formula to solve for r

  $r = \frac{0 \pm (\sqrt{-4})}{2}$

- Roots of the characteristic polynomial

  $r = (-\mathrm{I}, \mathrm{I})$

- 1st solution of the homogeneous ODE

  $y_1(t) = \cos(t)$

- 2nd solution of the homogeneous ODE

  $y_2(t) = \sin(t)$

- General solution of the ODE

  $y = c_1 y_1(t) + c_2 y_2(t) + y_p(t)$

- Substitute in solutions of the homogeneous ODE

  $y = c_1 \cos(t) + c_2 \sin(t) + y_p(t)$

□ Find a particular solution $y_p(t)$ of the ODE

- ◦ Use variation of parameters to find y_p here $f(t)$ is the forcing function

  $\left[y_p(t) = -y_1(t)\left(\int \frac{y_2(t)f(t)}{W(y_1(t),y_2(t))}dt\right) + y_2(t)\left(\int \frac{y_1(t)f(t)}{W(y_1(t),y_2(t))}dt\right), f(t) = 2\sec\left(\frac{t}{2}\right)\right]$

- ◦ Wronskian of solutions of the homogeneous equation

  $W(y_1(t), y_2(t)) = \begin{bmatrix} \cos(t) & \sin(t) \\ -\sin(t) & \cos(t) \end{bmatrix}$

- ◦ Compute Wronskian

  $W(y_1(t), y_2(t)) = 1$

- ◦ Substitute functions into equation for $y_p(t)$

$$y_p(t) = -4\cos(t)\left(\int \sin\left(\frac{t}{2}\right)dt\right) + 2\sin(t)\left(\int \cos(t)\sec\left(\frac{t}{2}\right)dt\right)$$

- Compute integrals

$$y_p(t) = \cos\left(\frac{t}{2}\right)\left(-8\ln\left(\sec\left(\frac{t}{2}\right) + \tan\left(\frac{t}{2}\right)\right)\sin\left(\frac{t}{2}\right) + 8\right)$$

- Substitute particular solution into general solution to ODE

$$y = c_1\cos(t) + c_2\sin(t) + \cos\left(\frac{t}{2}\right)\left(-8\ln\left(\sec\left(\frac{t}{2}\right) + \tan\left(\frac{t}{2}\right)\right)\sin\left(\frac{t}{2}\right) + 8\right)$$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 2; linear nonhomogeneous with symmetry [0,1]
trying a double symmetry of the form [xi=0, eta=F(x)]
-> Try solving first the homogeneous part of the ODE
   checking if the LODE has constant coefficients
   <- constant coefficients successful
<- solving first the homogeneous part of the ODE successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 33

```
dsolve(diff(y(t),t$2)+y(t) = 2*sec(t/2),y(t), singsol=all)
```

$$y(t) = -4\sin(t)\ln\left(\sec\left(\frac{t}{2}\right) + \tan\left(\frac{t}{2}\right)\right) + c_2\sin(t) + \cos(t)c_1 + 8\cos\left(\frac{t}{2}\right)$$

✓ Solution by Mathematica

Time used: 0.114 (sec). Leaf size: 35

```
DSolve[y''[t]+y[t]== 2*Sec[t/2],y[t],t,IncludeSingularSolutions -> True]
```

$$y(t) \to -4\sin(t)\text{arctanh}\left(\sin\left(\frac{t}{2}\right)\right) + 8\cos\left(\frac{t}{2}\right) + c_1\cos(t) + c_2\sin(t)$$

10.10 problem 10

Internal problem ID [692]
Internal file name [`OUTPUT/692_Sunday_June_05_2022_01_47_03_AM_2706801/index.tex`]

Book: Elementary differential equations and boundary value problems, 10th ed., Boyce and DiPrima
Section: Chapter 3, Second order linear equations, section 3.6, Variation of Parameters. page 190
Problem number: 10.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_linear_constant_coeff", "linear_second_order_ode_solved_by_an_integrating_factor"**

Maple gives the following as the ode type

```
[[_2nd_order, _linear, _nonhomogeneous]]
```

$$\boxed{y'' - 2y' + y = \frac{e^t}{t^2 + 1}}$$

10.10.1 Solving as second order linear constant coeff ode

This is second order non-homogeneous ODE. In standard form the ODE is

$$Ay''(t) + By'(t) + Cy(t) = f(t)$$

Where $A = 1, B = -2, C = 1, f(t) = \frac{e^t}{t^2+1}$. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(t) + By'(t) + Cy(t) = 0$, and y_p is a particular solution to the non-homogeneous ODE $Ay''(t) + By'(t) + Cy(t) = f(t)$.

y_h is the solution to

$$y'' - 2y' + y = 0$$

This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$Ay''(t) + By'(t) + Cy(t) = 0$$

Where in the above $A = 1, B = -2, C = 1$. Let the solution be $y = e^{\lambda t}$. Substituting this into the ODE gives

$$\lambda^2 e^{\lambda t} - 2\lambda e^{\lambda t} + e^{\lambda t} = 0 \tag{1}$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda t}$ gives

$$\lambda^2 - 2\lambda + 1 = 0 \tag{2}$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form.Using the quadratic formula

$$\lambda_{1,2} = \frac{-B}{2A} \pm \frac{1}{2A}\sqrt{B^2 - 4AC}$$

Substituting $A = 1, B = -2, C = 1$ into the above gives

$$\begin{aligned}\lambda_{1,2} &= \frac{2}{(2)(1)} \pm \frac{1}{(2)(1)}\sqrt{(-2)^2 - (4)(1)(1)} \\ &= 1\end{aligned}$$

Hence this is the case of a double root $\lambda_{1,2} = -1$. Therefore the solution is

$$y = c_1 e^t + c_2 t e^t \tag{1}$$

Therefore the homogeneous solution y_h is

$$y_h = c_1 e^t + c_2 t e^t$$

The particular solution y_p can be found using either the method of undetermined coefficients, or the method of variation of parameters. The method of variation of parameters will be used as it is more general and can be used when the coefficients of the ODE depend on t as well. Let

$$y_p(t) = u_1 y_1 + u_2 y_2 \tag{1}$$

Where u_1, u_2 to be determined, and y_1, y_2 are the two basis solutions (the two linearly independent solutions of the homogeneous ODE) found earlier when solving the homogeneous ODE as

$$\begin{aligned}y_1 &= e^t \\ y_2 &= t e^t\end{aligned}$$

In the Variation of parameters u_1, u_2 are found using

$$u_1 = -\int \frac{y_2 f(t)}{aW(t)} \tag{2}$$

$$u_2 = \int \frac{y_1 f(t)}{aW(t)} \tag{3}$$

Where $W(t)$ is the Wronskian and a is the coefficient in front of y'' in the given ODE. The Wronskian is given by $W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$. Hence

$$W = \begin{vmatrix} e^t & t\,e^t \\ \frac{d}{dt}(e^t) & \frac{d}{dt}(t\,e^t) \end{vmatrix}$$

Which gives

$$W = \begin{vmatrix} e^t & t\,e^t \\ e^t & t\,e^t + e^t \end{vmatrix}$$

Therefore

$$W = (e^t)\,(t\,e^t + e^t) - (t\,e^t)\,(e^t)$$

Which simplifies to

$$W = e^{2t}$$

Which simplifies to

$$W = e^{2t}$$

Therefore Eq. (2) becomes

$$u_1 = -\int \frac{\frac{t\,e^{2t}}{t^2+1}}{e^{2t}}\,dt$$

Which simplifies to

$$u_1 = -\int \frac{t}{t^2+1} dt$$

Hence

$$u_1 = -\frac{\ln(t^2+1)}{2}$$

And Eq. (3) becomes

$$u_2 = \int \frac{\frac{e^{2t}}{t^2+1}}{e^{2t}}\, dt$$

Which simplifies to

$$u_2 = \int \frac{1}{t^2+1} dt$$

Hence

$$u_2 = \arctan(t)$$

Therefore the particular solution, from equation (1) is

$$y_p(t) = -\frac{\ln(t^2+1)\, e^t}{2} + \arctan(t)\, t\, e^t$$

Which simplifies to

$$y_p(t) = e^t \left(-\frac{\ln(t^2+1)}{2} + t \arctan(t)\right)$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \left(c_1 e^t + c_2 t\, e^t\right) + \left(e^t \left(-\frac{\ln(t^2+1)}{2} + t \arctan(t)\right)\right) \end{aligned}$$

Which simplifies to

$$y = e^t (c_2 t + c_1) + e^t \left(-\frac{\ln(t^2+1)}{2} + t \arctan(t)\right)$$

Summary

The solution(s) found are the following

$$y = e^t (c_2 t + c_1) + e^t \left(-\frac{\ln(t^2+1)}{2} + t \arctan(t)\right) \tag{1}$$

Verification of solutions

$$y = e^t (c_2 t + c_1) + e^t \left(-\frac{\ln(t^2+1)}{2} + t \arctan(t)\right)$$

Verified OK.

10.10.2 Solving as linear second order ode solved by an integrating factor ode

The ode satisfies this form

$$y'' + p(t)\, y' + \frac{\left(p(t)^2 + p'(t)\right) y}{2} = f(t)$$

Where $p(t) = -2$. Therefore, there is an integrating factor given by

$$\begin{aligned} M(x) &= e^{\frac{1}{2}\int p\,dx} \\ &= e^{\int -2\,dx} \\ &= \mathrm{e}^{-t} \end{aligned}$$

Multiplying both sides of the ODE by the integrating factor $M(x)$ makes the left side of the ODE a complete differential

$$\begin{aligned} (M(x)y)'' &= \frac{\mathrm{e}^{-t}\mathrm{e}^{t}}{t^2+1} \\ \left(\mathrm{e}^{-t}y\right)'' &= \frac{\mathrm{e}^{-t}\mathrm{e}^{t}}{t^2+1} \end{aligned}$$

Integrating once gives

$$\left(\mathrm{e}^{-t}y\right)' = \arctan(t) + c_1$$

Integrating again gives

$$\left(\mathrm{e}^{-t}y\right) = c_1 t + t\arctan(t) - \frac{\ln(t^2+1)}{2} + c_2$$

Hence the solution is

$$y = \frac{c_1 t + t\arctan(t) - \frac{\ln(t^2+1)}{2} + c_2}{\mathrm{e}^{-t}}$$

Or

$$y = c_1 t\,\mathrm{e}^{t} + t\arctan(t)\,\mathrm{e}^{t} + c_2\mathrm{e}^{t} - \frac{\mathrm{e}^{t}\ln(t^2+1)}{2}$$

Summary

The solution(s) found are the following

$$y = c_1 t\,\mathrm{e}^{t} + t\arctan(t)\,\mathrm{e}^{t} + c_2\mathrm{e}^{t} - \frac{\mathrm{e}^{t}\ln(t^2+1)}{2} \tag{1}$$

Verification of solutions

$$y = c_1 t\,\mathrm{e}^t + t\arctan(t)\,\mathrm{e}^t + c_2\mathrm{e}^t - \frac{\mathrm{e}^t \ln(t^2+1)}{2}$$

Verified OK.

10.10.3 Solving using Kovacic algorithm

Writing the ode as

$$y'' - 2y' + y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1 \\ B &= -2 \\ C &= 1 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(t) = ye^{\int \frac{B}{2A}\,dt}$$

Then (2) becomes

$$z''(t) = rz(t) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{0}{1} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= 0 \\ t &= 1 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(t) = 0 \tag{7}$$

Equation (7) is now solved. After finding $z(t)$ then y is found using the inverse transformation

$$y = z(t)\, e^{-\int \frac{B}{2A}\, dt}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 492: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 0 - -\infty \\ &= \infty \end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is $infinity$ then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Since $r = 0$ is not a function of t, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$z_1(t) = 1$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$
\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dt} \\
&= z_1 e^{-\int \frac{1}{2}\frac{-2}{1}\,dt} \\
&= z_1 e^{t} \\
&= z_1 \left(\mathrm{e}^{t}\right)
\end{aligned}
$$

Which simplifies to

$$
y_1 = \mathrm{e}^{t}
$$

The second solution y_2 to the original ode is found using reduction of order

$$
y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dt}}{y_1^2}\,dt
$$

Substituting gives

$$
\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{-2}{1}\,dt}}{(y_1)^2}\,dt \\
&= y_1 \int \frac{e^{2t}}{(y_1)^2}\,dt \\
&= y_1(t)
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1\left(\mathrm{e}^{t}\right) + c_2\left(\mathrm{e}^{t}(t)\right)
\end{aligned}
$$

This is second order nonhomogeneous ODE. Let the solution be

$$
y = y_h + y_p
$$

Where y_h is the solution to the homogeneous ODE $Ay''(t) + By'(t) + Cy(t) = 0$, and y_p is a particular solution to the nonhomogeneous ODE $Ay''(t) + By'(t) + Cy(t) = f(t)$. y_h is the solution to

$$
y'' - 2y' + y = 0
$$

The homogeneous solution is found using the Kovacic algorithm which results in

$$y_h = c_1 e^t + c_2 t\,e^t$$

The particular solution y_p can be found using either the method of undetermined coefficients, or the method of variation of parameters. The method of variation of parameters will be used as it is more general and can be used when the coefficients of the ODE depend on t as well. Let

$$y_p(t) = u_1 y_1 + u_2 y_2 \tag{1}$$

Where u_1, u_2 to be determined, and y_1, y_2 are the two basis solutions (the two linearly independent solutions of the homogeneous ODE) found earlier when solving the homogeneous ODE as

$$y_1 = e^t$$
$$y_2 = t\,e^t$$

In the Variation of parameters u_1, u_2 are found using

$$u_1 = -\int \frac{y_2 f(t)}{aW(t)} \tag{2}$$

$$u_2 = \int \frac{y_1 f(t)}{aW(t)} \tag{3}$$

Where $W(t)$ is the Wronskian and a is the coefficient in front of y'' in the given ODE. The Wronskian is given by $W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$. Hence

$$W = \begin{vmatrix} e^t & t\,e^t \\ \frac{d}{dt}(e^t) & \frac{d}{dt}(t\,e^t) \end{vmatrix}$$

Which gives

$$W = \begin{vmatrix} e^t & t\,e^t \\ e^t & t\,e^t + e^t \end{vmatrix}$$

Therefore

$$W = (e^t)\,(t\,e^t + e^t) - (t\,e^t)\,(e^t)$$

Which simplifies to

$$W = \mathrm{e}^{2t}$$

Which simplifies to

$$W = \mathrm{e}^{2t}$$

Therefore Eq. (2) becomes

$$u_1 = -\left(\int \frac{\frac{t\,\mathrm{e}^{2t}}{t^2+1}}{\mathrm{e}^{2t}}dt\right)$$

Which simplifies to

$$u_1 = -\left(\int \frac{t}{t^2+1}dt\right)$$

Hence

$$u_1 = -\frac{\ln(t^2+1)}{2}$$

And Eq. (3) becomes

$$u_2 = \int \frac{\frac{\mathrm{e}^{2t}}{t^2+1}}{\mathrm{e}^{2t}}dt$$

Which simplifies to

$$u_2 = \int \frac{1}{t^2+1}dt$$

Hence

$$u_2 = \arctan(t)$$

Therefore the particular solution, from equation (1) is

$$y_p(t) = -\frac{\mathrm{e}^t \ln(t^2+1)}{2} + t\arctan(t)\,\mathrm{e}^t$$

Which simplifies to

$$y_p(t) = \mathrm{e}^t\left(-\frac{\ln(t^2+1)}{2} + t\arctan(t)\right)$$

Therefore the general solution is

$$
\begin{aligned}
y &= y_h + y_p \\
&= (c_1 \mathrm{e}^t + c_2 t\,\mathrm{e}^t) + \left(\mathrm{e}^t \left(-\frac{\ln(t^2+1)}{2} + t\arctan(t)\right)\right)
\end{aligned}
$$

Which simplifies to

$$
y = \mathrm{e}^t (c_2 t + c_1) + \mathrm{e}^t \left(-\frac{\ln(t^2+1)}{2} + t\arctan(t)\right)
$$

Summary

The solution(s) found are the following

$$
y = \mathrm{e}^t (c_2 t + c_1) + \mathrm{e}^t \left(-\frac{\ln(t^2+1)}{2} + t\arctan(t)\right) \tag{1}
$$

Verification of solutions

$$
y = \mathrm{e}^t (c_2 t + c_1) + \mathrm{e}^t \left(-\frac{\ln(t^2+1)}{2} + t\arctan(t)\right)
$$

Verified OK.

10.10.4 Maple step by step solution

Let's solve

$y'' - 2y' + y = \frac{\mathrm{e}^t}{t^2+1}$

- Highest derivative means the order of the ODE is 2

  y''

- Characteristic polynomial of homogeneous ODE

  $r^2 - 2r + 1 = 0$

- Factor the characteristic polynomial

  $(r-1)^2 = 0$

- Root of the characteristic polynomial

  $r = 1$

- 1st solution of the homogeneous ODE

$y_1(t) = e^t$

- Repeated root, multiply $y_1(t)$ by t to ensure linear independence

  $y_2(t) = t\,e^t$

- General solution of the ODE

  $y = c_1 y_1(t) + c_2 y_2(t) + y_p(t)$

- Substitute in solutions of the homogeneous ODE

  $y = c_1 e^t + c_2 t\,e^t + y_p(t)$

☐ Find a particular solution $y_p(t)$ of the ODE

  - Use variation of parameters to find y_p here $f(t)$ is the forcing function

    $\left[y_p(t) = -y_1(t)\left(\int \frac{y_2(t)f(t)}{W(y_1(t),y_2(t))}dt\right) + y_2(t)\left(\int \frac{y_1(t)f(t)}{W(y_1(t),y_2(t))}dt\right), f(t) = \frac{e^t}{t^2+1}\right]$

  - Wronskian of solutions of the homogeneous equation

    $W(y_1(t), y_2(t)) = \begin{bmatrix} e^t & t\,e^t \\ e^t & t\,e^t + e^t \end{bmatrix}$

  - Compute Wronskian

    $W(y_1(t), y_2(t)) = e^{2t}$

  - Substitute functions into equation for $y_p(t)$

    $y_p(t) = e^t\left(-\left(\int \frac{t}{t^2+1}dt\right) + \left(\int \frac{1}{t^2+1}dt\right)t\right)$

  - Compute integrals

    $y_p(t) = -\frac{e^t(-2t\arctan(t)+\ln(t^2+1))}{2}$

- Substitute particular solution into general solution to ODE

  $y = c_1 e^t + c_2 t\,e^t - \frac{e^t(-2t\arctan(t)+\ln(t^2+1))}{2}$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 2; linear nonhomogeneous with symmetry [0,1]
trying a double symmetry of the form [xi=0, eta=F(x)]
-> Try solving first the homogeneous part of the ODE
   checking if the LODE has constant coefficients
   <- constant coefficients successful
<- solving first the homogeneous part of the ODE successful`
```

✓ Solution by Maple
Time used: 0.015 (sec). Leaf size: 24

```
dsolve(diff(y(t),t$2)-2*diff(y(t),t)+y(t) = exp(t)/(1+t^2),y(t), singsol=all)
```

$$y(t) = e^{t}\left(c_2 + c_1 t - \frac{\ln(t^2+1)}{2} + t\arctan(t)\right)$$

✓ Solution by Mathematica
Time used: 0.029 (sec). Leaf size: 35

```
DSolve[y''[t]-2*y'[t]+y[t] == Exp[t]/(1+t^2),y[t],t,IncludeSingularSolutions -> True]
```

$$y(t) \to \frac{1}{2}e^t\left(2t\arctan(t) - \log\left(t^2+1\right) + 2(c_2 t + c_1)\right)$$

10.11 problem 11

Internal problem ID [693]
Internal file name [`OUTPUT/693_Sunday_June_05_2022_01_47_04_AM_3995225/index.tex`]

Book: Elementary differential equations and boundary value problems, 10th ed., Boyce and DiPrima
Section: Chapter 3, Second order linear equations, section 3.6, Variation of Parameters. page 190
Problem number: 11.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_linear_constant_coeff"**

Maple gives the following as the ode type

```
[[_2nd_order, _linear, _nonhomogeneous]]
```

$$\boxed{y'' - 5y' + 6y = g(t)}$$

10.11.1 Solving as second order linear constant coeff ode

This is second order non-homogeneous ODE. In standard form the ODE is

$$Ay''(t) + By'(t) + Cy(t) = f(t)$$

Where $A = 1, B = -5, C = 6, f(t) = g(t)$. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(t) + By'(t) + Cy(t) = 0$, and y_p is a particular solution to the non-homogeneous ODE $Ay''(t) + By'(t) + Cy(t) = f(t)$. y_h is the solution to

$$y'' - 5y' + 6y = 0$$

This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$Ay''(t) + By'(t) + Cy(t) = 0$$

Where in the above $A = 1, B = -5, C = 6$. Let the solution be $y = e^{\lambda t}$. Substituting this into the ODE gives

$$\lambda^2 \mathrm{e}^{\lambda t} - 5\lambda \,\mathrm{e}^{\lambda t} + 6\,\mathrm{e}^{\lambda t} = 0 \tag{1}$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda t}$ gives

$$\lambda^2 - 5\lambda + 6 = 0 \tag{2}$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form.Using the quadratic formula

$$\lambda_{1,2} = \frac{-B}{2A} \pm \frac{1}{2A}\sqrt{B^2 - 4AC}$$

Substituting $A = 1, B = -5, C = 6$ into the above gives

$$\begin{aligned}
\lambda_{1,2} &= \frac{5}{(2)\,(1)} \pm \frac{1}{(2)\,(1)}\sqrt{-5^2 - (4)\,(1)\,(6)} \\
&= \frac{5}{2} \pm \frac{1}{2}
\end{aligned}$$

Hence

$$\begin{aligned}
\lambda_1 &= \frac{5}{2} + \frac{1}{2} \\
\lambda_2 &= \frac{5}{2} - \frac{1}{2}
\end{aligned}$$

Which simplifies to

$$\begin{aligned}
\lambda_1 &= 3 \\
\lambda_2 &= 2
\end{aligned}$$

Since roots are real and distinct, then the solution is

$$\begin{aligned}
y &= c_1 e^{\lambda_1 t} + c_2 e^{\lambda_2 t} \\
y &= c_1 e^{(3)t} + c_2 e^{(2)t}
\end{aligned}$$

Or

$$y = c_1 \mathrm{e}^{3t} + c_2 \mathrm{e}^{2t}$$

Therefore the homogeneous solution y_h is

$$y_h = c_1 \mathrm{e}^{3t} + c_2 \mathrm{e}^{2t}$$

The particular solution y_p can be found using either the method of undetermined coefficients, or the method of variation of parameters. The method of variation of parameters will be used as it is more general and can be used when the coefficients of the ODE depend on t as well. Let

$$y_p(t) = u_1 y_1 + u_2 y_2 \tag{1}$$

Where u_1, u_2 to be determined, and y_1, y_2 are the two basis solutions (the two linearly independent solutions of the homogeneous ODE) found earlier when solving the homogeneous ODE as

$$y_1 = \mathrm{e}^{3t}$$
$$y_2 = \mathrm{e}^{2t}$$

In the Variation of parameters u_1, u_2 are found using

$$u_1 = -\int \frac{y_2 f(t)}{aW(t)} \tag{2}$$

$$u_2 = \int \frac{y_1 f(t)}{aW(t)} \tag{3}$$

Where $W(t)$ is the Wronskian and a is the coefficient in front of y'' in the given ODE. The Wronskian is given by $W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$. Hence

$$W = \begin{vmatrix} \mathrm{e}^{3t} & \mathrm{e}^{2t} \\ \frac{d}{dt}\left(\mathrm{e}^{3t}\right) & \frac{d}{dt}\left(\mathrm{e}^{2t}\right) \end{vmatrix}$$

Which gives

$$W = \begin{vmatrix} \mathrm{e}^{3t} & \mathrm{e}^{2t} \\ 3\,\mathrm{e}^{3t} & 2\,\mathrm{e}^{2t} \end{vmatrix}$$

Therefore

$$W = \left(\mathrm{e}^{3t}\right)\left(2\,\mathrm{e}^{2t}\right) - \left(\mathrm{e}^{2t}\right)\left(3\,\mathrm{e}^{3t}\right)$$

Which simplifies to

$$W = -\mathrm{e}^{3t}\mathrm{e}^{2t}$$

Which simplifies to

$$W = -e^{5t}$$

Therefore Eq. (2) becomes

$$u_1 = -\int \frac{e^{2t} g(t)}{-e^{5t}}\, dt$$

Which simplifies to

$$u_1 = -\int -g(t)\, e^{-3t} dt$$

Hence

$$u_1 = -\left(\int_0^t -g(\alpha)\, e^{-3\alpha} d\alpha\right)$$

And Eq. (3) becomes

$$u_2 = \int \frac{e^{3t} g(t)}{-e^{5t}}\, dt$$

Which simplifies to

$$u_2 = \int -g(t)\, e^{-2t} dt$$

Hence

$$u_2 = \int_0^t -g(\alpha)\, e^{-2\alpha} d\alpha$$

Which simplifies to

$$u_1 = \int_0^t g(\alpha)\, e^{-3\alpha} d\alpha$$
$$u_2 = -\left(\int_0^t g(\alpha)\, e^{-2\alpha} d\alpha\right)$$

Therefore the particular solution, from equation (1) is

$$y_p(t) = \left(\int_0^t g(\alpha)\, e^{-3\alpha} d\alpha\right) e^{3t} - \left(\int_0^t g(\alpha)\, e^{-2\alpha} d\alpha\right) e^{2t}$$

Therefore the general solution is

$$
\begin{aligned}
y &= y_h + y_p \\
&= (c_1 \mathrm{e}^{3t} + c_2 \mathrm{e}^{2t}) + \left(\left(\int_0^t g(\alpha)\, \mathrm{e}^{-3\alpha} d\alpha\right) \mathrm{e}^{3t} - \left(\int_0^t g(\alpha)\, \mathrm{e}^{-2\alpha} d\alpha\right) \mathrm{e}^{2t}\right)
\end{aligned}
$$

Summary
The solution(s) found are the following

$$
y = c_1 \mathrm{e}^{3t} + c_2 \mathrm{e}^{2t} + \left(\int_0^t g(\alpha)\, \mathrm{e}^{-3\alpha} d\alpha\right) \mathrm{e}^{3t} - \left(\int_0^t g(\alpha)\, \mathrm{e}^{-2\alpha} d\alpha\right) \mathrm{e}^{2t} \tag{1}
$$

Verification of solutions

$$
y = c_1 \mathrm{e}^{3t} + c_2 \mathrm{e}^{2t} + \left(\int_0^t g(\alpha)\, \mathrm{e}^{-3\alpha} d\alpha\right) \mathrm{e}^{3t} - \left(\int_0^t g(\alpha)\, \mathrm{e}^{-2\alpha} d\alpha\right) \mathrm{e}^{2t}
$$

Verified OK.

10.11.2 Solving using Kovacic algorithm

Writing the ode as

$$
y'' - 5y' + 6y = 0 \tag{1}
$$
$$
Ay'' + By' + Cy = 0 \tag{2}
$$

Comparing (1) and (2) shows that

$$
\begin{aligned}
A &= 1 \\
B &= -5 \\
C &= 6
\end{aligned} \tag{3}
$$

Applying the Liouville transformation on the dependent variable gives

$$
z(t) = y e^{\int \frac{B}{2A}\, dt}
$$

Then (2) becomes

$$
z''(t) = r z(t) \tag{4}
$$

Where r is given by

$$
\begin{aligned}
r &= \frac{s}{t} \\
&= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2}
\end{aligned} \tag{5}
$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{1}{4} \tag{6}$$

Comparing the above to (5) shows that

$$s = 1$$
$$t = 4$$

Therefore eq. (4) becomes

$$z''(t) = \frac{z(t)}{4} \tag{7}$$

Equation (7) is now solved. After finding $z(t)$ then y is found using the inverse transformation

$$y = z(t)\, e^{-\int \frac{B}{2A}\, dt}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 494: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 0 - 0 \\ &= 0 \end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Since $r = \frac{1}{4}$ is not a function of t, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$z_1(t) = \mathrm{e}^{-\frac{t}{2}}$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$\begin{aligned} y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dt} \\ &= z_1 e^{-\int \frac{1}{2}\frac{-5}{1}\,dt} \\ &= z_1 e^{\frac{5t}{2}} \\ &= z_1 \left(\mathrm{e}^{\frac{5t}{2}}\right) \end{aligned}$$

Which simplifies to

$$y_1 = \mathrm{e}^{2t}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dt}}{y_1^2}\,dt$$

Substituting gives

$$\begin{aligned} y_2 &= y_1 \int \frac{e^{\int -\frac{-5}{1}\,dt}}{(y_1)^2}\,dt \\ &= y_1 \int \frac{e^{5t}}{(y_1)^2}\,dt \\ &= y_1 \left(\mathrm{e}^{t}\right) \end{aligned}$$

Therefore the solution is

$$\begin{aligned} y &= c_1 y_1 + c_2 y_2 \\ &= c_1 \left(\mathrm{e}^{2t}\right) + c_2 \left(\mathrm{e}^{2t} \left(\mathrm{e}^{t}\right)\right) \end{aligned}$$

This is second order nonhomogeneous ODE. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(t) + By'(t) + Cy(t) = 0$, and y_p is a particular solution to the nonhomogeneous ODE $Ay''(t) + By'(t) + Cy(t) = f(t)$. y_h is the solution to

$$y'' - 5y' + 6y = 0$$

The homogeneous solution is found using the Kovacic algorithm which results in

$$y_h = c_1 e^{2t} + c_2 e^{3t}$$

The particular solution y_p can be found using either the method of undetermined coefficients, or the method of variation of parameters. The method of variation of parameters will be used as it is more general and can be used when the coefficients of the ODE depend on t as well. Let

$$y_p(t) = u_1 y_1 + u_2 y_2 \tag{1}$$

Where u_1, u_2 to be determined, and y_1, y_2 are the two basis solutions (the two linearly independent solutions of the homogeneous ODE) found earlier when solving the homogeneous ODE as

$$y_1 = e^{2t}$$
$$y_2 = e^{3t}$$

In the Variation of parameters u_1, u_2 are found using

$$u_1 = -\int \frac{y_2 f(t)}{aW(t)} \tag{2}$$

$$u_2 = \int \frac{y_1 f(t)}{aW(t)} \tag{3}$$

Where $W(t)$ is the Wronskian and a is the coefficient in front of y'' in the given ODE. The Wronskian is given by $W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$. Hence

$$W = \begin{vmatrix} e^{2t} & e^{3t} \\ \frac{d}{dt}(e^{2t}) & \frac{d}{dt}(e^{3t}) \end{vmatrix}$$

Which gives

$$W = \begin{vmatrix} e^{2t} & e^{3t} \\ 2\,e^{2t} & 3\,e^{3t} \end{vmatrix}$$

Therefore

$$W = \left(e^{2t}\right)\left(3\,e^{3t}\right) - \left(e^{3t}\right)\left(2\,e^{2t}\right)$$

Which simplifies to

$$W = e^{3t} e^{2t}$$

Which simplifies to

$$W = e^{5t}$$

Therefore Eq. (2) becomes

$$u_1 = -\int \frac{e^{3t} g(t)}{e^{5t}}\, dt$$

Which simplifies to

$$u_1 = -\int g(t)\, e^{-2t} dt$$

Hence

$$u_1 = -\left(\int_0^t g(\alpha)\, e^{-2\alpha} d\alpha\right)$$

And Eq. (3) becomes

$$u_2 = \int \frac{e^{2t} g(t)}{e^{5t}}\, dt$$

Which simplifies to

$$u_2 = \int g(t)\, e^{-3t} dt$$

Hence

$$u_2 = \int_0^t g(\alpha)\, e^{-3\alpha} d\alpha$$

Therefore the particular solution, from equation (1) is

$$y_p(t) = -\left(\int_0^t g(\alpha)\,\mathrm{e}^{-2\alpha}d\alpha\right)\mathrm{e}^{2t} + \left(\int_0^t g(\alpha)\,\mathrm{e}^{-3\alpha}d\alpha\right)\mathrm{e}^{3t}$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \left(c_1\mathrm{e}^{2t} + c_2\mathrm{e}^{3t}\right) + \left(-\left(\int_0^t g(\alpha)\,\mathrm{e}^{-2\alpha}d\alpha\right)\mathrm{e}^{2t} + \left(\int_0^t g(\alpha)\,\mathrm{e}^{-3\alpha}d\alpha\right)\mathrm{e}^{3t}\right) \end{aligned}$$

Summary

The solution(s) found are the following

$$y = c_1\mathrm{e}^{2t} + c_2\mathrm{e}^{3t} - \left(\int_0^t g(\alpha)\,\mathrm{e}^{-2\alpha}d\alpha\right)\mathrm{e}^{2t} + \left(\int_0^t g(\alpha)\,\mathrm{e}^{-3\alpha}d\alpha\right)\mathrm{e}^{3t} \tag{1}$$

Verification of solutions

$$y = c_1\mathrm{e}^{2t} + c_2\mathrm{e}^{3t} - \left(\int_0^t g(\alpha)\,\mathrm{e}^{-2\alpha}d\alpha\right)\mathrm{e}^{2t} + \left(\int_0^t g(\alpha)\,\mathrm{e}^{-3\alpha}d\alpha\right)\mathrm{e}^{3t}$$

Verified OK.

10.11.3 Maple step by step solution

Let's solve

$y'' - 5y' + 6y = g(t)$

- Highest derivative means the order of the ODE is 2

  y''

- Characteristic polynomial of homogeneous ODE

  $r^2 - 5r + 6 = 0$

- Factor the characteristic polynomial

  $(r-2)(r-3) = 0$

- Roots of the characteristic polynomial

  $r = (2, 3)$

- 1st solution of the homogeneous ODE

$y_1(t) = e^{2t}$

- 2nd solution of the homogeneous ODE

  $y_2(t) = e^{3t}$

- General solution of the ODE

  $y = c_1 y_1(t) + c_2 y_2(t) + y_p(t)$

- Substitute in solutions of the homogeneous ODE

  $y = c_1 e^{2t} + c_2 e^{3t} + y_p(t)$

☐ Find a particular solution $y_p(t)$ of the ODE

  - Use variation of parameters to find y_p here $f(t)$ is the forcing function

    $\left[y_p(t) = -y_1(t)\left(\int \frac{y_2(t)f(t)}{W(y_1(t),y_2(t))}dt\right) + y_2(t)\left(\int \frac{y_1(t)f(t)}{W(y_1(t),y_2(t))}dt\right), f(t) = g(t)\right]$

  - Wronskian of solutions of the homogeneous equation

    $W(y_1(t), y_2(t)) = \begin{bmatrix} e^{2t} & e^{3t} \\ 2e^{2t} & 3e^{3t} \end{bmatrix}$

  - Compute Wronskian

    $W(y_1(t), y_2(t)) = e^{5t}$

  - Substitute functions into equation for $y_p(t)$

    $y_p(t) = -e^{2t}\left(\int g(t) e^{-2t} dt\right) + e^{3t}\left(\int g(t) e^{-3t} dt\right)$

  - Compute integrals

    $y_p(t) = -e^{2t}\left(\int g(t) e^{-2t} dt\right) + e^{3t}\left(\int g(t) e^{-3t} dt\right)$

- Substitute particular solution into general solution to ODE

  $y = c_1 e^{2t} + c_2 e^{3t} - e^{2t}\left(\int g(t) e^{-2t} dt\right) + e^{3t}\left(\int g(t) e^{-3t} dt\right)$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 2; linear nonhomogeneous with symmetry [0,1]
trying a double symmetry of the form [xi=0, eta=F(x)]
-> Try solving first the homogeneous part of the ODE
   checking if the LODE has constant coefficients
   <- constant coefficients successful
<- solving first the homogeneous part of the ODE successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 46

```
dsolve(diff(y(t),t$2)-5*diff(y(t),t)+6*y(t) = g(t),y(t), singsol=all)
```

$$y(t) = c_2 \mathrm{e}^{2t} + c_1 \mathrm{e}^{3t} - \left(\int g(t)\, \mathrm{e}^{-2t} dt\right) \mathrm{e}^{2t} + \left(\int g(t)\, \mathrm{e}^{-3t} dt\right) \mathrm{e}^{3t}$$

✓ Solution by Mathematica

Time used: 0.064 (sec). Leaf size: 59

```
DSolve[y''[t]-5*y'[t]+6*y[t] == g[t],y[t],t,IncludeSingularSolutions -> True]
```

$$y(t) \to e^{2t}\left(\int_1^t -e^{-2K[1]} g(K[1]) dK[1] + e^t \int_1^t e^{-3K[2]} g(K[2]) dK[2] + c_2 e^t + c_1\right)$$

10.12 problem 12

Internal problem ID [694]
Internal file name [`OUTPUT/694_Sunday_June_05_2022_01_47_06_AM_78756365/index.tex`]

Book: Elementary differential equations and boundary value problems, 10th ed., Boyce and DiPrima
Section: Chapter 3, Second order linear equations, section 3.6, Variation of Parameters. page 190
Problem number: 12.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_linear_constant_coeff"**

Maple gives the following as the ode type

```
[[_2nd_order, _linear, _nonhomogeneous]]
```

$$\boxed{y'' + 4y = g(t)}$$

10.12.1 Solving as second order linear constant coeff ode

This is second order non-homogeneous ODE. In standard form the ODE is

$$Ay''(t) + By'(t) + Cy(t) = f(t)$$

Where $A = 1, B = 0, C = 4, f(t) = g(t)$. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(t) + By'(t) + Cy(t) = 0$, and y_p is a particular solution to the non-homogeneous ODE $Ay''(t) + By'(t) + Cy(t) = f(t)$. y_h is the solution to

$$y'' + 4y = 0$$

This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$Ay''(t) + By'(t) + Cy(t) = 0$$

Where in the above $A = 1, B = 0, C = 4$. Let the solution be $y = e^{\lambda t}$. Substituting this into the ODE gives

$$\lambda^2 \mathrm{e}^{\lambda t} + 4\,\mathrm{e}^{\lambda t} = 0 \tag{1}$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda t}$ gives

$$\lambda^2 + 4 = 0 \tag{2}$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form.Using the quadratic formula

$$\lambda_{1,2} = \frac{-B}{2A} \pm \frac{1}{2A}\sqrt{B^2 - 4AC}$$

Substituting $A = 1, B = 0, C = 4$ into the above gives

$$\begin{aligned}\lambda_{1,2} &= \frac{0}{(2)\,(1)} \pm \frac{1}{(2)\,(1)}\sqrt{0^2 - (4)\,(1)\,(4)}\\ &= \pm 2i\end{aligned}$$

Hence

$$\begin{aligned}\lambda_1 &= +2i\\ \lambda_2 &= -2i\end{aligned}$$

Which simplifies to

$$\begin{aligned}\lambda_1 &= 2i\\ \lambda_2 &= -2i\end{aligned}$$

Since roots are complex conjugate of each others, then let the roots be

$$\lambda_{1,2} = \alpha \pm i\beta$$

Where $\alpha = 0$ and $\beta = 2$. Therefore the final solution, when using Euler relation, can be written as

$$y = e^{\alpha t}(c_1 \cos(\beta t) + c_2 \sin(\beta t))$$

Which becomes

$$y = e^0(c_1 \cos(2t) + c_2 \sin(2t))$$

Or

$$y = c_1 \cos(2t) + c_2 \sin(2t)$$

Therefore the homogeneous solution y_h is

$$y_h = c_1 \cos(2t) + c_2 \sin(2t)$$

The particular solution y_p can be found using either the method of undetermined coefficients, or the method of variation of parameters. The method of variation of parameters will be used as it is more general and can be used when the coefficients of the ODE depend on t as well. Let

$$y_p(t) = u_1 y_1 + u_2 y_2 \tag{1}$$

Where u_1, u_2 to be determined, and y_1, y_2 are the two basis solutions (the two linearly independent solutions of the homogeneous ODE) found earlier when solving the homogeneous ODE as

$$y_1 = \cos(2t)$$
$$y_2 = \sin(2t)$$

In the Variation of parameters u_1, u_2 are found using

$$u_1 = -\int \frac{y_2 f(t)}{aW(t)} \tag{2}$$
$$u_2 = \int \frac{y_1 f(t)}{aW(t)} \tag{3}$$

Where $W(t)$ is the Wronskian and a is the coefficient in front of y'' in the given ODE. The Wronskian is given by $W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$. Hence

$$W = \begin{vmatrix} \cos(2t) & \sin(2t) \\ \frac{d}{dt}(\cos(2t)) & \frac{d}{dt}(\sin(2t)) \end{vmatrix}$$

Which gives

$$W = \begin{vmatrix} \cos(2t) & \sin(2t) \\ -2\sin(2t) & 2\cos(2t) \end{vmatrix}$$

Therefore

$$W = (\cos(2t))(2\cos(2t)) - (\sin(2t))(-2\sin(2t))$$

Which simplifies to

$$W = 2\cos(2t)^2 + 2\sin(2t)^2$$

Which simplifies to

$$W = 2$$

Therefore Eq. (2) becomes

$$u_1 = -\int \frac{\sin(2t)\, g(t)}{2}\, dt$$

Which simplifies to

$$u_1 = -\int \frac{\sin(2t)\, g(t)}{2} dt$$

Hence

$$u_1 = -\left(\int_0^t \frac{\sin(2\alpha)\, g(\alpha)}{2} d\alpha\right)$$

And Eq. (3) becomes

$$u_2 = \int \frac{\cos(2t)\, g(t)}{2}\, dt$$

Which simplifies to

$$u_2 = \int \frac{\cos(2t)\, g(t)}{2} dt$$

Hence

$$u_2 = \int_0^t \frac{\cos(2\alpha)\, g(\alpha)}{2} d\alpha$$

Which simplifies to

$$u_1 = -\frac{\left(\int_0^t \sin(2\alpha)\, g(\alpha)\, d\alpha\right)}{2}$$

$$u_2 = \frac{\left(\int_0^t \cos(2\alpha)\, g(\alpha)\, d\alpha\right)}{2}$$

Therefore the particular solution, from equation (1) is

$$y_p(t) = -\frac{\left(\int_0^t \sin(2\alpha) g(\alpha) d\alpha\right) \cos(2t)}{2} + \frac{\left(\int_0^t \cos(2\alpha) g(\alpha) d\alpha\right) \sin(2t)}{2}$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= (c_1 \cos(2t) + c_2 \sin(2t)) \\ &+ \left(-\frac{\left(\int_0^t \sin(2\alpha) g(\alpha) d\alpha\right) \cos(2t)}{2} + \frac{\left(\int_0^t \cos(2\alpha) g(\alpha) d\alpha\right) \sin(2t)}{2}\right) \end{aligned}$$

Summary

The solution(s) found are the following

$$\begin{aligned} y = c_1 \cos(2t) + c_2 \sin(2t) - \frac{\left(\int_0^t \sin(2\alpha) g(\alpha) d\alpha\right) \cos(2t)}{2} \\ + \frac{\left(\int_0^t \cos(2\alpha) g(\alpha) d\alpha\right) \sin(2t)}{2} \end{aligned} \tag{1}$$

Verification of solutions

$$\begin{aligned} y = c_1 \cos(2t) + c_2 \sin(2t) - \frac{\left(\int_0^t \sin(2\alpha) g(\alpha) d\alpha\right) \cos(2t)}{2} \\ + \frac{\left(\int_0^t \cos(2\alpha) g(\alpha) d\alpha\right) \sin(2t)}{2} \end{aligned}$$

Verified OK.

10.12.2 Solving using Kovacic algorithm

Writing the ode as

$$y'' + 4y = 0 \tag{1}$$

$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1 \\ B &= 0 \\ C &= 4 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(t) = ye^{\int \frac{B}{2A}\, dt}$$

Then (2) becomes

$$z''(t) = rz(t) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{-4}{1} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= -4 \\ t &= 1 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(t) = -4z(t) \tag{7}$$

Equation (7) is now solved. After finding $z(t)$ then y is found using the inverse transformation

$$y = z(t)\, e^{-\int \frac{B}{2A}\, dt}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 496: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned}O(\infty) &= \deg(t) - \deg(s)\\ &= 0 - 0\\ &= 0\end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Since $r = -4$ is not a function of t, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$z_1(t) = \cos(2t)$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$y_1 = z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dt}$$

Since $B = 0$ then the above reduces to

$$\begin{aligned}y_1 &= z_1\\ &= \cos(2t)\end{aligned}$$

Which simplifies to

$$y_1 = \cos(2t)$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A} dt}}{y_1^2} dt$$

Since $B = 0$ then the above becomes

$$\begin{aligned} y_2 &= y_1 \int \frac{1}{y_1^2} dt \\ &= \cos(2t) \int \frac{1}{\cos(2t)^2} dt \\ &= \cos(2t) \left(\frac{\tan(2t)}{2}\right) \end{aligned}$$

Therefore the solution is

$$\begin{aligned} y &= c_1 y_1 + c_2 y_2 \\ &= c_1(\cos(2t)) + c_2\left(\cos(2t)\left(\frac{\tan(2t)}{2}\right)\right) \end{aligned}$$

This is second order nonhomogeneous ODE. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(t) + By'(t) + Cy(t) = 0$, and y_p is a particular solution to the nonhomogeneous ODE $Ay''(t) + By'(t) + Cy(t) = f(t)$. y_h is the solution to

$$y'' + 4y = 0$$

The homogeneous solution is found using the Kovacic algorithm which results in

$$y_h = c_1 \cos(2t) + \frac{c_2 \sin(2t)}{2}$$

The particular solution y_p can be found using either the method of undetermined coefficients, or the method of variation of parameters. The method of variation of

parameters will be used as it is more general and can be used when the coefficients of the ODE depend on t as well. Let

$$y_p(t) = u_1 y_1 + u_2 y_2 \tag{1}$$

Where u_1, u_2 to be determined, and y_1, y_2 are the two basis solutions (the two linearly independent solutions of the homogeneous ODE) found earlier when solving the homogeneous ODE as

$$y_1 = \cos(2t)$$
$$y_2 = \frac{\sin(2t)}{2}$$

In the Variation of parameters u_1, u_2 are found using

$$u_1 = -\int \frac{y_2 f(t)}{a W(t)} \tag{2}$$
$$u_2 = \int \frac{y_1 f(t)}{a W(t)} \tag{3}$$

Where $W(t)$ is the Wronskian and a is the coefficient in front of y'' in the given ODE. The Wronskian is given by $W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$. Hence

$$W = \begin{vmatrix} \cos(2t) & \frac{\sin(2t)}{2} \\ \frac{d}{dt}(\cos(2t)) & \frac{d}{dt}\left(\frac{\sin(2t)}{2}\right) \end{vmatrix}$$

Which gives

$$W = \begin{vmatrix} \cos(2t) & \frac{\sin(2t)}{2} \\ -2\sin(2t) & \cos(2t) \end{vmatrix}$$

Therefore

$$W = (\cos(2t))(\cos(2t)) - \left(\frac{\sin(2t)}{2}\right)(-2\sin(2t))$$

Which simplifies to

$$W = \cos(2t)^2 + \sin(2t)^2$$

Which simplifies to

$$W = 1$$

Therefore Eq. (2) becomes

$$u_1 = -\int \frac{\frac{\sin(2t)g(t)}{2}}{1} dt$$

Which simplifies to

$$u_1 = -\int \frac{\sin(2t) g(t)}{2} dt$$

Hence

$$u_1 = -\left(\int_0^t \frac{\sin(2\alpha) g(\alpha)}{2} d\alpha\right)$$

And Eq. (3) becomes

$$u_2 = \int \frac{\cos(2t) g(t)}{1} dt$$

Which simplifies to

$$u_2 = \int \cos(2t) g(t) dt$$

Hence

$$u_2 = \int_0^t \cos(2\alpha) g(\alpha) d\alpha$$

Which simplifies to

$$u_1 = -\frac{\left(\int_0^t \sin(2\alpha) g(\alpha) d\alpha\right)}{2}$$

$$u_2 = \int_0^t \cos(2\alpha) g(\alpha) d\alpha$$

Therefore the particular solution, from equation (1) is

$$y_p(t) = -\frac{\left(\int_0^t \sin(2\alpha) g(\alpha) d\alpha\right) \cos(2t)}{2} + \frac{\left(\int_0^t \cos(2\alpha) g(\alpha) d\alpha\right) \sin(2t)}{2}$$

Therefore the general solution is

$$
\begin{aligned}
y &= y_h + y_p \\
&= \left(c_1 \cos(2t) + \frac{c_2 \sin(2t)}{2}\right) \\
&+ \left(-\frac{\left(\int_0^t \sin(2\alpha)\, g(\alpha)\, d\alpha\right) \cos(2t)}{2} + \frac{\left(\int_0^t \cos(2\alpha)\, g(\alpha)\, d\alpha\right) \sin(2t)}{2}\right)
\end{aligned}
$$

Summary

The solution(s) found are the following

$$
\begin{aligned}
y = c_1 \cos(2t) + \frac{c_2 \sin(2t)}{2} - \frac{\left(\int_0^t \sin(2\alpha)\, g(\alpha)\, d\alpha\right) \cos(2t)}{2} \\
+ \frac{\left(\int_0^t \cos(2\alpha)\, g(\alpha)\, d\alpha\right) \sin(2t)}{2}
\end{aligned} \tag{1}
$$

Verification of solutions

$$
\begin{aligned}
y = c_1 \cos(2t) + \frac{c_2 \sin(2t)}{2} - \frac{\left(\int_0^t \sin(2\alpha)\, g(\alpha)\, d\alpha\right) \cos(2t)}{2} \\
+ \frac{\left(\int_0^t \cos(2\alpha)\, g(\alpha)\, d\alpha\right) \sin(2t)}{2}
\end{aligned}
$$

Verified OK.

10.12.3 Maple step by step solution

Let's solve

$y'' + 4y = g(t)$

- Highest derivative means the order of the ODE is 2

  y''

- Characteristic polynomial of homogeneous ODE

  $r^2 + 4 = 0$

- Use quadratic formula to solve for r

  $r = \frac{0 \pm (\sqrt{-16})}{2}$

- Roots of the characteristic polynomial

$r = (-2\,\mathrm{I}, 2\,\mathrm{I})$

- 1st solution of the homogeneous ODE

  $y_1(t) = \cos(2t)$

- 2nd solution of the homogeneous ODE

  $y_2(t) = \sin(2t)$

- General solution of the ODE

  $y = c_1 y_1(t) + c_2 y_2(t) + y_p(t)$

- Substitute in solutions of the homogeneous ODE

  $y = c_1 \cos(2t) + c_2 \sin(2t) + y_p(t)$

☐ Find a particular solution $y_p(t)$ of the ODE

  ◦ Use variation of parameters to find y_p here $f(t)$ is the forcing function

  $\left[y_p(t) = -y_1(t) \left(\int \frac{y_2(t) f(t)}{W(y_1(t), y_2(t))} dt\right) + y_2(t) \left(\int \frac{y_1(t) f(t)}{W(y_1(t), y_2(t))} dt\right), f(t) = g(t)\right]$

  ◦ Wronskian of solutions of the homogeneous equation

  $W(y_1(t), y_2(t)) = \begin{bmatrix} \cos(2t) & \sin(2t) \\ -2\sin(2t) & 2\cos(2t) \end{bmatrix}$

  ◦ Compute Wronskian

  $W(y_1(t), y_2(t)) = 2$

  ◦ Substitute functions into equation for $y_p(t)$

  $y_p(t) = -\frac{\cos(2t)\left(\int \sin(2t) g(t) dt\right)}{2} + \frac{\sin(2t)\left(\int \cos(2t) g(t) dt\right)}{2}$

  ◦ Compute integrals

  $y_p(t) = -\frac{\cos(2t)\left(\int \sin(2t) g(t) dt\right)}{2} + \frac{\sin(2t)\left(\int \cos(2t) g(t) dt\right)}{2}$

- Substitute particular solution into general solution to ODE

  $y = c_1 \cos(2t) + c_2 \sin(2t) - \frac{\cos(2t)\left(\int \sin(2t) g(t) dt\right)}{2} + \frac{\sin(2t)\left(\int \cos(2t) g(t) dt\right)}{2}$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 2; linear nonhomogeneous with symmetry [0,1]
trying a double symmetry of the form [xi=0, eta=F(x)]
-> Try solving first the homogeneous part of the ODE
   checking if the LODE has constant coefficients
   <- constant coefficients successful
<- solving first the homogeneous part of the ODE successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 47

```
dsolve(diff(y(t),t$2)+4*y(t) = g(t),y(t), singsol=all)
```

$$y(t) = c_2 \sin(2t) + c_1 \cos(2t) + \frac{\left(\int \cos(2t)\, g(t)\, dt\right) \sin(2t)}{2} - \frac{\left(\int \sin(2t)\, g(t)\, dt\right) \cos(2t)}{2}$$

✓ Solution by Mathematica

Time used: 0.093 (sec). Leaf size: 67

```
DSolve[y''[t]+4*y[t] == g[t],y[t],t,IncludeSingularSolutions -> True]
```

$$y(t) \to \cos(2t) \int_1^t -\cos(K[1]) g(K[1]) \sin(K[1]) dK[1] + \sin(2t) \int_1^t \frac{1}{2} \cos(2K[2]) g(K[2]) dK[2] + c_1 \cos(2t) + c_2 \sin(2t)$$

10.13 problem 13

Internal problem ID [695]
Internal file name [`OUTPUT/695_Sunday_June_05_2022_01_47_07_AM_67149286/index.tex`]

Book: Elementary differential equations and boundary value problems, 10th ed., Boyce and DiPrima
Section: Chapter 3, Second order linear equations, section 3.6, Variation of Parameters. page 190
Problem number: 13.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_euler_ode", "exact linear second order ode", "second_order_integrable_as_is"**

Maple gives the following as the ode type

`[[_2nd_order, _exact, _linear, _nonhomogeneous]]`

$$\boxed{t^2y'' - 2y = 3t^2 - 1}$$

10.13.1 Solving as second order euler ode ode

This is second order non-homogeneous ODE. In standard form the ODE is

$$Ay''(t) + By'(t) + Cy(t) = f(t)$$

Where $A = t^2, B = 0, C = -2, f(t) = 3t^2 - 1$. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(t) + By'(t) + Cy(t) = 0$, and y_p is a particular solution to the non-homogeneous ODE $Ay''(t) + By'(t) + Cy(t) = f(t)$. Solving for y_h from

$$t^2y'' - 2y = 0$$

This is Euler second order ODE. Let the solution be $y = t^r$, then $y' = rt^{r-1}$ and $y'' = r(r-1)t^{r-2}$. Substituting these back into the given ODE gives

$$t^2(r(r-1))t^{r-2} + 0rt^{r-1} - 2t^r = 0$$

Simplifying gives

$$r(r-1)\,t^r + 0\,t^r - 2t^r = 0$$

Since $t^r \neq 0$ then dividing throughout by t^r gives

$$r(r-1) + 0 - 2 = 0$$

Or

$$r^2 - r - 2 = 0 \tag{1}$$

Equation (1) is the characteristic equation. Its roots determine the form of the general solution. Using the quadratic equation the roots are

$$r_1 = -1$$
$$r_2 = 2$$

Since the roots are real and distinct, then the general solution is

$$y = c_1 y_1 + c_2 y_2$$

Where $y_1 = t^{r_1}$ and $y_2 = t^{r_2}$. Hence

$$y = \frac{c_1}{t} + c_2 t^2$$

Next, we find the particular solution to the ODE

$$t^2 y'' - 2y = 3t^2 - 1$$

The particular solution y_p can be found using either the method of undetermined coefficients, or the method of variation of parameters. The method of variation of parameters will be used as it is more general and can be used when the coefficients of the ODE depend on t as well. Let

$$y_p(t) = u_1 y_1 + u_2 y_2 \tag{1}$$

Where u_1, u_2 to be determined, and y_1, y_2 are the two basis solutions (the two linearly independent solutions of the homogeneous ODE) found earlier when solving the homogeneous ODE as

$$y_1 = \frac{1}{t}$$
$$y_2 = t^2$$

In the Variation of parameters u_1, u_2 are found using

$$u_1 = -\int \frac{y_2 f(t)}{a W(t)} \tag{2}$$

$$u_2 = \int \frac{y_1 f(t)}{a W(t)} \tag{3}$$

Where $W(t)$ is the Wronskian and a is the coefficient in front of y'' in the given ODE. The Wronskian is given by $W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$. Hence

$$W = \begin{vmatrix} \frac{1}{t} & t^2 \\ \frac{d}{dt}\left(\frac{1}{t}\right) & \frac{d}{dt}(t^2) \end{vmatrix}$$

Which gives

$$W = \begin{vmatrix} \frac{1}{t} & t^2 \\ -\frac{1}{t^2} & 2t \end{vmatrix}$$

Therefore

$$W = \left(\frac{1}{t}\right)(2t) - \left(t^2\right)\left(-\frac{1}{t^2}\right)$$

Which simplifies to

$$W = 3$$

Which simplifies to

$$W = 3$$

Therefore Eq. (2) becomes

$$u_1 = -\int \frac{t^2(3t^2 - 1)}{3t^2}\, dt$$

Which simplifies to

$$u_1 = -\int \left(t^2 - \frac{1}{3}\right) dt$$

Hence

$$u_1 = -\frac{1}{3}t^3 + \frac{1}{3}t$$

And Eq. (3) becomes

$$u_2 = \int \frac{\frac{3t^2-1}{t}}{3t^2}\, dt$$

Which simplifies to

$$u_2 = \int \frac{3t^2 - 1}{3t^3} dt$$

Hence

$$u_2 = \ln(t) + \frac{1}{6t^2}$$

Therefore the particular solution, from equation (1) is

$$y_p(t) = \frac{-\frac{1}{3}t^3 + \frac{1}{3}t}{t} + \left(\ln(t) + \frac{1}{6t^2}\right) t^2$$

Which simplifies to

$$y_p(t) = t^2 \ln(t) - \frac{t^2}{3} + \frac{1}{2}$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= t^2 \ln(t) - \frac{t^2}{3} + \frac{1}{2} + \frac{c_1}{t} + c_2 t^2 \end{aligned}$$

Summary
The solution(s) found are the following

$$y = t^2 \ln(t) - \frac{t^2}{3} + \frac{1}{2} + \frac{c_1}{t} + c_2 t^2 \tag{1}$$

Verification of solutions

$$y = t^2 \ln(t) - \frac{t^2}{3} + \frac{1}{2} + \frac{c_1}{t} + c_2 t^2$$

Verified OK.

10.13.2 Solving as second order integrable as is ode

Integrating both sides of the ODE w.r.t t gives

$$\int (t^2y'' - 2y)\,dt = \int (3t^2 - 1)\,dt$$
$$y't^2 - 2yt = t^3 - t + c_1$$

Which is now solved for y.

Entering Linear first order ODE solver. In canonical form a linear first order is

$$y' + p(t)y = q(t)$$

Where here

$$p(t) = -\frac{2}{t}$$
$$q(t) = \frac{t^3 + c_1 - t}{t^2}$$

Hence the ode is

$$y' - \frac{2y}{t} = \frac{t^3 + c_1 - t}{t^2}$$

The integrating factor μ is

$$\mu = \mathrm{e}^{\int -\frac{2}{t}dt}$$
$$= \frac{1}{t^2}$$

The ode becomes

$$\frac{\mathrm{d}}{\mathrm{d}t}(\mu y) = (\mu)\left(\frac{t^3 + c_1 - t}{t^2}\right)$$
$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{y}{t^2}\right) = \left(\frac{1}{t^2}\right)\left(\frac{t^3 + c_1 - t}{t^2}\right)$$
$$\mathrm{d}\left(\frac{y}{t^2}\right) = \left(\frac{t^3 + c_1 - t}{t^4}\right)\mathrm{d}t$$

Integrating gives

$$\frac{y}{t^2} = \int \frac{t^3 + c_1 - t}{t^4}\,\mathrm{d}t$$
$$\frac{y}{t^2} = \ln(t) - \frac{c_1}{3t^3} + \frac{1}{2t^2} + c_2$$

Dividing both sides by the integrating factor $\mu = \frac{1}{t^2}$ results in

$$y = t^2 \left(\ln(t) - \frac{c_1}{3t^3} + \frac{1}{2t^2}\right) + c_2 t^2$$

Summary
The solution(s) found are the following

$$y = t^2 \left(\ln(t) - \frac{c_1}{3t^3} + \frac{1}{2t^2}\right) + c_2 t^2 \tag{1}$$

Verification of solutions

$$y = t^2 \left(\ln(t) - \frac{c_1}{3t^3} + \frac{1}{2t^2}\right) + c_2 t^2$$

Verified OK.

10.13.3 Solving as type second_order_integrable_as_is (not using ABC version)

Writing the ode as

$$t^2 y'' - 2y = 3t^2 - 1$$

Integrating both sides of the ODE w.r.t t gives

$$\int (t^2 y'' - 2y)\, dt = \int (3t^2 - 1)\, dt$$
$$y' t^2 - 2yt = t^3 - t + c_1$$

Which is now solved for y.

Entering Linear first order ODE solver. In canonical form a linear first order is

$$y' + p(t)y = q(t)$$

Where here

$$p(t) = -\frac{2}{t}$$
$$q(t) = \frac{t^3 + c_1 - t}{t^2}$$

Hence the ode is

$$y' - \frac{2y}{t} = \frac{t^3 + c_1 - t}{t^2}$$

The integrating factor μ is

$$\mu = e^{\int -\frac{2}{t}dt}$$
$$= \frac{1}{t^2}$$

The ode becomes

$$\frac{d}{dt}(\mu y) = (\mu)\left(\frac{t^3 + c_1 - t}{t^2}\right)$$
$$\frac{d}{dt}\left(\frac{y}{t^2}\right) = \left(\frac{1}{t^2}\right)\left(\frac{t^3 + c_1 - t}{t^2}\right)$$
$$d\left(\frac{y}{t^2}\right) = \left(\frac{t^3 + c_1 - t}{t^4}\right) dt$$

Integrating gives

$$\frac{y}{t^2} = \int \frac{t^3 + c_1 - t}{t^4} dt$$
$$\frac{y}{t^2} = \ln(t) - \frac{c_1}{3t^3} + \frac{1}{2t^2} + c_2$$

Dividing both sides by the integrating factor $\mu = \frac{1}{t^2}$ results in

$$y = t^2\left(\ln(t) - \frac{c_1}{3t^3} + \frac{1}{2t^2}\right) + c_2 t^2$$

Summary

The solution(s) found are the following

$$y = t^2\left(\ln(t) - \frac{c_1}{3t^3} + \frac{1}{2t^2}\right) + c_2 t^2 \quad (1)$$

Verification of solutions

$$y = t^2\left(\ln(t) - \frac{c_1}{3t^3} + \frac{1}{2t^2}\right) + c_2 t^2$$

Verified OK.

10.13.4 Solving using Kovacic algorithm

Writing the ode as

$$t^2y'' - 2y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= t^2 \\ B &= 0 \\ C &= -2 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(t) = ye^{\int \frac{B}{2A}\, dt}$$

Then (2) becomes

$$z''(t) = rz(t) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{2}{t^2} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= 2 \\ t &= t^2 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(t) = \left(\frac{2}{t^2}\right) z(t) \tag{7}$$

Equation (7) is now solved. After finding $z(t)$ then y is found using the inverse transformation

$$y = z(t)\, e^{-\int \frac{B}{2A}\, dt}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 498: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 2 - 0 \\ &= 2 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = t^2$. There is a pole at $t = 0$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 2 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Since pole order is not larger than 2 and the order at ∞ is 2 then the necessary conditions for case three are met. Therefore

$$L = [1, 2, 4, 6, 12]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = \frac{2}{t^2}$$

For the pole at $t = 0$ let b be the coefficient of $\frac{1}{t^2}$ in the partial fractions decomposition of r given above. Therefore $b = 2$. Hence

$$
\begin{aligned}
[\sqrt{r}]_c &= 0 \\
\alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = 2 \\
\alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = -1
\end{aligned}
$$

Since the order of r at ∞ is 2 then $[\sqrt{r}]_\infty = 0$. Let b be the coefficient of $\frac{1}{t^2}$ in the Laurent series expansion of r at ∞. which can be found by dividing the leading coefficient of s by the leading coefficient of t from

$$
r = \frac{s}{t} = \frac{2}{t^2}
$$

Since the $\gcd(s,t) = 1$. This gives $b = 2$. Hence

$$
\begin{aligned}
[\sqrt{r}]_\infty &= 0 \\
\alpha_\infty^+ &= \frac{1}{2} + \sqrt{1+4b} = 2 \\
\alpha_\infty^- &= \frac{1}{2} - \sqrt{1+4b} = -1
\end{aligned}
$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$
r = \frac{2}{t^2}
$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
0	2	0	2	-1

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
2	0	2	-1

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$
d = \alpha_\infty^{s(\infty)} - \sum_{c \in \Gamma} \alpha_c^{s(c)}
$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c \in \Gamma \cup \infty}$ until such d is found to work in finding candidate ω.

Trying $\alpha_{\infty}^{-} = -1$ then

$$
\begin{aligned}
d &= \alpha_{\infty}^{-} - \left(\alpha_{c_1}^{-}\right) \\
&= -1 - (-1) \\
&= 0
\end{aligned}
$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$
\omega = \sum_{c \in \Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{t - c} \right) + s(\infty)[\sqrt{r}]_\infty
$$

The above gives

$$
\begin{aligned}
\omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^{-}}{t - c_1} \right) + (-)[\sqrt{r}]_\infty \\
&= -\frac{1}{t} + (-)(0) \\
&= -\frac{1}{t} \\
&= -\frac{1}{t}
\end{aligned}
$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(t)$ of degree $d = 0$ to solve the ode. The polynomial $p(t)$ needs to satisfy the equation

$$
p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right) p = 0 \tag{1A}
$$

Let

$$
p(t) = 1 \tag{2A}
$$

Substituting the above in eq. (1A) gives

$$
\begin{aligned}
(0) + 2\left(-\frac{1}{t}\right)(0) + \left(\left(\frac{1}{t^2}\right) + \left(-\frac{1}{t}\right)^2 - \left(\frac{2}{t^2}\right) \right) &= 0 \\
0 &= 0
\end{aligned}
$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z'' = rz$ is

$$
\begin{aligned}
z_1(t) &= p e^{\int \omega \, dt} \\
&= e^{\int -\frac{1}{t} dt} \\
&= \frac{1}{t}
\end{aligned}
$$

The first solution to the original ode in y is found from

$$y_1 = z_1 e^{\int -\frac{1}{2}\frac{B}{A}\, dt}$$

Since $B = 0$ then the above reduces to

$$\begin{aligned} y_1 &= z_1 \\ &= \frac{1}{t} \end{aligned}$$

Which simplifies to

$$y_1 = \frac{1}{t}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\, dt}}{y_1^2}\, dt$$

Since $B = 0$ then the above becomes

$$\begin{aligned} y_2 &= y_1 \int \frac{1}{y_1^2}\, dt \\ &= \frac{1}{t} \int \frac{1}{\frac{1}{t^2}}\, dt \\ &= \frac{1}{t}\left(\frac{t^3}{3}\right) \end{aligned}$$

Therefore the solution is

$$\begin{aligned} y &= c_1 y_1 + c_2 y_2 \\ &= c_1\left(\frac{1}{t}\right) + c_2\left(\frac{1}{t}\left(\frac{t^3}{3}\right)\right) \end{aligned}$$

This is second order nonhomogeneous ODE. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(t)+By'(t)+Cy(t)=0$, and y_p is a particular solution to the nonhomogeneous ODE $Ay''(t)+By'(t)+Cy(t)=f(t)$. y_h is the solution to

$$t^2y''-2y=0$$

The homogeneous solution is found using the Kovacic algorithm which results in

$$y_h=\frac{c_1}{t}+\frac{c_2t^2}{3}$$

The particular solution y_p can be found using either the method of undetermined coefficients, or the method of variation of parameters. The method of variation of parameters will be used as it is more general and can be used when the coefficients of the ODE depend on t as well. Let

$$y_p(t)=u_1y_1+u_2y_2 \tag{1}$$

Where u_1,u_2 to be determined, and y_1,y_2 are the two basis solutions (the two linearly independent solutions of the homogeneous ODE) found earlier when solving the homogeneous ODE as

$$y_1=\frac{1}{t}$$
$$y_2=\frac{t^2}{3}$$

In the Variation of parameters u_1,u_2 are found using

$$u_1=-\int\frac{y_2f(t)}{aW(t)} \tag{2}$$
$$u_2=\int\frac{y_1f(t)}{aW(t)} \tag{3}$$

Where $W(t)$ is the Wronskian and a is the coefficient in front of y'' in the given ODE. The Wronskian is given by $W=\begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$. Hence

$$W=\begin{vmatrix} \frac{1}{t} & \frac{t^2}{3} \\ \frac{d}{dt}\left(\frac{1}{t}\right) & \frac{d}{dt}\left(\frac{t^2}{3}\right) \end{vmatrix}$$

Which gives

$$W = \begin{vmatrix} \frac{1}{t} & \frac{t^2}{3} \\ -\frac{1}{t^2} & \frac{2t}{3} \end{vmatrix}$$

Therefore

$$W = \left(\frac{1}{t}\right)\left(\frac{2t}{3}\right) - \left(\frac{t^2}{3}\right)\left(-\frac{1}{t^2}\right)$$

Which simplifies to

$$W = 1$$

Which simplifies to

$$W = 1$$

Therefore Eq. (2) becomes

$$u_1 = -\int \frac{\frac{t^2(3t^2-1)}{3}}{t^2}\, dt$$

Which simplifies to

$$u_1 = -\int \left(t^2 - \frac{1}{3}\right) dt$$

Hence

$$u_1 = -\frac{1}{3}t^3 + \frac{1}{3}t$$

And Eq. (3) becomes

$$u_2 = \int \frac{\frac{3t^2-1}{t}}{t^2}\, dt$$

Which simplifies to

$$u_2 = \int \frac{3t^2 - 1}{t^3} dt$$

Hence

$$u_2 = 3\ln(t) + \frac{1}{2t^2}$$

Therefore the particular solution, from equation (1) is

$$y_p(t) = \frac{-\frac{1}{3}t^3 + \frac{1}{3}t}{t} + \frac{\left(3\ln(t) + \frac{1}{2t^2}\right)t^2}{3}$$

Which simplifies to

$$y_p(t) = t^2 \ln(t) - \frac{t^2}{3} + \frac{1}{2}$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \left(\frac{c_1}{t} + \frac{c_2 t^2}{3}\right) + \left(t^2 \ln(t) - \frac{t^2}{3} + \frac{1}{2}\right) \end{aligned}$$

Summary
The solution(s) found are the following

$$y = \frac{c_1}{t} + \frac{c_2 t^2}{3} + t^2 \ln(t) - \frac{t^2}{3} + \frac{1}{2} \tag{1}$$

Verification of solutions

$$y = \frac{c_1}{t} + \frac{c_2 t^2}{3} + t^2 \ln(t) - \frac{t^2}{3} + \frac{1}{2}$$

Verified OK.

10.13.5 Solving as exact linear second order ode ode

An ode of the form

$$p(t)\,y'' + q(t)\,y' + r(t)\,y = s(t)$$

is exact if

$$p''(t) - q'(t) + r(t) = 0 \tag{1}$$

For the given ode we have

$$\begin{aligned} p(x) &= t^2 \\ q(x) &= 0 \\ r(x) &= -2 \\ s(x) &= 3t^2 - 1 \end{aligned}$$

Hence

$$
\begin{aligned}
p''(x) &= 2 \\
q'(x) &= 0
\end{aligned}
$$

Therefore (1) becomes

$$
2 - (0) + (-2) = 0
$$

Hence the ode is exact. Since we now know the ode is exact, it can be written as

$$
(p(t)\, y' + (q(t) - p'(t))\, y)' = s(x)
$$

Integrating gives

$$
p(t)\, y' + (q(t) - p'(t))\, y = \int s(t)\, dt
$$

Substituting the above values for p, q, r, s gives

$$
y't^2 - 2yt = \int 3t^2 - 1\, dt
$$

We now have a first order ode to solve which is

$$
y't^2 - 2yt = t^3 + c_1 - t
$$

Entering Linear first order ODE solver. In canonical form a linear first order is

$$
y' + p(t)y = q(t)
$$

Where here

$$
\begin{aligned}
p(t) &= -\frac{2}{t} \\
q(t) &= \frac{t^3 + c_1 - t}{t^2}
\end{aligned}
$$

Hence the ode is

$$
y' - \frac{2y}{t} = \frac{t^3 + c_1 - t}{t^2}
$$

The integrating factor μ is

$$
\begin{aligned}
\mu &= e^{\int -\frac{2}{t}dt} \\
&= \frac{1}{t^2}
\end{aligned}
$$

The ode becomes

$$
\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}(\mu y) &= (\mu)\left(\frac{t^3+c_1-t}{t^2}\right) \\
\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{y}{t^2}\right) &= \left(\frac{1}{t^2}\right)\left(\frac{t^3+c_1-t}{t^2}\right) \\
\mathrm{d}\left(\frac{y}{t^2}\right) &= \left(\frac{t^3+c_1-t}{t^4}\right)\mathrm{d}t
\end{aligned}
$$

Integrating gives

$$
\begin{aligned}
\frac{y}{t^2} &= \int \frac{t^3+c_1-t}{t^4}\,\mathrm{d}t \\
\frac{y}{t^2} &= \ln(t) - \frac{c_1}{3t^3} + \frac{1}{2t^2} + c_2
\end{aligned}
$$

Dividing both sides by the integrating factor $\mu = \frac{1}{t^2}$ results in

$$
y = t^2\left(\ln(t) - \frac{c_1}{3t^3} + \frac{1}{2t^2}\right) + c_2 t^2
$$

Summary
The solution(s) found are the following

$$
y = t^2\left(\ln(t) - \frac{c_1}{3t^3} + \frac{1}{2t^2}\right) + c_2 t^2 \tag{1}
$$

Verification of solutions

$$
y = t^2\left(\ln(t) - \frac{c_1}{3t^3} + \frac{1}{2t^2}\right) + c_2 t^2
$$

Verified OK.

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
<- high order exact linear fully integrable successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 22

```
dsolve(t^2*diff(y(t),t$2)-2*y(t) = 3*t^2-1,y(t), singsol=all)
```

$$y(t) = t^2 c_2 + \frac{1}{2} + t^2 \ln(t) + \frac{c_1}{t}$$

✓ Solution by Mathematica

Time used: 0.015 (sec). Leaf size: 31

```
DSolve[t^2*y''[t]-2*y[t] == 3*t^2-1,y[t],t,IncludeSingularSolutions -> True]
```

$$y(t) \to t^2 \log(t) + \left(-\frac{1}{3} + c_2\right) t^2 + \frac{c_1}{t} + \frac{1}{2}$$

10.14 problem 14

Internal problem ID [696]
Internal file name [`OUTPUT/696_Sunday_June_05_2022_01_47_08_AM_14861358/index.tex`]

Book: Elementary differential equations and boundary value problems, 10th ed., Boyce and DiPrima
Section: Chapter 3, Second order linear equations, section 3.6, Variation of Parameters. page 190
Problem number: 14.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_change_of_variable_on_y_method_1", "second_order_change_of_variable_on_y_method_2"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

$$\boxed{t^2y'' - t(2+t)\,y' + (2+t)\,y = 2t^3}$$

10.14.1 Solving as second order change of variable on y method 1 ode

This is second order non-homogeneous ODE. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(t) + By'(t) + Cy(t) = 0$, and y_p is a particular solution to the non-homogeneous ODE $Ay''(t) + By'(t) + Cy(t) = f(t)$. y_h is the solution to

$$t^2y'' + \left(-t^2 - 2t\right)y' + (2+t)\,y = 0$$

In normal form the given ode is written as

$$y'' + p(t)\,y' + q(t)\,y = 0 \tag{2}$$

Where

$$
\begin{aligned}
p(t) &= \frac{-t^2-2t}{t^2} \\
q(t) &= \frac{2+t}{t^2}
\end{aligned}
$$

Calculating the Liouville ode invariant Q given by

$$
\begin{aligned}
Q &= q - \frac{p'}{2} - \frac{p^2}{4} \\
&= \frac{2+t}{t^2} - \frac{\left(\frac{-t^2-2t}{t^2}\right)'}{2} - \frac{\left(\frac{-t^2-2t}{t^2}\right)^2}{4} \\
&= \frac{2+t}{t^2} - \frac{\left(\frac{-2t-2}{t^2} - \frac{2(-t^2-2t)}{t^3}\right)}{2} - \frac{\left(\frac{(-t^2-2t)^2}{t^4}\right)}{4} \\
&= \frac{2+t}{t^2} - \left(\frac{-2t-2}{2t^2} - \frac{-t^2-2t}{t^3}\right) - \frac{(-t^2-2t)^2}{4t^4} \\
&= -\frac{1}{4}
\end{aligned}
$$

Since the Liouville ode invariant does not depend on the independent variable t then the transformation

$$
y = v(t)\, z(t) \tag{3}
$$

is used to change the original ode to a constant coefficients ode in v. In (3) the term $z(t)$ is given by

$$
\begin{aligned}
z(t) &= \mathrm{e}^{-\left(\int \frac{p(t)}{2} dt\right)} \\
&= e^{-\int \frac{\frac{-t^2-2t}{t^2}}{2}} \\
&= t\,\mathrm{e}^{\frac{t}{2}} \qquad (5)
\end{aligned}
$$

Hence (3) becomes

$$
y = v(t)\, t\, \mathrm{e}^{\frac{t}{2}} \tag{4}
$$

Applying this change of variable to the original ode results in

$$
-\mathrm{e}^{\frac{t}{2}}\left(-4v''(t) + v(t)\right) = 8
$$

Which is now solved for $v(t)$ Simplyfing the ode gives

$$
v''(t) - \frac{v(t)}{4} = 2\,\mathrm{e}^{-\frac{t}{2}}
$$

This is second order non-homogeneous ODE. In standard form the ODE is

$$Av''(t) + Bv'(t) + Cv(t) = f(t)$$

Where $A = 1, B = 0, C = -\frac{1}{4}, f(t) = 2\,\mathrm{e}^{-\frac{t}{2}}$. Let the solution be

$$v(t) = v_h + v_p$$

Where v_h is the solution to the homogeneous ODE $Av''(t) + Bv'(t) + Cv(t) = 0$, and v_p is a particular solution to the non-homogeneous ODE $Av''(t) + Bv'(t) + Cv(t) = f(t)$. v_h is the solution to

$$v''(t) - \frac{v(t)}{4} = 0$$

This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$Av''(t) + Bv'(t) + Cv(t) = 0$$

Where in the above $A = 1, B = 0, C = -\frac{1}{4}$. Let the solution be $v(t) = e^{\lambda t}$. Substituting this into the ODE gives

$$\lambda^2 \mathrm{e}^{\lambda t} - \frac{\mathrm{e}^{\lambda t}}{4} = 0 \tag{1}$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda t}$ gives

$$\lambda^2 - \frac{1}{4} = 0 \tag{2}$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form.Using the quadratic formula

$$\lambda_{1,2} = \frac{-B}{2A} \pm \frac{1}{2A}\sqrt{B^2 - 4AC}$$

Substituting $A = 1, B = 0, C = -\frac{1}{4}$ into the above gives

$$\begin{aligned}\lambda_{1,2} &= \frac{0}{(2)\,(1)} \pm \frac{1}{(2)\,(1)}\sqrt{0^2 - (4)\,(1)\left(-\frac{1}{4}\right)} \\ &= \pm\frac{1}{2}\end{aligned}$$

Hence

$$\lambda_1 = +\frac{1}{2}$$

$$\lambda_2 = -\frac{1}{2}$$

Which simplifies to

$$\lambda_1 = \frac{1}{2}$$
$$\lambda_2 = -\frac{1}{2}$$

Since roots are real and distinct, then the solution is

$$v(t) = c_1 e^{\lambda_1 t} + c_2 e^{\lambda_2 t}$$
$$v(t) = c_1 e^{\left(\frac{1}{2}\right)t} + c_2 e^{\left(-\frac{1}{2}\right)t}$$

Or

$$v(t) = c_1 \mathrm{e}^{\frac{t}{2}} + c_2 \mathrm{e}^{-\frac{t}{2}}$$

Therefore the homogeneous solution v_h is

$$v_h = c_1 \mathrm{e}^{\frac{t}{2}} + c_2 \mathrm{e}^{-\frac{t}{2}}$$

The particular solution is now found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$2\,\mathrm{e}^{-\frac{t}{2}}$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$\left[\left\{\mathrm{e}^{-\frac{t}{2}}\right\}\right]$$

While the set of the basis functions for the homogeneous solution found earlier is

$$\left\{\mathrm{e}^{-\frac{t}{2}}, \mathrm{e}^{\frac{t}{2}}\right\}$$

Since $\mathrm{e}^{-\frac{t}{2}}$ is duplicated in the UC_set, then this basis is multiplied by extra t. The UC_set becomes

$$\left[\left\{t\,\mathrm{e}^{-\frac{t}{2}}\right\}\right]$$

Since there was duplication between the basis functions in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis function in the above updated UC_set.

$$v_p = A_1 t\,\mathrm{e}^{-\frac{t}{2}}$$

The unknowns $\{A_1\}$ are found by substituting the above trial solution v_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$-A_1 e^{-\frac{t}{2}} = 2\,e^{-\frac{t}{2}}$$

Solving for the unknowns by comparing coefficients results in

$$[A_1 = -2]$$

Substituting the above back in the above trial solution v_p, gives the particular solution

$$v_p = -2t\,e^{-\frac{t}{2}}$$

Therefore the general solution is

$$\begin{aligned} v &= v_h + v_p \\ &= \left(c_1 e^{\frac{t}{2}} + c_2 e^{-\frac{t}{2}}\right) + \left(-2t\,e^{-\frac{t}{2}}\right) \end{aligned}$$

Now that $v(t)$ is known, then

$$\begin{aligned} y &= v(t)\,z(t) \\ &= \left(c_1 e^{\frac{t}{2}} + c_2 e^{-\frac{t}{2}} - 2t\,e^{-\frac{t}{2}}\right)(z(t)) \end{aligned} \tag{7}$$

But from (5)

$$z(t) = t\,e^{\frac{t}{2}}$$

Hence (7) becomes

$$y = \left(c_1 e^{\frac{t}{2}} + c_2 e^{-\frac{t}{2}} - 2t\,e^{-\frac{t}{2}}\right) t\,e^{\frac{t}{2}}$$

Therefore the homogeneous solution y_h is

$$y_h = \left(c_1 e^{\frac{t}{2}} + c_2 e^{-\frac{t}{2}} - 2t\,e^{-\frac{t}{2}}\right) t\,e^{\frac{t}{2}}$$

The particular solution y_p can be found using either the method of undetermined coefficients, or the method of variation of parameters. The method of variation of

parameters will be used as it is more general and can be used when the coefficients of the ODE depend on t as well. Let

$$y_p(t) = u_1 y_1 + u_2 y_2 \tag{1}$$

Where u_1, u_2 to be determined, and y_1, y_2 are the two basis solutions (the two linearly independent solutions of the homogeneous ODE) found earlier when solving the homogeneous ODE as

$$\begin{aligned} y_1 &= t\,\mathrm{e}^{t} \\ y_2 &= \mathrm{e}^{-\frac{t}{2}} t\,\mathrm{e}^{\frac{t}{2}} \end{aligned}$$

In the Variation of parameters u_1, u_2 are found using

$$u_1 = -\int \frac{y_2 f(t)}{aW(t)} \tag{2}$$

$$u_2 = \int \frac{y_1 f(t)}{aW(t)} \tag{3}$$

Where $W(t)$ is the Wronskian and a is the coefficient in front of y'' in the given ODE. The Wronskian is given by $W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$. Hence

$$W = \begin{vmatrix} t\,\mathrm{e}^{t} & \mathrm{e}^{-\frac{t}{2}} t\,\mathrm{e}^{\frac{t}{2}} \\ \frac{d}{dt}(t\,\mathrm{e}^{t}) & \frac{d}{dt}\left(\mathrm{e}^{-\frac{t}{2}} t\,\mathrm{e}^{\frac{t}{2}}\right) \end{vmatrix}$$

Which gives

$$W = \begin{vmatrix} t\,\mathrm{e}^{t} & \mathrm{e}^{-\frac{t}{2}} t\,\mathrm{e}^{\frac{t}{2}} \\ t\,\mathrm{e}^{t} + \mathrm{e}^{t} & \mathrm{e}^{-\frac{t}{2}} \mathrm{e}^{\frac{t}{2}} \end{vmatrix}$$

Therefore

$$W = \left(t\,\mathrm{e}^{t}\right)\left(\mathrm{e}^{-\frac{t}{2}} \mathrm{e}^{\frac{t}{2}}\right) - \left(\mathrm{e}^{-\frac{t}{2}} t\,\mathrm{e}^{\frac{t}{2}}\right)\left(t\,\mathrm{e}^{t} + \mathrm{e}^{t}\right)$$

Which simplifies to

$$W = -\mathrm{e}^{-\frac{t}{2}} \mathrm{e}^{\frac{3t}{2}} t^2$$

Which simplifies to

$$W = -t^2 \mathrm{e}^{t}$$

Therefore Eq. (2) becomes

$$u_1 = -\int \frac{2\,\mathrm{e}^{-\frac{t}{2}} t^4 \mathrm{e}^{\frac{t}{2}}}{-t^4 \mathrm{e}^t}\,dt$$

Which simplifies to

$$u_1 = -\int -2\,\mathrm{e}^{-t} dt$$

Hence

$$u_1 = -2\,\mathrm{e}^{-t}$$

And Eq. (3) becomes

$$u_2 = \int \frac{2t^4 \mathrm{e}^t}{-t^4 \mathrm{e}^t}\,dt$$

Which simplifies to

$$u_2 = \int (-2)\,dt$$

Hence

$$u_2 = -2t$$

Therefore the particular solution, from equation (1) is

$$y_p(t) = -2\,\mathrm{e}^{-t}\mathrm{e}^t t - 2t^2 \mathrm{e}^{-\frac{t}{2}}\mathrm{e}^{\frac{t}{2}}$$

Which simplifies to

$$y_p(t) = -2t^2 - 2t$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \left(\left(c_1 \mathrm{e}^{\frac{t}{2}} + c_2 \mathrm{e}^{-\frac{t}{2}} - 2t\,\mathrm{e}^{-\frac{t}{2}}\right) t\,\mathrm{e}^{\frac{t}{2}}\right) + \left(-2t^2 - 2t\right) \end{aligned}$$

Which simplifies to

$$y = t\left(c_1 \mathrm{e}^t + c_2 - 2t\right) - 2t^2 - 2t$$

Summary
The solution(s) found are the following

$$y = t\left(c_1 e^t + c_2 - 2t\right) - 2t^2 - 2t \tag{1}$$

Verification of solutions

$$y = t\left(c_1 e^t + c_2 - 2t\right) - 2t^2 - 2t$$

Verified OK.

10.14.2 Solving as second order change of variable on y method 2 ode

This is second order non-homogeneous ODE. In standard form the ODE is

$$Ay''(t) + By'(t) + Cy(t) = f(t)$$

Where $A = t^2, B = -t^2 - 2t, C = 2 + t, f(t) = 2t^3$. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(t) + By'(t) + Cy(t) = 0$, and y_p is a particular solution to the non-homogeneous ODE $Ay''(t) + By'(t) + Cy(t) = f(t)$. Solving for y_h from

$$t^2 y'' + \left(-t^2 - 2t\right) y' + (2 + t)\, y = 0$$

In normal form the ode

$$t^2 y'' + \left(-t^2 - 2t\right) y' + (2 + t)\, y = 0 \tag{1}$$

Becomes

$$y'' + p(t)\, y' + q(t)\, y = 0 \tag{2}$$

Where

$$\begin{aligned} p(t) &= \frac{-t-2}{t} \\ q(t) &= \frac{2+t}{t^2} \end{aligned}$$

Applying change of variables on the depndent variable $y = v(t)\, t^n$ to (2) gives the following ode where the dependent variables is $v(t)$ and not y.

$$v''(t) + \left(\frac{2n}{t} + p\right) v'(t) + \left(\frac{n(n-1)}{t^2} + \frac{np}{t} + q\right) v(t) = 0 \tag{3}$$

Let the coefficient of $v(t)$ above be zero. Hence

$$\frac{n(n-1)}{t^2} + \frac{np}{t} + q = 0 \tag{4}$$

Substituting the earlier values found for $p(t)$ and $q(t)$ into (4) gives

$$\frac{n(n-1)}{t^2} + \frac{n(-t-2)}{t^2} + \frac{2+t}{t^2} = 0 \tag{5}$$

Solving (5) for n gives

$$n = 1 \tag{6}$$

Substituting this value in (3) gives

$$\begin{aligned} v''(t) + \left(\frac{2}{t} + \frac{-t-2}{t}\right) v'(t) &= 0 \\ v''(t) - v'(t) &= 0 \end{aligned} \tag{7}$$

Using the substitution

$$u(t) = v'(t)$$

Then (7) becomes

$$u'(t) - u(t) = 0 \tag{8}$$

The above is now solved for $u(t)$. Integrating both sides gives

$$\begin{aligned} \int \frac{1}{u} du &= t + c_1 \\ \ln(u) &= t + c_1 \\ u &= e^{t+c_1} \\ u &= c_1 e^t \end{aligned}$$

Now that $u(t)$ is known, then

$$\begin{aligned} v'(t) &= u(t) \\ v(t) &= \int u(t)\, dt + c_2 \\ &= c_1 e^t + c_2 \end{aligned}$$

Hence

$$
\begin{aligned}
y &= v(t)\, t^n \\
&= \left(c_1 \mathrm{e}^t + c_2\right) t \\
&= \left(c_1 \mathrm{e}^t + c_2\right) t
\end{aligned}
$$

Now the particular solution to this ODE is found

$$
t^2 y'' + \left(-t^2 - 2t\right) y' + (2 + t)\, y = 2t^3
$$

The particular solution y_p can be found using either the method of undetermined coefficients, or the method of variation of parameters. The method of variation of parameters will be used as it is more general and can be used when the coefficients of the ODE depend on t as well. Let

$$
y_p(t) = u_1 y_1 + u_2 y_2 \tag{1}
$$

Where u_1, u_2 to be determined, and y_1, y_2 are the two basis solutions (the two linearly independent solutions of the homogeneous ODE) found earlier when solving the homogeneous ODE as

$$
\begin{aligned}
y_1 &= t \\
y_2 &= t\, \mathrm{e}^t
\end{aligned}
$$

In the Variation of parameters u_1, u_2 are found using

$$
u_1 = -\int \frac{y_2 f(t)}{a W(t)} \tag{2}
$$

$$
u_2 = \int \frac{y_1 f(t)}{a W(t)} \tag{3}
$$

Where $W(t)$ is the Wronskian and a is the coefficient in front of y'' in the given ODE. The Wronskian is given by $W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$. Hence

$$
W = \begin{vmatrix} t & t\, \mathrm{e}^t \\ \frac{d}{dt}(t) & \frac{d}{dt}(t\, \mathrm{e}^t) \end{vmatrix}
$$

Which gives

$$W = \begin{vmatrix} t & t\,\mathrm{e}^{t} \\ 1 & t\,\mathrm{e}^{t}+\mathrm{e}^{t} \end{vmatrix}$$

Therefore

$$W = (t)\left(t\,\mathrm{e}^{t}+\mathrm{e}^{t}\right)-\left(t\,\mathrm{e}^{t}\right)(1)$$

Which simplifies to

$$W = t^{2}\mathrm{e}^{t}$$

Which simplifies to

$$W = t^{2}\mathrm{e}^{t}$$

Therefore Eq. (2) becomes

$$u_1 = -\int \frac{2t^{4}\mathrm{e}^{t}}{t^{4}\mathrm{e}^{t}}\,dt$$

Which simplifies to

$$u_1 = -\int 2dt$$

Hence

$$u_1 = -2t$$

And Eq. (3) becomes

$$u_2 = \int \frac{2t^{4}}{t^{4}\mathrm{e}^{t}}\,dt$$

Which simplifies to

$$u_2 = \int 2\,\mathrm{e}^{-t}dt$$

Hence

$$u_2 = -2\,\mathrm{e}^{-t}$$

Therefore the particular solution, from equation (1) is

$$y_p(t) = -2t^{2}-2\,\mathrm{e}^{-t}\mathrm{e}^{t}t$$

Which simplifies to

$$y_p(t) = -2t^2 - 2t$$

Therefore the general solution is

$$\begin{aligned}
y &= y_h + y_p \\
&= \left(\left(c_1 \mathrm{e}^t + c_2\right) t\right) + \left(-2t^2 - 2t\right) \\
&= -2t^2 - 2t + \left(c_1 \mathrm{e}^t + c_2\right) t
\end{aligned}$$

Which simplifies to

$$y = t\left(c_1 \mathrm{e}^t + c_2 - 2t - 2\right)$$

Summary

The solution(s) found are the following

$$y = t\left(c_1 \mathrm{e}^t + c_2 - 2t - 2\right) \tag{1}$$

Verification of solutions

$$y = t\left(c_1 \mathrm{e}^t + c_2 - 2t - 2\right)$$

Verified OK.

10.14.3 Solving using Kovacic algorithm

Writing the ode as

$$t^2 y'' + \left(-t^2 - 2t\right) y' + (2 + t) y = 0 \tag{1}$$

$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned}
A &= t^2 \\
B &= -t^2 - 2t \\
C &= 2 + t
\end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(t) = y e^{\int \frac{B}{2A} \, dt}$$

Then (2) becomes

$$z''(t) = rz(t) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{1}{4} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= 1 \\ t &= 4 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(t) = \frac{z(t)}{4} \tag{7}$$

Equation (7) is now solved. After finding $z(t)$ then y is found using the inverse transformation

$$y = z(t)\, e^{-\int \frac{B}{2A}\, dt}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 499: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 0 - 0 \\ &= 0 \end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Since $r = \frac{1}{4}$ is not a function of t, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$z_1(t) = e^{-\frac{t}{2}}$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$\begin{aligned} y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dt} \\ &= z_1 e^{-\int \frac{1}{2}\frac{-t^2-2t}{t^2}\,dt} \\ &= z_1 e^{\frac{t}{2}+\ln(t)} \\ &= z_1\left(t\,e^{\frac{t}{2}}\right) \end{aligned}$$

Which simplifies to

$$y_1 = t$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dt}}{y_1^2}\,dt$$

Substituting gives

$$\begin{aligned} y_2 &= y_1 \int \frac{e^{\int -\frac{-t^2-2t}{t^2}\,dt}}{(y_1)^2}\,dt \\ &= y_1 \int \frac{e^{t+2\ln(t)}}{(y_1)^2}\,dt \\ &= y_1\left(e^{t}\right) \end{aligned}$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1(t) + c_2\left(t\left(\mathrm{e}^{t}\right)\right)
\end{aligned}
$$

This is second order nonhomogeneous ODE. Let the solution be

$$
y = y_h + y_p
$$

Where y_h is the solution to the homogeneous ODE $Ay''(t)+By'(t)+Cy(t)=0$, and y_p is a particular solution to the nonhomogeneous ODE $Ay''(t)+By'(t)+Cy(t)=f(t)$. y_h is the solution to

$$
t^2 y'' + \left(-t^2 - 2t\right) y' + (2+t)\, y = 0
$$

The homogeneous solution is found using the Kovacic algorithm which results in

$$
y_h = c_1 t + c_2 t\,\mathrm{e}^{t}
$$

The particular solution y_p can be found using either the method of undetermined coefficients, or the method of variation of parameters. The method of variation of parameters will be used as it is more general and can be used when the coefficients of the ODE depend on t as well. Let

$$
y_p(t) = u_1 y_1 + u_2 y_2 \tag{1}
$$

Where u_1, u_2 to be determined, and y_1, y_2 are the two basis solutions (the two linearly independent solutions of the homogeneous ODE) found earlier when solving the homogeneous ODE as

$$
\begin{aligned}
y_1 &= t \\
y_2 &= t\,\mathrm{e}^{t}
\end{aligned}
$$

In the Variation of parameters u_1, u_2 are found using

$$
u_1 = -\int \frac{y_2 f(t)}{aW(t)} \tag{2}
$$

$$
u_2 = \int \frac{y_1 f(t)}{aW(t)} \tag{3}
$$

Where $W(t)$ is the Wronskian and a is the coefficient in front of y'' in the given ODE.

The Wronskian is given by $W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$. Hence

$$W = \begin{vmatrix} t & t\,e^{t} \\ \frac{d}{dt}(t) & \frac{d}{dt}(t\,e^{t}) \end{vmatrix}$$

Which gives

$$W = \begin{vmatrix} t & t\,e^{t} \\ 1 & t\,e^{t} + e^{t} \end{vmatrix}$$

Therefore

$$W = (t)\left(t\,e^{t} + e^{t}\right) - \left(t\,e^{t}\right)(1)$$

Which simplifies to

$$W = t^{2}e^{t}$$

Which simplifies to

$$W = t^{2}e^{t}$$

Therefore Eq. (2) becomes

$$u_1 = -\int \frac{2t^{4}e^{t}}{t^{4}e^{t}}\,dt$$

Which simplifies to

$$u_1 = -\int 2dt$$

Hence

$$u_1 = -2t$$

And Eq. (3) becomes

$$u_2 = \int \frac{2t^{4}}{t^{4}e^{t}}\,dt$$

Which simplifies to

$$u_2 = \int 2\,e^{-t}dt$$

Hence

$$u_2 = -2\,\mathrm{e}^{-t}$$

Therefore the particular solution, from equation (1) is

$$y_p(t) = -2t^2 - 2\,\mathrm{e}^{-t}\mathrm{e}^t t$$

Which simplifies to

$$y_p(t) = -2t^2 - 2t$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \left(c_1 t + c_2 t\,\mathrm{e}^t\right) + \left(-2t^2 - 2t\right) \end{aligned}$$

Which simplifies to

$$y = t\left(c_1 + c_2 \mathrm{e}^t\right) - 2t^2 - 2t$$

Summary

The solution(s) found are the following

$$y = t\left(c_1 + c_2 \mathrm{e}^t\right) - 2t^2 - 2t \tag{1}$$

Verification of solutions

$$y = t\left(c_1 + c_2 \mathrm{e}^t\right) - 2t^2 - 2t$$

Verified OK.

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 2; linear nonhomogeneous with symmetry [0,1]
trying a double symmetry of the form [xi=0, eta=F(x)]
-> Try solving first the homogeneous part of the ODE
   checking if the LODE has constant coefficients
   checking if the LODE is of Euler type
   trying a symmetry of the form [xi=0, eta=F(x)]
   checking if the LODE is missing y
   -> Trying a Liouvillian solution using Kovacics algorithm
      A Liouvillian solution exists
      Reducible group (found an exponential solution)
      Reducible group (found another exponential solution)
   <- Kovacics algorithm successful
<- solving first the homogeneous part of the ODE successful`
```

✓ Solution by Maple

Time used: 0.281 (sec). Leaf size: 15

```
dsolve(t^2*diff(y(t),t$2)-t*(t+2)*diff(y(t),t)+(t+2)*y(t) = 2*t^3,y(t), singsol=all)
```

$$y(t) = t\left(e^{t} c_1 + c_2 - 2t\right)$$

✓ Solution by Mathematica

Time used: 0.025 (sec). Leaf size: 20

```
DSolve[t^2*y''[t]-t*(t+2)*y'[t]+(t+2)*y[t] == 2*t^3,y[t],t,IncludeSingularSolutions -> True]
```

$$y(t) \to t\left(-2t + c_2 e^t - 2 + c_1\right)$$

10.15 problem 15

Internal problem ID [697]
Internal file name [`OUTPUT/697_Sunday_June_05_2022_01_47_10_AM_66662417/index.tex`]

Book: Elementary differential equations and boundary value problems, 10th ed., Boyce and DiPrima
Section: Chapter 3, Second order linear equations, section 3.6, Variation of Parameters. page 190
Problem number: 15.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_ode_non_constant_coeff_transformation_on_B"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

$$\boxed{ty'' - (t+1)\,y' + y = \mathrm{e}^{2t}t^2}$$

10.15.1 Solving as second order ode non constant coeff transformation on B ode

Given an ode of the form

$$Ay'' + By' + Cy = F(t)$$

This method reduces the order ode the ODE by one by applying the transformation

$$y = Bv$$

This results in

$$\begin{aligned}
y' &= B'v + v'B \\
y'' &= B''v + B'v' + v''B + v'B' \\
&= v''B + 2v' + B' + B''v
\end{aligned}$$

And now the original ode becomes

$$
\begin{aligned}
A(v''B+2v'B'+B''v)+B(B'v+v'B)+CBv&=0\\
ABv''+\left(2AB'+B^2\right)v'+(AB''+BB'+CB)\,v&=0 \qquad (1)
\end{aligned}
$$

If the term $AB''+BB'+CB$ is zero, then this method works and can be used to solve

$$
ABv''+\left(2AB'+B^2\right)v'=0
$$

By Using $u=v'$ which reduces the order of the above ode to one. The new ode is

$$
ABu'+\left(2AB'+B^2\right)u=0
$$

The above ode is first order ode which is solved for u. Now a new ode $v'=u$ is solved for v as first order ode. Then the final solution is obtain from $y=Bv$.

This method works only if the term $AB''+BB'+CB$ is zero. The given ODE shows that

$$
\begin{aligned}
A&=t\\
B&=-t-1\\
C&=1\\
F&=\mathrm{e}^{2t}t^2
\end{aligned}
$$

The above shows that for this ode

$$
\begin{aligned}
AB''+BB'+CB&=(t)\,(0)+(-t-1)\,(-1)+(1)\,(-t-1)\\
&=0
\end{aligned}
$$

Hence the ode in v given in (1) now simplifies to

$$
-t(t+1)\,v''+\left(t^2+1\right)v'=0
$$

Now by applying $v'=u$ the above becomes

$$
-t(t+1)\,u'(t)+\left(t^2+1\right)u(t)=0
$$

Which is now solved for u. In canonical form the ODE is

$$
\begin{aligned}
u'&=F(t,u)\\
&=f(t)g(u)\\
&=\frac{(t^2+1)\,u}{t\,(t+1)}
\end{aligned}
$$

Where $f(t) = \frac{t^2+1}{t(t+1)}$ and $g(u) = u$. Integrating both sides gives

$$
\begin{aligned}
\frac{1}{u}\, du &= \frac{t^2+1}{t\,(t+1)}\, dt \\
\int \frac{1}{u}\, du &= \int \frac{t^2+1}{t\,(t+1)}\, dt \\
\ln(u) &= t + \ln(t) - 2\ln(t+1) + c_1 \\
u &= \mathrm{e}^{t+\ln(t)-2\ln(t+1)+c_1} \\
&= c_1 \mathrm{e}^{t+\ln(t)-2\ln(t+1)}
\end{aligned}
$$

Which simplifies to

$$
u(t) = \frac{c_1 \mathrm{e}^t t}{(t+1)^2}
$$

The ode for v now becomes

$$
\begin{aligned}
v' &= u \\
&= \frac{c_1 \mathrm{e}^t t}{(t+1)^2}
\end{aligned}
$$

Which is now solved for v. Integrating both sides gives

$$
\begin{aligned}
v(t) &= \int \frac{c_1 \mathrm{e}^t t}{(t+1)^2}\, \mathrm{d}t \\
&= \frac{c_1 \mathrm{e}^t}{t+1} + c_2
\end{aligned}
$$

Therefore the homogeneous solution is

$$
\begin{aligned}
y_h(t) &= Bv \\
&= (-t-1)\left(\frac{c_1 \mathrm{e}^t}{t+1} + c_2\right) \\
&= -c_1 \mathrm{e}^t - c_2(t+1)
\end{aligned}
$$

And now the particular solution $y_p(t)$ will be found. The particular solution y_p can be found using either the method of undetermined coefficients, or the method of variation of parameters. The method of variation of parameters will be used as it is more general and can be used when the coefficients of the ODE depend on t as well. Let

$$
y_p(t) = u_1 y_1 + u_2 y_2 \tag{1}
$$

Where u_1, u_2 to be determined, and y_1, y_2 are the two basis solutions (the two linearly independent solutions of the homogeneous ODE) found earlier when solving the homogeneous ODE as

$$
\begin{aligned}
y_1 &= -t - 1 \\
y_2 &= \mathrm{e}^t
\end{aligned}
$$

In the Variation of parameters u_1, u_2 are found using

$$u_1 = -\int \frac{y_2 f(t)}{a W(t)} \tag{2}$$

$$u_2 = \int \frac{y_1 f(t)}{a W(t)} \tag{3}$$

Where $W(t)$ is the Wronskian and a is the coefficient in front of y'' in the given ODE. The Wronskian is given by $W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$. Hence

$$W = \begin{vmatrix} -t - 1 & \mathrm{e}^t \\ \frac{d}{dt}(-t - 1) & \frac{d}{dt}(\mathrm{e}^t) \end{vmatrix}$$

Which gives

$$W = \begin{vmatrix} -t - 1 & \mathrm{e}^t \\ -1 & \mathrm{e}^t \end{vmatrix}$$

Therefore

$$W = (-t - 1)(\mathrm{e}^t) - (\mathrm{e}^t)(-1)$$

Which simplifies to

$$W = -t\,\mathrm{e}^t$$

Which simplifies to

$$W = -t\,\mathrm{e}^t$$

Therefore Eq. (2) becomes

$$u_1 = -\int \frac{\mathrm{e}^t \mathrm{e}^{2t} t^2}{-t^2 \mathrm{e}^t}\, dt$$

Which simplifies to

$$u_1 = -\int -e^{2t} dt$$

Hence

$$u_1 = \frac{e^{2t}}{2}$$

And Eq. (3) becomes

$$u_2 = \int \frac{(-t-1)\, e^{2t} t^2}{-t^2 e^t}\, dt$$

Which simplifies to

$$u_2 = \int e^t (t+1)\, dt$$

Hence

$$u_2 = t\, e^t$$

Therefore the particular solution, from equation (1) is

$$y_p(t) = \frac{(-t-1)\, e^{2t}}{2} + e^{2t} t$$

Which simplifies to

$$y_p(t) = \frac{e^{2t}(-1+t)}{2}$$

Hence the complete solution is

$$\begin{aligned} y(t) &= y_h + y_p \\ &= \left(-c_1 e^t - c_2(t+1)\right) + \left(\frac{e^{2t}(-1+t)}{2}\right) \\ &= -c_1 e^t - c_2(t+1) + \frac{e^{2t}(-1+t)}{2} \end{aligned}$$

Summary
The solution(s) found are the following

$$y = -c_1 e^t - c_2(t+1) + \frac{e^{2t}(-1+t)}{2} \tag{1}$$

Verification of solutions

$$y = -c_1 e^t - c_2(t+1) + \frac{e^{2t}(-1+t)}{2}$$

Verified OK.

10.15.2 Solving using Kovacic algorithm

Writing the ode as

$$ty'' + (-t-1)\,y' + y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= t \\ B &= -t-1 \\ C &= 1 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(t) = ye^{\int \frac{B}{2A}\,dt}$$

Then (2) becomes

$$z''(t) = rz(t) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{t^2 - 2t + 3}{4t^2} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= t^2 - 2t + 3 \\ t &= 4t^2 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(t) = \left(\frac{t^2 - 2t + 3}{4t^2}\right) z(t) \tag{7}$$

Equation (7) is now solved. After finding $z(t)$ then y is found using the inverse transformation

$$y = z(t)\, e^{-\int \frac{B}{2A}\, dt}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 500: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 2 - 2 \\ &= 0 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 4t^2$. There is a pole at $t = 0$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Therefore

$$L = [1, 2]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = \frac{1}{4} - \frac{1}{2t} + \frac{3}{4t^2}$$

For the pole at $t = 0$ let b be the coefficient of $\frac{1}{t^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{3}{4}$. Hence

$$\begin{aligned}
[\sqrt{r}]_c &= 0 \\
\alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{3}{2} \\
\alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = -\frac{1}{2}
\end{aligned}$$

Since the order of r at ∞ is $O_r(\infty) = 0$ then

$$v = \frac{-O_r(\infty)}{2} = \frac{0}{2} = 0$$

$[\sqrt{r}]_\infty$ is the sum of terms involving t^i for $0 \leq i \leq v$ in the Laurent series for $\sqrt{r}$ at ∞. Therefore

$$\begin{aligned}
[\sqrt{r}]_\infty &= \sum_{i=0}^{v} a_i t^i \\
&= \sum_{i=0}^{0} a_i t^i \qquad (8)
\end{aligned}$$

Let a be the coefficient of $t^v = t^0$ in the above sum. The Laurent series of $\sqrt{r}$ at ∞ is

$$\sqrt{r} \approx \frac{1}{2} - \frac{1}{2t} + \frac{1}{2t^2} + \frac{1}{2t^3} + \frac{1}{4t^4} - \frac{1}{4t^5} - \frac{3}{4t^6} - \frac{3}{4t^7} + \dots \qquad (9)$$

Comparing Eq. (9) with Eq. (8) shows that

$$a = \frac{1}{2}$$

From Eq. (9) the sum up to $v = 0$ gives

$$\begin{aligned}
[\sqrt{r}]_\infty &= \sum_{i=0}^{0} a_i t^i \\
&= \frac{1}{2} \qquad (10)
\end{aligned}$$

Now we need to find b, where b be the coefficient of $t^{v-1} = t^{-1} = \frac{1}{t}$ in r minus the coefficient of same term but in $\left([\sqrt{r}]_\infty\right)^2$ where $[\sqrt{r}]_\infty$ was found above in Eq (10). Hence

$$\left([\sqrt{r}]_\infty\right)^2 = \frac{1}{4}$$

This shows that the coefficient of $\frac{1}{t}$ in the above is 0. Now we need to find the coefficient of $\frac{1}{t}$ in r. How this is done depends on if $v = 0$ or not. Since $v = 0$ then starting from $r = \frac{s}{t}$ and doing long division in the form

$$r = Q + \frac{R}{t}$$

Where Q is the quotient and R is the remainder. Then the coefficient of $\frac{1}{t}$ in r will be the coefficient in R of the term in t of degree of t minus one, divided by the leading coefficient in t. Doing long division gives

$$\begin{aligned}
r &= \frac{s}{t} \\
&= \frac{t^2 - 2t + 3}{4t^2} \\
&= Q + \frac{R}{4t^2} \\
&= \left(\frac{1}{4}\right) + \left(\frac{-2t + 3}{4t^2}\right) \\
&= \frac{1}{4} + \frac{-2t + 3}{4t^2}
\end{aligned}$$

Since the degree of t is 2, then we see that the coefficient of the term t in the remainder R is -2. Dividing this by leading coefficient in t which is 4 gives $-\frac{1}{2}$. Now b can be found.

$$\begin{aligned}
b &= \left(-\frac{1}{2}\right) - (0) \\
&= -\frac{1}{2}
\end{aligned}$$

Hence

$$\begin{aligned}
[\sqrt{r}]_\infty &= \frac{1}{2} \\
\alpha_\infty^{+} &= \frac{1}{2}\left(\frac{b}{a} - v\right) &= \frac{1}{2}\left(\frac{-\frac{1}{2}}{\frac{1}{2}} - 0\right) &= -\frac{1}{2} \\
\alpha_\infty^{-} &= \frac{1}{2}\left(-\frac{b}{a} - v\right) &= \frac{1}{2}\left(-\frac{-\frac{1}{2}}{\frac{1}{2}} - 0\right) &= \frac{1}{2}
\end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = \frac{t^2 - 2t + 3}{4t^2}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
0	2	0	$\frac{3}{2}$	$-\frac{1}{2}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
0	$\frac{1}{2}$	$-\frac{1}{2}$	$\frac{1}{2}$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c \in \Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c \in \Gamma \cup \infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^+ = -\frac{1}{2}$ then

$$\begin{aligned} d &= \alpha_\infty^+ - (\alpha_{c_1}^-) \\ &= -\frac{1}{2} - \left(-\frac{1}{2}\right) \\ &= 0 \end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c \in \Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{t - c} \right) + s(\infty)[\sqrt{r}]_\infty$$

Substituting the above values in the above results in

$$\begin{aligned} \omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^-}{t - c_1} \right) + (+)[\sqrt{r}]_\infty \\ &= -\frac{1}{2t} + \left(\frac{1}{2}\right) \\ &= \frac{1}{2} - \frac{1}{2t} \\ &= \frac{-1 + t}{2t} \end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(t)$ of degree $d = 0$ to solve the ode. The polynomial $p(t)$ needs to satisfy the equation

$$p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right) p = 0 \tag{1A}$$

Let

$$p(t) = 1 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$(0) + 2\left(\frac{1}{2} - \frac{1}{2t}\right)(0) + \left(\left(\frac{1}{2t^2}\right) + \left(\frac{1}{2} - \frac{1}{2t}\right)^2 - \left(\frac{t^2 - 2t + 3}{4t^2}\right)\right) = 0$$
$$0 = 0$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned} z_1(t) &= p e^{\int \omega \, dt} \\ &= e^{\int \left(\frac{1}{2} - \frac{1}{2t}\right) dt} \\ &= \frac{e^{\frac{t}{2}}}{\sqrt{t}} \end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned} y_1 &= z_1 e^{\int -\frac{1}{2} \frac{B}{A} \, dt} \\ &= z_1 e^{-\int \frac{1}{2} \frac{-t-1}{t} \, dt} \\ &= z_1 e^{\frac{t}{2} + \frac{\ln(t)}{2}} \\ &= z_1 \left(\sqrt{t}\, e^{\frac{t}{2}}\right) \end{aligned}$$

Which simplifies to

$$y_1 = e^t$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A} \, dt}}{y_1^2} \, dt$$

Substituting gives

$$
\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{-t-1}{t} dt}}{(y_1)^2} dt \\
&= y_1 \int \frac{e^{t+\ln(t)}}{(y_1)^2} dt \\
&= y_1 \left(-\mathrm{e}^{-t}(t+1)\right)
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1 \left(\mathrm{e}^t\right) + c_2 \left(\mathrm{e}^t \left(-\mathrm{e}^{-t}(t+1)\right)\right)
\end{aligned}
$$

This is second order nonhomogeneous ODE. Let the solution be

$$
y = y_h + y_p
$$

Where y_h is the solution to the homogeneous ODE $Ay''(t) + By'(t) + Cy(t) = 0$, and y_p is a particular solution to the nonhomogeneous ODE $Ay''(t) + By'(t) + Cy(t) = f(t)$. y_h is the solution to

$$
ty'' + (-t-1)\, y' + y = 0
$$

The homogeneous solution is found using the Kovacic algorithm which results in

$$
y_h = c_1 \mathrm{e}^t + (-t-1)\, c_2
$$

The particular solution y_p can be found using either the method of undetermined coefficients, or the method of variation of parameters. The method of variation of parameters will be used as it is more general and can be used when the coefficients of the ODE depend on t as well. Let

$$
y_p(t) = u_1 y_1 + u_2 y_2 \tag{1}
$$

Where u_1, u_2 to be determined, and y_1, y_2 are the two basis solutions (the two linearly independent solutions of the homogeneous ODE) found earlier when solving the homogeneous ODE as

$$
\begin{aligned}
y_1 &= \mathrm{e}^t \\
y_2 &= -t-1
\end{aligned}
$$

In the Variation of parameters u_1, u_2 are found using

$$u_1 = -\int \frac{y_2 f(t)}{a W(t)} \tag{2}$$

$$u_2 = \int \frac{y_1 f(t)}{a W(t)} \tag{3}$$

Where $W(t)$ is the Wronskian and a is the coefficient in front of y'' in the given ODE. The Wronskian is given by $W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$. Hence

$$W = \begin{vmatrix} e^t & -t-1 \\ \frac{d}{dt}(e^t) & \frac{d}{dt}(-t-1) \end{vmatrix}$$

Which gives

$$W = \begin{vmatrix} e^t & -t-1 \\ e^t & -1 \end{vmatrix}$$

Therefore

$$W = \left(e^t\right)(-1) - (-t-1)\left(e^t\right)$$

Which simplifies to

$$W = t\,e^t$$

Which simplifies to

$$W = t\,e^t$$

Therefore Eq. (2) becomes

$$u_1 = -\int \frac{(-t-1)\,e^{2t} t^2}{t^2 e^t}\,dt$$

Which simplifies to

$$u_1 = -\int -e^t (t+1)\,dt$$

Hence

$$u_1 = t\,e^t$$

And Eq. (3) becomes

$$u_2 = \int \frac{e^t e^{2t} t^2}{t^2 e^t}\, dt$$

Which simplifies to

$$u_2 = \int e^{2t} dt$$

Hence

$$u_2 = \frac{e^{2t}}{2}$$

Therefore the particular solution, from equation (1) is

$$y_p(t) = \frac{(-t-1)\, e^{2t}}{2} + e^{2t} t$$

Which simplifies to

$$y_p(t) = \frac{e^{2t}(-1+t)}{2}$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= (c_1 e^t + (-t-1)\, c_2) + \left(\frac{e^{2t}(-1+t)}{2}\right) \end{aligned}$$

Which simplifies to

$$y = c_1 e^t - c_2(t+1) + \frac{e^{2t}(-1+t)}{2}$$

Summary

The solution(s) found are the following

$$y = c_1 e^t - c_2(t+1) + \frac{e^{2t}(-1+t)}{2} \tag{1}$$

Verification of solutions

$$y = c_1 e^t - c_2(t+1) + \frac{e^{2t}(-1+t)}{2}$$

Verified OK.

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 2; linear nonhomogeneous with symmetry [0,1]
trying a double symmetry of the form [xi=0, eta=F(x)]
-> Try solving first the homogeneous part of the ODE
   checking if the LODE has constant coefficients
   checking if the LODE is of Euler type
   trying a symmetry of the form [xi=0, eta=F(x)]
   checking if the LODE is missing y
   -> Trying a Liouvillian solution using Kovacics algorithm
      A Liouvillian solution exists
      Reducible group (found an exponential solution)
      Reducible group (found another exponential solution)
   <- Kovacics algorithm successful
<- solving first the homogeneous part of the ODE successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 23

```
dsolve(t*diff(y(t),t$2)-(1+t)*diff(y(t),t)+y(t) = t^2*exp(2*t),y(t), singsol=all)
```

$$y(t) = (t+1)\, c_2 + \mathrm{e}^t c_1 + \frac{(t-1)\, \mathrm{e}^{2t}}{2}$$

✓ Solution by Mathematica

Time used: 0.052 (sec). Leaf size: 31

```
DSolve[t*y''[t]-(1+t)*y'[t]+y[t] == t^2*Exp[2*t],y[t],t,IncludeSingularSolutions -> True]
```

$$y(t) \to \frac{1}{2} e^{2t}(t-1) + c_1 e^t - c_2(t+1)$$

10.16 problem 16

Internal problem ID [698]
Internal file name [`OUTPUT/698_Sunday_June_05_2022_01_47_12_AM_63605988/index.tex`]

Book: Elementary differential equations and boundary value problems, 10th ed., Boyce and DiPrima
Section: Chapter 3, Second order linear equations, section 3.6, Variation of Parameters. page 190
Problem number: 16.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_change_of_variable_on_y_method_2", "second_order_ode_non_constant_coeff_transformation_on_B"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

$$\boxed{(1-t)\, y'' + ty' - y = 2(-1+t)^2 \,\mathrm{e}^{-t}}$$

10.16.1 Solving as second order change of variable on y method 2 ode

This is second order non-homogeneous ODE. In standard form the ODE is

$$Ay''(t) + By'(t) + Cy(t) = f(t)$$

Where $A = 1-t, B = t, C = -1, f(t) = 2(-1+t)^2 \,\mathrm{e}^{-t}$. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(t) + By'(t) + Cy(t) = 0$, and y_p is a particular solution to the non-homogeneous ODE $Ay''(t) + By'(t) + Cy(t) = f(t)$. Solving for y_h from

$$(1-t)\, y'' + ty' - y = 0$$

In normal form the ode

$$(1-t)\,y''+ty'-y=0 \tag{1}$$

Becomes

$$y''+p(t)\,y'+q(t)\,y=0 \tag{2}$$

Where

$$\begin{aligned} p(t)&=-\frac{t}{-1+t}\\ q(t)&=\frac{1}{-1+t} \end{aligned}$$

Applying change of variables on the depndent variable $y=v(t)\,t^n$ to (2) gives the following ode where the dependent variables is $v(t)$ and not y.

$$v''(t)+\left(\frac{2n}{t}+p\right)v'(t)+\left(\frac{n(n-1)}{t^2}+\frac{np}{t}+q\right)v(t)=0 \tag{3}$$

Let the coefficient of $v(t)$ above be zero. Hence

$$\frac{n(n-1)}{t^2}+\frac{np}{t}+q=0 \tag{4}$$

Substituting the earlier values found for $p(t)$ and $q(t)$ into (4) gives

$$\frac{n(n-1)}{t^2}-\frac{n}{-1+t}+\frac{1}{-1+t}=0 \tag{5}$$

Solving (5) for n gives

$$n=1 \tag{6}$$

Substituting this value in (3) gives

$$\begin{aligned} v''(t)+\left(\frac{2}{t}-\frac{t}{-1+t}\right)v'(t)&=0\\ v''(t)+\left(\frac{2}{t}-\frac{t}{-1+t}\right)v'(t)&=0 \end{aligned} \tag{7}$$

Using the substitution

$$u(t)=v'(t)$$

Then (7) becomes

$$u'(t)+\left(\frac{2}{t}-\frac{t}{-1+t}\right)u(t)=0 \tag{8}$$

The above is now solved for $u(t)$. In canonical form the ODE is

$$\begin{aligned}
u' &= F(t,u)\\
&= f(t)g(u)\\
&= \frac{u(t^2-2t+2)}{t\,(-1+t)}
\end{aligned}$$

Where $f(t)=\frac{t^2-2t+2}{t(-1+t)}$ and $g(u)=u$. Integrating both sides gives

$$\begin{aligned}
\frac{1}{u}\,du &= \frac{t^2-2t+2}{t\,(-1+t)}\,dt\\
\int\frac{1}{u}\,du &= \int\frac{t^2-2t+2}{t\,(-1+t)}\,dt\\
\ln(u) &= t-2\ln(t)+\ln(-1+t)+c_1\\
u &= e^{t-2\ln(t)+\ln(-1+t)+c_1}\\
&= c_1 e^{t-2\ln(t)+\ln(-1+t)}
\end{aligned}$$

Which simplifies to

$$u(t)=c_1\left(-\frac{e^t}{t^2}+\frac{e^t}{t}\right)$$

Now that $u(t)$ is known, then

$$\begin{aligned}
v'(t) &= u(t)\\
v(t) &= \int u(t)\,dt+c_2\\
&= \frac{e^t c_1}{t}+c_2
\end{aligned}$$

Hence

$$\begin{aligned}
y &= v(t)\,t^n\\
&= \left(\frac{e^t c_1}{t}+c_2\right)t\\
&= c_1 e^t+c_2 t
\end{aligned}$$

Now the particular solution to this ODE is found

$$(1-t)\,y''+ty'-y=2(-1+t)^2\,\mathrm{e}^{-t}$$

The particular solution y_p can be found using either the method of undetermined coefficients, or the method of variation of parameters. The method of variation of parameters will be used as it is more general and can be used when the coefficients of the ODE depend on t as well. Let

$$y_p(t)=u_1y_1+u_2y_2 \tag{1}$$

Where u_1, u_2 to be determined, and y_1, y_2 are the two basis solutions (the two linearly independent solutions of the homogeneous ODE) found earlier when solving the homogeneous ODE as

$$\begin{aligned} y_1 &= t \\ y_2 &= \mathrm{e}^t \end{aligned}$$

In the Variation of parameters u_1, u_2 are found using

$$u_1=-\int \frac{y_2f(t)}{aW(t)} \tag{2}$$

$$u_2=\int \frac{y_1f(t)}{aW(t)} \tag{3}$$

Where $W(t)$ is the Wronskian and a is the coefficient in front of y'' in the given ODE. The Wronskian is given by $W=\begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$. Hence

$$W=\begin{vmatrix} t & \mathrm{e}^t \\ \frac{d}{dt}(t) & \frac{d}{dt}(\mathrm{e}^t) \end{vmatrix}$$

Which gives

$$W=\begin{vmatrix} t & \mathrm{e}^t \\ 1 & \mathrm{e}^t \end{vmatrix}$$

Therefore

$$W=(t)\left(\mathrm{e}^t\right)-\left(\mathrm{e}^t\right)(1)$$

Which simplifies to

$$W = t\,\mathrm{e}^{t} - \mathrm{e}^{t}$$

Which simplifies to

$$W = \mathrm{e}^{t}(-1+t)$$

Therefore Eq. (2) becomes

$$u_1 = -\int \frac{2\,\mathrm{e}^{t}(-1+t)^2\,\mathrm{e}^{-t}}{(1-t)\,\mathrm{e}^{t}\,(-1+t)}\,dt$$

Which simplifies to

$$u_1 = -\int -2\,\mathrm{e}^{-t}dt$$

Hence

$$u_1 = -2\,\mathrm{e}^{-t}$$

And Eq. (3) becomes

$$u_2 = \int \frac{2t(-1+t)^2\,\mathrm{e}^{-t}}{(1-t)\,\mathrm{e}^{t}\,(-1+t)}\,dt$$

Which simplifies to

$$u_2 = \int -2t\,\mathrm{e}^{-2t}dt$$

Hence

$$u_2 = \frac{(1+2t)\,\mathrm{e}^{-2t}}{2}$$

Therefore the particular solution, from equation (1) is

$$y_p(t) = -2t\,\mathrm{e}^{-t} + \frac{(1+2t)\,\mathrm{e}^{-2t}\mathrm{e}^{t}}{2}$$

Which simplifies to

$$y_p(t) = \mathrm{e}^{-t}\left(-t+\frac{1}{2}\right)$$

Therefore the general solution is

$$
\begin{aligned}
y &= y_h + y_p \\
&= \left(\left(\frac{e^t c_1}{t} + c_2\right) t\right) + \left(e^{-t}\left(-t + \frac{1}{2}\right)\right) \\
&= e^{-t}\left(-t + \frac{1}{2}\right) + \left(\frac{e^t c_1}{t} + c_2\right) t
\end{aligned}
$$

Which simplifies to

$$
y = e^{-t}\left(-t + \frac{1}{2}\right) + \left(\frac{e^t c_1}{t} + c_2\right) t
$$

Summary
The solution(s) found are the following

$$
y = e^{-t}\left(-t + \frac{1}{2}\right) + \left(\frac{e^t c_1}{t} + c_2\right) t \tag{1}
$$

Verification of solutions

$$
y = e^{-t}\left(-t + \frac{1}{2}\right) + \left(\frac{e^t c_1}{t} + c_2\right) t
$$

Verified OK.

10.16.2 Solving as second order ode non constant coeff transformation on B ode

Given an ode of the form

$$
Ay'' + By' + Cy = F(t)
$$

This method reduces the order ode the ODE by one by applying the transformation

$$
y = Bv
$$

This results in

$$
\begin{aligned}
y' &= B'v + v'B \\
y'' &= B''v + B'v' + v''B + v'B' \\
&= v''B + 2v' + B' + B''v
\end{aligned}
$$

And now the original ode becomes

$$
\begin{aligned}
A(v''B + 2v'B' + B''v) + B(B'v + v'B) + CBv &= 0 \\
ABv'' + \left(2AB' + B^2\right) v' + \left(AB'' + BB' + CB\right) v &= 0 \qquad (1)
\end{aligned}
$$

If the term $AB'' + BB' + CB$ is zero, then this method works and can be used to solve

$$
ABv'' + \left(2AB' + B^2\right) v' = 0
$$

By Using $u = v'$ which reduces the order of the above ode to one. The new ode is

$$
ABu' + \left(2AB' + B^2\right) u = 0
$$

The above ode is first order ode which is solved for u. Now a new ode $v' = u$ is solved for v as first order ode. Then the final solution is obtain from $y = Bv$.

This method works only if the term $AB'' + BB' + CB$ is zero. The given ODE shows that

$$
\begin{aligned}
A &= 1 - t \\
B &= t \\
C &= -1 \\
F &= 2(-1 + t)^2 \, \mathrm{e}^{-t}
\end{aligned}
$$

The above shows that for this ode

$$
\begin{aligned}
AB'' + BB' + CB &= (1 - t)\,(0) + (t)\,(1) + (-1)\,(t) \\
&= 0
\end{aligned}
$$

Hence the ode in v given in (1) now simplifies to

$$
-t(-1 + t)\, v'' + \left(t^2 - 2t + 2\right) v' = 0
$$

Now by applying $v' = u$ the above becomes

$$
\left(-t^2 + t\right) u'(t) + \left(t^2 - 2t + 2\right) u(t) = 0
$$

Which is now solved for u. In canonical form the ODE is

$$
\begin{aligned}
u' &= F(t, u) \\
&= f(t) g(u) \\
&= \frac{u(t^2 - 2t + 2)}{t\,(-1 + t)}
\end{aligned}
$$

Where $f(t) = \frac{t^2-2t+2}{t(-1+t)}$ and $g(u) = u$. Integrating both sides gives

$$
\begin{aligned}
\frac{1}{u}\,du &= \frac{t^2-2t+2}{t\,(-1+t)}\,dt \\
\int \frac{1}{u}\,du &= \int \frac{t^2-2t+2}{t\,(-1+t)}\,dt \\
\ln(u) &= t - 2\ln(t) + \ln(-1+t) + c_1 \\
u &= \mathrm{e}^{t-2\ln(t)+\ln(-1+t)+c_1} \\
&= c_1 \mathrm{e}^{t-2\ln(t)+\ln(-1+t)}
\end{aligned}
$$

Which simplifies to

$$
u(t) = c_1 \left(-\frac{\mathrm{e}^t}{t^2} + \frac{\mathrm{e}^t}{t}\right)
$$

The ode for v now becomes

$$
\begin{aligned}
v' &= u \\
&= c_1 \left(-\frac{\mathrm{e}^t}{t^2} + \frac{\mathrm{e}^t}{t}\right)
\end{aligned}
$$

Which is now solved for v. Integrating both sides gives

$$
\begin{aligned}
v(t) &= \int \frac{c_1 \mathrm{e}^t(-1+t)}{t^2}\,\mathrm{d}t \\
&= \frac{\mathrm{e}^t c_1}{t} + c_2
\end{aligned}
$$

Therefore the homogeneous solution is

$$
\begin{aligned}
y_h(t) &= Bv \\
&= (t)\left(\frac{\mathrm{e}^t c_1}{t} + c_2\right) \\
&= c_1 \mathrm{e}^t + c_2 t
\end{aligned}
$$

And now the particular solution $y_p(t)$ will be found. The particular solution y_p can be found using either the method of undetermined coefficients, or the method of variation of parameters. The method of variation of parameters will be used as it is more general and can be used when the coefficients of the ODE depend on t as well. Let

$$
y_p(t) = u_1 y_1 + u_2 y_2 \tag{1}
$$

Where u_1, u_2 to be determined, and y_1, y_2 are the two basis solutions (the two linearly independent solutions of the homogeneous ODE) found earlier when solving the homogeneous ODE as

$$y_1 = t$$
$$y_2 = e^t$$

In the Variation of parameters u_1, u_2 are found using

$$u_1 = -\int \frac{y_2 f(t)}{aW(t)} \tag{2}$$
$$u_2 = \int \frac{y_1 f(t)}{aW(t)} \tag{3}$$

Where $W(t)$ is the Wronskian and a is the coefficient in front of y'' in the given ODE. The Wronskian is given by $W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$. Hence

$$W = \begin{vmatrix} t & e^t \\ \frac{d}{dt}(t) & \frac{d}{dt}(e^t) \end{vmatrix}$$

Which gives

$$W = \begin{vmatrix} t & e^t \\ 1 & e^t \end{vmatrix}$$

Therefore

$$W = (t)\left(e^t\right) - \left(e^t\right)(1)$$

Which simplifies to

$$W = t\,e^t - e^t$$

Which simplifies to

$$W = e^t(-1+t)$$

Therefore Eq. (2) becomes

$$u_1 = -\int \frac{2\,e^t(-1+t)^2\,e^{-t}}{(1-t)\,e^t\,(-1+t)}\,dt$$

Which simplifies to

$$u_1 = -\int -2\,\mathrm{e}^{-t} dt$$

Hence

$$u_1 = -2\,\mathrm{e}^{-t}$$

And Eq. (3) becomes

$$u_2 = \int \frac{2t(-1+t)^2\,\mathrm{e}^{-t}}{(1-t)\,\mathrm{e}^{t}\,(-1+t)}\,dt$$

Which simplifies to

$$u_2 = \int -2t\,\mathrm{e}^{-2t} dt$$

Hence

$$u_2 = \frac{(1+2t)\,\mathrm{e}^{-2t}}{2}$$

Therefore the particular solution, from equation (1) is

$$y_p(t) = -2t\,\mathrm{e}^{-t} + \frac{(1+2t)\,\mathrm{e}^{-2t}\mathrm{e}^{t}}{2}$$

Which simplifies to

$$y_p(t) = \mathrm{e}^{-t}\left(-t + \frac{1}{2}\right)$$

Hence the complete solution is

$$\begin{aligned} y(t) &= y_h + y_p \\ &= (c_1\mathrm{e}^{t} + c_2 t) + \left(\mathrm{e}^{-t}\left(-t + \frac{1}{2}\right)\right) \\ &= -t\,\mathrm{e}^{-t} + \frac{\mathrm{e}^{-t}}{2} + c_1\mathrm{e}^{t} + c_2 t \end{aligned}$$

<u>Summary</u>
The solution(s) found are the following

$$y = -t\,\mathrm{e}^{-t} + \frac{\mathrm{e}^{-t}}{2} + c_1\mathrm{e}^{t} + c_2 t \tag{1}$$

Verification of solutions

$$y = -t\,\mathrm{e}^{-t} + \frac{\mathrm{e}^{-t}}{2} + c_1 \mathrm{e}^{t} + c_2 t$$

Verified OK.

10.16.3 Solving using Kovacic algorithm

Writing the ode as

$$(1-t)\,y'' + ty' - y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1-t \\ B &= t \\ C &= -1 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(t) = y e^{\int \frac{B}{2A}\,dt}$$

Then (2) becomes

$$z''(t) = rz(t) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{t^2 - 4t + 6}{4\left(-1+t\right)^2} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= t^2 - 4t + 6 \\ t &= 4(-1+t)^2 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(t) = \left(\frac{t^2 - 4t + 6}{4\left(-1+t\right)^2}\right) z(t) \tag{7}$$

Equation (7) is now solved. After finding $z(t)$ then y is found using the inverse transformation

$$y = z(t)\, e^{-\int \frac{B}{2A}\, dt}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 501: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 2 - 2 \\ &= 0 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 4(-1+t)^2$. There is a pole at $t = 1$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Therefore

$$L = [1, 2]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = \frac{1}{4} - \frac{1}{2(-1+t)} + \frac{3}{4(-1+t)^2}$$

For the pole at $t = 1$ let b be the coefficient of $\frac{1}{(-1+t)^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{3}{4}$. Hence

$$\begin{aligned}
[\sqrt{r}]_c &= 0 \\
\alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{3}{2} \\
\alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = -\frac{1}{2}
\end{aligned}$$

Since the order of r at ∞ is $O_r(\infty) = 0$ then

$$v = \frac{-O_r(\infty)}{2} = \frac{0}{2} = 0$$

$[\sqrt{r}]_\infty$ is the sum of terms involving t^i for $0 \leq i \leq v$ in the Laurent series for $\sqrt{r}$ at ∞. Therefore

$$\begin{aligned}
[\sqrt{r}]_\infty &= \sum_{i=0}^{v} a_i t^i \\
&= \sum_{i=0}^{0} a_i t^i
\end{aligned} \tag{8}$$

Let a be the coefficient of $t^v = t^0$ in the above sum. The Laurent series of $\sqrt{r}$ at ∞ is

$$\sqrt{r} \approx \frac{1}{2} - \frac{1}{2t} + \frac{1}{t^3} + \frac{11}{4t^4} + \frac{21}{4t^5} + \frac{15}{2t^6} + \frac{6}{t^7} - \frac{117}{16t^8} + \dots \tag{9}$$

Comparing Eq. (9) with Eq. (8) shows that

$$a = \frac{1}{2}$$

From Eq. (9) the sum up to $v = 0$ gives

$$\begin{aligned}
[\sqrt{r}]_\infty &= \sum_{i=0}^{0} a_i t^i \\
&= \frac{1}{2}
\end{aligned} \tag{10}$$

Now we need to find b, where b be the coefficient of $t^{v-1} = t^{-1} = \frac{1}{t}$ in r minus the coefficient of same term but in $\left([\sqrt{r}]_\infty\right)^2$ where $[\sqrt{r}]_\infty$ was found above in Eq (10). Hence

$$\left([\sqrt{r}]_\infty\right)^2 = \frac{1}{4}$$

This shows that the coefficient of $\frac{1}{t}$ in the above is 0. Now we need to find the coefficient of $\frac{1}{t}$ in r. How this is done depends on if $v = 0$ or not. Since $v = 0$ then starting from $r = \frac{s}{t}$ and doing long division in the form

$$r = Q + \frac{R}{t}$$

Where Q is the quotient and R is the remainder. Then the coefficient of $\frac{1}{t}$ in r will be the coefficient in R of the term in t of degree of t minus one, divided by the leading coefficient in t. Doing long division gives

$$\begin{aligned}
r &= \frac{s}{t} \\
&= \frac{t^2 - 4t + 6}{4t^2 - 8t + 4} \\
&= Q + \frac{R}{4t^2 - 8t + 4} \\
&= \left(\frac{1}{4}\right) + \left(\frac{-2t + 5}{4t^2 - 8t + 4}\right) \\
&= \frac{1}{4} + \frac{-2t + 5}{4t^2 - 8t + 4}
\end{aligned}$$

Since the degree of t is 2, then we see that the coefficient of the term t in the remainder R is -2. Dividing this by leading coefficient in t which is 4 gives $-\frac{1}{2}$. Now b can be found.

$$\begin{aligned}
b &= \left(-\frac{1}{2}\right) - (0) \\
&= -\frac{1}{2}
\end{aligned}$$

Hence

$$\begin{aligned}
[\sqrt{r}]_\infty &= \frac{1}{2} \\
\alpha_\infty^+ &= \frac{1}{2}\left(\frac{b}{a} - v\right) &= \frac{1}{2}\left(\frac{-\frac{1}{2}}{\frac{1}{2}} - 0\right) &= -\frac{1}{2} \\
\alpha_\infty^- &= \frac{1}{2}\left(-\frac{b}{a} - v\right) &= \frac{1}{2}\left(-\frac{-\frac{1}{2}}{\frac{1}{2}} - 0\right) &= \frac{1}{2}
\end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = \frac{t^2 - 4t + 6}{4\left(-1+t\right)^2}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
1	2	0	$\frac{3}{2}$	$-\frac{1}{2}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
0	$\frac{1}{2}$	$-\frac{1}{2}$	$\frac{1}{2}$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^+ = -\frac{1}{2}$ then

$$\begin{aligned}
d &= \alpha_\infty^+ - \left(\alpha_{c_1}^-\right) \\
&= -\frac{1}{2} - \left(-\frac{1}{2}\right) \\
&= 0
\end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c\in\Gamma}\left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{t-c}\right) + s(\infty)[\sqrt{r}]_\infty$$

Substituting the above values in the above results in

$$\begin{aligned}
\omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^-}{t-c_1}\right) + (+)[\sqrt{r}]_\infty \\
&= -\frac{1}{2\left(-1+t\right)} + \left(\frac{1}{2}\right) \\
&= -\frac{1}{2\left(-1+t\right)} + \frac{1}{2} \\
&= \frac{t-2}{2t-2}
\end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(t)$ of degree $d = 0$ to solve the ode. The polynomial $p(t)$ needs to satisfy the equation

$$p'' + 2\omega p' + (\omega' + \omega^2 - r)\, p = 0 \tag{1A}$$

Let

$$p(t) = 1 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$(0) + 2\left(-\frac{1}{2(-1+t)} + \frac{1}{2}\right)(0) + \left(\left(\frac{1}{2(-1+t)^2}\right) + \left(-\frac{1}{2(-1+t)} + \frac{1}{2}\right)^2 - \left(\frac{t^2 - 4t + 6}{4(-1+t)^2}\right)\right) = 0$$
$$0 = 0$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned} z_1(t) &= p e^{\int \omega \, dt} \\ &= e^{\int \left(-\frac{1}{2(-1+t)} + \frac{1}{2}\right) dt} \\ &= \frac{e^{\frac{t}{2}}}{\sqrt{-1+t}} \end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned} y_1 &= z_1 e^{\int -\frac{1}{2} \frac{B}{A} \, dt} \\ &= z_1 e^{-\int \frac{1}{2} \frac{t}{1-t} \, dt} \\ &= z_1 e^{\frac{t}{2} + \frac{\ln(-1+t)}{2}} \\ &= z_1 \left(\sqrt{-1+t}\, e^{\frac{t}{2}}\right) \end{aligned}$$

Which simplifies to

$$y_1 = e^t$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A} \, dt}}{y_1^2} \, dt$$

Substituting gives

$$
\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{t}{1-t}\, dt}}{(y_1)^2}\, dt \\
&= y_1 \int \frac{e^{t+\ln(-1+t)}}{(y_1)^2}\, dt \\
&= y_1 \left(-t\, \mathrm{e}^{-t}\right)
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1 \left(\mathrm{e}^{t}\right) + c_2 \left(\mathrm{e}^{t} \left(-t\, \mathrm{e}^{-t}\right)\right)
\end{aligned}
$$

This is second order nonhomogeneous ODE. Let the solution be

$$
y = y_h + y_p
$$

Where y_h is the solution to the homogeneous ODE $Ay''(t) + By'(t) + Cy(t) = 0$, and y_p is a particular solution to the nonhomogeneous ODE $Ay''(t) + By'(t) + Cy(t) = f(t)$. y_h is the solution to

$$
(1-t)\, y'' + ty' - y = 0
$$

The homogeneous solution is found using the Kovacic algorithm which results in

$$
y_h = c_1 \mathrm{e}^{t} - c_2 t
$$

The particular solution y_p can be found using either the method of undetermined coefficients, or the method of variation of parameters. The method of variation of parameters will be used as it is more general and can be used when the coefficients of the ODE depend on t as well. Let

$$
y_p(t) = u_1 y_1 + u_2 y_2 \tag{1}
$$

Where u_1, u_2 to be determined, and y_1, y_2 are the two basis solutions (the two linearly independent solutions of the homogeneous ODE) found earlier when solving the homogeneous ODE as

$$
\begin{aligned}
y_1 &= \mathrm{e}^{t} \\
y_2 &= -t
\end{aligned}
$$

In the Variation of parameters u_1, u_2 are found using

$$u_1 = -\int \frac{y_2 f(t)}{aW(t)} \tag{2}$$

$$u_2 = \int \frac{y_1 f(t)}{aW(t)} \tag{3}$$

Where $W(t)$ is the Wronskian and a is the coefficient in front of y'' in the given ODE. The Wronskian is given by $W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$. Hence

$$W = \begin{vmatrix} e^t & -t \\ \frac{d}{dt}(e^t) & \frac{d}{dt}(-t) \end{vmatrix}$$

Which gives

$$W = \begin{vmatrix} e^t & -t \\ e^t & -1 \end{vmatrix}$$

Therefore

$$W = (e^t)(-1) - (-t)(e^t)$$

Which simplifies to

$$W = t\,e^t - e^t$$

Which simplifies to

$$W = e^t(-1 + t)$$

Therefore Eq. (2) becomes

$$u_1 = -\int \frac{-2t(-1+t)^2 e^{-t}}{(1-t)\,e^t\,(-1+t)}\,dt$$

Which simplifies to

$$u_1 = -\int 2t\,e^{-2t} dt$$

Hence

$$u_1 = \frac{(1+2t)\,e^{-2t}}{2}$$

And Eq. (3) becomes

$$u_2 = \int \frac{2\,e^t(-1+t)^2\,e^{-t}}{(1-t)\,e^t\,(-1+t)}\,dt$$

Which simplifies to

$$u_2 = \int -2\,e^{-t} dt$$

Hence

$$u_2 = 2\,e^{-t}$$

Therefore the particular solution, from equation (1) is

$$y_p(t) = -2t\,e^{-t} + \frac{(1+2t)\,e^{-2t}e^t}{2}$$

Which simplifies to

$$y_p(t) = e^{-t}\left(-t + \frac{1}{2}\right)$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \left(c_1 e^t - c_2 t\right) + \left(e^{-t}\left(-t + \frac{1}{2}\right)\right) \end{aligned}$$

Summary

The solution(s) found are the following

$$y = c_1 e^t - c_2 t + e^{-t}\left(-t + \frac{1}{2}\right) \tag{1}$$

Verification of solutions

$$y = c_1 e^t - c_2 t + e^{-t}\left(-t + \frac{1}{2}\right)$$

Verified OK.

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 2; linear nonhomogeneous with symmetry [0,1]
trying a double symmetry of the form [xi=0, eta=F(x)]
-> Try solving first the homogeneous part of the ODE
   checking if the LODE has constant coefficients
   checking if the LODE is of Euler type
   trying a symmetry of the form [xi=0, eta=F(x)]
   checking if the LODE is missing y
   -> Trying a Liouvillian solution using Kovacics algorithm
      A Liouvillian solution exists
      Reducible group (found an exponential solution)
      Reducible group (found another exponential solution)
   <- Kovacics algorithm successful
<- solving first the homogeneous part of the ODE successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 25

```
dsolve((1-t)*diff(y(t),t$2)+t*diff(y(t),t)-y(t) = 2*(t-1)^2*exp(-t),y(t), singsol=all)
```

$$y(t) = c_2 t + e^t c_1 - t\,e^{-t} + \frac{e^{-t}}{2}$$

✓ Solution by Mathematica

Time used: 0.07 (sec). Leaf size: 30

```
DSolve[(1-t)*y''[t]+t*y'[t]-y[t] == 2*(t-1)^2*Exp[-t],y[t],t,IncludeSingularSolutions -> Tru
```

$$y(t) \to e^{-t}\left(\frac{1}{2} - t\right) + c_1 e^t - c_2 t$$

10.17 problem 17

Internal problem ID [699]
Internal file name [`OUTPUT/699_Sunday_June_05_2022_01_47_14_AM_49289387/index.tex`]

Book: Elementary differential equations and boundary value problems, 10th ed., Boyce and DiPrima
Section: Chapter 3, Second order linear equations, section 3.6, Variation of Parameters. page 190
Problem number: 17.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_euler_ode", "second_order_change_of_variable_on_x_method_1", "second_order_change_of_variable_on_x_method_2", "second_order_change_of_variable_on_y_method_2"**

Maple gives the following as the ode type

```
[[_2nd_order, _linear, _nonhomogeneous]]
```

$$\boxed{x^2y'' - 3y'x + 4y = \ln(x)\, x^2}$$

10.17.1 Solving as second order euler ode ode

This is second order non-homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = f(x)$$

Where $A = x^2, B = -3x, C = 4, f(x) = \ln(x)\, x^2$. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the non-homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = f(x)$.

Solving for y_h from

$$x^2y'' - 3y'x + 4y = 0$$

This is Euler second order ODE. Let the solution be $y = x^r$, then $y' = rx^{r-1}$ and $y'' = r(r-1)x^{r-2}$. Substituting these back into the given ODE gives

$$x^2(r(r-1))x^{r-2} - 3xrx^{r-1} + 4x^r = 0$$

Simplifying gives

$$r(r-1)\,x^r - 3r\,x^r + 4x^r = 0$$

Since $x^r \neq 0$ then dividing throughout by x^r gives

$$r(r-1) - 3r + 4 = 0$$

Or

$$r^2 - 4r + 4 = 0 \tag{1}$$

Equation (1) is the characteristic equation. Its roots determine the form of the general solution. Using the quadratic equation the roots are

$$r_1 = 2$$
$$r_2 = 2$$

Since the roots are equal, then the general solution is

$$y = c_1y_1 + c_2y_2$$

Where $y_1 = x^r$ and $y_2 = x^r \ln(x)$. Hence

$$y = c_1x^2 + c_2x^2 \ln(x)$$

Next, we find the particular solution to the ODE

$$x^2y'' - 3y'x + 4y = \ln(x)\,x^2$$

The particular solution y_p can be found using either the method of undetermined coefficients, or the method of variation of parameters. The method of variation of parameters will be used as it is more general and can be used when the coefficients of the ODE depend on x as well. Let

$$y_p(x) = u_1y_1 + u_2y_2 \tag{1}$$

Where u_1, u_2 to be determined, and y_1, y_2 are the two basis solutions (the two linearly independent solutions of the homogeneous ODE) found earlier when solving the homogeneous ODE as

$$y_1 = x^2$$
$$y_2 = \ln(x)\, x^2$$

In the Variation of parameters u_1, u_2 are found using

$$u_1 = -\int \frac{y_2 f(x)}{aW(x)} \tag{2}$$
$$u_2 = \int \frac{y_1 f(x)}{aW(x)} \tag{3}$$

Where $W(x)$ is the Wronskian and a is the coefficient in front of y'' in the given ODE. The Wronskian is given by $W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$. Hence

$$W = \begin{vmatrix} x^2 & \ln(x)\, x^2 \\ \frac{d}{dx}(x^2) & \frac{d}{dx}(\ln(x)\, x^2) \end{vmatrix}$$

Which gives

$$W = \begin{vmatrix} x^2 & \ln(x)\, x^2 \\ 2x & x + 2x\ln(x) \end{vmatrix}$$

Therefore

$$W = (x^2)\,(x + 2x\ln(x)) - (\ln(x)\, x^2)\,(2x)$$

Which simplifies to

$$W = x^3$$

Which simplifies to

$$W = x^3$$

Therefore Eq. (2) becomes

$$u_1 = -\int \frac{\ln(x)^2\, x^4}{x^5}\, dx$$

Which simplifies to

$$u_1 = -\int \frac{\ln (x)^2}{x} dx$$

Hence

$$u_1 = -\frac{\ln (x)^3}{3}$$

And Eq. (3) becomes

$$u_2 = \int \frac{x^4 \ln (x)}{x^5} dx$$

Which simplifies to

$$u_2 = \int \frac{\ln (x)}{x} dx$$

Hence

$$u_2 = \frac{\ln (x)^2}{2}$$

Therefore the particular solution, from equation (1) is

$$y_p(x) = \frac{\ln (x)^3 x^2}{6}$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= x^2 \left(\frac{\ln (x)^3}{6} + c_1 + c_2 \ln (x)\right) \end{aligned}$$

Summary

The solution(s) found are the following

$$y = x^2 \left(\frac{\ln (x)^3}{6} + c_1 + c_2 \ln (x)\right) \tag{1}$$

Verification of solutions

$$y = x^2 \left(\frac{\ln (x)^3}{6} + c_1 + c_2 \ln (x)\right)$$

Verified OK.

10.17.2 Solving as second order change of variable on x method 2 ode

This is second order non-homogeneous ODE. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the non-homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = f(x)$. y_h is the solution to

$$x^2y'' - 3y'x + 4y = 0$$

In normal form the ode

$$x^2y'' - 3y'x + 4y = 0 \tag{1}$$

Becomes

$$y'' + p(x)\, y' + q(x)\, y = 0 \tag{2}$$

Where

$$\begin{aligned} p(x) &= -\frac{3}{x} \\ q(x) &= \frac{4}{x^2} \end{aligned}$$

Applying change of variables $\tau = g(x)$ to (2) gives

$$\frac{d^2}{d\tau^2}y(\tau) + p_1\left(\frac{d}{d\tau}y(\tau)\right) + q_1y(\tau) = 0 \tag{3}$$

Where τ is the new independent variable, and

$$p_1(\tau) = \frac{\tau''(x) + p(x)\,\tau'(x)}{\tau'\,(x)^2} \tag{4}$$

$$q_1(\tau) = \frac{q(x)}{\tau'\,(x)^2} \tag{5}$$

Let $p_1 = 0$. Eq (4) simplifies to

$$\tau''(x) + p(x)\,\tau'(x) = 0$$

This ode is solved resulting in

$$\begin{aligned}\tau &= \int e^{-\left(\int p(x)dx\right)} dx \\ &= \int e^{-\left(\int -\frac{3}{x}dx\right)} dx \\ &= \int e^{3\ln(x)}\, dx \\ &= \int x^3 dx \\ &= \frac{x^4}{4} \end{aligned} \tag{6}$$

Using (6) to evaluate q_1 from (5) gives

$$\begin{aligned} q_1(\tau) &= \frac{q(x)}{\tau'(x)^2} \\ &= \frac{\frac{4}{x^2}}{x^6} \\ &= \frac{4}{x^8} \end{aligned} \tag{7}$$

Substituting the above in (3) and noting that now $p_1 = 0$ results in

$$\frac{d^2}{d\tau^2}y(\tau) + q_1 y(\tau) = 0$$
$$\frac{d^2}{d\tau^2}y(\tau) + \frac{4y(\tau)}{x^8} = 0$$

But in terms of τ

$$\frac{4}{x^8} = \frac{1}{4\tau^2}$$

Hence the above ode becomes

$$\frac{d^2}{d\tau^2}y(\tau) + \frac{y(\tau)}{4\tau^2} = 0$$

The above ode is now solved for $y(\tau)$. The ode can be written as

$$4\left(\frac{d^2}{d\tau^2}y(\tau)\right)\tau^2 + y(\tau) = 0$$

Which shows it is a Euler ODE. This is Euler second order ODE. Let the solution be $y(\tau) = \tau^r$, then $y' = r\tau^{r-1}$ and $y'' = r(r-1)\tau^{r-2}$. Substituting these back into the given ODE gives

$$4\tau^2(r(r-1))\tau^{r-2} + 0r\tau^{r-1} + \tau^r = 0$$

Simplifying gives

$$4r(r-1)\,\tau^r + 0\,\tau^r + \tau^r = 0$$

Since $\tau^r \neq 0$ then dividing throughout by τ^r gives

$$4r(r-1) + 0 + 1 = 0$$

Or

$$4r^2 - 4r + 1 = 0 \tag{1}$$

Equation (1) is the characteristic equation. Its roots determine the form of the general solution. Using the quadratic equation the roots are

$$r_1 = \frac{1}{2}$$
$$r_2 = \frac{1}{2}$$

Since the roots are equal, then the general solution is

$$y(\tau) = c_1 y_1 + c_2 y_2$$

Where $y_1 = \tau^r$ and $y_2 = \tau^r \ln(\tau)$. Hence

$$y(\tau) = c_1\sqrt{\tau} + c_2\sqrt{\tau}\,\ln(\tau)$$

The above solution is now transformed back to y using (6) which results in

$$y = \frac{(c_2 \ln(x^4) - 2c_2 \ln(2) + c_1)\sqrt{x^4}}{2}$$

Therefore the homogeneous solution y_h is

$$y_h = \frac{(c_2 \ln(x^4) - 2c_2 \ln(2) + c_1)\sqrt{x^4}}{2}$$

The particular solution y_p can be found using either the method of undetermined coefficients, or the method of variation of parameters. The method of variation of parameters will be used as it is more general and can be used when the coefficients of the ODE depend on x as well. Let

$$y_p(x) = u_1 y_1 + u_2 y_2 \tag{1}$$

Where u_1, u_2 to be determined, and y_1, y_2 are the two basis solutions (the two linearly independent solutions of the homogeneous ODE) found earlier when solving the homogeneous ODE as

$$y_1 = \sqrt{x^4}$$

$$y_2 = -\ln(2)\sqrt{x^4} + \frac{\ln(x^4)\sqrt{x^4}}{2}$$

In the Variation of parameters u_1, u_2 are found using

$$u_1 = -\int \frac{y_2 f(x)}{aW(x)} \tag{2}$$

$$u_2 = \int \frac{y_1 f(x)}{aW(x)} \tag{3}$$

Where $W(x)$ is the Wronskian and a is the coefficient in front of y'' in the given ODE. The Wronskian is given by $W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$. Hence

$$W = \begin{vmatrix} \sqrt{x^4} & -\ln(2)\sqrt{x^4} + \frac{\ln(x^4)\sqrt{x^4}}{2} \\ \frac{d}{dx}\left(\sqrt{x^4}\right) & \frac{d}{dx}\left(-\ln(2)\sqrt{x^4} + \frac{\ln(x^4)\sqrt{x^4}}{2}\right) \end{vmatrix}$$

Which gives

$$W = \begin{vmatrix} \sqrt{x^4} & -\ln(2)\sqrt{x^4} + \frac{\ln(x^4)\sqrt{x^4}}{2} \\ \frac{2x^3}{\sqrt{x^4}} & -\frac{2\ln(2)x^3}{\sqrt{x^4}} + \frac{2\sqrt{x^4}}{x} + \frac{\ln(x^4)x^3}{\sqrt{x^4}} \end{vmatrix}$$

Therefore

$$W = \left(\sqrt{x^4}\right)\left(-\frac{2\ln(2)x^3}{\sqrt{x^4}} + \frac{2\sqrt{x^4}}{x} + \frac{\ln(x^4)x^3}{\sqrt{x^4}}\right) - \left(-\ln(2)\sqrt{x^4} + \frac{\ln(x^4)\sqrt{x^4}}{2}\right)\left(\frac{2x^3}{\sqrt{x^4}}\right)$$

Which simplifies to

$$W = 2x^3$$

Which simplifies to

$$W = 2x^3$$

Therefore Eq. (2) becomes

$$u_1 = -\int \frac{\left(-\ln(2)\sqrt{x^4} + \frac{\ln(x^4)\sqrt{x^4}}{2}\right)\ln(x)\,x^2}{2x^5}dx$$

Which simplifies for $0 < x$ to

$$u_1 = -\int -\frac{(\ln(2) - 2\ln(x))\ln(x)}{2x}dx$$

Hence

$$u_1 = -\frac{\ln(x)^3}{3} + \frac{\ln(2)\ln(x)^2}{4}$$

And Eq. (3) becomes

$$u_2 = \int \frac{\sqrt{x^4}\,\ln(x)\,x^2}{2x^5}dx$$

Which simplifies for $0 < x$ to

$$u_2 = \int \frac{\ln(x)}{2x}dx$$

Hence

$$u_2 = \frac{\ln(x)^2}{4}$$

Which simplifies to

$$u_1 = -\frac{\ln(x)^2(4\ln(x) - 3\ln(2))}{12}$$

$$u_2 = \frac{\ln(x)^2}{4}$$

Therefore the particular solution, from equation (1) is

$$y_p(x) = -\frac{\ln(x)^2(4\ln(x) - 3\ln(2))\sqrt{x^4}}{12} + \frac{\ln(x)^2\left(-\ln(2)\sqrt{x^4} + \frac{\ln(x^4)\sqrt{x^4}}{2}\right)}{4}$$

Which simplifies to

$$y_p(x) = \frac{\ln(x)^3 x^2}{6}$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \left(\frac{(c_2 \ln(x^4) - 2c_2 \ln(2) + c_1)\sqrt{x^4}}{2}\right) + \left(\frac{\ln(x)^3 x^2}{6}\right) \end{aligned}$$

Summary
The solution(s) found are the following

$$y = \frac{(c_2 \ln(x^4) - 2c_2 \ln(2) + c_1)\sqrt{x^4}}{2} + \frac{\ln(x)^3 x^2}{6} \tag{1}$$

Verification of solutions

$$y = \frac{(c_2 \ln(x^4) - 2c_2 \ln(2) + c_1)\sqrt{x^4}}{2} + \frac{\ln(x)^3 x^2}{6}$$

Verified OK. `{0 < x}`

10.17.3 Solving as second order change of variable on x method 1 ode

This is second order non-homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = f(x)$$

Where $A = x^2, B = -3x, C = 4, f(x) = \ln(x)\, x^2$. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the non-homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = f(x)$. Solving for y_h from

$$x^2y'' - 3y'x + 4y = 0$$

In normal form the ode

$$x^2y'' - 3y'x + 4y = 0 \tag{1}$$

Becomes

$$y'' + p(x)\, y' + q(x)\, y = 0 \tag{2}$$

Where

$$
\begin{aligned}
p(x) &= -\frac{3}{x} \\
q(x) &= \frac{4}{x^2}
\end{aligned}
$$

Applying change of variables $\tau = g(x)$ to (2) results

$$
\frac{d^2}{d\tau^2} y(\tau) + p_1 \left(\frac{d}{d\tau} y(\tau)\right) + q_1 y(\tau) = 0 \tag{3}
$$

Where τ is the new independent variable, and

$$
p_1(\tau) = \frac{\tau''(x) + p(x)\,\tau'(x)}{\tau'(x)^2} \tag{4}
$$

$$
q_1(\tau) = \frac{q(x)}{\tau'(x)^2} \tag{5}
$$

Let $q_1 = c^2$ where c is some constant. Therefore from (5)

$$
\begin{aligned}
\tau' &= \frac{1}{c}\sqrt{q} \\
&= \frac{2\sqrt{\frac{1}{x^2}}}{c} \\
\tau'' &= -\frac{2}{c\sqrt{\frac{1}{x^2}}\, x^3}
\end{aligned} \tag{6}
$$

Substituting the above into (4) results in

$$
\begin{aligned}
p_1(\tau) &= \frac{\tau''(x) + p(x)\,\tau'(x)}{\tau'(x)^2} \\
&= \frac{-\frac{2}{c\sqrt{\frac{1}{x^2}}\, x^3} - \frac{3}{x}\frac{2\sqrt{\frac{1}{x^2}}}{c}}{\left(\frac{2\sqrt{\frac{1}{x^2}}}{c}\right)^2} \\
&= -2c
\end{aligned}
$$

Therefore ode (3) now becomes

$$
\begin{aligned}
y(\tau)'' + p_1 y(\tau)' + q_1 y(\tau) &= 0 \\
\frac{d^2}{d\tau^2} y(\tau) - 2c\left(\frac{d}{d\tau} y(\tau)\right) + c^2 y(\tau) &= 0
\end{aligned} \tag{7}
$$

The above ode is now solved for $y(\tau)$. Since the ode is now constant coefficients, it can be easily solved to give

$$y(\tau) = e^{c\tau} c_1$$

Now from (6)

$$\begin{aligned}\tau &= \int \frac{1}{c} \sqrt{q}\, dx \\ &= \frac{\int 2\sqrt{\frac{1}{x^2}}dx}{c} \\ &= \frac{2\sqrt{\frac{1}{x^2}}\, x \ln(x)}{c}\end{aligned}$$

Substituting the above into the solution obtained gives

$$y = c_1 x^2$$

Now the particular solution to this ODE is found

$$x^2 y'' - 3y'x + 4y = \ln(x)\, x^2$$

The particular solution y_p can be found using either the method of undetermined coefficients, or the method of variation of parameters. The method of variation of parameters will be used as it is more general and can be used when the coefficients of the ODE depend on x as well. Let

$$y_p(x) = u_1 y_1 + u_2 y_2 \tag{1}$$

Where u_1, u_2 to be determined, and y_1, y_2 are the two basis solutions (the two linearly independent solutions of the homogeneous ODE) found earlier when solving the homogeneous ODE as

$$\begin{aligned}y_1 &= \sqrt{x^4} \\ y_2 &= -\ln(2)\sqrt{x^4} + \frac{\ln(x^4)\sqrt{x^4}}{2}\end{aligned}$$

In the Variation of parameters u_1, u_2 are found using

$$u_1 = -\int \frac{y_2 f(x)}{a W(x)} \tag{2}$$

$$u_2 = \int \frac{y_1 f(x)}{a W(x)} \tag{3}$$

Where $W(x)$ is the Wronskian and a is the coefficient in front of y'' in the given ODE. The Wronskian is given by $W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$. Hence

$$W = \begin{vmatrix} \sqrt{x^4} & -\ln(2)\sqrt{x^4} + \frac{\ln(x^4)\sqrt{x^4}}{2} \\ \frac{d}{dx}\left(\sqrt{x^4}\right) & \frac{d}{dx}\left(-\ln(2)\sqrt{x^4} + \frac{\ln(x^4)\sqrt{x^4}}{2}\right) \end{vmatrix}$$

Which gives

$$W = \begin{vmatrix} \sqrt{x^4} & -\ln(2)\sqrt{x^4} + \frac{\ln(x^4)\sqrt{x^4}}{2} \\ \frac{2x^3}{\sqrt{x^4}} & -\frac{2\ln(2)x^3}{\sqrt{x^4}} + \frac{2\sqrt{x^4}}{x} + \frac{\ln(x^4)x^3}{\sqrt{x^4}} \end{vmatrix}$$

Therefore

$$W = \left(\sqrt{x^4}\right)\left(-\frac{2\ln(2)\,x^3}{\sqrt{x^4}} + \frac{2\sqrt{x^4}}{x} + \frac{\ln(x^4)\,x^3}{\sqrt{x^4}}\right) - \left(-\ln(2)\sqrt{x^4} + \frac{\ln(x^4)\sqrt{x^4}}{2}\right)\left(\frac{2x^3}{\sqrt{x^4}}\right)$$

Which simplifies to

$$W = 2x^3$$

Which simplifies to

$$W = 2x^3$$

Therefore Eq. (2) becomes

$$u_1 = -\int \frac{\left(-\ln(2)\sqrt{x^4} + \frac{\ln(x^4)\sqrt{x^4}}{2}\right)\ln(x)\,x^2}{2x^5}\,dx$$

Which simplifies for $0 < x$ to

$$u_1 = -\int -\frac{(\ln(2) - 2\ln(x))\ln(x)}{2x}\,dx$$

Hence

$$u_1 = -\frac{\ln(x)^3}{3} + \frac{\ln(2)\ln(x)^2}{4}$$

And Eq. (3) becomes

$$u_2 = \int \frac{\sqrt{x^4}\, \ln(x)\, x^2}{2x^5}\, dx$$

Which simplifies for $0 < x$ to

$$u_2 = \int \frac{\ln(x)}{2x} dx$$

Hence

$$u_2 = \frac{\ln(x)^2}{4}$$

Which simplifies to

$$u_1 = -\frac{\ln(x)^2 (4\ln(x) - 3\ln(2))}{12}$$

$$u_2 = \frac{\ln(x)^2}{4}$$

Therefore the particular solution, from equation (1) is

$$y_p(x) = -\frac{\ln(x)^2 (4\ln(x) - 3\ln(2))\sqrt{x^4}}{12} + \frac{\ln(x)^2 \left(-\ln(2)\sqrt{x^4} + \frac{\ln(x^4)\sqrt{x^4}}{2}\right)}{4}$$

Which simplifies to

$$y_p(x) = \frac{\ln(x)^3 x^2}{6}$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= (c_1 x^2) + \left(\frac{\ln(x)^3 x^2}{6}\right) \\ &= \frac{\ln(x)^3 x^2}{6} + c_1 x^2 \end{aligned}$$

Which simplifies to

$$y = x^2 \left(\frac{\ln(x)^3}{6} + c_1\right)$$

Summary

The solution(s) found are the following

$$y = x^2 \left(\frac{\ln(x)^3}{6} + c_1\right) \tag{1}$$

Verification of solutions

$$y = x^2 \left(\frac{\ln(x)^3}{6} + c_1\right)$$

Verified OK. `{0 < x}`

10.17.4 Solving as second order change of variable on y method 2 ode

This is second order non-homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = f(x)$$

Where $A = x^2, B = -3x, C = 4, f(x) = \ln(x)\, x^2$. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the non-homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = f(x)$. Solving for y_h from

$$x^2y'' - 3y'x + 4y = 0$$

In normal form the ode

$$x^2y'' - 3y'x + 4y = 0 \tag{1}$$

Becomes

$$y'' + p(x)\, y' + q(x)\, y = 0 \tag{2}$$

Where

$$p(x) = -\frac{3}{x}$$
$$q(x) = \frac{4}{x^2}$$

Applying change of variables on the depndent variable $y = v(x)\, x^n$ to (2) gives the following ode where the dependent variables is $v(x)$ and not y.

$$v''(x) + \left(\frac{2n}{x} + p\right) v'(x) + \left(\frac{n(n-1)}{x^2} + \frac{np}{x} + q\right) v(x) = 0 \tag{3}$$

Let the coefficient of $v(x)$ above be zero. Hence

$$\frac{n(n-1)}{x^2} + \frac{np}{x} + q = 0 \tag{4}$$

Substituting the earlier values found for $p(x)$ and $q(x)$ into (4) gives

$$\frac{n(n-1)}{x^2} - \frac{3n}{x^2} + \frac{4}{x^2} = 0 \tag{5}$$

Solving (5) for n gives

$$n = 2 \tag{6}$$

Substituting this value in (3) gives

$$\begin{aligned} v''(x) + \frac{v'(x)}{x} &= 0 \\ v''(x) + \frac{v'(x)}{x} &= 0 \end{aligned} \tag{7}$$

Using the substitution

$$u(x) = v'(x)$$

Then (7) becomes

$$u'(x) + \frac{u(x)}{x} = 0 \tag{8}$$

The above is now solved for $u(x)$. In canonical form the ODE is

$$\begin{aligned} u' &= F(x, u) \\ &= f(x)g(u) \\ &= -\frac{u}{x} \end{aligned}$$

Where $f(x) = -\frac{1}{x}$ and $g(u) = u$. Integrating both sides gives

$$\begin{aligned}\frac{1}{u}\,du &= -\frac{1}{x}\,dx\\ \int \frac{1}{u}\,du &= \int -\frac{1}{x}\,dx\\ \ln(u) &= -\ln(x) + c_1\\ u &= \mathrm{e}^{-\ln(x)+c_1}\\ &= \frac{c_1}{x}\end{aligned}$$

Now that $u(x)$ is known, then

$$\begin{aligned}v'(x) &= u(x)\\ v(x) &= \int u(x)\,dx + c_2\\ &= c_1 \ln(x) + c_2\end{aligned}$$

Hence

$$\begin{aligned}y &= v(x)\,x^n\\ &= (c_1 \ln(x) + c_2)\,x^2\\ &= (c_1 \ln(x) + c_2)\,x^2\end{aligned}$$

Now the particular solution to this ODE is found

$$x^2 y'' - 3y'x + 4y = \ln(x)\,x^2$$

The particular solution y_p can be found using either the method of undetermined coefficients, or the method of variation of parameters. The method of variation of parameters will be used as it is more general and can be used when the coefficients of the ODE depend on x as well. Let

$$y_p(x) = u_1 y_1 + u_2 y_2 \tag{1}$$

Where u_1, u_2 to be determined, and y_1, y_2 are the two basis solutions (the two linearly independent solutions of the homogeneous ODE) found earlier when solving the homogeneous ODE as

$$\begin{aligned}y_1 &= x^2\\ y_2 &= \ln(x)\,x^2\end{aligned}$$

In the Variation of parameters u_1, u_2 are found using

$$u_1 = -\int \frac{y_2 f(x)}{aW(x)} \tag{2}$$

$$u_2 = \int \frac{y_1 f(x)}{aW(x)} \tag{3}$$

Where $W(x)$ is the Wronskian and a is the coefficient in front of y'' in the given ODE. The Wronskian is given by $W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$. Hence

$$W = \begin{vmatrix} x^2 & \ln(x)\,x^2 \\ \frac{d}{dx}(x^2) & \frac{d}{dx}(\ln(x)\,x^2) \end{vmatrix}$$

Which gives

$$W = \begin{vmatrix} x^2 & \ln(x)\,x^2 \\ 2x & x + 2x\ln(x) \end{vmatrix}$$

Therefore

$$W = (x^2)(x + 2x\ln(x)) - (\ln(x)\,x^2)(2x)$$

Which simplifies to

$$W = x^3$$

Which simplifies to

$$W = x^3$$

Therefore Eq. (2) becomes

$$u_1 = -\int \frac{\ln(x)^2\,x^4}{x^5}\,dx$$

Which simplifies to

$$u_1 = -\int \frac{\ln(x)^2}{x}\,dx$$

Hence

$$u_1 = -\frac{\ln(x)^3}{3}$$

And Eq. (3) becomes

$$u_2 = \int \frac{x^4 \ln(x)}{x^5} dx$$

Which simplifies to

$$u_2 = \int \frac{\ln(x)}{x} dx$$

Hence

$$u_2 = \frac{\ln(x)^2}{2}$$

Therefore the particular solution, from equation (1) is

$$y_p(x) = \frac{\ln(x)^3 x^2}{6}$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \left((c_1 \ln(x) + c_2) x^2\right) + \left(\frac{\ln(x)^3 x^2}{6}\right) \\ &= \frac{\ln(x)^3 x^2}{6} + (c_1 \ln(x) + c_2) x^2 \end{aligned}$$

Which simplifies to

$$y = x^2 \left(\frac{\ln(x)^3}{6} + c_1 \ln(x) + c_2\right)$$

Summary

The solution(s) found are the following

$$y = x^2 \left(\frac{\ln(x)^3}{6} + c_1 \ln(x) + c_2\right) \tag{1}$$

Verification of solutions

$$y = x^2 \left(\frac{\ln(x)^3}{6} + c_1 \ln(x) + c_2\right)$$

Verified OK. `{0 < x}`

10.17.5 Solving using Kovacic algorithm

Writing the ode as

$$x^2y'' - 3y'x + 4y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= x^2 \\ B &= -3x \\ C &= 4 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{-1}{4x^2} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= -1 \\ t &= 4x^2 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = \left(-\frac{1}{4x^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\,dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0,1,2,4,6,8,\cdots\}$	$\{\cdots,-6,-4,-2,0,2,3,4,5,6,\cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1,2\}$	$\{2,3,4,5,6,7,\cdots\}$

Table 502: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 2 - 0 \\ &= 2 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 4x^2$. There is a pole at $x = 0$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 2 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Since pole order is not larger than 2 and the order at ∞ is 2 then the necessary conditions for case three are met. Therefore

$$L = [1, 2, 4, 6, 12]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = -\frac{1}{4x^2}$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = -\frac{1}{4}$. Hence

$$
\begin{aligned}
[\sqrt{r}]_c &= 0 \\
\alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{1}{2} \\
\alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = \frac{1}{2}
\end{aligned}
$$

Since the order of r at ∞ is 2 then $[\sqrt{r}]_\infty = 0$. Let b be the coefficient of $\frac{1}{x^2}$ in the Laurent series expansion of r at ∞. which can be found by dividing the leading coefficient of s by the leading coefficient of t from

$$
r = \frac{s}{t} = -\frac{1}{4x^2}
$$

Since the $\gcd(s,t) = 1$. This gives $b = -\frac{1}{4}$. Hence

$$
\begin{aligned}
[\sqrt{r}]_\infty &= 0 \\
\alpha_\infty^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{1}{2} \\
\alpha_\infty^- &= \frac{1}{2} - \sqrt{1+4b} = \frac{1}{2}
\end{aligned}
$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$
r = -\frac{1}{4x^2}
$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
0	2	0	$\frac{1}{2}$	$\frac{1}{2}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
2	0	$\frac{1}{2}$	$\frac{1}{2}$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$
d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}
$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω.

Trying $\alpha_{\infty}^{-} = \frac{1}{2}$ then

$$\begin{aligned} d &= \alpha_{\infty}^{-} - \left(\alpha_{c_1}^{-}\right) \\ &= \frac{1}{2} - \left(\frac{1}{2}\right) \\ &= 0 \end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c \in \Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x - c} \right) + s(\infty)[\sqrt{r}]_{\infty}$$

The above gives

$$\begin{aligned} \omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^{-}}{x - c_1} \right) + (-)[\sqrt{r}]_{\infty} \\ &= \frac{1}{2x} + (-)\,(0) \\ &= \frac{1}{2x} \\ &= \frac{1}{2x} \end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 0$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right) p = 0 \tag{1A}$$

Let

$$p(x) = 1 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$\begin{aligned} (0) + 2\left(\frac{1}{2x}\right)(0) + \left(\left(-\frac{1}{2x^2}\right) + \left(\frac{1}{2x}\right)^2 - \left(-\frac{1}{4x^2}\right)\right) &= 0 \\ 0 &= 0 \end{aligned}$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned} z_1(x) &= p e^{\int \omega \, dx} \\ &= e^{\int \frac{1}{2x} dx} \\ &= \sqrt{x} \end{aligned}$$

The first solution to the original ode in y is found from

$$
\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{-3x}{x^2}\,dx} \\
&= z_1 e^{\frac{3\ln(x)}{2}} \\
&= z_1 \left(x^{\frac{3}{2}}\right)
\end{aligned}
$$

Which simplifies to

$$y_1 = x^2$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx$$

Substituting gives

$$
\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{-3x}{x^2}\,dx}}{(y_1)^2}\,dx \\
&= y_1 \int \frac{e^{3\ln(x)}}{(y_1)^2}\,dx \\
&= y_1(\ln(x))
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1\left(x^2\right) + c_2\left(x^2(\ln(x))\right)
\end{aligned}
$$

This is second order nonhomogeneous ODE. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the nonhomogeneous ODE $Ay''(x) + By'(x) + Cy(x) = f(x)$. y_h is the solution to

$$x^2y'' - 3y'x + 4y = 0$$

The homogeneous solution is found using the Kovacic algorithm which results in

$$y_h = c_1 x^2 + c_2 x^2 \ln(x)$$

The particular solution y_p can be found using either the method of undetermined coefficients, or the method of variation of parameters. The method of variation of parameters will be used as it is more general and can be used when the coefficients of the ODE depend on x as well. Let

$$y_p(x) = u_1 y_1 + u_2 y_2 \tag{1}$$

Where u_1, u_2 to be determined, and y_1, y_2 are the two basis solutions (the two linearly independent solutions of the homogeneous ODE) found earlier when solving the homogeneous ODE as

$$y_1 = x^2$$
$$y_2 = \ln(x)\, x^2$$

In the Variation of parameters u_1, u_2 are found using

$$u_1 = -\int \frac{y_2 f(x)}{a W(x)} \tag{2}$$

$$u_2 = \int \frac{y_1 f(x)}{a W(x)} \tag{3}$$

Where $W(x)$ is the Wronskian and a is the coefficient in front of y'' in the given ODE. The Wronskian is given by $W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$. Hence

$$W = \begin{vmatrix} x^2 & \ln(x)\, x^2 \\ \frac{d}{dx}(x^2) & \frac{d}{dx}(\ln(x)\, x^2) \end{vmatrix}$$

Which gives

$$W = \begin{vmatrix} x^2 & \ln(x)\, x^2 \\ 2x & x + 2x\ln(x) \end{vmatrix}$$

Therefore

$$W = (x^2)(x + 2x\ln(x)) - (\ln(x)\, x^2)(2x)$$

Which simplifies to

$$W = x^3$$

Which simplifies to

$$W = x^3$$

Therefore Eq. (2) becomes

$$u_1 = -\int \frac{\ln (x)^2 x^4}{x^5} dx$$

Which simplifies to

$$u_1 = -\int \frac{\ln (x)^2}{x} dx$$

Hence

$$u_1 = -\frac{\ln (x)^3}{3}$$

And Eq. (3) becomes

$$u_2 = \int \frac{x^4 \ln (x)}{x^5} dx$$

Which simplifies to

$$u_2 = \int \frac{\ln (x)}{x} dx$$

Hence

$$u_2 = \frac{\ln (x)^2}{2}$$

Therefore the particular solution, from equation (1) is

$$y_p(x) = \frac{\ln (x)^3 x^2}{6}$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= (c_1 x^2 + c_2 x^2 \ln (x)) + \left(\frac{\ln (x)^3 x^2}{6}\right) \end{aligned}$$

Which simplifies to

$$y = x^2(c_1 + c_2 \ln(x)) + \frac{\ln(x)^3 x^2}{6}$$

Summary

The solution(s) found are the following

$$y = x^2(c_1 + c_2 \ln(x)) + \frac{\ln(x)^3 x^2}{6} \tag{1}$$

Verification of solutions

$$y = x^2(c_1 + c_2 \ln(x)) + \frac{\ln(x)^3 x^2}{6}$$

Verified OK. `{0 < x}`

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 2; linear nonhomogeneous with symmetry [0,1]
trying a double symmetry of the form [xi=0, eta=F(x)]
-> Try solving first the homogeneous part of the ODE
   checking if the LODE has constant coefficients
   checking if the LODE is of Euler type
   <- LODE of Euler type successful
<- solving first the homogeneous part of the ODE successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 20

```
dsolve(x^2*diff(y(x),x$2)-3*x*diff(y(x),x)+4*y(x) = x^2*ln(x),y(x), singsol=all)
```

$$y(x) = x^2 \left(c_2 + \ln(x) c_1 + \frac{\ln(x)^3}{6}\right)$$

✓ Solution by Mathematica

Time used: 0.027 (sec). Leaf size: 27

```
DSolve[x^2*y''[x]-3*x*y'[x]+4*y[x] == x^2*Log[x],y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \frac{1}{6}x^2\left(\log^3(x) + 12c_2\log(x) + 6c_1\right)$$

10.18 problem 20

Internal problem ID [700]
Internal file name [`OUTPUT/700_Sunday_June_05_2022_01_47_15_AM_75865028/index.tex`]

Book: Elementary differential equations and boundary value problems, 10th ed., Boyce and DiPrima
Section: Chapter 3, Second order linear equations, section 3.6, Variation of Parameters. page 190
Problem number: 20.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_bessel_ode", "second_order_change_of_variable_on_y_method_1"**

Maple gives the following as the ode type

```
[[_2nd_order, _linear, _nonhomogeneous]]
```

$$\boxed{x^2y''+y'x+\left(x^2-\frac{1}{4}\right)y=g(x)}$$

10.18.1 Solving as second order change of variable on y method 1 ode

This is second order non-homogeneous ODE. Let the solution be

$$y=y_h+y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x)=0$, and y_p is a particular solution to the non-homogeneous ODE $Ay''(x)+By'(x)+Cy(x)=f(x)$. y_h is the solution to

$$x^2y''+y'x+\left(x^2-\frac{1}{4}\right)y=0$$

In normal form the given ode is written as

$$y''+p(x)\,y'+q(x)\,y=0 \tag{2}$$

Where

$$
\begin{aligned}
p(x) &= \frac{1}{x} \\
q(x) &= \frac{x^2 - \frac{1}{4}}{x^2}
\end{aligned}
$$

Calculating the Liouville ode invariant Q given by

$$
\begin{aligned}
Q &= q - \frac{p'}{2} - \frac{p^2}{4} \\
&= \frac{x^2 - \frac{1}{4}}{x^2} - \frac{\left(\frac{1}{x}\right)'}{2} - \frac{\left(\frac{1}{x}\right)^2}{4} \\
&= \frac{x^2 - \frac{1}{4}}{x^2} - \frac{\left(-\frac{1}{x^2}\right)}{2} - \frac{\left(\frac{1}{x^2}\right)}{4} \\
&= \frac{x^2 - \frac{1}{4}}{x^2} - \left(-\frac{1}{2x^2}\right) - \frac{1}{4x^2} \\
&= 1
\end{aligned}
$$

Since the Liouville ode invariant does not depend on the independent variable x then the transformation

$$
y = v(x)\, z(x) \tag{3}
$$

is used to change the original ode to a constant coefficients ode in v. In (3) the term $z(x)$ is given by

$$
\begin{aligned}
z(x) &= \mathrm{e}^{-\left(\int \frac{p(x)}{2} dx\right)} \\
&= e^{-\int \frac{\frac{1}{x}}{2}} \\
&= \frac{1}{\sqrt{x}} \qquad (5)
\end{aligned}
$$

Hence (3) becomes

$$
y = \frac{v(x)}{\sqrt{x}} \tag{4}
$$

Applying this change of variable to the original ode results in

$$
x^{\frac{3}{2}}(v''(x) + v(x)) = g(x)
$$

Which is now solved for $v(x)$ Simplyfing the ode gives

$$
v''(x) + v(x) = \frac{g(x)}{x^{\frac{3}{2}}}
$$

This is second order non-homogeneous ODE. In standard form the ODE is

$$Av''(x) + Bv'(x) + Cv(x) = f(x)$$

Where $A = 1, B = 0, C = 1, f(x) = \frac{g(x)}{x^{\frac{3}{2}}}$. Let the solution be

$$v(x) = v_h + v_p$$

Where v_h is the solution to the homogeneous ODE $Av''(x)+Bv'(x)+Cv(x) = 0$, and v_p is a particular solution to the non-homogeneous ODE $Av''(x)+Bv'(x)+Cv(x) = f(x)$. v_h is the solution to

$$v''(x) + v(x) = 0$$

This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$Av''(x) + Bv'(x) + Cv(x) = 0$$

Where in the above $A = 1, B = 0, C = 1$. Let the solution be $v(x) = e^{\lambda x}$. Substituting this into the ODE gives

$$\lambda^2 \mathrm{e}^{\lambda x} + \mathrm{e}^{\lambda x} = 0 \tag{1}$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda x}$ gives

$$\lambda^2 + 1 = 0 \tag{2}$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form.Using the quadratic formula

$$\lambda_{1,2} = \frac{-B}{2A} \pm \frac{1}{2A}\sqrt{B^2 - 4AC}$$

Substituting $A = 1, B = 0, C = 1$ into the above gives

$$\begin{aligned}\lambda_{1,2} &= \frac{0}{(2)(1)} \pm \frac{1}{(2)(1)}\sqrt{0^2 - (4)(1)(1)}\\ &= \pm i\end{aligned}$$

Hence

$$\lambda_1 = +i$$
$$\lambda_2 = -i$$

Which simplifies to

$$\lambda_1 = i$$
$$\lambda_2 = -i$$

Since roots are complex conjugate of each others, then let the roots be

$$\lambda_{1,2} = \alpha \pm i\beta$$

Where $\alpha = 0$ and $\beta = 1$. Therefore the final solution, when using Euler relation, can be written as

$$v(x) = e^{\alpha x}(c_1 \cos(\beta x) + c_2 \sin(\beta x))$$

Which becomes

$$v(x) = e^0(c_1 \cos(x) + c_2 \sin(x))$$

Or

$$v(x) = c_1 \cos(x) + c_2 \sin(x)$$

Therefore the homogeneous solution v_h is

$$v_h = c_1 \cos(x) + c_2 \sin(x)$$

The particular solution v_p can be found using either the method of undetermined coefficients, or the method of variation of parameters. The method of variation of parameters will be used as it is more general and can be used when the coefficients of the ODE depend on x as well. Let

$$v_p(x) = u_1 v_1 + u_2 v_2 \tag{1}$$

Where u_1, u_2 to be determined, and v_1, v_2 are the two basis solutions (the two linearly independent solutions of the homogeneous ODE) found earlier when solving the homogeneous ODE as

$$v_1 = \cos(x)$$
$$v_2 = \sin(x)$$

In the Variation of parameters u_1, u_2 are found using

$$u_1 = -\int \frac{v_2 f(x)}{aW(x)} \tag{2}$$

$$u_2 = \int \frac{v_1 f(x)}{aW(x)} \tag{3}$$

Where $W(x)$ is the Wronskian and a is the coefficient in front of v'' in the given ODE. The Wronskian is given by $W = \begin{vmatrix} v_1 & v_2 \\ v_1' & v_2' \end{vmatrix}$. Hence

$$W = \begin{vmatrix} \cos(x) & \sin(x) \\ \frac{d}{dx}(\cos(x)) & \frac{d}{dx}(\sin(x)) \end{vmatrix}$$

Which gives

$$W = \begin{vmatrix} \cos(x) & \sin(x) \\ -\sin(x) & \cos(x) \end{vmatrix}$$

Therefore

$$W = (\cos(x))(\cos(x)) - (\sin(x))(-\sin(x))$$

Which simplifies to

$$W = \cos(x)^2 + \sin(x)^2$$

Which simplifies to

$$W = 1$$

Therefore Eq. (2) becomes

$$u_1 = -\int \frac{\frac{\sin(x)g(x)}{x^{\frac{3}{2}}}}{1} dx$$

Which simplifies to

$$u_1 = -\int \frac{\sin(x)\, g(x)}{x^{\frac{3}{2}}} dx$$

Hence

$$u_1 = -\left(\int_0^x \frac{\sin(\alpha)\, g(\alpha)}{\alpha^{\frac{3}{2}}} d\alpha\right)$$

And Eq. (3) becomes

$$u_2 = \int \frac{\frac{\cos(x)g(x)}{x^{\frac{3}{2}}}}{1} dx$$

Which simplifies to

$$u_2 = \int \frac{\cos(x)\, g(x)}{x^{\frac{3}{2}}} dx$$

Hence

$$u_2 = \int_0^x \frac{\cos(\alpha)\, g(\alpha)}{\alpha^{\frac{3}{2}}} d\alpha$$

Therefore the particular solution, from equation (1) is

$$v_p(x) = -\left(\int_0^x \frac{\sin(\alpha)\, g(\alpha)}{\alpha^{\frac{3}{2}}} d\alpha\right) \cos(x) + \left(\int_0^x \frac{\cos(\alpha)\, g(\alpha)}{\alpha^{\frac{3}{2}}} d\alpha\right) \sin(x)$$

Therefore the general solution is

$$\begin{aligned} v &= v_h + v_p \\ &= (c_1 \cos(x) + c_2 \sin(x)) \\ &\quad + \left(-\left(\int_0^x \frac{\sin(\alpha)\, g(\alpha)}{\alpha^{\frac{3}{2}}} d\alpha\right) \cos(x) + \left(\int_0^x \frac{\cos(\alpha)\, g(\alpha)}{\alpha^{\frac{3}{2}}} d\alpha\right) \sin(x)\right) \end{aligned}$$

Now that $v(x)$ is known, then

$$\begin{aligned} y &= v(x)\, z(x) \\ &= \left(c_1 \cos(x) + c_2 \sin(x) - \left(\int_0^x \frac{\sin(\alpha)\, g(\alpha)}{\alpha^{\frac{3}{2}}} d\alpha\right) \cos(x) + \left(\int_0^x \frac{\cos(\alpha)\, g(\alpha)}{\alpha^{\frac{3}{2}}} d\alpha\right) \sin(x)\right) (z(x)) \end{aligned} \tag{7}$$

But from (5)

$$z(x) = \frac{1}{\sqrt{x}}$$

Hence (7) becomes

$$y = \frac{c_1 \cos(x) + c_2 \sin(x) - \left(\int_0^x \frac{\sin(\alpha) g(\alpha)}{\alpha^{\frac{3}{2}}} d\alpha\right) \cos(x) + \left(\int_0^x \frac{\cos(\alpha) g(\alpha)}{\alpha^{\frac{3}{2}}} d\alpha\right) \sin(x)}{\sqrt{x}}$$

Therefore the homogeneous solution y_h is

$$y_h = \frac{c_1 \cos(x) + c_2 \sin(x) - \left(\int_0^x \frac{\sin(\alpha) g(\alpha)}{\alpha^{\frac{3}{2}}} d\alpha\right) \cos(x) + \left(\int_0^x \frac{\cos(\alpha) g(\alpha)}{\alpha^{\frac{3}{2}}} d\alpha\right) \sin(x)}{\sqrt{x}}$$

The particular solution y_p can be found using either the method of undetermined coefficients, or the method of variation of parameters. The method of variation of parameters will be used as it is more general and can be used when the coefficients of the ODE depend on x as well. Let

$$y_p(x) = u_1 y_1 + u_2 y_2 \tag{1}$$

Where u_1, u_2 to be determined, and y_1, y_2 are the two basis solutions (the two linearly independent solutions of the homogeneous ODE) found earlier when solving the homogeneous ODE as

$$y_1 = \frac{\cos(x)}{\sqrt{x}}$$
$$y_2 = \frac{\sin(x)}{\sqrt{x}}$$

In the Variation of parameters u_1, u_2 are found using

$$u_1 = -\int \frac{y_2 f(x)}{aW(x)} \tag{2}$$
$$u_2 = \int \frac{y_1 f(x)}{aW(x)} \tag{3}$$

Where $W(x)$ is the Wronskian and a is the coefficient in front of y'' in the given ODE. The Wronskian is given by $W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$. Hence

$$W = \begin{vmatrix} \frac{\cos(x)}{\sqrt{x}} & \frac{\sin(x)}{\sqrt{x}} \\ \frac{d}{dx}\left(\frac{\cos(x)}{\sqrt{x}}\right) & \frac{d}{dx}\left(\frac{\sin(x)}{\sqrt{x}}\right) \end{vmatrix}$$

Which gives

$$W = \begin{vmatrix} \frac{\cos(x)}{\sqrt{x}} & \frac{\sin(x)}{\sqrt{x}} \\ -\frac{\sin(x)}{\sqrt{x}} - \frac{\cos(x)}{2x^{\frac{3}{2}}} & \frac{\cos(x)}{\sqrt{x}} - \frac{\sin(x)}{2x^{\frac{3}{2}}} \end{vmatrix}$$

Therefore

$$W = \left(\frac{\cos(x)}{\sqrt{x}}\right)\left(\frac{\cos(x)}{\sqrt{x}} - \frac{\sin(x)}{2x^{\frac{3}{2}}}\right) - \left(\frac{\sin(x)}{\sqrt{x}}\right)\left(-\frac{\sin(x)}{\sqrt{x}} - \frac{\cos(x)}{2x^{\frac{3}{2}}}\right)$$

Which simplifies to

$$W = \frac{\cos(x)^2 + \sin(x)^2}{x}$$

Which simplifies to

$$W = \frac{1}{x}$$

Therefore Eq. (2) becomes

$$u_1 = -\left(\int \frac{\frac{\sin(x)g(x)}{\sqrt{x}}}{x} dx\right)$$

Which simplifies to

$$u_1 = -\left(\int \frac{\sin(x)\, g(x)}{x^{\frac{3}{2}}} dx\right)$$

Hence

$$u_1 = -\left(\int_0^x \frac{\sin(\alpha)\, g(\alpha)}{\alpha^{\frac{3}{2}}} d\alpha\right)$$

And Eq. (3) becomes

$$u_2 = \int \frac{\frac{\cos(x)g(x)}{\sqrt{x}}}{x} dx$$

Which simplifies to

$$u_2 = \int \frac{\cos(x)\, g(x)}{x^{\frac{3}{2}}} dx$$

Hence

$$u_2 = \int_0^x \frac{\cos(\alpha)\, g(\alpha)}{\alpha^{\frac{3}{2}}} d\alpha$$

Therefore the particular solution, from equation (1) is

$$y_p(x) = -\frac{\left(\int_0^x \frac{\sin(\alpha)g(\alpha)}{\alpha^{\frac{3}{2}}} d\alpha\right)\cos(x)}{\sqrt{x}} + \frac{\left(\int_0^x \frac{\cos(\alpha)g(\alpha)}{\alpha^{\frac{3}{2}}} d\alpha\right)\sin(x)}{\sqrt{x}}$$

Which simplifies to

$$y_p(x) = \frac{-\left(\int_0^x \frac{\sin(\alpha) g(\alpha)}{\alpha^{\frac{3}{2}}} d\alpha\right) \cos(x) + \left(\int_0^x \frac{\cos(\alpha) g(\alpha)}{\alpha^{\frac{3}{2}}} d\alpha\right) \sin(x)}{\sqrt{x}}$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \left(\frac{c_1 \cos(x) + c_2 \sin(x) - \left(\int_0^x \frac{\sin(\alpha) g(\alpha)}{\alpha^{\frac{3}{2}}} d\alpha\right) \cos(x) + \left(\int_0^x \frac{\cos(\alpha) g(\alpha)}{\alpha^{\frac{3}{2}}} d\alpha\right) \sin(x)}{\sqrt{x}}\right) \\ &\quad + \left(\frac{-\left(\int_0^x \frac{\sin(\alpha) g(\alpha)}{\alpha^{\frac{3}{2}}} d\alpha\right) \cos(x) + \left(\int_0^x \frac{\cos(\alpha) g(\alpha)}{\alpha^{\frac{3}{2}}} d\alpha\right) \sin(x)}{\sqrt{x}}\right) \end{aligned}$$

Summary

The solution(s) found are the following

$$\begin{aligned} y &= \frac{c_1 \cos(x) + c_2 \sin(x) - \left(\int_0^x \frac{\sin(\alpha) g(\alpha)}{\alpha^{\frac{3}{2}}} d\alpha\right) \cos(x) + \left(\int_0^x \frac{\cos(\alpha) g(\alpha)}{\alpha^{\frac{3}{2}}} d\alpha\right) \sin(x)}{\sqrt{x}} \\ &\quad + \frac{-\left(\int_0^x \frac{\sin(\alpha) g(\alpha)}{\alpha^{\frac{3}{2}}} d\alpha\right) \cos(x) + \left(\int_0^x \frac{\cos(\alpha) g(\alpha)}{\alpha^{\frac{3}{2}}} d\alpha\right) \sin(x)}{\sqrt{x}} \end{aligned} \tag{1}$$

Verification of solutions

$$\begin{aligned} y &= \frac{c_1 \cos(x) + c_2 \sin(x) - \left(\int_0^x \frac{\sin(\alpha) g(\alpha)}{\alpha^{\frac{3}{2}}} d\alpha\right) \cos(x) + \left(\int_0^x \frac{\cos(\alpha) g(\alpha)}{\alpha^{\frac{3}{2}}} d\alpha\right) \sin(x)}{\sqrt{x}} \\ &\quad + \frac{-\left(\int_0^x \frac{\sin(\alpha) g(\alpha)}{\alpha^{\frac{3}{2}}} d\alpha\right) \cos(x) + \left(\int_0^x \frac{\cos(\alpha) g(\alpha)}{\alpha^{\frac{3}{2}}} d\alpha\right) \sin(x)}{\sqrt{x}} \end{aligned}$$

Verified OK.

10.18.2 Solving as second order bessel ode ode

Writing the ode as

$$x^2y'' + y'x + \left(x^2 - \frac{1}{4}\right) y = g(x) \tag{1}$$

Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE and y_p is a particular solution to the non-homogeneous ODE. Bessel ode has the form

$$x^2y'' + y'x + \left(-n^2 + x^2\right) y = 0 \tag{2}$$

The generalized form of Bessel ode is given by Bowman (1958) as the following

$$x^2y'' + (1 - 2\alpha)\, xy' + \left(\beta^2\gamma^2x^{2\gamma} - n^2\gamma^2 + \alpha^2\right) y = 0 \tag{3}$$

With the standard solution

$$y = x^{\alpha}(c_1 \operatorname{BesselJ}(n, \beta\, x^{\gamma}) + c_2 \operatorname{BesselY}(n, \beta\, x^{\gamma})) \tag{4}$$

Comparing (3) to (1) and solving for α, β, n, γ gives

$$\begin{aligned}
\alpha &= 0 \\
\beta &= 1 \\
n &= -\frac{1}{2} \\
\gamma &= 1
\end{aligned}$$

Substituting all the above into (4) gives the solution as

$$y = \frac{c_1\sqrt{2}\, \cos(x)}{\sqrt{\pi}\, \sqrt{x}} + \frac{c_2\sqrt{2}\, \sin(x)}{\sqrt{\pi}\, \sqrt{x}}$$

Therefore the homogeneous solution y_h is

$$y_h = \frac{c_1\sqrt{2}\, \cos(x)}{\sqrt{\pi}\, \sqrt{x}} + \frac{c_2\sqrt{2}\, \sin(x)}{\sqrt{\pi}\, \sqrt{x}}$$

The particular solution y_p can be found using either the method of undetermined coefficients, or the method of variation of parameters. The method of variation of

parameters will be used as it is more general and can be used when the coefficients of the ODE depend on x as well. Let

$$y_p(x) = u_1 y_1 + u_2 y_2 \tag{1}$$

Where u_1, u_2 to be determined, and y_1, y_2 are the two basis solutions (the two linearly independent solutions of the homogeneous ODE) found earlier when solving the homogeneous ODE as

$$y_1 = \frac{\cos(x)}{\sqrt{x}}$$

$$y_2 = \frac{\sin(x)}{\sqrt{x}}$$

In the Variation of parameters u_1, u_2 are found using

$$u_1 = -\int \frac{y_2 f(x)}{aW(x)} \tag{2}$$

$$u_2 = \int \frac{y_1 f(x)}{aW(x)} \tag{3}$$

Where $W(x)$ is the Wronskian and a is the coefficient in front of y'' in the given ODE. The Wronskian is given by $W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$. Hence

$$W = \begin{vmatrix} \frac{\cos(x)}{\sqrt{x}} & \frac{\sin(x)}{\sqrt{x}} \\ \frac{d}{dx}\left(\frac{\cos(x)}{\sqrt{x}}\right) & \frac{d}{dx}\left(\frac{\sin(x)}{\sqrt{x}}\right) \end{vmatrix}$$

Which gives

$$W = \begin{vmatrix} \frac{\cos(x)}{\sqrt{x}} & \frac{\sin(x)}{\sqrt{x}} \\ -\frac{\sin(x)}{\sqrt{x}} - \frac{\cos(x)}{2x^{\frac{3}{2}}} & \frac{\cos(x)}{\sqrt{x}} - \frac{\sin(x)}{2x^{\frac{3}{2}}} \end{vmatrix}$$

Therefore

$$W = \left(\frac{\cos(x)}{\sqrt{x}}\right)\left(\frac{\cos(x)}{\sqrt{x}} - \frac{\sin(x)}{2x^{\frac{3}{2}}}\right) - \left(\frac{\sin(x)}{\sqrt{x}}\right)\left(-\frac{\sin(x)}{\sqrt{x}} - \frac{\cos(x)}{2x^{\frac{3}{2}}}\right)$$

Which simplifies to

$$W = \frac{\cos(x)^2 + \sin(x)^2}{x}$$

Which simplifies to

$$W = \frac{1}{x}$$

Therefore Eq. (2) becomes

$$u_1 = -\left(\int \frac{\frac{\sin(x) g(x)}{\sqrt{x}}}{x} dx\right)$$

Which simplifies to

$$u_1 = -\left(\int \frac{\sin(x) g(x)}{x^{\frac{3}{2}}} dx\right)$$

Hence

$$u_1 = -\left(\int_0^x \frac{\sin(\alpha) g(\alpha)}{\alpha^{\frac{3}{2}}} d\alpha\right)$$

And Eq. (3) becomes

$$u_2 = \int \frac{\frac{\cos(x) g(x)}{\sqrt{x}}}{x} dx$$

Which simplifies to

$$u_2 = \int \frac{\cos(x) g(x)}{x^{\frac{3}{2}}} dx$$

Hence

$$u_2 = \int_0^x \frac{\cos(\alpha) g(\alpha)}{\alpha^{\frac{3}{2}}} d\alpha$$

Therefore the particular solution, from equation (1) is

$$y_p(x) = -\frac{\left(\int_0^x \frac{\sin(\alpha) g(\alpha)}{\alpha^{\frac{3}{2}}} d\alpha\right) \cos(x)}{\sqrt{x}} + \frac{\left(\int_0^x \frac{\cos(\alpha) g(\alpha)}{\alpha^{\frac{3}{2}}} d\alpha\right) \sin(x)}{\sqrt{x}}$$

Which simplifies to

$$y_p(x) = \frac{-\left(\int_0^x \frac{\sin(\alpha)g(\alpha)}{\alpha^{\frac{3}{2}}}d\alpha\right)\cos(x) + \left(\int_0^x \frac{\cos(\alpha)g(\alpha)}{\alpha^{\frac{3}{2}}}d\alpha\right)\sin(x)}{\sqrt{x}}$$

Therefore the general solution is

$$\begin{aligned}
y &= y_h + y_p \\
&= \left(\frac{c_1\sqrt{2}\,\cos(x)}{\sqrt{\pi}\,\sqrt{x}} + \frac{c_2\sqrt{2}\,\sin(x)}{\sqrt{\pi}\,\sqrt{x}}\right) \\
&\quad + \left(\frac{-\left(\int_0^x \frac{\sin(\alpha)g(\alpha)}{\alpha^{\frac{3}{2}}}d\alpha\right)\cos(x) + \left(\int_0^x \frac{\cos(\alpha)g(\alpha)}{\alpha^{\frac{3}{2}}}d\alpha\right)\sin(x)}{\sqrt{x}}\right)
\end{aligned}$$

Summary

The solution(s) found are the following

$$\begin{aligned}
y = {}& \frac{c_1\sqrt{2}\,\cos(x)}{\sqrt{\pi}\,\sqrt{x}} + \frac{c_2\sqrt{2}\,\sin(x)}{\sqrt{\pi}\,\sqrt{x}} \\
&+ \frac{-\left(\int_0^x \frac{\sin(\alpha)g(\alpha)}{\alpha^{\frac{3}{2}}}d\alpha\right)\cos(x) + \left(\int_0^x \frac{\cos(\alpha)g(\alpha)}{\alpha^{\frac{3}{2}}}d\alpha\right)\sin(x)}{\sqrt{x}}
\end{aligned} \tag{1}$$

Verification of solutions

$$y = \frac{c_1\sqrt{2}\,\cos(x)}{\sqrt{\pi}\,\sqrt{x}} + \frac{c_2\sqrt{2}\,\sin(x)}{\sqrt{\pi}\,\sqrt{x}} + \frac{-\left(\int_0^x \frac{\sin(\alpha)g(\alpha)}{\alpha^{\frac{3}{2}}}d\alpha\right)\cos(x) + \left(\int_0^x \frac{\cos(\alpha)g(\alpha)}{\alpha^{\frac{3}{2}}}d\alpha\right)\sin(x)}{\sqrt{x}}$$

Verified OK.

10.18.3 Solving using Kovacic algorithm

Writing the ode as

$$x^2y'' + y'x + \left(x^2 - \frac{1}{4}\right)y = 0 \tag{1}$$

$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned}
A &= x^2 \\
B &= x \\
C &= x^2 - \frac{1}{4}
\end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{-1}{1} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= -1 \\ t &= 1 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = -z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\,dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 503: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 0 - 0 \\ &= 0 \end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Since $r = -1$ is not a function of x, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$z_1(x) = \cos(x)$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$\begin{aligned} y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx} \\ &= z_1 e^{-\int \frac{1}{2}\frac{x}{x^2}\,dx} \\ &= z_1 e^{-\frac{\ln(x)}{2}} \\ &= z_1 \left(\frac{1}{\sqrt{x}}\right) \end{aligned}$$

Which simplifies to

$$y_1 = \frac{\cos(x)}{\sqrt{x}}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\, dx}}{y_1^2}\, dx$$

Substituting gives

$$\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{x}{x^2}\, dx}}{(y_1)^2}\, dx \\
&= y_1 \int \frac{e^{-\ln(x)}}{(y_1)^2}\, dx \\
&= y_1(\tan(x))
\end{aligned}$$

Therefore the solution is

$$\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1 \left(\frac{\cos(x)}{\sqrt{x}}\right) + c_2 \left(\frac{\cos(x)}{\sqrt{x}}(\tan(x))\right)
\end{aligned}$$

This is second order nonhomogeneous ODE. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the nonhomogeneous ODE $Ay''(x) + By'(x) + Cy(x) = f(x)$. y_h is the solution to

$$x^2 y'' + y'x + \left(x^2 - \frac{1}{4}\right) y = 0$$

The homogeneous solution is found using the Kovacic algorithm which results in

$$y_h = \frac{c_1 \cos(x)}{\sqrt{x}} + \frac{c_2 \sin(x)}{\sqrt{x}}$$

The particular solution y_p can be found using either the method of undetermined coefficients, or the method of variation of parameters. The method of variation of

parameters will be used as it is more general and can be used when the coefficients of the ODE depend on x as well. Let

$$y_p(x) = u_1 y_1 + u_2 y_2 \tag{1}$$

Where u_1, u_2 to be determined, and y_1, y_2 are the two basis solutions (the two linearly independent solutions of the homogeneous ODE) found earlier when solving the homogeneous ODE as

$$y_1 = \frac{\cos(x)}{\sqrt{x}}$$

$$y_2 = \frac{\sin(x)}{\sqrt{x}}$$

In the Variation of parameters u_1, u_2 are found using

$$u_1 = -\int \frac{y_2 f(x)}{aW(x)} \tag{2}$$

$$u_2 = \int \frac{y_1 f(x)}{aW(x)} \tag{3}$$

Where $W(x)$ is the Wronskian and a is the coefficient in front of y'' in the given ODE. The Wronskian is given by $W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$. Hence

$$W = \begin{vmatrix} \frac{\cos(x)}{\sqrt{x}} & \frac{\sin(x)}{\sqrt{x}} \\ \frac{d}{dx}\left(\frac{\cos(x)}{\sqrt{x}}\right) & \frac{d}{dx}\left(\frac{\sin(x)}{\sqrt{x}}\right) \end{vmatrix}$$

Which gives

$$W = \begin{vmatrix} \frac{\cos(x)}{\sqrt{x}} & \frac{\sin(x)}{\sqrt{x}} \\ -\frac{\sin(x)}{\sqrt{x}} - \frac{\cos(x)}{2x^{\frac{3}{2}}} & \frac{\cos(x)}{\sqrt{x}} - \frac{\sin(x)}{2x^{\frac{3}{2}}} \end{vmatrix}$$

Therefore

$$W = \left(\frac{\cos(x)}{\sqrt{x}}\right)\left(\frac{\cos(x)}{\sqrt{x}} - \frac{\sin(x)}{2x^{\frac{3}{2}}}\right) - \left(\frac{\sin(x)}{\sqrt{x}}\right)\left(-\frac{\sin(x)}{\sqrt{x}} - \frac{\cos(x)}{2x^{\frac{3}{2}}}\right)$$

Which simplifies to

$$W = \frac{\cos(x)^2 + \sin(x)^2}{x}$$

Which simplifies to

$$W = \frac{1}{x}$$

Therefore Eq. (2) becomes

$$u_1 = -\left(\int \frac{\frac{\sin(x)g(x)}{\sqrt{x}}}{x}dx\right)$$

Which simplifies to

$$u_1 = -\left(\int \frac{\sin(x)\,g(x)}{x^{\frac{3}{2}}}dx\right)$$

Hence

$$u_1 = -\left(\int_0^x \frac{\sin(\alpha)\,g(\alpha)}{\alpha^{\frac{3}{2}}}d\alpha\right)$$

And Eq. (3) becomes

$$u_2 = \int \frac{\frac{\cos(x)g(x)}{\sqrt{x}}}{x}dx$$

Which simplifies to

$$u_2 = \int \frac{\cos(x)\,g(x)}{x^{\frac{3}{2}}}dx$$

Hence

$$u_2 = \int_0^x \frac{\cos(\alpha)\,g(\alpha)}{\alpha^{\frac{3}{2}}}d\alpha$$

Therefore the particular solution, from equation (1) is

$$y_p(x) = -\frac{\left(\int_0^x \frac{\sin(\alpha)g(\alpha)}{\alpha^{\frac{3}{2}}}d\alpha\right)\cos(x)}{\sqrt{x}} + \frac{\left(\int_0^x \frac{\cos(\alpha)g(\alpha)}{\alpha^{\frac{3}{2}}}d\alpha\right)\sin(x)}{\sqrt{x}}$$

Which simplifies to

$$y_p(x) = \frac{-\left(\int_0^x \frac{\sin(\alpha)g(\alpha)}{\alpha^{\frac{3}{2}}}d\alpha\right)\cos(x) + \left(\int_0^x \frac{\cos(\alpha)g(\alpha)}{\alpha^{\frac{3}{2}}}d\alpha\right)\sin(x)}{\sqrt{x}}$$

Therefore the general solution is

$$y = y_h + y_p$$
$$= \left(\frac{c_1\cos(x)}{\sqrt{x}} + \frac{c_2\sin(x)}{\sqrt{x}}\right) + \left(\frac{-\left(\int_0^x \frac{\sin(\alpha)g(\alpha)}{\alpha^{\frac{3}{2}}}d\alpha\right)\cos(x) + \left(\int_0^x \frac{\cos(\alpha)g(\alpha)}{\alpha^{\frac{3}{2}}}d\alpha\right)\sin(x)}{\sqrt{x}}\right)$$

Which simplifies to

$$y = \frac{c_1\cos(x) + c_2\sin(x)}{\sqrt{x}} + \frac{-\left(\int_0^x \frac{\sin(\alpha)g(\alpha)}{\alpha^{\frac{3}{2}}}d\alpha\right)\cos(x) + \left(\int_0^x \frac{\cos(\alpha)g(\alpha)}{\alpha^{\frac{3}{2}}}d\alpha\right)\sin(x)}{\sqrt{x}}$$

Summary

The solution(s) found are the following

$$y = \frac{c_1\cos(x) + c_2\sin(x)}{\sqrt{x}} + \frac{-\left(\int_0^x \frac{\sin(\alpha)g(\alpha)}{\alpha^{\frac{3}{2}}}d\alpha\right)\cos(x) + \left(\int_0^x \frac{\cos(\alpha)g(\alpha)}{\alpha^{\frac{3}{2}}}d\alpha\right)\sin(x)}{\sqrt{x}} \tag{1}$$

Verification of solutions

$$y = \frac{c_1\cos(x) + c_2\sin(x)}{\sqrt{x}} + \frac{-\left(\int_0^x \frac{\sin(\alpha)g(\alpha)}{\alpha^{\frac{3}{2}}}d\alpha\right)\cos(x) + \left(\int_0^x \frac{\cos(\alpha)g(\alpha)}{\alpha^{\frac{3}{2}}}d\alpha\right)\sin(x)}{\sqrt{x}}$$

Verified OK.

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 2; linear nonhomogeneous with symmetry [0,1]
trying a double symmetry of the form [xi=0, eta=F(x)]
-> Try solving first the homogeneous part of the ODE
   checking if the LODE has constant coefficients
   checking if the LODE is of Euler type
   trying a symmetry of the form [xi=0, eta=F(x)]
   checking if the LODE is missing y
   -> Trying a Liouvillian solution using Kovacics algorithm
      A Liouvillian solution exists
      Group is reducible or imprimitive
   <- Kovacics algorithm successful
<- solving first the homogeneous part of the ODE successful`
```

✓ Solution by Maple

Time used: 0.015 (sec). Leaf size: 44

```
dsolve(x^2*diff(y(x),x$2)+x*diff(y(x),x)+(x^2-25/100)*y(x) = g(x),y(x), singsol=all)
```

$$y(x) = \frac{\sin(x) c_2 + \cos(x) c_1 + \left(\int \frac{\cos(x) g(x)}{x^{\frac{3}{2}}} dx\right) \sin(x) - \left(\int \frac{\sin(x) g(x)}{x^{\frac{3}{2}}} dx\right) \cos(x)}{\sqrt{x}}$$

✓ Solution by Mathematica

Time used: 0.203 (sec). Leaf size: 107

```
DSolve[x^2*y''[x]+x*y'[x]+(x^2-25/100)*y[x] == g[x],y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \frac{e^{-ix}\left(2 \int_1^x \frac{ie^{iK[1]} g(K[1])}{2K[1]^{3/2}} dK[1] - ie^{2ix} \int_1^x \frac{e^{-iK[2]} g(K[2])}{K[2]^{3/2}} dK[2] - ic_2 e^{2ix} + 2c_1\right)}{2\sqrt{x}}$$

10.19 problem 29

Internal problem ID [701]
Internal file name [`OUTPUT/701_Sunday_June_05_2022_01_47_17_AM_20232498/index.tex`]

Book: Elementary differential equations and boundary value problems, 10th ed., Boyce and DiPrima
Section: Chapter 3, Second order linear equations, section 3.6, Variation of Parameters. page 190
Problem number: 29.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_euler_ode", "second_order_change_of_variable_on_x_method_1", "second_order_change_of_variable_on_x_method_2", "second_order_change_of_variable_on_y_method_2", "linear_second_order_ode_solved_by_an_integrating_factor", "second_order_ode_non_constant_coeff_transformation_on_B"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

$$\boxed{t^2y'' - 2ty' + 2y = 4t^2}$$

10.19.1 Solving as second order euler ode ode

This is second order non-homogeneous ODE. In standard form the ODE is

$$Ay''(t) + By'(t) + Cy(t) = f(t)$$

Where $A = t^2, B = -2t, C = 2, f(t) = 4t^2$. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(t) + By'(t) + Cy(t) = 0$, and y_p is a particular solution to the non-homogeneous ODE $Ay''(t) + By'(t) + Cy(t) = f(t)$. Solving for y_h from

$$t^2y'' - 2ty' + 2y = 0$$

This is Euler second order ODE. Let the solution be $y = t^r$, then $y' = rt^{r-1}$ and $y'' = r(r-1)t^{r-2}$. Substituting these back into the given ODE gives

$$t^2(r(r-1))t^{r-2} - 2trt^{r-1} + 2t^r = 0$$

Simplifying gives

$$r(r-1)\,t^r - 2r\,t^r + 2t^r = 0$$

Since $t^r \neq 0$ then dividing throughout by t^r gives

$$r(r-1) - 2r + 2 = 0$$

Or

$$r^2 - 3r + 2 = 0 \tag{1}$$

Equation (1) is the characteristic equation. Its roots determine the form of the general solution. Using the quadratic equation the roots are

$$r_1 = 1$$
$$r_2 = 2$$

Since the roots are real and distinct, then the general solution is

$$y = c_1y_1 + c_2y_2$$

Where $y_1 = t^{r_1}$ and $y_2 = t^{r_2}$. Hence

$$y = c_2t^2 + c_1t$$

Next, we find the particular solution to the ODE

$$t^2y'' - 2ty' + 2y = 4t^2$$

The particular solution y_p can be found using either the method of undetermined coefficients, or the method of variation of parameters. The method of variation of parameters will be used as it is more general and can be used when the coefficients of the ODE depend on t as well. Let

$$y_p(t) = u_1 y_1 + u_2 y_2 \tag{1}$$

Where u_1, u_2 to be determined, and y_1, y_2 are the two basis solutions (the two linearly independent solutions of the homogeneous ODE) found earlier when solving the homogeneous ODE as

$$y_1 = t$$
$$y_2 = t^2$$

In the Variation of parameters u_1, u_2 are found using

$$u_1 = -\int \frac{y_2 f(t)}{aW(t)} \tag{2}$$

$$u_2 = \int \frac{y_1 f(t)}{aW(t)} \tag{3}$$

Where $W(t)$ is the Wronskian and a is the coefficient in front of y'' in the given ODE. The Wronskian is given by $W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$. Hence

$$W = \begin{vmatrix} t & t^2 \\ \frac{d}{dt}(t) & \frac{d}{dt}(t^2) \end{vmatrix}$$

Which gives

$$W = \begin{vmatrix} t & t^2 \\ 1 & 2t \end{vmatrix}$$

Therefore

$$W = (t)(2t) - (t^2)(1)$$

Which simplifies to

$$W = t^2$$

Which simplifies to

$$W = t^2$$

Therefore Eq. (2) becomes

$$u_1 = -\int \frac{4t^4}{t^4}\, dt$$

Which simplifies to

$$u_1 = -\int 4dt$$

Hence

$$u_1 = -4t$$

And Eq. (3) becomes

$$u_2 = \int \frac{4t^3}{t^4}\, dt$$

Which simplifies to

$$u_2 = \int \frac{4}{t} dt$$

Hence

$$u_2 = 4\ln(t)$$

Therefore the particular solution, from equation (1) is

$$y_p(t) = -4t^2 + 4t^2 \ln(t)$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= -4t^2 + 4t^2 \ln(t) + c_2 t^2 + c_1 t \end{aligned}$$

Summary

The solution(s) found are the following

$$y = -4t^2 + 4t^2 \ln(t) + c_2 t^2 + c_1 t \tag{1}$$

Verification of solutions

$$y = -4t^2 + 4t^2 \ln(t) + c_2 t^2 + c_1 t$$

Verified OK.

10.19.2 Solving as linear second order ode solved by an integrating factor ode

The ode satisfies this form

$$y''+p(t)\,y'+\frac{\left(p(t)^{2}+p'(t)\right)y}{2}=f(t)$$

Where $p(t)=-\frac{2}{t}$. Therefore, there is an integrating factor given by

$$\begin{aligned}M(x)&=e^{\frac{1}{2}\int p\,dx}\\&=e^{\int -\frac{2}{t}\,dx}\\&=\frac{1}{t}\end{aligned}$$

Multiplying both sides of the ODE by the integrating factor $M(x)$ makes the left side of the ODE a complete differential

$$\begin{aligned}(M(x)y)''&=\frac{4}{t}\\\left(\frac{y}{t}\right)''&=\frac{4}{t}\end{aligned}$$

Integrating once gives

$$\left(\frac{y}{t}\right)'=4\ln(t)+c_1$$

Integrating again gives

$$\left(\frac{y}{t}\right)=t(4\ln(t)+c_1-4)+c_2$$

Hence the solution is

$$y=\frac{t(4\ln(t)+c_1-4)+c_2}{\frac{1}{t}}$$

Or

$$y=c_1t^2+4t^2\ln(t)+c_2t-4t^2$$

Summary

The solution(s) found are the following

$$y=c_1t^2+4t^2\ln(t)+c_2t-4t^2 \tag{1}$$

Verification of solutions

$$y=c_1t^2+4t^2\ln(t)+c_2t-4t^2$$

Verified OK.

10.19.3 Solving as second order change of variable on x method 2 ode

This is second order non-homogeneous ODE. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(t) + By'(t) + Cy(t) = 0$, and y_p is a particular solution to the non-homogeneous ODE $Ay''(t) + By'(t) + Cy(t) = f(t)$. y_h is the solution to

$$t^2y'' - 2ty' + 2y = 0$$

In normal form the ode

$$t^2y'' - 2ty' + 2y = 0 \tag{1}$$

Becomes

$$y'' + p(t)\, y' + q(t)\, y = 0 \tag{2}$$

Where

$$\begin{aligned} p(t) &= -\frac{2}{t} \\ q(t) &= \frac{2}{t^2} \end{aligned}$$

Applying change of variables $\tau = g(t)$ to (2) gives

$$\frac{d^2}{d\tau^2}y(\tau) + p_1\left(\frac{d}{d\tau}y(\tau)\right) + q_1 y(\tau) = 0 \tag{3}$$

Where τ is the new independent variable, and

$$p_1(\tau) = \frac{\tau''(t) + p(t)\, \tau'(t)}{\tau'(t)^2} \tag{4}$$

$$q_1(\tau) = \frac{q(t)}{\tau'(t)^2} \tag{5}$$

Let $p_1 = 0$. Eq (4) simplifies to

$$\tau''(t) + p(t)\, \tau'(t) = 0$$

This ode is solved resulting in

$$\begin{aligned}\tau &= \int e^{-\left(\int p(t)dt\right)} dt \\ &= \int e^{-\left(\int -\frac{2}{t}dt\right)} dt \\ &= \int e^{2\ln(t)}\, dt \\ &= \int t^2 dt \\ &= \frac{t^3}{3} \end{aligned} \tag{6}$$

Using (6) to evaluate q_1 from (5) gives

$$\begin{aligned} q_1(\tau) &= \frac{q(t)}{\tau'(t)^2} \\ &= \frac{\frac{2}{t^2}}{t^4} \\ &= \frac{2}{t^6} \end{aligned} \tag{7}$$

Substituting the above in (3) and noting that now $p_1 = 0$ results in

$$\begin{aligned} \frac{d^2}{d\tau^2}y(\tau) + q_1 y(\tau) &= 0 \\ \frac{d^2}{d\tau^2}y(\tau) + \frac{2y(\tau)}{t^6} &= 0 \end{aligned}$$

But in terms of τ

$$\frac{2}{t^6} = \frac{2}{9\tau^2}$$

Hence the above ode becomes

$$\frac{d^2}{d\tau^2}y(\tau) + \frac{2y(\tau)}{9\tau^2} = 0$$

The above ode is now solved for $y(\tau)$. The ode can be written as

$$9\left(\frac{d^2}{d\tau^2}y(\tau)\right)\tau^2 + 2y(\tau) = 0$$

Which shows it is a Euler ODE. This is Euler second order ODE. Let the solution be $y(\tau) = \tau^r$, then $y' = r\tau^{r-1}$ and $y'' = r(r-1)\tau^{r-2}$. Substituting these back into the given ODE gives

$$9\tau^2(r(r-1))\tau^{r-2} + 0r\tau^{r-1} + 2\tau^r = 0$$

Simplifying gives

$$9r(r-1)\,\tau^r + 0\,\tau^r + 2\tau^r = 0$$

Since $\tau^r \neq 0$ then dividing throughout by τ^r gives

$$9r(r-1) + 0 + 2 = 0$$

Or

$$9r^2 - 9r + 2 = 0 \tag{1}$$

Equation (1) is the characteristic equation. Its roots determine the form of the general solution. Using the quadratic equation the roots are

$$r_1 = \frac{1}{3}$$
$$r_2 = \frac{2}{3}$$

Since the roots are real and distinct, then the general solution is

$$y(\tau) = c_1 y_1 + c_2 y_2$$

Where $y_1 = \tau^{r_1}$ and $y_2 = \tau^{r_2}$. Hence

$$y(\tau) = c_1 \tau^{\frac{1}{3}} + c_2 \tau^{\frac{2}{3}}$$

The above solution is now transformed back to y using (6) which results in

$$y = \frac{c_1 3^{\frac{2}{3}} (t^3)^{\frac{1}{3}}}{3} + \frac{c_2 3^{\frac{1}{3}} (t^3)^{\frac{2}{3}}}{3}$$

Therefore the homogeneous solution y_h is

$$y_h = \frac{c_1 3^{\frac{2}{3}} (t^3)^{\frac{1}{3}}}{3} + \frac{c_2 3^{\frac{1}{3}} (t^3)^{\frac{2}{3}}}{3}$$

The particular solution y_p can be found using either the method of undetermined coefficients, or the method of variation of parameters. The method of variation of parameters will be used as it is more general and can be used when the coefficients of the ODE depend on t as well. Let

$$y_p(t) = u_1 y_1 + u_2 y_2 \tag{1}$$

Where u_1, u_2 to be determined, and y_1, y_2 are the two basis solutions (the two linearly independent solutions of the homogeneous ODE) found earlier when solving the homogeneous ODE as

$$y_1 = (t^3)^{\frac{1}{3}}$$

$$y_2 = (t^3)^{\frac{2}{3}}$$

In the Variation of parameters u_1, u_2 are found using

$$u_1 = -\int \frac{y_2 f(t)}{a W(t)} \tag{2}$$

$$u_2 = \int \frac{y_1 f(t)}{a W(t)} \tag{3}$$

Where $W(t)$ is the Wronskian and a is the coefficient in front of y'' in the given ODE. The Wronskian is given by $W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$. Hence

$$W = \begin{vmatrix} (t^3)^{\frac{1}{3}} & (t^3)^{\frac{2}{3}} \\ \frac{d}{dt}\left((t^3)^{\frac{1}{3}}\right) & \frac{d}{dt}\left((t^3)^{\frac{2}{3}}\right) \end{vmatrix}$$

Which gives

$$W = \begin{vmatrix} (t^3)^{\frac{1}{3}} & (t^3)^{\frac{2}{3}} \\ \frac{t^2}{(t^3)^{\frac{2}{3}}} & \frac{2t^2}{(t^3)^{\frac{1}{3}}} \end{vmatrix}$$

Therefore

$$W = \left((t^3)^{\frac{1}{3}}\right)\left(\frac{2t^2}{(t^3)^{\frac{1}{3}}}\right) - \left((t^3)^{\frac{2}{3}}\right)\left(\frac{t^2}{(t^3)^{\frac{2}{3}}}\right)$$

Which simplifies to

$$W = t^2$$

Which simplifies to

$$W = t^2$$

Therefore Eq. (2) becomes

$$u_1 = -\int \frac{4(t^3)^{\frac{2}{3}}\, t^2}{t^4}\, dt$$

Which simplifies to

$$u_1 = -\int \frac{4(t^3)^{\frac{2}{3}}}{t^2} dt$$

Hence

$$u_1 = -\frac{4(t^3)^{\frac{2}{3}}}{t}$$

And Eq. (3) becomes

$$u_2 = \int \frac{4(t^3)^{\frac{1}{3}}\, t^2}{t^4}\, dt$$

Which simplifies to

$$u_2 = \int \frac{4(t^3)^{\frac{1}{3}}}{t^2} dt$$

Hence

$$u_2 = \frac{4(t^3)^{\frac{1}{3}} \ln(t)}{t}$$

Therefore the particular solution, from equation (1) is

$$y_p(t) = -4t^2 + 4t^2 \ln(t)$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \left(\frac{c_1 3^{\frac{2}{3}} (t^3)^{\frac{1}{3}}}{3} + \frac{c_2 3^{\frac{1}{3}} (t^3)^{\frac{2}{3}}}{3}\right) + \left(-4t^2 + 4t^2 \ln(t)\right) \end{aligned}$$

Summary
The solution(s) found are the following

$$y = \frac{c_1 3^{\frac{2}{3}} (t^3)^{\frac{1}{3}}}{3} + \frac{c_2 3^{\frac{1}{3}} (t^3)^{\frac{2}{3}}}{3} - 4t^2 + 4t^2 \ln(t) \tag{1}$$

Verification of solutions

$$y = \frac{c_1 3^{\frac{2}{3}} (t^3)^{\frac{1}{3}}}{3} + \frac{c_2 3^{\frac{1}{3}} (t^3)^{\frac{2}{3}}}{3} - 4t^2 + 4t^2 \ln(t)$$

Verified OK.

10.19.4 Solving as second order change of variable on x method 1 ode

This is second order non-homogeneous ODE. In standard form the ODE is

$$Ay''(t) + By'(t) + Cy(t) = f(t)$$

Where $A = t^2, B = -2t, C = 2, f(t) = 4t^2$. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(t) + By'(t) + Cy(t) = 0$, and y_p is a particular solution to the non-homogeneous ODE $Ay''(t) + By'(t) + Cy(t) = f(t)$. Solving for y_h from

$$t^2y'' - 2ty' + 2y = 0$$

In normal form the ode

$$t^2y'' - 2ty' + 2y = 0 \tag{1}$$

Becomes

$$y'' + p(t)\, y' + q(t)\, y = 0 \tag{2}$$

Where

$$\begin{aligned} p(t) &= -\frac{2}{t} \\ q(t) &= \frac{2}{t^2} \end{aligned}$$

Applying change of variables $\tau = g(t)$ to (2) results

$$\frac{d^2}{d\tau^2} y(\tau) + p_1 \left(\frac{d}{d\tau} y(\tau)\right) + q_1 y(\tau) = 0 \tag{3}$$

Where τ is the new independent variable, and

$$p_1(\tau) = \frac{\tau''(t) + p(t)\, \tau'(t)}{\tau'(t)^2} \tag{4}$$

$$q_1(\tau) = \frac{q(t)}{\tau'(t)^2} \tag{5}$$

Let $q_1 = c^2$ where c is some constant. Therefore from (5)

$$\begin{aligned}\tau' &= \frac{1}{c}\sqrt{q} \\ &= \frac{\sqrt{2}\,\sqrt{\frac{1}{t^2}}}{c} \\ \tau'' &= -\frac{\sqrt{2}}{c\sqrt{\frac{1}{t^2}}\,t^3}\end{aligned} \tag{6}$$

Substituting the above into (4) results in

$$\begin{aligned}p_1(\tau) &= \frac{\tau''(t) + p(t)\,\tau'(t)}{\tau'(t)^2} \\ &= \frac{-\frac{\sqrt{2}}{c\sqrt{\frac{1}{t^2}}\,t^3} - \frac{2}{t}\frac{\sqrt{2}\,\sqrt{\frac{1}{t^2}}}{c}}{\left(\frac{\sqrt{2}\,\sqrt{\frac{1}{t^2}}}{c}\right)^2} \\ &= -\frac{3c\sqrt{2}}{2}\end{aligned}$$

Therefore ode (3) now becomes

$$\begin{aligned}y(\tau)'' + p_1 y(\tau)' + q_1 y(\tau) &= 0 \\ \frac{d^2}{d\tau^2}y(\tau) - \frac{3c\sqrt{2}\,\left(\frac{d}{d\tau}y(\tau)\right)}{2} + c^2 y(\tau) &= 0\end{aligned} \tag{7}$$

The above ode is now solved for $y(\tau)$. Since the ode is now constant coefficients, it can be easily solved to give

$$y(\tau) = \mathrm{e}^{\frac{3\sqrt{2}\,c\tau}{4}}\left(c_1 \cosh\left(\frac{\sqrt{2}\,c\tau}{4}\right) + ic_2 \sinh\left(\frac{\sqrt{2}\,c\tau}{4}\right)\right)$$

Now from (6)

$$\begin{aligned}\tau &= \int \frac{1}{c}\sqrt{q}\,dt \\ &= \frac{\int \sqrt{2}\,\sqrt{\frac{1}{t^2}}dt}{c} \\ &= \frac{\sqrt{2}\,\sqrt{\frac{1}{t^2}}\,t\ln(t)}{c}\end{aligned}$$

Substituting the above into the solution obtained gives

$$y = t^{\frac{3}{2}} \left(c_1 \cosh\left(\frac{\ln(t)}{2}\right) + i c_2 \sinh\left(\frac{\ln(t)}{2}\right) \right)$$

Now the particular solution to this ODE is found

$$t^2 y'' - 2ty' + 2y = 4t^2$$

The particular solution y_p can be found using either the method of undetermined coefficients, or the method of variation of parameters. The method of variation of parameters will be used as it is more general and can be used when the coefficients of the ODE depend on t as well. Let

$$y_p(t) = u_1 y_1 + u_2 y_2 \tag{1}$$

Where u_1, u_2 to be determined, and y_1, y_2 are the two basis solutions (the two linearly independent solutions of the homogeneous ODE) found earlier when solving the homogeneous ODE as

$$y_1 = \left(t^3\right)^{\frac{1}{3}}$$
$$y_2 = \left(t^3\right)^{\frac{2}{3}}$$

In the Variation of parameters u_1, u_2 are found using

$$u_1 = -\int \frac{y_2 f(t)}{a W(t)} \tag{2}$$
$$u_2 = \int \frac{y_1 f(t)}{a W(t)} \tag{3}$$

Where $W(t)$ is the Wronskian and a is the coefficient in front of y'' in the given ODE. The Wronskian is given by $W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$. Hence

$$W = \begin{vmatrix} \left(t^3\right)^{\frac{1}{3}} & \left(t^3\right)^{\frac{2}{3}} \\ \frac{d}{dt}\left(\left(t^3\right)^{\frac{1}{3}}\right) & \frac{d}{dt}\left(\left(t^3\right)^{\frac{2}{3}}\right) \end{vmatrix}$$

Which gives

$$W = \begin{vmatrix} (t^3)^{\frac{1}{3}} & (t^3)^{\frac{2}{3}} \\ \frac{t^2}{(t^3)^{\frac{2}{3}}} & \frac{2t^2}{(t^3)^{\frac{1}{3}}} \end{vmatrix}$$

Therefore

$$W = \left((t^3)^{\frac{1}{3}}\right)\left(\frac{2t^2}{(t^3)^{\frac{1}{3}}}\right) - \left((t^3)^{\frac{2}{3}}\right)\left(\frac{t^2}{(t^3)^{\frac{2}{3}}}\right)$$

Which simplifies to

$$W = t^2$$

Which simplifies to

$$W = t^2$$

Therefore Eq. (2) becomes

$$u_1 = -\int \frac{4(t^3)^{\frac{2}{3}}\, t^2}{t^4}\, dt$$

Which simplifies to

$$u_1 = -\int \frac{4(t^3)^{\frac{2}{3}}}{t^2} dt$$

Hence

$$u_1 = -\frac{4(t^3)^{\frac{2}{3}}}{t}$$

And Eq. (3) becomes

$$u_2 = \int \frac{4(t^3)^{\frac{1}{3}}\, t^2}{t^4}\, dt$$

Which simplifies to

$$u_2 = \int \frac{4(t^3)^{\frac{1}{3}}}{t^2} dt$$

Hence

$$u_2 = \frac{4(t^3)^{\frac{1}{3}} \ln(t)}{t}$$

Therefore the particular solution, from equation (1) is

$$y_p(t) = -4t^2 + 4t^2 \ln(t)$$

Therefore the general solution is

$$\begin{aligned}
y &= y_h + y_p \\
&= \left(t^{\frac{3}{2}}\left(c_1 \cosh\left(\frac{\ln(t)}{2}\right) + ic_2 \sinh\left(\frac{\ln(t)}{2}\right)\right)\right) + \left(-4t^2 + 4t^2 \ln(t)\right) \\
&= -4t^2 + 4t^2 \ln(t) + t^{\frac{3}{2}}\left(c_1 \cosh\left(\frac{\ln(t)}{2}\right) + ic_2 \sinh\left(\frac{\ln(t)}{2}\right)\right)
\end{aligned}$$

Which simplifies to

$$y = i \sinh\left(\frac{\ln(t)}{2}\right) t^{\frac{3}{2}} c_2 + \cosh\left(\frac{\ln(t)}{2}\right) t^{\frac{3}{2}} c_1 + 4t^2 \ln(t) - 4t^2$$

Summary
The solution(s) found are the following

$$y = i \sinh\left(\frac{\ln(t)}{2}\right) t^{\frac{3}{2}} c_2 + \cosh\left(\frac{\ln(t)}{2}\right) t^{\frac{3}{2}} c_1 + 4t^2 \ln(t) - 4t^2 \tag{1}$$

Verification of solutions

$$y = i \sinh\left(\frac{\ln(t)}{2}\right) t^{\frac{3}{2}} c_2 + \cosh\left(\frac{\ln(t)}{2}\right) t^{\frac{3}{2}} c_1 + 4t^2 \ln(t) - 4t^2$$

Verified OK.

10.19.5 Solving as second order change of variable on y method 2 ode

This is second order non-homogeneous ODE. In standard form the ODE is

$$Ay''(t) + By'(t) + Cy(t) = f(t)$$

Where $A = t^2, B = -2t, C = 2, f(t) = 4t^2$. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(t)+By'(t)+Cy(t)=0$, and y_p is a particular solution to the non-homogeneous ODE $Ay''(t)+By'(t)+Cy(t)=f(t)$. Solving for y_h from

$$t^2y''-2ty'+2y=0$$

In normal form the ode

$$t^2y''-2ty'+2y=0 \tag{1}$$

Becomes

$$y''+p(t)\,y'+q(t)\,y=0 \tag{2}$$

Where

$$\begin{aligned}p(t)&=-\frac{2}{t}\\ q(t)&=\frac{2}{t^2}\end{aligned}$$

Applying change of variables on the depndent variable $y=v(t)\,t^n$ to (2) gives the following ode where the dependent variables is $v(t)$ and not y.

$$v''(t)+\left(\frac{2n}{t}+p\right)v'(t)+\left(\frac{n(n-1)}{t^2}+\frac{np}{t}+q\right)v(t)=0 \tag{3}$$

Let the coefficient of $v(t)$ above be zero. Hence

$$\frac{n(n-1)}{t^2}+\frac{np}{t}+q=0 \tag{4}$$

Substituting the earlier values found for $p(t)$ and $q(t)$ into (4) gives

$$\frac{n(n-1)}{t^2}-\frac{2n}{t^2}+\frac{2}{t^2}=0 \tag{5}$$

Solving (5) for n gives

$$n=2 \tag{6}$$

Substituting this value in (3) gives

$$\begin{aligned}v''(t)+\frac{2v'(t)}{t}&=0\\ v''(t)+\frac{2v'(t)}{t}&=0 \qquad (7)\end{aligned}$$

Using the substitution

$$u(t) = v'(t)$$

Then (7) becomes

$$u'(t) + \frac{2u(t)}{t} = 0 \tag{8}$$

The above is now solved for $u(t)$. In canonical form the ODE is

$$\begin{aligned} u' &= F(t, u) \\ &= f(t)g(u) \\ &= -\frac{2u}{t} \end{aligned}$$

Where $f(t) = -\frac{2}{t}$ and $g(u) = u$. Integrating both sides gives

$$\begin{aligned} \frac{1}{u}\, du &= -\frac{2}{t}\, dt \\ \int \frac{1}{u}\, du &= \int -\frac{2}{t}\, dt \\ \ln(u) &= -2\ln(t) + c_1 \\ u &= \mathrm{e}^{-2\ln(t)+c_1} \\ &= \frac{c_1}{t^2} \end{aligned}$$

Now that $u(t)$ is known, then

$$\begin{aligned} v'(t) &= u(t) \\ v(t) &= \int u(t)\, dt + c_2 \\ &= -\frac{c_1}{t} + c_2 \end{aligned}$$

Hence

$$\begin{aligned} y &= v(t)\, t^n \\ &= \left(-\frac{c_1}{t} + c_2\right) t^2 \\ &= (c_2 t - c_1)\, t \end{aligned}$$

Now the particular solution to this ODE is found

$$t^2y'' - 2ty' + 2y = 4t^2$$

The particular solution y_p can be found using either the method of undetermined coefficients, or the method of variation of parameters. The method of variation of parameters will be used as it is more general and can be used when the coefficients of the ODE depend on t as well. Let

$$y_p(t) = u_1y_1 + u_2y_2 \tag{1}$$

Where u_1, u_2 to be determined, and y_1, y_2 are the two basis solutions (the two linearly independent solutions of the homogeneous ODE) found earlier when solving the homogeneous ODE as

$$y_1 = t$$
$$y_2 = t^2$$

In the Variation of parameters u_1, u_2 are found using

$$u_1 = -\int \frac{y_2 f(t)}{aW(t)} \tag{2}$$

$$u_2 = \int \frac{y_1 f(t)}{aW(t)} \tag{3}$$

Where $W(t)$ is the Wronskian and a is the coefficient in front of y'' in the given ODE. The Wronskian is given by $W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$. Hence

$$W = \begin{vmatrix} t & t^2 \\ \frac{d}{dt}(t) & \frac{d}{dt}(t^2) \end{vmatrix}$$

Which gives

$$W = \begin{vmatrix} t & t^2 \\ 1 & 2t \end{vmatrix}$$

Therefore

$$W = (t)(2t) - (t^2)(1)$$

Which simplifies to

$$W = t^2$$

Which simplifies to

$$W = t^2$$

Therefore Eq. (2) becomes

$$u_1 = -\int \frac{4t^4}{t^4} dt$$

Which simplifies to

$$u_1 = -\int 4dt$$

Hence

$$u_1 = -4t$$

And Eq. (3) becomes

$$u_2 = \int \frac{4t^3}{t^4} dt$$

Which simplifies to

$$u_2 = \int \frac{4}{t} dt$$

Hence

$$u_2 = 4 \ln (t)$$

Therefore the particular solution, from equation (1) is

$$y_p(t) = -4t^2 + 4t^2 \ln (t)$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \left(\left(-\frac{c_1}{t} + c_2\right) t^2\right) + \left(-4t^2 + 4t^2 \ln (t)\right) \\ &= -4t^2 + 4t^2 \ln (t) + \left(-\frac{c_1}{t} + c_2\right) t^2 \end{aligned}$$

Which simplifies to

$$y = t(4\ln(t)t + c_2t - c_1 - 4t)$$

Summary
The solution(s) found are the following

$$y = t(4\ln(t)t + c_2t - c_1 - 4t) \tag{1}$$

Verification of solutions

$$y = t(4\ln(t)t + c_2t - c_1 - 4t)$$

Verified OK.

10.19.6 Solving as second order ode non constant coeff transformation on B ode

Given an ode of the form

$$Ay'' + By' + Cy = F(t)$$

This method reduces the order ode the ODE by one by applying the transformation

$$y = Bv$$

This results in

$$\begin{aligned} y' &= B'v + v'B \\ y'' &= B''v + B'v' + v''B + v'B' \\ &= v''B + 2v' + B' + B''v \end{aligned}$$

And now the original ode becomes

$$\begin{aligned} A(v''B + 2v'B' + B''v) + B(B'v + v'B) + CBv &= 0 \\ ABv'' + (2AB' + B^2)v' + (AB'' + BB' + CB)v &= 0 \end{aligned} \tag{1}$$

If the term $AB'' + BB' + CB$ is zero, then this method works and can be used to solve

$$ABv'' + (2AB' + B^2)v' = 0$$

By Using $u = v'$ which reduces the order of the above ode to one. The new ode is

$$ABu' + (2AB' + B^2)u = 0$$

The above ode is first order ode which is solved for u. Now a new ode $v' = u$ is solved for v as first order ode. Then the final solution is obtain from $y = Bv$.

This method works only if the term $AB'' + BB' + CB$ is zero. The given ODE shows that

$$
\begin{aligned}
A &= t^2 \\
B &= -2t \\
C &= 2 \\
F &= 4t^2
\end{aligned}
$$

The above shows that for this ode

$$
\begin{aligned}
AB'' + BB' + CB &= \left(t^2\right)(0) + (-2t)(-2) + (2)(-2t) \\
&= 0
\end{aligned}
$$

Hence the ode in v given in (1) now simplifies to

$$
-2t^3 v'' + (0) v' = 0
$$

Now by applying $v' = u$ the above becomes

$$
-2t^3 u'(t) = 0
$$

Which is now solved for u. Integrating both sides gives

$$
\begin{aligned}
u(t) &= \int 0 \, \mathrm{d}t \\
&= c_1
\end{aligned}
$$

The ode for v now becomes

$$
\begin{aligned}
v' &= u \\
&= c_1
\end{aligned}
$$

Which is now solved for v. Integrating both sides gives

$$
\begin{aligned}
v(t) &= \int c_1 \, \mathrm{d}t \\
&= c_1 t + c_2
\end{aligned}
$$

Therefore the homogeneous solution is

$$
\begin{aligned}
y_h(t) &= Bv \\
&= (-2t)(c_1 t + c_2) \\
&= -2t(c_1 t + c_2)
\end{aligned}
$$

And now the particular solution $y_p(t)$ will be found. The particular solution y_p can be found using either the method of undetermined coefficients, or the method of variation of parameters. The method of variation of parameters will be used as it is more general and can be used when the coefficients of the ODE depend on t as well. Let

$$y_p(t) = u_1 y_1 + u_2 y_2 \tag{1}$$

Where u_1, u_2 to be determined, and y_1, y_2 are the two basis solutions (the two linearly independent solutions of the homogeneous ODE) found earlier when solving the homogeneous ODE as

$$y_1 = t$$
$$y_2 = t^2$$

In the Variation of parameters u_1, u_2 are found using

$$u_1 = -\int \frac{y_2 f(t)}{aW(t)} \tag{2}$$

$$u_2 = \int \frac{y_1 f(t)}{aW(t)} \tag{3}$$

Where $W(t)$ is the Wronskian and a is the coefficient in front of y'' in the given ODE. The Wronskian is given by $W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$. Hence

$$W = \begin{vmatrix} t & t^2 \\ \frac{d}{dt}(t) & \frac{d}{dt}(t^2) \end{vmatrix}$$

Which gives

$$W = \begin{vmatrix} t & t^2 \\ 1 & 2t \end{vmatrix}$$

Therefore

$$W = (t)(2t) - (t^2)(1)$$

Which simplifies to

$$W = t^2$$

Which simplifies to

$$W = t^2$$

Therefore Eq. (2) becomes

$$u_1 = -\int \frac{4t^4}{t^4}\,dt$$

Which simplifies to

$$u_1 = -\int 4dt$$

Hence

$$u_1 = -4t$$

And Eq. (3) becomes

$$u_2 = \int \frac{4t^3}{t^4}\,dt$$

Which simplifies to

$$u_2 = \int \frac{4}{t}dt$$

Hence

$$u_2 = 4\ln(t)$$

Therefore the particular solution, from equation (1) is

$$y_p(t) = -4t^2 + 4t^2 \ln(t)$$

Hence the complete solution is

$$\begin{aligned} y(t) &= y_h + y_p \\ &= (-2t(c_1 t + c_2)) + \left(-4t^2 + 4t^2 \ln(t)\right) \\ &= -2(-2\ln(t)\,t + (c_1 + 2)\,t + c_2)\,t \end{aligned}$$

Summary
The solution(s) found are the following

$$y = -2(-2\ln(t)\,t + (c_1 + 2)\,t + c_2)\,t \tag{1}$$

Verification of solutions

$$y = -2(-2\ln(t)\,t + (c_1 + 2)\,t + c_2)\,t$$

Verified OK.

10.19.7 Solving using Kovacic algorithm

Writing the ode as

$$t^2y'' - 2ty' + 2y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= t^2 \\ B &= -2t \\ C &= 2 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(t) = ye^{\int \frac{B}{2A}\,dt}$$

Then (2) becomes

$$z''(t) = rz(t) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{0}{1} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= 0 \\ t &= 1 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(t) = 0 \tag{7}$$

Equation (7) is now solved. After finding $z(t)$ then y is found using the inverse transformation

$$y = z(t)\,e^{-\int \frac{B}{2A}\,dt}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 504: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$
\begin{aligned}
O(\infty) &= \deg(t) - \deg(s) \\
&= 0 - -\infty \\
&= \infty
\end{aligned}
$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is $infinity$ then the necessary conditions for case one are met. Therefore

$$
L = [1]
$$

Since $r = 0$ is not a function of t, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$
z_1(t) = 1
$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$
\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2} \frac{B}{A} \, dt} \\
&= z_1 e^{-\int \frac{1}{2} \frac{-2t}{t^2} \, dt}
\end{aligned}
$$

$$
\begin{aligned}
&= z_1 e^{\ln(t)} \\
&= z_1(t)
\end{aligned}
$$

Which simplifies to

$$
y_1 = t
$$

The second solution y_2 to the original ode is found using reduction of order

$$
y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\, dt}}{y_1^2}\, dt
$$

Substituting gives

$$
\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{-2t}{t^2}\, dt}}{(y_1)^2}\, dt \\
&= y_1 \int \frac{e^{2\ln(t)}}{(y_1)^2}\, dt \\
&= y_1(t)
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1(t) + c_2(t(t))
\end{aligned}
$$

This is second order nonhomogeneous ODE. Let the solution be

$$
y = y_h + y_p
$$

Where y_h is the solution to the homogeneous ODE $Ay''(t) + By'(t) + Cy(t) = 0$, and y_p is a particular solution to the nonhomogeneous ODE $Ay''(t) + By'(t) + Cy(t) = f(t)$. y_h is the solution to

$$
t^2 y'' - 2ty' + 2y = 0
$$

The homogeneous solution is found using the Kovacic algorithm which results in

$$
y_h = c_2 t^2 + c_1 t
$$

The particular solution y_p can be found using either the method of undetermined coefficients, or the method of variation of parameters. The method of variation of parameters will be used as it is more general and can be used when the coefficients of the ODE depend on t as well. Let

$$y_p(t) = u_1 y_1 + u_2 y_2 \tag{1}$$

Where u_1, u_2 to be determined, and y_1, y_2 are the two basis solutions (the two linearly independent solutions of the homogeneous ODE) found earlier when solving the homogeneous ODE as

$$y_1 = t$$
$$y_2 = t^2$$

In the Variation of parameters u_1, u_2 are found using

$$u_1 = -\int \frac{y_2 f(t)}{aW(t)} \tag{2}$$

$$u_2 = \int \frac{y_1 f(t)}{aW(t)} \tag{3}$$

Where $W(t)$ is the Wronskian and a is the coefficient in front of y'' in the given ODE. The Wronskian is given by $W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$. Hence

$$W = \begin{vmatrix} t & t^2 \\ \frac{d}{dt}(t) & \frac{d}{dt}(t^2) \end{vmatrix}$$

Which gives

$$W = \begin{vmatrix} t & t^2 \\ 1 & 2t \end{vmatrix}$$

Therefore

$$W = (t)(2t) - (t^2)(1)$$

Which simplifies to

$$W = t^2$$

Which simplifies to

$$W = t^2$$

Therefore Eq. (2) becomes

$$u_1 = -\int \frac{4t^4}{t^4} \, dt$$

Which simplifies to

$$u_1 = -\int 4 dt$$

Hence

$$u_1 = -4t$$

And Eq. (3) becomes

$$u_2 = \int \frac{4t^3}{t^4} \, dt$$

Which simplifies to

$$u_2 = \int \frac{4}{t} dt$$

Hence

$$u_2 = 4 \ln(t)$$

Therefore the particular solution, from equation (1) is

$$y_p(t) = -4t^2 + 4t^2 \ln(t)$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= (c_2 t^2 + c_1 t) + (-4t^2 + 4t^2 \ln(t)) \end{aligned}$$

Which simplifies to

$$y = t(c_2 t + c_1) - 4t^2 + 4t^2 \ln(t)$$

Summary

The solution(s) found are the following

$$y = t(c_2t + c_1) - 4t^2 + 4t^2 \ln (t) \tag{1}$$

Verification of solutions

$$y = t(c_2t + c_1) - 4t^2 + 4t^2 \ln (t)$$

Verified OK.

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 2; linear nonhomogeneous with symmetry [0,1]
trying a double symmetry of the form [xi=0, eta=F(x)]
-> Try solving first the homogeneous part of the ODE
   checking if the LODE has constant coefficients
   checking if the LODE is of Euler type
   <- LODE of Euler type successful
<- solving first the homogeneous part of the ODE successful`
```

✓ Solution by Maple

Time used: 0.015 (sec). Leaf size: 18

```
dsolve(t^2*diff(y(t),t$2)-2*t*diff(y(t),t)+2*y(t) = 4*t^2,y(t), singsol=all)
```

$$y(t) = t(4t \ln (t) + (c_1 - 4) t + c_2)$$

✓ Solution by Mathematica

Time used: 0.013 (sec). Leaf size: 21

```
DSolve[t^2*y''[t]-2*t*y'[t]+2*y[t] ==4*t^2,y[t],t,IncludeSingularSolutions -> True]
```

$$y(t) \to t(4t \log(t) + (-4 + c_2)t + c_1)$$

10.20 problem 30

Internal problem ID [702]
Internal file name [`OUTPUT/702_Sunday_June_05_2022_01_47_18_AM_49530951/index.tex`]

Book: Elementary differential equations and boundary value problems, 10th ed., Boyce and DiPrima
Section: Chapter 3, Second order linear equations, section 3.6, Variation of Parameters. page 190
Problem number: 30.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_euler_ode", "exact linear second order ode", "second_order_integrable_as_is", "second_order_change_of_variable_on_x_method_1", "second_order_change_of_variable_on_x_method_2", "second_order_change_of_variable_on_y_method_2"**

Maple gives the following as the ode type

```
[[_2nd_order, _exact, _linear, _nonhomogeneous]]
```

$$\boxed{t^2y'' + 7ty' + 5y = t}$$

10.20.1 Solving as second order euler ode ode

This is second order non-homogeneous ODE. In standard form the ODE is

$$Ay''(t) + By'(t) + Cy(t) = f(t)$$

Where $A = t^2, B = 7t, C = 5, f(t) = t$. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(t) + By'(t) + Cy(t) = 0$, and y_p is a particular solution to the non-homogeneous ODE $Ay''(t) + By'(t) + Cy(t) = f(t)$. Solving for y_h from

$$t^2y'' + 7ty' + 5y = 0$$

This is Euler second order ODE. Let the solution be $y = t^r$, then $y' = rt^{r-1}$ and $y'' = r(r-1)t^{r-2}$. Substituting these back into the given ODE gives

$$t^2(r(r-1))t^{r-2} + 7trt^{r-1} + 5t^r = 0$$

Simplifying gives

$$r(r-1)\,t^r + 7r\,t^r + 5t^r = 0$$

Since $t^r \neq 0$ then dividing throughout by t^r gives

$$r(r-1) + 7r + 5 = 0$$

Or

$$r^2 + 6r + 5 = 0 \tag{1}$$

Equation (1) is the characteristic equation. Its roots determine the form of the general solution. Using the quadratic equation the roots are

$$r_1 = -5$$
$$r_2 = -1$$

Since the roots are real and distinct, then the general solution is

$$y = c_1y_1 + c_2y_2$$

Where $y_1 = t^{r_1}$ and $y_2 = t^{r_2}$. Hence

$$y = \frac{c_1}{t^5} + \frac{c_2}{t}$$

Next, we find the particular solution to the ODE

$$t^2y'' + 7ty' + 5y = t$$

The particular solution y_p can be found using either the method of undetermined coefficients, or the method of variation of parameters. The method of variation of parameters will be used as it is more general and can be used when the coefficients of the ODE depend on t as well. Let

$$y_p(t) = u_1 y_1 + u_2 y_2 \tag{1}$$

Where u_1, u_2 to be determined, and y_1, y_2 are the two basis solutions (the two linearly independent solutions of the homogeneous ODE) found earlier when solving the homogeneous ODE as

$$y_1 = \frac{1}{t^5}$$

$$y_2 = \frac{1}{t}$$

In the Variation of parameters u_1, u_2 are found using

$$u_1 = -\int \frac{y_2 f(t)}{a W(t)} \tag{2}$$

$$u_2 = \int \frac{y_1 f(t)}{a W(t)} \tag{3}$$

Where $W(t)$ is the Wronskian and a is the coefficient in front of y'' in the given ODE. The Wronskian is given by $W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$. Hence

$$W = \begin{vmatrix} \frac{1}{t^5} & \frac{1}{t} \\ \frac{d}{dt}\left(\frac{1}{t^5}\right) & \frac{d}{dt}\left(\frac{1}{t}\right) \end{vmatrix}$$

Which gives

$$W = \begin{vmatrix} \frac{1}{t^5} & \frac{1}{t} \\ -\frac{5}{t^6} & -\frac{1}{t^2} \end{vmatrix}$$

Therefore

$$W = \left(\frac{1}{t^5}\right)\left(-\frac{1}{t^2}\right) - \left(\frac{1}{t}\right)\left(-\frac{5}{t^6}\right)$$

Which simplifies to

$$W = \frac{4}{t^7}$$

Which simplifies to

$$W = \frac{4}{t^7}$$

Therefore Eq. (2) becomes

$$u_1 = -\int \frac{1}{\frac{4}{t^5}}\, dt$$

Which simplifies to

$$u_1 = -\int \frac{t^5}{4} dt$$

Hence

$$u_1 = -\frac{t^6}{24}$$

And Eq. (3) becomes

$$u_2 = \int \frac{\frac{1}{t^4}}{\frac{4}{t^5}}\, dt$$

Which simplifies to

$$u_2 = \int \frac{t}{4} dt$$

Hence

$$u_2 = \frac{t^2}{8}$$

Therefore the particular solution, from equation (1) is

$$y_p(t) = \frac{t}{12}$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \frac{t}{12} + \frac{c_1}{t^5} + \frac{c_2}{t} \end{aligned}$$

Summary
The solution(s) found are the following

$$y = \frac{t}{12} + \frac{c_1}{t^5} + \frac{c_2}{t} \tag{1}$$

Verification of solutions

$$y = \frac{t}{12} + \frac{c_1}{t^5} + \frac{c_2}{t}$$

Verified OK.

10.20.2 Solving as second order change of variable on x method 2 ode

This is second order non-homogeneous ODE. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(t) + By'(t) + Cy(t) = 0$, and y_p is a particular solution to the non-homogeneous ODE $Ay''(t) + By'(t) + Cy(t) = f(t)$. y_h is the solution to

$$t^2y'' + 7ty' + 5y = 0$$

In normal form the ode

$$t^2y'' + 7ty' + 5y = 0 \tag{1}$$

Becomes

$$y'' + p(t)\, y' + q(t)\, y = 0 \tag{2}$$

Where

$$p(t) = \frac{7}{t}$$
$$q(t) = \frac{5}{t^2}$$

Applying change of variables $\tau = g(t)$ to (2) gives

$$\frac{d^2}{d\tau^2}y(\tau) + p_1\left(\frac{d}{d\tau}y(\tau)\right) + q_1 y(\tau) = 0 \tag{3}$$

Where τ is the new independent variable, and

$$p_1(\tau) = \frac{\tau''(t) + p(t)\,\tau'(t)}{\tau'(t)^2} \tag{4}$$
$$q_1(\tau) = \frac{q(t)}{\tau'(t)^2} \tag{5}$$

Let $p_1 = 0$. Eq (4) simplifies to

$$\tau''(t) + p(t)\,\tau'(t) = 0$$

This ode is solved resulting in

$$\begin{aligned}
\tau &= \int e^{-\left(\int p(t)dt\right)} dt \\
&= \int e^{-\left(\int \frac{7}{t} dt\right)} dt \\
&= \int e^{-7\ln(t)}\, dt \\
&= \int \frac{1}{t^7} dt \\
&= -\frac{1}{6t^6}
\end{aligned} \tag{6}$$

Using (6) to evaluate q_1 from (5) gives

$$\begin{aligned}
q_1(\tau) &= \frac{q(t)}{\tau'(t)^2} \\
&= \frac{\frac{5}{t^2}}{\frac{1}{t^{14}}} \\
&= 5t^{12}
\end{aligned} \tag{7}$$

Substituting the above in (3) and noting that now $p_1 = 0$ results in

$$\frac{d^2}{d\tau^2} y(\tau) + q_1 y(\tau) = 0$$
$$\frac{d^2}{d\tau^2} y(\tau) + 5t^{12} y(\tau) = 0$$

But in terms of τ

$$5t^{12} = \frac{5}{36\tau^2}$$

Hence the above ode becomes

$$\frac{d^2}{d\tau^2}y(\tau) + \frac{5y(\tau)}{36\tau^2} = 0$$

The above ode is now solved for $y(\tau)$. The ode can be written as

$$36\left(\frac{d^2}{d\tau^2}y(\tau)\right)\tau^2 + 5y(\tau) = 0$$

Which shows it is a Euler ODE. This is Euler second order ODE. Let the solution be $y(\tau) = \tau^r$, then $y' = r\tau^{r-1}$ and $y'' = r(r-1)\tau^{r-2}$. Substituting these back into the given ODE gives

$$36\tau^2(r(r-1))\tau^{r-2} + 0r\tau^{r-1} + 5\tau^r = 0$$

Simplifying gives

$$36r(r-1)\,\tau^r + 0\,\tau^r + 5\tau^r = 0$$

Since $\tau^r \neq 0$ then dividing throughout by τ^r gives

$$36r(r-1) + 0 + 5 = 0$$

Or

$$36r^2 - 36r + 5 = 0 \tag{1}$$

Equation (1) is the characteristic equation. Its roots determine the form of the general solution. Using the quadratic equation the roots are

$$r_1 = \frac{1}{6}$$
$$r_2 = \frac{5}{6}$$

Since the roots are real and distinct, then the general solution is

$$y(\tau) = c_1y_1 + c_2y_2$$

Where $y_1 = \tau^{r_1}$ and $y_2 = \tau^{r_2}$. Hence

$$y(\tau) = c_1\tau^{\frac{1}{6}} + c_2\tau^{\frac{5}{6}}$$

The above solution is now transformed back to y using (6) which results in

$$y = \frac{c_1 6^{\frac{5}{6}}\left(-\frac{1}{t^6}\right)^{\frac{1}{6}}}{6} + \frac{c_2 6^{\frac{1}{6}}\left(-\frac{1}{t^6}\right)^{\frac{5}{6}}}{6}$$

Therefore the homogeneous solution y_h is

$$y_h = \frac{c_1 6^{\frac{5}{6}} \left(-\frac{1}{t^6}\right)^{\frac{1}{6}}}{6} + \frac{c_2 6^{\frac{1}{6}} \left(-\frac{1}{t^6}\right)^{\frac{5}{6}}}{6}$$

The particular solution y_p can be found using either the method of undetermined coefficients, or the method of variation of parameters. The method of variation of parameters will be used as it is more general and can be used when the coefficients of the ODE depend on t as well. Let

$$y_p(t) = u_1 y_1 + u_2 y_2 \tag{1}$$

Where u_1, u_2 to be determined, and y_1, y_2 are the two basis solutions (the two linearly independent solutions of the homogeneous ODE) found earlier when solving the homogeneous ODE as

$$y_1 = \left(-\frac{1}{t^6}\right)^{\frac{1}{6}}$$

$$y_2 = \left(-\frac{1}{t^6}\right)^{\frac{5}{6}}$$

In the Variation of parameters u_1, u_2 are found using

$$u_1 = -\int \frac{y_2 f(t)}{a W(t)} \tag{2}$$

$$u_2 = \int \frac{y_1 f(t)}{a W(t)} \tag{3}$$

Where $W(t)$ is the Wronskian and a is the coefficient in front of y'' in the given ODE. The Wronskian is given by $W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$. Hence

$$W = \begin{vmatrix} \left(-\frac{1}{t^6}\right)^{\frac{1}{6}} & \left(-\frac{1}{t^6}\right)^{\frac{5}{6}} \\ \frac{d}{dt}\left(\left(-\frac{1}{t^6}\right)^{\frac{1}{6}}\right) & \frac{d}{dt}\left(\left(-\frac{1}{t^6}\right)^{\frac{5}{6}}\right) \end{vmatrix}$$

Which gives

$$W = \begin{vmatrix} \left(-\frac{1}{t^6}\right)^{\frac{1}{6}} & \left(-\frac{1}{t^6}\right)^{\frac{5}{6}} \\ \frac{1}{\left(-\frac{1}{t^6}\right)^{\frac{5}{6}} t^7} & \frac{5}{\left(-\frac{1}{t^6}\right)^{\frac{1}{6}} t^7} \end{vmatrix}$$

Therefore

$$W = \left(\left(-\frac{1}{t^6}\right)^{\frac{1}{6}}\right)\left(\frac{5}{\left(-\frac{1}{t^6}\right)^{\frac{1}{6}} t^7}\right) - \left(\left(-\frac{1}{t^6}\right)^{\frac{5}{6}}\right)\left(\frac{1}{\left(-\frac{1}{t^6}\right)^{\frac{5}{6}} t^7}\right)$$

Which simplifies to

$$W = \frac{4}{t^7}$$

Which simplifies to

$$W = \frac{4}{t^7}$$

Therefore Eq. (2) becomes

$$u_1 = -\int \frac{\left(-\frac{1}{t^6}\right)^{\frac{5}{6}} t}{\frac{4}{t^5}}\,dt$$

Which simplifies to

$$u_1 = -\int \frac{\left(-\frac{1}{t^6}\right)^{\frac{5}{6}} t^6}{4} dt$$

Hence

$$u_1 = -\frac{\left(-\frac{1}{t^6}\right)^{\frac{5}{6}} t^7}{8}$$

And Eq. (3) becomes

$$u_2 = \int \frac{\left(-\frac{1}{t^6}\right)^{\frac{1}{6}} t}{\frac{4}{t^5}}\,dt$$

Which simplifies to

$$u_2 = \int \frac{\left(-\frac{1}{t^6}\right)^{\frac{1}{6}} t^6}{4} dt$$

Hence

$$u_2 = \frac{\left(-\frac{1}{t^6}\right)^{\frac{1}{6}} t^7}{24}$$

Therefore the particular solution, from equation (1) is

$$y_p(t) = \frac{t}{12}$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \left(\frac{c_1 6^{\frac{5}{6}} \left(-\frac{1}{t^6}\right)^{\frac{1}{6}}}{6} + \frac{c_2 6^{\frac{1}{6}} \left(-\frac{1}{t^6}\right)^{\frac{5}{6}}}{6}\right) + \left(\frac{t}{12}\right) \end{aligned}$$

Summary

The solution(s) found are the following

$$y = \frac{c_1 6^{\frac{5}{6}} \left(-\frac{1}{t^6}\right)^{\frac{1}{6}}}{6} + \frac{c_2 6^{\frac{1}{6}} \left(-\frac{1}{t^6}\right)^{\frac{5}{6}}}{6} + \frac{t}{12} \tag{1}$$

Verification of solutions

$$y = \frac{c_1 6^{\frac{5}{6}} \left(-\frac{1}{t^6}\right)^{\frac{1}{6}}}{6} + \frac{c_2 6^{\frac{1}{6}} \left(-\frac{1}{t^6}\right)^{\frac{5}{6}}}{6} + \frac{t}{12}$$

Verified OK.

10.20.3 Solving as second order change of variable on x method 1 ode

This is second order non-homogeneous ODE. In standard form the ODE is

$$Ay''(t) + By'(t) + Cy(t) = f(t)$$

Where $A = t^2, B = 7t, C = 5, f(t) = t$. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(t) + By'(t) + Cy(t) = 0$, and y_p is a particular solution to the non-homogeneous ODE $Ay''(t) + By'(t) + Cy(t) = f(t)$. Solving for y_h from

$$t^2 y'' + 7ty' + 5y = 0$$

In normal form the ode

$$t^2 y'' + 7ty' + 5y = 0 \tag{1}$$

Becomes

$$y'' + p(t)\, y' + q(t)\, y = 0 \tag{2}$$

Where

$$\begin{aligned} p(t) &= \frac{7}{t} \\ q(t) &= \frac{5}{t^2} \end{aligned}$$

Applying change of variables $\tau = g(t)$ to (2) results

$$\frac{d^2}{d\tau^2} y(\tau) + p_1 \left(\frac{d}{d\tau} y(\tau) \right) + q_1 y(\tau) = 0 \tag{3}$$

Where τ is the new independent variable, and

$$p_1(\tau) = \frac{\tau''(t) + p(t)\, \tau'(t)}{\tau'(t)^2} \tag{4}$$

$$q_1(\tau) = \frac{q(t)}{\tau'(t)^2} \tag{5}$$

Let $q_1 = c^2$ where c is some constant. Therefore from (5)

$$\begin{aligned} \tau' &= \frac{1}{c}\sqrt{q} \\ &= \frac{\sqrt{5}\,\sqrt{\frac{1}{t^2}}}{c} \\ \tau'' &= -\frac{\sqrt{5}}{c\sqrt{\frac{1}{t^2}}\, t^3} \end{aligned} \tag{6}$$

Substituting the above into (4) results in

$$\begin{aligned} p_1(\tau) &= \frac{\tau''(t) + p(t)\, \tau'(t)}{\tau'(t)^2} \\ &= \frac{-\frac{\sqrt{5}}{c\sqrt{\frac{1}{t^2}}\, t^3} + \frac{7}{t} \frac{\sqrt{5}\,\sqrt{\frac{1}{t^2}}}{c}}{\left(\frac{\sqrt{5}\,\sqrt{\frac{1}{t^2}}}{c}\right)^2} \\ &= \frac{6c\sqrt{5}}{5} \end{aligned}$$

Therefore ode (3) now becomes

$$
\begin{aligned}
y(\tau)'' + p_1 y(\tau)' + q_1 y(\tau) &= 0 \\
\frac{d^2}{d\tau^2} y(\tau) + \frac{6c\sqrt{5}\left(\frac{d}{d\tau} y(\tau)\right)}{5} + c^2 y(\tau) &= 0 \qquad (7)
\end{aligned}
$$

The above ode is now solved for $y(\tau)$. Since the ode is now constant coefficients, it can be easily solved to give

$$
y(\tau) = \mathrm{e}^{-\frac{3\sqrt{5}\, c\tau}{5}} \left(c_1 \cosh\left(\frac{2\sqrt{5}\, c\tau}{5}\right) + i c_2 \sinh\left(\frac{2\sqrt{5}\, c\tau}{5}\right) \right)
$$

Now from (6)

$$
\begin{aligned}
\tau &= \int \frac{1}{c} \sqrt{q}\, dt \\
&= \frac{\int \sqrt{5}\, \sqrt{\frac{1}{t^2}} dt}{c} \\
&= \frac{\sqrt{5}\, \sqrt{\frac{1}{t^2}}\, t \ln(t)}{c}
\end{aligned}
$$

Substituting the above into the solution obtained gives

$$
y = \frac{c_1 \cosh(2\ln(t)) + i c_2 \sinh(2\ln(t))}{t^3}
$$

Now the particular solution to this ODE is found

$$
t^2 y'' + 7ty' + 5y = t
$$

The particular solution y_p can be found using either the method of undetermined coefficients, or the method of variation of parameters. The method of variation of parameters will be used as it is more general and can be used when the coefficients of the ODE depend on t as well. Let

$$
y_p(t) = u_1 y_1 + u_2 y_2 \qquad (1)
$$

Where u_1, u_2 to be determined, and y_1, y_2 are the two basis solutions (the two linearly independent solutions of the homogeneous ODE) found earlier when solving the homogeneous ODE as

$$
y_1 = \left(-\frac{1}{t^6}\right)^{\frac{1}{6}}
$$

$$
y_2 = \left(-\frac{1}{t^6}\right)^{\frac{5}{6}}
$$

In the Variation of parameters u_1, u_2 are found using

$$u_1 = -\int \frac{y_2 f(t)}{aW(t)} \tag{2}$$

$$u_2 = \int \frac{y_1 f(t)}{aW(t)} \tag{3}$$

Where $W(t)$ is the Wronskian and a is the coefficient in front of y'' in the given ODE. The Wronskian is given by $W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$. Hence

$$W = \begin{vmatrix} \left(-\frac{1}{t^6}\right)^{\frac{1}{6}} & \left(-\frac{1}{t^6}\right)^{\frac{5}{6}} \\ \frac{d}{dt}\left(\left(-\frac{1}{t^6}\right)^{\frac{1}{6}}\right) & \frac{d}{dt}\left(\left(-\frac{1}{t^6}\right)^{\frac{5}{6}}\right) \end{vmatrix}$$

Which gives

$$W = \begin{vmatrix} \left(-\frac{1}{t^6}\right)^{\frac{1}{6}} & \left(-\frac{1}{t^6}\right)^{\frac{5}{6}} \\ \frac{1}{\left(-\frac{1}{t^6}\right)^{\frac{5}{6}} t^7} & \frac{5}{\left(-\frac{1}{t^6}\right)^{\frac{1}{6}} t^7} \end{vmatrix}$$

Therefore

$$W = \left(\left(-\frac{1}{t^6}\right)^{\frac{1}{6}}\right)\left(\frac{5}{\left(-\frac{1}{t^6}\right)^{\frac{1}{6}} t^7}\right) - \left(\left(-\frac{1}{t^6}\right)^{\frac{5}{6}}\right)\left(\frac{1}{\left(-\frac{1}{t^6}\right)^{\frac{5}{6}} t^7}\right)$$

Which simplifies to

$$W = \frac{4}{t^7}$$

Which simplifies to

$$W = \frac{4}{t^7}$$

Therefore Eq. (2) becomes

$$u_1 = -\int \frac{\left(-\frac{1}{t^6}\right)^{\frac{5}{6}} t}{\frac{4}{t^5}}\, dt$$

Which simplifies to

$$u_1 = -\int \frac{\left(-\frac{1}{t^6}\right)^{\frac{5}{6}} t^6}{4} dt$$

Hence

$$u_1 = -\frac{\left(-\frac{1}{t^6}\right)^{\frac{5}{6}} t^7}{8}$$

And Eq. (3) becomes

$$u_2 = \int \frac{\left(-\frac{1}{t^6}\right)^{\frac{1}{6}} t}{\frac{4}{t^5}} dt$$

Which simplifies to

$$u_2 = \int \frac{\left(-\frac{1}{t^6}\right)^{\frac{1}{6}} t^6}{4} dt$$

Hence

$$u_2 = \frac{\left(-\frac{1}{t^6}\right)^{\frac{1}{6}} t^7}{24}$$

Therefore the particular solution, from equation (1) is

$$y_p(t) = \frac{t}{12}$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \left(\frac{c_1 \cosh(2\ln(t)) + i c_2 \sinh(2\ln(t))}{t^3}\right) + \left(\frac{t}{12}\right) \\ &= \frac{t}{12} + \frac{c_1 \cosh(2\ln(t)) + i c_2 \sinh(2\ln(t))}{t^3} \end{aligned}$$

Which simplifies to

$$y = \frac{t^6 + (6ic_2 + 6c_1)\, t^4 - 6ic_2 + 6c_1}{12t^5}$$

Summary
The solution(s) found are the following

$$y = \frac{t^6 + (6ic_2 + 6c_1)\, t^4 - 6ic_2 + 6c_1}{12t^5} \tag{1}$$

Verification of solutions

$$y = \frac{t^6 + (6ic_2 + 6c_1)\, t^4 - 6ic_2 + 6c_1}{12t^5}$$

Verified OK.

10.20.4 Solving as second order change of variable on y method 2 ode

This is second order non-homogeneous ODE. In standard form the ODE is

$$Ay''(t) + By'(t) + Cy(t) = f(t)$$

Where $A = t^2, B = 7t, C = 5, f(t) = t$. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(t) + By'(t) + Cy(t) = 0$, and y_p is a particular solution to the non-homogeneous ODE $Ay''(t) + By'(t) + Cy(t) = f(t)$. Solving for y_h from

$$t^2y'' + 7ty' + 5y = 0$$

In normal form the ode

$$t^2y'' + 7ty' + 5y = 0 \tag{1}$$

Becomes

$$y'' + p(t)\, y' + q(t)\, y = 0 \tag{2}$$

Where

$$\begin{aligned} p(t) &= \frac{7}{t} \\ q(t) &= \frac{5}{t^2} \end{aligned}$$

Applying change of variables on the depndent variable $y = v(t)\, t^n$ to (2) gives the following ode where the dependent variables is $v(t)$ and not y.

$$v''(t) + \left(\frac{2n}{t} + p\right) v'(t) + \left(\frac{n(n-1)}{t^2} + \frac{np}{t} + q\right) v(t) = 0 \tag{3}$$

Let the coefficient of $v(t)$ above be zero. Hence

$$\frac{n(n-1)}{t^2} + \frac{np}{t} + q = 0 \tag{4}$$

Substituting the earlier values found for $p(t)$ and $q(t)$ into (4) gives

$$\frac{n(n-1)}{t^2} + \frac{7n}{t^2} + \frac{5}{t^2} = 0 \tag{5}$$

Solving (5) for n gives

$$n = -1 \tag{6}$$

Substituting this value in (3) gives

$$\begin{aligned} v''(t) + \frac{5v'(t)}{t} &= 0 \\ v''(t) + \frac{5v'(t)}{t} &= 0 \end{aligned} \tag{7}$$

Using the substitution

$$u(t) = v'(t)$$

Then (7) becomes

$$u'(t) + \frac{5u(t)}{t} = 0 \tag{8}$$

The above is now solved for $u(t)$. In canonical form the ODE is

$$\begin{aligned} u' &= F(t, u) \\ &= f(t)g(u) \\ &= -\frac{5u}{t} \end{aligned}$$

Where $f(t) = -\frac{5}{t}$ and $g(u) = u$. Integrating both sides gives

$$\begin{aligned} \frac{1}{u}\, du &= -\frac{5}{t}\, dt \\ \int \frac{1}{u}\, du &= \int -\frac{5}{t}\, dt \\ \ln(u) &= -5\ln(t) + c_1 \\ u &= e^{-5\ln(t) + c_1} \\ &= \frac{c_1}{t^5} \end{aligned}$$

Now that $u(t)$ is known, then

$$
\begin{aligned}
v'(t) &= u(t) \\
v(t) &= \int u(t)\, dt + c_2 \\
&= -\frac{c_1}{4t^4} + c_2
\end{aligned}
$$

Hence

$$
\begin{aligned}
y &= v(t)\, t^n \\
&= \frac{-\frac{c_1}{4t^4} + c_2}{t} \\
&= \frac{4c_2t^4 - c_1}{4t^5}
\end{aligned}
$$

Now the particular solution to this ODE is found

$$
t^2y'' + 7ty' + 5y = t
$$

The particular solution y_p can be found using either the method of undetermined coefficients, or the method of variation of parameters. The method of variation of parameters will be used as it is more general and can be used when the coefficients of the ODE depend on t as well. Let

$$
y_p(t) = u_1y_1 + u_2y_2 \tag{1}
$$

Where u_1, u_2 to be determined, and y_1, y_2 are the two basis solutions (the two linearly independent solutions of the homogeneous ODE) found earlier when solving the homogeneous ODE as

$$
\begin{aligned}
y_1 &= \frac{1}{t^5} \\
y_2 &= \frac{1}{t}
\end{aligned}
$$

In the Variation of parameters u_1, u_2 are found using

$$
u_1 = -\int \frac{y_2f(t)}{aW(t)} \tag{2}
$$

$$
u_2 = \int \frac{y_1f(t)}{aW(t)} \tag{3}
$$

Where $W(t)$ is the Wronskian and a is the coefficient in front of y'' in the given ODE.

The Wronskian is given by $W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$. Hence

$$W = \begin{vmatrix} \frac{1}{t^5} & \frac{1}{t} \\ \frac{d}{dt}\left(\frac{1}{t^5}\right) & \frac{d}{dt}\left(\frac{1}{t}\right) \end{vmatrix}$$

Which gives

$$W = \begin{vmatrix} \frac{1}{t^5} & \frac{1}{t} \\ -\frac{5}{t^6} & -\frac{1}{t^2} \end{vmatrix}$$

Therefore

$$W = \left(\frac{1}{t^5}\right)\left(-\frac{1}{t^2}\right) - \left(\frac{1}{t}\right)\left(-\frac{5}{t^6}\right)$$

Which simplifies to

$$W = \frac{4}{t^7}$$

Which simplifies to

$$W = \frac{4}{t^7}$$

Therefore Eq. (2) becomes

$$u_1 = -\int \frac{1}{\frac{4}{t^5}}\, dt$$

Which simplifies to

$$u_1 = -\int \frac{t^5}{4} dt$$

Hence

$$u_1 = -\frac{t^6}{24}$$

And Eq. (3) becomes

$$u_2 = \int \frac{\frac{1}{t^4}}{\frac{4}{t^5}}\, dt$$

Which simplifies to

$$u_2 = \int \frac{t}{4} dt$$

Hence

$$u_2 = \frac{t^2}{8}$$

Therefore the particular solution, from equation (1) is

$$y_p(t) = \frac{t}{12}$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \left(\frac{-\frac{c_1}{4t^4} + c_2}{t}\right) + \left(\frac{t}{12}\right) \\ &= \frac{t}{12} + \frac{-\frac{c_1}{4t^4} + c_2}{t} \end{aligned}$$

Which simplifies to

$$y = \frac{t}{12} + \frac{-\frac{c_1}{4t^4} + c_2}{t}$$

Summary

The solution(s) found are the following

$$y = \frac{t}{12} + \frac{-\frac{c_1}{4t^4} + c_2}{t} \tag{1}$$

Verification of solutions

$$y = \frac{t}{12} + \frac{-\frac{c_1}{4t^4} + c_2}{t}$$

Verified OK.

10.20.5 Solving as second order integrable as is ode

Integrating both sides of the ODE w.r.t t gives

$$\int (t^2y'' + 7ty' + 5y) \, dt = \int t dt$$
$$y't^2 + 5yt = \frac{t^2}{2} + c_1$$

Which is now solved for y.

Entering Linear first order ODE solver. In canonical form a linear first order is

$$y' + p(t)y = q(t)$$

Where here

$$p(t) = \frac{5}{t}$$
$$q(t) = \frac{t^2 + 2c_1}{2t^2}$$

Hence the ode is

$$y' + \frac{5y}{t} = \frac{t^2 + 2c_1}{2t^2}$$

The integrating factor μ is

$$\mu = e^{\int \frac{5}{t} dt}$$
$$= t^5$$

The ode becomes

$$\frac{d}{dt}(\mu y) = (\mu) \left(\frac{t^2 + 2c_1}{2t^2}\right)$$
$$\frac{d}{dt}(t^5 y) = (t^5) \left(\frac{t^2 + 2c_1}{2t^2}\right)$$
$$d(t^5 y) = \left(\frac{(t^2 + 2c_1) t^3}{2}\right) dt$$

Integrating gives

$$t^5 y = \int \frac{(t^2 + 2c_1) t^3}{2} dt$$
$$t^5 y = \frac{1}{12} t^6 + \frac{1}{4} c_1 t^4 + c_2$$

Dividing both sides by the integrating factor $\mu = t^5$ results in

$$y = \frac{\frac{1}{12}t^6 + \frac{1}{4}c_1t^4}{t^5} + \frac{c_2}{t^5}$$

which simplifies to

$$y = \frac{t^6 + 3c_1t^4 + 12c_2}{12t^5}$$

Summary

The solution(s) found are the following

$$y = \frac{t^6 + 3c_1t^4 + 12c_2}{12t^5} \tag{1}$$

Verification of solutions

$$y = \frac{t^6 + 3c_1t^4 + 12c_2}{12t^5}$$

Verified OK.

10.20.6 Solving as type second_order_integrable_as_is (not using ABC version)

Writing the ode as

$$t^2y'' + 7ty' + 5y = t$$

Integrating both sides of the ODE w.r.t t gives

$$\int (t^2y'' + 7ty' + 5y)\,dt = \int t dt$$

$$y't^2 + 5yt = \frac{t^2}{2} + c_1$$

Which is now solved for y.

Entering Linear first order ODE solver. In canonical form a linear first order is

$$y' + p(t)y = q(t)$$

Where here

$$p(t) = \frac{5}{t}$$

$$q(t) = \frac{t^2 + 2c_1}{2t^2}$$

Hence the ode is

$$y' + \frac{5y}{t} = \frac{t^2 + 2c_1}{2t^2}$$

The integrating factor μ is

$$\begin{aligned}\mu &= e^{\int \frac{5}{t} dt} \\ &= t^5\end{aligned}$$

The ode becomes

$$\begin{aligned}\frac{d}{dt}(\mu y) &= (\mu)\left(\frac{t^2 + 2c_1}{2t^2}\right) \\ \frac{d}{dt}(t^5 y) &= (t^5)\left(\frac{t^2 + 2c_1}{2t^2}\right) \\ d(t^5 y) &= \left(\frac{(t^2 + 2c_1) t^3}{2}\right) dt\end{aligned}$$

Integrating gives

$$\begin{aligned}t^5 y &= \int \frac{(t^2 + 2c_1) t^3}{2} dt \\ t^5 y &= \frac{1}{12} t^6 + \frac{1}{4} c_1 t^4 + c_2\end{aligned}$$

Dividing both sides by the integrating factor $\mu = t^5$ results in

$$y = \frac{\frac{1}{12} t^6 + \frac{1}{4} c_1 t^4}{t^5} + \frac{c_2}{t^5}$$

which simplifies to

$$y = \frac{t^6 + 3c_1 t^4 + 12c_2}{12t^5}$$

Summary

The solution(s) found are the following

$$y = \frac{t^6 + 3c_1 t^4 + 12c_2}{12t^5} \tag{1}$$

Verification of solutions

$$y = \frac{t^6 + 3c_1 t^4 + 12c_2}{12t^5}$$

Verified OK.

10.20.7 Solving using Kovacic algorithm

Writing the ode as

$$t^2y'' + 7ty' + 5y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= t^2 \\ B &= 7t \\ C &= 5 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(t) = ye^{\int \frac{B}{2A}\,dt}$$

Then (2) becomes

$$z''(t) = rz(t) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{15}{4t^2} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= 15 \\ t &= 4t^2 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(t) = \left(\frac{15}{4t^2}\right) z(t) \tag{7}$$

Equation (7) is now solved. After finding $z(t)$ then y is found using the inverse transformation

$$y = z(t)\, e^{-\int \frac{B}{2A}\,dt}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 505: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 2 - 0 \\ &= 2 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 4t^2$. There is a pole at $t = 0$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 2 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Since pole order is not larger than 2 and the order at ∞ is 2 then the necessary conditions for case three are met. Therefore

$$L = [1, 2, 4, 6, 12]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = \frac{15}{4t^2}$$

For the pole at $t = 0$ let b be the coefficient of $\frac{1}{t^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{15}{4}$. Hence

$$
\begin{aligned}
[\sqrt{r}]_c &= 0 \\
\alpha_c^{+} &= \frac{1}{2} + \sqrt{1 + 4b} = \frac{5}{2} \\
\alpha_c^{-} &= \frac{1}{2} - \sqrt{1 + 4b} = -\frac{3}{2}
\end{aligned}
$$

Since the order of r at ∞ is 2 then $[\sqrt{r}]_\infty = 0$. Let b be the coefficient of $\frac{1}{t^2}$ in the Laurent series expansion of r at ∞. which can be found by dividing the leading coefficient of s by the leading coefficient of t from

$$
r = \frac{s}{t} = \frac{15}{4t^2}
$$

Since the $\gcd(s, t) = 1$. This gives $b = \frac{15}{4}$. Hence

$$
\begin{aligned}
[\sqrt{r}]_\infty &= 0 \\
\alpha_\infty^{+} &= \frac{1}{2} + \sqrt{1 + 4b} = \frac{5}{2} \\
\alpha_\infty^{-} &= \frac{1}{2} - \sqrt{1 + 4b} = -\frac{3}{2}
\end{aligned}
$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$
r = \frac{15}{4t^2}
$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^{+}	α_c^{-}
0	2	0	$\frac{5}{2}$	$-\frac{3}{2}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^{+}	α_∞^{-}
2	0	$\frac{5}{2}$	$-\frac{3}{2}$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^{\pm}$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^{\pm}$ have also been found, the next step is to determine possible non negative integer d from these using

$$
d = \alpha_\infty^{s(\infty)} - \sum_{c \in \Gamma} \alpha_c^{s(c)}
$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^{\pm}$. This is done by trial over all set of families $s = (s(c))_{c \in \Gamma \cup \infty}$ until such d is found to work in finding candidate ω.

Trying $\alpha_\infty^- = -\frac{3}{2}$ then

$$\begin{aligned} d &= \alpha_\infty^- - \left(\alpha_{c_1}^-\right) \\ &= -\frac{3}{2} - \left(-\frac{3}{2}\right) \\ &= 0 \end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c \in \Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{t-c} \right) + s(\infty)[\sqrt{r}]_\infty$$

The above gives

$$\begin{aligned} \omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^-}{t-c_1} \right) + (-)[\sqrt{r}]_\infty \\ &= -\frac{3}{2t} + (-)\,(0) \\ &= -\frac{3}{2t} \\ &= -\frac{3}{2t} \end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(t)$ of degree $d = 0$ to solve the ode. The polynomial $p(t)$ needs to satisfy the equation

$$p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right) p = 0 \tag{1A}$$

Let

$$p(t) = 1 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$\begin{aligned} (0) + 2\left(-\frac{3}{2t}\right)(0) + \left(\left(\frac{3}{2t^2}\right) + \left(-\frac{3}{2t}\right)^2 - \left(\frac{15}{4t^2}\right)\right) &= 0 \\ 0 &= 0 \end{aligned}$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned} z_1(t) &= p e^{\int \omega \, dt} \\ &= e^{\int -\frac{3}{2t} dt} \\ &= \frac{1}{t^{\frac{3}{2}}} \end{aligned}$$

The first solution to the original ode in y is found from

$$
\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dt} \\
&= z_1 e^{-\int \frac{1}{2}\frac{7t}{t^2}\,dt} \\
&= z_1 e^{-\frac{7\ln(t)}{2}} \\
&= z_1\left(\frac{1}{t^{\frac{7}{2}}}\right)
\end{aligned}
$$

Which simplifies to

$$
y_1 = \frac{1}{t^5}
$$

The second solution y_2 to the original ode is found using reduction of order

$$
y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dt}}{y_1^2}\,dt
$$

Substituting gives

$$
\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{7t}{t^2}\,dt}}{(y_1)^2}\,dt \\
&= y_1 \int \frac{e^{-7\ln(t)}}{(y_1)^2}\,dt \\
&= y_1\left(\frac{t^4}{4}\right)
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1\left(\frac{1}{t^5}\right) + c_2\left(\frac{1}{t^5}\left(\frac{t^4}{4}\right)\right)
\end{aligned}
$$

This is second order nonhomogeneous ODE. Let the solution be

$$
y = y_h + y_p
$$

Where y_h is the solution to the homogeneous ODE $Ay''(t)+By'(t)+Cy(t)=0$, and y_p is a particular solution to the nonhomogeneous ODE $Ay''(t)+By'(t)+Cy(t)=f(t)$. y_h is the solution to

$$t^2y''+7ty'+5y=0$$

The homogeneous solution is found using the Kovacic algorithm which results in

$$y_h=\frac{c_1}{t^5}+\frac{c_2}{4t}$$

The particular solution y_p can be found using either the method of undetermined coefficients, or the method of variation of parameters. The method of variation of parameters will be used as it is more general and can be used when the coefficients of the ODE depend on t as well. Let

$$y_p(t)=u_1y_1+u_2y_2 \tag{1}$$

Where u_1,u_2 to be determined, and y_1,y_2 are the two basis solutions (the two linearly independent solutions of the homogeneous ODE) found earlier when solving the homogeneous ODE as

$$y_1=\frac{1}{t^5}$$

$$y_2=\frac{1}{4t}$$

In the Variation of parameters u_1,u_2 are found using

$$u_1=-\int\frac{y_2f(t)}{aW(t)} \tag{2}$$

$$u_2=\int\frac{y_1f(t)}{aW(t)} \tag{3}$$

Where $W(t)$ is the Wronskian and a is the coefficient in front of y'' in the given ODE. The Wronskian is given by $W=\begin{vmatrix}y_1 & y_2\\ y_1' & y_2'\end{vmatrix}$. Hence

$$W=\begin{vmatrix}\frac{1}{t^5} & \frac{1}{4t}\\ \frac{d}{dt}\left(\frac{1}{t^5}\right) & \frac{d}{dt}\left(\frac{1}{4t}\right)\end{vmatrix}$$

Which gives

$$W = \begin{vmatrix} \frac{1}{t^5} & \frac{1}{4t} \\ -\frac{5}{t^6} & -\frac{1}{4t^2} \end{vmatrix}$$

Therefore

$$W = \left(\frac{1}{t^5}\right)\left(-\frac{1}{4t^2}\right) - \left(\frac{1}{4t}\right)\left(-\frac{5}{t^6}\right)$$

Which simplifies to

$$W = \frac{1}{t^7}$$

Which simplifies to

$$W = \frac{1}{t^7}$$

Therefore Eq. (2) becomes

$$u_1 = -\int \frac{\frac{1}{4}}{\frac{1}{t^5}}\,dt$$

Which simplifies to

$$u_1 = -\int \frac{t^5}{4} dt$$

Hence

$$u_1 = -\frac{t^6}{24}$$

And Eq. (3) becomes

$$u_2 = \int \frac{\frac{1}{t^4}}{\frac{1}{t^5}}\,dt$$

Which simplifies to

$$u_2 = \int t dt$$

Hence

$$u_2 = \frac{t^2}{2}$$

Therefore the particular solution, from equation (1) is

$$y_p(t) = \frac{t}{12}$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \left(\frac{c_1}{t^5} + \frac{c_2}{4t}\right) + \left(\frac{t}{12}\right) \end{aligned}$$

Summary
The solution(s) found are the following

$$y = \frac{c_1}{t^5} + \frac{c_2}{4t} + \frac{t}{12} \tag{1}$$

Verification of solutions

$$y = \frac{c_1}{t^5} + \frac{c_2}{4t} + \frac{t}{12}$$

Verified OK.

10.20.8 Solving as exact linear second order ode ode

An ode of the form

$$p(t)\, y'' + q(t)\, y' + r(t)\, y = s(t)$$

is exact if

$$p''(t) - q'(t) + r(t) = 0 \tag{1}$$

For the given ode we have

$$\begin{aligned} p(x) &= t^2 \\ q(x) &= 7t \\ r(x) &= 5 \\ s(x) &= t \end{aligned}$$

Hence

$$\begin{aligned} p''(x) &= 2 \\ q'(x) &= 7 \end{aligned}$$

Therefore (1) becomes

$$2-(7)+(5)=0$$

Hence the ode is exact. Since we now know the ode is exact, it can be written as

$$(p(t)\,y'+(q(t)-p'(t))\,y)'=s(x)$$

Integrating gives

$$p(t)\,y'+(q(t)-p'(t))\,y=\int s(t)\;dt$$

Substituting the above values for p,q,r,s gives

$$y't^2+5yt=\int t\,dt$$

We now have a first order ode to solve which is

$$y't^2+5yt=\frac{t^2}{2}+c_1$$

Entering Linear first order ODE solver. In canonical form a linear first order is

$$y'+p(t)y=q(t)$$

Where here

$$\begin{aligned}p(t)&=\frac{5}{t}\\q(t)&=\frac{t^2+2c_1}{2t^2}\end{aligned}$$

Hence the ode is

$$y'+\frac{5y}{t}=\frac{t^2+2c_1}{2t^2}$$

The integrating factor μ is

$$\begin{aligned}\mu&=\mathrm{e}^{\int \frac{5}{t}dt}\\&=t^5\end{aligned}$$

The ode becomes

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}(\mu y)&=(\mu)\left(\frac{t^2+2c_1}{2t^2}\right)\\\frac{\mathrm{d}}{\mathrm{d}t}(t^5y)&=(t^5)\left(\frac{t^2+2c_1}{2t^2}\right)\\\mathrm{d}(t^5y)&=\left(\frac{(t^2+2c_1)\,t^3}{2}\right)\,\mathrm{d}t\end{aligned}$$

Integrating gives

$$t^5 y = \int \frac{(t^2 + 2c_1)\, t^3}{2}\, \mathrm{d}t$$
$$t^5 y = \frac{1}{12} t^6 + \frac{1}{4} c_1 t^4 + c_2$$

Dividing both sides by the integrating factor $\mu = t^5$ results in

$$y = \frac{\frac{1}{12} t^6 + \frac{1}{4} c_1 t^4}{t^5} + \frac{c_2}{t^5}$$

which simplifies to

$$y = \frac{t^6 + 3c_1 t^4 + 12c_2}{12t^5}$$

Summary
The solution(s) found are the following

$$y = \frac{t^6 + 3c_1 t^4 + 12c_2}{12t^5} \tag{1}$$

Verification of solutions

$$y = \frac{t^6 + 3c_1 t^4 + 12c_2}{12t^5}$$

Verified OK.

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
<- high order exact linear fully integrable successful`
```

✓ Solution by Maple
Time used: 0.0 (sec). Leaf size: 27

```
dsolve(t^2*diff(y(t),t$2)+7*t*diff(y(t),t)+5*y(t) = t,y(t), singsol=all)
```

$$y(t) = \frac{t^6 + 3c_1 t^4 - 4c_1^3 + 12c_2}{12t^5}$$

✓ Solution by Mathematica

Time used: 0.015 (sec). Leaf size: 23

```
DSolve[t^2*y''[t]+7*t*y'[t]+5*y[t] ==t,y[t],t,IncludeSingularSolutions -> True]
```

$$y(t) \to \frac{c_1}{t^5} + \frac{t}{12} + \frac{c_2}{t}$$

10.21 problem 31

Internal problem ID [703]
Internal file name [`OUTPUT/703_Sunday_June_05_2022_01_47_19_AM_67928080/index.tex`]

Book: Elementary differential equations and boundary value problems, 10th ed., Boyce and DiPrima
Section: Chapter 3, Second order linear equations, section 3.6, Variation of Parameters. page 190
Problem number: 31.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_ode_non_constant_coeff_transformation_on_B"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

$$\boxed{ty'' - (t+1)\,y' + y = e^{2t}t^2}$$

10.21.1 Solving as second order ode non constant coeff transformation on B ode

Given an ode of the form

$$Ay'' + By' + Cy = F(t)$$

This method reduces the order ode the ODE by one by applying the transformation

$$y = Bv$$

This results in

$$\begin{aligned} y' &= B'v + v'B \\ y'' &= B''v + B'v' + v''B + v'B' \\ &= v''B + 2v' + B' + B''v \end{aligned}$$

And now the original ode becomes

$$
\begin{aligned}
A(v''B + 2v'B' + B''v) + B(B'v + v'B) + CBv &= 0 \\
ABv'' + \left(2AB' + B^2\right)v' + (AB'' + BB' + CB)\,v &= 0 \qquad (1)
\end{aligned}
$$

If the term $AB'' + BB' + CB$ is zero, then this method works and can be used to solve

$$
ABv'' + \left(2AB' + B^2\right)v' = 0
$$

By Using $u = v'$ which reduces the order of the above ode to one. The new ode is

$$
ABu' + \left(2AB' + B^2\right)u = 0
$$

The above ode is first order ode which is solved for u. Now a new ode $v' = u$ is solved for v as first order ode. Then the final solution is obtain from $y = Bv$.

This method works only if the term $AB'' + BB' + CB$ is zero. The given ODE shows that

$$
\begin{aligned}
A &= t \\
B &= -t - 1 \\
C &= 1 \\
F &= \mathrm{e}^{2t}t^2
\end{aligned}
$$

The above shows that for this ode

$$
\begin{aligned}
AB'' + BB' + CB &= (t)\,(0) + (-t-1)\,(-1) + (1)\,(-t-1) \\
&= 0
\end{aligned}
$$

Hence the ode in v given in (1) now simplifies to

$$
-t(t+1)\,v'' + \left(t^2+1\right)v' = 0
$$

Now by applying $v' = u$ the above becomes

$$
-t(t+1)\,u'(t) + \left(t^2+1\right)u(t) = 0
$$

Which is now solved for u. In canonical form the ODE is

$$
\begin{aligned}
u' &= F(t,u) \\
&= f(t)g(u) \\
&= \frac{(t^2+1)\,u}{t\,(t+1)}
\end{aligned}
$$

Where $f(t) = \frac{t^2+1}{t(t+1)}$ and $g(u) = u$. Integrating both sides gives

$$
\begin{aligned}
\frac{1}{u} du &= \frac{t^2+1}{t(t+1)} dt \\
\int \frac{1}{u} du &= \int \frac{t^2+1}{t(t+1)} dt \\
\ln(u) &= t + \ln(t) - 2\ln(t+1) + c_1 \\
u &= e^{t+\ln(t)-2\ln(t+1)+c_1} \\
&= c_1 e^{t+\ln(t)-2\ln(t+1)}
\end{aligned}
$$

Which simplifies to

$$
u(t) = \frac{c_1 e^t t}{(t+1)^2}
$$

The ode for v now becomes

$$
\begin{aligned}
v' &= u \\
&= \frac{c_1 e^t t}{(t+1)^2}
\end{aligned}
$$

Which is now solved for v. Integrating both sides gives

$$
\begin{aligned}
v(t) &= \int \frac{c_1 e^t t}{(t+1)^2} dt \\
&= \frac{c_1 e^t}{t+1} + c_2
\end{aligned}
$$

Therefore the homogeneous solution is

$$
\begin{aligned}
y_h(t) &= Bv \\
&= (-t-1)\left(\frac{c_1 e^t}{t+1} + c_2\right) \\
&= -c_1 e^t - c_2(t+1)
\end{aligned}
$$

And now the particular solution $y_p(t)$ will be found. The particular solution y_p can be found using either the method of undetermined coefficients, or the method of variation of parameters. The method of variation of parameters will be used as it is more general and can be used when the coefficients of the ODE depend on t as well. Let

$$
y_p(t) = u_1 y_1 + u_2 y_2 \tag{1}
$$

Where u_1, u_2 to be determined, and y_1, y_2 are the two basis solutions (the two linearly independent solutions of the homogeneous ODE) found earlier when solving the homogeneous ODE as

$$y_1 = -t - 1$$
$$y_2 = e^t$$

In the Variation of parameters u_1, u_2 are found using

$$u_1 = -\int \frac{y_2 f(t)}{aW(t)} \tag{2}$$

$$u_2 = \int \frac{y_1 f(t)}{aW(t)} \tag{3}$$

Where $W(t)$ is the Wronskian and a is the coefficient in front of y'' in the given ODE. The Wronskian is given by $W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$. Hence

$$W = \begin{vmatrix} -t - 1 & e^t \\ \frac{d}{dt}(-t - 1) & \frac{d}{dt}(e^t) \end{vmatrix}$$

Which gives

$$W = \begin{vmatrix} -t - 1 & e^t \\ -1 & e^t \end{vmatrix}$$

Therefore

$$W = (-t - 1)(e^t) - (e^t)(-1)$$

Which simplifies to

$$W = -t\, e^t$$

Which simplifies to

$$W = -t\, e^t$$

Therefore Eq. (2) becomes

$$u_1 = -\int \frac{e^t e^{2t} t^2}{-t^2 e^t}\, dt$$

Which simplifies to

$$u_1 = -\int -\mathrm{e}^{2t} dt$$

Hence

$$u_1 = \frac{\mathrm{e}^{2t}}{2}$$

And Eq. (3) becomes

$$u_2 = \int \frac{(-t-1)\,\mathrm{e}^{2t} t^2}{-t^2 \mathrm{e}^t} dt$$

Which simplifies to

$$u_2 = \int \mathrm{e}^t (t+1)\, dt$$

Hence

$$u_2 = t\,\mathrm{e}^t$$

Therefore the particular solution, from equation (1) is

$$y_p(t) = \frac{(-t-1)\,\mathrm{e}^{2t}}{2} + \mathrm{e}^{2t} t$$

Which simplifies to

$$y_p(t) = \frac{\mathrm{e}^{2t}(-1+t)}{2}$$

Hence the complete solution is

$$\begin{aligned} y(t) &= y_h + y_p \\ &= \left(-c_1 \mathrm{e}^t - c_2(t+1)\right) + \left(\frac{\mathrm{e}^{2t}(-1+t)}{2}\right) \\ &= -c_1 \mathrm{e}^t - c_2(t+1) + \frac{\mathrm{e}^{2t}(-1+t)}{2} \end{aligned}$$

Summary

The solution(s) found are the following

$$y = -c_1 \mathrm{e}^t - c_2(t+1) + \frac{\mathrm{e}^{2t}(-1+t)}{2} \tag{1}$$

Verification of solutions

$$y = -c_1 e^t - c_2(t+1) + \frac{e^{2t}(-1+t)}{2}$$

Verified OK.

10.21.2 Solving using Kovacic algorithm

Writing the ode as

$$ty'' + (-t-1)\,y' + y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= t \\ B &= -t-1 \\ C &= 1 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(t) = ye^{\int \frac{B}{2A}\,dt}$$

Then (2) becomes

$$z''(t) = rz(t) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{t^2 - 2t + 3}{4t^2} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= t^2 - 2t + 3 \\ t &= 4t^2 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(t) = \left(\frac{t^2 - 2t + 3}{4t^2}\right) z(t) \tag{7}$$

Equation (7) is now solved. After finding $z(t)$ then y is found using the inverse transformation

$$y = z(t)\, e^{-\int \frac{B}{2A}\, dt}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 506: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 2 - 2 \\ &= 0 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 4t^2$. There is a pole at $t = 0$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Therefore

$$L = [1, 2]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = \frac{1}{4} - \frac{1}{2t} + \frac{3}{4t^2}$$

For the pole at $t = 0$ let b be the coefficient of $\frac{1}{t^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{3}{4}$. Hence

$$\begin{aligned}
[\sqrt{r}]_c &= 0 \\
\alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{3}{2} \\
\alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = -\frac{1}{2}
\end{aligned}$$

Since the order of r at ∞ is $O_r(\infty) = 0$ then

$$v = \frac{-O_r(\infty)}{2} = \frac{0}{2} = 0$$

$[\sqrt{r}]_\infty$ is the sum of terms involving t^i for $0 \le i \le v$ in the Laurent series for $\sqrt{r}$ at ∞. Therefore

$$\begin{aligned}
[\sqrt{r}]_\infty &= \sum_{i=0}^{v} a_i t^i \\
&= \sum_{i=0}^{0} a_i t^i
\end{aligned} \tag{8}$$

Let a be the coefficient of $t^v = t^0$ in the above sum. The Laurent series of $\sqrt{r}$ at ∞ is

$$\sqrt{r} \approx \frac{1}{2} - \frac{1}{2t} + \frac{1}{2t^2} + \frac{1}{2t^3} + \frac{1}{4t^4} - \frac{1}{4t^5} - \frac{3}{4t^6} - \frac{3}{4t^7} + \dots \tag{9}$$

Comparing Eq. (9) with Eq. (8) shows that

$$a = \frac{1}{2}$$

From Eq. (9) the sum up to $v = 0$ gives

$$\begin{aligned}
[\sqrt{r}]_\infty &= \sum_{i=0}^{0} a_i t^i \\
&= \frac{1}{2}
\end{aligned} \tag{10}$$

Now we need to find b, where b be the coefficient of $t^{v-1} = t^{-1} = \frac{1}{t}$ in r minus the coefficient of same term but in $\left([\sqrt{r}]_\infty\right)^2$ where $[\sqrt{r}]_\infty$ was found above in Eq (10). Hence

$$\left([\sqrt{r}]_\infty\right)^2 = \frac{1}{4}$$

This shows that the coefficient of $\frac{1}{t}$ in the above is 0. Now we need to find the coefficient of $\frac{1}{t}$ in r. How this is done depends on if $v = 0$ or not. Since $v = 0$ then starting from $r = \frac{s}{t}$ and doing long division in the form

$$r = Q + \frac{R}{t}$$

Where Q is the quotient and R is the remainder. Then the coefficient of $\frac{1}{t}$ in r will be the coefficient in R of the term in t of degree of t minus one, divided by the leading coefficient in t. Doing long division gives

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{t^2 - 2t + 3}{4t^2} \\ &= Q + \frac{R}{4t^2} \\ &= \left(\frac{1}{4}\right) + \left(\frac{-2t + 3}{4t^2}\right) \\ &= \frac{1}{4} + \frac{-2t + 3}{4t^2} \end{aligned}$$

Since the degree of t is 2, then we see that the coefficient of the term t in the remainder R is -2. Dividing this by leading coefficient in t which is 4 gives $-\frac{1}{2}$. Now b can be found.

$$\begin{aligned} b &= \left(-\frac{1}{2}\right) - (0) \\ &= -\frac{1}{2} \end{aligned}$$

Hence

$$\begin{aligned} [\sqrt{r}]_\infty &= \frac{1}{2} \\ \alpha_\infty^+ &= \frac{1}{2}\left(\frac{b}{a} - v\right) &&= \frac{1}{2}\left(\frac{-\frac{1}{2}}{\frac{1}{2}} - 0\right) &&= -\frac{1}{2} \\ \alpha_\infty^- &= \frac{1}{2}\left(-\frac{b}{a} - v\right) &&= \frac{1}{2}\left(-\frac{-\frac{1}{2}}{\frac{1}{2}} - 0\right) &&= \frac{1}{2} \end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = \frac{t^2 - 2t + 3}{4t^2}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
0	2	0	$\frac{3}{2}$	$-\frac{1}{2}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
0	$\frac{1}{2}$	$-\frac{1}{2}$	$\frac{1}{2}$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c \in \Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c \in \Gamma \cup \infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^+ = -\frac{1}{2}$ then

$$\begin{aligned} d &= \alpha_\infty^+ - (\alpha_{c_1}^-) \\ &= -\frac{1}{2} - \left(-\frac{1}{2}\right) \\ &= 0 \end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c \in \Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{t - c} \right) + s(\infty)[\sqrt{r}]_\infty$$

Substituting the above values in the above results in

$$\begin{aligned} \omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^-}{t - c_1} \right) + (+)[\sqrt{r}]_\infty \\ &= -\frac{1}{2t} + \left(\frac{1}{2}\right) \\ &= \frac{1}{2} - \frac{1}{2t} \\ &= \frac{-1 + t}{2t} \end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(t)$ of degree $d = 0$ to solve the ode. The polynomial $p(t)$ needs to satisfy the equation

$$p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right) p = 0 \tag{1A}$$

Let

$$p(t) = 1 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$(0) + 2\left(\frac{1}{2} - \frac{1}{2t}\right)(0) + \left(\left(\frac{1}{2t^2}\right) + \left(\frac{1}{2} - \frac{1}{2t}\right)^2 - \left(\frac{t^2 - 2t + 3}{4t^2}\right)\right) = 0$$
$$0 = 0$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned} z_1(t) &= p e^{\int \omega \, dt} \\ &= \mathrm{e}^{\int \left(\frac{1}{2} - \frac{1}{2t}\right) dt} \\ &= \frac{\mathrm{e}^{\frac{t}{2}}}{\sqrt{t}} \end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned} y_1 &= z_1 e^{\int -\frac{1}{2} \frac{B}{A} \, dt} \\ &= z_1 e^{-\int \frac{1}{2} \frac{-t-1}{t} \, dt} \\ &= z_1 e^{\frac{t}{2} + \frac{\ln(t)}{2}} \\ &= z_1 \left(\sqrt{t}\, \mathrm{e}^{\frac{t}{2}}\right) \end{aligned}$$

Which simplifies to

$$y_1 = \mathrm{e}^t$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A} \, dt}}{y_1^2} \, dt$$

Substituting gives

$$
\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{-t-1}{t} \, dt}}{(y_1)^2} \, dt \\
&= y_1 \int \frac{e^{t+\ln(t)}}{(y_1)^2} \, dt \\
&= y_1 \left(-\mathrm{e}^{-t}(t+1)\right)
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1 \left(\mathrm{e}^t\right) + c_2 \left(\mathrm{e}^t \left(-\mathrm{e}^{-t}(t+1)\right)\right)
\end{aligned}
$$

This is second order nonhomogeneous ODE. Let the solution be

$$
y = y_h + y_p
$$

Where y_h is the solution to the homogeneous ODE $Ay''(t) + By'(t) + Cy(t) = 0$, and y_p is a particular solution to the nonhomogeneous ODE $Ay''(t) + By'(t) + Cy(t) = f(t)$. y_h is the solution to

$$
ty'' + (-t-1)\, y' + y = 0
$$

The homogeneous solution is found using the Kovacic algorithm which results in

$$
y_h = c_1 \mathrm{e}^t + (-t-1)\, c_2
$$

The particular solution y_p can be found using either the method of undetermined coefficients, or the method of variation of parameters. The method of variation of parameters will be used as it is more general and can be used when the coefficients of the ODE depend on t as well. Let

$$
y_p(t) = u_1 y_1 + u_2 y_2 \tag{1}
$$

Where u_1, u_2 to be determined, and y_1, y_2 are the two basis solutions (the two linearly independent solutions of the homogeneous ODE) found earlier when solving the homogeneous ODE as

$$
\begin{aligned}
y_1 &= \mathrm{e}^t \\
y_2 &= -t - 1
\end{aligned}
$$

In the Variation of parameters u_1, u_2 are found using

$$u_1 = -\int \frac{y_2 f(t)}{aW(t)} \tag{2}$$

$$u_2 = \int \frac{y_1 f(t)}{aW(t)} \tag{3}$$

Where $W(t)$ is the Wronskian and a is the coefficient in front of y'' in the given ODE. The Wronskian is given by $W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$. Hence

$$W = \begin{vmatrix} e^t & -t-1 \\ \frac{d}{dt}(e^t) & \frac{d}{dt}(-t-1) \end{vmatrix}$$

Which gives

$$W = \begin{vmatrix} e^t & -t-1 \\ e^t & -1 \end{vmatrix}$$

Therefore

$$W = (e^t)(-1) - (-t-1)(e^t)$$

Which simplifies to

$$W = t\,e^t$$

Which simplifies to

$$W = t\,e^t$$

Therefore Eq. (2) becomes

$$u_1 = -\int \frac{(-t-1)\,e^{2t}t^2}{t^2 e^t}\,dt$$

Which simplifies to

$$u_1 = -\int -e^t(t+1)\,dt$$

Hence

$$u_1 = t\,e^t$$

And Eq. (3) becomes

$$u_2 = \int \frac{e^t e^{2t} t^2}{t^2 e^t}\, dt$$

Which simplifies to

$$u_2 = \int e^{2t} dt$$

Hence

$$u_2 = \frac{e^{2t}}{2}$$

Therefore the particular solution, from equation (1) is

$$y_p(t) = \frac{(-t-1)\, e^{2t}}{2} + e^{2t} t$$

Which simplifies to

$$y_p(t) = \frac{e^{2t}(-1+t)}{2}$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= (c_1 e^t + (-t-1)\, c_2) + \left(\frac{e^{2t}(-1+t)}{2}\right) \end{aligned}$$

Which simplifies to

$$y = c_1 e^t - c_2(t+1) + \frac{e^{2t}(-1+t)}{2}$$

Summary

The solution(s) found are the following

$$y = c_1 e^t - c_2(t+1) + \frac{e^{2t}(-1+t)}{2} \tag{1}$$

Verification of solutions

$$y = c_1 e^t - c_2(t+1) + \frac{e^{2t}(-1+t)}{2}$$

Verified OK.

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 2; linear nonhomogeneous with symmetry [0,1]
trying a double symmetry of the form [xi=0, eta=F(x)]
-> Try solving first the homogeneous part of the ODE
   checking if the LODE has constant coefficients
   checking if the LODE is of Euler type
   trying a symmetry of the form [xi=0, eta=F(x)]
   checking if the LODE is missing y
   -> Trying a Liouvillian solution using Kovacics algorithm
      A Liouvillian solution exists
      Reducible group (found an exponential solution)
      Reducible group (found another exponential solution)
   <- Kovacics algorithm successful
<- solving first the homogeneous part of the ODE successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 23

```
dsolve(t*diff(y(t),t$2)-(1+t)*diff(y(t),t)+y(t) = t^2*exp(2*t),y(t), singsol=all)
```

$$y(t) = (t+1)\,c_2 + e^t c_1 + \frac{(t-1)\,e^{2t}}{2}$$

✓ Solution by Mathematica

Time used: 0.033 (sec). Leaf size: 31

```
DSolve[t*y''[t]-(1+t)*y'[t]+y[t] ==t^2*Exp[2*t],y[t],t,IncludeSingularSolutions -> True]
```

$$y(t) \to \frac{1}{2}e^{2t}(t-1) + c_1 e^t - c_2(t+1)$$

10.22 problem 32

Internal problem ID [704]
Internal file name [`OUTPUT/704_Sunday_June_05_2022_01_47_21_AM_33934857/index.tex`]

Book: Elementary differential equations and boundary value problems, 10th ed., Boyce and DiPrima
Section: Chapter 3, Second order linear equations, section 3.6, Variation of Parameters. page 190
Problem number: 32.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_change_of_variable_on_y_method_2", "second_order_ode_non_constant_coeff_transformation_on_B"**

Maple gives the following as the ode type

`[[_2nd_order, _with_linear_symmetries]]`

$$\boxed{(1-t)\, y'' + t y' - y = 2(-1+t)\, \mathrm{e}^{-t}}$$

10.22.1 Solving as second order change of variable on y method 2 ode

This is second order non-homogeneous ODE. In standard form the ODE is

$$Ay''(t) + By'(t) + Cy(t) = f(t)$$

Where $A = 1-t, B = t, C = -1, f(t) = 2(-1+t)\, \mathrm{e}^{-t}$. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(t) + By'(t) + Cy(t) = 0$, and y_p is a particular solution to the non-homogeneous ODE $Ay''(t) + By'(t) + Cy(t) = f(t)$. Solving for y_h from

$$(1-t)\, y'' + t y' - y = 0$$

In normal form the ode

$$(1-t)\,y''+ty'-y=0 \tag{1}$$

Becomes

$$y''+p(t)\,y'+q(t)\,y=0 \tag{2}$$

Where

$$\begin{aligned}p(t)&=-\frac{t}{-1+t}\\ q(t)&=\frac{1}{-1+t}\end{aligned}$$

Applying change of variables on the depndent variable $y=v(t)\,t^n$ to (2) gives the following ode where the dependent variables is $v(t)$ and not y.

$$v''(t)+\left(\frac{2n}{t}+p\right)v'(t)+\left(\frac{n(n-1)}{t^2}+\frac{np}{t}+q\right)v(t)=0 \tag{3}$$

Let the coefficient of $v(t)$ above be zero. Hence

$$\frac{n(n-1)}{t^2}+\frac{np}{t}+q=0 \tag{4}$$

Substituting the earlier values found for $p(t)$ and $q(t)$ into (4) gives

$$\frac{n(n-1)}{t^2}-\frac{n}{-1+t}+\frac{1}{-1+t}=0 \tag{5}$$

Solving (5) for n gives

$$n=1 \tag{6}$$

Substituting this value in (3) gives

$$\begin{aligned}v''(t)+\left(\frac{2}{t}-\frac{t}{-1+t}\right)v'(t)&=0\\ v''(t)+\left(\frac{2}{t}-\frac{t}{-1+t}\right)v'(t)&=0\end{aligned} \tag{7}$$

Using the substitution

$$u(t)=v'(t)$$

Then (7) becomes

$$u'(t)+\left(\frac{2}{t}-\frac{t}{-1+t}\right)u(t)=0 \tag{8}$$

The above is now solved for $u(t)$. In canonical form the ODE is

$$\begin{aligned}u' &= F(t,u)\\ &= f(t)g(u)\\ &= \frac{u(t^2-2t+2)}{t(-1+t)}\end{aligned}$$

Where $f(t)=\frac{t^2-2t+2}{t(-1+t)}$ and $g(u)=u$. Integrating both sides gives

$$\begin{aligned}\frac{1}{u}\,du &= \frac{t^2-2t+2}{t(-1+t)}\,dt\\ \int\frac{1}{u}\,du &= \int\frac{t^2-2t+2}{t(-1+t)}\,dt\\ \ln(u) &= t-2\ln(t)+\ln(-1+t)+c_1\\ u &= e^{t-2\ln(t)+\ln(-1+t)+c_1}\\ &= c_1e^{t-2\ln(t)+\ln(-1+t)}\end{aligned}$$

Which simplifies to

$$u(t)=c_1\left(-\frac{e^t}{t^2}+\frac{e^t}{t}\right)$$

Now that $u(t)$ is known, then

$$\begin{aligned}v'(t) &= u(t)\\ v(t) &= \int u(t)\,dt+c_2\\ &= \frac{e^tc_1}{t}+c_2\end{aligned}$$

Hence

$$\begin{aligned}y &= v(t)\,t^n\\ &= \left(\frac{e^tc_1}{t}+c_2\right)t\\ &= c_1e^t+c_2t\end{aligned}$$

Now the particular solution to this ODE is found

$$(1-t)\,y'' + ty' - y = 2(-1+t)\,\mathrm{e}^{-t}$$

The particular solution y_p can be found using either the method of undetermined coefficients, or the method of variation of parameters. The method of variation of parameters will be used as it is more general and can be used when the coefficients of the ODE depend on t as well. Let

$$y_p(t) = u_1 y_1 + u_2 y_2 \tag{1}$$

Where u_1, u_2 to be determined, and y_1, y_2 are the two basis solutions (the two linearly independent solutions of the homogeneous ODE) found earlier when solving the homogeneous ODE as

$$y_1 = t$$
$$y_2 = \mathrm{e}^t$$

In the Variation of parameters u_1, u_2 are found using

$$u_1 = -\int \frac{y_2 f(t)}{aW(t)} \tag{2}$$

$$u_2 = \int \frac{y_1 f(t)}{aW(t)} \tag{3}$$

Where $W(t)$ is the Wronskian and a is the coefficient in front of y'' in the given ODE. The Wronskian is given by $W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$. Hence

$$W = \begin{vmatrix} t & \mathrm{e}^t \\ \frac{d}{dt}(t) & \frac{d}{dt}(\mathrm{e}^t) \end{vmatrix}$$

Which gives

$$W = \begin{vmatrix} t & \mathrm{e}^t \\ 1 & \mathrm{e}^t \end{vmatrix}$$

Therefore

$$W = (t)\left(\mathrm{e}^t\right) - \left(\mathrm{e}^t\right)(1)$$

Which simplifies to

$$W = t\,e^{t} - e^{t}$$

Which simplifies to

$$W = e^{t}(-1+t)$$

Therefore Eq. (2) becomes

$$u_1 = -\int \frac{2\,e^{t}(-1+t)\,e^{-t}}{(1-t)\,e^{t}\,(-1+t)}\,dt$$

Which simplifies to

$$u_1 = -\int -\frac{2\,e^{-t}}{-1+t}dt$$

Hence

$$u_1 = -2\,e^{-1}\,\text{expIntegral}_1\,(-1+t)$$

And Eq. (3) becomes

$$u_2 = \int \frac{2\,e^{-t}t(-1+t)}{(1-t)\,e^{t}\,(-1+t)}\,dt$$

Which simplifies to

$$u_2 = \int -\frac{2t\,e^{-2t}}{-1+t}dt$$

Hence

$$u_2 = e^{-2t} + 2\,e^{-2}\,\text{expIntegral}_1\,(2t-2)$$

Therefore the particular solution, from equation (1) is

$$y_p(t) = -2\,e^{-1}\,\text{expIntegral}_1\,(-1+t)\,t + \left(e^{-2t} + 2\,e^{-2}\,\text{expIntegral}_1\,(2t-2)\right)e^{t}$$

Which simplifies to

$$y_p(t) = -2\,e^{-1}\,\text{expIntegral}_1\,(-1+t)\,t + 2\,\text{expIntegral}_1\,(2t-2)\,e^{t-2} + e^{-t}$$

Therefore the general solution is

$$
\begin{aligned}
y &= y_h + y_p \\
&= \left(\left(\frac{e^t c_1}{t} + c_2\right) t\right) + \left(-2\,e^{-1} \operatorname{expIntegral}_1(-1+t)\,t + 2 \operatorname{expIntegral}_1(2t-2)\,e^{t-2} + e^{-t}\right) \\
&= -2\,e^{-1} \operatorname{expIntegral}_1(-1+t)\,t + 2 \operatorname{expIntegral}_1(2t-2)\,e^{t-2} + e^{-t} + \left(\frac{e^t c_1}{t} + c_2\right) t
\end{aligned}
$$

Which simplifies to

$$
y = -2\,e^{-1} \operatorname{expIntegral}_1(-1+t)\,t + 2 \operatorname{expIntegral}_1(2t-2)\,e^{t-2} + c_1 e^t + c_2 t + e^{-t}
$$

Summary

The solution(s) found are the following

$$
y = -2\,e^{-1} \operatorname{expIntegral}_1(-1+t)\,t + 2 \operatorname{expIntegral}_1(2t-2)\,e^{t-2} + c_1 e^t + c_2 t + e^{-t} \tag{1}
$$

Verification of solutions

$$
y = -2\,e^{-1} \operatorname{expIntegral}_1(-1+t)\,t + 2 \operatorname{expIntegral}_1(2t-2)\,e^{t-2} + c_1 e^t + c_2 t + e^{-t}
$$

Verified OK.

10.22.2 Solving as second order ode non constant coeff transformation on B ode

Given an ode of the form

$$
Ay'' + By' + Cy = F(t)
$$

This method reduces the order ode the ODE by one by applying the transformation

$$
y = Bv
$$

This results in

$$
\begin{aligned}
y' &= B'v + v'B \\
y'' &= B''v + B'v' + v''B + v'B' \\
&= v''B + 2v' + B' + B''v
\end{aligned}
$$

And now the original ode becomes

$$
\begin{aligned}
A(v''B + 2v'B' + B''v) + B(B'v + v'B) + CBv &= 0 \\
ABv'' + \left(2AB' + B^2\right) v' + \left(AB'' + BB' + CB\right) v &= 0
\end{aligned} \tag{1}
$$

If the term $AB'' + BB' + CB$ is zero, then this method works and can be used to solve

$$ABv'' + \left(2AB' + B^2\right) v' = 0$$

By Using $u = v'$ which reduces the order of the above ode to one. The new ode is

$$ABu' + \left(2AB' + B^2\right) u = 0$$

The above ode is first order ode which is solved for u. Now a new ode $v' = u$ is solved for v as first order ode. Then the final solution is obtain from $y = Bv$.

This method works only if the term $AB'' + BB' + CB$ is zero. The given ODE shows that

$$\begin{aligned}
A &= 1 - t \\
B &= t \\
C &= -1 \\
F &= 2(-1 + t)\,\mathrm{e}^{-t}
\end{aligned}$$

The above shows that for this ode

$$\begin{aligned}
AB'' + BB' + CB &= (1 - t)\,(0) + (t)\,(1) + (-1)\,(t) \\
&= 0
\end{aligned}$$

Hence the ode in v given in (1) now simplifies to

$$-t(-1 + t)\,v'' + \left(t^2 - 2t + 2\right) v' = 0$$

Now by applying $v' = u$ the above becomes

$$\left(-t^2 + t\right) u'(t) + \left(t^2 - 2t + 2\right) u(t) = 0$$

Which is now solved for u. In canonical form the ODE is

$$\begin{aligned}
u' &= F(t, u) \\
&= f(t)g(u) \\
&= \frac{u(t^2 - 2t + 2)}{t\,(-1 + t)}
\end{aligned}$$

Where $f(t) = \frac{t^2-2t+2}{t(-1+t)}$ and $g(u) = u$. Integrating both sides gives

$$
\begin{aligned}
\frac{1}{u}\, du &= \frac{t^2-2t+2}{t\,(-1+t)}\, dt \\
\int \frac{1}{u}\, du &= \int \frac{t^2-2t+2}{t\,(-1+t)}\, dt \\
\ln(u) &= t - 2\ln(t) + \ln(-1+t) + c_1 \\
u &= \mathrm{e}^{t-2\ln(t)+\ln(-1+t)+c_1} \\
&= c_1 \mathrm{e}^{t-2\ln(t)+\ln(-1+t)}
\end{aligned}
$$

Which simplifies to

$$
u(t) = c_1 \left(-\frac{\mathrm{e}^t}{t^2} + \frac{\mathrm{e}^t}{t}\right)
$$

The ode for v now becomes

$$
\begin{aligned}
v' &= u \\
&= c_1 \left(-\frac{\mathrm{e}^t}{t^2} + \frac{\mathrm{e}^t}{t}\right)
\end{aligned}
$$

Which is now solved for v. Integrating both sides gives

$$
\begin{aligned}
v(t) &= \int \frac{c_1 \mathrm{e}^t (-1+t)}{t^2}\, \mathrm{d}t \\
&= \frac{\mathrm{e}^t c_1}{t} + c_2
\end{aligned}
$$

Therefore the homogeneous solution is

$$
\begin{aligned}
y_h(t) &= Bv \\
&= (t)\left(\frac{\mathrm{e}^t c_1}{t} + c_2\right) \\
&= c_1 \mathrm{e}^t + c_2 t
\end{aligned}
$$

And now the particular solution $y_p(t)$ will be found. The particular solution y_p can be found using either the method of undetermined coefficients, or the method of variation of parameters. The method of variation of parameters will be used as it is more general and can be used when the coefficients of the ODE depend on t as well. Let

$$
y_p(t) = u_1 y_1 + u_2 y_2 \tag{1}
$$

Where u_1, u_2 to be determined, and y_1, y_2 are the two basis solutions (the two linearly independent solutions of the homogeneous ODE) found earlier when solving the homogeneous ODE as

$$y_1 = t$$
$$y_2 = e^t$$

In the Variation of parameters u_1, u_2 are found using

$$u_1 = -\int \frac{y_2 f(t)}{aW(t)} \tag{2}$$
$$u_2 = \int \frac{y_1 f(t)}{aW(t)} \tag{3}$$

Where $W(t)$ is the Wronskian and a is the coefficient in front of y'' in the given ODE. The Wronskian is given by $W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$. Hence

$$W = \begin{vmatrix} t & e^t \\ \frac{d}{dt}(t) & \frac{d}{dt}(e^t) \end{vmatrix}$$

Which gives

$$W = \begin{vmatrix} t & e^t \\ 1 & e^t \end{vmatrix}$$

Therefore

$$W = (t)\left(e^t\right) - \left(e^t\right)(1)$$

Which simplifies to

$$W = t\, e^t - e^t$$

Which simplifies to

$$W = e^t(-1+t)$$

Therefore Eq. (2) becomes

$$u_1 = -\int \frac{2\, e^t(-1+t)\, e^{-t}}{(1-t)\, e^t\, (-1+t)}\, dt$$

Which simplifies to

$$u_1 = -\int -\frac{2\,\mathrm{e}^{-t}}{-1+t}dt$$

Hence

$$u_1 = -2\,\mathrm{e}^{-1}\,\mathrm{expIntegral}_1\left(-1+t\right)$$

And Eq. (3) becomes

$$u_2 = \int \frac{2\,\mathrm{e}^{-t}t(-1+t)}{\left(1-t\right)\mathrm{e}^{t}\left(-1+t\right)}\,dt$$

Which simplifies to

$$u_2 = \int -\frac{2t\,\mathrm{e}^{-2t}}{-1+t}dt$$

Hence

$$u_2 = \mathrm{e}^{-2t} + 2\,\mathrm{e}^{-2}\,\mathrm{expIntegral}_1\left(2t-2\right)$$

Therefore the particular solution, from equation (1) is

$$y_p(t) = -2\,\mathrm{e}^{-1}\,\mathrm{expIntegral}_1\left(-1+t\right)t + \left(\mathrm{e}^{-2t} + 2\,\mathrm{e}^{-2}\,\mathrm{expIntegral}_1\left(2t-2\right)\right)\mathrm{e}^{t}$$

Which simplifies to

$$y_p(t) = -2\,\mathrm{e}^{-1}\,\mathrm{expIntegral}_1\left(-1+t\right)t + 2\ \mathrm{expIntegral}_1\left(2t-2\right)\mathrm{e}^{t-2} + \mathrm{e}^{-t}$$

Hence the complete solution is

$$\begin{aligned} y(t) &= y_h + y_p \\ &= \left(c_1\mathrm{e}^{t} + c_2t\right) + \left(-2\,\mathrm{e}^{-1}\,\mathrm{expIntegral}_1\left(-1+t\right)t + 2\ \mathrm{expIntegral}_1\left(2t-2\right)\mathrm{e}^{t-2} + \mathrm{e}^{-t}\right) \\ &= -2\,\mathrm{e}^{-1}\,\mathrm{expIntegral}_1\left(-1+t\right)t + 2\ \mathrm{expIntegral}_1\left(2t-2\right)\mathrm{e}^{t-2} + c_1\mathrm{e}^{t} + c_2t + \mathrm{e}^{-t} \end{aligned}$$

Summary
The solution(s) found are the following

$$y = -2\,\mathrm{e}^{-1}\,\mathrm{expIntegral}_1\left(-1+t\right)t + 2\ \mathrm{expIntegral}_1\left(2t-2\right)\mathrm{e}^{t-2} + c_1\mathrm{e}^{t} + c_2t + \mathrm{e}^{-t} \tag{1}$$

Verification of solutions

$$y = -2\,\mathrm{e}^{-1}\,\mathrm{expIntegral}_1\left(-1+t\right)t + 2\ \mathrm{expIntegral}_1\left(2t-2\right)\mathrm{e}^{t-2} + c_1\mathrm{e}^{t} + c_2t + \mathrm{e}^{-t}$$

Verified OK.

10.22.3 Solving using Kovacic algorithm

Writing the ode as

$$(1-t)\, y''+t y'-y=0 \tag{1}$$
$$A y''+B y'+C y=0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1-t \\ B &= t \\ C &= -1 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(t)=y e^{\int \frac{B}{2A}\, dt}$$

Then (2) becomes

$$z''(t)=r z(t) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB'-2BA'+B^2-4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r=\frac{t^2-4t+6}{4\,(-1+t)^2} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= t^2-4t+6 \\ t &= 4(-1+t)^2 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(t)=\left(\frac{t^2-4t+6}{4\,(-1+t)^2}\right) z(t) \tag{7}$$

Equation (7) is now solved. After finding $z(t)$ then y is found using the inverse transformation

$$y=z(t)\, e^{-\int \frac{B}{2A}\, dt}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0,1,2,4,6,8,\cdots\}$	$\{\cdots,-6,-4,-2,0,2,3,4,5,6,\cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1,2\}$	$\{2,3,4,5,6,7,\cdots\}$

Table 507: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 2 - 2 \\ &= 0 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 4(-1+t)^2$. There is a pole at $t = 1$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Therefore

$$L = [1, 2]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = \frac{1}{4} - \frac{1}{2(-1+t)} + \frac{3}{4(-1+t)^2}$$

For the pole at $t = 1$ let b be the coefficient of $\frac{1}{(-1+t)^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{3}{4}$. Hence

$$
\begin{aligned}
[\sqrt{r}]_c &= 0 \\
\alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{3}{2} \\
\alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = -\frac{1}{2}
\end{aligned}
$$

Since the order of r at ∞ is $O_r(\infty) = 0$ then

$$
v = \frac{-O_r(\infty)}{2} = \frac{0}{2} = 0
$$

$[\sqrt{r}]_\infty$ is the sum of terms involving t^i for $0 \leq i \leq v$ in the Laurent series for $\sqrt{r}$ at ∞. Therefore

$$
\begin{aligned}
[\sqrt{r}]_\infty &= \sum_{i=0}^{v} a_i t^i \\
&= \sum_{i=0}^{0} a_i t^i
\end{aligned}
\tag{8}
$$

Let a be the coefficient of $t^v = t^0$ in the above sum. The Laurent series of $\sqrt{r}$ at ∞ is

$$
\sqrt{r} \approx \frac{1}{2} - \frac{1}{2t} + \frac{1}{t^3} + \frac{11}{4t^4} + \frac{21}{4t^5} + \frac{15}{2t^6} + \frac{6}{t^7} - \frac{117}{16t^8} + \dots \tag{9}
$$

Comparing Eq. (9) with Eq. (8) shows that

$$
a = \frac{1}{2}
$$

From Eq. (9) the sum up to $v = 0$ gives

$$
\begin{aligned}
[\sqrt{r}]_\infty &= \sum_{i=0}^{0} a_i t^i \\
&= \frac{1}{2}
\end{aligned}
\tag{10}
$$

Now we need to find b, where b be the coefficient of $t^{v-1} = t^{-1} = \frac{1}{t}$ in r minus the coefficient of same term but in $\left([\sqrt{r}]_\infty\right)^2$ where $[\sqrt{r}]_\infty$ was found above in Eq (10). Hence

$$
\left([\sqrt{r}]_\infty\right)^2 = \frac{1}{4}
$$

This shows that the coefficient of $\frac{1}{t}$ in the above is 0. Now we need to find the coefficient of $\frac{1}{t}$ in r. How this is done depends on if $v = 0$ or not. Since $v = 0$ then starting from $r = \frac{s}{t}$ and doing long division in the form

$$r = Q + \frac{R}{t}$$

Where Q is the quotient and R is the remainder. Then the coefficient of $\frac{1}{t}$ in r will be the coefficient in R of the term in t of degree of t minus one, divided by the leading coefficient in t. Doing long division gives

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{t^2 - 4t + 6}{4t^2 - 8t + 4} \\ &= Q + \frac{R}{4t^2 - 8t + 4} \\ &= \left(\frac{1}{4}\right) + \left(\frac{-2t + 5}{4t^2 - 8t + 4}\right) \\ &= \frac{1}{4} + \frac{-2t + 5}{4t^2 - 8t + 4} \end{aligned}$$

Since the degree of t is 2, then we see that the coefficient of the term t in the remainder R is -2. Dividing this by leading coefficient in t which is 4 gives $-\frac{1}{2}$. Now b can be found.

$$\begin{aligned} b &= \left(-\frac{1}{2}\right) - (0) \\ &= -\frac{1}{2} \end{aligned}$$

Hence

$$\begin{aligned} [\sqrt{r}]_\infty &= \frac{1}{2} \\ \alpha_\infty^+ &= \frac{1}{2}\left(\frac{b}{a} - v\right) &= \frac{1}{2}\left(\frac{-\frac{1}{2}}{\frac{1}{2}} - 0\right) &= -\frac{1}{2} \\ \alpha_\infty^- &= \frac{1}{2}\left(-\frac{b}{a} - v\right) &= \frac{1}{2}\left(-\frac{-\frac{1}{2}}{\frac{1}{2}} - 0\right) &= \frac{1}{2} \end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = \frac{t^2 - 4t + 6}{4(-1 + t)^2}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
1	2	0	$\frac{3}{2}$	$-\frac{1}{2}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
0	$\frac{1}{2}$	$-\frac{1}{2}$	$\frac{1}{2}$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^+ = -\frac{1}{2}$ then

$$\begin{aligned}
d &= \alpha_\infty^+ - (\alpha_{c_1}^-) \\
&= -\frac{1}{2} - \left(-\frac{1}{2}\right) \\
&= 0
\end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c\in\Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{t-c} \right) + s(\infty)[\sqrt{r}]_\infty$$

Substituting the above values in the above results in

$$\begin{aligned}
\omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^-}{t-c_1} \right) + (+)[\sqrt{r}]_\infty \\
&= -\frac{1}{2(-1+t)} + \left(\frac{1}{2}\right) \\
&= -\frac{1}{2(-1+t)} + \frac{1}{2} \\
&= \frac{t-2}{2t-2}
\end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(t)$ of degree $d = 0$ to solve the ode. The polynomial $p(t)$ needs to satisfy the equation

$$p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right) p = 0 \tag{1A}$$

Let

$$p(t) = 1 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$(0) + 2\left(-\frac{1}{2(-1+t)} + \frac{1}{2}\right)(0) + \left(\left(\frac{1}{2(-1+t)^2}\right) + \left(-\frac{1}{2(-1+t)} + \frac{1}{2}\right)^2 - \left(\frac{t^2-4t+6}{4(-1+t)^2}\right)\right) = 0$$

$$0 = 0$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned} z_1(t) &= p e^{\int \omega \, dt} \\ &= \mathrm{e}^{\int \left(-\frac{1}{2(-1+t)} + \frac{1}{2}\right) dt} \\ &= \frac{\mathrm{e}^{\frac{t}{2}}}{\sqrt{-1+t}} \end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned} y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A} \, dt} \\ &= z_1 e^{-\int \frac{1}{2}\frac{t}{1-t} \, dt} \\ &= z_1 e^{\frac{t}{2} + \frac{\ln(-1+t)}{2}} \\ &= z_1 \left(\sqrt{-1+t}\,\mathrm{e}^{\frac{t}{2}}\right) \end{aligned}$$

Which simplifies to

$$y_1 = \mathrm{e}^t$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A} \, dt}}{y_1^2} \, dt$$

Substituting gives

$$\begin{aligned} y_2 &= y_1 \int \frac{e^{\int -\frac{t}{1-t} \, dt}}{(y_1)^2} \, dt \\ &= y_1 \int \frac{e^{t+\ln(-1+t)}}{(y_1)^2} \, dt \\ &= y_1 \left(-t\,\mathrm{e}^{-t}\right) \end{aligned}$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1 \left(\mathrm{e}^t\right) + c_2 \left(\mathrm{e}^t \left(-t\,\mathrm{e}^{-t}\right)\right)
\end{aligned}
$$

This is second order nonhomogeneous ODE. Let the solution be

$$
y = y_h + y_p
$$

Where y_h is the solution to the homogeneous ODE $Ay''(t) + By'(t) + Cy(t) = 0$, and y_p is a particular solution to the nonhomogeneous ODE $Ay''(t) + By'(t) + Cy(t) = f(t)$. y_h is the solution to

$$
(1-t)\, y'' + ty' - y = 0
$$

The homogeneous solution is found using the Kovacic algorithm which results in

$$
y_h = c_1 \mathrm{e}^t - c_2 t
$$

The particular solution y_p can be found using either the method of undetermined coefficients, or the method of variation of parameters. The method of variation of parameters will be used as it is more general and can be used when the coefficients of the ODE depend on t as well. Let

$$
y_p(t) = u_1 y_1 + u_2 y_2 \tag{1}
$$

Where u_1, u_2 to be determined, and y_1, y_2 are the two basis solutions (the two linearly independent solutions of the homogeneous ODE) found earlier when solving the homogeneous ODE as

$$
\begin{aligned}
y_1 &= \mathrm{e}^t \\
y_2 &= -t
\end{aligned}
$$

In the Variation of parameters u_1, u_2 are found using

$$
u_1 = -\int \frac{y_2 f(t)}{aW(t)} \tag{2}
$$

$$
u_2 = \int \frac{y_1 f(t)}{aW(t)} \tag{3}
$$

Where $W(t)$ is the Wronskian and a is the coefficient in front of y'' in the given ODE. The Wronskian is given by $W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$. Hence

$$W = \begin{vmatrix} \mathrm{e}^t & -t \\ \frac{d}{dt}(\mathrm{e}^t) & \frac{d}{dt}(-t) \end{vmatrix}$$

Which gives

$$W = \begin{vmatrix} \mathrm{e}^t & -t \\ \mathrm{e}^t & -1 \end{vmatrix}$$

Therefore

$$W = (\mathrm{e}^t)(-1) - (-t)(\mathrm{e}^t)$$

Which simplifies to

$$W = t\,\mathrm{e}^t - \mathrm{e}^t$$

Which simplifies to

$$W = \mathrm{e}^t(-1+t)$$

Therefore Eq. (2) becomes

$$u_1 = -\int \frac{-2\,\mathrm{e}^{-t} t(-1+t)}{(1-t)\,\mathrm{e}^t\,(-1+t)}\,dt$$

Which simplifies to

$$u_1 = -\int \frac{2t\,\mathrm{e}^{-2t}}{-1+t} dt$$

Hence

$$u_1 = \mathrm{e}^{-2t} + 2\,\mathrm{e}^{-2}\,\mathrm{expIntegral}_1\,(2t-2)$$

And Eq. (3) becomes

$$u_2 = \int \frac{2\,\mathrm{e}^t(-1+t)\,\mathrm{e}^{-t}}{(1-t)\,\mathrm{e}^t\,(-1+t)}\,dt$$

Which simplifies to

$$u_2 = \int -\frac{2\,\mathrm{e}^{-t}}{-1+t}dt$$

Hence

$$u_2 = 2\,\mathrm{e}^{-1}\,\mathrm{expIntegral}_1\left(-1+t\right)$$

Therefore the particular solution, from equation (1) is

$$y_p(t) = -2\,\mathrm{e}^{-1}\,\mathrm{expIntegral}_1\left(-1+t\right)t + \left(\mathrm{e}^{-2t} + 2\,\mathrm{e}^{-2}\,\mathrm{expIntegral}_1\left(2t-2\right)\right)\mathrm{e}^{t}$$

Which simplifies to

$$y_p(t) = -2\,\mathrm{e}^{-1}\,\mathrm{expIntegral}_1\left(-1+t\right)t + 2\,\mathrm{expIntegral}_1\left(2t-2\right)\mathrm{e}^{t-2} + \mathrm{e}^{-t}$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \left(c_1\mathrm{e}^{t} - c_2 t\right) + \left(-2\,\mathrm{e}^{-1}\,\mathrm{expIntegral}_1\left(-1+t\right)t + 2\,\mathrm{expIntegral}_1\left(2t-2\right)\mathrm{e}^{t-2} + \mathrm{e}^{-t}\right) \end{aligned}$$

Summary

The solution(s) found are the following

$$y = c_1\mathrm{e}^{t} - c_2 t - 2\,\mathrm{e}^{-1}\,\mathrm{expIntegral}_1\left(-1+t\right)t + 2\,\mathrm{expIntegral}_1\left(2t-2\right)\mathrm{e}^{t-2} + \mathrm{e}^{-t} \quad (1)$$

Verification of solutions

$$y = c_1\mathrm{e}^{t} - c_2 t - 2\,\mathrm{e}^{-1}\,\mathrm{expIntegral}_1\left(-1+t\right)t + 2\,\mathrm{expIntegral}_1\left(2t-2\right)\mathrm{e}^{t-2} + \mathrm{e}^{-t}$$

Verified OK.

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 2; linear nonhomogeneous with symmetry [0,1]
trying a double symmetry of the form [xi=0, eta=F(x)]
-> Try solving first the homogeneous part of the ODE
   checking if the LODE has constant coefficients
   checking if the LODE is of Euler type
   trying a symmetry of the form [xi=0, eta=F(x)]
   checking if the LODE is missing y
   -> Trying a Liouvillian solution using Kovacics algorithm
      A Liouvillian solution exists
      Reducible group (found an exponential solution)
      Reducible group (found another exponential solution)
   <- Kovacics algorithm successful
<- solving first the homogeneous part of the ODE successful`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 39

```
dsolve((1-t)*diff(y(t),t$2)+t*diff(y(t),t)-y(t) = 2*(t-1)*exp(-t),y(t), singsol=all)
```

$$y(t) = -2\,\mathrm{e}^{-1}\,\mathrm{expIntegral}_1\left(t-1\right)t + 2\,\mathrm{expIntegral}_1\left(2t-2\right)\mathrm{e}^{t-2} + \mathrm{e}^{t}c_1 + c_2 t + \mathrm{e}^{-t}$$

✓ Solution by Mathematica

Time used: 0.187 (sec). Leaf size: 47

```
DSolve[(1-t)*y''[t]+t*y'[t]-y[t] ==2*(t-1)*Exp[-t],y[t],t,IncludeSingularSolutions -> True]
```

$$y(t) \to -2e^{t-2}\,\mathrm{ExpIntegralEi}(2-2t) + \frac{2t\,\mathrm{ExpIntegralEi}(1-t)}{e} + e^{-t} + c_1 e^{t} - c_2 t$$

11 Chapter 3, Second order linear equations, 3.7 Mechanical and Electrical Vibrations. page 203

11.1 problem 28

Internal problem ID [705]
Internal file name [`OUTPUT/705_Sunday_June_05_2022_01_47_23_AM_80288768/index.tex`]

Book: Elementary differential equations and boundary value problems, 10th ed., Boyce and DiPrima
Section: Chapter 3, Second order linear equations, 3.7 Mechanical and Electrical Vibrations. page 203
Problem number: 28.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_linear_constant_coeff", "second_order_ode_can_be_made_integrable"**

Maple gives the following as the ode type

```
[[_2nd_order, _missing_x]]
```

$$\boxed{u'' + 2u = 0}$$

11.1.1 Solving as second order linear constant coeff ode

This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$Au''(t) + Bu'(t) + Cu(t) = 0$$

Where in the above $A = 1, B = 0, C = 2$. Let the solution be $u = e^{\lambda t}$. Substituting this into the ODE gives

$$\lambda^2 \mathrm{e}^{\lambda t} + 2\,\mathrm{e}^{\lambda t} = 0 \tag{1}$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda t}$ gives

$$\lambda^2 + 2 = 0 \tag{2}$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form.Using the quadratic formula

$$\lambda_{1,2} = \frac{-B}{2A} \pm \frac{1}{2A}\sqrt{B^2 - 4AC}$$

Substituting $A = 1, B = 0, C = 2$ into the above gives

$$\begin{aligned}\lambda_{1,2} &= \frac{0}{(2)(1)} \pm \frac{1}{(2)(1)}\sqrt{0^2 - (4)(1)(2)} \\ &= \pm i\sqrt{2}\end{aligned}$$

Hence

$$\begin{aligned}\lambda_1 &= +i\sqrt{2} \\ \lambda_2 &= -i\sqrt{2}\end{aligned}$$

Which simplifies to

$$\begin{aligned}\lambda_1 &= i\sqrt{2} \\ \lambda_2 &= -i\sqrt{2}\end{aligned}$$

Since roots are complex conjugate of each others, then let the roots be

$$\lambda_{1,2} = \alpha \pm i\beta$$

Where $\alpha = 0$ and $\beta = \sqrt{2}$. Therefore the final solution, when using Euler relation, can be written as

$$u = e^{\alpha t}(c_1 \cos(\beta t) + c_2 \sin(\beta t))$$

Which becomes

$$u = e^0\left(c_1 \cos\left(\sqrt{2}\, t\right) + c_2 \sin\left(\sqrt{2}\, t\right)\right)$$

Or

$$u = c_1 \cos\left(\sqrt{2}\, t\right) + c_2 \sin\left(\sqrt{2}\, t\right)$$

Summary

The solution(s) found are the following

$$u = c_1 \cos\left(\sqrt{2}\, t\right) + c_2 \sin\left(\sqrt{2}\, t\right) \tag{1}$$

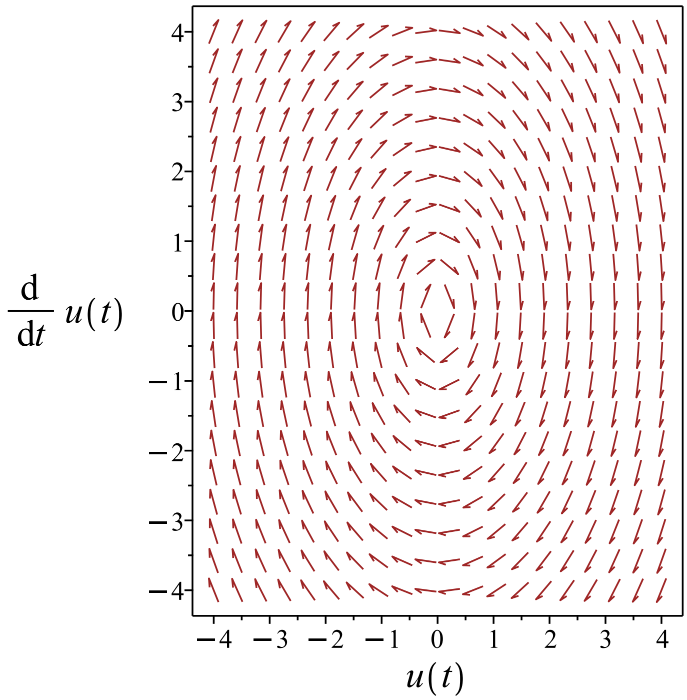


Figure 491: Slope field plot

Verification of solutions

$$u = c_1 \cos\left(\sqrt{2}\, t\right) + c_2 \sin\left(\sqrt{2}\, t\right)$$

Verified OK.

11.1.2 Solving as second order ode can be made integrable ode

Multiplying the ode by u' gives

$$u'u'' + 2u'u = 0$$

Integrating the above w.r.t t gives

$$\int (u'u'' + 2u'u)\, dt = 0$$

$$\frac{u'^2}{2} + u^2 = c_2$$

Which is now solved for u. Solving the given ode for u' results in 2 differential equations to solve. Each one of these will generate a solution. The equations generated are

$$u' = \sqrt{-2u^2 + 2c_1} \tag{1}$$

$$u' = -\sqrt{-2u^2 + 2c_1} \tag{2}$$

Now each one of the above ODE is solved.

Solving equation (1)

Integrating both sides gives

$$\int \frac{1}{\sqrt{-2u^2 + 2c_1}} du = \int dt$$

$$\frac{\sqrt{2}\, \arctan\left(\frac{\sqrt{2}\, u}{\sqrt{-2u^2+2c_1}}\right)}{2} = t + c_2$$

Solving equation (2)

Integrating both sides gives

$$\int -\frac{1}{\sqrt{-2u^2 + 2c_1}} du = \int dt$$

$$-\frac{\sqrt{2}\, \arctan\left(\frac{\sqrt{2}\, u}{\sqrt{-2u^2+2c_1}}\right)}{2} = t + c_3$$

Summary

The solution(s) found are the following

$$\frac{\sqrt{2}\, \arctan\left(\frac{\sqrt{2}\, u}{\sqrt{-2u^2+2c_1}}\right)}{2} = t + c_2 \tag{1}$$

$$-\frac{\sqrt{2}\, \arctan\left(\frac{\sqrt{2}\, u}{\sqrt{-2u^2+2c_1}}\right)}{2} = t + c_3 \tag{2}$$

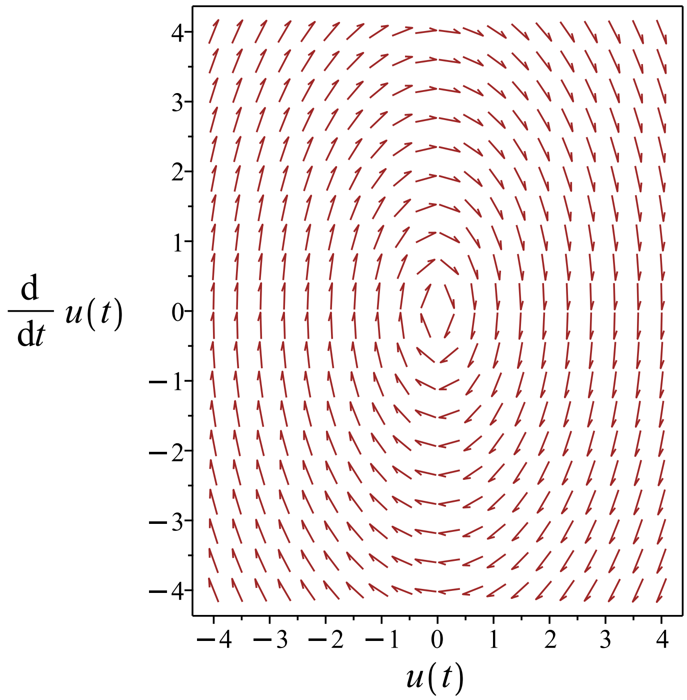


Figure 492: Slope field plot

Verification of solutions

$$\frac{\sqrt{2}\,\arctan\left(\frac{\sqrt{2}\,u}{\sqrt{-2u^2+2c_1}}\right)}{2} = t + c_2$$

Verified OK.

$$-\frac{\sqrt{2}\,\arctan\left(\frac{\sqrt{2}\,u}{\sqrt{-2u^2+2c_1}}\right)}{2} = t + c_3$$

Verified OK.

11.1.3 Solving using Kovacic algorithm

Writing the ode as

$$u'' + 2u = 0 \tag{1}$$

$$Au'' + Bu' + Cu = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1 \\ B &= 0 \\ C &= 2 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(t) = ue^{\int \frac{B}{2A}\,dt}$$

Then (2) becomes

$$z''(t) = rz(t) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{-2}{1} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= -2 \\ t &= 1 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(t) = -2z(t) \tag{7}$$

Equation (7) is now solved. After finding $z(t)$ then u is found using the inverse transformation

$$u = z(t)\, e^{-\int \frac{B}{2A}\,dt}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 508: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned}O(\infty) &= \deg(t) - \deg(s)\\ &= 0 - 0\\ &= 0\end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Since $r = -2$ is not a function of t, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$z_1(t) = \cos\left(\sqrt{2}\, t\right)$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in u is found from

$$u_1 = z_1 e^{\int -\frac{1}{2}\frac{B}{A}\, dt}$$

Since $B = 0$ then the above reduces to

$$\begin{aligned}u_1 &= z_1\\ &= \cos\left(\sqrt{2}\, t\right)\end{aligned}$$

Which simplifies to

$$u_1 = \cos\left(\sqrt{2}\, t\right)$$

The second solution u_2 to the original ode is found using reduction of order

$$u_2 = u_1 \int \frac{e^{\int -\frac{B}{A}\, dt}}{u_1^2}\, dt$$

Since $B = 0$ then the above becomes

$$\begin{aligned}
u_2 &= u_1 \int \frac{1}{u_1^2}\, dt \\
&= \cos\left(\sqrt{2}\, t\right) \int \frac{1}{\cos\left(\sqrt{2}\, t\right)^2}\, dt \\
&= \cos\left(\sqrt{2}\, t\right) \left(\frac{\sqrt{2}\, \tan\left(\sqrt{2}\, t\right)}{2}\right)
\end{aligned}$$

Therefore the solution is

$$\begin{aligned}
u &= c_1 u_1 + c_2 u_2 \\
&= c_1 \left(\cos\left(\sqrt{2}\, t\right)\right) + c_2 \left(\cos\left(\sqrt{2}\, t\right) \left(\frac{\sqrt{2}\, \tan\left(\sqrt{2}\, t\right)}{2}\right)\right)
\end{aligned}$$

Summary

The solution(s) found are the following

$$u = c_1 \cos\left(\sqrt{2}\, t\right) + \frac{c_2 \sqrt{2}\, \sin\left(\sqrt{2}\, t\right)}{2} \tag{1}$$

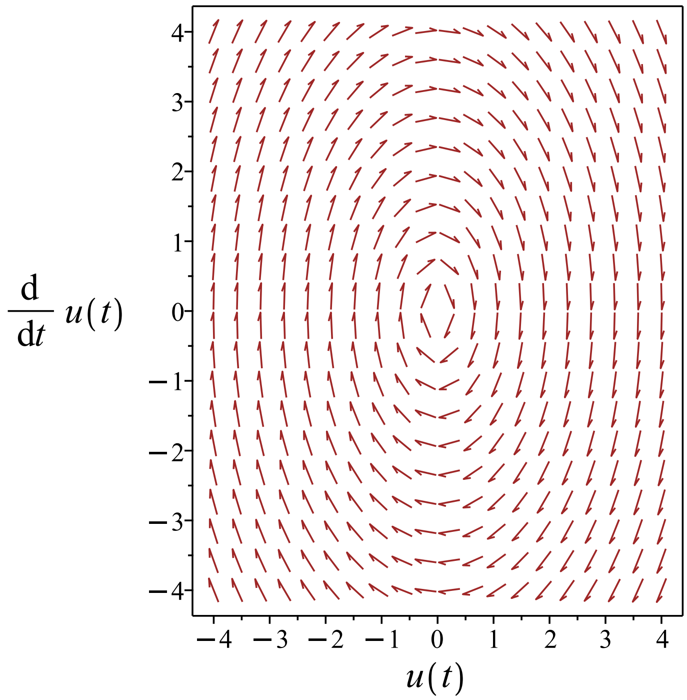


Figure 493: Slope field plot

Verification of solutions

$$u = c_1 \cos\left(\sqrt{2}\, t\right) + \frac{c_2 \sqrt{2}\, \sin\left(\sqrt{2}\, t\right)}{2}$$

Verified OK.

11.1.4 Maple step by step solution

Let's solve

$u'' + 2u = 0$

- Highest derivative means the order of the ODE is 2

  u''

- Characteristic polynomial of ODE

  $r^2 + 2 = 0$

- Use quadratic formula to solve for r

  $r = \frac{0 \pm (\sqrt{-8})}{2}$

- Roots of the characteristic polynomial

$$r = \left(-\mathrm{I}\sqrt{2}, \mathrm{I}\sqrt{2}\right)$$

- 1st solution of the ODE

$$u_1(t) = \cos\left(\sqrt{2}\, t\right)$$

- 2nd solution of the ODE

$$u_2(t) = \sin\left(\sqrt{2}\, t\right)$$

- General solution of the ODE

$$u = c_1 u_1(t) + c_2 u_2(t)$$

- Substitute in solutions

$$u = c_1 \cos\left(\sqrt{2}\, t\right) + c_2 \sin\left(\sqrt{2}\, t\right)$$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
<- constant coefficients successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 21

```
dsolve(diff(u(t),t$2)+2*u(t) = 0,u(t), singsol=all)
```

$$u(t) = c_1 \sin\left(t\sqrt{2}\right) + c_2 \cos\left(t\sqrt{2}\right)$$

✓ Solution by Mathematica

Time used: 0.017 (sec). Leaf size: 28

```
DSolve[u''[t]+2*u[t] ==0,u[t],t,IncludeSingularSolutions -> True]
```

$$u(t) \to c_1 \cos\left(\sqrt{2}t\right) + c_2 \sin\left(\sqrt{2}t\right)$$

11.2 problem 29

Internal problem ID [706]
Internal file name [`OUTPUT/706_Sunday_June_05_2022_01_47_24_AM_3139852/index.tex`]

Book: Elementary differential equations and boundary value problems, 10th ed., Boyce and DiPrima
Section: Chapter 3, Second order linear equations, 3.7 Mechanical and Electrical Vibrations. page 203
Problem number: 29.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_linear_constant_coeff"**

Maple gives the following as the ode type

```
[[_2nd_order, _missing_x]]
```

$$\boxed{u'' + \frac{u'}{4} + 2u = 0}$$

With initial conditions

$$[u(0) = 0, u'(0) = 2]$$

11.2.1 Existence and uniqueness analysis

This is a linear ODE. In canonical form it is written as

$$u'' + p(t)u' + q(t)u = F$$

Where here

$$p(t) = \frac{1}{4}$$
$$q(t) = 2$$
$$F = 0$$

Hence the ode is

$$u'' + \frac{u'}{4} + 2u = 0$$

The domain of $p(t) = \frac{1}{4}$ is

$$\{-\infty < t < \infty\}$$

And the point $t_0 = 0$ is inside this domain. The domain of $q(t) = 2$ is

$$\{-\infty < t < \infty\}$$

And the point $t_0 = 0$ is also inside this domain. Hence solution exists and is unique.

11.2.2 Solving as second order linear constant coeff ode

This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$Au''(t) + Bu'(t) + Cu(t) = 0$$

Where in the above $A = 1, B = \frac{1}{4}, C = 2$. Let the solution be $u = e^{\lambda t}$. Substituting this into the ODE gives

$$\lambda^2 e^{\lambda t} + \frac{\lambda\, e^{\lambda t}}{4} + 2\, e^{\lambda t} = 0 \tag{1}$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda t}$ gives

$$\lambda^2 + \frac{1}{4}\lambda + 2 = 0 \tag{2}$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form.Using the quadratic formula

$$\lambda_{1,2} = \frac{-B}{2A} \pm \frac{1}{2A}\sqrt{B^2 - 4AC}$$

Substituting $A = 1, B = \frac{1}{4}, C = 2$ into the above gives

$$\begin{aligned}\lambda_{1,2} &= \frac{-\frac{1}{4}}{(2)\,(1)} \pm \frac{1}{(2)\,(1)}\sqrt{\frac{1}{4}^2 - (4)\,(1)\,(2)}\\ &= -\frac{1}{8} \pm \frac{i\sqrt{127}}{8}\end{aligned}$$

Hence

$$\lambda_1 = -\frac{1}{8} + \frac{i\sqrt{127}}{8}$$
$$\lambda_2 = -\frac{1}{8} - \frac{i\sqrt{127}}{8}$$

Which simplifies to

$$\lambda_1 = -\frac{1}{8} + \frac{i\sqrt{127}}{8}$$
$$\lambda_2 = -\frac{1}{8} - \frac{i\sqrt{127}}{8}$$

Since roots are complex conjugate of each others, then let the roots be

$$\lambda_{1,2} = \alpha \pm i\beta$$

Where $\alpha = -\frac{1}{8}$ and $\beta = \frac{\sqrt{127}}{8}$. Therefore the final solution, when using Euler relation, can be written as

$$u = e^{\alpha t}(c_1 \cos(\beta t) + c_2 \sin(\beta t))$$

Which becomes

$$u = e^{-\frac{t}{8}}\left(c_1 \cos\left(\frac{\sqrt{127}\, t}{8}\right) + c_2 \sin\left(\frac{\sqrt{127}\, t}{8}\right)\right)$$

Initial conditions are used to solve for the constants of integration.

Looking at the above solution

$$u = \mathrm{e}^{-\frac{t}{8}}\left(c_1 \cos\left(\frac{\sqrt{127}\, t}{8}\right) + c_2 \sin\left(\frac{\sqrt{127}\, t}{8}\right)\right) \tag{1}$$

Initial conditions are now substituted in the above solution. This will generate the required equations to solve for the integration constants. substituting $u = 0$ and $t = 0$ in the above gives

$$0 = c_1 \tag{1A}$$

Taking derivative of the solution gives

$$u' = -\frac{\mathrm{e}^{-\frac{t}{8}}\left(c_1 \cos\left(\frac{\sqrt{127}\, t}{8}\right) + c_2 \sin\left(\frac{\sqrt{127}\, t}{8}\right)\right)}{8} + \mathrm{e}^{-\frac{t}{8}}\left(-\frac{c_1\sqrt{127}\, \sin\left(\frac{\sqrt{127}\, t}{8}\right)}{8} + \frac{c_2\sqrt{127}\, \cos\left(\frac{\sqrt{127}\, t}{8}\right)}{8}\right)$$

substituting $u' = 2$ and $t = 0$ in the above gives

$$2 = -\frac{c_1}{8} + \frac{\sqrt{127}\, c_2}{8} \tag{2A}$$

Equations {1A,2A} are now solved for $\{c_1, c_2\}$. Solving for the constants gives

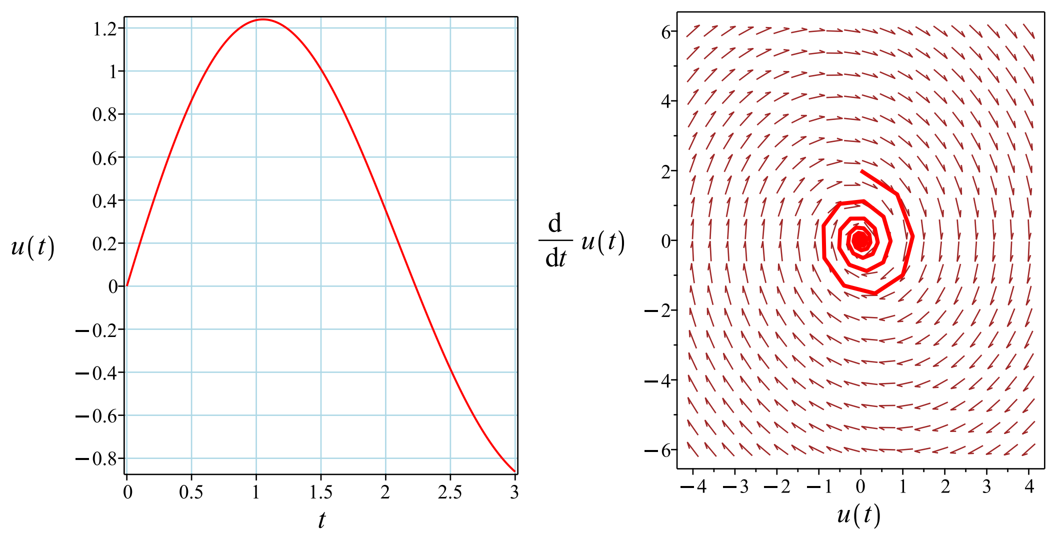
$$c_1 = 0$$
$$c_2 = \frac{16\sqrt{127}}{127}$$

Substituting these values back in above solution results in

$$u = \frac{16\sqrt{127}\, e^{-\frac{t}{8}} \sin\left(\frac{\sqrt{127}\, t}{8}\right)}{127}$$

Summary
The solution(s) found are the following

$$u = \frac{16\sqrt{127}\, e^{-\frac{t}{8}} \sin\left(\frac{\sqrt{127}\, t}{8}\right)}{127} \tag{1}$$



(a) Solution plot

(b) Slope field plot

Verification of solutions

$$u = \frac{16\sqrt{127}\, e^{-\frac{t}{8}} \sin\left(\frac{\sqrt{127}\, t}{8}\right)}{127}$$

Verified OK.

11.2.3 Solving using Kovacic algorithm

Writing the ode as

$$u'' + \frac{u'}{4} + 2u = 0 \tag{1}$$
$$Au'' + Bu' + Cu = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1 \\ B &= \frac{1}{4} \\ C &= 2 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(t) = ue^{\int \frac{B}{2A}\,dt}$$

Then (2) becomes

$$z''(t) = rz(t) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{-127}{64} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= -127 \\ t &= 64 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(t) = -\frac{127z(t)}{64} \tag{7}$$

Equation (7) is now solved. After finding $z(t)$ then u is found using the inverse transformation

$$u = z(t)\, e^{-\int \frac{B}{2A}\,dt}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 510: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$
\begin{aligned}
O(\infty) &= \deg(t) - \deg(s) \\
&= 0 - 0 \\
&= 0
\end{aligned}
$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Therefore

$$
L = [1]
$$

Since $r = -\frac{127}{64}$ is not a function of t, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$
z_1(t) = \cos\left(\frac{\sqrt{127}\, t}{8}\right)
$$

Using the above, the solution for the original ode can now be found. The first solution

to the original ode in u is found from

$$
\begin{aligned}
u_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dt} \\
&= z_1 e^{-\int \frac{1}{2}\frac{\frac{1}{4}}{1}\,dt} \\
&= z_1 e^{-\frac{t}{8}} \\
&= z_1\left(\mathrm{e}^{-\frac{t}{8}}\right)
\end{aligned}
$$

Which simplifies to

$$
u_1 = \mathrm{e}^{-\frac{t}{8}} \cos\left(\frac{\sqrt{127}\,t}{8}\right)
$$

The second solution u_2 to the original ode is found using reduction of order

$$
u_2 = u_1 \int \frac{e^{\int -\frac{B}{A}\,dt}}{u_1^2}\,dt
$$

Substituting gives

$$
\begin{aligned}
u_2 &= u_1 \int \frac{e^{\int -\frac{\frac{1}{4}}{1}\,dt}}{(u_1)^2}\,dt \\
&= u_1 \int \frac{e^{-\frac{t}{4}}}{(u_1)^2}\,dt \\
&= u_1\left(\frac{8\sqrt{127}\,\tan\left(\frac{\sqrt{127}\,t}{8}\right)}{127}\right)
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
u &= c_1 u_1 + c_2 u_2 \\
&= c_1\left(\mathrm{e}^{-\frac{t}{8}} \cos\left(\frac{\sqrt{127}\,t}{8}\right)\right) + c_2\left(\mathrm{e}^{-\frac{t}{8}} \cos\left(\frac{\sqrt{127}\,t}{8}\right)\left(\frac{8\sqrt{127}\,\tan\left(\frac{\sqrt{127}\,t}{8}\right)}{127}\right)\right)
\end{aligned}
$$

Initial conditions are used to solve for the constants of integration.

Looking at the above solution

$$u = c_1 e^{-\frac{t}{8}} \cos\left(\frac{\sqrt{127}\, t}{8}\right) + \frac{8 c_2 \sqrt{127}\, e^{-\frac{t}{8}} \sin\left(\frac{\sqrt{127}\, t}{8}\right)}{127} \tag{1}$$

Initial conditions are now substituted in the above solution. This will generate the required equations to solve for the integration constants. substituting $u = 0$ and $t = 0$ in the above gives

$$0 = c_1 \tag{1A}$$

Taking derivative of the solution gives

$$u' = -\frac{c_1 e^{-\frac{t}{8}} \cos\left(\frac{\sqrt{127}\, t}{8}\right)}{8} - \frac{c_1 e^{-\frac{t}{8}} \sqrt{127}\, \sin\left(\frac{\sqrt{127}\, t}{8}\right)}{8} - \frac{c_2 \sqrt{127}\, e^{-\frac{t}{8}} \sin\left(\frac{\sqrt{127}\, t}{8}\right)}{127} + c_2 e^{-\frac{t}{8}} \cos\left(\frac{\sqrt{127}\, t}{8}\right)$$

substituting $u' = 2$ and $t = 0$ in the above gives

$$2 = -\frac{c_1}{8} + c_2 \tag{2A}$$

Equations {1A,2A} are now solved for $\{c_1, c_2\}$. Solving for the constants gives

$$c_1 = 0$$
$$c_2 = 2$$

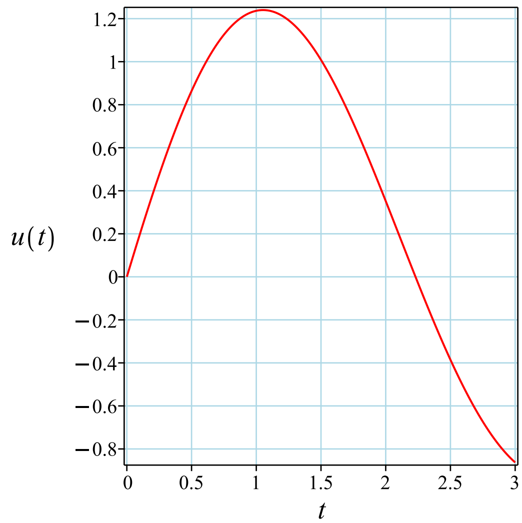
Substituting these values back in above solution results in

$$u = \frac{16\sqrt{127}\, e^{-\frac{t}{8}} \sin\left(\frac{\sqrt{127}\, t}{8}\right)}{127}$$

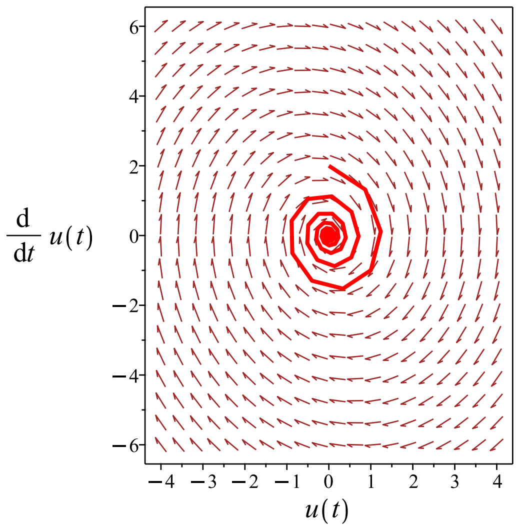
Summary

The solution(s) found are the following

$$u = \frac{16\sqrt{127}\, e^{-\frac{t}{8}} \sin\left(\frac{\sqrt{127}\, t}{8}\right)}{127} \tag{1}$$



(a) Solution plot



(b) Slope field plot

Verification of solutions

$$u = \frac{16\sqrt{127}\,\mathrm{e}^{-\frac{t}{8}} \sin\left(\frac{\sqrt{127}\,t}{8}\right)}{127}$$

Verified OK.

11.2.4 Maple step by step solution

Let's solve

$\left[u'' + \frac{u'}{4} + 2u = 0, u(0) = 0, u'\Big|_{\{t=0\}} = 2\right]$

- Highest derivative means the order of the ODE is 2

  u''

- Characteristic polynomial of ODE

  $r^2 + \frac{1}{4}r + 2 = 0$

- Use quadratic formula to solve for r

  $r = \frac{\left(-\frac{1}{4}\right) \pm \left(\sqrt{-\frac{127}{16}}\right)}{2}$

- Roots of the characteristic polynomial

  $r = \left(-\frac{1}{8} - \frac{\mathrm{I}\sqrt{127}}{8}, -\frac{1}{8} + \frac{\mathrm{I}\sqrt{127}}{8}\right)$

- 1st solution of the ODE

  $u_1(t) = e^{-\frac{t}{8}} \cos\left(\frac{\sqrt{127}\, t}{8}\right)$

- 2nd solution of the ODE

  $u_2(t) = e^{-\frac{t}{8}} \sin\left(\frac{\sqrt{127}\, t}{8}\right)$

- General solution of the ODE

  $u = c_1 u_1(t) + c_2 u_2(t)$

- Substitute in solutions

  $u = c_1 e^{-\frac{t}{8}} \cos\left(\frac{\sqrt{127}\, t}{8}\right) + c_2 \sin\left(\frac{\sqrt{127}\, t}{8}\right) e^{-\frac{t}{8}}$

☐ Check validity of solution $u = c_1 e^{-\frac{t}{8}} \cos\left(\frac{\sqrt{127}\, t}{8}\right) + c_2 \sin\left(\frac{\sqrt{127}\, t}{8}\right) e^{-\frac{t}{8}}$

  - Use initial condition $u(0) = 0$

    $0 = c_1$

  - Compute derivative of the solution

    $u' = -\frac{c_1 e^{-\frac{t}{8}} \cos\left(\frac{\sqrt{127}\, t}{8}\right)}{8} - \frac{c_1 e^{-\frac{t}{8}} \sqrt{127}\, \sin\left(\frac{\sqrt{127}\, t}{8}\right)}{8} + \frac{c_2 \sqrt{127}\, \cos\left(\frac{\sqrt{127}\, t}{8}\right) e^{-\frac{t}{8}}}{8} - \frac{c_2 \sin\left(\frac{\sqrt{127}\, t}{8}\right) e^{-\frac{t}{8}}}{8}$

  - Use the initial condition $u'\Big|_{\{t=0\}} = 2$

    $2 = -\frac{c_1}{8} + \frac{\sqrt{127}\, c_2}{8}$

  - Solve for c_1 and c_2

    $\left\{c_1 = 0, c_2 = \frac{16\sqrt{127}}{127}\right\}$

  - Substitute constant values into general solution and simplify

    $u = \frac{16\sqrt{127}\, e^{-\frac{t}{8}} \sin\left(\frac{\sqrt{127}\, t}{8}\right)}{127}$

- Solution to the IVP

  $u = \frac{16\sqrt{127}\, e^{-\frac{t}{8}} \sin\left(\frac{\sqrt{127}\, t}{8}\right)}{127}$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
<- constant coefficients successful`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 20

```
dsolve([diff(u(t),t$2)+1/4*diff(u(t),t)+2*u(t) = 0,u(0) = 0, D(u)(0) = 2],u(t), singsol=all)
```

$$u(t) = \frac{16\sqrt{127}\,\mathrm{e}^{-\frac{t}{8}}\sin\left(\frac{\sqrt{127}\,t}{8}\right)}{127}$$

✓ Solution by Mathematica

Time used: 0.025 (sec). Leaf size: 30

```
DSolve[{u''[t]+1/4*u'[t]+2*u[t] ==0,{u[0]==0,u'[0]==2}},u[t],t,IncludeSingularSolutions -> T
```

$$u(t) \to \frac{16e^{-t/8}\sin\left(\frac{\sqrt{127}t}{8}\right)}{\sqrt{127}}$$

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12.1 problem 21

Internal problem ID [707]
Internal file name [`OUTPUT/707_Sunday_June_05_2022_01_47_25_AM_44410193/index.tex`]

Book: Elementary differential equations and boundary value problems, 10th ed., Boyce and DiPrima
Section: Chapter 3, Second order linear equations, 3.7 Forced Vibrations. page 217
Problem number: 21.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_linear_constant_coeff"**

Maple gives the following as the ode type

`[[_2nd_order, _linear, _nonhomogeneous]]`

$$\boxed{u'' + \frac{u'}{8} + 4u = 3\cos\left(\frac{t}{4}\right)}$$

With initial conditions

$$[u(0) = 2, u'(0) = 0]$$

12.1.1 Existence and uniqueness analysis

This is a linear ODE. In canonical form it is written as

$$u'' + p(t)u' + q(t)u = F$$

Where here

$$p(t) = \frac{1}{8}$$
$$q(t) = 4$$
$$F = 3\cos\left(\frac{t}{4}\right)$$

Hence the ode is

$$u'' + \frac{u'}{8} + 4u = 3\cos\left(\frac{t}{4}\right)$$

The domain of $p(t) = \frac{1}{8}$ is

$$\{-\infty < t < \infty\}$$

And the point $t_0 = 0$ is inside this domain. The domain of $q(t) = 4$ is

$$\{-\infty < t < \infty\}$$

And the point $t_0 = 0$ is also inside this domain. The domain of $F = 3\cos\left(\frac{t}{4}\right)$ is

$$\{-\infty < t < \infty\}$$

And the point $t_0 = 0$ is also inside this domain. Hence solution exists and is unique.

12.1.2 Solving as second order linear constant coeff ode

This is second order non-homogeneous ODE. In standard form the ODE is

$$Au''(t) + Bu'(t) + Cu(t) = f(t)$$

Where $A = 1, B = \frac{1}{8}, C = 4, f(t) = 3\cos\left(\frac{t}{4}\right)$. Let the solution be

$$u = u_h + u_p$$

Where u_h is the solution to the homogeneous ODE $Au''(t)+Bu'(t)+Cu(t) = 0$, and u_p is a particular solution to the non-homogeneous ODE $Au''(t) + Bu'(t) + Cu(t) = f(t)$. u_h is the solution to

$$u'' + \frac{u'}{8} + 4u = 0$$

This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$Au''(t) + Bu'(t) + Cu(t) = 0$$

Where in the above $A = 1, B = \frac{1}{8}, C = 4$. Let the solution be $u = e^{\lambda t}$. Substituting this into the ODE gives

$$\lambda^2 e^{\lambda t} + \frac{\lambda e^{\lambda t}}{8} + 4 e^{\lambda t} = 0 \tag{1}$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda t}$ gives

$$\lambda^2 + \frac{1}{8}\lambda + 4 = 0 \tag{2}$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form.Using the quadratic formula

$$\lambda_{1,2} = \frac{-B}{2A} \pm \frac{1}{2A}\sqrt{B^2 - 4AC}$$

Substituting $A = 1, B = \frac{1}{8}, C = 4$ into the above gives

$$\begin{aligned}\lambda_{1,2} &= \frac{-\frac{1}{8}}{(2)(1)} \pm \frac{1}{(2)(1)}\sqrt{\frac{1}{8}^2 - (4)(1)(4)} \\ &= -\frac{1}{16} \pm \frac{i\sqrt{1023}}{16}\end{aligned}$$

Hence

$$\begin{aligned}\lambda_1 &= -\frac{1}{16} + \frac{i\sqrt{1023}}{16} \\ \lambda_2 &= -\frac{1}{16} - \frac{i\sqrt{1023}}{16}\end{aligned}$$

Which simplifies to

$$\lambda_1 = -\frac{1}{16} + \frac{i\sqrt{1023}}{16}$$

$$\lambda_2 = -\frac{1}{16} - \frac{i\sqrt{1023}}{16}$$

Since roots are complex conjugate of each others, then let the roots be

$$\lambda_{1,2} = \alpha \pm i\beta$$

Where $\alpha = -\frac{1}{16}$ and $\beta = \frac{\sqrt{1023}}{16}$. Therefore the final solution, when using Euler relation, can be written as

$$u = e^{\alpha t}(c_1 \cos(\beta t) + c_2 \sin(\beta t))$$

Which becomes

$$u = e^{-\frac{t}{16}}\left(c_1 \cos\left(\frac{\sqrt{1023}\, t}{16}\right) + c_2 \sin\left(\frac{\sqrt{1023}\, t}{16}\right)\right)$$

Therefore the homogeneous solution u_h is

$$u_h = e^{-\frac{t}{16}} \left(c_1 \cos\left(\frac{\sqrt{1023}\, t}{16}\right) + c_2 \sin\left(\frac{\sqrt{1023}\, t}{16}\right)\right)$$

The particular solution is now found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$3 \cos\left(\frac{t}{4}\right)$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$\left[\left\{\cos\left(\frac{t}{4}\right), \sin\left(\frac{t}{4}\right)\right\}\right]$$

While the set of the basis functions for the homogeneous solution found earlier is

$$\left\{e^{-\frac{t}{16}} \cos\left(\frac{\sqrt{1023}\, t}{16}\right), e^{-\frac{t}{16}} \sin\left(\frac{\sqrt{1023}\, t}{16}\right)\right\}$$

Since there is no duplication between the basis function in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis in the UC_set.

$$u_p = A_1 \cos\left(\frac{t}{4}\right) + A_2 \sin\left(\frac{t}{4}\right)$$

The unknowns $\{A_1, A_2\}$ are found by substituting the above trial solution u_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$\frac{63 A_1 \cos\left(\frac{t}{4}\right)}{16} + \frac{63 A_2 \sin\left(\frac{t}{4}\right)}{16} - \frac{A_1 \sin\left(\frac{t}{4}\right)}{32} + \frac{A_2 \cos\left(\frac{t}{4}\right)}{32} = 3 \cos\left(\frac{t}{4}\right)$$

Solving for the unknowns by comparing coefficients results in

$$\left[A_1 = \frac{12096}{15877}, A_2 = \frac{96}{15877}\right]$$

Substituting the above back in the above trial solution u_p, gives the particular solution

$$u_p = \frac{12096 \cos\left(\frac{t}{4}\right)}{15877} + \frac{96 \sin\left(\frac{t}{4}\right)}{15877}$$

Therefore the general solution is

$$
\begin{aligned}
u &= u_h + u_p \\
&= \left(\mathrm{e}^{-\frac{t}{16}} \left(c_1 \cos\left(\frac{\sqrt{1023}\, t}{16}\right) + c_2 \sin\left(\frac{\sqrt{1023}\, t}{16}\right)\right)\right) + \left(\frac{12096 \cos\left(\frac{t}{4}\right)}{15877} + \frac{96 \sin\left(\frac{t}{4}\right)}{15877}\right)
\end{aligned}
$$

Initial conditions are used to solve for the constants of integration.

Looking at the above solution

$$
u = \mathrm{e}^{-\frac{t}{16}} \left(c_1 \cos\left(\frac{\sqrt{1023}\, t}{16}\right) + c_2 \sin\left(\frac{\sqrt{1023}\, t}{16}\right)\right) + \frac{12096 \cos\left(\frac{t}{4}\right)}{15877} + \frac{96 \sin\left(\frac{t}{4}\right)}{15877} \tag{1}
$$

Initial conditions are now substituted in the above solution. This will generate the required equations to solve for the integration constants. substituting $u = 2$ and $t = 0$ in the above gives

$$
2 = c_1 + \frac{12096}{15877} \tag{1A}
$$

Taking derivative of the solution gives

$$
u' = -\frac{\mathrm{e}^{-\frac{t}{16}} \left(c_1 \cos\left(\frac{\sqrt{1023}\, t}{16}\right) + c_2 \sin\left(\frac{\sqrt{1023}\, t}{16}\right)\right)}{16} + \mathrm{e}^{-\frac{t}{16}} \left(-\frac{c_1 \sqrt{1023}\, \sin\left(\frac{\sqrt{1023}\, t}{16}\right)}{16} + \frac{c_2 \sqrt{1023}\, \cos\left(\frac{\sqrt{1023}\, t}{16}\right)}{16}
$$

substituting $u' = 0$ and $t = 0$ in the above gives

$$
0 = -\frac{c_1}{16} + \frac{\sqrt{1023}\, c_2}{16} + \frac{24}{15877} \tag{2A}
$$

Equations $\{1\text{A}, 2\text{A}\}$ are now solved for $\{c_1, c_2\}$. Solving for the constants gives

$$
\begin{aligned}
c_1 &= \frac{19658}{15877} \\
c_2 &= \frac{19274\sqrt{1023}}{16242171}
\end{aligned}
$$

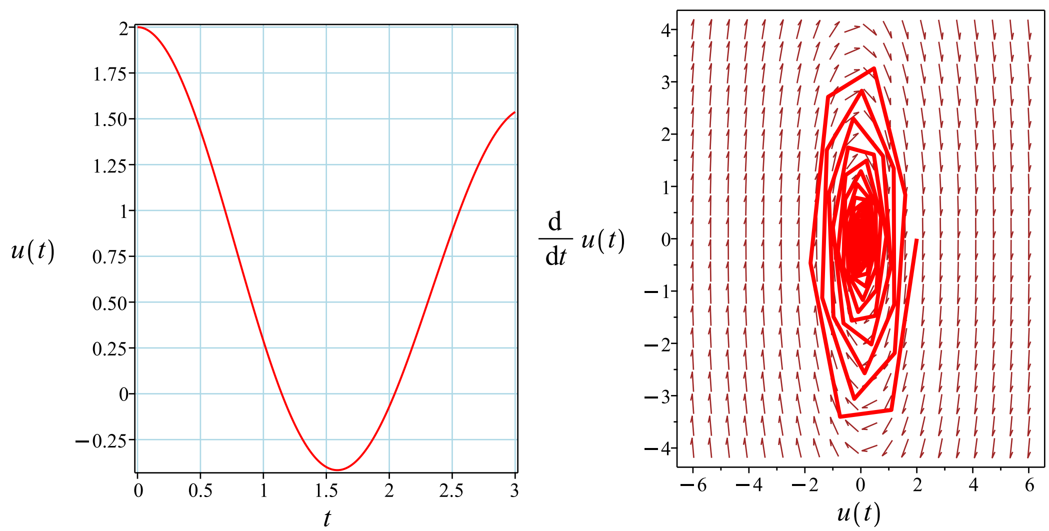
Substituting these values back in above solution results in

$$
u = \frac{19658\, \mathrm{e}^{-\frac{t}{16}} \cos\left(\frac{\sqrt{1023}\, t}{16}\right)}{15877} + \frac{19274\, \mathrm{e}^{-\frac{t}{16}} \sin\left(\frac{\sqrt{1023}\, t}{16}\right) \sqrt{1023}}{16242171} + \frac{12096 \cos\left(\frac{t}{4}\right)}{15877} + \frac{96 \sin\left(\frac{t}{4}\right)}{15877}
$$

Summary

The solution(s) found are the following

$$u = \frac{19658\, e^{-\frac{t}{16}} \cos\left(\frac{\sqrt{1023}\, t}{16}\right)}{15877} + \frac{19274\, e^{-\frac{t}{16}} \sin\left(\frac{\sqrt{1023}\, t}{16}\right) \sqrt{1023}}{16242171} + \frac{12096 \cos\left(\frac{t}{4}\right)}{15877} + \frac{96 \sin\left(\frac{t}{4}\right)}{15877} \tag{1}$$



(a) Solution plot (b) Slope field plot

Verification of solutions

$$u = \frac{19658\, e^{-\frac{t}{16}} \cos\left(\frac{\sqrt{1023}\, t}{16}\right)}{15877} + \frac{19274\, e^{-\frac{t}{16}} \sin\left(\frac{\sqrt{1023}\, t}{16}\right) \sqrt{1023}}{16242171} + \frac{12096 \cos\left(\frac{t}{4}\right)}{15877} + \frac{96 \sin\left(\frac{t}{4}\right)}{15877}$$

Verified OK.

12.1.3 Solving using Kovacic algorithm

Writing the ode as

$$u'' + \frac{u'}{8} + 4u = 0 \tag{1}$$

$$Au'' + Bu' + Cu = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1 \\ B &= \frac{1}{8} \\ C &= 4 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(t) = ue^{\int \frac{B}{2A}\,dt}$$

Then (2) becomes

$$z''(t) = rz(t) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{-1023}{256} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= -1023 \\ t &= 256 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(t) = -\frac{1023z(t)}{256} \tag{7}$$

Equation (7) is now solved. After finding $z(t)$ then u is found using the inverse transformation

$$u = z(t)\,e^{-\int \frac{B}{2A}\,dt}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 512: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 0 - 0 \\ &= 0 \end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Since $r = -\frac{1023}{256}$ is not a function of t, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$z_1(t) = \cos\left(\frac{\sqrt{1023}\, t}{16}\right)$$

Using the above, the solution for the original ode can now be found. The first solution

to the original ode in u is found from

$$
\begin{aligned}
u_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dt} \\
&= z_1 e^{-\int \frac{1}{2}\frac{\frac{1}{8}}{1}\,dt} \\
&= z_1 e^{-\frac{t}{16}} \\
&= z_1 \left(\mathrm{e}^{-\frac{t}{16}}\right)
\end{aligned}
$$

Which simplifies to

$$
u_1 = \mathrm{e}^{-\frac{t}{16}} \cos\left(\frac{\sqrt{1023}\,t}{16}\right)
$$

The second solution u_2 to the original ode is found using reduction of order

$$
u_2 = u_1 \int \frac{e^{\int -\frac{B}{A}\,dt}}{u_1^2}\,dt
$$

Substituting gives

$$
\begin{aligned}
u_2 &= u_1 \int \frac{e^{\int -\frac{\frac{1}{8}}{1}\,dt}}{(u_1)^2}\,dt \\
&= u_1 \int \frac{e^{-\frac{t}{8}}}{(u_1)^2}\,dt \\
&= u_1 \left(\frac{16\sqrt{1023}\,\tan\left(\frac{\sqrt{1023}\,t}{16}\right)}{1023}\right)
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
u &= c_1 u_1 + c_2 u_2 \\
&= c_1 \left(\mathrm{e}^{-\frac{t}{16}} \cos\left(\frac{\sqrt{1023}\,t}{16}\right)\right) + c_2 \left(\mathrm{e}^{-\frac{t}{16}} \cos\left(\frac{\sqrt{1023}\,t}{16}\right) \left(\frac{16\sqrt{1023}\,\tan\left(\frac{\sqrt{1023}\,t}{16}\right)}{1023}\right)\right)
\end{aligned}
$$

This is second order nonhomogeneous ODE. Let the solution be

$$
u = u_h + u_p
$$

Where u_h is the solution to the homogeneous ODE $Au''(t)+Bu'(t)+Cu(t)=0$, and u_p is a particular solution to the nonhomogeneous ODE $Au''(t)+Bu'(t)+Cu(t)=f(t)$. u_h is the solution to

$$u''+\frac{u'}{8}+4u=0$$

The homogeneous solution is found using the Kovacic algorithm which results in

$$u_h=c_1\mathrm{e}^{-\frac{t}{16}}\cos\left(\frac{\sqrt{1023}\,t}{16}\right)+\frac{16c_2\mathrm{e}^{-\frac{t}{16}}\sin\left(\frac{\sqrt{1023}\,t}{16}\right)\sqrt{1023}}{1023}$$

The particular solution is now found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$3\cos\left(\frac{t}{4}\right)$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$\left[\left\{\cos\left(\frac{t}{4}\right),\sin\left(\frac{t}{4}\right)\right\}\right]$$

While the set of the basis functions for the homogeneous solution found earlier is

$$\left\{\mathrm{e}^{-\frac{t}{16}}\cos\left(\frac{\sqrt{1023}\,t}{16}\right),\frac{16\,\mathrm{e}^{-\frac{t}{16}}\sin\left(\frac{\sqrt{1023}\,t}{16}\right)\sqrt{1023}}{1023}\right\}$$

Since there is no duplication between the basis function in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis in the UC_set.

$$u_p=A_1\cos\left(\frac{t}{4}\right)+A_2\sin\left(\frac{t}{4}\right)$$

The unknowns $\{A_1,A_2\}$ are found by substituting the above trial solution u_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$\frac{63A_1\cos\left(\frac{t}{4}\right)}{16}+\frac{63A_2\sin\left(\frac{t}{4}\right)}{16}-\frac{A_1\sin\left(\frac{t}{4}\right)}{32}+\frac{A_2\cos\left(\frac{t}{4}\right)}{32}=3\cos\left(\frac{t}{4}\right)$$

Solving for the unknowns by comparing coefficients results in

$$\left[A_1 = \frac{12096}{15877}, A_2 = \frac{96}{15877}\right]$$

Substituting the above back in the above trial solution u_p, gives the particular solution

$$u_p = \frac{12096 \cos\left(\frac{t}{4}\right)}{15877} + \frac{96 \sin\left(\frac{t}{4}\right)}{15877}$$

Therefore the general solution is

$$\begin{aligned} u &= u_h + u_p \\ &= \left(c_1 \mathrm{e}^{-\frac{t}{16}} \cos\left(\frac{\sqrt{1023}\, t}{16}\right) + \frac{16 c_2 \mathrm{e}^{-\frac{t}{16}} \sin\left(\frac{\sqrt{1023}\, t}{16}\right) \sqrt{1023}}{1023}\right) \\ &\quad + \left(\frac{12096 \cos\left(\frac{t}{4}\right)}{15877} + \frac{96 \sin\left(\frac{t}{4}\right)}{15877}\right) \end{aligned}$$

Initial conditions are used to solve for the constants of integration.

Looking at the above solution

$$u = c_1 \mathrm{e}^{-\frac{t}{16}} \cos\left(\frac{\sqrt{1023}\, t}{16}\right) + \frac{16 c_2 \mathrm{e}^{-\frac{t}{16}} \sin\left(\frac{\sqrt{1023}\, t}{16}\right) \sqrt{1023}}{1023} + \frac{12096 \cos\left(\frac{t}{4}\right)}{15877} + \frac{96 \sin\left(\frac{t}{4}\right)}{15877} \tag{1}$$

Initial conditions are now substituted in the above solution. This will generate the required equations to solve for the integration constants. substituting $u = 2$ and $t = 0$ in the above gives

$$2 = c_1 + \frac{12096}{15877} \tag{1A}$$

Taking derivative of the solution gives

$$u' = -\frac{c_1 \mathrm{e}^{-\frac{t}{16}} \cos\left(\frac{\sqrt{1023}\, t}{16}\right)}{16} - \frac{c_1 \mathrm{e}^{-\frac{t}{16}} \sqrt{1023} \sin\left(\frac{\sqrt{1023}\, t}{16}\right)}{16} - \frac{c_2 \mathrm{e}^{-\frac{t}{16}} \sin\left(\frac{\sqrt{1023}\, t}{16}\right) \sqrt{1023}}{1023} + c_2 \mathrm{e}^{-\frac{t}{16}} \cos\left(\frac{\sqrt{10}}{1}\right.$$

substituting $u' = 0$ and $t = 0$ in the above gives

$$0 = \frac{24}{15877} + c_2 - \frac{c_1}{16} \tag{2A}$$

Equations {1A,2A} are now solved for $\{c_1, c_2\}$. Solving for the constants gives

$$
\begin{aligned}
c_1 &= \frac{19658}{15877} \\
c_2 &= \frac{9637}{127016}
\end{aligned}
$$

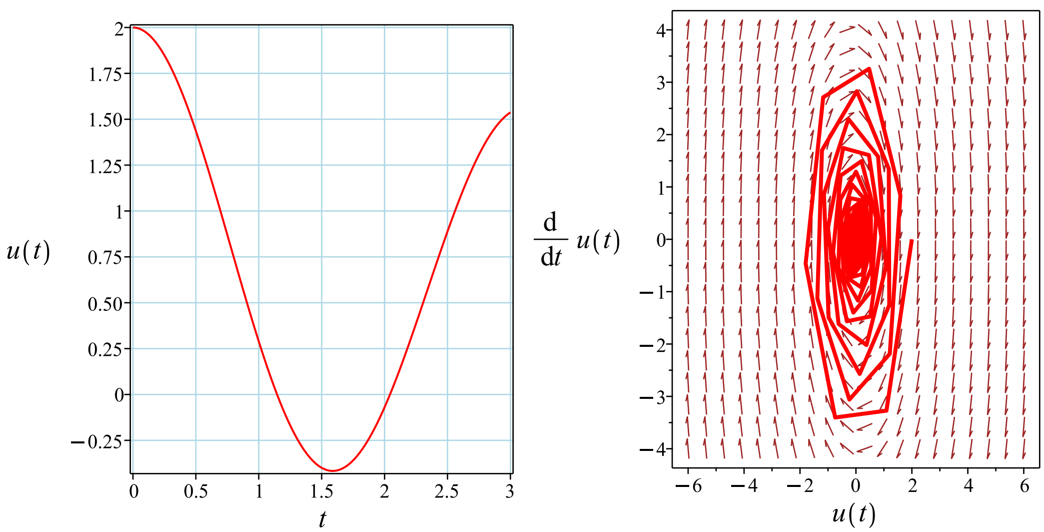
Substituting these values back in above solution results in

$$
u = \frac{19658\,\mathrm{e}^{-\frac{t}{16}} \cos\left(\frac{\sqrt{1023}\,t}{16}\right)}{15877} + \frac{19274\,\mathrm{e}^{-\frac{t}{16}} \sin\left(\frac{\sqrt{1023}\,t}{16}\right)\sqrt{1023}}{16242171} + \frac{12096\cos\left(\frac{t}{4}\right)}{15877} + \frac{96\sin\left(\frac{t}{4}\right)}{15877}
$$

Summary

The solution(s) found are the following

$$
u = \frac{19658\,\mathrm{e}^{-\frac{t}{16}} \cos\left(\frac{\sqrt{1023}\,t}{16}\right)}{15877} + \frac{19274\,\mathrm{e}^{-\frac{t}{16}} \sin\left(\frac{\sqrt{1023}\,t}{16}\right)\sqrt{1023}}{16242171} + \frac{12096\cos\left(\frac{t}{4}\right)}{15877} + \frac{96\sin\left(\frac{t}{4}\right)}{15877} \tag{1}
$$



(a) Solution plot

(b) Slope field plot

Verification of solutions

$$
u = \frac{19658\,\mathrm{e}^{-\frac{t}{16}} \cos\left(\frac{\sqrt{1023}\,t}{16}\right)}{15877} + \frac{19274\,\mathrm{e}^{-\frac{t}{16}} \sin\left(\frac{\sqrt{1023}\,t}{16}\right)\sqrt{1023}}{16242171} + \frac{12096\cos\left(\frac{t}{4}\right)}{15877} + \frac{96\sin\left(\frac{t}{4}\right)}{15877}
$$

Verified OK.

12.1.4 Maple step by step solution

Let's solve

$$\left[u''+\frac{u'}{8}+4u=3\cos\left(\frac{t}{4}\right), u(0)=2, u'\Big|_{\{t=0\}}=0\right]$$

- Highest derivative means the order of the ODE is 2

  u''

- Characteristic polynomial of homogeneous ODE

  $r^2+\frac{1}{8}r+4=0$

- Use quadratic formula to solve for r

  $r=\frac{\left(-\frac{1}{8}\right)\pm\left(\sqrt{-\frac{1023}{64}}\right)}{2}$

- Roots of the characteristic polynomial

  $r=\left(-\frac{1}{16}-\frac{\mathrm{I}\sqrt{1023}}{16}, -\frac{1}{16}+\frac{\mathrm{I}\sqrt{1023}}{16}\right)$

- 1st solution of the homogeneous ODE

  $u_1(t)=\mathrm{e}^{-\frac{t}{16}}\cos\left(\frac{\sqrt{1023}\,t}{16}\right)$

- 2nd solution of the homogeneous ODE

  $u_2(t)=\mathrm{e}^{-\frac{t}{16}}\sin\left(\frac{\sqrt{1023}\,t}{16}\right)$

- General solution of the ODE

  $u=c_1u_1(t)+c_2u_2(t)+u_p(t)$

- Substitute in solutions of the homogeneous ODE

  $u=c_1\mathrm{e}^{-\frac{t}{16}}\cos\left(\frac{\sqrt{1023}\,t}{16}\right)+\mathrm{e}^{-\frac{t}{16}}c_2\sin\left(\frac{\sqrt{1023}\,t}{16}\right)+u_p(t)$

□ Find a particular solution $u_p(t)$ of the ODE

  ◦ Use variation of parameters to find u_p here $f(t)$ is the forcing function

  $\left[u_p(t)=-u_1(t)\left(\int\frac{u_2(t)f(t)}{W(u_1(t),u_2(t))}dt\right)+u_2(t)\left(\int\frac{u_1(t)f(t)}{W(u_1(t),u_2(t))}dt\right), f(t)=3\cos\left(\frac{t}{4}\right)\right]$

  ◦ Wronskian of solutions of the homogeneous equation

  $W(u_1(t), u_2(t))=\left[\begin{array}{cc}\mathrm{e}^{-\frac{t}{16}}\cos\left(\frac{\sqrt{1023}\,t}{16}\right) & \mathrm{e}^{-\frac{t}{16}}\sin\left(\frac{\sqrt{1023}\,t}{16}\right)\\ -\frac{\mathrm{e}^{-\frac{t}{16}}\cos\left(\frac{\sqrt{1023}\,t}{16}\right)}{16}-\frac{\mathrm{e}^{-\frac{t}{16}}\sin\left(\frac{\sqrt{1023}\,t}{16}\right)\sqrt{1023}}{16} & -\frac{\mathrm{e}^{-\frac{t}{16}}\sin\left(\frac{\sqrt{1023}\,t}{16}\right)}{16}+\frac{\mathrm{e}^{-\frac{t}{16}}\sqrt{1023}\cos\left(\sqrt{10\ldots\right.}{16}\end{array}\right.$

- Compute Wronskian

  $W(u_1(t), u_2(t)) = \frac{\sqrt{1023}\,\mathrm{e}^{-\frac{t}{8}}}{16}$

- Substitute functions into equation for $u_p(t)$

  $u_p(t) = -\frac{16\,\mathrm{e}^{-\frac{t}{16}}\sqrt{1023}\left(\cos\left(\frac{\sqrt{1023}\,t}{16}\right)\left(\int \cos\left(\frac{t}{4}\right)\mathrm{e}^{\frac{t}{16}}\sin\left(\frac{\sqrt{1023}\,t}{16}\right)dt\right)-\sin\left(\frac{\sqrt{1023}\,t}{16}\right)\left(\int \cos\left(\frac{t}{4}\right)\mathrm{e}^{\frac{t}{16}}\cos\left(\frac{\sqrt{1023}\,t}{16}\right)dt\right)\right)}{341}$

- Compute integrals

  $u_p(t) = \frac{12096\cos\left(\frac{t}{4}\right)}{15877} + \frac{96\sin\left(\frac{t}{4}\right)}{15877}$

- Substitute particular solution into general solution to ODE

  $u = c_1\mathrm{e}^{-\frac{t}{16}}\cos\left(\frac{\sqrt{1023}\,t}{16}\right) + \mathrm{e}^{-\frac{t}{16}}c_2\sin\left(\frac{\sqrt{1023}\,t}{16}\right) + \frac{12096\cos\left(\frac{t}{4}\right)}{15877} + \frac{96\sin\left(\frac{t}{4}\right)}{15877}$

□ Check validity of solution $u = c_1\mathrm{e}^{-\frac{t}{16}}\cos\left(\frac{\sqrt{1023}\,t}{16}\right) + \mathrm{e}^{-\frac{t}{16}}c_2\sin\left(\frac{\sqrt{1023}\,t}{16}\right) + \frac{12096\cos\left(\frac{t}{4}\right)}{15877} + \frac{96\sin\left(\frac{t}{4}\right)}{15877}$

- Use initial condition $u(0) = 2$

  $2 = c_1 + \frac{12096}{15877}$

- Compute derivative of the solution

  $u' = -\frac{c_1\mathrm{e}^{-\frac{t}{16}}\cos\left(\frac{\sqrt{1023}\,t}{16}\right)}{16} - \frac{c_1\mathrm{e}^{-\frac{t}{16}}\sqrt{1023}\sin\left(\frac{\sqrt{1023}\,t}{16}\right)}{16} - \frac{\mathrm{e}^{-\frac{t}{16}}c_2\sin\left(\frac{\sqrt{1023}\,t}{16}\right)}{16} + \frac{\mathrm{e}^{-\frac{t}{16}}c_2\sqrt{1023}\cos\left(\frac{\sqrt{1023}\,t}{16}\right)}{16} - \frac{3024}{15}$

- Use the initial condition $u'\Big|_{\{t=0\}} = 0$

  $0 = -\frac{c_1}{16} + \frac{\sqrt{1023}\,c_2}{16} + \frac{24}{15877}$

- Solve for c_1 and c_2

  $\left\{c_1 = \frac{19658}{15877}, c_2 = \frac{19274\sqrt{1023}}{16242171}\right\}$

- Substitute constant values into general solution and simplify

  $u = \frac{19658\,\mathrm{e}^{-\frac{t}{16}}\cos\left(\frac{\sqrt{1023}\,t}{16}\right)}{15877} + \frac{19274\,\mathrm{e}^{-\frac{t}{16}}\sin\left(\frac{\sqrt{1023}\,t}{16}\right)\sqrt{1023}}{16242171} + \frac{12096\cos\left(\frac{t}{4}\right)}{15877} + \frac{96\sin\left(\frac{t}{4}\right)}{15877}$

- Solution to the IVP

  $u = \frac{19658\,\mathrm{e}^{-\frac{t}{16}}\cos\left(\frac{\sqrt{1023}\,t}{16}\right)}{15877} + \frac{19274\,\mathrm{e}^{-\frac{t}{16}}\sin\left(\frac{\sqrt{1023}\,t}{16}\right)\sqrt{1023}}{16242171} + \frac{12096\cos\left(\frac{t}{4}\right)}{15877} + \frac{96\sin\left(\frac{t}{4}\right)}{15877}$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 2; linear nonhomogeneous with symmetry [0,1]
trying a double symmetry of the form [xi=0, eta=F(x)]
-> Try solving first the homogeneous part of the ODE
   checking if the LODE has constant coefficients
   <- constant coefficients successful
<- solving first the homogeneous part of the ODE successful`
```

✓ Solution by Maple

Time used: 0.031 (sec). Leaf size: 46

```
dsolve([diff(u(t),t$2)+125/1000*diff(u(t),t)+4*u(t) = 3*cos(t/4),u(0) = 2, D(u)(0) = 0],u(t)
```

$$u(t) = \frac{19274\,\mathrm{e}^{-\frac{t}{16}}\sqrt{1023}\,\sin\left(\frac{\sqrt{1023}\,t}{16}\right)}{16242171} + \frac{19658\,\mathrm{e}^{-\frac{t}{16}}\cos\left(\frac{\sqrt{1023}\,t}{16}\right)}{15877} + \frac{96\sin\left(\frac{t}{4}\right)}{15877} + \frac{12096\cos\left(\frac{t}{4}\right)}{15877}$$

✓ Solution by Mathematica

Time used: 0.039 (sec). Leaf size: 71

```
DSolve[{u''[t]+125/1000*u'[t]+4*u[t] ==3*Cos[t/4],{u[0]==0,u'[0]==0}},u[t],t,IncludeSingular
```

$$u(t) \to \frac{32\left(1023\sin\left(\frac{t}{4}\right) - 130\sqrt{1023}e^{-t/16}\sin\left(\frac{\sqrt{1023}t}{16}\right) + 128898\cos\left(\frac{t}{4}\right) - 128898e^{-t/16}\cos\left(\frac{\sqrt{1023}t}{16}\right)\right)}{5414057}$$

12.2 problem 22

Internal problem ID [708]
Internal file name [`OUTPUT/708_Sunday_June_05_2022_01_47_27_AM_67339897/index.tex`]

Book: Elementary differential equations and boundary value problems, 10th ed., Boyce and DiPrima
Section: Chapter 3, Second order linear equations, 3.7 Forced Vibrations. page 217
Problem number: 22.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_linear_constant_coeff"**

Maple gives the following as the ode type

```
[[_2nd_order, _linear, _nonhomogeneous]]
```

$$\boxed{u'' + \frac{u'}{8} + 4u = 3\cos(2t)}$$

With initial conditions

$$[u(0) = 2, u'(0) = 0]$$

12.2.1 Existence and uniqueness analysis

This is a linear ODE. In canonical form it is written as

$$u'' + p(t)u' + q(t)u = F$$

Where here

$$\begin{aligned} p(t) &= \frac{1}{8} \\ q(t) &= 4 \\ F &= 3\cos(2t) \end{aligned}$$

Hence the ode is

$$u'' + \frac{u'}{8} + 4u = 3\cos(2t)$$

The domain of $p(t) = \frac{1}{8}$ is

$$\{-\infty < t < \infty\}$$

And the point $t_0 = 0$ is inside this domain. The domain of $q(t) = 4$ is

$$\{-\infty < t < \infty\}$$

And the point $t_0 = 0$ is also inside this domain. The domain of $F = 3\cos(2t)$ is

$$\{-\infty < t < \infty\}$$

And the point $t_0 = 0$ is also inside this domain. Hence solution exists and is unique.

12.2.2 Solving as second order linear constant coeff ode

This is second order non-homogeneous ODE. In standard form the ODE is

$$Au''(t) + Bu'(t) + Cu(t) = f(t)$$

Where $A = 1, B = \frac{1}{8}, C = 4, f(t) = 3\cos(2t)$. Let the solution be

$$u = u_h + u_p$$

Where u_h is the solution to the homogeneous ODE $Au''(t)+Bu'(t)+Cu(t) = 0$, and u_p is a particular solution to the non-homogeneous ODE $Au''(t) + Bu'(t) + Cu(t) = f(t)$. u_h is the solution to

$$u'' + \frac{u'}{8} + 4u = 0$$

This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$Au''(t) + Bu'(t) + Cu(t) = 0$$

Where in the above $A = 1, B = \frac{1}{8}, C = 4$. Let the solution be $u = e^{\lambda t}$. Substituting this into the ODE gives

$$\lambda^2 e^{\lambda t} + \frac{\lambda e^{\lambda t}}{8} + 4 e^{\lambda t} = 0 \tag{1}$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda t}$ gives

$$\lambda^2 + \frac{1}{8}\lambda + 4 = 0 \tag{2}$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form.Using the quadratic formula

$$\lambda_{1,2} = \frac{-B}{2A} \pm \frac{1}{2A}\sqrt{B^2 - 4AC}$$

Substituting $A = 1, B = \frac{1}{8}, C = 4$ into the above gives

$$\begin{aligned}\lambda_{1,2} &= \frac{-\frac{1}{8}}{(2)\,(1)} \pm \frac{1}{(2)\,(1)}\sqrt{\frac{1}{8}^2 - (4)\,(1)\,(4)} \\ &= -\frac{1}{16} \pm \frac{i\sqrt{1023}}{16}\end{aligned}$$

Hence

$$\begin{aligned}\lambda_1 &= -\frac{1}{16} + \frac{i\sqrt{1023}}{16} \\ \lambda_2 &= -\frac{1}{16} - \frac{i\sqrt{1023}}{16}\end{aligned}$$

Which simplifies to

$$\lambda_1 = -\frac{1}{16} + \frac{i\sqrt{1023}}{16}$$
$$\lambda_2 = -\frac{1}{16} - \frac{i\sqrt{1023}}{16}$$

Since roots are complex conjugate of each others, then let the roots be

$$\lambda_{1,2} = \alpha \pm i\beta$$

Where $\alpha = -\frac{1}{16}$ and $\beta = \frac{\sqrt{1023}}{16}$. Therefore the final solution, when using Euler relation, can be written as

$$u = e^{\alpha t}(c_1 \cos(\beta t) + c_2 \sin(\beta t))$$

Which becomes

$$u = e^{-\frac{t}{16}}\left(c_1 \cos\left(\frac{\sqrt{1023}\, t}{16}\right) + c_2 \sin\left(\frac{\sqrt{1023}\, t}{16}\right)\right)$$

Therefore the homogeneous solution u_h is

$$u_h = e^{-\frac{t}{16}} \left(c_1 \cos\left(\frac{\sqrt{1023}\, t}{16}\right) + c_2 \sin\left(\frac{\sqrt{1023}\, t}{16}\right)\right)$$

The particular solution is now found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$3\cos(2t)$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$[\{\cos(2t), \sin(2t)\}]$$

While the set of the basis functions for the homogeneous solution found earlier is

$$\left\{e^{-\frac{t}{16}} \cos\left(\frac{\sqrt{1023}\, t}{16}\right), e^{-\frac{t}{16}} \sin\left(\frac{\sqrt{1023}\, t}{16}\right)\right\}$$

Since there is no duplication between the basis function in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis in the UC_set.

$$u_p = A_1 \cos(2t) + A_2 \sin(2t)$$

The unknowns $\{A_1, A_2\}$ are found by substituting the above trial solution u_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$-\frac{A_1 \sin(2t)}{4} + \frac{A_2 \cos(2t)}{4} = 3\cos(2t)$$

Solving for the unknowns by comparing coefficients results in

$$[A_1 = 0, A_2 = 12]$$

Substituting the above back in the above trial solution u_p, gives the particular solution

$$u_p = 12\sin(2t)$$

Therefore the general solution is

$$\begin{aligned} u &= u_h + u_p \\ &= \left(e^{-\frac{t}{16}} \left(c_1 \cos\left(\frac{\sqrt{1023}\, t}{16}\right) + c_2 \sin\left(\frac{\sqrt{1023}\, t}{16}\right)\right)\right) + (12\sin(2t)) \end{aligned}$$

Initial conditions are used to solve for the constants of integration.

Looking at the above solution

$$u = e^{-\frac{t}{16}} \left(c_1 \cos\left(\frac{\sqrt{1023}\, t}{16}\right) + c_2 \sin\left(\frac{\sqrt{1023}\, t}{16}\right)\right) + 12 \sin(2t) \tag{1}$$

Initial conditions are now substituted in the above solution. This will generate the required equations to solve for the integration constants. substituting $u = 2$ and $t = 0$ in the above gives

$$2 = c_1 \tag{1A}$$

Taking derivative of the solution gives

$$u' = -\frac{e^{-\frac{t}{16}} \left(c_1 \cos\left(\frac{\sqrt{1023}\, t}{16}\right) + c_2 \sin\left(\frac{\sqrt{1023}\, t}{16}\right)\right)}{16} + e^{-\frac{t}{16}} \left(-\frac{c_1 \sqrt{1023}\, \sin\left(\frac{\sqrt{1023}\, t}{16}\right)}{16} + \frac{c_2 \sqrt{1023}\, \cos\left(\frac{\sqrt{1023}\, t}{16}\right)}{16}\right.$$

substituting $u' = 0$ and $t = 0$ in the above gives

$$0 = -\frac{c_1}{16} + \frac{\sqrt{1023}\, c_2}{16} + 24 \tag{2A}$$

Equations {1A,2A} are now solved for $\{c_1, c_2\}$. Solving for the constants gives

$$c_1 = 2$$
$$c_2 = -\frac{382\sqrt{1023}}{1023}$$

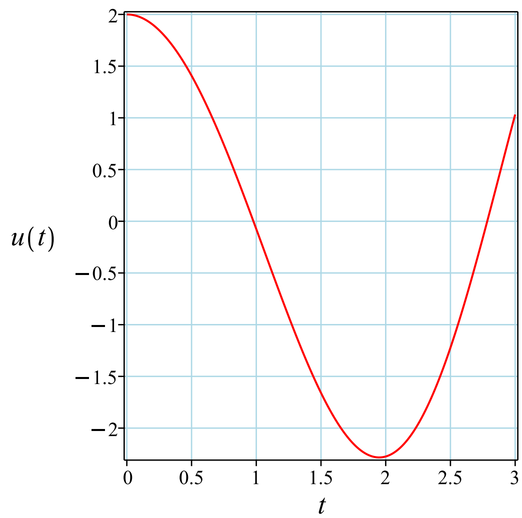
Substituting these values back in above solution results in

$$u = -\frac{382\, e^{-\frac{t}{16}} \sin\left(\frac{\sqrt{1023}\, t}{16}\right) \sqrt{1023}}{1023} + 2\, e^{-\frac{t}{16}} \cos\left(\frac{\sqrt{1023}\, t}{16}\right) + 12 \sin(2t)$$

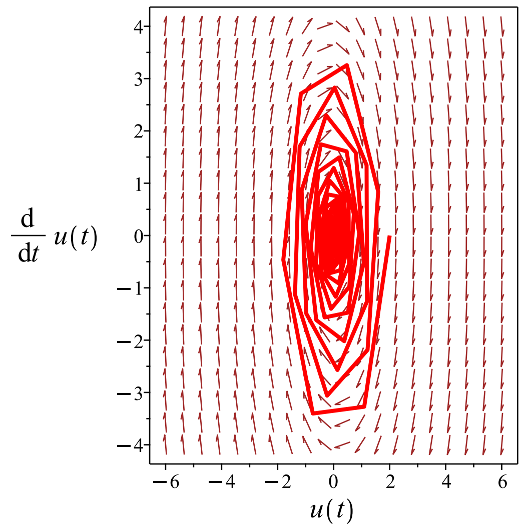
Summary

The solution(s) found are the following

$$u = -\frac{382\, e^{-\frac{t}{16}} \sin\left(\frac{\sqrt{1023}\, t}{16}\right) \sqrt{1023}}{1023} + 2\, e^{-\frac{t}{16}} \cos\left(\frac{\sqrt{1023}\, t}{16}\right) + 12 \sin(2t) \tag{1}$$



(a) Solution plot



(b) Slope field plot

Verification of solutions

$$u = -\frac{382\,\mathrm{e}^{-\frac{t}{16}} \sin\left(\frac{\sqrt{1023}\,t}{16}\right)\sqrt{1023}}{1023} + 2\,\mathrm{e}^{-\frac{t}{16}} \cos\left(\frac{\sqrt{1023}\,t}{16}\right) + 12\sin(2t)$$

Verified OK.

12.2.3 Solving using Kovacic algorithm

Writing the ode as

$$u'' + \frac{u'}{8} + 4u = 0 \tag{1}$$
$$Au'' + Bu' + Cu = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1 \\ B &= \frac{1}{8} \\ C &= 4 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(t) = ue^{\int \frac{B}{2A}\,dt}$$

Then (2) becomes

$$z''(t) = rz(t) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{-1023}{256} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= -1023 \\ t &= 256 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(t) = -\frac{1023z(t)}{256} \tag{7}$$

Equation (7) is now solved. After finding $z(t)$ then u is found using the inverse transformation

$$u = z(t)\, e^{-\int \frac{B}{2A}\, dt}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 514: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 0 - 0 \\ &= 0 \end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Since $r = -\frac{1023}{256}$ is not a function of t, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$z_1(t) = \cos\left(\frac{\sqrt{1023}\, t}{16}\right)$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in u is found from

$$\begin{aligned} u_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\, dt} \\ &= z_1 e^{-\int \frac{1}{2}\frac{\frac{1}{8}}{1}\, dt} \\ &= z_1 e^{-\frac{t}{16}} \\ &= z_1\left(\mathrm{e}^{-\frac{t}{16}}\right) \end{aligned}$$

Which simplifies to

$$u_1 = \mathrm{e}^{-\frac{t}{16}} \cos\left(\frac{\sqrt{1023}\, t}{16}\right)$$

The second solution u_2 to the original ode is found using reduction of order

$$u_2 = u_1 \int \frac{e^{\int -\frac{B}{A}\, dt}}{u_1^2}\, dt$$

Substituting gives

$$
\begin{aligned}
u_2 &= u_1 \int \frac{e^{\int -\frac{\frac{1}{8}}{1} dt}}{(u_1)^2} \, dt \\
&= u_1 \int \frac{e^{-\frac{t}{8}}}{(u_1)^2} \, dt \\
&= u_1 \left(\frac{16\sqrt{1023}\, \tan\left(\frac{\sqrt{1023}\, t}{16}\right)}{1023} \right)
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
u &= c_1 u_1 + c_2 u_2 \\
&= c_1 \left(\mathrm{e}^{-\frac{t}{16}} \cos\left(\frac{\sqrt{1023}\, t}{16}\right) \right) + c_2 \left(\mathrm{e}^{-\frac{t}{16}} \cos\left(\frac{\sqrt{1023}\, t}{16}\right) \left(\frac{16\sqrt{1023}\, \tan\left(\frac{\sqrt{1023}\, t}{16}\right)}{1023} \right) \right)
\end{aligned}
$$

This is second order nonhomogeneous ODE. Let the solution be

$$
u = u_h + u_p
$$

Where u_h is the solution to the homogeneous ODE $Au''(t) + Bu'(t) + Cu(t) = 0$, and u_p is a particular solution to the nonhomogeneous ODE $Au''(t) + Bu'(t) + Cu(t) = f(t)$. u_h is the solution to

$$
u'' + \frac{u'}{8} + 4u = 0
$$

The homogeneous solution is found using the Kovacic algorithm which results in

$$
u_h = c_1 \mathrm{e}^{-\frac{t}{16}} \cos\left(\frac{\sqrt{1023}\, t}{16}\right) + \frac{16 c_2 \mathrm{e}^{-\frac{t}{16}} \sin\left(\frac{\sqrt{1023}\, t}{16}\right) \sqrt{1023}}{1023}
$$

The particular solution is now found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$
3 \cos(2t)
$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$[\{\cos(2t), \sin(2t)\}]$$

While the set of the basis functions for the homogeneous solution found earlier is

$$\left\{ e^{-\frac{t}{16}} \cos\left(\frac{\sqrt{1023}\, t}{16}\right), \frac{16\, e^{-\frac{t}{16}} \sin\left(\frac{\sqrt{1023}\, t}{16}\right) \sqrt{1023}}{1023} \right\}$$

Since there is no duplication between the basis function in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis in the UC_set.

$$u_p = A_1 \cos(2t) + A_2 \sin(2t)$$

The unknowns $\{A_1, A_2\}$ are found by substituting the above trial solution u_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$-\frac{A_1 \sin(2t)}{4} + \frac{A_2 \cos(2t)}{4} = 3\cos(2t)$$

Solving for the unknowns by comparing coefficients results in

$$[A_1 = 0, A_2 = 12]$$

Substituting the above back in the above trial solution u_p, gives the particular solution

$$u_p = 12\sin(2t)$$

Therefore the general solution is

$$\begin{aligned} u &= u_h + u_p \\ &= \left(c_1 e^{-\frac{t}{16}} \cos\left(\frac{\sqrt{1023}\, t}{16}\right) + \frac{16 c_2 e^{-\frac{t}{16}} \sin\left(\frac{\sqrt{1023}\, t}{16}\right) \sqrt{1023}}{1023}\right) + (12\sin(2t)) \end{aligned}$$

Initial conditions are used to solve for the constants of integration.

Looking at the above solution

$$u = c_1 e^{-\frac{t}{16}} \cos\left(\frac{\sqrt{1023}\, t}{16}\right) + \frac{16 c_2 e^{-\frac{t}{16}} \sin\left(\frac{\sqrt{1023}\, t}{16}\right) \sqrt{1023}}{1023} + 12\sin(2t) \tag{1}$$

Initial conditions are now substituted in the above solution. This will generate the required equations to solve for the integration constants. substituting $u = 2$ and $t = 0$ in the above gives

$$2 = c_1 \tag{1A}$$

Taking derivative of the solution gives

$$u' = -\frac{c_1 \mathrm{e}^{-\frac{t}{16}} \cos\left(\frac{\sqrt{1023}\, t}{16}\right)}{16} - \frac{c_1 \mathrm{e}^{-\frac{t}{16}} \sqrt{1023}\, \sin\left(\frac{\sqrt{1023}\, t}{16}\right)}{16} - \frac{c_2 \mathrm{e}^{-\frac{t}{16}} \sin\left(\frac{\sqrt{1023}\, t}{16}\right) \sqrt{1023}}{1023} + c_2 \mathrm{e}^{-\frac{t}{16}} \cos\left(\frac{\sqrt{10}}{1}\right.$$

substituting $u' = 0$ and $t = 0$ in the above gives

$$0 = 24 - \frac{c_1}{16} + c_2 \tag{2A}$$

Equations {1A,2A} are now solved for $\{c_1, c_2\}$. Solving for the constants gives

$$c_1 = 2$$
$$c_2 = -\frac{191}{8}$$

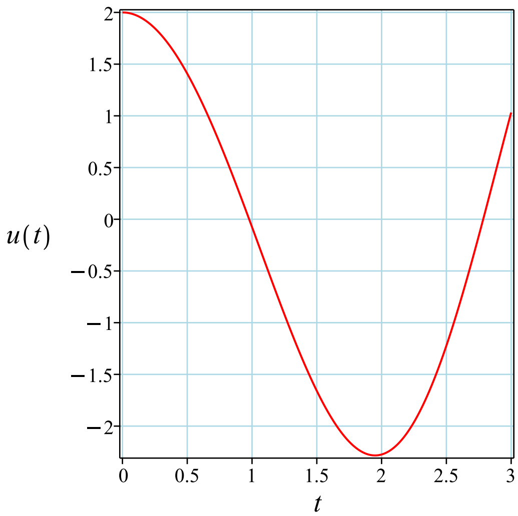
Substituting these values back in above solution results in

$$u = -\frac{382\, \mathrm{e}^{-\frac{t}{16}} \sin\left(\frac{\sqrt{1023}\, t}{16}\right) \sqrt{1023}}{1023} + 2\, \mathrm{e}^{-\frac{t}{16}} \cos\left(\frac{\sqrt{1023}\, t}{16}\right) + 12 \sin(2t)$$

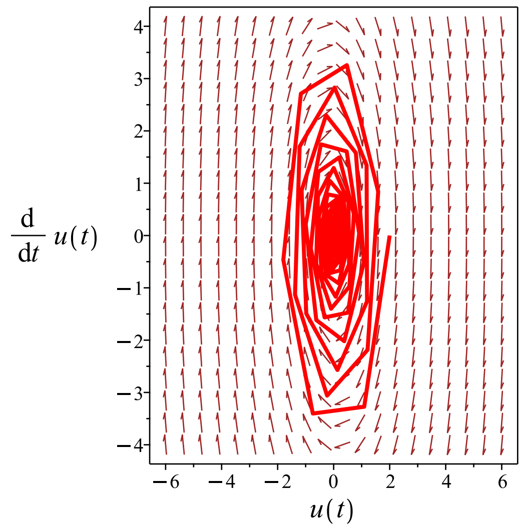
Summary

The solution(s) found are the following

$$u = -\frac{382\, \mathrm{e}^{-\frac{t}{16}} \sin\left(\frac{\sqrt{1023}\, t}{16}\right) \sqrt{1023}}{1023} + 2\, \mathrm{e}^{-\frac{t}{16}} \cos\left(\frac{\sqrt{1023}\, t}{16}\right) + 12 \sin(2t) \tag{1}$$



(a) Solution plot



(b) Slope field plot

<u>Verification of solutions</u>

$$u = -\frac{382\,\mathrm{e}^{-\frac{t}{16}} \sin\left(\frac{\sqrt{1023}\,t}{16}\right)\sqrt{1023}}{1023} + 2\,\mathrm{e}^{-\frac{t}{16}} \cos\left(\frac{\sqrt{1023}\,t}{16}\right) + 12\sin(2t)$$

Verified OK.

12.2.4 Maple step by step solution

Let's solve

$$\left[u'' + \frac{u'}{8} + 4u = 3\cos(2t)\,, u(0) = 2, u'\Big|_{\{t=0\}} = 0\right]$$

- Highest derivative means the order of the ODE is 2
  u''
- Characteristic polynomial of homogeneous ODE
  $r^2 + \frac{1}{8}r + 4 = 0$
- Use quadratic formula to solve for r
  $r = \frac{\left(-\frac{1}{8}\right)\pm\left(\sqrt{-\frac{1023}{64}}\right)}{2}$
- Roots of the characteristic polynomial

$r = \left(-\frac{1}{16} - \frac{\mathrm{I}\sqrt{1023}}{16}, -\frac{1}{16} + \frac{\mathrm{I}\sqrt{1023}}{16}\right)$

- 1st solution of the homogeneous ODE

  $u_1(t) = \mathrm{e}^{-\frac{t}{16}} \cos\left(\frac{\sqrt{1023}\, t}{16}\right)$

- 2nd solution of the homogeneous ODE

  $u_2(t) = \mathrm{e}^{-\frac{t}{16}} \sin\left(\frac{\sqrt{1023}\, t}{16}\right)$

- General solution of the ODE

  $u = c_1 u_1(t) + c_2 u_2(t) + u_p(t)$

- Substitute in solutions of the homogeneous ODE

  $u = c_1 \mathrm{e}^{-\frac{t}{16}} \cos\left(\frac{\sqrt{1023}\, t}{16}\right) + \mathrm{e}^{-\frac{t}{16}} c_2 \sin\left(\frac{\sqrt{1023}\, t}{16}\right) + u_p(t)$

☐ Find a particular solution $u_p(t)$ of the ODE

- ○ Use variation of parameters to find u_p here $f(t)$ is the forcing function

  $\left[u_p(t) = -u_1(t)\left(\int \frac{u_2(t) f(t)}{W(u_1(t),u_2(t))} dt\right) + u_2(t)\left(\int \frac{u_1(t) f(t)}{W(u_1(t),u_2(t))} dt\right), f(t) = 3\cos(2t)\right]$

- ○ Wronskian of solutions of the homogeneous equation

  $W(u_1(t), u_2(t)) = \begin{bmatrix} \mathrm{e}^{-\frac{t}{16}} \cos\left(\frac{\sqrt{1023}\, t}{16}\right) & \mathrm{e}^{-\frac{t}{16}} \sin\left(\frac{\sqrt{1023}\, t}{16}\right) \\ -\frac{\mathrm{e}^{-\frac{t}{16}} \cos\left(\frac{\sqrt{1023}\, t}{16}\right)}{16} - \frac{\mathrm{e}^{-\frac{t}{16}} \sin\left(\frac{\sqrt{1023}\, t}{16}\right)\sqrt{1023}}{16} & -\frac{\mathrm{e}^{-\frac{t}{16}} \sin\left(\frac{\sqrt{1023}\, t}{16}\right)}{16} + \frac{\mathrm{e}^{-\frac{t}{16}} \sqrt{1023} \cos\left(\frac{\sqrt{1023}\, t}{16}\right)}{16} \end{bmatrix}$

- ○ Compute Wronskian

  $W(u_1(t), u_2(t)) = \frac{\sqrt{1023}\, \mathrm{e}^{-\frac{t}{8}}}{16}$

- ○ Substitute functions into equation for $u_p(t)$

  $u_p(t) = -\frac{16\, \mathrm{e}^{-\frac{t}{16}} \sqrt{1023} \left(\cos\left(\frac{\sqrt{1023}\, t}{16}\right)\left(\int \cos(2t) \mathrm{e}^{\frac{t}{16}} \sin\left(\frac{\sqrt{1023}\, t}{16}\right) dt\right) - \sin\left(\frac{\sqrt{1023}\, t}{16}\right)\left(\int \cos(2t) \mathrm{e}^{\frac{t}{16}} \cos\left(\frac{\sqrt{1023}\, t}{16}\right) dt\right)\right)}{341}$

- ○ Compute integrals

  $u_p(t) = 12\sin(2t)$

- Substitute particular solution into general solution to ODE

  $u = c_1 \mathrm{e}^{-\frac{t}{16}} \cos\left(\frac{\sqrt{1023}\, t}{16}\right) + \mathrm{e}^{-\frac{t}{16}} c_2 \sin\left(\frac{\sqrt{1023}\, t}{16}\right) + 12\sin(2t)$

☐ Check validity of solution $u = c_1 \mathrm{e}^{-\frac{t}{16}} \cos\left(\frac{\sqrt{1023}\, t}{16}\right) + \mathrm{e}^{-\frac{t}{16}} c_2 \sin\left(\frac{\sqrt{1023}\, t}{16}\right) + 12\sin(2t)$

- ○ Use initial condition $u(0) = 2$

  $2 = c_1$

- Compute derivative of the solution

$$u' = -\frac{c_1 e^{-\frac{t}{16}} \cos\left(\frac{\sqrt{1023}\, t}{16}\right)}{16} - \frac{c_1 e^{-\frac{t}{16}} \sqrt{1023}\, \sin\left(\frac{\sqrt{1023}\, t}{16}\right)}{16} - \frac{e^{-\frac{t}{16}} c_2 \sin\left(\frac{\sqrt{1023}\, t}{16}\right)}{16} + \frac{e^{-\frac{t}{16}} c_2 \sqrt{1023}\, \cos\left(\frac{\sqrt{1023}\, t}{16}\right)}{16} + 24 c$$

- Use the initial condition $u'\Big|_{\{t=0\}} = 0$

$$0 = -\frac{c_1}{16} + \frac{\sqrt{1023}\, c_2}{16} + 24$$

- Solve for c_1 and c_2

$$\left\{c_1 = 2, c_2 = -\frac{382\sqrt{1023}}{1023}\right\}$$

- Substitute constant values into general solution and simplify

$$u = -\frac{382\, e^{-\frac{t}{16}} \sin\left(\frac{\sqrt{1023}\, t}{16}\right) \sqrt{1023}}{1023} + 2\, e^{-\frac{t}{16}} \cos\left(\frac{\sqrt{1023}\, t}{16}\right) + 12 \sin(2t)$$

- Solution to the IVP

$$u = -\frac{382\, e^{-\frac{t}{16}} \sin\left(\frac{\sqrt{1023}\, t}{16}\right) \sqrt{1023}}{1023} + 2\, e^{-\frac{t}{16}} \cos\left(\frac{\sqrt{1023}\, t}{16}\right) + 12 \sin(2t)$$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 2; linear nonhomogeneous with symmetry [0,1]
trying a double symmetry of the form [xi=0, eta=F(x)]
-> Try solving first the homogeneous part of the ODE
   checking if the LODE has constant coefficients
   <- constant coefficients successful
<- solving first the homogeneous part of the ODE successful`
```

✓ Solution by Maple

Time used: 0.032 (sec). Leaf size: 40

```
dsolve([diff(u(t),t$2)+125/1000*diff(u(t),t)+4*u(t) = 3*cos(2*t),u(0) = 2, D(u)(0) = 0],u(t)
```

$$u(t) = -\frac{382\, e^{-\frac{t}{16}} \sqrt{1023}\, \sin\left(\frac{\sqrt{1023}\, t}{16}\right)}{1023} + 2\, e^{-\frac{t}{16}} \cos\left(\frac{\sqrt{1023}\, t}{16}\right) + 12 \sin(2t)$$

✓ Solution by Mathematica

Time used: 0.03 (sec). Leaf size: 39

```
DSolve[{u''[t]+125/1000*u'[t]+4*u[t] ==3*Cos[2*t],{u[0]==0,u'[0]==0}},u[t],t,IncludeSingular
```

$$u(t) \rightarrow 12\sin(2t) - 128\sqrt{\frac{3}{341}}e^{-t/16}\sin\left(\frac{\sqrt{1023}t}{16}\right)$$

12.3 problem 23

Internal problem ID [709]
Internal file name [`OUTPUT/709_Sunday_June_05_2022_01_47_29_AM_56108713/index.tex`]

Book: Elementary differential equations and boundary value problems, 10th ed., Boyce and DiPrima
Section: Chapter 3, Second order linear equations, 3.7 Forced Vibrations. page 217
Problem number: 23.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_linear_constant_coeff"**

Maple gives the following as the ode type

```
[[_2nd_order, _linear, _nonhomogeneous]]
```

$$\boxed{u'' + \frac{u'}{8} + 4u = 3\cos(6t)}$$

With initial conditions

$$[u(0) = 2, u'(0) = 0]$$

12.3.1 Existence and uniqueness analysis

This is a linear ODE. In canonical form it is written as

$$u'' + p(t)u' + q(t)u = F$$

Where here

$$p(t) = \frac{1}{8}$$
$$q(t) = 4$$
$$F = 3\cos(6t)$$

Hence the ode is

$$u'' + \frac{u'}{8} + 4u = 3\cos(6t)$$

The domain of $p(t) = \frac{1}{8}$ is

$$\{-\infty < t < \infty\}$$

And the point $t_0 = 0$ is inside this domain. The domain of $q(t) = 4$ is

$$\{-\infty < t < \infty\}$$

And the point $t_0 = 0$ is also inside this domain. The domain of $F = 3\cos(6t)$ is

$$\{-\infty < t < \infty\}$$

And the point $t_0 = 0$ is also inside this domain. Hence solution exists and is unique.

12.3.2 Solving as second order linear constant coeff ode

This is second order non-homogeneous ODE. In standard form the ODE is

$$Au''(t) + Bu'(t) + Cu(t) = f(t)$$

Where $A = 1, B = \frac{1}{8}, C = 4, f(t) = 3\cos(6t)$. Let the solution be

$$u = u_h + u_p$$

Where u_h is the solution to the homogeneous ODE $Au''(t)+Bu'(t)+Cu(t) = 0$, and u_p is a particular solution to the non-homogeneous ODE $Au''(t) + Bu'(t) + Cu(t) = f(t)$. u_h is the solution to

$$u'' + \frac{u'}{8} + 4u = 0$$

This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$Au''(t) + Bu'(t) + Cu(t) = 0$$

Where in the above $A = 1, B = \frac{1}{8}, C = 4$. Let the solution be $u = e^{\lambda t}$. Substituting this into the ODE gives

$$\lambda^2 e^{\lambda t} + \frac{\lambda e^{\lambda t}}{8} + 4 e^{\lambda t} = 0 \tag{1}$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda t}$ gives

$$\lambda^2 + \frac{1}{8}\lambda + 4 = 0 \tag{2}$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form.Using the quadratic formula

$$\lambda_{1,2} = \frac{-B}{2A} \pm \frac{1}{2A}\sqrt{B^2 - 4AC}$$

Substituting $A = 1, B = \frac{1}{8}, C = 4$ into the above gives

$$\begin{aligned}\lambda_{1,2} &= \frac{-\frac{1}{8}}{(2)(1)} \pm \frac{1}{(2)(1)}\sqrt{\frac{1}{8}^2 - (4)(1)(4)}\\ &= -\frac{1}{16} \pm \frac{i\sqrt{1023}}{16}\end{aligned}$$

Hence

$$\begin{aligned}\lambda_1 &= -\frac{1}{16} + \frac{i\sqrt{1023}}{16}\\ \lambda_2 &= -\frac{1}{16} - \frac{i\sqrt{1023}}{16}\end{aligned}$$

Which simplifies to

$$\lambda_1 = -\frac{1}{16} + \frac{i\sqrt{1023}}{16}$$

$$\lambda_2 = -\frac{1}{16} - \frac{i\sqrt{1023}}{16}$$

Since roots are complex conjugate of each others, then let the roots be

$$\lambda_{1,2} = \alpha \pm i\beta$$

Where $\alpha = -\frac{1}{16}$ and $\beta = \frac{\sqrt{1023}}{16}$. Therefore the final solution, when using Euler relation, can be written as

$$u = e^{\alpha t}(c_1 \cos(\beta t) + c_2 \sin(\beta t))$$

Which becomes

$$u = e^{-\frac{t}{16}}\left(c_1 \cos\left(\frac{\sqrt{1023}\, t}{16}\right) + c_2 \sin\left(\frac{\sqrt{1023}\, t}{16}\right)\right)$$

Therefore the homogeneous solution u_h is

$$u_h = e^{-\frac{t}{16}} \left(c_1 \cos\left(\frac{\sqrt{1023}\, t}{16}\right) + c_2 \sin\left(\frac{\sqrt{1023}\, t}{16}\right)\right)$$

The particular solution is now found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$3 \cos(6t)$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$[\{\cos(6t), \sin(6t)\}]$$

While the set of the basis functions for the homogeneous solution found earlier is

$$\left\{e^{-\frac{t}{16}} \cos\left(\frac{\sqrt{1023}\, t}{16}\right), e^{-\frac{t}{16}} \sin\left(\frac{\sqrt{1023}\, t}{16}\right)\right\}$$

Since there is no duplication between the basis function in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis in the UC_set.

$$u_p = A_1 \cos(6t) + A_2 \sin(6t)$$

The unknowns $\{A_1, A_2\}$ are found by substituting the above trial solution u_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$-32A_1 \cos(6t) - 32A_2 \sin(6t) - \frac{3A_1 \sin(6t)}{4} + \frac{3A_2 \cos(6t)}{4} = 3 \cos(6t)$$

Solving for the unknowns by comparing coefficients results in

$$\left[A_1 = -\frac{1536}{16393}, A_2 = \frac{36}{16393}\right]$$

Substituting the above back in the above trial solution u_p, gives the particular solution

$$u_p = -\frac{1536 \cos(6t)}{16393} + \frac{36 \sin(6t)}{16393}$$

Therefore the general solution is

$$
\begin{aligned}
u &= u_h + u_p \\
&= \left(\mathrm{e}^{-\frac{t}{16}}\left(c_1 \cos\left(\frac{\sqrt{1023}\, t}{16}\right) + c_2 \sin\left(\frac{\sqrt{1023}\, t}{16}\right)\right)\right) + \left(-\frac{1536 \cos(6t)}{16393} + \frac{36 \sin(6t)}{16393}\right)
\end{aligned}
$$

Initial conditions are used to solve for the constants of integration.

Looking at the above solution

$$
u = \mathrm{e}^{-\frac{t}{16}}\left(c_1 \cos\left(\frac{\sqrt{1023}\, t}{16}\right) + c_2 \sin\left(\frac{\sqrt{1023}\, t}{16}\right)\right) - \frac{1536 \cos(6t)}{16393} + \frac{36 \sin(6t)}{16393} \tag{1}
$$

Initial conditions are now substituted in the above solution. This will generate the required equations to solve for the integration constants. substituting $u = 2$ and $t = 0$ in the above gives

$$
2 = c_1 - \frac{1536}{16393} \tag{1A}
$$

Taking derivative of the solution gives

$$
u' = -\frac{\mathrm{e}^{-\frac{t}{16}}\left(c_1 \cos\left(\frac{\sqrt{1023}\, t}{16}\right) + c_2 \sin\left(\frac{\sqrt{1023}\, t}{16}\right)\right)}{16} + \mathrm{e}^{-\frac{t}{16}}\left(-\frac{c_1\sqrt{1023}\, \sin\left(\frac{\sqrt{1023}\, t}{16}\right)}{16} + \frac{c_2\sqrt{1023}\, \cos\left(\frac{\sqrt{1023}\, t}{16}\right)}{16}\right.
$$

substituting $u' = 0$ and $t = 0$ in the above gives

$$
0 = -\frac{c_1}{16} + \frac{\sqrt{1023}\, c_2}{16} + \frac{216}{16393} \tag{2A}
$$

Equations {1A,2A} are now solved for $\{c_1, c_2\}$. Solving for the constants gives

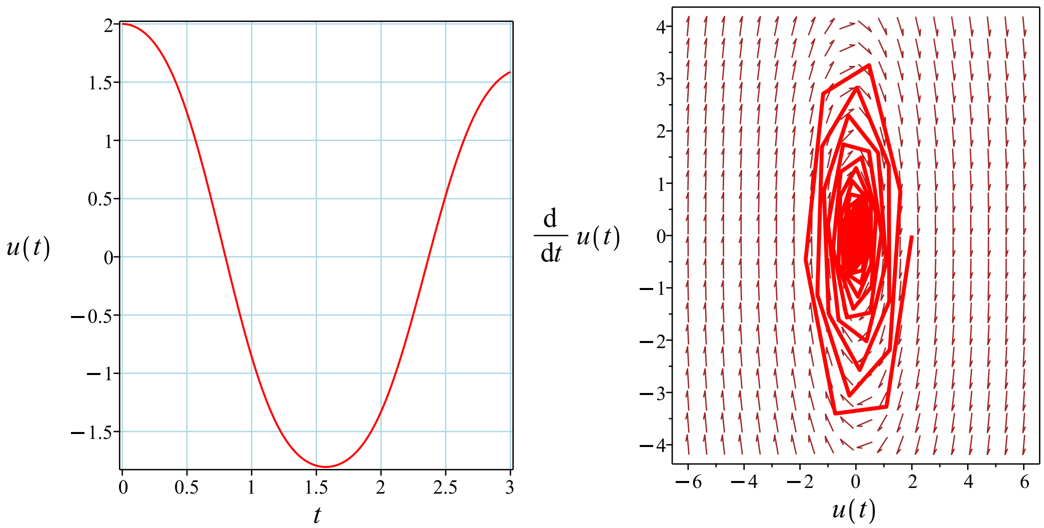
$$
\begin{aligned}
c_1 &= \frac{34322}{16393} \\
c_2 &= \frac{2806\sqrt{1023}}{1524549}
\end{aligned}
$$

Substituting these values back in above solution results in

$$
u = \frac{34322\, \mathrm{e}^{-\frac{t}{16}} \cos\left(\frac{\sqrt{1023}\, t}{16}\right)}{16393} + \frac{2806\, \mathrm{e}^{-\frac{t}{16}} \sin\left(\frac{\sqrt{1023}\, t}{16}\right) \sqrt{1023}}{1524549} - \frac{1536 \cos(6t)}{16393} + \frac{36 \sin(6t)}{16393}
$$

Summary
The solution(s) found are the following

$$u = \frac{34322\,e^{-\frac{t}{16}} \cos\left(\frac{\sqrt{1023}\,t}{16}\right)}{16393} + \frac{2806\,e^{-\frac{t}{16}} \sin\left(\frac{\sqrt{1023}\,t}{16}\right)\sqrt{1023}}{1524549} - \frac{1536\cos(6t)}{16393} + \frac{36\sin(6t)}{16393} \tag{1}$$



(a) Solution plot

(b) Slope field plot

Verification of solutions

$$u = \frac{34322\,e^{-\frac{t}{16}} \cos\left(\frac{\sqrt{1023}\,t}{16}\right)}{16393} + \frac{2806\,e^{-\frac{t}{16}} \sin\left(\frac{\sqrt{1023}\,t}{16}\right)\sqrt{1023}}{1524549} - \frac{1536\cos(6t)}{16393} + \frac{36\sin(6t)}{16393}$$

Verified OK.

12.3.3 Solving using Kovacic algorithm

Writing the ode as

$$u'' + \frac{u'}{8} + 4u = 0 \tag{1}$$

$$Au'' + Bu' + Cu = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$
\begin{aligned}
A &= 1 \\
B &= \frac{1}{8} \\
C &= 4
\end{aligned} \tag{3}
$$

Applying the Liouville transformation on the dependent variable gives

$$
z(t) = u e^{\int \frac{B}{2A} \, dt}
$$

Then (2) becomes

$$
z''(t) = r z(t) \tag{4}
$$

Where r is given by

$$
\begin{aligned}
r &= \frac{s}{t} \\
&= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2}
\end{aligned} \tag{5}
$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$
r = \frac{-1023}{256} \tag{6}
$$

Comparing the above to (5) shows that

$$
\begin{aligned}
s &= -1023 \\
t &= 256
\end{aligned}
$$

Therefore eq. (4) becomes

$$
z''(t) = -\frac{1023 z(t)}{256} \tag{7}
$$

Equation (7) is now solved. After finding $z(t)$ then u is found using the inverse transformation

$$
u = z(t) \, e^{-\int \frac{B}{2A} \, dt}
$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 516: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$
\begin{aligned}
O(\infty) &= \deg(t) - \deg(s) \\
&= 0 - 0 \\
&= 0
\end{aligned}
$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Therefore

$$
L = [1]
$$

Since $r = -\frac{1023}{256}$ is not a function of t, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$
z_1(t) = \cos\left(\frac{\sqrt{1023}\, t}{16}\right)
$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in u is found from

$$
\begin{aligned}
u_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\, dt} \\
&= z_1 e^{-\int \frac{1}{2}\frac{\frac{1}{8}}{1}\, dt} \\
&= z_1 e^{-\frac{t}{16}} \\
&= z_1 \left(\mathrm{e}^{-\frac{t}{16}}\right)
\end{aligned}
$$

Which simplifies to

$$u_1 = \mathrm{e}^{-\frac{t}{16}} \cos\left(\frac{\sqrt{1023}\, t}{16}\right)$$

The second solution u_2 to the original ode is found using reduction of order

$$u_2 = u_1 \int \frac{e^{\int -\frac{B}{A}\, dt}}{u_1^2}\, dt$$

Substituting gives

$$\begin{aligned}
u_2 &= u_1 \int \frac{e^{\int -\frac{\frac{1}{8}}{1}\, dt}}{(u_1)^2}\, dt \\
&= u_1 \int \frac{e^{-\frac{t}{8}}}{(u_1)^2}\, dt \\
&= u_1 \left(\frac{16\sqrt{1023}\, \tan\left(\frac{\sqrt{1023}\, t}{16}\right)}{1023}\right)
\end{aligned}$$

Therefore the solution is

$$\begin{aligned}
u &= c_1 u_1 + c_2 u_2 \\
&= c_1 \left(\mathrm{e}^{-\frac{t}{16}} \cos\left(\frac{\sqrt{1023}\, t}{16}\right)\right) + c_2 \left(\mathrm{e}^{-\frac{t}{16}} \cos\left(\frac{\sqrt{1023}\, t}{16}\right) \left(\frac{16\sqrt{1023}\, \tan\left(\frac{\sqrt{1023}\, t}{16}\right)}{1023}\right)\right)
\end{aligned}$$

This is second order nonhomogeneous ODE. Let the solution be

$$u = u_h + u_p$$

Where u_h is the solution to the homogeneous ODE $Au''(t)+Bu'(t)+Cu(t) = 0$, and u_p is a particular solution to the nonhomogeneous ODE $Au''(t) + Bu'(t) + Cu(t) = f(t)$. u_h is the solution to

$$u'' + \frac{u'}{8} + 4u = 0$$

The homogeneous solution is found using the Kovacic algorithm which results in

$$u_h = c_1 \mathrm{e}^{-\frac{t}{16}} \cos\left(\frac{\sqrt{1023}\, t}{16}\right) + \frac{16 c_2 \mathrm{e}^{-\frac{t}{16}} \sin\left(\frac{\sqrt{1023}\, t}{16}\right) \sqrt{1023}}{1023}$$

The particular solution is now found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$3\cos(6t)$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$[\{\cos(6t), \sin(6t)\}]$$

While the set of the basis functions for the homogeneous solution found earlier is

$$\left\{ e^{-\frac{t}{16}} \cos\left(\frac{\sqrt{1023}\, t}{16}\right), \frac{16\, e^{-\frac{t}{16}} \sin\left(\frac{\sqrt{1023}\, t}{16}\right) \sqrt{1023}}{1023} \right\}$$

Since there is no duplication between the basis function in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis in the UC_set.

$$u_p = A_1 \cos(6t) + A_2 \sin(6t)$$

The unknowns $\{A_1, A_2\}$ are found by substituting the above trial solution u_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$-32A_1 \cos(6t) - 32A_2 \sin(6t) - \frac{3A_1 \sin(6t)}{4} + \frac{3A_2 \cos(6t)}{4} = 3\cos(6t)$$

Solving for the unknowns by comparing coefficients results in

$$\left[A_1 = -\frac{1536}{16393}, A_2 = \frac{36}{16393}\right]$$

Substituting the above back in the above trial solution u_p, gives the particular solution

$$u_p = -\frac{1536\cos(6t)}{16393} + \frac{36\sin(6t)}{16393}$$

Therefore the general solution is

$$\begin{aligned} u &= u_h + u_p \\ &= \left(c_1 e^{-\frac{t}{16}} \cos\left(\frac{\sqrt{1023}\, t}{16}\right) + \frac{16c_2 e^{-\frac{t}{16}} \sin\left(\frac{\sqrt{1023}\, t}{16}\right) \sqrt{1023}}{1023}\right) \\ &\quad + \left(-\frac{1536\cos(6t)}{16393} + \frac{36\sin(6t)}{16393}\right) \end{aligned}$$

Initial conditions are used to solve for the constants of integration.

Looking at the above solution

$$u = c_1 \mathrm{e}^{-\frac{t}{16}} \cos\left(\frac{\sqrt{1023}\, t}{16}\right) + \frac{16 c_2 \mathrm{e}^{-\frac{t}{16}} \sin\left(\frac{\sqrt{1023}\, t}{16}\right) \sqrt{1023}}{1023} - \frac{1536 \cos\left(6t\right)}{16393} + \frac{36 \sin\left(6t\right)}{16393} \tag{1}$$

Initial conditions are now substituted in the above solution. This will generate the required equations to solve for the integration constants. substituting $u = 2$ and $t = 0$ in the above gives

$$2 = c_1 - \frac{1536}{16393} \tag{1A}$$

Taking derivative of the solution gives

$$u' = -\frac{c_1 \mathrm{e}^{-\frac{t}{16}} \cos\left(\frac{\sqrt{1023}\, t}{16}\right)}{16} - \frac{c_1 \mathrm{e}^{-\frac{t}{16}} \sqrt{1023}\, \sin\left(\frac{\sqrt{1023}\, t}{16}\right)}{16} - \frac{c_2 \mathrm{e}^{-\frac{t}{16}} \sin\left(\frac{\sqrt{1023}\, t}{16}\right) \sqrt{1023}}{1023} + c_2 \mathrm{e}^{-\frac{t}{16}} \cos\left(\frac{\sqrt{10}}{1}\right.$$

substituting $u' = 0$ and $t = 0$ in the above gives

$$0 = \frac{216}{16393} + c_2 - \frac{c_1}{16} \tag{2A}$$

Equations $\{1\text{A}, 2\text{A}\}$ are now solved for $\{c_1, c_2\}$. Solving for the constants gives

$$c_1 = \frac{34322}{16393}$$
$$c_2 = \frac{15433}{131144}$$

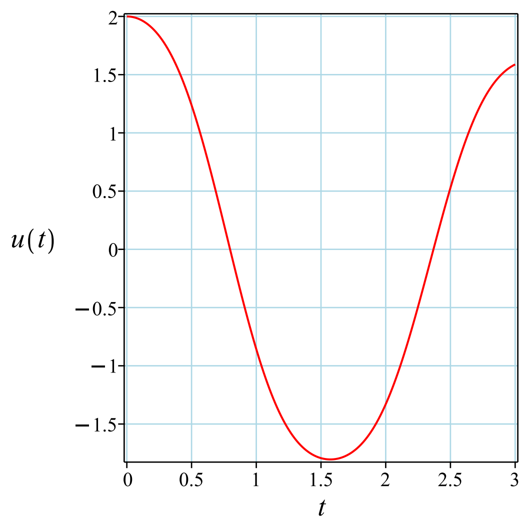
Substituting these values back in above solution results in

$$u = \frac{34322\, \mathrm{e}^{-\frac{t}{16}} \cos\left(\frac{\sqrt{1023}\, t}{16}\right)}{16393} + \frac{2806\, \mathrm{e}^{-\frac{t}{16}} \sin\left(\frac{\sqrt{1023}\, t}{16}\right) \sqrt{1023}}{1524549} - \frac{1536 \cos\left(6t\right)}{16393} + \frac{36 \sin\left(6t\right)}{16393}$$

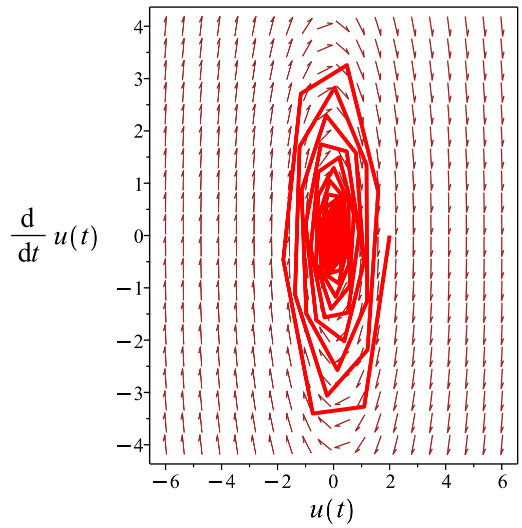
Summary

The solution(s) found are the following

$$u = \frac{34322\, \mathrm{e}^{-\frac{t}{16}} \cos\left(\frac{\sqrt{1023}\, t}{16}\right)}{16393} + \frac{2806\, \mathrm{e}^{-\frac{t}{16}} \sin\left(\frac{\sqrt{1023}\, t}{16}\right) \sqrt{1023}}{1524549} - \frac{1536 \cos\left(6t\right)}{16393} + \frac{36 \sin\left(6t\right)}{16393} \tag{1}$$



(a) Solution plot



(b) Slope field plot

Verification of solutions

$$u = \frac{34322\,\mathrm{e}^{-\frac{t}{16}} \cos\left(\frac{\sqrt{1023}\,t}{16}\right)}{16393} + \frac{2806\,\mathrm{e}^{-\frac{t}{16}} \sin\left(\frac{\sqrt{1023}\,t}{16}\right)\sqrt{1023}}{1524549} - \frac{1536\cos(6t)}{16393} + \frac{36\sin(6t)}{16393}$$

Verified OK.

12.3.4 Maple step by step solution

Let's solve

$$\left[u'' + \frac{u'}{8} + 4u = 3\cos(6t)\,, u(0) = 2, u'\Big|_{\{t=0\}} = 0\right]$$

- Highest derivative means the order of the ODE is 2

  u''

- Characteristic polynomial of homogeneous ODE

  $r^2 + \frac{1}{8}r + 4 = 0$

- Use quadratic formula to solve for r

  $r = \frac{\left(-\frac{1}{8}\right) \pm \left(\sqrt{-\frac{1023}{64}}\right)}{2}$

- Roots of the characteristic polynomial

  $r = \left(-\frac{1}{16} - \frac{\mathrm{I}\sqrt{1023}}{16}, -\frac{1}{16} + \frac{\mathrm{I}\sqrt{1023}}{16}\right)$

- 1st solution of the homogeneous ODE

  $u_1(t) = \mathrm{e}^{-\frac{t}{16}} \cos\left(\frac{\sqrt{1023}\, t}{16}\right)$

- 2nd solution of the homogeneous ODE

  $u_2(t) = \mathrm{e}^{-\frac{t}{16}} \sin\left(\frac{\sqrt{1023}\, t}{16}\right)$

- General solution of the ODE

  $u = c_1 u_1(t) + c_2 u_2(t) + u_p(t)$

- Substitute in solutions of the homogeneous ODE

  $u = c_1 \mathrm{e}^{-\frac{t}{16}} \cos\left(\frac{\sqrt{1023}\, t}{16}\right) + \mathrm{e}^{-\frac{t}{16}} c_2 \sin\left(\frac{\sqrt{1023}\, t}{16}\right) + u_p(t)$

- ☐ Find a particular solution $u_p(t)$ of the ODE
  - Use variation of parameters to find u_p here $f(t)$ is the forcing function

    $\left[u_p(t) = -u_1(t)\left(\int \frac{u_2(t) f(t)}{W(u_1(t), u_2(t))} dt\right) + u_2(t)\left(\int \frac{u_1(t) f(t)}{W(u_1(t), u_2(t))} dt\right), f(t) = 3\cos(6t)\right]$

  - Wronskian of solutions of the homogeneous equation

    $W(u_1(t), u_2(t)) = \begin{bmatrix} \mathrm{e}^{-\frac{t}{16}} \cos\left(\frac{\sqrt{1023}\, t}{16}\right) & \mathrm{e}^{-\frac{t}{16}} \sin\left(\frac{\sqrt{1023}\, t}{16}\right) \\ -\frac{\mathrm{e}^{-\frac{t}{16}} \cos\left(\frac{\sqrt{1023}\, t}{16}\right)}{16} - \frac{\mathrm{e}^{-\frac{t}{16}} \sin\left(\frac{\sqrt{1023}\, t}{16}\right)\sqrt{1023}}{16} & -\frac{\mathrm{e}^{-\frac{t}{16}} \sin\left(\frac{\sqrt{1023}\, t}{16}\right)}{16} + \frac{\mathrm{e}^{-\frac{t}{16}} \sqrt{1023} \cos\left(\frac{\sqrt{1023}\, t}{16}\right)}{16} \end{bmatrix}$

  - Compute Wronskian

    $W(u_1(t), u_2(t)) = \frac{\sqrt{1023}\, \mathrm{e}^{-\frac{t}{8}}}{16}$

  - Substitute functions into equation for $u_p(t)$

    $u_p(t) = -\frac{16\, \mathrm{e}^{-\frac{t}{16}} \sqrt{1023} \left(\cos\left(\frac{\sqrt{1023}\, t}{16}\right)\left(\int \cos(6t) \mathrm{e}^{\frac{t}{16}} \sin\left(\frac{\sqrt{1023}\, t}{16}\right) dt\right) - \sin\left(\frac{\sqrt{1023}\, t}{16}\right)\left(\int \cos(6t) \mathrm{e}^{\frac{t}{16}} \cos\left(\frac{\sqrt{1023}\, t}{16}\right) dt\right)\right)}{341}$

  - Compute integrals

    $u_p(t) = -\frac{1536 \cos(6t)}{16393} + \frac{36 \sin(6t)}{16393}$

- Substitute particular solution into general solution to ODE

  $u = c_1 \mathrm{e}^{-\frac{t}{16}} \cos\left(\frac{\sqrt{1023}\, t}{16}\right) + \mathrm{e}^{-\frac{t}{16}} c_2 \sin\left(\frac{\sqrt{1023}\, t}{16}\right) - \frac{1536 \cos(6t)}{16393} + \frac{36 \sin(6t)}{16393}$

- ☐ Check validity of solution $u = c_1 \mathrm{e}^{-\frac{t}{16}} \cos\left(\frac{\sqrt{1023}\, t}{16}\right) + \mathrm{e}^{-\frac{t}{16}} c_2 \sin\left(\frac{\sqrt{1023}\, t}{16}\right) - \frac{1536 \cos(6t)}{16393} + \frac{36 \sin(6t)}{16393}$
  - Use initial condition $u(0) = 2$

    $2 = c_1 - \frac{1536}{16393}$

  - Compute derivative of the solution

$$u' = -\frac{c_1 e^{-\frac{t}{16}} \cos\left(\frac{\sqrt{1023}\, t}{16}\right)}{16} - \frac{c_1 e^{-\frac{t}{16}} \sqrt{1023} \sin\left(\frac{\sqrt{1023}\, t}{16}\right)}{16} - \frac{e^{-\frac{t}{16}} c_2 \sin\left(\frac{\sqrt{1023}\, t}{16}\right)}{16} + \frac{e^{-\frac{t}{16}} c_2 \sqrt{1023} \cos\left(\frac{\sqrt{1023}\, t}{16}\right)}{16} + \frac{9216}{16}$$

- Use the initial condition $u'\Big|_{\{t=0\}} = 0$

$$0 = -\frac{c_1}{16} + \frac{\sqrt{1023}\, c_2}{16} + \frac{216}{16393}$$

- Solve for c_1 and c_2

$$\left\{c_1 = \frac{34322}{16393}, c_2 = \frac{2806\sqrt{1023}}{1524549}\right\}$$

- Substitute constant values into general solution and simplify

$$u = \frac{34322\, e^{-\frac{t}{16}} \cos\left(\frac{\sqrt{1023}\, t}{16}\right)}{16393} + \frac{2806\, e^{-\frac{t}{16}} \sin\left(\frac{\sqrt{1023}\, t}{16}\right)\sqrt{1023}}{1524549} - \frac{1536\cos(6t)}{16393} + \frac{36\sin(6t)}{16393}$$

- Solution to the IVP

$$u = \frac{34322\, e^{-\frac{t}{16}} \cos\left(\frac{\sqrt{1023}\, t}{16}\right)}{16393} + \frac{2806\, e^{-\frac{t}{16}} \sin\left(\frac{\sqrt{1023}\, t}{16}\right)\sqrt{1023}}{1524549} - \frac{1536\cos(6t)}{16393} + \frac{36\sin(6t)}{16393}$$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 2; linear nonhomogeneous with symmetry [0,1]
trying a double symmetry of the form [xi=0, eta=F(x)]
-> Try solving first the homogeneous part of the ODE
   checking if the LODE has constant coefficients
   <- constant coefficients successful
<- solving first the homogeneous part of the ODE successful`
```

✓ Solution by Maple

Time used: 0.031 (sec). Leaf size: 46

```
dsolve([diff(u(t),t$2)+125/1000*diff(u(t),t)+4*u(t) = 3*cos(6*t),u(0) = 2, D(u)(0) = 0],u(t)
```

$$u(t) = \frac{2806\, e^{-\frac{t}{16}} \sqrt{1023} \sin\left(\frac{\sqrt{1023}\, t}{16}\right)}{1524549} + \frac{34322\, e^{-\frac{t}{16}} \cos\left(\frac{\sqrt{1023}\, t}{16}\right)}{16393} + \frac{36\sin(6t)}{16393} - \frac{1536\cos(6t)}{16393}$$

✓ Solution by Mathematica

Time used: 0.032 (sec). Leaf size: 74

```
DSolve[{u''[t]+125/1000*u'[t]+4*u[t] ==3*Cos[6*t],{u[0]==0,u'[0]==0}},u[t],t,IncludeSingular
```

$$u(t) \to -\frac{4e^{-t/16}\left(-3069e^{t/16}\sin(6t)+160\sqrt{1023}\sin\left(\frac{\sqrt{1023}t}{16}\right)+130944e^{t/16}\cos(6t)-130944\cos\left(\frac{\sqrt{1023}t}{16}\right)\right)}{5590013}$$

12.4 problem 24

Internal problem ID [710]
Internal file name [OUTPUT/710_Sunday_June_05_2022_01_47_31_AM_72098390/index.tex]

Book: Elementary differential equations and boundary value problems, 10th ed., Boyce and DiPrima
Section: Chapter 3, Second order linear equations, 3.7 Forced Vibrations. page 217
Problem number: 24.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"unknown"**

Maple gives the following as the ode type

`[NONE]`

Unable to solve or complete the solution.

$$\boxed{u'' + u' + \frac{u^3}{5} = \cos(t)}$$

With initial conditions

$$[u(0) = 2, u'(0) = 0]$$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying 2nd order Liouville
trying 2nd order WeierstrassP
trying 2nd order JacobiSN
differential order: 2; trying a linearization to 3rd order
trying 2nd order ODE linearizable_by_differentiation
trying 2nd order, 2 integrating factors of the form mu(x,y)
trying differential order: 2; missing variables
-> trying 2nd order, dynamical_symmetries, fully reducible to Abel through one integrating f
   --- trying a change of variables {x -> y(x), y(x) -> x} and re-entering methods for dynam
   -> trying 2nd order, dynamical_symmetries, fully reducible to Abel through one integratin
trying 2nd order, integrating factors of the form mu(x,y)/(y)^n, only the singular cases
trying symmetries linear in x and y(x)
trying differential order: 2; exact nonlinear
trying 2nd order, integrating factor of the form mu(y)
trying 2nd order, integrating factor of the form mu(x,y)
trying 2nd order, integrating factor of the form mu(x,y)/(y)^n, only the general case
trying 2nd order, integrating factor of the form mu(y,y)
trying differential order: 2; mu polynomial in y
trying 2nd order, integrating factor of the form mu(x,y)
differential order: 2; looking for linear symmetries
-> trying 2nd order, the S-function method
   -> trying a change of variables {x -> y(x), y(x) -> x} and re-entering methods for the S-
   -> trying 2nd order, the S-function method
   -> trying 2nd order, No Point Symmetries Class V
      --- trying a change of variables {x -> y(x), y(x) -> x} and re-entering methods for dy
      -> trying 2nd order, No Point Symmetries Class V
   -> trying 2nd order, No Point Symmetries Class V
      --- trying a change of variables {x -> y(x), y(x) -> x} and re-entering methods for dy
      -> trying 2nd order, No Point Symmetries Class V
   -> trying 2nd order, No Point Symmetries Class V
      --- trying a change of variables {x -> y(x), y(x) -> x} and re-entering methods for dy
      -> trying 2nd order, No Point Symmetries Class V
trying 2nd order, integrating factor of the form mu(x,y)/(y)^n, only the general case
-> trying 2nd order, dynamical_symmetries, only a reduction of order through one integrating
   --- trying a change of variables {x -> y(x), y(x) -> x} and re-entering methods for dynam
   -> trying 2nd order, dynamical_symmetries, only a reduction of order through one integrat
--- Trying Lie symmetry methods, 2nd order ---
`, `-> Computing symmetries using: way = 3
`, `-> Computing symmetries using: way = 5
`, `-> Computing symmetries using: way = formal`
```

✗ Solution by Maple

```
dsolve([diff(u(t),t$2)+diff(u(t),t)+1/5*u(t)^3 = cos(t),u(0) = 2, D(u)(0) = 0],u(t), singsol
```

No solution found

✗ Solution by Mathematica

Time used: 0.0 (sec). Leaf size: 0

```
DSolve[{u''[t]+u'[t]+1/5*u[t]^3 ==3*Cos[t],{u[0]==0,u'[0]==0}},u[t],t,IncludeSingularSolutio
```

Not solved

13 Chapter 5.2, Series Solutions Near an Ordinary Point, Part I. page 263

13.1 problem 1

Internal problem ID [711]
Internal file name [`OUTPUT/711_Sunday_June_05_2022_01_47_34_AM_92887022/index.tex`]

Book: Elementary differential equations and boundary value problems, 10th ed., Boyce and DiPrima
Section: Chapter 5.2, Series Solutions Near an Ordinary Point, Part I. page 263
Problem number: 1.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second_order_linear_constant_coeff", "second_order_ode_can_be_made_integrable", "second order series method. Ordinary point", "second order series method. Taylor series method"**

Maple gives the following as the ode type

```
[[_2nd_order, _missing_x]]
```

$$\boxed{y'' - y = 0}$$

With the expansion point for the power series method at $x = 0$.

Solving ode using Taylor series method. This gives review on how the Taylor series method works for solving second order ode.

Let

$$y'' = f(x, y, y')$$

Assuming expansion is at $x_0 = 0$ (we can always shift the actual expansion point to 0 by change of variables) and assuming $f(x, y, y')$ is analytic at x_0 which must be the case for an ordinary point. Let initial conditions be $y(x_0) = y_0$ and $y'(x_0) = y'_0$. Using Taylor series gives

$$\begin{aligned} y(x) &= y(x_0) + (x - x_0)\, y'(x_0) + \frac{(x - x_0)^2}{2} y''(x_0) + \frac{(x - x_0)^3}{3!} y'''(x_0) + \cdots \\ &= y_0 + x y'_0 + \frac{x^2}{2} f|_{x_0, y_0, y'_0} + \frac{x^3}{3!} f'|_{x_0, y_0, y'_0} + \cdots \\ &= y_0 + x y'_0 + \sum_{n=0}^{\infty} \frac{x^{n+2}}{(n+2)\,!} \frac{d^n f}{dx^n}\bigg|_{x_0, y_0, y'_0} \end{aligned}$$

But

$$
\begin{aligned}
\frac{df}{dx} &= \frac{\partial f}{\partial x}\frac{dx}{dx} + \frac{\partial f}{\partial y}\frac{dy}{dx} + \frac{\partial f}{\partial y'}\frac{dy'}{dx} && (1) \\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} y' + \frac{\partial f}{\partial y'} y'' && (744) \\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} y' + \frac{\partial f}{\partial y'} f && (745) \\
\frac{d^2 f}{dx^2} &= \frac{d}{dx}\left(\frac{df}{dx}\right) \\
&= \frac{\partial}{\partial x}\left(\frac{df}{dx}\right) + \frac{\partial}{\partial y}\left(\frac{df}{dx}\right) y' + \frac{\partial}{\partial y'}\left(\frac{df}{dx}\right) f && (2) \\
\frac{d^3 f}{dx^3} &= \frac{d}{dx}\left(\frac{d^2 f}{dx^2}\right) \\
&= \frac{\partial}{\partial x}\left(\frac{d^2 f}{dx^2}\right) + \left(\frac{\partial}{\partial y}\frac{d^2 f}{dx^2}\right) y' + \frac{\partial}{\partial y'}\left(\frac{d^2 f}{dx^2}\right) f && (3) \\
&\vdots
\end{aligned}
$$

And so on. Hence if we name $F_0 = f(x,y,y')$ then the above can be written as

$$\begin{aligned}
F_0 &= f(x,y,y') \qquad (4)\\
F_1 &= \frac{df}{dx}\\
&= \frac{dF_0}{dx}\\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}y' + \frac{\partial f}{\partial y'}y''\\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}y' + \frac{\partial f}{\partial y'}f \qquad (5)\\
&= \frac{\partial F_0}{\partial x} + \frac{\partial F_0}{\partial y}y' + \frac{\partial F_0}{\partial y'}F_0\\
F_2 &= \frac{d}{dx}\left(\frac{d}{dx}f\right)\\
&= \frac{d}{dx}(F_1)\\
&= \frac{\partial}{\partial x}F_1 + \left(\frac{\partial F_1}{\partial y}\right)y' + \left(\frac{\partial F_1}{\partial y'}\right)y''\\
&= \frac{\partial}{\partial x}F_1 + \left(\frac{\partial F_1}{\partial y}\right)y' + \left(\frac{\partial F_1}{\partial y'}\right)F_0\\
&\vdots\\
F_n &= \frac{d}{dx}(F_{n-1})\\
&= \frac{\partial}{\partial x}F_{n-1} + \left(\frac{\partial F_{n-1}}{\partial y}\right)y' + \left(\frac{\partial F_{n-1}}{\partial y'}\right)y''\\
&= \frac{\partial}{\partial x}F_{n-1} + \left(\frac{\partial F_{n-1}}{\partial y}\right)y' + \left(\frac{\partial F_{n-1}}{\partial y'}\right)F_0 \qquad (6)
\end{aligned}$$

Therefore (6) can be used from now on along with

$$y(x) = y_0 + xy_0' + \sum_{n=0}^{\infty} \frac{x^{n+2}}{(n+2)!} F_n|_{x_0,y_0,y_0'} \qquad (7)$$

To find $y(x)$ series solution around $x=0$. Hence

$$
\begin{aligned}
F_0 &= y \\
F_1 &= \frac{dF_0}{dx} \\
&= \frac{\partial F_0}{\partial x} + \frac{\partial F_0}{\partial y} y' + \frac{\partial F_0}{\partial y'} F_0 \\
&= y' \\
F_2 &= \frac{dF_1}{dx} \\
&= \frac{\partial F_1}{\partial x} + \frac{\partial F_1}{\partial y} y' + \frac{\partial F_1}{\partial y'} F_1 \\
&= y \\
F_3 &= \frac{dF_2}{dx} \\
&= \frac{\partial F_2}{\partial x} + \frac{\partial F_2}{\partial y} y' + \frac{\partial F_2}{\partial y'} F_2 \\
&= y' \\
F_4 &= \frac{dF_3}{dx} \\
&= \frac{\partial F_3}{\partial x} + \frac{\partial F_3}{\partial y} y' + \frac{\partial F_3}{\partial y'} F_3 \\
&= y
\end{aligned}
$$

And so on. Evaluating all the above at initial conditions $x=0$ and $y(0)=y(0)$ and $y'(0)=y'(0)$ gives

$$
\begin{aligned}
F_0 &= y(0) \\
F_1 &= y'(0) \\
F_2 &= y(0) \\
F_3 &= y'(0) \\
F_4 &= y(0)
\end{aligned}
$$

Substituting all the above in (7) and simplifying gives the solution as

$$
y = \left(1 + \frac{1}{2}x^2 + \frac{1}{24}x^4 + \frac{1}{720}x^6\right) y(0) + \left(x + \frac{1}{6}x^3 + \frac{1}{120}x^5\right) y'(0) + O\left(x^6\right)
$$

Since the expansion point $x=0$ is an ordinary, we can also solve this using standard

power series Let the solution be represented as power series of the form

$$y = \sum_{n=0}^{\infty} a_n x^n$$

Then

$$y' = \sum_{n=1}^{\infty} n a_n x^{n-1}$$
$$y'' = \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2}$$

Substituting the above back into the ode gives

$$\sum_{n=2}^{\infty} n(n-1) a_n x^{n-2} = \sum_{n=0}^{\infty} a_n x^n \tag{1}$$

Which simplifies to

$$\left(\sum_{n=2}^{\infty} n(n-1) a_n x^{n-2}\right) + \sum_{n=0}^{\infty} (-a_n x^n) = 0 \tag{2}$$

The next step is to make all powers of x be n in each summation term. Going over each summation term above with power of x in it which is not already x^n and adjusting the power and the corresponding index gives

$$\sum_{n=2}^{\infty} n(n-1) a_n x^{n-2} = \sum_{n=0}^{\infty} (n+2) a_{n+2}(n+1) x^n$$

Substituting all the above in Eq (2) gives the following equation where now all powers of x are the same and equal to n.

$$\left(\sum_{n=0}^{\infty} (n+2) a_{n+2}(n+1) x^n\right) + \sum_{n=0}^{\infty} (-a_n x^n) = 0 \tag{3}$$

For $0 \leq n$, the recurrence equation is

$$(n+2) a_{n+2}(n+1) - a_n = 0 \tag{4}$$

Solving for a_{n+2}, gives

$$a_{n+2} = \frac{a_n}{(n+2)(n+1)} \tag{5}$$

For $n = 0$ the recurrence equation gives

$$2a_2 - a_0 = 0$$

Which after substituting the earlier terms found becomes

$$a_2 = \frac{a_0}{2}$$

For $n = 1$ the recurrence equation gives

$$6a_3 - a_1 = 0$$

Which after substituting the earlier terms found becomes

$$a_3 = \frac{a_1}{6}$$

For $n = 2$ the recurrence equation gives

$$12a_4 - a_2 = 0$$

Which after substituting the earlier terms found becomes

$$a_4 = \frac{a_0}{24}$$

For $n = 3$ the recurrence equation gives

$$20a_5 - a_3 = 0$$

Which after substituting the earlier terms found becomes

$$a_5 = \frac{a_1}{120}$$

For $n = 4$ the recurrence equation gives

$$30a_6 - a_4 = 0$$

Which after substituting the earlier terms found becomes

$$a_6 = \frac{a_0}{720}$$

For $n = 5$ the recurrence equation gives

$$42a_7 - a_5 = 0$$

Which after substituting the earlier terms found becomes

$$a_7 = \frac{a_1}{5040}$$

And so on. Therefore the solution is

$$\begin{aligned} y &= \sum_{n=0}^{\infty} a_n x^n \\ &= a_3 x^3 + a_2 x^2 + a_1 x + a_0 + \dots \end{aligned}$$

Substituting the values for a_n found above, the solution becomes

$$y = a_0 + a_1 x + \frac{1}{2} a_0 x^2 + \frac{1}{6} a_1 x^3 + \frac{1}{24} a_0 x^4 + \frac{1}{120} a_1 x^5 + \dots$$

Collecting terms, the solution becomes

$$y = \left(1 + \frac{1}{2}x^2 + \frac{1}{24}x^4\right) a_0 + \left(x + \frac{1}{6}x^3 + \frac{1}{120}x^5\right) a_1 + O(x^6) \tag{3}$$

At $x = 0$ the solution above becomes

$$y = \left(1 + \frac{1}{2}x^2 + \frac{1}{24}x^4\right) c_1 + \left(x + \frac{1}{6}x^3 + \frac{1}{120}x^5\right) c_2 + O(x^6)$$

Summary

The solution(s) found are the following

$$y = \left(1 + \frac{1}{2}x^2 + \frac{1}{24}x^4 + \frac{1}{720}x^6\right) y(0) + \left(x + \frac{1}{6}x^3 + \frac{1}{120}x^5\right) y'(0) + O(x^6) \tag{1}$$

$$y = \left(1 + \frac{1}{2}x^2 + \frac{1}{24}x^4\right) c_1 + \left(x + \frac{1}{6}x^3 + \frac{1}{120}x^5\right) c_2 + O(x^6) \tag{2}$$

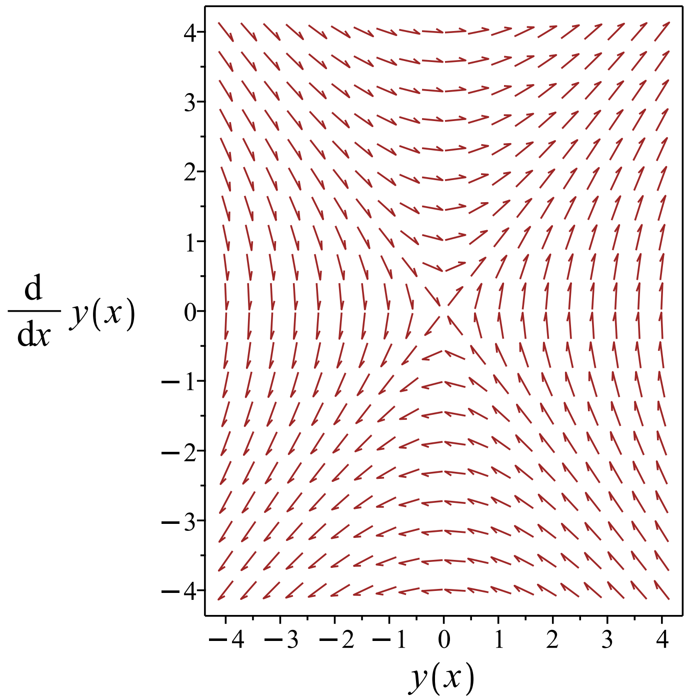


Figure 502: Slope field plot

Verification of solutions

$$y = \left(1 + \frac{1}{2}x^2 + \frac{1}{24}x^4 + \frac{1}{720}x^6\right) y(0) + \left(x + \frac{1}{6}x^3 + \frac{1}{120}x^5\right) y'(0) + O(x^6)$$

Verified OK.

$$y = \left(1 + \frac{1}{2}x^2 + \frac{1}{24}x^4\right) c_1 + \left(x + \frac{1}{6}x^3 + \frac{1}{120}x^5\right) c_2 + O(x^6)$$

Verified OK.

13.1.1 Maple step by step solution

Let's solve

$y'' = y$

- Highest derivative means the order of the ODE is 2

  y''

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y'' - y = 0$

- Characteristic polynomial of ODE

  $r^2 - 1 = 0$
- Factor the characteristic polynomial

  $(r-1)(r+1) = 0$
- Roots of the characteristic polynomial

  $r = (-1, 1)$
- 1st solution of the ODE

  $y_1(x) = e^{-x}$
- 2nd solution of the ODE

  $y_2(x) = e^{x}$
- General solution of the ODE

  $y = c_1 y_1(x) + c_2 y_2(x)$
- Substitute in solutions

  $y = c_1 e^{-x} + c_2 e^{x}$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
<- constant coefficients successful`
```

✓ Solution by Maple
Time used: 0.0 (sec). Leaf size: 34

```
Order:=6;
dsolve(diff(y(x),x$2)-y(x)=0,y(x),type='series',x=0);
```

$$y(x) = \left(1 + \frac{1}{2}x^2 + \frac{1}{24}x^4\right) y(0) + \left(x + \frac{1}{6}x^3 + \frac{1}{120}x^5\right) D(y)(0) + O(x^6)$$

✓ Solution by Mathematica

Time used: 0.001 (sec). Leaf size: 42

```
AsymptoticDSolveValue[y''[x]-y[x]==0,y[x],{x,0,5}]
```

$$y(x) \to c_2\left(\frac{x^5}{120}+\frac{x^3}{6}+x\right)+c_1\left(\frac{x^4}{24}+\frac{x^2}{2}+1\right)$$

13.2 problem 2

Internal problem ID [712]
Internal file name [`OUTPUT/712_Sunday_June_05_2022_01_47_35_AM_89001034/index.tex`]

Book: Elementary differential equations and boundary value problems, 10th ed., Boyce and DiPrima
Section: Chapter 5.2, Series Solutions Near an Ordinary Point, Part I. page 263
Problem number: 2.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"exact linear second order ode", "second_order_integrable_as_is", "second order series method. Ordinary point", "second order series method. Taylor series method"**

Maple gives the following as the ode type

```
[[_2nd_order, _exact, _linear, _homogeneous]]
```

$$\boxed{y'' - y'x - y = 0}$$

With the expansion point for the power series method at $x = 0$.

Solving ode using Taylor series method. This gives review on how the Taylor series method works for solving second order ode.

Let

$$y'' = f(x, y, y')$$

Assuming expansion is at $x_0 = 0$ (we can always shift the actual expansion point to 0 by change of variables) and assuming $f(x, y, y')$ is analytic at x_0 which must be the case for an ordinary point. Let initial conditions be $y(x_0) = y_0$ and $y'(x_0) = y'_0$. Using Taylor series gives

$$\begin{aligned} y(x) &= y(x_0) + (x - x_0)\, y'(x_0) + \frac{(x - x_0)^2}{2} y''(x_0) + \frac{(x - x_0)^3}{3!} y'''(x_0) + \cdots \\ &= y_0 + x y'_0 + \frac{x^2}{2} f|_{x_0, y_0, y'_0} + \frac{x^3}{3!} f'|_{x_0, y_0, y'_0} + \cdots \\ &= y_0 + x y'_0 + \sum_{n=0}^{\infty} \frac{x^{n+2}}{(n+2)!} \frac{d^n f}{dx^n}\bigg|_{x_0, y_0, y'_0} \end{aligned}$$

But

$$
\begin{aligned}
\frac{df}{dx} &= \frac{\partial f}{\partial x}\frac{dx}{dx} + \frac{\partial f}{\partial y}\frac{dy}{dx} + \frac{\partial f}{\partial y'}\frac{dy'}{dx} && (1) \\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}y' + \frac{\partial f}{\partial y'}y'' && (747) \\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}y' + \frac{\partial f}{\partial y'}f && (748) \\
\frac{d^2 f}{dx^2} &= \frac{d}{dx}\left(\frac{df}{dx}\right) \\
&= \frac{\partial}{\partial x}\left(\frac{df}{dx}\right) + \frac{\partial}{\partial y}\left(\frac{df}{dx}\right)y' + \frac{\partial}{\partial y'}\left(\frac{df}{dx}\right)f && (2) \\
\frac{d^3 f}{dx^3} &= \frac{d}{dx}\left(\frac{d^2 f}{dx^2}\right) \\
&= \frac{\partial}{\partial x}\left(\frac{d^2 f}{dx^2}\right) + \left(\frac{\partial}{\partial y}\frac{d^2 f}{dx^2}\right)y' + \frac{\partial}{\partial y'}\left(\frac{d^2 f}{dx^2}\right)f && (3) \\
&\vdots
\end{aligned}
$$

And so on. Hence if we name $F_0 = f(x,y,y')$ then the above can be written as

$$
\begin{aligned}
F_0 &= f(x,y,y') && (4)\\
F_1 &= \frac{df}{dx}\\
&= \frac{dF_0}{dx}\\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}y' + \frac{\partial f}{\partial y'}y''\\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}y' + \frac{\partial f}{\partial y'}f && (5)\\
&= \frac{\partial F_0}{\partial x} + \frac{\partial F_0}{\partial y}y' + \frac{\partial F_0}{\partial y'}F_0\\
F_2 &= \frac{d}{dx}\left(\frac{d}{dx}f\right)\\
&= \frac{d}{dx}(F_1)\\
&= \frac{\partial}{\partial x}F_1 + \left(\frac{\partial F_1}{\partial y}\right)y' + \left(\frac{\partial F_1}{\partial y'}\right)y''\\
&= \frac{\partial}{\partial x}F_1 + \left(\frac{\partial F_1}{\partial y}\right)y' + \left(\frac{\partial F_1}{\partial y'}\right)F_0\\
&\vdots\\
F_n &= \frac{d}{dx}(F_{n-1})\\
&= \frac{\partial}{\partial x}F_{n-1} + \left(\frac{\partial F_{n-1}}{\partial y}\right)y' + \left(\frac{\partial F_{n-1}}{\partial y'}\right)y''\\
&= \frac{\partial}{\partial x}F_{n-1} + \left(\frac{\partial F_{n-1}}{\partial y}\right)y' + \left(\frac{\partial F_{n-1}}{\partial y'}\right)F_0 && (6)
\end{aligned}
$$

Therefore (6) can be used from now on along with

$$
y(x) = y_0 + xy'_0 + \sum_{n=0}^{\infty} \frac{x^{n+2}}{(n+2)!} F_n\big|_{x_0,y_0,y'_0} \tag{7}
$$

To find $y(x)$ series solution around $x = 0$. Hence

$$
\begin{aligned}
F_0 &= y'x + y \\
F_1 &= \frac{dF_0}{dx} \\
&= \frac{\partial F_0}{\partial x} + \frac{\partial F_0}{\partial y} y' + \frac{\partial F_0}{\partial y'} F_0 \\
&= y'x^2 + yx + 2y' \\
F_2 &= \frac{dF_1}{dx} \\
&= \frac{\partial F_1}{\partial x} + \frac{\partial F_1}{\partial y} y' + \frac{\partial F_1}{\partial y'} F_1 \\
&= \left(x^3 + 5x\right) y' + y\left(x^2 + 3\right) \\
F_3 &= \frac{dF_2}{dx} \\
&= \frac{\partial F_2}{\partial x} + \frac{\partial F_2}{\partial y} y' + \frac{\partial F_2}{\partial y'} F_2 \\
&= \left(x^4 + 9x^2 + 8\right) y' + xy\left(x^2 + 7\right) \\
F_4 &= \frac{dF_3}{dx} \\
&= \frac{\partial F_3}{\partial x} + \frac{\partial F_3}{\partial y} y' + \frac{\partial F_3}{\partial y'} F_3 \\
&= \left(x^5 + 14x^3 + 33x\right) y' + y\left(x^4 + 12x^2 + 15\right)
\end{aligned}
$$

And so on. Evaluating all the above at initial conditions $x = 0$ and $y(0) = y(0)$ and $y'(0) = y'(0)$ gives

$$
\begin{aligned}
F_0 &= y(0) \\
F_1 &= 2y'(0) \\
F_2 &= 3y(0) \\
F_3 &= 8y'(0) \\
F_4 &= 15y(0)
\end{aligned}
$$

Substituting all the above in (7) and simplifying gives the solution as

$$
y = \left(1 + \frac{1}{2}x^2 + \frac{1}{8}x^4 + \frac{1}{48}x^6\right) y(0) + \left(x + \frac{1}{3}x^3 + \frac{1}{15}x^5\right) y'(0) + O\left(x^6\right)
$$

Since the expansion point $x = 0$ is an ordinary, we can also solve this using standard

power series Let the solution be represented as power series of the form

$$y=\sum_{n=0}^{\infty} a_n x^n$$

Then

$$y'=\sum_{n=1}^{\infty} n a_n x^{n-1}$$
$$y''=\sum_{n=2}^{\infty} n(n-1)\, a_n x^{n-2}$$

Substituting the above back into the ode gives

$$\sum_{n=2}^{\infty} n(n-1)\, a_n x^{n-2}=\left(\sum_{n=1}^{\infty} n a_n x^{n-1}\right) x+\left(\sum_{n=0}^{\infty} a_n x^n\right) \tag{1}$$

Which simplifies to

$$\left(\sum_{n=2}^{\infty} n(n-1)\, a_n x^{n-2}\right)+\sum_{n=1}^{\infty}\left(-n\, x^n a_n\right)+\sum_{n=0}^{\infty}\left(-a_n x^n\right)=0 \tag{2}$$

The next step is to make all powers of x be n in each summation term. Going over each summation term above with power of x in it which is not already x^n and adjusting the power and the corresponding index gives

$$\sum_{n=2}^{\infty} n(n-1)\, a_n x^{n-2}=\sum_{n=0}^{\infty}(n+2)\, a_{n+2}(n+1)\, x^n$$

Substituting all the above in Eq (2) gives the following equation where now all powers of x are the same and equal to n.

$$\left(\sum_{n=0}^{\infty}(n+2)\, a_{n+2}(n+1)\, x^n\right)+\sum_{n=1}^{\infty}\left(-n\, x^n a_n\right)+\sum_{n=0}^{\infty}\left(-a_n x^n\right)=0 \tag{3}$$

$n=0$ gives

$$2a_2-a_0=0$$

$$a_2=\frac{a_0}{2}$$

For $1 \leq n$, the recurrence equation is

$$(n+2)\, a_{n+2}(n+1) - n a_n - a_n = 0 \tag{4}$$

Solving for a_{n+2}, gives

$$a_{n+2} = \frac{a_n}{n+2} \tag{5}$$

For $n = 1$ the recurrence equation gives

$$6a_3 - 2a_1 = 0$$

Which after substituting the earlier terms found becomes

$$a_3 = \frac{a_1}{3}$$

For $n = 2$ the recurrence equation gives

$$12a_4 - 3a_2 = 0$$

Which after substituting the earlier terms found becomes

$$a_4 = \frac{a_0}{8}$$

For $n = 3$ the recurrence equation gives

$$20a_5 - 4a_3 = 0$$

Which after substituting the earlier terms found becomes

$$a_5 = \frac{a_1}{15}$$

For $n = 4$ the recurrence equation gives

$$30a_6 - 5a_4 = 0$$

Which after substituting the earlier terms found becomes

$$a_6 = \frac{a_0}{48}$$

For $n = 5$ the recurrence equation gives

$$42a_7 - 6a_5 = 0$$

Which after substituting the earlier terms found becomes

$$a_7 = \frac{a_1}{105}$$

And so on. Therefore the solution is

$$y = \sum_{n=0}^{\infty} a_n x^n$$
$$= a_3 x^3 + a_2 x^2 + a_1 x + a_0 + \dots$$

Substituting the values for a_n found above, the solution becomes

$$y = a_0 + a_1 x + \frac{1}{2} a_0 x^2 + \frac{1}{3} a_1 x^3 + \frac{1}{8} a_0 x^4 + \frac{1}{15} a_1 x^5 + \dots$$

Collecting terms, the solution becomes

$$y = \left(1 + \frac{1}{2}x^2 + \frac{1}{8}x^4\right) a_0 + \left(x + \frac{1}{3}x^3 + \frac{1}{15}x^5\right) a_1 + O(x^6) \tag{3}$$

At $x = 0$ the solution above becomes

$$y = \left(1 + \frac{1}{2}x^2 + \frac{1}{8}x^4\right) c_1 + \left(x + \frac{1}{3}x^3 + \frac{1}{15}x^5\right) c_2 + O(x^6)$$

Summary

The solution(s) found are the following

$$y = \left(1 + \frac{1}{2}x^2 + \frac{1}{8}x^4 + \frac{1}{48}x^6\right) y(0) + \left(x + \frac{1}{3}x^3 + \frac{1}{15}x^5\right) y'(0) + O(x^6) \tag{1}$$

$$y = \left(1 + \frac{1}{2}x^2 + \frac{1}{8}x^4\right) c_1 + \left(x + \frac{1}{3}x^3 + \frac{1}{15}x^5\right) c_2 + O(x^6) \tag{2}$$

Verification of solutions

$$y = \left(1 + \frac{1}{2}x^2 + \frac{1}{8}x^4 + \frac{1}{48}x^6\right) y(0) + \left(x + \frac{1}{3}x^3 + \frac{1}{15}x^5\right) y'(0) + O(x^6)$$

Verified OK.

$$y = \left(1 + \frac{1}{2}x^2 + \frac{1}{8}x^4\right) c_1 + \left(x + \frac{1}{3}x^3 + \frac{1}{15}x^5\right) c_2 + O(x^6)$$

Verified OK.

13.2.1 Maple step by step solution

Let's solve

$y'' = y'x + y$

- Highest derivative means the order of the ODE is 2

y''

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

$y'' - y'x - y = 0$

- Assume series solution for y

$y = \sum_{k=0}^{\infty} a_k x^k$

□ Rewrite DE with series expansions

  ◦ Convert $x \cdot y'$ to series expansion

  $x \cdot y' = \sum_{k=0}^{\infty} a_k k\, x^k$

  ◦ Convert y'' to series expansion

  $y'' = \sum_{k=2}^{\infty} a_k k(k-1)\, x^{k-2}$

  ◦ Shift index using $k->k+2$

  $y'' = \sum_{k=0}^{\infty} a_{k+2}(k+2)\,(k+1)\, x^k$

Rewrite DE with series expansions

$\sum_{k=0}^{\infty} \left(a_{k+2}(k+2)\,(k+1) - a_k(k+1)\right) x^k = 0$

- Each term in the series must be 0, giving the recursion relation

$(k+1)\left(a_{k+2}(k+2) - a_k\right) = 0$

- Recursion relation that defines the series solution to the ODE

$\left[y = \sum_{k=0}^{\infty} a_k x^k, a_{k+2} = \frac{a_k}{k+2}\right]$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
<- linear_1 successful`
```

✓ Solution by Maple

Time used: 0.015 (sec). Leaf size: 34

```
Order:=6;
dsolve(diff(y(x),x$2)-x*diff(y(x),x)-y(x)=0,y(x),type='series',x=0);
```

$$y(x) = \left(1+\frac{1}{2}x^2+\frac{1}{8}x^4\right)y(0)+\left(x+\frac{1}{3}x^3+\frac{1}{15}x^5\right)D(y)(0)+O(x^6)$$

✓ Solution by Mathematica

Time used: 0.001 (sec). Leaf size: 42

```
AsymptoticDSolveValue[y''[x]-x*y'[x]-y[x]==0,y[x],{x,0,5}]
```

$$y(x) \to c_2\left(\frac{x^5}{15}+\frac{x^3}{3}+x\right)+c_1\left(\frac{x^4}{8}+\frac{x^2}{2}+1\right)$$

13.3 problem 4

Internal problem ID [713]
Internal file name [`OUTPUT/713_Sunday_June_05_2022_01_47_36_AM_6480672/index.tex`]

Book: Elementary differential equations and boundary value problems, 10th ed., Boyce and DiPrima
Section: Chapter 5.2, Series Solutions Near an Ordinary Point, Part I. page 263
Problem number: 4.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second_order_bessel_ode", "second order series method. Ordinary point", "second order series method. Taylor series method"**

Maple gives the following as the ode type

```
[[_Emden, _Fowler]]
```

$$\boxed{y'' + k^2x^2y = 0}$$

With the expansion point for the power series method at $x = 0$.

Solving ode using Taylor series method. This gives review on how the Taylor series method works for solving second order ode.

Let

$$y'' = f(x, y, y')$$

Assuming expansion is at $x_0 = 0$ (we can always shift the actual expansion point to 0 by change of variables) and assuming $f(x, y, y')$ is analytic at x_0 which must be the case for an ordinary point. Let initial conditions be $y(x_0) = y_0$ and $y'(x_0) = y'_0$. Using Taylor series gives

$$\begin{aligned}
y(x) &= y(x_0) + (x - x_0)\, y'(x_0) + \frac{(x - x_0)^2}{2} y''(x_0) + \frac{(x - x_0)^3}{3!} y'''(x_0) + \cdots \\
&= y_0 + xy'_0 + \frac{x^2}{2} f|_{x_0,y_0,y'_0} + \frac{x^3}{3!} f'|_{x_0,y_0,y'_0} + \cdots \\
&= y_0 + xy'_0 + \sum_{n=0}^{\infty} \frac{x^{n+2}}{(n+2)!} \frac{d^n f}{dx^n}\bigg|_{x_0,y_0,y'_0}
\end{aligned}$$

But

$$\frac{df}{dx} = \frac{\partial f}{\partial x}\frac{dx}{dx} + \frac{\partial f}{\partial y}\frac{dy}{dx} + \frac{\partial f}{\partial y'}\frac{dy'}{dx} \tag{1}$$

$$= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}y' + \frac{\partial f}{\partial y'}y'' \tag{750}$$

$$= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}y' + \frac{\partial f}{\partial y'}f \tag{751}$$

$$\frac{d^2 f}{dx^2} = \frac{d}{dx}\left(\frac{df}{dx}\right)$$

$$= \frac{\partial}{\partial x}\left(\frac{df}{dx}\right) + \frac{\partial}{\partial y}\left(\frac{df}{dx}\right)y' + \frac{\partial}{\partial y'}\left(\frac{df}{dx}\right)f \tag{2}$$

$$\frac{d^3 f}{dx^3} = \frac{d}{dx}\left(\frac{d^2 f}{dx^2}\right)$$

$$= \frac{\partial}{\partial x}\left(\frac{d^2 f}{dx^2}\right) + \left(\frac{\partial}{\partial y}\frac{d^2 f}{dx^2}\right)y' + \frac{\partial}{\partial y'}\left(\frac{d^2 f}{dx^2}\right)f \tag{3}$$

$$\vdots$$

And so on. Hence if we name $F_0 = f(x,y,y')$ then the above can be written as

$$
\begin{aligned}
F_0 &= f(x,y,y') \qquad (4)\\
F_1 &= \frac{df}{dx}\\
&= \frac{dF_0}{dx}\\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}y' + \frac{\partial f}{\partial y'}y''\\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}y' + \frac{\partial f}{\partial y'}f \qquad (5)\\
&= \frac{\partial F_0}{\partial x} + \frac{\partial F_0}{\partial y}y' + \frac{\partial F_0}{\partial y'}F_0\\
F_2 &= \frac{d}{dx}\left(\frac{d}{dx}f\right)\\
&= \frac{d}{dx}(F_1)\\
&= \frac{\partial}{\partial x}F_1 + \left(\frac{\partial F_1}{\partial y}\right)y' + \left(\frac{\partial F_1}{\partial y'}\right)y''\\
&= \frac{\partial}{\partial x}F_1 + \left(\frac{\partial F_1}{\partial y}\right)y' + \left(\frac{\partial F_1}{\partial y'}\right)F_0\\
&\vdots\\
F_n &= \frac{d}{dx}(F_{n-1})\\
&= \frac{\partial}{\partial x}F_{n-1} + \left(\frac{\partial F_{n-1}}{\partial y}\right)y' + \left(\frac{\partial F_{n-1}}{\partial y'}\right)y''\\
&= \frac{\partial}{\partial x}F_{n-1} + \left(\frac{\partial F_{n-1}}{\partial y}\right)y' + \left(\frac{\partial F_{n-1}}{\partial y'}\right)F_0 \qquad (6)
\end{aligned}
$$

Therefore (6) can be used from now on along with

$$
y(x) = y_0 + xy'_0 + \sum_{n=0}^{\infty}\frac{x^{n+2}}{(n+2)!}F_n|_{x_0,y_0,y'_0} \qquad (7)
$$

To find $y(x)$ series solution around $x = 0$. Hence

$$
\begin{aligned}
F_0 &= -k^2x^2y \\
F_1 &= \frac{dF_0}{dx} \\
&= \frac{\partial F_0}{\partial x} + \frac{\partial F_0}{\partial y}y' + \frac{\partial F_0}{\partial y'}F_0 \\
&= -k^2x(2y + y'x) \\
F_2 &= \frac{dF_1}{dx} \\
&= \frac{\partial F_1}{\partial x} + \frac{\partial F_1}{\partial y}y' + \frac{\partial F_1}{\partial y'}F_1 \\
&= k^2\left(x^4k^2y - 4y'x - 2y\right) \\
F_3 &= \frac{dF_2}{dx} \\
&= \frac{\partial F_2}{\partial x} + \frac{\partial F_2}{\partial y}y' + \frac{\partial F_2}{\partial y'}F_2 \\
&= \left(\left(x^4k^2 - 6\right)y' + 8x^3k^2y\right)k^2 \\
F_4 &= \frac{dF_3}{dx} \\
&= \frac{\partial F_3}{\partial x} + \frac{\partial F_3}{\partial y}y' + \frac{\partial F_3}{\partial y'}F_3 \\
&= -\left(-12y'x + y\left(x^4k^2 - 30\right)\right)k^4x^2
\end{aligned}
$$

And so on. Evaluating all the above at initial conditions $x = 0$ and $y(0) = y(0)$ and $y'(0) = y'(0)$ gives

$$
\begin{aligned}
F_0 &= 0 \\
F_1 &= 0 \\
F_2 &= -2k^2y(0) \\
F_3 &= -6y'(0)\,k^2 \\
F_4 &= 0
\end{aligned}
$$

Substituting all the above in (7) and simplifying gives the solution as

$$
y = \left(1 - \frac{x^4k^2}{12}\right)y(0) + \left(x - \frac{1}{20}k^2x^5\right)y'(0) + O\left(x^6\right)
$$

Since the expansion point $x = 0$ is an ordinary, we can also solve this using standard

power series Let the solution be represented as power series of the form

$$y = \sum_{n=0}^{\infty} a_n x^n$$

Then

$$y' = \sum_{n=1}^{\infty} n a_n x^{n-1}$$
$$y'' = \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2}$$

Substituting the above back into the ode gives

$$\sum_{n=2}^{\infty} n(n-1) a_n x^{n-2} = -k^2 x^2 \left(\sum_{n=0}^{\infty} a_n x^n\right) \tag{1}$$

Which simplifies to

$$\left(\sum_{n=2}^{\infty} n(n-1) a_n x^{n-2}\right) + \left(\sum_{n=0}^{\infty} x^{n+2} k^2 a_n\right) = 0 \tag{2}$$

The next step is to make all powers of x be n in each summation term. Going over each summation term above with power of x in it which is not already x^n and adjusting the power and the corresponding index gives

$$\sum_{n=2}^{\infty} n(n-1) a_n x^{n-2} = \sum_{n=0}^{\infty} (n+2) a_{n+2} (n+1) x^n$$
$$\sum_{n=0}^{\infty} x^{n+2} k^2 a_n = \sum_{n=2}^{\infty} a_{n-2} k^2 x^n$$

Substituting all the above in Eq (2) gives the following equation where now all powers of x are the same and equal to n.

$$\left(\sum_{n=0}^{\infty} (n+2) a_{n+2} (n+1) x^n\right) + \left(\sum_{n=2}^{\infty} a_{n-2} k^2 x^n\right) = 0 \tag{3}$$

For $2 \leq n$, the recurrence equation is

$$(n+2) a_{n+2} (n+1) + a_{n-2} k^2 = 0 \tag{4}$$

Solving for a_{n+2}, gives

$$a_{n+2} = -\frac{a_{n-2}k^2}{(n+2)(n+1)} \tag{5}$$

For $n = 2$ the recurrence equation gives

$$a_0k^2 + 12a_4 = 0$$

Which after substituting the earlier terms found becomes

$$a_4 = -\frac{a_0k^2}{12}$$

For $n = 3$ the recurrence equation gives

$$a_1k^2 + 20a_5 = 0$$

Which after substituting the earlier terms found becomes

$$a_5 = -\frac{a_1k^2}{20}$$

For $n = 4$ the recurrence equation gives

$$a_2k^2 + 30a_6 = 0$$

Which after substituting the earlier terms found becomes

$$a_6 = 0$$

For $n = 5$ the recurrence equation gives

$$a_3k^2 + 42a_7 = 0$$

Which after substituting the earlier terms found becomes

$$a_7 = 0$$

And so on. Therefore the solution is

$$
\begin{aligned}
y &= \sum_{n=0}^{\infty} a_n x^n \\
&= a_3 x^3 + a_2 x^2 + a_1 x + a_0 + \dots
\end{aligned}
$$

Substituting the values for a_n found above, the solution becomes

$$
y = a_0 + a_1 x - \frac{1}{12} a_0 k^2 x^4 - \frac{1}{20} a_1 k^2 x^5 + \dots
$$

Collecting terms, the solution becomes

$$
y = \left(1 - \frac{x^4 k^2}{12}\right) a_0 + \left(x - \frac{1}{20} k^2 x^5\right) a_1 + O\left(x^6\right) \tag{3}
$$

At $x = 0$ the solution above becomes

$$
y = \left(1 - \frac{x^4 k^2}{12}\right) c_1 + \left(x - \frac{1}{20} k^2 x^5\right) c_2 + O\left(x^6\right)
$$

Summary

The solution(s) found are the following

$$
y = \left(1 - \frac{x^4 k^2}{12}\right) y(0) + \left(x - \frac{1}{20} k^2 x^5\right) y'(0) + O\left(x^6\right) \tag{1}
$$

$$
y = \left(1 - \frac{x^4 k^2}{12}\right) c_1 + \left(x - \frac{1}{20} k^2 x^5\right) c_2 + O\left(x^6\right) \tag{2}
$$

Verification of solutions

$$
y = \left(1 - \frac{x^4 k^2}{12}\right) y(0) + \left(x - \frac{1}{20} k^2 x^5\right) y'(0) + O\left(x^6\right)
$$

Verified OK.

$$
y = \left(1 - \frac{x^4 k^2}{12}\right) c_1 + \left(x - \frac{1}{20} k^2 x^5\right) c_2 + O\left(x^6\right)
$$

Verified OK.

13.3.1 Maple step by step solution

Let's solve

$y'' = -k^2x^2y$

- Highest derivative means the order of the ODE is 2

  y''

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y'' + k^2x^2y = 0$

- Assume series solution for y

  $y = \sum_{k=0}^{\infty} a_k x^k$

□ Rewrite ODE with series expansions

  - Convert $x^2 \cdot y$ to series expansion

    $x^2 \cdot y = \sum_{k=0}^{\infty} a_k x^{k+2}$

  - Shift index using $k- >k-2$

    $x^2 \cdot y = \sum_{k=2}^{\infty} a_{k-2} x^k$

  - Convert y'' to series expansion

    $y'' = \sum_{k=2}^{\infty} a_k k(k-1)\, x^{k-2}$

  - Shift index using $k- >k+2$

    $y'' = \sum_{k=0}^{\infty} a_{k+2}(k+2)\,(k+1)\, x^k$

  Rewrite ODE with series expansions

  $6a_3x + 2a_2 + \left(\sum_{k=2}^{\infty} \left(a_{k+2}(k+2)\,(k+1) + k^2a_{k-2}\right) x^k\right) = 0$

- The coefficients of each power of x must be 0

  $[2a_2 = 0, 6a_3 = 0]$

- Solve for the dependent coefficient(s)

  $\{a_2 = 0, a_3 = 0\}$

- Each term in the series must be 0, giving the recursion relation

  $(k^2 + 3k + 2)\, a_{k+2} + k^2a_{k-2} = 0$

- Shift index using $k->k+2$

  $$\left((k+2)^{2}+3k+8\right) a_{k+4}+k^{2} a_{k}=0$$

- Recursion relation that defines the series solution to the ODE

  $$\left[y=\sum_{k=0}^{\infty} a_{k} x^{k}, a_{k+4}=-\frac{k^{2} a_{k}}{k^{2}+7k+12}, a_{2}=0, a_{3}=0\right]$$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
<- No Liouvillian solutions exists
-> Trying a solution in terms of special functions:
   -> Bessel
   <- Bessel successful
<- special function solution successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 30

```
Order:=6;
dsolve(diff(y(x),x$2)+k^2*x^2*y(x)=0,y(x),type='series',x=0);
```

$$y(x)=\left(1-\frac{k^{2} x^{4}}{12}\right) y(0)+\left(x-\frac{1}{20} k^{2} x^{5}\right) D(y)(0)+O\left(x^{6}\right)$$

✓ Solution by Mathematica

Time used: 0.001 (sec). Leaf size: 34

```
AsymptoticDSolveValue[y''[x]+k^2*x^2*y[x]==0,y[x],{x,0,5}]
```

$$y(x)\to c_{2}\left(x-\frac{k^{2} x^{5}}{20}\right)+c_{1}\left(1-\frac{k^{2} x^{4}}{12}\right)$$

13.4 problem 5

Internal problem ID [714]
Internal file name [`OUTPUT/714_Sunday_June_05_2022_01_47_38_AM_69187182/index.tex`]

Book: Elementary differential equations and boundary value problems, 10th ed., Boyce and DiPrima
Section: Chapter 5.2, Series Solutions Near an Ordinary Point, Part I. page 263
Problem number: 5.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second order series method. Ordinary point", "second order series method. Taylor series method"**

Maple gives the following as the ode type

```
[[_Emden, _Fowler]]
```

$$\boxed{(1-x)\,y''+y=0}$$

With the expansion point for the power series method at $x=0$.

Solving ode using Taylor series method. This gives review on how the Taylor series method works for solving second order ode.

Let

$$y''=f(x,y,y')$$

Assuming expansion is at $x_0=0$ (we can always shift the actual expansion point to 0 by change of variables) and assuming $f(x,y,y')$ is analytic at x_0 which must be the case for an ordinary point. Let initial conditions be $y(x_0)=y_0$ and $y'(x_0)=y'_0$. Using Taylor series gives

$$\begin{aligned}
y(x) &= y(x_0)+(x-x_0)\,y'(x_0)+\frac{(x-x_0)^2}{2}y''(x_0)+\frac{(x-x_0)^3}{3!}y'''(x_0)+\cdots \\
&= y_0+xy'_0+\frac{x^2}{2}f|_{x_0,y_0,y'_0}+\frac{x^3}{3!}f'|_{x_0,y_0,y'_0}+\cdots \\
&= y_0+xy'_0+\sum_{n=0}^{\infty}\frac{x^{n+2}}{(n+2)\,!}\left.\frac{d^nf}{dx^n}\right|_{x_0,y_0,y'_0}
\end{aligned}$$

But

$$\frac{df}{dx} = \frac{\partial f}{\partial x}\frac{dx}{dx} + \frac{\partial f}{\partial y}\frac{dy}{dx} + \frac{\partial f}{\partial y'}\frac{dy'}{dx} \tag{1}$$

$$= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}y' + \frac{\partial f}{\partial y'}y'' \tag{753}$$

$$= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}y' + \frac{\partial f}{\partial y'}f \tag{754}$$

$$\frac{d^2 f}{dx^2} = \frac{d}{dx}\left(\frac{df}{dx}\right)$$

$$= \frac{\partial}{\partial x}\left(\frac{df}{dx}\right) + \frac{\partial}{\partial y}\left(\frac{df}{dx}\right)y' + \frac{\partial}{\partial y'}\left(\frac{df}{dx}\right)f \tag{2}$$

$$\frac{d^3 f}{dx^3} = \frac{d}{dx}\left(\frac{d^2 f}{dx^2}\right)$$

$$= \frac{\partial}{\partial x}\left(\frac{d^2 f}{dx^2}\right) + \left(\frac{\partial}{\partial y}\frac{d^2 f}{dx^2}\right)y' + \frac{\partial}{\partial y'}\left(\frac{d^2 f}{dx^2}\right)f \tag{3}$$

$$\vdots$$

And so on. Hence if we name $F_0 = f(x,y,y')$ then the above can be written as

$$
\begin{aligned}
F_0 &= f(x,y,y') \qquad (4)\\
F_1 &= \frac{df}{dx}\\
&= \frac{dF_0}{dx}\\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}y' + \frac{\partial f}{\partial y'}y''\\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}y' + \frac{\partial f}{\partial y'}f \qquad (5)\\
&= \frac{\partial F_0}{\partial x} + \frac{\partial F_0}{\partial y}y' + \frac{\partial F_0}{\partial y'}F_0\\
F_2 &= \frac{d}{dx}\left(\frac{d}{dx}f\right)\\
&= \frac{d}{dx}(F_1)\\
&= \frac{\partial}{\partial x}F_1 + \left(\frac{\partial F_1}{\partial y}\right)y' + \left(\frac{\partial F_1}{\partial y'}\right)y''\\
&= \frac{\partial}{\partial x}F_1 + \left(\frac{\partial F_1}{\partial y}\right)y' + \left(\frac{\partial F_1}{\partial y'}\right)F_0\\
&\vdots\\
F_n &= \frac{d}{dx}(F_{n-1})\\
&= \frac{\partial}{\partial x}F_{n-1} + \left(\frac{\partial F_{n-1}}{\partial y}\right)y' + \left(\frac{\partial F_{n-1}}{\partial y'}\right)y''\\
&= \frac{\partial}{\partial x}F_{n-1} + \left(\frac{\partial F_{n-1}}{\partial y}\right)y' + \left(\frac{\partial F_{n-1}}{\partial y'}\right)F_0 \qquad (6)
\end{aligned}
$$

Therefore (6) can be used from now on along with

$$
y(x) = y_0 + xy_0' + \sum_{n=0}^{\infty} \frac{x^{n+2}}{(n+2)\,!} F_n|_{x_0,y_0,y_0'} \qquad (7)
$$

To find $y(x)$ series solution around $x = 0$. Hence

$$
\begin{aligned}
F_0 &= \frac{y}{x-1} \\
F_1 &= \frac{dF_0}{dx} \\
&= \frac{\partial F_0}{\partial x} + \frac{\partial F_0}{\partial y} y' + \frac{\partial F_0}{\partial y'} F_0 \\
&= \frac{(x-1)\,y' - y}{(x-1)^2} \\
F_2 &= \frac{dF_1}{dx} \\
&= \frac{\partial F_1}{\partial x} + \frac{\partial F_1}{\partial y} y' + \frac{\partial F_1}{\partial y'} F_1 \\
&= \frac{(-2x+2)\,y' + (x+1)\,y}{(x-1)^3} \\
F_3 &= \frac{dF_2}{dx} \\
&= \frac{\partial F_2}{\partial x} + \frac{\partial F_2}{\partial y} y' + \frac{\partial F_2}{\partial y'} F_2 \\
&= \frac{(x^2+4x-5)\,y' + (-4x-2)\,y}{(x-1)^4} \\
F_4 &= \frac{dF_3}{dx} \\
&= \frac{\partial F_3}{\partial x} + \frac{\partial F_3}{\partial y} y' + \frac{\partial F_3}{\partial y'} F_3 \\
&= \frac{(-6x^2-12x+18)\,y' + y(x^2+16x+7)}{(x-1)^5}
\end{aligned}
$$

And so on. Evaluating all the above at initial conditions $x = 0$ and $y(0) = y(0)$ and $y'(0) = y'(0)$ gives

$$
\begin{aligned}
F_0 &= -y(0) \\
F_1 &= -y'(0) - y(0) \\
F_2 &= -y(0) - 2y'(0) \\
F_3 &= -2y(0) - 5y'(0) \\
F_4 &= -7y(0) - 18y'(0)
\end{aligned}
$$

Substituting all the above in (7) and simplifying gives the solution as

$$
\begin{aligned}
y = &\left(1-\frac{1}{2}x^{2}-\frac{1}{6}x^{3}-\frac{1}{24}x^{4}-\frac{1}{60}x^{5}-\frac{7}{720}x^{6}\right) y(0) \\
&+\left(x-\frac{1}{6}x^{3}-\frac{1}{12}x^{4}-\frac{1}{24}x^{5}-\frac{1}{40}x^{6}\right) y'(0)+O\left(x^{6}\right)
\end{aligned}
$$

Since the expansion point $x = 0$ is an ordinary, we can also solve this using standard power series The ode is normalized to be

$$
(1-x)\, y''+y = 0
$$

Let the solution be represented as power series of the form

$$
y = \sum_{n=0}^{\infty} a_{n} x^{n}
$$

Then

$$
\begin{aligned}
y' &= \sum_{n=1}^{\infty} n a_{n} x^{n-1} \\
y'' &= \sum_{n=2}^{\infty} n(n-1)\, a_{n} x^{n-2}
\end{aligned}
$$

Substituting the above back into the ode gives

$$
(1-x) \left(\sum_{n=2}^{\infty} n(n-1)\, a_{n} x^{n-2}\right)+\left(\sum_{n=0}^{\infty} a_{n} x^{n}\right) = 0 \tag{1}
$$

Which simplifies to

$$
\sum_{n =2}^{\infty}\left(-n\, x^{n-1} a_{n} (n-1)\right)+\left(\sum_{n=2}^{\infty} n(n-1)\, a_{n} x^{n-2}\right)+\left(\sum_{n=0}^{\infty} a_{n} x^{n}\right) = 0 \tag{2}
$$

The next step is to make all powers of x be n in each summation term. Going over each summation term above with power of x in it which is not already x^{n} and adjusting the power and the corresponding index gives

$$
\begin{aligned}
\sum_{n =2}^{\infty}\left(-n\, x^{n-1} a_{n} (n-1)\right) &= \sum_{n=1}^{\infty}\left(-(n+1)\, a_{n+1} n\, x^{n}\right) \\
\sum_{n =2}^{\infty} n(n-1)\, a_{n} x^{n-2} &= \sum_{n=0}^{\infty} (n+2)\, a_{n+2} (n+1)\, x^{n}
\end{aligned}
$$

Substituting all the above in Eq (2) gives the following equation where now all powers of x are the same and equal to n.

$$\sum_{n=1}^{\infty} \left(-(n+1) a_{n+1} n x^{n}\right) + \left(\sum_{n=0}^{\infty} (n+2) a_{n+2}(n+1) x^{n}\right) + \left(\sum_{n=0}^{\infty} a_{n} x^{n}\right) = 0 \quad (3)$$

$n = 0$ gives

$$2a_2 + a_0 = 0$$

$$a_2 = -\frac{a_0}{2}$$

For $1 \leq n$, the recurrence equation is

$$-(n+1) a_{n+1} n + (n+2) a_{n+2}(n+1) + a_n = 0 \quad (4)$$

Solving for a_{n+2}, gives

$$\begin{aligned} a_{n+2} &= \frac{n^2 a_{n+1} + n a_{n+1} - a_n}{(n+2)(n+1)} \\ &= -\frac{a_n}{(n+2)(n+1)} + \frac{(n^2+n) a_{n+1}}{(n+2)(n+1)} \end{aligned} \quad (5)$$

For $n = 1$ the recurrence equation gives

$$-2a_2 + 6a_3 + a_1 = 0$$

Which after substituting the earlier terms found becomes

$$a_3 = -\frac{a_0}{6} - \frac{a_1}{6}$$

For $n = 2$ the recurrence equation gives

$$-6a_3 + 12a_4 + a_2 = 0$$

Which after substituting the earlier terms found becomes

$$a_4 = -\frac{a_0}{24} - \frac{a_1}{12}$$

For $n = 3$ the recurrence equation gives

$$-12a_4 + 20a_5 + a_3 = 0$$

Which after substituting the earlier terms found becomes

$$a_5 = -\frac{a_0}{60} - \frac{a_1}{24}$$

For $n = 4$ the recurrence equation gives

$$-20a_5 + 30a_6 + a_4 = 0$$

Which after substituting the earlier terms found becomes

$$a_6 = -\frac{7a_0}{720} - \frac{a_1}{40}$$

For $n = 5$ the recurrence equation gives

$$-30a_6 + 42a_7 + a_5 = 0$$

Which after substituting the earlier terms found becomes

$$a_7 = -\frac{11a_0}{1680} - \frac{17a_1}{1008}$$

And so on. Therefore the solution is

$$\begin{aligned} y &= \sum_{n=0}^{\infty} a_n x^n \\ &= a_3 x^3 + a_2 x^2 + a_1 x + a_0 + \dots \end{aligned}$$

Substituting the values for a_n found above, the solution becomes

$$y = a_0 + a_1 x - \frac{a_0 x^2}{2} + \left(-\frac{a_0}{6} - \frac{a_1}{6}\right) x^3 + \left(-\frac{a_0}{24} - \frac{a_1}{12}\right) x^4 + \left(-\frac{a_0}{60} - \frac{a_1}{24}\right) x^5 + \dots$$

Collecting terms, the solution becomes

$$y = \left(1 - \frac{1}{2}x^2 - \frac{1}{6}x^3 - \frac{1}{24}x^4 - \frac{1}{60}x^5\right) a_0 + \left(x - \frac{1}{6}x^3 - \frac{1}{12}x^4 - \frac{1}{24}x^5\right) a_1 + O\left(x^6\right) \quad (3)$$

At $x = 0$ the solution above becomes

$$y = \left(1 - \frac{1}{2}x^2 - \frac{1}{6}x^3 - \frac{1}{24}x^4 - \frac{1}{60}x^5\right)c_1 + \left(x - \frac{1}{6}x^3 - \frac{1}{12}x^4 - \frac{1}{24}x^5\right)c_2 + O(x^6)$$

Summary
The solution(s) found are the following

$$\begin{aligned} y = &\left(1 - \frac{1}{2}x^2 - \frac{1}{6}x^3 - \frac{1}{24}x^4 - \frac{1}{60}x^5 - \frac{7}{720}x^6\right)y(0) \\ &+ \left(x - \frac{1}{6}x^3 - \frac{1}{12}x^4 - \frac{1}{24}x^5 - \frac{1}{40}x^6\right)y'(0) + O(x^6) \end{aligned} \tag{1}$$

$$y = \left(1 - \frac{1}{2}x^2 - \frac{1}{6}x^3 - \frac{1}{24}x^4 - \frac{1}{60}x^5\right)c_1 + \left(x - \frac{1}{6}x^3 - \frac{1}{12}x^4 - \frac{1}{24}x^5\right)c_2 + O(x^6) \tag{2}$$

Verification of solutions

$$\begin{aligned} y = &\left(1 - \frac{1}{2}x^2 - \frac{1}{6}x^3 - \frac{1}{24}x^4 - \frac{1}{60}x^5 - \frac{7}{720}x^6\right)y(0) \\ &+ \left(x - \frac{1}{6}x^3 - \frac{1}{12}x^4 - \frac{1}{24}x^5 - \frac{1}{40}x^6\right)y'(0) + O(x^6) \end{aligned}$$

Verified OK.

$$y = \left(1 - \frac{1}{2}x^2 - \frac{1}{6}x^3 - \frac{1}{24}x^4 - \frac{1}{60}x^5\right)c_1 + \left(x - \frac{1}{6}x^3 - \frac{1}{12}x^4 - \frac{1}{24}x^5\right)c_2 + O(x^6)$$

Verified OK.

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
<- No Liouvillian solutions exists
-> Trying a solution in terms of special functions:
   -> Bessel
   -> elliptic
   -> Legendre
   -> Whittaker
      -> hyper3: Equivalence to 1F1 under a power @ Moebius
   -> hypergeometric
      -> heuristic approach
      <- heuristic approach successful
   <- hypergeometric successful
<- special function solution successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 49

```
Order:=6;
dsolve((1-x)*diff(y(x),x$2)+y(x)=0,y(x),type='series',x=0);
```

$$y(x) = \left(1-\frac{1}{2}x^2-\frac{1}{6}x^3-\frac{1}{24}x^4-\frac{1}{60}x^5\right)y(0)+\left(x-\frac{1}{6}x^3-\frac{1}{12}x^4-\frac{1}{24}x^5\right)D(y)(0)+O(x^6)$$

✓ Solution by Mathematica

Time used: 0.001 (sec). Leaf size: 63

```
AsymptoticDSolveValue[(1-x)*y''[x]+y[x]==0,y[x],{x,0,5}]
```

$$y(x) \to c_2\left(-\frac{x^5}{24}-\frac{x^4}{12}-\frac{x^3}{6}+x\right)+c_1\left(-\frac{x^5}{60}-\frac{x^4}{24}-\frac{x^3}{6}-\frac{x^2}{2}+1\right)$$

13.5 problem 6

Internal problem ID [715]
Internal file name [`OUTPUT/715_Sunday_June_05_2022_01_47_39_AM_1741303/index.tex`]

Book: Elementary differential equations and boundary value problems, 10th ed., Boyce and DiPrima
Section: Chapter 5.2, Series Solutions Near an Ordinary Point, Part I. page 263
Problem number: 6.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second order series method. Ordinary point", "second order series method. Taylor series method"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

$$\boxed{(x^2+2)\,y''-y'x+4y=0}$$

With the expansion point for the power series method at $x=0$.

Solving ode using Taylor series method. This gives review on how the Taylor series method works for solving second order ode.

Let

$$y''=f(x,y,y')$$

Assuming expansion is at $x_0=0$ (we can always shift the actual expansion point to 0 by change of variables) and assuming $f(x,y,y')$ is analytic at x_0 which must be the case for an ordinary point. Let initial conditions be $y(x_0)=y_0$ and $y'(x_0)=y'_0$. Using Taylor series gives

$$\begin{aligned}
y(x) &= y(x_0)+(x-x_0)\,y'(x_0)+\frac{(x-x_0)^2}{2}y''(x_0)+\frac{(x-x_0)^3}{3!}y'''(x_0)+\cdots \\
&= y_0+xy'_0+\frac{x^2}{2}\,f|_{x_0,y_0,y'_0}+\frac{x^3}{3!}\,f'|_{x_0,y_0,y'_0}+\cdots \\
&= y_0+xy'_0+\sum_{n=0}^{\infty}\frac{x^{n+2}}{(n+2)\,!}\,\frac{d^nf}{dx^n}\bigg|_{x_0,y_0,y'_0}
\end{aligned}$$

But

$$\frac{df}{dx} = \frac{\partial f}{\partial x}\frac{dx}{dx} + \frac{\partial f}{\partial y}\frac{dy}{dx} + \frac{\partial f}{\partial y'}\frac{dy'}{dx} \tag{1}$$

$$= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}y' + \frac{\partial f}{\partial y'}y'' \tag{756}$$

$$= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}y' + \frac{\partial f}{\partial y'}f \tag{757}$$

$$\frac{d^2 f}{dx^2} = \frac{d}{dx}\left(\frac{df}{dx}\right)$$

$$= \frac{\partial}{\partial x}\left(\frac{df}{dx}\right) + \frac{\partial}{\partial y}\left(\frac{df}{dx}\right)y' + \frac{\partial}{\partial y'}\left(\frac{df}{dx}\right)f \tag{2}$$

$$\frac{d^3 f}{dx^3} = \frac{d}{dx}\left(\frac{d^2 f}{dx^2}\right)$$

$$= \frac{\partial}{\partial x}\left(\frac{d^2 f}{dx^2}\right) + \left(\frac{\partial}{\partial y}\frac{d^2 f}{dx^2}\right)y' + \frac{\partial}{\partial y'}\left(\frac{d^2 f}{dx^2}\right)f \tag{3}$$

$$\vdots$$

And so on. Hence if we name $F_0 = f(x,y,y')$ then the above can be written as

$$
\begin{aligned}
F_0 &= f(x,y,y') && (4)\\
F_1 &= \frac{df}{dx}\\
&= \frac{dF_0}{dx}\\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}y' + \frac{\partial f}{\partial y'}y''\\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}y' + \frac{\partial f}{\partial y'}f && (5)\\
&= \frac{\partial F_0}{\partial x} + \frac{\partial F_0}{\partial y}y' + \frac{\partial F_0}{\partial y'}F_0\\
F_2 &= \frac{d}{dx}\left(\frac{d}{dx}f\right)\\
&= \frac{d}{dx}(F_1)\\
&= \frac{\partial}{\partial x}F_1 + \left(\frac{\partial F_1}{\partial y}\right)y' + \left(\frac{\partial F_1}{\partial y'}\right)y''\\
&= \frac{\partial}{\partial x}F_1 + \left(\frac{\partial F_1}{\partial y}\right)y' + \left(\frac{\partial F_1}{\partial y'}\right)F_0\\
&\vdots\\
F_n &= \frac{d}{dx}(F_{n-1})\\
&= \frac{\partial}{\partial x}F_{n-1} + \left(\frac{\partial F_{n-1}}{\partial y}\right)y' + \left(\frac{\partial F_{n-1}}{\partial y'}\right)y''\\
&= \frac{\partial}{\partial x}F_{n-1} + \left(\frac{\partial F_{n-1}}{\partial y}\right)y' + \left(\frac{\partial F_{n-1}}{\partial y'}\right)F_0 && (6)
\end{aligned}
$$

Therefore (6) can be used from now on along with

$$
y(x) = y_0 + xy_0' + \sum_{n=0}^{\infty} \frac{x^{n+2}}{(n+2)!} F_n|_{x_0,y_0,y_0'} \tag{7}
$$

To find $y(x)$ series solution around $x=0$. Hence

$$
\begin{aligned}
F_0 &= \frac{y'x-4y}{x^2+2} \\
F_1 &= \frac{dF_0}{dx} \\
&= \frac{\partial F_0}{\partial x} + \frac{\partial F_0}{\partial y}y' + \frac{\partial F_0}{\partial y'}F_0 \\
&= \frac{-4y'x^2+4yx-6y'}{(x^2+2)^2} \\
F_2 &= \frac{dF_1}{dx} \\
&= \frac{\partial F_1}{\partial x} + \frac{\partial F_1}{\partial y}y' + \frac{\partial F_1}{\partial y'}F_1 \\
&= \frac{8x^3y'+4x^2y+10y'x+32y}{(x^2+2)^3} \\
F_3 &= \frac{dF_2}{dx} \\
&= \frac{\partial F_2}{\partial x} + \frac{\partial F_2}{\partial y}y' + \frac{\partial F_2}{\partial y'}F_2 \\
&= \frac{(-12x^4+48x^2+84)\,y'+(-48x^3-216x)\,y}{(x^2+2)^4} \\
F_4 &= \frac{dF_3}{dx} \\
&= \frac{\partial F_3}{\partial x} + \frac{\partial F_3}{\partial y}y' + \frac{\partial F_3}{\partial y'}F_3 \\
&= \frac{(-12x^5-648x^3-828x)\,y'+(288x^4+1032x^2-768)\,y}{(x^2+2)^5}
\end{aligned}
$$

And so on. Evaluating all the above at initial conditions $x=0$ and $y(0)=y(0)$ and $y'(0)=y'(0)$ gives

$$
\begin{aligned}
F_0 &= -2y(0) \\
F_1 &= -\frac{3y'(0)}{2} \\
F_2 &= 4y(0) \\
F_3 &= \frac{21y'(0)}{4} \\
F_4 &= -24y(0)
\end{aligned}
$$

Substituting all the above in (7) and simplifying gives the solution as

$$y = \left(1 - x^2 + \frac{1}{6}x^4 - \frac{1}{30}x^6\right) y(0) + \left(x - \frac{1}{4}x^3 + \frac{7}{160}x^5\right) y'(0) + O(x^6)$$

Since the expansion point $x = 0$ is an ordinary, we can also solve this using standard power series The ode is normalized to be

$$(x^2 + 2)\, y'' - y'x + 4y = 0$$

Let the solution be represented as power series of the form

$$y = \sum_{n=0}^{\infty} a_n x^n$$

Then

$$y' = \sum_{n=1}^{\infty} n a_n x^{n-1}$$
$$y'' = \sum_{n=2}^{\infty} n(n-1)\, a_n x^{n-2}$$

Substituting the above back into the ode gives

$$(x^2 + 2) \left(\sum_{n=2}^{\infty} n(n-1)\, a_n x^{n-2}\right) - \left(\sum_{n=1}^{\infty} n a_n x^{n-1}\right) x + 4\left(\sum_{n=0}^{\infty} a_n x^n\right) = 0 \tag{1}$$

Which simplifies to

$$\left(\sum_{n=2}^{\infty} x^n a_n n(n-1)\right) + \left(\sum_{n=2}^{\infty} 2n(n-1)\, a_n x^{n-2}\right) + \sum_{n=1}^{\infty} (-n a_n x^n) + \left(\sum_{n=0}^{\infty} 4a_n x^n\right) = 0 \tag{2}$$

The next step is to make all powers of x be n in each summation term. Going over each summation term above with power of x in it which is not already x^n and adjusting the power and the corresponding index gives

$$\sum_{n=2}^{\infty} 2n(n-1)\, a_n x^{n-2} = \sum_{n=0}^{\infty} 2(n+2)\, a_{n+2}(n+1)\, x^n$$

Substituting all the above in Eq (2) gives the following equation where now all powers of x are the same and equal to n.

$$\left(\sum_{n=2}^{\infty} x^n a_n n(n-1)\right) + \left(\sum_{n=0}^{\infty} 2(n+2)\, a_{n+2}(n+1)\, x^n\right) + \sum_{n=1}^{\infty} (-n a_n x^n) + \left(\sum_{n=0}^{\infty} 4 a_n x^n\right) = 0 \tag{3}$$

$n = 0$ gives

$$4a_2 + 4a_0 = 0$$

$$a_2 = -a_0$$

$n = 1$ gives

$$12a_3 + 3a_1 = 0$$

Which after substituting earlier equations, simplifies to

$$a_3 = -\frac{a_1}{4}$$

For $2 \leq n$, the recurrence equation is

$$n a_n(n-1) + 2(n+2)\, a_{n+2}(n+1) - n a_n + 4a_n = 0 \tag{4}$$

Solving for a_{n+2}, gives

$$a_{n+2} = -\frac{a_n(n^2 - 2n + 4)}{2\,(n+2)\,(n+1)} \tag{5}$$

For $n = 2$ the recurrence equation gives

$$4a_2 + 24a_4 = 0$$

Which after substituting the earlier terms found becomes

$$a_4 = \frac{a_0}{6}$$

For $n = 3$ the recurrence equation gives

$$7a_3 + 40a_5 = 0$$

Which after substituting the earlier terms found becomes

$$a_5 = \frac{7a_1}{160}$$

For $n = 4$ the recurrence equation gives

$$12a_4 + 60a_6 = 0$$

Which after substituting the earlier terms found becomes

$$a_6 = -\frac{a_0}{30}$$

For $n = 5$ the recurrence equation gives

$$19a_5 + 84a_7 = 0$$

Which after substituting the earlier terms found becomes

$$a_7 = -\frac{19a_1}{1920}$$

And so on. Therefore the solution is

$$\begin{aligned} y &= \sum_{n=0}^{\infty} a_n x^n \\ &= a_3 x^3 + a_2 x^2 + a_1 x + a_0 + \dots \end{aligned}$$

Substituting the values for a_n found above, the solution becomes

$$y = a_0 + a_1 x - a_0 x^2 - \frac{1}{4} a_1 x^3 + \frac{1}{6} a_0 x^4 + \frac{7}{160} a_1 x^5 + \dots$$

Collecting terms, the solution becomes

$$y = \left(1 - x^2 + \frac{1}{6} x^4\right) a_0 + \left(x - \frac{1}{4} x^3 + \frac{7}{160} x^5\right) a_1 + O(x^6) \tag{3}$$

At $x = 0$ the solution above becomes

$$y = \left(1 - x^2 + \frac{1}{6}x^4\right) c_1 + \left(x - \frac{1}{4}x^3 + \frac{7}{160}x^5\right) c_2 + O(x^6)$$

Summary

The solution(s) found are the following

$$y = \left(1 - x^2 + \frac{1}{6}x^4 - \frac{1}{30}x^6\right) y(0) + \left(x - \frac{1}{4}x^3 + \frac{7}{160}x^5\right) y'(0) + O(x^6) \tag{1}$$

$$y = \left(1 - x^2 + \frac{1}{6}x^4\right) c_1 + \left(x - \frac{1}{4}x^3 + \frac{7}{160}x^5\right) c_2 + O(x^6) \tag{2}$$

Verification of solutions

$$y = \left(1 - x^2 + \frac{1}{6}x^4 - \frac{1}{30}x^6\right) y(0) + \left(x - \frac{1}{4}x^3 + \frac{7}{160}x^5\right) y'(0) + O(x^6)$$

Verified OK.

$$y = \left(1 - x^2 + \frac{1}{6}x^4\right) c_1 + \left(x - \frac{1}{4}x^3 + \frac{7}{160}x^5\right) c_2 + O(x^6)$$

Verified OK.

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Group is reducible or imprimitive
<- Kovacics algorithm successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 34

```
Order:=6;
dsolve((2+x^2)*diff(y(x),x$2)-x*diff(y(x),x)+4*y(x)=0,y(x),type='series',x=0);
```

$$y(x) = \left(1 - x^2 + \frac{1}{6}x^4\right) y(0) + \left(x - \frac{1}{4}x^3 + \frac{7}{160}x^5\right) D(y)(0) + O(x^6)$$

✓ Solution by Mathematica

Time used: 0.001 (sec). Leaf size: 40

```
AsymptoticDSolveValue[(2+x^2)*y''[x]-x*y'[x]+4*y[x]==0,y[x],{x,0,5}]
```

$$y(x) \to c_2\left(\frac{7x^5}{160} - \frac{x^3}{4} + x\right) + c_1\left(\frac{x^4}{6} - x^2 + 1\right)$$

13.6 problem 7

Internal problem ID [716]
Internal file name [OUTPUT/716_Sunday_June_05_2022_01_47_41_AM_17236857/index.tex]

Book: Elementary differential equations and boundary value problems, 10th ed., Boyce and DiPrima
Section: Chapter 5.2, Series Solutions Near an Ordinary Point, Part I. page 263
Problem number: 7.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second order series method. Ordinary point", "second order series method. Taylor series method"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

$$\boxed{y'' + y'x + 2y = 0}$$

With the expansion point for the power series method at $x = 0$.

Solving ode using Taylor series method. This gives review on how the Taylor series method works for solving second order ode.

Let

$$y'' = f(x, y, y')$$

Assuming expansion is at $x_0 = 0$ (we can always shift the actual expansion point to 0 by change of variables) and assuming $f(x, y, y')$ is analytic at x_0 which must be the case for an ordinary point. Let initial conditions be $y(x_0) = y_0$ and $y'(x_0) = y'_0$. Using Taylor series gives

$$\begin{aligned}
y(x) &= y(x_0) + (x - x_0)\, y'(x_0) + \frac{(x - x_0)^2}{2} y''(x_0) + \frac{(x - x_0)^3}{3!} y'''(x_0) + \cdots \\
&= y_0 + x y'_0 + \frac{x^2}{2} f|_{x_0, y_0, y'_0} + \frac{x^3}{3!} f'|_{x_0, y_0, y'_0} + \cdots \\
&= y_0 + x y'_0 + \sum_{n=0}^{\infty} \frac{x^{n+2}}{(n+2)!} \frac{d^n f}{dx^n}\bigg|_{x_0, y_0, y'_0}
\end{aligned}$$

But

$$
\begin{aligned}
\frac{df}{dx} &= \frac{\partial f}{\partial x}\frac{dx}{dx} + \frac{\partial f}{\partial y}\frac{dy}{dx} + \frac{\partial f}{\partial y'}\frac{dy'}{dx} \qquad (1) \\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}y' + \frac{\partial f}{\partial y'}y'' \qquad (759) \\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}y' + \frac{\partial f}{\partial y'}f \qquad (760) \\
\frac{d^2 f}{dx^2} &= \frac{d}{dx}\left(\frac{df}{dx}\right) \\
&= \frac{\partial}{\partial x}\left(\frac{df}{dx}\right) + \frac{\partial}{\partial y}\left(\frac{df}{dx}\right)y' + \frac{\partial}{\partial y'}\left(\frac{df}{dx}\right)f \qquad (2) \\
\frac{d^3 f}{dx^3} &= \frac{d}{dx}\left(\frac{d^2 f}{dx^2}\right) \\
&= \frac{\partial}{\partial x}\left(\frac{d^2 f}{dx^2}\right) + \left(\frac{\partial}{\partial y}\frac{d^2 f}{dx^2}\right)y' + \frac{\partial}{\partial y'}\left(\frac{d^2 f}{dx^2}\right)f \qquad (3) \\
&\vdots
\end{aligned}
$$

And so on. Hence if we name $F_0 = f(x,y,y')$ then the above can be written as

$$
\begin{aligned}
F_0 &= f(x,y,y') \qquad (4)\\
F_1 &= \frac{df}{dx}\\
&= \frac{dF_0}{dx}\\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} y' + \frac{\partial f}{\partial y'} y''\\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} y' + \frac{\partial f}{\partial y'} f \qquad (5)\\
&= \frac{\partial F_0}{\partial x} + \frac{\partial F_0}{\partial y} y' + \frac{\partial F_0}{\partial y'} F_0\\
F_2 &= \frac{d}{dx}\left(\frac{d}{dx} f\right)\\
&= \frac{d}{dx}(F_1)\\
&= \frac{\partial}{\partial x} F_1 + \left(\frac{\partial F_1}{\partial y}\right) y' + \left(\frac{\partial F_1}{\partial y'}\right) y''\\
&= \frac{\partial}{\partial x} F_1 + \left(\frac{\partial F_1}{\partial y}\right) y' + \left(\frac{\partial F_1}{\partial y'}\right) F_0\\
&\vdots\\
F_n &= \frac{d}{dx}(F_{n-1})\\
&= \frac{\partial}{\partial x} F_{n-1} + \left(\frac{\partial F_{n-1}}{\partial y}\right) y' + \left(\frac{\partial F_{n-1}}{\partial y'}\right) y''\\
&= \frac{\partial}{\partial x} F_{n-1} + \left(\frac{\partial F_{n-1}}{\partial y}\right) y' + \left(\frac{\partial F_{n-1}}{\partial y'}\right) F_0 \qquad (6)
\end{aligned}
$$

Therefore (6) can be used from now on along with

$$
y(x) = y_0 + x y_0' + \sum_{n=0}^{\infty} \frac{x^{n+2}}{(n+2)!} F_n|_{x_0,y_0,y_0'} \qquad (7)
$$

To find $y(x)$ series solution around $x=0$. Hence

$$
\begin{aligned}
F_0 &= -y'x - 2y \\
F_1 &= \frac{dF_0}{dx} \\
&= \frac{\partial F_0}{\partial x} + \frac{\partial F_0}{\partial y} y' + \frac{\partial F_0}{\partial y'} F_0 \\
&= y'x^2 + 2yx - 3y' \\
F_2 &= \frac{dF_1}{dx} \\
&= \frac{\partial F_1}{\partial x} + \frac{\partial F_1}{\partial y} y' + \frac{\partial F_1}{\partial y'} F_1 \\
&= -x^3y' - 2x^2y + 7y'x + 8y \\
F_3 &= \frac{dF_2}{dx} \\
&= \frac{\partial F_2}{\partial x} + \frac{\partial F_2}{\partial y} y' + \frac{\partial F_2}{\partial y'} F_2 \\
&= \left(x^4 - 12x^2 + 15\right) y' + 2\left(x^3 - 9x\right) y \\
F_4 &= \frac{dF_3}{dx} \\
&= \frac{\partial F_3}{\partial x} + \frac{\partial F_3}{\partial y} y' + \frac{\partial F_3}{\partial y'} F_3 \\
&= \left(-x^5 + 18x^3 - 57x\right) y' - 2y\left(x^4 - 15x^2 + 24\right)
\end{aligned}
$$

And so on. Evaluating all the above at initial conditions $x=0$ and $y(0)=y(0)$ and $y'(0)=y'(0)$ gives

$$
\begin{aligned}
F_0 &= -2y(0) \\
F_1 &= -3y'(0) \\
F_2 &= 8y(0) \\
F_3 &= 15y'(0) \\
F_4 &= -48y(0)
\end{aligned}
$$

Substituting all the above in (7) and simplifying gives the solution as

$$
y = \left(1 - x^2 + \frac{1}{3}x^4 - \frac{1}{15}x^6\right) y(0) + \left(x - \frac{1}{2}x^3 + \frac{1}{8}x^5\right) y'(0) + O\left(x^6\right)
$$

Since the expansion point $x=0$ is an ordinary, we can also solve this using standard

power series Let the solution be represented as power series of the form

$$y = \sum_{n=0}^{\infty} a_n x^n$$

Then

$$y' = \sum_{n=1}^{\infty} n a_n x^{n-1}$$
$$y'' = \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2}$$

Substituting the above back into the ode gives

$$\sum_{n=2}^{\infty} n(n-1) a_n x^{n-2} = -\left(\sum_{n=1}^{\infty} n a_n x^{n-1}\right) x - 2\left(\sum_{n=0}^{\infty} a_n x^n\right) \tag{1}$$

Which simplifies to

$$\left(\sum_{n=2}^{\infty} n(n-1) a_n x^{n-2}\right) + \left(\sum_{n=1}^{\infty} n\, x^n a_n\right) + \left(\sum_{n=0}^{\infty} 2a_n x^n\right) = 0 \tag{2}$$

The next step is to make all powers of x be n in each summation term. Going over each summation term above with power of x in it which is not already x^n and adjusting the power and the corresponding index gives

$$\sum_{n=2}^{\infty} n(n-1) a_n x^{n-2} = \sum_{n=0}^{\infty} (n+2) a_{n+2}(n+1) x^n$$

Substituting all the above in Eq (2) gives the following equation where now all powers of x are the same and equal to n.

$$\left(\sum_{n=0}^{\infty} (n+2) a_{n+2}(n+1) x^n\right) + \left(\sum_{n=1}^{\infty} n\, x^n a_n\right) + \left(\sum_{n=0}^{\infty} 2a_n x^n\right) = 0 \tag{3}$$

$n = 0$ gives

$$2a_2 + 2a_0 = 0$$

$$a_2 = -a_0$$

For $1 \leq n$, the recurrence equation is

$$(n+2)\, a_{n+2}(n+1) + n a_n + 2 a_n = 0 \tag{4}$$

Solving for a_{n+2}, gives

$$a_{n+2} = -\frac{a_n}{n+1} \tag{5}$$

For $n = 1$ the recurrence equation gives

$$6a_3 + 3a_1 = 0$$

Which after substituting the earlier terms found becomes

$$a_3 = -\frac{a_1}{2}$$

For $n = 2$ the recurrence equation gives

$$12a_4 + 4a_2 = 0$$

Which after substituting the earlier terms found becomes

$$a_4 = \frac{a_0}{3}$$

For $n = 3$ the recurrence equation gives

$$20a_5 + 5a_3 = 0$$

Which after substituting the earlier terms found becomes

$$a_5 = \frac{a_1}{8}$$

For $n = 4$ the recurrence equation gives

$$30a_6 + 6a_4 = 0$$

Which after substituting the earlier terms found becomes

$$a_6 = -\frac{a_0}{15}$$

For $n = 5$ the recurrence equation gives

$$42a_7 + 7a_5 = 0$$

Which after substituting the earlier terms found becomes

$$a_7 = -\frac{a_1}{48}$$

And so on. Therefore the solution is

$$\begin{aligned} y &= \sum_{n=0}^{\infty} a_n x^n \\ &= a_3x^3 + a_2x^2 + a_1x + a_0 + \dots \end{aligned}$$

Substituting the values for a_n found above, the solution becomes

$$y = a_0 + a_1 x - a_0 x^2 - \frac{1}{2}a_1 x^3 + \frac{1}{3}a_0 x^4 + \frac{1}{8}a_1 x^5 + \dots$$

Collecting terms, the solution becomes

$$y = \left(1 - x^2 + \frac{1}{3}x^4\right) a_0 + \left(x - \frac{1}{2}x^3 + \frac{1}{8}x^5\right) a_1 + O(x^6) \tag{3}$$

At $x = 0$ the solution above becomes

$$y = \left(1 - x^2 + \frac{1}{3}x^4\right) c_1 + \left(x - \frac{1}{2}x^3 + \frac{1}{8}x^5\right) c_2 + O(x^6)$$

Summary
The solution(s) found are the following

$$y = \left(1 - x^2 + \frac{1}{3}x^4 - \frac{1}{15}x^6\right) y(0) + \left(x - \frac{1}{2}x^3 + \frac{1}{8}x^5\right) y'(0) + O(x^6) \tag{1}$$

$$y = \left(1 - x^2 + \frac{1}{3}x^4\right) c_1 + \left(x - \frac{1}{2}x^3 + \frac{1}{8}x^5\right) c_2 + O(x^6) \tag{2}$$

Verification of solutions

$$y = \left(1 - x^2 + \frac{1}{3}x^4 - \frac{1}{15}x^6\right) y(0) + \left(x - \frac{1}{2}x^3 + \frac{1}{8}x^5\right) y'(0) + O(x^6)$$

Verified OK.

$$y = \left(1 - x^2 + \frac{1}{3}x^4\right) c_1 + \left(x - \frac{1}{2}x^3 + \frac{1}{8}x^5\right) c_2 + O(x^6)$$

Verified OK.

13.6.1 Maple step by step solution

Let's solve

$y'' = -y'x - 2y$

- Highest derivative means the order of the ODE is 2

  y''

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y'' + y'x + 2y = 0$

- Assume series solution for y

  $y = \sum_{k=0}^{\infty} a_k x^k$

□ Rewrite DE with series expansions

  ◦ Convert $x \cdot y'$ to series expansion

    $x \cdot y' = \sum_{k=0}^{\infty} a_k k\, x^k$

  ◦ Convert y'' to series expansion

    $y'' = \sum_{k=2}^{\infty} a_k k(k-1)\, x^{k-2}$

  ◦ Shift index using $k- >k+2$

    $y'' = \sum_{k=0}^{\infty} a_{k+2}(k+2)\,(k+1)\, x^k$

  Rewrite DE with series expansions

  $\sum_{k=0}^{\infty} (a_{k+2}(k+2)\,(k+1) + a_k(k+2))\, x^k = 0$

- Each term in the series must be 0, giving the recursion relation

  $(k+2)\,(k a_{k+2} + a_k + a_{k+2}) = 0$

- Recursion relation that defines the series solution to the ODE

  $\left[y = \sum_{k=0}^{\infty} a_k x^k, a_{k+2} = -\frac{a_k}{k+1}\right]$

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Group is reducible, not completely reducible
<- Kovacics algorithm successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 34

```
Order:=6;
dsolve(diff(y(x),x$2)+x*diff(y(x),x)+2*y(x)=0,y(x),type='series',x=0);
```

$$y(x) = \left(1 - x^2 + \frac{1}{3}x^4\right) y(0) + \left(x - \frac{1}{2}x^3 + \frac{1}{8}x^5\right) D(y)(0) + O(x^6)$$

✓ Solution by Mathematica

Time used: 0.001 (sec). Leaf size: 40

```
AsymptoticDSolveValue[y''[x]+x*y'[x]+2*y[x]==0,y[x],{x,0,5}]
```

$$y(x) \to c_2\left(\frac{x^5}{8} - \frac{x^3}{2} + x\right) + c_1\left(\frac{x^4}{3} - x^2 + 1\right)$$

13.7 problem 9

Internal problem ID [717]
Internal file name [`OUTPUT/717_Sunday_June_05_2022_01_47_42_AM_81903652/index.tex`]

Book: Elementary differential equations and boundary value problems, 10th ed., Boyce and DiPrima
Section: Chapter 5.2, Series Solutions Near an Ordinary Point, Part I. page 263
Problem number: 9.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second order series method. Ordinary point", "second order series method. Taylor series method"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

$$\boxed{(x^2+1)\,y''-4y'x+6y=0}$$

With the expansion point for the power series method at $x=0$.

Solving ode using Taylor series method. This gives review on how the Taylor series method works for solving second order ode.

Let

$$y''=f(x,y,y')$$

Assuming expansion is at $x_0=0$ (we can always shift the actual expansion point to 0 by change of variables) and assuming $f(x,y,y')$ is analytic at x_0 which must be the case for an ordinary point. Let initial conditions be $y(x_0)=y_0$ and $y'(x_0)=y'_0$. Using Taylor series gives

$$\begin{aligned}
y(x) &= y(x_0)+(x-x_0)\,y'(x_0)+\frac{(x-x_0)^2}{2}y''(x_0)+\frac{(x-x_0)^3}{3!}y'''(x_0)+\cdots \\
&= y_0+xy'_0+\frac{x^2}{2}\,f|_{x_0,y_0,y'_0}+\frac{x^3}{3!}\,f'|_{x_0,y_0,y'_0}+\cdots \\
&= y_0+xy'_0+\sum_{n=0}^{\infty}\frac{x^{n+2}}{(n+2)\,!}\,\frac{d^n f}{dx^n}\bigg|_{x_0,y_0,y'_0}
\end{aligned}$$

But

$$\frac{df}{dx} = \frac{\partial f}{\partial x}\frac{dx}{dx} + \frac{\partial f}{\partial y}\frac{dy}{dx} + \frac{\partial f}{\partial y'}\frac{dy'}{dx} \tag{1}$$

$$= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}y' + \frac{\partial f}{\partial y'}y'' \tag{762}$$

$$= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}y' + \frac{\partial f}{\partial y'}f \tag{763}$$

$$\frac{d^2 f}{dx^2} = \frac{d}{dx}\left(\frac{df}{dx}\right)$$

$$= \frac{\partial}{\partial x}\left(\frac{df}{dx}\right) + \frac{\partial}{\partial y}\left(\frac{df}{dx}\right)y' + \frac{\partial}{\partial y'}\left(\frac{df}{dx}\right)f \tag{2}$$

$$\frac{d^3 f}{dx^3} = \frac{d}{dx}\left(\frac{d^2 f}{dx^2}\right)$$

$$= \frac{\partial}{\partial x}\left(\frac{d^2 f}{dx^2}\right) + \left(\frac{\partial}{\partial y}\frac{d^2 f}{dx^2}\right)y' + \frac{\partial}{\partial y'}\left(\frac{d^2 f}{dx^2}\right)f \tag{3}$$

$$\vdots$$

And so on. Hence if we name $F_0 = f(x,y,y')$ then the above can be written as

$$
\begin{aligned}
F_0 &= f(x,y,y') \qquad (4)\\
F_1 &= \frac{df}{dx}\\
&= \frac{dF_0}{dx}\\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}y' + \frac{\partial f}{\partial y'}y''\\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}y' + \frac{\partial f}{\partial y'}f \qquad (5)\\
&= \frac{\partial F_0}{\partial x} + \frac{\partial F_0}{\partial y}y' + \frac{\partial F_0}{\partial y'}F_0\\
F_2 &= \frac{d}{dx}\left(\frac{d}{dx}f\right)\\
&= \frac{d}{dx}(F_1)\\
&= \frac{\partial}{\partial x}F_1 + \left(\frac{\partial F_1}{\partial y}\right)y' + \left(\frac{\partial F_1}{\partial y'}\right)y''\\
&= \frac{\partial}{\partial x}F_1 + \left(\frac{\partial F_1}{\partial y}\right)y' + \left(\frac{\partial F_1}{\partial y'}\right)F_0\\
&\vdots\\
F_n &= \frac{d}{dx}(F_{n-1})\\
&= \frac{\partial}{\partial x}F_{n-1} + \left(\frac{\partial F_{n-1}}{\partial y}\right)y' + \left(\frac{\partial F_{n-1}}{\partial y'}\right)y''\\
&= \frac{\partial}{\partial x}F_{n-1} + \left(\frac{\partial F_{n-1}}{\partial y}\right)y' + \left(\frac{\partial F_{n-1}}{\partial y'}\right)F_0 \qquad (6)
\end{aligned}
$$

Therefore (6) can be used from now on along with

$$
y(x) = y_0 + xy_0' + \sum_{n=0}^{\infty} \frac{x^{n+2}}{(n+2)!} F_n|_{x_0,y_0,y_0'} \qquad (7)
$$

To find $y(x)$ series solution around $x = 0$. Hence

$$
\begin{aligned}
F_0 &= \frac{4y'x - 6y}{x^2 + 1} \\
F_1 &= \frac{dF_0}{dx} \\
&= \frac{\partial F_0}{\partial x} + \frac{\partial F_0}{\partial y} y' + \frac{\partial F_0}{\partial y'} F_0 \\
&= \frac{6y'x^2 - 12yx - 2y'}{\left(x^2 + 1\right)^2} \\
F_2 &= \frac{dF_1}{dx} \\
&= \frac{\partial F_1}{\partial x} + \frac{\partial F_1}{\partial y} y' + \frac{\partial F_1}{\partial y'} F_1 \\
&= 0 \\
F_3 &= \frac{dF_2}{dx} \\
&= \frac{\partial F_2}{\partial x} + \frac{\partial F_2}{\partial y} y' + \frac{\partial F_2}{\partial y'} F_2 \\
&= 0 \\
F_4 &= \frac{dF_3}{dx} \\
&= \frac{\partial F_3}{\partial x} + \frac{\partial F_3}{\partial y} y' + \frac{\partial F_3}{\partial y'} F_3 \\
&= 0
\end{aligned}
$$

And so on. Evaluating all the above at initial conditions $x = 0$ and $y(0) = y(0)$ and $y'(0) = y'(0)$ gives

$$
\begin{aligned}
F_0 &= -6y(0) \\
F_1 &= -2y'(0) \\
F_2 &= 0 \\
F_3 &= 0 \\
F_4 &= 0
\end{aligned}
$$

Substituting all the above in (7) and simplifying gives the solution as

$$
y = \left(-3x^2 + 1\right) y(0) + \left(-\frac{1}{3}x^3 + x\right) y'(0) + O\left(x^6\right)
$$

Since the expansion point $x = 0$ is an ordinary, we can also solve this using standard power series The ode is normalized to be

$$(x^2+1)\, y'' - 4y'x + 6y = 0$$

Let the solution be represented as power series of the form

$$y = \sum_{n=0}^{\infty} a_n x^n$$

Then

$$y' = \sum_{n=1}^{\infty} n a_n x^{n-1}$$
$$y'' = \sum_{n=2}^{\infty} n(n-1)\, a_n x^{n-2}$$

Substituting the above back into the ode gives

$$(x^2+1)\left(\sum_{n=2}^{\infty} n(n-1)\, a_n x^{n-2}\right) - 4\left(\sum_{n=1}^{\infty} n a_n x^{n-1}\right) x + 6\left(\sum_{n=0}^{\infty} a_n x^n\right) = 0 \tag{1}$$

Which simplifies to

$$\left(\sum_{n=2}^{\infty} x^n a_n n(n-1)\right) + \left(\sum_{n=2}^{\infty} n(n-1)\, a_n x^{n-2}\right) + \sum_{n=1}^{\infty} (-4n a_n x^n) + \left(\sum_{n=0}^{\infty} 6a_n x^n\right) = 0 \tag{2}$$

The next step is to make all powers of x be n in each summation term. Going over each summation term above with power of x in it which is not already x^n and adjusting the power and the corresponding index gives

$$\sum_{n=2}^{\infty} n(n-1)\, a_n x^{n-2} = \sum_{n=0}^{\infty} (n+2)\, a_{n+2}(n+1)\, x^n$$

Substituting all the above in Eq (2) gives the following equation where now all powers of x are the same and equal to n.

$$\begin{aligned}\left(\sum_{n=2}^{\infty} x^n a_n n(n-1)\right) + \left(\sum_{n=0}^{\infty} (n+2)\, a_{n+2}(n+1)\, x^n\right) \\ + \sum_{n=1}^{\infty} (-4n a_n x^n) + \left(\sum_{n=0}^{\infty} 6a_n x^n\right) = 0\end{aligned} \tag{3}$$

$n = 0$ gives

$$2a_2 + 6a_0 = 0$$

$$a_2 = -3a_0$$

$n = 1$ gives

$$6a_3 + 2a_1 = 0$$

Which after substituting earlier equations, simplifies to

$$a_3 = -\frac{a_1}{3}$$

For $2 \leq n$, the recurrence equation is

$$n a_n(n-1) + (n+2)\, a_{n+2}(n+1) - 4n a_n + 6a_n = 0 \tag{4}$$

Solving for a_{n+2}, gives

$$a_{n+2} = -\frac{a_n(n^2 - 5n + 6)}{(n+2)\,(n+1)} \tag{5}$$

For $n = 2$ the recurrence equation gives

$$12a_4 = 0$$

Which after substituting the earlier terms found becomes

$$a_4 = 0$$

For $n = 3$ the recurrence equation gives

$$20a_5 = 0$$

Which after substituting the earlier terms found becomes

$$a_5 = 0$$

For $n = 4$ the recurrence equation gives

$$2a_4 + 30a_6 = 0$$

Which after substituting the earlier terms found becomes

$$a_6 = 0$$

For $n = 5$ the recurrence equation gives

$$6a_5 + 42a_7 = 0$$

Which after substituting the earlier terms found becomes

$$a_7 = 0$$

And so on. Therefore the solution is

$$\begin{aligned} y &= \sum_{n=0}^{\infty} a_n x^n \\ &= a_3 x^3 + a_2 x^2 + a_1 x + a_0 + \dots \end{aligned}$$

Substituting the values for a_n found above, the solution becomes

$$y = a_0 + a_1 x - 3a_0 x^2 - \frac{1}{3} a_1 x^3 + \dots$$

Collecting terms, the solution becomes

$$y = (-3x^2 + 1) a_0 + \left(-\frac{1}{3}x^3 + x\right) a_1 + O(x^6) \tag{3}$$

At $x = 0$ the solution above becomes

$$y = (-3x^2 + 1) c_1 + \left(-\frac{1}{3}x^3 + x\right) c_2 + O(x^6)$$

Summary
The solution(s) found are the following

$$y = (-3x^2 + 1) y(0) + \left(-\frac{1}{3}x^3 + x\right) y'(0) + O(x^6) \tag{1}$$

$$y = (-3x^2 + 1) c_1 + \left(-\frac{1}{3}x^3 + x\right) c_2 + O(x^6) \tag{2}$$

Verification of solutions

$$y = (-3x^2 + 1)\, y(0) + \left(-\frac{1}{3}x^3 + x\right) y'(0) + O(x^6)$$

Verified OK.

$$y = (-3x^2 + 1)\, c_1 + \left(-\frac{1}{3}x^3 + x\right) c_2 + O(x^6)$$

Verified OK.

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
<- Kovacics algorithm successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 25

```
Order:=6;
dsolve((1+x^2)*diff(y(x),x$2)-4*x*diff(y(x),x)+6*y(x)=0,y(x),type='series',x=0);
```

$$y(x) = y(0) + D(y)(0)\, x - 3y(0)\, x^2 - \frac{D(y)(0)\, x^3}{3}$$

✓ Solution by Mathematica

Time used: 0.001 (sec). Leaf size: 26

```
AsymptoticDSolveValue[(1+x^2)*y''[x]-4*x*y'[x]+6*y[x]==0,y[x],{x,0,5}]
```

$$y(x) \to c_2\left(x - \frac{x^3}{3}\right) + c_1(1 - 3x^2)$$

13.8 problem 10

Internal problem ID [718]
Internal file name [`OUTPUT/718_Sunday_June_05_2022_01_47_43_AM_81935409/index.tex`]

Book: Elementary differential equations and boundary value problems, 10th ed., Boyce and DiPrima
Section: Chapter 5.2, Series Solutions Near an Ordinary Point, Part I. page 263
Problem number: 10.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"exact linear second order ode", "second_order_integrable_as_is", "second order series method. Ordinary point", "second order series method. Taylor series method"**

Maple gives the following as the ode type

```
[[_2nd_order, _exact, _linear, _homogeneous]]
```

$$\boxed{(-x^2+4)\,y''+2y=0}$$

With the expansion point for the power series method at $x=0$.

Solving ode using Taylor series method. This gives review on how the Taylor series method works for solving second order ode.

Let

$$y''=f(x,y,y')$$

Assuming expansion is at $x_0=0$ (we can always shift the actual expansion point to 0 by change of variables) and assuming $f(x,y,y')$ is analytic at x_0 which must be the case for an ordinary point. Let initial conditions be $y(x_0)=y_0$ and $y'(x_0)=y'_0$. Using Taylor series gives

$$\begin{aligned}
y(x) &= y(x_0)+(x-x_0)\,y'(x_0)+\frac{(x-x_0)^2}{2}y''(x_0)+\frac{(x-x_0)^3}{3!}y'''(x_0)+\cdots\\
&= y_0+xy'_0+\frac{x^2}{2}f|_{x_0,y_0,y'_0}+\frac{x^3}{3!}f'|_{x_0,y_0,y'_0}+\cdots\\
&= y_0+xy'_0+\sum_{n=0}^{\infty}\frac{x^{n+2}}{(n+2)!}\frac{d^nf}{dx^n}\bigg|_{x_0,y_0,y'_0}
\end{aligned}$$

But

$$\frac{df}{dx} = \frac{\partial f}{\partial x}\frac{dx}{dx} + \frac{\partial f}{\partial y}\frac{dy}{dx} + \frac{\partial f}{\partial y'}\frac{dy'}{dx} \tag{1}$$

$$= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}y' + \frac{\partial f}{\partial y'}y'' \tag{765}$$

$$= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}y' + \frac{\partial f}{\partial y'}f \tag{766}$$

$$\frac{d^2 f}{dx^2} = \frac{d}{dx}\left(\frac{df}{dx}\right)$$

$$= \frac{\partial}{\partial x}\left(\frac{df}{dx}\right) + \frac{\partial}{\partial y}\left(\frac{df}{dx}\right)y' + \frac{\partial}{\partial y'}\left(\frac{df}{dx}\right)f \tag{2}$$

$$\frac{d^3 f}{dx^3} = \frac{d}{dx}\left(\frac{d^2 f}{dx^2}\right)$$

$$= \frac{\partial}{\partial x}\left(\frac{d^2 f}{dx^2}\right) + \left(\frac{\partial}{\partial y}\frac{d^2 f}{dx^2}\right)y' + \frac{\partial}{\partial y'}\left(\frac{d^2 f}{dx^2}\right)f \tag{3}$$

$$\vdots$$

And so on. Hence if we name $F_0 = f(x, y, y')$ then the above can be written as

$$
\begin{align}
F_0 &= f(x, y, y') \tag{4}\\
F_1 &= \frac{df}{dx} \nonumber\\
&= \frac{dF_0}{dx} \nonumber\\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} y' + \frac{\partial f}{\partial y'} y'' \nonumber\\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} y' + \frac{\partial f}{\partial y'} f \tag{5}\\
&= \frac{\partial F_0}{\partial x} + \frac{\partial F_0}{\partial y} y' + \frac{\partial F_0}{\partial y'} F_0 \nonumber\\
F_2 &= \frac{d}{dx}\left(\frac{d}{dx} f\right) \nonumber\\
&= \frac{d}{dx}(F_1) \nonumber\\
&= \frac{\partial}{\partial x} F_1 + \left(\frac{\partial F_1}{\partial y}\right) y' + \left(\frac{\partial F_1}{\partial y'}\right) y'' \nonumber\\
&= \frac{\partial}{\partial x} F_1 + \left(\frac{\partial F_1}{\partial y}\right) y' + \left(\frac{\partial F_1}{\partial y'}\right) F_0 \nonumber\\
&\vdots \nonumber\\
F_n &= \frac{d}{dx}(F_{n-1}) \nonumber\\
&= \frac{\partial}{\partial x} F_{n-1} + \left(\frac{\partial F_{n-1}}{\partial y}\right) y' + \left(\frac{\partial F_{n-1}}{\partial y'}\right) y'' \nonumber\\
&= \frac{\partial}{\partial x} F_{n-1} + \left(\frac{\partial F_{n-1}}{\partial y}\right) y' + \left(\frac{\partial F_{n-1}}{\partial y'}\right) F_0 \tag{6}
\end{align}
$$

Therefore (6) can be used from now on along with

$$
y(x) = y_0 + x y_0' + \sum_{n=0}^{\infty} \frac{x^{n+2}}{(n+2)!} F_n|_{x_0, y_0, y_0'} \tag{7}
$$

To find $y(x)$ series solution around $x = 0$. Hence

$$
\begin{aligned}
F_0 &= \frac{2y}{x^2-4} \\
F_1 &= \frac{dF_0}{dx} \\
&= \frac{\partial F_0}{\partial x} + \frac{\partial F_0}{\partial y}y' + \frac{\partial F_0}{\partial y'}F_0 \\
&= \frac{2y'x^2 - 4yx - 8y'}{(x^2-4)^2} \\
F_2 &= \frac{dF_1}{dx} \\
&= \frac{\partial F_1}{\partial x} + \frac{\partial F_1}{\partial y}y' + \frac{\partial F_1}{\partial y'}F_1 \\
&= -\frac{8(y'x^2 - 2yx - 4y')\,x}{(x^2-4)^3} \\
F_3 &= \frac{dF_2}{dx} \\
&= \frac{\partial F_2}{\partial x} + \frac{\partial F_2}{\partial y}y' + \frac{\partial F_2}{\partial y'}F_2 \\
&= \frac{40\left(x^2+\frac{4}{5}\right)\left(-2yx+\left(x^2-4\right)y'\right)}{(x^2-4)^4} \\
F_4 &= \frac{dF_3}{dx} \\
&= \frac{\partial F_3}{\partial x} + \frac{\partial F_3}{\partial y}y' + \frac{\partial F_3}{\partial y'}F_3 \\
&= -\frac{240\left(x^2+\frac{12}{5}\right)\left(-2yx+\left(x^2-4\right)y'\right)x}{(x^2-4)^5}
\end{aligned}
$$

And so on. Evaluating all the above at initial conditions $x = 0$ and $y(0) = y(0)$ and $y'(0) = y'(0)$ gives

$$
\begin{aligned}
F_0 &= -\frac{y(0)}{2} \\
F_1 &= -\frac{y'(0)}{2} \\
F_2 &= 0 \\
F_3 &= -\frac{y'(0)}{2} \\
F_4 &= 0
\end{aligned}
$$

Substituting all the above in (7) and simplifying gives the solution as

$$y = \left(1-\frac{x^2}{4}\right) y(0)+\left(x-\frac{1}{12}x^3-\frac{1}{240}x^5\right) y'(0)+O(x^6)$$

Since the expansion point $x = 0$ is an ordinary, we can also solve this using standard power series The ode is normalized to be

$$(-x^2+4)\, y''+2y = 0$$

Let the solution be represented as power series of the form

$$y = \sum_{n=0}^{\infty} a_n x^n$$

Then

$$y' = \sum_{n=1}^{\infty} n a_n x^{n-1}$$
$$y'' = \sum_{n=2}^{\infty} n(n-1)\, a_n x^{n-2}$$

Substituting the above back into the ode gives

$$(-x^2+4)\left(\sum_{n=2}^{\infty} n(n-1)\, a_n x^{n-2}\right)+2\left(\sum_{n=0}^{\infty} a_n x^n\right) = 0 \tag{1}$$

Which simplifies to

$$\sum_{n=2}^{\infty}\left(-x^n a_n n(n-1)\right)+\left(\sum_{n=2}^{\infty} 4n(n-1)\, a_n x^{n-2}\right)+\left(\sum_{n=0}^{\infty} 2a_n x^n\right) = 0 \tag{2}$$

The next step is to make all powers of x be n in each summation term. Going over each summation term above with power of x in it which is not already x^n and adjusting the power and the corresponding index gives

$$\sum_{n=2}^{\infty} 4n(n-1)\, a_n x^{n-2} = \sum_{n=0}^{\infty} 4(n+2)\, a_{n+2}(n+1)\, x^n$$

Substituting all the above in Eq (2) gives the following equation where now all powers of x are the same and equal to n.

$$\sum_{n=2}^{\infty}\left(-x^n a_n n(n-1)\right)+\left(\sum_{n=0}^{\infty} 4(n+2)\, a_{n+2}(n+1)\, x^n\right)+\left(\sum_{n=0}^{\infty} 2a_n x^n\right) = 0 \tag{3}$$

$n = 0$ gives

$$8a_2 + 2a_0 = 0$$

$$a_2 = -\frac{a_0}{4}$$

$n = 1$ gives

$$24a_3 + 2a_1 = 0$$

Which after substituting earlier equations, simplifies to

$$a_3 = -\frac{a_1}{12}$$

For $2 \leq n$, the recurrence equation is

$$-na_n(n-1) + 4(n+2)\,a_{n+2}(n+1) + 2a_n = 0 \tag{4}$$

Solving for a_{n+2}, gives

$$a_{n+2} = \frac{(n-2)\,a_n}{4n+8} \tag{5}$$

For $n = 2$ the recurrence equation gives

$$48a_4 = 0$$

Which after substituting the earlier terms found becomes

$$a_4 = 0$$

For $n = 3$ the recurrence equation gives

$$-4a_3 + 80a_5 = 0$$

Which after substituting the earlier terms found becomes

$$a_5 = -\frac{a_1}{240}$$

For $n = 4$ the recurrence equation gives

$$-10a_4 + 120a_6 = 0$$

Which after substituting the earlier terms found becomes

$$a_6 = 0$$

For $n = 5$ the recurrence equation gives

$$-18a_5 + 168a_7 = 0$$

Which after substituting the earlier terms found becomes

$$a_7 = -\frac{a_1}{2240}$$

And so on. Therefore the solution is

$$\begin{aligned} y &= \sum_{n=0}^{\infty} a_n x^n \\ &= a_3x^3 + a_2x^2 + a_1x + a_0 + \dots \end{aligned}$$

Substituting the values for a_n found above, the solution becomes

$$y = a_0 + a_1x - \frac{1}{4}a_0x^2 - \frac{1}{12}a_1x^3 - \frac{1}{240}a_1x^5 + \dots$$

Collecting terms, the solution becomes

$$y = \left(1 - \frac{x^2}{4}\right)a_0 + \left(x - \frac{1}{12}x^3 - \frac{1}{240}x^5\right)a_1 + O(x^6) \tag{3}$$

At $x = 0$ the solution above becomes

$$y = \left(1 - \frac{x^2}{4}\right)c_1 + \left(x - \frac{1}{12}x^3 - \frac{1}{240}x^5\right)c_2 + O(x^6)$$

Summary

The solution(s) found are the following

$$y = \left(1 - \frac{x^2}{4}\right)y(0) + \left(x - \frac{1}{12}x^3 - \frac{1}{240}x^5\right)y'(0) + O(x^6) \tag{1}$$

$$y = \left(1 - \frac{x^2}{4}\right)c_1 + \left(x - \frac{1}{12}x^3 - \frac{1}{240}x^5\right)c_2 + O(x^6) \tag{2}$$

Verification of solutions

$$y = \left(1-\frac{x^2}{4}\right) y(0) + \left(x-\frac{1}{12}x^3-\frac{1}{240}x^5\right) y'(0) + O(x^6)$$

Verified OK.

$$y = \left(1-\frac{x^2}{4}\right) c_1 + \left(x-\frac{1}{12}x^3-\frac{1}{240}x^5\right) c_2 + O(x^6)$$

Verified OK.

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
<- linear_1 successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 29

```
Order:=6;
dsolve((4-x^2)*diff(y(x),x$2)+2*y(x)=0,y(x),type='series',x=0);
```

$$y(x) = \left(1-\frac{x^2}{4}\right) y(0) + \left(x-\frac{1}{12}x^3-\frac{1}{240}x^5\right) D(y)(0) + O(x^6)$$

✓ Solution by Mathematica

Time used: 0.001 (sec). Leaf size: 35

```
AsymptoticDSolveValue[(4-x^2)*y''[x]+2*y[x]==0,y[x],{x,0,5}]
```

$$y(x) \to c_1\left(1-\frac{x^2}{4}\right) + c_2\left(-\frac{x^5}{240}-\frac{x^3}{12}+x\right)$$

13.9 problem 11

Internal problem ID [719]
Internal file name [`OUTPUT/719_Sunday_June_05_2022_01_47_44_AM_50694559/index.tex`]

Book: Elementary differential equations and boundary value problems, 10th ed., Boyce and DiPrima
Section: Chapter 5.2, Series Solutions Near an Ordinary Point, Part I. page 263
Problem number: 11.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"exact linear second order ode", "second_order_integrable_as_is", "second order series method. Ordinary point", "second order series method. Taylor series method"**

Maple gives the following as the ode type

```
[[_2nd_order, _exact, _linear, _homogeneous]]
```

$$\boxed{(-x^2+3)\, y''-3y'x-y=0}$$

With the expansion point for the power series method at $x=0$.

Solving ode using Taylor series method. This gives review on how the Taylor series method works for solving second order ode.

Let

$$y''=f(x,y,y')$$

Assuming expansion is at $x_0=0$ (we can always shift the actual expansion point to 0 by change of variables) and assuming $f(x,y,y')$ is analytic at x_0 which must be the case for an ordinary point. Let initial conditions be $y(x_0)=y_0$ and $y'(x_0)=y'_0$. Using Taylor series gives

$$\begin{aligned}
y(x) &= y(x_0)+(x-x_0)\, y'(x_0)+\frac{(x-x_0)^2}{2}y''(x_0)+\frac{(x-x_0)^3}{3!}y'''(x_0)+\cdots \\
&= y_0+xy'_0+\frac{x^2}{2}\, f|_{x_0,y_0,y'_0}+\frac{x^3}{3!}\, f'|_{x_0,y_0,y'_0}+\cdots \\
&= y_0+xy'_0+\sum_{n=0}^{\infty}\frac{x^{n+2}}{(n+2)\,!}\, \frac{d^n f}{dx^n}\bigg|_{x_0,y_0,y'_0}
\end{aligned}$$

But

$$\frac{df}{dx} = \frac{\partial f}{\partial x}\frac{dx}{dx} + \frac{\partial f}{\partial y}\frac{dy}{dx} + \frac{\partial f}{\partial y'}\frac{dy'}{dx} \tag{1}$$

$$= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}y' + \frac{\partial f}{\partial y'}y'' \tag{768}$$

$$= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}y' + \frac{\partial f}{\partial y'}f \tag{769}$$

$$\frac{d^2 f}{dx^2} = \frac{d}{dx}\left(\frac{df}{dx}\right)$$

$$= \frac{\partial}{\partial x}\left(\frac{df}{dx}\right) + \frac{\partial}{\partial y}\left(\frac{df}{dx}\right) y' + \frac{\partial}{\partial y'}\left(\frac{df}{dx}\right) f \tag{2}$$

$$\frac{d^3 f}{dx^3} = \frac{d}{dx}\left(\frac{d^2 f}{dx^2}\right)$$

$$= \frac{\partial}{\partial x}\left(\frac{d^2 f}{dx^2}\right) + \left(\frac{\partial}{\partial y}\frac{d^2 f}{dx^2}\right) y' + \frac{\partial}{\partial y'}\left(\frac{d^2 f}{dx^2}\right) f \tag{3}$$

$$\vdots$$

And so on. Hence if we name $F_0 = f(x,y,y')$ then the above can be written as

$$
\begin{aligned}
F_0 &= f(x,y,y') \qquad (4)\\
F_1 &= \frac{df}{dx}\\
&= \frac{dF_0}{dx}\\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}y' + \frac{\partial f}{\partial y'}y''\\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}y' + \frac{\partial f}{\partial y'}f \qquad (5)\\
&= \frac{\partial F_0}{\partial x} + \frac{\partial F_0}{\partial y}y' + \frac{\partial F_0}{\partial y'}F_0\\
F_2 &= \frac{d}{dx}\left(\frac{d}{dx}f\right)\\
&= \frac{d}{dx}(F_1)\\
&= \frac{\partial}{\partial x}F_1 + \left(\frac{\partial F_1}{\partial y}\right)y' + \left(\frac{\partial F_1}{\partial y'}\right)y''\\
&= \frac{\partial}{\partial x}F_1 + \left(\frac{\partial F_1}{\partial y}\right)y' + \left(\frac{\partial F_1}{\partial y'}\right)F_0\\
&\vdots\\
F_n &= \frac{d}{dx}(F_{n-1})\\
&= \frac{\partial}{\partial x}F_{n-1} + \left(\frac{\partial F_{n-1}}{\partial y}\right)y' + \left(\frac{\partial F_{n-1}}{\partial y'}\right)y''\\
&= \frac{\partial}{\partial x}F_{n-1} + \left(\frac{\partial F_{n-1}}{\partial y}\right)y' + \left(\frac{\partial F_{n-1}}{\partial y'}\right)F_0 \qquad (6)
\end{aligned}
$$

Therefore (6) can be used from now on along with

$$
y(x) = y_0 + xy_0' + \sum_{n=0}^{\infty} \frac{x^{n+2}}{(n+2)!} F_n|_{x_0,y_0,y_0'} \qquad (7)
$$

To find $y(x)$ series solution around $x=0$. Hence

$$
\begin{aligned}
F_0 &= -\frac{3y'x+y}{x^2-3} \\
F_1 &= \frac{dF_0}{dx} \\
&= \frac{\partial F_0}{\partial x}+\frac{\partial F_0}{\partial y}y'+\frac{\partial F_0}{\partial y'}F_0 \\
&= \frac{11y'x^2+5yx+12y'}{\left(x^2-3\right)^2} \\
F_2 &= \frac{dF_1}{dx} \\
&= \frac{\partial F_1}{\partial x}+\frac{\partial F_1}{\partial y}y'+\frac{\partial F_1}{\partial y'}F_1 \\
&= \frac{-50x^3y'-26x^2y-165y'x-27y}{\left(x^2-3\right)^3} \\
F_3 &= \frac{dF_2}{dx} \\
&= \frac{\partial F_2}{\partial x}+\frac{\partial F_2}{\partial y}y'+\frac{\partial F_2}{\partial y'}F_2 \\
&= \frac{\left(274x^4+1821x^2+576\right)y'+\left(154x^3+483x\right)y}{\left(x^2-3\right)^4} \\
F_4 &= \frac{dF_3}{dx} \\
&= \frac{\partial F_3}{\partial x}+\frac{\partial F_3}{\partial y}y'+\frac{\partial F_3}{\partial y'}F_3 \\
&= \frac{\left(-1764x^5-19656x^3-18711x\right)y'+\left(-1044x^4-6588x^2-2025\right)y}{\left(x^2-3\right)^5}
\end{aligned}
$$

And so on. Evaluating all the above at initial conditions $x=0$ and $y(0)=y(0)$ and $y'(0)=y'(0)$ gives

$$
\begin{aligned}
F_0 &= \frac{y(0)}{3} \\
F_1 &= \frac{4y'(0)}{3} \\
F_2 &= y(0) \\
F_3 &= \frac{64y'(0)}{9} \\
F_4 &= \frac{25y(0)}{3}
\end{aligned}
$$

Substituting all the above in (7) and simplifying gives the solution as

$$y = \left(1+\frac{1}{6}x^2+\frac{1}{24}x^4+\frac{5}{432}x^6\right) y(0)+\left(x+\frac{2}{9}x^3+\frac{8}{135}x^5\right) y'(0)+O(x^6)$$

Since the expansion point $x = 0$ is an ordinary, we can also solve this using standard power series The ode is normalized to be

$$\left(-x^2+3\right) y''-3y'x-y = 0$$

Let the solution be represented as power series of the form

$$y = \sum_{n=0}^{\infty} a_n x^n$$

Then

$$y' = \sum_{n=1}^{\infty} n a_n x^{n-1}$$
$$y'' = \sum_{n=2}^{\infty} n(n-1)\, a_n x^{n-2}$$

Substituting the above back into the ode gives

$$\left(-x^2+3\right)\left(\sum_{n=2}^{\infty} n(n-1)\, a_n x^{n-2}\right)-3\left(\sum_{n=1}^{\infty} n a_n x^{n-1}\right) x-\left(\sum_{n=0}^{\infty} a_n x^n\right) = 0 \quad (1)$$

Which simplifies to

$$\sum_{n=2}^{\infty}\left(-x^n a_n n(n-1)\right)+\left(\sum_{n=2}^{\infty} 3n(n-1)\, a_n x^{n-2}\right)+\sum_{n=1}^{\infty}\left(-3n a_n x^n\right)+\sum_{n=0}^{\infty}\left(-a_n x^n\right) = 0 \quad (2)$$

The next step is to make all powers of x be n in each summation term. Going over each summation term above with power of x in it which is not already x^n and adjusting the power and the corresponding index gives

$$\sum_{n=2}^{\infty} 3n(n-1)\, a_n x^{n-2} = \sum_{n=0}^{\infty} 3(n+2)\, a_{n+2}(n+1)\, x^n$$

Substituting all the above in Eq (2) gives the following equation where now all powers of x are the same and equal to n.

$$\sum_{n=2}^{\infty} \left(-x^n a_n n(n-1)\right) + \left(\sum_{n=0}^{\infty} 3(n+2)\, a_{n+2}(n+1)\, x^n\right) + \sum_{n=1}^{\infty} \left(-3n a_n x^n\right) + \sum_{n=0}^{\infty} \left(-a_n x^n\right) = 0 \tag{3}$$

$n = 0$ gives

$$6a_2 - a_0 = 0$$

$$a_2 = \frac{a_0}{6}$$

$n = 1$ gives

$$18a_3 - 4a_1 = 0$$

Which after substituting earlier equations, simplifies to

$$a_3 = \frac{2a_1}{9}$$

For $2 \leq n$, the recurrence equation is

$$-n a_n(n-1) + 3(n+2)\, a_{n+2}(n+1) - 3n a_n - a_n = 0 \tag{4}$$

Solving for a_{n+2}, gives

$$a_{n+2} = \frac{(n+1)\, a_n}{3n+6} \tag{5}$$

For $n = 2$ the recurrence equation gives

$$-9a_2 + 36a_4 = 0$$

Which after substituting the earlier terms found becomes

$$a_4 = \frac{a_0}{24}$$

For $n = 3$ the recurrence equation gives

$$-16a_3 + 60a_5 = 0$$

Which after substituting the earlier terms found becomes

$$a_5 = \frac{8a_1}{135}$$

For $n = 4$ the recurrence equation gives

$$-25a_4 + 90a_6 = 0$$

Which after substituting the earlier terms found becomes

$$a_6 = \frac{5a_0}{432}$$

For $n = 5$ the recurrence equation gives

$$-36a_5 + 126a_7 = 0$$

Which after substituting the earlier terms found becomes

$$a_7 = \frac{16a_1}{945}$$

And so on. Therefore the solution is

$$\begin{aligned} y &= \sum_{n=0}^{\infty} a_n x^n \\ &= a_3 x^3 + a_2 x^2 + a_1 x + a_0 + \dots \end{aligned}$$

Substituting the values for a_n found above, the solution becomes

$$y = a_0 + a_1 x + \frac{1}{6} a_0 x^2 + \frac{2}{9} a_1 x^3 + \frac{1}{24} a_0 x^4 + \frac{8}{135} a_1 x^5 + \dots$$

Collecting terms, the solution becomes

$$y = \left(1 + \frac{1}{6}x^2 + \frac{1}{24}x^4\right) a_0 + \left(x + \frac{2}{9}x^3 + \frac{8}{135}x^5\right) a_1 + O(x^6) \tag{3}$$

At $x = 0$ the solution above becomes

$$y = \left(1+\frac{1}{6}x^2+\frac{1}{24}x^4\right)c_1+\left(x+\frac{2}{9}x^3+\frac{8}{135}x^5\right)c_2+O(x^6)$$

Summary

The solution(s) found are the following

$$y = \left(1+\frac{1}{6}x^2+\frac{1}{24}x^4+\frac{5}{432}x^6\right)y(0)+\left(x+\frac{2}{9}x^3+\frac{8}{135}x^5\right)y'(0)+O(x^6) \quad (1)$$

$$y = \left(1+\frac{1}{6}x^2+\frac{1}{24}x^4\right)c_1+\left(x+\frac{2}{9}x^3+\frac{8}{135}x^5\right)c_2+O(x^6) \quad (2)$$

Verification of solutions

$$y = \left(1+\frac{1}{6}x^2+\frac{1}{24}x^4+\frac{5}{432}x^6\right)y(0)+\left(x+\frac{2}{9}x^3+\frac{8}{135}x^5\right)y'(0)+O(x^6)$$

Verified OK.

$$y = \left(1+\frac{1}{6}x^2+\frac{1}{24}x^4\right)c_1+\left(x+\frac{2}{9}x^3+\frac{8}{135}x^5\right)c_2+O(x^6)$$

Verified OK.

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
<- linear_1 successful`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 34

```
Order:=6;
dsolve((3-x^2)*diff(y(x),x$2)-3*x*diff(y(x),x)-y(x)=0,y(x),type='series',x=0);
```

$$y(x) = \left(1+\frac{1}{6}x^2+\frac{1}{24}x^4\right)y(0)+\left(x+\frac{2}{9}x^3+\frac{8}{135}x^5\right)D(y)(0)+O(x^6)$$

✓ Solution by Mathematica

Time used: 0.001 (sec). Leaf size: 70

```
AsymptoticDSolveValue[(3-x^2)*y''[x]-3*y'[x]-y[x]==0,y[x],{x,0,5}]
```

$$y(x) \to c_1\left(\frac{13x^5}{1080}+\frac{x^4}{36}+\frac{x^3}{18}+\frac{x^2}{6}+1\right)+c_2\left(\frac{49x^5}{1080}+\frac{7x^4}{72}+\frac{2x^3}{9}+\frac{x^2}{2}+x\right)$$

13.10 problem 12

Internal problem ID [720]
Internal file name [`OUTPUT/720_Sunday_June_05_2022_01_47_45_AM_90164067/index.tex`]

Book: Elementary differential equations and boundary value problems, 10th ed., Boyce and DiPrima
Section: Chapter 5.2, Series Solutions Near an Ordinary Point, Part I. page 263
Problem number: 12.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second order series method. Ordinary point", "second_order_change_of_variable_on_y_method_2", "second order series method. Taylor series method", "second_order_ode_non_constant_coeff_transformation_on_B"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

$$\boxed{(1-x)\,y''+y'x-y=0}$$

With the expansion point for the power series method at $x=0$.

Solving ode using Taylor series method. This gives review on how the Taylor series method works for solving second order ode.

Let

$$y''=f(x,y,y')$$

Assuming expansion is at $x_0=0$ (we can always shift the actual expansion point to 0 by change of variables) and assuming $f(x,y,y')$ is analytic at x_0 which must be the case for an ordinary point. Let initial conditions be $y(x_0)=y_0$ and $y'(x_0)=y'_0$. Using

Taylor series gives

$$
\begin{aligned}
y(x) &= y(x_0) + (x - x_0)\, y'(x_0) + \frac{(x-x_0)^2}{2} y''(x_0) + \frac{(x-x_0)^3}{3!} y'''(x_0) + \cdots \\
&= y_0 + x y_0' + \frac{x^2}{2} f|_{x_0,y_0,y_0'} + \frac{x^3}{3!} f'|_{x_0,y_0,y_0'} + \cdots \\
&= y_0 + x y_0' + \sum_{n=0}^{\infty} \frac{x^{n+2}}{(n+2)\,!} \frac{d^n f}{dx^n}\bigg|_{x_0,y_0,y_0'}
\end{aligned}
$$

But

$$
\begin{aligned}
\frac{df}{dx} &= \frac{\partial f}{\partial x}\frac{dx}{dx} + \frac{\partial f}{\partial y}\frac{dy}{dx} + \frac{\partial f}{\partial y'}\frac{dy'}{dx} && (1)\\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} y' + \frac{\partial f}{\partial y'} y'' && (771)\\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} y' + \frac{\partial f}{\partial y'} f && (772)\\
\frac{d^2 f}{dx^2} &= \frac{d}{dx}\left(\frac{df}{dx}\right) \\
&= \frac{\partial}{\partial x}\left(\frac{df}{dx}\right) + \frac{\partial}{\partial y}\left(\frac{df}{dx}\right) y' + \frac{\partial}{\partial y'}\left(\frac{df}{dx}\right) f && (2)\\
\frac{d^3 f}{dx^3} &= \frac{d}{dx}\left(\frac{d^2 f}{dx^2}\right) \\
&= \frac{\partial}{\partial x}\left(\frac{d^2 f}{dx^2}\right) + \left(\frac{\partial}{\partial y}\frac{d^2 f}{dx^2}\right) y' + \frac{\partial}{\partial y'}\left(\frac{d^2 f}{dx^2}\right) f && (3)\\
&\vdots
\end{aligned}
$$

And so on. Hence if we name $F_0 = f(x,y,y')$ then the above can be written as

$$
\begin{aligned}
F_0 &= f(x,y,y') \qquad (4)\\
F_1 &= \frac{df}{dx}\\
&= \frac{dF_0}{dx}\\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}y' + \frac{\partial f}{\partial y'}y''\\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}y' + \frac{\partial f}{\partial y'}f \qquad (5)\\
&= \frac{\partial F_0}{\partial x} + \frac{\partial F_0}{\partial y}y' + \frac{\partial F_0}{\partial y'}F_0\\
F_2 &= \frac{d}{dx}\left(\frac{d}{dx}f\right)\\
&= \frac{d}{dx}(F_1)\\
&= \frac{\partial}{\partial x}F_1 + \left(\frac{\partial F_1}{\partial y}\right)y' + \left(\frac{\partial F_1}{\partial y'}\right)y''\\
&= \frac{\partial}{\partial x}F_1 + \left(\frac{\partial F_1}{\partial y}\right)y' + \left(\frac{\partial F_1}{\partial y'}\right)F_0\\
&\vdots\\
F_n &= \frac{d}{dx}(F_{n-1})\\
&= \frac{\partial}{\partial x}F_{n-1} + \left(\frac{\partial F_{n-1}}{\partial y}\right)y' + \left(\frac{\partial F_{n-1}}{\partial y'}\right)y''\\
&= \frac{\partial}{\partial x}F_{n-1} + \left(\frac{\partial F_{n-1}}{\partial y}\right)y' + \left(\frac{\partial F_{n-1}}{\partial y'}\right)F_0 \qquad (6)
\end{aligned}
$$

Therefore (6) can be used from now on along with

$$
y(x) = y_0 + xy'_0 + \sum_{n=0}^{\infty} \frac{x^{n+2}}{(n+2)\,!} F_n|_{x_0,y_0,y'_0} \qquad (7)
$$

To find $y(x)$ series solution around $x=0$. Hence

$$
\begin{aligned}
F_0 &= \frac{-y+y'x}{x-1} \\
F_1 &= \frac{dF_0}{dx} \\
&= \frac{\partial F_0}{\partial x} + \frac{\partial F_0}{\partial y} y' + \frac{\partial F_0}{\partial y'} F_0 \\
&= \frac{-y+y'x}{x-1} \\
F_2 &= \frac{dF_1}{dx} \\
&= \frac{\partial F_1}{\partial x} + \frac{\partial F_1}{\partial y} y' + \frac{\partial F_1}{\partial y'} F_1 \\
&= \frac{-y+y'x}{x-1} \\
F_3 &= \frac{dF_2}{dx} \\
&= \frac{\partial F_2}{\partial x} + \frac{\partial F_2}{\partial y} y' + \frac{\partial F_2}{\partial y'} F_2 \\
&= \frac{-y+y'x}{x-1} \\
F_4 &= \frac{dF_3}{dx} \\
&= \frac{\partial F_3}{\partial x} + \frac{\partial F_3}{\partial y} y' + \frac{\partial F_3}{\partial y'} F_3 \\
&= \frac{-y+y'x}{x-1}
\end{aligned}
$$

And so on. Evaluating all the above at initial conditions $x=0$ and $y(0)=y(0)$ and $y'(0)=y'(0)$ gives

$$
\begin{aligned}
F_0 &= y(0) \\
F_1 &= y(0) \\
F_2 &= y(0) \\
F_3 &= y(0) \\
F_4 &= y(0)
\end{aligned}
$$

Substituting all the above in (7) and simplifying gives the solution as

$$
y = \left(1 + \frac{1}{2}x^2 + \frac{1}{6}x^3 + \frac{1}{24}x^4 + \frac{1}{120}x^5 + \frac{1}{720}x^6\right) y(0) + y'(0)\, x + O\left(x^6\right)
$$

Since the expansion point $x = 0$ is an ordinary, we can also solve this using standard power series The ode is normalized to be

$$(1 - x) y'' + y'x - y = 0$$

Let the solution be represented as power series of the form

$$y = \sum_{n=0}^{\infty} a_n x^n$$

Then

$$y' = \sum_{n=1}^{\infty} n a_n x^{n-1}$$
$$y'' = \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2}$$

Substituting the above back into the ode gives

$$(1 - x) \left(\sum_{n=2}^{\infty} n(n-1) a_n x^{n-2}\right) + \left(\sum_{n=1}^{\infty} n a_n x^{n-1}\right) x - \left(\sum_{n=0}^{\infty} a_n x^n\right) = 0 \tag{1}$$

Which simplifies to

$$\sum_{n=2}^{\infty} \left(-n\, x^{n-1} a_n (n-1)\right) + \left(\sum_{n=2}^{\infty} n(n-1) a_n x^{n-2}\right) + \left(\sum_{n=1}^{\infty} n a_n x^n\right) + \sum_{n=0}^{\infty} (-a_n x^n) = 0 \tag{2}$$

The next step is to make all powers of x be n in each summation term. Going over each summation term above with power of x in it which is not already x^n and adjusting the power and the corresponding index gives

$$\sum_{n=2}^{\infty} \left(-n\, x^{n-1} a_n (n-1)\right) = \sum_{n=1}^{\infty} \left(-(n+1) a_{n+1} n\, x^n\right)$$
$$\sum_{n=2}^{\infty} n(n-1) a_n x^{n-2} = \sum_{n=0}^{\infty} (n+2) a_{n+2} (n+1) x^n$$

Substituting all the above in Eq (2) gives the following equation where now all powers of x are the same and equal to n.

$$
\sum_{n=1}^{\infty} \left(-(n+1) a_{n+1} n x^{n}\right) + \left(\sum_{n=0}^{\infty} (n+2) a_{n+2} (n+1) x^{n}\right) + \left(\sum_{n=1}^{\infty} n a_{n} x^{n}\right) + \sum_{n=0}^{\infty} \left(-a_{n} x^{n}\right) = 0 \tag{3}
$$

$n = 0$ gives

$$
2a_2 - a_0 = 0
$$

$$
a_2 = \frac{a_0}{2}
$$

For $1 \leq n$, the recurrence equation is

$$
-(n+1) a_{n+1} n + (n+2) a_{n+2} (n+1) + n a_n - a_n = 0 \tag{4}
$$

Solving for a_{n+2}, gives

$$
\begin{aligned}
a_{n+2} &= \frac{n^2 a_{n+1} - n a_n + n a_{n+1} + a_n}{(n+2)(n+1)} \\
&= \frac{(-n+1) a_n}{(n+2)(n+1)} + \frac{(n^2+n) a_{n+1}}{(n+2)(n+1)}
\end{aligned} \tag{5}
$$

For $n = 1$ the recurrence equation gives

$$
-2a_2 + 6a_3 = 0
$$

Which after substituting the earlier terms found becomes

$$
a_3 = \frac{a_0}{6}
$$

For $n = 2$ the recurrence equation gives

$$
-6a_3 + 12a_4 + a_2 = 0
$$

Which after substituting the earlier terms found becomes

$$a_4 = \frac{a_0}{24}$$

For $n = 3$ the recurrence equation gives

$$-12a_4 + 20a_5 + 2a_3 = 0$$

Which after substituting the earlier terms found becomes

$$a_5 = \frac{a_0}{120}$$

For $n = 4$ the recurrence equation gives

$$-20a_5 + 30a_6 + 3a_4 = 0$$

Which after substituting the earlier terms found becomes

$$a_6 = \frac{a_0}{720}$$

For $n = 5$ the recurrence equation gives

$$-30a_6 + 42a_7 + 4a_5 = 0$$

Which after substituting the earlier terms found becomes

$$a_7 = \frac{a_0}{5040}$$

And so on. Therefore the solution is

$$\begin{aligned} y &= \sum_{n=0}^{\infty} a_n x^n \\ &= a_3 x^3 + a_2 x^2 + a_1 x + a_0 + \dots \end{aligned}$$

Substituting the values for a_n found above, the solution becomes

$$y = a_0 + a_1 x + \frac{1}{2} a_0 x^2 + \frac{1}{6} a_0 x^3 + \frac{1}{24} a_0 x^4 + \frac{1}{120} a_0 x^5 + \dots$$

Collecting terms, the solution becomes

$$y = \left(1+\frac{1}{2}x^2+\frac{1}{6}x^3+\frac{1}{24}x^4+\frac{1}{120}x^5\right)a_0+a_1x+O\left(x^6\right) \tag{3}$$

At $x=0$ the solution above becomes

$$y = \left(1+\frac{1}{2}x^2+\frac{1}{6}x^3+\frac{1}{24}x^4+\frac{1}{120}x^5\right)c_1+c_2x+O\left(x^6\right)$$

<u>Summary</u>
The solution(s) found are the following

$$y = \left(1+\frac{1}{2}x^2+\frac{1}{6}x^3+\frac{1}{24}x^4+\frac{1}{120}x^5+\frac{1}{720}x^6\right)y(0)+y'(0)\,x+O\left(x^6\right) \tag{1}$$

$$y = \left(1+\frac{1}{2}x^2+\frac{1}{6}x^3+\frac{1}{24}x^4+\frac{1}{120}x^5\right)c_1+c_2x+O\left(x^6\right) \tag{2}$$

<u>Verification of solutions</u>

$$y = \left(1+\frac{1}{2}x^2+\frac{1}{6}x^3+\frac{1}{24}x^4+\frac{1}{120}x^5+\frac{1}{720}x^6\right)y(0)+y'(0)\,x+O\left(x^6\right)$$

Verified OK.

$$y = \left(1+\frac{1}{2}x^2+\frac{1}{6}x^3+\frac{1}{24}x^4+\frac{1}{120}x^5\right)c_1+c_2x+O\left(x^6\right)$$

Verified OK.

13.10.1 Maple step by step solution

Let's solve

$(1-x)\,y''+y'x-y=0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y''=-\frac{y}{x-1}+\frac{xy'}{x-1}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y''-\frac{xy'}{x-1}+\frac{y}{x-1}=0$

- ☐ Check to see if $x_0 = 1$ is a regular singular point
  - Define functions
    $\left[P_2(x) = -\frac{x}{x-1}, P_3(x) = \frac{1}{x-1}\right]$
  - $(x-1)\cdot P_2(x)$ is analytic at $x = 1$
    $$\left((x-1)\cdot P_2(x)\right)\bigg|_{x=1} = -1$$
  - $(x-1)^2\cdot P_3(x)$ is analytic at $x = 1$
    $$\left((x-1)^2\cdot P_3(x)\right)\bigg|_{x=1} = 0$$
  - $x = 1$is a regular singular point
    Check to see if $x_0 = 1$ is a regular singular point
    $x_0 = 1$
- Multiply by denominators
  $(x-1)\,y'' - y'x + y = 0$
- Change variables using $x = u + 1$ so that the regular singular point is at $u = 0$
  $$u\left(\frac{d^2}{du^2}y(u)\right) + (-u-1)\left(\frac{d}{du}y(u)\right) + y(u) = 0$$
- Assume series solution for $y(u)$
  $$y(u) = \sum_{k=0}^{\infty} a_k u^{k+r}$$
- ☐ Rewrite ODE with series expansions
  - Convert $u^m\cdot\left(\frac{d}{du}y(u)\right)$ to series expansion for $m = 0..1$
    $$u^m\cdot\left(\frac{d}{du}y(u)\right) = \sum_{k=0}^{\infty} a_k(k+r)\,u^{k+r-1+m}$$
  - Shift index using $k- >k+1-m$
    $$u^m\cdot\left(\frac{d}{du}y(u)\right) = \sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\,u^{k+r}$$
  - Convert $u\cdot\left(\frac{d^2}{du^2}y(u)\right)$ to series expansion
    $$u\cdot\left(\frac{d^2}{du^2}y(u)\right) = \sum_{k=0}^{\infty} a_k(k+r)\,(k+r-1)\,u^{k+r-1}$$
  - Shift index using $k- >k+1$

$$u \cdot \left(\frac{d^2}{du^2}y(u)\right) = \sum_{k=-1}^{\infty} a_{k+1}(k+1+r)(k+r)\,u^{k+r}$$

Rewrite ODE with series expansions

$$a_0 r(-2+r)\,u^{-1+r} + \left(\sum_{k=0}^{\infty} \left(a_{k+1}(k+1+r)(k+r-1) - a_k(k+r-1)\right) u^{k+r}\right) = 0$$

- $a_0$cannot be 0 by assumption, giving the indicial equation

$$r(-2+r) = 0$$

- Values of r that satisfy the indicial equation

$$r \in \{0, 2\}$$

- Each term in the series must be 0, giving the recursion relation

$$(k+r-1)\left(a_{k+1}(k+1+r) - a_k\right) = 0$$

- Recursion relation that defines series solution to ODE

$$a_{k+1} = \frac{a_k}{k+1+r}$$

- Recursion relation for $r = 0$

$$a_{k+1} = \frac{a_k}{k+1}$$

- Solution for $r = 0$

$$\left[y(u) = \sum_{k=0}^{\infty} a_k u^k, a_{k+1} = \frac{a_k}{k+1}\right]$$

- Revert the change of variables $u = x - 1$

$$\left[y = \sum_{k=0}^{\infty} a_k (x-1)^k, a_{k+1} = \frac{a_k}{k+1}\right]$$

- Recursion relation for $r = 2$

$$a_{k+1} = \frac{a_k}{k+3}$$

- Solution for $r = 2$

$$\left[y(u) = \sum_{k=0}^{\infty} a_k u^{k+2}, a_{k+1} = \frac{a_k}{k+3}\right]$$

- Revert the change of variables $u = x - 1$

$$\left[y = \sum_{k=0}^{\infty} a_k (x-1)^{k+2}, a_{k+1} = \frac{a_k}{k+3}\right]$$

- Combine solutions and rename parameters

$$\left[y = \left(\sum_{k=0}^{\infty} a_k (x-1)^k\right) + \left(\sum_{k=0}^{\infty} b_k (x-1)^{k+2}\right), a_{k+1} = \frac{a_k}{k+1}, b_{k+1} = \frac{b_k}{k+3}\right]$$

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Reducible group (found another exponential solution)
<- Kovacics algorithm successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 34

```
Order:=6;
dsolve((1-x)*diff(y(x),x$2)+x*diff(y(x),x)-y(x)=0,y(x),type='series',x=0);
```

$$y(x) = \left(1 + \frac{1}{2}x^2 + \frac{1}{6}x^3 + \frac{1}{24}x^4 + \frac{1}{120}x^5\right) y(0) + D(y)(0)\,x + O(x^6)$$

✓ Solution by Mathematica

Time used: 0.001 (sec). Leaf size: 41

```
AsymptoticDSolveValue[(1-x)*y''[x]+x*y'[x]-y[x]==0,y[x],{x,0,5}]
```

$$y(x) \to c_1\left(\frac{x^5}{120} + \frac{x^4}{24} + \frac{x^3}{6} + \frac{x^2}{2} + 1\right) + c_2x$$

13.11 problem 13

Internal problem ID [721]
Internal file name [OUTPUT/721_Sunday_June_05_2022_01_47_46_AM_4048532/index.tex]

Book: Elementary differential equations and boundary value problems, 10th ed., Boyce and DiPrima
Section: Chapter 5.2, Series Solutions Near an Ordinary Point, Part I. page 263
Problem number: 13.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second order series method. Ordinary point", "second order series method. Taylor series method"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

$$\boxed{2y'' + y'x + 3y = 0}$$

With the expansion point for the power series method at $x = 0$.

Solving ode using Taylor series method. This gives review on how the Taylor series method works for solving second order ode.

Let

$$y'' = f(x, y, y')$$

Assuming expansion is at $x_0 = 0$ (we can always shift the actual expansion point to 0 by change of variables) and assuming $f(x, y, y')$ is analytic at x_0 which must be the case for an ordinary point. Let initial conditions be $y(x_0) = y_0$ and $y'(x_0) = y'_0$. Using Taylor series gives

$$\begin{aligned} y(x) &= y(x_0) + (x - x_0)\, y'(x_0) + \frac{(x - x_0)^2}{2} y''(x_0) + \frac{(x - x_0)^3}{3!} y'''(x_0) + \cdots \\ &= y_0 + xy'_0 + \frac{x^2}{2} f|_{x_0,y_0,y'_0} + \frac{x^3}{3!} f'|_{x_0,y_0,y'_0} + \cdots \\ &= y_0 + xy'_0 + \sum_{n=0}^{\infty} \frac{x^{n+2}}{(n+2)\,!} \frac{d^n f}{dx^n}\bigg|_{x_0,y_0,y'_0} \end{aligned}$$

But

$$\frac{df}{dx} = \frac{\partial f}{\partial x}\frac{dx}{dx} + \frac{\partial f}{\partial y}\frac{dy}{dx} + \frac{\partial f}{\partial y'}\frac{dy'}{dx} \tag{1}$$

$$= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}y' + \frac{\partial f}{\partial y'}y'' \tag{774}$$

$$= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}y' + \frac{\partial f}{\partial y'}f \tag{775}$$

$$\frac{d^2 f}{dx^2} = \frac{d}{dx}\left(\frac{df}{dx}\right)$$

$$= \frac{\partial}{\partial x}\left(\frac{df}{dx}\right) + \frac{\partial}{\partial y}\left(\frac{df}{dx}\right)y' + \frac{\partial}{\partial y'}\left(\frac{df}{dx}\right)f \tag{2}$$

$$\frac{d^3 f}{dx^3} = \frac{d}{dx}\left(\frac{d^2 f}{dx^2}\right)$$

$$= \frac{\partial}{\partial x}\left(\frac{d^2 f}{dx^2}\right) + \left(\frac{\partial}{\partial y}\frac{d^2 f}{dx^2}\right)y' + \frac{\partial}{\partial y'}\left(\frac{d^2 f}{dx^2}\right)f \tag{3}$$

$$\vdots$$

And so on. Hence if we name $F_0 = f(x,y,y')$ then the above can be written as

$$
\begin{aligned}
F_0 &= f(x,y,y') \qquad (4)\\
F_1 &= \frac{df}{dx}\\
&= \frac{dF_0}{dx}\\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}y' + \frac{\partial f}{\partial y'}y''\\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}y' + \frac{\partial f}{\partial y'}f \qquad (5)\\
&= \frac{\partial F_0}{\partial x} + \frac{\partial F_0}{\partial y}y' + \frac{\partial F_0}{\partial y'}F_0\\
F_2 &= \frac{d}{dx}\left(\frac{d}{dx}f\right)\\
&= \frac{d}{dx}(F_1)\\
&= \frac{\partial}{\partial x}F_1 + \left(\frac{\partial F_1}{\partial y}\right)y' + \left(\frac{\partial F_1}{\partial y'}\right)y''\\
&= \frac{\partial}{\partial x}F_1 + \left(\frac{\partial F_1}{\partial y}\right)y' + \left(\frac{\partial F_1}{\partial y'}\right)F_0\\
&\vdots\\
F_n &= \frac{d}{dx}(F_{n-1})\\
&= \frac{\partial}{\partial x}F_{n-1} + \left(\frac{\partial F_{n-1}}{\partial y}\right)y' + \left(\frac{\partial F_{n-1}}{\partial y'}\right)y''\\
&= \frac{\partial}{\partial x}F_{n-1} + \left(\frac{\partial F_{n-1}}{\partial y}\right)y' + \left(\frac{\partial F_{n-1}}{\partial y'}\right)F_0 \qquad (6)
\end{aligned}
$$

Therefore (6) can be used from now on along with

$$y(x) = y_0 + xy_0' + \sum_{n=0}^{\infty} \frac{x^{n+2}}{(n+2)!} F_n|_{x_0,y_0,y_0'} \qquad (7)$$

To find $y(x)$ series solution around $x=0$. Hence

$$
\begin{aligned}
F_0 &= -\frac{y'x}{2} - \frac{3y}{2} \\
F_1 &= \frac{dF_0}{dx} \\
&= \frac{\partial F_0}{\partial x} + \frac{\partial F_0}{\partial y} y' + \frac{\partial F_0}{\partial y'} F_0 \\
&= \frac{y'x^2}{4} + \frac{3yx}{4} - 2y' \\
F_2 &= \frac{dF_1}{dx} \\
&= \frac{\partial F_1}{\partial x} + \frac{\partial F_1}{\partial y} y' + \frac{\partial F_1}{\partial y'} F_1 \\
&= \frac{(-x^3+18x)\, y'}{8} + \frac{(-3x^2+30)\, y}{8} \\
F_3 &= \frac{dF_2}{dx} \\
&= \frac{\partial F_2}{\partial x} + \frac{\partial F_2}{\partial y} y' + \frac{\partial F_2}{\partial y'} F_2 \\
&= \frac{(x^4-30x^2+96)\, y'}{16} + \frac{3xy(x^2-22)}{16} \\
F_4 &= \frac{dF_3}{dx} \\
&= \frac{\partial F_3}{\partial x} + \frac{\partial F_3}{\partial y} y' + \frac{\partial F_3}{\partial y'} F_3 \\
&= \frac{(-x^5+44x^3-348x)\, y'}{32} - \frac{3y(x^4-36x^2+140)}{32}
\end{aligned}
$$

And so on. Evaluating all the above at initial conditions $x=0$ and $y(0)=y(0)$ and $y'(0)=y'(0)$ gives

$$
\begin{aligned}
F_0 &= -\frac{3y(0)}{2} \\
F_1 &= -2y'(0) \\
F_2 &= \frac{15y(0)}{4} \\
F_3 &= 6y'(0) \\
F_4 &= -\frac{105y(0)}{8}
\end{aligned}
$$

Substituting all the above in (7) and simplifying gives the solution as

$$
y = \left(1 - \frac{3}{4}x^2 + \frac{5}{32}x^4 - \frac{7}{384}x^6\right) y(0) + \left(x - \frac{1}{3}x^3 + \frac{1}{20}x^5\right) y'(0) + O\left(x^6\right)
$$

Since the expansion point $x = 0$ is an ordinary, we can also solve this using standard power series Let the solution be represented as power series of the form

$$y = \sum_{n=0}^{\infty} a_n x^n$$

Then

$$y' = \sum_{n=1}^{\infty} n a_n x^{n-1}$$
$$y'' = \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2}$$

Substituting the above back into the ode gives

$$\sum_{n=2}^{\infty} n(n-1) a_n x^{n-2} = -\frac{\left(\sum_{n=1}^{\infty} n a_n x^{n-1}\right) x}{2} - \frac{3\left(\sum_{n=0}^{\infty} a_n x^n\right)}{2} \tag{1}$$

Which simplifies to

$$\left(\sum_{n=2}^{\infty} 2n(n-1) a_n x^{n-2}\right) + \left(\sum_{n=1}^{\infty} n\, x^n a_n\right) + \left(\sum_{n=0}^{\infty} 3a_n x^n\right) = 0 \tag{2}$$

The next step is to make all powers of x be n in each summation term. Going over each summation term above with power of x in it which is not already x^n and adjusting the power and the corresponding index gives

$$\sum_{n=2}^{\infty} 2n(n-1) a_n x^{n-2} = \sum_{n=0}^{\infty} 2(n+2) a_{n+2}(n+1) x^n$$

Substituting all the above in Eq (2) gives the following equation where now all powers of x are the same and equal to n.

$$\left(\sum_{n=0}^{\infty} 2(n+2) a_{n+2}(n+1) x^n\right) + \left(\sum_{n=1}^{\infty} n\, x^n a_n\right) + \left(\sum_{n=0}^{\infty} 3a_n x^n\right) = 0 \tag{3}$$

$n = 0$ gives

$$4a_2 + 3a_0 = 0$$

$$
a_2 = -\frac{3a_0}{4}
$$

For $1 \leq n$, the recurrence equation is

$$
2(n+2)\, a_{n+2}(n+1) + n a_n + 3a_n = 0 \tag{4}
$$

Solving for a_{n+2}, gives

$$
a_{n+2} = -\frac{a_n(n+3)}{2\,(n+2)\,(n+1)} \tag{5}
$$

For $n = 1$ the recurrence equation gives

$$
12a_3 + 4a_1 = 0
$$

Which after substituting the earlier terms found becomes

$$
a_3 = -\frac{a_1}{3}
$$

For $n = 2$ the recurrence equation gives

$$
24a_4 + 5a_2 = 0
$$

Which after substituting the earlier terms found becomes

$$
a_4 = \frac{5a_0}{32}
$$

For $n = 3$ the recurrence equation gives

$$
40a_5 + 6a_3 = 0
$$

Which after substituting the earlier terms found becomes

$$
a_5 = \frac{a_1}{20}
$$

For $n = 4$ the recurrence equation gives

$$
60a_6 + 7a_4 = 0
$$

Which after substituting the earlier terms found becomes

$$a_6 = -\frac{7a_0}{384}$$

For $n = 5$ the recurrence equation gives

$$84a_7 + 8a_5 = 0$$

Which after substituting the earlier terms found becomes

$$a_7 = -\frac{a_1}{210}$$

And so on. Therefore the solution is

$$\begin{aligned} y &= \sum_{n=0}^{\infty} a_n x^n \\ &= a_3 x^3 + a_2 x^2 + a_1 x + a_0 + \dots \end{aligned}$$

Substituting the values for a_n found above, the solution becomes

$$y = a_0 + a_1 x - \frac{3}{4} a_0 x^2 - \frac{1}{3} a_1 x^3 + \frac{5}{32} a_0 x^4 + \frac{1}{20} a_1 x^5 + \dots$$

Collecting terms, the solution becomes

$$y = \left(1 - \frac{3}{4} x^2 + \frac{5}{32} x^4\right) a_0 + \left(x - \frac{1}{3} x^3 + \frac{1}{20} x^5\right) a_1 + O(x^6) \tag{3}$$

At $x = 0$ the solution above becomes

$$y = \left(1 - \frac{3}{4} x^2 + \frac{5}{32} x^4\right) c_1 + \left(x - \frac{1}{3} x^3 + \frac{1}{20} x^5\right) c_2 + O(x^6)$$

Summary

The solution(s) found are the following

$$y = \left(1 - \frac{3}{4} x^2 + \frac{5}{32} x^4 - \frac{7}{384} x^6\right) y(0) + \left(x - \frac{1}{3} x^3 + \frac{1}{20} x^5\right) y'(0) + O(x^6) \tag{1}$$

$$y = \left(1 - \frac{3}{4} x^2 + \frac{5}{32} x^4\right) c_1 + \left(x - \frac{1}{3} x^3 + \frac{1}{20} x^5\right) c_2 + O(x^6) \tag{2}$$

Verification of solutions

$$y = \left(1-\frac{3}{4}x^{2}+\frac{5}{32}x^{4}-\frac{7}{384}x^{6}\right) y(0)+\left(x-\frac{1}{3}x^{3}+\frac{1}{20}x^{5}\right) y'(0)+O\left(x^{6}\right)$$

Verified OK.

$$y = \left(1-\frac{3}{4}x^{2}+\frac{5}{32}x^{4}\right) c_{1}+\left(x-\frac{1}{3}x^{3}+\frac{1}{20}x^{5}\right) c_{2}+O\left(x^{6}\right)$$

Verified OK.

13.11.1 Maple step by step solution

Let's solve

$y'' = -\frac{y' x}{2}-\frac{3y}{2}$

- Highest derivative means the order of the ODE is 2

  y''

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y''+\frac{y' x}{2}+\frac{3y}{2} = 0$

- Multiply by denominators

  $2y''+y' x+3y = 0$

- Assume series solution for y

  $y = \sum_{k=0}^{\infty} a_{k} x^{k}$

□ Rewrite DE with series expansions

  - Convert $x \cdot y'$ to series expansion

    $x \cdot y' = \sum_{k=0}^{\infty} a_{k} k\, x^{k}$

  - Convert y'' to series expansion

    $y'' = \sum_{k=2}^{\infty} a_{k} k(k-1)\, x^{k-2}$

  - Shift index using $k->k+2$

    $y'' = \sum_{k=0}^{\infty} a_{k+2}(k+2)\,(k+1)\, x^{k}$

  Rewrite DE with series expansions

$$\sum_{k=0}^{\infty} \left(2a_{k+2}(k+2)\,(k+1)+a_k(k+3)\right) x^k = 0$$

- Each term in the series must be 0, giving the recursion relation

$$\left(2k^2+6k+4\right) a_{k+2}+a_k(k+3)=0$$

- Recursion relation that defines the series solution to the ODE

$$\left[y=\sum_{k=0}^{\infty} a_k x^k, a_{k+2}=-\frac{a_k(k+3)}{2(k^2+3k+2)}\right]$$

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Group is reducible, not completely reducible
   Solution has integrals. Trying a special function solution free of integrals...
   -> Trying a solution in terms of special functions:
      -> Bessel
      -> elliptic
      -> Legendre
      <- Kummer successful
   <- special function solution successful
      -> Trying to convert hypergeometric functions to elementary form...
      <- elementary form could result into a too large expression - returning special functi
   <- Kovacics algorithm successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 34

```
Order:=6;
dsolve(2*diff(y(x),x$2)+x*diff(y(x),x)+3*y(x)=0,y(x),type='series',x=0);
```

$$y(x) = \left(1 - \frac{3}{4}x^2 + \frac{5}{32}x^4\right) y(0) + \left(x - \frac{1}{3}x^3 + \frac{1}{20}x^5\right) D(y)(0) + O(x^6)$$

✓ Solution by Mathematica

Time used: 0.001 (sec). Leaf size: 42

```
AsymptoticDSolveValue[2*y''[x]+x*y'[x]+3*y[x]==0,y[x],{x,0,5}]
```

$$y(x) \to c_2\left(\frac{x^5}{20} - \frac{x^3}{3} + x\right) + c_1\left(\frac{5x^4}{32} - \frac{3x^2}{4} + 1\right)$$

13.12 problem 15

Internal problem ID [722]
Internal file name [`OUTPUT/722_Sunday_June_05_2022_01_47_48_AM_32860525/index.tex`]

Book: Elementary differential equations and boundary value problems, 10th ed., Boyce and DiPrima
Section: Chapter 5.2, Series Solutions Near an Ordinary Point, Part I. page 263
Problem number: 15.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"exact linear second order ode", "second_order_integrable_as_is", "second order series method. Ordinary point", "second order series method. Taylor series method"**

Maple gives the following as the ode type

```
[[_2nd_order, _exact, _linear, _homogeneous]]
```

$$\boxed{y'' - y'x - y = 0}$$

With initial conditions

$$[y(0) = 2, y'(0) = 1]$$

With the expansion point for the power series method at $x = 0$.

13.12.1 Existence and uniqueness analysis

This is a linear ODE. In canonical form it is written as

$$y'' + p(x)y' + q(x)y = F$$

Where here

$$\begin{aligned} p(x) &= -x \\ q(x) &= -1 \\ F &= 0 \end{aligned}$$

Hence the ode is

$$y'' - y'x - y = 0$$

The domain of $p(x) = -x$ is

$$\{-\infty < x < \infty\}$$

And the point $x_0 = 0$ is inside this domain. The domain of $q(x) = -1$ is

$$\{-\infty < x < \infty\}$$

And the point $x_0 = 0$ is also inside this domain. Hence solution exists and is unique.

Solving ode using Taylor series method. This gives review on how the Taylor series method works for solving second order ode.

Let

$$y'' = f(x, y, y')$$

Assuming expansion is at $x_0 = 0$ (we can always shift the actual expansion point to 0 by change of variables) and assuming $f(x, y, y')$ is analytic at x_0 which must be the case for an ordinary point. Let initial conditions be $y(x_0) = y_0$ and $y'(x_0) = y'_0$. Using Taylor series gives

$$\begin{aligned}
y(x) &= y(x_0) + (x - x_0)\, y'(x_0) + \frac{(x - x_0)^2}{2} y''(x_0) + \frac{(x - x_0)^3}{3!} y'''(x_0) + \cdots \\
&= y_0 + x y'_0 + \frac{x^2}{2} f|_{x_0, y_0, y'_0} + \frac{x^3}{3!} f'|_{x_0, y_0, y'_0} + \cdots \\
&= y_0 + x y'_0 + \sum_{n=0}^{\infty} \frac{x^{n+2}}{(n+2)\,!} \left.\frac{d^n f}{dx^n}\right|_{x_0, y_0, y'_0}
\end{aligned}$$

But

$$\begin{aligned}
\frac{df}{dx} &= \frac{\partial f}{\partial x}\frac{dx}{dx} + \frac{\partial f}{\partial y}\frac{dy}{dx} + \frac{\partial f}{\partial y'}\frac{dy'}{dx} && (1)\\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}y' + \frac{\partial f}{\partial y'}y'' && (777)\\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}y' + \frac{\partial f}{\partial y'}f && (778)\\
\frac{d^2 f}{dx^2} &= \frac{d}{dx}\left(\frac{df}{dx}\right)\\
&= \frac{\partial}{\partial x}\left(\frac{df}{dx}\right) + \frac{\partial}{\partial y}\left(\frac{df}{dx}\right)y' + \frac{\partial}{\partial y'}\left(\frac{df}{dx}\right)f && (2)\\
\frac{d^3 f}{dx^3} &= \frac{d}{dx}\left(\frac{d^2 f}{dx^2}\right)\\
&= \frac{\partial}{\partial x}\left(\frac{d^2 f}{dx^2}\right) + \left(\frac{\partial}{\partial y}\frac{d^2 f}{dx^2}\right)y' + \frac{\partial}{\partial y'}\left(\frac{d^2 f}{dx^2}\right)f && (3)\\
&\vdots
\end{aligned}$$

And so on. Hence if we name $F_0 = f(x,y,y')$ then the above can be written as

$$\begin{aligned}
F_0 &= f(x,y,y') \qquad (4)\\
F_1 &= \frac{df}{dx}\\
&= \frac{dF_0}{dx}\\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}y' + \frac{\partial f}{\partial y'}y''\\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}y' + \frac{\partial f}{\partial y'}f \qquad (5)\\
&= \frac{\partial F_0}{\partial x} + \frac{\partial F_0}{\partial y}y' + \frac{\partial F_0}{\partial y'}F_0\\
F_2 &= \frac{d}{dx}\left(\frac{d}{dx}f\right)\\
&= \frac{d}{dx}(F_1)\\
&= \frac{\partial}{\partial x}F_1 + \left(\frac{\partial F_1}{\partial y}\right)y' + \left(\frac{\partial F_1}{\partial y'}\right)y''\\
&= \frac{\partial}{\partial x}F_1 + \left(\frac{\partial F_1}{\partial y}\right)y' + \left(\frac{\partial F_1}{\partial y'}\right)F_0\\
&\vdots\\
F_n &= \frac{d}{dx}(F_{n-1})\\
&= \frac{\partial}{\partial x}F_{n-1} + \left(\frac{\partial F_{n-1}}{\partial y}\right)y' + \left(\frac{\partial F_{n-1}}{\partial y'}\right)y''\\
&= \frac{\partial}{\partial x}F_{n-1} + \left(\frac{\partial F_{n-1}}{\partial y}\right)y' + \left(\frac{\partial F_{n-1}}{\partial y'}\right)F_0 \qquad (6)
\end{aligned}$$

Therefore (6) can be used from now on along with

$$y(x) = y_0 + xy_0' + \sum_{n=0}^{\infty} \frac{x^{n+2}}{(n+2)!} F_n|_{x_0,y_0,y_0'} \tag{7}$$

To find $y(x)$ series solution around $x = 0$. Hence

$$
\begin{aligned}
F_0 &= y'x + y \\
F_1 &= \frac{dF_0}{dx} \\
&= \frac{\partial F_0}{\partial x} + \frac{\partial F_0}{\partial y} y' + \frac{\partial F_0}{\partial y'} F_0 \\
&= y'x^2 + yx + 2y' \\
F_2 &= \frac{dF_1}{dx} \\
&= \frac{\partial F_1}{\partial x} + \frac{\partial F_1}{\partial y} y' + \frac{\partial F_1}{\partial y'} F_1 \\
&= \left(x^3 + 5x\right) y' + y\left(x^2 + 3\right) \\
F_3 &= \frac{dF_2}{dx} \\
&= \frac{\partial F_2}{\partial x} + \frac{\partial F_2}{\partial y} y' + \frac{\partial F_2}{\partial y'} F_2 \\
&= \left(x^4 + 9x^2 + 8\right) y' + xy\left(x^2 + 7\right) \\
F_4 &= \frac{dF_3}{dx} \\
&= \frac{\partial F_3}{\partial x} + \frac{\partial F_3}{\partial y} y' + \frac{\partial F_3}{\partial y'} F_3 \\
&= \left(x^5 + 14x^3 + 33x\right) y' + y\left(x^4 + 12x^2 + 15\right)
\end{aligned}
$$

And so on. Evaluating all the above at initial conditions $x = 0$ and $y(0) = 2$ and $y'(0) = 1$ gives

$$
\begin{aligned}
F_0 &= 2 \\
F_1 &= 2 \\
F_2 &= 6 \\
F_3 &= 8 \\
F_4 &= 30
\end{aligned}
$$

Substituting all the above in (7) and simplifying gives the solution as

$$
y = x^2 + x + 2 + \frac{x^3}{3} + \frac{x^4}{4} + \frac{x^5}{15} + \frac{x^6}{24} + O\left(x^6\right)
$$

$$
y = x^2 + x + 2 + \frac{x^3}{3} + \frac{x^4}{4} + \frac{x^5}{15} + \frac{x^6}{24} + O\left(x^6\right)
$$

Since the expansion point $x = 0$ is an ordinary, we can also solve this using standard power series Let the solution be represented as power series of the form

$$y = \sum_{n=0}^{\infty} a_n x^n$$

Then

$$y' = \sum_{n=1}^{\infty} n a_n x^{n-1}$$
$$y'' = \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2}$$

Substituting the above back into the ode gives

$$\sum_{n=2}^{\infty} n(n-1) a_n x^{n-2} = \left(\sum_{n=1}^{\infty} n a_n x^{n-1}\right) x + \left(\sum_{n=0}^{\infty} a_n x^n\right) \tag{1}$$

Which simplifies to

$$\left(\sum_{n=2}^{\infty} n(n-1) a_n x^{n-2}\right) + \sum_{n=1}^{\infty} (-n\, x^n a_n) + \sum_{n=0}^{\infty} (-a_n x^n) = 0 \tag{2}$$

The next step is to make all powers of x be n in each summation term. Going over each summation term above with power of x in it which is not already x^n and adjusting the power and the corresponding index gives

$$\sum_{n=2}^{\infty} n(n-1) a_n x^{n-2} = \sum_{n=0}^{\infty} (n+2) a_{n+2}(n+1) x^n$$

Substituting all the above in Eq (2) gives the following equation where now all powers of x are the same and equal to n.

$$\left(\sum_{n=0}^{\infty} (n+2) a_{n+2}(n+1) x^n\right) + \sum_{n=1}^{\infty} (-n\, x^n a_n) + \sum_{n=0}^{\infty} (-a_n x^n) = 0 \tag{3}$$

$n = 0$ gives

$$2a_2 - a_0 = 0$$

$$a_2 = \frac{a_0}{2}$$

For $1 \leq n$, the recurrence equation is

$$(n+2)\, a_{n+2}(n+1) - n a_n - a_n = 0 \tag{4}$$

Solving for a_{n+2}, gives

$$a_{n+2} = \frac{a_n}{n+2} \tag{5}$$

For $n = 1$ the recurrence equation gives

$$6a_3 - 2a_1 = 0$$

Which after substituting the earlier terms found becomes

$$a_3 = \frac{a_1}{3}$$

For $n = 2$ the recurrence equation gives

$$12a_4 - 3a_2 = 0$$

Which after substituting the earlier terms found becomes

$$a_4 = \frac{a_0}{8}$$

For $n = 3$ the recurrence equation gives

$$20a_5 - 4a_3 = 0$$

Which after substituting the earlier terms found becomes

$$a_5 = \frac{a_1}{15}$$

For $n = 4$ the recurrence equation gives

$$30a_6 - 5a_4 = 0$$

Which after substituting the earlier terms found becomes

$$a_6 = \frac{a_0}{48}$$

For $n = 5$ the recurrence equation gives

$$42a_7 - 6a_5 = 0$$

Which after substituting the earlier terms found becomes

$$a_7 = \frac{a_1}{105}$$

And so on. Therefore the solution is

$$\begin{aligned} y &= \sum_{n=0}^{\infty} a_n x^n \\ &= a_3 x^3 + a_2 x^2 + a_1 x + a_0 + \dots \end{aligned}$$

Substituting the values for a_n found above, the solution becomes

$$y = a_0 + a_1 x + \frac{1}{2} a_0 x^2 + \frac{1}{3} a_1 x^3 + \frac{1}{8} a_0 x^4 + \frac{1}{15} a_1 x^5 + \dots$$

Collecting terms, the solution becomes

$$y = \left(1 + \frac{1}{2}x^2 + \frac{1}{8}x^4\right) a_0 + \left(x + \frac{1}{3}x^3 + \frac{1}{15}x^5\right) a_1 + O(x^6) \tag{3}$$

At $x = 0$ the solution above becomes

$$y = \left(1 + \frac{1}{2}x^2 + \frac{1}{8}x^4\right) c_1 + \left(x + \frac{1}{3}x^3 + \frac{1}{15}x^5\right) c_2 + O(x^6)$$

$$y = 2 + x^2 + \frac{x^4}{4} + x + \frac{x^3}{3} + \frac{x^5}{15} + O(x^6)$$

<u>Summary</u>

The solution(s) found are the following

$$y = x^2 + x + 2 + \frac{x^3}{3} + \frac{x^4}{4} + \frac{x^5}{15} + \frac{x^6}{24} + O(x^6) \tag{1}$$

$$y = 2 + x^2 + \frac{x^4}{4} + x + \frac{x^3}{3} + \frac{x^5}{15} + O(x^6) \tag{2}$$

Verification of solutions

$$y = x^2 + x + 2 + \frac{x^3}{3} + \frac{x^4}{4} + \frac{x^5}{15} + \frac{x^6}{24} + O(x^6)$$

Verified OK.

$$y = 2 + x^2 + \frac{x^4}{4} + x + \frac{x^3}{3} + \frac{x^5}{15} + O(x^6)$$

Verified OK.

13.12.2 Maple step by step solution

Let's solve

$\left[y'' = y'x + y, y(0) = 2, y'\Big|_{\{x=0\}} = 1\right]$

- Highest derivative means the order of the ODE is 2

  y''

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y'' - y'x - y = 0$

- Assume series solution for y

  $y = \sum_{k=0}^{\infty} a_k x^k$

□ Rewrite DE with series expansions

  ◦ Convert $x \cdot y'$ to series expansion

  $x \cdot y' = \sum_{k=0}^{\infty} a_k k\, x^k$

  ◦ Convert y'' to series expansion

  $y'' = \sum_{k=2}^{\infty} a_k k(k-1)\, x^{k-2}$

  ◦ Shift index using $k->k+2$

  $y'' = \sum_{k=0}^{\infty} a_{k+2}(k+2)\,(k+1)\, x^k$

  Rewrite DE with series expansions

  $\sum_{k=0}^{\infty} \left(a_{k+2}(k+2)\,(k+1) - a_k(k+1)\right) x^k = 0$

- Each term in the series must be 0, giving the recursion relation

  $(k+1)(a_{k+2}(k+2)-a_k)=0$

- Recursion relation that defines the series solution to the ODE

  $\left[y=\sum_{k=0}^{\infty} a_k x^k, a_{k+2}=\frac{a_k}{k+2}\right]$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
<- linear_1 successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 20

```
Order:=6;
dsolve([diff(y(x),x$2)-x*diff(y(x),x)-y(x)=0,y(0) = 2, D(y)(0) = 1],y(x),type='series',x=0);
```

$$y(x)=2+x+x^2+\frac{1}{3}x^3+\frac{1}{4}x^4+\frac{1}{15}x^5+\mathrm{O}\left(x^6\right)$$

✓ Solution by Mathematica

Time used: 0.001 (sec). Leaf size: 30

```
AsymptoticDSolveValue[{y''[x]-x*y'[x]-y[x]==0,{y[0]==2,y'[0]==1}},y[x],{x,0,5}]
```

$$y(x)\to\frac{x^5}{15}+\frac{x^4}{4}+\frac{x^3}{3}+x^2+x+2$$

13.13 problem 16

Internal problem ID [723]
Internal file name [`OUTPUT/723_Sunday_June_05_2022_01_47_50_AM_35648494/index.tex`]

Book: Elementary differential equations and boundary value problems, 10th ed., Boyce and DiPrima
Section: Chapter 5.2, Series Solutions Near an Ordinary Point, Part I. page 263
Problem number: 16.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second order series method. Ordinary point", "second order series method. Taylor series method"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

$$\boxed{(x^2+2)\,y''-y'x+4y=0}$$

With initial conditions

$$[y(0)=-1,y'(0)=3]$$

With the expansion point for the power series method at $x=0$.

13.13.1 Existence and uniqueness analysis

This is a linear ODE. In canonical form it is written as

$$y''+p(x)y'+q(x)y=F$$

Where here

$$\begin{aligned}
p(x)&=-\frac{x}{x^2+2}\\
q(x)&=\frac{4}{x^2+2}\\
F&=0
\end{aligned}$$

Hence the ode is

$$y'' - \frac{xy'}{x^2+2} + \frac{4y}{x^2+2} = 0$$

The domain of $p(x) = -\frac{x}{x^2+2}$ is

$$\{-\infty < x < \infty\}$$

And the point $x_0 = 0$ is inside this domain. The domain of $q(x) = \frac{4}{x^2+2}$ is

$$\{-\infty < x < \infty\}$$

And the point $x_0 = 0$ is also inside this domain. Hence solution exists and is unique.

Solving ode using Taylor series method. This gives review on how the Taylor series method works for solving second order ode.

Let

$$y'' = f(x, y, y')$$

Assuming expansion is at $x_0 = 0$ (we can always shift the actual expansion point to 0 by change of variables) and assuming $f(x, y, y')$ is analytic at x_0 which must be the case for an ordinary point. Let initial conditions be $y(x_0) = y_0$ and $y'(x_0) = y'_0$. Using Taylor series gives

$$\begin{aligned}
y(x) &= y(x_0) + (x - x_0)\, y'(x_0) + \frac{(x - x_0)^2}{2} y''(x_0) + \frac{(x - x_0)^3}{3!} y'''(x_0) + \cdots \\
&= y_0 + x y'_0 + \frac{x^2}{2} f|_{x_0, y_0, y'_0} + \frac{x^3}{3!} f'|_{x_0, y_0, y'_0} + \cdots \\
&= y_0 + x y'_0 + \sum_{n=0}^{\infty} \frac{x^{n+2}}{(n+2)!} \frac{d^n f}{dx^n}\bigg|_{x_0, y_0, y'_0}
\end{aligned}$$

But

$$\frac{df}{dx} = \frac{\partial f}{\partial x}\frac{dx}{dx} + \frac{\partial f}{\partial y}\frac{dy}{dx} + \frac{\partial f}{\partial y'}\frac{dy'}{dx} \tag{1}$$

$$= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}y' + \frac{\partial f}{\partial y'}y'' \tag{780}$$

$$= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}y' + \frac{\partial f}{\partial y'}f \tag{781}$$

$$\frac{d^2 f}{dx^2} = \frac{d}{dx}\left(\frac{df}{dx}\right)$$

$$= \frac{\partial}{\partial x}\left(\frac{df}{dx}\right) + \frac{\partial}{\partial y}\left(\frac{df}{dx}\right) y' + \frac{\partial}{\partial y'}\left(\frac{df}{dx}\right) f \tag{2}$$

$$\frac{d^3 f}{dx^3} = \frac{d}{dx}\left(\frac{d^2 f}{dx^2}\right)$$

$$= \frac{\partial}{\partial x}\left(\frac{d^2 f}{dx^2}\right) + \left(\frac{\partial}{\partial y}\frac{d^2 f}{dx^2}\right) y' + \frac{\partial}{\partial y'}\left(\frac{d^2 f}{dx^2}\right) f \tag{3}$$

$$\vdots$$

And so on. Hence if we name $F_0 = f(x,y,y')$ then the above can be written as

$$
\begin{aligned}
F_0 &= f(x,y,y') \qquad (4)\\
F_1 &= \frac{df}{dx}\\
&= \frac{dF_0}{dx}\\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}y' + \frac{\partial f}{\partial y'}y''\\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}y' + \frac{\partial f}{\partial y'}f \qquad (5)\\
&= \frac{\partial F_0}{\partial x} + \frac{\partial F_0}{\partial y}y' + \frac{\partial F_0}{\partial y'}F_0\\
F_2 &= \frac{d}{dx}\left(\frac{d}{dx}f\right)\\
&= \frac{d}{dx}(F_1)\\
&= \frac{\partial}{\partial x}F_1 + \left(\frac{\partial F_1}{\partial y}\right)y' + \left(\frac{\partial F_1}{\partial y'}\right)y''\\
&= \frac{\partial}{\partial x}F_1 + \left(\frac{\partial F_1}{\partial y}\right)y' + \left(\frac{\partial F_1}{\partial y'}\right)F_0\\
&\vdots\\
F_n &= \frac{d}{dx}(F_{n-1})\\
&= \frac{\partial}{\partial x}F_{n-1} + \left(\frac{\partial F_{n-1}}{\partial y}\right)y' + \left(\frac{\partial F_{n-1}}{\partial y'}\right)y''\\
&= \frac{\partial}{\partial x}F_{n-1} + \left(\frac{\partial F_{n-1}}{\partial y}\right)y' + \left(\frac{\partial F_{n-1}}{\partial y'}\right)F_0 \qquad (6)
\end{aligned}
$$

Therefore (6) can be used from now on along with

$$
y(x) = y_0 + xy'_0 + \sum_{n=0}^{\infty}\frac{x^{n+2}}{(n+2)\,!}F_n|_{x_0,y_0,y'_0} \qquad (7)
$$

To find $y(x)$ series solution around $x=0$. Hence

$$
\begin{aligned}
F_0 &= \frac{y'x-4y}{x^2+2} \\
F_1 &= \frac{dF_0}{dx} \\
&= \frac{\partial F_0}{\partial x}+\frac{\partial F_0}{\partial y}y'+\frac{\partial F_0}{\partial y'}F_0 \\
&= \frac{-4y'x^2+4yx-6y'}{\left(x^2+2\right)^2} \\
F_2 &= \frac{dF_1}{dx} \\
&= \frac{\partial F_1}{\partial x}+\frac{\partial F_1}{\partial y}y'+\frac{\partial F_1}{\partial y'}F_1 \\
&= \frac{8x^3y'+4x^2y+10y'x+32y}{\left(x^2+2\right)^3} \\
F_3 &= \frac{dF_2}{dx} \\
&= \frac{\partial F_2}{\partial x}+\frac{\partial F_2}{\partial y}y'+\frac{\partial F_2}{\partial y'}F_2 \\
&= \frac{\left(-12x^4+48x^2+84\right)y'+\left(-48x^3-216x\right)y}{\left(x^2+2\right)^4} \\
F_4 &= \frac{dF_3}{dx} \\
&= \frac{\partial F_3}{\partial x}+\frac{\partial F_3}{\partial y}y'+\frac{\partial F_3}{\partial y'}F_3 \\
&= \frac{\left(-12x^5-648x^3-828x\right)y'+\left(288x^4+1032x^2-768\right)y}{\left(x^2+2\right)^5}
\end{aligned}
$$

And so on. Evaluating all the above at initial conditions $x=0$ and $y(0)=-1$ and $y'(0)=3$ gives

$$
\begin{aligned}
F_0 &= 2 \\
F_1 &= -\frac{9}{2} \\
F_2 &= -4 \\
F_3 &= \frac{63}{4} \\
F_4 &= 24
\end{aligned}
$$

Substituting all the above in (7) and simplifying gives the solution as

$$y = x^2 + 3x - 1 - \frac{3x^3}{4} - \frac{x^4}{6} + \frac{21x^5}{160} + \frac{x^6}{30} + O(x^6)$$

$$y = x^2 + 3x - 1 - \frac{3x^3}{4} - \frac{x^4}{6} + \frac{21x^5}{160} + \frac{x^6}{30} + O(x^6)$$

Since the expansion point $x = 0$ is an ordinary, we can also solve this using standard power series The ode is normalized to be

$$(x^2 + 2)\, y'' - y'x + 4y = 0$$

Let the solution be represented as power series of the form

$$y = \sum_{n=0}^{\infty} a_n x^n$$

Then

$$y' = \sum_{n=1}^{\infty} n a_n x^{n-1}$$
$$y'' = \sum_{n=2}^{\infty} n(n-1)\, a_n x^{n-2}$$

Substituting the above back into the ode gives

$$(x^2 + 2)\left(\sum_{n=2}^{\infty} n(n-1)\, a_n x^{n-2}\right) - \left(\sum_{n=1}^{\infty} n a_n x^{n-1}\right) x + 4\left(\sum_{n=0}^{\infty} a_n x^n\right) = 0 \quad (1)$$

Which simplifies to

$$\left(\sum_{n=2}^{\infty} x^n a_n n(n-1)\right) + \left(\sum_{n=2}^{\infty} 2n(n-1)\, a_n x^{n-2}\right) + \sum_{n=1}^{\infty} (-n a_n x^n) + \left(\sum_{n=0}^{\infty} 4 a_n x^n\right) = 0 \quad (2)$$

The next step is to make all powers of x be n in each summation term. Going over each summation term above with power of x in it which is not already x^n and adjusting the power and the corresponding index gives

$$\sum_{n=2}^{\infty} 2n(n-1)\, a_n x^{n-2} = \sum_{n=0}^{\infty} 2(n+2)\, a_{n+2}(n+1)\, x^n$$

Substituting all the above in Eq (2) gives the following equation where now all powers of x are the same and equal to n.

$$\left(\sum_{n=2}^{\infty} x^{n} a_{n} n(n-1)\right)+\left(\sum_{n=0}^{\infty} 2(n+2) a_{n+2}(n+1) x^{n}\right) + \sum_{n=1}^{\infty}\left(-n a_{n} x^{n}\right)+\left(\sum_{n=0}^{\infty} 4 a_{n} x^{n}\right)=0 \tag{3}$$

$n=0$ gives

$$4a_2+4a_0=0$$

$$a_2=-a_0$$

$n=1$ gives

$$12a_3+3a_1=0$$

Which after substituting earlier equations, simplifies to

$$a_3=-\frac{a_1}{4}$$

For $2\leq n$, the recurrence equation is

$$n a_{n}(n-1)+2(n+2) a_{n+2}(n+1)-n a_{n}+4 a_{n}=0 \tag{4}$$

Solving for a_{n+2}, gives

$$a_{n+2}=-\frac{a_{n}\left(n^{2}-2 n+4\right)}{2\left(n+2\right)\left(n+1\right)} \tag{5}$$

For $n=2$ the recurrence equation gives

$$4a_2+24a_4=0$$

Which after substituting the earlier terms found becomes

$$a_4=\frac{a_0}{6}$$

For $n = 3$ the recurrence equation gives

$$7a_3 + 40a_5 = 0$$

Which after substituting the earlier terms found becomes

$$a_5 = \frac{7a_1}{160}$$

For $n = 4$ the recurrence equation gives

$$12a_4 + 60a_6 = 0$$

Which after substituting the earlier terms found becomes

$$a_6 = -\frac{a_0}{30}$$

For $n = 5$ the recurrence equation gives

$$19a_5 + 84a_7 = 0$$

Which after substituting the earlier terms found becomes

$$a_7 = -\frac{19a_1}{1920}$$

And so on. Therefore the solution is

$$\begin{aligned} y &= \sum_{n=0}^{\infty} a_n x^n \\ &= a_3 x^3 + a_2 x^2 + a_1 x + a_0 + \dots \end{aligned}$$

Substituting the values for a_n found above, the solution becomes

$$y = a_0 + a_1 x - a_0 x^2 - \frac{1}{4} a_1 x^3 + \frac{1}{6} a_0 x^4 + \frac{7}{160} a_1 x^5 + \dots$$

Collecting terms, the solution becomes

$$y = \left(1 - x^2 + \frac{1}{6} x^4\right) a_0 + \left(x - \frac{1}{4} x^3 + \frac{7}{160} x^5\right) a_1 + O(x^6) \tag{3}$$

At $x=0$ the solution above becomes

$$y = \left(1 - x^2 + \frac{1}{6}x^4\right)c_1 + \left(x - \frac{1}{4}x^3 + \frac{7}{160}x^5\right)c_2 + O(x^6)$$

$$y = -1 + x^2 - \frac{x^4}{6} + 3x - \frac{3x^3}{4} + \frac{21x^5}{160} + O(x^6)$$

Summary
The solution(s) found are the following

$$y = x^2 + 3x - 1 - \frac{3x^3}{4} - \frac{x^4}{6} + \frac{21x^5}{160} + \frac{x^6}{30} + O(x^6) \tag{1}$$

$$y = -1 + x^2 - \frac{x^4}{6} + 3x - \frac{3x^3}{4} + \frac{21x^5}{160} + O(x^6) \tag{2}$$

Verification of solutions

$$y = x^2 + 3x - 1 - \frac{3x^3}{4} - \frac{x^4}{6} + \frac{21x^5}{160} + \frac{x^6}{30} + O(x^6)$$

Verified OK.

$$y = -1 + x^2 - \frac{x^4}{6} + 3x - \frac{3x^3}{4} + \frac{21x^5}{160} + O(x^6)$$

Verified OK.

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Group is reducible or imprimitive
<- Kovacics algorithm successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 20

```
Order:=6;
dsolve([(2+x^2)*diff(y(x),x$2)-x*diff(y(x),x)+4*y(x)=0,y(0) = -1, D(y)(0) = 3],y(x),type='se
```

$$y(x) = -1 + 3x + x^2 - \frac{3}{4}x^3 - \frac{1}{6}x^4 + \frac{21}{160}x^5 + \mathrm{O}\left(x^6\right)$$

✓ Solution by Mathematica

Time used: 0.001 (sec). Leaf size: 32

```
AsymptoticDSolveValue[{(2+x^2)*y''[x]-x*y'[x]+4*y[x]==0,{y[0]==-1,y'[0]==3}},y[x],{x,0,5}]
```

$$y(x) \to \frac{21x^5}{160} - \frac{x^4}{6} - \frac{3x^3}{4} + x^2 + 3x - 1$$

13.14 problem 17

Internal problem ID [724]
Internal file name [OUTPUT/724_Sunday_June_05_2022_01_47_52_AM_6286473/index.tex]

Book: Elementary differential equations and boundary value problems, 10th ed., Boyce and DiPrima
Section: Chapter 5.2, Series Solutions Near an Ordinary Point, Part I. page 263
Problem number: 17.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second order series method. Ordinary point", "second order series method. Taylor series method"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

$$\boxed{y'' + y'x + 2y = 0}$$

With initial conditions

$$[y(0) = 4, y'(0) = -1]$$

With the expansion point for the power series method at $x = 0$.

13.14.1 Existence and uniqueness analysis

This is a linear ODE. In canonical form it is written as

$$y'' + p(x)y' + q(x)y = F$$

Where here

$$\begin{aligned} p(x) &= x \\ q(x) &= 2 \\ F &= 0 \end{aligned}$$

Hence the ode is

$$y'' + y'x + 2y = 0$$

The domain of $p(x) = x$ is

$$\{-\infty < x < \infty\}$$

And the point $x_0 = 0$ is inside this domain. The domain of $q(x) = 2$ is

$$\{-\infty < x < \infty\}$$

And the point $x_0 = 0$ is also inside this domain. Hence solution exists and is unique.

Solving ode using Taylor series method. This gives review on how the Taylor series method works for solving second order ode.

Let

$$y'' = f(x, y, y')$$

Assuming expansion is at $x_0 = 0$ (we can always shift the actual expansion point to 0 by change of variables) and assuming $f(x, y, y')$ is analytic at x_0 which must be the case for an ordinary point. Let initial conditions be $y(x_0) = y_0$ and $y'(x_0) = y'_0$. Using Taylor series gives

$$\begin{aligned}
y(x) &= y(x_0) + (x - x_0)\, y'(x_0) + \frac{(x - x_0)^2}{2} y''(x_0) + \frac{(x - x_0)^3}{3!} y'''(x_0) + \cdots \\
&= y_0 + x y'_0 + \frac{x^2}{2} f|_{x_0, y_0, y'_0} + \frac{x^3}{3!} f'|_{x_0, y_0, y'_0} + \cdots \\
&= y_0 + x y'_0 + \sum_{n=0}^{\infty} \frac{x^{n+2}}{(n+2)!} \left.\frac{d^n f}{dx^n}\right|_{x_0, y_0, y'_0}
\end{aligned}$$

But

$$\frac{df}{dx} = \frac{\partial f}{\partial x}\frac{dx}{dx} + \frac{\partial f}{\partial y}\frac{dy}{dx} + \frac{\partial f}{\partial y'}\frac{dy'}{dx} \tag{1}$$

$$= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}y' + \frac{\partial f}{\partial y'}y'' \tag{783}$$

$$= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}y' + \frac{\partial f}{\partial y'}f \tag{784}$$

$$\frac{d^2 f}{dx^2} = \frac{d}{dx}\left(\frac{df}{dx}\right)$$

$$= \frac{\partial}{\partial x}\left(\frac{df}{dx}\right) + \frac{\partial}{\partial y}\left(\frac{df}{dx}\right)y' + \frac{\partial}{\partial y'}\left(\frac{df}{dx}\right)f \tag{2}$$

$$\frac{d^3 f}{dx^3} = \frac{d}{dx}\left(\frac{d^2 f}{dx^2}\right)$$

$$= \frac{\partial}{\partial x}\left(\frac{d^2 f}{dx^2}\right) + \left(\frac{\partial}{\partial y}\frac{d^2 f}{dx^2}\right)y' + \frac{\partial}{\partial y'}\left(\frac{d^2 f}{dx^2}\right)f \tag{3}$$

$$\vdots$$

And so on. Hence if we name $F_0 = f(x, y, y')$ then the above can be written as

$$
\begin{aligned}
F_0 &= f(x, y, y') \qquad (4)\\
F_1 &= \frac{df}{dx}\\
&= \frac{dF_0}{dx}\\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} y' + \frac{\partial f}{\partial y'} y''\\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} y' + \frac{\partial f}{\partial y'} f \qquad (5)\\
&= \frac{\partial F_0}{\partial x} + \frac{\partial F_0}{\partial y} y' + \frac{\partial F_0}{\partial y'} F_0\\
F_2 &= \frac{d}{dx}\left(\frac{d}{dx} f\right)\\
&= \frac{d}{dx}(F_1)\\
&= \frac{\partial}{\partial x} F_1 + \left(\frac{\partial F_1}{\partial y}\right) y' + \left(\frac{\partial F_1}{\partial y'}\right) y''\\
&= \frac{\partial}{\partial x} F_1 + \left(\frac{\partial F_1}{\partial y}\right) y' + \left(\frac{\partial F_1}{\partial y'}\right) F_0\\
&\vdots\\
F_n &= \frac{d}{dx}(F_{n-1})\\
&= \frac{\partial}{\partial x} F_{n-1} + \left(\frac{\partial F_{n-1}}{\partial y}\right) y' + \left(\frac{\partial F_{n-1}}{\partial y'}\right) y''\\
&= \frac{\partial}{\partial x} F_{n-1} + \left(\frac{\partial F_{n-1}}{\partial y}\right) y' + \left(\frac{\partial F_{n-1}}{\partial y'}\right) F_0 \qquad (6)
\end{aligned}
$$

Therefore (6) can be used from now on along with

$$
y(x) = y_0 + x y_0' + \sum_{n=0}^{\infty} \frac{x^{n+2}}{(n+2)!} F_n|_{x_0, y_0, y_0'} \tag{7}
$$

To find $y(x)$ series solution around $x = 0$. Hence

$$
\begin{aligned}
F_0 &= -y'x - 2y \\
F_1 &= \frac{dF_0}{dx} \\
&= \frac{\partial F_0}{\partial x} + \frac{\partial F_0}{\partial y}y' + \frac{\partial F_0}{\partial y'}F_0 \\
&= y'x^2 + 2yx - 3y' \\
F_2 &= \frac{dF_1}{dx} \\
&= \frac{\partial F_1}{\partial x} + \frac{\partial F_1}{\partial y}y' + \frac{\partial F_1}{\partial y'}F_1 \\
&= -x^3y' - 2x^2y + 7y'x + 8y \\
F_3 &= \frac{dF_2}{dx} \\
&= \frac{\partial F_2}{\partial x} + \frac{\partial F_2}{\partial y}y' + \frac{\partial F_2}{\partial y'}F_2 \\
&= \left(x^4 - 12x^2 + 15\right)y' + 2\left(x^3 - 9x\right)y \\
F_4 &= \frac{dF_3}{dx} \\
&= \frac{\partial F_3}{\partial x} + \frac{\partial F_3}{\partial y}y' + \frac{\partial F_3}{\partial y'}F_3 \\
&= \left(-x^5 + 18x^3 - 57x\right)y' - 2y\left(x^4 - 15x^2 + 24\right)
\end{aligned}
$$

And so on. Evaluating all the above at initial conditions $x = 0$ and $y(0) = 4$ and $y'(0) = -1$ gives

$$
\begin{aligned}
F_0 &= -8 \\
F_1 &= 3 \\
F_2 &= 32 \\
F_3 &= -15 \\
F_4 &= -192
\end{aligned}
$$

Substituting all the above in (7) and simplifying gives the solution as

$$
y = -4x^2 - x + 4 + \frac{x^3}{2} + \frac{4x^4}{3} - \frac{x^5}{8} - \frac{4x^6}{15} + O\left(x^6\right)
$$

$$
y = -4x^2 - x + 4 + \frac{x^3}{2} + \frac{4x^4}{3} - \frac{x^5}{8} - \frac{4x^6}{15} + O\left(x^6\right)
$$

Since the expansion point $x = 0$ is an ordinary, we can also solve this using standard power series Let the solution be represented as power series of the form

$$y = \sum_{n=0}^{\infty} a_n x^n$$

Then

$$y' = \sum_{n=1}^{\infty} n a_n x^{n-1}$$
$$y'' = \sum_{n=2}^{\infty} n(n-1)\, a_n x^{n-2}$$

Substituting the above back into the ode gives

$$\sum_{n=2}^{\infty} n(n-1)\, a_n x^{n-2} = -\left(\sum_{n=1}^{\infty} n a_n x^{n-1}\right) x - 2\left(\sum_{n=0}^{\infty} a_n x^n\right) \tag{1}$$

Which simplifies to

$$\left(\sum_{n=2}^{\infty} n(n-1)\, a_n x^{n-2}\right) + \left(\sum_{n=1}^{\infty} n\, x^n a_n\right) + \left(\sum_{n=0}^{\infty} 2 a_n x^n\right) = 0 \tag{2}$$

The next step is to make all powers of x be n in each summation term. Going over each summation term above with power of x in it which is not already x^n and adjusting the power and the corresponding index gives

$$\sum_{n\,=2}^{\infty} n(n-1)\, a_n x^{n-2} = \sum_{n=0}^{\infty} (n+2)\, a_{n+2}(n+1)\, x^n$$

Substituting all the above in Eq (2) gives the following equation where now all powers of x are the same and equal to n.

$$\left(\sum_{n=0}^{\infty} (n+2)\, a_{n+2}(n+1)\, x^n\right) + \left(\sum_{n=1}^{\infty} n\, x^n a_n\right) + \left(\sum_{n=0}^{\infty} 2 a_n x^n\right) = 0 \tag{3}$$

$n = 0$ gives

$$2a_2 + 2a_0 = 0$$

$$a_2 = -a_0$$

For $1 \leq n$, the recurrence equation is

$$(n+2)\, a_{n+2}(n+1) + n a_n + 2a_n = 0 \tag{4}$$

Solving for a_{n+2}, gives

$$a_{n+2} = -\frac{a_n}{n+1} \tag{5}$$

For $n = 1$ the recurrence equation gives

$$6a_3 + 3a_1 = 0$$

Which after substituting the earlier terms found becomes

$$a_3 = -\frac{a_1}{2}$$

For $n = 2$ the recurrence equation gives

$$12a_4 + 4a_2 = 0$$

Which after substituting the earlier terms found becomes

$$a_4 = \frac{a_0}{3}$$

For $n = 3$ the recurrence equation gives

$$20a_5 + 5a_3 = 0$$

Which after substituting the earlier terms found becomes

$$a_5 = \frac{a_1}{8}$$

For $n = 4$ the recurrence equation gives

$$30a_6 + 6a_4 = 0$$

Which after substituting the earlier terms found becomes

$$a_6 = -\frac{a_0}{15}$$

For $n = 5$ the recurrence equation gives

$$42a_7 + 7a_5 = 0$$

Which after substituting the earlier terms found becomes

$$a_7 = -\frac{a_1}{48}$$

And so on. Therefore the solution is

$$\begin{aligned} y &= \sum_{n=0}^{\infty} a_n x^n \\ &= a_3 x^3 + a_2 x^2 + a_1 x + a_0 + \dots \end{aligned}$$

Substituting the values for a_n found above, the solution becomes

$$y = a_0 + a_1 x - a_0 x^2 - \frac{1}{2} a_1 x^3 + \frac{1}{3} a_0 x^4 + \frac{1}{8} a_1 x^5 + \dots$$

Collecting terms, the solution becomes

$$y = \left(1 - x^2 + \frac{1}{3} x^4\right) a_0 + \left(x - \frac{1}{2} x^3 + \frac{1}{8} x^5\right) a_1 + O(x^6) \tag{3}$$

At $x = 0$ the solution above becomes

$$y = \left(1 - x^2 + \frac{1}{3} x^4\right) c_1 + \left(x - \frac{1}{2} x^3 + \frac{1}{8} x^5\right) c_2 + O(x^6)$$

$$y = 4 - 4x^2 + \frac{4x^4}{3} - x + \frac{x^3}{2} - \frac{x^5}{8} + O(x^6)$$

Summary

The solution(s) found are the following

$$y = -4x^2 - x + 4 + \frac{x^3}{2} + \frac{4x^4}{3} - \frac{x^5}{8} - \frac{4x^6}{15} + O(x^6) \tag{1}$$

$$y = 4 - 4x^2 + \frac{4x^4}{3} - x + \frac{x^3}{2} - \frac{x^5}{8} + O(x^6) \tag{2}$$

Verification of solutions

$$y = -4x^2 - x + 4 + \frac{x^3}{2} + \frac{4x^4}{3} - \frac{x^5}{8} - \frac{4x^6}{15} + O(x^6)$$

Verified OK.

$$y = 4 - 4x^2 + \frac{4x^4}{3} - x + \frac{x^3}{2} - \frac{x^5}{8} + O(x^6)$$

Verified OK.

13.14.2 Maple step by step solution

Let's solve

$$\left[y'' = -y'x - 2y, y(0) = 4, y'\Big|_{\{x=0\}} = -1\right]$$

- Highest derivative means the order of the ODE is 2

  y''

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y'' + y'x + 2y = 0$

- Assume series solution for y

  $y = \sum_{k=0}^{\infty} a_k x^k$

□ Rewrite DE with series expansions

  ∘ Convert $x \cdot y'$ to series expansion

    $x \cdot y' = \sum_{k=0}^{\infty} a_k k\, x^k$

  ∘ Convert y'' to series expansion

    $y'' = \sum_{k=2}^{\infty} a_k k(k-1)\, x^{k-2}$

  ∘ Shift index using $k- >k+2$

    $y'' = \sum_{k=0}^{\infty} a_{k+2}(k+2)\,(k+1)\, x^k$

  Rewrite DE with series expansions

  $\sum_{k=0}^{\infty} \left(a_{k+2}(k+2)\,(k+1) + a_k(k+2)\right) x^k = 0$

- Each term in the series must be 0, giving the recursion relation

  $(k+2)(ka_{k+2}+a_k+a_{k+2})=0$

- Recursion relation that defines the series solution to the ODE

  $\left[y=\sum_{k=0}^{\infty} a_k x^k, a_{k+2}=-\frac{a_k}{k+1}\right]$

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Group is reducible, not completely reducible
<- Kovacics algorithm successful`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 20

```
Order:=6;
dsolve([diff(y(x),x$2)+x*diff(y(x),x)+2*y(x)=0,y(0) = 4, D(y)(0) = -1],y(x),type='series',x=
```

$$y(x)=4-x-4x^2+\frac{1}{2}x^3+\frac{4}{3}x^4-\frac{1}{8}x^5+\mathrm{O}\left(x^6\right)$$

✓ Solution by Mathematica

Time used: 0.001 (sec). Leaf size: 34

```
AsymptoticDSolveValue[{y''[x]+x*y'[x]+2*y[x]==0,{y[0]==4,y'[0]==-1}},y[x],{x,0,5}]
```

$$y(x)\to -\frac{x^5}{8}+\frac{4x^4}{3}+\frac{x^3}{2}-4x^2-x+4$$

13.15 problem 18

Internal problem ID [725]
Internal file name [OUTPUT/725_Sunday_June_05_2022_01_47_54_AM_53815552/index.tex]

Book: Elementary differential equations and boundary value problems, 10th ed., Boyce and DiPrima
Section: Chapter 5.2, Series Solutions Near an Ordinary Point, Part I. page 263
Problem number: 18.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second order series method. Ordinary point", "second_order_change_of_variable_on_y_method_2", "second order series method. Taylor series method", "second_order_ode_non_constant_coeff_transformation_on_B"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

$$\boxed{(1-x)\,y''+y'x-y=0}$$

With initial conditions

$$[y(0)=-3, y'(0)=2]$$

With the expansion point for the power series method at $x=0$.

13.15.1 Existence and uniqueness analysis

This is a linear ODE. In canonical form it is written as

$$y''+p(x)y'+q(x)y=F$$

Where here

$$\begin{aligned}p(x)&=\frac{x}{1-x}\\ q(x)&=-\frac{1}{1-x}\\ F&=0\end{aligned}$$

Hence the ode is

$$y'' + \frac{xy'}{1-x} - \frac{y}{1-x} = 0$$

The domain of $p(x) = \frac{x}{1-x}$ is

$$\{x < 1 \lor 1 < x\}$$

And the point $x_0 = 0$ is inside this domain. The domain of $q(x) = -\frac{1}{1-x}$ is

$$\{x < 1 \lor 1 < x\}$$

And the point $x_0 = 0$ is also inside this domain. Hence solution exists and is unique.

Solving ode using Taylor series method. This gives review on how the Taylor series method works for solving second order ode.

Let

$$y'' = f(x, y, y')$$

Assuming expansion is at $x_0 = 0$ (we can always shift the actual expansion point to 0 by change of variables) and assuming $f(x, y, y')$ is analytic at x_0 which must be the case for an ordinary point. Let initial conditions be $y(x_0) = y_0$ and $y'(x_0) = y'_0$. Using Taylor series gives

$$\begin{aligned} y(x) &= y(x_0) + (x - x_0)\, y'(x_0) + \frac{(x - x_0)^2}{2} y''(x_0) + \frac{(x - x_0)^3}{3!} y'''(x_0) + \cdots \\ &= y_0 + xy'_0 + \frac{x^2}{2} f|_{x_0,y_0,y'_0} + \frac{x^3}{3!} f'|_{x_0,y_0,y'_0} + \cdots \\ &= y_0 + xy'_0 + \sum_{n=0}^{\infty} \frac{x^{n+2}}{(n+2)\,!} \frac{d^n f}{dx^n}\bigg|_{x_0,y_0,y'_0} \end{aligned}$$

But

$$\frac{df}{dx} = \frac{\partial f}{\partial x}\frac{dx}{dx} + \frac{\partial f}{\partial y}\frac{dy}{dx} + \frac{\partial f}{\partial y'}\frac{dy'}{dx} \tag{1}$$

$$= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}y' + \frac{\partial f}{\partial y'}y'' \tag{786}$$

$$= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}y' + \frac{\partial f}{\partial y'}f \tag{787}$$

$$\frac{d^2 f}{dx^2} = \frac{d}{dx}\left(\frac{df}{dx}\right)$$

$$= \frac{\partial}{\partial x}\left(\frac{df}{dx}\right) + \frac{\partial}{\partial y}\left(\frac{df}{dx}\right)y' + \frac{\partial}{\partial y'}\left(\frac{df}{dx}\right)f \tag{2}$$

$$\frac{d^3 f}{dx^3} = \frac{d}{dx}\left(\frac{d^2 f}{dx^2}\right)$$

$$= \frac{\partial}{\partial x}\left(\frac{d^2 f}{dx^2}\right) + \left(\frac{\partial}{\partial y}\frac{d^2 f}{dx^2}\right)y' + \frac{\partial}{\partial y'}\left(\frac{d^2 f}{dx^2}\right)f \tag{3}$$

$$\vdots$$

And so on. Hence if we name $F_0 = f(x, y, y')$ then the above can be written as

$$
\begin{aligned}
F_0 &= f(x, y, y') \qquad (4)\\
F_1 &= \frac{df}{dx}\\
&= \frac{dF_0}{dx}\\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}y' + \frac{\partial f}{\partial y'}y''\\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}y' + \frac{\partial f}{\partial y'}f \qquad (5)\\
&= \frac{\partial F_0}{\partial x} + \frac{\partial F_0}{\partial y}y' + \frac{\partial F_0}{\partial y'}F_0\\
F_2 &= \frac{d}{dx}\left(\frac{d}{dx}f\right)\\
&= \frac{d}{dx}(F_1)\\
&= \frac{\partial}{\partial x}F_1 + \left(\frac{\partial F_1}{\partial y}\right)y' + \left(\frac{\partial F_1}{\partial y'}\right)y''\\
&= \frac{\partial}{\partial x}F_1 + \left(\frac{\partial F_1}{\partial y}\right)y' + \left(\frac{\partial F_1}{\partial y'}\right)F_0\\
&\vdots\\
F_n &= \frac{d}{dx}(F_{n-1})\\
&= \frac{\partial}{\partial x}F_{n-1} + \left(\frac{\partial F_{n-1}}{\partial y}\right)y' + \left(\frac{\partial F_{n-1}}{\partial y'}\right)y''\\
&= \frac{\partial}{\partial x}F_{n-1} + \left(\frac{\partial F_{n-1}}{\partial y}\right)y' + \left(\frac{\partial F_{n-1}}{\partial y'}\right)F_0 \qquad (6)
\end{aligned}
$$

Therefore (6) can be used from now on along with

$$
y(x) = y_0 + xy_0' + \sum_{n=0}^{\infty} \frac{x^{n+2}}{(n+2)!} F_n|_{x_0, y_0, y_0'} \qquad (7)
$$

To find $y(x)$ series solution around $x = 0$. Hence

$$
\begin{aligned}
F_0 &= \frac{-y + y'x}{x - 1} \\
F_1 &= \frac{dF_0}{dx} \\
&= \frac{\partial F_0}{\partial x} + \frac{\partial F_0}{\partial y} y' + \frac{\partial F_0}{\partial y'} F_0 \\
&= \frac{-y + y'x}{x - 1} \\
F_2 &= \frac{dF_1}{dx} \\
&= \frac{\partial F_1}{\partial x} + \frac{\partial F_1}{\partial y} y' + \frac{\partial F_1}{\partial y'} F_1 \\
&= \frac{-y + y'x}{x - 1} \\
F_3 &= \frac{dF_2}{dx} \\
&= \frac{\partial F_2}{\partial x} + \frac{\partial F_2}{\partial y} y' + \frac{\partial F_2}{\partial y'} F_2 \\
&= \frac{-y + y'x}{x - 1} \\
F_4 &= \frac{dF_3}{dx} \\
&= \frac{\partial F_3}{\partial x} + \frac{\partial F_3}{\partial y} y' + \frac{\partial F_3}{\partial y'} F_3 \\
&= \frac{-y + y'x}{x - 1}
\end{aligned}
$$

And so on. Evaluating all the above at initial conditions $x = 0$ and $y(0) = -3$ and $y'(0) = 2$ gives

$$
\begin{aligned}
F_0 &= -3 \\
F_1 &= -3 \\
F_2 &= -3 \\
F_3 &= -3 \\
F_4 &= -3
\end{aligned}
$$

Substituting all the above in (7) and simplifying gives the solution as

$$
y = -3 + 2x - \frac{3x^2}{2} - \frac{x^3}{2} - \frac{x^4}{8} - \frac{x^5}{40} - \frac{x^6}{240} + O(x^6)
$$

$$
y=-3+2x-\frac{3x^{2}}{2}-\frac{x^{3}}{2}-\frac{x^{4}}{8}-\frac{x^{5}}{40}-\frac{x^{6}}{240}+O\left(x^{6}\right)
$$

Since the expansion point $x=0$ is an ordinary, we can also solve this using standard power series The ode is normalized to be

$$
(1-x)\,y''+y'x-y=0
$$

Let the solution be represented as power series of the form

$$
y=\sum_{n=0}^{\infty}a_{n}x^{n}
$$

Then

$$
\begin{aligned}
y' &= \sum_{n=1}^{\infty}n a_{n}x^{n-1}\\
y'' &= \sum_{n=2}^{\infty}n(n-1)\,a_{n}x^{n-2}
\end{aligned}
$$

Substituting the above back into the ode gives

$$
(1-x)\left(\sum_{n=2}^{\infty}n(n-1)\,a_{n}x^{n-2}\right)+\left(\sum_{n=1}^{\infty}n a_{n}x^{n-1}\right)x-\left(\sum_{n=0}^{\infty}a_{n}x^{n}\right)=0 \tag{1}
$$

Which simplifies to

$$
\sum_{n=2}^{\infty}\left(-n\,x^{n-1}a_{n}(n-1)\right)+\left(\sum_{n=2}^{\infty}n(n-1)\,a_{n}x^{n-2}\right)+\left(\sum_{n=1}^{\infty}n a_{n}x^{n}\right)+\sum_{n=0}^{\infty}\left(-a_{n}x^{n}\right)=0 \tag{2}
$$

The next step is to make all powers of x be n in each summation term. Going over each summation term above with power of x in it which is not already x^{n} and adjusting the power and the corresponding index gives

$$
\begin{aligned}
\sum_{n=2}^{\infty}\left(-n\,x^{n-1}a_{n}(n-1)\right) &= \sum_{n=1}^{\infty}\left(-(n+1)\,a_{n+1}n\,x^{n}\right)\\
\sum_{n=2}^{\infty}n(n-1)\,a_{n}x^{n-2} &= \sum_{n=0}^{\infty}(n+2)\,a_{n+2}(n+1)\,x^{n}
\end{aligned}
$$

Substituting all the above in Eq (2) gives the following equation where now all powers of x are the same and equal to n.

$$
\begin{aligned}
\sum_{n=1}^{\infty} \left(-(n+1)\, a_{n+1} n\, x^{n}\right) + \left(\sum_{n=0}^{\infty} (n+2)\, a_{n+2}(n+1)\, x^{n}\right) \\
+ \left(\sum_{n=1}^{\infty} n a_{n} x^{n}\right) + \sum_{n=0}^{\infty} \left(-a_{n} x^{n}\right) = 0
\end{aligned} \tag{3}
$$

$n = 0$ gives

$$
2a_{2} - a_{0} = 0
$$

$$
a_{2} = \frac{a_{0}}{2}
$$

For $1 \leq n$, the recurrence equation is

$$
-(n+1)\, a_{n+1} n + (n+2)\, a_{n+2}(n+1) + n a_{n} - a_{n} = 0 \tag{4}
$$

Solving for a_{n+2}, gives

$$
\begin{aligned}
a_{n+2} &= \frac{n^{2} a_{n+1} - n a_{n} + n a_{n+1} + a_{n}}{(n+2)\,(n+1)} \\
&= \frac{(-n+1)\, a_{n}}{(n+2)\,(n+1)} + \frac{(n^{2}+n)\, a_{n+1}}{(n+2)\,(n+1)}
\end{aligned} \tag{5}
$$

For $n = 1$ the recurrence equation gives

$$
-2a_{2} + 6a_{3} = 0
$$

Which after substituting the earlier terms found becomes

$$
a_{3} = \frac{a_{0}}{6}
$$

For $n = 2$ the recurrence equation gives

$$
-6a_{3} + 12a_{4} + a_{2} = 0
$$

Which after substituting the earlier terms found becomes

$$a_4 = \frac{a_0}{24}$$

For $n = 3$ the recurrence equation gives

$$-12a_4 + 20a_5 + 2a_3 = 0$$

Which after substituting the earlier terms found becomes

$$a_5 = \frac{a_0}{120}$$

For $n = 4$ the recurrence equation gives

$$-20a_5 + 30a_6 + 3a_4 = 0$$

Which after substituting the earlier terms found becomes

$$a_6 = \frac{a_0}{720}$$

For $n = 5$ the recurrence equation gives

$$-30a_6 + 42a_7 + 4a_5 = 0$$

Which after substituting the earlier terms found becomes

$$a_7 = \frac{a_0}{5040}$$

And so on. Therefore the solution is

$$\begin{aligned} y &= \sum_{n=0}^{\infty} a_n x^n \\ &= a_3 x^3 + a_2 x^2 + a_1 x + a_0 + \dots \end{aligned}$$

Substituting the values for a_n found above, the solution becomes

$$y = a_0 + a_1 x + \frac{1}{2} a_0 x^2 + \frac{1}{6} a_0 x^3 + \frac{1}{24} a_0 x^4 + \frac{1}{120} a_0 x^5 + \dots$$

Collecting terms, the solution becomes

$$y = \left(1+\frac{1}{2}x^{2}+\frac{1}{6}x^{3}+\frac{1}{24}x^{4}+\frac{1}{120}x^{5}\right) a_{0}+a_{1}x+O\left(x^{6}\right) \tag{3}$$

At $x = 0$ the solution above becomes

$$y = \left(1+\frac{1}{2}x^{2}+\frac{1}{6}x^{3}+\frac{1}{24}x^{4}+\frac{1}{120}x^{5}\right) c_{1}+c_{2}x+O\left(x^{6}\right)$$

$$y = -3-\frac{3x^{2}}{2}-\frac{x^{3}}{2}-\frac{x^{4}}{8}-\frac{x^{5}}{40}+2x+O\left(x^{6}\right)$$

Summary

The solution(s) found are the following

$$y = -3+2x-\frac{3x^{2}}{2}-\frac{x^{3}}{2}-\frac{x^{4}}{8}-\frac{x^{5}}{40}-\frac{x^{6}}{240}+O\left(x^{6}\right) \tag{1}$$

$$y = -3-\frac{3x^{2}}{2}-\frac{x^{3}}{2}-\frac{x^{4}}{8}-\frac{x^{5}}{40}+2x+O\left(x^{6}\right) \tag{2}$$

Verification of solutions

$$y = -3+2x-\frac{3x^{2}}{2}-\frac{x^{3}}{2}-\frac{x^{4}}{8}-\frac{x^{5}}{40}-\frac{x^{6}}{240}+O\left(x^{6}\right)$$

Verified OK.

$$y = -3-\frac{3x^{2}}{2}-\frac{x^{3}}{2}-\frac{x^{4}}{8}-\frac{x^{5}}{40}+2x+O\left(x^{6}\right)$$

Verified OK.

13.15.2 Maple step by step solution

Let's solve

$$\left[(1-x)\, y''+y'x-y=0, y(0)=-3, y'\Big|_{\{x=0\}}=2\right]$$

- Highest derivative means the order of the ODE is 2

  y''
- Isolate 2nd derivative

$y'' = -\frac{y}{x-1} + \frac{xy'}{x-1}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y'' - \frac{xy'}{x-1} + \frac{y}{x-1} = 0$

☐ Check to see if $x_0 = 1$ is a regular singular point

- ○ Define functions

  $\left[P_2(x) = -\frac{x}{x-1}, P_3(x) = \frac{1}{x-1}\right]$

- ○ $(x-1) \cdot P_2(x)$ is analytic at $x = 1$

  $\left((x-1) \cdot P_2(x)\right)\Big|_{x=1} = -1$

- ○ $(x-1)^2 \cdot P_3(x)$ is analytic at $x = 1$

  $\left((x-1)^2 \cdot P_3(x)\right)\Big|_{x=1} = 0$

- ○ $x = 1$is a regular singular point

  Check to see if $x_0 = 1$ is a regular singular point

  $x_0 = 1$

- Multiply by denominators

  $(x-1)\, y'' - y'x + y = 0$

- Change variables using $x = u + 1$ so that the regular singular point is at $u = 0$

  $u\left(\frac{d^2}{du^2}y(u)\right) + (-u-1)\left(\frac{d}{du}y(u)\right) + y(u) = 0$

- Assume series solution for $y(u)$

  $y(u) = \sum_{k=0}^{\infty} a_k u^{k+r}$

☐ Rewrite ODE with series expansions

- ○ Convert $u^m \cdot \left(\frac{d}{du}y(u)\right)$ to series expansion for $m = 0..1$

  $u^m \cdot \left(\frac{d}{du}y(u)\right) = \sum_{k=0}^{\infty} a_k (k+r)\, u^{k+r-1+m}$

- ○ Shift index using $k- >k+1-m$

  $u^m \cdot \left(\frac{d}{du}y(u)\right) = \sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\, u^{k+r}$

- ○ Convert $u \cdot \left(\frac{d^2}{du^2}y(u)\right)$ to series expansion

$$u \cdot \left(\frac{d^2}{du^2}y(u)\right) = \sum_{k=0}^{\infty} a_k(k+r)(k+r-1)\,u^{k+r-1}$$

- Shift index using $k->k+1$

$$u \cdot \left(\frac{d^2}{du^2}y(u)\right) = \sum_{k=-1}^{\infty} a_{k+1}(k+1+r)(k+r)\,u^{k+r}$$

Rewrite ODE with series expansions

$$a_0 r(-2+r)\,u^{-1+r} + \left(\sum_{k=0}^{\infty} \left(a_{k+1}(k+1+r)(k+r-1) - a_k(k+r-1)\right) u^{k+r}\right) = 0$$

- $a_0$cannot be 0 by assumption, giving the indicial equation

$$r(-2+r) = 0$$

- Values of r that satisfy the indicial equation

$$r \in \{0, 2\}$$

- Each term in the series must be 0, giving the recursion relation

$$(k+r-1)\left(a_{k+1}(k+1+r) - a_k\right) = 0$$

- Recursion relation that defines series solution to ODE

$$a_{k+1} = \frac{a_k}{k+1+r}$$

- Recursion relation for $r = 0$

$$a_{k+1} = \frac{a_k}{k+1}$$

- Solution for $r = 0$

$$\left[y(u) = \sum_{k=0}^{\infty} a_k u^k, a_{k+1} = \frac{a_k}{k+1}\right]$$

- Revert the change of variables $u = x-1$

$$\left[y = \sum_{k=0}^{\infty} a_k(x-1)^k, a_{k+1} = \frac{a_k}{k+1}\right]$$

- Recursion relation for $r = 2$

$$a_{k+1} = \frac{a_k}{k+3}$$

- Solution for $r = 2$

$$\left[y(u) = \sum_{k=0}^{\infty} a_k u^{k+2}, a_{k+1} = \frac{a_k}{k+3}\right]$$

- Revert the change of variables $u = x-1$

$$\left[y = \sum_{k=0}^{\infty} a_k(x-1)^{k+2}, a_{k+1} = \frac{a_k}{k+3}\right]$$

- Combine solutions and rename parameters

$$\left[y = \left(\sum_{k=0}^{\infty} a_k (x-1)^k\right) + \left(\sum_{k=0}^{\infty} b_k (x-1)^{k+2}\right), a_{k+1} = \frac{a_k}{k+1}, b_{k+1} = \frac{b_k}{k+3}\right]$$

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Reducible group (found another exponential solution)
<- Kovacics algorithm successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 20

```
Order:=6;
dsolve([(1-x)*diff(y(x),x$2)+x*diff(y(x),x)-y(x)=0,y(0) = -3, D(y)(0) = 2],y(x),type='series
```

$$y(x) = -3 + 2x - \frac{3}{2}x^2 - \frac{1}{2}x^3 - \frac{1}{8}x^4 - \frac{1}{40}x^5 + \mathrm{O}\left(x^6\right)$$

✓ Solution by Mathematica

Time used: 0.001 (sec). Leaf size: 36

```
AsymptoticDSolveValue[{(1-x)*y''[x]+x*y'[x]-y[x]==0,{y[0]==-3,y'[0]==2}},y[x],{x,0,5}]
```

$$y(x) \to -\frac{x^5}{40} - \frac{x^4}{8} - \frac{x^3}{2} - \frac{3x^2}{2} + 2x - 3$$

13.16 problem 21

Internal problem ID [726]
Internal file name [`OUTPUT/726_Sunday_June_05_2022_01_47_56_AM_30999466/index.tex`]

Book: Elementary differential equations and boundary value problems, 10th ed., Boyce and DiPrima
Section: Chapter 5.2, Series Solutions Near an Ordinary Point, Part I. page 263
Problem number: 21.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second order series method. Ordinary point", "second order series method. Taylor series method"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

$$\boxed{y'' - 2y'x + \lambda y = 0}$$

With the expansion point for the power series method at $x = 0$.

Solving ode using Taylor series method. This gives review on how the Taylor series method works for solving second order ode.

Let

$$y'' = f(x, y, y')$$

Assuming expansion is at $x_0 = 0$ (we can always shift the actual expansion point to 0 by change of variables) and assuming $f(x, y, y')$ is analytic at x_0 which must be the case for an ordinary point. Let initial conditions be $y(x_0) = y_0$ and $y'(x_0) = y'_0$. Using Taylor series gives

$$\begin{aligned} y(x) &= y(x_0) + (x - x_0)\, y'(x_0) + \frac{(x - x_0)^2}{2} y''(x_0) + \frac{(x - x_0)^3}{3!} y'''(x_0) + \cdots \\ &= y_0 + x y'_0 + \frac{x^2}{2} f|_{x_0, y_0, y'_0} + \frac{x^3}{3!} f'|_{x_0, y_0, y'_0} + \cdots \\ &= y_0 + x y'_0 + \sum_{n=0}^{\infty} \frac{x^{n+2}}{(n+2)\,!} \frac{d^n f}{dx^n}\bigg|_{x_0, y_0, y'_0} \end{aligned}$$

But

$$
\begin{aligned}
\frac{df}{dx} &= \frac{\partial f}{\partial x}\frac{dx}{dx} + \frac{\partial f}{\partial y}\frac{dy}{dx} + \frac{\partial f}{\partial y'}\frac{dy'}{dx} && (1)\\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}y' + \frac{\partial f}{\partial y'}y'' && (789)\\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}y' + \frac{\partial f}{\partial y'}f && (790)\\
\frac{d^2 f}{dx^2} &= \frac{d}{dx}\left(\frac{df}{dx}\right)\\
&= \frac{\partial}{\partial x}\left(\frac{df}{dx}\right) + \frac{\partial}{\partial y}\left(\frac{df}{dx}\right)y' + \frac{\partial}{\partial y'}\left(\frac{df}{dx}\right)f && (2)\\
\frac{d^3 f}{dx^3} &= \frac{d}{dx}\left(\frac{d^2 f}{dx^2}\right)\\
&= \frac{\partial}{\partial x}\left(\frac{d^2 f}{dx^2}\right) + \left(\frac{\partial}{\partial y}\frac{d^2 f}{dx^2}\right)y' + \frac{\partial}{\partial y'}\left(\frac{d^2 f}{dx^2}\right)f && (3)\\
&\vdots
\end{aligned}
$$

And so on. Hence if we name $F_0 = f(x,y,y')$ then the above can be written as

$$
\begin{aligned}
F_0 &= f(x,y,y') &(4)\\
F_1 &= \frac{df}{dx}\\
&= \frac{dF_0}{dx}\\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}y' + \frac{\partial f}{\partial y'}y''\\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}y' + \frac{\partial f}{\partial y'}f &(5)\\
&= \frac{\partial F_0}{\partial x} + \frac{\partial F_0}{\partial y}y' + \frac{\partial F_0}{\partial y'}F_0\\
F_2 &= \frac{d}{dx}\left(\frac{d}{dx}f\right)\\
&= \frac{d}{dx}(F_1)\\
&= \frac{\partial}{\partial x}F_1 + \left(\frac{\partial F_1}{\partial y}\right)y' + \left(\frac{\partial F_1}{\partial y'}\right)y''\\
&= \frac{\partial}{\partial x}F_1 + \left(\frac{\partial F_1}{\partial y}\right)y' + \left(\frac{\partial F_1}{\partial y'}\right)F_0\\
&\vdots\\
F_n &= \frac{d}{dx}(F_{n-1})\\
&= \frac{\partial}{\partial x}F_{n-1} + \left(\frac{\partial F_{n-1}}{\partial y}\right)y' + \left(\frac{\partial F_{n-1}}{\partial y'}\right)y''\\
&= \frac{\partial}{\partial x}F_{n-1} + \left(\frac{\partial F_{n-1}}{\partial y}\right)y' + \left(\frac{\partial F_{n-1}}{\partial y'}\right)F_0 &(6)
\end{aligned}
$$

Therefore (6) can be used from now on along with

$$
y(x) = y_0 + xy_0' + \sum_{n=0}^{\infty} \frac{x^{n+2}}{(n+2)\,!} F_n|_{x_0,y_0,y_0'} \tag{7}
$$

To find $y(x)$ series solution around $x=0$. Hence

$$
\begin{aligned}
F_0 &= 2y'x-\lambda y\\
F_1 &= \frac{dF_0}{dx}\\
&= \frac{\partial F_0}{\partial x}+\frac{\partial F_0}{\partial y}y'+\frac{\partial F_0}{\partial y'}F_0\\
&= \left(4x^2-\lambda+2\right)y'-2y\lambda x\\
F_2 &= \frac{dF_1}{dx}\\
&= \frac{\partial F_1}{\partial x}+\frac{\partial F_1}{\partial y}y'+\frac{\partial F_1}{\partial y'}F_1\\
&= \left(8x^3-4\lambda x+12x\right)y'-4\lambda\left(x^2-\frac{\lambda}{4}+1\right)y\\
F_3 &= \frac{dF_2}{dx}\\
&= \frac{\partial F_2}{\partial x}+\frac{\partial F_2}{\partial y}y'+\frac{\partial F_2}{\partial y'}F_2\\
&= \left(\lambda^2+\left(-12x^2-8\right)\lambda+16x^4+48x^2+12\right)y'-8\left(x^2-\frac{\lambda}{2}+\frac{5}{2}\right)x\lambda y\\
F_4 &= \frac{dF_3}{dx}\\
&= \frac{\partial F_3}{\partial x}+\frac{\partial F_3}{\partial y}y'+\frac{\partial F_3}{\partial y'}F_3\\
&= \left(6x\,\lambda^2+\left(-32x^3-60x\right)\lambda+32x^5+160x^3+120x\right)y'-16\left(\frac{\lambda^2}{16}+\left(-\frac{3x^2}{4}-\frac{3}{4}\right)\lambda+x^4+\frac{9x^2}{2}+2\right)
\end{aligned}
$$

And so on. Evaluating all the above at initial conditions $x=0$ and $y(0)=y(0)$ and $y'(0)=y'(0)$ gives

$$
\begin{aligned}
F_0 &= -y(0)\,\lambda\\
F_1 &= -y'(0)\,\lambda+2y'(0)\\
F_2 &= y(0)\,\lambda^2-4y(0)\,\lambda\\
F_3 &= y'(0)\,\lambda^2-8y'(0)\,\lambda+12y'(0)\\
F_4 &= -y(0)\,\lambda^3+12y(0)\,\lambda^2-32y(0)\,\lambda
\end{aligned}
$$

Substituting all the above in (7) and simplifying gives the solution as

$$
\begin{aligned}
y = &\left(1-\frac{1}{2}\lambda\,x^2+\frac{1}{24}x^4\lambda^2-\frac{1}{6}x^4\lambda-\frac{1}{720}x^6\lambda^3+\frac{1}{60}x^6\lambda^2-\frac{2}{45}x^6\lambda\right)y(0)\\
&+\left(x-\frac{1}{6}x^3\lambda+\frac{1}{3}x^3+\frac{1}{120}x^5\lambda^2-\frac{1}{15}x^5\lambda+\frac{1}{10}x^5\right)y'(0)+O\left(x^6\right)
\end{aligned}
$$

Since the expansion point $x = 0$ is an ordinary, we can also solve this using standard power series Let the solution be represented as power series of the form

$$y = \sum_{n=0}^{\infty} a_n x^n$$

Then

$$y' = \sum_{n=1}^{\infty} n a_n x^{n-1}$$
$$y'' = \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2}$$

Substituting the above back into the ode gives

$$\sum_{n=2}^{\infty} n(n-1) a_n x^{n-2} = 2\left(\sum_{n=1}^{\infty} n a_n x^{n-1}\right) x - \lambda\left(\sum_{n=0}^{\infty} a_n x^n\right) \tag{1}$$

Which simplifies to

$$\left(\sum_{n=2}^{\infty} n(n-1) a_n x^{n-2}\right) + \sum_{n=1}^{\infty} (-2n\, x^n a_n) + \left(\sum_{n=0}^{\infty} \lambda a_n x^n\right) = 0 \tag{2}$$

The next step is to make all powers of x be n in each summation term. Going over each summation term above with power of x in it which is not already x^n and adjusting the power and the corresponding index gives

$$\sum_{n=2}^{\infty} n(n-1) a_n x^{n-2} = \sum_{n=0}^{\infty} (n+2) a_{n+2}(n+1) x^n$$

Substituting all the above in Eq (2) gives the following equation where now all powers of x are the same and equal to n.

$$\left(\sum_{n=0}^{\infty} (n+2) a_{n+2}(n+1) x^n\right) + \sum_{n=1}^{\infty} (-2n\, x^n a_n) + \left(\sum_{n=0}^{\infty} \lambda a_n x^n\right) = 0 \tag{3}$$

$n = 0$ gives

$$\lambda a_0 + 2a_2 = 0$$

$$a_2 = -\frac{\lambda a_0}{2}$$

For $1 \leq n$, the recurrence equation is

$$(n+2)\, a_{n+2}(n+1) - 2n a_n + \lambda a_n = 0 \tag{4}$$

Solving for a_{n+2}, gives

$$a_{n+2} = -\frac{a_n(\lambda - 2n)}{(n+2)\,(n+1)} \tag{5}$$

For $n = 1$ the recurrence equation gives

$$\lambda a_1 - 2a_1 + 6a_3 = 0$$

Which after substituting the earlier terms found becomes

$$a_3 = -\frac{1}{6}\lambda a_1 + \frac{1}{3}a_1$$

For $n = 2$ the recurrence equation gives

$$\lambda a_2 - 4a_2 + 12a_4 = 0$$

Which after substituting the earlier terms found becomes

$$a_4 = \frac{1}{24}\lambda^2 a_0 - \frac{1}{6}\lambda a_0$$

For $n = 3$ the recurrence equation gives

$$\lambda a_3 - 6a_3 + 20a_5 = 0$$

Which after substituting the earlier terms found becomes

$$a_5 = \frac{1}{120}\lambda^2 a_1 - \frac{1}{15}\lambda a_1 + \frac{1}{10}a_1$$

For $n = 4$ the recurrence equation gives

$$\lambda a_4 - 8a_4 + 30a_6 = 0$$

Which after substituting the earlier terms found becomes

$$a_6 = -\frac{1}{720}\lambda^3 a_0 + \frac{1}{60}\lambda^2 a_0 - \frac{2}{45}\lambda a_0$$

For $n = 5$ the recurrence equation gives

$$\lambda a_5 - 10a_5 + 42a_7 = 0$$

Which after substituting the earlier terms found becomes

$$a_7 = -\frac{1}{5040}\lambda^3 a_1 + \frac{1}{280}\lambda^2 a_1 - \frac{23}{1260}\lambda a_1 + \frac{1}{42}a_1$$

And so on. Therefore the solution is

$$\begin{aligned} y &= \sum_{n=0}^{\infty} a_n x^n \\ &= a_3 x^3 + a_2 x^2 + a_1 x + a_0 + \dots \end{aligned}$$

Substituting the values for a_n found above, the solution becomes

$$\begin{aligned} y = a_0 + a_1 x - \frac{\lambda a_0 x^2}{2} + \left(-\frac{1}{6}\lambda a_1 + \frac{1}{3}a_1\right)x^3 \\ + \left(\frac{1}{24}\lambda^2 a_0 - \frac{1}{6}\lambda a_0\right)x^4 + \left(\frac{1}{120}\lambda^2 a_1 - \frac{1}{15}\lambda a_1 + \frac{1}{10}a_1\right)x^5 + \dots \end{aligned}$$

Collecting terms, the solution becomes

$$\begin{aligned} y = \left(1 - \frac{\lambda x^2}{2} + \left(\frac{1}{24}\lambda^2 - \frac{1}{6}\lambda\right)x^4\right)a_0 \\ + \left(x + \left(-\frac{\lambda}{6} + \frac{1}{3}\right)x^3 + \left(\frac{1}{120}\lambda^2 - \frac{1}{15}\lambda + \frac{1}{10}\right)x^5\right)a_1 + O(x^6) \end{aligned} \tag{3}$$

At $x = 0$ the solution above becomes

$$\begin{aligned} y = \left(1 - \frac{\lambda x^2}{2} + \left(\frac{1}{24}\lambda^2 - \frac{1}{6}\lambda\right)x^4\right)c_1 \\ + \left(x + \left(-\frac{\lambda}{6} + \frac{1}{3}\right)x^3 + \left(\frac{1}{120}\lambda^2 - \frac{1}{15}\lambda + \frac{1}{10}\right)x^5\right)c_2 + O(x^6) \end{aligned}$$

Summary

The solution(s) found are the following

$$y = \left(1-\frac{1}{2}\lambda\,x^{2}+\frac{1}{24}x^{4}\lambda^{2}-\frac{1}{6}x^{4}\lambda-\frac{1}{720}x^{6}\lambda^{3}+\frac{1}{60}x^{6}\lambda^{2}-\frac{2}{45}x^{6}\lambda\right)y(0) + \left(x-\frac{1}{6}x^{3}\lambda+\frac{1}{3}x^{3}+\frac{1}{120}x^{5}\lambda^{2}-\frac{1}{15}x^{5}\lambda+\frac{1}{10}x^{5}\right)y'(0)+O\left(x^{6}\right) \tag{1}$$

$$y = \left(1-\frac{\lambda\,x^{2}}{2}+\left(\frac{1}{24}\lambda^{2}-\frac{1}{6}\lambda\right)x^{4}\right)c_{1} + \left(x+\left(-\frac{\lambda}{6}+\frac{1}{3}\right)x^{3}+\left(\frac{1}{120}\lambda^{2}-\frac{1}{15}\lambda+\frac{1}{10}\right)x^{5}\right)c_{2}+O\left(x^{6}\right) \tag{2}$$

Verification of solutions

$$y = \left(1-\frac{1}{2}\lambda\,x^{2}+\frac{1}{24}x^{4}\lambda^{2}-\frac{1}{6}x^{4}\lambda-\frac{1}{720}x^{6}\lambda^{3}+\frac{1}{60}x^{6}\lambda^{2}-\frac{2}{45}x^{6}\lambda\right)y(0) + \left(x-\frac{1}{6}x^{3}\lambda+\frac{1}{3}x^{3}+\frac{1}{120}x^{5}\lambda^{2}-\frac{1}{15}x^{5}\lambda+\frac{1}{10}x^{5}\right)y'(0)+O\left(x^{6}\right)$$

Verified OK.

$$y = \left(1-\frac{\lambda\,x^{2}}{2}+\left(\frac{1}{24}\lambda^{2}-\frac{1}{6}\lambda\right)x^{4}\right)c_{1} + \left(x+\left(-\frac{\lambda}{6}+\frac{1}{3}\right)x^{3}+\left(\frac{1}{120}\lambda^{2}-\frac{1}{15}\lambda+\frac{1}{10}\right)x^{5}\right)c_{2}+O\left(x^{6}\right)$$

Verified OK.

13.16.1 Maple step by step solution

Let's solve

$y'' = 2y'x - \lambda y$

- Highest derivative means the order of the ODE is 2

  y''

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y'' - 2y'x + \lambda y = 0$

- Assume series solution for y

  $y = \sum_{k=0}^{\infty} a_k x^k$

□ Rewrite DE with series expansions

- Convert $x \cdot y'$ to series expansion

  $$x \cdot y' = \sum_{k=0}^{\infty} a_k k\, x^k$$

- Convert y'' to series expansion

  $$y'' = \sum_{k=2}^{\infty} a_k k(k-1)\, x^{k-2}$$

- Shift index using $k- >k+2$

  $$y'' = \sum_{k=0}^{\infty} a_{k+2}(k+2)\,(k+1)\, x^k$$

  Rewrite DE with series expansions

  $$\sum_{k=0}^{\infty} \left(a_{k+2}(k+2)\,(k+1) - a_k(2k-\lambda)\right) x^k = 0$$

- Each term in the series must be 0, giving the recursion relation

  $$(k^2+3k+2)\, a_{k+2} - 2\left(k - \frac{\lambda}{2}\right) a_k = 0$$

- Recursion relation that defines the series solution to the ODE

  $$\left[y = \sum_{k=0}^{\infty} a_k x^k, a_{k+2} = \frac{(2k-\lambda)a_k}{k^2+3k+2}\right]$$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
<- No Liouvillian solutions exists
-> Trying a solution in terms of special functions:
   -> Bessel
   -> elliptic
   -> Legendre
   -> Kummer
      -> hyper3: Equivalence to 1F1 under a power @ Moebius
      <- hyper3 successful: received ODE is equivalent to the 1F1 ODE
   <- Kummer successful
<- special function solution successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 63

```
Order:=6;
dsolve(diff(y(x),x$2)-2*x*diff(y(x),x)+lambda*y(x)=0,y(x),type='series',x=0);
```

$$y(x) = \left(1 - \frac{\lambda\, x^2}{2} + \frac{\lambda(\lambda-4)\, x^4}{24}\right) y(0) + \left(x - \frac{(\lambda-2)\, x^3}{6} + \frac{(\lambda-2)\,(-6+\lambda)\, x^5}{120}\right) D(y)\,(0) + O\left(x^6\right)$$

✓ Solution by Mathematica

Time used: 0.001 (sec). Leaf size: 80

```
AsymptoticDSolveValue[y''[x]-2*x*y'[x]+\[Lambda]*y[x]==0,y[x],{x,0,5}]
```

$$y(x) \to c_2\left(\frac{\lambda^2 x^5}{120} - \frac{\lambda x^5}{15} + \frac{x^5}{10} - \frac{\lambda x^3}{6} + \frac{x^3}{3} + x\right) + c_1\left(\frac{\lambda^2 x^4}{24} - \frac{\lambda x^4}{6} - \frac{\lambda x^2}{2} + 1\right)$$

13.17 problem 23

Internal problem ID [727]
Internal file name [`OUTPUT/727_Sunday_June_05_2022_01_47_57_AM_94347532/index.tex`]

Book: Elementary differential equations and boundary value problems, 10th ed., Boyce and DiPrima
Section: Chapter 5.2, Series Solutions Near an Ordinary Point, Part I. page 263
Problem number: 23.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"exact linear second order ode", "second_order_integrable_as_is", "second order series method. Ordinary point", "second order series method. Taylor series method"**

Maple gives the following as the ode type

```
[[_2nd_order, _exact, _linear, _homogeneous]]
```

$$\boxed{y'' - y'x - y = 0}$$

With initial conditions

$$[y(0) = 1, y'(0) = 0]$$

With the expansion point for the power series method at $x = 0$.

13.17.1 Existence and uniqueness analysis

This is a linear ODE. In canonical form it is written as

$$y'' + p(x)y' + q(x)y = F$$

Where here

$$\begin{aligned} p(x) &= -x \\ q(x) &= -1 \\ F &= 0 \end{aligned}$$

Hence the ode is

$$y'' - y'x - y = 0$$

The domain of $p(x) = -x$ is

$$\{-\infty < x < \infty\}$$

And the point $x_0 = 0$ is inside this domain. The domain of $q(x) = -1$ is

$$\{-\infty < x < \infty\}$$

And the point $x_0 = 0$ is also inside this domain. Hence solution exists and is unique.

Solving ode using Taylor series method. This gives review on how the Taylor series method works for solving second order ode.

Let

$$y'' = f(x, y, y')$$

Assuming expansion is at $x_0 = 0$ (we can always shift the actual expansion point to 0 by change of variables) and assuming $f(x, y, y')$ is analytic at x_0 which must be the case for an ordinary point. Let initial conditions be $y(x_0) = y_0$ and $y'(x_0) = y'_0$. Using Taylor series gives

$$\begin{aligned}
y(x) &= y(x_0) + (x - x_0)\, y'(x_0) + \frac{(x - x_0)^2}{2} y''(x_0) + \frac{(x - x_0)^3}{3!} y'''(x_0) + \cdots \\
&= y_0 + x y'_0 + \frac{x^2}{2} f|_{x_0, y_0, y'_0} + \frac{x^3}{3!} f'|_{x_0, y_0, y'_0} + \cdots \\
&= y_0 + x y'_0 + \sum_{n=0}^{\infty} \frac{x^{n+2}}{(n+2)!} \frac{d^n f}{dx^n}\bigg|_{x_0, y_0, y'_0}
\end{aligned}$$

But

$$\begin{aligned}\frac{df}{dx} &= \frac{\partial f}{\partial x}\frac{dx}{dx} + \frac{\partial f}{\partial y}\frac{dy}{dx} + \frac{\partial f}{\partial y'}\frac{dy'}{dx} && (1)\\ &= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}y' + \frac{\partial f}{\partial y'}y'' && (792)\\ &= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}y' + \frac{\partial f}{\partial y'}f && (793)\\ \frac{d^2 f}{dx^2} &= \frac{d}{dx}\left(\frac{df}{dx}\right)\\ &= \frac{\partial}{\partial x}\left(\frac{df}{dx}\right) + \frac{\partial}{\partial y}\left(\frac{df}{dx}\right)y' + \frac{\partial}{\partial y'}\left(\frac{df}{dx}\right)f && (2)\\ \frac{d^3 f}{dx^3} &= \frac{d}{dx}\left(\frac{d^2 f}{dx^2}\right)\\ &= \frac{\partial}{\partial x}\left(\frac{d^2 f}{dx^2}\right) + \left(\frac{\partial}{\partial y}\frac{d^2 f}{dx^2}\right)y' + \frac{\partial}{\partial y'}\left(\frac{d^2 f}{dx^2}\right)f && (3)\\ &\vdots\end{aligned}$$

And so on. Hence if we name $F_0 = f(x,y,y')$ then the above can be written as

$$\begin{aligned} F_0 &= f(x,y,y') \qquad (4)\\ F_1 &= \frac{df}{dx}\\ &= \frac{dF_0}{dx}\\ &= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}y' + \frac{\partial f}{\partial y'}y''\\ &= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}y' + \frac{\partial f}{\partial y'}f \qquad (5)\\ &= \frac{\partial F_0}{\partial x} + \frac{\partial F_0}{\partial y}y' + \frac{\partial F_0}{\partial y'}F_0\\ F_2 &= \frac{d}{dx}\left(\frac{d}{dx}f\right)\\ &= \frac{d}{dx}(F_1)\\ &= \frac{\partial}{\partial x}F_1 + \left(\frac{\partial F_1}{\partial y}\right)y' + \left(\frac{\partial F_1}{\partial y'}\right)y''\\ &= \frac{\partial}{\partial x}F_1 + \left(\frac{\partial F_1}{\partial y}\right)y' + \left(\frac{\partial F_1}{\partial y'}\right)F_0\\ &\vdots\\ F_n &= \frac{d}{dx}(F_{n-1})\\ &= \frac{\partial}{\partial x}F_{n-1} + \left(\frac{\partial F_{n-1}}{\partial y}\right)y' + \left(\frac{\partial F_{n-1}}{\partial y'}\right)y''\\ &= \frac{\partial}{\partial x}F_{n-1} + \left(\frac{\partial F_{n-1}}{\partial y}\right)y' + \left(\frac{\partial F_{n-1}}{\partial y'}\right)F_0 \qquad (6) \end{aligned}$$

Therefore (6) can be used from now on along with

$$y(x) = y_0 + xy_0' + \sum_{n=0}^{\infty} \frac{x^{n+2}}{(n+2)!} F_n|_{x_0,y_0,y_0'} \tag{7}$$

To find $y(x)$ series solution around $x = 0$. Hence

$$
\begin{aligned}
F_0 &= y'x + y \\
F_1 &= \frac{dF_0}{dx} \\
&= \frac{\partial F_0}{\partial x} + \frac{\partial F_0}{\partial y} y' + \frac{\partial F_0}{\partial y'} F_0 \\
&= y'x^2 + yx + 2y' \\
F_2 &= \frac{dF_1}{dx} \\
&= \frac{\partial F_1}{\partial x} + \frac{\partial F_1}{\partial y} y' + \frac{\partial F_1}{\partial y'} F_1 \\
&= \left(x^3 + 5x\right) y' + y\left(x^2 + 3\right) \\
F_3 &= \frac{dF_2}{dx} \\
&= \frac{\partial F_2}{\partial x} + \frac{\partial F_2}{\partial y} y' + \frac{\partial F_2}{\partial y'} F_2 \\
&= \left(x^4 + 9x^2 + 8\right) y' + xy\left(x^2 + 7\right) \\
F_4 &= \frac{dF_3}{dx} \\
&= \frac{\partial F_3}{\partial x} + \frac{\partial F_3}{\partial y} y' + \frac{\partial F_3}{\partial y'} F_3 \\
&= \left(x^5 + 14x^3 + 33x\right) y' + y\left(x^4 + 12x^2 + 15\right)
\end{aligned}
$$

And so on. Evaluating all the above at initial conditions $x = 0$ and $y(0) = 1$ and $y'(0) = 0$ gives

$$
\begin{aligned}
F_0 &= 1 \\
F_1 &= 0 \\
F_2 &= 3 \\
F_3 &= 0 \\
F_4 &= 15
\end{aligned}
$$

Substituting all the above in (7) and simplifying gives the solution as

$$
y = 1 + \frac{x^2}{2} + \frac{x^4}{8} + \frac{x^6}{48} + O\left(x^6\right)
$$

$$
y = 1 + \frac{x^2}{2} + \frac{x^4}{8} + \frac{x^6}{48} + O\left(x^6\right)
$$

Since the expansion point $x = 0$ is an ordinary, we can also solve this using standard power series Let the solution be represented as power series of the form

$$y = \sum_{n=0}^{\infty} a_n x^n$$

Then

$$y' = \sum_{n=1}^{\infty} n a_n x^{n-1}$$
$$y'' = \sum_{n=2}^{\infty} n(n-1)\, a_n x^{n-2}$$

Substituting the above back into the ode gives

$$\sum_{n=2}^{\infty} n(n-1)\, a_n x^{n-2} = \left(\sum_{n=1}^{\infty} n a_n x^{n-1}\right) x + \left(\sum_{n=0}^{\infty} a_n x^n\right) \tag{1}$$

Which simplifies to

$$\left(\sum_{n=2}^{\infty} n(n-1)\, a_n x^{n-2}\right) + \sum_{n=1}^{\infty} (-n\, x^n a_n) + \sum_{n=0}^{\infty} (-a_n x^n) = 0 \tag{2}$$

The next step is to make all powers of x be n in each summation term. Going over each summation term above with power of x in it which is not already x^n and adjusting the power and the corresponding index gives

$$\sum_{n=2}^{\infty} n(n-1)\, a_n x^{n-2} = \sum_{n=0}^{\infty} (n+2)\, a_{n+2}(n+1)\, x^n$$

Substituting all the above in Eq (2) gives the following equation where now all powers of x are the same and equal to n.

$$\left(\sum_{n=0}^{\infty} (n+2)\, a_{n+2}(n+1)\, x^n\right) + \sum_{n=1}^{\infty} (-n\, x^n a_n) + \sum_{n=0}^{\infty} (-a_n x^n) = 0 \tag{3}$$

$n = 0$ gives

$$2a_2 - a_0 = 0$$

$$a_2 = \frac{a_0}{2}$$

For $1 \leq n$, the recurrence equation is

$$(n+2)\, a_{n+2}(n+1) - n a_n - a_n = 0 \tag{4}$$

Solving for a_{n+2}, gives

$$a_{n+2} = \frac{a_n}{n+2} \tag{5}$$

For $n = 1$ the recurrence equation gives

$$6a_3 - 2a_1 = 0$$

Which after substituting the earlier terms found becomes

$$a_3 = \frac{a_1}{3}$$

For $n = 2$ the recurrence equation gives

$$12a_4 - 3a_2 = 0$$

Which after substituting the earlier terms found becomes

$$a_4 = \frac{a_0}{8}$$

For $n = 3$ the recurrence equation gives

$$20a_5 - 4a_3 = 0$$

Which after substituting the earlier terms found becomes

$$a_5 = \frac{a_1}{15}$$

For $n = 4$ the recurrence equation gives

$$30a_6 - 5a_4 = 0$$

Which after substituting the earlier terms found becomes

$$a_6 = \frac{a_0}{48}$$

For $n = 5$ the recurrence equation gives

$$42a_7 - 6a_5 = 0$$

Which after substituting the earlier terms found becomes

$$a_7 = \frac{a_1}{105}$$

And so on. Therefore the solution is

$$\begin{aligned} y &= \sum_{n=0}^{\infty} a_n x^n \\ &= a_3 x^3 + a_2 x^2 + a_1 x + a_0 + \dots \end{aligned}$$

Substituting the values for a_n found above, the solution becomes

$$y = a_0 + a_1 x + \frac{1}{2} a_0 x^2 + \frac{1}{3} a_1 x^3 + \frac{1}{8} a_0 x^4 + \frac{1}{15} a_1 x^5 + \dots$$

Collecting terms, the solution becomes

$$y = \left(1 + \frac{1}{2} x^2 + \frac{1}{8} x^4\right) a_0 + \left(x + \frac{1}{3} x^3 + \frac{1}{15} x^5\right) a_1 + O(x^6) \tag{3}$$

At $x = 0$ the solution above becomes

$$y = \left(1 + \frac{1}{2} x^2 + \frac{1}{8} x^4\right) c_1 + \left(x + \frac{1}{3} x^3 + \frac{1}{15} x^5\right) c_2 + O(x^6)$$

$$y = 1 + \frac{x^2}{2} + \frac{x^4}{8} + O(x^6)$$

<u>Summary</u>

The solution(s) found are the following

$$y = 1 + \frac{x^2}{2} + \frac{x^4}{8} + \frac{x^6}{48} + O(x^6) \tag{1}$$

$$y = 1 + \frac{x^2}{2} + \frac{x^4}{8} + O(x^6) \tag{2}$$

Verification of solutions

$$y = 1 + \frac{x^2}{2} + \frac{x^4}{8} + \frac{x^6}{48} + O(x^6)$$

Verified OK.

$$y = 1 + \frac{x^2}{2} + \frac{x^4}{8} + O(x^6)$$

Verified OK.

13.17.2 Maple step by step solution

Let's solve

$\left[y'' = y'x + y, y(0) = 1, y'\Big|_{\{x=0\}} = 0\right]$

- Highest derivative means the order of the ODE is 2

  y''

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y'' - y'x - y = 0$

- Assume series solution for y

  $y = \sum_{k=0}^{\infty} a_k x^k$

□ Rewrite DE with series expansions

  ◦ Convert $x \cdot y'$ to series expansion

  $x \cdot y' = \sum_{k=0}^{\infty} a_k k\, x^k$

  ◦ Convert y'' to series expansion

  $y'' = \sum_{k=2}^{\infty} a_k k(k-1)\, x^{k-2}$

  ◦ Shift index using $k- >k+2$

  $y'' = \sum_{k=0}^{\infty} a_{k+2}(k+2)(k+1)\, x^k$

  Rewrite DE with series expansions

  $\sum_{k=0}^{\infty} \left(a_{k+2}(k+2)(k+1) - a_k(k+1)\right) x^k = 0$

- Each term in the series must be 0, giving the recursion relation

  $(k+1)(a_{k+2}(k+2)-a_k)=0$

- Recursion relation that defines the series solution to the ODE

  $\left[y=\sum_{k=0}^{\infty} a_k x^k, a_{k+2}=\frac{a_k}{k+2}\right]$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
<- linear_1 successful`
```

✓ Solution by Maple

Time used: 0.015 (sec). Leaf size: 14

```
Order:=6;
dsolve([diff(y(x),x$2)-x*diff(y(x),x)-y(x)=0,y(0) = 1, D(y)(0) = 0],y(x),type='series',x=0);
```

$$y(x)=1+\frac{1}{2}x^2+\frac{1}{8}x^4+\mathrm{O}\left(x^6\right)$$

✓ Solution by Mathematica

Time used: 0.001 (sec). Leaf size: 19

```
AsymptoticDSolveValue[{y''[x]-x*y'[x]-y[x]==0,{y[0]==1,y'[0]==0}},y[x],{x,0,5}]
```

$$y(x)\to\frac{x^4}{8}+\frac{x^2}{2}+1$$

13.18 problem 24

Internal problem ID [728]
Internal file name [`OUTPUT/728_Sunday_June_05_2022_01_47_59_AM_5624398/index.tex`]

Book: Elementary differential equations and boundary value problems, 10th ed., Boyce and DiPrima
Section: Chapter 5.2, Series Solutions Near an Ordinary Point, Part I. page 263
Problem number: 24.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second order series method. Ordinary point", "second order series method. Taylor series method"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

$$\boxed{(x^2+2)\,y''-y'x+4y=0}$$

With initial conditions

$$[y(0)=1, y'(0)=0]$$

With the expansion point for the power series method at $x=0$.

13.18.1 Existence and uniqueness analysis

This is a linear ODE. In canonical form it is written as

$$y''+p(x)y'+q(x)y=F$$

Where here

$$\begin{aligned} p(x) &= -\frac{x}{x^2+2} \\ q(x) &= \frac{4}{x^2+2} \\ F &= 0 \end{aligned}$$

Hence the ode is

$$y'' - \frac{xy'}{x^2+2} + \frac{4y}{x^2+2} = 0$$

The domain of $p(x) = -\frac{x}{x^2+2}$ is

$$\{-\infty < x < \infty\}$$

And the point $x_0 = 0$ is inside this domain. The domain of $q(x) = \frac{4}{x^2+2}$ is

$$\{-\infty < x < \infty\}$$

And the point $x_0 = 0$ is also inside this domain. Hence solution exists and is unique.

Solving ode using Taylor series method. This gives review on how the Taylor series method works for solving second order ode.

Let

$$y'' = f(x, y, y')$$

Assuming expansion is at $x_0 = 0$ (we can always shift the actual expansion point to 0 by change of variables) and assuming $f(x, y, y')$ is analytic at x_0 which must be the case for an ordinary point. Let initial conditions be $y(x_0) = y_0$ and $y'(x_0) = y'_0$. Using Taylor series gives

$$\begin{aligned}
y(x) &= y(x_0) + (x - x_0)\, y'(x_0) + \frac{(x - x_0)^2}{2} y''(x_0) + \frac{(x - x_0)^3}{3!} y'''(x_0) + \cdots \\
&= y_0 + x y'_0 + \frac{x^2}{2} f|_{x_0, y_0, y'_0} + \frac{x^3}{3!} f'|_{x_0, y_0, y'_0} + \cdots \\
&= y_0 + x y'_0 + \sum_{n=0}^{\infty} \frac{x^{n+2}}{(n+2)!} \frac{d^n f}{dx^n}\bigg|_{x_0, y_0, y'_0}
\end{aligned}$$

But

$$\begin{aligned}
\frac{df}{dx} &= \frac{\partial f}{\partial x}\frac{dx}{dx} + \frac{\partial f}{\partial y}\frac{dy}{dx} + \frac{\partial f}{\partial y'}\frac{dy'}{dx} && (1)\\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}y' + \frac{\partial f}{\partial y'}y'' && (795)\\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}y' + \frac{\partial f}{\partial y'}f && (796)\\
\frac{d^2 f}{dx^2} &= \frac{d}{dx}\left(\frac{df}{dx}\right)\\
&= \frac{\partial}{\partial x}\left(\frac{df}{dx}\right) + \frac{\partial}{\partial y}\left(\frac{df}{dx}\right)y' + \frac{\partial}{\partial y'}\left(\frac{df}{dx}\right)f && (2)\\
\frac{d^3 f}{dx^3} &= \frac{d}{dx}\left(\frac{d^2 f}{dx^2}\right)\\
&= \frac{\partial}{\partial x}\left(\frac{d^2 f}{dx^2}\right) + \left(\frac{\partial}{\partial y}\frac{d^2 f}{dx^2}\right)y' + \frac{\partial}{\partial y'}\left(\frac{d^2 f}{dx^2}\right)f && (3)\\
&\vdots
\end{aligned}$$

And so on. Hence if we name $F_0 = f(x, y, y')$ then the above can be written as

$$
\begin{aligned}
F_0 &= f(x,y,y') \qquad (4)\\
F_1 &= \frac{df}{dx}\\
&= \frac{dF_0}{dx}\\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} y' + \frac{\partial f}{\partial y'} y''\\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} y' + \frac{\partial f}{\partial y'} f \qquad (5)\\
&= \frac{\partial F_0}{\partial x} + \frac{\partial F_0}{\partial y} y' + \frac{\partial F_0}{\partial y'} F_0\\
F_2 &= \frac{d}{dx}\left(\frac{d}{dx} f\right)\\
&= \frac{d}{dx}(F_1)\\
&= \frac{\partial}{\partial x} F_1 + \left(\frac{\partial F_1}{\partial y}\right) y' + \left(\frac{\partial F_1}{\partial y'}\right) y''\\
&= \frac{\partial}{\partial x} F_1 + \left(\frac{\partial F_1}{\partial y}\right) y' + \left(\frac{\partial F_1}{\partial y'}\right) F_0\\
&\vdots\\
F_n &= \frac{d}{dx}(F_{n-1})\\
&= \frac{\partial}{\partial x} F_{n-1} + \left(\frac{\partial F_{n-1}}{\partial y}\right) y' + \left(\frac{\partial F_{n-1}}{\partial y'}\right) y''\\
&= \frac{\partial}{\partial x} F_{n-1} + \left(\frac{\partial F_{n-1}}{\partial y}\right) y' + \left(\frac{\partial F_{n-1}}{\partial y'}\right) F_0 \qquad (6)
\end{aligned}
$$

Therefore (6) can be used from now on along with

$$
y(x) = y_0 + x y_0' + \sum_{n=0}^{\infty} \frac{x^{n+2}}{(n+2)!} F_n\big|_{x_0,y_0,y_0'} \qquad (7)
$$

To find $y(x)$ series solution around $x = 0$. Hence

$$
\begin{aligned}
F_0 &= \frac{y'x - 4y}{x^2 + 2} \\
F_1 &= \frac{dF_0}{dx} \\
&= \frac{\partial F_0}{\partial x} + \frac{\partial F_0}{\partial y} y' + \frac{\partial F_0}{\partial y'} F_0 \\
&= \frac{-4y'x^2 + 4yx - 6y'}{(x^2 + 2)^2} \\
F_2 &= \frac{dF_1}{dx} \\
&= \frac{\partial F_1}{\partial x} + \frac{\partial F_1}{\partial y} y' + \frac{\partial F_1}{\partial y'} F_1 \\
&= \frac{8x^3y' + 4x^2y + 10y'x + 32y}{(x^2 + 2)^3} \\
F_3 &= \frac{dF_2}{dx} \\
&= \frac{\partial F_2}{\partial x} + \frac{\partial F_2}{\partial y} y' + \frac{\partial F_2}{\partial y'} F_2 \\
&= \frac{(-12x^4 + 48x^2 + 84)\, y' + (-48x^3 - 216x)\, y}{(x^2 + 2)^4} \\
F_4 &= \frac{dF_3}{dx} \\
&= \frac{\partial F_3}{\partial x} + \frac{\partial F_3}{\partial y} y' + \frac{\partial F_3}{\partial y'} F_3 \\
&= \frac{(-12x^5 - 648x^3 - 828x)\, y' + (288x^4 + 1032x^2 - 768)\, y}{(x^2 + 2)^5}
\end{aligned}
$$

And so on. Evaluating all the above at initial conditions $x = 0$ and $y(0) = 1$ and $y'(0) = 0$ gives

$$
\begin{aligned}
F_0 &= -2 \\
F_1 &= 0 \\
F_2 &= 4 \\
F_3 &= 0 \\
F_4 &= -24
\end{aligned}
$$

Substituting all the above in (7) and simplifying gives the solution as

$$
y = 1 - x^2 + \frac{x^4}{6} - \frac{x^6}{30} + O(x^6)
$$

$$
y = 1 - x^2 + \frac{x^4}{6} - \frac{x^6}{30} + O(x^6)
$$

Since the expansion point $x = 0$ is an ordinary, we can also solve this using standard power series The ode is normalized to be

$$
(x^2 + 2)\, y'' - y'x + 4y = 0
$$

Let the solution be represented as power series of the form

$$
y = \sum_{n=0}^{\infty} a_n x^n
$$

Then

$$
y' = \sum_{n=1}^{\infty} n a_n x^{n-1}
$$

$$
y'' = \sum_{n=2}^{\infty} n(n-1)\, a_n x^{n-2}
$$

Substituting the above back into the ode gives

$$
(x^2 + 2)\left(\sum_{n=2}^{\infty} n(n-1)\, a_n x^{n-2}\right) - \left(\sum_{n=1}^{\infty} n a_n x^{n-1}\right) x + 4\left(\sum_{n=0}^{\infty} a_n x^n\right) = 0 \tag{1}
$$

Which simplifies to

$$
\left(\sum_{n=2}^{\infty} x^n a_n n(n-1)\right) + \left(\sum_{n=2}^{\infty} 2n(n-1)\, a_n x^{n-2}\right) + \sum_{n=1}^{\infty} (-n a_n x^n) + \left(\sum_{n=0}^{\infty} 4a_n x^n\right) = 0 \tag{2}
$$

The next step is to make all powers of x be n in each summation term. Going over each summation term above with power of x in it which is not already x^n and adjusting the power and the corresponding index gives

$$
\sum_{n=2}^{\infty} 2n(n-1)\, a_n x^{n-2} = \sum_{n=0}^{\infty} 2(n+2)\, a_{n+2}(n+1)\, x^n
$$

Substituting all the above in Eq (2) gives the following equation where now all powers of x are the same and equal to n.

$$
\begin{aligned}
&\left(\sum_{n=2}^{\infty} x^n a_n n(n-1)\right) + \left(\sum_{n=0}^{\infty} 2(n+2)\, a_{n+2}(n+1)\, x^n\right) \\
&+ \sum_{n=1}^{\infty} (-n a_n x^n) + \left(\sum_{n=0}^{\infty} 4a_n x^n\right) = 0
\end{aligned} \tag{3}
$$

$n = 0$ gives

$$4a_2 + 4a_0 = 0$$

$$a_2 = -a_0$$

$n = 1$ gives

$$12a_3 + 3a_1 = 0$$

Which after substituting earlier equations, simplifies to

$$a_3 = -\frac{a_1}{4}$$

For $2 \leq n$, the recurrence equation is

$$na_n(n-1) + 2(n+2)\, a_{n+2}(n+1) - na_n + 4a_n = 0 \tag{4}$$

Solving for a_{n+2}, gives

$$a_{n+2} = -\frac{a_n(n^2 - 2n + 4)}{2\,(n+2)\,(n+1)} \tag{5}$$

For $n = 2$ the recurrence equation gives

$$4a_2 + 24a_4 = 0$$

Which after substituting the earlier terms found becomes

$$a_4 = \frac{a_0}{6}$$

For $n = 3$ the recurrence equation gives

$$7a_3 + 40a_5 = 0$$

Which after substituting the earlier terms found becomes

$$a_5 = \frac{7a_1}{160}$$

For $n = 4$ the recurrence equation gives

$$12a_4 + 60a_6 = 0$$

Which after substituting the earlier terms found becomes

$$a_6 = -\frac{a_0}{30}$$

For $n = 5$ the recurrence equation gives

$$19a_5 + 84a_7 = 0$$

Which after substituting the earlier terms found becomes

$$a_7 = -\frac{19a_1}{1920}$$

And so on. Therefore the solution is

$$\begin{aligned} y &= \sum_{n=0}^{\infty} a_n x^n \\ &= a_3 x^3 + a_2 x^2 + a_1 x + a_0 + \dots \end{aligned}$$

Substituting the values for a_n found above, the solution becomes

$$y = a_0 + a_1 x - a_0 x^2 - \frac{1}{4} a_1 x^3 + \frac{1}{6} a_0 x^4 + \frac{7}{160} a_1 x^5 + \dots$$

Collecting terms, the solution becomes

$$y = \left(1 - x^2 + \frac{1}{6}x^4\right) a_0 + \left(x - \frac{1}{4}x^3 + \frac{7}{160}x^5\right) a_1 + O(x^6) \tag{3}$$

At $x = 0$ the solution above becomes

$$y = \left(1 - x^2 + \frac{1}{6}x^4\right) c_1 + \left(x - \frac{1}{4}x^3 + \frac{7}{160}x^5\right) c_2 + O(x^6)$$

$$y = 1 - x^2 + \frac{x^4}{6} + O(x^6)$$

Summary
The solution(s) found are the following

$$y = 1 - x^2 + \frac{x^4}{6} - \frac{x^6}{30} + O(x^6) \tag{1}$$

$$y = 1 - x^2 + \frac{x^4}{6} + O(x^6) \tag{2}$$

Verification of solutions

$$y = 1 - x^2 + \frac{x^4}{6} - \frac{x^6}{30} + O(x^6)$$

Verified OK.

$$y = 1 - x^2 + \frac{x^4}{6} + O(x^6)$$

Verified OK.

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Group is reducible or imprimitive
<- Kovacics algorithm successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 14

```
Order:=6;
dsolve([(2+x^2)*diff(y(x),x$2)-x*diff(y(x),x)+4*y(x)=0,y(0) = 1, D(y)(0) = 0],y(x),type='ser
```

$$y(x) = 1 - x^2 + \frac{1}{6}x^4 + O(x^6)$$

✓ Solution by Mathematica

Time used: 0.001 (sec). Leaf size: 17

```
AsymptoticDSolveValue[{(2+x^2)*y''[x]-x*y'[x]+4*y[x]==0,{y[0]==1,y'[0]==0}},y[x],{x,0,5}]
```

$$y(x) \to \frac{x^4}{6} - x^2 + 1$$

13.19 problem 25

Internal problem ID [729]
Internal file name [`OUTPUT/729_Sunday_June_05_2022_01_48_01_AM_21971426/index.tex`]

Book: Elementary differential equations and boundary value problems, 10th ed., Boyce and DiPrima
Section: Chapter 5.2, Series Solutions Near an Ordinary Point, Part I. page 263
Problem number: 25.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second order series method. Ordinary point", "second order series method. Taylor series method"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

$$\boxed{y'' + y'x + 2y = 0}$$

With initial conditions

$$[y(0) = 0, y'(0) = 1]$$

With the expansion point for the power series method at $x = 0$.

13.19.1 Existence and uniqueness analysis

This is a linear ODE. In canonical form it is written as

$$y'' + p(x)y' + q(x)y = F$$

Where here

$$\begin{aligned} p(x) &= x \\ q(x) &= 2 \\ F &= 0 \end{aligned}$$

Hence the ode is

$$y'' + y'x + 2y = 0$$

The domain of $p(x) = x$ is

$$\{-\infty < x < \infty\}$$

And the point $x_0 = 0$ is inside this domain. The domain of $q(x) = 2$ is

$$\{-\infty < x < \infty\}$$

And the point $x_0 = 0$ is also inside this domain. Hence solution exists and is unique.

Solving ode using Taylor series method. This gives review on how the Taylor series method works for solving second order ode.

Let

$$y'' = f(x, y, y')$$

Assuming expansion is at $x_0 = 0$ (we can always shift the actual expansion point to 0 by change of variables) and assuming $f(x, y, y')$ is analytic at x_0 which must be the case for an ordinary point. Let initial conditions be $y(x_0) = y_0$ and $y'(x_0) = y'_0$. Using Taylor series gives

$$\begin{aligned}
y(x) &= y(x_0) + (x - x_0)\, y'(x_0) + \frac{(x - x_0)^2}{2} y''(x_0) + \frac{(x - x_0)^3}{3!} y'''(x_0) + \cdots \\
&= y_0 + x y'_0 + \frac{x^2}{2} f|_{x_0, y_0, y'_0} + \frac{x^3}{3!} f'|_{x_0, y_0, y'_0} + \cdots \\
&= y_0 + x y'_0 + \sum_{n=0}^{\infty} \frac{x^{n+2}}{(n+2)!} \left.\frac{d^n f}{dx^n}\right|_{x_0, y_0, y'_0}
\end{aligned}$$

But

$$\begin{aligned}
\frac{df}{dx} &= \frac{\partial f}{\partial x}\frac{dx}{dx} + \frac{\partial f}{\partial y}\frac{dy}{dx} + \frac{\partial f}{\partial y'}\frac{dy'}{dx} && (1) \\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}y' + \frac{\partial f}{\partial y'}y'' && (798) \\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}y' + \frac{\partial f}{\partial y'}f && (799) \\
\frac{d^2 f}{dx^2} &= \frac{d}{dx}\left(\frac{df}{dx}\right) \\
&= \frac{\partial}{\partial x}\left(\frac{df}{dx}\right) + \frac{\partial}{\partial y}\left(\frac{df}{dx}\right)y' + \frac{\partial}{\partial y'}\left(\frac{df}{dx}\right)f && (2) \\
\frac{d^3 f}{dx^3} &= \frac{d}{dx}\left(\frac{d^2 f}{dx^2}\right) \\
&= \frac{\partial}{\partial x}\left(\frac{d^2 f}{dx^2}\right) + \left(\frac{\partial}{\partial y}\frac{d^2 f}{dx^2}\right)y' + \frac{\partial}{\partial y'}\left(\frac{d^2 f}{dx^2}\right)f && (3) \\
&\vdots
\end{aligned}$$

And so on. Hence if we name $F_0 = f(x,y,y')$ then the above can be written as

$$
\begin{aligned}
F_0 &= f(x,y,y') \qquad (4)\\
F_1 &= \frac{df}{dx}\\
&= \frac{dF_0}{dx}\\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}y' + \frac{\partial f}{\partial y'}y''\\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}y' + \frac{\partial f}{\partial y'}f \qquad (5)\\
&= \frac{\partial F_0}{\partial x} + \frac{\partial F_0}{\partial y}y' + \frac{\partial F_0}{\partial y'}F_0\\
F_2 &= \frac{d}{dx}\left(\frac{d}{dx}f\right)\\
&= \frac{d}{dx}(F_1)\\
&= \frac{\partial}{\partial x}F_1 + \left(\frac{\partial F_1}{\partial y}\right)y' + \left(\frac{\partial F_1}{\partial y'}\right)y''\\
&= \frac{\partial}{\partial x}F_1 + \left(\frac{\partial F_1}{\partial y}\right)y' + \left(\frac{\partial F_1}{\partial y'}\right)F_0\\
&\vdots\\
F_n &= \frac{d}{dx}(F_{n-1})\\
&= \frac{\partial}{\partial x}F_{n-1} + \left(\frac{\partial F_{n-1}}{\partial y}\right)y' + \left(\frac{\partial F_{n-1}}{\partial y'}\right)y''\\
&= \frac{\partial}{\partial x}F_{n-1} + \left(\frac{\partial F_{n-1}}{\partial y}\right)y' + \left(\frac{\partial F_{n-1}}{\partial y'}\right)F_0 \qquad (6)
\end{aligned}
$$

Therefore (6) can be used from now on along with

$$y(x) = y_0 + xy_0' + \sum_{n=0}^{\infty}\frac{x^{n+2}}{(n+2)!}F_n|_{x_0,y_0,y_0'} \tag{7}$$

To find $y(x)$ series solution around $x = 0$. Hence

$$
\begin{aligned}
F_0 &= -y'x - 2y \\
F_1 &= \frac{dF_0}{dx} \\
&= \frac{\partial F_0}{\partial x} + \frac{\partial F_0}{\partial y} y' + \frac{\partial F_0}{\partial y'} F_0 \\
&= y'x^2 + 2yx - 3y' \\
F_2 &= \frac{dF_1}{dx} \\
&= \frac{\partial F_1}{\partial x} + \frac{\partial F_1}{\partial y} y' + \frac{\partial F_1}{\partial y'} F_1 \\
&= -x^3y' - 2x^2y + 7y'x + 8y \\
F_3 &= \frac{dF_2}{dx} \\
&= \frac{\partial F_2}{\partial x} + \frac{\partial F_2}{\partial y} y' + \frac{\partial F_2}{\partial y'} F_2 \\
&= \left(x^4 - 12x^2 + 15\right) y' + 2\left(x^3 - 9x\right) y \\
F_4 &= \frac{dF_3}{dx} \\
&= \frac{\partial F_3}{\partial x} + \frac{\partial F_3}{\partial y} y' + \frac{\partial F_3}{\partial y'} F_3 \\
&= \left(-x^5 + 18x^3 - 57x\right) y' - 2y\left(x^4 - 15x^2 + 24\right)
\end{aligned}
$$

And so on. Evaluating all the above at initial conditions $x = 0$ and $y(0) = 0$ and $y'(0) = 1$ gives

$$
\begin{aligned}
F_0 &= 0 \\
F_1 &= -3 \\
F_2 &= 0 \\
F_3 &= 15 \\
F_4 &= 0
\end{aligned}
$$

Substituting all the above in (7) and simplifying gives the solution as

$$
y = x - \frac{x^3}{2} + \frac{x^5}{8} + O\left(x^6\right)
$$

$$
y = x - \frac{x^3}{2} + \frac{x^5}{8} + O\left(x^6\right)
$$

Since the expansion point $x = 0$ is an ordinary, we can also solve this using standard power series Let the solution be represented as power series of the form

$$y = \sum_{n=0}^{\infty} a_n x^n$$

Then

$$y' = \sum_{n=1}^{\infty} n a_n x^{n-1}$$
$$y'' = \sum_{n=2}^{\infty} n(n-1)\, a_n x^{n-2}$$

Substituting the above back into the ode gives

$$\sum_{n=2}^{\infty} n(n-1)\, a_n x^{n-2} = -\left(\sum_{n=1}^{\infty} n a_n x^{n-1}\right) x - 2\left(\sum_{n=0}^{\infty} a_n x^n\right) \tag{1}$$

Which simplifies to

$$\left(\sum_{n=2}^{\infty} n(n-1)\, a_n x^{n-2}\right) + \left(\sum_{n=1}^{\infty} n\, x^n a_n\right) + \left(\sum_{n=0}^{\infty} 2 a_n x^n\right) = 0 \tag{2}$$

The next step is to make all powers of x be n in each summation term. Going over each summation term above with power of x in it which is not already x^n and adjusting the power and the corresponding index gives

$$\sum_{n=2}^{\infty} n(n-1)\, a_n x^{n-2} = \sum_{n=0}^{\infty} (n+2)\, a_{n+2}(n+1)\, x^n$$

Substituting all the above in Eq (2) gives the following equation where now all powers of x are the same and equal to n.

$$\left(\sum_{n=0}^{\infty} (n+2)\, a_{n+2}(n+1)\, x^n\right) + \left(\sum_{n=1}^{\infty} n\, x^n a_n\right) + \left(\sum_{n=0}^{\infty} 2 a_n x^n\right) = 0 \tag{3}$$

$n = 0$ gives

$$2a_2 + 2a_0 = 0$$

$$a_2 = -a_0$$

For $1 \leq n$, the recurrence equation is

$$(n+2)\, a_{n+2}(n+1) + n a_n + 2a_n = 0 \tag{4}$$

Solving for a_{n+2}, gives

$$a_{n+2} = -\frac{a_n}{n+1} \tag{5}$$

For $n = 1$ the recurrence equation gives

$$6a_3 + 3a_1 = 0$$

Which after substituting the earlier terms found becomes

$$a_3 = -\frac{a_1}{2}$$

For $n = 2$ the recurrence equation gives

$$12a_4 + 4a_2 = 0$$

Which after substituting the earlier terms found becomes

$$a_4 = \frac{a_0}{3}$$

For $n = 3$ the recurrence equation gives

$$20a_5 + 5a_3 = 0$$

Which after substituting the earlier terms found becomes

$$a_5 = \frac{a_1}{8}$$

For $n = 4$ the recurrence equation gives

$$30a_6 + 6a_4 = 0$$

Which after substituting the earlier terms found becomes

$$a_6 = -\frac{a_0}{15}$$

For $n = 5$ the recurrence equation gives

$$42a_7 + 7a_5 = 0$$

Which after substituting the earlier terms found becomes

$$a_7 = -\frac{a_1}{48}$$

And so on. Therefore the solution is

$$\begin{aligned} y &= \sum_{n=0}^{\infty} a_n x^n \\ &= a_3 x^3 + a_2 x^2 + a_1 x + a_0 + \dots \end{aligned}$$

Substituting the values for a_n found above, the solution becomes

$$y = a_0 + a_1 x - a_0 x^2 - \frac{1}{2} a_1 x^3 + \frac{1}{3} a_0 x^4 + \frac{1}{8} a_1 x^5 + \dots$$

Collecting terms, the solution becomes

$$y = \left(1 - x^2 + \frac{1}{3} x^4\right) a_0 + \left(x - \frac{1}{2} x^3 + \frac{1}{8} x^5\right) a_1 + O(x^6) \tag{3}$$

At $x = 0$ the solution above becomes

$$y = \left(1 - x^2 + \frac{1}{3} x^4\right) c_1 + \left(x - \frac{1}{2} x^3 + \frac{1}{8} x^5\right) c_2 + O(x^6)$$

$$y = x - \frac{x^3}{2} + \frac{x^5}{8} + O(x^6)$$

Summary
The solution(s) found are the following

$$y = x - \frac{x^3}{2} + \frac{x^5}{8} + O(x^6) \tag{1}$$

$$y = x - \frac{x^3}{2} + \frac{x^5}{8} + O(x^6) \tag{2}$$

Verification of solutions

$$y = x - \frac{x^3}{2} + \frac{x^5}{8} + O(x^6)$$

Verified OK.

$$y = x - \frac{x^3}{2} + \frac{x^5}{8} + O(x^6)$$

Verified OK.

13.19.2 Maple step by step solution

Let's solve

$\left[y'' = -y'x - 2y, y(0) = 0, y'\Big|_{\{x=0\}} = 1\right]$

- Highest derivative means the order of the ODE is 2

  y''

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y'' + y'x + 2y = 0$

- Assume series solution for y

  $y = \sum_{k=0}^{\infty} a_k x^k$

□ Rewrite DE with series expansions

  ○ Convert $x \cdot y'$ to series expansion

  $x \cdot y' = \sum_{k=0}^{\infty} a_k k\, x^k$

  ○ Convert y'' to series expansion

  $y'' = \sum_{k=2}^{\infty} a_k k(k-1)\, x^{k-2}$

  ○ Shift index using $k- >k+2$

  $y'' = \sum_{k=0}^{\infty} a_{k+2}(k+2)\,(k+1)\, x^k$

  Rewrite DE with series expansions

  $\sum_{k=0}^{\infty} (a_{k+2}(k+2)\,(k+1) + a_k(k+2))\, x^k = 0$

- Each term in the series must be 0, giving the recursion relation

  $$(k+2)(ka_{k+2}+a_k+a_{k+2})=0$$

- Recursion relation that defines the series solution to the ODE

  $$\left[y=\sum_{k=0}^{\infty} a_k x^k, a_{k+2}=-\frac{a_k}{k+1}\right]$$

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Group is reducible, not completely reducible
<- Kovacics algorithm successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 14

```
Order:=6;
dsolve([diff(y(x),x$2)+x*diff(y(x),x)+2*y(x)=0,y(0) = 0, D(y)(0) = 1],y(x),type='series',x=0
```

$$y(x)=x-\frac{1}{2}x^3+\frac{1}{8}x^5+\mathrm{O}\left(x^6\right)$$

✓ Solution by Mathematica

Time used: 0.001 (sec). Leaf size: 19

```
AsymptoticDSolveValue[{y''[x]+x*y'[x]+2*y[x]==0,{y[0]==0,y'[0]==1}},y[x],{x,0,5}]
```

$$y(x)\to\frac{x^5}{8}-\frac{x^3}{2}+x$$

13.20 problem 26

Internal problem ID [730]
Internal file name [`OUTPUT/730_Sunday_June_05_2022_01_48_03_AM_91788621/index.tex`]

Book: Elementary differential equations and boundary value problems, 10th ed., Boyce and DiPrima
Section: Chapter 5.2, Series Solutions Near an Ordinary Point, Part I. page 263
Problem number: 26.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second order series method. Ordinary point", "second order series method. Taylor series method"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

$$\boxed{(-x^2+4)\, y''+y'x+2y=0}$$

With initial conditions

$$[y(0)=0, y'(0)=1]$$

With the expansion point for the power series method at $x=0$.

13.20.1 Existence and uniqueness analysis

This is a linear ODE. In canonical form it is written as

$$y''+p(x)y'+q(x)y=F$$

Where here

$$\begin{aligned}
p(x) &= \frac{x}{-x^2+4} \\
q(x) &= \frac{2}{-x^2+4} \\
F &= 0
\end{aligned}$$

Hence the ode is

$$y'' + \frac{xy'}{-x^2+4} + \frac{2y}{-x^2+4} = 0$$

The domain of $p(x) = \frac{x}{-x^2+4}$ is

$$\{-\infty \le x < -2, -2 < x < 2, 2 < x \le \infty\}$$

And the point $x_0 = 0$ is inside this domain. The domain of $q(x) = \frac{2}{-x^2+4}$ is

$$\{-\infty \le x < -2, -2 < x < 2, 2 < x \le \infty\}$$

And the point $x_0 = 0$ is also inside this domain. Hence solution exists and is unique.

Solving ode using Taylor series method. This gives review on how the Taylor series method works for solving second order ode.

Let

$$y'' = f(x, y, y')$$

Assuming expansion is at $x_0 = 0$ (we can always shift the actual expansion point to 0 by change of variables) and assuming $f(x, y, y')$ is analytic at x_0 which must be the case for an ordinary point. Let initial conditions be $y(x_0) = y_0$ and $y'(x_0) = y'_0$. Using Taylor series gives

$$\begin{aligned}
y(x) &= y(x_0) + (x - x_0)\, y'(x_0) + \frac{(x - x_0)^2}{2} y''(x_0) + \frac{(x - x_0)^3}{3!} y'''(x_0) + \cdots \\
&= y_0 + x y'_0 + \frac{x^2}{2} f|_{x_0, y_0, y'_0} + \frac{x^3}{3!} f'|_{x_0, y_0, y'_0} + \cdots \\
&= y_0 + x y'_0 + \sum_{n=0}^{\infty} \frac{x^{n+2}}{(n+2)!} \frac{d^n f}{dx^n}\bigg|_{x_0, y_0, y'_0}
\end{aligned}$$

But

$$\frac{df}{dx} = \frac{\partial f}{\partial x}\frac{dx}{dx} + \frac{\partial f}{\partial y}\frac{dy}{dx} + \frac{\partial f}{\partial y'}\frac{dy'}{dx} \tag{1}$$

$$= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}y' + \frac{\partial f}{\partial y'}y'' \tag{801}$$

$$= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}y' + \frac{\partial f}{\partial y'}f \tag{802}$$

$$\frac{d^2 f}{dx^2} = \frac{d}{dx}\left(\frac{df}{dx}\right)$$

$$= \frac{\partial}{\partial x}\left(\frac{df}{dx}\right) + \frac{\partial}{\partial y}\left(\frac{df}{dx}\right)y' + \frac{\partial}{\partial y'}\left(\frac{df}{dx}\right)f \tag{2}$$

$$\frac{d^3 f}{dx^3} = \frac{d}{dx}\left(\frac{d^2 f}{dx^2}\right)$$

$$= \frac{\partial}{\partial x}\left(\frac{d^2 f}{dx^2}\right) + \left(\frac{\partial}{\partial y}\frac{d^2 f}{dx^2}\right)y' + \frac{\partial}{\partial y'}\left(\frac{d^2 f}{dx^2}\right)f \tag{3}$$

$$\vdots$$

And so on. Hence if we name $F_0 = f(x, y, y')$ then the above can be written as

$$
\begin{aligned}
F_0 &= f(x, y, y') &(4)\\
F_1 &= \frac{df}{dx}\\
&= \frac{dF_0}{dx}\\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} y' + \frac{\partial f}{\partial y'} y''\\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} y' + \frac{\partial f}{\partial y'} f &(5)\\
&= \frac{\partial F_0}{\partial x} + \frac{\partial F_0}{\partial y} y' + \frac{\partial F_0}{\partial y'} F_0\\
F_2 &= \frac{d}{dx}\left(\frac{d}{dx} f\right)\\
&= \frac{d}{dx}(F_1)\\
&= \frac{\partial}{\partial x} F_1 + \left(\frac{\partial F_1}{\partial y}\right) y' + \left(\frac{\partial F_1}{\partial y'}\right) y''\\
&= \frac{\partial}{\partial x} F_1 + \left(\frac{\partial F_1}{\partial y}\right) y' + \left(\frac{\partial F_1}{\partial y'}\right) F_0\\
&\vdots\\
F_n &= \frac{d}{dx}(F_{n-1})\\
&= \frac{\partial}{\partial x} F_{n-1} + \left(\frac{\partial F_{n-1}}{\partial y}\right) y' + \left(\frac{\partial F_{n-1}}{\partial y'}\right) y''\\
&= \frac{\partial}{\partial x} F_{n-1} + \left(\frac{\partial F_{n-1}}{\partial y}\right) y' + \left(\frac{\partial F_{n-1}}{\partial y'}\right) F_0 &(6)
\end{aligned}
$$

Therefore (6) can be used from now on along with

$$
y(x) = y_0 + x y_0' + \sum_{n=0}^{\infty} \frac{x^{n+2}}{(n+2)!} F_n\big|_{x_0, y_0, y_0'} \tag{7}
$$

To find $y(x)$ series solution around $x = 0$. Hence

$$
\begin{aligned}
F_0 &= \frac{2y + y'x}{x^2 - 4} \\
F_1 &= \frac{dF_0}{dx} \\
&= \frac{\partial F_0}{\partial x} + \frac{\partial F_0}{\partial y} y' + \frac{\partial F_0}{\partial y'} F_0 \\
&= \frac{2y'x^2 - 2yx - 12y'}{(x^2 - 4)^2} \\
F_2 &= \frac{dF_1}{dx} \\
&= \frac{\partial F_1}{\partial x} + \frac{\partial F_1}{\partial y} y' + \frac{\partial F_1}{\partial y'} F_1 \\
&= \frac{-4x^3y' + 10x^2y + 28y'x - 16y}{(x^2 - 4)^3} \\
F_3 &= \frac{dF_2}{dx} \\
&= \frac{\partial F_2}{\partial x} + \frac{\partial F_2}{\partial y} y' + \frac{\partial F_2}{\partial y'} F_2 \\
&= \frac{(18x^4 - 120x^2 - 48)\, y' + (-48x^3 + 72x)\, y}{(x^2 - 4)^4} \\
F_4 &= \frac{dF_3}{dx} \\
&= \frac{\partial F_3}{\partial x} + \frac{\partial F_3}{\partial y} y' + \frac{\partial F_3}{\partial y'} F_3 \\
&= \frac{(-102x^5 + 576x^3 + 1008x)\, y' + (276x^4 - 168x^2 - 384)\, y}{(x^2 - 4)^5}
\end{aligned}
$$

And so on. Evaluating all the above at initial conditions $x = 0$ and $y(0) = 0$ and $y'(0) = 1$ gives

$$
\begin{aligned}
F_0 &= 0 \\
F_1 &= -\frac{3}{4} \\
F_2 &= 0 \\
F_3 &= -\frac{3}{16} \\
F_4 &= 0
\end{aligned}
$$

Substituting all the above in (7) and simplifying gives the solution as

$$y = x - \frac{x^3}{8} - \frac{x^5}{640} + O(x^6)$$

$$y = x - \frac{x^3}{8} - \frac{x^5}{640} + O(x^6)$$

Since the expansion point $x = 0$ is an ordinary, we can also solve this using standard power series The ode is normalized to be

$$(-x^2 + 4)\, y'' + y'x + 2y = 0$$

Let the solution be represented as power series of the form

$$y = \sum_{n=0}^{\infty} a_n x^n$$

Then

$$y' = \sum_{n=1}^{\infty} n a_n x^{n-1}$$
$$y'' = \sum_{n=2}^{\infty} n(n-1)\, a_n x^{n-2}$$

Substituting the above back into the ode gives

$$(-x^2 + 4)\left(\sum_{n=2}^{\infty} n(n-1)\, a_n x^{n-2}\right) + \left(\sum_{n=1}^{\infty} n a_n x^{n-1}\right) x + 2\left(\sum_{n=0}^{\infty} a_n x^n\right) = 0 \quad (1)$$

Which simplifies to

$$\sum_{n=2}^{\infty} (-x^n a_n n(n-1)) + \left(\sum_{n=2}^{\infty} 4n(n-1)\, a_n x^{n-2}\right) + \left(\sum_{n=1}^{\infty} n a_n x^n\right) + \left(\sum_{n=0}^{\infty} 2a_n x^n\right) = 0 \quad (2)$$

The next step is to make all powers of x be n in each summation term. Going over each summation term above with power of x in it which is not already x^n and adjusting the power and the corresponding index gives

$$\sum_{n=2}^{\infty} 4n(n-1)\, a_n x^{n-2} = \sum_{n=0}^{\infty} 4(n+2)\, a_{n+2}(n+1)\, x^n$$

Substituting all the above in Eq (2) gives the following equation where now all powers of x are the same and equal to n.

$$\sum_{n=2}^{\infty} \left(-x^n a_n n(n-1)\right) + \left(\sum_{n=0}^{\infty} 4(n+2)\, a_{n+2}(n+1)\, x^n\right) + \left(\sum_{n=1}^{\infty} n a_n x^n\right) + \left(\sum_{n=0}^{\infty} 2a_n x^n\right) = 0 \tag{3}$$

$n = 0$ gives

$$8a_2 + 2a_0 = 0$$

$$a_2 = -\frac{a_0}{4}$$

$n = 1$ gives

$$24a_3 + 3a_1 = 0$$

Which after substituting earlier equations, simplifies to

$$a_3 = -\frac{a_1}{8}$$

For $2 \leq n$, the recurrence equation is

$$-n a_n(n-1) + 4(n+2)\, a_{n+2}(n+1) + n a_n + 2a_n = 0 \tag{4}$$

Solving for a_{n+2}, gives

$$a_{n+2} = \frac{a_n(n^2 - 2n - 2)}{4\,(n+2)\,(n+1)} \tag{5}$$

For $n = 2$ the recurrence equation gives

$$2a_2 + 48a_4 = 0$$

Which after substituting the earlier terms found becomes

$$a_4 = \frac{a_0}{96}$$

For $n = 3$ the recurrence equation gives

$$-a_3 + 80a_5 = 0$$

Which after substituting the earlier terms found becomes

$$a_5 = -\frac{a_1}{640}$$

For $n = 4$ the recurrence equation gives

$$-6a_4 + 120a_6 = 0$$

Which after substituting the earlier terms found becomes

$$a_6 = \frac{a_0}{1920}$$

For $n = 5$ the recurrence equation gives

$$-13a_5 + 168a_7 = 0$$

Which after substituting the earlier terms found becomes

$$a_7 = -\frac{13a_1}{107520}$$

And so on. Therefore the solution is

$$\begin{aligned} y &= \sum_{n=0}^{\infty} a_n x^n \\ &= a_3 x^3 + a_2 x^2 + a_1 x + a_0 + \dots \end{aligned}$$

Substituting the values for a_n found above, the solution becomes

$$y = a_0 + a_1 x - \frac{1}{4} a_0 x^2 - \frac{1}{8} a_1 x^3 + \frac{1}{96} a_0 x^4 - \frac{1}{640} a_1 x^5 + \dots$$

Collecting terms, the solution becomes

$$y = \left(1 - \frac{1}{4}x^2 + \frac{1}{96}x^4\right) a_0 + \left(x - \frac{1}{8}x^3 - \frac{1}{640}x^5\right) a_1 + O(x^6) \tag{3}$$

At $x = 0$ the solution above becomes

$$y = \left(1 - \frac{1}{4}x^2 + \frac{1}{96}x^4\right)c_1 + \left(x - \frac{1}{8}x^3 - \frac{1}{640}x^5\right)c_2 + O(x^6)$$

$$y = x - \frac{x^3}{8} - \frac{x^5}{640} + O(x^6)$$

Summary

The solution(s) found are the following

$$y = x - \frac{x^3}{8} - \frac{x^5}{640} + O(x^6) \tag{1}$$

$$y = x - \frac{x^3}{8} - \frac{x^5}{640} + O(x^6) \tag{2}$$

Verification of solutions

$$y = x - \frac{x^3}{8} - \frac{x^5}{640} + O(x^6)$$

Verified OK.

$$y = x - \frac{x^3}{8} - \frac{x^5}{640} + O(x^6)$$

Verified OK.

13.20.2 Maple step by step solution

Let's solve

$$\left[(-x^2 + 4)\,y'' + y'x + 2y = 0, y(0) = 0, y'\Big|_{\{x=0\}} = 1\right]$$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y'' = \frac{xy'}{x^2-4} + \frac{2y}{x^2-4}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y'' - \frac{xy'}{x^2-4} - \frac{2y}{x^2-4} = 0$

□ Check to see if x_0 is a regular singular point

- Define functions

  $\left[P_2(x) = -\frac{x}{x^2-4}, P_3(x) = -\frac{2}{x^2-4}\right]$

- $(2+x) \cdot P_2(x)$ is analytic at $x=-2$

  $\left((2+x) \cdot P_2(x)\right)\Big|_{x=-2} = -\frac{1}{2}$

- $(2+x)^2 \cdot P_3(x)$ is analytic at $x=-2$

  $\left((2+x)^2 \cdot P_3(x)\right)\Big|_{x=-2} = 0$

- $x=-2$is a regular singular point

  Check to see if x_0 is a regular singular point

  $x_0 = -2$

- Multiply by denominators

  $y''(x^2-4) - y'x - 2y = 0$

- Change variables using $x = u-2$ so that the regular singular point is at $u=0$

  $(u^2-4u)\left(\frac{d^2}{du^2}y(u)\right) + (-u+2)\left(\frac{d}{du}y(u)\right) - 2y(u) = 0$

- Assume series solution for $y(u)$

  $y(u) = \sum_{k=0}^{\infty} a_k u^{k+r}$

☐ Rewrite ODE with series expansions

- Convert $u^m \cdot \left(\frac{d}{du}y(u)\right)$ to series expansion for $m = 0..1$

  $u^m \cdot \left(\frac{d}{du}y(u)\right) = \sum_{k=0}^{\infty} a_k(k+r)\, u^{k+r-1+m}$

- Shift index using $k->k+1-m$

  $u^m \cdot \left(\frac{d}{du}y(u)\right) = \sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\, u^{k+r}$

- Convert $u^m \cdot \left(\frac{d^2}{du^2}y(u)\right)$ to series expansion for $m = 1..2$

  $u^m \cdot \left(\frac{d^2}{du^2}y(u)\right) = \sum_{k=0}^{\infty} a_k(k+r)\,(k+r-1)\, u^{k+r-2+m}$

- Shift index using $k->k+2-m$

  $u^m \cdot \left(\frac{d^2}{du^2}y(u)\right) = \sum_{k=-2+m}^{\infty} a_{k+2-m}(k+2-m+r)\,(k+1-m+r)\, u^{k+r}$

Rewrite ODE with series expansions

$$-2a_0r(-3+2r)\,u^{-1+r}+\left(\sum_{k=0}^{\infty}\left(-2a_{k+1}(k+1+r)\,(2k-1+2r)+a_k(k^2+2kr+r^2-2k-2r-\right.\right.$$

- $a_0$cannot be 0 by assumption, giving the indicial equation

$$-2r(-3+2r)=0$$

- Values of r that satisfy the indicial equation

$$r\in\left\{0,\frac{3}{2}\right\}$$

- Each term in the series must be 0, giving the recursion relation

$$-4\left(k-\frac{1}{2}+r\right)(k+1+r)\,a_{k+1}+\left(k^2+(2r-2)\,k+r^2-2r-2\right)a_k=0$$

- Recursion relation that defines series solution to ODE

$$a_{k+1}=\frac{(k^2+2kr+r^2-2k-2r-2)\,a_k}{2(2k-1+2r)\,(k+1+r)}$$

- Recursion relation for $r=0$

$$a_{k+1}=\frac{(k^2-2k-2)\,a_k}{2(2k-1)\,(k+1)}$$

- Solution for $r=0$

$$\left[y(u)=\sum_{k=0}^{\infty}a_ku^k,a_{k+1}=\frac{(k^2-2k-2)\,a_k}{2(2k-1)\,(k+1)}\right]$$

- Revert the change of variables $u=2+x$

$$\left[y=\sum_{k=0}^{\infty}a_k(2+x)^k\,,a_{k+1}=\frac{(k^2-2k-2)\,a_k}{2(2k-1)\,(k+1)}\right]$$

- Recursion relation for $r=\frac{3}{2}$

$$a_{k+1}=\frac{\left(k^2+k-\frac{11}{4}\right)a_k}{2(2k+2)\left(k+\frac{5}{2}\right)}$$

- Solution for $r=\frac{3}{2}$

$$\left[y(u)=\sum_{k=0}^{\infty}a_ku^{k+\frac{3}{2}},a_{k+1}=\frac{\left(k^2+k-\frac{11}{4}\right)a_k}{2(2k+2)\left(k+\frac{5}{2}\right)}\right]$$

- Revert the change of variables $u=2+x$

$$\left[y=\sum_{k=0}^{\infty}a_k(2+x)^{k+\frac{3}{2}}\,,a_{k+1}=\frac{\left(k^2+k-\frac{11}{4}\right)a_k}{2(2k+2)\left(k+\frac{5}{2}\right)}\right]$$

- Combine solutions and rename parameters

$$\left[y=\left(\sum_{k=0}^{\infty}a_k(2+x)^k\right)+\left(\sum_{k=0}^{\infty}b_k(2+x)^{k+\frac{3}{2}}\right),a_{k+1}=\frac{(k^2-2k-2)\,a_k}{2(2k-1)\,(k+1)},b_{k+1}=\frac{\left(k^2+k-\frac{11}{4}\right)b_k}{2(2k+2)\left(k+\frac{5}{2}\right)}\right]$$

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Group is reducible or imprimitive
   Solution has integrals. Trying a special function solution free of integrals...
   -> Trying a solution in terms of special functions:
      -> Bessel
      -> elliptic
      -> Legendre
      <- Legendre successful
   <- special function solution successful
      -> Trying to convert hypergeometric functions to elementary form...
      <- elementary form could result into a too large expression - returning special functi
   <- Kovacics algorithm successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 14

```
Order:=6;
dsolve([(4-x^2)*diff(y(x),x$2)+x*diff(y(x),x)+2*y(x)=0,y(0) = 0, D(y)(0) = 1],y(x),type='ser
```

$$y(x)=x-\frac{1}{8}x^{3}-\frac{1}{640}x^{5}+\mathrm{O}\left(x^{6}\right)$$

✓ Solution by Mathematica

Time used: 0.001 (sec). Leaf size: 19

```
AsymptoticDSolveValue[{(4-x^2)*y''[x]+x*y'[x]+2*y[x]==0,{y[0]==0,y'[0]==1}},y[x],{x,0,5}]
```

$$y(x)\to -\frac{x^5}{640}-\frac{x^3}{8}+x$$

13.21 problem 27

Internal problem ID [731]
Internal file name [`OUTPUT/731_Sunday_June_05_2022_01_48_05_AM_12870666/index.tex`]

Book: Elementary differential equations and boundary value problems, 10th ed., Boyce and DiPrima
Section: Chapter 5.2, Series Solutions Near an Ordinary Point, Part I. page 263
Problem number: 27.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second_order_bessel_ode", "second order series method. Ordinary point", "second order series method. Taylor series method"**

Maple gives the following as the ode type

```
[[_Emden, _Fowler]]
```

$$\boxed{y'' + x^2y = 0}$$

With initial conditions

$$[y(0) = 1, y'(0) = 0]$$

With the expansion point for the power series method at $x = 0$.

13.21.1 Existence and uniqueness analysis

This is a linear ODE. In canonical form it is written as

$$y'' + p(x)y' + q(x)y = F$$

Where here

$$\begin{aligned} p(x) &= 0 \\ q(x) &= x^2 \\ F &= 0 \end{aligned}$$

Hence the ode is

$$y'' + x^2 y = 0$$

The domain of $p(x) = 0$ is

$$\{-\infty < x < \infty\}$$

And the point $x_0 = 0$ is inside this domain. The domain of $q(x) = x^2$ is

$$\{-\infty < x < \infty\}$$

And the point $x_0 = 0$ is also inside this domain. Hence solution exists and is unique.

Solving ode using Taylor series method. This gives review on how the Taylor series method works for solving second order ode.

Let

$$y'' = f(x, y, y')$$

Assuming expansion is at $x_0 = 0$ (we can always shift the actual expansion point to 0 by change of variables) and assuming $f(x, y, y')$ is analytic at x_0 which must be the case for an ordinary point. Let initial conditions be $y(x_0) = y_0$ and $y'(x_0) = y'_0$. Using Taylor series gives

$$\begin{aligned} y(x) &= y(x_0) + (x - x_0)\, y'(x_0) + \frac{(x - x_0)^2}{2} y''(x_0) + \frac{(x - x_0)^3}{3!} y'''(x_0) + \cdots \\ &= y_0 + x y'_0 + \frac{x^2}{2} f|_{x_0, y_0, y'_0} + \frac{x^3}{3!} f'|_{x_0, y_0, y'_0} + \cdots \\ &= y_0 + x y'_0 + \sum_{n=0}^{\infty} \frac{x^{n+2}}{(n+2)\,!} \frac{d^n f}{dx^n}\bigg|_{x_0, y_0, y'_0} \end{aligned}$$

But

$$\frac{df}{dx} = \frac{\partial f}{\partial x}\frac{dx}{dx} + \frac{\partial f}{\partial y}\frac{dy}{dx} + \frac{\partial f}{\partial y'}\frac{dy'}{dx} \tag{1}$$

$$= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}y' + \frac{\partial f}{\partial y'}y'' \tag{804}$$

$$= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}y' + \frac{\partial f}{\partial y'}f \tag{805}$$

$$\frac{d^2 f}{dx^2} = \frac{d}{dx}\left(\frac{df}{dx}\right)$$

$$= \frac{\partial}{\partial x}\left(\frac{df}{dx}\right) + \frac{\partial}{\partial y}\left(\frac{df}{dx}\right)y' + \frac{\partial}{\partial y'}\left(\frac{df}{dx}\right)f \tag{2}$$

$$\frac{d^3 f}{dx^3} = \frac{d}{dx}\left(\frac{d^2 f}{dx^2}\right)$$

$$= \frac{\partial}{\partial x}\left(\frac{d^2 f}{dx^2}\right) + \left(\frac{\partial}{\partial y}\frac{d^2 f}{dx^2}\right)y' + \frac{\partial}{\partial y'}\left(\frac{d^2 f}{dx^2}\right)f \tag{3}$$

$$\vdots$$

And so on. Hence if we name $F_0 = f(x, y, y')$ then the above can be written as

$$
\begin{aligned}
F_0 &= f(x,y,y') &(4)\\
F_1 &= \frac{df}{dx}\\
&= \frac{dF_0}{dx}\\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}y' + \frac{\partial f}{\partial y'}y''\\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}y' + \frac{\partial f}{\partial y'}f &(5)\\
&= \frac{\partial F_0}{\partial x} + \frac{\partial F_0}{\partial y}y' + \frac{\partial F_0}{\partial y'}F_0\\
F_2 &= \frac{d}{dx}\left(\frac{d}{dx}f\right)\\
&= \frac{d}{dx}(F_1)\\
&= \frac{\partial}{\partial x}F_1 + \left(\frac{\partial F_1}{\partial y}\right)y' + \left(\frac{\partial F_1}{\partial y'}\right)y''\\
&= \frac{\partial}{\partial x}F_1 + \left(\frac{\partial F_1}{\partial y}\right)y' + \left(\frac{\partial F_1}{\partial y'}\right)F_0\\
&\vdots\\
F_n &= \frac{d}{dx}(F_{n-1})\\
&= \frac{\partial}{\partial x}F_{n-1} + \left(\frac{\partial F_{n-1}}{\partial y}\right)y' + \left(\frac{\partial F_{n-1}}{\partial y'}\right)y''\\
&= \frac{\partial}{\partial x}F_{n-1} + \left(\frac{\partial F_{n-1}}{\partial y}\right)y' + \left(\frac{\partial F_{n-1}}{\partial y'}\right)F_0 &(6)
\end{aligned}
$$

Therefore (6) can be used from now on along with

$$
y(x) = y_0 + xy_0' + \sum_{n=0}^{\infty} \frac{x^{n+2}}{(n+2)!} F_n|_{x_0,y_0,y_0'} \tag{7}
$$

To find $y(x)$ series solution around $x = 0$. Hence

$$
\begin{aligned}
F_0 &= -x^2 y \\
F_1 &= \frac{dF_0}{dx} \\
&= \frac{\partial F_0}{\partial x} + \frac{\partial F_0}{\partial y} y' + \frac{\partial F_0}{\partial y'} F_0 \\
&= -(2y + y'x)\, x \\
F_2 &= \frac{dF_1}{dx} \\
&= \frac{\partial F_1}{\partial x} + \frac{\partial F_1}{\partial y} y' + \frac{\partial F_1}{\partial y'} F_1 \\
&= y x^4 - 4y'x - 2y \\
F_3 &= \frac{dF_2}{dx} \\
&= \frac{\partial F_2}{\partial x} + \frac{\partial F_2}{\partial y} y' + \frac{\partial F_2}{\partial y'} F_2 \\
&= y' x^4 + 8y x^3 - 6y' \\
F_4 &= \frac{dF_3}{dx} \\
&= \frac{\partial F_3}{\partial x} + \frac{\partial F_3}{\partial y} y' + \frac{\partial F_3}{\partial y'} F_3 \\
&= 12x^3 y' - x^2 y\left(x^4 - 30\right)
\end{aligned}
$$

And so on. Evaluating all the above at initial conditions $x = 0$ and $y(0) = 1$ and $y'(0) = 0$ gives

$$
\begin{aligned}
F_0 &= 0 \\
F_1 &= 0 \\
F_2 &= -2 \\
F_3 &= 0 \\
F_4 &= 0
\end{aligned}
$$

Substituting all the above in (7) and simplifying gives the solution as

$$
y = 1 - \frac{x^4}{12} + O\left(x^6\right)
$$

$$
y = 1 - \frac{x^4}{12} + O\left(x^6\right)
$$

Since the expansion point $x = 0$ is an ordinary, we can also solve this using standard power series Let the solution be represented as power series of the form

$$y = \sum_{n=0}^{\infty} a_n x^n$$

Then

$$y' = \sum_{n=1}^{\infty} n a_n x^{n-1}$$
$$y'' = \sum_{n=2}^{\infty} n(n-1)\, a_n x^{n-2}$$

Substituting the above back into the ode gives

$$\sum_{n=2}^{\infty} n(n-1)\, a_n x^{n-2} = -x^2 \left(\sum_{n=0}^{\infty} a_n x^n\right) \tag{1}$$

Which simplifies to

$$\left(\sum_{n=2}^{\infty} n(n-1)\, a_n x^{n-2}\right) + \left(\sum_{n=0}^{\infty} x^{n+2} a_n\right) = 0 \tag{2}$$

The next step is to make all powers of x be n in each summation term. Going over each summation term above with power of x in it which is not already x^n and adjusting the power and the corresponding index gives

$$\sum_{n=2}^{\infty} n(n-1)\, a_n x^{n-2} = \sum_{n=0}^{\infty} (n+2)\, a_{n+2}(n+1)\, x^n$$

$$\sum_{n=0}^{\infty} x^{n+2} a_n = \sum_{n=2}^{\infty} a_{n-2} x^n$$

Substituting all the above in Eq (2) gives the following equation where now all powers of x are the same and equal to n.

$$\left(\sum_{n=0}^{\infty} (n+2)\, a_{n+2}(n+1)\, x^n\right) + \left(\sum_{n=2}^{\infty} a_{n-2} x^n\right) = 0 \tag{3}$$

For $2 \le n$, the recurrence equation is

$$(n+2)\, a_{n+2}(n+1) + a_{n-2} = 0 \tag{4}$$

Solving for a_{n+2}, gives

$$a_{n+2} = -\frac{a_{n-2}}{(n+2)(n+1)} \tag{5}$$

For $n = 2$ the recurrence equation gives

$$12a_4 + a_0 = 0$$

Which after substituting the earlier terms found becomes

$$a_4 = -\frac{a_0}{12}$$

For $n = 3$ the recurrence equation gives

$$20a_5 + a_1 = 0$$

Which after substituting the earlier terms found becomes

$$a_5 = -\frac{a_1}{20}$$

For $n = 4$ the recurrence equation gives

$$30a_6 + a_2 = 0$$

Which after substituting the earlier terms found becomes

$$a_6 = 0$$

For $n = 5$ the recurrence equation gives

$$42a_7 + a_3 = 0$$

Which after substituting the earlier terms found becomes

$$a_7 = 0$$

And so on. Therefore the solution is

$$\begin{aligned} y &= \sum_{n=0}^{\infty} a_n x^n \\ &= a_3x^3 + a_2x^2 + a_1x + a_0 + \dots \end{aligned}$$

Substituting the values for a_n found above, the solution becomes

$$y = a_0 + a_1 x - \frac{1}{12} a_0 x^4 - \frac{1}{20} a_1 x^5 + \dots$$

Collecting terms, the solution becomes

$$y = \left(1 - \frac{x^4}{12}\right) a_0 + \left(x - \frac{1}{20} x^5\right) a_1 + O(x^6) \tag{3}$$

At $x = 0$ the solution above becomes

$$y = \left(1 - \frac{x^4}{12}\right) c_1 + \left(x - \frac{1}{20} x^5\right) c_2 + O(x^6)$$

$$y = 1 - \frac{x^4}{12} + O(x^6)$$

Summary

The solution(s) found are the following

$$y = 1 - \frac{x^4}{12} + O(x^6) \tag{1}$$

$$y = 1 - \frac{x^4}{12} + O(x^6) \tag{2}$$

Verification of solutions

$$y = 1 - \frac{x^4}{12} + O(x^6)$$

Verified OK.

$$y = 1 - \frac{x^4}{12} + O(x^6)$$

Verified OK.

13.21.2 Maple step by step solution

Let's solve

$$\left[y'' = -x^2 y, y(0) = 1, y'\Big|_{\{x=0\}} = 0\right]$$

- Highest derivative means the order of the ODE is 2

  y''

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y'' + x^2 y = 0$

- Assume series solution for y

  $$y = \sum_{k=0}^{\infty} a_k x^k$$

□ Rewrite ODE with series expansions

  ○ Convert $x^2 \cdot y$ to series expansion

  $$x^2 \cdot y = \sum_{k=0}^{\infty} a_k x^{k+2}$$

  ○ Shift index using $k- >k-2$

  $$x^2 \cdot y = \sum_{k=2}^{\infty} a_{k-2} x^k$$

  ○ Convert y'' to series expansion

  $$y'' = \sum_{k=2}^{\infty} a_k k(k-1)\, x^{k-2}$$

  ○ Shift index using $k- >k+2$

  $$y'' = \sum_{k=0}^{\infty} a_{k+2}(k+2)\,(k+1)\, x^k$$

  Rewrite ODE with series expansions

  $$6a_3 x + 2a_2 + \left(\sum_{k=2}^{\infty} \left(a_{k+2}(k+2)\,(k+1) + a_{k-2}\right) x^k\right) = 0$$

- The coefficients of each power of x must be 0

  $[2a_2 = 0, 6a_3 = 0]$

- Solve for the dependent coefficient(s)

  $\{a_2 = 0, a_3 = 0\}$

- Each term in the series must be 0, giving the recursion relation

$$\left(k^2+3k+2\right)a_{k+2}+a_{k-2}=0$$

- Shift index using $k- >k+2$

$$\left((k+2)^2+3k+8\right)a_{k+4}+a_k=0$$

- Recursion relation that defines the series solution to the ODE

$$\left[y=\sum_{k=0}^{\infty}a_k x^k, a_{k+4}=-\frac{a_k}{k^2+7k+12}, a_2=0, a_3=0\right]$$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
<- No Liouvillian solutions exists
-> Trying a solution in terms of special functions:
   -> Bessel
   <- Bessel successful
<- special function solution successful`
```

✓ Solution by Maple

Time used: 0.282 (sec). Leaf size: 12

```
Order:=6;
dsolve([diff(y(x),x$2)+x^2*y(x)=0,y(0) = 1, D(y)(0) = 0],y(x),type='series',x=0);
```

$$y(x)=1-\frac{1}{12}x^4+\mathrm{O}\left(x^6\right)$$

✓ Solution by Mathematica

Time used: 0.001 (sec). Leaf size: 12

```
AsymptoticDSolveValue[{y''[x]+x^2*y[x]==0,{y[0]==1,y'[0]==0}},y[x],{x,0,5}]
```

$$y(x) \to 1 - \frac{x^4}{12}$$

13.22 problem 28

Internal problem ID [732]
Internal file name [`OUTPUT/732_Sunday_June_05_2022_01_48_08_AM_85496591/index.tex`]

Book: Elementary differential equations and boundary value problems, 10th ed., Boyce and DiPrima
Section: Chapter 5.2, Series Solutions Near an Ordinary Point, Part I. page 263
Problem number: 28.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second order series method. Ordinary point", "second order series method. Taylor series method"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries], [_2nd_order, _linear, `
   _with_symmetry_[0,F(x)]`]]
```

$$\boxed{(1-x)\,y''+y'x-2y=0}$$

With initial conditions

$$[y(0)=0, y'(0)=1]$$

With the expansion point for the power series method at $x=0$.

13.22.1 Existence and uniqueness analysis

This is a linear ODE. In canonical form it is written as

$$y''+p(x)y'+q(x)y=F$$

Where here

$$\begin{aligned}p(x)&=\frac{x}{1-x}\\ q(x)&=-\frac{2}{1-x}\\ F&=0\end{aligned}$$

Hence the ode is

$$y'' + \frac{xy'}{1-x} - \frac{2y}{1-x} = 0$$

The domain of $p(x) = \frac{x}{1-x}$ is

$$\{x < 1 \lor 1 < x\}$$

And the point $x_0 = 0$ is inside this domain. The domain of $q(x) = -\frac{2}{1-x}$ is

$$\{x < 1 \lor 1 < x\}$$

And the point $x_0 = 0$ is also inside this domain. Hence solution exists and is unique.

Solving ode using Taylor series method. This gives review on how the Taylor series method works for solving second order ode.

Let

$$y'' = f(x, y, y')$$

Assuming expansion is at $x_0 = 0$ (we can always shift the actual expansion point to 0 by change of variables) and assuming $f(x, y, y')$ is analytic at x_0 which must be the case for an ordinary point. Let initial conditions be $y(x_0) = y_0$ and $y'(x_0) = y'_0$. Using Taylor series gives

$$\begin{aligned}
y(x) &= y(x_0) + (x - x_0)\, y'(x_0) + \frac{(x - x_0)^2}{2} y''(x_0) + \frac{(x - x_0)^3}{3!} y'''(x_0) + \cdots \\
&= y_0 + x y'_0 + \frac{x^2}{2} f|_{x_0, y_0, y'_0} + \frac{x^3}{3!} f'|_{x_0, y_0, y'_0} + \cdots \\
&= y_0 + x y'_0 + \sum_{n=0}^{\infty} \frac{x^{n+2}}{(n+2)\,!} \frac{d^n f}{dx^n}\bigg|_{x_0, y_0, y'_0}
\end{aligned}$$

But

$$\frac{df}{dx} = \frac{\partial f}{\partial x}\frac{dx}{dx} + \frac{\partial f}{\partial y}\frac{dy}{dx} + \frac{\partial f}{\partial y'}\frac{dy'}{dx} \tag{1}$$

$$= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}y' + \frac{\partial f}{\partial y'}y'' \tag{807}$$

$$= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}y' + \frac{\partial f}{\partial y'}f \tag{808}$$

$$\frac{d^2 f}{dx^2} = \frac{d}{dx}\left(\frac{df}{dx}\right)$$

$$= \frac{\partial}{\partial x}\left(\frac{df}{dx}\right) + \frac{\partial}{\partial y}\left(\frac{df}{dx}\right)y' + \frac{\partial}{\partial y'}\left(\frac{df}{dx}\right)f \tag{2}$$

$$\frac{d^3 f}{dx^3} = \frac{d}{dx}\left(\frac{d^2 f}{dx^2}\right)$$

$$= \frac{\partial}{\partial x}\left(\frac{d^2 f}{dx^2}\right) + \left(\frac{\partial}{\partial y}\frac{d^2 f}{dx^2}\right)y' + \frac{\partial}{\partial y'}\left(\frac{d^2 f}{dx^2}\right)f \tag{3}$$

$$\vdots$$

And so on. Hence if we name $F_0 = f(x,y,y')$ then the above can be written as

$$
\begin{aligned}
F_0 &= f(x,y,y') \qquad (4)\\
F_1 &= \frac{df}{dx}\\
&= \frac{dF_0}{dx}\\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}y' + \frac{\partial f}{\partial y'}y''\\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}y' + \frac{\partial f}{\partial y'}f \qquad (5)\\
&= \frac{\partial F_0}{\partial x} + \frac{\partial F_0}{\partial y}y' + \frac{\partial F_0}{\partial y'}F_0\\
F_2 &= \frac{d}{dx}\left(\frac{d}{dx}f\right)\\
&= \frac{d}{dx}(F_1)\\
&= \frac{\partial}{\partial x}F_1 + \left(\frac{\partial F_1}{\partial y}\right)y' + \left(\frac{\partial F_1}{\partial y'}\right)y''\\
&= \frac{\partial}{\partial x}F_1 + \left(\frac{\partial F_1}{\partial y}\right)y' + \left(\frac{\partial F_1}{\partial y'}\right)F_0\\
&\vdots\\
F_n &= \frac{d}{dx}(F_{n-1})\\
&= \frac{\partial}{\partial x}F_{n-1} + \left(\frac{\partial F_{n-1}}{\partial y}\right)y' + \left(\frac{\partial F_{n-1}}{\partial y'}\right)y''\\
&= \frac{\partial}{\partial x}F_{n-1} + \left(\frac{\partial F_{n-1}}{\partial y}\right)y' + \left(\frac{\partial F_{n-1}}{\partial y'}\right)F_0 \qquad (6)
\end{aligned}
$$

Therefore (6) can be used from now on along with

$$
y(x) = y_0 + xy_0' + \sum_{n=0}^{\infty} \frac{x^{n+2}}{(n+2)\,!}\, F_n|_{x_0,y_0,y_0'} \qquad (7)
$$

To find $y(x)$ series solution around $x = 0$. Hence

$$
\begin{aligned}
F_0 &= \frac{y'x - 2y}{x - 1} \\
F_1 &= \frac{dF_0}{dx} \\
&= \frac{\partial F_0}{\partial x} + \frac{\partial F_0}{\partial y} y' + \frac{\partial F_0}{\partial y'} F_0 \\
&= \frac{(x - 1) y' - 2y}{x - 1} \\
F_2 &= \frac{dF_1}{dx} \\
&= \frac{\partial F_1}{\partial x} + \frac{\partial F_1}{\partial y} y' + \frac{\partial F_1}{\partial y'} F_1 \\
&= \frac{((x - 1) y' - 2y) (-2 + x)}{(x - 1)^2} \\
F_3 &= \frac{dF_2}{dx} \\
&= \frac{\partial F_2}{\partial x} + \frac{\partial F_2}{\partial y} y' + \frac{\partial F_2}{\partial y'} F_2 \\
&= \frac{(x^2 - 4x + 5) ((x - 1) y' - 2y)}{(x - 1)^3} \\
F_4 &= \frac{dF_3}{dx} \\
&= \frac{\partial F_3}{\partial x} + \frac{\partial F_3}{\partial y} y' + \frac{\partial F_3}{\partial y'} F_3 \\
&= \frac{(x^3 - 6x^2 + 15x - 16) ((x - 1) y' - 2y)}{(x - 1)^4}
\end{aligned}
$$

And so on. Evaluating all the above at initial conditions $x = 0$ and $y(0) = 0$ and $y'(0) = 1$ gives

$$
\begin{aligned}
F_0 &= 0 \\
F_1 &= 1 \\
F_2 &= 2 \\
F_3 &= 5 \\
F_4 &= 16
\end{aligned}
$$

Substituting all the above in (7) and simplifying gives the solution as

$$
y = x + \frac{x^3}{6} + \frac{x^4}{12} + \frac{x^5}{24} + \frac{x^6}{45} + O(x^6)
$$

$$y = x + \frac{x^3}{6} + \frac{x^4}{12} + \frac{x^5}{24} + \frac{x^6}{45} + O(x^6)$$

Since the expansion point $x = 0$ is an ordinary, we can also solve this using standard power series The ode is normalized to be

$$(1 - x)\, y'' + y'x - 2y = 0$$

Let the solution be represented as power series of the form

$$y = \sum_{n=0}^{\infty} a_n x^n$$

Then

$$y' = \sum_{n=1}^{\infty} n a_n x^{n-1}$$
$$y'' = \sum_{n=2}^{\infty} n(n-1)\, a_n x^{n-2}$$

Substituting the above back into the ode gives

$$(1 - x) \left(\sum_{n=2}^{\infty} n(n-1)\, a_n x^{n-2}\right) + \left(\sum_{n=1}^{\infty} n a_n x^{n-1}\right) x - 2\left(\sum_{n=0}^{\infty} a_n x^n\right) = 0 \qquad (1)$$

Which simplifies to

$$\sum_{n=2}^{\infty} \left(-n\, x^{n-1} a_n (n-1)\right) + \left(\sum_{n=2}^{\infty} n(n-1)\, a_n x^{n-2}\right) + \left(\sum_{n=1}^{\infty} n a_n x^n\right) + \sum_{n=0}^{\infty} (-2 a_n x^n) = 0 \qquad (2)$$

The next step is to make all powers of x be n in each summation term. Going over each summation term above with power of x in it which is not already x^n and adjusting the power and the corresponding index gives

$$\sum_{n=2}^{\infty} \left(-n\, x^{n-1} a_n (n-1)\right) = \sum_{n=1}^{\infty} \left(-(n+1)\, a_{n+1} n\, x^n\right)$$
$$\sum_{n=2}^{\infty} n(n-1)\, a_n x^{n-2} = \sum_{n=0}^{\infty} (n+2)\, a_{n+2} (n+1)\, x^n$$

Substituting all the above in Eq (2) gives the following equation where now all powers of x are the same and equal to n.

$$
\sum_{n=1}^{\infty} \left(-(n+1)\, a_{n+1} n\, x^n\right) + \left(\sum_{n=0}^{\infty} (n+2)\, a_{n+2} (n+1)\, x^n\right) + \left(\sum_{n=1}^{\infty} n a_n x^n\right) + \sum_{n=0}^{\infty} \left(-2 a_n x^n\right) = 0 \tag{3}
$$

$n = 0$ gives

$$2a_2 - 2a_0 = 0$$

$$a_2 = a_0$$

For $1 \leq n$, the recurrence equation is

$$
-(n+1)\, a_{n+1} n + (n+2)\, a_{n+2} (n+1) + n a_n - 2a_n = 0 \tag{4}
$$

Solving for a_{n+2}, gives

$$
\begin{aligned}
a_{n+2} &= \frac{n^2 a_{n+1} - n a_n + n a_{n+1} + 2a_n}{(n+2)\,(n+1)} \\
&= \frac{(-n+2)\, a_n}{(n+2)\,(n+1)} + \frac{(n^2+n)\, a_{n+1}}{(n+2)\,(n+1)}
\end{aligned} \tag{5}
$$

For $n = 1$ the recurrence equation gives

$$-2a_2 + 6a_3 - a_1 = 0$$

Which after substituting the earlier terms found becomes

$$a_3 = \frac{a_0}{3} + \frac{a_1}{6}$$

For $n = 2$ the recurrence equation gives

$$-6a_3 + 12a_4 = 0$$

Which after substituting the earlier terms found becomes

$$a_4 = \frac{a_0}{6} + \frac{a_1}{12}$$

For $n = 3$ the recurrence equation gives

$$-12a_4 + 20a_5 + a_3 = 0$$

Which after substituting the earlier terms found becomes

$$a_5 = \frac{a_0}{12} + \frac{a_1}{24}$$

For $n = 4$ the recurrence equation gives

$$-20a_5 + 30a_6 + 2a_4 = 0$$

Which after substituting the earlier terms found becomes

$$a_6 = \frac{2a_0}{45} + \frac{a_1}{45}$$

For $n = 5$ the recurrence equation gives

$$-30a_6 + 42a_7 + 3a_5 = 0$$

Which after substituting the earlier terms found becomes

$$a_7 = \frac{13a_0}{504} + \frac{13a_1}{1008}$$

And so on. Therefore the solution is

$$\begin{aligned} y &= \sum_{n=0}^{\infty} a_n x^n \\ &= a_3x^3 + a_2x^2 + a_1x + a_0 + \dots \end{aligned}$$

Substituting the values for a_n found above, the solution becomes

$$y = a_0 + a_1x + a_0x^2 + \left(\frac{a_0}{3} + \frac{a_1}{6}\right)x^3 + \left(\frac{a_0}{6} + \frac{a_1}{12}\right)x^4 + \left(\frac{a_0}{12} + \frac{a_1}{24}\right)x^5 + \dots$$

Collecting terms, the solution becomes

$$y = \left(1+x^2+\frac{1}{3}x^3+\frac{1}{6}x^4+\frac{1}{12}x^5\right)a_0+\left(x+\frac{1}{6}x^3+\frac{1}{12}x^4+\frac{1}{24}x^5\right)a_1+O(x^6) \tag{3}$$

At $x = 0$ the solution above becomes

$$y = \left(1+x^2+\frac{1}{3}x^3+\frac{1}{6}x^4+\frac{1}{12}x^5\right)c_1+\left(x+\frac{1}{6}x^3+\frac{1}{12}x^4+\frac{1}{24}x^5\right)c_2+O(x^6)$$

$$y = x+\frac{x^3}{6}+\frac{x^4}{12}+\frac{x^5}{24}+O(x^6)$$

Summary

The solution(s) found are the following

$$y = x+\frac{x^3}{6}+\frac{x^4}{12}+\frac{x^5}{24}+\frac{x^6}{45}+O(x^6) \tag{1}$$

$$y = x+\frac{x^3}{6}+\frac{x^4}{12}+\frac{x^5}{24}+O(x^6) \tag{2}$$

Verification of solutions

$$y = x+\frac{x^3}{6}+\frac{x^4}{12}+\frac{x^5}{24}+\frac{x^6}{45}+O(x^6)$$

Verified OK.

$$y = x+\frac{x^3}{6}+\frac{x^4}{12}+\frac{x^5}{24}+O(x^6)$$

Verified OK.

13.22.2 Maple step by step solution

Let's solve

$$\left[(1-x)\,y''+y'x-2y=0, y(0)=0, y'\Big|_{\{x=0\}}=1\right]$$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

$y'' = -\frac{2y}{x-1} + \frac{xy'}{x-1}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

$y'' - \frac{xy'}{x-1} + \frac{2y}{x-1} = 0$

☐ Check to see if $x_0 = 1$ is a regular singular point

- Define functions

$\left[P_2(x) = -\frac{x}{x-1}, P_3(x) = \frac{2}{x-1}\right]$

- $(x-1) \cdot P_2(x)$ is analytic at $x = 1$

$\left((x-1) \cdot P_2(x)\right)\bigg|_{x=1} = -1$

- $(x-1)^2 \cdot P_3(x)$ is analytic at $x = 1$

$\left((x-1)^2 \cdot P_3(x)\right)\bigg|_{x=1} = 0$

- $x = 1$is a regular singular point

Check to see if $x_0 = 1$ is a regular singular point

$x_0 = 1$

- Multiply by denominators

$(x-1)\, y'' - y'x + 2y = 0$

- Change variables using $x = u + 1$ so that the regular singular point is at $u = 0$

$u\left(\frac{d^2}{du^2}y(u)\right) + (-u-1)\left(\frac{d}{du}y(u)\right) + 2y(u) = 0$

- Assume series solution for $y(u)$

$y(u) = \sum_{k=0}^{\infty} a_k u^{k+r}$

☐ Rewrite ODE with series expansions

- Convert $u^m \cdot \left(\frac{d}{du}y(u)\right)$ to series expansion for $m = 0..1$

$u^m \cdot \left(\frac{d}{du}y(u)\right) = \sum_{k=0}^{\infty} a_k(k+r)\, u^{k+r-1+m}$

- Shift index using $k- >k+1-m$

$u^m \cdot \left(\frac{d}{du}y(u)\right) = \sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\, u^{k+r}$

- Convert $u \cdot \left(\frac{d^2}{du^2}y(u)\right)$ to series expansion

$$u \cdot \left(\frac{d^2}{du^2}y(u)\right) = \sum_{k=0}^{\infty} a_k(k+r)(k+r-1)\,u^{k+r-1}$$

- Shift index using $k->k+1$

  $$u \cdot \left(\frac{d^2}{du^2}y(u)\right) = \sum_{k=-1}^{\infty} a_{k+1}(k+1+r)(k+r)\,u^{k+r}$$

  Rewrite ODE with series expansions

  $$a_0 r(-2+r)\,u^{-1+r} + \left(\sum_{k=0}^{\infty} \left(a_{k+1}(k+1+r)(k+r-1) - a_k(k+r-2)\right) u^{k+r}\right) = 0$$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $$r(-2+r) = 0$$

- Values of r that satisfy the indicial equation

  $$r \in \{0, 2\}$$

- Each term in the series must be 0, giving the recursion relation

  $$a_{k+1}(k+1+r)(k+r-1) - a_k(k+r-2) = 0$$

- Recursion relation that defines series solution to ODE

  $$a_{k+1} = \frac{a_k(k+r-2)}{(k+1+r)(k+r-1)}$$

- Recursion relation for $r=0$; series terminates at $k=2$

  $$a_{k+1} = \frac{a_k(k-2)}{(k+1)(k-1)}$$

- Series not valid for $r=0$, division by 0 in the recursion relation at $k=1$

  $$a_{k+1} = \frac{a_k(k-2)}{(k+1)(k-1)}$$

- Recursion relation for $r=2$

  $$a_{k+1} = \frac{a_k k}{(k+3)(k+1)}$$

- Solution for $r=2$

  $$\left[y(u) = \sum_{k=0}^{\infty} a_k u^{k+2}, a_{k+1} = \frac{a_k k}{(k+3)(k+1)}\right]$$

- Revert the change of variables $u = x-1$

  $$\left[y = \sum_{k=0}^{\infty} a_k(x-1)^{k+2}, a_{k+1} = \frac{a_k k}{(k+3)(k+1)}\right]$$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
<- linear_1 successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 16

```
Order:=6;
dsolve([(1-x)*diff(y(x),x$2)+x*diff(y(x),x)-2*y(x)=0,y(0) = 0, D(y)(0) = 1],y(x),type='serie
```

$$y(x) = x + \frac{1}{6}x^3 + \frac{1}{12}x^4 + \frac{1}{24}x^5 + \mathrm{O}\left(x^6\right)$$

✓ Solution by Mathematica

Time used: 0.001 (sec). Leaf size: 26

```
AsymptoticDSolveValue[{(1-x)*y''[x]+x*y'[x]-2*y[x]==0,{y[0]==0,y'[0]==1}},y[x],{x,0,5}]
```

$$y(x) \to \frac{x^5}{24} + \frac{x^4}{12} + \frac{x^3}{6} + x$$

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14.1 problem 1

Internal problem ID [733]
Internal file name [`OUTPUT/733_Sunday_June_05_2022_01_48_10_AM_75056564/index.tex`]

Book: Elementary differential equations and boundary value problems, 10th ed., Boyce and DiPrima
Section: Chapter 5.3, Series Solutions Near an Ordinary Point, Part II. page 269
Problem number: 1.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"exact linear second order ode", "second_order_integrable_as_is", "second order series method. Ordinary point", "second order series method. Taylor series method"**

Maple gives the following as the ode type

```
[[_2nd_order, _exact, _linear, _homogeneous]]
```

$$\boxed{y'' + y'x + y = 0}$$

With initial conditions

$$[y(0) = 1, y'(0) = 0]$$

With the expansion point for the power series method at $x = 0$.

14.1.1 Existence and uniqueness analysis

This is a linear ODE. In canonical form it is written as

$$y'' + p(x)y' + q(x)y = F$$

Where here

$$\begin{aligned} p(x) &= x \\ q(x) &= 1 \\ F &= 0 \end{aligned}$$

Hence the ode is

$$y'' + y'x + y = 0$$

The domain of $p(x) = x$ is

$$\{-\infty < x < \infty\}$$

And the point $x_0 = 0$ is inside this domain. The domain of $q(x) = 1$ is

$$\{-\infty < x < \infty\}$$

And the point $x_0 = 0$ is also inside this domain. Hence solution exists and is unique.

Solving ode using Taylor series method. This gives review on how the Taylor series method works for solving second order ode.

Let

$$y'' = f(x, y, y')$$

Assuming expansion is at $x_0 = 0$ (we can always shift the actual expansion point to 0 by change of variables) and assuming $f(x, y, y')$ is analytic at x_0 which must be the case for an ordinary point. Let initial conditions be $y(x_0) = y_0$ and $y'(x_0) = y'_0$. Using Taylor series gives

$$\begin{aligned}
y(x) &= y(x_0) + (x - x_0)\, y'(x_0) + \frac{(x - x_0)^2}{2} y''(x_0) + \frac{(x - x_0)^3}{3!} y'''(x_0) + \cdots \\
&= y_0 + x y'_0 + \frac{x^2}{2} f|_{x_0, y_0, y'_0} + \frac{x^3}{3!} f'|_{x_0, y_0, y'_0} + \cdots \\
&= y_0 + x y'_0 + \sum_{n=0}^{\infty} \frac{x^{n+2}}{(n+2)!} \frac{d^n f}{dx^n}\bigg|_{x_0, y_0, y'_0}
\end{aligned}$$

But

$$\begin{aligned}
\frac{df}{dx} &= \frac{\partial f}{\partial x}\frac{dx}{dx} + \frac{\partial f}{\partial y}\frac{dy}{dx} + \frac{\partial f}{\partial y'}\frac{dy'}{dx} && (1)\\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}y' + \frac{\partial f}{\partial y'}y'' && (810)\\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}y' + \frac{\partial f}{\partial y'}f && (811)\\
\frac{d^2 f}{dx^2} &= \frac{d}{dx}\left(\frac{df}{dx}\right)\\
&= \frac{\partial}{\partial x}\left(\frac{df}{dx}\right) + \frac{\partial}{\partial y}\left(\frac{df}{dx}\right)y' + \frac{\partial}{\partial y'}\left(\frac{df}{dx}\right)f && (2)\\
\frac{d^3 f}{dx^3} &= \frac{d}{dx}\left(\frac{d^2 f}{dx^2}\right)\\
&= \frac{\partial}{\partial x}\left(\frac{d^2 f}{dx^2}\right) + \left(\frac{\partial}{\partial y}\frac{d^2 f}{dx^2}\right)y' + \frac{\partial}{\partial y'}\left(\frac{d^2 f}{dx^2}\right)f && (3)\\
&\vdots
\end{aligned}$$

And so on. Hence if we name $F_0 = f(x,y,y')$ then the above can be written as

$$
\begin{aligned}
F_0 &= f(x,y,y') \qquad (4)\\
F_1 &= \frac{df}{dx}\\
&= \frac{dF_0}{dx}\\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}y' + \frac{\partial f}{\partial y'}y''\\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}y' + \frac{\partial f}{\partial y'}f \qquad (5)\\
&= \frac{\partial F_0}{\partial x} + \frac{\partial F_0}{\partial y}y' + \frac{\partial F_0}{\partial y'}F_0\\
F_2 &= \frac{d}{dx}\left(\frac{d}{dx}f\right)\\
&= \frac{d}{dx}(F_1)\\
&= \frac{\partial}{\partial x}F_1 + \left(\frac{\partial F_1}{\partial y}\right)y' + \left(\frac{\partial F_1}{\partial y'}\right)y''\\
&= \frac{\partial}{\partial x}F_1 + \left(\frac{\partial F_1}{\partial y}\right)y' + \left(\frac{\partial F_1}{\partial y'}\right)F_0\\
&\vdots\\
F_n &= \frac{d}{dx}(F_{n-1})\\
&= \frac{\partial}{\partial x}F_{n-1} + \left(\frac{\partial F_{n-1}}{\partial y}\right)y' + \left(\frac{\partial F_{n-1}}{\partial y'}\right)y''\\
&= \frac{\partial}{\partial x}F_{n-1} + \left(\frac{\partial F_{n-1}}{\partial y}\right)y' + \left(\frac{\partial F_{n-1}}{\partial y'}\right)F_0 \qquad (6)
\end{aligned}
$$

Therefore (6) can be used from now on along with

$$
y(x) = y_0 + xy_0' + \sum_{n=0}^{\infty} \frac{x^{n+2}}{(n+2)!} F_n\big|_{x_0,y_0,y_0'} \tag{7}
$$

To find $y(x)$ series solution around $x = 0$. Hence

$$
\begin{aligned}
F_0 &= -y'x - y \\
F_1 &= \frac{dF_0}{dx} \\
&= \frac{\partial F_0}{\partial x} + \frac{\partial F_0}{\partial y} y' + \frac{\partial F_0}{\partial y'} F_0 \\
&= y'x^2 + yx - 2y' \\
F_2 &= \frac{dF_1}{dx} \\
&= \frac{\partial F_1}{\partial x} + \frac{\partial F_1}{\partial y} y' + \frac{\partial F_1}{\partial y'} F_1 \\
&= -x^3y' - x^2y + 5y'x + 3y \\
F_3 &= \frac{dF_2}{dx} \\
&= \frac{\partial F_2}{\partial x} + \frac{\partial F_2}{\partial y} y' + \frac{\partial F_2}{\partial y'} F_2 \\
&= \left(x^4 - 9x^2 + 8\right) y' + xy\left(x^2 - 7\right) \\
F_4 &= \frac{dF_3}{dx} \\
&= \frac{\partial F_3}{\partial x} + \frac{\partial F_3}{\partial y} y' + \frac{\partial F_3}{\partial y'} F_3 \\
&= \left(-x^5 + 14x^3 - 33x\right) y' - y\left(x^4 - 12x^2 + 15\right)
\end{aligned}
$$

And so on. Evaluating all the above at initial conditions $x = 0$ and $y(0) = 1$ and $y'(0) = 0$ gives

$$
\begin{aligned}
F_0 &= -1 \\
F_1 &= 0 \\
F_2 &= 3 \\
F_3 &= 0 \\
F_4 &= -15
\end{aligned}
$$

Substituting all the above in (7) and simplifying gives the solution as

$$
y = -\frac{x^2}{2} + 1 + \frac{x^4}{8} - \frac{x^6}{48} + O\left(x^6\right)
$$

$$
y = -\frac{x^2}{2} + 1 + \frac{x^4}{8} - \frac{x^6}{48} + O\left(x^6\right)
$$

Since the expansion point $x = 0$ is an ordinary, we can also solve this using standard power series Let the solution be represented as power series of the form

$$y = \sum_{n=0}^{\infty} a_n x^n$$

Then

$$y' = \sum_{n=1}^{\infty} n a_n x^{n-1}$$
$$y'' = \sum_{n=2}^{\infty} n(n-1)\, a_n x^{n-2}$$

Substituting the above back into the ode gives

$$\sum_{n=2}^{\infty} n(n-1)\, a_n x^{n-2} = -\left(\sum_{n=1}^{\infty} n a_n x^{n-1}\right) x - \left(\sum_{n=0}^{\infty} a_n x^n\right) \tag{1}$$

Which simplifies to

$$\left(\sum_{n=2}^{\infty} n(n-1)\, a_n x^{n-2}\right) + \left(\sum_{n=1}^{\infty} n\, x^n a_n\right) + \left(\sum_{n=0}^{\infty} a_n x^n\right) = 0 \tag{2}$$

The next step is to make all powers of x be n in each summation term. Going over each summation term above with power of x in it which is not already x^n and adjusting the power and the corresponding index gives

$$\sum_{n\,=2}^{\infty} n(n-1)\, a_n x^{n-2} = \sum_{n=0}^{\infty} (n+2)\, a_{n+2} (n+1)\, x^n$$

Substituting all the above in Eq (2) gives the following equation where now all powers of x are the same and equal to n.

$$\left(\sum_{n=0}^{\infty} (n+2)\, a_{n+2} (n+1)\, x^n\right) + \left(\sum_{n=1}^{\infty} n\, x^n a_n\right) + \left(\sum_{n=0}^{\infty} a_n x^n\right) = 0 \tag{3}$$

$n = 0$ gives

$$2a_2 + a_0 = 0$$

$$a_2 = -\frac{a_0}{2}$$

For $1 \leq n$, the recurrence equation is

$$(n+2)\, a_{n+2}(n+1) + n a_n + a_n = 0 \tag{4}$$

Solving for a_{n+2}, gives

$$a_{n+2} = -\frac{a_n}{n+2} \tag{5}$$

For $n = 1$ the recurrence equation gives

$$6a_3 + 2a_1 = 0$$

Which after substituting the earlier terms found becomes

$$a_3 = -\frac{a_1}{3}$$

For $n = 2$ the recurrence equation gives

$$12a_4 + 3a_2 = 0$$

Which after substituting the earlier terms found becomes

$$a_4 = \frac{a_0}{8}$$

For $n = 3$ the recurrence equation gives

$$20a_5 + 4a_3 = 0$$

Which after substituting the earlier terms found becomes

$$a_5 = \frac{a_1}{15}$$

For $n = 4$ the recurrence equation gives

$$30a_6 + 5a_4 = 0$$

Which after substituting the earlier terms found becomes

$$a_6 = -\frac{a_0}{48}$$

For $n = 5$ the recurrence equation gives

$$42a_7 + 6a_5 = 0$$

Which after substituting the earlier terms found becomes

$$a_7 = -\frac{a_1}{105}$$

And so on. Therefore the solution is

$$\begin{aligned} y &= \sum_{n=0}^{\infty} a_n x^n \\ &= a_3x^3 + a_2x^2 + a_1x + a_0 + \dots \end{aligned}$$

Substituting the values for a_n found above, the solution becomes

$$y = a_0 + a_1x - \frac{1}{2}a_0x^2 - \frac{1}{3}a_1x^3 + \frac{1}{8}a_0x^4 + \frac{1}{15}a_1x^5 + \dots$$

Collecting terms, the solution becomes

$$y = \left(-\frac{1}{2}x^2 + 1 + \frac{1}{8}x^4\right)a_0 + \left(x - \frac{1}{3}x^3 + \frac{1}{15}x^5\right)a_1 + O(x^6) \tag{3}$$

At $x = 0$ the solution above becomes

$$y = \left(-\frac{1}{2}x^2 + 1 + \frac{1}{8}x^4\right)c_1 + \left(x - \frac{1}{3}x^3 + \frac{1}{15}x^5\right)c_2 + O(x^6)$$

$$y = -\frac{x^2}{2} + 1 + \frac{x^4}{8} + O(x^6)$$

Summary
The solution(s) found are the following

$$y = -\frac{x^2}{2} + 1 + \frac{x^4}{8} - \frac{x^6}{48} + O(x^6) \tag{1}$$

$$y = -\frac{x^2}{2} + 1 + \frac{x^4}{8} + O(x^6) \tag{2}$$

Verification of solutions

$$y = -\frac{x^2}{2} + 1 + \frac{x^4}{8} - \frac{x^6}{48} + O(x^6)$$

Verified OK.

$$y = -\frac{x^2}{2} + 1 + \frac{x^4}{8} + O(x^6)$$

Verified OK.

14.1.2 Maple step by step solution

Let's solve

$\left[y'' = -y'x - y, y(0) = 1, y'\Big|_{\{x=0\}} = 0\right]$

- Highest derivative means the order of the ODE is 2

  y''

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y'' + y'x + y = 0$

- Assume series solution for y

  $y = \sum_{k=0}^{\infty} a_k x^k$

□ Rewrite DE with series expansions

  ◦ Convert $x \cdot y'$ to series expansion

  $x \cdot y' = \sum_{k=0}^{\infty} a_k k\, x^k$

  ◦ Convert y'' to series expansion

  $y'' = \sum_{k=2}^{\infty} a_k k(k-1)\, x^{k-2}$

  ◦ Shift index using $k- >k+2$

  $y'' = \sum_{k=0}^{\infty} a_{k+2}(k+2)\,(k+1)\, x^k$

  Rewrite DE with series expansions

  $\sum_{k=0}^{\infty} \left(a_{k+2}(k+2)\,(k+1) + a_k(k+1)\right) x^k = 0$

- Each term in the series must be 0, giving the recursion relation

  $(k+1)(a_{k+2}(k+2)+a_k)=0$

- Recursion relation that defines the series solution to the ODE

  $\left[y=\sum_{k=0}^{\infty} a_k x^k, a_{k+2}=-\frac{a_k}{k+2}\right]$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
<- linear_1 successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 14

```
Order:=6;
dsolve([diff(y(x),x$2)+x*diff(y(x),x)+y(x)=0,y(0) = 1, D(y)(0) = 0],y(x),type='series',x=0);
```

$$y(x)=1-\frac{1}{2}x^2+\frac{1}{8}x^4+\mathrm{O}\left(x^6\right)$$

✓ Solution by Mathematica

Time used: 0.001 (sec). Leaf size: 19

```
AsymptoticDSolveValue[{y''[x]+x*y'[x]+y[x]==0,{y[0]==1,y'[0]==0}},y[x],{x,0,5}]
```

$$y(x)\to\frac{x^4}{8}-\frac{x^2}{2}+1$$

14.2 problem 2

Internal problem ID [734]
Internal file name [`OUTPUT/734_Sunday_June_05_2022_01_48_12_AM_11361222/index.tex`]

Book: Elementary differential equations and boundary value problems, 10th ed., Boyce and DiPrima
Section: Chapter 5.3, Series Solutions Near an Ordinary Point, Part II. page 269
Problem number: 2.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"exact linear second order ode", "second_order_integrable_as_is", "second order series method. Ordinary point", "second order series method. Taylor series method"**

Maple gives the following as the ode type

```
[[_2nd_order, _exact, _linear, _homogeneous]]
```

$$\boxed{y'' + \sin(x)\, y' + \cos(x)\, y = 0}$$

With initial conditions

$$[y(0) = 0, y'(0) = 1]$$

With the expansion point for the power series method at $x = 0$.

14.2.1 Existence and uniqueness analysis

This is a linear ODE. In canonical form it is written as

$$y'' + p(x)y' + q(x)y = F$$

Where here

$$\begin{aligned} p(x) &= \sin(x) \\ q(x) &= \cos(x) \\ F &= 0 \end{aligned}$$

Hence the ode is

$$y'' + \sin(x)\, y' + \cos(x)\, y = 0$$

The domain of $p(x) = \sin(x)$ is

$$\{-\infty < x < \infty\}$$

And the point $x_0 = 0$ is inside this domain. The domain of $q(x) = \cos(x)$ is

$$\{-\infty < x < \infty\}$$

And the point $x_0 = 0$ is also inside this domain. Hence solution exists and is unique.

Solving ode using Taylor series method. This gives review on how the Taylor series method works for solving second order ode.

Let

$$y'' = f(x, y, y')$$

Assuming expansion is at $x_0 = 0$ (we can always shift the actual expansion point to 0 by change of variables) and assuming $f(x, y, y')$ is analytic at x_0 which must be the case for an ordinary point. Let initial conditions be $y(x_0) = y_0$ and $y'(x_0) = y'_0$. Using Taylor series gives

$$\begin{aligned}
y(x) &= y(x_0) + (x - x_0)\, y'(x_0) + \frac{(x - x_0)^2}{2} y''(x_0) + \frac{(x - x_0)^3}{3!} y'''(x_0) + \cdots \\
&= y_0 + x y'_0 + \frac{x^2}{2} f|_{x_0, y_0, y'_0} + \frac{x^3}{3!} f'|_{x_0, y_0, y'_0} + \cdots \\
&= y_0 + x y'_0 + \sum_{n=0}^{\infty} \frac{x^{n+2}}{(n+2)!} \left.\frac{d^n f}{dx^n}\right|_{x_0, y_0, y'_0}
\end{aligned}$$

But

$$\frac{df}{dx} = \frac{\partial f}{\partial x}\frac{dx}{dx} + \frac{\partial f}{\partial y}\frac{dy}{dx} + \frac{\partial f}{\partial y'}\frac{dy'}{dx} \tag{1}$$

$$= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}y' + \frac{\partial f}{\partial y'}y'' \tag{813}$$

$$= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}y' + \frac{\partial f}{\partial y'}f \tag{814}$$

$$\frac{d^2 f}{dx^2} = \frac{d}{dx}\left(\frac{df}{dx}\right)$$

$$= \frac{\partial}{\partial x}\left(\frac{df}{dx}\right) + \frac{\partial}{\partial y}\left(\frac{df}{dx}\right)y' + \frac{\partial}{\partial y'}\left(\frac{df}{dx}\right)f \tag{2}$$

$$\frac{d^3 f}{dx^3} = \frac{d}{dx}\left(\frac{d^2 f}{dx^2}\right)$$

$$= \frac{\partial}{\partial x}\left(\frac{d^2 f}{dx^2}\right) + \left(\frac{\partial}{\partial y}\frac{d^2 f}{dx^2}\right)y' + \frac{\partial}{\partial y'}\left(\frac{d^2 f}{dx^2}\right)f \tag{3}$$

$$\vdots$$

And so on. Hence if we name $F_0 = f(x,y,y')$ then the above can be written as

$$
\begin{aligned}
F_0 &= f(x,y,y') && (4)\\
F_1 &= \frac{df}{dx}\\
&= \frac{dF_0}{dx}\\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}y' + \frac{\partial f}{\partial y'}y''\\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}y' + \frac{\partial f}{\partial y'}f && (5)\\
&= \frac{\partial F_0}{\partial x} + \frac{\partial F_0}{\partial y}y' + \frac{\partial F_0}{\partial y'}F_0\\
F_2 &= \frac{d}{dx}\left(\frac{d}{dx}f\right)\\
&= \frac{d}{dx}(F_1)\\
&= \frac{\partial}{\partial x}F_1 + \left(\frac{\partial F_1}{\partial y}\right)y' + \left(\frac{\partial F_1}{\partial y'}\right)y''\\
&= \frac{\partial}{\partial x}F_1 + \left(\frac{\partial F_1}{\partial y}\right)y' + \left(\frac{\partial F_1}{\partial y'}\right)F_0\\
&\vdots\\
F_n &= \frac{d}{dx}(F_{n-1})\\
&= \frac{\partial}{\partial x}F_{n-1} + \left(\frac{\partial F_{n-1}}{\partial y}\right)y' + \left(\frac{\partial F_{n-1}}{\partial y'}\right)y''\\
&= \frac{\partial}{\partial x}F_{n-1} + \left(\frac{\partial F_{n-1}}{\partial y}\right)y' + \left(\frac{\partial F_{n-1}}{\partial y'}\right)F_0 && (6)
\end{aligned}
$$

Therefore (6) can be used from now on along with

$$
y(x) = y_0 + xy_0' + \sum_{n=0}^{\infty} \frac{x^{n+2}}{(n+2)!} F_n|_{x_0,y_0,y_0'} \tag{7}
$$

To find $y(x)$ series solution around $x = 0$. Hence

$$
\begin{aligned}
F_0 &= -\sin(x)\, y' - \cos(x)\, y \\
F_1 &= \frac{dF_0}{dx} \\
&= \frac{\partial F_0}{\partial x} + \frac{\partial F_0}{\partial y} y' + \frac{\partial F_0}{\partial y'} F_0 \\
&= \left(\sin(x)^2 - 2\cos(x)\right) y' + y \sin(x) \left(\cos(x) + 1\right) \\
F_2 &= \frac{dF_1}{dx} \\
&= \frac{\partial F_1}{\partial x} + \frac{\partial F_1}{\partial y} y' + \frac{\partial F_1}{\partial y'} F_1 \\
&= \left(\cos(x)^2 + 5\cos(x) + 2\right) \sin(x)\, y' + \left(\cos(x)^3 + 4\cos(x)^2 - 1\right) y \\
F_3 &= \frac{dF_2}{dx} \\
&= \frac{\partial F_2}{\partial x} + \frac{\partial F_2}{\partial y} y' + \frac{\partial F_2}{\partial y'} F_2 \\
&= \left(\cos(x)^4 + 9\cos(x)^3 + 15\cos(x)^2 - 5\cos(x) - 8\right) y' - \cos(x)\sin(x)\, y \left(\cos(x)^2 + 8\cos(x) + 10\right) \\
F_4 &= \frac{dF_3}{dx} \\
&= \frac{\partial F_3}{\partial x} + \frac{\partial F_3}{\partial y} y' + \frac{\partial F_3}{\partial y'} F_3 \\
&= -\sin(x) \left(\cos(x)^4 + 14\cos(x)^3 + 50\cos(x)^2 + 35\cos(x) - 13\right) y' - y \left(\cos(x)^5 + 13\cos(x)^4 + 39\cos
\end{aligned}
$$

And so on. Evaluating all the above at initial conditions $x = 0$ and $y(0) = 0$ and $y'(0) = 1$ gives

$$
\begin{aligned}
F_0 &= 0 \\
F_1 &= -2 \\
F_2 &= 0 \\
F_3 &= 12 \\
F_4 &= 0
\end{aligned}
$$

Substituting all the above in (7) and simplifying gives the solution as

$$
y = -\frac{x^3}{3} + x + \frac{x^5}{10} + O\left(x^6\right)
$$

$$
y = -\frac{x^3}{3} + x + \frac{x^5}{10} + O\left(x^6\right)
$$

Since the expansion point $x = 0$ is an ordinary, we can also solve this using standard power series Let the solution be represented as power series of the form

$$y = \sum_{n=0}^{\infty} a_n x^n$$

Then

$$\begin{aligned} y' &= \sum_{n=1}^{\infty} n a_n x^{n-1} \\ y'' &= \sum_{n=2}^{\infty} n(n-1)\, a_n x^{n-2} \end{aligned}$$

Substituting the above back into the ode gives

$$\sum_{n=2}^{\infty} n(n-1)\, a_n x^{n-2} = -\sin(x) \left(\sum_{n=1}^{\infty} n a_n x^{n-1}\right) - \cos(x) \left(\sum_{n=0}^{\infty} a_n x^n\right) \tag{1}$$

Expanding $\sin(x)$ as Taylor series around $x = 0$ and keeping only the first 6 terms gives

$$\begin{aligned} \sin(x) &= x - \frac{1}{6}x^3 + \frac{1}{120}x^5 - \frac{1}{5040}x^7 + \dots \\ &= x - \frac{1}{6}x^3 + \frac{1}{120}x^5 - \frac{1}{5040}x^7 \end{aligned}$$

Expanding $\cos(x)$ as Taylor series around $x = 0$ and keeping only the first 6 terms gives

$$\begin{aligned} \cos(x) &= -\frac{1}{2}x^2 + 1 + \frac{1}{24}x^4 - \frac{1}{720}x^6 + \dots \\ &= -\frac{1}{2}x^2 + 1 + \frac{1}{24}x^4 - \frac{1}{720}x^6 \end{aligned}$$

Hence the ODE in Eq (1) becomes

$$\begin{aligned} &\left(\sum_{n=2}^{\infty} n(n-1)\, a_n x^{n-2}\right) + \left(x - \frac{1}{6}x^3 + \frac{1}{120}x^5 - \frac{1}{5040}x^7\right) \left(\sum_{n=1}^{\infty} n a_n x^{n-1}\right) \\ &+ \left(-\frac{1}{2}x^2 + 1 + \frac{1}{24}x^4 - \frac{1}{720}x^6\right) \left(\sum_{n=0}^{\infty} a_n x^n\right) = 0 \end{aligned}$$

Expanding the second term in (1) gives

$$\left(\sum_{n=2}^{\infty} n(n-1)\, a_n x^{n-2}\right) + x \cdot \left(\sum_{n=1}^{\infty} n a_n x^{n-1}\right) - \frac{x^3}{6} \cdot \left(\sum_{n=1}^{\infty} n a_n x^{n-1}\right) + \frac{x^5}{120} \cdot \left(\sum_{n=1}^{\infty} n a_n x^{n-1}\right) - \frac{x^7}{5040} \cdot \left(\sum_{n=1}^{\infty} n a_n x^{n-1}\right) + \left(-\frac{1}{2}x^2 + 1 + \frac{1}{24}x^4 - \frac{1}{720}x^6\right) \left(\sum_{n=0}^{\infty} a_n x^n\right) = 0$$

Expanding the third term in (1) gives

$$\left(\sum_{n=2}^{\infty} n(n-1)\, a_n x^{n-2}\right) + x \cdot \left(\sum_{n=1}^{\infty} n a_n x^{n-1}\right) - \frac{x^3}{6} \cdot \left(\sum_{n=1}^{\infty} n a_n x^{n-1}\right) + \frac{x^5}{120} \cdot \left(\sum_{n=1}^{\infty} n a_n x^{n-1}\right) - \frac{x^7}{5040} \cdot \left(\sum_{n=1}^{\infty} n a_n x^{n-1}\right) + -\frac{x^2}{2} \cdot \left(\sum_{n=0}^{\infty} a_n x^n\right) + 1 \cdot \left(\sum_{n=0}^{\infty} a_n x^n\right) + \frac{x^4}{24} \cdot \left(\sum_{n=0}^{\infty} a_n x^n\right) - \frac{x^6}{720} \cdot \left(\sum_{n=0}^{\infty} a_n x^n\right) = 0$$

Which simplifies to

$$\left(\sum_{n=2}^{\infty} n(n-1)\, a_n x^{n-2}\right) + \left(\sum_{n=1}^{\infty} n a_n x^n\right) + \sum_{n=1}^{\infty} \left(-\frac{n\, x^{n+2} a_n}{6}\right) + \left(\sum_{n=1}^{\infty} \frac{n\, x^{n+4} a_n}{120}\right) + \sum_{n=1}^{\infty} \left(-\frac{n\, x^{n+6} a_n}{5040}\right) + \sum_{n=0}^{\infty} \left(-\frac{x^{n+2} a_n}{2}\right) + \left(\sum_{n=0}^{\infty} a_n x^n\right) + \left(\sum_{n=0}^{\infty} \frac{x^{n+4} a_n}{24}\right) + \sum_{n=0}^{\infty} \left(-\frac{x^{n+6} a_n}{720}\right) = 0 \tag{2}$$

The next step is to make all powers of x be n in each summation term. Going over each summation term above with power of x in it which is not already x^n and adjusting the power and the corresponding index gives

$$\sum_{n=2}^{\infty} n(n-1)\, a_n x^{n-2} = \sum_{n=0}^{\infty} (n+2)\, a_{n+2}(n+1)\, x^n$$

$$\sum_{n=1}^{\infty} \left(-\frac{n\, x^{n+2} a_n}{6}\right) = \sum_{n=3}^{\infty} \left(-\frac{(n-2)\, a_{n-2} x^n}{6}\right)$$

$$\sum_{n=1}^{\infty} \frac{n\,x^{n+4} a_n}{120} = \sum_{n=5}^{\infty} \frac{(n-4)\,a_{n-4} x^n}{120}$$

$$\sum_{n=1}^{\infty} \left(-\frac{n\,x^{n+6} a_n}{5040}\right) = \sum_{n=7}^{\infty} \left(-\frac{(n-6)\,a_{n-6} x^n}{5040}\right)$$

$$\sum_{n=0}^{\infty} \left(-\frac{x^{n+2} a_n}{2}\right) = \sum_{n=2}^{\infty} \left(-\frac{a_{n-2} x^n}{2}\right)$$

$$\sum_{n=0}^{\infty} \frac{x^{n+4} a_n}{24} = \sum_{n=4}^{\infty} \frac{a_{n-4} x^n}{24}$$

$$\sum_{n=0}^{\infty} \left(-\frac{x^{n+6} a_n}{720}\right) = \sum_{n=6}^{\infty} \left(-\frac{a_{n-6} x^n}{720}\right)$$

Substituting all the above in Eq (2) gives the following equation where now all powers of x are the same and equal to n.

$$\begin{split} \left(\sum_{n=0}^{\infty} (n+2)\,a_{n+2}(n+1)\,x^n\right) + \left(\sum_{n=1}^{\infty} n a_n x^n\right) + \sum_{n=3}^{\infty} \left(-\frac{(n-2)\,a_{n-2} x^n}{6}\right) \\ + \left(\sum_{n=5}^{\infty} \frac{(n-4)\,a_{n-4} x^n}{120}\right) + \sum_{n=7}^{\infty} \left(-\frac{(n-6)\,a_{n-6} x^n}{5040}\right) + \sum_{n=2}^{\infty} \left(-\frac{a_{n-2} x^n}{2}\right) \\ + \left(\sum_{n=0}^{\infty} a_n x^n\right) + \left(\sum_{n=4}^{\infty} \frac{a_{n-4} x^n}{24}\right) + \sum_{n=6}^{\infty} \left(-\frac{a_{n-6} x^n}{720}\right) = 0 \end{split} \tag{3}$$

$n = 0$ gives

$$2a_2 + a_0 = 0$$

$$a_2 = -\frac{a_0}{2}$$

$n = 1$ gives

$$6a_3 + 2a_1 = 0$$

Which after substituting earlier equations, simplifies to

$$a_3 = -\frac{a_1}{3}$$

$n = 2$ gives

$$12a_4 + 3a_2 - \frac{a_0}{2} = 0$$

Which after substituting earlier equations, simplifies to

$$a_4 = \frac{a_0}{6}$$

$n = 3$ gives

$$20a_5 + 4a_3 - \frac{2a_1}{3} = 0$$

Which after substituting earlier equations, simplifies to

$$a_5 = \frac{a_1}{10}$$

$n = 4$ gives

$$30a_6 + 5a_4 - \frac{5a_2}{6} + \frac{a_0}{24} = 0$$

Which after substituting earlier equations, simplifies to

$$a_6 = -\frac{31a_0}{720}$$

$n = 5$ gives

$$42a_7 + 6a_5 - a_3 + \frac{a_1}{20} = 0$$

Which after substituting earlier equations, simplifies to

$$a_7 = -\frac{59a_1}{2520}$$

For $7 \leq n$, the recurrence equation is

$$\begin{aligned}(n+2)\, a_{n+2}(n+1) + n a_n - \frac{(n-2)\, a_{n-2}}{6} + \frac{(n-4)\, a_{n-4}}{120} \\ - \frac{(n-6)\, a_{n-6}}{5040} - \frac{a_{n-2}}{2} + a_n + \frac{a_{n-4}}{24} - \frac{a_{n-6}}{720} = 0\end{aligned} \tag{4}$$

Solving for a_{n+2}, gives

$$
\begin{aligned}
a_{n+2} &= -\frac{5040a_n - a_{n-6} + 42a_{n-4} - 840a_{n-2}}{5040\,(n+2)} \\
&= -\frac{a_n}{n+2} + \frac{a_{n-6}}{5040n + 10080} - \frac{a_{n-4}}{120\,(n+2)} + \frac{a_{n-2}}{6n+12}
\end{aligned} \tag{5}
$$

And so on. Therefore the solution is

$$
\begin{aligned}
y &= \sum_{n=0}^{\infty} a_n x^n \\
&= a_3x^3 + a_2x^2 + a_1x + a_0 + \dots
\end{aligned}
$$

Substituting the values for a_n found above, the solution becomes

$$
y = a_0 + a_1x - \frac{1}{2}a_0x^2 - \frac{1}{3}a_1x^3 + \frac{1}{6}a_0x^4 + \frac{1}{10}a_1x^5 + \dots
$$

Collecting terms, the solution becomes

$$
y = \left(1 - \frac{1}{2}x^2 + \frac{1}{6}x^4\right)a_0 + \left(-\frac{1}{3}x^3 + x + \frac{1}{10}x^5\right)a_1 + O\left(x^6\right) \tag{3}
$$

At $x = 0$ the solution above becomes

$$
y = \left(1 - \frac{1}{2}x^2 + \frac{1}{6}x^4\right)c_1 + \left(-\frac{1}{3}x^3 + x + \frac{1}{10}x^5\right)c_2 + O\left(x^6\right)
$$

$$
y = -\frac{x^3}{3} + x + \frac{x^5}{10} + O\left(x^6\right)
$$

Summary
The solution(s) found are the following

$$
y = -\frac{x^3}{3} + x + \frac{x^5}{10} + O\left(x^6\right) \tag{1}
$$

$$
y = -\frac{x^3}{3} + x + \frac{x^5}{10} + O\left(x^6\right) \tag{2}
$$

Verification of solutions

$$y = -\frac{x^3}{3} + x + \frac{x^5}{10} + O(x^6)$$

Verified OK.

$$y = -\frac{x^3}{3} + x + \frac{x^5}{10} + O(x^6)$$

Verified OK.

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a symmetry of the form [xi=0, eta=F(x)]
   One independent solution has integrals. Trying a hypergeometric solution free of integral
   -> hyper3: Equivalence to 2F1, 1F1 or 0F1 under a power @ Moebius
No hypergeometric solution was found.
<- linear_1 successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 14

```
Order:=6;
dsolve([diff(y(x),x$2)+sin(x)*diff(y(x),x)+cos(x)*y(x)=0,y(0) = 0, D(y)(0) = 1],y(x),type='s
```

$$y(x) = x - \frac{1}{3}x^3 + \frac{1}{10}x^5 + \mathrm{O}\left(x^6\right)$$

✓ Solution by Mathematica

Time used: 0.001 (sec). Leaf size: 19

```
AsymptoticDSolveValue[{y''[x]+Sin[x]*y'[x]+Cos[x]*y[x]==0,{y[0]==0,y'[0]==1}},y[x],{x,0,5}]
```

$$y(x) \to \frac{x^5}{10} - \frac{x^3}{3} + x$$

14.3 problem 3

Internal problem ID [735]
Internal file name [OUTPUT/735_Sunday_June_05_2022_01_48_14_AM_17099555/index.tex]

Book: Elementary differential equations and boundary value problems, 10th ed., Boyce and DiPrima
Section: Chapter 5.3, Series Solutions Near an Ordinary Point, Part II. page 269
Problem number: 3.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second order series method. Ordinary point", "second order series method. Taylor series method"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

$$\boxed{x^2y'' + (x+1)\,y' + 3\ln(x)\,y = 0}$$

With initial conditions

$$[y(1) = 2, y'(1) = 0]$$

With the expansion point for the power series method at $x = 1$.

14.3.1 Existence and uniqueness analysis

This is a linear ODE. In canonical form it is written as

$$y'' + p(x)y' + q(x)y = F$$

Where here

$$\begin{aligned}
p(x) &= \frac{x+1}{x^2} \\
q(x) &= \frac{3\ln(x)}{x^2} \\
F &= 0
\end{aligned}$$

Hence the ode is

$$y'' + \frac{(x+1)\, y'}{x^2} + \frac{3\ln(x)\, y}{x^2} = 0$$

The domain of $p(x) = \frac{x+1}{x^2}$ is

$$\{x < 0 \lor 0 < x\}$$

And the point $x_0 = 1$ is inside this domain. The domain of $q(x) = \frac{3\ln(x)}{x^2}$ is

$$\{0 < x\}$$

And the point $x_0 = 1$ is also inside this domain. Hence solution exists and is unique.

The ode does not have its expansion point at $x = 0$, therefore to simplify the computation of power series expansion, change of variable is made on the independent variable to shift the initial conditions and the expasion point back to zero. The new ode is then solved more easily since the expansion point is now at zero. The solution converted back to the original independent variable. Let

$$t = x - 1$$

The ode is converted to be in terms of the new independent variable t. This results in

$$(t+1)^2 \left(\frac{d^2}{dt^2} y(t)\right) + (2+t) \left(\frac{d}{dt} y(t)\right) + 3\ln(t+1)\, y(t) = 0$$

With its expansion point and initial conditions now at $t = 0$. With initial conditions now becoming

$$y(0) = 2$$
$$y'(0) = 0$$

The transformed ODE is now solved. Solving ode using Taylor series method. This gives review on how the Taylor series method works for solving second order ode.

Let

$$y'' = f(x, y, y')$$

Assuming expansion is at $x_0 = 0$ (we can always shift the actual expansion point to 0 by change of variables) and assuming $f(x, y, y')$ is analytic at x_0 which must be the case for an ordinary point. Let initial conditions be $y(x_0) = y_0$ and $y'(x_0) = y'_0$. Using Taylor series gives

$$\begin{aligned}
y(x) &= y(x_0) + (x - x_0)\, y'(x_0) + \frac{(x-x_0)^2}{2} y''(x_0) + \frac{(x-x_0)^3}{3!} y'''(x_0) + \cdots \\
&= y_0 + x y'_0 + \frac{x^2}{2} f|_{x_0,y_0,y'_0} + \frac{x^3}{3!} f'|_{x_0,y_0,y'_0} + \cdots \\
&= y_0 + x y'_0 + \sum_{n=0}^{\infty} \frac{x^{n+2}}{(n+2)\,!} \left.\frac{d^n f}{dx^n}\right|_{x_0,y_0,y'_0}
\end{aligned}$$

But

$$\begin{aligned}
\frac{df}{dx} &= \frac{\partial f}{\partial x}\frac{dx}{dx} + \frac{\partial f}{\partial y}\frac{dy}{dx} + \frac{\partial f}{\partial y'}\frac{dy'}{dx} && (1) \\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} y' + \frac{\partial f}{\partial y'} y'' && (816) \\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} y' + \frac{\partial f}{\partial y'} f && (817) \\
\frac{d^2 f}{dx^2} &= \frac{d}{dx}\left(\frac{df}{dx}\right) \\
&= \frac{\partial}{\partial x}\left(\frac{df}{dx}\right) + \frac{\partial}{\partial y}\left(\frac{df}{dx}\right) y' + \frac{\partial}{\partial y'}\left(\frac{df}{dx}\right) f && (2) \\
\frac{d^3 f}{dx^3} &= \frac{d}{dx}\left(\frac{d^2 f}{dx^2}\right) \\
&= \frac{\partial}{\partial x}\left(\frac{d^2 f}{dx^2}\right) + \left(\frac{\partial}{\partial y}\frac{d^2 f}{dx^2}\right) y' + \frac{\partial}{\partial y'}\left(\frac{d^2 f}{dx^2}\right) f && (3) \\
&\vdots
\end{aligned}$$

And so on. Hence if we name $F_0 = f(x,y,y')$ then the above can be written as

$$
\begin{aligned}
F_0 &= f(x,y,y') &(4)\\
F_1 &= \frac{df}{dx}\\
&= \frac{dF_0}{dx}\\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}y' + \frac{\partial f}{\partial y'}y''\\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}y' + \frac{\partial f}{\partial y'}f &(5)\\
&= \frac{\partial F_0}{\partial x} + \frac{\partial F_0}{\partial y}y' + \frac{\partial F_0}{\partial y'}F_0\\
F_2 &= \frac{d}{dx}\left(\frac{d}{dx}f\right)\\
&= \frac{d}{dx}(F_1)\\
&= \frac{\partial}{\partial x}F_1 + \left(\frac{\partial F_1}{\partial y}\right)y' + \left(\frac{\partial F_1}{\partial y'}\right)y''\\
&= \frac{\partial}{\partial x}F_1 + \left(\frac{\partial F_1}{\partial y}\right)y' + \left(\frac{\partial F_1}{\partial y'}\right)F_0\\
&\vdots\\
F_n &= \frac{d}{dx}(F_{n-1})\\
&= \frac{\partial}{\partial x}F_{n-1} + \left(\frac{\partial F_{n-1}}{\partial y}\right)y' + \left(\frac{\partial F_{n-1}}{\partial y'}\right)y''\\
&= \frac{\partial}{\partial x}F_{n-1} + \left(\frac{\partial F_{n-1}}{\partial y}\right)y' + \left(\frac{\partial F_{n-1}}{\partial y'}\right)F_0 &(6)
\end{aligned}
$$

Therefore (6) can be used from now on along with

$$
y(x) = y_0 + xy'_0 + \sum_{n=0}^{\infty} \frac{x^{n+2}}{(n+2)!} F_n|_{x_0,y_0,y'_0} \tag{7}
$$

To find $y(x)$ series solution around $x = 0$. Hence

$$
\begin{aligned}
F_0 &= -\frac{t\left(\frac{d}{dt}y(t)\right) + 3\ln\left(t+1\right)y(t) + 2\frac{d}{dt}y(t)}{\left(t+1\right)^2} \\
F_1 &= \frac{dF_0}{dt} \\
&= \frac{\partial F_0}{\partial t} + \frac{\partial F_0}{\partial y}\frac{d}{dt}y(t) + \frac{\partial F_0}{\partial \frac{d}{dt}y\left(t\right)}F_0 \\
&= \frac{\left(-3(t+1)^2\left(\frac{d}{dt}y(t)\right) + (9t+12)\,y(t)\right)\ln\left(t+1\right) + \left(2t^2+8t+7\right)\left(\frac{d}{dt}y(t)\right) - 3(t+1)\,y(t)}{\left(t+1\right)^4} \\
F_2 &= \frac{dF_1}{dt} \\
&= \frac{\partial F_1}{\partial t} + \frac{\partial F_1}{\partial y}\frac{d}{dt}y(t) + \frac{\partial F_1}{\partial \frac{d}{dt}y\left(t\right)}F_1 \\
&= \frac{9(t+1)^2\,y(t)\ln\left(t+1\right)^2 + \left(18(t+1)^2\left(t+\frac{4}{3}\right)\left(\frac{d}{dt}y(t)\right) + \left(-33t^2-90t-60\right)y(t)\right)\ln\left(t+1\right) + (-12t}{\left(t+1\right)^6} \\
F_3 &= \frac{dF_2}{dt} \\
&= \frac{\partial F_2}{\partial t} + \frac{\partial F_2}{\partial y}\frac{d}{dt}y(t) + \frac{\partial F_2}{\partial \frac{d}{dt}y\left(t\right)}F_2 \\
&= \frac{9(t+1)^2\left((t+1)^2\left(\frac{d}{dt}y(t)\right) + (-10t-12)\,y(t)\right)\ln\left(t+1\right)^2 + \left(\left(-105t^4-492t^3-855t^2-654t-186\right.\right.}{} \\
F_4 &= \frac{dF_3}{dt} \\
&= \frac{\partial F_3}{\partial t} + \frac{\partial F_3}{\partial y}\frac{d}{dt}y(t) + \frac{\partial F_3}{\partial \frac{d}{dt}y\left(t\right)}F_3 \\
&= \frac{-27y(t)\left(t+1\right)^4\ln\left(t+1\right)^3 - 135(t+1)^2\left(\left(t+\frac{6}{5}\right)(t+1)^2\left(\frac{d}{dt}y(t)\right) - \frac{17\left(t^2+\frac{206}{85}t+\frac{124}{85}\right)y(t)}{3}\right)\ln\left(t+1\right)^2 +}{}
\end{aligned}
$$

And so on. Evaluating all the above at initial conditions $t = 0$ and $y(0) = 2$ and $y'(0) = 0$ gives

$$
\begin{aligned}
F_0 &= 0 \\
F_1 &= -6 \\
F_2 &= 42 \\
F_3 &= -294 \\
F_4 &= 2376
\end{aligned}
$$

Substituting all the above in (7) and simplifying gives the solution as

$$y(t) = -t^3 + 2 + \frac{7t^4}{4} - \frac{49t^5}{20} + \frac{33t^6}{10} + O(t^6)$$

$$y(t) = -t^3 + 2 + \frac{7t^4}{4} - \frac{49t^5}{20} + \frac{33t^6}{10} + O(t^6)$$

Since the expansion point $t = 0$ is an ordinary, we can also solve this using standard power series The ode is normalized to be

$$(t^2 + 2t + 1)\left(\frac{d^2}{dt^2}y(t)\right) + (2 + t)\left(\frac{d}{dt}y(t)\right) + 3\ln(t + 1)\,y(t) = 0$$

Let the solution be represented as power series of the form

$$y(t) = \sum_{n=0}^{\infty} a_n t^n$$

Then

$$\begin{aligned}
\frac{d}{dt}y(t) &= \sum_{n=1}^{\infty} n a_n t^{n-1} \\
\frac{d^2}{dt^2}y(t) &= \sum_{n=2}^{\infty} n(n-1)\,a_n t^{n-2}
\end{aligned}$$

Substituting the above back into the ode gives

$$(t^2 + 2t + 1)\left(\sum_{n=2}^{\infty} n(n-1)\,a_n t^{n-2}\right) + (2 + t)\left(\sum_{n=1}^{\infty} n a_n t^{n-1}\right) + 3\ln(t + 1)\left(\sum_{n=0}^{\infty} a_n t^n\right) = 0 \tag{1}$$

Expanding $3\ln(t + 1)$ as Taylor series around $t = 0$ and keeping only the first 6 terms gives

$$\begin{aligned}
3\ln(t + 1) &= 3t - \frac{3}{2}t^2 + t^3 - \frac{3}{4}t^4 + \frac{3}{5}t^5 - \frac{1}{2}t^6 + \dots \\
&= 3t - \frac{3}{2}t^2 + t^3 - \frac{3}{4}t^4 + \frac{3}{5}t^5 - \frac{1}{2}t^6
\end{aligned}$$

Hence the ODE in Eq (1) becomes

$$
\left(t^{2}+2t+1\right)\left(\sum_{n=2}^{\infty}n(n-1)\,a_{n}t^{n-2}\right)+(2+t)\left(\sum_{n=1}^{\infty}n a_{n}t^{n-1}\right)
+\left(3t-\frac{3}{2}t^{2}+t^{3}-\frac{3}{4}t^{4}+\frac{3}{5}t^{5}-\frac{1}{2}t^{6}\right)\left(\sum_{n=0}^{\infty}a_{n}t^{n}\right)=0
$$

Expanding the third term in (1) gives

$$
\left(t^{2}+2t+1\right)\left(\sum_{n=2}^{\infty}n(n-1)\,a_{n}t^{n-2}\right)+(2+t)\left(\sum_{n=1}^{\infty}n a_{n}t^{n-1}\right)
+3t\cdot\left(\sum_{n=0}^{\infty}a_{n}t^{n}\right)-\frac{3t^{2}}{2}\cdot\left(\sum_{n=0}^{\infty}a_{n}t^{n}\right)+t^{3}\cdot\left(\sum_{n=0}^{\infty}a_{n}t^{n}\right)-\frac{3t^{4}}{4}
\cdot\left(\sum_{n=0}^{\infty}a_{n}t^{n}\right)+\frac{3t^{5}}{5}\cdot\left(\sum_{n=0}^{\infty}a_{n}t^{n}\right)-\frac{t^{6}}{2}\cdot\left(\sum_{n=0}^{\infty}a_{n}t^{n}\right)=0
$$

Which simplifies to

$$
\begin{aligned}
&\left(\sum_{n=2}^{\infty}t^{n}a_{n}n(n-1)\right)+\left(\sum_{n=2}^{\infty}2n\,t^{n-1}a_{n}(n-1)\right)\\
&+\left(\sum_{n=2}^{\infty}n(n-1)\,a_{n}t^{n-2}\right)+\left(\sum_{n=1}^{\infty}2n a_{n}t^{n-1}\right)+\left(\sum_{n=1}^{\infty}n a_{n}t^{n}\right)\\
&+\left(\sum_{n=0}^{\infty}3t^{1+n}a_{n}\right)+\sum_{n=0}^{\infty}\left(-\frac{3t^{n+2}a_{n}}{2}\right)+\left(\sum_{n=0}^{\infty}t^{n+3}a_{n}\right)\\
&+\sum_{n=0}^{\infty}\left(-\frac{3t^{n+4}a_{n}}{4}\right)+\left(\sum_{n=0}^{\infty}\frac{3t^{n+5}a_{n}}{5}\right)+\sum_{n=0}^{\infty}\left(-\frac{t^{n+6}a_{n}}{2}\right)=0
\end{aligned}
\tag{2}
$$

The next step is to make all powers of t be n in each summation term. Going over each summation term above with power of t in it which is not already t^n and adjusting the power and the corresponding index gives

$$
\begin{aligned}
\sum_{n=2}^{\infty}2n\,t^{n-1}a_{n}(n-1)&=\sum_{n=1}^{\infty}2(1+n)\,a_{1+n}n\,t^{n}\\
\sum_{n=2}^{\infty}n(n-1)\,a_{n}t^{n-2}&=\sum_{n=0}^{\infty}(n+2)\,a_{n+2}(1+n)\,t^{n}
\end{aligned}
$$

$$\sum_{n=1}^{\infty} 2n a_n t^{n-1} = \sum_{n=0}^{\infty} 2(1+n) a_{1+n} t^n$$

$$\sum_{n=0}^{\infty} 3t^{1+n} a_n = \sum_{n=1}^{\infty} 3a_{n-1} t^n$$

$$\sum_{n=0}^{\infty} \left(-\frac{3t^{n+2} a_n}{2}\right) = \sum_{n=2}^{\infty} \left(-\frac{3a_{n-2} t^n}{2}\right)$$

$$\sum_{n=0}^{\infty} t^{n+3} a_n = \sum_{n=3}^{\infty} a_{n-3} t^n$$

$$\sum_{n=0}^{\infty} \left(-\frac{3t^{n+4} a_n}{4}\right) = \sum_{n=4}^{\infty} \left(-\frac{3a_{n-4} t^n}{4}\right)$$

$$\sum_{n=0}^{\infty} \frac{3t^{n+5} a_n}{5} = \sum_{n=5}^{\infty} \frac{3a_{n-5} t^n}{5}$$

$$\sum_{n=0}^{\infty} \left(-\frac{t^{n+6} a_n}{2}\right) = \sum_{n=6}^{\infty} \left(-\frac{a_{n-6} t^n}{2}\right)$$

Substituting all the above in Eq (2) gives the following equation where now all powers of t are the same and equal to n.

$$\begin{aligned} &\left(\sum_{n=2}^{\infty} t^n a_n n(n-1)\right) + \left(\sum_{n=1}^{\infty} 2(1+n) a_{1+n} n\, t^n\right) \\ &+ \left(\sum_{n=0}^{\infty} (n+2) a_{n+2}(1+n) t^n\right) + \left(\sum_{n=0}^{\infty} 2(1+n) a_{1+n} t^n\right) \\ &+ \left(\sum_{n=1}^{\infty} n a_n t^n\right) + \left(\sum_{n=1}^{\infty} 3a_{n-1} t^n\right) + \sum_{n=2}^{\infty} \left(-\frac{3a_{n-2} t^n}{2}\right) + \left(\sum_{n=3}^{\infty} a_{n-3} t^n\right) \\ &+ \sum_{n=4}^{\infty} \left(-\frac{3a_{n-4} t^n}{4}\right) + \left(\sum_{n=5}^{\infty} \frac{3a_{n-5} t^n}{5}\right) + \sum_{n=6}^{\infty} \left(-\frac{a_{n-6} t^n}{2}\right) = 0 \end{aligned} \tag{3}$$

$n = 0$ gives

$$2a_2 + 2a_1 = 0$$

$$a_2 = -a_1$$

$n = 1$ gives

$$8a_2 + 6a_3 + a_1 + 3a_0 = 0$$

Which after substituting earlier equations, simplifies to

$$a_3 = -\frac{a_0}{2} + \frac{7a_1}{6}$$

$n = 2$ gives

$$4a_2 + 18a_3 + 12a_4 + 3a_1 - \frac{3a_0}{2} = 0$$

Which after substituting earlier equations, simplifies to

$$a_4 = \frac{7a_0}{8} - \frac{5a_1}{3}$$

$n = 3$ gives

$$9a_3 + 32a_4 + 20a_5 + 3a_2 - \frac{3a_1}{2} + a_0 = 0$$

Which after substituting earlier equations, simplifies to

$$a_5 = -\frac{49a_0}{40} + \frac{71a_1}{30}$$

$n = 4$ gives

$$16a_4 + 50a_5 + 30a_6 + 3a_3 - \frac{3a_2}{2} + a_1 - \frac{3a_0}{4} = 0$$

Which after substituting earlier equations, simplifies to

$$a_6 = \frac{33a_0}{20} - \frac{293a_1}{90}$$

$n = 5$ gives

$$25a_5 + 72a_6 + 42a_7 + 3a_4 - \frac{3a_3}{2} + a_2 - \frac{3a_1}{4} + \frac{3a_0}{5} = 0$$

Which after substituting earlier equations, simplifies to

$$a_7 = -\frac{1843a_0}{840} + \frac{1378a_1}{315}$$

For $6 \leq n$, the recurrence equation is

$$na_n(n-1) + 2(1+n)\, a_{1+n} n + (n+2)\, a_{n+2}(1+n) + 2(1+n)\, a_{1+n} + na_n + 3a_{n-1} - \frac{3a_{n-2}}{2} + a_{n-3} - \frac{3a_{n-4}}{4} + \frac{3a_{n-5}}{5} - \frac{a_{n-6}}{2} = 0 \tag{4}$$

Solving for a_{n+2}, gives

$$a_{n+2} = -\frac{20n^2 a_n + 40n^2 a_{1+n} + 80n a_{1+n} + 40a_{1+n} - 10a_{n-6} + 12a_{n-5} - 15a_{n-4} + 20a_{n-3} - 30a_{n-2} + 60a_{n-1}}{20\,(n+2)\,(1+n)}$$

$$(5) \quad = -\frac{n^2 a_n}{(n+2)\,(1+n)} - \frac{(40n^2 + 80n + 40)\, a_{1+n}}{20\,(n+2)\,(1+n)} + \frac{a_{n-6}}{2\,(n+2)\,(1+n)} - \frac{3a_{n-5}}{5\,(n+2)\,(1+n)} + \frac{3a_{n-4}}{4\,(n+2)\,(1+n)} - \frac{a_{n-3}}{(n+2)\,(1+n)} + \frac{3a_{n-2}}{2\,(n+2)\,(1+n)} - \frac{3a_{n-1}}{(n+2)\,(1+n)}$$

And so on. Therefore the solution is

$$y(t) = \sum_{n=0}^{\infty} a_n t^n = a_3 t^3 + a_2 t^2 + a_1 t + a_0 + \dots$$

Substituting the values for a_n found above, the solution becomes

$$y(t) = a_0 + a_1 t - a_1 t^2 + \left(-\frac{a_0}{2} + \frac{7a_1}{6}\right) t^3 + \left(\frac{7a_0}{8} - \frac{5a_1}{3}\right) t^4 + \left(-\frac{49a_0}{40} + \frac{71a_1}{30}\right) t^5 + \dots$$

Collecting terms, the solution becomes

$$y(t) = \left(1 - \frac{1}{2}t^3 + \frac{7}{8}t^4 - \frac{49}{40}t^5\right) a_0 + \left(t - t^2 + \frac{7}{6}t^3 - \frac{5}{3}t^4 + \frac{71}{30}t^5\right) a_1 + O(t^6) \tag{3}$$

At $t = 0$ the solution above becomes

$$y(t) = \left(1 - \frac{1}{2}t^3 + \frac{7}{8}t^4 - \frac{49}{40}t^5\right) c_1 + \left(t - t^2 + \frac{7}{6}t^3 - \frac{5}{3}t^4 + \frac{71}{30}t^5\right) c_2 + O(t^6)$$

$$y(t) = -t^3 + 2 + \frac{7t^4}{4} - \frac{49t^5}{20} + O(t^6)$$

Replacing t in the above with the original independent variable $x$using $t = x-1$ results in

$$y = -(x-1)^{3}+2+\frac{7(x-1)^{4}}{4}-\frac{49(x-1)^{5}}{20}+\frac{33(x-1)^{6}}{10}+O\left((x-1)^{6}\right)$$

Summary

The solution(s) found are the following

$$y = -(x-1)^{3}+2+\frac{7(x-1)^{4}}{4}-\frac{49(x-1)^{5}}{20}+\frac{33(x-1)^{6}}{10}+O\left((x-1)^{6}\right) \tag{1}$$

Verification of solutions

$$y = -(x-1)^{3}+2+\frac{7(x-1)^{4}}{4}-\frac{49(x-1)^{5}}{20}+\frac{33(x-1)^{6}}{10}+O\left((x-1)^{6}\right)$$

Verified OK.

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Heun: Equivalence to the GHE or one of its 4 confluent cases under a power @ Moebius
-> trying a solution of the form r0(x) * Y + r1(x) * Y where Y = exp(int(r(x), dx)) * 2F1([a
-> Trying changes of variables to rationalize or make the ODE simpler
   trying a symmetry of the form [xi=0, eta=F(x)]
   checking if the LODE is missing y
   -> Heun: Equivalence to the GHE or one of its 4 confluent cases under a power @ Moebius
   -> trying a solution of the form r0(x) * Y + r1(x) * Y where Y = exp(int(r(x), dx)) * 2F1
      trying a symmetry of the form [xi=0, eta=F(x)]
      trying 2nd order exact linear
      trying symmetries linear in x and y(x)
      trying to convert to a linear ODE with constant coefficients
<- unable to find a useful change of variables
   trying a symmetry of the form [xi=0, eta=F(x)]
   trying differential order: 2; exact nonlinear
   trying symmetries linear in x and y(x)
   trying to convert to a linear ODE with constant coefficients
   trying 2nd order, integrating factor of the form mu(x,y)
   trying a symmetry of the form [xi=0, eta=F(x)]
   checking if the LODE is missing y
   -> Heun: Equivalence to the GHE or one of its 4 confluent cases under a power @ Moebius
   -> trying a solution of the form r0(x) * Y + r1(x) * Y where Y = exp(int(r(x), dx)) * 2F1
   -> Trying changes of variables to rationalize or make the ODE simpler
      trying a symmetry of the form [xi=0, eta=F(x)]
      checking if the LODE is missing y
      -> Heun: Equivalence to the GHE or one of its 4 confluent cases under a power @ Moebiu
      -> trying a solution of the form r0(x) * Y + r1(x) * Y where Y = exp(int(r(x), dx)) *
         trying a symmetry of the form [xi=0, eta=F(x)]
         trying 2nd order exact linear
         trying symmetries linear in x and y(x)
         trying to convert to a linear ODE with constant coefficients
   <- unable to find a useful change of variables
      trying a symmetry of the form [xi=0, eta=F(x)]
   trying to convert to an ODE of Bessel type
   -> trying reduction of order to Riccati
      trying Riccati sub-methods:
         trying Riccati_symmetries
         -> trying a symmetry pattern of the form [F(x)*G(y), 0]
         -> trying a symmetry pattern of the form [0, F(x)*G(y)]
         -> trying a symmetry pattern of the form [F(x),G(x)*y+H(x)]
--- Trying Lie symmetry methods, 2nd order ---
`, `-> Computing symmetries using: way = 3`[0, y]
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 16

```
Order:=6;
dsolve([x^2*diff(y(x),x$2)+(1+x)*diff(y(x),x)+3*ln(x)*y(x)=0,y(1) = 2, D(y)(1) = 0],y(x),typ
```

$$y(x) = 2 - (x-1)^3 + \frac{7}{4}(x-1)^4 - \frac{49}{20}(x-1)^5 + \mathrm{O}\left((x-1)^6\right)$$

✓ Solution by Mathematica

Time used: 0.001 (sec). Leaf size: 30

```
AsymptoticDSolveValue[{x^2*y''[x]+(1+x)*y'[x]+3*Log[x]*y[x]==0,{y[1]==2,y'[1]==0}},y[x],{x,1
```

$$y(x) \to -\frac{49}{20}(x-1)^5 + \frac{7}{4}(x-1)^4 - (x-1)^3 + 2$$

14.4 problem 4

Internal problem ID [736]
Internal file name [`OUTPUT/736_Sunday_June_05_2022_01_48_18_AM_81415202/index.tex`]

Book: Elementary differential equations and boundary value problems, 10th ed., Boyce and DiPrima
Section: Chapter 5.3, Series Solutions Near an Ordinary Point, Part II. page 269
Problem number: 4.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second order series method. Ordinary point", "second order series method. Taylor series method"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

$$\boxed{y'' + y'x^2 + \sin(x)\, y = 0}$$

With initial conditions

$$[y(0) = a_0, y'(0) = a_1]$$

With the expansion point for the power series method at $x = 0$.

14.4.1 Existence and uniqueness analysis

This is a linear ODE. In canonical form it is written as

$$y'' + p(x)y' + q(x)y = F$$

Where here

$$\begin{aligned} p(x) &= x^2 \\ q(x) &= \sin(x) \\ F &= 0 \end{aligned}$$

Hence the ode is

$$y'' + y'x^2 + \sin(x)\, y = 0$$

The domain of $p(x) = x^2$ is

$$\{-\infty < x < \infty\}$$

And the point $x_0 = 0$ is inside this domain. The domain of $q(x) = \sin(x)$ is

$$\{-\infty < x < \infty\}$$

And the point $x_0 = 0$ is also inside this domain. Hence solution exists and is unique.

Solving ode using Taylor series method. This gives review on how the Taylor series method works for solving second order ode.

Let

$$y'' = f(x, y, y')$$

Assuming expansion is at $x_0 = 0$ (we can always shift the actual expansion point to 0 by change of variables) and assuming $f(x, y, y')$ is analytic at x_0 which must be the case for an ordinary point. Let initial conditions be $y(x_0) = y_0$ and $y'(x_0) = y_0'$. Using Taylor series gives

$$\begin{aligned}
y(x) &= y(x_0) + (x - x_0)\, y'(x_0) + \frac{(x - x_0)^2}{2} y''(x_0) + \frac{(x - x_0)^3}{3!} y'''(x_0) + \cdots \\
&= y_0 + x y_0' + \frac{x^2}{2} f|_{x_0, y_0, y_0'} + \frac{x^3}{3!} f'|_{x_0, y_0, y_0'} + \cdots \\
&= y_0 + x y_0' + \sum_{n=0}^{\infty} \frac{x^{n+2}}{(n+2)!} \frac{d^n f}{dx^n}\bigg|_{x_0, y_0, y_0'}
\end{aligned}$$

But

$$\begin{aligned}
\frac{df}{dx} &= \frac{\partial f}{\partial x}\frac{dx}{dx} + \frac{\partial f}{\partial y}\frac{dy}{dx} + \frac{\partial f}{\partial y'}\frac{dy'}{dx} && (1)\\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}y' + \frac{\partial f}{\partial y'}y'' && (819)\\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}y' + \frac{\partial f}{\partial y'}f && (820)\\
\frac{d^2 f}{dx^2} &= \frac{d}{dx}\left(\frac{df}{dx}\right)\\
&= \frac{\partial}{\partial x}\left(\frac{df}{dx}\right) + \frac{\partial}{\partial y}\left(\frac{df}{dx}\right)y' + \frac{\partial}{\partial y'}\left(\frac{df}{dx}\right)f && (2)\\
\frac{d^3 f}{dx^3} &= \frac{d}{dx}\left(\frac{d^2 f}{dx^2}\right)\\
&= \frac{\partial}{\partial x}\left(\frac{d^2 f}{dx^2}\right) + \left(\frac{\partial}{\partial y}\frac{d^2 f}{dx^2}\right)y' + \frac{\partial}{\partial y'}\left(\frac{d^2 f}{dx^2}\right)f && (3)\\
&\vdots
\end{aligned}$$

And so on. Hence if we name $F_0 = f(x,y,y')$ then the above can be written as

$$
\begin{aligned}
F_0 &= f(x,y,y') && (4)\\
F_1 &= \frac{df}{dx}\\
&= \frac{dF_0}{dx}\\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}y' + \frac{\partial f}{\partial y'}y''\\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}y' + \frac{\partial f}{\partial y'}f && (5)\\
&= \frac{\partial F_0}{\partial x} + \frac{\partial F_0}{\partial y}y' + \frac{\partial F_0}{\partial y'}F_0\\
F_2 &= \frac{d}{dx}\left(\frac{d}{dx}f\right)\\
&= \frac{d}{dx}(F_1)\\
&= \frac{\partial}{\partial x}F_1 + \left(\frac{\partial F_1}{\partial y}\right)y' + \left(\frac{\partial F_1}{\partial y'}\right)y''\\
&= \frac{\partial}{\partial x}F_1 + \left(\frac{\partial F_1}{\partial y}\right)y' + \left(\frac{\partial F_1}{\partial y'}\right)F_0\\
&\vdots\\
F_n &= \frac{d}{dx}(F_{n-1})\\
&= \frac{\partial}{\partial x}F_{n-1} + \left(\frac{\partial F_{n-1}}{\partial y}\right)y' + \left(\frac{\partial F_{n-1}}{\partial y'}\right)y''\\
&= \frac{\partial}{\partial x}F_{n-1} + \left(\frac{\partial F_{n-1}}{\partial y}\right)y' + \left(\frac{\partial F_{n-1}}{\partial y'}\right)F_0 && (6)
\end{aligned}
$$

Therefore (6) can be used from now on along with

$$
y(x) = y_0 + xy_0' + \sum_{n=0}^{\infty} \frac{x^{n+2}}{(n+2)!} F_n|_{x_0,y_0,y_0'} \tag{7}
$$

To find $y(x)$ series solution around $x = 0$. Hence

$$
\begin{aligned}
F_0 &= -y' x^2 - \sin(x) y \\
F_1 &= \frac{dF_0}{dx} \\
&= \frac{\partial F_0}{\partial x} + \frac{\partial F_0}{\partial y} y' + \frac{\partial F_0}{\partial y'} F_0 \\
&= \left(x^4 - 2x - \sin(x)\right) y' + y\left(\sin(x) x^2 - \cos(x)\right) \\
F_2 &= \frac{dF_1}{dx} \\
&= \frac{\partial F_1}{\partial x} + \frac{\partial F_1}{\partial y} y' + \frac{\partial F_1}{\partial y'} F_1 \\
&= \left(-x^6 + 6x^3 + 2\sin(x) x^2 - 2\cos(x) - 2\right) y' - y\left(-\sin(x)^2 + \left(x^4 - 4x - 1\right)\sin(x) - x^2\cos(x)\right) \\
F_3 &= \frac{dF_2}{dx} \\
&= \frac{\partial F_2}{\partial x} + \frac{\partial F_2}{\partial y} y' + \frac{\partial F_2}{\partial y'} F_2 \\
&= \left(\sin(x)^2 + \left(-3x^4 + 8x + 3\right)\sin(x) + x^2\left(x^6 - 12x^3 + 5\cos(x) + 20\right)\right) y' + \left(-2\sin(x)^2 x^2 + \left(x^6 - 1\right.\right. \\
F_4 &= \frac{dF_3}{dx} \\
&= \frac{\partial F_3}{\partial x} + \frac{\partial F_3}{\partial y} y' + \frac{\partial F_3}{\partial y'} F_3 \\
&= \left(-3\sin(x)^2 x^2 + \left(4x^6 - 30x^3 - 9x^2 + 6\cos(x) + 14\right)\sin(x) + \left(-9x^4 + 24x + 4\right)\cos(x) - x^{10} + 20\right.
\end{aligned}
$$

And so on. Evaluating all the above at initial conditions $x = 0$ and $y(0) = a_0$ and $y'(0) = a_1$ gives

$$
\begin{aligned}
F_0 &= 0 \\
F_1 &= -a_0 \\
F_2 &= -4a_1 \\
F_3 &= a_0 \\
F_4 &= 4a_1 + 16a_0
\end{aligned}
$$

Substituting all the above in (7) and simplifying gives the solution as

$$
y = x a_1 + a_0 - \frac{a_0 x^3}{6} - \frac{a_1 x^4}{6} + \frac{a_0 x^5}{120} + \frac{x^6 a_0}{45} + \frac{x^6 a_1}{180} + O\left(x^6\right)
$$

$$
y = x a_1 + a_0 - \frac{a_0 x^3}{6} - \frac{a_1 x^4}{6} + \frac{a_0 x^5}{120} + \frac{x^6 a_0}{45} + \frac{x^6 a_1}{180} + O\left(x^6\right)
$$

Since the expansion point $x = 0$ is an ordinary, we can also solve this using standard power series Let the solution be represented as power series of the form

$$y = \sum_{n=0}^{\infty} a_n x^n$$

Then

$$y' = \sum_{n=1}^{\infty} n a_n x^{n-1}$$
$$y'' = \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2}$$

Substituting the above back into the ode gives

$$\sum_{n=2}^{\infty} n(n-1) a_n x^{n-2} = -\left(\sum_{n=1}^{\infty} n a_n x^{n-1}\right) x^2 - \sin(x) \left(\sum_{n=0}^{\infty} a_n x^n\right) \tag{1}$$

Expanding $\sin(x)$ as Taylor series around $x = 0$ and keeping only the first 6 terms gives

$$\begin{aligned}\sin(x) &= x - \frac{1}{6}x^3 + \frac{1}{120}x^5 - \frac{1}{5040}x^7 + \dots \\ &= x - \frac{1}{6}x^3 + \frac{1}{120}x^5 - \frac{1}{5040}x^7\end{aligned}$$

Hence the ODE in Eq (1) becomes

$$\left(\sum_{n=2}^{\infty} n(n-1) a_n x^{n-2}\right) + \left(\sum_{n=1}^{\infty} n a_n x^{n-1}\right) x^2 + \left(x - \frac{1}{6}x^3 + \frac{1}{120}x^5 - \frac{1}{5040}x^7\right) \left(\sum_{n=0}^{\infty} a_n x^n\right) = 0$$

Expanding the third term in (1) gives

$$\left(\sum_{n=2}^{\infty} n(n-1) a_n x^{n-2}\right) + \left(\sum_{n=1}^{\infty} n a_n x^{n-1}\right) x^2 + x \cdot \left(\sum_{n=0}^{\infty} a_n x^n\right) - \frac{x^3}{6} \cdot \left(\sum_{n=0}^{\infty} a_n x^n\right) + \frac{x^5}{120} \cdot \left(\sum_{n=0}^{\infty} a_n x^n\right) - \frac{x^7}{5040} \cdot \left(\sum_{n=0}^{\infty} a_n x^n\right) = 0$$

Which simplifies to

$$\left(\sum_{n=2}^{\infty} n(n-1)\, a_n x^{n-2}\right) + \left(\sum_{n=1}^{\infty} n\, x^{1+n} a_n\right) + \left(\sum_{n=0}^{\infty} x^{1+n} a_n\right) + \sum_{n=0}^{\infty} \left(-\frac{x^{n+3} a_n}{6}\right) + \left(\sum_{n=0}^{\infty} \frac{x^{n+5} a_n}{120}\right) + \sum_{n=0}^{\infty} \left(-\frac{x^{n+7} a_n}{5040}\right) = 0 \tag{2}$$

The next step is to make all powers of x be n in each summation term. Going over each summation term above with power of x in it which is not already x^n and adjusting the power and the corresponding index gives

$$\sum_{n=2}^{\infty} n(n-1)\, a_n x^{n-2} = \sum_{n=0}^{\infty} (n+2)\, a_{n+2}(1+n)\, x^n$$

$$\sum_{n=1}^{\infty} n\, x^{1+n} a_n = \sum_{n=2}^{\infty} (n-1)\, a_{n-1} x^n$$

$$\sum_{n=0}^{\infty} x^{1+n} a_n = \sum_{n=1}^{\infty} a_{n-1} x^n$$

$$\sum_{n=0}^{\infty} \left(-\frac{x^{n+3} a_n}{6}\right) = \sum_{n=3}^{\infty} \left(-\frac{a_{n-3} x^n}{6}\right)$$

$$\sum_{n=0}^{\infty} \frac{x^{n+5} a_n}{120} = \sum_{n=5}^{\infty} \frac{a_{n-5} x^n}{120}$$

$$\sum_{n=0}^{\infty} \left(-\frac{x^{n+7} a_n}{5040}\right) = \sum_{n=7}^{\infty} \left(-\frac{a_{n-7} x^n}{5040}\right)$$

Substituting all the above in Eq (2) gives the following equation where now all powers of x are the same and equal to n.

$$\left(\sum_{n=0}^{\infty} (n+2)\, a_{n+2}(1+n)\, x^n\right) + \left(\sum_{n=2}^{\infty} (n-1)\, a_{n-1} x^n\right) + \left(\sum_{n=1}^{\infty} a_{n-1} x^n\right) + \sum_{n=3}^{\infty} \left(-\frac{a_{n-3} x^n}{6}\right) + \left(\sum_{n=5}^{\infty} \frac{a_{n-5} x^n}{120}\right) + \sum_{n=7}^{\infty} \left(-\frac{a_{n-7} x^n}{5040}\right) = 0 \tag{3}$$

$n = 1$ gives

$$6a_3 + a_0 = 0$$

Which after substituting earlier equations, simplifies to

$$a_3 = -\frac{a_0}{6}$$

$n = 2$ gives

$$12a_4 + 2a_1 = 0$$

Which after substituting earlier equations, simplifies to

$$a_4 = -\frac{a_1}{6}$$

$n = 3$ gives

$$20a_5 + 3a_2 - \frac{a_0}{6} = 0$$

Which after substituting earlier equations, simplifies to

$$a_5 = \frac{a_0}{120}$$

$n = 4$ gives

$$30a_6 + 4a_3 - \frac{a_1}{6} = 0$$

Which after substituting earlier equations, simplifies to

$$a_6 = \frac{a_0}{45} + \frac{a_1}{180}$$

$n = 5$ gives

$$42a_7 + 5a_4 - \frac{a_2}{6} + \frac{a_0}{120} = 0$$

Which after substituting earlier equations, simplifies to

$$a_7 = -\frac{a_0}{5040} + \frac{5a_1}{252}$$

For $7 \leq n$, the recurrence equation is

$$(n+2)\, a_{n+2}(1+n) + (n-1)\, a_{n-1} + a_{n-1} - \frac{a_{n-3}}{6} + \frac{a_{n-5}}{120} - \frac{a_{n-7}}{5040} = 0 \tag{4}$$

Solving for a_{n+2}, gives

$$\begin{aligned} a_{n+2} &= -\frac{5040 n a_{n-1} - a_{n-7} + 42 a_{n-5} - 840 a_{n-3}}{5040\,(n+2)\,(1+n)} \\ &= \frac{a_{n-7}}{5040\,(n+2)\,(1+n)} - \frac{a_{n-5}}{120\,(n+2)\,(1+n)} + \frac{a_{n-3}}{6\,(n+2)\,(1+n)} - \frac{n a_{n-1}}{(n+2)\,(1+n)} \end{aligned} \tag{5}$$

And so on. Therefore the solution is

$$\begin{aligned} y &= \sum_{n=0}^{\infty} a_n x^n \\ &= a_3 x^3 + a_2 x^2 + a_1 x + a_0 + \dots \end{aligned}$$

Substituting the values for a_n found above, the solution becomes

$$y = a_0 + a_1 x - \frac{1}{6} a_0 x^3 - \frac{1}{6} a_1 x^4 + \frac{1}{120} a_0 x^5 + \dots$$

Collecting terms, the solution becomes

$$y = \left(1 - \frac{1}{6}x^3 + \frac{1}{120}x^5\right) a_0 + \left(x - \frac{1}{6}x^4\right) a_1 + O(x^6) \tag{3}$$

At $x = 0$ the solution above becomes

$$y = \left(1 - \frac{1}{6}x^3 + \frac{1}{120}x^5\right) c_1 + \left(x - \frac{1}{6}x^4\right) c_2 + O(x^6)$$

$$y = a_0 - \frac{a_0 x^3}{6} + \frac{a_0 x^5}{120} + x a_1 - \frac{a_1 x^4}{6} + O(x^6)$$

Summary

The solution(s) found are the following

$$y = x a_1 + a_0 - \frac{a_0 x^3}{6} - \frac{a_1 x^4}{6} + \frac{a_0 x^5}{120} + \frac{x^6 a_0}{45} + \frac{x^6 a_1}{180} + O(x^6) \tag{1}$$

$$y = a_0 - \frac{a_0 x^3}{6} + \frac{a_0 x^5}{120} + x a_1 - \frac{a_1 x^4}{6} + O(x^6) \tag{2}$$

Verification of solutions

$$y = xa_1 + a_0 - \frac{a_0 x^3}{6} - \frac{a_1 x^4}{6} + \frac{a_0 x^5}{120} + \frac{x^6 a_0}{45} + \frac{x^6 a_1}{180} + O(x^6)$$

Verified OK.

$$y = a_0 - \frac{a_0 x^3}{6} + \frac{a_0 x^5}{120} + xa_1 - \frac{a_1 x^4}{6} + O(x^6)$$

Verified OK.

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Heun: Equivalence to the GHE or one of its 4 confluent cases under a power @ Moebius
-> trying a solution of the form r0(x) * Y + r1(x) * Y where Y = exp(int(r(x), dx)) * 2F1([a
-> Trying changes of variables to rationalize or make the ODE simpler
   trying a symmetry of the form [xi=0, eta=F(x)]
   checking if the LODE is missing y
   -> Heun: Equivalence to the GHE or one of its 4 confluent cases under a power @ Moebius
   -> trying a solution of the form r0(x) * Y + r1(x) * Y where Y = exp(int(r(x), dx)) * 2F1
      trying a symmetry of the form [xi=0, eta=F(x)]
      trying 2nd order exact linear
      trying symmetries linear in x and y(x)
      trying to convert to a linear ODE with constant coefficients
   trying a symmetry of the form [xi=0, eta=F(x)]
   checking if the LODE is missing y
   -> Heun: Equivalence to the GHE or one of its 4 confluent cases under a power @ Moebius
   -> trying a solution of the form r0(x) * Y + r1(x) * Y where Y = exp(int(r(x), dx)) * 2F1
      trying a symmetry of the form [xi=0, eta=F(x)]
      trying 2nd order exact linear
      trying symmetries linear in x and y(x)
      trying to convert to a linear ODE with constant coefficients
      -> trying with_periodic_functions in the coefficients
         --- Trying Lie symmetry methods, 2nd order ---
         `, `-> Computing symmetries using: way = 5`[0, u]
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 24

```
Order:=6;
dsolve([diff(y(x),x$2)+x^2*diff(y(x),x)+sin(x)*y(x)=0,y(0) = a__0, D(y)(0) = a__1],y(x),type
```

$$y(x) = a_0 + a_1 x - \frac{1}{6} a_0 x^3 - \frac{1}{6} a_1 x^4 + \frac{1}{120} a_0 x^5 + \mathrm{O}\left(x^6\right)$$

✓ Solution by Mathematica

Time used: 0.001 (sec). Leaf size: 32

```
AsymptoticDSolveValue[{y''[x]+x^2*y'[x]+Sin[x]*y[x]==0,{y[0]==a0,y'[0]==a1}},y[x],{x,0,5}]
```

$$y(x) \to \frac{\mathrm{a0} x^5}{120} - \frac{\mathrm{a0} x^3}{6} + \mathrm{a0} - \frac{\mathrm{a1} x^4}{6} + \mathrm{a1} x$$

14.5 problem 5. case $x_0 = 0$

Internal problem ID [737]
Internal file name [`OUTPUT/737_Sunday_June_05_2022_01_48_21_AM_72114671/index.tex`]

Book: Elementary differential equations and boundary value problems, 10th ed., Boyce and DiPrima
Section: Chapter 5.3, Series Solutions Near an Ordinary Point, Part II. page 269
Problem number: 5. case $x_0 = 0$.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second_order_airy", "second order series method. Ordinary point", "second order series method. Taylor series method"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

$$\boxed{y'' + 4y' + 6yx = 0}$$

With the expansion point for the power series method at $x = 0$.

Solving ode using Taylor series method. This gives review on how the Taylor series method works for solving second order ode.

Let

$$y'' = f(x, y, y')$$

Assuming expansion is at $x_0 = 0$ (we can always shift the actual expansion point to 0 by change of variables) and assuming $f(x, y, y')$ is analytic at x_0 which must be the case for an ordinary point. Let initial conditions be $y(x_0) = y_0$ and $y'(x_0) = y_0'$. Using Taylor series gives

$$\begin{aligned} y(x) &= y(x_0) + (x - x_0)\, y'(x_0) + \frac{(x - x_0)^2}{2} y''(x_0) + \frac{(x - x_0)^3}{3!} y'''(x_0) + \cdots \\ &= y_0 + x y_0' + \frac{x^2}{2} f|_{x_0, y_0, y_0'} + \frac{x^3}{3!} f'|_{x_0, y_0, y_0'} + \cdots \\ &= y_0 + x y_0' + \sum_{n=0}^{\infty} \frac{x^{n+2}}{(n+2)!} \frac{d^n f}{dx^n}\bigg|_{x_0, y_0, y_0'} \end{aligned}$$

But

$$\begin{aligned}
\frac{df}{dx} &= \frac{\partial f}{\partial x}\frac{dx}{dx} + \frac{\partial f}{\partial y}\frac{dy}{dx} + \frac{\partial f}{\partial y'}\frac{dy'}{dx} && (1) \\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}y' + \frac{\partial f}{\partial y'}y'' && (822) \\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}y' + \frac{\partial f}{\partial y'}f && (823) \\
\frac{d^2 f}{dx^2} &= \frac{d}{dx}\left(\frac{df}{dx}\right) \\
&= \frac{\partial}{\partial x}\left(\frac{df}{dx}\right) + \frac{\partial}{\partial y}\left(\frac{df}{dx}\right)y' + \frac{\partial}{\partial y'}\left(\frac{df}{dx}\right)f && (2) \\
\frac{d^3 f}{dx^3} &= \frac{d}{dx}\left(\frac{d^2 f}{dx^2}\right) \\
&= \frac{\partial}{\partial x}\left(\frac{d^2 f}{dx^2}\right) + \left(\frac{\partial}{\partial y}\frac{d^2 f}{dx^2}\right)y' + \frac{\partial}{\partial y'}\left(\frac{d^2 f}{dx^2}\right)f && (3) \\
&\vdots
\end{aligned}$$

And so on. Hence if we name $F_0 = f(x,y,y')$ then the above can be written as

$$
\begin{aligned}
F_0 &= f(x,y,y') \qquad (4)\\
F_1 &= \frac{df}{dx}\\
&= \frac{dF_0}{dx}\\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} y' + \frac{\partial f}{\partial y'} y''\\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} y' + \frac{\partial f}{\partial y'} f \qquad (5)\\
&= \frac{\partial F_0}{\partial x} + \frac{\partial F_0}{\partial y} y' + \frac{\partial F_0}{\partial y'} F_0\\
F_2 &= \frac{d}{dx}\left(\frac{d}{dx} f\right)\\
&= \frac{d}{dx}(F_1)\\
&= \frac{\partial}{\partial x} F_1 + \left(\frac{\partial F_1}{\partial y}\right) y' + \left(\frac{\partial F_1}{\partial y'}\right) y''\\
&= \frac{\partial}{\partial x} F_1 + \left(\frac{\partial F_1}{\partial y}\right) y' + \left(\frac{\partial F_1}{\partial y'}\right) F_0\\
&\vdots\\
F_n &= \frac{d}{dx}(F_{n-1})\\
&= \frac{\partial}{\partial x} F_{n-1} + \left(\frac{\partial F_{n-1}}{\partial y}\right) y' + \left(\frac{\partial F_{n-1}}{\partial y'}\right) y''\\
&= \frac{\partial}{\partial x} F_{n-1} + \left(\frac{\partial F_{n-1}}{\partial y}\right) y' + \left(\frac{\partial F_{n-1}}{\partial y'}\right) F_0 \qquad (6)
\end{aligned}
$$

Therefore (6) can be used from now on along with

$$
y(x) = y_0 + x y_0' + \sum_{n=0}^{\infty} \frac{x^{n+2}}{(n+2)!} F_n|_{x_0,y_0,y_0'} \tag{7}
$$

To find $y(x)$ series solution around $x=0$. Hence

$$
\begin{aligned}
F_0 &= -4y' - 6yx \\
F_1 &= \frac{dF_0}{dx} \\
&= \frac{\partial F_0}{\partial x} + \frac{\partial F_0}{\partial y} y' + \frac{\partial F_0}{\partial y'} F_0 \\
&= (-6x + 16)\, y' + (24x - 6)\, y \\
F_2 &= \frac{dF_1}{dx} \\
&= \frac{\partial F_1}{\partial x} + \frac{\partial F_1}{\partial y} y' + \frac{\partial F_1}{\partial y'} F_1 \\
&= (48x - 76)\, y' + 36\left(x^2 - \frac{8}{3}x + \frac{2}{3}\right) y \\
F_3 &= \frac{dF_2}{dx} \\
&= \frac{\partial F_2}{\partial x} + \frac{\partial F_2}{\partial y} y' + \frac{\partial F_2}{\partial y'} F_2 \\
&= \left(36x^2 - 288x + 376\right) y' - 288\left(x^2 - \frac{11}{6}x + \frac{1}{3}\right) y \\
F_4 &= \frac{dF_3}{dx} \\
&= \frac{\partial F_3}{\partial x} + \frac{\partial F_3}{\partial y} y' + \frac{\partial F_3}{\partial y'} F_3 \\
&= \left(-432x^2 + 1752x - 1888\right) y' - 216\left(x^3 - 8x^2 + \frac{118}{9}x - \frac{22}{9}\right) y
\end{aligned}
$$

And so on. Evaluating all the above at initial conditions $x=0$ and $y(0)=y(0)$ and $y'(0)=y'(0)$ gives

$$
\begin{aligned}
F_0 &= -4y'(0) \\
F_1 &= 16y'(0) - 6y(0) \\
F_2 &= -76y'(0) + 24y(0) \\
F_3 &= 376y'(0) - 96y(0) \\
F_4 &= -1888y'(0) + 528y(0)
\end{aligned}
$$

Substituting all the above in (7) and simplifying gives the solution as

$$
\begin{aligned}
y = &\left(1 - x^3 + x^4 - \frac{4}{5}x^5 + \frac{11}{15}x^6\right) y(0) \\
&+ \left(x - 2x^2 + \frac{8}{3}x^3 - \frac{19}{6}x^4 + \frac{47}{15}x^5 - \frac{118}{45}x^6\right) y'(0) + O\left(x^6\right)
\end{aligned}
$$

Since the expansion point $x = 0$ is an ordinary, we can also solve this using standard power series Let the solution be represented as power series of the form

$$y = \sum_{n=0}^{\infty} a_n x^n$$

Then

$$y' = \sum_{n=1}^{\infty} n a_n x^{n-1}$$
$$y'' = \sum_{n=2}^{\infty} n(n-1)\, a_n x^{n-2}$$

Substituting the above back into the ode gives

$$\sum_{n=2}^{\infty} n(n-1)\, a_n x^{n-2} = -4\left(\sum_{n=1}^{\infty} n a_n x^{n-1}\right) - 6\left(\sum_{n=0}^{\infty} a_n x^n\right) x \tag{1}$$

Which simplifies to

$$\left(\sum_{n=2}^{\infty} n(n-1)\, a_n x^{n-2}\right) + \left(\sum_{n=1}^{\infty} 4n a_n x^{n-1}\right) + \left(\sum_{n=0}^{\infty} 6x^{1+n} a_n\right) = 0 \tag{2}$$

The next step is to make all powers of x be n in each summation term. Going over each summation term above with power of x in it which is not already x^n and adjusting the power and the corresponding index gives

$$\sum_{n=2}^{\infty} n(n-1)\, a_n x^{n-2} = \sum_{n=0}^{\infty} (n+2)\, a_{n+2}(1+n)\, x^n$$

$$\sum_{n=1}^{\infty} 4n a_n x^{n-1} = \sum_{n=0}^{\infty} 4(1+n)\, a_{1+n} x^n$$

$$\sum_{n=0}^{\infty} 6x^{1+n} a_n = \sum_{n=1}^{\infty} 6a_{n-1} x^n$$

Substituting all the above in Eq (2) gives the following equation where now all powers of x are the same and equal to n.

$$\left(\sum_{n=0}^{\infty} (n+2)\, a_{n+2}(1+n)\, x^n\right) + \left(\sum_{n=0}^{\infty} 4(1+n)\, a_{1+n} x^n\right) + \left(\sum_{n=1}^{\infty} 6a_{n-1} x^n\right) = 0 \tag{3}$$

$n = 0$ gives

$$2a_2 + 4a_1 = 0$$

$$a_2 = -2a_1$$

For $1 \leq n$, the recurrence equation is

$$(n+2)\, a_{n+2}(1+n) + 4(1+n)\, a_{1+n} + 6a_{n-1} = 0 \tag{4}$$

Solving for a_{n+2}, gives

$$\begin{aligned} a_{n+2} &= -\frac{2(2na_{1+n} + 2a_{1+n} + 3a_{n-1})}{(n+2)\,(1+n)} \\ &= -\frac{2(2n+2)\, a_{1+n}}{(n+2)\,(1+n)} - \frac{6a_{n-1}}{(n+2)\,(1+n)} \end{aligned} \tag{5}$$

For $n = 1$ the recurrence equation gives

$$6a_3 + 8a_2 + 6a_0 = 0$$

Which after substituting the earlier terms found becomes

$$a_3 = \frac{8a_1}{3} - a_0$$

For $n = 2$ the recurrence equation gives

$$12a_4 + 12a_3 + 6a_1 = 0$$

Which after substituting the earlier terms found becomes

$$a_4 = -\frac{19a_1}{6} + a_0$$

For $n = 3$ the recurrence equation gives

$$20a_5 + 16a_4 + 6a_2 = 0$$

Which after substituting the earlier terms found becomes

$$a_5 = \frac{47a_1}{15} - \frac{4a_0}{5}$$

For $n = 4$ the recurrence equation gives

$$30a_6 + 20a_5 + 6a_3 = 0$$

Which after substituting the earlier terms found becomes

$$a_6 = -\frac{118a_1}{45} + \frac{11a_0}{15}$$

For $n = 5$ the recurrence equation gives

$$42a_7 + 24a_6 + 6a_4 = 0$$

Which after substituting the earlier terms found becomes

$$a_7 = \frac{1229a_1}{630} - \frac{59a_0}{105}$$

And so on. Therefore the solution is

$$\begin{aligned} y &= \sum_{n=0}^{\infty} a_n x^n \\ &= a_3 x^3 + a_2 x^2 + a_1 x + a_0 + \dots \end{aligned}$$

Substituting the values for a_n found above, the solution becomes

$$y = a_0 + a_1 x - 2a_1 x^2 + \left(\frac{8a_1}{3} - a_0\right) x^3 + \left(-\frac{19a_1}{6} + a_0\right) x^4 + \left(\frac{47a_1}{15} - \frac{4a_0}{5}\right) x^5 + \dots$$

Collecting terms, the solution becomes

$$y = \left(1 - x^3 + x^4 - \frac{4}{5}x^5\right) a_0 + \left(x - 2x^2 + \frac{8}{3}x^3 - \frac{19}{6}x^4 + \frac{47}{15}x^5\right) a_1 + O(x^6) \quad (3)$$

At $x = 0$ the solution above becomes

$$y = \left(1 - x^3 + x^4 - \frac{4}{5}x^5\right) c_1 + \left(x - 2x^2 + \frac{8}{3}x^3 - \frac{19}{6}x^4 + \frac{47}{15}x^5\right) c_2 + O(x^6)$$

Summary
The solution(s) found are the following

$$
\begin{align*}
y &= \left(1-x^{3}+x^{4}-\frac{4}{5}x^{5}+\frac{11}{15}x^{6}\right) y(0) \\
&\quad+\left(x-2x^{2}+\frac{8}{3}x^{3}-\frac{19}{6}x^{4}+\frac{47}{15}x^{5}-\frac{118}{45}x^{6}\right) y'(0)+O\left(x^{6}\right) \tag{1}
\end{align*}
$$

$$
y = \left(1-x^{3}+x^{4}-\frac{4}{5}x^{5}\right) c_{1}+\left(x-2x^{2}+\frac{8}{3}x^{3}-\frac{19}{6}x^{4}+\frac{47}{15}x^{5}\right) c_{2}+O\left(x^{6}\right) \tag{2}
$$

Verification of solutions

$$
\begin{align*}
y &= \left(1-x^{3}+x^{4}-\frac{4}{5}x^{5}+\frac{11}{15}x^{6}\right) y(0) \\
&\quad+\left(x-2x^{2}+\frac{8}{3}x^{3}-\frac{19}{6}x^{4}+\frac{47}{15}x^{5}-\frac{118}{45}x^{6}\right) y'(0)+O\left(x^{6}\right)
\end{align*}
$$

Verified OK.

$$
y = \left(1-x^{3}+x^{4}-\frac{4}{5}x^{5}\right) c_{1}+\left(x-2x^{2}+\frac{8}{3}x^{3}-\frac{19}{6}x^{4}+\frac{47}{15}x^{5}\right) c_{2}+O\left(x^{6}\right)
$$

Verified OK.

14.5.1 Maple step by step solution

Let's solve

$y'' = -4y' - 6yx$

- Highest derivative means the order of the ODE is 2

  y''

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y'' + 4y' + 6yx = 0$

- Assume series solution for y

  $y = \sum_{k=0}^{\infty} a_k x^k$

□ Rewrite ODE with series expansions

  ◦ Convert $x \cdot y$ to series expansion

  $x \cdot y = \sum_{k=0}^{\infty} a_k x^{k+1}$

- Shift index using $k->k-1$

  $$x \cdot y = \sum_{k=1}^{\infty} a_{k-1} x^k$$

- Convert y' to series expansion

  $$y' = \sum_{k=1}^{\infty} a_k k \, x^{k-1}$$

- Shift index using $k->k+1$

  $$y' = \sum_{k=0}^{\infty} a_{k+1}(k+1) \, x^k$$

- Convert y'' to series expansion

  $$y'' = \sum_{k=2}^{\infty} a_k k(k-1) \, x^{k-2}$$

- Shift index using $k->k+2$

  $$y'' = \sum_{k=0}^{\infty} a_{k+2}(k+2)\,(k+1) \, x^k$$

  Rewrite ODE with series expansions

  $$2a_2 + 4a_1 + \left(\sum_{k=1}^{\infty} \left(a_{k+2}(k+2)\,(k+1) + 4a_{k+1}(k+1) + 6a_{k-1}\right) x^k\right) = 0$$

- Each term must be 0

  $$2a_2 + 4a_1 = 0$$

- Each term in the series must be 0, giving the recursion relation

  $$(k^2 + 3k + 2)\, a_{k+2} + 4a_{k+1}k + 6a_{k-1} + 4a_{k+1} = 0$$

- Shift index using $k->k+1$

  $$\left((k+1)^2 + 3k + 5\right) a_{k+3} + 4a_{k+2}(k+1) + 6a_k + 4a_{k+2} = 0$$

- Recursion relation that defines the series solution to the ODE

  $$\left[y = \sum_{k=0}^{\infty} a_k x^k, a_{k+3} = -\frac{2(2ka_{k+2} + 3a_k + 4a_{k+2})}{k^2+5k+6}, 2a_2 + 4a_1 = 0\right]$$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
<- No Liouvillian solutions exists
-> Trying a solution in terms of special functions:
   -> Bessel
   <- Bessel successful
<- special function solution successful`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 47

```
Order:=6;
dsolve(diff(y(x),x$2)+4*diff(y(x),x)+6*x*y(x)=0,y(x),type='series',x=0);
```

$$y(x) = \left(1 - x^3 + x^4 - \frac{4}{5}x^5\right) y(0) + \left(x - 2x^2 + \frac{8}{3}x^3 - \frac{19}{6}x^4 + \frac{47}{15}x^5\right) D(y)(0) + O(x^6)$$

✓ Solution by Mathematica

Time used: 0.001 (sec). Leaf size: 55

```
AsymptoticDSolveValue[y''[x]+4*y'[x]+6*x*y[x]==0,y[x],{x,0,5}]
```

$$y(x) \to c_1\left(-\frac{4x^5}{5} + x^4 - x^3 + 1\right) + c_2\left(\frac{47x^5}{15} - \frac{19x^4}{6} + \frac{8x^3}{3} - 2x^2 + x\right)$$

14.6 problem 5. case $x_0 = 4$

Internal problem ID [738]
Internal file name [`OUTPUT/738_Sunday_June_05_2022_01_48_23_AM_64748739/index.tex`]

Book: Elementary differential equations and boundary value problems, 10th ed., Boyce and DiPrima
Section: Chapter 5.3, Series Solutions Near an Ordinary Point, Part II. page 269
Problem number: 5. case $x_0 = 4$.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second order series method. Ordinary point", "second order series method. Taylor series method"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

$$\boxed{y'' + 4y' + 6yx = 0}$$

With the expansion point for the power series method at $x = 4$.

The ode does not have its expansion point at $x = 0$, therefore to simplify the computation of power series expansion, change of variable is made on the independent variable to shift the initial conditions and the expasion point back to zero. The new ode is then solved more easily since the expansion point is now at zero. The solution converted back to the original independent variable. Let

$$t = x - 4$$

The ode is converted to be in terms of the new independent variable t. This results in

$$\frac{d^2}{dt^2}y(t) + 4\frac{d}{dt}y(t) + 6y(t)\,(t + 4) = 0$$

With its expansion point and initial conditions now at $t = 0$. The transformed ODE is now solved. Solving ode using Taylor series method. This gives review on how the Taylor series method works for solving second order ode.

Let

$$y'' = f(x, y, y')$$

Assuming expansion is at $x_0 = 0$ (we can always shift the actual expansion point to 0 by change of variables) and assuming $f(x, y, y')$ is analytic at x_0 which must be the case for an ordinary point. Let initial conditions be $y(x_0) = y_0$ and $y'(x_0) = y_0'$. Using Taylor series gives

$$\begin{aligned}
y(x) &= y(x_0) + (x - x_0)\, y'(x_0) + \frac{(x - x_0)^2}{2} y''(x_0) + \frac{(x - x_0)^3}{3!} y'''(x_0) + \cdots \\
&= y_0 + x y_0' + \frac{x^2}{2} f|_{x_0, y_0, y_0'} + \frac{x^3}{3!} f'|_{x_0, y_0, y_0'} + \cdots \\
&= y_0 + x y_0' + \sum_{n=0}^{\infty} \frac{x^{n+2}}{(n+2)!} \left. \frac{d^n f}{dx^n} \right|_{x_0, y_0, y_0'}
\end{aligned}$$

But

$$\begin{aligned}
\frac{df}{dx} &= \frac{\partial f}{\partial x}\frac{dx}{dx} + \frac{\partial f}{\partial y}\frac{dy}{dx} + \frac{\partial f}{\partial y'}\frac{dy'}{dx} && (1) \\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} y' + \frac{\partial f}{\partial y'} y'' && (825) \\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} y' + \frac{\partial f}{\partial y'} f && (826) \\
\frac{d^2 f}{dx^2} &= \frac{d}{dx}\left(\frac{df}{dx}\right) \\
&= \frac{\partial}{\partial x}\left(\frac{df}{dx}\right) + \frac{\partial}{\partial y}\left(\frac{df}{dx}\right) y' + \frac{\partial}{\partial y'}\left(\frac{df}{dx}\right) f && (2) \\
\frac{d^3 f}{dx^3} &= \frac{d}{dx}\left(\frac{d^2 f}{dx^2}\right) \\
&= \frac{\partial}{\partial x}\left(\frac{d^2 f}{dx^2}\right) + \left(\frac{\partial}{\partial y}\frac{d^2 f}{dx^2}\right) y' + \frac{\partial}{\partial y'}\left(\frac{d^2 f}{dx^2}\right) f && (3) \\
&\vdots
\end{aligned}$$

And so on. Hence if we name $F_0 = f(x,y,y')$ then the above can be written as

$$
\begin{aligned}
F_0 &= f(x,y,y') && (4)\\
F_1 &= \frac{df}{dx}\\
&= \frac{dF_0}{dx}\\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}y' + \frac{\partial f}{\partial y'}y''\\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}y' + \frac{\partial f}{\partial y'}f && (5)\\
&= \frac{\partial F_0}{\partial x} + \frac{\partial F_0}{\partial y}y' + \frac{\partial F_0}{\partial y'}F_0\\
F_2 &= \frac{d}{dx}\left(\frac{d}{dx}f\right)\\
&= \frac{d}{dx}(F_1)\\
&= \frac{\partial}{\partial x}F_1 + \left(\frac{\partial F_1}{\partial y}\right)y' + \left(\frac{\partial F_1}{\partial y'}\right)y''\\
&= \frac{\partial}{\partial x}F_1 + \left(\frac{\partial F_1}{\partial y}\right)y' + \left(\frac{\partial F_1}{\partial y'}\right)F_0\\
&\vdots\\
F_n &= \frac{d}{dx}(F_{n-1})\\
&= \frac{\partial}{\partial x}F_{n-1} + \left(\frac{\partial F_{n-1}}{\partial y}\right)y' + \left(\frac{\partial F_{n-1}}{\partial y'}\right)y''\\
&= \frac{\partial}{\partial x}F_{n-1} + \left(\frac{\partial F_{n-1}}{\partial y}\right)y' + \left(\frac{\partial F_{n-1}}{\partial y'}\right)F_0 && (6)
\end{aligned}
$$

Therefore (6) can be used from now on along with

$$
y(x) = y_0 + xy_0' + \sum_{n=0}^{\infty} \frac{x^{n+2}}{(n+2)!} F_n|_{x_0,y_0,y_0'} \tag{7}
$$

To find $y(x)$ series solution around $x = 0$. Hence

$$
\begin{aligned}
F_0 &= -6y(t)\,t - 24y(t) - 4\frac{d}{dt}y(t) \\
F_1 &= \frac{dF_0}{dt} \\
&= \frac{\partial F_0}{\partial t} + \frac{\partial F_0}{\partial y}\frac{d}{dt}y(t) + \frac{\partial F_0}{\partial \frac{d}{dt}y(t)}F_0 \\
&= (-6t - 8)\left(\frac{d}{dt}y(t)\right) + (24t + 90)\,y(t) \\
F_2 &= \frac{dF_1}{dt} \\
&= \frac{\partial F_1}{\partial t} + \frac{\partial F_1}{\partial y}\frac{d}{dt}y(t) + \frac{\partial F_1}{\partial \frac{d}{dt}y(t)}F_1 \\
&= (48t + 116)\left(\frac{d}{dt}y(t)\right) + 36y(t)\left(t^2 + \frac{16}{3}t + 6\right) \\
F_3 &= \frac{dF_2}{dt} \\
&= \frac{\partial F_2}{\partial t} + \frac{\partial F_2}{\partial y}\frac{d}{dt}y(t) + \frac{\partial F_2}{\partial \frac{d}{dt}y(t)}F_2 \\
&= \left(36t^2 - 200\right)\left(\frac{d}{dt}y(t)\right) - 288\left(t^2 + \frac{37}{6}t + 9\right)y(t) \\
F_4 &= \frac{dF_3}{dt} \\
&= \frac{\partial F_3}{\partial t} + \frac{\partial F_3}{\partial y}\frac{d}{dt}y(t) + \frac{\partial F_3}{\partial \frac{d}{dt}y(t)}F_3 \\
&= \left(-432t^2 - 1704t - 1792\right)\left(\frac{d}{dt}y(t)\right) - 216\left(t^3 + 4t^2 - \frac{26}{9}t - 14\right)y(t)
\end{aligned}
$$

And so on. Evaluating all the above at initial conditions $t = 0$ and $y(0) = y(0)$ and $y'(0) = y'(0)$ gives

$$
\begin{aligned}
F_0 &= -24y(0) - 4y'(0) \\
F_1 &= -8y'(0) + 90y(0) \\
F_2 &= 116y'(0) + 216y(0) \\
F_3 &= -200y'(0) - 2592y(0) \\
F_4 &= -1792y'(0) + 3024y(0)
\end{aligned}
$$

Substituting all the above in (7) and simplifying gives the solution as

$$
\begin{aligned}
y(t) &= \left(1-12t^{2}+15t^{3}+9t^{4}-\frac{108}{5}t^{5}+\frac{21}{5}t^{6}\right)y(0) \\
&+\left(t-2t^{2}-\frac{4}{3}t^{3}+\frac{29}{6}t^{4}-\frac{5}{3}t^{5}-\frac{112}{45}t^{6}\right)y'(0)+O\left(t^{6}\right)
\end{aligned}
$$

Since the expansion point $t = 0$ is an ordinary, we can also solve this using standard power series Let the solution be represented as power series of the form

$$
y(t) = \sum_{n=0}^{\infty} a_n t^n
$$

Then

$$
\begin{aligned}
\frac{d}{dt}y(t) &= \sum_{n=1}^{\infty} n a_n t^{n-1} \\
\frac{d^2}{dt^2}y(t) &= \sum_{n=2}^{\infty} n(n-1)\, a_n t^{n-2}
\end{aligned}
$$

Substituting the above back into the ode gives

$$
\sum_{n=2}^{\infty} n(n-1)\, a_n t^{n-2} = -6\left(\sum_{n=0}^{\infty} a_n t^n\right)t - 24\left(\sum_{n=0}^{\infty} a_n t^n\right) - 4\left(\sum_{n=1}^{\infty} n a_n t^{n-1}\right) \tag{1}
$$

Which simplifies to

$$
\left(\sum_{n=2}^{\infty} n(n-1)\, a_n t^{n-2}\right) + \left(\sum_{n=1}^{\infty} 4n a_n t^{n-1}\right) + \left(\sum_{n=0}^{\infty} 6t^{1+n} a_n\right) + \left(\sum_{n=0}^{\infty} 24 a_n t^n\right) = 0 \tag{2}
$$

The next step is to make all powers of t be n in each summation term. Going over each summation term above with power of t in it which is not already t^n and adjusting the power and the corresponding index gives

$$
\begin{aligned}
\sum_{n=2}^{\infty} n(n-1)\, a_n t^{n-2} &= \sum_{n=0}^{\infty} (n+2)\, a_{n+2}(1+n)\, t^n \\
\sum_{n=1}^{\infty} 4n a_n t^{n-1} &= \sum_{n=0}^{\infty} 4(1+n)\, a_{1+n} t^n \\
\sum_{n=0}^{\infty} 6t^{1+n} a_n &= \sum_{n=1}^{\infty} 6a_{n-1} t^n
\end{aligned}
$$

Substituting all the above in Eq (2) gives the following equation where now all powers of t are the same and equal to n.

$$\left(\sum_{n=0}^{\infty} (n+2) a_{n+2}(1+n) t^n\right) + \left(\sum_{n=0}^{\infty} 4(1+n) a_{1+n} t^n\right) + \left(\sum_{n=1}^{\infty} 6a_{n-1} t^n\right) + \left(\sum_{n=0}^{\infty} 24a_n t^n\right) = 0 \tag{3}$$

$n = 0$ gives

$$2a_2 + 4a_1 + 24a_0 = 0$$

$$a_2 = -12a_0 - 2a_1$$

For $1 \leq n$, the recurrence equation is

$$(n+2) a_{n+2}(1+n) + 4(1+n) a_{1+n} + 6a_{n-1} + 24a_n = 0 \tag{4}$$

Solving for a_{n+2}, gives

$$\begin{aligned} a_{n+2} &= -\frac{2(2na_{1+n} + 12a_n + 2a_{1+n} + 3a_{n-1})}{(n+2)(1+n)} \\ &= -\frac{24a_n}{(n+2)(1+n)} - \frac{2(2n+2) a_{1+n}}{(n+2)(1+n)} - \frac{6a_{n-1}}{(n+2)(1+n)} \end{aligned} \tag{5}$$

For $n = 1$ the recurrence equation gives

$$6a_3 + 8a_2 + 6a_0 + 24a_1 = 0$$

Which after substituting the earlier terms found becomes

$$a_3 = 15a_0 - \frac{4a_1}{3}$$

For $n = 2$ the recurrence equation gives

$$12a_4 + 12a_3 + 6a_1 + 24a_2 = 0$$

Which after substituting the earlier terms found becomes

$$a_4 = 9a_0 + \frac{29a_1}{6}$$

For $n = 3$ the recurrence equation gives

$$20a_5 + 16a_4 + 6a_2 + 24a_3 = 0$$

Which after substituting the earlier terms found becomes

$$a_5 = -\frac{108a_0}{5} - \frac{5a_1}{3}$$

For $n = 4$ the recurrence equation gives

$$30a_6 + 20a_5 + 6a_3 + 24a_4 = 0$$

Which after substituting the earlier terms found becomes

$$a_6 = \frac{21a_0}{5} - \frac{112a_1}{45}$$

For $n = 5$ the recurrence equation gives

$$42a_7 + 24a_6 + 6a_4 + 24a_5 = 0$$

Which after substituting the earlier terms found becomes

$$a_7 = \frac{303a_0}{35} + \frac{1061a_1}{630}$$

And so on. Therefore the solution is

$$\begin{aligned} y(t) &= \sum_{n=0}^{\infty} a_n t^n \\ &= a_3 t^3 + a_2 t^2 + a_1 t + a_0 + \dots \end{aligned}$$

Substituting the values for a_n found above, the solution becomes

$$\begin{aligned} y(t) = a_0 + a_1 t + (-12a_0 - 2a_1)\, t^2 + \left(15a_0 - \frac{4a_1}{3}\right) t^3 \\ + \left(9a_0 + \frac{29a_1}{6}\right) t^4 + \left(-\frac{108a_0}{5} - \frac{5a_1}{3}\right) t^5 + \dots \end{aligned}$$

Collecting terms, the solution becomes

$$y(t) = \left(1-12t^2+15t^3+9t^4-\frac{108}{5}t^5\right)a_0+\left(t-2t^2-\frac{4}{3}t^3+\frac{29}{6}t^4-\frac{5}{3}t^5\right)a_1+O(t^6) \quad (3)$$

At $t=0$ the solution above becomes

$$y(t) = \left(1-12t^2+15t^3+9t^4-\frac{108}{5}t^5\right)c_1+\left(t-2t^2-\frac{4}{3}t^3+\frac{29}{6}t^4-\frac{5}{3}t^5\right)c_2+O(t^6)$$

Replacing t in the above with the original independent variable $x$susing $t = x-4$ results in

$$\begin{aligned} y = {} & \left(1-12(x-4)^2+15(x-4)^3+9(x-4)^4-\frac{108(x-4)^5}{5}+\frac{21(x-4)^6}{5}\right)y(4) \\ & + \left(x-4-2(x-4)^2-\frac{4(x-4)^3}{3}+\frac{29(x-4)^4}{6}-\frac{5(x-4)^5}{3}-\frac{112(x-4)^6}{45}\right)y'(4) \\ & + O\left((x-4)^6\right) \end{aligned}$$

Summary

The solution(s) found are the following

$$\begin{aligned} y = {} & \left(1-12(x-4)^2+15(x-4)^3+9(x-4)^4-\frac{108(x-4)^5}{5}+\frac{21(x-4)^6}{5}\right)y(4) \\ & + \left(x-4-2(x-4)^2-\frac{4(x-4)^3}{3}+\frac{29(x-4)^4}{6}-\frac{5(x-4)^5}{3}-\frac{112(x-4)^6}{45}\right)y'(4) \\ & + O\left((x-4)^6\right) \end{aligned} \quad (1)$$

Verification of solutions

$$\begin{aligned} y = {} & \left(1-12(x-4)^2+15(x-4)^3+9(x-4)^4-\frac{108(x-4)^5}{5}+\frac{21(x-4)^6}{5}\right)y(4) \\ & + \left(x-4-2(x-4)^2-\frac{4(x-4)^3}{3}+\frac{29(x-4)^4}{6}-\frac{5(x-4)^5}{3}-\frac{112(x-4)^6}{45}\right)y'(4) \\ & + O\left((x-4)^6\right) \end{aligned}$$

Verified OK.

14.6.1 Maple step by step solution

Let's solve

$y'' + 4y' + 6yx = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Assume series solution for y

  $y = \sum_{k=0}^{\infty} a_k x^k$

□ Rewrite ODE with series expansions

  - Convert $x \cdot y$ to series expansion

    $x \cdot y = \sum_{k=0}^{\infty} a_k x^{k+1}$

  - Shift index using $k- >k - 1$

    $x \cdot y = \sum_{k=1}^{\infty} a_{k-1} x^k$

  - Convert y' to series expansion

    $y' = \sum_{k=1}^{\infty} a_k k\, x^{k-1}$

  - Shift index using $k- >k + 1$

    $y' = \sum_{k=0}^{\infty} a_{k+1}(k+1)\, x^k$

  - Convert y'' to series expansion

    $y'' = \sum_{k=2}^{\infty} a_k k(k-1)\, x^{k-2}$

  - Shift index using $k- >k + 2$

    $y'' = \sum_{k=0}^{\infty} a_{k+2}(k+2)\,(k+1)\, x^k$

  Rewrite ODE with series expansions

  $$2a_2 + 4a_1 + \left(\sum_{k=1}^{\infty} \left(a_{k+2}(k+2)\,(k+1) + 4a_{k+1}(k+1) + 6a_{k-1}\right) x^k\right) = 0$$

- Each term must be 0

  $2a_2 + 4a_1 = 0$

- Each term in the series must be 0, giving the recursion relation

  $(k^2+3k+2)\,a_{k+2}+4a_{k+1}k+6a_{k-1}+4a_{k+1}=0$
- Shift index using $k->k+1$

  $\left((k+1)^2+3k+5\right)a_{k+3}+4a_{k+2}(k+1)+6a_k+4a_{k+2}=0$
- Recursion relation that defines the series solution to the ODE

  $\left[y=\sum_{k=0}^{\infty}a_k x^k, a_{k+3}=-\frac{2(2ka_{k+2}+3a_k+4a_{k+2})}{k^2+5k+6}, 2a_2+4a_1=0\right]$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
<- No Liouvillian solutions exists
-> Trying a solution in terms of special functions:
   -> Bessel
   <- Bessel successful
<- special function solution successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 54

```
Order:=6;
dsolve(diff(y(x),x$2)+4*diff(y(x),x)+6*x*y(x)=0,y(x),type='series',x=4);
```

$$y(x)=\left(1-12(x-4)^2+15(x-4)^3+9(x-4)^4-\frac{108(x-4)^5}{5}\right)y(4)+\left(x-4-2(x-4)^2-\frac{4(x-4)^3}{3}+\frac{29(x-4)^4}{6}-\frac{5(x-4)^5}{3}\right)D(y)(4)+O(x^6)$$

✓ Solution by Mathematica

Time used: 0.001 (sec). Leaf size: 79

```
AsymptoticDSolveValue[y''[x]+4*y'[x]+6*x*y[x]==0,y[x],{x,4,5}]
```

$$
\begin{aligned}
y(x) \to\ & c_1 \left(-\frac{108}{5}(x-4)^5 + 9(x-4)^4 + 15(x-4)^3 - 12(x-4)^2 + 1 \right) \\
& + c_2 \left(-\frac{5}{3}(x-4)^5 + \frac{29}{6}(x-4)^4 - \frac{4}{3}(x-4)^3 - 2(x-4)^2 + x - 4 \right)
\end{aligned}
$$

14.7 problem 6. case $x_0 = 0$

Internal problem ID [739]
Internal file name [`OUTPUT/739_Sunday_June_05_2022_01_48_24_AM_30076032/index.tex`]

Book: Elementary differential equations and boundary value problems, 10th ed., Boyce and DiPrima
Section: Chapter 5.3, Series Solutions Near an Ordinary Point, Part II. page 269
Problem number: 6. case $x_0 = 0$.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second order series method. Ordinary point", "second order series method. Taylor series method"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

$$\boxed{\left(x^2 - 2x - 3\right) y'' + y'x + 4y = 0}$$

With the expansion point for the power series method at $x = 0$.

Solving ode using Taylor series method. This gives review on how the Taylor series method works for solving second order ode.

Let

$$y'' = f(x, y, y')$$

Assuming expansion is at $x_0 = 0$ (we can always shift the actual expansion point to 0 by change of variables) and assuming $f(x, y, y')$ is analytic at x_0 which must be the case for an ordinary point. Let initial conditions be $y(x_0) = y_0$ and $y'(x_0) = y'_0$. Using Taylor series gives

$$\begin{aligned}
y(x) &= y(x_0) + (x - x_0)\, y'(x_0) + \frac{(x - x_0)^2}{2} y''(x_0) + \frac{(x - x_0)^3}{3!} y'''(x_0) + \cdots \\
&= y_0 + x y'_0 + \frac{x^2}{2} f|_{x_0, y_0, y'_0} + \frac{x^3}{3!} f'|_{x_0, y_0, y'_0} + \cdots \\
&= y_0 + x y'_0 + \sum_{n=0}^{\infty} \frac{x^{n+2}}{(n+2)\,!} \frac{d^n f}{dx^n}\bigg|_{x_0, y_0, y'_0}
\end{aligned}$$

But

$$\frac{df}{dx} = \frac{\partial f}{\partial x}\frac{dx}{dx} + \frac{\partial f}{\partial y}\frac{dy}{dx} + \frac{\partial f}{\partial y'}\frac{dy'}{dx} \tag{1}$$

$$= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}y' + \frac{\partial f}{\partial y'}y'' \tag{828}$$

$$= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}y' + \frac{\partial f}{\partial y'}f \tag{829}$$

$$\frac{d^2 f}{dx^2} = \frac{d}{dx}\left(\frac{df}{dx}\right)$$

$$= \frac{\partial}{\partial x}\left(\frac{df}{dx}\right) + \frac{\partial}{\partial y}\left(\frac{df}{dx}\right)y' + \frac{\partial}{\partial y'}\left(\frac{df}{dx}\right)f \tag{2}$$

$$\frac{d^3 f}{dx^3} = \frac{d}{dx}\left(\frac{d^2 f}{dx^2}\right)$$

$$= \frac{\partial}{\partial x}\left(\frac{d^2 f}{dx^2}\right) + \left(\frac{\partial}{\partial y}\frac{d^2 f}{dx^2}\right)y' + \frac{\partial}{\partial y'}\left(\frac{d^2 f}{dx^2}\right)f \tag{3}$$

$$\vdots$$

And so on. Hence if we name $F_0 = f(x,y,y')$ then the above can be written as

$$
\begin{aligned}
F_0 &= f(x,y,y') && (4)\\
F_1 &= \frac{df}{dx}\\
&= \frac{dF_0}{dx}\\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}y' + \frac{\partial f}{\partial y'}y''\\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}y' + \frac{\partial f}{\partial y'}f && (5)\\
&= \frac{\partial F_0}{\partial x} + \frac{\partial F_0}{\partial y}y' + \frac{\partial F_0}{\partial y'}F_0\\
F_2 &= \frac{d}{dx}\left(\frac{d}{dx}f\right)\\
&= \frac{d}{dx}(F_1)\\
&= \frac{\partial}{\partial x}F_1 + \left(\frac{\partial F_1}{\partial y}\right)y' + \left(\frac{\partial F_1}{\partial y'}\right)y''\\
&= \frac{\partial}{\partial x}F_1 + \left(\frac{\partial F_1}{\partial y}\right)y' + \left(\frac{\partial F_1}{\partial y'}\right)F_0\\
&\vdots\\
F_n &= \frac{d}{dx}(F_{n-1})\\
&= \frac{\partial}{\partial x}F_{n-1} + \left(\frac{\partial F_{n-1}}{\partial y}\right)y' + \left(\frac{\partial F_{n-1}}{\partial y'}\right)y''\\
&= \frac{\partial}{\partial x}F_{n-1} + \left(\frac{\partial F_{n-1}}{\partial y}\right)y' + \left(\frac{\partial F_{n-1}}{\partial y'}\right)F_0 && (6)
\end{aligned}
$$

Therefore (6) can be used from now on along with

$$
y(x) = y_0 + xy'_0 + \sum_{n=0}^{\infty} \frac{x^{n+2}}{(n+2)!} F_n|_{x_0,y_0,y'_0} \tag{7}
$$

To find $y(x)$ series solution around $x=0$. Hence

$$
\begin{aligned}
F_0 &= -\frac{y'x+4y}{x^2-2x-3} \\
F_1 &= \frac{dF_0}{dx} \\
&= \frac{\partial F_0}{\partial x}+\frac{\partial F_0}{\partial y}y'+\frac{\partial F_0}{\partial y'}F_0 \\
&= \frac{(-2x^2+8x+15)\,y'+(12x-8)\,y}{(x^2-2x-3)^2} \\
F_2 &= \frac{dF_1}{dx} \\
&= \frac{\partial F_1}{\partial x}+\frac{\partial F_1}{\partial y}y'+\frac{\partial F_1}{\partial y'}F_1 \\
&= \frac{(18x^3-64x^2-67x+60)\,y'-28y\left(x^2-\frac{6}{7}x+\frac{32}{7}\right)}{(x^2-2x-3)^3} \\
F_3 &= \frac{dF_2}{dx} \\
&= \frac{\partial F_2}{\partial x}+\frac{\partial F_2}{\partial y}y'+\frac{\partial F_2}{\partial y'}F_2 \\
&= \frac{(-100x^4+400x^3+20x^2-120x+945)\,y'+40\left(x^3+2x^2+\frac{65}{2}x-27\right)y}{(x^2-2x-3)^4} \\
F_4 &= \frac{dF_3}{dx} \\
&= \frac{\partial F_3}{\partial x}+\frac{\partial F_3}{\partial y}y'+\frac{\partial F_3}{\partial y'}F_3 \\
&= \frac{(540x^5-2400x^4+2880x^3-6480x^2-11085x+11160)\,y'+200y\left(x^4-10x^3-\frac{461}{10}x^2+\frac{411}{5}x-\frac{408}{5}\right)}{(x^2-2x-3)^5}
\end{aligned}
$$

And so on. Evaluating all the above at initial conditions $x=0$ and $y(0)=y(0)$ and $y'(0)=y'(0)$ gives

$$
\begin{aligned}
F_0 &= \frac{4y(0)}{3} \\
F_1 &= -\frac{8y(0)}{9}+\frac{5y'(0)}{3} \\
F_2 &= \frac{128y(0)}{27}-\frac{20y'(0)}{9} \\
F_3 &= -\frac{40y(0)}{3}+\frac{35y'(0)}{3} \\
F_4 &= \frac{5440y(0)}{81}-\frac{1240y'(0)}{27}
\end{aligned}
$$

Substituting all the above in (7) and simplifying gives the solution as

$$y = \left(1+\frac{2}{3}x^2-\frac{4}{27}x^3+\frac{16}{81}x^4-\frac{1}{9}x^5+\frac{68}{729}x^6\right)y(0) + \left(x+\frac{5}{18}x^3-\frac{5}{54}x^4+\frac{7}{72}x^5-\frac{31}{486}x^6\right)y'(0)+O(x^6)$$

Since the expansion point $x = 0$ is an ordinary, we can also solve this using standard power series The ode is normalized to be

$$(x^2-2x-3)\,y''+y'x+4y=0$$

Let the solution be represented as power series of the form

$$y=\sum_{n=0}^{\infty} a_n x^n$$

Then

$$y' = \sum_{n=1}^{\infty} n a_n x^{n-1}$$
$$y'' = \sum_{n=2}^{\infty} n(n-1)\,a_n x^{n-2}$$

Substituting the above back into the ode gives

$$(x^2-2x-3)\left(\sum_{n=2}^{\infty} n(n-1)\,a_n x^{n-2}\right)+\left(\sum_{n=1}^{\infty} n a_n x^{n-1}\right)x+4\left(\sum_{n=0}^{\infty} a_n x^n\right)=0 \quad (1)$$

Which simplifies to

$$\left(\sum_{n=2}^{\infty} x^n a_n n(n-1)\right)+\sum_{n=2}^{\infty}\left(-2n\,x^{n-1}a_n(n-1)\right) + \sum_{n=2}^{\infty}\left(-3n(n-1)\,a_n x^{n-2}\right)+\left(\sum_{n=1}^{\infty} n a_n x^n\right)+\left(\sum_{n=0}^{\infty} 4a_n x^n\right)=0 \quad (2)$$

The next step is to make all powers of x be n in each summation term. Going over each summation term above with power of x in it which is not already x^n and adjusting the power and the corresponding index gives

$$\sum_{n=2}^{\infty} \left(-2n\, x^{n-1} a_n (n-1)\right) = \sum_{n=1}^{\infty} \left(-2(n+1)\, a_{n+1} n\, x^n\right)$$

$$\sum_{n=2}^{\infty} \left(-3n(n-1)\, a_n x^{n-2}\right) = \sum_{n=0}^{\infty} \left(-3(n+2)\, a_{n+2} (n+1)\, x^n\right)$$

Substituting all the above in Eq (2) gives the following equation where now all powers of x are the same and equal to n.

$$\left(\sum_{n=2}^{\infty} x^n a_n n(n-1)\right) + \sum_{n=1}^{\infty} \left(-2(n+1)\, a_{n+1} n\, x^n\right) + \sum_{n=0}^{\infty} \left(-3(n+2)\, a_{n+2}(n+1)\, x^n\right) + \left(\sum_{n=1}^{\infty} n a_n x^n\right) + \left(\sum_{n=0}^{\infty} 4 a_n x^n\right) = 0 \tag{3}$$

$n = 0$ gives

$$-6a_2 + 4a_0 = 0$$

$$a_2 = \frac{2a_0}{3}$$

$n = 1$ gives

$$-4a_2 - 18a_3 + 5a_1 = 0$$

Which after substituting earlier equations, simplifies to

$$a_3 = -\frac{4a_0}{27} + \frac{5a_1}{18}$$

For $2 \leq n$, the recurrence equation is

$$n a_n (n-1) - 2(n+1)\, a_{n+1} n - 3(n+2)\, a_{n+2}(n+1) + n a_n + 4a_n = 0 \tag{4}$$

Solving for a_{n+2}, gives

$$\begin{aligned} a_{n+2} &= \frac{n^2 a_n - 2n^2 a_{n+1} - 2n a_{n+1} + 4a_n}{3\,(n+2)\,(n+1)} \\ &= \frac{(n^2+4)\, a_n}{3\,(n+2)\,(n+1)} + \frac{(-2n^2-2n)\, a_{n+1}}{3\,(n+2)\,(n+1)} \end{aligned} \tag{5}$$

For $n = 2$ the recurrence equation gives

$$8a_2 - 12a_3 - 36a_4 = 0$$

Which after substituting the earlier terms found becomes

$$a_4 = \frac{16a_0}{81} - \frac{5a_1}{54}$$

For $n = 3$ the recurrence equation gives

$$13a_3 - 24a_4 - 60a_5 = 0$$

Which after substituting the earlier terms found becomes

$$a_5 = -\frac{a_0}{9} + \frac{7a_1}{72}$$

For $n = 4$ the recurrence equation gives

$$20a_4 - 40a_5 - 90a_6 = 0$$

Which after substituting the earlier terms found becomes

$$a_6 = \frac{68a_0}{729} - \frac{31a_1}{486}$$

For $n = 5$ the recurrence equation gives

$$29a_5 - 60a_6 - 126a_7 = 0$$

Which after substituting the earlier terms found becomes

$$a_7 = -\frac{2143a_0}{30618} + \frac{4307a_1}{81648}$$

And so on. Therefore the solution is

$$\begin{aligned} y &= \sum_{n=0}^{\infty} a_n x^n \\ &= a_3x^3 + a_2x^2 + a_1x + a_0 + \dots \end{aligned}$$

Substituting the values for a_n found above, the solution becomes

$$y = a_0 + a_1 x + \frac{2a_0 x^2}{3} + \left(-\frac{4a_0}{27} + \frac{5a_1}{18}\right)x^3 + \left(\frac{16a_0}{81} - \frac{5a_1}{54}\right)x^4 + \left(-\frac{a_0}{9} + \frac{7a_1}{72}\right)x^5 + \dots$$

Collecting terms, the solution becomes

$$y = \left(1 + \frac{2}{3}x^2 - \frac{4}{27}x^3 + \frac{16}{81}x^4 - \frac{1}{9}x^5\right)a_0 + \left(x + \frac{5}{18}x^3 - \frac{5}{54}x^4 + \frac{7}{72}x^5\right)a_1 + O(x^6) \quad (3)$$

At $x = 0$ the solution above becomes

$$y = \left(1 + \frac{2}{3}x^2 - \frac{4}{27}x^3 + \frac{16}{81}x^4 - \frac{1}{9}x^5\right)c_1 + \left(x + \frac{5}{18}x^3 - \frac{5}{54}x^4 + \frac{7}{72}x^5\right)c_2 + O(x^6)$$

Summary

The solution(s) found are the following

$$y = \left(1 + \frac{2}{3}x^2 - \frac{4}{27}x^3 + \frac{16}{81}x^4 - \frac{1}{9}x^5 + \frac{68}{729}x^6\right)y(0) + \left(x + \frac{5}{18}x^3 - \frac{5}{54}x^4 + \frac{7}{72}x^5 - \frac{31}{486}x^6\right)y'(0) + O(x^6) \quad (1)$$

$$y = \left(1 + \frac{2}{3}x^2 - \frac{4}{27}x^3 + \frac{16}{81}x^4 - \frac{1}{9}x^5\right)c_1 + \left(x + \frac{5}{18}x^3 - \frac{5}{54}x^4 + \frac{7}{72}x^5\right)c_2 + O(x^6) \quad (2)$$

Verification of solutions

$$y = \left(1 + \frac{2}{3}x^2 - \frac{4}{27}x^3 + \frac{16}{81}x^4 - \frac{1}{9}x^5 + \frac{68}{729}x^6\right)y(0) + \left(x + \frac{5}{18}x^3 - \frac{5}{54}x^4 + \frac{7}{72}x^5 - \frac{31}{486}x^6\right)y'(0) + O(x^6)$$

Verified OK.

$$y = \left(1 + \frac{2}{3}x^2 - \frac{4}{27}x^3 + \frac{16}{81}x^4 - \frac{1}{9}x^5\right)c_1 + \left(x + \frac{5}{18}x^3 - \frac{5}{54}x^4 + \frac{7}{72}x^5\right)c_2 + O(x^6)$$

Verified OK.

14.7.1 Maple step by step solution

Let's solve

$(x^2-2x-3)\,y''+y'x+4y=0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y''=-\frac{4y}{x^2-2x-3}-\frac{xy'}{x^2-2x-3}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y''+\frac{xy'}{x^2-2x-3}+\frac{4y}{x^2-2x-3}=0$

□ Check to see if x_0 is a regular singular point

  - Define functions

    $\left[P_2(x)=\frac{x}{x^2-2x-3}, P_3(x)=\frac{4}{x^2-2x-3}\right]$

  - $(x+1)\cdot P_2(x)$ is analytic at $x=-1$

    $\left((x+1)\cdot P_2(x)\right)\Big|_{x=-1}=\frac{1}{4}$

  - $(x+1)^2\cdot P_3(x)$ is analytic at $x=-1$

    $\left((x+1)^2\cdot P_3(x)\right)\Big|_{x=-1}=0$

  - $x=-1$is a regular singular point

  Check to see if x_0 is a regular singular point

  $x_0=-1$

- Multiply by denominators

  $(x^2-2x-3)\,y''+y'x+4y=0$

- Change variables using $x=u-1$ so that the regular singular point is at $u=0$

  $(u^2-4u)\left(\frac{d^2}{du^2}y(u)\right)+(u-1)\left(\frac{d}{du}y(u)\right)+4y(u)=0$

- Assume series solution for $y(u)$

  $y(u)=\sum_{k=0}^{\infty} a_k u^{k+r}$

□ Rewrite ODE with series expansions

  - Convert $u^m\cdot\left(\frac{d}{du}y(u)\right)$ to series expansion for $m=0..1$

$$u^m \cdot \left(\frac{d}{du}y(u)\right) = \sum_{k=0}^{\infty} a_k(k+r)\, u^{k+r-1+m}$$

- Shift index using $k- >k+1-m$

$$u^m \cdot \left(\frac{d}{du}y(u)\right) = \sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\, u^{k+r}$$

- Convert $u^m \cdot \left(\frac{d^2}{du^2}y(u)\right)$ to series expansion for $m = 1..2$

$$u^m \cdot \left(\frac{d^2}{du^2}y(u)\right) = \sum_{k=0}^{\infty} a_k(k+r)\,(k+r-1)\, u^{k+r-2+m}$$

- Shift index using $k- >k+2-m$

$$u^m \cdot \left(\frac{d^2}{du^2}y(u)\right) = \sum_{k=-2+m}^{\infty} a_{k+2-m}(k+2-m+r)\,(k+1-m+r)\, u^{k+r}$$

Rewrite ODE with series expansions

$$-a_0r(-3+4r)\, u^{-1+r} + \left(\sum_{k=0}^{\infty} \left(-a_{k+1}(k+1+r)\,(4k+1+4r) + a_k(k^2+2kr+r^2+4)\right) u^{k+r}\right) =$$

- $a_0$cannot be 0 by assumption, giving the indicial equation

$$-r(-3+4r) = 0$$

- Values of r that satisfy the indicial equation

$$r \in \left\{0, \frac{3}{4}\right\}$$

- Each term in the series must be 0, giving the recursion relation

$$-4\left(k+\frac{1}{4}+r\right)(k+1+r)\, a_{k+1} + a_k(k^2+2kr+r^2+4) = 0$$

- Recursion relation that defines series solution to ODE

$$a_{k+1} = \frac{a_k\,(k^2+2kr+r^2+4)}{(4k+1+4r)(k+1+r)}$$

- Recursion relation for $r = 0$

$$a_{k+1} = \frac{a_k\,(k^2+4)}{(4k+1)(k+1)}$$

- Solution for $r = 0$

$$\left[y(u) = \sum_{k=0}^{\infty} a_k u^k, a_{k+1} = \frac{a_k\,(k^2+4)}{(4k+1)(k+1)}\right]$$

- Revert the change of variables $u = x+1$

$$\left[y = \sum_{k=0}^{\infty} a_k(x+1)^k, a_{k+1} = \frac{a_k\,(k^2+4)}{(4k+1)(k+1)}\right]$$

- Recursion relation for $r = \frac{3}{4}$

$$a_{k+1} = \frac{a_k \left(k^2+\frac{3}{2}k+\frac{73}{16}\right)}{(4k+4)\left(k+\frac{7}{4}\right)}$$

- Solution for $r = \frac{3}{4}$

$$\left[y(u) = \sum_{k=0}^{\infty} a_k u^{k+\frac{3}{4}}, a_{k+1} = \frac{a_k \left(k^2+\frac{3}{2}k+\frac{73}{16}\right)}{(4k+4)\left(k+\frac{7}{4}\right)}\right]$$

- Revert the change of variables $u = x + 1$

$$\left[y = \sum_{k=0}^{\infty} a_k (x+1)^{k+\frac{3}{4}}, a_{k+1} = \frac{a_k \left(k^2+\frac{3}{2}k+\frac{73}{16}\right)}{(4k+4)\left(k+\frac{7}{4}\right)}\right]$$

- Combine solutions and rename parameters

$$\left[y = \left(\sum_{k=0}^{\infty} a_k (x+1)^k\right) + \left(\sum_{k=0}^{\infty} b_k (x+1)^{k+\frac{3}{4}}\right), a_{k+1} = \frac{a_k (k^2+4)}{(4k+1)(k+1)}, b_{k+1} = \frac{b_k \left(k^2+\frac{3}{2}k+\frac{73}{16}\right)}{(4k+4)\left(k+\frac{7}{4}\right)}\right]$$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
<- No Liouvillian solutions exists
-> Trying a solution in terms of special functions:
   -> Bessel
   -> elliptic
   -> Legendre
   -> Kummer
      -> hyper3: Equivalence to 1F1 under a power @ Moebius
   -> hypergeometric
      -> heuristic approach
      <- heuristic approach successful
   <- hypergeometric successful
<- special function solution successful`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 49

```
Order:=6;
dsolve((x^2-2*x-3)*diff(y(x),x$2)+x*diff(y(x),x)+4*y(x)=0,y(x),type='series',x=0);
```

$$y(x) = \left(1+\frac{2}{3}x^2-\frac{4}{27}x^3+\frac{16}{81}x^4-\frac{1}{9}x^5\right)y(0) + \left(x+\frac{5}{18}x^3-\frac{5}{54}x^4+\frac{7}{72}x^5\right)D(y)(0)+O(x^6)$$

✓ Solution by Mathematica

Time used: 0.001 (sec). Leaf size: 63

```
AsymptoticDSolveValue[(x^2-2*x-3)*y''[x]+x*y'[x]+4*y[x]==0,y[x],{x,0,5}]
```

$$y(x) \to c_2\left(\frac{7x^5}{72}-\frac{5x^4}{54}+\frac{5x^3}{18}+x\right)+c_1\left(-\frac{x^5}{9}+\frac{16x^4}{81}-\frac{4x^3}{27}+\frac{2x^2}{3}+1\right)$$

14.8 problem 6. case $x_0 = 4$ only

Internal problem ID [740]
Internal file name [OUTPUT/740_Sunday_June_05_2022_01_48_25_AM_34591398/index.tex]

Book: Elementary differential equations and boundary value problems, 10th ed., Boyce and DiPrima
Section: Chapter 5.3, Series Solutions Near an Ordinary Point, Part II. page 269
Problem number: 6. case $x_0 = 4$ only.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second order series method. Ordinary point", "second order series method. Taylor series method"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

$$\boxed{(x^2 - 2x - 3)\, y'' + y'x + 4y = 0}$$

With the expansion point for the power series method at $x = 4$.

The ode does not have its expansion point at $x = 0$, therefore to simplify the computation of power series expansion, change of variable is made on the independent variable to shift the initial conditions and the expasion point back to zero. The new ode is then solved more easily since the expansion point is now at zero. The solution converted back to the original independent variable. Let

$$t = x - 4$$

The ode is converted to be in terms of the new independent variable t. This results in

$$\left((t+4)^2 - 2t - 11\right)\left(\frac{d^2}{dt^2}y(t)\right) + \left(\frac{d}{dt}y(t)\right)(t+4) + 4y(t) = 0$$

With its expansion point and initial conditions now at $t = 0$. The transformed ODE is now solved. Solving ode using Taylor series method. This gives review on how the Taylor series method works for solving second order ode.

Let

$$y'' = f(x, y, y')$$

Assuming expansion is at $x_0 = 0$ (we can always shift the actual expansion point to 0 by change of variables) and assuming $f(x, y, y')$ is analytic at x_0 which must be the case for an ordinary point. Let initial conditions be $y(x_0) = y_0$ and $y'(x_0) = y'_0$. Using Taylor series gives

$$\begin{aligned}
y(x) &= y(x_0) + (x - x_0)\, y'(x_0) + \frac{(x - x_0)^2}{2} y''(x_0) + \frac{(x - x_0)^3}{3!} y'''(x_0) + \cdots \\
&= y_0 + x y'_0 + \frac{x^2}{2} f|_{x_0, y_0, y'_0} + \frac{x^3}{3!} f'|_{x_0, y_0, y'_0} + \cdots \\
&= y_0 + x y'_0 + \sum_{n=0}^{\infty} \frac{x^{n+2}}{(n+2)\,!} \frac{d^n f}{dx^n}\bigg|_{x_0, y_0, y'_0}
\end{aligned}$$

But

$$\begin{aligned}
\frac{df}{dx} &= \frac{\partial f}{\partial x}\frac{dx}{dx} + \frac{\partial f}{\partial y}\frac{dy}{dx} + \frac{\partial f}{\partial y'}\frac{dy'}{dx} && (1) \\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} y' + \frac{\partial f}{\partial y'} y'' && (831) \\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} y' + \frac{\partial f}{\partial y'} f && (832) \\
\frac{d^2 f}{dx^2} &= \frac{d}{dx}\left(\frac{df}{dx}\right) \\
&= \frac{\partial}{\partial x}\left(\frac{df}{dx}\right) + \frac{\partial}{\partial y}\left(\frac{df}{dx}\right) y' + \frac{\partial}{\partial y'}\left(\frac{df}{dx}\right) f && (2) \\
\frac{d^3 f}{dx^3} &= \frac{d}{dx}\left(\frac{d^2 f}{dx^2}\right) \\
&= \frac{\partial}{\partial x}\left(\frac{d^2 f}{dx^2}\right) + \left(\frac{\partial}{\partial y}\frac{d^2 f}{dx^2}\right) y' + \frac{\partial}{\partial y'}\left(\frac{d^2 f}{dx^2}\right) f && (3) \\
&\vdots
\end{aligned}$$

And so on. Hence if we name $F_0 = f(x,y,y')$ then the above can be written as

$$\begin{aligned}
F_0 &= f(x,y,y') \qquad (4)\\
F_1 &= \frac{df}{dx}\\
&= \frac{dF_0}{dx}\\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}y' + \frac{\partial f}{\partial y'}y''\\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}y' + \frac{\partial f}{\partial y'}f \qquad (5)\\
&= \frac{\partial F_0}{\partial x} + \frac{\partial F_0}{\partial y}y' + \frac{\partial F_0}{\partial y'}F_0\\
F_2 &= \frac{d}{dx}\left(\frac{d}{dx}f\right)\\
&= \frac{d}{dx}(F_1)\\
&= \frac{\partial}{\partial x}F_1 + \left(\frac{\partial F_1}{\partial y}\right)y' + \left(\frac{\partial F_1}{\partial y'}\right)y''\\
&= \frac{\partial}{\partial x}F_1 + \left(\frac{\partial F_1}{\partial y}\right)y' + \left(\frac{\partial F_1}{\partial y'}\right)F_0\\
&\vdots\\
F_n &= \frac{d}{dx}(F_{n-1})\\
&= \frac{\partial}{\partial x}F_{n-1} + \left(\frac{\partial F_{n-1}}{\partial y}\right)y' + \left(\frac{\partial F_{n-1}}{\partial y'}\right)y''\\
&= \frac{\partial}{\partial x}F_{n-1} + \left(\frac{\partial F_{n-1}}{\partial y}\right)y' + \left(\frac{\partial F_{n-1}}{\partial y'}\right)F_0 \qquad (6)
\end{aligned}$$

Therefore (6) can be used from now on along with

$$y(x) = y_0 + xy_0' + \sum_{n=0}^{\infty} \frac{x^{n+2}}{(n+2)\,!} F_n|_{x_0,y_0,y_0'} \qquad (7)$$

To find $y(x)$ series solution around $x = 0$. Hence

$$
\begin{aligned}
F_0 &= -\frac{t\left(\frac{d}{dt}y(t)\right) + 4\frac{d}{dt}y(t) + 4y(t)}{t^2+6t+5} \\
F_1 &= \frac{dF_0}{dt} \\
&= \frac{\partial F_0}{\partial t} + \frac{\partial F_0}{\partial y}\frac{d}{dt}y(t) + \frac{\partial F_0}{\partial \frac{d}{dt}y(t)}F_0 \\
&= \frac{(-2t^2-8t+15)\left(\frac{d}{dt}y(t)\right) + (12t+40)\,y(t)}{(t^2+6t+5)^2} \\
F_2 &= \frac{dF_1}{dt} \\
&= \frac{\partial F_1}{\partial t} + \frac{\partial F_1}{\partial y}\frac{d}{dt}y(t) + \frac{\partial F_1}{\partial \frac{d}{dt}y(t)}F_1 \\
&= \frac{(18t^3+152t^2+285t-80)\left(\frac{d}{dt}y(t)\right) - 28\left(t^2+\frac{50}{7}t+\frac{120}{7}\right)y(t)}{(t^2+6t+5)^3} \\
F_3 &= \frac{dF_2}{dt} \\
&= \frac{\partial F_2}{\partial t} + \frac{\partial F_2}{\partial y}\frac{d}{dt}y(t) + \frac{\partial F_2}{\partial \frac{d}{dt}y(t)}F_2 \\
&= \frac{(-100t^4-1200t^3-4780t^2-6360t+785)\left(\frac{d}{dt}y(t)\right) + 40\left(t^3+14t^2+\frac{193}{2}t+199\right)y(t)}{(t^2+6t+5)^4} \\
F_4 &= \frac{dF_3}{dt} \\
&= \frac{\partial F_3}{\partial t} + \frac{\partial F_3}{\partial y}\frac{d}{dt}y(t) + \frac{\partial F_3}{\partial \frac{d}{dt}y(t)}F_3 \\
&= \frac{(540t^5+8400t^4+50880t^3+143280t^2+152115t-13980)\left(\frac{d}{dt}y(t)\right) + 200\left(t^4+6t^3-\frac{701}{10}t^2-\frac{2553}{5}t - \right.}{(t^2+6t+5)^5}
\end{aligned}
$$

And so on. Evaluating all the above at initial conditions $t = 0$ and $y(0) = y(0)$ and

$y'(0) = y'(0)$ gives

$$\begin{aligned}
F_0 &= -\frac{4y(0)}{5} - \frac{4y'(0)}{5} \\
F_1 &= \frac{8y(0)}{5} + \frac{3y'(0)}{5} \\
F_2 &= -\frac{96y(0)}{25} - \frac{16y'(0)}{25} \\
F_3 &= \frac{1592y(0)}{125} + \frac{157y'(0)}{125} \\
F_4 &= -\frac{34976y(0)}{625} - \frac{2796y'(0)}{625}
\end{aligned}$$

Substituting all the above in (7) and simplifying gives the solution as

$$\begin{aligned}
y(t) = &\left(1 - \frac{2}{5}t^2 + \frac{4}{15}t^3 - \frac{4}{25}t^4 + \frac{199}{1875}t^5 - \frac{2186}{28125}t^6\right) y(0) \\
&+ \left(t - \frac{2}{5}t^2 + \frac{1}{10}t^3 - \frac{2}{75}t^4 + \frac{157}{15000}t^5 - \frac{233}{37500}t^6\right) y'(0) + O(t^6)
\end{aligned}$$

Since the expansion point $t = 0$ is an ordinary, we can also solve this using standard power series The ode is normalized to be

$$\left(\frac{d^2}{dt^2}y(t)\right)(t^2 + 6t + 5) + \left(\frac{d}{dt}y(t)\right)(t + 4) + 4y(t) = 0$$

Let the solution be represented as power series of the form

$$y(t) = \sum_{n=0}^{\infty} a_n t^n$$

Then

$$\begin{aligned}
\frac{d}{dt}y(t) &= \sum_{n=1}^{\infty} n a_n t^{n-1} \\
\frac{d^2}{dt^2}y(t) &= \sum_{n=2}^{\infty} n(n-1)\, a_n t^{n-2}
\end{aligned}$$

Substituting the above back into the ode gives

$$\left(\sum_{n=2}^{\infty} n(n-1)\, a_n t^{n-2}\right)(t^2 + 6t + 5) + \left(\sum_{n=1}^{\infty} n a_n t^{n-1}\right)(t + 4) + 4\left(\sum_{n=0}^{\infty} a_n t^n\right) = 0 \quad (1)$$

Which simplifies to

$$
\begin{gathered}
\left(\sum_{n=2}^{\infty} t^{n} a_{n} n(n-1)\right)+\left(\sum_{n=2}^{\infty} 6 n\, t^{n-1} a_{n}(n-1)\right)+\left(\sum_{n=2}^{\infty} 5 n(n-1)\, a_{n} t^{n-2}\right) \\
+\left(\sum_{n=1}^{\infty} n a_{n} t^{n}\right)+\left(\sum_{n=1}^{\infty} 4 n a_{n} t^{n-1}\right)+\left(\sum_{n=0}^{\infty} 4 a_{n} t^{n}\right)=0
\end{gathered} \tag{2}
$$

The next step is to make all powers of t be n in each summation term. Going over each summation term above with power of t in it which is not already t^n and adjusting the power and the corresponding index gives

$$
\begin{aligned}
\sum_{n=2}^{\infty} 6 n\, t^{n-1} a_{n}(n-1) &= \sum_{n=1}^{\infty} 6(n+1)\, a_{n+1} n\, t^{n} \\
\sum_{n=2}^{\infty} 5 n(n-1)\, a_{n} t^{n-2} &= \sum_{n=0}^{\infty} 5(n+2)\, a_{n+2}(n+1)\, t^{n} \\
\sum_{n=1}^{\infty} 4 n a_{n} t^{n-1} &= \sum_{n=0}^{\infty} 4(n+1)\, a_{n+1} t^{n}
\end{aligned}
$$

Substituting all the above in Eq (2) gives the following equation where now all powers of t are the same and equal to n.

$$
\begin{gathered}
\left(\sum_{n=2}^{\infty} t^{n} a_{n} n(n-1)\right)+\left(\sum_{n=1}^{\infty} 6(n+1)\, a_{n+1} n\, t^{n}\right)+\left(\sum_{n=0}^{\infty} 5(n+2)\, a_{n+2}(n+1)\, t^{n}\right) \\
+\left(\sum_{n=1}^{\infty} n a_{n} t^{n}\right)+\left(\sum_{n=0}^{\infty} 4(n+1)\, a_{n+1} t^{n}\right)+\left(\sum_{n=0}^{\infty} 4 a_{n} t^{n}\right)=0
\end{gathered} \tag{3}
$$

$n = 0$ gives

$$
10 a_{2}+4 a_{1}+4 a_{0}=0
$$

$$
a_{2}=-\frac{2 a_{0}}{5}-\frac{2 a_{1}}{5}
$$

$n = 1$ gives

$$
20 a_{2}+30 a_{3}+5 a_{1}=0
$$

Which after substituting earlier equations, simplifies to

$$a_3 = \frac{4a_0}{15} + \frac{a_1}{10}$$

For $2 \leq n$, the recurrence equation is

$$n a_n (n-1) + 6(n+1) a_{n+1} n + 5(n+2) a_{n+2} (n+1) + n a_n + 4(n+1) a_{n+1} + 4a_n = 0 \quad (4)$$

Solving for a_{n+2}, gives

$$\begin{aligned} a_{n+2} &= -\frac{n^2 a_n + 6n^2 a_{n+1} + 10n a_{n+1} + 4a_n + 4a_{n+1}}{5\,(n+2)\,(n+1)} \\ &= -\frac{(n^2+4)\,a_n}{5\,(n+2)\,(n+1)} - \frac{(6n^2+10n+4)\,a_{n+1}}{5\,(n+2)\,(n+1)} \end{aligned} \quad (5)$$

For $n = 2$ the recurrence equation gives

$$8a_2 + 48a_3 + 60a_4 = 0$$

Which after substituting the earlier terms found becomes

$$a_4 = -\frac{4a_0}{25} - \frac{2a_1}{75}$$

For $n = 3$ the recurrence equation gives

$$13a_3 + 88a_4 + 100a_5 = 0$$

Which after substituting the earlier terms found becomes

$$a_5 = \frac{199a_0}{1875} + \frac{157a_1}{15000}$$

For $n = 4$ the recurrence equation gives

$$20a_4 + 140a_5 + 150a_6 = 0$$

Which after substituting the earlier terms found becomes

$$a_6 = -\frac{2186a_0}{28125} - \frac{233a_1}{37500}$$

For $n=5$ the recurrence equation gives

$$29a_5 + 204a_6 + 210a_7 = 0$$

Which after substituting the earlier terms found becomes

$$a_7 = \frac{39931a_0}{656250} + \frac{72299a_1}{15750000}$$

And so on. Therefore the solution is

$$\begin{aligned} y(t) &= \sum_{n=0}^{\infty} a_n t^n \\ &= a_3 t^3 + a_2 t^2 + a_1 t + a_0 + \dots \end{aligned}$$

Substituting the values for a_n found above, the solution becomes

$$\begin{aligned} y(t) = a_0 + a_1 t + \left(-\frac{2a_0}{5} - \frac{2a_1}{5}\right)t^2 + \left(\frac{4a_0}{15} + \frac{a_1}{10}\right)t^3 \\ + \left(-\frac{4a_0}{25} - \frac{2a_1}{75}\right)t^4 + \left(\frac{199a_0}{1875} + \frac{157a_1}{15000}\right)t^5 + \dots \end{aligned}$$

Collecting terms, the solution becomes

$$\begin{aligned} y(t) = \left(1 - \frac{2}{5}t^2 + \frac{4}{15}t^3 - \frac{4}{25}t^4 + \frac{199}{1875}t^5\right)a_0 \\ + \left(t - \frac{2}{5}t^2 + \frac{1}{10}t^3 - \frac{2}{75}t^4 + \frac{157}{15000}t^5\right)a_1 + O(t^6) \end{aligned} \tag{3}$$

At $t=0$ the solution above becomes

$$y(t) = \left(1 - \frac{2}{5}t^2 + \frac{4}{15}t^3 - \frac{4}{25}t^4 + \frac{199}{1875}t^5\right)c_1 + \left(t - \frac{2}{5}t^2 + \frac{1}{10}t^3 - \frac{2}{75}t^4 + \frac{157}{15000}t^5\right)c_2 + O(t^6)$$

Replacing t in the above with the original independent variable $x$susing $t = x - 4$ results in

$$\begin{aligned} y = &\left(1 - \frac{2(x-4)^2}{5} + \frac{4(x-4)^3}{15} - \frac{4(x-4)^4}{25} + \frac{199(x-4)^5}{1875} - \frac{2186(x-4)^6}{28125}\right)y(4) \\ &+ \left(x - 4 - \frac{2(x-4)^2}{5} + \frac{(x-4)^3}{10} - \frac{2(x-4)^4}{75} + \frac{157(x-4)^5}{15000} - \frac{233(x-4)^6}{37500}\right)y'(4) \\ &+ O\left((x-4)^6\right) \end{aligned}$$

Summary
The solution(s) found are the following

$$y = \left(1-\frac{2(x-4)^{2}}{5}+\frac{4(x-4)^{3}}{15}-\frac{4(x-4)^{4}}{25}+\frac{199(x-4)^{5}}{1875}-\frac{2186(x-4)^{6}}{28125}\right)y(4) + \left(x-4-\frac{2(x-4)^{2}}{5}+\frac{(x-4)^{3}}{10}-\frac{2(x-4)^{4}}{75}+\frac{157(x-4)^{5}}{15000}-\frac{233(x-4)^{6}}{37500}\right)y'(4) + O\left((x-4)^{6}\right) \tag{1}$$

Verification of solutions

$$y = \left(1-\frac{2(x-4)^{2}}{5}+\frac{4(x-4)^{3}}{15}-\frac{4(x-4)^{4}}{25}+\frac{199(x-4)^{5}}{1875}-\frac{2186(x-4)^{6}}{28125}\right)y(4) + \left(x-4-\frac{2(x-4)^{2}}{5}+\frac{(x-4)^{3}}{10}-\frac{2(x-4)^{4}}{75}+\frac{157(x-4)^{5}}{15000}-\frac{233(x-4)^{6}}{37500}\right)y'(4) + O\left((x-4)^{6}\right)$$

Verified OK.

14.8.1 Maple step by step solution

Let's solve

$(x^2-2x-3)\,y''+y'x+4y=0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y''=-\frac{4y}{x^2-2x-3}-\frac{xy'}{x^2-2x-3}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y''+\frac{xy'}{x^2-2x-3}+\frac{4y}{x^2-2x-3}=0$

☐ Check to see if x_0 is a regular singular point

  ◦ Define functions

    $\left[P_2(x)=\frac{x}{x^2-2x-3}, P_3(x)=\frac{4}{x^2-2x-3}\right]$

  ◦ $(x+1)\cdot P_2(x)$ is analytic at $x=-1$

    $\left((x+1)\cdot P_2(x)\right)\Big|_{x=-1}=\frac{1}{4}$

- $(x+1)^2 \cdot P_3(x)$ is analytic at $x=-1$

  $\left((x+1)^2 \cdot P_3(x)\right)\Big|_{x=-1} = 0$

- $x=-1$is a regular singular point

  Check to see if x_0 is a regular singular point

  $x_0 = -1$

- Multiply by denominators

  $(x^2-2x-3)\,y''+y'x+4y=0$

- Change variables using $x=u-1$ so that the regular singular point is at $u=0$

  $(u^2-4u)\left(\frac{d^2}{du^2}y(u)\right)+(u-1)\left(\frac{d}{du}y(u)\right)+4y(u)=0$

- Assume series solution for $y(u)$

  $y(u)=\sum_{k=0}^{\infty} a_k u^{k+r}$

□ Rewrite ODE with series expansions

- Convert $u^m \cdot \left(\frac{d}{du}y(u)\right)$ to series expansion for $m=0..1$

  $u^m \cdot \left(\frac{d}{du}y(u)\right)=\sum_{k=0}^{\infty} a_k(k+r)\,u^{k+r-1+m}$

- Shift index using $k->k+1-m$

  $u^m \cdot \left(\frac{d}{du}y(u)\right)=\sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\,u^{k+r}$

- Convert $u^m \cdot \left(\frac{d^2}{du^2}y(u)\right)$ to series expansion for $m=1..2$

  $u^m \cdot \left(\frac{d^2}{du^2}y(u)\right)=\sum_{k=0}^{\infty} a_k(k+r)\,(k+r-1)\,u^{k+r-2+m}$

- Shift index using $k->k+2-m$

  $u^m \cdot \left(\frac{d^2}{du^2}y(u)\right)=\sum_{k=-2+m}^{\infty} a_{k+2-m}(k+2-m+r)\,(k+1-m+r)\,u^{k+r}$

  Rewrite ODE with series expansions

  $-a_0r(-3+4r)\,u^{-1+r}+\left(\sum_{k=0}^{\infty}\left(-a_{k+1}(k+1+r)\,(4k+1+4r)+a_k(k^2+2kr+r^2+4)\right)u^{k+r}\right)=$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $-r(-3+4r)=0$

- Values of r that satisfy the indicial equation

  $r \in \left\{0, \frac{3}{4}\right\}$

- Each term in the series must be 0, giving the recursion relation

  $-4\left(k+\frac{1}{4}+r\right)(k+1+r)\, a_{k+1}+a_k\left(k^2+2kr+r^2+4\right)=0$

- Recursion relation that defines series solution to ODE

  $a_{k+1}=\frac{a_k\left(k^2+2kr+r^2+4\right)}{(4k+1+4r)(k+1+r)}$

- Recursion relation for $r=0$

  $a_{k+1}=\frac{a_k\left(k^2+4\right)}{(4k+1)(k+1)}$

- Solution for $r=0$

  $\left[y(u)=\sum_{k=0}^{\infty} a_k u^k, a_{k+1}=\frac{a_k\left(k^2+4\right)}{(4k+1)(k+1)}\right]$

- Revert the change of variables $u=x+1$

  $\left[y=\sum_{k=0}^{\infty} a_k(x+1)^k, a_{k+1}=\frac{a_k\left(k^2+4\right)}{(4k+1)(k+1)}\right]$

- Recursion relation for $r=\frac{3}{4}$

  $a_{k+1}=\frac{a_k\left(k^2+\frac{3}{2}k+\frac{73}{16}\right)}{(4k+4)\left(k+\frac{7}{4}\right)}$

- Solution for $r=\frac{3}{4}$

  $\left[y(u)=\sum_{k=0}^{\infty} a_k u^{k+\frac{3}{4}}, a_{k+1}=\frac{a_k\left(k^2+\frac{3}{2}k+\frac{73}{16}\right)}{(4k+4)\left(k+\frac{7}{4}\right)}\right]$

- Revert the change of variables $u=x+1$

  $\left[y=\sum_{k=0}^{\infty} a_k(x+1)^{k+\frac{3}{4}}, a_{k+1}=\frac{a_k\left(k^2+\frac{3}{2}k+\frac{73}{16}\right)}{(4k+4)\left(k+\frac{7}{4}\right)}\right]$

- Combine solutions and rename parameters

  $\left[y=\left(\sum_{k=0}^{\infty} a_k(x+1)^k\right)+\left(\sum_{k=0}^{\infty} b_k(x+1)^{k+\frac{3}{4}}\right), a_{k+1}=\frac{a_k\left(k^2+4\right)}{(4k+1)(k+1)}, b_{k+1}=\frac{b_k\left(k^2+\frac{3}{2}k+\frac{73}{16}\right)}{(4k+4)\left(k+\frac{7}{4}\right)}\right]$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
<- No Liouvillian solutions exists
-> Trying a solution in terms of special functions:
   -> Bessel
   -> elliptic
   -> Legendre
   -> Kummer
      -> hyper3: Equivalence to 1F1 under a power @ Moebius
   -> hypergeometric
      -> heuristic approach
      <- heuristic approach successful
   <- hypergeometric successful
<- special function solution successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 54

```
Order:=6;
dsolve((x^2-2*x-3)*diff(y(x),x$2)+x*diff(y(x),x)+4*y(x)=0,y(x),type='series',x=4);
```

$$y(x) = \left(1-\frac{2(x-4)^2}{5}+\frac{4(x-4)^3}{15}-\frac{4(x-4)^4}{25}+\frac{199(x-4)^5}{1875}\right)y(4) + \left(x-4-\frac{2(x-4)^2}{5}+\frac{(x-4)^3}{10}-\frac{2(x-4)^4}{75}+\frac{157(x-4)^5}{15000}\right)D(y)(4)+O(x^6)$$

✓ Solution by Mathematica

Time used: 0.001 (sec). Leaf size: 87

```
AsymptoticDSolveValue[(x^2-2*x-3)*y''[x]+x*y'[x]+4*y[x]==0,y[x],{x,4,5}]
```

$$y(x) \to c_1 \left(\frac{199(x-4)^5}{1875} - \frac{4}{25}(x-4)^4 + \frac{4}{15}(x-4)^3 - \frac{2}{5}(x-4)^2 + 1 \right) + c_2 \left(\frac{157(x-4)^5}{15000} - \frac{2}{75}(x-4)^4 + \frac{1}{10}(x-4)^3 - \frac{2}{5}(x-4)^2 + x - 4 \right)$$

14.9 problem 6. case $x_0 = -4$

Internal problem ID [741]
Internal file name [OUTPUT/741_Sunday_June_05_2022_01_48_27_AM_34703844/index.tex]

Book: Elementary differential equations and boundary value problems, 10th ed., Boyce and DiPrima
Section: Chapter 5.3, Series Solutions Near an Ordinary Point, Part II. page 269
Problem number: 6. case $x_0 = -4$.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second order series method. Ordinary point", "second order series method. Taylor series method"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

$$\boxed{(x^2 - 2x - 3)\, y'' + y'x + 4y = 0}$$

With the expansion point for the power series method at $x = -4$.

The ode does not have its expansion point at $x = 0$, therefore to simplify the computation of power series expansion, change of variable is made on the independent variable to shift the initial conditions and the expasion point back to zero. The new ode is then solved more easily since the expansion point is now at zero. The solution converted back to the original independent variable. Let

$$t = x + 4$$

The ode is converted to be in terms of the new independent variable t. This results in

$$\left((t-4)^2 - 2t + 5\right)\left(\frac{d^2}{dt^2}y(t)\right) + \left(\frac{d}{dt}y(t)\right)(t-4) + 4y(t) = 0$$

With its expansion point and initial conditions now at $t = 0$. The transformed ODE is now solved. Solving ode using Taylor series method. This gives review on how the Taylor series method works for solving second order ode.

Let

$$y'' = f(x, y, y')$$

Assuming expansion is at $x_0 = 0$ (we can always shift the actual expansion point to 0 by change of variables) and assuming $f(x, y, y')$ is analytic at x_0 which must be the case for an ordinary point. Let initial conditions be $y(x_0) = y_0$ and $y'(x_0) = y'_0$. Using Taylor series gives

$$\begin{aligned}
y(x) &= y(x_0) + (x - x_0)\, y'(x_0) + \frac{(x - x_0)^2}{2} y''(x_0) + \frac{(x - x_0)^3}{3!} y'''(x_0) + \cdots \\
&= y_0 + x y'_0 + \frac{x^2}{2} f|_{x_0, y_0, y'_0} + \frac{x^3}{3!} f'|_{x_0, y_0, y'_0} + \cdots \\
&= y_0 + x y'_0 + \sum_{n=0}^{\infty} \frac{x^{n+2}}{(n+2)!} \frac{d^n f}{dx^n}\bigg|_{x_0, y_0, y'_0}
\end{aligned}$$

But

$$\begin{aligned}
\frac{df}{dx} &= \frac{\partial f}{\partial x}\frac{dx}{dx} + \frac{\partial f}{\partial y}\frac{dy}{dx} + \frac{\partial f}{\partial y'}\frac{dy'}{dx} && (1) \\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} y' + \frac{\partial f}{\partial y'} y'' && (834) \\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} y' + \frac{\partial f}{\partial y'} f && (835) \\
\frac{d^2 f}{dx^2} &= \frac{d}{dx}\left(\frac{df}{dx}\right) \\
&= \frac{\partial}{\partial x}\left(\frac{df}{dx}\right) + \frac{\partial}{\partial y}\left(\frac{df}{dx}\right) y' + \frac{\partial}{\partial y'}\left(\frac{df}{dx}\right) f && (2) \\
\frac{d^3 f}{dx^3} &= \frac{d}{dx}\left(\frac{d^2 f}{dx^2}\right) \\
&= \frac{\partial}{\partial x}\left(\frac{d^2 f}{dx^2}\right) + \left(\frac{\partial}{\partial y}\frac{d^2 f}{dx^2}\right) y' + \frac{\partial}{\partial y'}\left(\frac{d^2 f}{dx^2}\right) f && (3) \\
&\ \ \vdots
\end{aligned}$$

And so on. Hence if we name $F_0 = f(x,y,y')$ then the above can be written as

$$
\begin{aligned}
F_0 &= f(x,y,y') \qquad (4)\\
F_1 &= \frac{df}{dx}\\
&= \frac{dF_0}{dx}\\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}y' + \frac{\partial f}{\partial y'}y''\\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}y' + \frac{\partial f}{\partial y'}f \qquad (5)\\
&= \frac{\partial F_0}{\partial x} + \frac{\partial F_0}{\partial y}y' + \frac{\partial F_0}{\partial y'}F_0\\
F_2 &= \frac{d}{dx}\left(\frac{d}{dx}f\right)\\
&= \frac{d}{dx}(F_1)\\
&= \frac{\partial}{\partial x}F_1 + \left(\frac{\partial F_1}{\partial y}\right)y' + \left(\frac{\partial F_1}{\partial y'}\right)y''\\
&= \frac{\partial}{\partial x}F_1 + \left(\frac{\partial F_1}{\partial y}\right)y' + \left(\frac{\partial F_1}{\partial y'}\right)F_0\\
&\vdots\\
F_n &= \frac{d}{dx}(F_{n-1})\\
&= \frac{\partial}{\partial x}F_{n-1} + \left(\frac{\partial F_{n-1}}{\partial y}\right)y' + \left(\frac{\partial F_{n-1}}{\partial y'}\right)y''\\
&= \frac{\partial}{\partial x}F_{n-1} + \left(\frac{\partial F_{n-1}}{\partial y}\right)y' + \left(\frac{\partial F_{n-1}}{\partial y'}\right)F_0 \qquad (6)
\end{aligned}
$$

Therefore (6) can be used from now on along with

$$
y(x) = y_0 + xy_0' + \sum_{n=0}^{\infty} \frac{x^{n+2}}{(n+2)!} F_n|_{x_0,y_0,y_0'} \qquad (7)
$$

To find $y(x)$ series solution around $x=0$. Hence

$$
\begin{aligned}
F_0 &= -\frac{t\left(\frac{d}{dt}y(t)\right)-4\frac{d}{dt}y(t)+4y(t)}{t^2-10t+21} \\
F_1 &= \frac{dF_0}{dt} \\
&= \frac{\partial F_0}{\partial t}+\frac{\partial F_0}{\partial y}\frac{d}{dt}y(t)+\frac{\partial F_0}{\partial \frac{d}{dt}y\left(t\right)}F_0 \\
&= \frac{\left(-2t^2+24t-49\right)\left(\frac{d}{dt}y(t)\right)+\left(12t-56\right)y(t)}{\left(t-3\right)^2\left(t-7\right)^2} \\
F_2 &= \frac{dF_1}{dt} \\
&= \frac{\partial F_1}{\partial t}+\frac{\partial F_1}{\partial y}\frac{d}{dt}y(t)+\frac{\partial F_1}{\partial \frac{d}{dt}y\left(t\right)}F_1 \\
&= \frac{\left(18t^3-280t^2+1309t-1848\right)\left(\frac{d}{dt}y(t)\right)-28\left(t^2-\frac{62}{7}t+24\right)y(t)}{\left(t-3\right)^3\left(t-7\right)^3} \\
F_3 &= \frac{dF_2}{dt} \\
&= \frac{\partial F_2}{\partial t}+\frac{\partial F_2}{\partial y}\frac{d}{dt}y(t)+\frac{\partial F_2}{\partial \frac{d}{dt}y\left(t\right)}F_2 \\
&= \frac{\left(-100t^4+2000t^3-14380t^2+44520t-49455\right)\left(\frac{d}{dt}y(t)\right)+40y(t)\left(t^3-10t^2+\frac{129}{2}t-189\right)}{\left(t-3\right)^4\left(t-7\right)^4} \\
F_4 &= \frac{dF_3}{dt} \\
&= \frac{\partial F_3}{\partial t}+\frac{\partial F_3}{\partial y}\frac{d}{dt}y(t)+\frac{\partial F_3}{\partial \frac{d}{dt}y\left(t\right)}F_3 \\
&= \frac{\left(540t^5-13200t^4+127680t^3-617040t^2+1484595t-1399860\right)\left(\frac{d}{dt}y(t)\right)+200\left(t^4-26t^3+\frac{1699}{10}t^2-\right.}{\left(t-3\right)^5\left(t-7\right)^5}
\end{aligned}
$$

And so on. Evaluating all the above at initial conditions $t=0$ and $y(0)=y(0)$ and

$y'(0) = y'(0)$ gives

$$
\begin{aligned}
F_0 &= -\frac{4y(0)}{21} + \frac{4y'(0)}{21} \\
F_1 &= -\frac{8y(0)}{63} - \frac{y'(0)}{9} \\
F_2 &= -\frac{32y(0)}{441} - \frac{88y'(0)}{441} \\
F_3 &= -\frac{40y(0)}{1029} - \frac{785y'(0)}{3087} \\
F_4 &= -\frac{800y(0)}{64827} - \frac{22220y'(0)}{64827}
\end{aligned}
$$

Substituting all the above in (7) and simplifying gives the solution as

$$
\begin{aligned}
y(t) = {} & \left(1 - \frac{2}{21}t^2 - \frac{4}{189}t^3 - \frac{4}{1323}t^4 - \frac{1}{3087}t^5 - \frac{10}{583443}t^6\right) y(0) \\
& + \left(t + \frac{2}{21}t^2 - \frac{1}{54}t^3 - \frac{11}{1323}t^4 - \frac{157}{74088}t^5 - \frac{1111}{2333772}t^6\right) y'(0) + O(t^6)
\end{aligned}
$$

Since the expansion point $t = 0$ is an ordinary, we can also solve this using standard power series The ode is normalized to be

$$
\left(\frac{d^2}{dt^2}y(t)\right)\left(t^2 - 10t + 21\right) + \left(\frac{d}{dt}y(t)\right)(t - 4) + 4y(t) = 0
$$

Let the solution be represented as power series of the form

$$
y(t) = \sum_{n=0}^{\infty} a_n t^n
$$

Then

$$
\begin{aligned}
\frac{d}{dt}y(t) &= \sum_{n=1}^{\infty} n a_n t^{n-1} \\
\frac{d^2}{dt^2}y(t) &= \sum_{n=2}^{\infty} n(n-1)\, a_n t^{n-2}
\end{aligned}
$$

Substituting the above back into the ode gives

$$
\left(\sum_{n=2}^{\infty} n(n-1)\, a_n t^{n-2}\right)\left(t^2 - 10t + 21\right) + \left(\sum_{n=1}^{\infty} n a_n t^{n-1}\right)(t - 4) + 4\left(\sum_{n=0}^{\infty} a_n t^n\right) = 0 \tag{1}
$$

Which simplifies to

$$
\begin{aligned}
&\left(\sum_{n=2}^{\infty} t^{n} a_{n} n(n-1)\right)+\sum_{n=2}^{\infty}\left(-10 n\, t^{n-1} a_{n}(n-1)\right)+\left(\sum_{n=2}^{\infty} 21 n(n-1)\, a_{n} t^{n-2}\right) \\
&+\left(\sum_{n=1}^{\infty} n a_{n} t^{n}\right)+\sum_{n=1}^{\infty}\left(-4 n a_{n} t^{n-1}\right)+\left(\sum_{n=0}^{\infty} 4 a_{n} t^{n}\right)=0
\end{aligned} \tag{2}
$$

The next step is to make all powers of t be n in each summation term. Going over each summation term above with power of t in it which is not already t^n and adjusting the power and the corresponding index gives

$$
\begin{aligned}
\sum_{n=2}^{\infty}\left(-10 n\, t^{n-1} a_{n}(n-1)\right) &= \sum_{n=1}^{\infty}\left(-10(n+1)\, a_{n+1} n\, t^{n}\right) \\
\sum_{n=2}^{\infty} 21 n(n-1)\, a_{n} t^{n-2} &= \sum_{n=0}^{\infty} 21(n+2)\, a_{n+2}(n+1)\, t^{n} \\
\sum_{n=1}^{\infty}\left(-4 n a_{n} t^{n-1}\right) &= \sum_{n=0}^{\infty}\left(-4(n+1)\, a_{n+1} t^{n}\right)
\end{aligned}
$$

Substituting all the above in Eq (2) gives the following equation where now all powers of t are the same and equal to n.

$$
\begin{aligned}
&\left(\sum_{n=2}^{\infty} t^{n} a_{n} n(n-1)\right)+\sum_{n=1}^{\infty}\left(-10(n+1)\, a_{n+1} n\, t^{n}\right) \\
&+\left(\sum_{n=0}^{\infty} 21(n+2)\, a_{n+2}(n+1)\, t^{n}\right)+\left(\sum_{n=1}^{\infty} n a_{n} t^{n}\right) \\
&+\sum_{n=0}^{\infty}\left(-4(n+1)\, a_{n+1} t^{n}\right)+\left(\sum_{n=0}^{\infty} 4 a_{n} t^{n}\right)=0
\end{aligned} \tag{3}
$$

$n = 0$ gives

$$
42 a_{2}-4 a_{1}+4 a_{0}=0
$$

$$
a_{2}=-\frac{2 a_{0}}{21}+\frac{2 a_{1}}{21}
$$

$n=1$ gives

$$-28a_2 + 126a_3 + 5a_1 = 0$$

Which after substituting earlier equations, simplifies to

$$a_3 = -\frac{4a_0}{189} - \frac{a_1}{54}$$

For $2 \leq n$, the recurrence equation is

$$na_n(n-1) - 10(n+1)\,a_{n+1}n + 21(n+2)\,a_{n+2}(n+1) + na_n - 4(n+1)\,a_{n+1} + 4a_n = 0 \quad (4)$$

Solving for a_{n+2}, gives

$$\begin{aligned} a_{n+2} &= -\frac{n^2a_n - 10n^2a_{n+1} - 14na_{n+1} + 4a_n - 4a_{n+1}}{21\,(n+2)\,(n+1)} \\ &= -\frac{(n^2+4)\,a_n}{21\,(n+2)\,(n+1)} - \frac{(-10n^2 - 14n - 4)\,a_{n+1}}{21\,(n+2)\,(n+1)} \end{aligned} \quad (5)$$

For $n=2$ the recurrence equation gives

$$8a_2 - 72a_3 + 252a_4 = 0$$

Which after substituting the earlier terms found becomes

$$a_4 = -\frac{4a_0}{1323} - \frac{11a_1}{1323}$$

For $n=3$ the recurrence equation gives

$$13a_3 - 136a_4 + 420a_5 = 0$$

Which after substituting the earlier terms found becomes

$$a_5 = -\frac{a_0}{3087} - \frac{157a_1}{74088}$$

For $n=4$ the recurrence equation gives

$$20a_4 - 220a_5 + 630a_6 = 0$$

Which after substituting the earlier terms found becomes

$$a_6 = -\frac{10a_0}{583443} - \frac{1111a_1}{2333772}$$

For $n = 5$ the recurrence equation gives

$$29a_5 - 324a_6 + 882a_7 = 0$$

Which after substituting the earlier terms found becomes

$$a_7 = \frac{83a_0}{19059138} - \frac{48121a_1}{457419312}$$

And so on. Therefore the solution is

$$\begin{aligned} y(t) &= \sum_{n=0}^{\infty} a_n t^n \\ &= a_3 t^3 + a_2 t^2 + a_1 t + a_0 + \dots \end{aligned}$$

Substituting the values for a_n found above, the solution becomes

$$\begin{aligned} y(t) = a_0 + a_1 t + \left(-\frac{2a_0}{21} + \frac{2a_1}{21}\right) t^2 + \left(-\frac{4a_0}{189} - \frac{a_1}{54}\right) t^3 \\ + \left(-\frac{4a_0}{1323} - \frac{11a_1}{1323}\right) t^4 + \left(-\frac{a_0}{3087} - \frac{157a_1}{74088}\right) t^5 + \dots \end{aligned}$$

Collecting terms, the solution becomes

$$\begin{aligned} y(t) = \left(1 - \frac{2}{21}t^2 - \frac{4}{189}t^3 - \frac{4}{1323}t^4 - \frac{1}{3087}t^5\right) a_0 \\ + \left(t + \frac{2}{21}t^2 - \frac{1}{54}t^3 - \frac{11}{1323}t^4 - \frac{157}{74088}t^5\right) a_1 + O(t^6) \end{aligned} \tag{3}$$

At $t = 0$ the solution above becomes

$$\begin{aligned} y(t) = \left(1 - \frac{2}{21}t^2 - \frac{4}{189}t^3 - \frac{4}{1323}t^4 - \frac{1}{3087}t^5\right) c_1 \\ + \left(t + \frac{2}{21}t^2 - \frac{1}{54}t^3 - \frac{11}{1323}t^4 - \frac{157}{74088}t^5\right) c_2 + O(t^6) \end{aligned}$$

Replacing t in the above with the original independent variable $x$using $t = x + 4$ results in

$$y = \left(1 - \frac{2(x+4)^2}{21} - \frac{4(x+4)^3}{189} - \frac{4(x+4)^4}{1323} - \frac{(x+4)^5}{3087} - \frac{10(x+4)^6}{583443}\right) y(-4) + \left(x+4+\frac{2(x+4)^2}{21} - \frac{(x+4)^3}{54} - \frac{11(x+4)^4}{1323} - \frac{157(x+4)^5}{74088} - \frac{1111(x+4)^6}{2333772}\right) y'(-4) + O\left((x+4)^6\right)$$

Summary

The solution(s) found are the following

$$y = \left(1 - \frac{2(x+4)^2}{21} - \frac{4(x+4)^3}{189} - \frac{4(x+4)^4}{1323} - \frac{(x+4)^5}{3087} - \frac{10(x+4)^6}{583443}\right) y(-4) + \left(x+4+\frac{2(x+4)^2}{21} - \frac{(x+4)^3}{54} - \frac{11(x+4)^4}{1323} - \frac{157(x+4)^5}{74088} - \frac{1111(x+4)^6}{2333772}\right) y'(-4) + O\left((x+4)^6\right) \tag{1}$$

Verification of solutions

$$y = \left(1 - \frac{2(x+4)^2}{21} - \frac{4(x+4)^3}{189} - \frac{4(x+4)^4}{1323} - \frac{(x+4)^5}{3087} - \frac{10(x+4)^6}{583443}\right) y(-4) + \left(x+4+\frac{2(x+4)^2}{21} - \frac{(x+4)^3}{54} - \frac{11(x+4)^4}{1323} - \frac{157(x+4)^5}{74088} - \frac{1111(x+4)^6}{2333772}\right) y'(-4) + O\left((x+4)^6\right)$$

Verified OK.

14.9.1 Maple step by step solution

Let's solve

$(x^2 - 2x - 3)\, y'' + y'x + 4y = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

$y'' = -\frac{4y}{x^2-2x-3} - \frac{xy'}{x^2-2x-3}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y'' + \frac{xy'}{x^2-2x-3} + \frac{4y}{x^2-2x-3} = 0$

☐ Check to see if x_0 is a regular singular point

  - Define functions

    $\left[P_2(x) = \frac{x}{x^2-2x-3}, P_3(x) = \frac{4}{x^2-2x-3}\right]$

  - $(x+1) \cdot P_2(x)$ is analytic at $x = -1$

    $\left((x+1) \cdot P_2(x)\right)\Big|_{x=-1} = \frac{1}{4}$

  - $(x+1)^2 \cdot P_3(x)$ is analytic at $x = -1$

    $\left((x+1)^2 \cdot P_3(x)\right)\Big|_{x=-1} = 0$

  - $x = -1$is a regular singular point

    Check to see if x_0 is a regular singular point

    $x_0 = -1$

- Multiply by denominators

  $(x^2 - 2x - 3)\, y'' + y'x + 4y = 0$

- Change variables using $x = u - 1$ so that the regular singular point is at $u = 0$

  $(u^2 - 4u)\left(\frac{d^2}{du^2}y(u)\right) + (u-1)\left(\frac{d}{du}y(u)\right) + 4y(u) = 0$

- Assume series solution for $y(u)$

  $y(u) = \sum_{k=0}^{\infty} a_k u^{k+r}$

☐ Rewrite ODE with series expansions

  - Convert $u^m \cdot \left(\frac{d}{du}y(u)\right)$ to series expansion for $m = 0..1$

    $u^m \cdot \left(\frac{d}{du}y(u)\right) = \sum_{k=0}^{\infty} a_k (k+r)\, u^{k+r-1+m}$

  - Shift index using $k- >k+1-m$

    $u^m \cdot \left(\frac{d}{du}y(u)\right) = \sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\, u^{k+r}$

  - Convert $u^m \cdot \left(\frac{d^2}{du^2}y(u)\right)$ to series expansion for $m = 1..2$

$$u^m \cdot \left(\frac{d^2}{du^2}y(u)\right) = \sum_{k=0}^{\infty} a_k(k+r)\,(k+r-1)\,u^{k+r-2+m}$$

- Shift index using $k- > k+2-m$

$$u^m \cdot \left(\frac{d^2}{du^2}y(u)\right) = \sum_{k=-2+m}^{\infty} a_{k+2-m}(k+2-m+r)\,(k+1-m+r)\,u^{k+r}$$

Rewrite ODE with series expansions

$$-a_0 r(-3+4r)\,u^{-1+r} + \left(\sum_{k=0}^{\infty} \left(-a_{k+1}(k+1+r)\,(4k+1+4r) + a_k(k^2+2kr+r^2+4)\right) u^{k+r}\right) =$$

- $a_0$cannot be 0 by assumption, giving the indicial equation

$$-r(-3+4r) = 0$$

- Values of r that satisfy the indicial equation

$$r \in \left\{0, \frac{3}{4}\right\}$$

- Each term in the series must be 0, giving the recursion relation

$$-4\left(k+\frac{1}{4}+r\right)(k+1+r)\,a_{k+1} + a_k(k^2+2kr+r^2+4) = 0$$

- Recursion relation that defines series solution to ODE

$$a_{k+1} = \frac{a_k(k^2+2kr+r^2+4)}{(4k+1+4r)(k+1+r)}$$

- Recursion relation for $r = 0$

$$a_{k+1} = \frac{a_k(k^2+4)}{(4k+1)(k+1)}$$

- Solution for $r = 0$

$$\left[y(u) = \sum_{k=0}^{\infty} a_k u^k, a_{k+1} = \frac{a_k(k^2+4)}{(4k+1)(k+1)}\right]$$

- Revert the change of variables $u = x+1$

$$\left[y = \sum_{k=0}^{\infty} a_k(x+1)^k, a_{k+1} = \frac{a_k(k^2+4)}{(4k+1)(k+1)}\right]$$

- Recursion relation for $r = \frac{3}{4}$

$$a_{k+1} = \frac{a_k\left(k^2+\frac{3}{2}k+\frac{73}{16}\right)}{(4k+4)\left(k+\frac{7}{4}\right)}$$

- Solution for $r = \frac{3}{4}$

$$\left[y(u) = \sum_{k=0}^{\infty} a_k u^{k+\frac{3}{4}}, a_{k+1} = \frac{a_k\left(k^2+\frac{3}{2}k+\frac{73}{16}\right)}{(4k+4)\left(k+\frac{7}{4}\right)}\right]$$

- Revert the change of variables $u = x+1$

$$\left[y = \sum_{k=0}^{\infty} a_k (x+1)^{k+\frac{3}{4}}, a_{k+1} = \frac{a_k \left(k^2+\frac{3}{2}k+\frac{73}{16}\right)}{(4k+4)\left(k+\frac{7}{4}\right)}\right]$$

- Combine solutions and rename parameters

$$\left[y = \left(\sum_{k=0}^{\infty} a_k (x+1)^k\right) + \left(\sum_{k=0}^{\infty} b_k (x+1)^{k+\frac{3}{4}}\right), a_{k+1} = \frac{a_k (k^2+4)}{(4k+1)(k+1)}, b_{k+1} = \frac{b_k \left(k^2+\frac{3}{2}k+\frac{73}{16}\right)}{(4k+4)\left(k+\frac{7}{4}\right)}\right]$$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
<- No Liouvillian solutions exists
-> Trying a solution in terms of special functions:
   -> Bessel
   -> elliptic
   -> Legendre
   -> Kummer
      -> hyper3: Equivalence to 1F1 under a power @ Moebius
   -> hypergeometric
      -> heuristic approach
      <- heuristic approach successful
   <- hypergeometric successful
<- special function solution successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 54

```
Order:=6;
dsolve((x^2-2*x-3)*diff(y(x),x$2)+x*diff(y(x),x)+4*y(x)=0,y(x),type='series',x=-4);
```

$$y(x) = \left(1-\frac{2(x+4)^2}{21}-\frac{4(x+4)^3}{189}-\frac{4(x+4)^4}{1323}-\frac{(x+4)^5}{3087}\right)y(-4) + \left(x+4+\frac{2(x+4)^2}{21}-\frac{(x+4)^3}{54}-\frac{11(x+4)^4}{1323}-\frac{157(x+4)^5}{74088}\right)D(y)(-4)+O(x^6)$$

✓ Solution by Mathematica

Time used: 0.001 (sec). Leaf size: 87

```
AsymptoticDSolveValue[(x^2-2*x-3)*y''[x]+x*y'[x]+4*y[x]==0,y[x],{x,-4,5}]
```

$$y(x) \to c_1\left(-\frac{(x+4)^5}{3087}-\frac{4(x+4)^4}{1323}-\frac{4}{189}(x+4)^3-\frac{2}{21}(x+4)^2+1\right) + c_2\left(-\frac{157(x+4)^5}{74088}-\frac{11(x+4)^4}{1323}-\frac{1}{54}(x+4)^3+\frac{2}{21}(x+4)^2+x+4\right)$$

14.10 problem 7. case $x_0 = 0$

Internal problem ID [742]
Internal file name [`OUTPUT/742_Sunday_June_05_2022_01_48_29_AM_48339428/index.tex`]

Book: Elementary differential equations and boundary value problems, 10th ed., Boyce and DiPrima
Section: Chapter 5.3, Series Solutions Near an Ordinary Point, Part II. page 269
Problem number: 7. case $x_0 = 0$.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second order series method. Ordinary point", "second order series method. Taylor series method"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

$$\boxed{(x^3 + 1)\, y'' + 4y'x + y = 0}$$

With the expansion point for the power series method at $x = 0$.

Solving ode using Taylor series method. This gives review on how the Taylor series method works for solving second order ode.

Let

$$y'' = f(x, y, y')$$

Assuming expansion is at $x_0 = 0$ (we can always shift the actual expansion point to 0 by change of variables) and assuming $f(x, y, y')$ is analytic at x_0 which must be the case for an ordinary point. Let initial conditions be $y(x_0) = y_0$ and $y'(x_0) = y'_0$. Using Taylor series gives

$$\begin{aligned}
y(x) &= y(x_0) + (x - x_0)\, y'(x_0) + \frac{(x - x_0)^2}{2} y''(x_0) + \frac{(x - x_0)^3}{3!} y'''(x_0) + \cdots \\
&= y_0 + xy'_0 + \frac{x^2}{2} f|_{x_0,y_0,y'_0} + \frac{x^3}{3!} f'|_{x_0,y_0,y'_0} + \cdots \\
&= y_0 + xy'_0 + \sum_{n=0}^{\infty} \frac{x^{n+2}}{(n+2)!} \frac{d^n f}{dx^n}\bigg|_{x_0,y_0,y'_0}
\end{aligned}$$

But

$$\begin{aligned}
\frac{df}{dx} &= \frac{\partial f}{\partial x}\frac{dx}{dx} + \frac{\partial f}{\partial y}\frac{dy}{dx} + \frac{\partial f}{\partial y'}\frac{dy'}{dx} && (1)\\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}y' + \frac{\partial f}{\partial y'}y'' && (837)\\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}y' + \frac{\partial f}{\partial y'}f && (838)\\
\frac{d^2 f}{dx^2} &= \frac{d}{dx}\left(\frac{df}{dx}\right)\\
&= \frac{\partial}{\partial x}\left(\frac{df}{dx}\right) + \frac{\partial}{\partial y}\left(\frac{df}{dx}\right)y' + \frac{\partial}{\partial y'}\left(\frac{df}{dx}\right)f && (2)\\
\frac{d^3 f}{dx^3} &= \frac{d}{dx}\left(\frac{d^2 f}{dx^2}\right)\\
&= \frac{\partial}{\partial x}\left(\frac{d^2 f}{dx^2}\right) + \left(\frac{\partial}{\partial y}\frac{d^2 f}{dx^2}\right)y' + \frac{\partial}{\partial y'}\left(\frac{d^2 f}{dx^2}\right)f && (3)\\
&\vdots
\end{aligned}$$

And so on. Hence if we name $F_0 = f(x,y,y')$ then the above can be written as

$$
\begin{aligned}
F_0 &= f(x,y,y') \qquad (4)\\
F_1 &= \frac{df}{dx}\\
&= \frac{dF_0}{dx}\\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}y' + \frac{\partial f}{\partial y'}y''\\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}y' + \frac{\partial f}{\partial y'}f \qquad (5)\\
&= \frac{\partial F_0}{\partial x} + \frac{\partial F_0}{\partial y}y' + \frac{\partial F_0}{\partial y'}F_0\\
F_2 &= \frac{d}{dx}\left(\frac{d}{dx}f\right)\\
&= \frac{d}{dx}(F_1)\\
&= \frac{\partial}{\partial x}F_1 + \left(\frac{\partial F_1}{\partial y}\right)y' + \left(\frac{\partial F_1}{\partial y'}\right)y''\\
&= \frac{\partial}{\partial x}F_1 + \left(\frac{\partial F_1}{\partial y}\right)y' + \left(\frac{\partial F_1}{\partial y'}\right)F_0\\
&\vdots\\
F_n &= \frac{d}{dx}(F_{n-1})\\
&= \frac{\partial}{\partial x}F_{n-1} + \left(\frac{\partial F_{n-1}}{\partial y}\right)y' + \left(\frac{\partial F_{n-1}}{\partial y'}\right)y''\\
&= \frac{\partial}{\partial x}F_{n-1} + \left(\frac{\partial F_{n-1}}{\partial y}\right)y' + \left(\frac{\partial F_{n-1}}{\partial y'}\right)F_0 \qquad (6)
\end{aligned}
$$

Therefore (6) can be used from now on along with

$$
y(x) = y_0 + xy_0' + \sum_{n=0}^{\infty}\frac{x^{n+2}}{(n+2)!}F_n|_{x_0,y_0,y_0'} \qquad (7)
$$

To find $y(x)$ series solution around $x = 0$. Hence

$$
\begin{aligned}
F_0 &= -\frac{4y'x + y}{x^3 + 1} \\
F_1 &= \frac{dF_0}{dx} \\
&= \frac{\partial F_0}{\partial x} + \frac{\partial F_0}{\partial y} y' + \frac{\partial F_0}{\partial y'} F_0 \\
&= \frac{(7x^3 + 16x^2 - 5)\, y' + 3\left(x + \frac{4}{3}\right) xy}{(x^3 + 1)^2} \\
F_2 &= \frac{dF_1}{dx} \\
&= \frac{\partial F_1}{\partial x} + \frac{\partial F_1}{\partial y} y' + \frac{\partial F_1}{\partial y'} F_1 \\
&= \frac{(-18x^5 - 88x^4 - 64x^3 + 54x^2 + 56x)\, y' - 12\left(x^4 + \frac{9}{4}x^3 + \frac{4}{3}x^2 - \frac{1}{2}x - \frac{3}{4}\right) y}{(x^3 + 1)^3} \\
F_3 &= \frac{dF_2}{dx} \\
&= \frac{\partial F_2}{\partial x} + \frac{\partial F_2}{\partial y} y' + \frac{\partial F_2}{\partial y'} F_2 \\
&= \frac{(60x^7 + 485x^6 + 720x^5 - 218x^4 - 1034x^3 - 432x^2 + 114x + 65)\, y' + 60\left(x^6 + 3x^5 + \frac{10}{3}x^4 - \frac{8}{15}x^3 - \cdots\right.}{(x^3 + 1)^4} \\
F_4 &= \frac{dF_3}{dx} \\
&= \frac{\partial F_3}{\partial x} + \frac{\partial F_3}{\partial y} y' + \frac{\partial F_3}{\partial y'} F_3 \\
&= \frac{(-240x^9 - 2970x^8 - 6780x^7 - 688x^6 + 13052x^5 + 12168x^4 - 424x^3 - 4554x^2 - 1212x + 120)\, y' - \cdots}{(x^3 + 1)^5}
\end{aligned}
$$

And so on. Evaluating all the above at initial conditions $x = 0$ and $y(0) = y(0)$ and $y'(0) = y'(0)$ gives

$$
\begin{aligned}
F_0 &= -y(0) \\
F_1 &= -5y'(0) \\
F_2 &= 9y(0) \\
F_3 &= 6y(0) + 65y'(0) \\
F_4 &= -153y(0) + 120y'(0)
\end{aligned}
$$

Substituting all the above in (7) and simplifying gives the solution as

$$
y = \left(1 - \frac{1}{2}x^2 + \frac{3}{8}x^4 + \frac{1}{20}x^5 - \frac{17}{80}x^6\right) y(0) + \left(x - \frac{5}{6}x^3 + \frac{13}{24}x^5 + \frac{1}{6}x^6\right) y'(0) + O(x^6)
$$

Since the expansion point $x = 0$ is an ordinary, we can also solve this using standard power series The ode is normalized to be

$$
(x^3 + 1) y'' + 4y'x + y = 0
$$

Let the solution be represented as power series of the form

$$
y = \sum_{n=0}^{\infty} a_n x^n
$$

Then

$$
\begin{aligned}
y' &= \sum_{n=1}^{\infty} n a_n x^{n-1} \\
y'' &= \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2}
\end{aligned}
$$

Substituting the above back into the ode gives

$$
(x^3 + 1) \left(\sum_{n=2}^{\infty} n(n-1) a_n x^{n-2}\right) + 4\left(\sum_{n=1}^{\infty} n a_n x^{n-1}\right) x + \left(\sum_{n=0}^{\infty} a_n x^n\right) = 0 \tag{1}
$$

Which simplifies to

$$
\left(\sum_{n=2}^{\infty} n\, x^{1+n} a_n (n-1)\right) + \left(\sum_{n=2}^{\infty} n(n-1) a_n x^{n-2}\right) + \left(\sum_{n=1}^{\infty} 4n a_n x^n\right) + \left(\sum_{n=0}^{\infty} a_n x^n\right) = 0 \tag{2}
$$

The next step is to make all powers of x be n in each summation term. Going over each summation term above with power of x in it which is not already x^n and adjusting the power and the corresponding index gives

$$
\begin{aligned}
\sum_{n=2}^{\infty} n\, x^{1+n} a_n (n-1) &= \sum_{n=3}^{\infty} (n-1) a_{n-1} (n-2) x^n \\
\sum_{n=2}^{\infty} n(n-1) a_n x^{n-2} &= \sum_{n=0}^{\infty} (n+2) a_{n+2} (1+n) x^n
\end{aligned}
$$

Substituting all the above in Eq (2) gives the following equation where now all powers of x are the same and equal to n.

$$\left(\sum_{n=3}^{\infty} (n-1)\, a_{n-1}(n-2)\, x^{n}\right)+\left(\sum_{n=0}^{\infty} (n+2)\, a_{n+2}(1+n)\, x^{n}\right) \\ +\left(\sum_{n=1}^{\infty} 4n a_{n} x^{n}\right)+\left(\sum_{n=0}^{\infty} a_{n} x^{n}\right)=0 \tag{3}$$

$n=0$ gives

$$2a_{2}+a_{0}=0$$

$$a_{2}=-\frac{a_{0}}{2}$$

$n=1$ gives

$$6a_{3}+5a_{1}=0$$

Which after substituting earlier equations, simplifies to

$$a_{3}=-\frac{5a_{1}}{6}$$

$n=2$ gives

$$12a_{4}+9a_{2}=0$$

Which after substituting earlier equations, simplifies to

$$12a_{4}-\frac{9a_{0}}{2}=0$$

Or

$$a_{4}=\frac{3a_{0}}{8}$$

For $3\leq n$, the recurrence equation is

$$(n-1)\, a_{n-1}(n-2)+(n+2)\, a_{n+2}(1+n)+4n a_{n}+a_{n}=0 \tag{4}$$

Solving for a_{n+2}, gives

$$
\begin{aligned}
a_{n+2} &= -\frac{n^2 a_{n-1} + 4n a_n - 3n a_{n-1} + a_n + 2a_{n-1}}{(n+2)(1+n)} \\
&= -\frac{(4n+1)a_n}{(n+2)(1+n)} - \frac{(n^2-3n+2)a_{n-1}}{(n+2)(1+n)}
\end{aligned} \tag{5}
$$

For $n = 3$ the recurrence equation gives

$$
2a_2 + 20a_5 + 13a_3 = 0
$$

Which after substituting the earlier terms found becomes

$$
a_5 = \frac{a_0}{20} + \frac{13a_1}{24}
$$

For $n = 4$ the recurrence equation gives

$$
6a_3 + 30a_6 + 17a_4 = 0
$$

Which after substituting the earlier terms found becomes

$$
a_6 = \frac{a_1}{6} - \frac{17a_0}{80}
$$

For $n = 5$ the recurrence equation gives

$$
12a_4 + 42a_7 + 21a_5 = 0
$$

Which after substituting the earlier terms found becomes

$$
a_7 = -\frac{37a_0}{280} - \frac{13a_1}{48}
$$

And so on. Therefore the solution is

$$
\begin{aligned}
y &= \sum_{n=0}^{\infty} a_n x^n \\
&= a_3 x^3 + a_2 x^2 + a_1 x + a_0 + \dots
\end{aligned}
$$

Substituting the values for a_n found above, the solution becomes

$$y = a_0 + a_1 x - \frac{a_0 x^2}{2} - \frac{5 a_1 x^3}{6} + \frac{3 a_0 x^4}{8} + \left(\frac{a_0}{20} + \frac{13 a_1}{24}\right) x^5 + \dots$$

Collecting terms, the solution becomes

$$y = \left(1 - \frac{1}{2}x^2 + \frac{3}{8}x^4 + \frac{1}{20}x^5\right) a_0 + \left(x - \frac{5}{6}x^3 + \frac{13}{24}x^5\right) a_1 + O(x^6) \tag{3}$$

At $x = 0$ the solution above becomes

$$y = \left(1 - \frac{1}{2}x^2 + \frac{3}{8}x^4 + \frac{1}{20}x^5\right) c_1 + \left(x - \frac{5}{6}x^3 + \frac{13}{24}x^5\right) c_2 + O(x^6)$$

Summary

The solution(s) found are the following

$$y = \left(1 - \frac{1}{2}x^2 + \frac{3}{8}x^4 + \frac{1}{20}x^5 - \frac{17}{80}x^6\right) y(0) + \left(x - \frac{5}{6}x^3 + \frac{13}{24}x^5 + \frac{1}{6}x^6\right) y'(0) + O(x^6) \tag{1}$$

$$y = \left(1 - \frac{1}{2}x^2 + \frac{3}{8}x^4 + \frac{1}{20}x^5\right) c_1 + \left(x - \frac{5}{6}x^3 + \frac{13}{24}x^5\right) c_2 + O(x^6) \tag{2}$$

Verification of solutions

$$y = \left(1 - \frac{1}{2}x^2 + \frac{3}{8}x^4 + \frac{1}{20}x^5 - \frac{17}{80}x^6\right) y(0) + \left(x - \frac{5}{6}x^3 + \frac{13}{24}x^5 + \frac{1}{6}x^6\right) y'(0) + O(x^6)$$

Verified OK.

$$y = \left(1 - \frac{1}{2}x^2 + \frac{3}{8}x^4 + \frac{1}{20}x^5\right) c_1 + \left(x - \frac{5}{6}x^3 + \frac{13}{24}x^5\right) c_2 + O(x^6)$$

Verified OK.

14.10.1 Maple step by step solution

Let's solve

$(x^3 + 1) y'' + 4y'x + y = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y'' = -\frac{4xy'}{x^3+1} - \frac{y}{x^3+1}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y'' + \frac{4xy'}{x^3+1} + \frac{y}{x^3+1} = 0$

□ Check to see if x_0 is a regular singular point

  - Define functions

    $\left[P_2(x) = \frac{4x}{x^3+1}, P_3(x) = \frac{1}{x^3+1}\right]$

  - $(x+1) \cdot P_2(x)$ is analytic at $x = -1$

    $\left((x+1) \cdot P_2(x)\right)\Big|_{x=-1} = -\frac{4}{3}$

  - $(x+1)^2 \cdot P_3(x)$ is analytic at $x = -1$

    $\left((x+1)^2 \cdot P_3(x)\right)\Big|_{x=-1} = 0$

  - $x = -1$is a regular singular point

    Check to see if x_0 is a regular singular point

    $x_0 = -1$

- Multiply by denominators

  $(x^3+1)\, y'' + 4y'x + y = 0$

- Change variables using $x = u - 1$ so that the regular singular point is at $u = 0$

  $(u^3 - 3u^2 + 3u)\left(\frac{d^2}{du^2}y(u)\right) + (4u - 4)\left(\frac{d}{du}y(u)\right) + y(u) = 0$

- Assume series solution for $y(u)$

  $y(u) = \sum_{k=0}^{\infty} a_k u^{k+r}$

□ Rewrite ODE with series expansions

  - Convert $u^m \cdot \left(\frac{d}{du}y(u)\right)$ to series expansion for $m = 0..1$

    $u^m \cdot \left(\frac{d}{du}y(u)\right) = \sum_{k=0}^{\infty} a_k(k+r)\, u^{k+r-1+m}$

  - Shift index using $k- >k+1-m$

    $u^m \cdot \left(\frac{d}{du}y(u)\right) = \sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\, u^{k+r}$

  - Convert $u^m \cdot \left(\frac{d^2}{du^2}y(u)\right)$ to series expansion for $m = 1..3$

$$u^{m} \cdot \left(\frac{d^{2}}{d u^{2}} y(u)\right) = \sum_{k=0}^{\infty} a_{k}(k+r)\,(k+r-1)\, u^{k+r-2+m}$$

- Shift index using $k->k+2-m$

$$u^{m} \cdot \left(\frac{d^{2}}{d u^{2}} y(u)\right) = \sum_{k=-2+m}^{\infty} a_{k+2-m}(k+2-m+r)\,(k+1-m+r)\, u^{k+r}$$

Rewrite ODE with series expansions

$$a_{0} r(-7+3r)\, u^{-1+r} + \left(a_{1}(1+r)\,(-4+3r) - a_{0}(3r^{2}-7r-1)\right) u^{r} + \left(\sum_{k=1}^{\infty} \left(a_{k+1}(k+1+r)\,(3k -\right.\right.$$

- a_{0}cannot be 0 by assumption, giving the indicial equation

$$r(-7+3r) = 0$$

- Values of r that satisfy the indicial equation

$$r \in \left\{0, \frac{7}{3}\right\}$$

- Each term must be 0

$$a_{1}(1+r)\,(-4+3r) - a_{0}(3r^{2}-7r-1) = 0$$

- Each term in the series must be 0, giving the recursion relation

$$(-3a_{k}+a_{k-1}+3a_{k+1})\, k^{2} + \left((-6a_{k}+2a_{k-1}+6a_{k+1})\, r + 7a_{k} - 3a_{k-1} - a_{k+1}\right) k + (-3a_{k}+a_{k-1}$$

- Shift index using $k->k+1$

$$(-3a_{k+1}+a_{k}+3a_{k+2})\,(k+1)^{2} + \left((-6a_{k+1}+2a_{k}+6a_{k+2})\, r + 7a_{k+1} - 3a_{k} - a_{k+2}\right)(k+1) + (-$$

- Recursion relation that defines series solution to ODE

$$a_{k+2} = -\frac{k^{2} a_{k} - 3k^{2} a_{k+1} + 2kr a_{k} - 6kr a_{k+1} + r^{2} a_{k} - 3r^{2} a_{k+1} - k a_{k} + k a_{k+1} - r a_{k} + r a_{k+1} + 5a_{k+1}}{3k^{2}+6kr+3r^{2}+5k+5r-2}$$

- Recursion relation for $r = 0$

$$a_{k+2} = -\frac{k^{2} a_{k} - 3k^{2} a_{k+1} - k a_{k} + k a_{k+1} + 5a_{k+1}}{3k^{2}+5k-2}$$

- Solution for $r = 0$

$$\left[y(u) = \sum_{k=0}^{\infty} a_{k} u^{k}, a_{k+2} = -\frac{k^{2} a_{k} - 3k^{2} a_{k+1} - k a_{k} + k a_{k+1} + 5a_{k+1}}{3k^{2}+5k-2}, -4a_{1} + a_{0} = 0\right]$$

- Revert the change of variables $u = x+1$

$$\left[y = \sum_{k=0}^{\infty} a_{k}(x+1)^{k}, a_{k+2} = -\frac{k^{2} a_{k} - 3k^{2} a_{k+1} - k a_{k} + k a_{k+1} + 5a_{k+1}}{3k^{2}+5k-2}, -4a_{1} + a_{0} = 0\right]$$

- Recursion relation for $r = \frac{7}{3}$

$$a_{k+2} = -\frac{k^{2} a_{k} - 3k^{2} a_{k+1} + \frac{11}{3} k a_{k} - 13k a_{k+1} + \frac{28}{9} a_{k} - 9a_{k+1}}{3k^{2}+19k+26}$$

- Solution for $r = \frac{7}{3}$

$$\left[y(u)=\sum_{k=0}^{\infty} a_k u^{k+\frac{7}{3}}, a_{k+2}=-\frac{k^2 a_k-3k^2 a_{k+1}+\frac{11}{3} k a_k-13 k a_{k+1}+\frac{28}{9} a_k-9 a_{k+1}}{3k^2+19k+26}, 10a_1+a_0=0\right]$$

- Revert the change of variables $u=x+1$

$$\left[y=\sum_{k=0}^{\infty} a_k (x+1)^{k+\frac{7}{3}}, a_{k+2}=-\frac{k^2 a_k-3k^2 a_{k+1}+\frac{11}{3} k a_k-13 k a_{k+1}+\frac{28}{9} a_k-9 a_{k+1}}{3k^2+19k+26}, 10a_1+a_0=0\right]$$

- Combine solutions and rename parameters

$$\left[y=\left(\sum_{k=0}^{\infty} a_k (x+1)^{k}\right)+\left(\sum_{k=0}^{\infty} b_k (x+1)^{k+\frac{7}{3}}\right), a_{k+2}=-\frac{k^2 a_k-3k^2 a_{k+1}-k a_k+k a_{k+1}+5 a_{k+1}}{3k^2+5k-2}, -4a_1+a_0\right.$$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
<- No Liouvillian solutions exists
-> Trying a solution in terms of special functions:
   -> Bessel
   -> elliptic
   -> Legendre
   -> Kummer
      -> hyper3: Equivalence to 1F1 under a power @ Moebius
   -> hypergeometric
      -> heuristic approach
      -> hyper3: Equivalence to 2F1, 1F1 or 0F1 under a power @ Moebius
   -> Mathieu
      -> Equivalence to the rational form of Mathieu ODE under a power @ Moebius
trying a solution in terms of MeijerG functions
-> Heun: Equivalence to the GHE or one of its 4 confluent cases under a power @ Moebius
<- Heun successful: received ODE is equivalent to the  HeunG  ODE, case  a <> 0, e <> 0, g <
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 39

```
Order:=6;
dsolve((1+x^3)*diff(y(x),x$2)+4*x*diff(y(x),x)+y(x)=0,y(x),type='series',x=0);
```

$$y(x) = \left(1-\frac{1}{2}x^2+\frac{3}{8}x^4+\frac{1}{20}x^5\right)y(0)+\left(x-\frac{5}{6}x^3+\frac{13}{24}x^5\right)D(y)(0)+O(x^6)$$

✓ Solution by Mathematica

Time used: 0.001 (sec). Leaf size: 49

```
AsymptoticDSolveValue[(1+x^3)*y''[x]+4*x*y'[x]+y[x]==0,y[x],{x,0,5}]
```

$$y(x) \to c_2\left(\frac{13x^5}{24}-\frac{5x^3}{6}+x\right)+c_1\left(\frac{x^5}{20}+\frac{3x^4}{8}-\frac{x^2}{2}+1\right)$$

14.11 problem 7. case $x_0 = 2$

Internal problem ID [743]
Internal file name [`OUTPUT/743_Sunday_June_05_2022_01_48_32_AM_81422129/index.tex`]

Book: Elementary differential equations and boundary value problems, 10th ed., Boyce and DiPrima
Section: Chapter 5.3, Series Solutions Near an Ordinary Point, Part II. page 269
Problem number: 7. case $x_0 = 2$.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second order series method. Ordinary point", "second order series method. Taylor series method"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

$$\boxed{(x^3+1)\, y'' + 4y'x + y = 0}$$

With the expansion point for the power series method at $x = 2$.

The ode does not have its expansion point at $x = 0$, therefore to simplify the computation of power series expansion, change of variable is made on the independent variable to shift the initial conditions and the expasion point back to zero. The new ode is then solved more easily since the expansion point is now at zero. The solution converted back to the original independent variable. Let

$$t = -2 + x$$

The ode is converted to be in terms of the new independent variable t. This results in

$$((2+t)^3+1)\left(\frac{d^2}{dt^2}y(t)\right) + 4(2+t)\left(\frac{d}{dt}y(t)\right) + y(t) = 0$$

With its expansion point and initial conditions now at $t = 0$. The transformed ODE is now solved. Solving ode using Taylor series method. This gives review on how the Taylor series method works for solving second order ode.

Let

$$y'' = f(x, y, y')$$

Assuming expansion is at $x_0 = 0$ (we can always shift the actual expansion point to 0 by change of variables) and assuming $f(x, y, y')$ is analytic at x_0 which must be the case for an ordinary point. Let initial conditions be $y(x_0) = y_0$ and $y'(x_0) = y'_0$. Using Taylor series gives

$$\begin{aligned}
y(x) &= y(x_0) + (x - x_0)\, y'(x_0) + \frac{(x - x_0)^2}{2} y''(x_0) + \frac{(x - x_0)^3}{3!} y'''(x_0) + \cdots \\
&= y_0 + x y'_0 + \frac{x^2}{2} f|_{x_0, y_0, y'_0} + \frac{x^3}{3!} f'|_{x_0, y_0, y'_0} + \cdots \\
&= y_0 + x y'_0 + \sum_{n=0}^{\infty} \frac{x^{n+2}}{(n+2)!} \frac{d^n f}{dx^n}\bigg|_{x_0, y_0, y'_0}
\end{aligned}$$

But

$$\begin{aligned}
\frac{df}{dx} &= \frac{\partial f}{\partial x}\frac{dx}{dx} + \frac{\partial f}{\partial y}\frac{dy}{dx} + \frac{\partial f}{\partial y'}\frac{dy'}{dx} && (1) \\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} y' + \frac{\partial f}{\partial y'} y'' && (840) \\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} y' + \frac{\partial f}{\partial y'} f && (841) \\
\frac{d^2 f}{dx^2} &= \frac{d}{dx}\left(\frac{df}{dx}\right) \\
&= \frac{\partial}{\partial x}\left(\frac{df}{dx}\right) + \frac{\partial}{\partial y}\left(\frac{df}{dx}\right) y' + \frac{\partial}{\partial y'}\left(\frac{df}{dx}\right) f && (2) \\
\frac{d^3 f}{dx^3} &= \frac{d}{dx}\left(\frac{d^2 f}{dx^2}\right) \\
&= \frac{\partial}{\partial x}\left(\frac{d^2 f}{dx^2}\right) + \left(\frac{\partial}{\partial y}\frac{d^2 f}{dx^2}\right) y' + \frac{\partial}{\partial y'}\left(\frac{d^2 f}{dx^2}\right) f && (3) \\
&\vdots
\end{aligned}$$

And so on. Hence if we name $F_0 = f(x,y,y')$ then the above can be written as

$$
\begin{aligned}
F_0 &= f(x,y,y') \qquad (4)\\
F_1 &= \frac{df}{dx}\\
&= \frac{dF_0}{dx}\\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} y' + \frac{\partial f}{\partial y'} y''\\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} y' + \frac{\partial f}{\partial y'} f \qquad (5)\\
&= \frac{\partial F_0}{\partial x} + \frac{\partial F_0}{\partial y} y' + \frac{\partial F_0}{\partial y'} F_0\\
F_2 &= \frac{d}{dx}\left(\frac{d}{dx} f\right)\\
&= \frac{d}{dx}(F_1)\\
&= \frac{\partial}{\partial x} F_1 + \left(\frac{\partial F_1}{\partial y}\right) y' + \left(\frac{\partial F_1}{\partial y'}\right) y''\\
&= \frac{\partial}{\partial x} F_1 + \left(\frac{\partial F_1}{\partial y}\right) y' + \left(\frac{\partial F_1}{\partial y'}\right) F_0\\
&\vdots\\
F_n &= \frac{d}{dx}(F_{n-1})\\
&= \frac{\partial}{\partial x} F_{n-1} + \left(\frac{\partial F_{n-1}}{\partial y}\right) y' + \left(\frac{\partial F_{n-1}}{\partial y'}\right) y''\\
&= \frac{\partial}{\partial x} F_{n-1} + \left(\frac{\partial F_{n-1}}{\partial y}\right) y' + \left(\frac{\partial F_{n-1}}{\partial y'}\right) F_0 \qquad (6)
\end{aligned}
$$

Therefore (6) can be used from now on along with

$$
y(x) = y_0 + x y_0' + \sum_{n=0}^{\infty} \frac{x^{n+2}}{(n+2)!} F_n|_{x_0,y_0,y_0'} \qquad (7)
$$

To find $y(x)$ series solution around $x = 0$. Hence

$$F_0 = -\frac{4t\left(\frac{d}{dt}y(t)\right) + 8\frac{d}{dt}y(t) + y(t)}{t^3 + 6t^2 + 12t + 9}$$

$$\begin{aligned}
F_1 &= \frac{dF_0}{dt} \\
&= \frac{\partial F_0}{\partial t} + \frac{\partial F_0}{\partial y}\frac{d}{dt}y(t) + \frac{\partial F_0}{\partial \frac{d}{dt}y(t)}F_0 \\
&= \frac{(7t^3 + 58t^2 + 148t + 115)\left(\frac{d}{dt}y(t)\right) + 3\left(t + \frac{10}{3}\right)(2 + t)\, y(t)}{(t + 3)^2\,(t^2 + 3t + 3)^2}
\end{aligned}$$

$$\begin{aligned}
F_2 &= \frac{dF_1}{dt} \\
&= \frac{\partial F_1}{\partial t} + \frac{\partial F_1}{\partial y}\frac{d}{dt}y(t) + \frac{\partial F_1}{\partial \frac{d}{dt}y(t)}F_1 \\
&= \frac{(-18t^5 - 268t^4 - 1488t^3 - 3882t^2 - 4752t - 2168)\left(\frac{d}{dt}y(t)\right) - 12\left(t^4 + \frac{41}{4}t^3 + \frac{233}{6}t^2 + \frac{383}{6}t + \frac{451}{12}\right)y(t}{(t + 3)^3\,(t^2 + 3t + 3)^3}
\end{aligned}$$

$$\begin{aligned}
F_3 &= \frac{dF_2}{dt} \\
&= \frac{\partial F_2}{\partial t} + \frac{\partial F_2}{\partial y}\frac{d}{dt}y(t) + \frac{\partial F_2}{\partial \frac{d}{dt}y(t)}F_2 \\
&= \frac{(60t^7 + 1325t^6 + 11580t^5 + 52882t^4 + 137222t^3 + 202452t^2 + 156602t + 48565)\left(\frac{d}{dt}y(t)\right) + 60\left(t^6 + 1\right.}{(t + 3)^4\,(t^2 + 3t + 3)^4}
\end{aligned}$$

$$\begin{aligned}
F_4 &= \frac{dF_3}{dt} \\
&= \frac{\partial F_3}{\partial t} + \frac{\partial F_3}{\partial y}\frac{d}{dt}y(t) + \frac{\partial F_3}{\partial \frac{d}{dt}y(t)}F_3 \\
&= \frac{(-240t^9 - 7290t^8 - 88860t^7 - 589528t^6 - 2379124t^5 - 6091072t^4 - 9900360t^3 - 9820346t^2 - 5354}{(t +}
\end{aligned}$$

And so on. Evaluating all the above at initial conditions $t = 0$ and $y(0) = y(0)$ and

$y'(0) = y'(0)$ gives

$$\begin{aligned}
F_0 &= -\frac{y(0)}{9} - \frac{8y'(0)}{9} \\
F_1 &= \frac{20y(0)}{81} + \frac{115y'(0)}{81} \\
F_2 &= -\frac{451y(0)}{729} - \frac{2168y'(0)}{729} \\
F_3 &= \frac{11510y(0)}{6561} + \frac{48565y'(0)}{6561} \\
F_4 &= -\frac{322189y(0)}{59049} - \frac{1206632y'(0)}{59049}
\end{aligned}$$

Substituting all the above in (7) and simplifying gives the solution as

$$\begin{aligned}
y(t) = {} & \left(1 - \frac{1}{18}t^2 + \frac{10}{243}t^3 - \frac{451}{17496}t^4 + \frac{1151}{78732}t^5 - \frac{322189}{42515280}t^6\right) y(0) \\
& + \left(t - \frac{4}{9}t^2 + \frac{115}{486}t^3 - \frac{271}{2187}t^4 + \frac{9713}{157464}t^5 - \frac{150829}{5314410}t^6\right) y'(0) + O(t^6)
\end{aligned}$$

Since the expansion point $t = 0$ is an ordinary, we can also solve this using standard power series The ode is normalized to be

$$\left(\frac{d^2}{dt^2}y(t)\right)(t^3 + 6t^2 + 12t + 9) + (8 + 4t)\left(\frac{d}{dt}y(t)\right) + y(t) = 0$$

Let the solution be represented as power series of the form

$$y(t) = \sum_{n=0}^{\infty} a_n t^n$$

Then

$$\begin{aligned}
\frac{d}{dt}y(t) &= \sum_{n=1}^{\infty} n a_n t^{n-1} \\
\frac{d^2}{dt^2}y(t) &= \sum_{n=2}^{\infty} n(n-1)\, a_n t^{n-2}
\end{aligned}$$

Substituting the above back into the ode gives

$$\left(\sum_{n=2}^{\infty} n(n-1)\, a_n t^{n-2}\right)(t^3 + 6t^2 + 12t + 9) + (8 + 4t)\left(\sum_{n=1}^{\infty} n a_n t^{n-1}\right) + \left(\sum_{n=0}^{\infty} a_n t^n\right) = 0 \tag{1}$$

Which simplifies to

$$\left(\sum_{n=2}^{\infty} n\,t^{1+n} a_n(n-1)\right)+\left(\sum_{n=2}^{\infty} 6t^n a_n n(n-1)\right)+\left(\sum_{n=2}^{\infty} 12n\,t^{n-1} a_n(n-1)\right) \tag{2}$$
$$+\left(\sum_{n=2}^{\infty} 9n(n-1)\,a_n t^{n-2}\right)+\left(\sum_{n=1}^{\infty} 8n a_n t^{n-1}\right)+\left(\sum_{n=1}^{\infty} 4n a_n t^n\right)+\left(\sum_{n=0}^{\infty} a_n t^n\right)$$
$$=0$$

The next step is to make all powers of t be n in each summation term. Going over each summation term above with power of t in it which is not already t^n and adjusting the power and the corresponding index gives

$$\sum_{n=2}^{\infty} n\,t^{1+n} a_n(n-1)=\sum_{n=3}^{\infty}(n-1)\,a_{n-1}(n-2)\,t^n$$

$$\sum_{n=2}^{\infty} 12n\,t^{n-1} a_n(n-1)=\sum_{n=1}^{\infty} 12(1+n)\,a_{1+n} n\,t^n$$

$$\sum_{n=2}^{\infty} 9n(n-1)\,a_n t^{n-2}=\sum_{n=0}^{\infty} 9(n+2)\,a_{n+2}(1+n)\,t^n$$

$$\sum_{n=1}^{\infty} 8n a_n t^{n-1}=\sum_{n=0}^{\infty} 8(1+n)\,a_{1+n} t^n$$

Substituting all the above in Eq (2) gives the following equation where now all powers of t are the same and equal to n.

$$\left(\sum_{n=3}^{\infty}(n-1)\,a_{n-1}(n-2)\,t^n\right)+\left(\sum_{n=2}^{\infty} 6t^n a_n n(n-1)\right) \tag{3}$$
$$+\left(\sum_{n=1}^{\infty} 12(1+n)\,a_{1+n} n\,t^n\right)+\left(\sum_{n=0}^{\infty} 9(n+2)\,a_{n+2}(1+n)\,t^n\right)$$
$$+\left(\sum_{n=0}^{\infty} 8(1+n)\,a_{1+n} t^n\right)+\left(\sum_{n=1}^{\infty} 4n a_n t^n\right)+\left(\sum_{n=0}^{\infty} a_n t^n\right)=0$$

$n=0$ gives

$$18a_2+8a_1+a_0=0$$

$$a_2 = -\frac{a_0}{18} - \frac{4a_1}{9}$$

$n = 1$ gives

$$40a_2 + 54a_3 + 5a_1 = 0$$

Which after substituting earlier equations, simplifies to

$$a_3 = \frac{10a_0}{243} + \frac{115a_1}{486}$$

$n = 2$ gives

$$21a_2 + 96a_3 + 108a_4 = 0$$

Which after substituting earlier equations, simplifies to

$$\frac{451a_0}{162} + \frac{1084a_1}{81} + 108a_4 = 0$$

Or

$$a_4 = -\frac{451a_0}{17496} - \frac{271a_1}{2187}$$

For $3 \leq n$, the recurrence equation is

$$\begin{aligned}(n-1)\, a_{n-1}(n-2) + 6na_n(n-1) + 12(1+n)\, a_{1+n}n \\ + 9(n+2)\, a_{n+2}(1+n) + 8(1+n)\, a_{1+n} + 4na_n + a_n = 0\end{aligned} \tag{4}$$

Solving for a_{n+2}, gives

$$\begin{aligned}a_{n+2} &= -\frac{6n^2a_n + 12n^2a_{1+n} + n^2a_{n-1} - 2na_n + 20na_{1+n} - 3na_{n-1} + a_n + 8a_{1+n} + 2a_{n-1}}{9\,(n+2)\,(1+n)} \\ &= -\frac{(6n^2 - 2n + 1)\, a_n}{9\,(n+2)\,(1+n)} - \frac{(12n^2 + 20n + 8)\, a_{1+n}}{9\,(n+2)\,(1+n)} - \frac{(n^2 - 3n + 2)\, a_{n-1}}{9\,(n+2)\,(1+n)}\end{aligned} \tag{5}$$

For $n = 3$ the recurrence equation gives

$$2a_2 + 49a_3 + 176a_4 + 180a_5 = 0$$

Which after substituting the earlier terms found becomes

$$a_5 = \frac{1151a_0}{78732} + \frac{9713a_1}{157464}$$

For $n = 4$ the recurrence equation gives

$$6a_3 + 89a_4 + 280a_5 + 270a_6 = 0$$

Which after substituting the earlier terms found becomes

$$a_6 = -\frac{322189a_0}{42515280} - \frac{150829a_1}{5314410}$$

For $n = 5$ the recurrence equation gives

$$12a_4 + 141a_5 + 408a_6 + 378a_7 = 0$$

Which after substituting the earlier terms found becomes

$$a_7 = \frac{4747261a_0}{1339231320} + \frac{30958471a_1}{2678462640}$$

And so on. Therefore the solution is

$$\begin{aligned} y(t) &= \sum_{n=0}^{\infty} a_n t^n \\ &= a_3 t^3 + a_2 t^2 + a_1 t + a_0 + \dots \end{aligned}$$

Substituting the values for a_n found above, the solution becomes

$$\begin{aligned} y(t) = a_0 + a_1 t + \left(-\frac{a_0}{18} - \frac{4a_1}{9}\right) t^2 + \left(\frac{10a_0}{243} + \frac{115a_1}{486}\right) t^3 \\ + \left(-\frac{451a_0}{17496} - \frac{271a_1}{2187}\right) t^4 + \left(\frac{1151a_0}{78732} + \frac{9713a_1}{157464}\right) t^5 + \dots \end{aligned}$$

Collecting terms, the solution becomes

$$\begin{aligned} y(t) = \left(1 - \frac{1}{18}t^2 + \frac{10}{243}t^3 - \frac{451}{17496}t^4 + \frac{1151}{78732}t^5\right) a_0 \\ + \left(t - \frac{4}{9}t^2 + \frac{115}{486}t^3 - \frac{271}{2187}t^4 + \frac{9713}{157464}t^5\right) a_1 + O(t^6) \end{aligned} \tag{3}$$

At $t = 0$ the solution above becomes

$$y(t) = \left(1 - \frac{1}{18}t^2 + \frac{10}{243}t^3 - \frac{451}{17496}t^4 + \frac{1151}{78732}t^5\right)c_1 + \left(t - \frac{4}{9}t^2 + \frac{115}{486}t^3 - \frac{271}{2187}t^4 + \frac{9713}{157464}t^5\right)c_2 + O(t^6)$$

Replacing t in the above with the original independent variable $x$using $t = -2 + x$ results in

$$y = \left(1 - \frac{(-2+x)^2}{18} + \frac{10(-2+x)^3}{243} - \frac{451(-2+x)^4}{17496} + \frac{1151(-2+x)^5}{78732} - \frac{322189(-2+x)^6}{42515280}\right)y(2) + \left(-2 + x - \frac{4(-2+x)^2}{9} + \frac{115(-2+x)^3}{486} - \frac{271(-2+x)^4}{2187} + \frac{9713(-2+x)^5}{157464} - \frac{150829(-2+x)^6}{5314410}\right)y'(2) + O\left((-2+x)^6\right)$$

Summary
The solution(s) found are the following

$$y = \left(1 - \frac{(-2+x)^2}{18} + \frac{10(-2+x)^3}{243} - \frac{451(-2+x)^4}{17496} + \frac{1151(-2+x)^5}{78732} - \frac{322189(-2+x)^6}{42515280}\right)y(2) + \left(-2 + x - \frac{4(-2+x)^2}{9} + \frac{115(-2+x)^3}{486} - \frac{271(-2+x)^4}{2187} + \frac{9713(-2+x)^5}{157464} - \frac{150829(-2+x)^6}{5314410}\right)y'(2) + O\left((-2+x)^6\right) \tag{1}$$

Verification of solutions

$$y = \left(1 - \frac{(-2+x)^2}{18} + \frac{10(-2+x)^3}{243} - \frac{451(-2+x)^4}{17496} + \frac{1151(-2+x)^5}{78732} - \frac{322189(-2+x)^6}{42515280}\right)y(2) + \left(-2 + x - \frac{4(-2+x)^2}{9} + \frac{115(-2+x)^3}{486} - \frac{271(-2+x)^4}{2187} + \frac{9713(-2+x)^5}{157464} - \frac{150829(-2+x)^6}{5314410}\right)y'(2) + O\left((-2+x)^6\right)$$

Verified OK.

14.11.1 Maple step by step solution

Let's solve

$(x^3+1)\,y''+4y'x+y=0$

- Highest derivative means the order of the ODE is 2

y''

- Isolate 2nd derivative

$y''=-\frac{4xy'}{x^3+1}-\frac{y}{x^3+1}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

$y''+\frac{4xy'}{x^3+1}+\frac{y}{x^3+1}=0$

□ Check to see if x_0 is a regular singular point

- ◦ Define functions

$\left[P_2(x)=\frac{4x}{x^3+1}, P_3(x)=\frac{1}{x^3+1}\right]$

- ◦ $(x+1)\cdot P_2(x)$ is analytic at $x=-1$

$\left((x+1)\cdot P_2(x)\right)\Big|_{x=-1}=-\frac{4}{3}$

- ◦ $(x+1)^2\cdot P_3(x)$ is analytic at $x=-1$

$\left((x+1)^2\cdot P_3(x)\right)\Big|_{x=-1}=0$

- ◦ $x=-1$is a regular singular point

Check to see if x_0 is a regular singular point

$x_0=-1$

- Multiply by denominators

$(x^3+1)\,y''+4y'x+y=0$

- Change variables using $x=u-1$ so that the regular singular point is at $u=0$

$(u^3-3u^2+3u)\left(\frac{d^2}{du^2}y(u)\right)+(4u-4)\left(\frac{d}{du}y(u)\right)+y(u)=0$

- Assume series solution for $y(u)$

$y(u)=\sum_{k=0}^{\infty}a_k u^{k+r}$

□ Rewrite ODE with series expansions

- ◦ Convert $u^m\cdot\left(\frac{d}{du}y(u)\right)$ to series expansion for $m=0..1$

$u^{m}\cdot \left(\frac{d}{d u}y \left(u \right)\right)=\sum_{k =0}^{\infty}a_{k} \left(k +r \right) u^{k +r -1+m}$

- Shift index using $k->k +1-m$

$u^{m}\cdot \left(\frac{d}{d u}y \left(u \right)\right)=\sum_{k =-1+m}^{\infty}a_{k +1-m} \left(k +1-m +r \right) u^{k +r}$

- Convert $u^{m}\cdot \left(\frac{d^{2}}{d u^{2}}y \left(u \right)\right)$ to series expansion for $m =1..3$

$u^{m}\cdot \left(\frac{d^{2}}{d u^{2}}y \left(u \right)\right)=\sum_{k =0}^{\infty}a_{k} \left(k +r \right) \left(k +r -1\right) u^{k +r -2+m}$

- Shift index using $k->k +2-m$

$u^{m}\cdot \left(\frac{d^{2}}{d u^{2}}y \left(u \right)\right)=\sum_{k =-2+m}^{\infty}a_{k +2-m} \left(k +2-m +r \right) \left(k +1-m +r \right) u^{k +r}$

Rewrite ODE with series expansions

$a_{0} r \left(-7+3 r \right) u^{-1+r}+\left(a_{1} \left(1+r \right) \left(-4+3 r \right)-a_{0} \left(3 r^{2}-7 r -1\right)\right) u^{r}+\left(\sum_{k =1}^{\infty}\left(a_{k +1} \left(k +1+r \right) \left(3 k -\right.\right.\right.$

- a_{0}cannot be 0 by assumption, giving the indicial equation

$r \left(-7+3 r \right)=0$

- Values of r that satisfy the indicial equation

$r \in \left\{0, \frac{7}{3}\right\}$

- Each term must be 0

$a_{1} \left(1+r \right) \left(-4+3 r \right)-a_{0} \left(3 r^{2}-7 r -1\right)=0$

- Each term in the series must be 0, giving the recursion relation

$\left(-3 a_{k}+a_{k -1}+3 a_{k +1}\right) k^{2}+\left(\left(-6 a_{k}+2 a_{k -1}+6 a_{k +1}\right) r +7 a_{k}-3 a_{k -1}-a_{k +1}\right) k +\left(-3 a_{k}+a_{k -1}\right.$

- Shift index using $k->k +1$

$\left(-3 a_{k +1}+a_{k}+3 a_{k +2}\right) \left(k +1\right)^{2}+\left(\left(-6 a_{k +1}+2 a_{k}+6 a_{k +2}\right) r +7 a_{k +1}-3 a_{k}-a_{k +2}\right) \left(k +1\right)+\left(-\right.$

- Recursion relation that defines series solution to ODE

$a_{k +2}=-\frac{k^{2} a_{k}-3 k^{2} a_{k +1}+2 k r a_{k}-6 k r a_{k +1}+r^{2} a_{k}-3 r^{2} a_{k +1}-k a_{k}+k a_{k +1}-r a_{k}+r a_{k +1}+5 a_{k +1}}{3 k^{2}+6 k r +3 r^{2}+5 k +5 r -2}$

- Recursion relation for $r =0$

$a_{k +2}=-\frac{k^{2} a_{k}-3 k^{2} a_{k +1}-k a_{k}+k a_{k +1}+5 a_{k +1}}{3 k^{2}+5 k -2}$

- Solution for $r =0$

$\left[y \left(u \right)=\sum_{k =0}^{\infty}a_{k} u^{k}, a_{k +2}=-\frac{k^{2} a_{k}-3 k^{2} a_{k +1}-k a_{k}+k a_{k +1}+5 a_{k +1}}{3 k^{2}+5 k -2}, -4 a_{1}+a_{0}=0\right]$

- Revert the change of variables $u = x+1$

$$\left[y = \sum_{k=0}^{\infty} a_k (x+1)^k , a_{k+2} = -\frac{k^2 a_k - 3k^2 a_{k+1} - k a_k + k a_{k+1} + 5 a_{k+1}}{3k^2+5k-2}, -4a_1 + a_0 = 0\right]$$

- Recursion relation for $r = \frac{7}{3}$

$$a_{k+2} = -\frac{k^2 a_k - 3k^2 a_{k+1} + \frac{11}{3} k a_k - 13 k a_{k+1} + \frac{28}{9} a_k - 9 a_{k+1}}{3k^2+19k+26}$$

- Solution for $r = \frac{7}{3}$

$$\left[y(u) = \sum_{k=0}^{\infty} a_k u^{k+\frac{7}{3}}, a_{k+2} = -\frac{k^2 a_k - 3k^2 a_{k+1} + \frac{11}{3} k a_k - 13 k a_{k+1} + \frac{28}{9} a_k - 9 a_{k+1}}{3k^2+19k+26}, 10a_1 + a_0 = 0\right]$$

- Revert the change of variables $u = x+1$

$$\left[y = \sum_{k=0}^{\infty} a_k (x+1)^{k+\frac{7}{3}} , a_{k+2} = -\frac{k^2 a_k - 3k^2 a_{k+1} + \frac{11}{3} k a_k - 13 k a_{k+1} + \frac{28}{9} a_k - 9 a_{k+1}}{3k^2+19k+26}, 10a_1 + a_0 = 0\right]$$

- Combine solutions and rename parameters

$$\left[y = \left(\sum_{k=0}^{\infty} a_k (x+1)^k\right) + \left(\sum_{k=0}^{\infty} b_k (x+1)^{k+\frac{7}{3}}\right), a_{k+2} = -\frac{k^2 a_k - 3k^2 a_{k+1} - k a_k + k a_{k+1} + 5 a_{k+1}}{3k^2+5k-2}, -4a_1 + a_0\right.$$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
<- No Liouvillian solutions exists
-> Trying a solution in terms of special functions:
   -> Bessel
   -> elliptic
   -> Legendre
   -> Kummer
      -> hyper3: Equivalence to 1F1 under a power @ Moebius
   -> hypergeometric
      -> heuristic approach
      -> hyper3: Equivalence to 2F1, 1F1 or 0F1 under a power @ Moebius
   -> Mathieu
      -> Equivalence to the rational form of Mathieu ODE under a power @ Moebius
trying a solution in terms of MeijerG functions
-> Heun: Equivalence to the GHE or one of its 4 confluent cases under a power @ Moebius
<- Heun successful: received ODE is equivalent to the  HeunG  ODE, case  a <> 0, e <> 0, g <
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 54

```
Order:=6;
dsolve((1+x^3)*diff(y(x),x$2)+4*x*diff(y(x),x)+y(x)=0,y(x),type='series',x=2);
```

$$y(x) = \left(1-\frac{(-2+x)^2}{18}+\frac{10(-2+x)^3}{243}-\frac{451(-2+x)^4}{17496}+\frac{1151(-2+x)^5}{78732}\right)y(2) + \left(-2+x-\frac{4(-2+x)^2}{9}+\frac{115(-2+x)^3}{486}-\frac{271(-2+x)^4}{2187} + \frac{9713(-2+x)^5}{157464}\right)D(y)(2)+O(x^6)$$

✓ Solution by Mathematica

Time used: 0.001 (sec). Leaf size: 87

```
AsymptoticDSolveValue[(1+x^3)*y''[x]+4*x*y'[x]+y[x]==0,y[x],{x,2,5}]
```

$$y(x) \to c_1\left(\frac{1151(x-2)^5}{78732}-\frac{451(x-2)^4}{17496}+\frac{10}{243}(x-2)^3-\frac{1}{18}(x-2)^2+1\right)+c_2\left(\frac{9713(x-2)^5}{157464}-\frac{271(x-2)^4}{2187}+\frac{115}{486}(x-2)^3-\frac{4}{9}(x-2)^2+x-2\right)$$

14.12 problem 8

Internal problem ID [744]
Internal file name [OUTPUT/744_Sunday_June_05_2022_01_48_34_AM_11668954/index.tex]

Book: Elementary differential equations and boundary value problems, 10th ed., Boyce and DiPrima
Section: Chapter 5.3, Series Solutions Near an Ordinary Point, Part II. page 269
Problem number: 8.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second order series method. Ordinary point", "second order series method. Taylor series method"**

Maple gives the following as the ode type

```
[[_Emden, _Fowler]]
```

$$\boxed{xy'' + y = 0}$$

With the expansion point for the power series method at $x = 1$.

The ode does not have its expansion point at $x = 0$, therefore to simplify the computation of power series expansion, change of variable is made on the independent variable to shift the initial conditions and the expasion point back to zero. The new ode is then solved more easily since the expansion point is now at zero. The solution converted back to the original independent variable. Let

$$t = x - 1$$

The ode is converted to be in terms of the new independent variable t. This results in

$$\left(\frac{d^2}{dt^2}y(t)\right)(t+1) + y(t) = 0$$

With its expansion point and initial conditions now at $t = 0$. The transformed ODE is now solved. Solving ode using Taylor series method. This gives review on how the Taylor series method works for solving second order ode.

Let

$$y'' = f(x, y, y')$$

Assuming expansion is at $x_0 = 0$ (we can always shift the actual expansion point to 0 by change of variables) and assuming $f(x, y, y')$ is analytic at x_0 which must be the case for an ordinary point. Let initial conditions be $y(x_0) = y_0$ and $y'(x_0) = y'_0$. Using Taylor series gives

$$\begin{aligned}
y(x) &= y(x_0) + (x - x_0)\, y'(x_0) + \frac{(x - x_0)^2}{2} y''(x_0) + \frac{(x - x_0)^3}{3!} y'''(x_0) + \cdots \\
&= y_0 + x y'_0 + \frac{x^2}{2} f|_{x_0, y_0, y'_0} + \frac{x^3}{3!} f'|_{x_0, y_0, y'_0} + \cdots \\
&= y_0 + x y'_0 + \sum_{n=0}^{\infty} \frac{x^{n+2}}{(n+2)!} \frac{d^n f}{dx^n}\bigg|_{x_0, y_0, y'_0}
\end{aligned}$$

But

$$\begin{aligned}
\frac{df}{dx} &= \frac{\partial f}{\partial x}\frac{dx}{dx} + \frac{\partial f}{\partial y}\frac{dy}{dx} + \frac{\partial f}{\partial y'}\frac{dy'}{dx} && (1)\\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} y' + \frac{\partial f}{\partial y'} y'' && (843)\\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} y' + \frac{\partial f}{\partial y'} f && (844)\\
\frac{d^2 f}{dx^2} &= \frac{d}{dx}\left(\frac{df}{dx}\right) \\
&= \frac{\partial}{\partial x}\left(\frac{df}{dx}\right) + \frac{\partial}{\partial y}\left(\frac{df}{dx}\right) y' + \frac{\partial}{\partial y'}\left(\frac{df}{dx}\right) f && (2)\\
\frac{d^3 f}{dx^3} &= \frac{d}{dx}\left(\frac{d^2 f}{dx^2}\right) \\
&= \frac{\partial}{\partial x}\left(\frac{d^2 f}{dx^2}\right) + \left(\frac{\partial}{\partial y}\frac{d^2 f}{dx^2}\right) y' + \frac{\partial}{\partial y'}\left(\frac{d^2 f}{dx^2}\right) f && (3)\\
&\vdots
\end{aligned}$$

And so on. Hence if we name $F_0 = f(x,y,y')$ then the above can be written as

$$
\begin{aligned}
F_0 &= f(x,y,y') && (4)\\
F_1 &= \frac{df}{dx}\\
&= \frac{dF_0}{dx}\\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}y' + \frac{\partial f}{\partial y'}y''\\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}y' + \frac{\partial f}{\partial y'}f && (5)\\
&= \frac{\partial F_0}{\partial x} + \frac{\partial F_0}{\partial y}y' + \frac{\partial F_0}{\partial y'}F_0\\
F_2 &= \frac{d}{dx}\left(\frac{d}{dx}f\right)\\
&= \frac{d}{dx}(F_1)\\
&= \frac{\partial}{\partial x}F_1 + \left(\frac{\partial F_1}{\partial y}\right)y' + \left(\frac{\partial F_1}{\partial y'}\right)y''\\
&= \frac{\partial}{\partial x}F_1 + \left(\frac{\partial F_1}{\partial y}\right)y' + \left(\frac{\partial F_1}{\partial y'}\right)F_0\\
&\vdots\\
F_n &= \frac{d}{dx}(F_{n-1})\\
&= \frac{\partial}{\partial x}F_{n-1} + \left(\frac{\partial F_{n-1}}{\partial y}\right)y' + \left(\frac{\partial F_{n-1}}{\partial y'}\right)y''\\
&= \frac{\partial}{\partial x}F_{n-1} + \left(\frac{\partial F_{n-1}}{\partial y}\right)y' + \left(\frac{\partial F_{n-1}}{\partial y'}\right)F_0 && (6)
\end{aligned}
$$

Therefore (6) can be used from now on along with

$$
y(x) = y_0 + xy_0' + \sum_{n=0}^{\infty} \frac{x^{n+2}}{(n+2)\,!} F_n|_{x_0,y_0,y_0'} \tag{7}
$$

To find $y(x)$ series solution around $x = 0$. Hence

$$
\begin{aligned}
F_0 &= -\frac{y(t)}{t+1} \\
F_1 &= \frac{dF_0}{dt} \\
&= \frac{\partial F_0}{\partial t} + \frac{\partial F_0}{\partial y}\frac{d}{dt}y(t) + \frac{\partial F_0}{\partial \frac{d}{dt}y(t)}F_0 \\
&= \frac{(-t-1)\left(\frac{d}{dt}y(t)\right) + y(t)}{(t+1)^2} \\
F_2 &= \frac{dF_1}{dt} \\
&= \frac{\partial F_1}{\partial t} + \frac{\partial F_1}{\partial y}\frac{d}{dt}y(t) + \frac{\partial F_1}{\partial \frac{d}{dt}y(t)}F_1 \\
&= \frac{(2t+2)\left(\frac{d}{dt}y(t)\right) + y(t)(-1+t)}{(t+1)^3} \\
F_3 &= \frac{dF_2}{dt} \\
&= \frac{\partial F_2}{\partial t} + \frac{\partial F_2}{\partial y}\frac{d}{dt}y(t) + \frac{\partial F_2}{\partial \frac{d}{dt}y(t)}F_2 \\
&= \frac{(t^2-4t-5)\left(\frac{d}{dt}y(t)\right) + (-4t+2)\,y(t)}{(t+1)^4} \\
F_4 &= \frac{dF_3}{dt} \\
&= \frac{\partial F_3}{\partial t} + \frac{\partial F_3}{\partial y}\frac{d}{dt}y(t) + \frac{\partial F_3}{\partial \frac{d}{dt}y(t)}F_3 \\
&= \frac{(-6t^2+12t+18)\left(\frac{d}{dt}y(t)\right) - y(t)\,(t^2-16t+7)}{(t+1)^5}
\end{aligned}
$$

And so on. Evaluating all the above at initial conditions $t = 0$ and $y(0) = y(0)$ and $y'(0) = y'(0)$ gives

$$
\begin{aligned}
F_0 &= -y(0) \\
F_1 &= -y'(0) + y(0) \\
F_2 &= -y(0) + 2y'(0) \\
F_3 &= 2y(0) - 5y'(0) \\
F_4 &= -7y(0) + 18y'(0)
\end{aligned}
$$

Substituting all the above in (7) and simplifying gives the solution as

$$y(t) = \left(1 - \frac{1}{2}t^2 + \frac{1}{6}t^3 - \frac{1}{24}t^4 + \frac{1}{60}t^5 - \frac{7}{720}t^6\right) y(0) + \left(t - \frac{1}{6}t^3 + \frac{1}{12}t^4 - \frac{1}{24}t^5 + \frac{1}{40}t^6\right) y'(0) + O(t^6)$$

Since the expansion point $t = 0$ is an ordinary, we can also solve this using standard power series The ode is normalized to be

$$\left(\frac{d^2}{dt^2}y(t)\right)(t+1) + y(t) = 0$$

Let the solution be represented as power series of the form

$$y(t) = \sum_{n=0}^{\infty} a_n t^n$$

Then

$$\frac{d}{dt}y(t) = \sum_{n=1}^{\infty} n a_n t^{n-1}$$

$$\frac{d^2}{dt^2}y(t) = \sum_{n=2}^{\infty} n(n-1) a_n t^{n-2}$$

Substituting the above back into the ode gives

$$\left(\sum_{n=2}^{\infty} n(n-1) a_n t^{n-2}\right)(t+1) + \left(\sum_{n=0}^{\infty} a_n t^n\right) = 0 \tag{1}$$

Which simplifies to

$$\left(\sum_{n=2}^{\infty} n\, t^{n-1} a_n (n-1)\right) + \left(\sum_{n=2}^{\infty} n(n-1) a_n t^{n-2}\right) + \left(\sum_{n=0}^{\infty} a_n t^n\right) = 0 \tag{2}$$

The next step is to make all powers of t be n in each summation term. Going over each summation term above with power of t in it which is not already t^n and adjusting the power and the corresponding index gives

$$\sum_{n=2}^{\infty} n\, t^{n-1} a_n (n-1) = \sum_{n=1}^{\infty} (n+1) a_{n+1} n\, t^n$$

$$\sum_{n=2}^{\infty} n(n-1) a_n t^{n-2} = \sum_{n=0}^{\infty} (n+2) a_{n+2} (n+1)\, t^n$$

Substituting all the above in Eq (2) gives the following equation where now all powers of t are the same and equal to n.

$$\left(\sum_{n=1}^{\infty} (n+1)\, a_{n+1} n\, t^{n}\right) + \left(\sum_{n=0}^{\infty} (n+2)\, a_{n+2} (n+1)\, t^{n}\right) + \left(\sum_{n=0}^{\infty} a_{n} t^{n}\right) = 0 \tag{3}$$

$n = 0$ gives

$$2a_2 + a_0 = 0$$

$$a_2 = -\frac{a_0}{2}$$

For $1 \leq n$, the recurrence equation is

$$(n+1)\, a_{n+1} n + (n+2)\, a_{n+2} (n+1) + a_n = 0 \tag{4}$$

Solving for a_{n+2}, gives

$$\begin{aligned} a_{n+2} &= -\frac{n^2 a_{n+1} + n a_{n+1} + a_n}{(n+2)\,(n+1)} \\ &= -\frac{a_n}{(n+2)\,(n+1)} - \frac{(n^2+n)\, a_{n+1}}{(n+2)\,(n+1)} \end{aligned} \tag{5}$$

For $n = 1$ the recurrence equation gives

$$2a_2 + 6a_3 + a_1 = 0$$

Which after substituting the earlier terms found becomes

$$a_3 = \frac{a_0}{6} - \frac{a_1}{6}$$

For $n = 2$ the recurrence equation gives

$$6a_3 + 12a_4 + a_2 = 0$$

Which after substituting the earlier terms found becomes

$$a_4 = -\frac{a_0}{24} + \frac{a_1}{12}$$

For $n = 3$ the recurrence equation gives

$$12a_4 + 20a_5 + a_3 = 0$$

Which after substituting the earlier terms found becomes

$$a_5 = \frac{a_0}{60} - \frac{a_1}{24}$$

For $n = 4$ the recurrence equation gives

$$20a_5 + 30a_6 + a_4 = 0$$

Which after substituting the earlier terms found becomes

$$a_6 = -\frac{7a_0}{720} + \frac{a_1}{40}$$

For $n = 5$ the recurrence equation gives

$$30a_6 + 42a_7 + a_5 = 0$$

Which after substituting the earlier terms found becomes

$$a_7 = \frac{11a_0}{1680} - \frac{17a_1}{1008}$$

And so on. Therefore the solution is

$$\begin{aligned} y(t) &= \sum_{n=0}^{\infty} a_n t^n \\ &= a_3 t^3 + a_2 t^2 + a_1 t + a_0 + \dots \end{aligned}$$

Substituting the values for a_n found above, the solution becomes

$$y(t) = a_0 + a_1 t - \frac{a_0 t^2}{2} + \left(\frac{a_0}{6} - \frac{a_1}{6}\right) t^3 + \left(-\frac{a_0}{24} + \frac{a_1}{12}\right) t^4 + \left(\frac{a_0}{60} - \frac{a_1}{24}\right) t^5 + \dots$$

Collecting terms, the solution becomes

$$y(t) = \left(1 - \frac{1}{2}t^2 + \frac{1}{6}t^3 - \frac{1}{24}t^4 + \frac{1}{60}t^5\right) a_0 + \left(t - \frac{1}{6}t^3 + \frac{1}{12}t^4 - \frac{1}{24}t^5\right) a_1 + O(t^6) \quad (3)$$

At $t = 0$ the solution above becomes

$$y(t) = \left(1 - \frac{1}{2}t^2 + \frac{1}{6}t^3 - \frac{1}{24}t^4 + \frac{1}{60}t^5\right)c_1 + \left(t - \frac{1}{6}t^3 + \frac{1}{12}t^4 - \frac{1}{24}t^5\right)c_2 + O(t^6)$$

Replacing t in the above with the original independent variable $x$using $t = x-1$ results in

$$\begin{aligned} y = &\left(1 - \frac{(x-1)^2}{2} + \frac{(x-1)^3}{6} - \frac{(x-1)^4}{24} + \frac{(x-1)^5}{60} - \frac{7(x-1)^6}{720}\right)y(1) \\ &+ \left(x - 1 - \frac{(x-1)^3}{6} + \frac{(x-1)^4}{12} - \frac{(x-1)^5}{24} + \frac{(x-1)^6}{40}\right)y'(1) + O\left((x-1)^6\right) \end{aligned}$$

Summary

The solution(s) found are the following

$$\begin{aligned} y = &\left(1 - \frac{(x-1)^2}{2} + \frac{(x-1)^3}{6} - \frac{(x-1)^4}{24} + \frac{(x-1)^5}{60} - \frac{7(x-1)^6}{720}\right)y(1) \\ &+ \left(x - 1 - \frac{(x-1)^3}{6} + \frac{(x-1)^4}{12} - \frac{(x-1)^5}{24} + \frac{(x-1)^6}{40}\right)y'(1) + O\left((x-1)^6\right) \end{aligned} \tag{1}$$

Verification of solutions

$$\begin{aligned} y = &\left(1 - \frac{(x-1)^2}{2} + \frac{(x-1)^3}{6} - \frac{(x-1)^4}{24} + \frac{(x-1)^5}{60} - \frac{7(x-1)^6}{720}\right)y(1) \\ &+ \left(x - 1 - \frac{(x-1)^3}{6} + \frac{(x-1)^4}{12} - \frac{(x-1)^5}{24} + \frac{(x-1)^6}{40}\right)y'(1) + O\left((x-1)^6\right) \end{aligned}$$

Verified OK.

14.12.1 Maple step by step solution

Let's solve

$y''x + y = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

$y'' = -\frac{y}{x}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

$y'' + \frac{y}{x} = 0$

□ Check to see if $x_0 = 0$ is a regular singular point

- ◦ Define functions

$\left[P_2(x) = 0, P_3(x) = \frac{1}{x}\right]$

- ◦ $x \cdot P_2(x)$ is analytic at $x = 0$

$\left(x \cdot P_2(x)\right)\Big|_{x=0} = 0$

- ◦ $x^2 \cdot P_3(x)$ is analytic at $x = 0$

$\left(x^2 \cdot P_3(x)\right)\Big|_{x=0} = 0$

- ◦ $x = 0$is a regular singular point

Check to see if $x_0 = 0$ is a regular singular point

$x_0 = 0$

- Multiply by denominators

$y''x + y = 0$

- Assume series solution for y

$y = \sum_{k=0}^{\infty} a_k x^{k+r}$

□ Rewrite ODE with series expansions

- ◦ Convert $x \cdot y''$ to series expansion

$x \cdot y'' = \sum_{k=0}^{\infty} a_k(k+r)(k+r-1)\, x^{k+r-1}$

- ◦ Shift index using $k- > k+1$

$x \cdot y'' = \sum_{k=-1}^{\infty} a_{k+1}(k+1+r)(k+r)\, x^{k+r}$

Rewrite ODE with series expansions

$a_0 r(-1+r)\, x^{-1+r} + \left(\sum_{k=0}^{\infty} \left(a_{k+1}(k+1+r)(k+r) + a_k\right) x^{k+r}\right) = 0$

- $a_0$cannot be 0 by assumption, giving the indicial equation

$r(-1+r) = 0$

- Values of r that satisfy the indicial equation

  $r \in \{0, 1\}$

- Each term in the series must be 0, giving the recursion relation

  $a_{k+1}(k+1+r)(k+r) + a_k = 0$

- Recursion relation that defines series solution to ODE

  $a_{k+1} = -\frac{a_k}{(k+1+r)(k+r)}$

- Recursion relation for $r = 0$

  $a_{k+1} = -\frac{a_k}{(k+1)k}$

- Solution for $r = 0$

  $\left[y = \sum_{k=0}^{\infty} a_k x^k, a_{k+1} = -\frac{a_k}{(k+1)k}\right]$

- Recursion relation for $r = 1$

  $a_{k+1} = -\frac{a_k}{(k+2)(k+1)}$

- Solution for $r = 1$

  $\left[y = \sum_{k=0}^{\infty} a_k x^{k+1}, a_{k+1} = -\frac{a_k}{(k+2)(k+1)}\right]$

- Combine solutions and rename parameters

  $\left[y = \left(\sum_{k=0}^{\infty} a_k x^k\right) + \left(\sum_{k=0}^{\infty} b_k x^{k+1}\right), a_{k+1} = -\frac{a_k}{(k+1)k}, b_{k+1} = -\frac{b_k}{(k+2)(k+1)}\right]$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
<- No Liouvillian solutions exists
-> Trying a solution in terms of special functions:
   -> Bessel
   <- Bessel successful
<- special function solution successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 49

```
Order:=6;
dsolve(x*diff(y(x),x$2)+y(x)=0,y(x),type='series',x=1);
```

$$y(x) = \left(1 - \frac{(x-1)^2}{2} + \frac{(x-1)^3}{6} - \frac{(x-1)^4}{24} + \frac{(x-1)^5}{60}\right) y(1) + \left(x - 1 - \frac{(x-1)^3}{6} + \frac{(x-1)^4}{12} - \frac{(x-1)^5}{24}\right) D(y)(1) + O(x^6)$$

✓ Solution by Mathematica

Time used: 0.001 (sec). Leaf size: 78

```
AsymptoticDSolveValue[x*y''[x]+y[x]==0,y[x],{x,1,5}]
```

$$y(x) \to c_1 \left(\frac{1}{60}(x-1)^5 - \frac{1}{24}(x-1)^4 + \frac{1}{6}(x-1)^3 - \frac{1}{2}(x-1)^2 + 1\right) + c_2 \left(-\frac{1}{24}(x-1)^5 + \frac{1}{12}(x-1)^4 - \frac{1}{6}(x-1)^3 + x - 1\right)$$

14.13 problem 10

Internal problem ID [745]
Internal file name [OUTPUT/745_Sunday_June_05_2022_01_48_36_AM_47024459/index.tex]

Book: Elementary differential equations and boundary value problems, 10th ed., Boyce and DiPrima
Section: Chapter 5.3, Series Solutions Near an Ordinary Point, Part II. page 269
Problem number: 10.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second order series method. Ordinary point", "second_order_change_of_variable_on_x_method_1", "second_order_change_of_variable_on_x_method_2", "second order series method. Taylor series method"**

Maple gives the following as the ode type

```
[_Gegenbauer, [_2nd_order, _linear, `_with_symmetry_[0,F(x)]`]]
```

$$\boxed{(-x^2+1)\, y'' - y'x + \alpha^2 y = 0}$$

With the expansion point for the power series method at $x = 0$.

Solving ode using Taylor series method. This gives review on how the Taylor series method works for solving second order ode.

Let

$$y'' = f(x, y, y')$$

Assuming expansion is at $x_0 = 0$ (we can always shift the actual expansion point to 0 by change of variables) and assuming $f(x, y, y')$ is analytic at x_0 which must be the case for an ordinary point. Let initial conditions be $y(x_0) = y_0$ and $y'(x_0) = y'_0$. Using

Taylor series gives

$$\begin{aligned}
y(x) &= y(x_0) + (x - x_0)\, y'(x_0) + \frac{(x-x_0)^2}{2} y''(x_0) + \frac{(x-x_0)^3}{3!} y'''(x_0) + \cdots \\
&= y_0 + x y_0' + \frac{x^2}{2} f|_{x_0,y_0,y_0'} + \frac{x^3}{3!} f'|_{x_0,y_0,y_0'} + \cdots \\
&= y_0 + x y_0' + \sum_{n=0}^{\infty} \frac{x^{n+2}}{(n+2)\,!} \frac{d^n f}{dx^n}\bigg|_{x_0,y_0,y_0'}
\end{aligned}$$

But

$$\begin{aligned}
\frac{df}{dx} &= \frac{\partial f}{\partial x}\frac{dx}{dx} + \frac{\partial f}{\partial y}\frac{dy}{dx} + \frac{\partial f}{\partial y'}\frac{dy'}{dx} && (1) \\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} y' + \frac{\partial f}{\partial y'} y'' && (846) \\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} y' + \frac{\partial f}{\partial y'} f && (847) \\
\frac{d^2 f}{dx^2} &= \frac{d}{dx}\left(\frac{df}{dx}\right) \\
&= \frac{\partial}{\partial x}\left(\frac{df}{dx}\right) + \frac{\partial}{\partial y}\left(\frac{df}{dx}\right) y' + \frac{\partial}{\partial y'}\left(\frac{df}{dx}\right) f && (2) \\
\frac{d^3 f}{dx^3} &= \frac{d}{dx}\left(\frac{d^2 f}{dx^2}\right) \\
&= \frac{\partial}{\partial x}\left(\frac{d^2 f}{dx^2}\right) + \left(\frac{\partial}{\partial y}\frac{d^2 f}{dx^2}\right) y' + \frac{\partial}{\partial y'}\left(\frac{d^2 f}{dx^2}\right) f && (3) \\
&\vdots
\end{aligned}$$

And so on. Hence if we name $F_0 = f(x,y,y')$ then the above can be written as

$$\begin{aligned}
F_0 &= f(x,y,y') \qquad (4)\\
F_1 &= \frac{df}{dx}\\
&= \frac{dF_0}{dx}\\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}y' + \frac{\partial f}{\partial y'}y''\\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}y' + \frac{\partial f}{\partial y'}f \qquad (5)\\
&= \frac{\partial F_0}{\partial x} + \frac{\partial F_0}{\partial y}y' + \frac{\partial F_0}{\partial y'}F_0\\
F_2 &= \frac{d}{dx}\left(\frac{d}{dx}f\right)\\
&= \frac{d}{dx}(F_1)\\
&= \frac{\partial}{\partial x}F_1 + \left(\frac{\partial F_1}{\partial y}\right)y' + \left(\frac{\partial F_1}{\partial y'}\right)y''\\
&= \frac{\partial}{\partial x}F_1 + \left(\frac{\partial F_1}{\partial y}\right)y' + \left(\frac{\partial F_1}{\partial y'}\right)F_0\\
&\vdots\\
F_n &= \frac{d}{dx}(F_{n-1})\\
&= \frac{\partial}{\partial x}F_{n-1} + \left(\frac{\partial F_{n-1}}{\partial y}\right)y' + \left(\frac{\partial F_{n-1}}{\partial y'}\right)y''\\
&= \frac{\partial}{\partial x}F_{n-1} + \left(\frac{\partial F_{n-1}}{\partial y}\right)y' + \left(\frac{\partial F_{n-1}}{\partial y'}\right)F_0 \qquad (6)
\end{aligned}$$

Therefore (6) can be used from now on along with

$$y(x) = y_0 + xy_0' + \sum_{n=0}^{\infty}\frac{x^{n+2}}{(n+2)!}F_n|_{x_0,y_0,y_0'} \qquad (7)$$

To find $y(x)$ series solution around $x = 0$. Hence

$$
\begin{aligned}
F_0 &= \frac{\alpha^2 y - y' x}{x^2 - 1} \\
F_1 &= \frac{dF_0}{dx} \\
&= \frac{\partial F_0}{\partial x} + \frac{\partial F_0}{\partial y} y' + \frac{\partial F_0}{\partial y'} F_0 \\
&= \frac{\left(\left(\alpha^2 + 2\right) x^2 - \alpha^2 + 1\right) y' - 3 y \alpha^2 x}{\left(x^2 - 1\right)^2} \\
F_2 &= \frac{dF_1}{dx} \\
&= \frac{\partial F_1}{\partial x} + \frac{\partial F_1}{\partial y} y' + \frac{\partial F_1}{\partial y'} F_1 \\
&= \frac{\left(-6\alpha^2 x^3 + 6\alpha^2 x - 6x^3 - 9x\right) y' + \left(\left(\alpha^2 + 11\right) x^2 - \alpha^2 + 4\right) \alpha^2 y}{\left(x^2 - 1\right)^3} \\
F_3 &= \frac{dF_2}{dx} \\
&= \frac{\partial F_2}{\partial x} + \frac{\partial F_2}{\partial y} y' + \frac{\partial F_2}{\partial y'} F_2 \\
&= \frac{\left(\left(\alpha^4 + 35\alpha^2 + 24\right) x^4 + \left(-2\alpha^4 - 25\alpha^2 + 72\right) x^2 + \alpha^4 - 10\alpha^2 + 9\right) y' - 10\left(\left(\alpha^2 + 5\right) x^2 - \alpha^2 + \frac{11}{2}\right) x \alpha}{\left(x^2 - 1\right)^4} \\
F_4 &= \frac{dF_3}{dx} \\
&= \frac{\partial F_3}{\partial x} + \frac{\partial F_3}{\partial y} y' + \frac{\partial F_3}{\partial y'} F_3 \\
&= -\frac{15\left(\left(\left(\alpha^4 + 15\alpha^2 + 8\right) x^4 + \left(-2\alpha^4 - 2\alpha^2 + 40\right) x^2 + \alpha^4 - 13\alpha^2 + 15\right) x y' - \frac{\left(\left(\alpha^4 + 85\alpha^2 + 274\right) x^4 + \left(-2\alpha^4 - 65\right.\right.}{1}}{\left(x^2 - 1\right)^6}
\end{aligned}
$$

And so on. Evaluating all the above at initial conditions $x = 0$ and $y(0) = y(0)$ and $y'(0) = y'(0)$ gives

$$
\begin{aligned}
F_0 &= -y(0)\,\alpha^2 \\
F_1 &= -y'(0)\,\alpha^2 + y'(0) \\
F_2 &= y(0)\,\alpha^4 - 4y(0)\,\alpha^2 \\
F_3 &= y'(0)\,\alpha^4 - 10y'(0)\,\alpha^2 + 9y'(0) \\
F_4 &= -y(0)\,\alpha^6 + 20y(0)\,\alpha^4 - 64y(0)\,\alpha^2
\end{aligned}
$$

Substituting all the above in (7) and simplifying gives the solution as

$$y = \left(1 - \frac{1}{2}\alpha^2 x^2 + \frac{1}{24}\alpha^4 x^4 - \frac{1}{6}\alpha^2 x^4 - \frac{1}{720}x^6\alpha^6 + \frac{1}{36}x^6\alpha^4 - \frac{4}{45}x^6\alpha^2\right) y(0) + \left(x - \frac{1}{6}\alpha^2 x^3 + \frac{1}{6}x^3 + \frac{1}{120}\alpha^4 x^5 - \frac{1}{12}\alpha^2 x^5 + \frac{3}{40}x^5\right) y'(0) + O(x^6)$$

Since the expansion point $x = 0$ is an ordinary, we can also solve this using standard power series The ode is normalized to be

$$(-x^2 + 1)\, y'' - y'x + \alpha^2 y = 0$$

Let the solution be represented as power series of the form

$$y = \sum_{n=0}^{\infty} a_n x^n$$

Then

$$y' = \sum_{n=1}^{\infty} n a_n x^{n-1}$$
$$y'' = \sum_{n=2}^{\infty} n(n-1)\, a_n x^{n-2}$$

Substituting the above back into the ode gives

$$(-x^2 + 1)\left(\sum_{n=2}^{\infty} n(n-1)\, a_n x^{n-2}\right) - \left(\sum_{n=1}^{\infty} n a_n x^{n-1}\right) x + \alpha^2 \left(\sum_{n=0}^{\infty} a_n x^n\right) = 0 \quad (1)$$

Which simplifies to

$$\sum_{n=2}^{\infty} (-x^n a_n n(n-1)) + \left(\sum_{n=2}^{\infty} n(n-1)\, a_n x^{n-2}\right) + \sum_{n=1}^{\infty} (-n a_n x^n) + \left(\sum_{n=0}^{\infty} \alpha^2 a_n x^n\right) = 0 \quad (2)$$

The next step is to make all powers of x be n in each summation term. Going over each summation term above with power of x in it which is not already x^n and adjusting the power and the corresponding index gives

$$\sum_{n=2}^{\infty} n(n-1)\, a_n x^{n-2} = \sum_{n=0}^{\infty} (n+2)\, a_{n+2}(n+1)\, x^n$$

Substituting all the above in Eq (2) gives the following equation where now all powers of x are the same and equal to n.

$$\sum_{n=2}^{\infty} \left(-x^n a_n n(n-1)\right) + \left(\sum_{n=0}^{\infty} (n+2)\, a_{n+2}(n+1)\, x^n\right) + \sum_{n=1}^{\infty} (-n a_n x^n) + \left(\sum_{n=0}^{\infty} \alpha^2 a_n x^n\right) = 0 \tag{3}$$

$n = 0$ gives

$$a_0\alpha^2 + 2a_2 = 0$$

$$a_2 = -\frac{a_0\alpha^2}{2}$$

$n = 1$ gives

$$a_1\alpha^2 - a_1 + 6a_3 = 0$$

Which after substituting earlier equations, simplifies to

$$a_3 = -\frac{1}{6}a_1\alpha^2 + \frac{1}{6}a_1$$

For $2 \leq n$, the recurrence equation is

$$-na_n(n-1) + (n+2)\, a_{n+2}(n+1) - na_n + a_n\alpha^2 = 0 \tag{4}$$

Solving for a_{n+2}, gives

$$a_{n+2} = -\frac{a_n(\alpha^2 - n^2)}{(n+2)\,(n+1)} \tag{5}$$

For $n = 2$ the recurrence equation gives

$$a_2\alpha^2 - 4a_2 + 12a_4 = 0$$

Which after substituting the earlier terms found becomes

$$a_4 = \frac{1}{24}\alpha^4 a_0 - \frac{1}{6}a_0\alpha^2$$

For $n = 3$ the recurrence equation gives

$$a_3\alpha^2 - 9a_3 + 20a_5 = 0$$

Which after substituting the earlier terms found becomes

$$a_5 = \frac{1}{120}\alpha^4 a_1 - \frac{1}{12}a_1\alpha^2 + \frac{3}{40}a_1$$

For $n = 4$ the recurrence equation gives

$$a_4\alpha^2 - 16a_4 + 30a_6 = 0$$

Which after substituting the earlier terms found becomes

$$a_6 = -\frac{1}{720}\alpha^6 a_0 + \frac{1}{36}\alpha^4 a_0 - \frac{4}{45}a_0\alpha^2$$

For $n = 5$ the recurrence equation gives

$$a_5\alpha^2 - 25a_5 + 42a_7 = 0$$

Which after substituting the earlier terms found becomes

$$a_7 = -\frac{1}{5040}\alpha^6 a_1 + \frac{1}{144}\alpha^4 a_1 - \frac{37}{720}a_1\alpha^2 + \frac{5}{112}a_1$$

And so on. Therefore the solution is

$$\begin{aligned} y &= \sum_{n=0}^{\infty} a_n x^n \\ &= a_3x^3 + a_2x^2 + a_1x + a_0 + \dots \end{aligned}$$

Substituting the values for a_n found above, the solution becomes

$$\begin{aligned} y = a_0 + a_1x - \frac{a_0\alpha^2x^2}{2} + \left(-\frac{1}{6}a_1\alpha^2 + \frac{1}{6}a_1\right)x^3 \\ + \left(\frac{1}{24}\alpha^4 a_0 - \frac{1}{6}a_0\alpha^2\right)x^4 + \left(\frac{1}{120}\alpha^4 a_1 - \frac{1}{12}a_1\alpha^2 + \frac{3}{40}a_1\right)x^5 + \dots \end{aligned}$$

Collecting terms, the solution becomes

$$y = \left(1 - \frac{\alpha^2 x^2}{2} + \left(\frac{1}{24}\alpha^4 - \frac{1}{6}\alpha^2\right)x^4\right)a_0 + \left(x + \left(-\frac{\alpha^2}{6} + \frac{1}{6}\right)x^3 + \left(\frac{1}{120}\alpha^4 - \frac{1}{12}\alpha^2 + \frac{3}{40}\right)x^5\right)a_1 + O(x^6) \tag{3}$$

At $x = 0$ the solution above becomes

$$y = \left(1 - \frac{\alpha^2 x^2}{2} + \left(\frac{1}{24}\alpha^4 - \frac{1}{6}\alpha^2\right)x^4\right)c_1 + \left(x + \left(-\frac{\alpha^2}{6} + \frac{1}{6}\right)x^3 + \left(\frac{1}{120}\alpha^4 - \frac{1}{12}\alpha^2 + \frac{3}{40}\right)x^5\right)c_2 + O(x^6)$$

Summary

The solution(s) found are the following

$$y = \left(1 - \frac{1}{2}\alpha^2 x^2 + \frac{1}{24}\alpha^4 x^4 - \frac{1}{6}\alpha^2 x^4 - \frac{1}{720}x^6\alpha^6 + \frac{1}{36}x^6\alpha^4 - \frac{4}{45}x^6\alpha^2\right)y(0) + \left(x - \frac{1}{6}\alpha^2 x^3 + \frac{1}{6}x^3 + \frac{1}{120}\alpha^4 x^5 - \frac{1}{12}\alpha^2 x^5 + \frac{3}{40}x^5\right)y'(0) + O(x^6) \tag{1}$$

$$y = \left(1 - \frac{\alpha^2 x^2}{2} + \left(\frac{1}{24}\alpha^4 - \frac{1}{6}\alpha^2\right)x^4\right)c_1 + \left(x + \left(-\frac{\alpha^2}{6} + \frac{1}{6}\right)x^3 + \left(\frac{1}{120}\alpha^4 - \frac{1}{12}\alpha^2 + \frac{3}{40}\right)x^5\right)c_2 + O(x^6) \tag{2}$$

Verification of solutions

$$y = \left(1 - \frac{1}{2}\alpha^2 x^2 + \frac{1}{24}\alpha^4 x^4 - \frac{1}{6}\alpha^2 x^4 - \frac{1}{720}x^6\alpha^6 + \frac{1}{36}x^6\alpha^4 - \frac{4}{45}x^6\alpha^2\right)y(0) + \left(x - \frac{1}{6}\alpha^2 x^3 + \frac{1}{6}x^3 + \frac{1}{120}\alpha^4 x^5 - \frac{1}{12}\alpha^2 x^5 + \frac{3}{40}x^5\right)y'(0) + O(x^6)$$

Verified OK.

$$y = \left(1 - \frac{\alpha^2 x^2}{2} + \left(\frac{1}{24}\alpha^4 - \frac{1}{6}\alpha^2\right)x^4\right)c_1 + \left(x + \left(-\frac{\alpha^2}{6} + \frac{1}{6}\right)x^3 + \left(\frac{1}{120}\alpha^4 - \frac{1}{12}\alpha^2 + \frac{3}{40}\right)x^5\right)c_2 + O(x^6)$$

Verified OK.

14.13.1 Maple step by step solution

Let's solve

$(-x^2+1)\, y'' - y'x + \alpha^2 y = 0$

- Highest derivative means the order of the ODE is 2

y''

- Isolate 2nd derivative

$y'' = -\frac{x y'}{x^2-1} + \frac{\alpha^2 y}{x^2-1}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

$y'' + \frac{x y'}{x^2-1} - \frac{\alpha^2 y}{x^2-1} = 0$

□ Check to see if x_0 is a regular singular point

- ○ Define functions

$\left[P_2(x) = \frac{x}{x^2-1}, P_3(x) = -\frac{\alpha^2}{x^2-1}\right]$

- ○ $(x+1)\cdot P_2(x)$ is analytic at $x=-1$

$\left((x+1)\cdot P_2(x)\right)\bigg|_{x=-1} = \frac{1}{2}$

- ○ $(x+1)^2\cdot P_3(x)$ is analytic at $x=-1$

$\left((x+1)^2\cdot P_3(x)\right)\bigg|_{x=-1} = 0$

- ○ $x=-1$is a regular singular point

Check to see if x_0 is a regular singular point

$x_0 = -1$

- Multiply by denominators

$(x^2-1)\, y'' + y'x - \alpha^2 y = 0$

- Change variables using $x = u-1$ so that the regular singular point is at $u=0$

$(u^2-2u)\left(\frac{d^2}{du^2}y(u)\right) + (u-1)\left(\frac{d}{du}y(u)\right) - \alpha^2 y(u) = 0$

- Assume series solution for $y(u)$

$y(u) = \sum_{k=0}^{\infty} a_k u^{k+r}$

□ Rewrite ODE with series expansions

- Convert $u^m \cdot \left(\frac{d}{du}y(u)\right)$ to series expansion for $m = 0..1$

  $$u^m \cdot \left(\frac{d}{du}y(u)\right) = \sum_{k=0}^{\infty} a_k(k+r)\, u^{k+r-1+m}$$

- Shift index using $k->k+1-m$

  $$u^m \cdot \left(\frac{d}{du}y(u)\right) = \sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\, u^{k+r}$$

- Convert $u^m \cdot \left(\frac{d^2}{du^2}y(u)\right)$ to series expansion for $m = 1..2$

  $$u^m \cdot \left(\frac{d^2}{du^2}y(u)\right) = \sum_{k=0}^{\infty} a_k(k+r)\,(k+r-1)\, u^{k+r-2+m}$$

- Shift index using $k->k+2-m$

  $$u^m \cdot \left(\frac{d^2}{du^2}y(u)\right) = \sum_{k=-2+m}^{\infty} a_{k+2-m}(k+2-m+r)\,(k+1-m+r)\, u^{k+r}$$

  Rewrite ODE with series expansions

  $$-a_0 r(-1+2r)\, u^{-1+r} + \left(\sum_{k=0}^{\infty} \left(-a_{1+k}(1+k+r)\,(1+2k+2r) - a_k(\alpha+k+r)\,(\alpha-k-r)\right) u^{k+r}\right)$$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $$-r(-1+2r) = 0$$

- Values of r that satisfy the indicial equation

  $$r \in \left\{0, \frac{1}{2}\right\}$$

- Each term in the series must be 0, giving the recursion relation

  $$-2\left(\frac{1}{2}+k+r\right)(1+k+r)\, a_{1+k} + a_k(\alpha+k+r)\,(k+r-\alpha) = 0$$

- Recursion relation that defines series solution to ODE

  $$a_{1+k} = -\frac{a_k(\alpha+k+r)(\alpha-k-r)}{(1+2k+2r)(1+k+r)}$$

- Recursion relation for $r = 0$

  $$a_{1+k} = -\frac{a_k(\alpha+k)(\alpha-k)}{(1+2k)(1+k)}$$

- Solution for $r = 0$

  $$\left[y(u) = \sum_{k=0}^{\infty} a_k u^k, a_{1+k} = -\frac{a_k(\alpha+k)(\alpha-k)}{(1+2k)(1+k)}\right]$$

- Revert the change of variables $u = x+1$

  $$\left[y = \sum_{k=0}^{\infty} a_k(x+1)^k, a_{1+k} = -\frac{a_k(\alpha+k)(\alpha-k)}{(1+2k)(1+k)}\right]$$

- Recursion relation for $r = \frac{1}{2}$

  $a_{1+k} = -\frac{a_k \left(\alpha+k+\frac{1}{2}\right) \left(\alpha-k-\frac{1}{2}\right)}{(2+2k) \left(\frac{3}{2}+k\right)}$

- Solution for $r = \frac{1}{2}$

  $$\left[y(u) = \sum_{k=0}^{\infty} a_k u^{k+\frac{1}{2}}, a_{1+k} = -\frac{a_k \left(\alpha+k+\frac{1}{2}\right) \left(\alpha-k-\frac{1}{2}\right)}{(2+2k) \left(\frac{3}{2}+k\right)}\right]$$

- Revert the change of variables $u = x + 1$

  $$\left[y = \sum_{k=0}^{\infty} a_k (x+1)^{k+\frac{1}{2}}, a_{1+k} = -\frac{a_k \left(\alpha+k+\frac{1}{2}\right) \left(\alpha-k-\frac{1}{2}\right)}{(2+2k) \left(\frac{3}{2}+k\right)}\right]$$

- Combine solutions and rename parameters

  $$\left[y = \left(\sum_{k=0}^{\infty} a_k (x+1)^k\right) + \left(\sum_{k=0}^{\infty} b_k (x+1)^{k+\frac{1}{2}}\right), a_{k+1} = -\frac{a_k(\alpha+k)(\alpha-k)}{(1+2k)(k+1)}, b_{k+1} = -\frac{b_k \left(\alpha+k+\frac{1}{2}\right) \left(\alpha-k-\frac{1}{2}\right)}{(2+2k) \left(\frac{3}{2}+k\right)}\right]$$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
<- linear_1 successful`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 71

```
Order:=6;
dsolve((1-x^2)*diff(y(x),x$2)-x*diff(y(x),x)+alpha^2*y(x)=0,y(x),type='series',x=0);
```

$$y(x) = \left(1 - \frac{\alpha^2 x^2}{2} + \frac{\alpha^2 (\alpha^2 - 4)\, x^4}{24}\right) y(0) + \left(x - \frac{(\alpha^2 - 1)\, x^3}{6} + \frac{(\alpha^4 - 10\alpha^2 + 9)\, x^5}{120}\right) D(y)\,(0) + O\left(x^6\right)$$

✓ Solution by Mathematica

Time used: 0.001 (sec). Leaf size: 88

```
AsymptoticDSolveValue[(1-x^2)*y''[x]-x*y'[x]+\[Alpha]^2*y[x]==0,y[x],{x,0,5}]
```

$$y(x) \to c_2\left(\frac{\alpha^4x^5}{120} - \frac{\alpha^2x^5}{12} + \frac{3x^5}{40} - \frac{\alpha^2x^3}{6} + \frac{x^3}{6} + x\right) + c_1\left(\frac{\alpha^4x^4}{24} - \frac{\alpha^2x^4}{6} - \frac{\alpha^2x^2}{2} + 1\right)$$

14.14 problem 16

Internal problem ID [746]
Internal file name [`OUTPUT/746_Sunday_June_05_2022_01_48_37_AM_93764472/index.tex`]

Book: Elementary differential equations and boundary value problems, 10th ed., Boyce and DiPrima
Section: Chapter 5.3, Series Solutions Near an Ordinary Point, Part II. page 269
Problem number: 16.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"first order ode series method. Ordinary point", "first order ode series method. Taylor series method"**

Maple gives the following as the ode type

```
[_quadrature]
```

$$\boxed{y' - y = 0}$$

With the expansion point for the power series method at $x = 0$.

14.14.1 Solving as series ode

Solving ode using Taylor series method. This gives review on how the Taylor series method works for solving first order ode. Let

$$y' = f(x, y)$$

Where $f(x, y)$ is analytic at expansion point x_0. We can always shift to $x_0 = 0$ if x_0 is not zero. So from now we assume $x_0 = 0$. Assume also that $y(x_0) = y_0$. Using Taylor series

$$\begin{aligned} y(x) &= y(x_0) + (x - x_0)\, y'(x_0) + \frac{(x - x_0)^2}{2} y''(x_0) + \frac{(x - x_0)^3}{3!} y''(x_0) + \cdots \\ &= y_0 + xf + \frac{x^2}{2} \frac{df}{dx}\bigg|_{x_0,y_0} + \frac{x^3}{3!} \frac{d^2 f}{dx^2}\bigg|_{x_0,y_0} + \cdots \\ &= y_0 + \sum_{n=0}^{\infty} \frac{x^{n+1}}{(n+1)\,!} \frac{d^n f}{dx^n}\bigg|_{x_0,y_0} \end{aligned}$$

But

$$\frac{df}{dx} = \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} f \tag{1}$$

$$\frac{d^2 f}{dx^2} = \frac{d}{dx}\left(\frac{df}{dx}\right)$$

$$= \frac{\partial}{\partial x}\left(\frac{df}{dx}\right) + \frac{\partial}{\partial y}\left(\frac{df}{dx}\right) f \tag{2}$$

$$\frac{d^3 f}{dx^3} = \frac{d}{dx}\left(\frac{d^2 f}{dx^2}\right)$$

$$= \frac{\partial}{\partial x}\left(\frac{d^2 f}{dx^2}\right) + \left(\frac{\partial}{\partial y}\frac{d^2 f}{dx^2}\right) f \tag{3}$$

$$\vdots$$

And so on. Hence if we name $F_0 = f(x,y)$ then the above can be written as

$$F_0 = f(x,y) \tag{4}$$

$$F_n = \frac{d}{dx}(F_{n-1})$$

$$= \frac{\partial}{\partial x} F_{n-1} + \left(\frac{\partial F_{n-1}}{\partial y}\right) F_0 \tag{5}$$

For example, for $n = 1$ we see that

$$\begin{aligned} F_1 &= \frac{d}{dx}(F_0) \\ &= \frac{\partial}{\partial x} F_0 + \left(\frac{\partial F_0}{\partial y}\right) F_0 \\ &= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} f \end{aligned}$$

Which is (1). And when $n = 2$

$$\begin{aligned} F_2 &= \frac{d}{dx}(F_1) \\ &= \frac{\partial}{\partial x} F_1 + \left(\frac{\partial F_1}{\partial y}\right) F_0 \\ &= \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} f\right) + \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} f\right) f \\ &= \frac{\partial}{\partial x}\left(\frac{df}{dx}\right) + \frac{\partial}{\partial y}\left(\frac{df}{dx}\right) f \end{aligned}$$

Which is (2) and so on. Therefore (4,5) can be used from now on along with

$$y(x) = y_0 + \sum_{n=0}^{\infty} \frac{x^{n+1}}{(n+1)\,!} F_n|_{x_0,y_0} \tag{6}$$

Hence

$$\begin{aligned}
F_0 &= y \\
F_1 &= \frac{dF_0}{dx} \\
&= \frac{\partial F_0}{\partial x} + \frac{\partial F_0}{\partial y} F_0 \\
&= y \\
F_2 &= \frac{dF_1}{dx} \\
&= \frac{\partial F_1}{\partial x} + \frac{\partial F_1}{\partial y} F_1 \\
&= y \\
F_3 &= \frac{dF_2}{dx} \\
&= \frac{\partial F_2}{\partial x} + \frac{\partial F_2}{\partial y} F_2 \\
&= y \\
F_4 &= \frac{dF_3}{dx} \\
&= \frac{\partial F_3}{\partial x} + \frac{\partial F_3}{\partial y} F_3 \\
&= y
\end{aligned}$$

And so on. Evaluating all the above at initial conditions $x(0) = 0$ and $y(0) = y(0)$ gives

$$\begin{aligned}
F_0 &= y(0) \\
F_1 &= y(0) \\
F_2 &= y(0) \\
F_3 &= y(0) \\
F_4 &= y(0)
\end{aligned}$$

Substituting all the above in (6) and simplifying gives the solution as

$$y = \left(1 + x + \frac{1}{2}x^2 + \frac{1}{6}x^3 + \frac{1}{24}x^4 + \frac{1}{120}x^5\right) y(0) + O(x^6)$$

Since $x = 0$ is also an ordinary point, then standard power series can also be used. Writing the ODE as

$$\begin{aligned} y' + q(x)y &= p(x) \\ y' - y &= 0 \end{aligned}$$

Where

$$\begin{aligned} q(x) &= -1 \\ p(x) &= 0 \end{aligned}$$

Next, the type of the expansion point $x = 0$ is determined. This point can be an ordinary point, a regular singular point (also called removable singularity), or irregular singular point (also called non-removable singularity or essential singularity). When $x = 0$ is an ordinary point, then the standard power series is used. If the point is a regular singular point, Frobenius series is used instead. Irregular singular point requires more advanced methods (asymptotic methods) and is not supported now. Hopefully this will be added in the future. $x = 0$ is called an ordinary point $q(x)$ has a Taylor series expansion around the point $x = 0$. $x = 0$ is called a regular singular point if $q(x)$ is not not analytic at $x = 0$ but $xq(x)$ has Taylor series expansion. And finally, $x = 0$ is an irregular singular point if the point is not ordinary and not regular singular. This is the most complicated case. Now the expansion point $x = 0$ is checked to see if it is an ordinary point or not. Let the solution be represented as power series of the form

$$y = \sum_{n=0}^{\infty} a_n x^n$$

Then

$$y' = \sum_{n=1}^{\infty} n a_n x^{n-1}$$

Substituting the above back into the ode gives

$$\left(\sum_{n=1}^{\infty} n a_n x^{n-1}\right) - \left(\sum_{n=0}^{\infty} a_n x^n\right) = 0 \tag{1}$$

Which simplifies to

$$\left(\sum_{n=1}^{\infty} n a_n x^{n-1}\right) + \sum_{n=0}^{\infty} (-a_n x^n) = 0 \tag{2}$$

The next step is to make all powers of x be n in each summation term. Going over each summation term above with power of x in it which is not already x^n and adjusting the power and the corresponding index gives

$$\sum_{n=1}^{\infty} n a_n x^{n-1} = \sum_{n=0}^{\infty} (n+1) a_{n+1} x^n$$

Substituting all the above in Eq (2) gives the following equation where now all powers of x are the same and equal to n.

$$\left(\sum_{n=0}^{\infty} (n+1) a_{n+1} x^n\right) + \sum_{n=0}^{\infty} (-a_n x^n) = 0 \tag{3}$$

For $0 \leq n$, the recurrence equation is

$$(n+1) a_{n+1} - a_n = 0 \tag{4}$$

Solving for a_{n+1}, gives

$$a_{n+1} = \frac{a_n}{n+1} \tag{5}$$

For $n = 0$ the recurrence equation gives

$$a_1 - a_0 = 0$$

Which after substituting the earlier terms found becomes

$$a_1 = a_0$$

For $n = 1$ the recurrence equation gives

$$2a_2 - a_1 = 0$$

Which after substituting the earlier terms found becomes

$$a_2 = \frac{a_0}{2}$$

For $n = 2$ the recurrence equation gives

$$3a_3 - a_2 = 0$$

Which after substituting the earlier terms found becomes

$$a_3 = \frac{a_0}{6}$$

For $n = 3$ the recurrence equation gives

$$4a_4 - a_3 = 0$$

Which after substituting the earlier terms found becomes

$$a_4 = \frac{a_0}{24}$$

For $n = 4$ the recurrence equation gives

$$5a_5 - a_4 = 0$$

Which after substituting the earlier terms found becomes

$$a_5 = \frac{a_0}{120}$$

For $n = 5$ the recurrence equation gives

$$6a_6 - a_5 = 0$$

Which after substituting the earlier terms found becomes

$$a_6 = \frac{a_0}{720}$$

And so on. Therefore the solution is

$$\begin{aligned} y &= \sum_{n=0}^{\infty} a_n x^n \\ &= a_3 x^3 + a_2 x^2 + a_1 x + a_0 + \dots \end{aligned}$$

Substituting the values for a_n found above, the solution becomes

$$y = a_0 + a_0 x + \frac{1}{2} a_0 x^2 + \frac{1}{6} a_0 x^3 + \frac{1}{24} a_0 x^4 + \frac{1}{120} a_0 x^5 + \dots$$

Collecting terms, the solution becomes

$$y = \left(1 + x + \frac{1}{2}x^2 + \frac{1}{6}x^3 + \frac{1}{24}x^4 + \frac{1}{120}x^5\right) a_0 + O\left(x^6\right) \tag{3}$$

Summary
The solution(s) found are the following

$$y = \left(1 + x + \frac{1}{2}x^2 + \frac{1}{6}x^3 + \frac{1}{24}x^4 + \frac{1}{120}x^5\right) y(0) + O\left(x^6\right) \tag{1}$$

$$y = \left(1 + x + \frac{1}{2}x^2 + \frac{1}{6}x^3 + \frac{1}{24}x^4 + \frac{1}{120}x^5\right) c_1 + O\left(x^6\right) \tag{2}$$

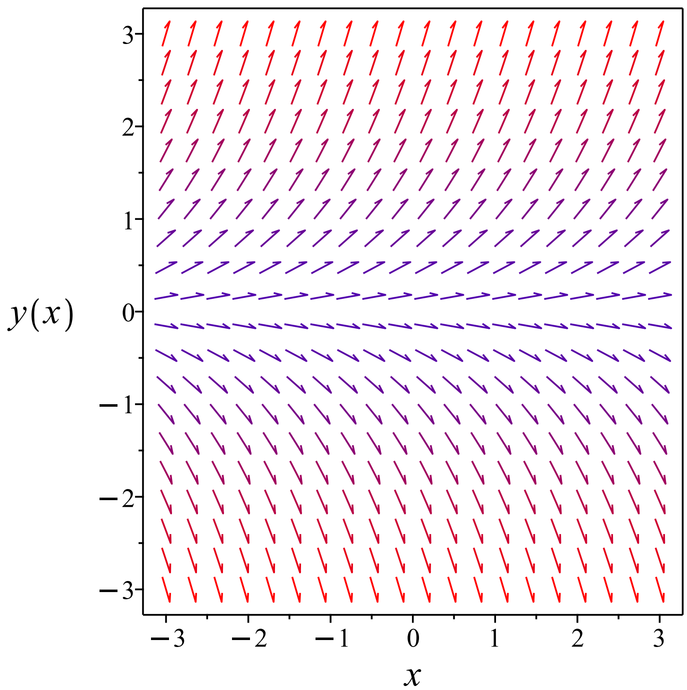


Figure 503: Slope field plot

Verification of solutions

$$y = \left(1 + x + \frac{1}{2}x^2 + \frac{1}{6}x^3 + \frac{1}{24}x^4 + \frac{1}{120}x^5\right) y(0) + O\left(x^6\right)$$

Verified OK.

$$y = \left(1 + x + \frac{1}{2}x^2 + \frac{1}{6}x^3 + \frac{1}{24}x^4 + \frac{1}{120}x^5\right) c_1 + O\left(x^6\right)$$

Verified OK.

14.14.2 Maple step by step solution

Let's solve

$y' - y = 0$

- Highest derivative means the order of the ODE is 1

  y'

- Separate variables

  $\frac{y'}{y} = 1$

- Integrate both sides with respect to x

  $\int \frac{y'}{y} dx = \int 1 dx + c_1$

- Evaluate integral

  $\ln(y) = x + c_1$

- Solve for y

  $y = e^{x+c_1}$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
<- 1st order linear successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 34

```
Order:=6;
dsolve(diff(y(x),x)-y(x)=0,y(x),type='series',x=0);
```

$$y(x) = \left(1 + x + \frac{1}{2}x^2 + \frac{1}{6}x^3 + \frac{1}{24}x^4 + \frac{1}{120}x^5\right) y(0) + O(x^6)$$

✓ Solution by Mathematica

Time used: 0.001 (sec). Leaf size: 37

```
AsymptoticDSolveValue[y'[x]-y[x]==0,y[x],{x,0,5}]
```

$$y(x) \to c_1\left(\frac{x^5}{120} + \frac{x^4}{24} + \frac{x^3}{6} + \frac{x^2}{2} + x + 1\right)$$

14.15 problem 17

Internal problem ID [747]
Internal file name [`OUTPUT/747_Sunday_June_05_2022_01_48_38_AM_46508026/index.tex`]

Book: Elementary differential equations and boundary value problems, 10th ed., Boyce and DiPrima
Section: Chapter 5.3, Series Solutions Near an Ordinary Point, Part II. page 269
Problem number: 17.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"first order ode series method. Ordinary point", "first order ode series method. Taylor series method"**

Maple gives the following as the ode type

```
[_separable]
```

$$\boxed{-yx + y' = 0}$$

With the expansion point for the power series method at $x = 0$.

14.15.1 Solving as series ode

Solving ode using Taylor series method. This gives review on how the Taylor series method works for solving first order ode. Let

$$y' = f(x, y)$$

Where $f(x, y)$ is analytic at expansion point x_0. We can always shift to $x_0 = 0$ if x_0 is not zero. So from now we assume $x_0 = 0$. Assume also that $y(x_0) = y_0$. Using Taylor series

$$\begin{aligned}
y(x) &= y(x_0) + (x - x_0)\, y'(x_0) + \frac{(x - x_0)^2}{2} y''(x_0) + \frac{(x - x_0)^3}{3!} y''(x_0) + \cdots \\
&= y_0 + xf + \frac{x^2}{2} \frac{df}{dx}\bigg|_{x_0, y_0} + \frac{x^3}{3!} \frac{d^2 f}{dx^2}\bigg|_{x_0, y_0} + \cdots \\
&= y_0 + \sum_{n=0}^{\infty} \frac{x^{n+1}}{(n+1)\,!} \frac{d^n f}{dx^n}\bigg|_{x_0, y_0}
\end{aligned}$$

But

$$
\begin{aligned}
\frac{df}{dx} &= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} f && (1) \\
\frac{d^2 f}{dx^2} &= \frac{d}{dx}\left(\frac{df}{dx}\right) \\
&= \frac{\partial}{\partial x}\left(\frac{df}{dx}\right) + \frac{\partial}{\partial y}\left(\frac{df}{dx}\right) f && (2) \\
\frac{d^3 f}{dx^3} &= \frac{d}{dx}\left(\frac{d^2 f}{dx^2}\right) \\
&= \frac{\partial}{\partial x}\left(\frac{d^2 f}{dx^2}\right) + \left(\frac{\partial}{\partial y}\frac{d^2 f}{dx^2}\right) f && (3) \\
&\vdots
\end{aligned}
$$

And so on. Hence if we name $F_0 = f(x,y)$ then the above can be written as

$$
\begin{aligned}
F_0 &= f(x,y) && (4) \\
F_n &= \frac{d}{dx}(F_{n-1}) \\
&= \frac{\partial}{\partial x}F_{n-1} + \left(\frac{\partial F_{n-1}}{\partial y}\right) F_0 && (5)
\end{aligned}
$$

For example, for $n = 1$ we see that

$$
\begin{aligned}
F_1 &= \frac{d}{dx}(F_0) \\
&= \frac{\partial}{\partial x}F_0 + \left(\frac{\partial F_0}{\partial y}\right) F_0 \\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} f
\end{aligned}
$$

Which is (1). And when $n = 2$

$$
\begin{aligned}
F_2 &= \frac{d}{dx}(F_1) \\
&= \frac{\partial}{\partial x}F_1 + \left(\frac{\partial F_1}{\partial y}\right) F_0 \\
&= \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} f\right) + \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} f\right) f \\
&= \frac{\partial}{\partial x}\left(\frac{df}{dx}\right) + \frac{\partial}{\partial y}\left(\frac{df}{dx}\right) f
\end{aligned}
$$

Which is (2) and so on. Therefore (4,5) can be used from now on along with

$$y(x) = y_0 + \sum_{n=0}^{\infty} \frac{x^{n+1}}{(n+1)!} F_n|_{x_0,y_0} \tag{6}$$

Hence

$$\begin{aligned}
F_0 &= yx \\
F_1 &= \frac{dF_0}{dx} \\
&= \frac{\partial F_0}{\partial x} + \frac{\partial F_0}{\partial y} F_0 \\
&= y(x^2+1) \\
F_2 &= \frac{dF_1}{dx} \\
&= \frac{\partial F_1}{\partial x} + \frac{\partial F_1}{\partial y} F_1 \\
&= yx(x^2+3) \\
F_3 &= \frac{dF_2}{dx} \\
&= \frac{\partial F_2}{\partial x} + \frac{\partial F_2}{\partial y} F_2 \\
&= y(x^4+6x^2+3) \\
F_4 &= \frac{dF_3}{dx} \\
&= \frac{\partial F_3}{\partial x} + \frac{\partial F_3}{\partial y} F_3 \\
&= yx(x^4+10x^2+15)
\end{aligned}$$

And so on. Evaluating all the above at initial conditions $x(0) = 0$ and $y(0) = y(0)$ gives

$$\begin{aligned}
F_0 &= 0 \\
F_1 &= y(0) \\
F_2 &= 0 \\
F_3 &= 3y(0) \\
F_4 &= 0
\end{aligned}$$

Substituting all the above in (6) and simplifying gives the solution as

$$y = \left(1 + \frac{1}{2}x^2 + \frac{1}{8}x^4\right) y(0) + O(x^6)$$

Since $x = 0$ is also an ordinary point, then standard power series can also be used. Writing the ODE as

$$
\begin{aligned}
y' + q(x)y &= p(x) \\
-yx + y' &= 0
\end{aligned}
$$

Where

$$
\begin{aligned}
q(x) &= -x \\
p(x) &= 0
\end{aligned}
$$

Next, the type of the expansion point $x = 0$ is determined. This point can be an ordinary point, a regular singular point (also called removable singularity), or irregular singular point (also called non-removable singularity or essential singularity). When $x = 0$ is an ordinary point, then the standard power series is used. If the point is a regular singular point, Frobenius series is used instead. Irregular singular point requires more advanced methods (asymptotic methods) and is not supported now. Hopefully this will be added in the future. $x = 0$ is called an ordinary point $q(x)$ has a Taylor series expansion around the point $x = 0$. $x = 0$ is called a regular singular point if $q(x)$ is not not analytic at $x = 0$ but $xq(x)$ has Taylor series expansion. And finally, $x = 0$ is an irregular singular point if the point is not ordinary and not regular singular. This is the most complicated case. Now the expansion point $x = 0$ is checked to see if it is an ordinary point or not. Let the solution be represented as power series of the form

$$
y = \sum_{n=0}^{\infty} a_n x^n
$$

Then

$$
y' = \sum_{n=1}^{\infty} n a_n x^{n-1}
$$

Substituting the above back into the ode gives

$$
-\left(\sum_{n=0}^{\infty} a_n x^n\right) x + \left(\sum_{n=1}^{\infty} n a_n x^{n-1}\right) = 0 \tag{1}
$$

Which simplifies to

$$
\left(\sum_{n=1}^{\infty} n a_n x^{n-1}\right) + \sum_{n=0}^{\infty} \left(-x^{1+n} a_n\right) = 0 \tag{2}
$$

The next step is to make all powers of x be n in each summation term. Going over each summation term above with power of x in it which is not already x^n and adjusting the power and the corresponding index gives

$$\sum_{n=1}^{\infty} n a_n x^{n-1} = \sum_{n=0}^{\infty} (1+n) a_{1+n} x^n$$

$$\sum_{n=0}^{\infty} \left(-x^{1+n} a_n\right) = \sum_{n=1}^{\infty} \left(-a_{n-1} x^n\right)$$

Substituting all the above in Eq (2) gives the following equation where now all powers of x are the same and equal to n.

$$\left(\sum_{n=0}^{\infty} (1+n) a_{1+n} x^n\right) + \sum_{n=1}^{\infty} \left(-a_{n-1} x^n\right) = 0 \tag{3}$$

For $1 \leq n$, the recurrence equation is

$$(1+n) a_{1+n} - a_{n-1} = 0 \tag{4}$$

Solving for a_{1+n}, gives

$$a_{1+n} = \frac{a_{n-1}}{1+n} \tag{5}$$

For $n = 1$ the recurrence equation gives

$$2a_2 - a_0 = 0$$

Which after substituting the earlier terms found becomes

$$a_2 = \frac{a_0}{2}$$

For $n = 2$ the recurrence equation gives

$$3a_3 - a_1 = 0$$

Which after substituting the earlier terms found becomes

$$a_3 = 0$$

For $n=3$ the recurrence equation gives

$$4a_4 - a_2 = 0$$

Which after substituting the earlier terms found becomes

$$a_4 = \frac{a_0}{8}$$

For $n=4$ the recurrence equation gives

$$5a_5 - a_3 = 0$$

Which after substituting the earlier terms found becomes

$$a_5 = 0$$

For $n=5$ the recurrence equation gives

$$6a_6 - a_4 = 0$$

Which after substituting the earlier terms found becomes

$$a_6 = \frac{a_0}{48}$$

And so on. Therefore the solution is

$$\begin{aligned} y &= \sum_{n=0}^{\infty} a_n x^n \\ &= a_3 x^3 + a_2 x^2 + a_1 x + a_0 + \dots \end{aligned}$$

Substituting the values for a_n found above, the solution becomes

$$y = a_0 + \frac{1}{2} a_0 x^2 + \frac{1}{8} a_0 x^4 + \dots$$

Collecting terms, the solution becomes

$$y = \left(1 + \frac{1}{2} x^2 + \frac{1}{8} x^4\right) a_0 + O(x^6) \tag{3}$$

Summary

The solution(s) found are the following

$$y = \left(1+\frac{1}{2}x^2+\frac{1}{8}x^4\right) y(0)+O\left(x^6\right) \tag{1}$$

$$y = \left(1+\frac{1}{2}x^2+\frac{1}{8}x^4\right) c_1+O\left(x^6\right) \tag{2}$$

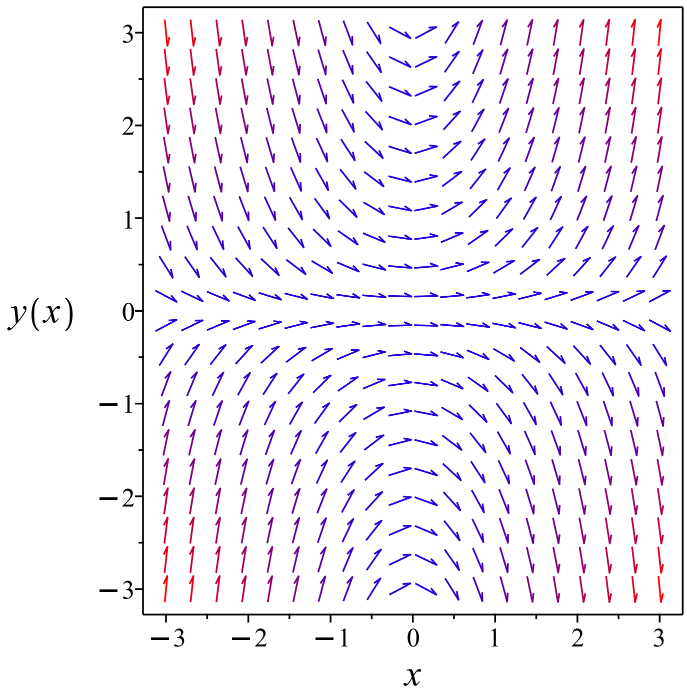


Figure 504: Slope field plot

Verification of solutions

$$y = \left(1+\frac{1}{2}x^2+\frac{1}{8}x^4\right) y(0)+O\left(x^6\right)$$

Verified OK.

$$y = \left(1+\frac{1}{2}x^2+\frac{1}{8}x^4\right) c_1+O\left(x^6\right)$$

Verified OK.

14.15.2 Maple step by step solution

Let's solve

$-yx + y' = 0$

- Highest derivative means the order of the ODE is 1

  y'

- Separate variables

  $\frac{y'}{y} = x$

- Integrate both sides with respect to x

  $\int \frac{y'}{y} dx = \int x dx + c_1$

- Evaluate integral

  $\ln(y) = \frac{x^2}{2} + c_1$

- Solve for y

  $y = \mathrm{e}^{\frac{x^2}{2} + c_1}$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
<- 1st order linear successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 21

```
Order:=6;
dsolve(diff(y(x),x)-x*y(x)=0,y(x),type='series',x=0);
```

$$y(x) = \left(1 + \frac{1}{2}x^2 + \frac{1}{8}x^4\right) y(0) + O(x^6)$$

✓ Solution by Mathematica

Time used: 0.001 (sec). Leaf size: 22

```
AsymptoticDSolveValue[y'[x]-x*y[x]==0,y[x],{x,0,5}]
```

$$y(x) \to c_1\left(\frac{x^4}{8}+\frac{x^2}{2}+1\right)$$

14.16 problem 19

Internal problem ID [748]
Internal file name [OUTPUT/748_Sunday_June_05_2022_01_48_39_AM_74598292/index.tex]

Book: Elementary differential equations and boundary value problems, 10th ed., Boyce and DiPrima
Section: Chapter 5.3, Series Solutions Near an Ordinary Point, Part II. page 269
Problem number: 19.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"first order ode series method. Ordinary point", "first order ode series method. Taylor series method"**

Maple gives the following as the ode type

`[_separable]`

$$\boxed{(1-x)\,y'-y=0}$$

With the expansion point for the power series method at $x=0$.

14.16.1 Solving as series ode

Solving ode using Taylor series method. This gives review on how the Taylor series method works for solving first order ode. Let

$$y'=f(x,y)$$

Where $f(x,y)$ is analytic at expansion point x_0. We can always shift to $x_0=0$ if x_0 is not zero. So from now we assume $x_0=0$. Assume also that $y(x_0)=y_0$. Using Taylor series

$$\begin{aligned}
y(x) &= y(x_0)+(x-x_0)\,y'(x_0)+\frac{(x-x_0)^2}{2}y''(x_0)+\frac{(x-x_0)^3}{3!}y''(x_0)+\cdots \\
&= y_0+xf+\frac{x^2}{2}\frac{df}{dx}\bigg|_{x_0,y_0}+\frac{x^3}{3!}\frac{d^2f}{dx^2}\bigg|_{x_0,y_0}+\cdots \\
&= y_0+\sum_{n=0}^{\infty}\frac{x^{n+1}}{(n+1)\,!}\frac{d^nf}{dx^n}\bigg|_{x_0,y_0}
\end{aligned}$$

But

$$
\begin{aligned}
\frac{df}{dx} &= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} f && (1)\\
\frac{d^2 f}{dx^2} &= \frac{d}{dx}\left(\frac{df}{dx}\right)\\
&= \frac{\partial}{\partial x}\left(\frac{df}{dx}\right) + \frac{\partial}{\partial y}\left(\frac{df}{dx}\right) f && (2)\\
\frac{d^3 f}{dx^3} &= \frac{d}{dx}\left(\frac{d^2 f}{dx^2}\right)\\
&= \frac{\partial}{\partial x}\left(\frac{d^2 f}{dx^2}\right) + \left(\frac{\partial}{\partial y}\frac{d^2 f}{dx^2}\right) f && (3)\\
&\vdots
\end{aligned}
$$

And so on. Hence if we name $F_0 = f(x, y)$ then the above can be written as

$$
\begin{aligned}
F_0 &= f(x, y) && (4)\\
F_n &= \frac{d}{dx}(F_{n-1})\\
&= \frac{\partial}{\partial x} F_{n-1} + \left(\frac{\partial F_{n-1}}{\partial y}\right) F_0 && (5)
\end{aligned}
$$

For example, for $n = 1$ we see that

$$
\begin{aligned}
F_1 &= \frac{d}{dx}(F_0)\\
&= \frac{\partial}{\partial x} F_0 + \left(\frac{\partial F_0}{\partial y}\right) F_0\\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} f
\end{aligned}
$$

Which is (1). And when $n = 2$

$$
\begin{aligned}
F_2 &= \frac{d}{dx}(F_1)\\
&= \frac{\partial}{\partial x} F_1 + \left(\frac{\partial F_1}{\partial y}\right) F_0\\
&= \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} f\right) + \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} f\right) f\\
&= \frac{\partial}{\partial x}\left(\frac{df}{dx}\right) + \frac{\partial}{\partial y}\left(\frac{df}{dx}\right) f
\end{aligned}
$$

Which is (2) and so on. Therefore (4,5) can be used from now on along with

$$y(x) = y_0 + \sum_{n=0}^{\infty} \frac{x^{n+1}}{(n+1)!} F_n|_{x_0,y_0} \tag{6}$$

Hence

$$\begin{aligned}
F_0 &= -\frac{y}{x-1} \\
F_1 &= \frac{dF_0}{dx} \\
&= \frac{\partial F_0}{\partial x} + \frac{\partial F_0}{\partial y} F_0 \\
&= \frac{2y}{(x-1)^2} \\
F_2 &= \frac{dF_1}{dx} \\
&= \frac{\partial F_1}{\partial x} + \frac{\partial F_1}{\partial y} F_1 \\
&= -\frac{6y}{(x-1)^3} \\
F_3 &= \frac{dF_2}{dx} \\
&= \frac{\partial F_2}{\partial x} + \frac{\partial F_2}{\partial y} F_2 \\
&= \frac{24y}{(x-1)^4} \\
F_4 &= \frac{dF_3}{dx} \\
&= \frac{\partial F_3}{\partial x} + \frac{\partial F_3}{\partial y} F_3 \\
&= -\frac{120y}{(x-1)^5}
\end{aligned}$$

And so on. Evaluating all the above at initial conditions $x(0) = 0$ and $y(0) = y(0)$ gives

$$\begin{aligned}
F_0 &= y(0) \\
F_1 &= 2y(0) \\
F_2 &= 6y(0) \\
F_3 &= 24y(0) \\
F_4 &= 120y(0)
\end{aligned}$$

Substituting all the above in (6) and simplifying gives the solution as

$$y = (x^5 + x^4 + x^3 + x^2 + x + 1)\, y(0) + O(x^6)$$

Since $x = 0$ is also an ordinary point, then standard power series can also be used. Writing the ODE as

$$\begin{aligned} y' + q(x)y &= p(x) \\ y' + \frac{y}{x-1} &= 0 \end{aligned}$$

Where

$$\begin{aligned} q(x) &= \frac{1}{x-1} \\ p(x) &= 0 \end{aligned}$$

Next, the type of the expansion point $x = 0$ is determined. This point can be an ordinary point, a regular singular point (also called removable singularity), or irregular singular point (also called non-removable singularity or essential singularity). When $x = 0$ is an ordinary point, then the standard power series is used. If the point is a regular singular point, Frobenius series is used instead. Irregular singular point requires more advanced methods (asymptotic methods) and is not supported now. Hopefully this will be added in the future. $x = 0$ is called an ordinary point $q(x)$ has a Taylor series expansion around the point $x = 0$. $x = 0$ is called a regular singular point if $q(x)$ is not not analytic at $x = 0$ but $xq(x)$ has Taylor series expansion. And finally, $x = 0$ is an irregular singular point if the point is not ordinary and not regular singular. This is the most complicated case. Now the expansion point $x = 0$ is checked to see if it is an ordinary point or not. Now the ode is normalized by writing it as

$$(x-1)\, y' + y = 0$$

Let the solution be represented as power series of the form

$$y = \sum_{n=0}^{\infty} a_n x^n$$

Then

$$y' = \sum_{n=1}^{\infty} n a_n x^{n-1}$$

Substituting the above back into the ode gives

$$(x-1)\left(\sum_{n=1}^{\infty} n a_n x^{n-1}\right)+\left(\sum_{n=0}^{\infty} a_n x^n\right)=0 \tag{1}$$

Which simplifies to

$$\left(\sum_{n=1}^{\infty} n a_n x^n\right)+\sum_{n=1}^{\infty}\left(-n a_n x^{n-1}\right)+\left(\sum_{n=0}^{\infty} a_n x^n\right)=0 \tag{2}$$

The next step is to make all powers of x be n in each summation term. Going over each summation term above with power of x in it which is not already x^n and adjusting the power and the corresponding index gives

$$\sum_{n=1}^{\infty}\left(-n a_n x^{n-1}\right)=\sum_{n=0}^{\infty}\left(-(n+1) a_{n+1} x^n\right)$$

Substituting all the above in Eq (2) gives the following equation where now all powers of x are the same and equal to n.

$$\left(\sum_{n=1}^{\infty} n a_n x^n\right)+\sum_{n=0}^{\infty}\left(-(n+1) a_{n+1} x^n\right)+\left(\sum_{n=0}^{\infty} a_n x^n\right)=0 \tag{3}$$

$n=0$ gives

$$-a_1+a_0=0$$
$$a_1=a_0$$

For $1 \leq n$, the recurrence equation is

$$n a_n-(n+1) a_{n+1}+a_n=0 \tag{4}$$

Solving for a_{n+1}, gives

$$a_{n+1}=a_n \tag{5}$$

For $n=1$ the recurrence equation gives

$$2 a_1-2 a_2=0$$

Which after substituting the earlier terms found becomes

$$a_2 = a_0$$

For $n = 2$ the recurrence equation gives

$$3a_2 - 3a_3 = 0$$

Which after substituting the earlier terms found becomes

$$a_3 = a_0$$

For $n = 3$ the recurrence equation gives

$$4a_3 - 4a_4 = 0$$

Which after substituting the earlier terms found becomes

$$a_4 = a_0$$

For $n = 4$ the recurrence equation gives

$$5a_4 - 5a_5 = 0$$

Which after substituting the earlier terms found becomes

$$a_5 = a_0$$

For $n = 5$ the recurrence equation gives

$$6a_5 - 6a_6 = 0$$

Which after substituting the earlier terms found becomes

$$a_6 = a_0$$

And so on. Therefore the solution is

$$\begin{aligned} y &= \sum_{n=0}^{\infty} a_n x^n \\ &= a_3 x^3 + a_2 x^2 + a_1 x + a_0 + \dots \end{aligned}$$

Substituting the values for a_n found above, the solution becomes

$$y = a_0x^5 + a_0x^4 + a_0x^3 + a_0x^2 + a_0x + a_0 + \dots$$

Collecting terms, the solution becomes

$$y = (x^5 + x^4 + x^3 + x^2 + x + 1) a_0 + O(x^6) \tag{3}$$

Summary

The solution(s) found are the following

$$y = (x^5 + x^4 + x^3 + x^2 + x + 1) y(0) + O(x^6) \tag{1}$$

$$y = (x^5 + x^4 + x^3 + x^2 + x + 1) c_1 + O(x^6) \tag{2}$$

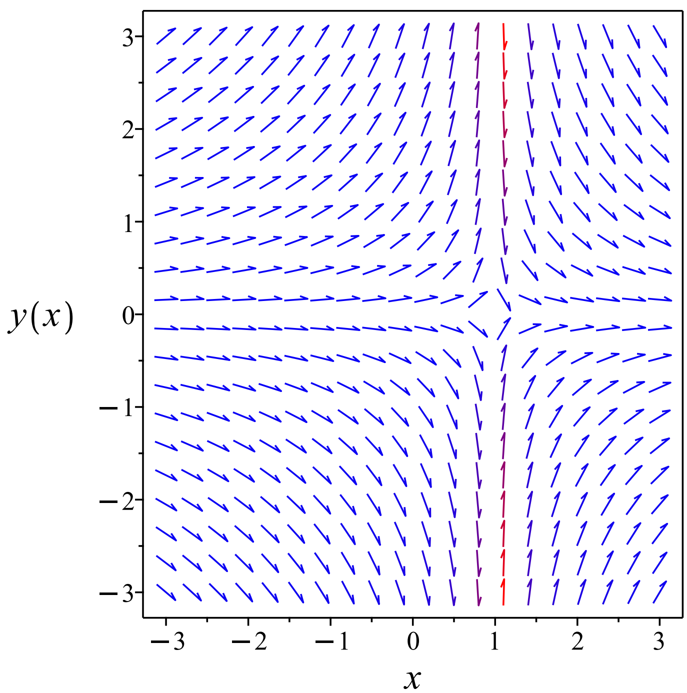


Figure 505: Slope field plot

Verification of solutions

$$y = (x^5 + x^4 + x^3 + x^2 + x + 1) y(0) + O(x^6)$$

Verified OK.

$$y = (x^5 + x^4 + x^3 + x^2 + x + 1) c_1 + O(x^6)$$

Verified OK.

14.16.2 Maple step by step solution

Let's solve

$(x-1)\,y'+y=0$

- Highest derivative means the order of the ODE is 1

  y'

- Integrate both sides with respect to x

  $\int \left(\left(x-1\right) y'+y\right)dx =\int 0d x +c_{1}$

- Evaluate integral

  $y(x-1)=c_{1}$

- Solve for y

  $y=\frac{c_{1}}{x-1}$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
<- 1st order linear successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 26

```
Order:=6;
dsolve((1-x)*diff(y(x),x)=y(x),y(x),type='series',x=0);
```

$$y(x)=\left(x^{5}+x^{4}+x^{3}+x^{2}+x+1\right) y(0)+O\left(x^{6}\right)$$

✓ Solution by Mathematica

Time used: 0.001 (sec). Leaf size: 21

```
AsymptoticDSolveValue[(1-x)*y'[x]==y[x],y[x],{x,0,5}]
```

$$y(x)\to c_1\left(x^5+x^4+x^3+x^2+x+1\right)$$

14.17 problem 22

Internal problem ID [749]
Internal file name [OUTPUT/749_Sunday_June_05_2022_01_48_40_AM_29896868/index.tex]

Book: Elementary differential equations and boundary value problems, 10th ed., Boyce and DiPrima
Section: Chapter 5.3, Series Solutions Near an Ordinary Point, Part II. page 269
Problem number: 22.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second order series method. Ordinary point", "second order series method. Taylor series method"**

Maple gives the following as the ode type

`[_Gegenbauer]`

$$\boxed{\left(-x^2+1\right) y''-2y'x+\alpha(\alpha+1)\, y=0}$$

With the expansion point for the power series method at $x=0$.

Solving ode using Taylor series method. This gives review on how the Taylor series method works for solving second order ode.

Let

$$y''=f(x,y,y')$$

Assuming expansion is at $x_0=0$ (we can always shift the actual expansion point to 0 by change of variables) and assuming $f(x,y,y')$ is analytic at x_0 which must be the case for an ordinary point. Let initial conditions be $y(x_0)=y_0$ and $y'(x_0)=y'_0$. Using Taylor series gives

$$\begin{aligned} y(x) &= y(x_0)+(x-x_0)\, y'(x_0)+\frac{(x-x_0)^2}{2}y''(x_0)+\frac{(x-x_0)^3}{3!}y'''(x_0)+\cdots \\ &= y_0+xy'_0+\frac{x^2}{2}\, f|_{x_0,y_0,y'_0}+\frac{x^3}{3!}\, f'|_{x_0,y_0,y'_0}+\cdots \\ &= y_0+xy'_0+\sum_{n=0}^{\infty}\frac{x^{n+2}}{(n+2)\,!}\, \frac{d^n f}{dx^n}\bigg|_{x_0,y_0,y'_0} \end{aligned}$$

But

$$\frac{df}{dx} = \frac{\partial f}{\partial x}\frac{dx}{dx} + \frac{\partial f}{\partial y}\frac{dy}{dx} + \frac{\partial f}{\partial y'}\frac{dy'}{dx} \tag{1}$$

$$= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}y' + \frac{\partial f}{\partial y'}y'' \tag{852}$$

$$= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}y' + \frac{\partial f}{\partial y'}f \tag{853}$$

$$\frac{d^2 f}{dx^2} = \frac{d}{dx}\left(\frac{df}{dx}\right)$$

$$= \frac{\partial}{\partial x}\left(\frac{df}{dx}\right) + \frac{\partial}{\partial y}\left(\frac{df}{dx}\right)y' + \frac{\partial}{\partial y'}\left(\frac{df}{dx}\right)f \tag{2}$$

$$\frac{d^3 f}{dx^3} = \frac{d}{dx}\left(\frac{d^2 f}{dx^2}\right)$$

$$= \frac{\partial}{\partial x}\left(\frac{d^2 f}{dx^2}\right) + \left(\frac{\partial}{\partial y}\frac{d^2 f}{dx^2}\right)y' + \frac{\partial}{\partial y'}\left(\frac{d^2 f}{dx^2}\right)f \tag{3}$$

$$\vdots$$

And so on. Hence if we name $F_0 = f(x,y,y')$ then the above can be written as

$$
\begin{aligned}
F_0 &= f(x,y,y') \qquad (4)\\
F_1 &= \frac{df}{dx}\\
&= \frac{dF_0}{dx}\\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}y' + \frac{\partial f}{\partial y'}y''\\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}y' + \frac{\partial f}{\partial y'}f \qquad (5)\\
&= \frac{\partial F_0}{\partial x} + \frac{\partial F_0}{\partial y}y' + \frac{\partial F_0}{\partial y'}F_0\\
F_2 &= \frac{d}{dx}\left(\frac{d}{dx}f\right)\\
&= \frac{d}{dx}(F_1)\\
&= \frac{\partial}{\partial x}F_1 + \left(\frac{\partial F_1}{\partial y}\right)y' + \left(\frac{\partial F_1}{\partial y'}\right)y''\\
&= \frac{\partial}{\partial x}F_1 + \left(\frac{\partial F_1}{\partial y}\right)y' + \left(\frac{\partial F_1}{\partial y'}\right)F_0\\
&\vdots\\
F_n &= \frac{d}{dx}(F_{n-1})\\
&= \frac{\partial}{\partial x}F_{n-1} + \left(\frac{\partial F_{n-1}}{\partial y}\right)y' + \left(\frac{\partial F_{n-1}}{\partial y'}\right)y''\\
&= \frac{\partial}{\partial x}F_{n-1} + \left(\frac{\partial F_{n-1}}{\partial y}\right)y' + \left(\frac{\partial F_{n-1}}{\partial y'}\right)F_0 \qquad (6)
\end{aligned}
$$

Therefore (6) can be used from now on along with

$$
y(x) = y_0 + xy_0' + \sum_{n=0}^{\infty}\frac{x^{n+2}}{(n+2)!}F_n|_{x_0,y_0,y_0'} \tag{7}
$$

To find $y(x)$ series solution around $x=0$. Hence

$$
\begin{aligned}
F_0 &= \frac{\alpha^2 y + \alpha y - 2y' x}{x^2-1} \\
F_1 &= \frac{dF_0}{dx} \\
&= \frac{\partial F_0}{\partial x} + \frac{\partial F_0}{\partial y} y' + \frac{\partial F_0}{\partial y'} F_0 \\
&= \frac{\left(\alpha^2 x^2 + \alpha\, x^2 - \alpha^2 + 6x^2 - \alpha + 2\right) y' - 4y\alpha x(\alpha+1)}{\left(x^2-1\right)^2} \\
F_2 &= \frac{dF_1}{dx} \\
&= \frac{\partial F_1}{\partial x} + \frac{\partial F_1}{\partial y} y' + \frac{\partial F_1}{\partial y'} F_1 \\
&= -\frac{8(x+1)\,(x-1)\left(x\left(\left(\alpha^2+\alpha+3\right)x^2 - \alpha^2 - \alpha + 3\right) y' - \frac{(\alpha+1)\left(\left(\alpha^2+\alpha+18\right)x^2 - \alpha^2 - \alpha + 6\right)\alpha y}{8}\right)}{\left(x^2-1\right)^4} \\
F_3 &= \frac{dF_2}{dx} \\
&= \frac{\partial F_2}{\partial x} + \frac{\partial F_2}{\partial y} y' + \frac{\partial F_2}{\partial y'} F_2 \\
&= \frac{\left(\left(\left(\alpha^4 + 2\alpha^3 + 59\alpha^2 + 58\alpha + 120\right)x^4 + \left(-2\alpha^4 - 4\alpha^3 - 46\alpha^2 - 44\alpha + 240\right)x^2 + \alpha^4 + 2\alpha^3 - 13\alpha^2 - \right.\right.}{\left(x^2-1\right)^5} \\
F_4 &= \frac{dF_3}{dx} \\
&= \frac{\partial F_3}{\partial x} + \frac{\partial F_3}{\partial y} y' + \frac{\partial F_3}{\partial y'} F_3 \\
&= -\frac{18\left(\left(40 + \left(x^4 - 2x^2 + 1\right)\alpha^4 + 2\left(x^4 - 2x^2 + 1\right)\alpha^3 + \left(-\frac{26}{3}x^2 - 17 + \frac{77}{3}x^4\right)\alpha^2 + 2\left(-9 - \frac{10}{3}x^2 + \frac{37}{3}\right.\right.\right.}{}
\end{aligned}
$$

And so on. Evaluating all the above at initial conditions $x=0$ and $y(0)=y(0)$ and $y'(0)=y'(0)$ gives

$$
\begin{aligned}
F_0 &= -y(0)\,\alpha(\alpha+1) \\
F_1 &= -y'(0)\,\alpha^2 - y'(0)\,\alpha + 2y'(0) \\
F_2 &= y(0)\,\alpha^4 + 2y(0)\,\alpha^3 - 5y(0)\,\alpha^2 - 6y(0)\,\alpha \\
F_3 &= y'(0)\,\alpha^4 + 2y'(0)\,\alpha^3 - 13y'(0)\,\alpha^2 - 14y'(0)\,\alpha + 24y'(0) \\
F_4 &= -y(0)\,\alpha^6 - 3y(0)\,\alpha^5 + 23y(0)\,\alpha^4 + 51y(0)\,\alpha^3 - 94y(0)\,\alpha^2 - 120y(0)\,\alpha
\end{aligned}
$$

Substituting all the above in (7) and simplifying gives the solution as

$$y=\left(1-\frac{1}{2}\alpha^{2}x^{2}-\frac{1}{2}\alpha\,x^{2}+\frac{1}{24}\alpha^{4}x^{4}+\frac{1}{12}\alpha^{3}x^{4}-\frac{5}{24}\alpha^{2}x^{4}-\frac{1}{4}\alpha\,x^{4}-\frac{1}{720}x^{6}\alpha^{6}-\frac{1}{240}x^{6}\alpha^{5}\right.$$
$$\left.+\frac{23}{720}x^{6}\alpha^{4}+\frac{17}{240}x^{6}\alpha^{3}-\frac{47}{360}x^{6}\alpha^{2}-\frac{1}{6}x^{6}\alpha\right)y(0)$$
$$+\left(x-\frac{1}{6}\alpha^{2}x^{3}-\frac{1}{6}\alpha\,x^{3}+\frac{1}{3}x^{3}+\frac{1}{120}\alpha^{4}x^{5}+\frac{1}{60}x^{5}\alpha^{3}-\frac{13}{120}\alpha^{2}x^{5}-\frac{7}{60}x^{5}\alpha+\frac{1}{5}x^{5}\right)y'(0)$$
$$+O(x^{6})$$

Since the expansion point $x=0$ is an ordinary, we can also solve this using standard power series The ode is normalized to be

$$\left(-x^{2}+1\right)y''-2y'x+\left(\alpha^{2}+\alpha\right)y=0$$

Let the solution be represented as power series of the form

$$y=\sum_{n=0}^{\infty}a_{n}x^{n}$$

Then

$$y'=\sum_{n=1}^{\infty}n a_{n}x^{n-1}$$
$$y''=\sum_{n=2}^{\infty}n(n-1)\,a_{n}x^{n-2}$$

Substituting the above back into the ode gives

$$\left(-x^{2}+1\right)\left(\sum_{n=2}^{\infty}n(n-1)\,a_{n}x^{n-2}\right)-2\left(\sum_{n=1}^{\infty}n a_{n}x^{n-1}\right)x+\left(\alpha^{2}+\alpha\right)\left(\sum_{n=0}^{\infty}a_{n}x^{n}\right)=0 \tag{1}$$

Which simplifies to

$$\sum_{n=2}^{\infty}\left(-x^{n}a_{n}n(n-1)\right)+\left(\sum_{n=2}^{\infty}n(n-1)\,a_{n}x^{n-2}\right)+\sum_{n=1}^{\infty}\left(-2n a_{n}x^{n}\right)+\left(\sum_{n=0}^{\infty}\left(\alpha^{2}+\alpha\right)a_{n}x^{n}\right)=0 \tag{2}$$

The next step is to make all powers of x be n in each summation term. Going over each summation term above with power of x in it which is not already x^n and adjusting the power and the corresponding index gives

$$\sum_{n=2}^{\infty} n(n-1) a_n x^{n-2} = \sum_{n=0}^{\infty} (n+2) a_{n+2}(n+1) x^n$$

Substituting all the above in Eq (2) gives the following equation where now all powers of x are the same and equal to n.

$$\sum_{n=2}^{\infty} (-x^n a_n n(n-1)) + \left(\sum_{n=0}^{\infty} (n+2) a_{n+2}(n+1) x^n\right) + \sum_{n=1}^{\infty} (-2n a_n x^n) + \left(\sum_{n=0}^{\infty} (\alpha^2+\alpha) a_n x^n\right) = 0 \tag{3}$$

$n = 0$ gives

$$2a_2 + a_0\alpha(\alpha+1) = 0$$

$$a_2 = -\frac{1}{2}a_0\alpha^2 - \frac{1}{2}a_0\alpha$$

$n = 1$ gives

$$6a_3 - 2a_1 + a_1\alpha(\alpha+1) = 0$$

Which after substituting earlier equations, simplifies to

$$a_3 = -\frac{1}{6}a_1\alpha^2 - \frac{1}{6}a_1\alpha + \frac{1}{3}a_1$$

For $2 \leq n$, the recurrence equation is

$$-na_n(n-1) + (n+2) a_{n+2}(n+1) - 2na_n + a_n\alpha(\alpha+1) = 0 \tag{4}$$

Solving for a_{n+2}, gives

$$a_{n+2} = -\frac{a_n(\alpha^2 - n^2 + \alpha - n)}{(n+2)(n+1)} \tag{5}$$

For $n=2$ the recurrence equation gives

$$-6a_2+12a_4+a_2\alpha(\alpha+1)=0$$

Which after substituting the earlier terms found becomes

$$a_4=-\frac{5}{24}a_0\alpha^2-\frac{1}{4}a_0\alpha+\frac{1}{24}a_0\alpha^4+\frac{1}{12}a_0\alpha^3$$

For $n=3$ the recurrence equation gives

$$-12a_3+20a_5+a_3\alpha(\alpha+1)=0$$

Which after substituting the earlier terms found becomes

$$a_5=-\frac{13}{120}a_1\alpha^2-\frac{7}{60}a_1\alpha+\frac{1}{5}a_1+\frac{1}{120}a_1\alpha^4+\frac{1}{60}a_1\alpha^3$$

For $n=4$ the recurrence equation gives

$$-20a_4+30a_6+a_4\alpha(\alpha+1)=0$$

Which after substituting the earlier terms found becomes

$$a_6=-\frac{47}{360}a_0\alpha^2-\frac{1}{6}a_0\alpha+\frac{23}{720}a_0\alpha^4+\frac{17}{240}a_0\alpha^3-\frac{1}{720}a_0\alpha^6-\frac{1}{240}a_0\alpha^5$$

For $n=5$ the recurrence equation gives

$$-30a_5+42a_7+a_5\alpha(\alpha+1)=0$$

Which after substituting the earlier terms found becomes

$$a_7=-\frac{5}{63}a_1\alpha^2-\frac{37}{420}a_1\alpha+\frac{1}{7}a_1+\frac{41}{5040}a_1\alpha^4+\frac{29}{1680}a_1\alpha^3-\frac{1}{5040}a_1\alpha^6-\frac{1}{1680}a_1\alpha^5$$

And so on. Therefore the solution is

$$\begin{aligned}y&=\sum_{n=0}^{\infty}a_nx^n\\&=a_3x^3+a_2x^2+a_1x+a_0+\dots\end{aligned}$$

Substituting the values for a_n found above, the solution becomes

$$
\begin{aligned}
y = a_0 + a_1 x &+ \left(-\frac{1}{2} a_0 \alpha^2 - \frac{1}{2} a_0 \alpha\right) x^2 + \left(-\frac{1}{6} a_1 \alpha^2 - \frac{1}{6} a_1 \alpha + \frac{1}{3} a_1\right) x^3 \\
&+ \left(-\frac{5}{24} a_0 \alpha^2 - \frac{1}{4} a_0 \alpha + \frac{1}{24} a_0 \alpha^4 + \frac{1}{12} a_0 \alpha^3\right) x^4 \\
&+ \left(-\frac{13}{120} a_1 \alpha^2 - \frac{7}{60} a_1 \alpha + \frac{1}{5} a_1 + \frac{1}{120} a_1 \alpha^4 + \frac{1}{60} a_1 \alpha^3\right) x^5 + \dots
\end{aligned}
$$

Collecting terms, the solution becomes

$$
\begin{aligned}
y = {} & \left(1 + \left(-\frac{1}{2}\alpha^2 - \frac{1}{2}\alpha\right) x^2 + \left(-\frac{5}{24}\alpha^2 - \frac{1}{4}\alpha + \frac{1}{24}\alpha^4 + \frac{1}{12}\alpha^3\right) x^4\right) a_0 + \Bigg(x \\
& + \left(-\frac{1}{6}\alpha^2 - \frac{1}{6}\alpha + \frac{1}{3}\right) x^3 + \left(-\frac{13}{120}\alpha^2 - \frac{7}{60}\alpha + \frac{1}{5} + \frac{1}{120}\alpha^4 + \frac{1}{60}\alpha^3\right) x^5\Bigg) a_1 + O(x^6)
\end{aligned} \tag{3}
$$

At $x = 0$ the solution above becomes

$$
\begin{aligned}
y = {} & \left(1 + \left(-\frac{1}{2}\alpha^2 - \frac{1}{2}\alpha\right) x^2 + \left(-\frac{5}{24}\alpha^2 - \frac{1}{4}\alpha + \frac{1}{24}\alpha^4 + \frac{1}{12}\alpha^3\right) x^4\right) c_1 + \Bigg(x \\
& + \left(-\frac{1}{6}\alpha^2 - \frac{1}{6}\alpha + \frac{1}{3}\right) x^3 + \left(-\frac{13}{120}\alpha^2 - \frac{7}{60}\alpha + \frac{1}{5} + \frac{1}{120}\alpha^4 + \frac{1}{60}\alpha^3\right) x^5\Bigg) c_2 + O(x^6)
\end{aligned}
$$

<u>Summary</u>

The solution(s) found are the following

$$
\begin{aligned}
y = {} & \Bigg(1 - \frac{1}{2}\alpha^2 x^2 - \frac{1}{2}\alpha\, x^2 + \frac{1}{24}\alpha^4 x^4 + \frac{1}{12}\alpha^3 x^4 - \frac{5}{24}\alpha^2 x^4 - \frac{1}{4}\alpha\, x^4 - \frac{1}{720} x^6 \alpha^6 - \frac{1}{240} x^6 \alpha^5 \\
& + \frac{23}{720} x^6 \alpha^4 + \frac{17}{240} x^6 \alpha^3 - \frac{47}{360} x^6 \alpha^2 - \frac{1}{6} x^6 \alpha\Bigg) y(0) + \Bigg(x - \frac{1}{6}\alpha^2 x^3 - \frac{1}{6}\alpha\, x^3 + \frac{1}{3} x^3 \\
& + \frac{1}{120}\alpha^4 x^5 + \frac{1}{60} x^5 \alpha^3 - \frac{13}{120}\alpha^2 x^5 - \frac{7}{60} x^5 \alpha + \frac{1}{5} x^5\Bigg) y'(0) + O(x^6)
\end{aligned} \tag{1}
$$

$$
\begin{aligned}
y = {} & \left(1 + \left(-\frac{1}{2}\alpha^2 - \frac{1}{2}\alpha\right) x^2 + \left(-\frac{5}{24}\alpha^2 - \frac{1}{4}\alpha + \frac{1}{24}\alpha^4 + \frac{1}{12}\alpha^3\right) x^4\right) c_1 \\
& + \left(x + \left(-\frac{1}{6}\alpha^2 - \frac{1}{6}\alpha + \frac{1}{3}\right) x^3 + \left(-\frac{13}{120}\alpha^2 - \frac{7}{60}\alpha + \frac{1}{5} + \frac{1}{120}\alpha^4 + \frac{1}{60}\alpha^3\right) x^5\right) c_2 \\
& + O(x^6)
\end{aligned} \tag{2}
$$

Verification of solutions

$$y = \left(1-\frac{1}{2}\alpha^2 x^2-\frac{1}{2}\alpha\, x^2+\frac{1}{24}\alpha^4 x^4+\frac{1}{12}\alpha^3 x^4-\frac{5}{24}\alpha^2 x^4-\frac{1}{4}\alpha\, x^4-\frac{1}{720}x^6\alpha^6-\frac{1}{240}x^6\alpha^5 + \frac{23}{720}x^6\alpha^4+\frac{17}{240}x^6\alpha^3-\frac{47}{360}x^6\alpha^2-\frac{1}{6}x^6\alpha\right) y(0) + \left(x-\frac{1}{6}\alpha^2 x^3-\frac{1}{6}\alpha\, x^3+\frac{1}{3}x^3+\frac{1}{120}\alpha^4 x^5+\frac{1}{60}x^5\alpha^3-\frac{13}{120}\alpha^2 x^5-\frac{7}{60}x^5\alpha+\frac{1}{5}x^5\right) y'(0) + O(x^6)$$

Verified OK.

$$y = \left(1+\left(-\frac{1}{2}\alpha^2-\frac{1}{2}\alpha\right)x^2+\left(-\frac{5}{24}\alpha^2-\frac{1}{4}\alpha+\frac{1}{24}\alpha^4+\frac{1}{12}\alpha^3\right)x^4\right)c_1+\left(x + \left(-\frac{1}{6}\alpha^2-\frac{1}{6}\alpha+\frac{1}{3}\right)x^3+\left(-\frac{13}{120}\alpha^2-\frac{7}{60}\alpha+\frac{1}{5}+\frac{1}{120}\alpha^4+\frac{1}{60}\alpha^3\right)x^5\right)c_2+O(x^6)$$

Verified OK.

14.17.1 Maple step by step solution

Let's solve

$(-x^2+1)\,y''-2y'x+(\alpha^2+\alpha)\,y=0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y''=\frac{\alpha(\alpha+1)y}{x^2-1}-\frac{2xy'}{x^2-1}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y''+\frac{2xy'}{x^2-1}-\frac{\alpha(\alpha+1)y}{x^2-1}=0$

□ Check to see if x_0 is a regular singular point

- ○ Define functions

  $\left[P_2(x)=\frac{2x}{x^2-1}, P_3(x)=-\frac{\alpha(\alpha+1)}{x^2-1}\right]$

- ○ $(x+1)\cdot P_2(x)$ is analytic at $x=-1$

  $\left((x+1)\cdot P_2(x)\right)\Big|_{x=-1}=1$

- ○ $(x+1)^2\cdot P_3(x)$ is analytic at $x=-1$

$$\left((x+1)^2 \cdot P_3(x)\right)\Big|_{x=-1} = 0$$

- $x = -1$is a regular singular point

  Check to see if x_0 is a regular singular point

  $x_0 = -1$

- Multiply by denominators

  $(x^2-1)\,y'' + 2y'x - \alpha(\alpha+1)\,y = 0$

- Change variables using $x = u-1$ so that the regular singular point is at $u = 0$

  $(u^2-2u)\left(\frac{d^2}{du^2}y(u)\right) + (2u-2)\left(\frac{d}{du}y(u)\right) + (-\alpha^2-\alpha)\,y(u) = 0$

- Assume series solution for $y(u)$

  $y(u) = \sum_{k=0}^{\infty} a_k u^{k+r}$

□ Rewrite ODE with series expansions

- Convert $u^m \cdot \left(\frac{d}{du}y(u)\right)$ to series expansion for $m = 0..1$

  $u^m \cdot \left(\frac{d}{du}y(u)\right) = \sum_{k=0}^{\infty} a_k(k+r)\,u^{k+r-1+m}$

- Shift index using $k- >k+1-m$

  $u^m \cdot \left(\frac{d}{du}y(u)\right) = \sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\,u^{k+r}$

- Convert $u^m \cdot \left(\frac{d^2}{du^2}y(u)\right)$ to series expansion for $m = 1..2$

  $u^m \cdot \left(\frac{d^2}{du^2}y(u)\right) = \sum_{k=0}^{\infty} a_k(k+r)\,(k+r-1)\,u^{k+r-2+m}$

- Shift index using $k- >k+2-m$

  $u^m \cdot \left(\frac{d^2}{du^2}y(u)\right) = \sum_{k=-2+m}^{\infty} a_{k+2-m}(k+2-m+r)\,(k+1-m+r)\,u^{k+r}$

  Rewrite ODE with series expansions

  $-2a_0r^2u^{-1+r} + \left(\sum_{k=0}^{\infty} \left(-2a_{k+1}(k+1+r)^2 - a_k(r+1+k+\alpha)\,(-r-k+\alpha)\right)u^{k+r}\right) = 0$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $-2r^2 = 0$

- Values of r that satisfy the indicial equation

$r = 0$

- Each term in the series must be 0, giving the recursion relation

  $-2a_{k+1}(k+1)^2 + a_k(1+k+\alpha)(k-\alpha) = 0$

- Recursion relation that defines series solution to ODE

  $a_{k+1} = -\frac{a_k(1+k+\alpha)(-k+\alpha)}{2(k+1)^2}$

- Recursion relation for $r = 0$

  $a_{k+1} = -\frac{a_k(1+k+\alpha)(-k+\alpha)}{2(k+1)^2}$

- Solution for $r = 0$

  $\left[y(u) = \sum_{k=0}^{\infty} a_k u^k, a_{k+1} = -\frac{a_k(1+k+\alpha)(-k+\alpha)}{2(k+1)^2}\right]$

- Revert the change of variables $u = x + 1$

  $\left[y = \sum_{k=0}^{\infty} a_k(x+1)^k, a_{k+1} = -\frac{a_k(1+k+\alpha)(-k+\alpha)}{2(k+1)^2}\right]$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
<- No Liouvillian solutions exists
-> Trying a solution in terms of special functions:
   -> Bessel
   -> elliptic
   -> Legendre
   <- Legendre successful
<- special function solution successful`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 101

```
Order:=6;
dsolve((1-x^2)*diff(y(x),x$2)-2*x*diff(y(x),x)+alpha*(alpha+1)*y(x)=0,y(x),type='series',x=0
```

$$y(x) = \left(1 - \frac{\alpha(1+\alpha)\, x^2}{2} + \frac{\alpha(\alpha^3 + 2\alpha^2 - 5\alpha - 6)\, x^4}{24}\right) y(0) + \left(x - \frac{(\alpha^2 + \alpha - 2)\, x^3}{6} + \frac{(\alpha^4 + 2\alpha^3 - 13\alpha^2 - 14\alpha + 24)\, x^5}{120}\right) D(y)(0) + O(x^6)$$

✓ Solution by Mathematica

Time used: 0.001 (sec). Leaf size: 127

```
AsymptoticDSolveValue[(1-x^2)*y''[x]-2*x*y'[x]+\[Alpha]*(\[Alpha]+1)*y[x]==0,y[x],{x,0,5}]
```

$$y(x) \to c_2\left(\frac{1}{60}\left(-\alpha^2 - \alpha\right) x^5 - \frac{1}{120}\left(-\alpha^2 - \alpha\right)\left(\alpha^2 + \alpha\right) x^5 - \frac{1}{10}\left(\alpha^2 + \alpha\right) x^5 + \frac{x^5}{5} - \frac{1}{6}\left(\alpha^2 + \alpha\right) x^3 + \frac{x^3}{3} + x\right) + c_1\left(\frac{1}{24}\left(\alpha^2 + \alpha\right)^2 x^4 - \frac{1}{4}\left(\alpha^2 + \alpha\right) x^4 - \frac{1}{2}\left(\alpha^2 + \alpha\right) x^2 + 1\right)$$

15 Chapter 7.5, Homogeneous Linear Systems with Constant Coefficients. page 407

15.1 problem 30

Internal problem ID [750]
Internal file name [`OUTPUT/750_Sunday_June_05_2022_01_48_42_AM_80033784/index.tex`]

Book: Elementary differential equations and boundary value problems, 10th ed., Boyce and DiPrima
Section: Chapter 7.5, Homogeneous Linear Systems with Constant Coefficients. page 407
Problem number: 30.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"system of linear ODEs"**

Solve

$$x_1'(t) = -\frac{x_1(t)}{10} + \frac{3x_2(t)}{40}$$
$$x_2'(t) = \frac{x_1(t)}{10} - \frac{x_2(t)}{5}$$

With initial conditions

$$[x_1(0) = -17, x_2(0) = -21]$$

15.1.1 Solution using Matrix exponential method

In this method, we will assume we have found the matrix exponential e^{At} allready. There are different methods to determine this but will not be shown here. This is a system of linear ODE's given as

$$\vec{x}'(t) = A\,\vec{x}(t)$$

Or

$$\begin{bmatrix} x_1'(t) \\ x_2'(t) \end{bmatrix} = \begin{bmatrix} -\frac{1}{10} & \frac{3}{40} \\ \frac{1}{10} & -\frac{1}{5} \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}$$

For the above matrix A, the matrix exponential can be found to be

$$e^{At} = \begin{bmatrix} \frac{e^{-\frac{t}{4}}}{4} + \frac{3\,e^{-\frac{t}{20}}}{4} & \frac{3\,e^{-\frac{t}{20}}}{8} - \frac{3\,e^{-\frac{t}{4}}}{8} \\ \frac{e^{-\frac{t}{20}}}{2} - \frac{e^{-\frac{t}{4}}}{2} & \frac{3\,e^{-\frac{t}{4}}}{4} + \frac{e^{-\frac{t}{20}}}{4} \end{bmatrix}$$

Therefore the homogeneous solution is

$$
\begin{aligned}
\vec{x}_h(t) &= e^{At}\vec{x}_0 \\
&= \begin{bmatrix} \frac{e^{-\frac{t}{4}}}{4}+\frac{3\,e^{-\frac{t}{20}}}{4} & \frac{3\,e^{-\frac{t}{20}}}{8}-\frac{3\,e^{-\frac{t}{4}}}{8} \\ \frac{e^{-\frac{t}{20}}}{2}-\frac{e^{-\frac{t}{4}}}{2} & \frac{3\,e^{-\frac{t}{4}}}{4}+\frac{e^{-\frac{t}{20}}}{4} \end{bmatrix} \begin{bmatrix} -17 \\ -21 \end{bmatrix} \\
&= \begin{bmatrix} \frac{29\,e^{-\frac{t}{4}}}{8}-\frac{165\,e^{-\frac{t}{20}}}{8} \\ -\frac{55\,e^{-\frac{t}{20}}}{4}-\frac{29\,e^{-\frac{t}{4}}}{4} \end{bmatrix}
\end{aligned}
$$

Since no forcing function is given, then the final solution is $\vec{x}_h(t)$ above.

15.1.2 Solution using explicit Eigenvalue and Eigenvector method

This is a system of linear ODE's given as

$$
\vec{x}'(t) = A\,\vec{x}(t)
$$

Or

$$
\begin{bmatrix} x_1'(t) \\ x_2'(t) \end{bmatrix} = \begin{bmatrix} -\frac{1}{10} & \frac{3}{40} \\ \frac{1}{10} & -\frac{1}{5} \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}
$$

The first step is find the homogeneous solution. We start by finding the eigenvalues of A. This is done by solving the following equation for the eigenvalues λ

$$
\det(A - \lambda I) = 0
$$

Expanding gives

$$
\det\left(\begin{bmatrix} -\frac{1}{10} & \frac{3}{40} \\ \frac{1}{10} & -\frac{1}{5} \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right) = 0
$$

Therefore

$$
\det\left(\begin{bmatrix} -\frac{1}{10}-\lambda & \frac{3}{40} \\ \frac{1}{10} & -\frac{1}{5}-\lambda \end{bmatrix}\right) = 0
$$

Which gives the characteristic equation

$$
\lambda^2 + \frac{3}{10}\lambda + \frac{1}{80} = 0
$$

The roots of the above are the eigenvalues.

$$\lambda_1 = -\frac{1}{4}$$
$$\lambda_2 = -\frac{1}{20}$$

This table summarises the above result

eigenvalue	algebraic multiplicity	type of eigenvalue
$-\frac{1}{4}$	1	real eigenvalue
$-\frac{1}{20}$	1	real eigenvalue

Now the eigenvector for each eigenvalue are found.

Considering the eigenvalue $\lambda_1 = -\frac{1}{4}$

We need to solve $A\vec{v} = \lambda\vec{v}$ or $(A - \lambda I)\vec{v} = \vec{0}$ which becomes

$$\left(\begin{bmatrix} -\frac{1}{10} & \frac{3}{40} \\ \frac{1}{10} & -\frac{1}{5} \end{bmatrix} - \left(-\frac{1}{4}\right)\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right)\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$
$$\begin{bmatrix} \frac{3}{20} & \frac{3}{40} \\ \frac{1}{10} & \frac{1}{20} \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Now forward elimination is applied to solve for the eigenvector $\vec{v}$. The augmented matrix is

$$\left[\begin{array}{cc|c} \frac{3}{20} & \frac{3}{40} & 0 \\ \frac{1}{10} & \frac{1}{20} & 0 \end{array}\right]$$

$$R_2 = R_2 - \frac{2R_1}{3} \Longrightarrow \left[\begin{array}{cc|c} \frac{3}{20} & \frac{3}{40} & 0 \\ 0 & 0 & 0 \end{array}\right]$$

Therefore the system in Echelon form is

$$\begin{bmatrix} \frac{3}{20} & \frac{3}{40} \\ 0 & 0 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

The free variables are $\{v_2\}$ and the leading variables are $\{v_1\}$. Let $v_2 = t$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\left\{v_1 = -\frac{t}{2}\right\}$

Hence the solution is

$$\begin{bmatrix}-\frac{t}{2}\\ t\end{bmatrix}=\begin{bmatrix}-\frac{t}{2}\\ t\end{bmatrix}$$

Since there is one free Variable, we have found one eigenvector associated with this eigenvalue. The above can be written as

$$\begin{bmatrix}-\frac{t}{2}\\ t\end{bmatrix}=t\begin{bmatrix}-\frac{1}{2}\\ 1\end{bmatrix}$$

Let $t=1$ the eigenvector becomes

$$\begin{bmatrix}-\frac{t}{2}\\ t\end{bmatrix}=\begin{bmatrix}-\frac{1}{2}\\ 1\end{bmatrix}$$

Which is normalized to

$$\begin{bmatrix}-\frac{t}{2}\\ t\end{bmatrix}=\begin{bmatrix}-1\\ 2\end{bmatrix}$$

Considering the eigenvalue $\lambda_2=-\frac{1}{20}$

We need to solve $A\vec{v}=\lambda\vec{v}$ or $(A-\lambda I)\vec{v}=\vec{0}$ which becomes

$$\left(\begin{bmatrix}-\frac{1}{10} & \frac{3}{40}\\ \frac{1}{10} & -\frac{1}{5}\end{bmatrix}-\left(-\frac{1}{20}\right)\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}\right)\begin{bmatrix}v_1\\ v_2\end{bmatrix}=\begin{bmatrix}0\\ 0\end{bmatrix}$$

$$\begin{bmatrix}-\frac{1}{20} & \frac{3}{40}\\ \frac{1}{10} & -\frac{3}{20}\end{bmatrix}\begin{bmatrix}v_1\\ v_2\end{bmatrix}=\begin{bmatrix}0\\ 0\end{bmatrix}$$

Now forward elimination is applied to solve for the eigenvector $\vec{v}$. The augmented matrix is

$$\left[\begin{array}{cc|c}-\frac{1}{20} & \frac{3}{40} & 0\\ \frac{1}{10} & -\frac{3}{20} & 0\end{array}\right]$$

$$R_2=R_2+2R_1\Longrightarrow\left[\begin{array}{cc|c}-\frac{1}{20} & \frac{3}{40} & 0\\ 0 & 0 & 0\end{array}\right]$$

Therefore the system in Echelon form is

$$\begin{bmatrix}-\frac{1}{20} & \frac{3}{40}\\ 0 & 0\end{bmatrix}\begin{bmatrix}v_1\\ v_2\end{bmatrix}=\begin{bmatrix}0\\ 0\end{bmatrix}$$

The free variables are $\{v_2\}$ and the leading variables are $\{v_1\}$. Let $v_2 = t$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\left\{v_1 = \frac{3t}{2}\right\}$

Hence the solution is

$$\begin{bmatrix} \frac{3t}{2} \\ t \end{bmatrix} = \begin{bmatrix} \frac{3t}{2} \\ t \end{bmatrix}$$

Since there is one free Variable, we have found one eigenvector associated with this eigenvalue. The above can be written as

$$\begin{bmatrix} \frac{3t}{2} \\ t \end{bmatrix} = t \begin{bmatrix} \frac{3}{2} \\ 1 \end{bmatrix}$$

Let $t = 1$ the eigenvector becomes

$$\begin{bmatrix} \frac{3t}{2} \\ t \end{bmatrix} = \begin{bmatrix} \frac{3}{2} \\ 1 \end{bmatrix}$$

Which is normalized to

$$\begin{bmatrix} \frac{3t}{2} \\ t \end{bmatrix} = \begin{bmatrix} 3 \\ 2 \end{bmatrix}$$

The following table gives a summary of this result. It shows for each eigenvalue the algebraic multiplicity m, and its geometric multiplicity k and the eigenvectors associated with the eigenvalue. If $m > k$ then the eigenvalue is defective which means the number of normal linearly independent eigenvectors associated with this eigenvalue (called the geometric multiplicity k) does not equal the algebraic multiplicity m, and we need to determine an additional $m - k$ generalized eigenvectors for this eigenvalue.

	multiplicity			
eigenvalue	algebraic m	geometric k	defective?	eigenvectors
$-\frac{1}{4}$	1	1	No	$\begin{bmatrix} -\frac{1}{2} \\ 1 \end{bmatrix}$
$-\frac{1}{20}$	1	1	No	$\begin{bmatrix} \frac{3}{2} \\ 1 \end{bmatrix}$

Now that we found the eigenvalues and associated eigenvectors, we will go over each eigenvalue and generate the solution basis. The only problem we need to take care of is if the eigenvalue is defective. Since eigenvalue $-\frac{1}{4}$ is real and distinct then the corresponding eigenvector solution is

$$\begin{aligned}\vec{x}_1(t) &= \vec{v}_1 e^{-\frac{t}{4}} \\ &= \left[\begin{array}{c} -\frac{1}{2} \\ 1 \end{array}\right] e^{-\frac{t}{4}}\end{aligned}$$

Since eigenvalue $-\frac{1}{20}$ is real and distinct then the corresponding eigenvector solution is

$$\begin{aligned}\vec{x}_2(t) &= \vec{v}_2 e^{-\frac{t}{20}} \\ &= \left[\begin{array}{c} \frac{3}{2} \\ 1 \end{array}\right] e^{-\frac{t}{20}}\end{aligned}$$

Therefore the final solution is

$$\vec{x}_h(t) = c_1 \vec{x}_1(t) + c_2 \vec{x}_2(t)$$

Which is written as

$$\left[\begin{array}{c} x_1(t) \\ x_2(t) \end{array}\right] = c_1 \left[\begin{array}{c} -\frac{\mathrm{e}^{-\frac{t}{4}}}{2} \\ \mathrm{e}^{-\frac{t}{4}} \end{array}\right] + c_2 \left[\begin{array}{c} \frac{3\,\mathrm{e}^{-\frac{t}{20}}}{2} \\ \mathrm{e}^{-\frac{t}{20}} \end{array}\right]$$

Which becomes

$$\left[\begin{array}{c} x_1(t) \\ x_2(t) \end{array}\right] = \left[\begin{array}{c} -\frac{c_1 \mathrm{e}^{-\frac{t}{4}}}{2} + \frac{3 c_2 \mathrm{e}^{-\frac{t}{20}}}{2} \\ c_1 \mathrm{e}^{-\frac{t}{4}} + c_2 \mathrm{e}^{-\frac{t}{20}} \end{array}\right]$$

Since initial conditions are given, the solution above needs to be updated by solving for the constants of integrations using the given initial conditions

$$\left[\begin{array}{c} x_1(0) = -17 \\ x_2(0) = -21 \end{array}\right] \tag{1}$$

Substituting initial conditions into the above solution at $t = 0$ gives

$$\left[\begin{array}{c} -17 \\ -21 \end{array}\right] = \left[\begin{array}{c} -\frac{c_1}{2} + \frac{3 c_2}{2} \\ c_1 + c_2 \end{array}\right]$$

Solving for the constants of integrations gives

$$\left[c_1 = -\frac{29}{4}, c_2 = -\frac{55}{4}\right]$$

Substituting these constants back in original solution in Eq. (1) gives

$$\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = \begin{bmatrix} \frac{29\,\mathrm{e}^{-\frac{t}{4}}}{8} - \frac{165\,\mathrm{e}^{-\frac{t}{20}}}{8} \\ -\frac{55\,\mathrm{e}^{-\frac{t}{20}}}{4} - \frac{29\,\mathrm{e}^{-\frac{t}{4}}}{4} \end{bmatrix}$$

The following is the phase plot of the system.

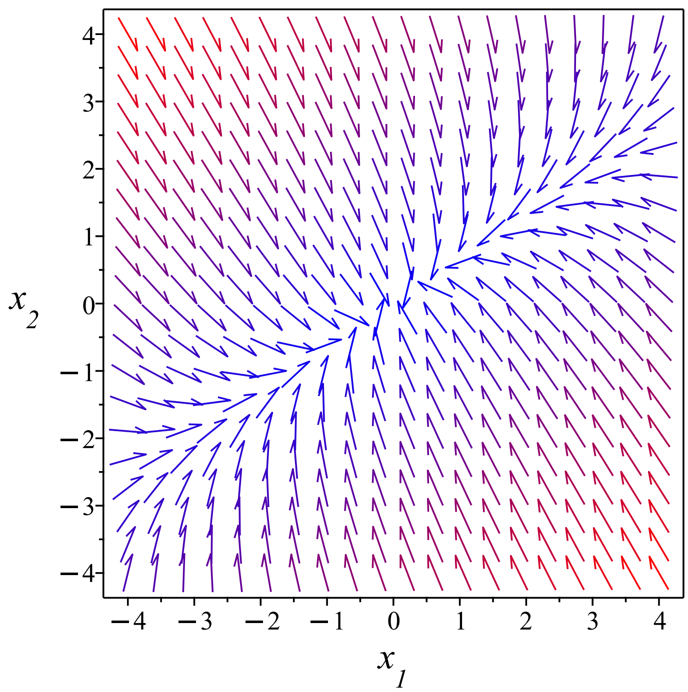
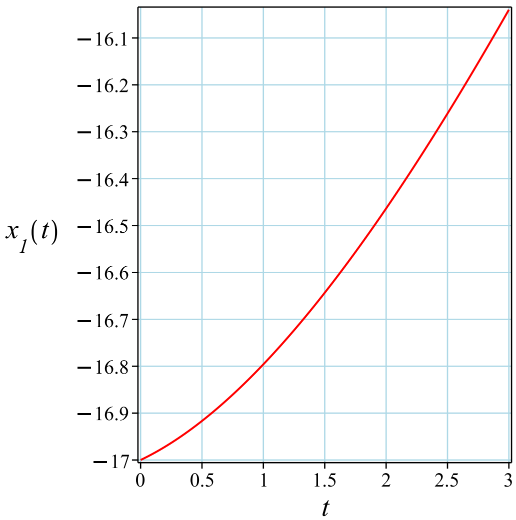
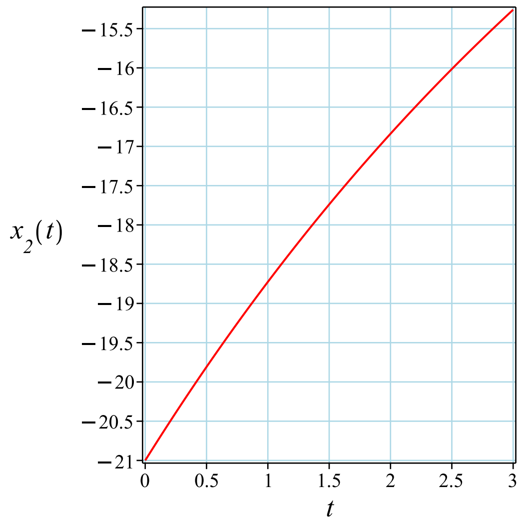


Figure 506: Phase plot

The following are plots of each solution.





✓ Solution by Maple

Time used: 0.031 (sec). Leaf size: 34

```
dsolve([diff(x__1(t),t) = -1/10*x__1(t)+3/40*x__2(t), diff(x__2(t),t) = 1/10*x__1(t)-1/5*x__
```

$$x_1(t) = -\frac{165\,e^{-\frac{t}{20}}}{8} + \frac{29\,e^{-\frac{t}{4}}}{8}$$
$$x_2(t) = -\frac{55\,e^{-\frac{t}{20}}}{4} - \frac{29\,e^{-\frac{t}{4}}}{4}$$

✓ Solution by Mathematica

Time used: 0.007 (sec). Leaf size: 52

```
DSolve[{x1'[t]==-1/10*x1[t]+3/40*x2[t],x2'[t]==1/10*x1[t]-1/5*x2[t]},{x1[0]==-17,x2[0]==-21}
```

$$\mathrm{x1}(t) \to \frac{1}{8}e^{-t/4}\left(29 - 165e^{t/5}\right)$$
$$\mathrm{x2}(t) \to -\frac{1}{4}e^{-t/4}\left(55e^{t/5} + 29\right)$$

16 Chapter 7.6, Complex Eigenvalues. page 417

16.1 problem 1

Internal problem ID [751]
Internal file name [`OUTPUT/751_Sunday_June_05_2022_01_48_43_AM_43130140/index.tex`]

Book: Elementary differential equations and boundary value problems, 10th ed., Boyce and DiPrima
Section: Chapter 7.6, Complex Eigenvalues. page 417
Problem number: 1.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"system of linear ODEs"**

Solve

$$
\begin{aligned}
x_1'(t) &= 3x_1(t) - 2x_2(t) \\
x_2'(t) &= 4x_1(t) - x_2(t)
\end{aligned}
$$

16.1.1 Solution using Matrix exponential method

In this method, we will assume we have found the matrix exponential e^{At} allready. There are different methods to determine this but will not be shown here. This is a system of linear ODE's given as

$$
\vec{x}'(t) = A\,\vec{x}(t)
$$

Or

$$
\begin{bmatrix} x_1'(t) \\ x_2'(t) \end{bmatrix} = \begin{bmatrix} 3 & -2 \\ 4 & -1 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}
$$

For the above matrix A, the matrix exponential can be found to be

$$
\begin{aligned}
e^{At} &= \begin{bmatrix} e^t \cos(2t) + e^t \sin(2t) & -e^t \sin(2t) \\ 2\,e^t \sin(2t) & e^t \cos(2t) - e^t \sin(2t) \end{bmatrix} \\
&= \begin{bmatrix} e^t(\cos(2t) + \sin(2t)) & -e^t \sin(2t) \\ 2\,e^t \sin(2t) & e^t(\cos(2t) - \sin(2t)) \end{bmatrix}
\end{aligned}
$$

Therefore the homogeneous solution is

$$
\begin{aligned}
\vec{x}_h(t) &= e^{At}\vec{c} \\
&= \begin{bmatrix} e^{t}(\cos(2t)+\sin(2t)) & -e^{t}\sin(2t) \\ 2\,e^{t}\sin(2t) & e^{t}(\cos(2t)-\sin(2t)) \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} \\
&= \begin{bmatrix} e^{t}(\cos(2t)+\sin(2t))\,c_1 - e^{t}\sin(2t)\,c_2 \\ 2\,e^{t}\sin(2t)\,c_1 + e^{t}(\cos(2t)-\sin(2t))\,c_2 \end{bmatrix} \\
&= \begin{bmatrix} ((-c_2+c_1)\sin(2t)+c_1\cos(2t))\,e^{t} \\ e^{t}(2c_1-c_2)\sin(2t)+e^{t}\cos(2t)\,c_2 \end{bmatrix}
\end{aligned}
$$

Since no forcing function is given, then the final solution is $\vec{x}_h(t)$ above.

16.1.2 Solution using explicit Eigenvalue and Eigenvector method

This is a system of linear ODE's given as

$$\vec{x}'(t) = A\,\vec{x}(t)$$

Or

$$\begin{bmatrix} x_1'(t) \\ x_2'(t) \end{bmatrix} = \begin{bmatrix} 3 & -2 \\ 4 & -1 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}$$

The first step is find the homogeneous solution. We start by finding the eigenvalues of A. This is done by solving the following equation for the eigenvalues λ

$$\det(A-\lambda I) = 0$$

Expanding gives

$$\det\left(\begin{bmatrix} 3 & -2 \\ 4 & -1 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right) = 0$$

Therefore

$$\det\left(\begin{bmatrix} 3-\lambda & -2 \\ 4 & -1-\lambda \end{bmatrix}\right) = 0$$

Which gives the characteristic equation

$$\lambda^2 - 2\lambda + 5 = 0$$

The roots of the above are the eigenvalues.

$$\lambda_1 = 1 + 2i$$
$$\lambda_2 = 1 - 2i$$

This table summarises the above result

eigenvalue	algebraic multiplicity	type of eigenvalue
$1 + 2i$	1	complex eigenvalue
$1 - 2i$	1	complex eigenvalue

Now the eigenvector for each eigenvalue are found.

Considering the eigenvalue $\lambda_1 = 1 - 2i$

We need to solve $A\vec{v} = \lambda\vec{v}$ or $(A - \lambda I)\vec{v} = \vec{0}$ which becomes

$$\left(\begin{bmatrix} 3 & -2 \\ 4 & -1 \end{bmatrix} - (1 - 2i)\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right)\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$
$$\begin{bmatrix} 2 + 2i & -2 \\ 4 & -2 + 2i \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Now forward elimination is applied to solve for the eigenvector $\vec{v}$. The augmented matrix is

$$\left[\begin{array}{cc|c} 2 + 2i & -2 & 0 \\ 4 & -2 + 2i & 0 \end{array}\right]$$

$$R_2 = R_2 + (-1 + i)\,R_1 \Longrightarrow \left[\begin{array}{cc|c} 2 + 2i & -2 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

Therefore the system in Echelon form is

$$\begin{bmatrix} 2 + 2i & -2 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

The free variables are $\{v_2\}$ and the leading variables are $\{v_1\}$. Let $v_2 = t$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\left\{v_1 = \left(\frac{1}{2} - \frac{i}{2}\right) t\right\}$

Hence the solution is

$$\begin{bmatrix} \left(\frac{1}{2} - \frac{I}{2}\right) t \\ t \end{bmatrix} = \begin{bmatrix} \left(\frac{1}{2} - \frac{i}{2}\right) t \\ t \end{bmatrix}$$

Since there is one free Variable, we have found one eigenvector associated with this eigenvalue. The above can be written as

$$\begin{bmatrix} \left(\frac{1}{2} - \frac{I}{2}\right) t \\ t \end{bmatrix} = t \begin{bmatrix} \frac{1}{2} - \frac{i}{2} \\ 1 \end{bmatrix}$$

Let $t = 1$ the eigenvector becomes

$$\begin{bmatrix} \left(\frac{1}{2} - \frac{I}{2}\right) t \\ t \end{bmatrix} = \begin{bmatrix} \frac{1}{2} - \frac{i}{2} \\ 1 \end{bmatrix}$$

Which is normalized to

$$\begin{bmatrix} \left(\frac{1}{2} - \frac{I}{2}\right) t \\ t \end{bmatrix} = \begin{bmatrix} 1 - i \\ 2 \end{bmatrix}$$

Considering the eigenvalue $\lambda_2 = 1 + 2i$

We need to solve $A\vec{v} = \lambda \vec{v}$ or $(A - \lambda I)\vec{v} = \vec{0}$ which becomes

$$\left(\begin{bmatrix} 3 & -2 \\ 4 & -1 \end{bmatrix} - (1 + 2i) \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right) \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 2 - 2i & -2 \\ 4 & -2 - 2i \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Now forward elimination is applied to solve for the eigenvector $\vec{v}$. The augmented matrix is

$$\left[\begin{array}{cc|c} 2 - 2i & -2 & 0 \\ 4 & -2 - 2i & 0 \end{array}\right]$$

$$R_2 = R_2 + (-1 - i) R_1 \Longrightarrow \left[\begin{array}{cc|c} 2 - 2i & -2 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

Therefore the system in Echelon form is

$$\begin{bmatrix} 2-2i & -2 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

The free variables are $\{v_2\}$ and the leading variables are $\{v_1\}$. Let $v_2 = t$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\left\{v_1 = \left(\frac{1}{2} + \frac{i}{2}\right) t\right\}$

Hence the solution is

$$\begin{bmatrix} \left(\frac{1}{2} + \frac{I}{2}\right) t \\ t \end{bmatrix} = \begin{bmatrix} \left(\frac{1}{2} + \frac{i}{2}\right) t \\ t \end{bmatrix}$$

Since there is one free Variable, we have found one eigenvector associated with this eigenvalue. The above can be written as

$$\begin{bmatrix} \left(\frac{1}{2} + \frac{I}{2}\right) t \\ t \end{bmatrix} = t \begin{bmatrix} \frac{1}{2} + \frac{i}{2} \\ 1 \end{bmatrix}$$

Let $t = 1$ the eigenvector becomes

$$\begin{bmatrix} \left(\frac{1}{2} + \frac{I}{2}\right) t \\ t \end{bmatrix} = \begin{bmatrix} \frac{1}{2} + \frac{i}{2} \\ 1 \end{bmatrix}$$

Which is normalized to

$$\begin{bmatrix} \left(\frac{1}{2} + \frac{I}{2}\right) t \\ t \end{bmatrix} = \begin{bmatrix} 1+i \\ 2 \end{bmatrix}$$

The following table gives a summary of this result. It shows for each eigenvalue the algebraic multiplicity m, and its geometric multiplicity k and the eigenvectors associated with the eigenvalue. If $m > k$ then the eigenvalue is defective which means the number of normal linearly independent eigenvectors associated with this eigenvalue (called the geometric multiplicity k) does not equal the algebraic multiplicity m, and we need to determine an additional $m - k$ generalized eigenvectors for this eigenvalue.

eigenvalue	multiplicity algebraic m	geometric k	defective?	eigenvectors
$1+2i$	1	1	No	$\begin{bmatrix} \frac{1}{2}+\frac{i}{2} \\ 1 \end{bmatrix}$
$1-2i$	1	1	No	$\begin{bmatrix} \frac{1}{2}-\frac{i}{2} \\ 1 \end{bmatrix}$

Now that we found the eigenvalues and associated eigenvectors, we will go over each eigenvalue and generate the solution basis. The only problem we need to take care of is if the eigenvalue is defective. Therefore the final solution is

$$\vec{x}_h(t) = c_1\vec{x}_1(t) + c_2\vec{x}_2(t)$$

Which is written as

$$\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = c_1 \begin{bmatrix} \left(\frac{1}{2}+\frac{i}{2}\right) \mathrm{e}^{(1+2i)t} \\ \mathrm{e}^{(1+2i)t} \end{bmatrix} + c_2 \begin{bmatrix} \left(\frac{1}{2}-\frac{i}{2}\right) \mathrm{e}^{(1-2i)t} \\ \mathrm{e}^{(1-2i)t} \end{bmatrix}$$

Which becomes

$$\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = \begin{bmatrix} \left(\frac{1}{2}+\frac{i}{2}\right) c_1 \mathrm{e}^{(1+2i)t} + \left(\frac{1}{2}-\frac{i}{2}\right) c_2 \mathrm{e}^{(1-2i)t} \\ c_1 \mathrm{e}^{(1+2i)t} + c_2 \mathrm{e}^{(1-2i)t} \end{bmatrix}$$

The following is the phase plot of the system.

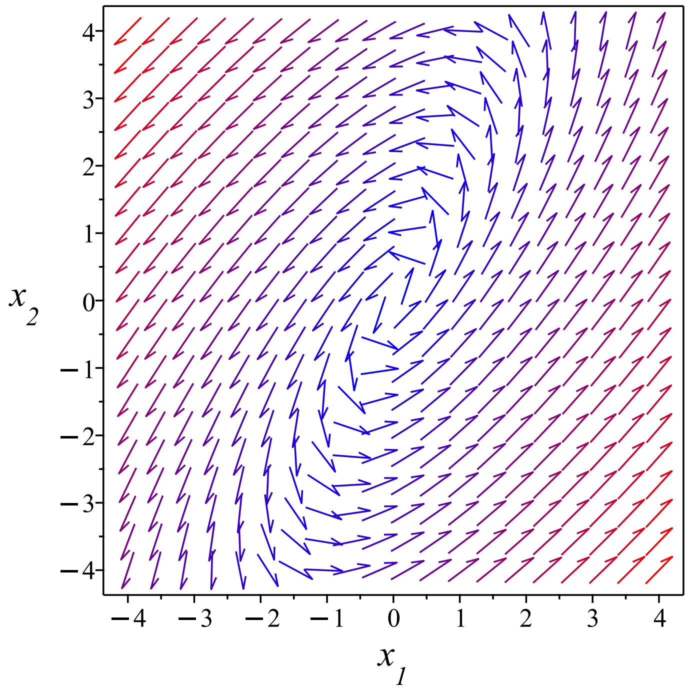


Figure 507: Phase plot

16.1.3 Maple step by step solution

Let's solve

$[x_1'(t) = 3x_1(t) - 2x_2(t), x_2'(t) = 4x_1(t) - x_2(t)]$

- Define vector

$$\underline{x^{\rightarrow}}(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}$$

- Convert system into a vector equation

$$\underline{x^{\rightarrow}}'(t) = \begin{bmatrix} 3 & -2 \\ 4 & -1 \end{bmatrix} \cdot \underline{x^{\rightarrow}}(t) + \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

- System to solve

$$\underline{x^{\rightarrow}}'(t) = \begin{bmatrix} 3 & -2 \\ 4 & -1 \end{bmatrix} \cdot \underline{x^{\rightarrow}}(t)$$

- Define the coefficient matrix

$$A = \begin{bmatrix} 3 & -2 \\ 4 & -1 \end{bmatrix}$$

- Rewrite the system as

$$\underline{x^{\rightarrow}}'(t) = A \cdot \underline{x^{\rightarrow}}(t)$$

- To solve the system, find the eigenvalues and eigenvectors of A
- Eigenpairs of A

$$\left[\left[1-2\,\mathrm{I}, \begin{bmatrix} \frac{1}{2}-\frac{\mathrm{I}}{2} \\ 1 \end{bmatrix}\right], \left[1+2\,\mathrm{I}, \begin{bmatrix} \frac{1}{2}+\frac{\mathrm{I}}{2} \\ 1 \end{bmatrix}\right]\right]$$

- Consider complex eigenpair, complex conjugate eigenvalue can be ignored

$$\left[1-2\,\mathrm{I}, \begin{bmatrix} \frac{1}{2}-\frac{\mathrm{I}}{2} \\ 1 \end{bmatrix}\right]$$

- Solution from eigenpair

$$\mathrm{e}^{(1-2\,\mathrm{I})t} \cdot \begin{bmatrix} \frac{1}{2}-\frac{\mathrm{I}}{2} \\ 1 \end{bmatrix}$$

- Use Euler identity to write solution in terms of $\sin$ and $\cos$

$$\mathrm{e}^{t} \cdot (\cos(2t) - \mathrm{I}\sin(2t)) \cdot \begin{bmatrix} \frac{1}{2}-\frac{\mathrm{I}}{2} \\ 1 \end{bmatrix}$$

- Simplify expression

$$\mathrm{e}^{t} \cdot \begin{bmatrix} \left(\frac{1}{2}-\frac{\mathrm{I}}{2}\right)(\cos(2t) - \mathrm{I}\sin(2t)) \\ \cos(2t) - \mathrm{I}\sin(2t) \end{bmatrix}$$

- Both real and imaginary parts are solutions to the homogeneous system

$$\left[\underline{x^{\rightarrow}}_{1}(t) = \mathrm{e}^{t} \cdot \begin{bmatrix} \frac{\cos(2t)}{2} - \frac{\sin(2t)}{2} \\ \cos(2t) \end{bmatrix}, \underline{x^{\rightarrow}}_{2}(t) = \mathrm{e}^{t} \cdot \begin{bmatrix} -\frac{\sin(2t)}{2} - \frac{\cos(2t)}{2} \\ -\sin(2t) \end{bmatrix}\right]$$

- General solution to the system of ODEs

$$\underline{x^{\rightarrow}} = c_1 \underline{x^{\rightarrow}}_{1}(t) + c_2 \underline{x^{\rightarrow}}_{2}(t)$$

- Substitute solutions into the general solution

$$\underline{x^{\rightarrow}} = c_1 e^t \cdot \begin{bmatrix} \frac{\cos(2t)}{2} - \frac{\sin(2t)}{2} \\ \cos(2t) \end{bmatrix} + c_2 e^t \cdot \begin{bmatrix} -\frac{\sin(2t)}{2} - \frac{\cos(2t)}{2} \\ -\sin(2t) \end{bmatrix}$$

- Substitute in vector of dependent variables

$$\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = \begin{bmatrix} \frac{e^t((c_1-c_2)\cos(2t)-\sin(2t)(c_1+c_2))}{2} \\ e^t(-c_2\sin(2t)+c_1\cos(2t)) \end{bmatrix}$$

- Solution to the system of ODEs

$$\left\{x_1(t) = \frac{e^t((c_1-c_2)\cos(2t)-\sin(2t)(c_1+c_2))}{2}, x_2(t) = e^t(-c_2\sin(2t)+c_1\cos(2t))\right\}$$

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 56

```
dsolve([diff(x__1(t),t)=3*x__1(t)-2*x__2(t),diff(x__2(t),t)=4*x__1(t)-1*x__2(t)],singsol=all
```

$$x_1(t) = e^t(c_1\sin(2t)+c_2\cos(2t))$$
$$x_2(t) = -e^t(c_1\cos(2t)-c_2\cos(2t)-c_1\sin(2t)-c_2\sin(2t))$$

✓ Solution by Mathematica

Time used: 0.005 (sec). Leaf size: 58

```
DSolve[{x1'[t]==3*x1[t]-2*x2[t],x2'[t]==4*x1[t]-1*x2[t]},{x1[t],x2[t]},t,IncludeSingularSolu
```

$$\text{x1}(t) \to e^t(c_1\cos(2t)+(c_1-c_2)\sin(2t))$$
$$\text{x2}(t) \to e^t(c_2\cos(2t)+(2c_1-c_2)\sin(2t))$$

16.2 problem 2

Internal problem ID [752]
Internal file name [`OUTPUT/752_Sunday_June_05_2022_01_48_44_AM_37520237/index.tex`]

Book: Elementary differential equations and boundary value problems, 10th ed., Boyce and DiPrima
Section: Chapter 7.6, Complex Eigenvalues. page 417
Problem number: 2.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"system of linear ODEs"**

Solve

$$x_1'(t) = -x_1(t) - 4x_2(t)$$
$$x_2'(t) = x_1(t) - x_2(t)$$

16.2.1 Solution using Matrix exponential method

In this method, we will assume we have found the matrix exponential e^{At} allready. There are different methods to determine this but will not be shown here. This is a system of linear ODE's given as

$$\vec{x}'(t) = A\,\vec{x}(t)$$

Or

$$\begin{bmatrix} x_1'(t) \\ x_2'(t) \end{bmatrix} = \begin{bmatrix} -1 & -4 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}$$

For the above matrix A, the matrix exponential can be found to be

$$e^{At} = \begin{bmatrix} \mathrm{e}^{-t}\cos(2t) & -2\,\mathrm{e}^{-t}\sin(2t) \\ \frac{\mathrm{e}^{-t}\sin(2t)}{2} & \mathrm{e}^{-t}\cos(2t) \end{bmatrix}$$

Therefore the homogeneous solution is

$$
\begin{aligned}
\vec{x}_h(t) &= e^{At}\vec{c} \\
&= \begin{bmatrix} \mathrm{e}^{-t}\cos(2t) & -2\,\mathrm{e}^{-t}\sin(2t) \\ \frac{\mathrm{e}^{-t}\sin(2t)}{2} & \mathrm{e}^{-t}\cos(2t) \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} \\
&= \begin{bmatrix} \mathrm{e}^{-t}\cos(2t)\,c_1 - 2\,\mathrm{e}^{-t}\sin(2t)\,c_2 \\ \frac{\mathrm{e}^{-t}\sin(2t)c_1}{2} + \mathrm{e}^{-t}\cos(2t)\,c_2 \end{bmatrix} \\
&= \begin{bmatrix} \mathrm{e}^{-t}(\cos(2t)\,c_1 - 2\sin(2t)\,c_2) \\ \frac{\mathrm{e}^{-t}(\sin(2t)c_1 + 2\cos(2t)c_2)}{2} \end{bmatrix}
\end{aligned}
$$

Since no forcing function is given, then the final solution is $\vec{x}_h(t)$ above.

16.2.2 Solution using explicit Eigenvalue and Eigenvector method

This is a system of linear ODE's given as

$$\vec{x}'(t) = A\,\vec{x}(t)$$

Or

$$\begin{bmatrix} x_1'(t) \\ x_2'(t) \end{bmatrix} = \begin{bmatrix} -1 & -4 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}$$

The first step is find the homogeneous solution. We start by finding the eigenvalues of A. This is done by solving the following equation for the eigenvalues λ

$$\det(A - \lambda I) = 0$$

Expanding gives

$$\det\left(\begin{bmatrix} -1 & -4 \\ 1 & -1 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right) = 0$$

Therefore

$$\det\left(\begin{bmatrix} -1-\lambda & -4 \\ 1 & -1-\lambda \end{bmatrix}\right) = 0$$

Which gives the characteristic equation

$$\lambda^2 + 2\lambda + 5 = 0$$

The roots of the above are the eigenvalues.

$$\lambda_1 = -1 + 2i$$
$$\lambda_2 = -1 - 2i$$

This table summarises the above result

eigenvalue	algebraic multiplicity	type of eigenvalue
$-1 - 2i$	1	complex eigenvalue
$-1 + 2i$	1	complex eigenvalue

Now the eigenvector for each eigenvalue are found.

Considering the eigenvalue $\lambda_1 = -1 - 2i$

We need to solve $A\vec{v} = \lambda\vec{v}$ or $(A - \lambda I)\vec{v} = \vec{0}$ which becomes

$$\left(\begin{bmatrix} -1 & -4 \\ 1 & -1 \end{bmatrix} - (-1 - 2i)\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right)\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$
$$\begin{bmatrix} 2i & -4 \\ 1 & 2i \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Now forward elimination is applied to solve for the eigenvector $\vec{v}$. The augmented matrix is

$$\left[\begin{array}{cc|c} 2i & -4 & 0 \\ 1 & 2i & 0 \end{array}\right]$$

$$R_2 = R_2 + \frac{iR_1}{2} \Longrightarrow \left[\begin{array}{cc|c} 2i & -4 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

Therefore the system in Echelon form is

$$\begin{bmatrix} 2i & -4 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

The free variables are $\{v_2\}$ and the leading variables are $\{v_1\}$. Let $v_2 = t$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\{v_1 = -2it\}$

Hence the solution is

$$\begin{bmatrix} -2\,\mathrm{I}\,t \\ t \end{bmatrix} = \begin{bmatrix} -2it \\ t \end{bmatrix}$$

Since there is one free Variable, we have found one eigenvector associated with this eigenvalue. The above can be written as

$$\begin{bmatrix} -2\,\mathrm{I}\,t \\ t \end{bmatrix} = t \begin{bmatrix} -2i \\ 1 \end{bmatrix}$$

Let $t = 1$ the eigenvector becomes

$$\begin{bmatrix} -2\,\mathrm{I}\,t \\ t \end{bmatrix} = \begin{bmatrix} -2i \\ 1 \end{bmatrix}$$

Considering the eigenvalue $\lambda_2 = -1 + 2i$

We need to solve $A\vec{v} = \lambda\vec{v}$ or $(A - \lambda I)\vec{v} = \vec{0}$ which becomes

$$\left(\begin{bmatrix} -1 & -4 \\ 1 & -1 \end{bmatrix} - (-1+2i) \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right) \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} -2i & -4 \\ 1 & -2i \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Now forward elimination is applied to solve for the eigenvector $\vec{v}$. The augmented matrix is

$$\left[\begin{array}{cc|c} -2i & -4 & 0 \\ 1 & -2i & 0 \end{array}\right]$$

$$R_2 = R_2 - \frac{iR_1}{2} \Longrightarrow \left[\begin{array}{cc|c} -2i & -4 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

Therefore the system in Echelon form is

$$\begin{bmatrix} -2i & -4 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

The free variables are $\{v_2\}$ and the leading variables are $\{v_1\}$. Let $v_2 = t$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\{v_1 = 2it\}$

Hence the solution is

$$\begin{bmatrix} 2\,\mathrm{I}t \\ t \end{bmatrix} = \begin{bmatrix} 2it \\ t \end{bmatrix}$$

Since there is one free Variable, we have found one eigenvector associated with this eigenvalue. The above can be written as

$$\begin{bmatrix} 2\,\mathrm{I}t \\ t \end{bmatrix} = t \begin{bmatrix} 2i \\ 1 \end{bmatrix}$$

Let $t = 1$ the eigenvector becomes

$$\begin{bmatrix} 2\,\mathrm{I}t \\ t \end{bmatrix} = \begin{bmatrix} 2i \\ 1 \end{bmatrix}$$

The following table gives a summary of this result. It shows for each eigenvalue the algebraic multiplicity m, and its geometric multiplicity k and the eigenvectors associated with the eigenvalue. If $m > k$ then the eigenvalue is defective which means the number of normal linearly independent eigenvectors associated with this eigenvalue (called the geometric multiplicity k) does not equal the algebraic multiplicity m, and we need to determine an additional $m - k$ generalized eigenvectors for this eigenvalue.

	multiplicity			
eigenvalue	algebraic m	geometric k	defective?	eigenvectors
$-1 + 2i$	1	1	No	$\begin{bmatrix} 2i \\ 1 \end{bmatrix}$
$-1 - 2i$	1	1	No	$\begin{bmatrix} -2i \\ 1 \end{bmatrix}$

Now that we found the eigenvalues and associated eigenvectors, we will go over each eigenvalue and generate the solution basis. The only problem we need to take care of is if the eigenvalue is defective. Therefore the final solution is

$$\vec{x}_h(t) = c_1 \vec{x}_1(t) + c_2 \vec{x}_2(t)$$

Which is written as

$$\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = c_1 \begin{bmatrix} 2i\,e^{(-1+2i)t} \\ e^{(-1+2i)t} \end{bmatrix} + c_2 \begin{bmatrix} -2i\,e^{(-1-2i)t} \\ e^{(-1-2i)t} \end{bmatrix}$$

Which becomes

$$\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = \begin{bmatrix} -2i\left(c_2 e^{(-1-2i)t} - c_1 e^{(-1+2i)t}\right) \\ c_1 e^{(-1+2i)t} + c_2 e^{(-1-2i)t} \end{bmatrix}$$

The following is the phase plot of the system.

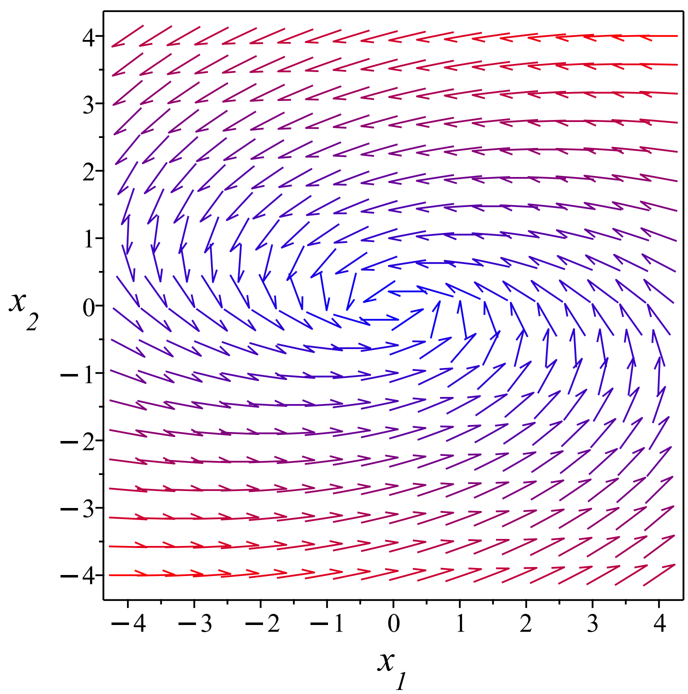


Figure 508: Phase plot

16.2.3 Maple step by step solution

Let's solve

$[x_1'(t) = -x_1(t) - 4x_2(t)\,, x_2'(t) = x_1(t) - x_2(t)]$

- Define vector

$$\underline{x^{\rightarrow}}(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}$$

- Convert system into a vector equation

$$\underline{x^{\rightarrow}}'(t) = \begin{bmatrix} -1 & -4 \\ 1 & -1 \end{bmatrix} \cdot \underline{x^{\rightarrow}}(t) + \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

- System to solve

$$\underline{x^{\rightarrow}}'(t) = \begin{bmatrix} -1 & -4 \\ 1 & -1 \end{bmatrix} \cdot \underline{x^{\rightarrow}}(t)$$

- Define the coefficient matrix

$$A = \begin{bmatrix} -1 & -4 \\ 1 & -1 \end{bmatrix}$$

- Rewrite the system as

$$\underline{x^{\rightarrow}}'(t) = A \cdot \underline{x^{\rightarrow}}(t)$$

- To solve the system, find the eigenvalues and eigenvectors of A
- Eigenpairs of A

$$\left[\left[-1-2\mathrm{I}, \begin{bmatrix} -2\mathrm{I} \\ 1 \end{bmatrix}\right], \left[-1+2\mathrm{I}, \begin{bmatrix} 2\mathrm{I} \\ 1 \end{bmatrix}\right]\right]$$

- Consider complex eigenpair, complex conjugate eigenvalue can be ignored

$$\left[-1-2\mathrm{I}, \begin{bmatrix} -2\mathrm{I} \\ 1 \end{bmatrix}\right]$$

- Solution from eigenpair

$$\mathrm{e}^{(-1-2\mathrm{I})t} \cdot \begin{bmatrix} -2\mathrm{I} \\ 1 \end{bmatrix}$$

- Use Euler identity to write solution in terms of $\sin$ and $\cos$

$$\mathrm{e}^{-t} \cdot (\cos(2t) - \mathrm{I}\sin(2t)) \cdot \begin{bmatrix} -2\mathrm{I} \\ 1 \end{bmatrix}$$

- Simplify expression

$$e^{-t} \cdot \begin{bmatrix} -2\,\mathrm{I}(\cos(2t) - \mathrm{I}\sin(2t)) \\ \cos(2t) - \mathrm{I}\sin(2t) \end{bmatrix}$$

- Both real and imaginary parts are solutions to the homogeneous system

$$\left[\underline{x}^{\rightarrow}{}_{1}(t) = e^{-t} \cdot \begin{bmatrix} -2\sin(2t) \\ \cos(2t) \end{bmatrix}, \underline{x}^{\rightarrow}{}_{2}(t) = e^{-t} \cdot \begin{bmatrix} -2\cos(2t) \\ -\sin(2t) \end{bmatrix}\right]$$

- General solution to the system of ODEs

$$\underline{x}^{\rightarrow} = c_1 \underline{x}^{\rightarrow}{}_{1}(t) + c_2 \underline{x}^{\rightarrow}{}_{2}(t)$$

- Substitute solutions into the general solution

$$\underline{x}^{\rightarrow} = c_1 e^{-t} \cdot \begin{bmatrix} -2\sin(2t) \\ \cos(2t) \end{bmatrix} + c_2 e^{-t} \cdot \begin{bmatrix} -2\cos(2t) \\ -\sin(2t) \end{bmatrix}$$

- Substitute in vector of dependent variables

$$\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = \begin{bmatrix} -2\,e^{-t}(c_1 \sin(2t) + c_2 \cos(2t)) \\ e^{-t}(-c_2 \sin(2t) + c_1 \cos(2t)) \end{bmatrix}$$

- Solution to the system of ODEs

$$\{x_1(t) = -2\,e^{-t}(c_1 \sin(2t) + c_2 \cos(2t)), x_2(t) = e^{-t}(-c_2 \sin(2t) + c_1 \cos(2t))\}$$

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 46

```
dsolve([diff(x__1(t),t)=-1*x__1(t)-4*x__2(t),diff(x__2(t),t)=1*x__1(t)-1*x__2(t)],singsol=al
```

$$x_1(t) = e^{-t}(c_1 \sin(2t) + c_2 \cos(2t))$$
$$x_2(t) = -\frac{e^{-t}(c_1 \cos(2t) - c_2 \sin(2t))}{2}$$

✓ Solution by Mathematica

Time used: 0.005 (sec). Leaf size: 55

```
DSolve[{x1'[t]==-1*x1[t]-4*x2[t],x2'[t]==1*x1[t]-1*x2[t]},{x1[t],x2[t]},t,IncludeSingularSol
```

$$\mathrm{x1}(t) \to e^{-t}(c_1 \cos(2t) - 2c_2 \sin(2t))$$
$$\mathrm{x2}(t) \to \frac{1}{2}e^{-t}(2c_2 \cos(2t) + c_1 \sin(2t))$$

16.3 problem 3

Internal problem ID [753]
Internal file name [`OUTPUT/753_Sunday_June_05_2022_01_48_46_AM_75742661/index.tex`]

Book: Elementary differential equations and boundary value problems, 10th ed., Boyce and DiPrima
Section: Chapter 7.6, Complex Eigenvalues. page 417
Problem number: 3.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"system of linear ODEs"**

Solve

$$x_1'(t) = 2x_1(t) - 5x_2(t)$$
$$x_2'(t) = x_1(t) - 2x_2(t)$$

16.3.1 Solution using Matrix exponential method

In this method, we will assume we have found the matrix exponential e^{At} allready. There are different methods to determine this but will not be shown here. This is a system of linear ODE's given as

$$\vec{x}'(t) = A\,\vec{x}(t)$$

Or

$$\begin{bmatrix} x_1'(t) \\ x_2'(t) \end{bmatrix} = \begin{bmatrix} 2 & -5 \\ 1 & -2 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}$$

For the above matrix A, the matrix exponential can be found to be

$$e^{At} = \begin{bmatrix} \cos(t) + 2\sin(t) & -5\sin(t) \\ \sin(t) & \cos(t) - 2\sin(t) \end{bmatrix}$$

Therefore the homogeneous solution is

$$
\begin{aligned}
\vec{x}_h(t) &= e^{At}\vec{c} \\
&= \begin{bmatrix} \cos(t) + 2\sin(t) & -5\sin(t) \\ \sin(t) & \cos(t) - 2\sin(t) \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} \\
&= \begin{bmatrix} (\cos(t) + 2\sin(t))\, c_1 - 5\sin(t)\, c_2 \\ \sin(t)\, c_1 + (\cos(t) - 2\sin(t))\, c_2 \end{bmatrix} \\
&= \begin{bmatrix} (2c_1 - 5c_2)\sin(t) + c_1\cos(t) \\ (c_1 - 2c_2)\sin(t) + c_2\cos(t) \end{bmatrix}
\end{aligned}
$$

Since no forcing function is given, then the final solution is $\vec{x}_h(t)$ above.

16.3.2 Solution using explicit Eigenvalue and Eigenvector method

This is a system of linear ODE's given as

$$
\vec{x}'(t) = A\,\vec{x}(t)
$$

Or

$$
\begin{bmatrix} x_1'(t) \\ x_2'(t) \end{bmatrix} = \begin{bmatrix} 2 & -5 \\ 1 & -2 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}
$$

The first step is find the homogeneous solution. We start by finding the eigenvalues of A. This is done by solving the following equation for the eigenvalues λ

$$
\det(A - \lambda I) = 0
$$

Expanding gives

$$
\det\left(\begin{bmatrix} 2 & -5 \\ 1 & -2 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right) = 0
$$

Therefore

$$
\det\left(\begin{bmatrix} 2 - \lambda & -5 \\ 1 & -2 - \lambda \end{bmatrix} \right) = 0
$$

Which gives the characteristic equation

$$\lambda^2 + 1 = 0$$

The roots of the above are the eigenvalues.

$$\lambda_1 = i$$
$$\lambda_2 = -i$$

This table summarises the above result

eigenvalue	algebraic multiplicity	type of eigenvalue
$-i$	1	complex eigenvalue
i	1	complex eigenvalue

Now the eigenvector for each eigenvalue are found.

Considering the eigenvalue $\lambda_1 = -i$

We need to solve $A\vec{v} = \lambda\vec{v}$ or $(A - \lambda I)\vec{v} = \vec{0}$ which becomes

$$\left(\begin{bmatrix} 2 & -5 \\ 1 & -2 \end{bmatrix} - (-i)\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right)\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$
$$\begin{bmatrix} 2+i & -5 \\ 1 & -2+i \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Now forward elimination is applied to solve for the eigenvector $\vec{v}$. The augmented matrix is

$$\left[\begin{array}{cc|c} 2+i & -5 & 0 \\ 1 & -2+i & 0 \end{array}\right]$$

$$R_2 = R_2 + \left(-\frac{2}{5} + \frac{i}{5}\right) R_1 \Longrightarrow \left[\begin{array}{cc|c} 2+i & -5 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

Therefore the system in Echelon form is

$$\begin{bmatrix} 2+i & -5 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

The free variables are $\{v_2\}$ and the leading variables are $\{v_1\}$. Let $v_2 = t$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\{v_1 = (2-i)\,t\}$

Hence the solution is

$$\begin{bmatrix} (2-\mathrm{I})\,t \\ t \end{bmatrix} = \begin{bmatrix} (2-i)\,t \\ t \end{bmatrix}$$

Since there is one free Variable, we have found one eigenvector associated with this eigenvalue. The above can be written as

$$\begin{bmatrix} (2-\mathrm{I})\,t \\ t \end{bmatrix} = t \begin{bmatrix} 2-i \\ 1 \end{bmatrix}$$

Let $t = 1$ the eigenvector becomes

$$\begin{bmatrix} (2-\mathrm{I})\,t \\ t \end{bmatrix} = \begin{bmatrix} 2-i \\ 1 \end{bmatrix}$$

Considering the eigenvalue $\lambda_2 = i$

We need to solve $A\vec{v} = \lambda\vec{v}$ or $(A-\lambda I)\vec{v} = \vec{0}$ which becomes

$$\left(\begin{bmatrix} 2 & -5 \\ 1 & -2 \end{bmatrix} - (i) \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right) \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 2-i & -5 \\ 1 & -2-i \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Now forward elimination is applied to solve for the eigenvector $\vec{v}$. The augmented matrix is

$$\left[\begin{array}{cc|c} 2-i & -5 & 0 \\ 1 & -2-i & 0 \end{array}\right]$$

$$R_2 = R_2 + \left(-\frac{2}{5} - \frac{i}{5}\right) R_1 \implies \left[\begin{array}{cc|c} 2-i & -5 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

Therefore the system in Echelon form is

$$\begin{bmatrix} 2-i & -5 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

The free variables are $\{v_2\}$ and the leading variables are $\{v_1\}$. Let $v_2 = t$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\{v_1 = (2+i)\,t\}$

Hence the solution is

$$\begin{bmatrix} (2+\mathrm{I})\,t \\ t \end{bmatrix} = \begin{bmatrix} (2+i)\,t \\ t \end{bmatrix}$$

Since there is one free Variable, we have found one eigenvector associated with this eigenvalue. The above can be written as

$$\begin{bmatrix} (2+\mathrm{I})\,t \\ t \end{bmatrix} = t \begin{bmatrix} 2+i \\ 1 \end{bmatrix}$$

Let $t = 1$ the eigenvector becomes

$$\begin{bmatrix} (2+\mathrm{I})\,t \\ t \end{bmatrix} = \begin{bmatrix} 2+i \\ 1 \end{bmatrix}$$

The following table gives a summary of this result. It shows for each eigenvalue the algebraic multiplicity m, and its geometric multiplicity k and the eigenvectors associated with the eigenvalue. If $m > k$ then the eigenvalue is defective which means the number of normal linearly independent eigenvectors associated with this eigenvalue (called the geometric multiplicity k) does not equal the algebraic multiplicity m, and we need to determine an additional $m - k$ generalized eigenvectors for this eigenvalue.

	multiplicity			
eigenvalue	algebraic m	geometric k	defective?	eigenvectors
i	1	1	No	$\begin{bmatrix} 2+i \\ 1 \end{bmatrix}$
$-i$	1	1	No	$\begin{bmatrix} 2-i \\ 1 \end{bmatrix}$

Now that we found the eigenvalues and associated eigenvectors, we will go over each eigenvalue and generate the solution basis. The only problem we need to take care of is if the eigenvalue is defective. Therefore the final solution is

$$\vec{x}_h(t) = c_1\vec{x}_1(t) + c_2\vec{x}_2(t)$$

Which is written as

$$\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = c_1 \begin{bmatrix} (2+i)\,\mathrm{e}^{it} \\ \mathrm{e}^{it} \end{bmatrix} + c_2 \begin{bmatrix} (2-i)\,\mathrm{e}^{-it} \\ \mathrm{e}^{-it} \end{bmatrix}$$

Which becomes

$$\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = \begin{bmatrix} (2+i)\,c_1\mathrm{e}^{it} + (2-i)\,c_2\mathrm{e}^{-it} \\ c_1\mathrm{e}^{it} + c_2\mathrm{e}^{-it} \end{bmatrix}$$

The following is the phase plot of the system.

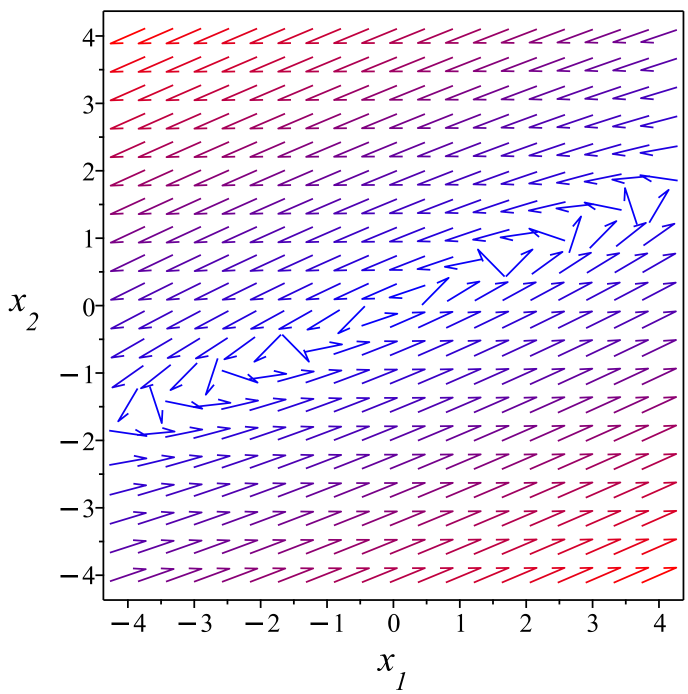


Figure 509: Phase plot

16.3.3 Maple step by step solution

Let's solve

$[x_1'(t) = 2x_1(t) - 5x_2(t)\,, x_2'(t) = x_1(t) - 2x_2(t)]$

- Define vector

$$\underline{x}^{\rightarrow}(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}$$

- Convert system into a vector equation

$$\underline{x}^{\rightarrow \prime}(t) = \begin{bmatrix} 2 & -5 \\ 1 & -2 \end{bmatrix} \cdot \underline{x}^{\rightarrow}(t) + \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

- System to solve

$$\underline{x}^{\rightarrow \prime}(t) = \begin{bmatrix} 2 & -5 \\ 1 & -2 \end{bmatrix} \cdot \underline{x}^{\rightarrow}(t)$$

- Define the coefficient matrix

$$A = \begin{bmatrix} 2 & -5 \\ 1 & -2 \end{bmatrix}$$

- Rewrite the system as

$$\underline{x}^{\rightarrow \prime}(t) = A \cdot \underline{x}^{\rightarrow}(t)$$

- To solve the system, find the eigenvalues and eigenvectors of A
- Eigenpairs of A

$$\left[\left[-\mathrm{I}, \begin{bmatrix} 2-\mathrm{I} \\ 1 \end{bmatrix}\right], \left[\mathrm{I}, \begin{bmatrix} 2+\mathrm{I} \\ 1 \end{bmatrix}\right]\right]$$

- Consider complex eigenpair, complex conjugate eigenvalue can be ignored

$$\left[-\mathrm{I}, \begin{bmatrix} 2-\mathrm{I} \\ 1 \end{bmatrix}\right]$$

- Solution from eigenpair

$$\mathrm{e}^{-\mathrm{I}t} \cdot \begin{bmatrix} 2-\mathrm{I} \\ 1 \end{bmatrix}$$

- Use Euler identity to write solution in terms of sin and cos

$$(\cos(t) - \mathrm{I}\sin(t)) \cdot \begin{bmatrix} 2-\mathrm{I} \\ 1 \end{bmatrix}$$

- Simplify expression

$$\left[\begin{array}{c}(2-\mathrm{I})\left(\cos\left(t\right)-\mathrm{I}\sin\left(t\right)\right)\\ \cos\left(t\right)-\mathrm{I}\sin\left(t\right)\end{array}\right]$$

- Both real and imaginary parts are solutions to the homogeneous system

$$\left[\underline{x}^{\rightarrow}{}_{1}(t)=\left[\begin{array}{c}2\cos\left(t\right)-\sin\left(t\right)\\ \cos\left(t\right)\end{array}\right],\underline{x}^{\rightarrow}{}_{2}(t)=\left[\begin{array}{c}-\cos\left(t\right)-2\sin\left(t\right)\\ -\sin\left(t\right)\end{array}\right]\right]$$

- General solution to the system of ODEs

$$\underline{x}^{\rightarrow}=c_{1}\underline{x}^{\rightarrow}{}_{1}(t)+c_{2}\underline{x}^{\rightarrow}{}_{2}(t)$$

- Substitute solutions into the general solution

$$\underline{x}^{\rightarrow}=\left[\begin{array}{c}c_{2}(-\cos\left(t\right)-2\sin\left(t\right))+c_{1}(2\cos\left(t\right)-\sin\left(t\right))\\ -c_{2}\sin\left(t\right)+c_{1}\cos\left(t\right)\end{array}\right]$$

- Substitute in vector of dependent variables

$$\left[\begin{array}{c}x_{1}(t)\\ x_{2}(t)\end{array}\right]=\left[\begin{array}{c}\cos\left(t\right)\left(2c_{1}-c_{2}\right)-\sin\left(t\right)\left(c_{1}+2c_{2}\right)\\ -c_{2}\sin\left(t\right)+c_{1}\cos\left(t\right)\end{array}\right]$$

- Solution to the system of ODEs

$$\{x_{1}(t)=\cos\left(t\right)\left(2c_{1}-c_{2}\right)-\sin\left(t\right)\left(c_{1}+2c_{2}\right),x_{2}(t)=-c_{2}\sin\left(t\right)+c_{1}\cos\left(t\right)\}$$

✓ Solution by Maple

Time used: 0.015 (sec). Leaf size: 38

```
dsolve([diff(x__1(t),t)=2*x__1(t)-5*x__2(t),diff(x__2(t),t)=1*x__1(t)-2*x__2(t)],singsol=all
```

$$x_{1}(t)=c_{1}\sin\left(t\right)+c_{2}\cos\left(t\right)$$
$$x_{2}(t)=-\frac{c_{1}\cos\left(t\right)}{5}+\frac{c_{2}\sin\left(t\right)}{5}+\frac{2c_{1}\sin\left(t\right)}{5}+\frac{2c_{2}\cos\left(t\right)}{5}$$

✓ Solution by Mathematica

Time used: 0.005 (sec). Leaf size: 41

```
DSolve[{x1'[t]==2*x1[t]-5*x2[t],x2'[t]==1*x1[t]-2*x2[t]},{x1[t],x2[t]},t,IncludeSingularSolu
```

$$\mathrm{x1}(t)\to c_{1}(2\sin(t)+\cos(t))-5c_{2}\sin(t)$$
$$\mathrm{x2}(t)\to c_{2}\cos(t)+(c_{1}-2c_{2})\sin(t)$$

16.4 problem 4

Internal problem ID [754]
Internal file name [`OUTPUT/754_Sunday_June_05_2022_01_48_47_AM_27946383/index.tex`]

Book: Elementary differential equations and boundary value problems, 10th ed., Boyce and DiPrima
Section: Chapter 7.6, Complex Eigenvalues. page 417
Problem number: 4.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"system of linear ODEs"**

Solve

$$x_1'(t) = 2x_1(t) - \frac{5x_2(t)}{2}$$
$$x_2'(t) = \frac{9x_1(t)}{5} - x_2(t)$$

16.4.1 Solution using Matrix exponential method

In this method, we will assume we have found the matrix exponential e^{At} allready. There are different methods to determine this but will not be shown here. This is a system of linear ODE's given as

$$\vec{x}'(t) = A\,\vec{x}(t)$$

Or

$$\begin{bmatrix} x_1'(t) \\ x_2'(t) \end{bmatrix} = \begin{bmatrix} 2 & -\frac{5}{2} \\ \frac{9}{5} & -1 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}$$

For the above matrix A, the matrix exponential can be found to be

$$
e^{At} = \begin{bmatrix} e^{\frac{t}{2}}\cos\left(\frac{3t}{2}\right)+e^{\frac{t}{2}}\sin\left(\frac{3t}{2}\right) & -\frac{5\,e^{\frac{t}{2}}\sin\left(\frac{3t}{2}\right)}{3} \\ \frac{6\,e^{\frac{t}{2}}\sin\left(\frac{3t}{2}\right)}{5} & e^{\frac{t}{2}}\cos\left(\frac{3t}{2}\right)-e^{\frac{t}{2}}\sin\left(\frac{3t}{2}\right) \end{bmatrix}
$$

$$
= \begin{bmatrix} e^{\frac{t}{2}}\left(\cos\left(\frac{3t}{2}\right)+\sin\left(\frac{3t}{2}\right)\right) & -\frac{5\,e^{\frac{t}{2}}\sin\left(\frac{3t}{2}\right)}{3} \\ \frac{6\,e^{\frac{t}{2}}\sin\left(\frac{3t}{2}\right)}{5} & e^{\frac{t}{2}}\left(\cos\left(\frac{3t}{2}\right)-\sin\left(\frac{3t}{2}\right)\right) \end{bmatrix}
$$

Therefore the homogeneous solution is

$$
\begin{aligned}
\vec{x}_h(t) &= e^{At}\vec{c} \\
&= \begin{bmatrix} e^{\frac{t}{2}}\left(\cos\left(\frac{3t}{2}\right)+\sin\left(\frac{3t}{2}\right)\right) & -\frac{5\,e^{\frac{t}{2}}\sin\left(\frac{3t}{2}\right)}{3} \\ \frac{6\,e^{\frac{t}{2}}\sin\left(\frac{3t}{2}\right)}{5} & e^{\frac{t}{2}}\left(\cos\left(\frac{3t}{2}\right)-\sin\left(\frac{3t}{2}\right)\right) \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} \\
&= \begin{bmatrix} e^{\frac{t}{2}}\left(\cos\left(\frac{3t}{2}\right)+\sin\left(\frac{3t}{2}\right)\right)c_1 - \frac{5\,e^{\frac{t}{2}}\sin\left(\frac{3t}{2}\right)c_2}{3} \\ \frac{6\,e^{\frac{t}{2}}\sin\left(\frac{3t}{2}\right)c_1}{5} + e^{\frac{t}{2}}\left(\cos\left(\frac{3t}{2}\right)-\sin\left(\frac{3t}{2}\right)\right)c_2 \end{bmatrix} \\
&= \begin{bmatrix} \left(\left(c_1-\frac{5c_2}{3}\right)\sin\left(\frac{3t}{2}\right)+c_1\cos\left(\frac{3t}{2}\right)\right)e^{\frac{t}{2}} \\ \frac{e^{\frac{t}{2}}(6c_1-5c_2)\sin\left(\frac{3t}{2}\right)}{5} + e^{\frac{t}{2}}\cos\left(\frac{3t}{2}\right)c_2 \end{bmatrix}
\end{aligned}
$$

Since no forcing function is given, then the final solution is $\vec{x}_h(t)$ above.

16.4.2 Solution using explicit Eigenvalue and Eigenvector method

This is a system of linear ODE's given as

$$
\vec{x}'(t) = A\,\vec{x}(t)
$$

Or

$$
\begin{bmatrix} x_1'(t) \\ x_2'(t) \end{bmatrix} = \begin{bmatrix} 2 & -\frac{5}{2} \\ \frac{9}{5} & -1 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}
$$

The first step is find the homogeneous solution. We start by finding the eigenvalues of A. This is done by solving the following equation for the eigenvalues λ

$$
\det\left(A-\lambda I\right) = 0
$$

Expanding gives

$$\det\left(\begin{bmatrix} 2 & -\frac{5}{2} \\ \frac{9}{5} & -1 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right) = 0$$

Therefore

$$\det\left(\begin{bmatrix} 2-\lambda & -\frac{5}{2} \\ \frac{9}{5} & -1-\lambda \end{bmatrix}\right) = 0$$

Which gives the characteristic equation

$$\lambda^2 - \lambda + \frac{5}{2} = 0$$

The roots of the above are the eigenvalues.

$$\lambda_1 = \frac{1}{2} + \frac{3i}{2}$$
$$\lambda_2 = \frac{1}{2} - \frac{3i}{2}$$

This table summarises the above result

eigenvalue	algebraic multiplicity	type of eigenvalue
$\frac{1}{2} - \frac{3i}{2}$	1	complex eigenvalue
$\frac{1}{2} + \frac{3i}{2}$	1	complex eigenvalue

Now the eigenvector for each eigenvalue are found.

Considering the eigenvalue $\lambda_1 = \frac{1}{2} - \frac{3i}{2}$

We need to solve $A\vec{v} = \lambda\vec{v}$ or $(A - \lambda I)\vec{v} = \vec{0}$ which becomes

$$\left(\begin{bmatrix} 2 & -\frac{5}{2} \\ \frac{9}{5} & -1 \end{bmatrix} - \left(\frac{1}{2} - \frac{3i}{2}\right)\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right)\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$
$$\begin{bmatrix} \frac{3}{2} + \frac{3i}{2} & -\frac{5}{2} \\ \frac{9}{5} & -\frac{3}{2} + \frac{3i}{2} \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Now forward elimination is applied to solve for the eigenvector $\vec{v}$. The augmented matrix is

$$\left[\begin{array}{cc|c} \frac{3}{2} + \frac{3i}{2} & -\frac{5}{2} & 0 \\ \frac{9}{5} & -\frac{3}{2} + \frac{3i}{2} & 0 \end{array}\right]$$

$$R_2 = R_2 + \left(-\frac{3}{5} + \frac{3i}{5}\right) R_1 \Longrightarrow \left[\begin{array}{cc|c} \frac{3}{2} + \frac{3i}{2} & -\frac{5}{2} & 0 \\ 0 & 0 & 0 \end{array}\right]$$

Therefore the system in Echelon form is

$$\begin{bmatrix} \frac{3}{2} + \frac{3i}{2} & -\frac{5}{2} \\ 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

The free variables are $\{v_2\}$ and the leading variables are $\{v_1\}$. Let $v_2 = t$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\left\{v_1 = \left(\frac{5}{6} - \frac{5i}{6}\right) t\right\}$

Hence the solution is

$$\begin{bmatrix} \left(\frac{5}{6} - \frac{5\,\mathrm{I}}{6}\right) t \\ t \end{bmatrix} = \begin{bmatrix} \left(\frac{5}{6} - \frac{5i}{6}\right) t \\ t \end{bmatrix}$$

Since there is one free Variable, we have found one eigenvector associated with this eigenvalue. The above can be written as

$$\begin{bmatrix} \left(\frac{5}{6} - \frac{5\,\mathrm{I}}{6}\right) t \\ t \end{bmatrix} = t \begin{bmatrix} \frac{5}{6} - \frac{5i}{6} \\ 1 \end{bmatrix}$$

Let $t = 1$ the eigenvector becomes

$$\begin{bmatrix} \left(\frac{5}{6} - \frac{5\,\mathrm{I}}{6}\right) t \\ t \end{bmatrix} = \begin{bmatrix} \frac{5}{6} - \frac{5i}{6} \\ 1 \end{bmatrix}$$

Which is normalized to

$$\begin{bmatrix} \left(\frac{5}{6} - \frac{5\,\mathrm{I}}{6}\right) t \\ t \end{bmatrix} = \begin{bmatrix} 5 - 5i \\ 6 \end{bmatrix}$$

Considering the eigenvalue $\lambda_2 = \frac{1}{2} + \frac{3i}{2}$

We need to solve $A\vec{v} = \lambda\vec{v}$ or $(A - \lambda I)\vec{v} = \vec{0}$ which becomes

$$\left(\begin{bmatrix} 2 & -\frac{5}{2} \\ \frac{9}{5} & -1 \end{bmatrix} - \left(\frac{1}{2} + \frac{3i}{2}\right)\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right)\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} \frac{3}{2} - \frac{3i}{2} & -\frac{5}{2} \\ \frac{9}{5} & -\frac{3}{2} - \frac{3i}{2} \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Now forward elimination is applied to solve for the eigenvector $\vec{v}$. The augmented matrix is

$$\left[\begin{array}{cc|c} \frac{3}{2}-\frac{3i}{2} & -\frac{5}{2} & 0 \\ \frac{9}{5} & -\frac{3}{2}-\frac{3i}{2} & 0 \end{array}\right]$$

$$R_2 = R_2 + \left(-\frac{3}{5}-\frac{3i}{5}\right) R_1 \Longrightarrow \left[\begin{array}{cc|c} \frac{3}{2}-\frac{3i}{2} & -\frac{5}{2} & 0 \\ 0 & 0 & 0 \end{array}\right]$$

Therefore the system in Echelon form is

$$\begin{bmatrix} \frac{3}{2}-\frac{3i}{2} & -\frac{5}{2} \\ 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

The free variables are $\{v_2\}$ and the leading variables are $\{v_1\}$. Let $v_2 = t$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\left\{v_1 = \left(\frac{5}{6}+\frac{5i}{6}\right)t\right\}$

Hence the solution is

$$\begin{bmatrix} \left(\frac{5}{6}+\frac{5\,\mathrm{I}}{6}\right)t \\ t \end{bmatrix} = \begin{bmatrix} \left(\frac{5}{6}+\frac{5i}{6}\right)t \\ t \end{bmatrix}$$

Since there is one free Variable, we have found one eigenvector associated with this eigenvalue. The above can be written as

$$\begin{bmatrix} \left(\frac{5}{6}+\frac{5\,\mathrm{I}}{6}\right)t \\ t \end{bmatrix} = t \begin{bmatrix} \frac{5}{6}+\frac{5i}{6} \\ 1 \end{bmatrix}$$

Let $t = 1$ the eigenvector becomes

$$\begin{bmatrix} \left(\frac{5}{6}+\frac{5\,\mathrm{I}}{6}\right)t \\ t \end{bmatrix} = \begin{bmatrix} \frac{5}{6}+\frac{5i}{6} \\ 1 \end{bmatrix}$$

Which is normalized to

$$\begin{bmatrix} \left(\frac{5}{6}+\frac{5\,\mathrm{I}}{6}\right)t \\ t \end{bmatrix} = \begin{bmatrix} 5+5i \\ 6 \end{bmatrix}$$

The following table gives a summary of this result. It shows for each eigenvalue the algebraic multiplicity m, and its geometric multiplicity k and the eigenvectors associated with the eigenvalue. If $m > k$ then the eigenvalue is defective which means the number of normal linearly independent eigenvectors associated with this eigenvalue (called the geometric multiplicity k) does not equal the algebraic multiplicity m, and we need to determine an additional $m - k$ generalized eigenvectors for this eigenvalue.

eigenvalue	multiplicity algebraic m	multiplicity geometric k	defective?	eigenvectors
$\frac{1}{2}+\frac{3i}{2}$	1	1	No	$\begin{bmatrix}\frac{5}{6}+\frac{5i}{6}\\ 1\end{bmatrix}$
$\frac{1}{2}-\frac{3i}{2}$	1	1	No	$\begin{bmatrix}\frac{5}{6}-\frac{5i}{6}\\ 1\end{bmatrix}$

Now that we found the eigenvalues and associated eigenvectors, we will go over each eigenvalue and generate the solution basis. The only problem we need to take care of is if the eigenvalue is defective. Therefore the final solution is

$$\vec{x}_h(t) = c_1\vec{x}_1(t) + c_2\vec{x}_2(t)$$

Which is written as

$$\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = c_1 \begin{bmatrix} \left(\frac{5}{6}+\frac{5i}{6}\right) e^{\left(\frac{1}{2}+\frac{3i}{2}\right)t} \\ e^{\left(\frac{1}{2}+\frac{3i}{2}\right)t} \end{bmatrix} + c_2 \begin{bmatrix} \left(\frac{5}{6}-\frac{5i}{6}\right) e^{\left(\frac{1}{2}-\frac{3i}{2}\right)t} \\ e^{\left(\frac{1}{2}-\frac{3i}{2}\right)t} \end{bmatrix}$$

Which becomes

$$\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = \begin{bmatrix} \left(\frac{5}{6}+\frac{5i}{6}\right) c_1 e^{\left(\frac{1}{2}+\frac{3i}{2}\right)t} + \left(\frac{5}{6}-\frac{5i}{6}\right) c_2 e^{\left(\frac{1}{2}-\frac{3i}{2}\right)t} \\ c_1 e^{\left(\frac{1}{2}+\frac{3i}{2}\right)t} + c_2 e^{\left(\frac{1}{2}-\frac{3i}{2}\right)t} \end{bmatrix}$$

The following is the phase plot of the system.

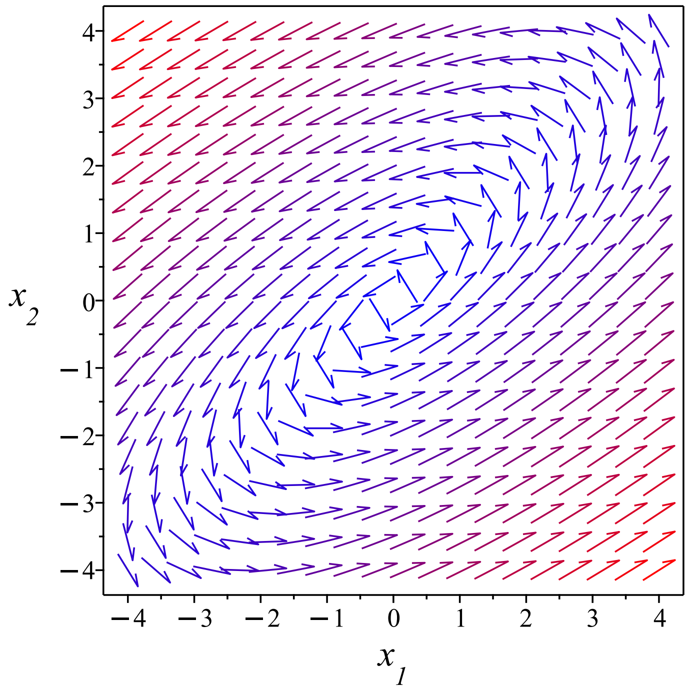


Figure 510: Phase plot

16.4.3 Maple step by step solution

Let's solve

$$\left[x_1'(t) = 2x_1(t) - \frac{5x_2(t)}{2}, x_2'(t) = \frac{9x_1(t)}{5} - x_2(t)\right]$$

- Define vector

$$\underline{x}^{\rightarrow}(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}$$

- Convert system into a vector equation

$$\underline{x}^{\rightarrow\prime}(t) = \begin{bmatrix} 2 & -\frac{5}{2} \\ \frac{9}{5} & -1 \end{bmatrix} \cdot \underline{x}^{\rightarrow}(t) + \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

- System to solve

$$\underline{x}^{\rightarrow\prime}(t) = \begin{bmatrix} 2 & -\frac{5}{2} \\ \frac{9}{5} & -1 \end{bmatrix} \cdot \underline{x}^{\rightarrow}(t)$$

- Define the coefficient matrix

$$A = \left[\begin{array}{cc} 2 & -\frac{5}{2} \\ \frac{9}{5} & -1 \end{array}\right]$$

- Rewrite the system as

$${\underline{x}^{\rightarrow}}'(t) = A \cdot \underline{x}^{\rightarrow}(t)$$

- To solve the system, find the eigenvalues and eigenvectors of A
- Eigenpairs of A

$$\left[\left[\frac{1}{2}-\frac{3\,\mathrm{I}}{2}, \left[\begin{array}{c} \frac{5}{6}-\frac{5\,\mathrm{I}}{6} \\ 1 \end{array}\right]\right], \left[\frac{1}{2}+\frac{3\,\mathrm{I}}{2}, \left[\begin{array}{c} \frac{5}{6}+\frac{5\,\mathrm{I}}{6} \\ 1 \end{array}\right]\right]\right]$$

- Consider complex eigenpair, complex conjugate eigenvalue can be ignored

$$\left[\frac{1}{2}-\frac{3\,\mathrm{I}}{2}, \left[\begin{array}{c} \frac{5}{6}-\frac{5\,\mathrm{I}}{6} \\ 1 \end{array}\right]\right]$$

- Solution from eigenpair

$$\mathrm{e}^{\left(\frac{1}{2}-\frac{3\,\mathrm{I}}{2}\right)t} \cdot \left[\begin{array}{c} \frac{5}{6}-\frac{5\,\mathrm{I}}{6} \\ 1 \end{array}\right]$$

- Use Euler identity to write solution in terms of $\sin$ and $\cos$

$$\mathrm{e}^{\frac{t}{2}} \cdot \left(\cos\left(\frac{3t}{2}\right) - \mathrm{I}\sin\left(\frac{3t}{2}\right)\right) \cdot \left[\begin{array}{c} \frac{5}{6}-\frac{5\,\mathrm{I}}{6} \\ 1 \end{array}\right]$$

- Simplify expression

$$\mathrm{e}^{\frac{t}{2}} \cdot \left[\begin{array}{c} \left(\frac{5}{6}-\frac{5\,\mathrm{I}}{6}\right)\left(\cos\left(\frac{3t}{2}\right) - \mathrm{I}\sin\left(\frac{3t}{2}\right)\right) \\ \cos\left(\frac{3t}{2}\right) - \mathrm{I}\sin\left(\frac{3t}{2}\right) \end{array}\right]$$

- Both real and imaginary parts are solutions to the homogeneous system

$$\left[\underline{x}^{\rightarrow}{}_{1}(t) = \mathrm{e}^{\frac{t}{2}} \cdot \left[\begin{array}{c} \frac{5\cos\left(\frac{3t}{2}\right)}{6} - \frac{5\sin\left(\frac{3t}{2}\right)}{6} \\ \cos\left(\frac{3t}{2}\right) \end{array}\right], \underline{x}^{\rightarrow}{}_{2}(t) = \mathrm{e}^{\frac{t}{2}} \cdot \left[\begin{array}{c} -\frac{5\sin\left(\frac{3t}{2}\right)}{6} - \frac{5\cos\left(\frac{3t}{2}\right)}{6} \\ -\sin\left(\frac{3t}{2}\right) \end{array}\right]\right]$$

- General solution to the system of ODEs

$$\underline{x}^{\rightarrow} = c_1 \underline{x}^{\rightarrow}{}_{1}(t) + c_2 \underline{x}^{\rightarrow}{}_{2}(t)$$

- Substitute solutions into the general solution

$$\overrightarrow{x} = c_1 e^{\frac{t}{2}} \cdot \begin{bmatrix} \frac{5\cos\left(\frac{3t}{2}\right)}{6} - \frac{5\sin\left(\frac{3t}{2}\right)}{6} \\ \cos\left(\frac{3t}{2}\right) \end{bmatrix} + c_2 e^{\frac{t}{2}} \cdot \begin{bmatrix} -\frac{5\sin\left(\frac{3t}{2}\right)}{6} - \frac{5\cos\left(\frac{3t}{2}\right)}{6} \\ -\sin\left(\frac{3t}{2}\right) \end{bmatrix}$$

- Substitute in vector of dependent variables

$$\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = \begin{bmatrix} \frac{5\left(\left(c_1-c_2\right)\cos\left(\frac{3t}{2}\right)-\sin\left(\frac{3t}{2}\right)\left(c_1+c_2\right)\right)e^{\frac{t}{2}}}{6} \\ e^{\frac{t}{2}}\left(c_1\cos\left(\frac{3t}{2}\right) - c_2\sin\left(\frac{3t}{2}\right)\right) \end{bmatrix}$$

- Solution to the system of ODEs

$$\left\{x_1(t) = \frac{5\left(\left(c_1-c_2\right)\cos\left(\frac{3t}{2}\right)-\sin\left(\frac{3t}{2}\right)\left(c_1+c_2\right)\right)e^{\frac{t}{2}}}{6}, x_2(t) = e^{\frac{t}{2}}\left(c_1\cos\left(\frac{3t}{2}\right) - c_2\sin\left(\frac{3t}{2}\right)\right)\right\}$$

✓ Solution by Maple

Time used: 0.015 (sec). Leaf size: 58

```
dsolve([diff(x__1(t),t)=2*x__1(t)-5/2*x__2(t),diff(x__2(t),t)=9/5*x__1(t)-1*x__2(t)],singsol
```

$$x_1(t) = e^{\frac{t}{2}}\left(\sin\left(\frac{3t}{2}\right)c_1 + \cos\left(\frac{3t}{2}\right)c_2\right)$$
$$x_2(t) = \frac{3\,e^{\frac{t}{2}}\left(\sin\left(\frac{3t}{2}\right)c_1 + \sin\left(\frac{3t}{2}\right)c_2 - \cos\left(\frac{3t}{2}\right)c_1 + \cos\left(\frac{3t}{2}\right)c_2\right)}{5}$$

✓ Solution by Mathematica

Time used: 0.006 (sec). Leaf size: 84

```
DSolve[{x1'[t]==2*x1[t]-5/2*x2[t],x2'[t]==9/5*x1[t]-1*x2[t]},{x1[t],x2[t]},t,IncludeSingular
```

$$\text{x1}(t) \to \frac{1}{3}e^{t/2}\left(3c_1\cos\left(\frac{3t}{2}\right) + (3c_1 - 5c_2)\sin\left(\frac{3t}{2}\right)\right)$$
$$\text{x2}(t) \to \frac{1}{5}e^{t/2}\left(5c_2\cos\left(\frac{3t}{2}\right) + (6c_1 - 5c_2)\sin\left(\frac{3t}{2}\right)\right)$$

16.5 problem 5

Internal problem ID [755]
Internal file name [`OUTPUT/755_Sunday_June_05_2022_01_48_49_AM_29534035/index.tex`]

Book: Elementary differential equations and boundary value problems, 10th ed., Boyce and DiPrima
Section: Chapter 7.6, Complex Eigenvalues. page 417
Problem number: 5.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"system of linear ODEs"**

Solve

$$
\begin{aligned}
x_1'(t) &= x_1(t) - x_2(t) \\
x_2'(t) &= 5x_1(t) - 3x_2(t)
\end{aligned}
$$

16.5.1 Solution using Matrix exponential method

In this method, we will assume we have found the matrix exponential e^{At} allready. There are different methods to determine this but will not be shown here. This is a system of linear ODE's given as

$$\vec{x}'(t) = A\,\vec{x}(t)$$

Or

$$\begin{bmatrix} x_1'(t) \\ x_2'(t) \end{bmatrix} = \begin{bmatrix} 1 & -1 \\ 5 & -3 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}$$

For the above matrix A, the matrix exponential can be found to be

$$
\begin{aligned}
e^{At} &= \begin{bmatrix} e^{-t}\cos(t) + 2\,e^{-t}\sin(t) & -e^{-t}\sin(t) \\ 5\,e^{-t}\sin(t) & e^{-t}\cos(t) - 2\,e^{-t}\sin(t) \end{bmatrix} \\
&= \begin{bmatrix} e^{-t}(\cos(t) + 2\sin(t)) & -e^{-t}\sin(t) \\ 5\,e^{-t}\sin(t) & e^{-t}(\cos(t) - 2\sin(t)) \end{bmatrix}
\end{aligned}
$$

Therefore the homogeneous solution is

$$
\begin{aligned}
\vec{x}_h(t) &= e^{At}\vec{c} \\
&= \begin{bmatrix} e^{-t}(\cos(t)+2\sin(t)) & -e^{-t}\sin(t) \\ 5\,e^{-t}\sin(t) & e^{-t}(\cos(t)-2\sin(t)) \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} \\
&= \begin{bmatrix} e^{-t}(\cos(t)+2\sin(t))\,c_1 - e^{-t}\sin(t)\,c_2 \\ 5\,e^{-t}\sin(t)\,c_1 + e^{-t}(\cos(t)-2\sin(t))\,c_2 \end{bmatrix} \\
&= \begin{bmatrix} ((2c_1-c_2)\sin(t)+c_1\cos(t))\,e^{-t} \\ ((5c_1-2c_2)\sin(t)+c_2\cos(t))\,e^{-t} \end{bmatrix}
\end{aligned}
$$

Since no forcing function is given, then the final solution is $\vec{x}_h(t)$ above.

16.5.2 Solution using explicit Eigenvalue and Eigenvector method

This is a system of linear ODE's given as

$$\vec{x}'(t) = A\,\vec{x}(t)$$

Or

$$\begin{bmatrix} x_1'(t) \\ x_2'(t) \end{bmatrix} = \begin{bmatrix} 1 & -1 \\ 5 & -3 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}$$

The first step is find the homogeneous solution. We start by finding the eigenvalues of A. This is done by solving the following equation for the eigenvalues λ

$$\det(A-\lambda I)=0$$

Expanding gives

$$\det\left(\begin{bmatrix} 1 & -1 \\ 5 & -3 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right) = 0$$

Therefore

$$\det\left(\begin{bmatrix} 1-\lambda & -1 \\ 5 & -3-\lambda \end{bmatrix}\right) = 0$$

Which gives the characteristic equation

$$\lambda^2 + 2\lambda + 2 = 0$$

The roots of the above are the eigenvalues.

$$\lambda_1 = -1 + i$$
$$\lambda_2 = -1 - i$$

This table summarises the above result

eigenvalue	algebraic multiplicity	type of eigenvalue
$-1 - i$	1	complex eigenvalue
$-1 + i$	1	complex eigenvalue

Now the eigenvector for each eigenvalue are found.

Considering the eigenvalue $\lambda_1 = -1 - i$

We need to solve $A\vec{v} = \lambda\vec{v}$ or $(A - \lambda I)\vec{v} = \vec{0}$ which becomes

$$\left(\begin{bmatrix} 1 & -1 \\ 5 & -3 \end{bmatrix} - (-1 - i)\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right)\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$
$$\begin{bmatrix} 2 + i & -1 \\ 5 & -2 + i \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Now forward elimination is applied to solve for the eigenvector $\vec{v}$. The augmented matrix is

$$\left[\begin{array}{cc|c} 2 + i & -1 & 0 \\ 5 & -2 + i & 0 \end{array}\right]$$

$$R_2 = R_2 + (-2 + i)\, R_1 \Longrightarrow \left[\begin{array}{cc|c} 2 + i & -1 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

Therefore the system in Echelon form is

$$\begin{bmatrix} 2 + i & -1 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

The free variables are $\{v_2\}$ and the leading variables are $\{v_1\}$. Let $v_2 = t$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\left\{v_1 = \left(\frac{2}{5} - \frac{i}{5}\right) t\right\}$

Hence the solution is

$$\begin{bmatrix} \left(\frac{2}{5} - \frac{I}{5}\right) t \\ t \end{bmatrix} = \begin{bmatrix} \left(\frac{2}{5} - \frac{i}{5}\right) t \\ t \end{bmatrix}$$

Since there is one free Variable, we have found one eigenvector associated with this eigenvalue. The above can be written as

$$\begin{bmatrix} \left(\frac{2}{5} - \frac{I}{5}\right) t \\ t \end{bmatrix} = t \begin{bmatrix} \frac{2}{5} - \frac{i}{5} \\ 1 \end{bmatrix}$$

Let $t = 1$ the eigenvector becomes

$$\begin{bmatrix} \left(\frac{2}{5} - \frac{I}{5}\right) t \\ t \end{bmatrix} = \begin{bmatrix} \frac{2}{5} - \frac{i}{5} \\ 1 \end{bmatrix}$$

Which is normalized to

$$\begin{bmatrix} \left(\frac{2}{5} - \frac{I}{5}\right) t \\ t \end{bmatrix} = \begin{bmatrix} 2 - i \\ 5 \end{bmatrix}$$

Considering the eigenvalue $\lambda_2 = -1 + i$

We need to solve $A\vec{v} = \lambda\vec{v}$ or $(A - \lambda I)\vec{v} = \vec{0}$ which becomes

$$\left(\begin{bmatrix} 1 & -1 \\ 5 & -3 \end{bmatrix} - (-1 + i) \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right) \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 2 - i & -1 \\ 5 & -2 - i \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Now forward elimination is applied to solve for the eigenvector $\vec{v}$. The augmented matrix is

$$\left[\begin{array}{cc|c} 2 - i & -1 & 0 \\ 5 & -2 - i & 0 \end{array}\right]$$

$$R_2 = R_2 + (-2 - i) R_1 \Longrightarrow \left[\begin{array}{cc|c} 2 - i & -1 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

Therefore the system in Echelon form is

$$\begin{bmatrix} 2-i & -1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

The free variables are $\{v_2\}$ and the leading variables are $\{v_1\}$. Let $v_2 = t$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\left\{v_1 = \left(\frac{2}{5} + \frac{i}{5}\right) t\right\}$

Hence the solution is

$$\begin{bmatrix} \left(\frac{2}{5} + \frac{I}{5}\right) t \\ t \end{bmatrix} = \begin{bmatrix} \left(\frac{2}{5} + \frac{i}{5}\right) t \\ t \end{bmatrix}$$

Since there is one free Variable, we have found one eigenvector associated with this eigenvalue. The above can be written as

$$\begin{bmatrix} \left(\frac{2}{5} + \frac{I}{5}\right) t \\ t \end{bmatrix} = t \begin{bmatrix} \frac{2}{5} + \frac{i}{5} \\ 1 \end{bmatrix}$$

Let $t = 1$ the eigenvector becomes

$$\begin{bmatrix} \left(\frac{2}{5} + \frac{I}{5}\right) t \\ t \end{bmatrix} = \begin{bmatrix} \frac{2}{5} + \frac{i}{5} \\ 1 \end{bmatrix}$$

Which is normalized to

$$\begin{bmatrix} \left(\frac{2}{5} + \frac{I}{5}\right) t \\ t \end{bmatrix} = \begin{bmatrix} 2+i \\ 5 \end{bmatrix}$$

The following table gives a summary of this result. It shows for each eigenvalue the algebraic multiplicity m, and its geometric multiplicity k and the eigenvectors associated with the eigenvalue. If $m > k$ then the eigenvalue is defective which means the number of normal linearly independent eigenvectors associated with this eigenvalue (called the geometric multiplicity k) does not equal the algebraic multiplicity m, and we need to determine an additional $m - k$ generalized eigenvectors for this eigenvalue.

eigenvalue	multiplicity algebraic m	multiplicity geometric k	defective?	eigenvectors
$-1+i$	1	1	No	$\begin{bmatrix} \frac{2}{5}+\frac{i}{5} \\ 1 \end{bmatrix}$
$-1-i$	1	1	No	$\begin{bmatrix} \frac{2}{5}-\frac{i}{5} \\ 1 \end{bmatrix}$

Now that we found the eigenvalues and associated eigenvectors, we will go over each eigenvalue and generate the solution basis. The only problem we need to take care of is if the eigenvalue is defective. Therefore the final solution is

$$\vec{x}_h(t) = c_1\vec{x}_1(t) + c_2\vec{x}_2(t)$$

Which is written as

$$\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = c_1 \begin{bmatrix} \left(\frac{2}{5}+\frac{i}{5}\right) \mathrm{e}^{(-1+i)t} \\ \mathrm{e}^{(-1+i)t} \end{bmatrix} + c_2 \begin{bmatrix} \left(\frac{2}{5}-\frac{i}{5}\right) \mathrm{e}^{(-1-i)t} \\ \mathrm{e}^{(-1-i)t} \end{bmatrix}$$

Which becomes

$$\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = \begin{bmatrix} \left(\frac{2}{5}+\frac{i}{5}\right) c_1 \mathrm{e}^{(-1+i)t} + \left(\frac{2}{5}-\frac{i}{5}\right) c_2 \mathrm{e}^{(-1-i)t} \\ c_1 \mathrm{e}^{(-1+i)t} + c_2 \mathrm{e}^{(-1-i)t} \end{bmatrix}$$

The following is the phase plot of the system.

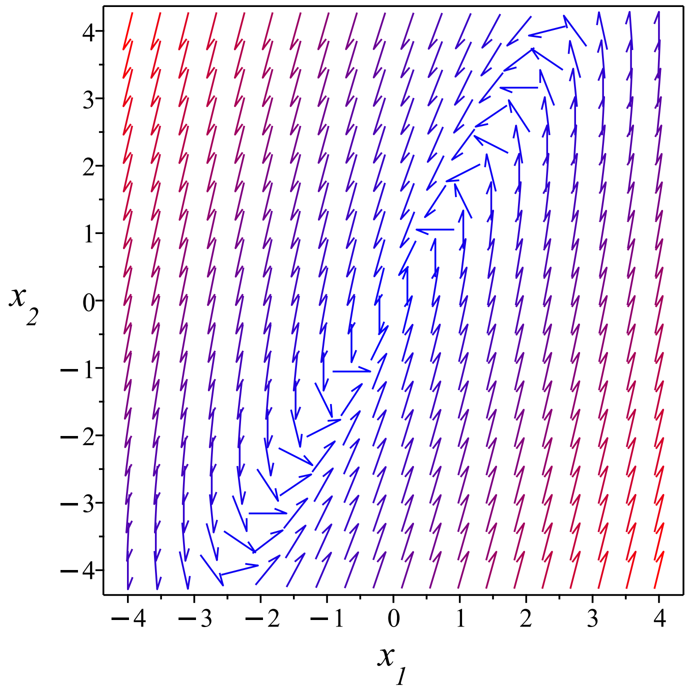


Figure 511: Phase plot

16.5.3 Maple step by step solution

Let's solve

$[x_1'(t) = x_1(t) - x_2(t), x_2'(t) = 5x_1(t) - 3x_2(t)]$

- Define vector

$$\underline{x}^{\rightarrow}(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}$$

- Convert system into a vector equation

$$\underline{x}^{\rightarrow}{}'(t) = \begin{bmatrix} 1 & -1 \\ 5 & -3 \end{bmatrix} \cdot \underline{x}^{\rightarrow}(t) + \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

- System to solve

$$\underline{x}^{\rightarrow}{}'(t) = \begin{bmatrix} 1 & -1 \\ 5 & -3 \end{bmatrix} \cdot \underline{x}^{\rightarrow}(t)$$

- Define the coefficient matrix

$$A = \begin{bmatrix} 1 & -1 \\ 5 & -3 \end{bmatrix}$$

- Rewrite the system as

$$\underline{x}^{\rightarrow}{}'(t) = A \cdot \underline{x}^{\rightarrow}(t)$$

- To solve the system, find the eigenvalues and eigenvectors of A
- Eigenpairs of A

$$\left[\left[-1-\mathrm{I}, \begin{bmatrix} \frac{2}{5}-\frac{\mathrm{I}}{5} \\ 1 \end{bmatrix}\right], \left[-1+\mathrm{I}, \begin{bmatrix} \frac{2}{5}+\frac{\mathrm{I}}{5} \\ 1 \end{bmatrix}\right]\right]$$

- Consider complex eigenpair, complex conjugate eigenvalue can be ignored

$$\left[-1-\mathrm{I}, \begin{bmatrix} \frac{2}{5}-\frac{\mathrm{I}}{5} \\ 1 \end{bmatrix}\right]$$

- Solution from eigenpair

$$\mathrm{e}^{(-1-\mathrm{I})t} \cdot \begin{bmatrix} \frac{2}{5}-\frac{\mathrm{I}}{5} \\ 1 \end{bmatrix}$$

- Use Euler identity to write solution in terms of $\sin$ and $\cos$

$$\mathrm{e}^{-t} \cdot (\cos(t) - \mathrm{I}\sin(t)) \cdot \begin{bmatrix} \frac{2}{5}-\frac{\mathrm{I}}{5} \\ 1 \end{bmatrix}$$

- Simplify expression

$$\mathrm{e}^{-t} \cdot \begin{bmatrix} \left(\frac{2}{5}-\frac{\mathrm{I}}{5}\right)(\cos(t) - \mathrm{I}\sin(t)) \\ \cos(t) - \mathrm{I}\sin(t) \end{bmatrix}$$

- Both real and imaginary parts are solutions to the homogeneous system

$$\left[\underline{x}^{\rightarrow}{}_{1}(t) = \mathrm{e}^{-t} \cdot \begin{bmatrix} \frac{2\cos(t)}{5} - \frac{\sin(t)}{5} \\ \cos(t) \end{bmatrix}, \underline{x}^{\rightarrow}{}_{2}(t) = \mathrm{e}^{-t} \cdot \begin{bmatrix} -\frac{2\sin(t)}{5} - \frac{\cos(t)}{5} \\ -\sin(t) \end{bmatrix}\right]$$

- General solution to the system of ODEs

$$\underline{x}^{\rightarrow} = c_1 \underline{x}^{\rightarrow}{}_{1}(t) + c_2 \underline{x}^{\rightarrow}{}_{2}(t)$$

- Substitute solutions into the general solution

$$\underline{x^{\rightarrow}} = c_1 \mathrm{e}^{-t} \cdot \begin{bmatrix} \frac{2\cos(t)}{5} - \frac{\sin(t)}{5} \\ \cos(t) \end{bmatrix} + c_2 \mathrm{e}^{-t} \cdot \begin{bmatrix} -\frac{2\sin(t)}{5} - \frac{\cos(t)}{5} \\ -\sin(t) \end{bmatrix}$$

- Substitute in vector of dependent variables

$$\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = \begin{bmatrix} \frac{2\left(\left(c_1 - \frac{c_2}{2}\right)\cos(t) - \frac{\sin(t)(c_1+2c_2)}{2}\right)\mathrm{e}^{-t}}{5} \\ \mathrm{e}^{-t}(-c_2 \sin(t) + c_1 \cos(t)) \end{bmatrix}$$

- Solution to the system of ODEs

$$\left\{x_1(t) = \frac{2\left(\left(c_1 - \frac{c_2}{2}\right)\cos(t) - \frac{\sin(t)(c_1+2c_2)}{2}\right)\mathrm{e}^{-t}}{5}, x_2(t) = \mathrm{e}^{-t}(-c_2 \sin(t) + c_1 \cos(t))\right\}$$

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 48

```
dsolve([diff(x__1(t),t)=1*x__1(t)-1*x__2(t),diff(x__2(t),t)=5*x__1(t)-3*x__2(t)],singsol=all
```

$$x_1(t) = \mathrm{e}^{-t}(c_1 \sin(t) + c_2 \cos(t))$$
$$x_2(t) = -\mathrm{e}^{-t}(c_1 \cos(t) - 2c_2 \cos(t) - 2c_1 \sin(t) - c_2 \sin(t))$$

✓ Solution by Mathematica

Time used: 0.005 (sec). Leaf size: 56

```
DSolve[{x1'[t]==1*x1[t]-1*x2[t],x2'[t]==5*x1[t]-3*x2[t]},{x1[t],x2[t]},t,IncludeSingularSolu
```

$$\text{x1}(t) \to e^{-t}(c_1 \cos(t) + (2c_1 - c_2)\sin(t))$$
$$\text{x2}(t) \to e^{-t}(c_2 \cos(t) + (5c_1 - 2c_2)\sin(t))$$

16.6 problem 6

Internal problem ID [756]
Internal file name [`OUTPUT/756_Sunday_June_05_2022_01_48_51_AM_70838050/index.tex`]

Book: Elementary differential equations and boundary value problems, 10th ed., Boyce and DiPrima
Section: Chapter 7.6, Complex Eigenvalues. page 417
Problem number: 6.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"system of linear ODEs"**

Solve

$$x_1'(t) = x_1(t) + 2x_2(t)$$
$$x_2'(t) = -5x_1(t) - x_2(t)$$

16.6.1 Solution using Matrix exponential method

In this method, we will assume we have found the matrix exponential e^{At} allready. There are different methods to determine this but will not be shown here. This is a system of linear ODE's given as

$$\vec{x}'(t) = A\,\vec{x}(t)$$

Or

$$\begin{bmatrix} x_1'(t) \\ x_2'(t) \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ -5 & -1 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}$$

For the above matrix A, the matrix exponential can be found to be

$$e^{At} = \begin{bmatrix} \cos(3t) + \frac{\sin(3t)}{3} & \frac{2\sin(3t)}{3} \\ -\frac{5\sin(3t)}{3} & \cos(3t) - \frac{\sin(3t)}{3} \end{bmatrix}$$

Therefore the homogeneous solution is

$$
\begin{aligned}
\vec{x}_h(t) &= e^{At}\vec{c} \\
&= \begin{bmatrix} \cos(3t) + \frac{\sin(3t)}{3} & \frac{2\sin(3t)}{3} \\ -\frac{5\sin(3t)}{3} & \cos(3t) - \frac{\sin(3t)}{3} \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} \\
&= \begin{bmatrix} \left(\cos(3t) + \frac{\sin(3t)}{3}\right) c_1 + \frac{2\sin(3t)c_2}{3} \\ -\frac{5\sin(3t)c_1}{3} + \left(\cos(3t) - \frac{\sin(3t)}{3}\right) c_2 \end{bmatrix} \\
&= \begin{bmatrix} \frac{\sin(3t)(c_1+2c_2)}{3} + c_1\cos(3t) \\ \frac{(-5c_1-c_2)\sin(3t)}{3} + c_2\cos(3t) \end{bmatrix}
\end{aligned}
$$

Since no forcing function is given, then the final solution is $\vec{x}_h(t)$ above.

16.6.2 Solution using explicit Eigenvalue and Eigenvector method

This is a system of linear ODE's given as

$$\vec{x}'(t) = A\,\vec{x}(t)$$

Or

$$\begin{bmatrix} x_1'(t) \\ x_2'(t) \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ -5 & -1 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}$$

The first step is find the homogeneous solution. We start by finding the eigenvalues of A. This is done by solving the following equation for the eigenvalues λ

$$\det(A - \lambda I) = 0$$

Expanding gives

$$\det\left(\begin{bmatrix} 1 & 2 \\ -5 & -1 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right) = 0$$

Therefore

$$\det\left(\begin{bmatrix} 1-\lambda & 2 \\ -5 & -1-\lambda \end{bmatrix}\right) = 0$$

Which gives the characteristic equation

$$\lambda^2 + 9 = 0$$

The roots of the above are the eigenvalues.

$$\lambda_1 = 3i$$
$$\lambda_2 = -3i$$

This table summarises the above result

eigenvalue	algebraic multiplicity	type of eigenvalue
$-3i$	1	complex eigenvalue
$3i$	1	complex eigenvalue

Now the eigenvector for each eigenvalue are found.

Considering the eigenvalue $\lambda_1 = -3i$

We need to solve $A\vec{v} = \lambda\vec{v}$ or $(A - \lambda I)\vec{v} = \vec{0}$ which becomes

$$\left(\begin{bmatrix} 1 & 2 \\ -5 & -1 \end{bmatrix} - (-3i)\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right)\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$
$$\begin{bmatrix} 1+3i & 2 \\ -5 & -1+3i \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Now forward elimination is applied to solve for the eigenvector $\vec{v}$. The augmented matrix is

$$\left[\begin{array}{cc|c} 1+3i & 2 & 0 \\ -5 & -1+3i & 0 \end{array}\right]$$

$$R_2 = R_2 + \left(\frac{1}{2} - \frac{3i}{2}\right) R_1 \Longrightarrow \left[\begin{array}{cc|c} 1+3i & 2 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

Therefore the system in Echelon form is

$$\begin{bmatrix} 1+3i & 2 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

The free variables are $\{v_2\}$ and the leading variables are $\{v_1\}$. Let $v_2 = t$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\left\{v_1 = \left(-\frac{1}{5} + \frac{3i}{5}\right) t\right\}$

Hence the solution is

$$\begin{bmatrix} \left(-\frac{1}{5} + \frac{3\,\mathrm{I}}{5}\right) t \\ t \end{bmatrix} = \begin{bmatrix} \left(-\frac{1}{5} + \frac{3i}{5}\right) t \\ t \end{bmatrix}$$

Since there is one free Variable, we have found one eigenvector associated with this eigenvalue. The above can be written as

$$\begin{bmatrix} \left(-\frac{1}{5} + \frac{3\,\mathrm{I}}{5}\right) t \\ t \end{bmatrix} = t \begin{bmatrix} -\frac{1}{5} + \frac{3i}{5} \\ 1 \end{bmatrix}$$

Let $t = 1$ the eigenvector becomes

$$\begin{bmatrix} \left(-\frac{1}{5} + \frac{3\,\mathrm{I}}{5}\right) t \\ t \end{bmatrix} = \begin{bmatrix} -\frac{1}{5} + \frac{3i}{5} \\ 1 \end{bmatrix}$$

Which is normalized to

$$\begin{bmatrix} \left(-\frac{1}{5} + \frac{3\,\mathrm{I}}{5}\right) t \\ t \end{bmatrix} = \begin{bmatrix} -1 + 3i \\ 5 \end{bmatrix}$$

Considering the eigenvalue $\lambda_2 = 3i$

We need to solve $A\vec{v} = \lambda\vec{v}$ or $(A - \lambda I)\vec{v} = \vec{0}$ which becomes

$$\left(\begin{bmatrix} 1 & 2 \\ -5 & -1 \end{bmatrix} - (3i) \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right) \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 - 3i & 2 \\ -5 & -1 - 3i \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Now forward elimination is applied to solve for the eigenvector $\vec{v}$. The augmented matrix is

$$\left[\begin{array}{cc|c} 1 - 3i & 2 & 0 \\ -5 & -1 - 3i & 0 \end{array}\right]$$

$$R_2 = R_2 + \left(\frac{1}{2} + \frac{3i}{2}\right) R_1 \Longrightarrow \left[\begin{array}{cc|c} 1-3i & 2 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

Therefore the system in Echelon form is

$$\begin{bmatrix} 1-3i & 2 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

The free variables are $\{v_2\}$ and the leading variables are $\{v_1\}$. Let $v_2 = t$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\left\{v_1 = \left(-\frac{1}{5} - \frac{3i}{5}\right) t\right\}$

Hence the solution is

$$\begin{bmatrix} \left(-\frac{1}{5} - \frac{3\,\mathrm{I}}{5}\right) t \\ t \end{bmatrix} = \begin{bmatrix} \left(-\frac{1}{5} - \frac{3i}{5}\right) t \\ t \end{bmatrix}$$

Since there is one free Variable, we have found one eigenvector associated with this eigenvalue. The above can be written as

$$\begin{bmatrix} \left(-\frac{1}{5} - \frac{3\,\mathrm{I}}{5}\right) t \\ t \end{bmatrix} = t \begin{bmatrix} -\frac{1}{5} - \frac{3i}{5} \\ 1 \end{bmatrix}$$

Let $t = 1$ the eigenvector becomes

$$\begin{bmatrix} \left(-\frac{1}{5} - \frac{3\,\mathrm{I}}{5}\right) t \\ t \end{bmatrix} = \begin{bmatrix} -\frac{1}{5} - \frac{3i}{5} \\ 1 \end{bmatrix}$$

Which is normalized to

$$\begin{bmatrix} \left(-\frac{1}{5} - \frac{3\,\mathrm{I}}{5}\right) t \\ t \end{bmatrix} = \begin{bmatrix} -1-3i \\ 5 \end{bmatrix}$$

The following table gives a summary of this result. It shows for each eigenvalue the algebraic multiplicity m, and its geometric multiplicity k and the eigenvectors associated with the eigenvalue. If $m > k$ then the eigenvalue is defective which means the number of normal linearly independent eigenvectors associated with this eigenvalue (called the geometric multiplicity k) does not equal the algebraic multiplicity m, and we need to determine an additional $m - k$ generalized eigenvectors for this eigenvalue.

	multiplicity			
eigenvalue	algebraic m	geometric k	defective?	eigenvectors
$3i$	1	1	No	$\begin{bmatrix} -\frac{1}{5}-\frac{3i}{5} \\ 1 \end{bmatrix}$
$-3i$	1	1	No	$\begin{bmatrix} -\frac{1}{5}+\frac{3i}{5} \\ 1 \end{bmatrix}$

Now that we found the eigenvalues and associated eigenvectors, we will go over each eigenvalue and generate the solution basis. The only problem we need to take care of is if the eigenvalue is defective. Therefore the final solution is

$$\vec{x}_h(t) = c_1\vec{x}_1(t) + c_2\vec{x}_2(t)$$

Which is written as

$$\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = c_1 \begin{bmatrix} \left(-\frac{1}{5}-\frac{3i}{5}\right) e^{3it} \\ e^{3it} \end{bmatrix} + c_2 \begin{bmatrix} \left(-\frac{1}{5}+\frac{3i}{5}\right) e^{-3it} \\ e^{-3it} \end{bmatrix}$$

Which becomes

$$\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = \begin{bmatrix} \left(-\frac{1}{5}-\frac{3i}{5}\right) c_1 e^{3it} + \left(-\frac{1}{5}+\frac{3i}{5}\right) c_2 e^{-3it} \\ c_1 e^{3it} + c_2 e^{-3it} \end{bmatrix}$$

The following is the phase plot of the system.

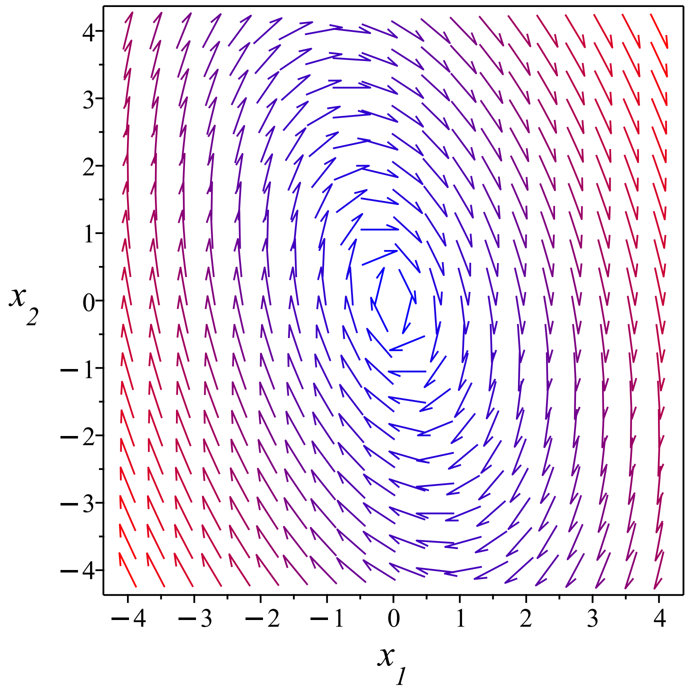


Figure 512: Phase plot

16.6.3 Maple step by step solution

Let's solve

$[x_1'(t) = x_1(t) + 2x_2(t), x_2'(t) = -5x_1(t) - x_2(t)]$

- Define vector

$$\underline{x}^{\rightarrow}(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}$$

- Convert system into a vector equation

$$\underline{x}^{\rightarrow}{}'(t) = \begin{bmatrix} 1 & 2 \\ -5 & -1 \end{bmatrix} \cdot \underline{x}^{\rightarrow}(t) + \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

- System to solve

$$\underline{x}^{\rightarrow}{}'(t) = \begin{bmatrix} 1 & 2 \\ -5 & -1 \end{bmatrix} \cdot \underline{x}^{\rightarrow}(t)$$

- Define the coefficient matrix

$$A = \begin{bmatrix} 1 & 2 \\ -5 & -1 \end{bmatrix}$$

- Rewrite the system as

$$\underline{x^{\rightarrow}}'(t) = A \cdot \underline{x^{\rightarrow}}(t)$$

- To solve the system, find the eigenvalues and eigenvectors of A
- Eigenpairs of A

$$\left[\left[-3\,\mathrm{I}, \begin{bmatrix} -\frac{1}{5}+\frac{3\,\mathrm{I}}{5} \\ 1 \end{bmatrix}\right], \left[3\,\mathrm{I}, \begin{bmatrix} -\frac{1}{5}-\frac{3\,\mathrm{I}}{5} \\ 1 \end{bmatrix}\right]\right]$$

- Consider complex eigenpair, complex conjugate eigenvalue can be ignored

$$\left[-3\,\mathrm{I}, \begin{bmatrix} -\frac{1}{5}+\frac{3\,\mathrm{I}}{5} \\ 1 \end{bmatrix}\right]$$

- Solution from eigenpair

$$\mathrm{e}^{-3\,\mathrm{I}t} \cdot \begin{bmatrix} -\frac{1}{5}+\frac{3\,\mathrm{I}}{5} \\ 1 \end{bmatrix}$$

- Use Euler identity to write solution in terms of sin and cos

$$(\cos(3t) - \mathrm{I}\sin(3t)) \cdot \begin{bmatrix} -\frac{1}{5}+\frac{3\,\mathrm{I}}{5} \\ 1 \end{bmatrix}$$

- Simplify expression

$$\begin{bmatrix} \left(-\frac{1}{5}+\frac{3\,\mathrm{I}}{5}\right)(\cos(3t) - \mathrm{I}\sin(3t)) \\ \cos(3t) - \mathrm{I}\sin(3t) \end{bmatrix}$$

- Both real and imaginary parts are solutions to the homogeneous system

$$\left[\underline{x^{\rightarrow}}_1(t) = \begin{bmatrix} -\frac{\cos(3t)}{5}+\frac{3\sin(3t)}{5} \\ \cos(3t) \end{bmatrix}, \underline{x^{\rightarrow}}_2(t) = \begin{bmatrix} \frac{\sin(3t)}{5}+\frac{3\cos(3t)}{5} \\ -\sin(3t) \end{bmatrix}\right]$$

- General solution to the system of ODEs

$$\underline{x^{\rightarrow}} = c_1 \underline{x^{\rightarrow}}_1(t) + c_2 \underline{x^{\rightarrow}}_2(t)$$

- Substitute solutions into the general solution

$$\vec{x} = \begin{bmatrix} c_2\left(\frac{\sin(3t)}{5}+\frac{3\cos(3t)}{5}\right)+c_1\left(-\frac{\cos(3t)}{5}+\frac{3\sin(3t)}{5}\right) \\ -c_2\sin(3t)+c_1\cos(3t) \end{bmatrix}$$

- Substitute in vector of dependent variables

$$\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = \begin{bmatrix} \frac{(-c_1+3c_2)\cos(3t)}{5}+\frac{3\left(c_1+\frac{c_2}{3}\right)\sin(3t)}{5} \\ -c_2\sin(3t)+c_1\cos(3t) \end{bmatrix}$$

- Solution to the system of ODEs

$$\left\{x_1(t)=\frac{(-c_1+3c_2)\cos(3t)}{5}+\frac{3\left(c_1+\frac{c_2}{3}\right)\sin(3t)}{5}, x_2(t)=-c_2\sin(3t)+c_1\cos(3t)\right\}$$

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 50

```
dsolve([diff(x__1(t),t)=1*x__1(t)+2*x__2(t),diff(x__2(t),t)=-5*x__1(t)-1*x__2(t)],singsol=al
```

$$x_1(t) = c_1\sin(3t)+c_2\cos(3t)$$
$$x_2(t) = \frac{3c_1\cos(3t)}{2}-\frac{3c_2\sin(3t)}{2}-\frac{c_1\sin(3t)}{2}-\frac{c_2\cos(3t)}{2}$$

✓ Solution by Mathematica

Time used: 0.004 (sec). Leaf size: 54

```
DSolve[{x1'[t]==1*x1[t]+2*x2[t],x2'[t]==-5*x1[t]-1*x2[t]},{x1[t],x2[t]},t,IncludeSingularSol
```

$$\text{x1}(t) \to c_1\cos(3t)+\frac{1}{3}(c_1+2c_2)\sin(3t)$$
$$\text{x2}(t) \to c_2\cos(3t)-\frac{1}{3}(5c_1+c_2)\sin(3t)$$

16.7 problem 7

Internal problem ID [757]
Internal file name [`OUTPUT/757_Sunday_June_05_2022_01_48_52_AM_27514033/index.tex`]

Book: Elementary differential equations and boundary value problems, 10th ed., Boyce and DiPrima
Section: Chapter 7.6, Complex Eigenvalues. page 417
Problem number: 7.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"system of linear ODEs"**

Solve

$$
\begin{aligned}
x_1'(t) &= x_1(t) \\
x_2'(t) &= 2x_1(t) + x_2(t) - 2x_3(t) \\
x_3'(t) &= 3x_1(t) + 2x_2(t) + x_3(t)
\end{aligned}
$$

16.7.1 Solution using Matrix exponential method

In this method, we will assume we have found the matrix exponential e^{At} allready. There are different methods to determine this but will not be shown here. This is a system of linear ODE's given as

$$
\vec{x}'(t) = A\,\vec{x}(t)
$$

Or

$$
\begin{bmatrix} x_1'(t) \\ x_2'(t) \\ x_3'(t) \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & -2 \\ 3 & 2 & 1 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{bmatrix}
$$

For the above matrix A, the matrix exponential can be found to be

$$e^{At} = \begin{bmatrix} e^{t} & 0 & 0 \\ \frac{3\,e^{t}\cos(2t)}{2} + e^{t}\sin(2t) - \frac{3\,e^{t}}{2} & e^{t}\cos(2t) & -e^{t}\sin(2t) \\ -e^{t}\cos(2t) + \frac{3\,e^{t}\sin(2t)}{2} + e^{t} & e^{t}\sin(2t) & e^{t}\cos(2t) \end{bmatrix}$$

$$= \begin{bmatrix} e^{t} & 0 & 0 \\ \frac{e^{t}(-3+3\cos(2t)+2\sin(2t))}{2} & e^{t}\cos(2t) & -e^{t}\sin(2t) \\ -\frac{e^{t}(-2+2\cos(2t)-3\sin(2t))}{2} & e^{t}\sin(2t) & e^{t}\cos(2t) \end{bmatrix}$$

Therefore the homogeneous solution is

$$\begin{aligned}
\vec{x}_h(t) &= e^{At}\vec{c} \\
&= \begin{bmatrix} e^{t} & 0 & 0 \\ \frac{e^{t}(-3+3\cos(2t)+2\sin(2t))}{2} & e^{t}\cos(2t) & -e^{t}\sin(2t) \\ -\frac{e^{t}(-2+2\cos(2t)-3\sin(2t))}{2} & e^{t}\sin(2t) & e^{t}\cos(2t) \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} \\
&= \begin{bmatrix} e^{t}c_1 \\ \frac{e^{t}(-3+3\cos(2t)+2\sin(2t))c_1}{2} + e^{t}\cos(2t)\,c_2 - e^{t}\sin(2t)\,c_3 \\ -\frac{e^{t}(-2+2\cos(2t)-3\sin(2t))c_1}{2} + e^{t}\sin(2t)\,c_2 + e^{t}\cos(2t)\,c_3 \end{bmatrix} \\
&= \begin{bmatrix} e^{t}c_1 \\ \frac{3\,e^{t}\left(\left(c_1+\frac{2c_2}{3}\right)\cos(2t)+\frac{2(c_1-c_3)\sin(2t)}{3}-c_1\right)}{2} \\ -\left((c_1-c_3)\cos(2t) + \left(-\frac{3c_1}{2}-c_2\right)\sin(2t) - c_1\right)e^{t} \end{bmatrix}
\end{aligned}$$

Since no forcing function is given, then the final solution is $\vec{x}_h(t)$ above.

16.7.2 Solution using explicit Eigenvalue and Eigenvector method

This is a system of linear ODE's given as

$$\vec{x}'(t) = A\,\vec{x}(t)$$

Or

$$\begin{bmatrix} x_1'(t) \\ x_2'(t) \\ x_3'(t) \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & -2 \\ 3 & 2 & 1 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{bmatrix}$$

The first step is find the homogeneous solution. We start by finding the eigenvalues of A. This is done by solving the following equation for the eigenvalues λ

$$\det(A - \lambda I) = 0$$

Expanding gives

$$\det\left(\begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & -2 \\ 3 & 2 & 1 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\right) = 0$$

Therefore

$$\det\left(\begin{bmatrix} 1-\lambda & 0 & 0 \\ 2 & 1-\lambda & -2 \\ 3 & 2 & 1-\lambda \end{bmatrix}\right) = 0$$

Which gives the characteristic equation

$$\lambda^3 - 3\lambda^2 + 7\lambda - 5 = 0$$

The roots of the above are the eigenvalues.

$$\begin{aligned} \lambda_1 &= 1 \\ \lambda_2 &= 1 + 2i \\ \lambda_3 &= 1 - 2i \end{aligned}$$

This table summarises the above result

eigenvalue	algebraic multiplicity	type of eigenvalue
1	1	real eigenvalue
$1 + 2i$	1	complex eigenvalue
$1 - 2i$	1	complex eigenvalue

Now the eigenvector for each eigenvalue are found.

Considering the eigenvalue $\lambda_1 = 1$

We need to solve $A\vec{v} = \lambda\vec{v}$ or $(A - \lambda I)\vec{v} = \vec{0}$ which becomes

$$\left(\begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & -2 \\ 3 & 2 & 1 \end{bmatrix} - (1) \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \right) \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 0 & 0 & 0 \\ 2 & 0 & -2 \\ 3 & 2 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Now forward elimination is applied to solve for the eigenvector $\vec{v}$. The augmented matrix is

$$\left[\begin{array}{ccc|c} 0 & 0 & 0 & 0 \\ 2 & 0 & -2 & 0 \\ 3 & 2 & 0 & 0 \end{array}\right]$$

Since the current pivot $A(1,1)$ is zero, then the current pivot row is replaced with a row with a non-zero pivot. Swapping row 1 and row 2 gives

$$\left[\begin{array}{ccc|c} 2 & 0 & -2 & 0 \\ 0 & 0 & 0 & 0 \\ 3 & 2 & 0 & 0 \end{array}\right]$$

$$R_3 = R_3 - \frac{3R_1}{2} \Longrightarrow \left[\begin{array}{ccc|c} 2 & 0 & -2 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 2 & 3 & 0 \end{array}\right]$$

Since the current pivot $A(2,2)$ is zero, then the current pivot row is replaced with a row with a non-zero pivot. Swapping row 2 and row 3 gives

$$\left[\begin{array}{ccc|c} 2 & 0 & -2 & 0 \\ 0 & 2 & 3 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

Therefore the system in Echelon form is

$$\begin{bmatrix} 2 & 0 & -2 \\ 0 & 2 & 3 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

The free variables are $\{v_3\}$ and the leading variables are $\{v_1, v_2\}$. Let $v_3 = t$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\left\{v_1 = t, v_2 = -\frac{3t}{2}\right\}$

Hence the solution is

$$\begin{bmatrix} t \\ -\frac{3t}{2} \\ t \end{bmatrix} = \begin{bmatrix} t \\ -\frac{3t}{2} \\ t \end{bmatrix}$$

Since there is one free Variable, we have found one eigenvector associated with this eigenvalue. The above can be written as

$$\begin{bmatrix} t \\ -\frac{3t}{2} \\ t \end{bmatrix} = t \begin{bmatrix} 1 \\ -\frac{3}{2} \\ 1 \end{bmatrix}$$

Let $t = 1$ the eigenvector becomes

$$\begin{bmatrix} t \\ -\frac{3t}{2} \\ t \end{bmatrix} = \begin{bmatrix} 1 \\ -\frac{3}{2} \\ 1 \end{bmatrix}$$

Which is normalized to

$$\begin{bmatrix} t \\ -\frac{3t}{2} \\ t \end{bmatrix} = \begin{bmatrix} 2 \\ -3 \\ 2 \end{bmatrix}$$

Considering the eigenvalue $\lambda_2 = 1 - 2i$

We need to solve $A\vec{v} = \lambda\vec{v}$ or $(A - \lambda I)\vec{v} = \vec{0}$ which becomes

$$\left(\begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & -2 \\ 3 & 2 & 1 \end{bmatrix} - (1 - 2i) \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \right) \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 2i & 0 & 0 \\ 2 & 2i & -2 \\ 3 & 2 & 2i \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Now forward elimination is applied to solve for the eigenvector $\vec{v}$. The augmented matrix is

$$\left[\begin{array}{ccc|c} 2i & 0 & 0 & 0 \\ 2 & 2i & -2 & 0 \\ 3 & 2 & 2i & 0 \end{array}\right]$$

$$R_2 = iR_1 + R_2 \Longrightarrow \left[\begin{array}{ccc|c} 2i & 0 & 0 & 0 \\ 0 & 2i & -2 & 0 \\ 3 & 2 & 2i & 0 \end{array}\right]$$

$$R_3 = R_3 + \frac{3iR_1}{2} \Longrightarrow \left[\begin{array}{ccc|c} 2i & 0 & 0 & 0 \\ 0 & 2i & -2 & 0 \\ 0 & 2 & 2i & 0 \end{array}\right]$$

$$R_3 = iR_2 + R_3 \Longrightarrow \left[\begin{array}{ccc|c} 2i & 0 & 0 & 0 \\ 0 & 2i & -2 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

Therefore the system in Echelon form is

$$\begin{bmatrix} 2i & 0 & 0 \\ 0 & 2i & -2 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

The free variables are $\{v_3\}$ and the leading variables are $\{v_1, v_2\}$. Let $v_3 = t$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\{v_1 = 0, v_2 = -it\}$

Hence the solution is

$$\begin{bmatrix} 0 \\ \text{-I}\,t \\ t \end{bmatrix} = \begin{bmatrix} 0 \\ -it \\ t \end{bmatrix}$$

Since there is one free Variable, we have found one eigenvector associated with this eigenvalue. The above can be written as

$$\begin{bmatrix} 0 \\ \text{-I}\,t \\ t \end{bmatrix} = t \begin{bmatrix} 0 \\ -i \\ 1 \end{bmatrix}$$

Let $t = 1$ the eigenvector becomes

$$\begin{bmatrix} 0 \\ \text{-I}t \\ t \end{bmatrix} = \begin{bmatrix} 0 \\ -i \\ 1 \end{bmatrix}$$

Considering the eigenvalue $\lambda_3 = 1 + 2i$

We need to solve $A\vec{v} = \lambda\vec{v}$ or $(A - \lambda I)\vec{v} = \vec{0}$ which becomes

$$\left(\begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & -2 \\ 3 & 2 & 1 \end{bmatrix} - (1+2i) \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \right) \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} -2i & 0 & 0 \\ 2 & -2i & -2 \\ 3 & 2 & -2i \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Now forward elimination is applied to solve for the eigenvector $\vec{v}$. The augmented matrix is

$$\left[\begin{array}{ccc|c} -2i & 0 & 0 & 0 \\ 2 & -2i & -2 & 0 \\ 3 & 2 & -2i & 0 \end{array}\right]$$

$$R_2 = -iR_1 + R_2 \Longrightarrow \left[\begin{array}{ccc|c} -2i & 0 & 0 & 0 \\ 0 & -2i & -2 & 0 \\ 3 & 2 & -2i & 0 \end{array}\right]$$

$$R_3 = R_3 - \frac{3iR_1}{2} \Longrightarrow \left[\begin{array}{ccc|c} -2i & 0 & 0 & 0 \\ 0 & -2i & -2 & 0 \\ 0 & 2 & -2i & 0 \end{array}\right]$$

$$R_3 = -iR_2 + R_3 \Longrightarrow \left[\begin{array}{ccc|c} -2i & 0 & 0 & 0 \\ 0 & -2i & -2 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

Therefore the system in Echelon form is

$$\begin{bmatrix} -2i & 0 & 0 \\ 0 & -2i & -2 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

The free variables are $\{v_3\}$ and the leading variables are $\{v_1, v_2\}$. Let $v_3 = t$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\{v_1 = 0, v_2 = it\}$

Hence the solution is

$$\begin{bmatrix} 0 \\ \mathrm{I}t \\ t \end{bmatrix} = \begin{bmatrix} 0 \\ it \\ t \end{bmatrix}$$

Since there is one free Variable, we have found one eigenvector associated with this eigenvalue. The above can be written as

$$\begin{bmatrix} 0 \\ \mathrm{I}t \\ t \end{bmatrix} = t \begin{bmatrix} 0 \\ i \\ 1 \end{bmatrix}$$

Let $t = 1$ the eigenvector becomes

$$\begin{bmatrix} 0 \\ \mathrm{I}t \\ t \end{bmatrix} = \begin{bmatrix} 0 \\ i \\ 1 \end{bmatrix}$$

The following table gives a summary of this result. It shows for each eigenvalue the algebraic multiplicity m, and its geometric multiplicity k and the eigenvectors associated with the eigenvalue. If $m > k$ then the eigenvalue is defective which means the number of normal linearly independent eigenvectors associated with this eigenvalue (called the geometric multiplicity k) does not equal the algebraic multiplicity m, and we need to determine an additional $m - k$ generalized eigenvectors for this eigenvalue.

eigenvalue	multiplicity		defective?	eigenvectors
	algebraic m	geometric k		
1	1	1	No	$\begin{bmatrix} 1 \\ -\frac{3}{2} \\ 1 \end{bmatrix}$
$1+2i$	1	1	No	$\begin{bmatrix} 0 \\ i \\ 1 \end{bmatrix}$
$1-2i$	1	1	No	$\begin{bmatrix} 0 \\ -i \\ 1 \end{bmatrix}$

Now that we found the eigenvalues and associated eigenvectors, we will go over each eigenvalue and generate the solution basis. The only problem we need to take care of is if the eigenvalue is defective. Since eigenvalue 1 is real and distinct then the corresponding eigenvector solution is

$$\begin{aligned} \vec{x}_1(t) &= \vec{v}_1 e^t \\ &= \begin{bmatrix} 1 \\ -\frac{3}{2} \\ 1 \end{bmatrix} e^t \end{aligned}$$

Therefore the final solution is

$$\vec{x}_h(t) = c_1\vec{x}_1(t) + c_2\vec{x}_2(t) + c_3\vec{x}_3(t)$$

Which is written as

$$\begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{bmatrix} = c_1 \begin{bmatrix} \mathrm{e}^t \\ -\frac{3\,\mathrm{e}^t}{2} \\ \mathrm{e}^t \end{bmatrix} + c_2 \begin{bmatrix} 0 \\ i\mathrm{e}^{(1+2i)t} \\ \mathrm{e}^{(1+2i)t} \end{bmatrix} + c_3 \begin{bmatrix} 0 \\ -i\mathrm{e}^{(1-2i)t} \\ \mathrm{e}^{(1-2i)t} \end{bmatrix}$$

Which becomes

$$\begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{bmatrix} = \begin{bmatrix} c_1 e^t \\ -\frac{3c_1 e^t}{2} + i c_2 e^{(1+2i)t} - i c_3 e^{(1-2i)t} \\ c_1 e^t + c_2 e^{(1+2i)t} + c_3 e^{(1-2i)t} \end{bmatrix}$$

16.7.3 Maple step by step solution

Let's solve

$[x_1'(t) = x_1(t), x_2'(t) = 2x_1(t) + x_2(t) - 2x_3(t), x_3'(t) = 3x_1(t) + 2x_2(t) + x_3(t)]$

- Define vector

$$\underline{\vec{x}}(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{bmatrix}$$

- Convert system into a vector equation

$$\underline{\vec{x}}'(t) = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & -2 \\ 3 & 2 & 1 \end{bmatrix} \cdot \underline{\vec{x}}(t) + \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

- System to solve

$$\underline{\vec{x}}'(t) = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & -2 \\ 3 & 2 & 1 \end{bmatrix} \cdot \underline{\vec{x}}(t)$$

- Define the coefficient matrix

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & -2 \\ 3 & 2 & 1 \end{bmatrix}$$

- Rewrite the system as

$\underline{\vec{x}}'(t) = A \cdot \underline{\vec{x}}(t)$

- To solve the system, find the eigenvalues and eigenvectors of A
- Eigenpairs of A

$$\left[\left[1, \begin{bmatrix} 1 \\ -\frac{3}{2} \\ 1 \end{bmatrix}\right], \left[1-2\,\mathrm{I}, \begin{bmatrix} 0 \\ -\mathrm{I} \\ 1 \end{bmatrix}\right], \left[1+2\,\mathrm{I}, \begin{bmatrix} 0 \\ \mathrm{I} \\ 1 \end{bmatrix}\right]\right]$$

- Consider eigenpair

$$\left[1, \begin{bmatrix} 1 \\ -\frac{3}{2} \\ 1 \end{bmatrix}\right]$$

- Solution to homogeneous system from eigenpair

$$\underline{\vec{x}}_{1} = \mathrm{e}^{t} \cdot \begin{bmatrix} 1 \\ -\frac{3}{2} \\ 1 \end{bmatrix}$$

- Consider complex eigenpair, complex conjugate eigenvalue can be ignored

$$\left[1-2\,\mathrm{I}, \begin{bmatrix} 0 \\ -\mathrm{I} \\ 1 \end{bmatrix}\right]$$

- Solution from eigenpair

$$\mathrm{e}^{(1-2\,\mathrm{I})t} \cdot \begin{bmatrix} 0 \\ -\mathrm{I} \\ 1 \end{bmatrix}$$

- Use Euler identity to write solution in terms of $\sin$ and $\cos$

$$\mathrm{e}^{t} \cdot (\cos(2t) - \mathrm{I}\sin(2t)) \cdot \begin{bmatrix} 0 \\ -\mathrm{I} \\ 1 \end{bmatrix}$$

- Simplify expression

$$\mathrm{e}^{t} \cdot \begin{bmatrix} 0 \\ -\mathrm{I}(\cos(2t) - \mathrm{I}\sin(2t)) \\ \cos(2t) - \mathrm{I}\sin(2t) \end{bmatrix}$$

- Both real and imaginary parts are solutions to the homogeneous system

$$\left[\vec{x}_{2}(t) = e^{t} \cdot \begin{bmatrix} 0 \\ -\sin(2t) \\ \cos(2t) \end{bmatrix}, \vec{x}_{3}(t) = e^{t} \cdot \begin{bmatrix} 0 \\ -\cos(2t) \\ -\sin(2t) \end{bmatrix}\right]$$

- General solution to the system of ODEs

$$\vec{x} = c_1 \vec{x}_1 + c_2 \vec{x}_2(t) + c_3 \vec{x}_3(t)$$

- Substitute solutions into the general solution

$$\vec{x} = c_1 e^{t} \cdot \begin{bmatrix} 1 \\ -\frac{3}{2} \\ 1 \end{bmatrix} + c_2 e^{t} \cdot \begin{bmatrix} 0 \\ -\sin(2t) \\ \cos(2t) \end{bmatrix} + c_3 e^{t} \cdot \begin{bmatrix} 0 \\ -\cos(2t) \\ -\sin(2t) \end{bmatrix}$$

- Substitute in vector of dependent variables

$$\begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{bmatrix} = \begin{bmatrix} c_1 e^{t} \\ -\frac{e^{t}(2c_3\cos(2t)+2c_2\sin(2t)+3c_1)}{2} \\ e^{t}(c_1 + c_2\cos(2t) - c_3\sin(2t)) \end{bmatrix}$$

- Solution to the system of ODEs

$$\left\{x_1(t) = c_1 e^{t}, x_2(t) = -\frac{e^{t}(2c_3\cos(2t)+2c_2\sin(2t)+3c_1)}{2}, x_3(t) = e^{t}(c_1 + c_2\cos(2t) - c_3\sin(2t))\right\}$$

✓ Solution by Maple

Time used: 0.109 (sec). Leaf size: 73

```
dsolve([diff(x__1(t),t)=1*x__1(t)+0*x__2(t)+0*x__3(t),diff(x__2(t),t)=2*x__1(t)+1*x__2(t)-2*
```

$$x_1(t) = c_3 e^{t}$$

$$x_2(t) = \frac{e^{t}(2c_1\cos(2t) - 3c_3\cos(2t) + 2c_2\sin(2t) - 3c_3)}{2}$$

$$x_3(t) = -\frac{e^{t}(2c_2\cos(2t) - 2c_1\sin(2t) + 3c_3\sin(2t) - 2c_3)}{2}$$

✓ Solution by Mathematica

Time used: 0.007 (sec). Leaf size: 95

```
DSolve[{x1'[t]==1*x1[t]+0*x2[t]+0*x3[t],x2'[t]==2*x1[t]+1*x2[t]-2*x3[t],x3'[t]==3*x1[t]+2*x2
```

$$\begin{aligned}
\text{x1}(t) &\to c_1 e^t \\
\text{x2}(t) &\to \frac{1}{2} e^t ((3c_1 + 2c_2)\cos(2t) + 2(c_1 - c_3)\sin(2t) - 3c_1) \\
\text{x3}(t) &\to \frac{1}{2} e^t (-2(c_1 - c_3)\cos(2t) + (3c_1 + 2c_2)\sin(2t) + 2c_1)
\end{aligned}$$

16.8 problem 8

Internal problem ID [758]
Internal file name [`OUTPUT/758_Sunday_June_05_2022_01_48_54_AM_69288655/index.tex`]

Book: Elementary differential equations and boundary value problems, 10th ed., Boyce and DiPrima
Section: Chapter 7.6, Complex Eigenvalues. page 417
Problem number: 8.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"system of linear ODEs"**

Solve

$$
\begin{aligned}
x_1'(t) &= -3x_1(t) + 2x_3(t)\\
x_2'(t) &= x_1(t) - x_2(t)\\
x_3'(t) &= -2x_1(t) - x_2(t)
\end{aligned}
$$

16.8.1 Solution using Matrix exponential method

In this method, we will assume we have found the matrix exponential e^{At} allready. There are different methods to determine this but will not be shown here. This is a system of linear ODE's given as

$$\vec{x}'(t) = A\,\vec{x}(t)$$

Or

$$\begin{bmatrix} x_1'(t) \\ x_2'(t) \\ x_3'(t) \end{bmatrix} = \begin{bmatrix} -3 & 0 & 2 \\ 1 & -1 & 0 \\ -2 & -1 & 0 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{bmatrix}$$

For the above matrix A, the matrix exponential can be found to be

$$e^{At} = \begin{bmatrix} \frac{2\,\mathrm{e}^{-2t}}{3} + \frac{\mathrm{e}^{-t}\cos\left(\sqrt{2}\,t\right)}{3} - \frac{2\sqrt{2}\,\mathrm{e}^{-t}\sin\left(\sqrt{2}\,t\right)}{3} & -\frac{\sqrt{2}\,\mathrm{e}^{-t}\sin\left(\sqrt{2}\,t\right)}{3} + \frac{2\,\mathrm{e}^{-t}\cos\left(\sqrt{2}\,t\right)}{3} - \frac{2\,\mathrm{e}^{-2t}}{3} & \frac{2\,\mathrm{e}^{-t}\cos\left(\sqrt{2}\,t\right)}{3} + \cdots \\ \frac{2\,\mathrm{e}^{-t}\cos\left(\sqrt{2}\,t\right)}{3} + \frac{\sqrt{2}\,\mathrm{e}^{-t}\sin\left(\sqrt{2}\,t\right)}{6} - \frac{2\,\mathrm{e}^{-2t}}{3} & \frac{2\,\mathrm{e}^{-2t}}{3} + \frac{\mathrm{e}^{-t}\cos\left(\sqrt{2}\,t\right)}{3} + \frac{\sqrt{2}\,\mathrm{e}^{-t}\sin\left(\sqrt{2}\,t\right)}{3} & -\frac{2\,\mathrm{e}^{-t}\cos\left(\sqrt{2}\,t\right)}{3} + \cdots \\ -\frac{\mathrm{e}^{-t}\cos\left(\sqrt{2}\,t\right)}{3} + \frac{\mathrm{e}^{-2t}}{3} - \frac{5\sqrt{2}\,\mathrm{e}^{-t}\sin\left(\sqrt{2}\,t\right)}{6} & -\frac{2\sqrt{2}\,\mathrm{e}^{-t}\sin\left(\sqrt{2}\,t\right)}{3} + \frac{\mathrm{e}^{-t}\cos\left(\sqrt{2}\,t\right)}{3} - \frac{\mathrm{e}^{-2t}}{3} & \frac{4\,\mathrm{e}^{-t}\cos\left(\sqrt{2}\,t\right)}{3} + \cdots \end{bmatrix}$$

Therefore the homogeneous solution is

$$\vec{x}_h(t) = e^{At}\vec{c}$$

$$= \begin{bmatrix} \frac{2\,\mathrm{e}^{-2t}}{3} + \frac{\mathrm{e}^{-t}\cos\left(\sqrt{2}\,t\right)}{3} - \frac{2\sqrt{2}\,\mathrm{e}^{-t}\sin\left(\sqrt{2}\,t\right)}{3} & -\frac{\sqrt{2}\,\mathrm{e}^{-t}\sin\left(\sqrt{2}\,t\right)}{3} + \frac{2\,\mathrm{e}^{-t}\cos\left(\sqrt{2}\,t\right)}{3} - \frac{2\,\mathrm{e}^{-2t}}{3} & \frac{2\,\mathrm{e}^{-t}\cos\left(\sqrt{2}\,t\right)}{3} + \cdots \\ \frac{2\,\mathrm{e}^{-t}\cos\left(\sqrt{2}\,t\right)}{3} + \frac{\sqrt{2}\,\mathrm{e}^{-t}\sin\left(\sqrt{2}\,t\right)}{6} - \frac{2\,\mathrm{e}^{-2t}}{3} & \frac{2\,\mathrm{e}^{-2t}}{3} + \frac{\mathrm{e}^{-t}\cos\left(\sqrt{2}\,t\right)}{3} + \frac{\sqrt{2}\,\mathrm{e}^{-t}\sin\left(\sqrt{2}\,t\right)}{3} & -\frac{2\,\mathrm{e}^{-t}\cos\left(\sqrt{2}\,t\right)}{3} + \cdots \\ -\frac{\mathrm{e}^{-t}\cos\left(\sqrt{2}\,t\right)}{3} + \frac{\mathrm{e}^{-2t}}{3} - \frac{5\sqrt{2}\,\mathrm{e}^{-t}\sin\left(\sqrt{2}\,t\right)}{6} & -\frac{2\sqrt{2}\,\mathrm{e}^{-t}\sin\left(\sqrt{2}\,t\right)}{3} + \frac{\mathrm{e}^{-t}\cos\left(\sqrt{2}\,t\right)}{3} - \frac{\mathrm{e}^{-2t}}{3} & \frac{4\,\mathrm{e}^{-t}\cos\left(\sqrt{2}\,t\right)}{3} + \cdots \end{bmatrix}$$

$$= \begin{bmatrix} \left(\frac{2\,\mathrm{e}^{-2t}}{3} + \frac{\mathrm{e}^{-t}\cos\left(\sqrt{2}\,t\right)}{3} - \frac{2\sqrt{2}\,\mathrm{e}^{-t}\sin\left(\sqrt{2}\,t\right)}{3}\right)c_1 + \left(-\frac{\sqrt{2}\,\mathrm{e}^{-t}\sin\left(\sqrt{2}\,t\right)}{3} + \frac{2\,\mathrm{e}^{-t}\cos\left(\sqrt{2}\,t\right)}{3} - \frac{2\,\mathrm{e}^{-2t}}{3}\right)c_2 + \left(\cdots\right. \\ \left(\frac{2\,\mathrm{e}^{-t}\cos\left(\sqrt{2}\,t\right)}{3} + \frac{\sqrt{2}\,\mathrm{e}^{-t}\sin\left(\sqrt{2}\,t\right)}{6} - \frac{2\,\mathrm{e}^{-2t}}{3}\right)c_1 + \left(\frac{2\,\mathrm{e}^{-2t}}{3} + \frac{\mathrm{e}^{-t}\cos\left(\sqrt{2}\,t\right)}{3} + \frac{\sqrt{2}\,\mathrm{e}^{-t}\sin\left(\sqrt{2}\,t\right)}{3}\right)c_2 + \left(-\cdots\right. \\ \left(-\frac{\mathrm{e}^{-t}\cos\left(\sqrt{2}\,t\right)}{3} + \frac{\mathrm{e}^{-2t}}{3} - \frac{5\sqrt{2}\,\mathrm{e}^{-t}\sin\left(\sqrt{2}\,t\right)}{6}\right)c_1 + \left(-\frac{2\sqrt{2}\,\mathrm{e}^{-t}\sin\left(\sqrt{2}\,t\right)}{3} + \frac{\mathrm{e}^{-t}\cos\left(\sqrt{2}\,t\right)}{3} - \frac{\mathrm{e}^{-2t}}{3}\right)c_2 + \left(\cdots\right. \end{bmatrix}$$

$$= \begin{bmatrix} \frac{\mathrm{e}^{-t}(c_1+2c_2+2c_3)\cos\left(\sqrt{2}\,t\right)}{3} - \frac{2\,\mathrm{e}^{-t}\left(c_1+\frac{c_2}{2}-c_3\right)\sqrt{2}\,\sin\left(\sqrt{2}\,t\right)}{3} + \frac{2\,\mathrm{e}^{-2t}(c_1-c_2-c_3)}{3} \\ \frac{2\,\mathrm{e}^{-t}\left(c_1+\frac{c_2}{2}-c_3\right)\cos\left(\sqrt{2}\,t\right)}{3} + \frac{\sqrt{2}\,\mathrm{e}^{-t}(c_1+2c_2+2c_3)\sin\left(\sqrt{2}\,t\right)}{6} - \frac{2\,\mathrm{e}^{-2t}(c_1-c_2-c_3)}{3} \\ -\frac{\mathrm{e}^{-t}(c_1-c_2-4c_3)\cos\left(\sqrt{2}\,t\right)}{3} - \frac{5\left(c_1+\frac{4c_2}{5}-\frac{2c_3}{5}\right)\mathrm{e}^{-t}\sqrt{2}\,\sin\left(\sqrt{2}\,t\right)}{6} + \frac{\mathrm{e}^{-2t}(c_1-c_2-c_3)}{3} \end{bmatrix}$$

Since no forcing function is given, then the final solution is $\vec{x}_h(t)$ above.

16.8.2 Solution using explicit Eigenvalue and Eigenvector method

This is a system of linear ODE's given as

$$\vec{x}'(t) = A\,\vec{x}(t)$$

Or

$$\begin{bmatrix} x_1'(t) \\ x_2'(t) \\ x_3'(t) \end{bmatrix} = \begin{bmatrix} -3 & 0 & 2 \\ 1 & -1 & 0 \\ -2 & -1 & 0 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{bmatrix}$$

The first step is find the homogeneous solution. We start by finding the eigenvalues of A. This is done by solving the following equation for the eigenvalues λ

$$\det(A - \lambda I) = 0$$

Expanding gives

$$\det\left(\begin{bmatrix} -3 & 0 & 2 \\ 1 & -1 & 0 \\ -2 & -1 & 0 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\right) = 0$$

Therefore

$$\det\left(\begin{bmatrix} -3-\lambda & 0 & 2 \\ 1 & -1-\lambda & 0 \\ -2 & -1 & -\lambda \end{bmatrix}\right) = 0$$

Which gives the characteristic equation

$$\lambda^3 + 4\lambda^2 + 7\lambda + 6 = 0$$

The roots of the above are the eigenvalues.

$$\lambda_1 = -1 + i\sqrt{2}$$
$$\lambda_2 = -1 - i\sqrt{2}$$
$$\lambda_3 = -2$$

This table summarises the above result

eigenvalue	algebraic multiplicity	type of eigenvalue
$-1 + i\sqrt{2}$	1	complex eigenvalue
-2	1	real eigenvalue
$-1 - i\sqrt{2}$	1	complex eigenvalue

Now the eigenvector for each eigenvalue are found.

Considering the eigenvalue $\lambda_1 = -2$

We need to solve $A\vec{v} = \lambda\vec{v}$ or $(A - \lambda I)\vec{v} = \vec{0}$ which becomes

$$\left(\begin{bmatrix} -3 & 0 & 2 \\ 1 & -1 & 0 \\ -2 & -1 & 0 \end{bmatrix} - (-2) \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \right) \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} -1 & 0 & 2 \\ 1 & 1 & 0 \\ -2 & -1 & 2 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Now forward elimination is applied to solve for the eigenvector $\vec{v}$. The augmented matrix is

$$\left[\begin{array}{ccc|c} -1 & 0 & 2 & 0 \\ 1 & 1 & 0 & 0 \\ -2 & -1 & 2 & 0 \end{array}\right]$$

$$R_2 = R_2 + R_1 \Longrightarrow \left[\begin{array}{ccc|c} -1 & 0 & 2 & 0 \\ 0 & 1 & 2 & 0 \\ -2 & -1 & 2 & 0 \end{array}\right]$$

$$R_3 = R_3 - 2R_1 \Longrightarrow \left[\begin{array}{ccc|c} -1 & 0 & 2 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & -1 & -2 & 0 \end{array}\right]$$

$$R_3 = R_3 + R_2 \Longrightarrow \left[\begin{array}{ccc|c} -1 & 0 & 2 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

Therefore the system in Echelon form is

$$\begin{bmatrix} -1 & 0 & 2 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

The free variables are $\{v_3\}$ and the leading variables are $\{v_1, v_2\}$. Let $v_3 = t$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\{v_1 = 2t, v_2 = -2t\}$

Hence the solution is

$$\begin{bmatrix} 2t \\ -2t \\ t \end{bmatrix} = \begin{bmatrix} 2t \\ -2t \\ t \end{bmatrix}$$

Since there is one free Variable, we have found one eigenvector associated with this eigenvalue. The above can be written as

$$\begin{bmatrix} 2t \\ -2t \\ t \end{bmatrix} = t \begin{bmatrix} 2 \\ -2 \\ 1 \end{bmatrix}$$

Let $t = 1$ the eigenvector becomes

$$\begin{bmatrix} 2t \\ -2t \\ t \end{bmatrix} = \begin{bmatrix} 2 \\ -2 \\ 1 \end{bmatrix}$$

Considering the eigenvalue $\lambda_2 = -1 - i\sqrt{2}$

We need to solve $A\vec{v} = \lambda\vec{v}$ or $(A - \lambda I)\vec{v} = \vec{0}$ which becomes

$$\left(\begin{bmatrix} -3 & 0 & 2 \\ 1 & -1 & 0 \\ -2 & -1 & 0 \end{bmatrix} - \left(-1 - i\sqrt{2}\right) \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \right) \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} -2 + i\sqrt{2} & 0 & 2 \\ 1 & i\sqrt{2} & 0 \\ -2 & -1 & 1 + i\sqrt{2} \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Now forward elimination is applied to solve for the eigenvector $\vec{v}$. The augmented matrix is

$$\left[\begin{array}{ccc|c} -2 + i\sqrt{2} & 0 & 2 & 0 \\ 1 & i\sqrt{2} & 0 & 0 \\ -2 & -1 & 1 + i\sqrt{2} & 0 \end{array}\right]$$

$$R_2 = R_2 - \frac{R_1}{-2+i\sqrt{2}} \implies \left[\begin{array}{ccc|c} -2+i\sqrt{2} & 0 & 2 & 0 \\ 0 & i\sqrt{2} & -\frac{2}{-2+i\sqrt{2}} & 0 \\ -2 & -1 & 1+i\sqrt{2} & 0 \end{array}\right]$$

$$R_3 = R_3 + \frac{2R_1}{-2+i\sqrt{2}} \implies \left[\begin{array}{ccc|c} -2+i\sqrt{2} & 0 & 2 & 0 \\ 0 & i\sqrt{2} & -\frac{2}{-2+i\sqrt{2}} & 0 \\ 0 & -1 & -\frac{\sqrt{2}}{2i+\sqrt{2}} & 0 \end{array}\right]$$

$$R_3 = R_3 - \frac{i\sqrt{2}\,R_2}{2} \implies \left[\begin{array}{ccc|c} -2+i\sqrt{2} & 0 & 2 & 0 \\ 0 & i\sqrt{2} & -\frac{2}{-2+i\sqrt{2}} & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

Therefore the system in Echelon form is

$$\begin{bmatrix} -2+i\sqrt{2} & 0 & 2 \\ 0 & i\sqrt{2} & -\frac{2}{-2+i\sqrt{2}} \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

The free variables are $\{v_3\}$ and the leading variables are $\{v_1, v_2\}$. Let $v_3 = t$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\left\{v_1 = -\frac{2t}{-2+i\sqrt{2}}, v_2 = -\frac{t}{1+i\sqrt{2}}\right\}$

Hence the solution is

$$\begin{bmatrix} -\frac{2t}{-2+\mathrm{I}\sqrt{2}} \\ -\frac{t}{1+\mathrm{I}\sqrt{2}} \\ t \end{bmatrix} = \begin{bmatrix} -\frac{2t}{-2+i\sqrt{2}} \\ -\frac{t}{1+i\sqrt{2}} \\ t \end{bmatrix}$$

Since there is one free Variable, we have found one eigenvector associated with this eigenvalue. The above can be written as

$$\begin{bmatrix} -\frac{2t}{-2+\mathrm{I}\sqrt{2}} \\ -\frac{t}{1+\mathrm{I}\sqrt{2}} \\ t \end{bmatrix} = t \begin{bmatrix} -\frac{2}{-2+i\sqrt{2}} \\ -\frac{1}{1+i\sqrt{2}} \\ 1 \end{bmatrix}$$

Let $t = 1$ the eigenvector becomes

$$\begin{bmatrix} -\frac{2t}{-2+\mathrm{I}\sqrt{2}} \\ -\frac{t}{1+\mathrm{I}\sqrt{2}} \\ t \end{bmatrix} = \begin{bmatrix} -\frac{2}{-2+i\sqrt{2}} \\ -\frac{1}{1+i\sqrt{2}} \\ 1 \end{bmatrix}$$

Which is normalized to

$$\begin{bmatrix} -\frac{2t}{-2+\mathrm{I}\sqrt{2}} \\ -\frac{t}{1+\mathrm{I}\sqrt{2}} \\ t \end{bmatrix} = \begin{bmatrix} -\frac{2}{-2+i\sqrt{2}} \\ -\frac{1}{1+i\sqrt{2}} \\ 1 \end{bmatrix}$$

Considering the eigenvalue $\lambda_3 = -1 + i\sqrt{2}$

We need to solve $A\vec{v} = \lambda\vec{v}$ or $(A - \lambda I)\vec{v} = \vec{0}$ which becomes

$$\left(\begin{bmatrix} -3 & 0 & 2 \\ 1 & -1 & 0 \\ -2 & -1 & 0 \end{bmatrix} - \left(-1 + i\sqrt{2}\right) \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \right) \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} -2 - i\sqrt{2} & 0 & 2 \\ 1 & -i\sqrt{2} & 0 \\ -2 & -1 & 1 - i\sqrt{2} \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Now forward elimination is applied to solve for the eigenvector $\vec{v}$. The augmented matrix is

$$\left[\begin{array}{ccc|c} -2 - i\sqrt{2} & 0 & 2 & 0 \\ 1 & -i\sqrt{2} & 0 & 0 \\ -2 & -1 & 1 - i\sqrt{2} & 0 \end{array}\right]$$

$$R_2 = R_2 - \frac{R_1}{-2 - i\sqrt{2}} \Longrightarrow \left[\begin{array}{ccc|c} -2 - i\sqrt{2} & 0 & 2 & 0 \\ 0 & -i\sqrt{2} & \frac{2}{2+i\sqrt{2}} & 0 \\ -2 & -1 & 1 - i\sqrt{2} & 0 \end{array}\right]$$

$$R_3 = R_3 + \frac{2R_1}{-2 - i\sqrt{2}} \Longrightarrow \left[\begin{array}{ccc|c} -2 - i\sqrt{2} & 0 & 2 & 0 \\ 0 & -i\sqrt{2} & \frac{2}{2+i\sqrt{2}} & 0 \\ 0 & -1 & \frac{\sqrt{2}}{2i-\sqrt{2}} & 0 \end{array}\right]$$

$$R_3 = R_3 + \frac{i\sqrt{2}\,R_2}{2} \Longrightarrow \left[\begin{array}{ccc|c} -2-i\sqrt{2} & 0 & 2 & 0 \\ 0 & -i\sqrt{2} & \frac{2}{2+i\sqrt{2}} & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

Therefore the system in Echelon form is

$$\begin{bmatrix} -2-i\sqrt{2} & 0 & 2 \\ 0 & -i\sqrt{2} & \frac{2}{2+i\sqrt{2}} \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

The free variables are $\{v_3\}$ and the leading variables are $\{v_1, v_2\}$. Let $v_3 = t$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\left\{v_1 = \frac{2t}{2+i\sqrt{2}}, v_2 = \frac{t}{-1+i\sqrt{2}}\right\}$

Hence the solution is

$$\begin{bmatrix} \frac{2t}{2+\mathrm{I}\sqrt{2}} \\ \frac{t}{-1+\mathrm{I}\sqrt{2}} \\ t \end{bmatrix} = \begin{bmatrix} \frac{2t}{2+i\sqrt{2}} \\ \frac{t}{-1+i\sqrt{2}} \\ t \end{bmatrix}$$

Since there is one free Variable, we have found one eigenvector associated with this eigenvalue. The above can be written as

$$\begin{bmatrix} \frac{2t}{2+\mathrm{I}\sqrt{2}} \\ \frac{t}{-1+\mathrm{I}\sqrt{2}} \\ t \end{bmatrix} = t \begin{bmatrix} \frac{2}{2+i\sqrt{2}} \\ \frac{1}{-1+i\sqrt{2}} \\ 1 \end{bmatrix}$$

Let $t = 1$ the eigenvector becomes

$$\begin{bmatrix} \frac{2t}{2+\mathrm{I}\sqrt{2}} \\ \frac{t}{-1+\mathrm{I}\sqrt{2}} \\ t \end{bmatrix} = \begin{bmatrix} \frac{2}{2+i\sqrt{2}} \\ \frac{1}{-1+i\sqrt{2}} \\ 1 \end{bmatrix}$$

Which is normalized to

$$\begin{bmatrix} \frac{2t}{2+\mathrm{I}\sqrt{2}} \\ \frac{t}{-1+\mathrm{I}\sqrt{2}} \\ t \end{bmatrix} = \begin{bmatrix} \frac{2}{2+i\sqrt{2}} \\ \frac{1}{-1+i\sqrt{2}} \\ 1 \end{bmatrix}$$

The following table gives a summary of this result. It shows for each eigenvalue the algebraic multiplicity m, and its geometric multiplicity k and the eigenvectors associated with the eigenvalue. If $m > k$ then the eigenvalue is defective which means the number of normal linearly independent eigenvectors associated with this eigenvalue (called the geometric multiplicity k) does not equal the algebraic multiplicity m, and we need to determine an additional $m - k$ generalized eigenvectors for this eigenvalue.

	multiplicity			
eigenvalue	algebraic m	geometric k	defective?	eigenvectors
$-1+i\sqrt{2}$	1	1	No	$\begin{bmatrix} \frac{2}{2+i\sqrt{2}} \\ \frac{i\left(-2+i\sqrt{2}\right)\sqrt{2}}{6} \\ 1 \end{bmatrix}$
$-1-i\sqrt{2}$	1	1	No	$\begin{bmatrix} \frac{2}{2-i\sqrt{2}} \\ -\frac{i\left(-2-i\sqrt{2}\right)\sqrt{2}}{6} \\ 1 \end{bmatrix}$
-2	1	1	No	$\begin{bmatrix} 2 \\ -2 \\ 1 \end{bmatrix}$

Now that we found the eigenvalues and associated eigenvectors, we will go over each eigenvalue and generate the solution basis. The only problem we need to take care of is if the eigenvalue is defective. Since eigenvalue -2 is real and distinct then the corresponding eigenvector solution is

$$\begin{aligned} \vec{x}_1(t) &= \vec{v}_1 e^{-2t} \\ &= \begin{bmatrix} 2 \\ -2 \\ 1 \end{bmatrix} e^{-2t} \end{aligned}$$

Therefore the final solution is

$$\vec{x}_h(t) = c_1\vec{x}_1(t) + c_2\vec{x}_2(t) + c_3\vec{x}_3(t)$$

Which is written as

$$\begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{bmatrix} = c_1 \begin{bmatrix} \frac{2\,\mathrm{e}^{\left(-1+i\sqrt{2}\right)t}}{2+i\sqrt{2}} \\ \frac{i\mathrm{e}^{\left(-1+i\sqrt{2}\right)t}\left(-2+i\sqrt{2}\right)\sqrt{2}}{6} \\ \mathrm{e}^{\left(-1+i\sqrt{2}\right)t} \end{bmatrix} + c_2 \begin{bmatrix} \frac{2\,\mathrm{e}^{\left(-1-i\sqrt{2}\right)t}}{2-i\sqrt{2}} \\ -\frac{i\mathrm{e}^{\left(-1-i\sqrt{2}\right)t}\left(-2-i\sqrt{2}\right)\sqrt{2}}{6} \\ \mathrm{e}^{\left(-1-i\sqrt{2}\right)t} \end{bmatrix} + c_3 \begin{bmatrix} 2\,\mathrm{e}^{-2t} \\ -2\,\mathrm{e}^{-2t} \\ \mathrm{e}^{-2t} \end{bmatrix}$$

Which becomes

$$\begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{bmatrix} = \begin{bmatrix} \frac{\left(2+i\sqrt{2}\right)c_2\mathrm{e}^{-\left(1+i\sqrt{2}\right)t}}{3} + \frac{c_1\left(2-i\sqrt{2}\right)\mathrm{e}^{\left(-1+i\sqrt{2}\right)t}}{3} + 2c_3\mathrm{e}^{-2t} \\ \frac{\left(-1+i\sqrt{2}\right)c_2\mathrm{e}^{-\left(1+i\sqrt{2}\right)t}}{3} + \frac{c_1\left(-1-i\sqrt{2}\right)\mathrm{e}^{\left(-1+i\sqrt{2}\right)t}}{3} - 2c_3\mathrm{e}^{-2t} \\ c_1\mathrm{e}^{\left(-1+i\sqrt{2}\right)t} + c_2\mathrm{e}^{-\left(1+i\sqrt{2}\right)t} + c_3\mathrm{e}^{-2t} \end{bmatrix}$$

16.8.3 Maple step by step solution

Let's solve

$[x_1'(t) = -3x_1(t) + 2x_3(t)\,, x_2'(t) = x_1(t) - x_2(t)\,, x_3'(t) = -2x_1(t) - x_2(t)]$

- Define vector

$$\underline{x}^{\rightarrow}(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{bmatrix}$$

- Convert system into a vector equation

$$\underline{x}^{\rightarrow}{}'(t) = \begin{bmatrix} -3 & 0 & 2 \\ 1 & -1 & 0 \\ -2 & -1 & 0 \end{bmatrix} \cdot \underline{x}^{\rightarrow}(t) + \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

- System to solve

$$\underline{x}^{\rightarrow}{}'(t) = \begin{bmatrix} -3 & 0 & 2 \\ 1 & -1 & 0 \\ -2 & -1 & 0 \end{bmatrix} \cdot \underline{x}^{\rightarrow}(t)$$

- Define the coefficient matrix

$$A = \begin{bmatrix} -3 & 0 & 2 \\ 1 & -1 & 0 \\ -2 & -1 & 0 \end{bmatrix}$$

- Rewrite the system as

$$\underline{x}^{\rightarrow}{}'(t) = A \cdot \underline{x}^{\rightarrow}(t)$$

- To solve the system, find the eigenvalues and eigenvectors of A
- Eigenpairs of A

$$\left[\left[-2, \begin{bmatrix} 2 \\ -2 \\ 1 \end{bmatrix}\right], \left[-1 - \mathrm{I}\sqrt{2}, \begin{bmatrix} \frac{2}{2-\mathrm{I}\sqrt{2}} \\ -\frac{\mathrm{I}}{6}\left(-2 - \mathrm{I}\sqrt{2}\right)\sqrt{2} \\ 1 \end{bmatrix}\right], \left[-1 + \mathrm{I}\sqrt{2}, \begin{bmatrix} \frac{2}{2+\mathrm{I}\sqrt{2}} \\ \frac{\mathrm{I}}{6}\left(-2 + \mathrm{I}\sqrt{2}\right)\sqrt{2} \\ 1 \end{bmatrix}\right]\right]$$

- Consider eigenpair

$$\left[-2, \begin{bmatrix} 2 \\ -2 \\ 1 \end{bmatrix}\right]$$

- Solution to homogeneous system from eigenpair

$$\underline{x}^{\rightarrow}{}_{1} = \mathrm{e}^{-2t} \cdot \begin{bmatrix} 2 \\ -2 \\ 1 \end{bmatrix}$$

- Consider complex eigenpair, complex conjugate eigenvalue can be ignored

$$\left[-1 - \mathrm{I}\sqrt{2}, \begin{bmatrix} \frac{2}{2-\mathrm{I}\sqrt{2}} \\ -\frac{\mathrm{I}}{6}\left(-2 - \mathrm{I}\sqrt{2}\right)\sqrt{2} \\ 1 \end{bmatrix}\right]$$

- Solution from eigenpair

$$\mathrm{e}^{\left(-1-\mathrm{I}\sqrt{2}\right)t} \cdot \begin{bmatrix} \frac{2}{2-\mathrm{I}\sqrt{2}} \\ -\frac{\mathrm{I}}{6}\left(-2 - \mathrm{I}\sqrt{2}\right)\sqrt{2} \\ 1 \end{bmatrix}$$

- Use Euler identity to write solution in terms of $\sin$ and $\cos$

$$e^{-t}\cdot\left(\cos\left(\sqrt{2}\,t\right)-I\sin\left(\sqrt{2}\,t\right)\right)\cdot\left[\begin{array}{c}\frac{2}{2-I\sqrt{2}}\\ -\frac{I}{6}\left(-2-I\sqrt{2}\right)\sqrt{2}\\ 1\end{array}\right]$$

- Simplify expression

$$e^{-t}\cdot\left[\begin{array}{c}\frac{2\left(\cos\left(\sqrt{2}\,t\right)-I\sin\left(\sqrt{2}\,t\right)\right)}{2-I\sqrt{2}}\\ -\frac{I}{6}\left(\cos\left(\sqrt{2}\,t\right)-I\sin\left(\sqrt{2}\,t\right)\right)\left(-2-I\sqrt{2}\right)\sqrt{2}\\ \cos\left(\sqrt{2}\,t\right)-I\sin\left(\sqrt{2}\,t\right)\end{array}\right]$$

- Both real and imaginary parts are solutions to the homogeneous system

$$\left[\underline{\vec{x}}_{2}(t)=e^{-t}\cdot\left[\begin{array}{c}\frac{2\cos\left(\sqrt{2}\,t\right)}{3}+\frac{\sqrt{2}\,\sin\left(\sqrt{2}\,t\right)}{3}\\ \frac{\sqrt{2}\left(-\cos\left(\sqrt{2}\,t\right)\sqrt{2}+2\sin\left(\sqrt{2}\,t\right)\right)}{6}\\ \cos\left(\sqrt{2}\,t\right)\end{array}\right],\underline{\vec{x}}_{3}(t)=e^{-t}\cdot\left[\begin{array}{c}\frac{\cos\left(\sqrt{2}\,t\right)\sqrt{2}}{3}-\frac{2\sin\left(\sqrt{2}\,t\right)}{3}\\ -\frac{\sqrt{2}\left(-2\cos\left(\sqrt{2}\,t\right)-\sqrt{2}\,\sin\left(\sqrt{2}\,t\right)\right)}{6}\\ -\sin\left(\sqrt{2}\,t\right)\end{array}\right.\right.$$

- General solution to the system of ODEs

$$\underline{\vec{x}}=c_1\underline{\vec{x}}_{1}+c_2\underline{\vec{x}}_{2}(t)+c_3\underline{\vec{x}}_{3}(t)$$

- Substitute solutions into the general solution

$$\underline{\vec{x}}=c_1e^{-2t}\cdot\left[\begin{array}{c}2\\ -2\\ 1\end{array}\right]+c_2e^{-t}\cdot\left[\begin{array}{c}\frac{2\cos\left(\sqrt{2}\,t\right)}{3}+\frac{\sqrt{2}\,\sin\left(\sqrt{2}\,t\right)}{3}\\ \frac{\sqrt{2}\left(-\cos\left(\sqrt{2}\,t\right)\sqrt{2}+2\sin\left(\sqrt{2}\,t\right)\right)}{6}\\ \cos\left(\sqrt{2}\,t\right)\end{array}\right]+c_3e^{-t}\cdot\left[\begin{array}{c}\frac{\cos\left(\sqrt{2}\,t\right)\sqrt{2}}{3}-\frac{2\sin\left(\sqrt{2}\,t\right)}{3}\\ -\frac{\sqrt{2}\left(-2\cos\left(\sqrt{2}\,t\right)-\sqrt{2}\,\sin\left(\sqrt{2}\,t\right)\right)}{6}\\ -\sin\left(\sqrt{2}\,t\right)\end{array}\right.$$

- Substitute in vector of dependent variables

$$\left[\begin{array}{c}x_1(t)\\ x_2(t)\\ x_3(t)\end{array}\right]=\left[\begin{array}{c}\frac{2\left(\frac{c_3\sqrt{2}}{2}+c_2\right)e^{-t}\cos\left(\sqrt{2}\,t\right)}{3}+\frac{e^{-t}\left(\sqrt{2}\,c_2-2c_3\right)\sin\left(\sqrt{2}\,t\right)}{3}+2c_1e^{-2t}\\ -\frac{e^{-t}\left(-c_3\sqrt{2}+c_2\right)\cos\left(\sqrt{2}\,t\right)}{3}+\frac{e^{-t}\left(\sqrt{2}\,c_2+c_3\right)\sin\left(\sqrt{2}\,t\right)}{3}-2c_1e^{-2t}\\ c_1e^{-2t}+c_2e^{-t}\cos\left(\sqrt{2}\,t\right)-c_3e^{-t}\sin\left(\sqrt{2}\,t\right)\end{array}\right]$$

- Solution to the system of ODEs

$$\left\{x_1(t)=\frac{2\left(\frac{c_3\sqrt{2}}{2}+c_2\right)e^{-t}\cos\left(\sqrt{2}\,t\right)}{3}+\frac{e^{-t}\left(\sqrt{2}\,c_2-2c_3\right)\sin\left(\sqrt{2}\,t\right)}{3}+2c_1e^{-2t},x_2(t)=-\frac{e^{-t}\left(-c_3\sqrt{2}+c_2\right)\cos\left(\sqrt{2}\,t\right)}{3}+\right.$$

✓ Solution by Maple

Time used: 0.031 (sec). Leaf size: 146

```
dsolve([diff(x__1(t),t)=-3*x__1(t)+0*x__2(t)+2*x__3(t),diff(x__2(t),t)=1*x__1(t)-1*x__2(t)-0
```

$$x_1(t) = c_1 e^{-2t} + c_2 e^{-t} \sin\left(\sqrt{2}\, t\right) + c_3 e^{-t} \cos\left(\sqrt{2}\, t\right)$$

$$x_2(t) = -c_1 e^{-2t} - \frac{c_2 e^{-t}\sqrt{2}\, \cos\left(\sqrt{2}\, t\right)}{2} + \frac{c_3 e^{-t}\sqrt{2}\, \sin\left(\sqrt{2}\, t\right)}{2}$$

$$x_3(t) = \frac{c_1 e^{-2t}}{2} + c_2 e^{-t} \sin\left(\sqrt{2}\, t\right) + \frac{c_2 e^{-t}\sqrt{2}\, \cos\left(\sqrt{2}\, t\right)}{2} + c_3 e^{-t} \cos\left(\sqrt{2}\, t\right) - \frac{c_3 e^{-t}\sqrt{2}\, \sin\left(\sqrt{2}\, t\right)}{2}$$

✓ Solution by Mathematica

Time used: 0.038 (sec). Leaf size: 235

```
DSolve[{x1'[t]==-3*x1[t]+0*x2[t]+2*x3[t],x2'[t]==1*x1[t]-1*x2[t]-0*x3[t],x3'[t]==-2*x1[t]-1*
```

$$\text{x1}(t) \to \frac{1}{3} e^{-2t}\left((c_1 + 2(c_2 + c_3))e^t \cos\left(\sqrt{2}t\right) - \sqrt{2}(2c_1 + c_2 - 2c_3)e^t \sin\left(\sqrt{2}t\right) + 2(c_1 - c_2 - c_3)\right)$$

$$\text{x2}(t) \to \frac{1}{6} e^{-2t}\left(2(2c_1 + c_2 - 2c_3)e^t \cos\left(\sqrt{2}t\right) + \sqrt{2}(c_1 + 2(c_2 + c_3))e^t \sin\left(\sqrt{2}t\right) + 4(-c_1 + c_2 + c_3)\right)$$

$$\text{x3}(t) \to \frac{1}{6} e^{-2t}\left(-2(c_1 - c_2 - 4c_3)e^t \cos\left(\sqrt{2}t\right) - \sqrt{2}(5c_1 + 4c_2 - 2c_3)e^t \sin\left(\sqrt{2}t\right) + 2(c_1 - c_2 - c_3)\right)$$

16.9 problem 9

Internal problem ID [759]
Internal file name [OUTPUT/759_Sunday_June_05_2022_01_48_57_AM_24403091/index.tex]

Book: Elementary differential equations and boundary value problems, 10th ed., Boyce and DiPrima
Section: Chapter 7.6, Complex Eigenvalues. page 417
Problem number: 9.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"system of linear ODEs"**

Solve

$$x_1'(t) = x_1(t) - 5x_2(t)$$
$$x_2'(t) = x_1(t) - 3x_2(t)$$

With initial conditions

$$[x_1(0) = 1, x_2(0) = 1]$$

16.9.1 Solution using Matrix exponential method

In this method, we will assume we have found the matrix exponential e^{At} allready. There are different methods to determine this but will not be shown here. This is a system of linear ODE's given as

$$\vec{x}'(t) = A\,\vec{x}(t)$$

Or

$$\begin{bmatrix} x_1'(t) \\ x_2'(t) \end{bmatrix} = \begin{bmatrix} 1 & -5 \\ 1 & -3 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}$$

For the above matrix A, the matrix exponential can be found to be

$$e^{At} = \begin{bmatrix} e^{-t}\cos(t) + 2\,e^{-t}\sin(t) & -5\,e^{-t}\sin(t) \\ e^{-t}\sin(t) & e^{-t}\cos(t) - 2\,e^{-t}\sin(t) \end{bmatrix}$$
$$= \begin{bmatrix} e^{-t}(\cos(t) + 2\sin(t)) & -5\,e^{-t}\sin(t) \\ e^{-t}\sin(t) & e^{-t}(\cos(t) - 2\sin(t)) \end{bmatrix}$$

Therefore the homogeneous solution is

$$\begin{aligned}
\vec{x}_h(t) &= e^{At}\vec{x}_0 \\
&= \begin{bmatrix} e^{-t}(\cos(t)+2\sin(t)) & -5\,e^{-t}\sin(t) \\ e^{-t}\sin(t) & e^{-t}(\cos(t)-2\sin(t)) \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} \\
&= \begin{bmatrix} e^{-t}(\cos(t)+2\sin(t)) - 5\,e^{-t}\sin(t) \\ e^{-t}\sin(t) + e^{-t}(\cos(t)-2\sin(t)) \end{bmatrix} \\
&= \begin{bmatrix} (-3\sin(t)+\cos(t))\,e^{-t} \\ e^{-t}(-\sin(t)+\cos(t)) \end{bmatrix}
\end{aligned}$$

Since no forcing function is given, then the final solution is $\vec{x}_h(t)$ above.

16.9.2 Solution using explicit Eigenvalue and Eigenvector method

This is a system of linear ODE's given as

$$\vec{x}'(t) = A\,\vec{x}(t)$$

Or

$$\begin{bmatrix} x_1'(t) \\ x_2'(t) \end{bmatrix} = \begin{bmatrix} 1 & -5 \\ 1 & -3 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}$$

The first step is find the homogeneous solution. We start by finding the eigenvalues of A. This is done by solving the following equation for the eigenvalues λ

$$\det(A - \lambda I) = 0$$

Expanding gives

$$\det\left(\begin{bmatrix} 1 & -5 \\ 1 & -3 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right) = 0$$

Therefore

$$\det\left(\begin{bmatrix} 1-\lambda & -5 \\ 1 & -3-\lambda \end{bmatrix}\right) = 0$$

Which gives the characteristic equation

$$\lambda^2 + 2\lambda + 2 = 0$$

The roots of the above are the eigenvalues.

$$\lambda_1 = -1 + i$$
$$\lambda_2 = -1 - i$$

This table summarises the above result

eigenvalue	algebraic multiplicity	type of eigenvalue
$-1 - i$	1	complex eigenvalue
$-1 + i$	1	complex eigenvalue

Now the eigenvector for each eigenvalue are found.

Considering the eigenvalue $\lambda_1 = -1 - i$

We need to solve $A\vec{v} = \lambda\vec{v}$ or $(A - \lambda I)\vec{v} = \vec{0}$ which becomes

$$\left(\begin{bmatrix} 1 & -5 \\ 1 & -3 \end{bmatrix} - (-1 - i)\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right)\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$
$$\begin{bmatrix} 2 + i & -5 \\ 1 & -2 + i \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Now forward elimination is applied to solve for the eigenvector $\vec{v}$. The augmented matrix is

$$\left[\begin{array}{cc|c} 2 + i & -5 & 0 \\ 1 & -2 + i & 0 \end{array}\right]$$

$$R_2 = R_2 + \left(-\frac{2}{5} + \frac{i}{5}\right)R_1 \Longrightarrow \left[\begin{array}{cc|c} 2 + i & -5 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

Therefore the system in Echelon form is

$$\begin{bmatrix} 2 + i & -5 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

The free variables are $\{v_2\}$ and the leading variables are $\{v_1\}$. Let $v_2 = t$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\{v_1 = (2-i)\,t\}$

Hence the solution is

$$\begin{bmatrix} (2-\mathrm{I})\,t \\ t \end{bmatrix} = \begin{bmatrix} (2-i)\,t \\ t \end{bmatrix}$$

Since there is one free Variable, we have found one eigenvector associated with this eigenvalue. The above can be written as

$$\begin{bmatrix} (2-\mathrm{I})\,t \\ t \end{bmatrix} = t \begin{bmatrix} 2-i \\ 1 \end{bmatrix}$$

Let $t = 1$ the eigenvector becomes

$$\begin{bmatrix} (2-\mathrm{I})\,t \\ t \end{bmatrix} = \begin{bmatrix} 2-i \\ 1 \end{bmatrix}$$

Considering the eigenvalue $\lambda_2 = -1 + i$

We need to solve $A\vec{v} = \lambda\vec{v}$ or $(A - \lambda I)\vec{v} = \vec{0}$ which becomes

$$\left(\begin{bmatrix} 1 & -5 \\ 1 & -3 \end{bmatrix} - (-1+i) \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right) \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 2-i & -5 \\ 1 & -2-i \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Now forward elimination is applied to solve for the eigenvector $\vec{v}$. The augmented matrix is

$$\left[\begin{array}{cc|c} 2-i & -5 & 0 \\ 1 & -2-i & 0 \end{array}\right]$$

$$R_2 = R_2 + \left(-\frac{2}{5} - \frac{i}{5}\right) R_1 \Longrightarrow \left[\begin{array}{cc|c} 2-i & -5 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

Therefore the system in Echelon form is

$$\begin{bmatrix} 2-i & -5 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

The free variables are $\{v_2\}$ and the leading variables are $\{v_1\}$. Let $v_2 = t$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\{v_1 = (2+i)\,t\}$

Hence the solution is

$$\begin{bmatrix} (2+\mathrm{I})\,t \\ t \end{bmatrix} = \begin{bmatrix} (2+i)\,t \\ t \end{bmatrix}$$

Since there is one free Variable, we have found one eigenvector associated with this eigenvalue. The above can be written as

$$\begin{bmatrix} (2+\mathrm{I})\,t \\ t \end{bmatrix} = t \begin{bmatrix} 2+i \\ 1 \end{bmatrix}$$

Let $t = 1$ the eigenvector becomes

$$\begin{bmatrix} (2+\mathrm{I})\,t \\ t \end{bmatrix} = \begin{bmatrix} 2+i \\ 1 \end{bmatrix}$$

The following table gives a summary of this result. It shows for each eigenvalue the algebraic multiplicity m, and its geometric multiplicity k and the eigenvectors associated with the eigenvalue. If $m > k$ then the eigenvalue is defective which means the number of normal linearly independent eigenvectors associated with this eigenvalue (called the geometric multiplicity k) does not equal the algebraic multiplicity m, and we need to determine an additional $m-k$ generalized eigenvectors for this eigenvalue.

	multiplicity			
eigenvalue	algebraic m	geometric k	defective?	eigenvectors
$-1+i$	1	1	No	$\begin{bmatrix} 2+i \\ 1 \end{bmatrix}$
$-1-i$	1	1	No	$\begin{bmatrix} 2-i \\ 1 \end{bmatrix}$

Now that we found the eigenvalues and associated eigenvectors, we will go over each eigenvalue and generate the solution basis. The only problem we need to take care of is if the eigenvalue is defective. Therefore the final solution is

$$\vec{x}_h(t) = c_1\vec{x}_1(t) + c_2\vec{x}_2(t)$$

Which is written as

$$\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = c_1 \begin{bmatrix} (2+i)\,\mathrm{e}^{(-1+i)t} \\ \mathrm{e}^{(-1+i)t} \end{bmatrix} + c_2 \begin{bmatrix} (2-i)\,\mathrm{e}^{(-1-i)t} \\ \mathrm{e}^{(-1-i)t} \end{bmatrix}$$

Which becomes

$$\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = \begin{bmatrix} (2+i)\,c_1\mathrm{e}^{(-1+i)t} + (2-i)\,c_2\mathrm{e}^{(-1-i)t} \\ c_1\mathrm{e}^{(-1+i)t} + c_2\mathrm{e}^{(-1-i)t} \end{bmatrix}$$

Since initial conditions are given, the solution above needs to be updated by solving for the constants of integrations using the given initial conditions

$$\begin{bmatrix} x_1(0) = 1 \\ x_2(0) = 1 \end{bmatrix} \tag{1}$$

Substituting initial conditions into the above solution at $t = 0$ gives

$$\begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} (2+i)\,c_1 + (2-i)\,c_2 \\ c_1 + c_2 \end{bmatrix}$$

Solving for the constants of integrations gives

$$\begin{bmatrix} c_1 = \frac{1}{2} + \frac{i}{2} \\ c_2 = \frac{1}{2} - \frac{i}{2} \end{bmatrix}$$

Substituting these constants back in original solution in Eq. (1) gives

$$\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = \begin{bmatrix} \left(\frac{1}{2} + \frac{3i}{2}\right)\mathrm{e}^{(-1+i)t} + \left(\frac{1}{2} - \frac{3i}{2}\right)\mathrm{e}^{(-1-i)t} \\ \left(\frac{1}{2} + \frac{i}{2}\right)\mathrm{e}^{(-1+i)t} + \left(\frac{1}{2} - \frac{i}{2}\right)\mathrm{e}^{(-1-i)t} \end{bmatrix}$$

The following is the phase plot of the system.

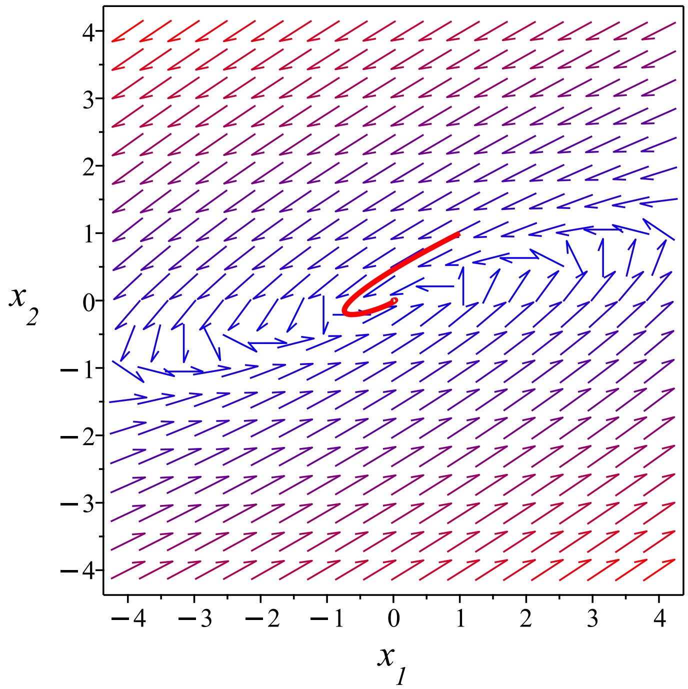


Figure 513: Phase plot

The following are plots of each solution.

✓ Solution by Maple

Time used: 0.125 (sec). Leaf size: 35

```
dsolve([diff(x__1(t),t) = x__1(t)-5*x__2(t), diff(x__2(t),t) = x__1(t)-3*x__2(t), x__1(0) =
```

$$x_1(t) = e^{-t}(-3\sin(t) + \cos(t))$$
$$x_2(t) = \frac{e^{-t}(5\cos(t) - 5\sin(t))}{5}$$

✓ Solution by Mathematica

Time used: 0.006 (sec). Leaf size: 34

```
DSolve[{x1'[t]==1*x1[t]-5*x2[t],x2'[t]==1*x1[t]-3*x2[t]},{x1[0]==1,x2[0]==1},{x1[t],x2[t]},t
```

$$\text{x1}(t) \to e^{-t}(\cos(t) - 3\sin(t))$$
$$\text{x2}(t) \to e^{-t}(\cos(t) - \sin(t))$$

16.10 problem 10

Internal problem ID [760]
Internal file name [`OUTPUT/760_Sunday_June_05_2022_01_48_59_AM_20065990/index.tex`]

Book: Elementary differential equations and boundary value problems, 10th ed., Boyce and DiPrima
Section: Chapter 7.6, Complex Eigenvalues. page 417
Problem number: 10.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"system of linear ODEs"**

Solve

$$
\begin{aligned}
x_1'(t) &= -3x_1(t) + 2x_2(t) \\
x_2'(t) &= -x_1(t) - x_2(t)
\end{aligned}
$$

With initial conditions

$$
[x_1(0) = 1, x_2(0) = -2]
$$

16.10.1 Solution using Matrix exponential method

In this method, we will assume we have found the matrix exponential e^{At} allready. There are different methods to determine this but will not be shown here. This is a system of linear ODE's given as

$$
\vec{x}'(t) = A\,\vec{x}(t)
$$

Or

$$
\begin{bmatrix} x_1'(t) \\ x_2'(t) \end{bmatrix} = \begin{bmatrix} -3 & 2 \\ -1 & -1 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}
$$

For the above matrix A, the matrix exponential can be found to be

$$
\begin{aligned}
e^{At} &= \begin{bmatrix} e^{-2t}\cos(t) - e^{-2t}\sin(t) & 2\,e^{-2t}\sin(t) \\ -e^{-2t}\sin(t) & e^{-2t}\cos(t) + e^{-2t}\sin(t) \end{bmatrix} \\
&= \begin{bmatrix} e^{-2t}(-\sin(t) + \cos(t)) & 2\,e^{-2t}\sin(t) \\ -e^{-2t}\sin(t) & e^{-2t}(\cos(t) + \sin(t)) \end{bmatrix}
\end{aligned}
$$

Therefore the homogeneous solution is

$$
\begin{aligned}
\vec{x}_h(t) &= e^{At}\vec{x}_0 \\
&= \begin{bmatrix} e^{-2t}(-\sin(t)+\cos(t)) & 2\,e^{-2t}\sin(t) \\ -e^{-2t}\sin(t) & e^{-2t}(\cos(t)+\sin(t)) \end{bmatrix} \begin{bmatrix} 1 \\ -2 \end{bmatrix} \\
&= \begin{bmatrix} e^{-2t}(-\sin(t)+\cos(t)) - 4\,e^{-2t}\sin(t) \\ -e^{-2t}\sin(t) - 2\,e^{-2t}(\cos(t)+\sin(t)) \end{bmatrix} \\
&= \begin{bmatrix} e^{-2t}(-5\sin(t)+\cos(t)) \\ e^{-2t}(-3\sin(t)-2\cos(t)) \end{bmatrix}
\end{aligned}
$$

Since no forcing function is given, then the final solution is $\vec{x}_h(t)$ above.

16.10.2 Solution using explicit Eigenvalue and Eigenvector method

This is a system of linear ODE's given as

$$\vec{x}'(t) = A\,\vec{x}(t)$$

Or

$$\begin{bmatrix} x_1'(t) \\ x_2'(t) \end{bmatrix} = \begin{bmatrix} -3 & 2 \\ -1 & -1 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}$$

The first step is find the homogeneous solution. We start by finding the eigenvalues of A. This is done by solving the following equation for the eigenvalues λ

$$\det(A - \lambda I) = 0$$

Expanding gives

$$\det\left(\begin{bmatrix} -3 & 2 \\ -1 & -1 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right) = 0$$

Therefore

$$\det\left(\begin{bmatrix} -3-\lambda & 2 \\ -1 & -1-\lambda \end{bmatrix}\right) = 0$$

Which gives the characteristic equation

$$\lambda^2 + 4\lambda + 5 = 0$$

The roots of the above are the eigenvalues.

$$\lambda_1 = -2 + i$$
$$\lambda_2 = -2 - i$$

This table summarises the above result

eigenvalue	algebraic multiplicity	type of eigenvalue
$-2+i$	1	complex eigenvalue
$-2-i$	1	complex eigenvalue

Now the eigenvector for each eigenvalue are found.

Considering the eigenvalue $\lambda_1 = -2 - i$

We need to solve $A\vec{v} = \lambda\vec{v}$ or $(A - \lambda I)\vec{v} = \vec{0}$ which becomes

$$\left(\begin{bmatrix} -3 & 2 \\ -1 & -1 \end{bmatrix} - (-2 - i)\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right)\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$
$$\begin{bmatrix} -1+i & 2 \\ -1 & 1+i \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Now forward elimination is applied to solve for the eigenvector $\vec{v}$. The augmented matrix is

$$\left[\begin{array}{cc|c} -1+i & 2 & 0 \\ -1 & 1+i & 0 \end{array}\right]$$

$$R_2 = R_2 + \left(-\frac{1}{2} - \frac{i}{2}\right)R_1 \Longrightarrow \left[\begin{array}{cc|c} -1+i & 2 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

Therefore the system in Echelon form is

$$\begin{bmatrix} -1+i & 2 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

The free variables are $\{v_2\}$ and the leading variables are $\{v_1\}$. Let $v_2 = t$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\{v_1 = (1+i)\,t\}$

Hence the solution is

$$\begin{bmatrix} (1+\mathrm{I})\,t \\ t \end{bmatrix} = \begin{bmatrix} (1+i)\,t \\ t \end{bmatrix}$$

Since there is one free Variable, we have found one eigenvector associated with this eigenvalue. The above can be written as

$$\begin{bmatrix} (1+\mathrm{I})\,t \\ t \end{bmatrix} = t \begin{bmatrix} 1+i \\ 1 \end{bmatrix}$$

Let $t = 1$ the eigenvector becomes

$$\begin{bmatrix} (1+\mathrm{I})\,t \\ t \end{bmatrix} = \begin{bmatrix} 1+i \\ 1 \end{bmatrix}$$

Considering the eigenvalue $\lambda_2 = -2+i$

We need to solve $A\vec{v} = \lambda\vec{v}$ or $(A-\lambda I)\vec{v} = \vec{0}$ which becomes

$$\left(\begin{bmatrix} -3 & 2 \\ -1 & -1 \end{bmatrix} - (-2+i) \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right) \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} -1-i & 2 \\ -1 & 1-i \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Now forward elimination is applied to solve for the eigenvector $\vec{v}$. The augmented matrix is

$$\left[\begin{array}{cc|c} -1-i & 2 & 0 \\ -1 & 1-i & 0 \end{array}\right]$$

$$R_2 = R_2 + \left(-\frac{1}{2} + \frac{i}{2}\right) R_1 \Longrightarrow \left[\begin{array}{cc|c} -1-i & 2 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

Therefore the system in Echelon form is

$$\begin{bmatrix} -1-i & 2 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

The free variables are $\{v_2\}$ and the leading variables are $\{v_1\}$. Let $v_2 = t$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\{v_1 = (1 - i)\, t\}$

Hence the solution is

$$\begin{bmatrix} (1 - \mathrm{I})\, t \\ t \end{bmatrix} = \begin{bmatrix} (1 - i)\, t \\ t \end{bmatrix}$$

Since there is one free Variable, we have found one eigenvector associated with this eigenvalue. The above can be written as

$$\begin{bmatrix} (1 - \mathrm{I})\, t \\ t \end{bmatrix} = t \begin{bmatrix} 1 - i \\ 1 \end{bmatrix}$$

Let $t = 1$ the eigenvector becomes

$$\begin{bmatrix} (1 - \mathrm{I})\, t \\ t \end{bmatrix} = \begin{bmatrix} 1 - i \\ 1 \end{bmatrix}$$

The following table gives a summary of this result. It shows for each eigenvalue the algebraic multiplicity m, and its geometric multiplicity k and the eigenvectors associated with the eigenvalue. If $m > k$ then the eigenvalue is defective which means the number of normal linearly independent eigenvectors associated with this eigenvalue (called the geometric multiplicity k) does not equal the algebraic multiplicity m, and we need to determine an additional $m - k$ generalized eigenvectors for this eigenvalue.

	multiplicity			
eigenvalue	algebraic m	geometric k	defective?	eigenvectors
$-2 + i$	1	1	No	$\begin{bmatrix} 1 - i \\ 1 \end{bmatrix}$
$-2 - i$	1	1	No	$\begin{bmatrix} 1 + i \\ 1 \end{bmatrix}$

Now that we found the eigenvalues and associated eigenvectors, we will go over each eigenvalue and generate the solution basis. The only problem we need to take care of is if the eigenvalue is defective. Therefore the final solution is

$$\vec{x}_h(t) = c_1 \vec{x}_1(t) + c_2 \vec{x}_2(t)$$

Which is written as

$$\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = c_1 \begin{bmatrix} (1-i)\,\mathrm{e}^{(-2+i)t} \\ \mathrm{e}^{(-2+i)t} \end{bmatrix} + c_2 \begin{bmatrix} (1+i)\,\mathrm{e}^{(-2-i)t} \\ \mathrm{e}^{(-2-i)t} \end{bmatrix}$$

Which becomes

$$\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = \begin{bmatrix} (1-i)\,c_1\mathrm{e}^{(-2+i)t} + (1+i)\,c_2\mathrm{e}^{(-2-i)t} \\ c_1\mathrm{e}^{(-2+i)t} + c_2\mathrm{e}^{(-2-i)t} \end{bmatrix}$$

Since initial conditions are given, the solution above needs to be updated by solving for the constants of integrations using the given initial conditions

$$\begin{bmatrix} x_1(0) = 1 \\ x_2(0) = -2 \end{bmatrix} \tag{1}$$

Substituting initial conditions into the above solution at $t = 0$ gives

$$\begin{bmatrix} 1 \\ -2 \end{bmatrix} = \begin{bmatrix} (1-i)\,c_1 + (1+i)\,c_2 \\ c_1 + c_2 \end{bmatrix}$$

Solving for the constants of integrations gives

$$\begin{bmatrix} c_1 = -1 + \frac{3i}{2} \\ c_2 = -1 - \frac{3i}{2} \end{bmatrix}$$

Substituting these constants back in original solution in Eq. (1) gives

$$\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = \begin{bmatrix} \left(\frac{1}{2} + \frac{5i}{2}\right)\mathrm{e}^{(-2+i)t} + \left(\frac{1}{2} - \frac{5i}{2}\right)\mathrm{e}^{(-2-i)t} \\ \left(-1 + \frac{3i}{2}\right)\mathrm{e}^{(-2+i)t} + \left(-1 - \frac{3i}{2}\right)\mathrm{e}^{(-2-i)t} \end{bmatrix}$$

The following is the phase plot of the system.

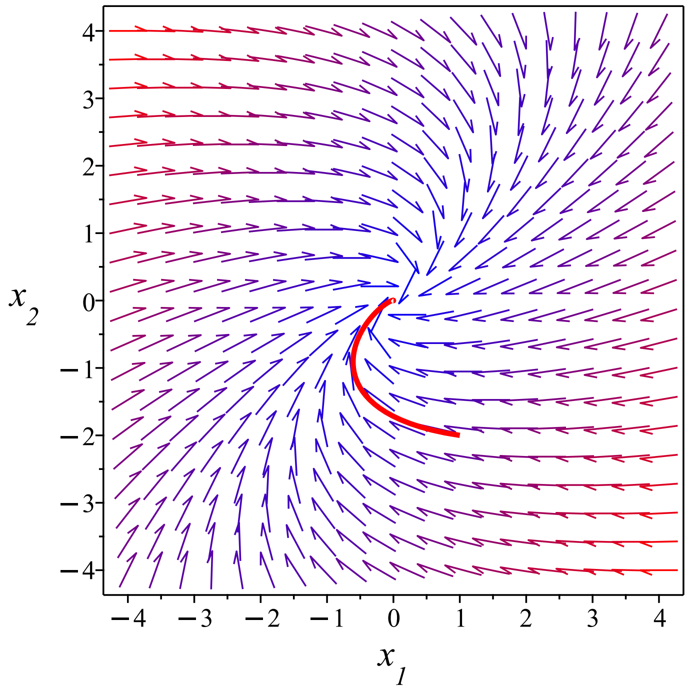


Figure 514: Phase plot

The following are plots of each solution.

✓ Solution by Maple

Time used: 0.015 (sec). Leaf size: 35

```
dsolve([diff(x__1(t),t) = -3*x__1(t)+2*x__2(t), diff(x__2(t),t) = -x__1(t)-x__2(t), x__1(0)
```

$$x_1(t) = e^{-2t}(-5\sin(t) + \cos(t))$$
$$x_2(t) = \frac{e^{-2t}(-6\sin(t) - 4\cos(t))}{2}$$

✓ Solution by Mathematica

Time used: 0.006 (sec). Leaf size: 27

```
DSolve[{x1'[t]==-3*x1[t]+2*x2[t],x2'[t]==-1*x1[t]-1*x2[t]},{x1[0]==1,x2[0]==1},{x1[t],x2[t]}
```

$$\text{x1}(t) \to e^{-2t}(\sin(t) + \cos(t))$$
$$\text{x2}(t) \to e^{-2t}\cos(t)$$

16.11 problem 11

Internal problem ID [761]
Internal file name [`OUTPUT/761_Sunday_June_05_2022_01_49_01_AM_57682055/index.tex`]

Book: Elementary differential equations and boundary value problems, 10th ed., Boyce and DiPrima
Section: Chapter 7.6, Complex Eigenvalues. page 417
Problem number: 11.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"system of linear ODEs"**

Solve

$$
\begin{aligned}
x_1'(t) &= \frac{3x_1(t)}{4} - 2x_2(t) \\
x_2'(t) &= x_1(t) - \frac{5x_2(t)}{4}
\end{aligned}
$$

16.11.1 Solution using Matrix exponential method

In this method, we will assume we have found the matrix exponential e^{At} allready. There are different methods to determine this but will not be shown here. This is a system of linear ODE's given as

$$\vec{x}'(t) = A\,\vec{x}(t)$$

Or

$$\begin{bmatrix} x_1'(t) \\ x_2'(t) \end{bmatrix} = \begin{bmatrix} \frac{3}{4} & -2 \\ 1 & -\frac{5}{4} \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}$$

For the above matrix A, the matrix exponential can be found to be

$$
\begin{aligned}
e^{At} &= \begin{bmatrix} e^{-\frac{t}{4}}\cos(t) + e^{-\frac{t}{4}}\sin(t) & -2\,e^{-\frac{t}{4}}\sin(t) \\ e^{-\frac{t}{4}}\sin(t) & e^{-\frac{t}{4}}\cos(t) - e^{-\frac{t}{4}}\sin(t) \end{bmatrix} \\
&= \begin{bmatrix} e^{-\frac{t}{4}}(\cos(t) + \sin(t)) & -2\,e^{-\frac{t}{4}}\sin(t) \\ e^{-\frac{t}{4}}\sin(t) & e^{-\frac{t}{4}}(-\sin(t) + \cos(t)) \end{bmatrix}
\end{aligned}
$$

Therefore the homogeneous solution is

$$
\begin{aligned}
\vec{x}_h(t) &= e^{At}\vec{c} \\
&= \begin{bmatrix} e^{-\frac{t}{4}}(\cos(t)+\sin(t)) & -2\,e^{-\frac{t}{4}}\sin(t) \\ e^{-\frac{t}{4}}\sin(t) & e^{-\frac{t}{4}}(-\sin(t)+\cos(t)) \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} \\
&= \begin{bmatrix} e^{-\frac{t}{4}}(\cos(t)+\sin(t))\,c_1 - 2\,e^{-\frac{t}{4}}\sin(t)\,c_2 \\ e^{-\frac{t}{4}}\sin(t)\,c_1 + e^{-\frac{t}{4}}(-\sin(t)+\cos(t))\,c_2 \end{bmatrix} \\
&= \begin{bmatrix} ((c_1-2c_2)\sin(t)+c_1\cos(t))\,e^{-\frac{t}{4}} \\ ((-c_2+c_1)\sin(t)+c_2\cos(t))\,e^{-\frac{t}{4}} \end{bmatrix}
\end{aligned}
$$

Since no forcing function is given, then the final solution is $\vec{x}_h(t)$ above.

16.11.2 Solution using explicit Eigenvalue and Eigenvector method

This is a system of linear ODE's given as

$$\vec{x}'(t) = A\,\vec{x}(t)$$

Or

$$\begin{bmatrix} x_1'(t) \\ x_2'(t) \end{bmatrix} = \begin{bmatrix} \frac{3}{4} & -2 \\ 1 & -\frac{5}{4} \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}$$

The first step is find the homogeneous solution. We start by finding the eigenvalues of A. This is done by solving the following equation for the eigenvalues λ

$$\det(A-\lambda I) = 0$$

Expanding gives

$$\det\left(\begin{bmatrix} \frac{3}{4} & -2 \\ 1 & -\frac{5}{4} \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right) = 0$$

Therefore

$$\det\left(\begin{bmatrix} \frac{3}{4}-\lambda & -2 \\ 1 & -\frac{5}{4}-\lambda \end{bmatrix}\right) = 0$$

Which gives the characteristic equation

$$\lambda^2 + \frac{1}{2}\lambda + \frac{17}{16} = 0$$

The roots of the above are the eigenvalues.

$$\lambda_1 = -\frac{1}{4} + i$$
$$\lambda_2 = -\frac{1}{4} - i$$

This table summarises the above result

eigenvalue	algebraic multiplicity	type of eigenvalue
$-\frac{1}{4} - i$	1	complex eigenvalue
$-\frac{1}{4} + i$	1	complex eigenvalue

Now the eigenvector for each eigenvalue are found.

Considering the eigenvalue $\lambda_1 = -\frac{1}{4} - i$

We need to solve $A\vec{v} = \lambda\vec{v}$ or $(A - \lambda I)\vec{v} = \vec{0}$ which becomes

$$\left(\begin{bmatrix} \frac{3}{4} & -2 \\ 1 & -\frac{5}{4} \end{bmatrix} - \left(-\frac{1}{4} - i\right)\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right)\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$
$$\begin{bmatrix} 1+i & -2 \\ 1 & -1+i \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Now forward elimination is applied to solve for the eigenvector $\vec{v}$. The augmented matrix is

$$\left[\begin{array}{cc|c} 1+i & -2 & 0 \\ 1 & -1+i & 0 \end{array}\right]$$

$$R_2 = R_2 + \left(-\frac{1}{2} + \frac{i}{2}\right)R_1 \Longrightarrow \left[\begin{array}{cc|c} 1+i & -2 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

Therefore the system in Echelon form is

$$\begin{bmatrix} 1+i & -2 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

The free variables are $\{v_2\}$ and the leading variables are $\{v_1\}$. Let $v_2 = t$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\{v_1 = (1-i)\,t\}$

Hence the solution is

$$\begin{bmatrix} (1-\mathrm{I})\,t \\ t \end{bmatrix} = \begin{bmatrix} (1-i)\,t \\ t \end{bmatrix}$$

Since there is one free Variable, we have found one eigenvector associated with this eigenvalue. The above can be written as

$$\begin{bmatrix} (1-\mathrm{I})\,t \\ t \end{bmatrix} = t \begin{bmatrix} 1-i \\ 1 \end{bmatrix}$$

Let $t = 1$ the eigenvector becomes

$$\begin{bmatrix} (1-\mathrm{I})\,t \\ t \end{bmatrix} = \begin{bmatrix} 1-i \\ 1 \end{bmatrix}$$

Considering the eigenvalue $\lambda_2 = -\frac{1}{4} + i$

We need to solve $A\vec{v} = \lambda\vec{v}$ or $(A - \lambda I)\vec{v} = \vec{0}$ which becomes

$$\left(\begin{bmatrix} \frac{3}{4} & -2 \\ 1 & -\frac{5}{4} \end{bmatrix} - \left(-\frac{1}{4} + i\right) \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right) \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 1-i & -2 \\ 1 & -1-i \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Now forward elimination is applied to solve for the eigenvector $\vec{v}$. The augmented matrix is

$$\left[\begin{array}{cc|c} 1-i & -2 & 0 \\ 1 & -1-i & 0 \end{array}\right]$$

$$R_2 = R_2 + \left(-\frac{1}{2} - \frac{i}{2}\right) R_1 \Longrightarrow \left[\begin{array}{cc|c} 1-i & -2 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

Therefore the system in Echelon form is

$$\begin{bmatrix} 1-i & -2 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

The free variables are $\{v_2\}$ and the leading variables are $\{v_1\}$. Let $v_2 = t$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\{v_1 = (1+i)\,t\}$

Hence the solution is

$$\begin{bmatrix} (1+\mathrm{I})\,t \\ t \end{bmatrix} = \begin{bmatrix} (1+i)\,t \\ t \end{bmatrix}$$

Since there is one free Variable, we have found one eigenvector associated with this eigenvalue. The above can be written as

$$\begin{bmatrix} (1+\mathrm{I})\,t \\ t \end{bmatrix} = t \begin{bmatrix} 1+i \\ 1 \end{bmatrix}$$

Let $t = 1$ the eigenvector becomes

$$\begin{bmatrix} (1+\mathrm{I})\,t \\ t \end{bmatrix} = \begin{bmatrix} 1+i \\ 1 \end{bmatrix}$$

The following table gives a summary of this result. It shows for each eigenvalue the algebraic multiplicity m, and its geometric multiplicity k and the eigenvectors associated with the eigenvalue. If $m > k$ then the eigenvalue is defective which means the number of normal linearly independent eigenvectors associated with this eigenvalue (called the geometric multiplicity k) does not equal the algebraic multiplicity m, and we need to determine an additional $m - k$ generalized eigenvectors for this eigenvalue.

	multiplicity			
eigenvalue	algebraic m	geometric k	defective?	eigenvectors
$-\frac{1}{4}+i$	1	1	No	$\begin{bmatrix} 1+i \\ 1 \end{bmatrix}$
$-\frac{1}{4}-i$	1	1	No	$\begin{bmatrix} 1-i \\ 1 \end{bmatrix}$

Now that we found the eigenvalues and associated eigenvectors, we will go over each eigenvalue and generate the solution basis. The only problem we need to take care of is if the eigenvalue is defective. Therefore the final solution is

$$\vec{x}_h(t) = c_1\vec{x}_1(t) + c_2\vec{x}_2(t)$$

Which is written as

$$\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = c_1 \begin{bmatrix} (1+i)\, e^{\left(-\frac{1}{4}+i\right)t} \\ e^{\left(-\frac{1}{4}+i\right)t} \end{bmatrix} + c_2 \begin{bmatrix} (1-i)\, e^{\left(-\frac{1}{4}-i\right)t} \\ e^{\left(-\frac{1}{4}-i\right)t} \end{bmatrix}$$

Which becomes

$$\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = \begin{bmatrix} (1+i)\, c_1 e^{\left(-\frac{1}{4}+i\right)t} + (1-i)\, c_2 e^{\left(-\frac{1}{4}-i\right)t} \\ c_1 e^{\left(-\frac{1}{4}+i\right)t} + c_2 e^{\left(-\frac{1}{4}-i\right)t} \end{bmatrix}$$

The following is the phase plot of the system.

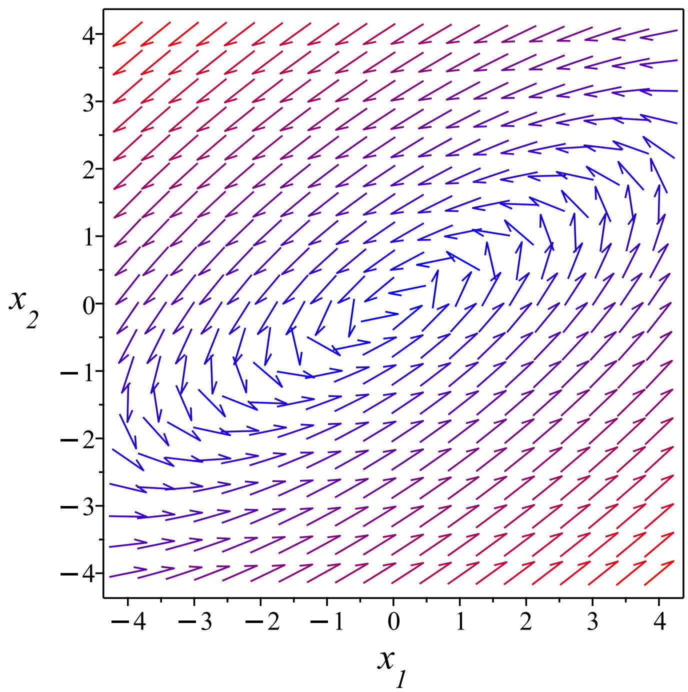


Figure 515: Phase plot

16.11.3 Maple step by step solution

Let's solve

$\left[x_1'(t) = \frac{3x_1(t)}{4} - 2x_2(t)\,, x_2'(t) = x_1(t) - \frac{5x_2(t)}{4}\right]$

- Define vector

$$\underline{x}^{\rightarrow}(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}$$

- Convert system into a vector equation

$$\underline{x}^{\rightarrow}{}'(t) = \begin{bmatrix} \frac{3}{4} & -2 \\ 1 & -\frac{5}{4} \end{bmatrix} \cdot \underline{x}^{\rightarrow}(t) + \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

- System to solve

$$\underline{x}^{\rightarrow}{}'(t) = \begin{bmatrix} \frac{3}{4} & -2 \\ 1 & -\frac{5}{4} \end{bmatrix} \cdot \underline{x}^{\rightarrow}(t)$$

- Define the coefficient matrix

$$A = \begin{bmatrix} \frac{3}{4} & -2 \\ 1 & -\frac{5}{4} \end{bmatrix}$$

- Rewrite the system as

$$\underline{x}^{\rightarrow}{}'(t) = A \cdot \underline{x}^{\rightarrow}(t)$$

- To solve the system, find the eigenvalues and eigenvectors of A
- Eigenpairs of A

$$\left[\left[-\frac{1}{4} - \mathrm{I}, \begin{bmatrix} 1 - \mathrm{I} \\ 1 \end{bmatrix}\right], \left[-\frac{1}{4} + \mathrm{I}, \begin{bmatrix} 1 + \mathrm{I} \\ 1 \end{bmatrix}\right]\right]$$

- Consider complex eigenpair, complex conjugate eigenvalue can be ignored

$$\left[-\frac{1}{4} - \mathrm{I}, \begin{bmatrix} 1 - \mathrm{I} \\ 1 \end{bmatrix}\right]$$

- Solution from eigenpair

$$\mathrm{e}^{\left(-\frac{1}{4} - \mathrm{I}\right)t} \cdot \begin{bmatrix} 1 - \mathrm{I} \\ 1 \end{bmatrix}$$

- Use Euler identity to write solution in terms of $\sin$ and $\cos$

$$\mathrm{e}^{-\frac{t}{4}} \cdot (\cos(t) - \mathrm{I}\sin(t)) \cdot \begin{bmatrix} 1 - \mathrm{I} \\ 1 \end{bmatrix}$$

- Simplify expression

$$e^{-\frac{t}{4}} \cdot \left[\begin{array}{c} (1-\mathrm{I})(\cos(t)-\mathrm{I}\sin(t)) \\ \cos(t)-\mathrm{I}\sin(t) \end{array}\right]$$

- Both real and imaginary parts are solutions to the homogeneous system

$$\left[\underline{\overrightarrow{x}}_{1}(t) = e^{-\frac{t}{4}} \cdot \left[\begin{array}{c} -\sin(t)+\cos(t) \\ \cos(t) \end{array}\right], \underline{\overrightarrow{x}}_{2}(t) = e^{-\frac{t}{4}} \cdot \left[\begin{array}{c} -\cos(t)-\sin(t) \\ -\sin(t) \end{array}\right]\right]$$

- General solution to the system of ODEs

$$\underline{\overrightarrow{x}} = c_1 \underline{\overrightarrow{x}}_{1}(t) + c_2 \underline{\overrightarrow{x}}_{2}(t)$$

- Substitute solutions into the general solution

$$\underline{\overrightarrow{x}} = c_1 e^{-\frac{t}{4}} \cdot \left[\begin{array}{c} -\sin(t)+\cos(t) \\ \cos(t) \end{array}\right] + c_2 e^{-\frac{t}{4}} \cdot \left[\begin{array}{c} -\cos(t)-\sin(t) \\ -\sin(t) \end{array}\right]$$

- Substitute in vector of dependent variables

$$\left[\begin{array}{c} x_1(t) \\ x_2(t) \end{array}\right] = \left[\begin{array}{c} e^{-\frac{t}{4}}(\cos(t)(c_1-c_2)-\sin(t)(c_1+c_2)) \\ e^{-\frac{t}{4}}(-c_2\sin(t)+c_1\cos(t)) \end{array}\right]$$

- Solution to the system of ODEs

$$\left\{x_1(t) = e^{-\frac{t}{4}}(\cos(t)(c_1-c_2)-\sin(t)(c_1+c_2)), x_2(t) = e^{-\frac{t}{4}}(-c_2\sin(t)+c_1\cos(t))\right\}$$

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 46

```
dsolve([diff(x__1(t),t)=3/4*x__1(t)-2*x__2(t),diff(x__2(t),t)=1*x__1(t)-5/4*x__2(t)],singsol
```

$$x_1(t) = e^{-\frac{t}{4}}(c_1\sin(t)+c_2\cos(t))$$
$$x_2(t) = \frac{e^{-\frac{t}{4}}(c_1\sin(t)+c_2\sin(t)-c_1\cos(t)+c_2\cos(t))}{2}$$

✓ Solution by Mathematica

Time used: 0.005 (sec). Leaf size: 56

```
DSolve[{x1'[t]==3/4*x1[t]-2*x2[t],x2'[t]==1*x1[t]-5/4*x2[t]},{x1[t],x2[t]},t,IncludeSingular
```

$$\mathrm{x1}(t) \to e^{-t/4}(c_1\cos(t)+(c_1-2c_2)\sin(t))$$
$$\mathrm{x2}(t) \to e^{-t/4}(c_2\cos(t)+(c_1-c_2)\sin(t))$$

16.12 problem 12

Internal problem ID [762]
Internal file name [`OUTPUT/762_Sunday_June_05_2022_01_49_02_AM_3546516/index.tex`]

Book: Elementary differential equations and boundary value problems, 10th ed., Boyce and DiPrima
Section: Chapter 7.6, Complex Eigenvalues. page 417
Problem number: 12.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"system of linear ODEs"**

Solve

$$x_1'(t) = -\frac{4x_1(t)}{5} + 2x_2(t)$$
$$x_2'(t) = -x_1(t) + \frac{6x_2(t)}{5}$$

16.12.1 Solution using Matrix exponential method

In this method, we will assume we have found the matrix exponential e^{At} allready. There are different methods to determine this but will not be shown here. This is a system of linear ODE's given as

$$\vec{x}'(t) = A\,\vec{x}(t)$$

Or

$$\begin{bmatrix} x_1'(t) \\ x_2'(t) \end{bmatrix} = \begin{bmatrix} -\frac{4}{5} & 2 \\ -1 & \frac{6}{5} \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}$$

For the above matrix A, the matrix exponential can be found to be

$$\begin{aligned} e^{At} &= \begin{bmatrix} e^{\frac{t}{5}} \cos(t) - e^{\frac{t}{5}} \sin(t) & 2\, e^{\frac{t}{5}} \sin(t) \\ -e^{\frac{t}{5}} \sin(t) & e^{\frac{t}{5}} \cos(t) + e^{\frac{t}{5}} \sin(t) \end{bmatrix} \\ &= \begin{bmatrix} e^{\frac{t}{5}} (-\sin(t) + \cos(t)) & 2\, e^{\frac{t}{5}} \sin(t) \\ -e^{\frac{t}{5}} \sin(t) & e^{\frac{t}{5}} (\cos(t) + \sin(t)) \end{bmatrix} \end{aligned}$$

Therefore the homogeneous solution is

$$
\begin{aligned}
\vec{x}_h(t) &= e^{At}\vec{c} \\
&= \begin{bmatrix} e^{\frac{t}{5}}(-\sin(t)+\cos(t)) & 2\,e^{\frac{t}{5}}\sin(t) \\ -e^{\frac{t}{5}}\sin(t) & e^{\frac{t}{5}}(\cos(t)+\sin(t)) \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} \\
&= \begin{bmatrix} e^{\frac{t}{5}}(-\sin(t)+\cos(t))\,c_1 + 2\,e^{\frac{t}{5}}\sin(t)\,c_2 \\ -e^{\frac{t}{5}}\sin(t)\,c_1 + e^{\frac{t}{5}}(\cos(t)+\sin(t))\,c_2 \end{bmatrix} \\
&= \begin{bmatrix} ((-c_1+2c_2)\sin(t)+c_1\cos(t))\,e^{\frac{t}{5}} \\ -((-c_2+c_1)\sin(t)-c_2\cos(t))\,e^{\frac{t}{5}} \end{bmatrix}
\end{aligned}
$$

Since no forcing function is given, then the final solution is $\vec{x}_h(t)$ above.

16.12.2 Solution using explicit Eigenvalue and Eigenvector method

This is a system of linear ODE's given as

$$\vec{x}'(t) = A\,\vec{x}(t)$$

Or

$$\begin{bmatrix} x_1'(t) \\ x_2'(t) \end{bmatrix} = \begin{bmatrix} -\frac{4}{5} & 2 \\ -1 & \frac{6}{5} \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}$$

The first step is find the homogeneous solution. We start by finding the eigenvalues of A. This is done by solving the following equation for the eigenvalues λ

$$\det(A-\lambda I) = 0$$

Expanding gives

$$\det\left(\begin{bmatrix} -\frac{4}{5} & 2 \\ -1 & \frac{6}{5} \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right) = 0$$

Therefore

$$\det\left(\begin{bmatrix} -\frac{4}{5}-\lambda & 2 \\ -1 & \frac{6}{5}-\lambda \end{bmatrix}\right) = 0$$

Which gives the characteristic equation

$$\lambda^2 - \frac{2}{5}\lambda + \frac{26}{25} = 0$$

The roots of the above are the eigenvalues.

$$\lambda_1 = \frac{1}{5} + i$$
$$\lambda_2 = \frac{1}{5} - i$$

This table summarises the above result

eigenvalue	algebraic multiplicity	type of eigenvalue
$\frac{1}{5} - i$	1	complex eigenvalue
$\frac{1}{5} + i$	1	complex eigenvalue

Now the eigenvector for each eigenvalue are found.

Considering the eigenvalue $\lambda_1 = \frac{1}{5} - i$

We need to solve $A\vec{v} = \lambda\vec{v}$ or $(A - \lambda I)\vec{v} = \vec{0}$ which becomes

$$\left(\begin{bmatrix} -\frac{4}{5} & 2 \\ -1 & \frac{6}{5} \end{bmatrix} - \left(\frac{1}{5} - i\right)\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right)\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$
$$\begin{bmatrix} -1+i & 2 \\ -1 & 1+i \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Now forward elimination is applied to solve for the eigenvector $\vec{v}$. The augmented matrix is

$$\left[\begin{array}{cc|c} -1+i & 2 & 0 \\ -1 & 1+i & 0 \end{array}\right]$$

$$R_2 = R_2 + \left(-\frac{1}{2} - \frac{i}{2}\right)R_1 \Longrightarrow \left[\begin{array}{cc|c} -1+i & 2 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

Therefore the system in Echelon form is

$$\begin{bmatrix} -1+i & 2 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

The free variables are $\{v_2\}$ and the leading variables are $\{v_1\}$. Let $v_2 = t$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\{v_1 = (1+i)\,t\}$

Hence the solution is

$$\begin{bmatrix} (1+\mathrm{I})\,t \\ t \end{bmatrix} = \begin{bmatrix} (1+i)\,t \\ t \end{bmatrix}$$

Since there is one free Variable, we have found one eigenvector associated with this eigenvalue. The above can be written as

$$\begin{bmatrix} (1+\mathrm{I})\,t \\ t \end{bmatrix} = t \begin{bmatrix} 1+i \\ 1 \end{bmatrix}$$

Let $t = 1$ the eigenvector becomes

$$\begin{bmatrix} (1+\mathrm{I})\,t \\ t \end{bmatrix} = \begin{bmatrix} 1+i \\ 1 \end{bmatrix}$$

Considering the eigenvalue $\lambda_2 = \frac{1}{5} + i$

We need to solve $A\vec{v} = \lambda\vec{v}$ or $(A - \lambda I)\vec{v} = \vec{0}$ which becomes

$$\left(\begin{bmatrix} -\frac{4}{5} & 2 \\ -1 & \frac{6}{5} \end{bmatrix} - \left(\frac{1}{5} + i\right) \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right) \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} -1-i & 2 \\ -1 & 1-i \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Now forward elimination is applied to solve for the eigenvector $\vec{v}$. The augmented matrix is

$$\left[\begin{array}{cc|c} -1-i & 2 & 0 \\ -1 & 1-i & 0 \end{array}\right]$$

$$R_2 = R_2 + \left(-\frac{1}{2} + \frac{i}{2}\right) R_1 \Longrightarrow \left[\begin{array}{cc|c} -1-i & 2 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

Therefore the system in Echelon form is

$$\begin{bmatrix} -1-i & 2 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

The free variables are $\{v_2\}$ and the leading variables are $\{v_1\}$. Let $v_2 = t$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\{v_1 = (1-i)\,t\}$

Hence the solution is

$$\begin{bmatrix} (1-\mathrm{I})\,t \\ t \end{bmatrix} = \begin{bmatrix} (1-i)\,t \\ t \end{bmatrix}$$

Since there is one free Variable, we have found one eigenvector associated with this eigenvalue. The above can be written as

$$\begin{bmatrix} (1-\mathrm{I})\,t \\ t \end{bmatrix} = t \begin{bmatrix} 1-i \\ 1 \end{bmatrix}$$

Let $t = 1$ the eigenvector becomes

$$\begin{bmatrix} (1-\mathrm{I})\,t \\ t \end{bmatrix} = \begin{bmatrix} 1-i \\ 1 \end{bmatrix}$$

The following table gives a summary of this result. It shows for each eigenvalue the algebraic multiplicity m, and its geometric multiplicity k and the eigenvectors associated with the eigenvalue. If $m > k$ then the eigenvalue is defective which means the number of normal linearly independent eigenvectors associated with this eigenvalue (called the geometric multiplicity k) does not equal the algebraic multiplicity m, and we need to determine an additional $m - k$ generalized eigenvectors for this eigenvalue.

	multiplicity			
eigenvalue	algebraic m	geometric k	defective?	eigenvectors
$\frac{1}{5} + i$	1	1	No	$\begin{bmatrix} 1-i \\ 1 \end{bmatrix}$
$\frac{1}{5} - i$	1	1	No	$\begin{bmatrix} 1+i \\ 1 \end{bmatrix}$

Now that we found the eigenvalues and associated eigenvectors, we will go over each eigenvalue and generate the solution basis. The only problem we need to take care of is if the eigenvalue is defective. Therefore the final solution is

$$\vec{x}_h(t) = c_1\vec{x}_1(t) + c_2\vec{x}_2(t)$$

Which is written as

$$\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = c_1 \begin{bmatrix} (1-i)\,e^{\left(\frac{1}{5}+i\right)t} \\ e^{\left(\frac{1}{5}+i\right)t} \end{bmatrix} + c_2 \begin{bmatrix} (1+i)\,e^{\left(\frac{1}{5}-i\right)t} \\ e^{\left(\frac{1}{5}-i\right)t} \end{bmatrix}$$

Which becomes

$$\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = \begin{bmatrix} (1-i)\,c_1 e^{\left(\frac{1}{5}+i\right)t} + (1+i)\,c_2 e^{\left(\frac{1}{5}-i\right)t} \\ c_1 e^{\left(\frac{1}{5}+i\right)t} + c_2 e^{\left(\frac{1}{5}-i\right)t} \end{bmatrix}$$

The following is the phase plot of the system.

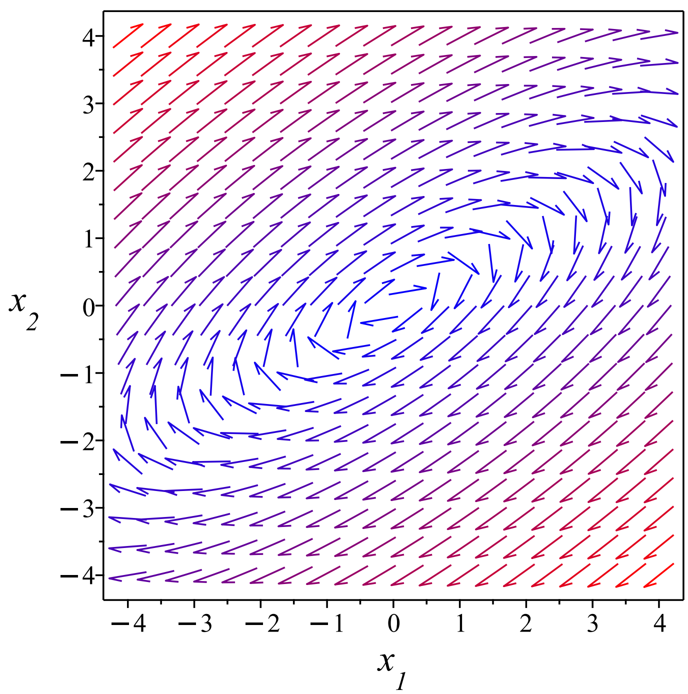


Figure 516: Phase plot

16.12.3 Maple step by step solution

Let's solve

$$\left[x_1'(t) = -\frac{4x_1(t)}{5} + 2x_2(t)\,,\, x_2'(t) = -x_1(t) + \frac{6x_2(t)}{5}\right]$$

- Define vector

$$\underline{x}^{\rightarrow}(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}$$

- Convert system into a vector equation

$$\underline{x}^{\rightarrow}{}'(t) = \begin{bmatrix} -\frac{4}{5} & 2 \\ -1 & \frac{6}{5} \end{bmatrix} \cdot \underline{x}^{\rightarrow}(t) + \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

- System to solve

$$\underline{x}^{\rightarrow}{}'(t) = \begin{bmatrix} -\frac{4}{5} & 2 \\ -1 & \frac{6}{5} \end{bmatrix} \cdot \underline{x}^{\rightarrow}(t)$$

- Define the coefficient matrix

$$A = \begin{bmatrix} -\frac{4}{5} & 2 \\ -1 & \frac{6}{5} \end{bmatrix}$$

- Rewrite the system as

$$\underline{x}^{\rightarrow}{}'(t) = A \cdot \underline{x}^{\rightarrow}(t)$$

- To solve the system, find the eigenvalues and eigenvectors of A
- Eigenpairs of A

$$\left[\left[\frac{1}{5} - \mathrm{I}, \begin{bmatrix} 1+\mathrm{I} \\ 1 \end{bmatrix}\right], \left[\frac{1}{5} + \mathrm{I}, \begin{bmatrix} 1-\mathrm{I} \\ 1 \end{bmatrix}\right]\right]$$

- Consider complex eigenpair, complex conjugate eigenvalue can be ignored

$$\left[\frac{1}{5} - \mathrm{I}, \begin{bmatrix} 1+\mathrm{I} \\ 1 \end{bmatrix}\right]$$

- Solution from eigenpair

$$\mathrm{e}^{\left(\frac{1}{5}-\mathrm{I}\right)t} \cdot \begin{bmatrix} 1+\mathrm{I} \\ 1 \end{bmatrix}$$

- Use Euler identity to write solution in terms of $\sin$ and $\cos$

$$\mathrm{e}^{\frac{t}{5}} \cdot (\cos(t) - \mathrm{I}\sin(t)) \cdot \begin{bmatrix} 1+\mathrm{I} \\ 1 \end{bmatrix}$$

- Simplify expression

$$e^{\frac{t}{5}} \cdot \begin{bmatrix} (1+I)(\cos(t) - I\sin(t)) \\ \cos(t) - I\sin(t) \end{bmatrix}$$

- Both real and imaginary parts are solutions to the homogeneous system

$$\left[\underline{x}^{\rightarrow}{}_1(t) = e^{\frac{t}{5}} \cdot \begin{bmatrix} \cos(t) + \sin(t) \\ \cos(t) \end{bmatrix}, \underline{x}^{\rightarrow}{}_2(t) = e^{\frac{t}{5}} \cdot \begin{bmatrix} -\sin(t) + \cos(t) \\ -\sin(t) \end{bmatrix}\right]$$

- General solution to the system of ODEs

$$\underline{x}^{\rightarrow} = c_1 \underline{x}^{\rightarrow}{}_1(t) + c_2 \underline{x}^{\rightarrow}{}_2(t)$$

- Substitute solutions into the general solution

$$\underline{x}^{\rightarrow} = c_1 e^{\frac{t}{5}} \cdot \begin{bmatrix} \cos(t) + \sin(t) \\ \cos(t) \end{bmatrix} + c_2 e^{\frac{t}{5}} \cdot \begin{bmatrix} -\sin(t) + \cos(t) \\ -\sin(t) \end{bmatrix}$$

- Substitute in vector of dependent variables

$$\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = \begin{bmatrix} (\cos(t)(c_1 + c_2) + (c_1 - c_2)\sin(t)) e^{\frac{t}{5}} \\ e^{\frac{t}{5}}(-c_2 \sin(t) + c_1 \cos(t)) \end{bmatrix}$$

- Solution to the system of ODEs

$$\left\{x_1(t) = (\cos(t)(c_1 + c_2) + (c_1 - c_2)\sin(t)) e^{\frac{t}{5}}, x_2(t) = e^{\frac{t}{5}}(-c_2 \sin(t) + c_1 \cos(t))\right\}$$

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 46

```
dsolve([diff(x__1(t),t)=-4/5*x__1(t)+2*x__2(t),diff(x__2(t),t)=-1*x__1(t)+6/5*x__2(t)],sings
```

$$x_1(t) = e^{\frac{t}{5}}(c_1 \sin(t) + c_2 \cos(t))$$
$$x_2(t) = \frac{e^{\frac{t}{5}}(c_1 \sin(t) - c_2 \sin(t) + c_1 \cos(t) + c_2 \cos(t))}{2}$$

✓ Solution by Mathematica

Time used: 0.005 (sec). Leaf size: 56

```
DSolve[{x1'[t]==-4/5*x1[t]+2*x2[t],x2'[t]==-1*x1[t]+6/5*x2[t]},{x1[t],x2[t]},t,IncludeSingul
```

$$\text{x1}(t) \rightarrow e^{t/5}(c_1 \cos(t) - (c_1 - 2c_2)\sin(t))$$
$$\text{x2}(t) \rightarrow e^{t/5}(c_2(\sin(t) + \cos(t)) - c_1 \sin(t))$$

16.13 problem 23

Internal problem ID [763]
Internal file name [`OUTPUT/763_Sunday_June_05_2022_01_49_04_AM_12891500/index.tex`]

Book: Elementary differential equations and boundary value problems, 10th ed., Boyce and DiPrima
Section: Chapter 7.6, Complex Eigenvalues. page 417
Problem number: 23.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"system of linear ODEs"**

Solve

$$
\begin{aligned}
x_1'(t) &= -\frac{x_1(t)}{4} + x_2(t) \\
x_2'(t) &= -x_1(t) - \frac{x_2(t)}{4} \\
x_3'(t) &= -\frac{x_3(t)}{4}
\end{aligned}
$$

16.13.1 Solution using Matrix exponential method

In this method, we will assume we have found the matrix exponential e^{At} allready. There are different methods to determine this but will not be shown here. This is a system of linear ODE's given as

$$\vec{x}'(t) = A\,\vec{x}(t)$$

Or

$$\begin{bmatrix} x_1'(t) \\ x_2'(t) \\ x_3'(t) \end{bmatrix} = \begin{bmatrix} -\frac{1}{4} & 1 & 0 \\ -1 & -\frac{1}{4} & 0 \\ 0 & 0 & -\frac{1}{4} \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{bmatrix}$$

For the above matrix A, the matrix exponential can be found to be

$$e^{At} = \begin{bmatrix} e^{-\frac{t}{4}}\cos(t) & e^{-\frac{t}{4}}\sin(t) & 0 \\ -e^{-\frac{t}{4}}\sin(t) & e^{-\frac{t}{4}}\cos(t) & 0 \\ 0 & 0 & e^{-\frac{t}{4}} \end{bmatrix}$$

Therefore the homogeneous solution is

$$\begin{aligned} \vec{x}_h(t) &= e^{At}\vec{c} \\ &= \begin{bmatrix} e^{-\frac{t}{4}}\cos(t) & e^{-\frac{t}{4}}\sin(t) & 0 \\ -e^{-\frac{t}{4}}\sin(t) & e^{-\frac{t}{4}}\cos(t) & 0 \\ 0 & 0 & e^{-\frac{t}{4}} \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} \\ &= \begin{bmatrix} e^{-\frac{t}{4}}\cos(t)\,c_1 + e^{-\frac{t}{4}}\sin(t)\,c_2 \\ -e^{-\frac{t}{4}}\sin(t)\,c_1 + e^{-\frac{t}{4}}\cos(t)\,c_2 \\ e^{-\frac{t}{4}}c_3 \end{bmatrix} \\ &= \begin{bmatrix} e^{-\frac{t}{4}}(\cos(t)\,c_1 + \sin(t)\,c_2) \\ e^{-\frac{t}{4}}(-\sin(t)\,c_1 + \cos(t)\,c_2) \\ e^{-\frac{t}{4}}c_3 \end{bmatrix} \end{aligned}$$

Since no forcing function is given, then the final solution is $\vec{x}_h(t)$ above.

16.13.2 Solution using explicit Eigenvalue and Eigenvector method

This is a system of linear ODE's given as

$$\vec{x}'(t) = A\,\vec{x}(t)$$

Or

$$\begin{bmatrix} x_1'(t) \\ x_2'(t) \\ x_3'(t) \end{bmatrix} = \begin{bmatrix} -\frac{1}{4} & 1 & 0 \\ -1 & -\frac{1}{4} & 0 \\ 0 & 0 & -\frac{1}{4} \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{bmatrix}$$

The first step is find the homogeneous solution. We start by finding the eigenvalues of A. This is done by solving the following equation for the eigenvalues λ

$$\det(A - \lambda I) = 0$$

Expanding gives

$$\det\left(\begin{bmatrix} -\frac{1}{4} & 1 & 0 \\ -1 & -\frac{1}{4} & 0 \\ 0 & 0 & -\frac{1}{4} \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\right) = 0$$

Therefore

$$\det\left(\begin{bmatrix} -\frac{1}{4} - \lambda & 1 & 0 \\ -1 & -\frac{1}{4} - \lambda & 0 \\ 0 & 0 & -\frac{1}{4} - \lambda \end{bmatrix}\right) = 0$$

Which gives the characteristic equation

$$\lambda^3 + \frac{3}{4}\lambda^2 + \frac{19}{16}\lambda + \frac{17}{64} = 0$$

The roots of the above are the eigenvalues.

$$\lambda_1 = -\frac{1}{4} + i$$
$$\lambda_2 = -\frac{1}{4} - i$$
$$\lambda_3 = -\frac{1}{4}$$

This table summarises the above result

eigenvalue	algebraic multiplicity	type of eigenvalue
$-\frac{1}{4} - i$	1	complex eigenvalue
$-\frac{1}{4}$	1	real eigenvalue
$-\frac{1}{4} + i$	1	complex eigenvalue

Now the eigenvector for each eigenvalue are found.

Considering the eigenvalue $\lambda_1 = -\frac{1}{4}$

We need to solve $A\vec{v} = \lambda\vec{v}$ or $(A - \lambda I)\vec{v} = \vec{0}$ which becomes

$$\left(\begin{bmatrix} -\frac{1}{4} & 1 & 0 \\ -1 & -\frac{1}{4} & 0 \\ 0 & 0 & -\frac{1}{4} \end{bmatrix} - \left(-\frac{1}{4}\right)\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\right)\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Now forward elimination is applied to solve for the eigenvector $\vec{v}$. The augmented matrix is

$$\left[\begin{array}{ccc|c} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

Since the current pivot $A(1,1)$ is zero, then the current pivot row is replaced with a row with a non-zero pivot. Swapping row 1 and row 2 gives

$$\left[\begin{array}{ccc|c} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

Therefore the system in Echelon form is

$$\begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

The free variables are $\{v_3\}$ and the leading variables are $\{v_1, v_2\}$. Let $v_3 = t$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\{v_1 = 0, v_2 = 0\}$

Hence the solution is

$$\begin{bmatrix} 0 \\ 0 \\ t \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ t \end{bmatrix}$$

Since there is one free Variable, we have found one eigenvector associated with this eigenvalue. The above can be written as

$$\begin{bmatrix} 0 \\ 0 \\ t \end{bmatrix} = t \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

Let $t = 1$ the eigenvector becomes

$$\begin{bmatrix} 0 \\ 0 \\ t \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

Considering the eigenvalue $\lambda_2 = -\frac{1}{4} - i$

We need to solve $A\vec{v} = \lambda\vec{v}$ or $(A - \lambda I)\vec{v} = \vec{0}$ which becomes

$$\left(\begin{bmatrix} -\frac{1}{4} & 1 & 0 \\ -1 & -\frac{1}{4} & 0 \\ 0 & 0 & -\frac{1}{4} \end{bmatrix} - \left(-\frac{1}{4} - i\right) \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \right) \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} i & 1 & 0 \\ -1 & i & 0 \\ 0 & 0 & i \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Now forward elimination is applied to solve for the eigenvector $\vec{v}$. The augmented matrix is

$$\left[\begin{array}{ccc|c} i & 1 & 0 & 0 \\ -1 & i & 0 & 0 \\ 0 & 0 & i & 0 \end{array}\right]$$

$$R_2 = -iR_1 + R_2 \Longrightarrow \left[\begin{array}{ccc|c} i & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & i & 0 \end{array}\right]$$

Since the current pivot $A(2,3)$ is zero, then the current pivot row is replaced with a row with a non-zero pivot. Swapping row 2 and row 3 gives

$$\left[\begin{array}{ccc|c} i & 1 & 0 & 0 \\ 0 & 0 & i & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

Therefore the system in Echelon form is

$$\begin{bmatrix} i & 1 & 0 \\ 0 & 0 & i \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

The free variables are $\{v_2\}$ and the leading variables are $\{v_1, v_3\}$. Let $v_2 = t$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\{v_1 = it, v_3 = 0\}$

Hence the solution is

$$\begin{bmatrix} \mathrm{I}t \\ t \\ 0 \end{bmatrix} = \begin{bmatrix} it \\ t \\ 0 \end{bmatrix}$$

Since there is one free Variable, we have found one eigenvector associated with this eigenvalue. The above can be written as

$$\begin{bmatrix} \mathrm{I}t \\ t \\ 0 \end{bmatrix} = t \begin{bmatrix} i \\ 1 \\ 0 \end{bmatrix}$$

Let $t = 1$ the eigenvector becomes

$$\begin{bmatrix} \mathrm{I}t \\ t \\ 0 \end{bmatrix} = \begin{bmatrix} i \\ 1 \\ 0 \end{bmatrix}$$

Considering the eigenvalue $\lambda_3 = -\frac{1}{4} + i$

We need to solve $A\vec{v} = \lambda\vec{v}$ or $(A - \lambda I)\vec{v} = \vec{0}$ which becomes

$$\left(\begin{bmatrix} -\frac{1}{4} & 1 & 0 \\ -1 & -\frac{1}{4} & 0 \\ 0 & 0 & -\frac{1}{4} \end{bmatrix} - \left(-\frac{1}{4} + i\right) \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \right) \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} -i & 1 & 0 \\ -1 & -i & 0 \\ 0 & 0 & -i \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Now forward elimination is applied to solve for the eigenvector $\vec{v}$. The augmented matrix is

$$\left[\begin{array}{ccc|c} -i & 1 & 0 & 0 \\ -1 & -i & 0 & 0 \\ 0 & 0 & -i & 0 \end{array}\right]$$

$$R_2 = iR_1 + R_2 \Longrightarrow \left[\begin{array}{ccc|c} -i & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -i & 0 \end{array}\right]$$

Since the current pivot $A(2,3)$ is zero, then the current pivot row is replaced with a row with a non-zero pivot. Swapping row 2 and row 3 gives

$$\left[\begin{array}{ccc|c} -i & 1 & 0 & 0 \\ 0 & 0 & -i & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

Therefore the system in Echelon form is

$$\begin{bmatrix} -i & 1 & 0 \\ 0 & 0 & -i \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

The free variables are $\{v_2\}$ and the leading variables are $\{v_1, v_3\}$. Let $v_2 = t$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\{v_1 = -it, v_3 = 0\}$

Hence the solution is

$$\begin{bmatrix} \text{-I}\,t \\ t \\ 0 \end{bmatrix} = \begin{bmatrix} -it \\ t \\ 0 \end{bmatrix}$$

Since there is one free Variable, we have found one eigenvector associated with this eigenvalue. The above can be written as

$$\begin{bmatrix} \text{-I}\,t \\ t \\ 0 \end{bmatrix} = t \begin{bmatrix} -i \\ 1 \\ 0 \end{bmatrix}$$

Let $t = 1$ the eigenvector becomes

$$\begin{bmatrix} \text{-I}\,t \\ t \\ 0 \end{bmatrix} = \begin{bmatrix} -i \\ 1 \\ 0 \end{bmatrix}$$

The following table gives a summary of this result. It shows for each eigenvalue the algebraic multiplicity m, and its geometric multiplicity k and the eigenvectors associated with the eigenvalue. If $m > k$ then the eigenvalue is defective which means the number of normal linearly independent eigenvectors associated with this eigenvalue (called the geometric multiplicity k) does not equal the algebraic multiplicity m, and we need to determine an additional $m - k$ generalized eigenvectors for this eigenvalue.

	multiplicity			
eigenvalue	algebraic m	geometric k	defective?	eigenvectors
$-\frac{1}{4}+i$	1	1	No	$\begin{bmatrix} -i \\ 1 \\ 0 \end{bmatrix}$
$-\frac{1}{4}-i$	1	1	No	$\begin{bmatrix} i \\ 1 \\ 0 \end{bmatrix}$
$-\frac{1}{4}$	1	1	No	$\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$

Now that we found the eigenvalues and associated eigenvectors, we will go over each eigenvalue and generate the solution basis. The only problem we need to take care of is if the eigenvalue is defective. Since eigenvalue $-\frac{1}{4}$ is real and distinct then the corresponding eigenvector solution is

$$\begin{aligned}\vec{x}_1(t) &= \vec{v}_1 e^{-\frac{t}{4}}\\ &= \begin{bmatrix}0\\0\\1\end{bmatrix} e^{-\frac{t}{4}}\end{aligned}$$

Therefore the final solution is

$$\vec{x}_h(t) = c_1\vec{x}_1(t) + c_2\vec{x}_2(t) + c_3\vec{x}_3(t)$$

Which is written as

$$\begin{bmatrix}x_1(t)\\x_2(t)\\x_3(t)\end{bmatrix} = c_1\begin{bmatrix}-i\mathrm{e}^{\left(-\frac{1}{4}+i\right)t}\\ \mathrm{e}^{\left(-\frac{1}{4}+i\right)t}\\ 0\end{bmatrix} + c_2\begin{bmatrix}i\mathrm{e}^{\left(-\frac{1}{4}-i\right)t}\\ \mathrm{e}^{\left(-\frac{1}{4}-i\right)t}\\ 0\end{bmatrix} + c_3\begin{bmatrix}0\\0\\ \mathrm{e}^{-\frac{t}{4}}\end{bmatrix}$$

Which becomes

$$\begin{bmatrix}x_1(t)\\x_2(t)\\x_3(t)\end{bmatrix} = \begin{bmatrix}-i\left(c_1\mathrm{e}^{\left(-\frac{1}{4}+i\right)t} - c_2\mathrm{e}^{\left(-\frac{1}{4}-i\right)t}\right)\\ c_1\mathrm{e}^{\left(-\frac{1}{4}+i\right)t} + c_2\mathrm{e}^{\left(-\frac{1}{4}-i\right)t}\\ c_3\mathrm{e}^{-\frac{t}{4}}\end{bmatrix}$$

16.13.3 Maple step by step solution

Let's solve

$\left[x_1'(t) = -\frac{x_1(t)}{4} + x_2(t), x_2'(t) = -x_1(t) - \frac{x_2(t)}{4}, x_3'(t) = -\frac{x_3(t)}{4}\right]$

- Define vector

$$\underline{x}^{\rightarrow}(t) = \begin{bmatrix}x_1(t)\\x_2(t)\\x_3(t)\end{bmatrix}$$

- Convert system into a vector equation

$$\underline{\vec{x}}'(t) = \begin{bmatrix} -\frac{1}{4} & 1 & 0 \\ -1 & -\frac{1}{4} & 0 \\ 0 & 0 & -\frac{1}{4} \end{bmatrix} \cdot \underline{\vec{x}}(t) + \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

- System to solve

$$\underline{\vec{x}}'(t) = \begin{bmatrix} -\frac{1}{4} & 1 & 0 \\ -1 & -\frac{1}{4} & 0 \\ 0 & 0 & -\frac{1}{4} \end{bmatrix} \cdot \underline{\vec{x}}(t)$$

- Define the coefficient matrix

$$A = \begin{bmatrix} -\frac{1}{4} & 1 & 0 \\ -1 & -\frac{1}{4} & 0 \\ 0 & 0 & -\frac{1}{4} \end{bmatrix}$$

- Rewrite the system as

$$\underline{\vec{x}}'(t) = A \cdot \underline{\vec{x}}(t)$$

- To solve the system, find the eigenvalues and eigenvectors of A
- Eigenpairs of A

$$\left[\left[-\frac{1}{4}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}\right], \left[-\frac{1}{4} - \mathrm{I}, \begin{bmatrix} \mathrm{I} \\ 1 \\ 0 \end{bmatrix}\right], \left[-\frac{1}{4} + \mathrm{I}, \begin{bmatrix} -\mathrm{I} \\ 1 \\ 0 \end{bmatrix}\right]\right]$$

- Consider eigenpair

$$\left[-\frac{1}{4}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}\right]$$

- Solution to homogeneous system from eigenpair

$$\underline{\vec{x}}_1 = \mathrm{e}^{-\frac{t}{4}} \cdot \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

- Consider complex eigenpair, complex conjugate eigenvalue can be ignored

$$\left[-\frac{1}{4}-\mathrm{I}, \left[\begin{array}{c}\mathrm{I} \\ 1 \\ 0\end{array}\right]\right]$$

- Solution from eigenpair

$$\mathrm{e}^{\left(-\frac{1}{4}-\mathrm{I}\right)t} \cdot \left[\begin{array}{c}\mathrm{I} \\ 1 \\ 0\end{array}\right]$$

- Use Euler identity to write solution in terms of $\sin$ and $\cos$

$$\mathrm{e}^{-\frac{t}{4}} \cdot (\cos(t) - \mathrm{I}\sin(t)) \cdot \left[\begin{array}{c}\mathrm{I} \\ 1 \\ 0\end{array}\right]$$

- Simplify expression

$$\mathrm{e}^{-\frac{t}{4}} \cdot \left[\begin{array}{c}\mathrm{I}(\cos(t) - \mathrm{I}\sin(t)) \\ \cos(t) - \mathrm{I}\sin(t) \\ 0\end{array}\right]$$

- Both real and imaginary parts are solutions to the homogeneous system

$$\left[\underline{\vec{x}}_{2}(t) = \mathrm{e}^{-\frac{t}{4}} \cdot \left[\begin{array}{c}\sin(t) \\ \cos(t) \\ 0\end{array}\right], \underline{\vec{x}}_{3}(t) = \mathrm{e}^{-\frac{t}{4}} \cdot \left[\begin{array}{c}\cos(t) \\ -\sin(t) \\ 0\end{array}\right]\right]$$

- General solution to the system of ODEs

$$\underline{\vec{x}} = c_1 \underline{\vec{x}}_{1} + c_2 \underline{\vec{x}}_{2}(t) + c_3 \underline{\vec{x}}_{3}(t)$$

- Substitute solutions into the general solution

$$\underline{\vec{x}} = c_1 \mathrm{e}^{-\frac{t}{4}} \cdot \left[\begin{array}{c}0 \\ 0 \\ 1\end{array}\right] + c_2 \mathrm{e}^{-\frac{t}{4}} \cdot \left[\begin{array}{c}\sin(t) \\ \cos(t) \\ 0\end{array}\right] + c_3 \mathrm{e}^{-\frac{t}{4}} \cdot \left[\begin{array}{c}\cos(t) \\ -\sin(t) \\ 0\end{array}\right]$$

- Substitute in vector of dependent variables

$$\begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{bmatrix} = \begin{bmatrix} e^{-\frac{t}{4}}(c_2 \sin(t) + c_3 \cos(t)) \\ e^{-\frac{t}{4}}(c_2 \cos(t) - c_3 \sin(t)) \\ c_1 e^{-\frac{t}{4}} \end{bmatrix}$$

- Solution to the system of ODEs

$$\left\{x_1(t) = e^{-\frac{t}{4}}(c_2 \sin(t) + c_3 \cos(t)), x_2(t) = e^{-\frac{t}{4}}(c_2 \cos(t) - c_3 \sin(t)), x_3(t) = c_1 e^{-\frac{t}{4}}\right\}$$

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 47

```
dsolve([diff(x__1(t),t)=-1/4*x__1(t)+1*x__2(t)+0*x__3(t),diff(x__2(t),t)=-1*x__1(t)-1/4*x__2
```

$$x_1(t) = e^{-\frac{t}{4}}(c_1 \sin(t) + c_2 \cos(t))$$
$$x_2(t) = -e^{-\frac{t}{4}}(c_2 \sin(t) - c_1 \cos(t))$$
$$x_3(t) = c_3 e^{-\frac{t}{4}}$$

✓ Solution by Mathematica

Time used: 0.023 (sec). Leaf size: 110

```
DSolve[{x1'[t]==-1/4*x1[t]+1*x2[t]+0*x3[t],x2'[t]==-1*x1[t]-1/4*x2[t]+0*x3[t],x3'[t]==0*x1[t
```

$$\text{x1}(t) \to e^{-t/4}(c_1 \cos(t) + c_2 \sin(t))$$
$$\text{x2}(t) \to e^{-t/4}(c_2 \cos(t) - c_1 \sin(t))$$
$$\text{x3}(t) \to c_3 e^{-t/4}$$
$$\text{x1}(t) \to e^{-t/4}(c_1 \cos(t) + c_2 \sin(t))$$
$$\text{x2}(t) \to e^{-t/4}(c_2 \cos(t) - c_1 \sin(t))$$
$$\text{x3}(t) \to 0$$

16.14 problem 24

Internal problem ID [764]
Internal file name [`OUTPUT/764_Sunday_June_05_2022_01_49_05_AM_58372908/index.tex`]

Book: Elementary differential equations and boundary value problems, 10th ed., Boyce and DiPrima
Section: Chapter 7.6, Complex Eigenvalues. page 417
Problem number: 24.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"system of linear ODEs"**

Solve

$$x_1'(t) = -\frac{x_1(t)}{4} + x_2(t)$$
$$x_2'(t) = -x_1(t) - \frac{x_2(t)}{4}$$
$$x_3'(t) = \frac{x_3(t)}{10}$$

16.14.1 Solution using Matrix exponential method

In this method, we will assume we have found the matrix exponential e^{At} allready. There are different methods to determine this but will not be shown here. This is a system of linear ODE's given as

$$\vec{x}'(t) = A\,\vec{x}(t)$$

Or

$$\begin{bmatrix} x_1'(t) \\ x_2'(t) \\ x_3'(t) \end{bmatrix} = \begin{bmatrix} -\frac{1}{4} & 1 & 0 \\ -1 & -\frac{1}{4} & 0 \\ 0 & 0 & \frac{1}{10} \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{bmatrix}$$

For the above matrix A, the matrix exponential can be found to be

$$e^{At} = \begin{bmatrix} e^{-\frac{t}{4}} \cos(t) & e^{-\frac{t}{4}} \sin(t) & 0 \\ -e^{-\frac{t}{4}} \sin(t) & e^{-\frac{t}{4}} \cos(t) & 0 \\ 0 & 0 & e^{\frac{t}{10}} \end{bmatrix}$$

Therefore the homogeneous solution is

$$\begin{aligned} \vec{x}_h(t) &= e^{At}\vec{c} \\ &= \begin{bmatrix} e^{-\frac{t}{4}} \cos(t) & e^{-\frac{t}{4}} \sin(t) & 0 \\ -e^{-\frac{t}{4}} \sin(t) & e^{-\frac{t}{4}} \cos(t) & 0 \\ 0 & 0 & e^{\frac{t}{10}} \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} \\ &= \begin{bmatrix} e^{-\frac{t}{4}} \cos(t) c_1 + e^{-\frac{t}{4}} \sin(t) c_2 \\ -e^{-\frac{t}{4}} \sin(t) c_1 + e^{-\frac{t}{4}} \cos(t) c_2 \\ e^{\frac{t}{10}} c_3 \end{bmatrix} \\ &= \begin{bmatrix} e^{-\frac{t}{4}} (\cos(t) c_1 + \sin(t) c_2) \\ e^{-\frac{t}{4}} (-\sin(t) c_1 + \cos(t) c_2) \\ e^{\frac{t}{10}} c_3 \end{bmatrix} \end{aligned}$$

Since no forcing function is given, then the final solution is $\vec{x}_h(t)$ above.

16.14.2 Solution using explicit Eigenvalue and Eigenvector method

This is a system of linear ODE's given as

$$\vec{x}'(t) = A\,\vec{x}(t)$$

Or

$$\begin{bmatrix} x_1'(t) \\ x_2'(t) \\ x_3'(t) \end{bmatrix} = \begin{bmatrix} -\frac{1}{4} & 1 & 0 \\ -1 & -\frac{1}{4} & 0 \\ 0 & 0 & \frac{1}{10} \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{bmatrix}$$

The first step is find the homogeneous solution. We start by finding the eigenvalues of A. This is done by solving the following equation for the eigenvalues λ

$$\det(A - \lambda I) = 0$$

Expanding gives

$$\det\left(\begin{bmatrix} -\frac{1}{4} & 1 & 0 \\ -1 & -\frac{1}{4} & 0 \\ 0 & 0 & \frac{1}{10} \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\right) = 0$$

Therefore

$$\det\left(\begin{bmatrix} -\frac{1}{4} - \lambda & 1 & 0 \\ -1 & -\frac{1}{4} - \lambda & 0 \\ 0 & 0 & \frac{1}{10} - \lambda \end{bmatrix}\right) = 0$$

Which gives the characteristic equation

$$\lambda^3 + \frac{2}{5}\lambda^2 + \frac{81}{80}\lambda - \frac{17}{160} = 0$$

The roots of the above are the eigenvalues.

$$\begin{aligned} \lambda_1 &= -\frac{1}{4} + i \\ \lambda_2 &= -\frac{1}{4} - i \\ \lambda_3 &= \frac{1}{10} \end{aligned}$$

This table summarises the above result

eigenvalue	algebraic multiplicity	type of eigenvalue
$-\frac{1}{4} - i$	1	complex eigenvalue
$-\frac{1}{4} + i$	1	complex eigenvalue
$\frac{1}{10}$	1	real eigenvalue

Now the eigenvector for each eigenvalue are found.

Considering the eigenvalue $\lambda_1 = \frac{1}{10}$

We need to solve $A\vec{v} = \lambda\vec{v}$ or $(A - \lambda I)\vec{v} = \vec{0}$ which becomes

$$\left(\begin{bmatrix} -\frac{1}{4} & 1 & 0 \\ -1 & -\frac{1}{4} & 0 \\ 0 & 0 & \frac{1}{10} \end{bmatrix} - \left(\frac{1}{10}\right) \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \right) \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} -\frac{7}{20} & 1 & 0 \\ -1 & -\frac{7}{20} & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Now forward elimination is applied to solve for the eigenvector $\vec{v}$. The augmented matrix is

$$\left[\begin{array}{ccc|c} -\frac{7}{20} & 1 & 0 & 0 \\ -1 & -\frac{7}{20} & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$$R_2 = R_2 - \frac{20R_1}{7} \Longrightarrow \left[\begin{array}{ccc|c} -\frac{7}{20} & 1 & 0 & 0 \\ 0 & -\frac{449}{140} & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

Therefore the system in Echelon form is

$$\begin{bmatrix} -\frac{7}{20} & 1 & 0 \\ 0 & -\frac{449}{140} & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

The free variables are $\{v_3\}$ and the leading variables are $\{v_1, v_2\}$. Let $v_3 = t$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\{v_1 = 0, v_2 = 0\}$

Hence the solution is

$$\begin{bmatrix} 0 \\ 0 \\ t \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ t \end{bmatrix}$$

Since there is one free Variable, we have found one eigenvector associated with this eigenvalue. The above can be written as

$$\begin{bmatrix} 0 \\ 0 \\ t \end{bmatrix} = t \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

Let $t = 1$ the eigenvector becomes

$$\begin{bmatrix} 0 \\ 0 \\ t \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

Considering the eigenvalue $\lambda_2 = -\frac{1}{4} - i$

We need to solve $A\vec{v} = \lambda\vec{v}$ or $(A - \lambda I)\vec{v} = \vec{0}$ which becomes

$$\left(\begin{bmatrix} -\frac{1}{4} & 1 & 0 \\ -1 & -\frac{1}{4} & 0 \\ 0 & 0 & \frac{1}{10} \end{bmatrix} - \left(-\frac{1}{4} - i\right) \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \right) \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} i & 1 & 0 \\ -1 & i & 0 \\ 0 & 0 & \frac{7}{20} + i \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Now forward elimination is applied to solve for the eigenvector $\vec{v}$. The augmented matrix is

$$\left[\begin{array}{ccc|c} i & 1 & 0 & 0 \\ -1 & i & 0 & 0 \\ 0 & 0 & \frac{7}{20} + i & 0 \end{array}\right]$$

$$R_2 = -iR_1 + R_2 \Longrightarrow \left[\begin{array}{ccc|c} i & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{7}{20} + i & 0 \end{array}\right]$$

Since the current pivot $A(2,3)$ is zero, then the current pivot row is replaced with a row with a non-zero pivot. Swapping row 2 and row 3 gives

$$\left[\begin{array}{ccc|c} i & 1 & 0 & 0 \\ 0 & 0 & \frac{7}{20}+i & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

Therefore the system in Echelon form is

$$\begin{bmatrix} i & 1 & 0 \\ 0 & 0 & \frac{7}{20}+i \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

The free variables are $\{v_2\}$ and the leading variables are $\{v_1, v_3\}$. Let $v_2 = t$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\{v_1 = it, v_3 = 0\}$

Hence the solution is

$$\begin{bmatrix} \mathrm{I}t \\ t \\ 0 \end{bmatrix} = \begin{bmatrix} it \\ t \\ 0 \end{bmatrix}$$

Since there is one free Variable, we have found one eigenvector associated with this eigenvalue. The above can be written as

$$\begin{bmatrix} \mathrm{I}t \\ t \\ 0 \end{bmatrix} = t \begin{bmatrix} i \\ 1 \\ 0 \end{bmatrix}$$

Let $t = 1$ the eigenvector becomes

$$\begin{bmatrix} \mathrm{I}t \\ t \\ 0 \end{bmatrix} = \begin{bmatrix} i \\ 1 \\ 0 \end{bmatrix}$$

Considering the eigenvalue $\lambda_3 = -\frac{1}{4} + i$

We need to solve $A\vec{v} = \lambda\vec{v}$ or $(A - \lambda I)\vec{v} = \vec{0}$ which becomes

$$\left(\begin{bmatrix} -\frac{1}{4} & 1 & 0 \\ -1 & -\frac{1}{4} & 0 \\ 0 & 0 & \frac{1}{10} \end{bmatrix} - \left(-\frac{1}{4} + i\right)\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\right)\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} -i & 1 & 0 \\ -1 & -i & 0 \\ 0 & 0 & \frac{7}{20} - i \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Now forward elimination is applied to solve for the eigenvector $\vec{v}$. The augmented matrix is

$$\left[\begin{array}{ccc|c} -i & 1 & 0 & 0 \\ -1 & -i & 0 & 0 \\ 0 & 0 & \frac{7}{20} - i & 0 \end{array}\right]$$

$$R_2 = iR_1 + R_2 \Longrightarrow \left[\begin{array}{ccc|c} -i & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{7}{20} - i & 0 \end{array}\right]$$

Since the current pivot $A(2,3)$ is zero, then the current pivot row is replaced with a row with a non-zero pivot. Swapping row 2 and row 3 gives

$$\left[\begin{array}{ccc|c} -i & 1 & 0 & 0 \\ 0 & 0 & \frac{7}{20} - i & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

Therefore the system in Echelon form is

$$\begin{bmatrix} -i & 1 & 0 \\ 0 & 0 & \frac{7}{20} - i \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

The free variables are $\{v_2\}$ and the leading variables are $\{v_1, v_3\}$. Let $v_2 = t$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\{v_1 = -it, v_3 = 0\}$

Hence the solution is

$$\begin{bmatrix} \text{-I}\,t \\ t \\ 0 \end{bmatrix} = \begin{bmatrix} -it \\ t \\ 0 \end{bmatrix}$$

Since there is one free Variable, we have found one eigenvector associated with this eigenvalue. The above can be written as

$$\begin{bmatrix} \text{-I}\,t \\ t \\ 0 \end{bmatrix} = t \begin{bmatrix} -i \\ 1 \\ 0 \end{bmatrix}$$

Let $t = 1$ the eigenvector becomes

$$\begin{bmatrix} \text{-I}\,t \\ t \\ 0 \end{bmatrix} = \begin{bmatrix} -i \\ 1 \\ 0 \end{bmatrix}$$

The following table gives a summary of this result. It shows for each eigenvalue the algebraic multiplicity m, and its geometric multiplicity k and the eigenvectors associated with the eigenvalue. If $m > k$ then the eigenvalue is defective which means the number of normal linearly independent eigenvectors associated with this eigenvalue (called the geometric multiplicity k) does not equal the algebraic multiplicity m, and we need to determine an additional $m - k$ generalized eigenvectors for this eigenvalue.

	multiplicity			
eigenvalue	algebraic m	geometric k	defective?	eigenvectors
$-\frac{1}{4}+i$	1	1	No	$\begin{bmatrix} -i \\ 1 \\ 0 \end{bmatrix}$
$-\frac{1}{4}-i$	1	1	No	$\begin{bmatrix} i \\ 1 \\ 0 \end{bmatrix}$
$\frac{1}{10}$	1	1	No	$\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$

Now that we found the eigenvalues and associated eigenvectors, we will go over each eigenvalue and generate the solution basis. The only problem we need to take care of is if the eigenvalue is defective. Since eigenvalue $\frac{1}{10}$ is real and distinct then the corresponding eigenvector solution is

$$
\begin{aligned}
\vec{x}_1(t) &= \vec{v}_1 e^{\frac{t}{10}} \\
&= \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} e^{\frac{t}{10}}
\end{aligned}
$$

Therefore the final solution is

$$
\vec{x}_h(t) = c_1\vec{x}_1(t) + c_2\vec{x}_2(t) + c_3\vec{x}_3(t)
$$

Which is written as

$$
\begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{bmatrix} = c_1 \begin{bmatrix} -i\mathrm{e}^{\left(-\frac{1}{4}+i\right)t} \\ \mathrm{e}^{\left(-\frac{1}{4}+i\right)t} \\ 0 \end{bmatrix} + c_2 \begin{bmatrix} i\mathrm{e}^{\left(-\frac{1}{4}-i\right)t} \\ \mathrm{e}^{\left(-\frac{1}{4}-i\right)t} \\ 0 \end{bmatrix} + c_3 \begin{bmatrix} 0 \\ 0 \\ \mathrm{e}^{\frac{t}{10}} \end{bmatrix}
$$

Which becomes

$$
\begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{bmatrix} = \begin{bmatrix} i\left(c_2\mathrm{e}^{\left(-\frac{1}{4}-i\right)t} - c_1\mathrm{e}^{\left(-\frac{1}{4}+i\right)t}\right) \\ c_1\mathrm{e}^{\left(-\frac{1}{4}+i\right)t} + c_2\mathrm{e}^{\left(-\frac{1}{4}-i\right)t} \\ c_3\mathrm{e}^{\frac{t}{10}} \end{bmatrix}
$$

16.14.3 Maple step by step solution

Let's solve

$\left[x_1'(t) = -\frac{x_1(t)}{4} + x_2(t), x_2'(t) = -x_1(t) - \frac{x_2(t)}{4}, x_3'(t) = \frac{x_3(t)}{10}\right]$

- Define vector

$$
\underline{x}^{\rightarrow}(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{bmatrix}
$$

- Convert system into a vector equation

$$\underline{x}^{\rightarrow\prime}(t) = \begin{bmatrix} -\frac{1}{4} & 1 & 0 \\ -1 & -\frac{1}{4} & 0 \\ 0 & 0 & \frac{1}{10} \end{bmatrix} \cdot \underline{x}^{\rightarrow}(t) + \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

- System to solve

$$\underline{x}^{\rightarrow\prime}(t) = \begin{bmatrix} -\frac{1}{4} & 1 & 0 \\ -1 & -\frac{1}{4} & 0 \\ 0 & 0 & \frac{1}{10} \end{bmatrix} \cdot \underline{x}^{\rightarrow}(t)$$

- Define the coefficient matrix

$$A = \begin{bmatrix} -\frac{1}{4} & 1 & 0 \\ -1 & -\frac{1}{4} & 0 \\ 0 & 0 & \frac{1}{10} \end{bmatrix}$$

- Rewrite the system as

$$\underline{x}^{\rightarrow\prime}(t) = A \cdot \underline{x}^{\rightarrow}(t)$$

- To solve the system, find the eigenvalues and eigenvectors of A
- Eigenpairs of A

$$\left[\left[\frac{1}{10}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}\right], \left[-\frac{1}{4} - \mathrm{I}, \begin{bmatrix} \mathrm{I} \\ 1 \\ 0 \end{bmatrix}\right], \left[-\frac{1}{4} + \mathrm{I}, \begin{bmatrix} -\mathrm{I} \\ 1 \\ 0 \end{bmatrix}\right]\right]$$

- Consider eigenpair

$$\left[\frac{1}{10}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}\right]$$

- Solution to homogeneous system from eigenpair

$$\underline{x}^{\rightarrow}{}_{1} = \mathrm{e}^{\frac{t}{10}} \cdot \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

- Consider complex eigenpair, complex conjugate eigenvalue can be ignored

$$\left[-\frac{1}{4}-\mathrm{I}, \left[\begin{array}{c}\mathrm{I}\\ 1\\ 0\end{array}\right]\right]$$

- Solution from eigenpair

$$e^{\left(-\frac{1}{4}-\mathrm{I}\right)t}\cdot\left[\begin{array}{c}\mathrm{I}\\ 1\\ 0\end{array}\right]$$

- Use Euler identity to write solution in terms of $\sin$ and $\cos$

$$e^{-\frac{t}{4}}\cdot\left(\cos\left(t\right)-\mathrm{I}\sin\left(t\right)\right)\cdot\left[\begin{array}{c}\mathrm{I}\\ 1\\ 0\end{array}\right]$$

- Simplify expression

$$e^{-\frac{t}{4}}\cdot\left[\begin{array}{c}\mathrm{I}\left(\cos\left(t\right)-\mathrm{I}\sin\left(t\right)\right)\\ \cos\left(t\right)-\mathrm{I}\sin\left(t\right)\\ 0\end{array}\right]$$

- Both real and imaginary parts are solutions to the homogeneous system

$$\left[\underline{\vec{x}}_{2}(t)=e^{-\frac{t}{4}}\cdot\left[\begin{array}{c}\sin\left(t\right)\\ \cos\left(t\right)\\ 0\end{array}\right], \underline{\vec{x}}_{3}(t)=e^{-\frac{t}{4}}\cdot\left[\begin{array}{c}\cos\left(t\right)\\ -\sin\left(t\right)\\ 0\end{array}\right]\right]$$

- General solution to the system of ODEs

$$\underline{\vec{x}}=c_{1}\underline{\vec{x}}_{1}+c_{2}\underline{\vec{x}}_{2}(t)+c_{3}\underline{\vec{x}}_{3}(t)$$

- Substitute solutions into the general solution

$$\underline{\vec{x}}=c_{1}e^{\frac{t}{10}}\cdot\left[\begin{array}{c}0\\ 0\\ 1\end{array}\right]+c_{2}e^{-\frac{t}{4}}\cdot\left[\begin{array}{c}\sin\left(t\right)\\ \cos\left(t\right)\\ 0\end{array}\right]+c_{3}e^{-\frac{t}{4}}\cdot\left[\begin{array}{c}\cos\left(t\right)\\ -\sin\left(t\right)\\ 0\end{array}\right]$$

- Substitute in vector of dependent variables

$$\begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{bmatrix} = \begin{bmatrix} e^{-\frac{t}{4}}(c_2\sin(t)+c_3\cos(t)) \\ e^{-\frac{t}{4}}(c_2\cos(t)-c_3\sin(t)) \\ c_1e^{\frac{t}{10}} \end{bmatrix}$$

- Solution to the system of ODEs

$$\left\{x_1(t)=e^{-\frac{t}{4}}(c_2\sin(t)+c_3\cos(t)), x_2(t)=e^{-\frac{t}{4}}(c_2\cos(t)-c_3\sin(t)), x_3(t)=c_1e^{\frac{t}{10}}\right\}$$

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 47

```
dsolve([diff(x__1(t),t)=-1/4*x__1(t)+1*x__2(t)+0*x__3(t),diff(x__2(t),t)=-1*x__1(t)-1/4*x__2
```

$$x_1(t)=e^{-\frac{t}{4}}(c_1\sin(t)+c_2\cos(t))$$
$$x_2(t)=-e^{-\frac{t}{4}}(c_2\sin(t)-c_1\cos(t))$$
$$x_3(t)=c_3e^{\frac{t}{10}}$$

✓ Solution by Mathematica

Time used: 0.021 (sec). Leaf size: 110

```
DSolve[{x1'[t]==-1/4*x1[t]+1*x2[t]+0*x3[t],x2'[t]==-1*x1[t]-1/4*x2[t]+0*x3[t],x3'[t]==0*x1[t
```

$$\text{x1}(t)\to e^{-t/4}(c_1\cos(t)+c_2\sin(t))$$
$$\text{x2}(t)\to e^{-t/4}(c_2\cos(t)-c_1\sin(t))$$
$$\text{x3}(t)\to c_3e^{t/10}$$
$$\text{x1}(t)\to e^{-t/4}(c_1\cos(t)+c_2\sin(t))$$
$$\text{x2}(t)\to e^{-t/4}(c_2\cos(t)-c_1\sin(t))$$
$$\text{x3}(t)\to 0$$

16.15 problem 25

Internal problem ID [765]
Internal file name [`OUTPUT/765_Sunday_June_05_2022_01_49_07_AM_11650177/index.tex`]

Book: Elementary differential equations and boundary value problems, 10th ed., Boyce and DiPrima
Section: Chapter 7.6, Complex Eigenvalues. page 417
Problem number: 25.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"system of linear ODEs"**

Solve

$$
\begin{aligned}
x_1'(t) &= -\frac{x_1(t)}{2} - \frac{x_2(t)}{8} \\
x_2'(t) &= 2x_1(t) - \frac{x_2(t)}{2}
\end{aligned}
$$

16.15.1 Solution using Matrix exponential method

In this method, we will assume we have found the matrix exponential e^{At} allready. There are different methods to determine this but will not be shown here. This is a system of linear ODE's given as

$$\vec{x}'(t) = A\,\vec{x}(t)$$

Or

$$\begin{bmatrix} x_1'(t) \\ x_2'(t) \end{bmatrix} = \begin{bmatrix} -\frac{1}{2} & -\frac{1}{8} \\ 2 & -\frac{1}{2} \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}$$

For the above matrix A, the matrix exponential can be found to be

$$e^{At} = \begin{bmatrix} \mathrm{e}^{-\frac{t}{2}} \cos\left(\frac{t}{2}\right) & -\frac{\mathrm{e}^{-\frac{t}{2}} \sin\left(\frac{t}{2}\right)}{4} \\ 4\,\mathrm{e}^{-\frac{t}{2}} \sin\left(\frac{t}{2}\right) & \mathrm{e}^{-\frac{t}{2}} \cos\left(\frac{t}{2}\right) \end{bmatrix}$$

Therefore the homogeneous solution is

$$
\begin{aligned}
\vec{x}_h(t) &= e^{At}\vec{c} \\
&= \begin{bmatrix} \mathrm{e}^{-\frac{t}{2}}\cos\left(\frac{t}{2}\right) & -\frac{\mathrm{e}^{-\frac{t}{2}}\sin\left(\frac{t}{2}\right)}{4} \\ 4\,\mathrm{e}^{-\frac{t}{2}}\sin\left(\frac{t}{2}\right) & \mathrm{e}^{-\frac{t}{2}}\cos\left(\frac{t}{2}\right) \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} \\
&= \begin{bmatrix} \mathrm{e}^{-\frac{t}{2}}\cos\left(\frac{t}{2}\right)c_1 - \frac{\mathrm{e}^{-\frac{t}{2}}\sin\left(\frac{t}{2}\right)c_2}{4} \\ 4\,\mathrm{e}^{-\frac{t}{2}}\sin\left(\frac{t}{2}\right)c_1 + \mathrm{e}^{-\frac{t}{2}}\cos\left(\frac{t}{2}\right)c_2 \end{bmatrix} \\
&= \begin{bmatrix} \mathrm{e}^{-\frac{t}{2}}\cos\left(\frac{t}{2}\right)c_1 - \frac{\mathrm{e}^{-\frac{t}{2}}\sin\left(\frac{t}{2}\right)c_2}{4} \\ \mathrm{e}^{-\frac{t}{2}}\left(4\sin\left(\frac{t}{2}\right)c_1 + \cos\left(\frac{t}{2}\right)c_2\right) \end{bmatrix}
\end{aligned}
$$

Since no forcing function is given, then the final solution is $\vec{x}_h(t)$ above.

16.15.2 Solution using explicit Eigenvalue and Eigenvector method

This is a system of linear ODE's given as

$$
\vec{x}'(t) = A\,\vec{x}(t)
$$

Or

$$
\begin{bmatrix} x_1'(t) \\ x_2'(t) \end{bmatrix} = \begin{bmatrix} -\frac{1}{2} & -\frac{1}{8} \\ 2 & -\frac{1}{2} \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}
$$

The first step is find the homogeneous solution. We start by finding the eigenvalues of A. This is done by solving the following equation for the eigenvalues λ

$$
\det(A - \lambda I) = 0
$$

Expanding gives

$$
\det\left(\begin{bmatrix} -\frac{1}{2} & -\frac{1}{8} \\ 2 & -\frac{1}{2} \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right) = 0
$$

Therefore

$$
\det\left(\begin{bmatrix} -\frac{1}{2} - \lambda & -\frac{1}{8} \\ 2 & -\frac{1}{2} - \lambda \end{bmatrix}\right) = 0
$$

Which gives the characteristic equation

$$\lambda^2+\lambda+\frac{1}{2}=0$$

The roots of the above are the eigenvalues.

$$\lambda_1=-\frac{1}{2}+\frac{i}{2}$$
$$\lambda_2=-\frac{1}{2}-\frac{i}{2}$$

This table summarises the above result

eigenvalue	algebraic multiplicity	type of eigenvalue
$-\frac{1}{2}+\frac{i}{2}$	1	complex eigenvalue
$-\frac{1}{2}-\frac{i}{2}$	1	complex eigenvalue

Now the eigenvector for each eigenvalue are found.

Considering the eigenvalue $\lambda_1=-\frac{1}{2}-\frac{i}{2}$

We need to solve $A\vec{v}=\lambda\vec{v}$ or $(A-\lambda I)\vec{v}=\vec{0}$ which becomes

$$\left(\begin{bmatrix}-\frac{1}{2} & -\frac{1}{8}\\ 2 & -\frac{1}{2}\end{bmatrix}-\left(-\frac{1}{2}-\frac{i}{2}\right)\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}\right)\begin{bmatrix}v_1\\ v_2\end{bmatrix}=\begin{bmatrix}0\\ 0\end{bmatrix}$$
$$\begin{bmatrix}\frac{i}{2} & -\frac{1}{8}\\ 2 & \frac{i}{2}\end{bmatrix}\begin{bmatrix}v_1\\ v_2\end{bmatrix}=\begin{bmatrix}0\\ 0\end{bmatrix}$$

Now forward elimination is applied to solve for the eigenvector $\vec{v}$. The augmented matrix is

$$\left[\begin{array}{cc|c}\frac{i}{2} & -\frac{1}{8} & 0\\ 2 & \frac{i}{2} & 0\end{array}\right]$$

$$R_2=4iR_1+R_2\Longrightarrow\left[\begin{array}{cc|c}\frac{i}{2} & -\frac{1}{8} & 0\\ 0 & 0 & 0\end{array}\right]$$

Therefore the system in Echelon form is

$$\begin{bmatrix}\frac{i}{2} & -\frac{1}{8}\\ 0 & 0\end{bmatrix}\begin{bmatrix}v_1\\ v_2\end{bmatrix}=\begin{bmatrix}0\\ 0\end{bmatrix}$$

The free variables are $\{v_2\}$ and the leading variables are $\{v_1\}$. Let $v_2 = t$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\left\{v_1 = -\frac{it}{4}\right\}$

Hence the solution is

$$\begin{bmatrix} -\frac{I}{4}t \\ t \end{bmatrix} = \begin{bmatrix} -\frac{it}{4} \\ t \end{bmatrix}$$

Since there is one free Variable, we have found one eigenvector associated with this eigenvalue. The above can be written as

$$\begin{bmatrix} -\frac{I}{4}t \\ t \end{bmatrix} = t \begin{bmatrix} -\frac{i}{4} \\ 1 \end{bmatrix}$$

Let $t = 1$ the eigenvector becomes

$$\begin{bmatrix} -\frac{I}{4}t \\ t \end{bmatrix} = \begin{bmatrix} -\frac{i}{4} \\ 1 \end{bmatrix}$$

Which is normalized to

$$\begin{bmatrix} -\frac{I}{4}t \\ t \end{bmatrix} = \begin{bmatrix} -i \\ 4 \end{bmatrix}$$

Considering the eigenvalue $\lambda_2 = -\frac{1}{2} + \frac{i}{2}$

We need to solve $A\vec{v} = \lambda\vec{v}$ or $(A - \lambda I)\vec{v} = \vec{0}$ which becomes

$$\left(\begin{bmatrix} -\frac{1}{2} & -\frac{1}{8} \\ 2 & -\frac{1}{2} \end{bmatrix} - \left(-\frac{1}{2} + \frac{i}{2}\right) \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right) \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} -\frac{i}{2} & -\frac{1}{8} \\ 2 & -\frac{i}{2} \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Now forward elimination is applied to solve for the eigenvector $\vec{v}$. The augmented matrix is

$$\left[\begin{array}{cc|c} -\frac{i}{2} & -\frac{1}{8} & 0 \\ 2 & -\frac{i}{2} & 0 \end{array}\right]$$

$$R_2 = -4iR_1 + R_2 \Longrightarrow \left[\begin{array}{cc|c} -\frac{i}{2} & -\frac{1}{8} & 0 \\ 0 & 0 & 0 \end{array}\right]$$

Therefore the system in Echelon form is

$$\begin{bmatrix} -\frac{i}{2} & -\frac{1}{8} \\ 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

The free variables are $\{v_2\}$ and the leading variables are $\{v_1\}$. Let $v_2 = t$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\left\{v_1 = \frac{it}{4}\right\}$

Hence the solution is

$$\begin{bmatrix} \frac{\mathrm{I}}{4}t \\ t \end{bmatrix} = \begin{bmatrix} \frac{it}{4} \\ t \end{bmatrix}$$

Since there is one free Variable, we have found one eigenvector associated with this eigenvalue. The above can be written as

$$\begin{bmatrix} \frac{\mathrm{I}}{4}t \\ t \end{bmatrix} = t \begin{bmatrix} \frac{i}{4} \\ 1 \end{bmatrix}$$

Let $t = 1$ the eigenvector becomes

$$\begin{bmatrix} \frac{\mathrm{I}}{4}t \\ t \end{bmatrix} = \begin{bmatrix} \frac{i}{4} \\ 1 \end{bmatrix}$$

Which is normalized to

$$\begin{bmatrix} \frac{\mathrm{I}}{4}t \\ t \end{bmatrix} = \begin{bmatrix} i \\ 4 \end{bmatrix}$$

The following table gives a summary of this result. It shows for each eigenvalue the algebraic multiplicity m, and its geometric multiplicity k and the eigenvectors associated with the eigenvalue. If $m > k$ then the eigenvalue is defective which means the number of normal linearly independent eigenvectors associated with this eigenvalue (called the geometric multiplicity k) does not equal the algebraic multiplicity m, and we need to determine an additional $m - k$ generalized eigenvectors for this eigenvalue.

eigenvalue	multiplicity		defective?	eigenvectors
	algebraic m	geometric k		
$-\frac{1}{2}+\frac{i}{2}$	1	1	No	$\left[\begin{array}{c} \frac{i}{4} \\ 1 \end{array}\right]$
$-\frac{1}{2}-\frac{i}{2}$	1	1	No	$\left[\begin{array}{c} -\frac{i}{4} \\ 1 \end{array}\right]$

Now that we found the eigenvalues and associated eigenvectors, we will go over each eigenvalue and generate the solution basis. The only problem we need to take care of is if the eigenvalue is defective. Therefore the final solution is

$$\vec{x}_h(t) = c_1\vec{x}_1(t) + c_2\vec{x}_2(t)$$

Which is written as

$$\left[\begin{array}{c} x_1(t) \\ x_2(t) \end{array}\right] = c_1 \left[\begin{array}{c} \frac{i \mathrm{e}^{\left(-\frac{1}{2}+\frac{i}{2}\right)t}}{4} \\ \mathrm{e}^{\left(-\frac{1}{2}+\frac{i}{2}\right)t} \end{array}\right] + c_2 \left[\begin{array}{c} -\frac{i \mathrm{e}^{\left(-\frac{1}{2}-\frac{i}{2}\right)t}}{4} \\ \mathrm{e}^{\left(-\frac{1}{2}-\frac{i}{2}\right)t} \end{array}\right]$$

Which becomes

$$\left[\begin{array}{c} x_1(t) \\ x_2(t) \end{array}\right] = \left[\begin{array}{c} -\frac{i\left(c_2 \mathrm{e}^{\left(-\frac{1}{2}-\frac{i}{2}\right)t} - c_1 \mathrm{e}^{\left(-\frac{1}{2}+\frac{i}{2}\right)t}\right)}{4} \\ c_1 \mathrm{e}^{\left(-\frac{1}{2}+\frac{i}{2}\right)t} + c_2 \mathrm{e}^{\left(-\frac{1}{2}-\frac{i}{2}\right)t} \end{array}\right]$$

The following is the phase plot of the system.

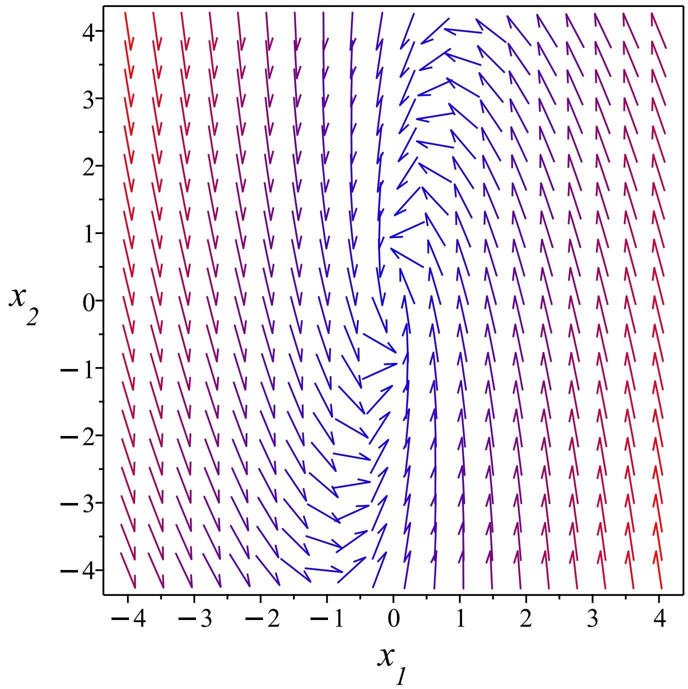


Figure 517: Phase plot

16.15.3 Maple step by step solution

Let's solve

$\left[x_1'(t) = -\frac{x_1(t)}{2} - \frac{x_2(t)}{8}, x_2'(t) = 2x_1(t) - \frac{x_2(t)}{2}\right]$

- Define vector

$$\underline{x^{\rightarrow}}(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}$$

- Convert system into a vector equation

$$\underline{x^{\rightarrow}}'(t) = \begin{bmatrix} -\frac{1}{2} & -\frac{1}{8} \\ 2 & -\frac{1}{2} \end{bmatrix} \cdot \underline{x^{\rightarrow}}(t) + \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

- System to solve

$$\underline{x^{\rightarrow}}'(t) = \begin{bmatrix} -\frac{1}{2} & -\frac{1}{8} \\ 2 & -\frac{1}{2} \end{bmatrix} \cdot \underline{x^{\rightarrow}}(t)$$

- Define the coefficient matrix

$$A = \begin{bmatrix} -\frac{1}{2} & -\frac{1}{8} \\ 2 & -\frac{1}{2} \end{bmatrix}$$

- Rewrite the system as

$$\underline{x}^{\rightarrow\prime}(t) = A \cdot \underline{x}^{\rightarrow}(t)$$

- To solve the system, find the eigenvalues and eigenvectors of A
- Eigenpairs of A

$$\left[\left[-\frac{1}{2} - \frac{\mathrm{I}}{2}, \begin{bmatrix} -\frac{\mathrm{I}}{4} \\ 1 \end{bmatrix}\right], \left[-\frac{1}{2} + \frac{\mathrm{I}}{2}, \begin{bmatrix} \frac{\mathrm{I}}{4} \\ 1 \end{bmatrix}\right]\right]$$

- Consider complex eigenpair, complex conjugate eigenvalue can be ignored

$$\left[-\frac{1}{2} - \frac{\mathrm{I}}{2}, \begin{bmatrix} -\frac{\mathrm{I}}{4} \\ 1 \end{bmatrix}\right]$$

- Solution from eigenpair

$$\mathrm{e}^{\left(-\frac{1}{2}-\frac{\mathrm{I}}{2}\right)t} \cdot \begin{bmatrix} -\frac{\mathrm{I}}{4} \\ 1 \end{bmatrix}$$

- Use Euler identity to write solution in terms of $\sin$ and $\cos$

$$\mathrm{e}^{-\frac{t}{2}} \cdot \left(\cos\left(\frac{t}{2}\right) - \mathrm{I}\sin\left(\frac{t}{2}\right)\right) \cdot \begin{bmatrix} -\frac{\mathrm{I}}{4} \\ 1 \end{bmatrix}$$

- Simplify expression

$$\mathrm{e}^{-\frac{t}{2}} \cdot \begin{bmatrix} -\frac{\mathrm{I}}{4}\left(\cos\left(\frac{t}{2}\right) - \mathrm{I}\sin\left(\frac{t}{2}\right)\right) \\ \cos\left(\frac{t}{2}\right) - \mathrm{I}\sin\left(\frac{t}{2}\right) \end{bmatrix}$$

- Both real and imaginary parts are solutions to the homogeneous system

$$\left[\underline{x}^{\rightarrow}{}_1(t) = \mathrm{e}^{-\frac{t}{2}} \cdot \begin{bmatrix} -\frac{\sin\left(\frac{t}{2}\right)}{4} \\ \cos\left(\frac{t}{2}\right) \end{bmatrix}, \underline{x}^{\rightarrow}{}_2(t) = \mathrm{e}^{-\frac{t}{2}} \cdot \begin{bmatrix} -\frac{\cos\left(\frac{t}{2}\right)}{4} \\ -\sin\left(\frac{t}{2}\right) \end{bmatrix}\right]$$

- General solution to the system of ODEs

$$\underline{x}^{\rightarrow} = c_1 \underline{x}^{\rightarrow}{}_1(t) + c_2 \underline{x}^{\rightarrow}{}_2(t)$$

- Substitute solutions into the general solution

$$\underline{\overrightarrow{x}} = c_1 e^{-\frac{t}{2}} \cdot \begin{bmatrix} -\frac{\sin\left(\frac{t}{2}\right)}{4} \\ \cos\left(\frac{t}{2}\right) \end{bmatrix} + c_2 e^{-\frac{t}{2}} \cdot \begin{bmatrix} -\frac{\cos\left(\frac{t}{2}\right)}{4} \\ -\sin\left(\frac{t}{2}\right) \end{bmatrix}$$

- Substitute in vector of dependent variables

$$\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = \begin{bmatrix} -\frac{e^{-\frac{t}{2}}\left(c_1 \sin\left(\frac{t}{2}\right) + c_2 \cos\left(\frac{t}{2}\right)\right)}{4} \\ e^{-\frac{t}{2}}\left(c_1 \cos\left(\frac{t}{2}\right) - c_2 \sin\left(\frac{t}{2}\right)\right) \end{bmatrix}$$

- Solution to the system of ODEs

$$\left\{x_1(t) = -\frac{e^{-\frac{t}{2}}\left(c_1 \sin\left(\frac{t}{2}\right) + c_2 \cos\left(\frac{t}{2}\right)\right)}{4}, x_2(t) = e^{-\frac{t}{2}}\left(c_1 \cos\left(\frac{t}{2}\right) - c_2 \sin\left(\frac{t}{2}\right)\right)\right\}$$

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 46

```
dsolve([diff(x__1(t),t)=-1/2*x__1(t)-1/8*x__2(t),diff(x__2(t),t)=2*x__1(t)-1/2*x__2(t)],sing
```

$$x_1(t) = e^{-\frac{t}{2}}\left(c_2 \cos\left(\frac{t}{2}\right) + c_1 \sin\left(\frac{t}{2}\right)\right)$$
$$x_2(t) = -4\,e^{-\frac{t}{2}}\left(\cos\left(\frac{t}{2}\right) c_1 - \sin\left(\frac{t}{2}\right) c_2\right)$$

✓ Solution by Mathematica

Time used: 0.005 (sec). Leaf size: 68

```
DSolve[{x1'[t]==-1/2*x1[t]-1/8*x2[t],x2'[t]==2*x1[t]-1/2*x2[t]},{x1[t],x2[t]},t,IncludeSingu
```

$$\text{x1}(t) \to \frac{1}{4} e^{-t/2}\left(4c_1 \cos\left(\frac{t}{2}\right) - c_2 \sin\left(\frac{t}{2}\right)\right)$$
$$\text{x2}(t) \to e^{-t/2}\left(c_2 \cos\left(\frac{t}{2}\right) + 4c_1 \sin\left(\frac{t}{2}\right)\right)$$

17 Chapter 7.8, Repeated Eigenvalues. page 436

17.1 problem 1

Internal problem ID [766]
Internal file name [`OUTPUT/766_Sunday_June_05_2022_01_49_09_AM_14401963/index.tex`]

Book: Elementary differential equations and boundary value problems, 10th ed., Boyce and DiPrima
Section: Chapter 7.8, Repeated Eigenvalues. page 436
Problem number: 1.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"system of linear ODEs"**

Solve

$$x_1'(t) = 3x_1(t) - 4x_2(t)$$
$$x_2'(t) = x_1(t) - x_2(t)$$

17.1.1 Solution using Matrix exponential method

In this method, we will assume we have found the matrix exponential e^{At} allready. There are different methods to determine this but will not be shown here. This is a system of linear ODE's given as

$$\vec{x}'(t) = A\,\vec{x}(t)$$

Or

$$\begin{bmatrix} x_1'(t) \\ x_2'(t) \end{bmatrix} = \begin{bmatrix} 3 & -4 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}$$

For the above matrix A, the matrix exponential can be found to be

$$e^{At} = \begin{bmatrix} e^t(1+2t) & -4t\,e^t \\ t\,e^t & e^t(1-2t) \end{bmatrix}$$

Therefore the homogeneous solution is

$$\begin{aligned}
\vec{x}_h(t) &= e^{At}\vec{c} \\
&= \begin{bmatrix} e^t(1+2t) & -4t\,e^t \\ t\,e^t & e^t(1-2t) \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} \\
&= \begin{bmatrix} e^t(1+2t)\,c_1 - 4t\,e^t c_2 \\ t\,e^t c_1 + e^t(1-2t)\,c_2 \end{bmatrix} \\
&= \begin{bmatrix} e^t(2tc_1 - 4c_2t + c_1) \\ e^t(tc_1 - 2c_2t + c_2) \end{bmatrix}
\end{aligned}$$

Since no forcing function is given, then the final solution is $\vec{x}_h(t)$ above.

17.1.2 Solution using explicit Eigenvalue and Eigenvector method

This is a system of linear ODE's given as

$$\vec{x}'(t) = A\,\vec{x}(t)$$

Or

$$\begin{bmatrix} x_1'(t) \\ x_2'(t) \end{bmatrix} = \begin{bmatrix} 3 & -4 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}$$

The first step is find the homogeneous solution. We start by finding the eigenvalues of A. This is done by solving the following equation for the eigenvalues λ

$$\det(A - \lambda I) = 0$$

Expanding gives

$$\det\left(\begin{bmatrix} 3 & -4 \\ 1 & -1 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right) = 0$$

Therefore

$$\det\left(\begin{bmatrix} 3-\lambda & -4 \\ 1 & -1-\lambda \end{bmatrix}\right) = 0$$

Which gives the characteristic equation

$$\lambda^2 - 2\lambda + 1 = 0$$

The roots of the above are the eigenvalues.

$$\lambda_1 = 1$$

This table summarises the above result

eigenvalue	algebraic multiplicity	type of eigenvalue
1	1	real eigenvalue

Now the eigenvector for each eigenvalue are found.

Considering the eigenvalue $\lambda_1 = 1$

We need to solve $A\vec{v} = \lambda\vec{v}$ or $(A - \lambda I)\vec{v} = \vec{0}$ which becomes

$$\left(\begin{bmatrix} 3 & -4 \\ 1 & -1 \end{bmatrix} - (1)\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right)\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$
$$\begin{bmatrix} 2 & -4 \\ 1 & -2 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Now forward elimination is applied to solve for the eigenvector $\vec{v}$. The augmented matrix is

$$\left[\begin{array}{cc|c} 2 & -4 & 0 \\ 1 & -2 & 0 \end{array}\right]$$

$$R_2 = R_2 - \frac{R_1}{2} \Longrightarrow \left[\begin{array}{cc|c} 2 & -4 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

Therefore the system in Echelon form is

$$\begin{bmatrix} 2 & -4 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

The free variables are $\{v_2\}$ and the leading variables are $\{v_1\}$. Let $v_2 = t$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\{v_1 = 2t\}$

Hence the solution is

$$\begin{bmatrix} 2t \\ t \end{bmatrix} = \begin{bmatrix} 2t \\ t \end{bmatrix}$$

Since there is one free Variable, we have found one eigenvector associated with this eigenvalue. The above can be written as

$$\begin{bmatrix} 2t \\ t \end{bmatrix} = t \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

Let $t = 1$ the eigenvector becomes

$$\begin{bmatrix} 2t \\ t \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

The following table gives a summary of this result. It shows for each eigenvalue the algebraic multiplicity m, and its geometric multiplicity k and the eigenvectors associated with the eigenvalue. If $m > k$ then the eigenvalue is defective which means the number of normal linearly independent eigenvectors associated with this eigenvalue (called the geometric multiplicity k) does not equal the algebraic multiplicity m, and we need to determine an additional $m - k$ generalized eigenvectors for this eigenvalue.

	multiplicity			
eigenvalue	algebraic m	geometric k	defective?	eigenvectors
1	2	1	Yes	$\begin{bmatrix} 2 \\ 1 \end{bmatrix}$

Now that we found the eigenvalues and associated eigenvectors, we will go over each eigenvalue and generate the solution basis. The only problem we need to take care of is if the eigenvalue is defective. eigenvalue 1 is real and repated eigenvalue of multiplicity 2.There are two possible cases that can happen. This is illustrated in this diagram

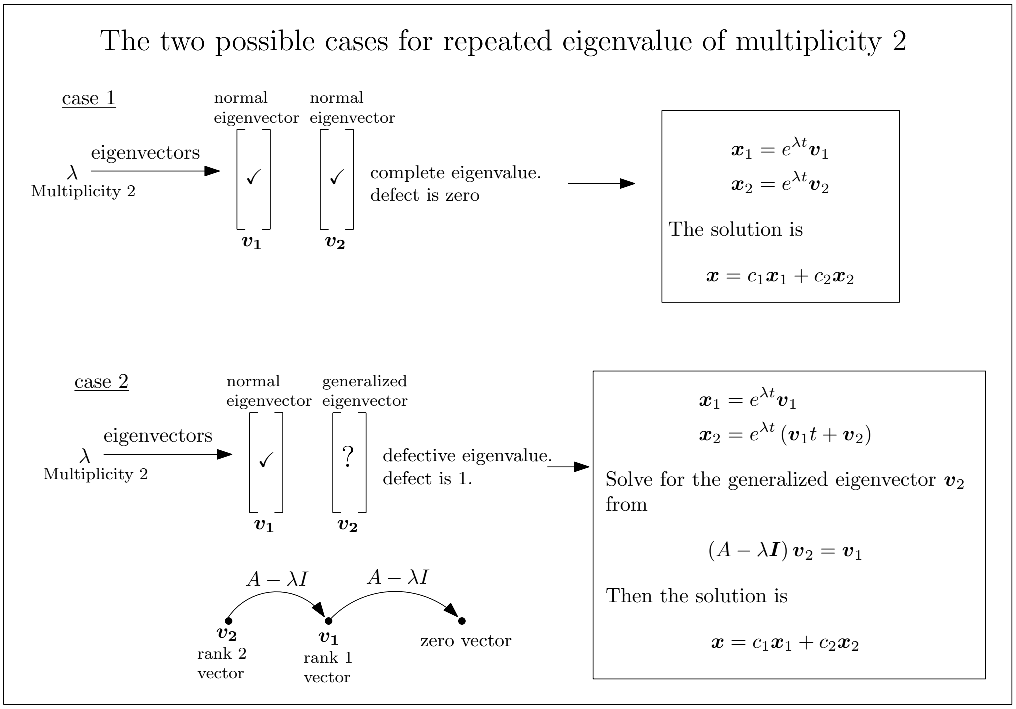


Figure 518: Possible case for repeated λ of multiplicity 2

This eigenvalue has algebraic multiplicity of 2, and geometric multiplicity 1, therefore this is defective eigenvalue. The defect is 1. This falls into case 2 shown above. We need to generate the missing additonal generalized eigevector $\vec{v}_2$ by solving

$$(A - \lambda I)\,\vec{v}_2 = \vec{v}_1$$

Where $\vec{v}_1$ is the normal (rank 1) eigenvector found above. Hence we need to solve

$$\left(\begin{bmatrix} 3 & -4 \\ 1 & -1 \end{bmatrix} - (1)\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right)\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$
$$\begin{bmatrix} 2 & -4 \\ 1 & -2 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

Solving for $\vec{v}_2$ gives

$$\vec{v}_2 = \begin{bmatrix} 3 \\ 1 \end{bmatrix}$$

We have found two generalized eigenvectors for eigenvalue 1. Therefore the two basis solution associated with this eigenvalue are

$$\begin{aligned}\vec{x}_1(t) &= \vec{v}_1 e^{\lambda t} \\ &= \begin{bmatrix} 2 \\ 1 \end{bmatrix} \mathrm{e}^{t} \\ &= \begin{bmatrix} 2\,\mathrm{e}^{t} \\ \mathrm{e}^{t} \end{bmatrix}\end{aligned}$$

And

$$\begin{aligned}\vec{x}_2(t) &= (\vec{v}_1 t + \vec{v}_2)\, e^{\lambda t} \\ &= \left(\begin{bmatrix} 2 \\ 1 \end{bmatrix} t + \begin{bmatrix} 3 \\ 1 \end{bmatrix}\right) \mathrm{e}^{t} \\ &= \begin{bmatrix} \mathrm{e}^{t}(2t+3) \\ \mathrm{e}^{t}(t+1) \end{bmatrix}\end{aligned}$$

Therefore the final solution is

$$\vec{x}_h(t) = c_1 \vec{x}_1(t) + c_2 \vec{x}_2(t)$$

Which is written as

$$\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = c_1 \begin{bmatrix} 2\,\mathrm{e}^{t} \\ \mathrm{e}^{t} \end{bmatrix} + c_2 \begin{bmatrix} \mathrm{e}^{t}(2t+3) \\ \mathrm{e}^{t}(t+1) \end{bmatrix}$$

Which becomes

$$\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = \begin{bmatrix} ((2t+3)\,c_2 + 2c_1)\,\mathrm{e}^{t} \\ \mathrm{e}^{t}(c_2 t + c_1 + c_2) \end{bmatrix}$$

The following is the phase plot of the system.

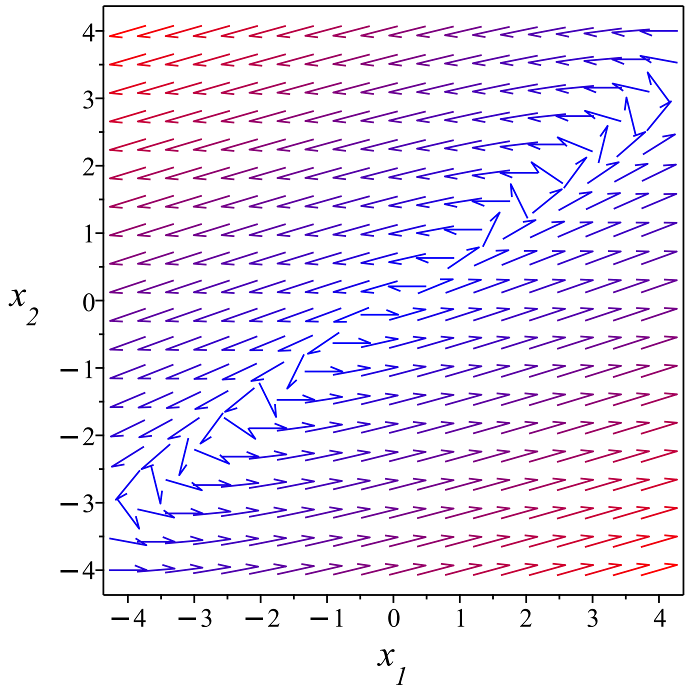


Figure 519: Phase plot

17.1.3 Maple step by step solution

Let's solve

$[x_1'(t) = 3x_1(t) - 4x_2(t), x_2'(t) = x_1(t) - x_2(t)]$

- Define vector

$$\underline{x}^{\rightarrow}(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}$$

- Convert system into a vector equation

$$\underline{x}^{\rightarrow}{}'(t) = \begin{bmatrix} 3 & -4 \\ 1 & -1 \end{bmatrix} \cdot \underline{x}^{\rightarrow}(t) + \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

- System to solve

$$\underline{x}^{\rightarrow}{}'(t) = \begin{bmatrix} 3 & -4 \\ 1 & -1 \end{bmatrix} \cdot \underline{x}^{\rightarrow}(t)$$

- Define the coefficient matrix

$$A = \begin{bmatrix} 3 & -4 \\ 1 & -1 \end{bmatrix}$$

- Rewrite the system as

  $$\underline{x^{\rightarrow}}'(t) = A \cdot \underline{x^{\rightarrow}}(t)$$

- To solve the system, find the eigenvalues and eigenvectors of A
- Eigenpairs of A

  $$\left[\left[1, \begin{bmatrix} 2 \\ 1 \end{bmatrix}\right], \left[1, \begin{bmatrix} 0 \\ 0 \end{bmatrix}\right]\right]$$

- Consider eigenpair, with eigenvalue of algebraic multiplicity 2

  $$\left[1, \begin{bmatrix} 2 \\ 1 \end{bmatrix}\right]$$

- First solution from eigenvalue 1

  $$\underline{x^{\rightarrow}}_1(t) = e^t \cdot \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

- Form of the 2nd homogeneous solution where $\vec{p}$ is to be solved for, $\lambda = 1$ is the eigenvalue, and

  $$\underline{x^{\rightarrow}}_2(t) = e^{\lambda t}\left(t\vec{v} + \vec{p}\right)$$

- Note that the t multiplying $\vec{v}$ makes this solution linearly independent to the 1st solution obtai
- Substitute $\underline{x^{\rightarrow}}_2(t)$ into the homogeneous system

  $$\lambda e^{\lambda t}\left(t\vec{v} + \vec{p}\right) + e^{\lambda t}\vec{v} = \left(e^{\lambda t} A\right) \cdot \left(t\vec{v} + \vec{p}\right)$$

- Use the fact that $\vec{v}$ is an eigenvector of A

  $$\lambda e^{\lambda t}\left(t\vec{v} + \vec{p}\right) + e^{\lambda t}\vec{v} = e^{\lambda t}\left(\lambda t\vec{v} + A \cdot \vec{p}\right)$$

- Simplify equation

  $$\lambda\vec{p} + \vec{v} = A \cdot \vec{p}$$

- Make use of the identity matrix I

  $$(\lambda \cdot I) \cdot \vec{p} + \vec{v} = A \cdot \vec{p}$$

- Condition $\vec{p}$ must meet for $\underline{x^{\rightarrow}}_2(t)$ to be a solution to the homogeneous system

  $$(A - \lambda \cdot I) \cdot \vec{p} = \vec{v}$$

- Choose $\vec{p}$ to use in the second solution to the homogeneous system from eigenvalue 1

$$\left(\begin{bmatrix} 3 & -4 \\ 1 & -1 \end{bmatrix} - 1 \cdot \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right) \cdot \vec{p} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

- Choice of $\vec{p}$

$$\vec{p} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

- Second solution from eigenvalue 1

$$\vec{x}_2(t) = e^{t} \cdot \left(t \cdot \begin{bmatrix} 2 \\ 1 \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \end{bmatrix}\right)$$

- General solution to the system of ODEs

$$\vec{x} = c_1 \vec{x}_1(t) + c_2 \vec{x}_2(t)$$

- Substitute solutions into the general solution

$$\vec{x} = c_1 e^{t} \cdot \begin{bmatrix} 2 \\ 1 \end{bmatrix} + c_2 e^{t} \cdot \left(t \cdot \begin{bmatrix} 2 \\ 1 \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \end{bmatrix}\right)$$

- Substitute in vector of dependent variables

$$\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = \begin{bmatrix} e^{t}(2c_2 t + 2c_1 + c_2) \\ e^{t}(c_2 t + c_1) \end{bmatrix}$$

- Solution to the system of ODEs

$$\{x_1(t) = e^{t}(2c_2 t + 2c_1 + c_2), x_2(t) = e^{t}(c_2 t + c_1)\}$$

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 31

```
dsolve([diff(x__1(t),t)=3*x__1(t)-4*x__2(t),diff(x__2(t),t)=1*x__1(t)-1*x__2(t)],singsol=all
```

$$x_1(t) = e^{t}(c_2 t + c_1)$$
$$x_2(t) = \frac{e^{t}(2c_2 t + 2c_1 - c_2)}{4}$$

✓ Solution by Mathematica

Time used: 0.002 (sec). Leaf size: 41

```
DSolve[{x1'[t]==3*x1[t]-4*x2[t],x2'[t]==1*x1[t]-1*x2[t]},{x1[t],x2[t]},t,IncludeSingularSolu
```

$$x1(t) \to e^t(2c_1t - 4c_2t + c_1)$$
$$x2(t) \to e^t((c_1 - 2c_2)t + c_2)$$

17.2 problem 2

Internal problem ID [767]
Internal file name [`OUTPUT/767_Sunday_June_05_2022_01_49_10_AM_87296969/index.tex`]

Book: Elementary differential equations and boundary value problems, 10th ed., Boyce and DiPrima
Section: Chapter 7.8, Repeated Eigenvalues. page 436
Problem number: 2.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"system of linear ODEs"**

Solve

$$x_1'(t) = 4x_1(t) - 2x_2(t)$$
$$x_2'(t) = 8x_1(t) - 4x_2(t)$$

17.2.1 Solution using Matrix exponential method

In this method, we will assume we have found the matrix exponential e^{At} allready. There are different methods to determine this but will not be shown here. This is a system of linear ODE's given as

$$\vec{x}'(t) = A\,\vec{x}(t)$$

Or

$$\begin{bmatrix} x_1'(t) \\ x_2'(t) \end{bmatrix} = \begin{bmatrix} 4 & -2 \\ 8 & -4 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}$$

For the above matrix A, the matrix exponential can be found to be

$$e^{At} = \begin{bmatrix} 4t+1 & -2t \\ 8t & 1-4t \end{bmatrix}$$

Therefore the homogeneous solution is

$$
\begin{aligned}
\vec{x}_h(t) &= e^{At}\vec{c} \\
&= \begin{bmatrix} 4t+1 & -2t \\ 8t & 1-4t \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} \\
&= \begin{bmatrix} (4t+1)\,c_1 - 2tc_2 \\ 8tc_1 + (1-4t)\,c_2 \end{bmatrix} \\
&= \begin{bmatrix} (4c_1 - 2c_2)\,t + c_1 \\ (8c_1 - 4c_2)\,t + c_2 \end{bmatrix}
\end{aligned}
$$

Since no forcing function is given, then the final solution is $\vec{x}_h(t)$ above.

17.2.2 Solution using explicit Eigenvalue and Eigenvector method

This is a system of linear ODE's given as

$$
\vec{x}'(t) = A\,\vec{x}(t)
$$

Or

$$
\begin{bmatrix} x_1'(t) \\ x_2'(t) \end{bmatrix} = \begin{bmatrix} 4 & -2 \\ 8 & -4 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}
$$

The first step is find the homogeneous solution. We start by finding the eigenvalues of A. This is done by solving the following equation for the eigenvalues λ

$$
\det\left(A - \lambda I\right) = 0
$$

Expanding gives

$$
\det\left(\begin{bmatrix} 4 & -2 \\ 8 & -4 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right) = 0
$$

Therefore

$$
\det\left(\begin{bmatrix} 4-\lambda & -2 \\ 8 & -4-\lambda \end{bmatrix}\right) = 0
$$

Which gives the characteristic equation

$$\lambda^2 = 0$$

The roots of the above are the eigenvalues.

$$\lambda_1 = 0$$

This table summarises the above result

eigenvalue	algebraic multiplicity	type of eigenvalue
0	1	real eigenvalue

Now the eigenvector for each eigenvalue are found.

Considering the eigenvalue $\lambda_1 = 0$

We need to solve $A\vec{v} = \lambda\vec{v}$ or $(A - \lambda I)\vec{v} = \vec{0}$ which becomes

$$\left(\begin{bmatrix} 4 & -2 \\ 8 & -4 \end{bmatrix} - (0)\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right)\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 4 & -2 \\ 8 & -4 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Now forward elimination is applied to solve for the eigenvector $\vec{v}$. The augmented matrix is

$$\left[\begin{array}{cc|c} 4 & -2 & 0 \\ 8 & -4 & 0 \end{array}\right]$$

$$R_2 = R_2 - 2R_1 \Longrightarrow \left[\begin{array}{cc|c} 4 & -2 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

Therefore the system in Echelon form is

$$\begin{bmatrix} 4 & -2 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

The free variables are $\{v_2\}$ and the leading variables are $\{v_1\}$. Let $v_2 = t$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\left\{v_1 = \frac{t}{2}\right\}$

Hence the solution is

$$\begin{bmatrix} \frac{t}{2} \\ t \end{bmatrix} = \begin{bmatrix} \frac{t}{2} \\ t \end{bmatrix}$$

Since there is one free Variable, we have found one eigenvector associated with this eigenvalue. The above can be written as

$$\begin{bmatrix} \frac{t}{2} \\ t \end{bmatrix} = t \begin{bmatrix} \frac{1}{2} \\ 1 \end{bmatrix}$$

Let $t = 1$ the eigenvector becomes

$$\begin{bmatrix} \frac{t}{2} \\ t \end{bmatrix} = \begin{bmatrix} \frac{1}{2} \\ 1 \end{bmatrix}$$

Which is normalized to

$$\begin{bmatrix} \frac{t}{2} \\ t \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

The following table gives a summary of this result. It shows for each eigenvalue the algebraic multiplicity m, and its geometric multiplicity k and the eigenvectors associated with the eigenvalue. If $m > k$ then the eigenvalue is defective which means the number of normal linearly independent eigenvectors associated with this eigenvalue (called the geometric multiplicity k) does not equal the algebraic multiplicity m, and we need to determine an additional $m - k$ generalized eigenvectors for this eigenvalue.

	multiplicity			
eigenvalue	algebraic m	geometric k	defective?	eigenvectors
0	2	1	Yes	$\begin{bmatrix} \frac{1}{2} \\ 1 \end{bmatrix}$

Now that we found the eigenvalues and associated eigenvectors, we will go over each eigenvalue and generate the solution basis. The only problem we need to take care of is if the eigenvalue is defective. eigenvalue 0 is real and repated eigenvalue of multiplicity 2.There are two possible cases that can happen. This is illustrated in this diagram

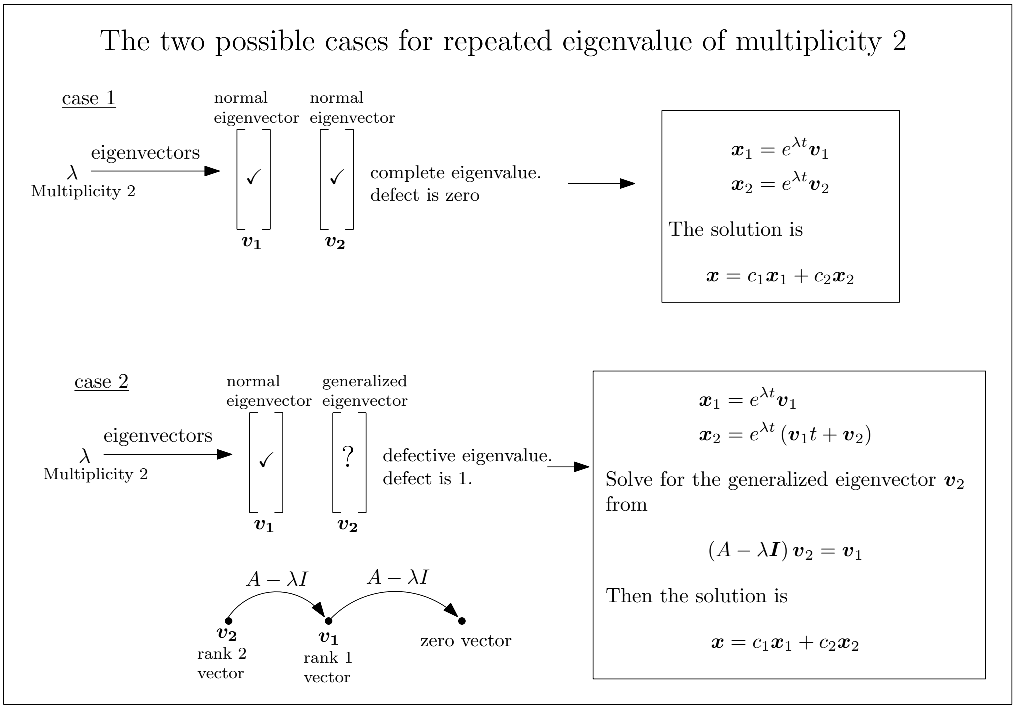


Figure 520: Possible case for repeated λ of multiplicity 2

This eigenvalue has algebraic multiplicity of 2, and geometric multiplicity 1, therefore this is defective eigenvalue. The defect is 1. This falls into case 2 shown above. We need to generate the missing additonal generalized eigevector $\vec{v}_2$ by solving

$$(A - \lambda I)\,\vec{v}_2 = \vec{v}_1$$

Where $\vec{v}_1$ is the normal (rank 1) eigenvector found above. Hence we need to solve

$$\left(\begin{bmatrix} 4 & -2 \\ 8 & -4 \end{bmatrix} - (0)\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right)\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} \frac{1}{2} \\ 1 \end{bmatrix}$$

$$\begin{bmatrix} 4 & -2 \\ 8 & -4 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} \frac{1}{2} \\ 1 \end{bmatrix}$$

Solving for $\vec{v}_2$ gives

$$\vec{v}_2 = \begin{bmatrix} 1 \\ \frac{7}{4} \end{bmatrix}$$

We have found two generalized eigenvectors for eigenvalue 0. Therefore the two basis solution associated with this eigenvalue are

$$
\begin{aligned}
\vec{x}_1(t) &= \vec{v}_1 e^{\lambda t} \\
&= \begin{bmatrix} \frac{1}{2} \\ 1 \end{bmatrix} 1 \\
&= \begin{bmatrix} \frac{1}{2} \\ 1 \end{bmatrix}
\end{aligned}
$$

And

$$
\begin{aligned}
\vec{x}_2(t) &= (\vec{v}_1 t + \vec{v}_2) e^{\lambda t} \\
&= \left(\begin{bmatrix} \frac{1}{2} \\ 1 \end{bmatrix} t + \begin{bmatrix} 1 \\ \frac{7}{4} \end{bmatrix} \right) 1 \\
&= \begin{bmatrix} \frac{t}{2} + 1 \\ t + \frac{7}{4} \end{bmatrix}
\end{aligned}
$$

Therefore the final solution is

$$
\vec{x}_h(t) = c_1 \vec{x}_1(t) + c_2 \vec{x}_2(t)
$$

Which is written as

$$
\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = c_1 \begin{bmatrix} \frac{1}{2} \\ 1 \end{bmatrix} + c_2 \begin{bmatrix} \frac{t}{2} + 1 \\ t + \frac{7}{4} \end{bmatrix}
$$

Which becomes

$$
\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = \begin{bmatrix} \frac{1}{2} c_1 + \frac{1}{2} c_2 t + c_2 \\ c_1 + c_2 t + \frac{7}{4} c_2 \end{bmatrix}
$$

The following is the phase plot of the system.

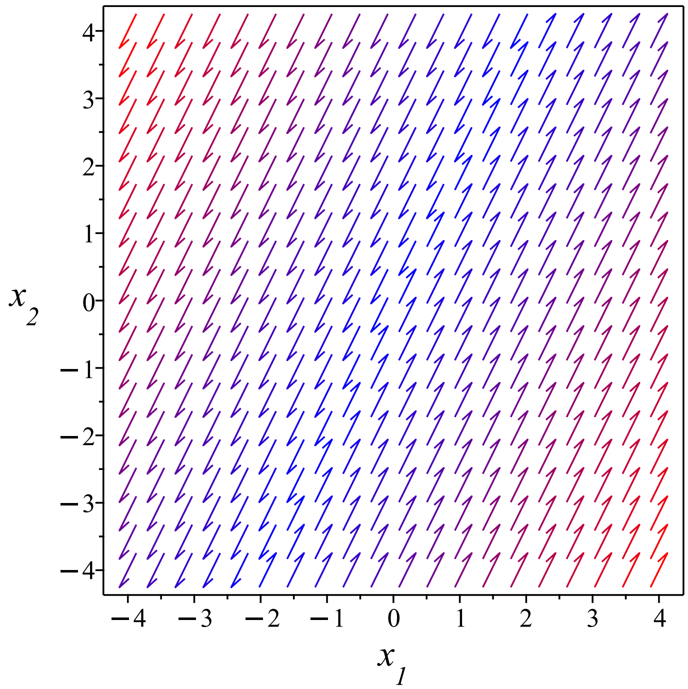


Figure 521: Phase plot

17.2.3 Maple step by step solution

Let's solve

$[x_1'(t) = 4x_1(t) - 2x_2(t), x_2'(t) = 8x_1(t) - 4x_2(t)]$

- Define vector

$$\underline{x}^{\rightarrow}(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}$$

- Convert system into a vector equation

$$\underline{x}^{\rightarrow}{}'(t) = \begin{bmatrix} 4 & -2 \\ 8 & -4 \end{bmatrix} \cdot \underline{x}^{\rightarrow}(t) + \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

- System to solve

$$\underline{x}^{\rightarrow}{}'(t) = \begin{bmatrix} 4 & -2 \\ 8 & -4 \end{bmatrix} \cdot \underline{x}^{\rightarrow}(t)$$

- Define the coefficient matrix

$$A = \begin{bmatrix} 4 & -2 \\ 8 & -4 \end{bmatrix}$$

- Rewrite the system as

$$\underline{\vec{x}}'(t) = A \cdot \underline{\vec{x}}(t)$$

- To solve the system, find the eigenvalues and eigenvectors of A
- Eigenpairs of A

$$\left[\left[0, \begin{bmatrix} \frac{1}{2} \\ 1 \end{bmatrix}\right], \left[0, \begin{bmatrix} 0 \\ 0 \end{bmatrix}\right]\right]$$

- Consider eigenpair

$$\left[0, \begin{bmatrix} \frac{1}{2} \\ 1 \end{bmatrix}\right]$$

- Solution to homogeneous system from eigenpair

$$\underline{\vec{x}}_1 = \begin{bmatrix} \frac{1}{2} \\ 1 \end{bmatrix}$$

- Consider eigenpair

$$\left[0, \begin{bmatrix} 0 \\ 0 \end{bmatrix}\right]$$

- Solution to homogeneous system from eigenpair

$$\underline{\vec{x}}_2 = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

- General solution to the system of ODEs

$$\underline{\vec{x}} = c_1 \underline{\vec{x}}_1 + c_2 \underline{\vec{x}}_2$$

- Substitute solutions into the general solution

$$\underline{\vec{x}} = \begin{bmatrix} \frac{c_1}{2} \\ c_1 \end{bmatrix}$$

- Substitute in vector of dependent variables

$$\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = \begin{bmatrix} \frac{c_1}{2} \\ c_1 \end{bmatrix}$$

- Solution to the system of ODEs

  $\left\{x_1(t) = \frac{c_1}{2}, x_2(t) = c_1\right\}$

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 24

```
dsolve([diff(x__1(t),t)=4*x__1(t)-2*x__2(t),diff(x__2(t),t)=8*x__1(t)-4*x__2(t)],singsol=all
```

$$x_1(t) = c_1 t + c_2$$
$$x_2(t) = -\frac{1}{2}c_1 + 2c_1 t + 2c_2$$

✓ Solution by Mathematica

Time used: 0.002 (sec). Leaf size: 34

```
DSolve[{x1'[t]==4*x1[t]-2*x2[t],x2'[t]==8*x1[t]-4*x2[t]},{x1[t],x2[t]},t,IncludeSingularSolu
```

$$\mathrm{x1}(t) \to 4c_1 t - 2c_2 t + c_1$$
$$\mathrm{x2}(t) \to 8c_1 t - 4c_2 t + c_2$$

17.3 problem 3

Internal problem ID [768]
Internal file name [`OUTPUT/768_Sunday_June_05_2022_01_49_11_AM_71145901/index.tex`]

Book: Elementary differential equations and boundary value problems, 10th ed., Boyce and DiPrima
Section: Chapter 7.8, Repeated Eigenvalues. page 436
Problem number: 3.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"system of linear ODEs"**

Solve

$$x_1'(t) = -\frac{3x_1(t)}{2} + x_2(t)$$
$$x_2'(t) = -\frac{x_1(t)}{4} - \frac{x_2(t)}{2}$$

17.3.1 Solution using Matrix exponential method

In this method, we will assume we have found the matrix exponential e^{At} allready. There are different methods to determine this but will not be shown here. This is a system of linear ODE's given as

$$\vec{x}'(t) = A\,\vec{x}(t)$$

Or

$$\begin{bmatrix} x_1'(t) \\ x_2'(t) \end{bmatrix} = \begin{bmatrix} -\frac{3}{2} & 1 \\ -\frac{1}{4} & -\frac{1}{2} \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}$$

For the above matrix A, the matrix exponential can be found to be

$$e^{At} = \begin{bmatrix} e^{-t}\left(-\frac{t}{2}+1\right) & t\,e^{-t} \\ -\frac{t\,e^{-t}}{4} & e^{-t}\left(\frac{t}{2}+1\right) \end{bmatrix}$$

Therefore the homogeneous solution is

$$
\begin{aligned}
\vec{x}_h(t) &= e^{At}\vec{c} \\
&= \begin{bmatrix} e^{-t}\left(-\frac{t}{2}+1\right) & t\,e^{-t} \\ -\frac{t\,e^{-t}}{4} & e^{-t}\left(\frac{t}{2}+1\right) \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} \\
&= \begin{bmatrix} e^{-t}\left(-\frac{t}{2}+1\right) c_1 + t\,e^{-t} c_2 \\ -\frac{t\,e^{-t} c_1}{4} + e^{-t}\left(\frac{t}{2}+1\right) c_2 \end{bmatrix} \\
&= \begin{bmatrix} e^{-t}\left(-\frac{1}{2} t c_1 + c_1 + c_2 t\right) \\ -\frac{((-2t-4)c_2 + t c_1)e^{-t}}{4} \end{bmatrix}
\end{aligned}
$$

Since no forcing function is given, then the final solution is $\vec{x}_h(t)$ above.

17.3.2 Solution using explicit Eigenvalue and Eigenvector method

This is a system of linear ODE's given as

$$
\vec{x}'(t) = A\,\vec{x}(t)
$$

Or

$$
\begin{bmatrix} x_1'(t) \\ x_2'(t) \end{bmatrix} = \begin{bmatrix} -\frac{3}{2} & 1 \\ -\frac{1}{4} & -\frac{1}{2} \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}
$$

The first step is find the homogeneous solution. We start by finding the eigenvalues of A. This is done by solving the following equation for the eigenvalues λ

$$
\det\left(A - \lambda I\right) = 0
$$

Expanding gives

$$
\det\left(\begin{bmatrix} -\frac{3}{2} & 1 \\ -\frac{1}{4} & -\frac{1}{2} \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right) = 0
$$

Therefore

$$
\det\left(\begin{bmatrix} -\frac{3}{2} - \lambda & 1 \\ -\frac{1}{4} & -\frac{1}{2} - \lambda \end{bmatrix}\right) = 0
$$

Which gives the characteristic equation

$$\lambda^2 + 2\lambda + 1 = 0$$

The roots of the above are the eigenvalues.

$$\lambda_1 = -1$$

This table summarises the above result

eigenvalue	algebraic multiplicity	type of eigenvalue
-1	1	real eigenvalue

Now the eigenvector for each eigenvalue are found.

Considering the eigenvalue $\lambda_1 = -1$

We need to solve $A\vec{v} = \lambda\vec{v}$ or $(A - \lambda I)\vec{v} = \vec{0}$ which becomes

$$\left(\begin{bmatrix} -\frac{3}{2} & 1 \\ -\frac{1}{4} & -\frac{1}{2} \end{bmatrix} - (-1)\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right)\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$
$$\begin{bmatrix} -\frac{1}{2} & 1 \\ -\frac{1}{4} & \frac{1}{2} \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Now forward elimination is applied to solve for the eigenvector $\vec{v}$. The augmented matrix is

$$\left[\begin{array}{cc|c} -\frac{1}{2} & 1 & 0 \\ -\frac{1}{4} & \frac{1}{2} & 0 \end{array}\right]$$

$$R_2 = R_2 - \frac{R_1}{2} \Longrightarrow \left[\begin{array}{cc|c} -\frac{1}{2} & 1 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

Therefore the system in Echelon form is

$$\begin{bmatrix} -\frac{1}{2} & 1 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

The free variables are $\{v_2\}$ and the leading variables are $\{v_1\}$. Let $v_2 = t$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\{v_1 = 2t\}$

Hence the solution is

$$\begin{bmatrix} 2t \\ t \end{bmatrix} = \begin{bmatrix} 2t \\ t \end{bmatrix}$$

Since there is one free Variable, we have found one eigenvector associated with this eigenvalue. The above can be written as

$$\begin{bmatrix} 2t \\ t \end{bmatrix} = t \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

Let $t = 1$ the eigenvector becomes

$$\begin{bmatrix} 2t \\ t \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

The following table gives a summary of this result. It shows for each eigenvalue the algebraic multiplicity m, and its geometric multiplicity k and the eigenvectors associated with the eigenvalue. If $m > k$ then the eigenvalue is defective which means the number of normal linearly independent eigenvectors associated with this eigenvalue (called the geometric multiplicity k) does not equal the algebraic multiplicity m, and we need to determine an additional $m - k$ generalized eigenvectors for this eigenvalue.

	multiplicity			
eigenvalue	algebraic m	geometric k	defective?	eigenvectors
-1	2	1	Yes	$\begin{bmatrix} 2 \\ 1 \end{bmatrix}$

Now that we found the eigenvalues and associated eigenvectors, we will go over each eigenvalue and generate the solution basis. The only problem we need to take care of is if the eigenvalue is defective. eigenvalue -1 is real and repated eigenvalue of multiplicity 2.There are two possible cases that can happen. This is illustrated in this diagram

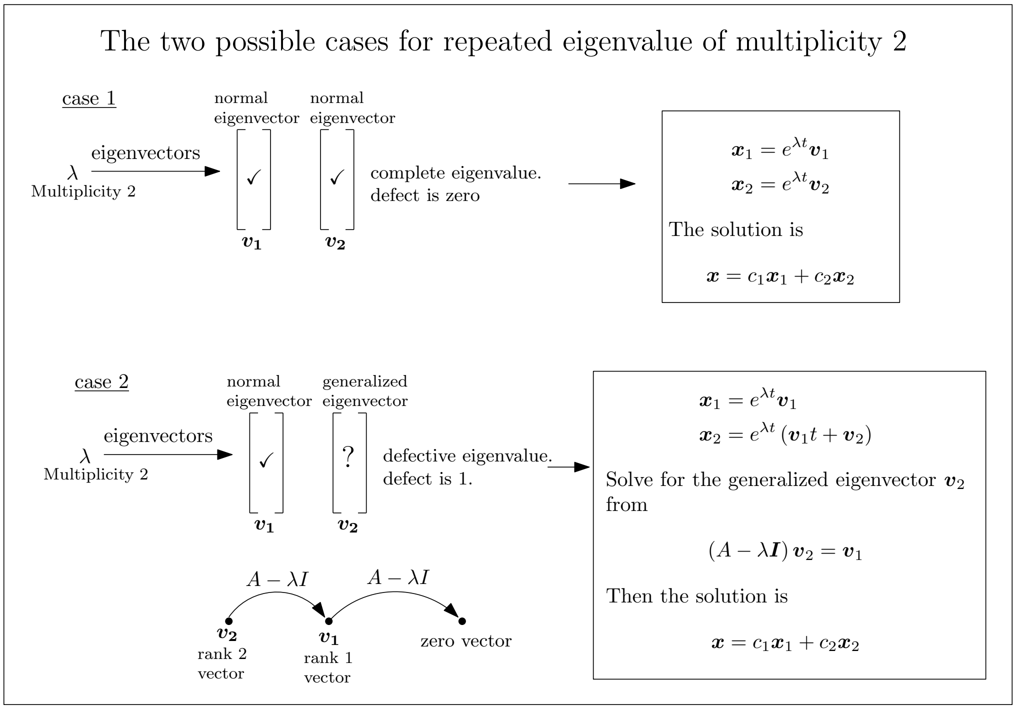


Figure 522: Possible case for repeated λ of multiplicity 2

This eigenvalue has algebraic multiplicity of 2, and geometric multiplicity 1, therefore this is defective eigenvalue. The defect is 1. This falls into case 2 shown above. We need to generate the missing additonal generalized eigevector $\vec{v}_2$ by solving

$$(A - \lambda I)\,\vec{v}_2 = \vec{v}_1$$

Where $\vec{v}_1$ is the normal (rank 1) eigenvector found above. Hence we need to solve

$$\left(\begin{bmatrix} -\frac{3}{2} & 1 \\ -\frac{1}{4} & -\frac{1}{2} \end{bmatrix} - (-1)\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right)\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$
$$\begin{bmatrix} -\frac{1}{2} & 1 \\ -\frac{1}{4} & \frac{1}{2} \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

Solving for $\vec{v}_2$ gives

$$\vec{v}_2 = \begin{bmatrix} -2 \\ 1 \end{bmatrix}$$

We have found two generalized eigenvectors for eigenvalue -1. Therefore the two basis solution associated with this eigenvalue are

$$\begin{aligned}
\vec{x}_1(t) &= \vec{v}_1 e^{\lambda t} \\
&= \begin{bmatrix} 2 \\ 1 \end{bmatrix} \mathrm{e}^{-t} \\
&= \begin{bmatrix} 2\,\mathrm{e}^{-t} \\ \mathrm{e}^{-t} \end{bmatrix}
\end{aligned}$$

And

$$\begin{aligned}
\vec{x}_2(t) &= (\vec{v}_1 t + \vec{v}_2)\, e^{\lambda t} \\
&= \left(\begin{bmatrix} 2 \\ 1 \end{bmatrix} t + \begin{bmatrix} -2 \\ 1 \end{bmatrix} \right) \mathrm{e}^{-t} \\
&= \begin{bmatrix} 2(-1+t)\,\mathrm{e}^{-t} \\ \mathrm{e}^{-t}(t+1) \end{bmatrix}
\end{aligned}$$

Therefore the final solution is

$$\vec{x}_h(t) = c_1 \vec{x}_1(t) + c_2 \vec{x}_2(t)$$

Which is written as

$$\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = c_1 \begin{bmatrix} 2\,\mathrm{e}^{-t} \\ \mathrm{e}^{-t} \end{bmatrix} + c_2 \begin{bmatrix} \mathrm{e}^{-t}(2t-2) \\ \mathrm{e}^{-t}(t+1) \end{bmatrix}$$

Which becomes

$$\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = \begin{bmatrix} 2((-1+t)\,c_2 + c_1)\,\mathrm{e}^{-t} \\ \mathrm{e}^{-t}(c_2 t + c_1 + c_2) \end{bmatrix}$$

The following is the phase plot of the system.

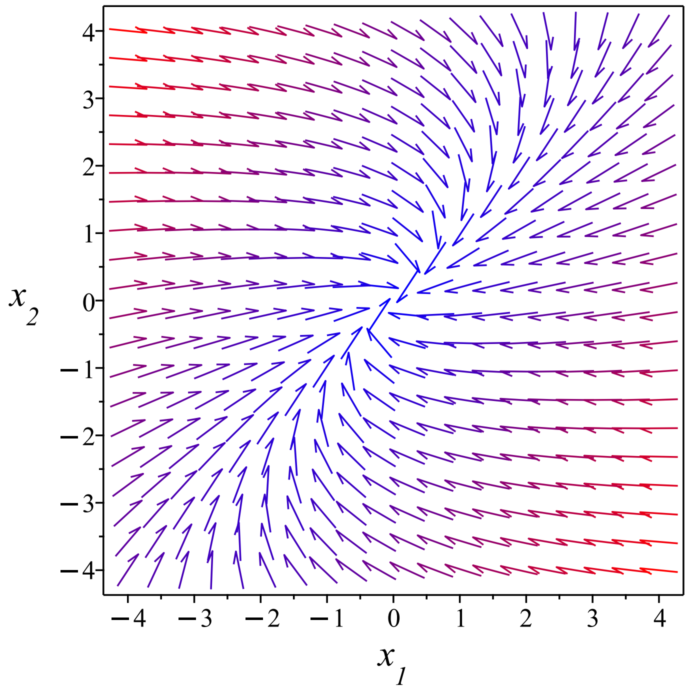


Figure 523: Phase plot

17.3.3 Maple step by step solution

Let's solve

$$\left[x_1'(t)=-\frac{3x_1(t)}{2}+x_2(t), x_2'(t)=-\frac{x_1(t)}{4}-\frac{x_2(t)}{2}\right]$$

- Define vector

$$\underline{x}^{\rightarrow}(t)=\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}$$

- Convert system into a vector equation

$$\underline{x}^{\rightarrow\prime}(t)=\begin{bmatrix} -\frac{3}{2} & 1 \\ -\frac{1}{4} & -\frac{1}{2} \end{bmatrix}\cdot\underline{x}^{\rightarrow}(t)+\begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

- System to solve

$$\underline{x}^{\rightarrow\prime}(t)=\begin{bmatrix} -\frac{3}{2} & 1 \\ -\frac{1}{4} & -\frac{1}{2} \end{bmatrix}\cdot\underline{x}^{\rightarrow}(t)$$

- Define the coefficient matrix

$$A = \begin{bmatrix} -\frac{3}{2} & 1 \\ -\frac{1}{4} & -\frac{1}{2} \end{bmatrix}$$

- Rewrite the system as

  $$\underline{\vec{x}}'(t) = A \cdot \underline{\vec{x}}(t)$$

- To solve the system, find the eigenvalues and eigenvectors of A
- Eigenpairs of A

  $$\left[\left[-1, \begin{bmatrix} 2 \\ 1 \end{bmatrix}\right], \left[-1, \begin{bmatrix} 0 \\ 0 \end{bmatrix}\right]\right]$$

- Consider eigenpair, with eigenvalue of algebraic multiplicity 2

  $$\left[-1, \begin{bmatrix} 2 \\ 1 \end{bmatrix}\right]$$

- First solution from eigenvalue -1

  $$\underline{\vec{x}}_1(t) = \mathrm{e}^{-t} \cdot \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

- Form of the 2nd homogeneous solution where $\vec{p}$ is to be solved for, $\lambda = -1$ is the eigenvalue, an

  $$\underline{\vec{x}}_2(t) = \mathrm{e}^{\lambda t}\left(t\vec{v} + \vec{p}\right)$$

- Note that the t multiplying $\vec{v}$ makes this solution linearly independent to the 1st solution obtai
- Substitute $\underline{\vec{x}}_2(t)$ into the homogeneous system

  $$\lambda\,\mathrm{e}^{\lambda t}\left(t\vec{v} + \vec{p}\right) + \mathrm{e}^{\lambda t}\vec{v} = \left(\mathrm{e}^{\lambda t} A\right) \cdot \left(t\vec{v} + \vec{p}\right)$$

- Use the fact that $\vec{v}$ is an eigenvector of A

  $$\lambda\,\mathrm{e}^{\lambda t}\left(t\vec{v} + \vec{p}\right) + \mathrm{e}^{\lambda t}\vec{v} = \mathrm{e}^{\lambda t}\left(\lambda t\vec{v} + A \cdot \vec{p}\right)$$

- Simplify equation

  $$\lambda\vec{p} + \vec{v} = A \cdot \vec{p}$$

- Make use of the identity matrix I

  $$(\lambda \cdot I) \cdot \vec{p} + \vec{v} = A \cdot \vec{p}$$

- Condition $\vec{p}$ must meet for $\underline{\vec{x}}_2(t)$ to be a solution to the homogeneous system

  $$(A - \lambda \cdot I) \cdot \vec{p} = \vec{v}$$

- Choose $\overrightarrow{p}$ to use in the second solution to the homogeneous system from eigenvalue -1

$$\left(\left[\begin{array}{cc}-\frac{3}{2} & 1 \\ -\frac{1}{4} & -\frac{1}{2}\end{array}\right]-(-1)\cdot\left[\begin{array}{cc}1 & 0 \\ 0 & 1\end{array}\right]\right)\cdot\overrightarrow{p}=\left[\begin{array}{c}2 \\ 1\end{array}\right]$$

- Choice of $\overrightarrow{p}$

$$\overrightarrow{p}=\left[\begin{array}{c}-4 \\ 0\end{array}\right]$$

- Second solution from eigenvalue -1

$$\overrightarrow{x}_{2}(t)=\mathrm{e}^{-t}\cdot\left(t\cdot\left[\begin{array}{c}2 \\ 1\end{array}\right]+\left[\begin{array}{c}-4 \\ 0\end{array}\right]\right)$$

- General solution to the system of ODEs

$$\overrightarrow{x}=c_1\overrightarrow{x}_{1}(t)+c_2\overrightarrow{x}_{2}(t)$$

- Substitute solutions into the general solution

$$\overrightarrow{x}=c_1\mathrm{e}^{-t}\cdot\left[\begin{array}{c}2 \\ 1\end{array}\right]+c_2\mathrm{e}^{-t}\cdot\left(t\cdot\left[\begin{array}{c}2 \\ 1\end{array}\right]+\left[\begin{array}{c}-4 \\ 0\end{array}\right]\right)$$

- Substitute in vector of dependent variables

$$\left[\begin{array}{c}x_1(t) \\ x_2(t)\end{array}\right]=\left[\begin{array}{c}2((t-2)\,c_2+c_1)\,\mathrm{e}^{-t} \\ (c_2t+c_1)\,\mathrm{e}^{-t}\end{array}\right]$$

- Solution to the system of ODEs

$$\{x_1(t)=2((t-2)\,c_2+c_1)\,\mathrm{e}^{-t},x_2(t)=(c_2t+c_1)\,\mathrm{e}^{-t}\}$$

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 32

```
dsolve([diff(x__1(t),t)=-3/2*x__1(t)+1*x__2(t),diff(x__2(t),t)=-1/4*x__1(t)-1/2*x__2(t)],sin
```

$$x_1(t)=\mathrm{e}^{-t}(c_2t+c_1)$$

$$x_2(t)=\frac{\mathrm{e}^{-t}(c_2t+c_1+2c_2)}{2}$$

✓ Solution by Mathematica

Time used: 0.002 (sec). Leaf size: 54

```
DSolve[{x1'[t]==-3/2*x1[t]+1*x2[t],x2'[t]==-1/4*x1[t]-1/2*x2[t]},{x1[t],x2[t]},t,IncludeSing
```

$$\text{x1}(t) \to \frac{1}{2}e^{-t}(2c_2t - c_1(t-2))$$
$$\text{x2}(t) \to \frac{1}{4}e^{-t}(c_1(-t) + 2c_2t + 4c_2)$$

17.4 problem 4

Internal problem ID [769]
Internal file name [`OUTPUT/769_Sunday_June_05_2022_01_49_12_AM_14655444/index.tex`]

Book: Elementary differential equations and boundary value problems, 10th ed., Boyce and DiPrima
Section: Chapter 7.8, Repeated Eigenvalues. page 436
Problem number: 4.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"system of linear ODEs"**

Solve

$$x_1'(t) = -3x_1(t) + \frac{5x_2(t)}{2}$$
$$x_2'(t) = -\frac{5x_1(t)}{2} + 2x_2(t)$$

17.4.1 Solution using Matrix exponential method

In this method, we will assume we have found the matrix exponential e^{At} allready. There are different methods to determine this but will not be shown here. This is a system of linear ODE's given as

$$\vec{x}'(t) = A\,\vec{x}(t)$$

Or

$$\begin{bmatrix} x_1'(t) \\ x_2'(t) \end{bmatrix} = \begin{bmatrix} -3 & \frac{5}{2} \\ -\frac{5}{2} & 2 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}$$

For the above matrix A, the matrix exponential can be found to be

$$e^{At} = \begin{bmatrix} e^{-\frac{t}{2}}\left(1 - \frac{5t}{2}\right) & \frac{5t\,e^{-\frac{t}{2}}}{2} \\ -\frac{5t\,e^{-\frac{t}{2}}}{2} & e^{-\frac{t}{2}}\left(1 + \frac{5t}{2}\right) \end{bmatrix}$$

Therefore the homogeneous solution is

$$
\begin{aligned}
\vec{x}_h(t) &= e^{At}\vec{c} \\
&= \begin{bmatrix} e^{-\frac{t}{2}}\left(1-\frac{5t}{2}\right) & \frac{5t\,e^{-\frac{t}{2}}}{2} \\ -\frac{5t\,e^{-\frac{t}{2}}}{2} & e^{-\frac{t}{2}}\left(1+\frac{5t}{2}\right) \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} \\
&= \begin{bmatrix} e^{-\frac{t}{2}}\left(1-\frac{5t}{2}\right)c_1 + \frac{5t\,e^{-\frac{t}{2}}c_2}{2} \\ -\frac{5t\,e^{-\frac{t}{2}}c_1}{2} + e^{-\frac{t}{2}}\left(1+\frac{5t}{2}\right)c_2 \end{bmatrix} \\
&= \begin{bmatrix} \frac{((2-5t)c_1+5c_2t)e^{-\frac{t}{2}}}{2} \\ \frac{((5t+2)c_2-5tc_1)e^{-\frac{t}{2}}}{2} \end{bmatrix}
\end{aligned}
$$

Since no forcing function is given, then the final solution is $\vec{x}_h(t)$ above.

17.4.2 Solution using explicit Eigenvalue and Eigenvector method

This is a system of linear ODE's given as

$$
\vec{x}'(t) = A\,\vec{x}(t)
$$

Or

$$
\begin{bmatrix} x_1'(t) \\ x_2'(t) \end{bmatrix} = \begin{bmatrix} -3 & \frac{5}{2} \\ -\frac{5}{2} & 2 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}
$$

The first step is find the homogeneous solution. We start by finding the eigenvalues of A. This is done by solving the following equation for the eigenvalues λ

$$
\det\left(A - \lambda I\right) = 0
$$

Expanding gives

$$
\det\left(\begin{bmatrix} -3 & \frac{5}{2} \\ -\frac{5}{2} & 2 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right) = 0
$$

Therefore

$$
\det\left(\begin{bmatrix} -3-\lambda & \frac{5}{2} \\ -\frac{5}{2} & 2-\lambda \end{bmatrix}\right) = 0
$$

Which gives the characteristic equation

$$\lambda^2 + \lambda + \frac{1}{4} = 0$$

The roots of the above are the eigenvalues.

$$\lambda_1 = -\frac{1}{2}$$

This table summarises the above result

eigenvalue	algebraic multiplicity	type of eigenvalue
$-\frac{1}{2}$	1	real eigenvalue

Now the eigenvector for each eigenvalue are found.

Considering the eigenvalue $\lambda_1 = -\frac{1}{2}$

We need to solve $A\vec{v} = \lambda\vec{v}$ or $(A - \lambda I)\vec{v} = \vec{0}$ which becomes

$$\left(\begin{bmatrix} -3 & \frac{5}{2} \\ -\frac{5}{2} & 2 \end{bmatrix} - \left(-\frac{1}{2}\right)\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right)\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} -\frac{5}{2} & \frac{5}{2} \\ -\frac{5}{2} & \frac{5}{2} \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Now forward elimination is applied to solve for the eigenvector $\vec{v}$. The augmented matrix is

$$\left[\begin{array}{cc|c} -\frac{5}{2} & \frac{5}{2} & 0 \\ -\frac{5}{2} & \frac{5}{2} & 0 \end{array}\right]$$

$$R_2 = R_2 - R_1 \Longrightarrow \left[\begin{array}{cc|c} -\frac{5}{2} & \frac{5}{2} & 0 \\ 0 & 0 & 0 \end{array}\right]$$

Therefore the system in Echelon form is

$$\begin{bmatrix} -\frac{5}{2} & \frac{5}{2} \\ 0 & 0 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

The free variables are $\{v_2\}$ and the leading variables are $\{v_1\}$. Let $v_2 = t$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\{v_1 = t\}$

Hence the solution is

$$\begin{bmatrix} t \\ t \end{bmatrix} = \begin{bmatrix} t \\ t \end{bmatrix}$$

Since there is one free Variable, we have found one eigenvector associated with this eigenvalue. The above can be written as

$$\begin{bmatrix} t \\ t \end{bmatrix} = t \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

Let $t = 1$ the eigenvector becomes

$$\begin{bmatrix} t \\ t \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

The following table gives a summary of this result. It shows for each eigenvalue the algebraic multiplicity m, and its geometric multiplicity k and the eigenvectors associated with the eigenvalue. If $m > k$ then the eigenvalue is defective which means the number of normal linearly independent eigenvectors associated with this eigenvalue (called the geometric multiplicity k) does not equal the algebraic multiplicity m, and we need to determine an additional $m - k$ generalized eigenvectors for this eigenvalue.

	multiplicity			
eigenvalue	algebraic m	geometric k	defective?	eigenvectors
$-\frac{1}{2}$	2	1	Yes	$\begin{bmatrix} 1 \\ 1 \end{bmatrix}$

Now that we found the eigenvalues and associated eigenvectors, we will go over each eigenvalue and generate the solution basis. The only problem we need to take care of is if the eigenvalue is defective. eigenvalue $-\frac{1}{2}$ is real and repated eigenvalue of multiplicity 2.There are two possible cases that can happen. This is illustrated in this diagram

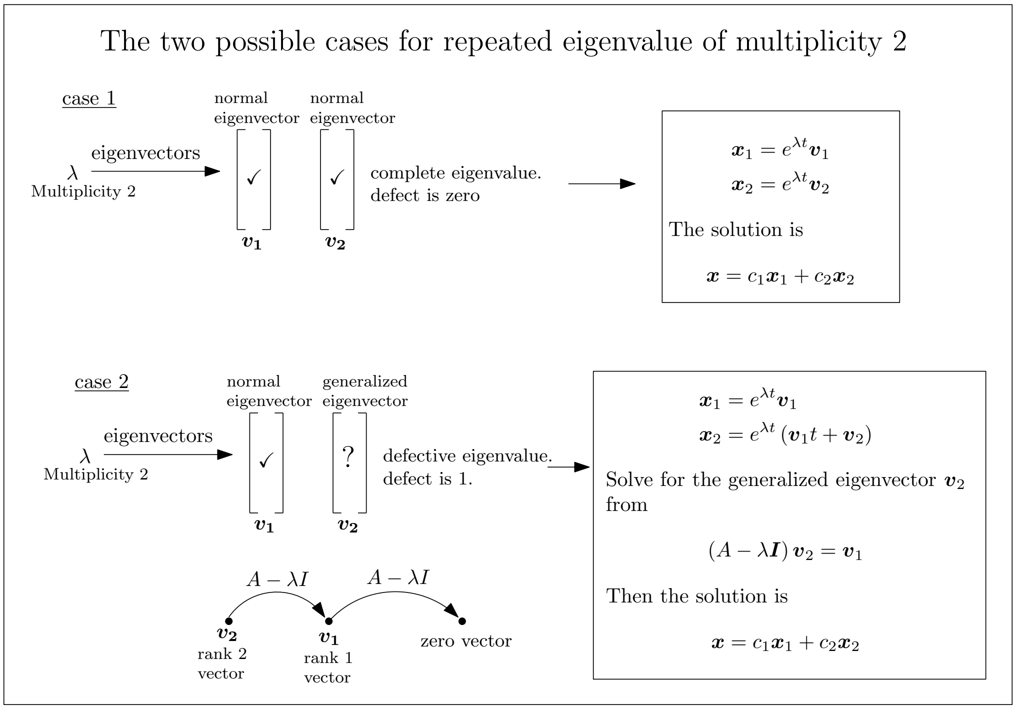


Figure 524: Possible case for repeated λ of multiplicity 2

This eigenvalue has algebraic multiplicity of 2, and geometric multiplicity 1, therefore this is defective eigenvalue. The defect is 1. This falls into case 2 shown above. We need to generate the missing additonal generalized eigevector $\vec{v}_2$ by solving

$$(A - \lambda I)\,\vec{v}_2 = \vec{v}_1$$

Where $\vec{v}_1$ is the normal (rank 1) eigenvector found above. Hence we need to solve

$$\left(\begin{bmatrix} -3 & \frac{5}{2} \\ -\frac{5}{2} & 2 \end{bmatrix} - \left(-\frac{1}{2}\right)\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right)\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$
$$\begin{bmatrix} -\frac{5}{2} & \frac{5}{2} \\ -\frac{5}{2} & \frac{5}{2} \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

Solving for $\vec{v}_2$ gives

$$\vec{v}_2 = \begin{bmatrix} \frac{3}{5} \\ 1 \end{bmatrix}$$

We have found two generalized eigenvectors for eigenvalue $-\frac{1}{2}$. Therefore the two basis solution associated with this eigenvalue are

$$
\begin{aligned}
\vec{x}_1(t) &= \vec{v}_1 e^{\lambda t} \\
&= \begin{bmatrix} 1 \\ 1 \end{bmatrix} \mathrm{e}^{-\frac{t}{2}} \\
&= \begin{bmatrix} \mathrm{e}^{-\frac{t}{2}} \\ \mathrm{e}^{-\frac{t}{2}} \end{bmatrix}
\end{aligned}
$$

And

$$
\begin{aligned}
\vec{x}_2(t) &= (\vec{v}_1 t + \vec{v}_2)\, e^{\lambda t} \\
&= \left(\begin{bmatrix} 1 \\ 1 \end{bmatrix} t + \begin{bmatrix} \frac{3}{5} \\ 1 \end{bmatrix} \right) \mathrm{e}^{-\frac{t}{2}} \\
&= \begin{bmatrix} \frac{\mathrm{e}^{-\frac{t}{2}}(5t+3)}{5} \\ \mathrm{e}^{-\frac{t}{2}}(t+1) \end{bmatrix}
\end{aligned}
$$

Therefore the final solution is

$$
\vec{x}_h(t) = c_1 \vec{x}_1(t) + c_2 \vec{x}_2(t)
$$

Which is written as

$$
\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = c_1 \begin{bmatrix} \mathrm{e}^{-\frac{t}{2}} \\ \mathrm{e}^{-\frac{t}{2}} \end{bmatrix} + c_2 \begin{bmatrix} \mathrm{e}^{-\frac{t}{2}} \left(t + \frac{3}{5}\right) \\ \mathrm{e}^{-\frac{t}{2}}(t+1) \end{bmatrix}
$$

Which becomes

$$
\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = \begin{bmatrix} \mathrm{e}^{-\frac{t}{2}} \left(c_1 + c_2 t + \frac{3}{5} c_2\right) \\ \mathrm{e}^{-\frac{t}{2}} (c_2 t + c_1 + c_2) \end{bmatrix}
$$

The following is the phase plot of the system.

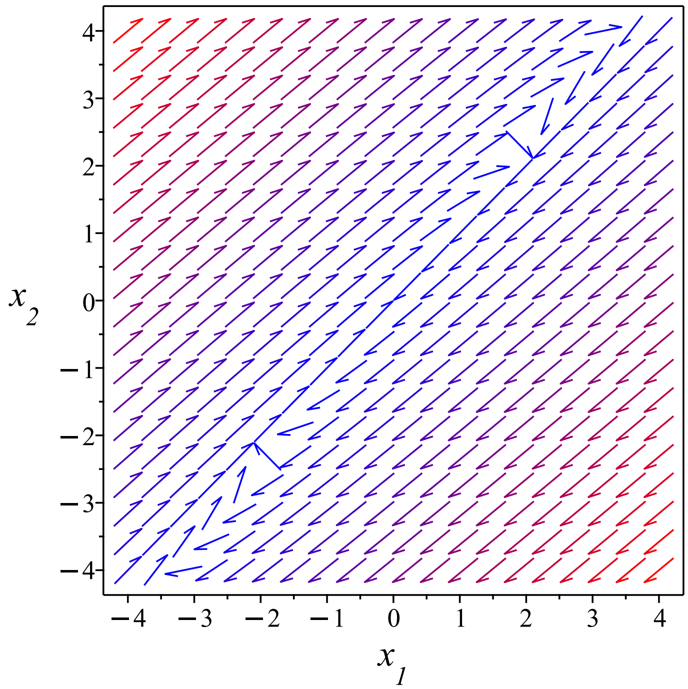


Figure 525: Phase plot

17.4.3 Maple step by step solution

Let's solve

$$\left[x_1'(t) = -3x_1(t) + \frac{5x_2(t)}{2}, x_2'(t) = -\frac{5x_1(t)}{2} + 2x_2(t)\right]$$

- Define vector

$$\underline{x}^{\rightarrow}(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}$$

- Convert system into a vector equation

$$\underline{x}^{\rightarrow\prime}(t) = \begin{bmatrix} -3 & \frac{5}{2} \\ -\frac{5}{2} & 2 \end{bmatrix} \cdot \underline{x}^{\rightarrow}(t) + \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

- System to solve

$$\underline{x}^{\rightarrow\prime}(t) = \begin{bmatrix} -3 & \frac{5}{2} \\ -\frac{5}{2} & 2 \end{bmatrix} \cdot \underline{x}^{\rightarrow}(t)$$

- Define the coefficient matrix

$$A = \begin{bmatrix} -3 & \frac{5}{2} \\ -\frac{5}{2} & 2 \end{bmatrix}$$

- Rewrite the system as

$$\underline{x^{\rightarrow}}'(t) = A \cdot \underline{x^{\rightarrow}}(t)$$

- To solve the system, find the eigenvalues and eigenvectors of A
- Eigenpairs of A

$$\left[\left[-\frac{1}{2}, \begin{bmatrix} 1 \\ 1 \end{bmatrix}\right], \left[-\frac{1}{2}, \begin{bmatrix} 0 \\ 0 \end{bmatrix}\right]\right]$$

- Consider eigenpair, with eigenvalue of algebraic multiplicity 2

$$\left[-\frac{1}{2}, \begin{bmatrix} 1 \\ 1 \end{bmatrix}\right]$$

- First solution from eigenvalue $-\frac{1}{2}$

$$\underline{x^{\rightarrow}}_1(t) = \mathrm{e}^{-\frac{t}{2}} \cdot \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

- Form of the 2nd homogeneous solution where $\vec{p}$ is to be solved for, $\lambda = -\frac{1}{2}$ is the eigenvalue, an

$$\underline{x^{\rightarrow}}_2(t) = \mathrm{e}^{\lambda t}\left(t\vec{v} + \vec{p}\right)$$

- Note that the t multiplying $\vec{v}$ makes this solution linearly independent to the 1st solution obtai
- Substitute $\underline{x^{\rightarrow}}_2(t)$ into the homogeneous system

$$\lambda\,\mathrm{e}^{\lambda t}\left(t\vec{v} + \vec{p}\right) + \mathrm{e}^{\lambda t}\vec{v} = \left(\mathrm{e}^{\lambda t}A\right) \cdot \left(t\vec{v} + \vec{p}\right)$$

- Use the fact that $\vec{v}$ is an eigenvector of A

$$\lambda\,\mathrm{e}^{\lambda t}\left(t\vec{v} + \vec{p}\right) + \mathrm{e}^{\lambda t}\vec{v} = \mathrm{e}^{\lambda t}\left(\lambda t\vec{v} + A \cdot \vec{p}\right)$$

- Simplify equation

$$\lambda\vec{p} + \vec{v} = A \cdot \vec{p}$$

- Make use of the identity matrix I

$$(\lambda \cdot I) \cdot \vec{p} + \vec{v} = A \cdot \vec{p}$$

- Condition $\vec{p}$ must meet for $\underline{x^{\rightarrow}}_2(t)$ to be a solution to the homogeneous system

$$(A - \lambda \cdot I) \cdot \vec{p} = \vec{v}$$

- Choose $\overrightarrow{p}$ to use in the second solution to the homogeneous system from eigenvalue $-\frac{1}{2}$

$$\left(\left[\begin{array}{cc}-3 & \frac{5}{2} \\ -\frac{5}{2} & 2\end{array}\right]--\frac{1}{2}\cdot\left[\begin{array}{cc}1 & 0 \\ 0 & 1\end{array}\right]\right)\cdot\overrightarrow{p}=\left[\begin{array}{c}1 \\ 1\end{array}\right]$$

- Choice of $\overrightarrow{p}$

$$\overrightarrow{p}=\left[\begin{array}{c}-\frac{2}{5} \\ 0\end{array}\right]$$

- Second solution from eigenvalue $-\frac{1}{2}$

$$\overrightarrow{x}_{2}(t)=\mathrm{e}^{-\frac{t}{2}}\cdot\left(t\cdot\left[\begin{array}{c}1 \\ 1\end{array}\right]+\left[\begin{array}{c}-\frac{2}{5} \\ 0\end{array}\right]\right)$$

- General solution to the system of ODEs

$$\overrightarrow{x}=c_1\overrightarrow{x}_{1}(t)+c_2\overrightarrow{x}_{2}(t)$$

- Substitute solutions into the general solution

$$\overrightarrow{x}=c_1\mathrm{e}^{-\frac{t}{2}}\cdot\left[\begin{array}{c}1 \\ 1\end{array}\right]+c_2\mathrm{e}^{-\frac{t}{2}}\cdot\left(t\cdot\left[\begin{array}{c}1 \\ 1\end{array}\right]+\left[\begin{array}{c}-\frac{2}{5} \\ 0\end{array}\right]\right)$$

- Substitute in vector of dependent variables

$$\left[\begin{array}{c}x_1(t) \\ x_2(t)\end{array}\right]=\left[\begin{array}{c}\mathrm{e}^{-\frac{t}{2}}\left(c_1+c_2t-\frac{2}{5}c_2\right) \\ \mathrm{e}^{-\frac{t}{2}}(c_2t+c_1)\end{array}\right]$$

- Solution to the system of ODEs

$$\left\{x_1(t)=\mathrm{e}^{-\frac{t}{2}}\left(c_1+c_2t-\frac{2}{5}c_2\right),x_2(t)=\mathrm{e}^{-\frac{t}{2}}(c_2t+c_1)\right\}$$

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 35

```
dsolve([diff(x__1(t),t)=-3*x__1(t)+5/2*x__2(t),diff(x__2(t),t)=-5/2*x__1(t)+2*x__2(t)],sings
```

$$x_1(t)=\mathrm{e}^{-\frac{t}{2}}(c_2t+c_1)$$
$$x_2(t)=\frac{\mathrm{e}^{-\frac{t}{2}}(5c_2t+5c_1+2c_2)}{5}$$

✓ Solution by Mathematica

Time used: 0.002 (sec). Leaf size: 59

```
DSolve[{x1'[t]==-3*x1[t]+5/2*x2[t],x2'[t]==-5/2*x1[t]+2*x2[t]},{x1[t],x2[t]},t,IncludeSingul
```

$$\text{x1}(t) \to \frac{1}{2}e^{-t/2}(c_1(2-5t)+5c_2t)$$
$$\text{x2}(t) \to \frac{1}{2}e^{-t/2}(-5c_1t+5c_2t+2c_2)$$

17.5 problem 5

Internal problem ID [770]
Internal file name [`OUTPUT/770_Sunday_June_05_2022_01_49_14_AM_32078012/index.tex`]

Book: Elementary differential equations and boundary value problems, 10th ed., Boyce and DiPrima
Section: Chapter 7.8, Repeated Eigenvalues. page 436
Problem number: 5.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"system of linear ODEs"**

Solve

$$
\begin{aligned}
x_1'(t) &= x_1(t) + x_2(t) + x_3(t)\\
x_2'(t) &= 2x_1(t) + x_2(t) - x_3(t)\\
x_3'(t) &= -x_2(t) + x_3(t)
\end{aligned}
$$

17.5.1 Solution using Matrix exponential method

In this method, we will assume we have found the matrix exponential e^{At} allready. There are different methods to determine this but will not be shown here. This is a system of linear ODE's given as

$$\vec{x}'(t) = A\,\vec{x}(t)$$

Or

$$
\begin{bmatrix} x_1'(t) \\ x_2'(t) \\ x_3'(t) \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 \\ 2 & 1 & -1 \\ 0 & -1 & 1 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{bmatrix}
$$

For the above matrix A, the matrix exponential can be found to be

$$
e^{At} = \begin{bmatrix} \frac{e^{-t}}{3} + \frac{2\,e^{2t}}{3} & \frac{e^{2t}}{3} - \frac{e^{-t}}{3} & \frac{e^{2t}}{3} - \frac{e^{-t}}{3} \\ \frac{(6t+4)e^{2t}}{9} - \frac{4\,e^{-t}}{9} & \frac{(3t+5)e^{2t}}{9} + \frac{4\,e^{-t}}{9} & \frac{e^{2t}(3t-4)}{9} + \frac{4\,e^{-t}}{9} \\ \frac{(-6t+2)e^{2t}}{9} - \frac{2\,e^{-t}}{9} & \frac{(-3t-2)e^{2t}}{9} + \frac{2\,e^{-t}}{9} & \frac{(-3t+7)e^{2t}}{9} + \frac{2\,e^{-t}}{9} \end{bmatrix}
$$

Therefore the homogeneous solution is

$$\vec{x}_h(t) = e^{At}\vec{c}$$

$$= \begin{bmatrix} \frac{e^{-t}}{3} + \frac{2e^{2t}}{3} & \frac{e^{2t}}{3} - \frac{e^{-t}}{3} & \frac{e^{2t}}{3} - \frac{e^{-t}}{3} \\ \frac{(6t+4)e^{2t}}{9} - \frac{4e^{-t}}{9} & \frac{(3t+5)e^{2t}}{9} + \frac{4e^{-t}}{9} & \frac{e^{2t}(3t-4)}{9} + \frac{4e^{-t}}{9} \\ \frac{(-6t+2)e^{2t}}{9} - \frac{2e^{-t}}{9} & \frac{(-3t-2)e^{2t}}{9} + \frac{2e^{-t}}{9} & \frac{(-3t+7)e^{2t}}{9} + \frac{2e^{-t}}{9} \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix}$$

$$= \begin{bmatrix} \left(\frac{e^{-t}}{3} + \frac{2e^{2t}}{3}\right)c_1 + \left(\frac{e^{2t}}{3} - \frac{e^{-t}}{3}\right)c_2 + \left(\frac{e^{2t}}{3} - \frac{e^{-t}}{3}\right)c_3 \\ \left(\frac{(6t+4)e^{2t}}{9} - \frac{4e^{-t}}{9}\right)c_1 + \left(\frac{(3t+5)e^{2t}}{9} + \frac{4e^{-t}}{9}\right)c_2 + \left(\frac{e^{2t}(3t-4)}{9} + \frac{4e^{-t}}{9}\right)c_3 \\ \left(\frac{(-6t+2)e^{2t}}{9} - \frac{2e^{-t}}{9}\right)c_1 + \left(\frac{(-3t-2)e^{2t}}{9} + \frac{2e^{-t}}{9}\right)c_2 + \left(\frac{(-3t+7)e^{2t}}{9} + \frac{2e^{-t}}{9}\right)c_3 \end{bmatrix}$$

$$= \begin{bmatrix} \frac{(c_1-c_2-c_3)e^{-t}}{3} + \frac{2\left(c_1+\frac{c_2}{2}+\frac{c_3}{2}\right)e^{2t}}{3} \\ \frac{((6t+4)c_1+(3t+5)c_2+c_3(3t-4))e^{2t}}{9} - \frac{4(c_1-c_2-c_3)e^{-t}}{9} \\ \frac{((-6t+2)c_1+(-3t-2)c_2+(-3t+7)c_3)e^{2t}}{9} - \frac{2(c_1-c_2-c_3)e^{-t}}{9} \end{bmatrix}$$

Since no forcing function is given, then the final solution is $\vec{x}_h(t)$ above.

17.5.2 Solution using explicit Eigenvalue and Eigenvector method

This is a system of linear ODE's given as

$$\vec{x}'(t) = A\,\vec{x}(t)$$

Or

$$\begin{bmatrix} x_1'(t) \\ x_2'(t) \\ x_3'(t) \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 \\ 2 & 1 & -1 \\ 0 & -1 & 1 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{bmatrix}$$

The first step is find the homogeneous solution. We start by finding the eigenvalues of A. This is done by solving the following equation for the eigenvalues λ

$$\det(A - \lambda I) = 0$$

Expanding gives

$$\det\left(\begin{bmatrix} 1 & 1 & 1 \\ 2 & 1 & -1 \\ 0 & -1 & 1 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\right) = 0$$

Therefore

$$\det\left(\begin{bmatrix} 1-\lambda & 1 & 1 \\ 2 & 1-\lambda & -1 \\ 0 & -1 & 1-\lambda \end{bmatrix}\right) = 0$$

Which gives the characteristic equation

$$\lambda^3 - 3\lambda^2 + 4 = 0$$

The roots of the above are the eigenvalues.

$$\lambda_1 = -1$$
$$\lambda_2 = 2$$

This table summarises the above result

eigenvalue	algebraic multiplicity	type of eigenvalue
-1	1	real eigenvalue
2	1	real eigenvalue

Now the eigenvector for each eigenvalue are found.

Considering the eigenvalue $\lambda_1 = -1$

We need to solve $A\vec{v} = \lambda\vec{v}$ or $(A - \lambda I)\vec{v} = \vec{0}$ which becomes

$$\left(\begin{bmatrix} 1 & 1 & 1 \\ 2 & 1 & -1 \\ 0 & -1 & 1 \end{bmatrix} - (-1)\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\right)\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$
$$\begin{bmatrix} 2 & 1 & 1 \\ 2 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Now forward elimination is applied to solve for the eigenvector $\vec{v}$. The augmented matrix is

$$\left[\begin{array}{ccc|c} 2 & 1 & 1 & 0 \\ 2 & 2 & -1 & 0 \\ 0 & -1 & 2 & 0 \end{array}\right]$$

$$R_2 = R_2 - R_1 \Longrightarrow \left[\begin{array}{ccc|c} 2 & 1 & 1 & 0 \\ 0 & 1 & -2 & 0 \\ 0 & -1 & 2 & 0 \end{array}\right]$$

$$R_3 = R_3 + R_2 \Longrightarrow \left[\begin{array}{ccc|c} 2 & 1 & 1 & 0 \\ 0 & 1 & -2 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

Therefore the system in Echelon form is

$$\begin{bmatrix} 2 & 1 & 1 \\ 0 & 1 & -2 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

The free variables are $\{v_3\}$ and the leading variables are $\{v_1, v_2\}$. Let $v_3 = t$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\left\{v_1 = -\frac{3t}{2}, v_2 = 2t\right\}$

Hence the solution is

$$\begin{bmatrix} -\frac{3t}{2} \\ 2t \\ t \end{bmatrix} = \begin{bmatrix} -\frac{3t}{2} \\ 2t \\ t \end{bmatrix}$$

Since there is one free Variable, we have found one eigenvector associated with this eigenvalue. The above can be written as

$$\begin{bmatrix} -\frac{3t}{2} \\ 2t \\ t \end{bmatrix} = t \begin{bmatrix} -\frac{3}{2} \\ 2 \\ 1 \end{bmatrix}$$

Let $t = 1$ the eigenvector becomes

$$\begin{bmatrix} -\frac{3t}{2} \\ 2t \\ t \end{bmatrix} = \begin{bmatrix} -\frac{3}{2} \\ 2 \\ 1 \end{bmatrix}$$

Which is normalized to

$$\begin{bmatrix} -\frac{3t}{2} \\ 2t \\ t \end{bmatrix} = \begin{bmatrix} -3 \\ 4 \\ 2 \end{bmatrix}$$

Considering the eigenvalue $\lambda_2 = 2$

We need to solve $A\vec{v} = \lambda\vec{v}$ or $(A - \lambda I)\vec{v} = \vec{0}$ which becomes

$$\left(\begin{bmatrix} 1 & 1 & 1 \\ 2 & 1 & -1 \\ 0 & -1 & 1 \end{bmatrix} - (2) \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \right) \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} -1 & 1 & 1 \\ 2 & -1 & -1 \\ 0 & -1 & -1 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Now forward elimination is applied to solve for the eigenvector $\vec{v}$. The augmented matrix is

$$\left[\begin{array}{ccc|c} -1 & 1 & 1 & 0 \\ 2 & -1 & -1 & 0 \\ 0 & -1 & -1 & 0 \end{array}\right]$$

$$R_2 = R_2 + 2R_1 \Longrightarrow \left[\begin{array}{ccc|c} -1 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & -1 & -1 & 0 \end{array}\right]$$

$$R_3 = R_3 + R_2 \Longrightarrow \left[\begin{array}{ccc|c} -1 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

Therefore the system in Echelon form is

$$\begin{bmatrix} -1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

The free variables are $\{v_3\}$ and the leading variables are $\{v_1, v_2\}$. Let $v_3 = t$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\{v_1 = 0, v_2 = -t\}$

Hence the solution is

$$\begin{bmatrix} 0 \\ -t \\ t \end{bmatrix} = \begin{bmatrix} 0 \\ -t \\ t \end{bmatrix}$$

Since there is one free Variable, we have found one eigenvector associated with this eigenvalue. The above can be written as

$$\begin{bmatrix} 0 \\ -t \\ t \end{bmatrix} = t \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix}$$

Let $t = 1$ the eigenvector becomes

$$\begin{bmatrix} 0 \\ -t \\ t \end{bmatrix} = \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix}$$

The following table gives a summary of this result. It shows for each eigenvalue the algebraic multiplicity m, and its geometric multiplicity k and the eigenvectors associated with the eigenvalue. If $m > k$ then the eigenvalue is defective which means the number of normal linearly independent eigenvectors associated with this eigenvalue (called the geometric multiplicity k) does not equal the algebraic multiplicity m, and we need to determine an additional $m - k$ generalized eigenvectors for this eigenvalue.

	multiplicity			
eigenvalue	algebraic m	geometric k	defective?	eigenvectors
-1	1	1	No	$\begin{bmatrix} -\frac{3}{2} \\ 2 \\ 1 \end{bmatrix}$
2	2	1	Yes	$\begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix}$

Now that we found the eigenvalues and associated eigenvectors, we will go over each eigenvalue and generate the solution basis. The only problem we need to take care of is if the eigenvalue is defective. Since eigenvalue -1 is real and distinct then the corresponding eigenvector solution is

$$
\begin{aligned}
\vec{x}_1(t) &= \vec{v}_1 e^{-t} \\
&= \begin{bmatrix} -\frac{3}{2} \\ 2 \\ 1 \end{bmatrix} e^{-t}
\end{aligned}
$$

eigenvalue 2 is real and repated eigenvalue of multiplicity 2.There are two possible cases that can happen. This is illustrated in this diagram

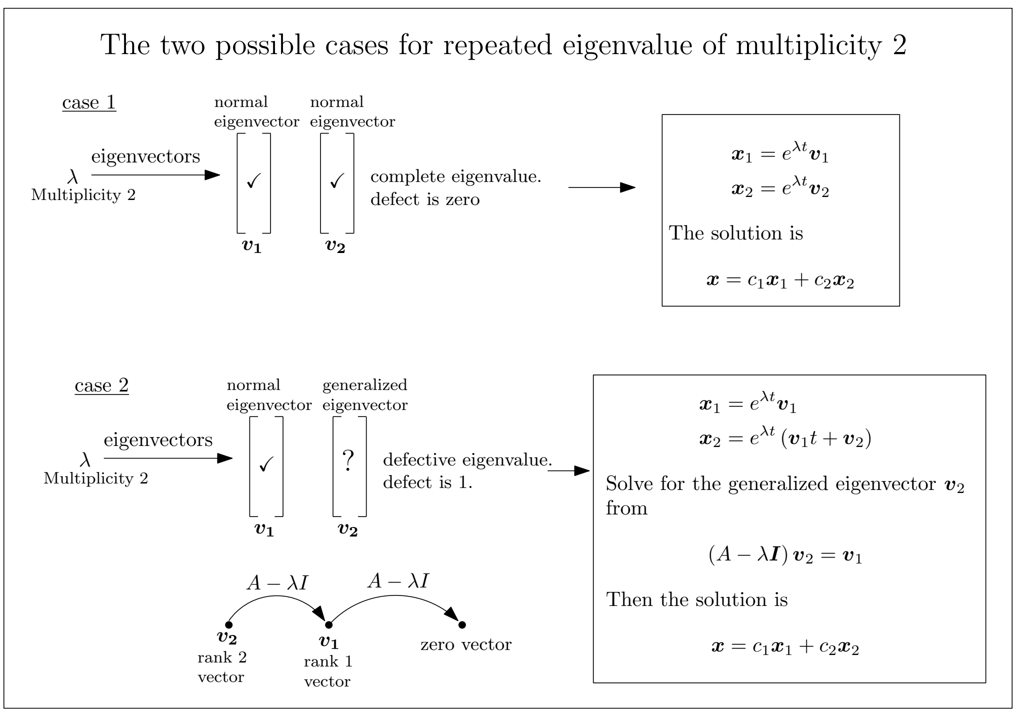


Figure 526: Possible case for repeated λ of multiplicity 2

This eigenvalue has algebraic multiplicity of 2, and geometric multiplicity 1, therefore this is defective eigenvalue. The defect is 1. This falls into case 2 shown above. We need to generate the missing additonal generalized eigevector $\vec{v}_2$ by solving

$$
(A - \lambda I) \vec{v}_2 = \vec{v}_1
$$

Where $\vec{v}_1$ is the normal (rank 1) eigenvector found above. Hence we need to solve

$$\left(\begin{bmatrix} 1 & 1 & 1 \\ 2 & 1 & -1 \\ 0 & -1 & 1 \end{bmatrix} - (2)\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\right)\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix}$$

$$\begin{bmatrix} -1 & 1 & 1 \\ 2 & -1 & -1 \\ 0 & -1 & -1 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix}$$

Solving for $\vec{v}_2$ gives

$$\vec{v}_2 = \begin{bmatrix} -1 \\ -2 \\ 1 \end{bmatrix}$$

We have found two generalized eigenvectors for eigenvalue 2. Therefore the two basis solution associated with this eigenvalue are

$$\begin{aligned} \vec{x}_2(t) &= \vec{v}_1 e^{\lambda t} \\ &= \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix} \mathrm{e}^{2t} \\ &= \begin{bmatrix} 0 \\ -\mathrm{e}^{2t} \\ \mathrm{e}^{2t} \end{bmatrix} \end{aligned}$$

And

$$\begin{aligned} \vec{x}_3(t) &= (\vec{v}_1 t + \vec{v}_2)\, e^{\lambda t} \\ &= \left(\begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix} t + \begin{bmatrix} -1 \\ -2 \\ 1 \end{bmatrix}\right) \mathrm{e}^{2t} \\ &= \begin{bmatrix} -\mathrm{e}^{2t} \\ -\mathrm{e}^{2t}(2+t) \\ \mathrm{e}^{2t}(t+1) \end{bmatrix} \end{aligned}$$

Therefore the final solution is

$$\vec{x}_h(t) = c_1\vec{x}_1(t) + c_2\vec{x}_2(t) + c_3\vec{x}_3(t)$$

Which is written as

$$\begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{bmatrix} = c_1 \begin{bmatrix} -\frac{3\,\mathrm{e}^{-t}}{2} \\ 2\,\mathrm{e}^{-t} \\ \mathrm{e}^{-t} \end{bmatrix} + c_2 \begin{bmatrix} 0 \\ -\mathrm{e}^{2t} \\ \mathrm{e}^{2t} \end{bmatrix} + c_3 \begin{bmatrix} -\mathrm{e}^{2t} \\ \mathrm{e}^{2t}(-t-2) \\ \mathrm{e}^{2t}(t+1) \end{bmatrix}$$

Which becomes

$$\begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{bmatrix} = \begin{bmatrix} -\frac{3c_1\mathrm{e}^{-t}}{2} - c_3\mathrm{e}^{2t} \\ ((-t-2)\,c_3 - c_2)\,\mathrm{e}^{2t} + 2c_1\mathrm{e}^{-t} \\ ((t+1)\,c_3 + c_2)\,\mathrm{e}^{2t} + c_1\mathrm{e}^{-t} \end{bmatrix}$$

17.5.3 Maple step by step solution

Let's solve

$[x_1'(t) = x_1(t) + x_2(t) + x_3(t)\,, x_2'(t) = 2x_1(t) + x_2(t) - x_3(t)\,, x_3'(t) = -x_2(t) + x_3(t)]$

- Define vector

$$\underline{x}^{\rightarrow}(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{bmatrix}$$

- Convert system into a vector equation

$$\underline{x}^{\rightarrow\prime}(t) = \begin{bmatrix} 1 & 1 & 1 \\ 2 & 1 & -1 \\ 0 & -1 & 1 \end{bmatrix} \cdot \underline{x}^{\rightarrow}(t) + \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

- System to solve

$$\underline{x}^{\rightarrow\prime}(t) = \begin{bmatrix} 1 & 1 & 1 \\ 2 & 1 & -1 \\ 0 & -1 & 1 \end{bmatrix} \cdot \underline{x}^{\rightarrow}(t)$$

- Define the coefficient matrix

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 2 & 1 & -1 \\ 0 & -1 & 1 \end{bmatrix}$$

- Rewrite the system as

  $\underline{x}^{\rightarrow\prime}(t) = A \cdot \underline{x}^{\rightarrow}(t)$

- To solve the system, find the eigenvalues and eigenvectors of A
- Eigenpairs of A

  $$\left[\left[-1, \begin{bmatrix} -\frac{3}{2} \\ 2 \\ 1 \end{bmatrix}\right], \left[2, \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix}\right], \left[2, \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}\right]\right]$$

- Consider eigenpair

  $$\left[-1, \begin{bmatrix} -\frac{3}{2} \\ 2 \\ 1 \end{bmatrix}\right]$$

- Solution to homogeneous system from eigenpair

  $$\underline{x}^{\rightarrow}{}_{1} = e^{-t} \cdot \begin{bmatrix} -\frac{3}{2} \\ 2 \\ 1 \end{bmatrix}$$

- Consider eigenpair, with eigenvalue of algebraic multiplicity 2

  $$\left[2, \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix}\right]$$

- First solution from eigenvalue 2

  $$\underline{x}^{\rightarrow}{}_{2}(t) = e^{2t} \cdot \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix}$$

- Form of the 2nd homogeneous solution where $\vec{p}$ is to be solved for, $\lambda = 2$ is the eigenvalue, and

$\underline{\vec{x}}_{3}(t) = \mathrm{e}^{\lambda t}\left(t\vec{v} + \vec{p}\right)$

- Note that the t multiplying $\vec{v}$ makes this solution linearly independent to the 1st solution obtai
- Substitute $\underline{\vec{x}}_{3}(t)$ into the homogeneous system

  $\lambda\,\mathrm{e}^{\lambda t}\left(t\vec{v} + \vec{p}\right) + \mathrm{e}^{\lambda t}\vec{v} = \left(\mathrm{e}^{\lambda t}A\right)\cdot\left(t\vec{v} + \vec{p}\right)$
- Use the fact that $\vec{v}$ is an eigenvector of A

  $\lambda\,\mathrm{e}^{\lambda t}\left(t\vec{v} + \vec{p}\right) + \mathrm{e}^{\lambda t}\vec{v} = \mathrm{e}^{\lambda t}\left(\lambda t\vec{v} + A\cdot\vec{p}\right)$
- Simplify equation

  $\lambda\vec{p} + \vec{v} = A\cdot\vec{p}$
- Make use of the identity matrix I

  $(\lambda\cdot I)\cdot\vec{p} + \vec{v} = A\cdot\vec{p}$
- Condition $\vec{p}$ must meet for $\underline{\vec{x}}_{3}(t)$ to be a solution to the homogeneous system

  $(A - \lambda\cdot I)\cdot\vec{p} = \vec{v}$
- Choose $\vec{p}$ to use in the second solution to the homogeneous system from eigenvalue 2

  $$\left(\begin{bmatrix} 1 & 1 & 1 \\ 2 & 1 & -1 \\ 0 & -1 & 1 \end{bmatrix} - 2\cdot\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\right)\cdot\vec{p} = \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix}$$
- Choice of $\vec{p}$

  $$\vec{p} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$
- Second solution from eigenvalue 2

  $$\underline{\vec{x}}_{3}(t) = \mathrm{e}^{2t}\cdot\left(t\cdot\begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}\right)$$
- General solution to the system of ODEs

  $\underline{\vec{x}} = c_1\underline{\vec{x}}_{1} + c_2\underline{\vec{x}}_{2}(t) + c_3\underline{\vec{x}}_{3}(t)$
- Substitute solutions into the general solution

$$\underline{\overrightarrow{x}} = c_1 e^{-t} \cdot \begin{bmatrix} -\frac{3}{2} \\ 2 \\ 1 \end{bmatrix} + c_2 e^{2t} \cdot \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix} + c_3 e^{2t} \cdot \left(t \cdot \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \right)$$

- Substitute in vector of dependent variables

$$\begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{bmatrix} = \begin{bmatrix} -\frac{3c_1 e^{-t}}{2} \\ (-c_3 t - c_2) e^{2t} + 2c_1 e^{-t} \\ (c_3 t + c_2) e^{2t} + c_1 e^{-t} \end{bmatrix}$$

- Solution to the system of ODEs

$$\left\{x_1(t) = -\frac{3c_1 e^{-t}}{2}, x_2(t) = (-c_3 t - c_2) e^{2t} + 2c_1 e^{-t}, x_3(t) = (c_3 t + c_2) e^{2t} + c_1 e^{-t}\right\}$$

✓ Solution by Maple

Time used: 0.031 (sec). Leaf size: 76

```
dsolve([diff(x__1(t),t)=1*x__1(t)+1*x__2(t)+1*x__3(t),diff(x__2(t),t)=2*x__1(t)+1*x__2(t)-1*
```

$$x_1(t) = -\frac{3\,e^{-t} c_1}{2} - c_3 e^{2t}$$
$$x_2(t) = 2\,e^{-t} c_1 - c_2 e^{2t} - e^{2t} c_3 t - c_3 e^{2t}$$
$$x_3(t) = e^{-t} c_1 + c_2 e^{2t} + e^{2t} c_3 t$$

✓ Solution by Mathematica

Time used: 0.01 (sec). Leaf size: 164

```
DSolve[{x1'[t]==1*x1[t]+1*x2[t]+1*x3[t],x2'[t]==2*x1[t]+1*x2[t]-1*x3[t],x3'[t]==0*x1[t]-1*x2
```

$$\text{x1}(t) \to \frac{1}{3} e^{-t} \left(c_1 \left(2e^{3t} + 1\right) + (c_2 + c_3) \left(e^{3t} - 1\right)\right)$$
$$\text{x2}(t) \to \frac{1}{9} e^{-t} \left(c_1 \left(e^{3t}(6t + 4) - 4\right) + c_2 \left(e^{3t}(3t + 5) + 4\right) + c_3 \left(e^{3t}(3t - 4) + 4\right)\right)$$
$$\text{x3}(t) \to \frac{1}{9} e^{-t} \left(c_1 \left(e^{3t}(2 - 6t) - 2\right) + c_2 \left(2 - e^{3t}(3t + 2)\right) - c_3 \left(e^{3t}(3t - 7) - 2\right)\right)$$

17.6 problem 6

Internal problem ID [771]
Internal file name [`OUTPUT/771_Sunday_June_05_2022_01_49_16_AM_29590752/index.tex`]

Book: Elementary differential equations and boundary value problems, 10th ed., Boyce and DiPrima
Section: Chapter 7.8, Repeated Eigenvalues. page 436
Problem number: 6.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"system of linear ODEs"**

Solve

$$
\begin{aligned}
x_1'(t) &= x_2(t) + x_3(t) \\
x_2'(t) &= x_1(t) + x_3(t) \\
x_3'(t) &= x_1(t) + x_2(t)
\end{aligned}
$$

17.6.1 Solution using Matrix exponential method

In this method, we will assume we have found the matrix exponential e^{At} allready. There are different methods to determine this but will not be shown here. This is a system of linear ODE's given as

$$\vec{x}'(t) = A\,\vec{x}(t)$$

Or

$$\begin{bmatrix} x_1'(t) \\ x_2'(t) \\ x_3'(t) \end{bmatrix} = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{bmatrix}$$

For the above matrix A, the matrix exponential can be found to be

$$e^{At} = \begin{bmatrix} \frac{2\,\mathrm{e}^{-t}}{3} + \frac{\mathrm{e}^{2t}}{3} & \frac{\mathrm{e}^{2t}}{3} - \frac{\mathrm{e}^{-t}}{3} & \frac{\mathrm{e}^{2t}}{3} - \frac{\mathrm{e}^{-t}}{3} \\ \frac{\mathrm{e}^{2t}}{3} - \frac{\mathrm{e}^{-t}}{3} & \frac{2\,\mathrm{e}^{-t}}{3} + \frac{\mathrm{e}^{2t}}{3} & \frac{\mathrm{e}^{2t}}{3} - \frac{\mathrm{e}^{-t}}{3} \\ \frac{\mathrm{e}^{2t}}{3} - \frac{\mathrm{e}^{-t}}{3} & \frac{\mathrm{e}^{2t}}{3} - \frac{\mathrm{e}^{-t}}{3} & \frac{2\,\mathrm{e}^{-t}}{3} + \frac{\mathrm{e}^{2t}}{3} \end{bmatrix}$$

Therefore the homogeneous solution is

$$
\begin{aligned}
\vec{x}_h(t) &= e^{At}\vec{c} \\
&= \begin{bmatrix} \frac{2\,\mathrm{e}^{-t}}{3}+\frac{\mathrm{e}^{2t}}{3} & \frac{\mathrm{e}^{2t}}{3}-\frac{\mathrm{e}^{-t}}{3} & \frac{\mathrm{e}^{2t}}{3}-\frac{\mathrm{e}^{-t}}{3} \\ \frac{\mathrm{e}^{2t}}{3}-\frac{\mathrm{e}^{-t}}{3} & \frac{2\,\mathrm{e}^{-t}}{3}+\frac{\mathrm{e}^{2t}}{3} & \frac{\mathrm{e}^{2t}}{3}-\frac{\mathrm{e}^{-t}}{3} \\ \frac{\mathrm{e}^{2t}}{3}-\frac{\mathrm{e}^{-t}}{3} & \frac{\mathrm{e}^{2t}}{3}-\frac{\mathrm{e}^{-t}}{3} & \frac{2\,\mathrm{e}^{-t}}{3}+\frac{\mathrm{e}^{2t}}{3} \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} \\
&= \begin{bmatrix} \left(\frac{2\,\mathrm{e}^{-t}}{3}+\frac{\mathrm{e}^{2t}}{3}\right)c_1+\left(\frac{\mathrm{e}^{2t}}{3}-\frac{\mathrm{e}^{-t}}{3}\right)c_2+\left(\frac{\mathrm{e}^{2t}}{3}-\frac{\mathrm{e}^{-t}}{3}\right)c_3 \\ \left(\frac{\mathrm{e}^{2t}}{3}-\frac{\mathrm{e}^{-t}}{3}\right)c_1+\left(\frac{2\,\mathrm{e}^{-t}}{3}+\frac{\mathrm{e}^{2t}}{3}\right)c_2+\left(\frac{\mathrm{e}^{2t}}{3}-\frac{\mathrm{e}^{-t}}{3}\right)c_3 \\ \left(\frac{\mathrm{e}^{2t}}{3}-\frac{\mathrm{e}^{-t}}{3}\right)c_1+\left(\frac{\mathrm{e}^{2t}}{3}-\frac{\mathrm{e}^{-t}}{3}\right)c_2+\left(\frac{2\,\mathrm{e}^{-t}}{3}+\frac{\mathrm{e}^{2t}}{3}\right)c_3 \end{bmatrix} \\
&= \begin{bmatrix} \frac{(2c_1-c_2-c_3)\mathrm{e}^{-t}}{3}+\frac{\mathrm{e}^{2t}(c_1+c_2+c_3)}{3} \\ \frac{(-c_1+2c_2-c_3)\mathrm{e}^{-t}}{3}+\frac{\mathrm{e}^{2t}(c_1+c_2+c_3)}{3} \\ \frac{(-c_1-c_2+2c_3)\mathrm{e}^{-t}}{3}+\frac{\mathrm{e}^{2t}(c_1+c_2+c_3)}{3} \end{bmatrix}
\end{aligned}
$$

Since no forcing function is given, then the final solution is $\vec{x}_h(t)$ above.

17.6.2 Solution using explicit Eigenvalue and Eigenvector method

This is a system of linear ODE's given as

$$\vec{x}'(t) = A\,\vec{x}(t)$$

Or

$$\begin{bmatrix} x_1'(t) \\ x_2'(t) \\ x_3'(t) \end{bmatrix} = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{bmatrix}$$

The first step is find the homogeneous solution. We start by finding the eigenvalues of A. This is done by solving the following equation for the eigenvalues λ

$$\det\left(A-\lambda I\right)=0$$

Expanding gives

$$\det\left(\begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\right) = 0$$

Therefore

$$\det\left(\begin{bmatrix}-\lambda & 1 & 1\\ 1 & -\lambda & 1\\ 1 & 1 & -\lambda\end{bmatrix}\right) = 0$$

Which gives the characteristic equation

$$\lambda^3 - 3\lambda - 2 = 0$$

The roots of the above are the eigenvalues.

$$\lambda_1 = 2$$
$$\lambda_2 = -1$$

This table summarises the above result

eigenvalue	algebraic multiplicity	type of eigenvalue
-1	1	real eigenvalue
2	1	real eigenvalue

Now the eigenvector for each eigenvalue are found.

Considering the eigenvalue $\lambda_1 = -1$

We need to solve $A\vec{v} = \lambda\vec{v}$ or $(A - \lambda I)\vec{v} = \vec{0}$ which becomes

$$\left(\begin{bmatrix}0 & 1 & 1\\ 1 & 0 & 1\\ 1 & 1 & 0\end{bmatrix} - (-1)\begin{bmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{bmatrix}\right)\begin{bmatrix}v_1\\ v_2\\ v_3\end{bmatrix} = \begin{bmatrix}0\\ 0\\ 0\end{bmatrix}$$

$$\begin{bmatrix}1 & 1 & 1\\ 1 & 1 & 1\\ 1 & 1 & 1\end{bmatrix}\begin{bmatrix}v_1\\ v_2\\ v_3\end{bmatrix} = \begin{bmatrix}0\\ 0\\ 0\end{bmatrix}$$

Now forward elimination is applied to solve for the eigenvector $\vec{v}$. The augmented matrix is

$$\left[\begin{array}{ccc|c}1 & 1 & 1 & 0\\ 1 & 1 & 1 & 0\\ 1 & 1 & 1 & 0\end{array}\right]$$

$$R_2 = R_2 - R_1 \Longrightarrow \left[\begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 \end{array}\right]$$

$$R_3 = R_3 - R_1 \Longrightarrow \left[\begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

Therefore the system in Echelon form is

$$\begin{bmatrix} 1 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

The free variables are $\{v_2, v_3\}$ and the leading variables are $\{v_1\}$. Let $v_2 = t$. Let $v_3 = s$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\{v_1 = -t - s\}$

Hence the solution is

$$\begin{bmatrix} -t - s \\ t \\ s \end{bmatrix} = \begin{bmatrix} -t - s \\ t \\ s \end{bmatrix}$$

Since there are two free Variable, we have found two eigenvectors associated with this eigenvalue. The above can be written as

$$\begin{aligned} \begin{bmatrix} -t - s \\ t \\ s \end{bmatrix} &= \begin{bmatrix} -t \\ t \\ 0 \end{bmatrix} + \begin{bmatrix} -s \\ 0 \\ s \end{bmatrix} \\ &= t \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} + s \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} \end{aligned}$$

By letting $t = 1$ and $s = 1$ then the above becomes

$$\begin{bmatrix} -t - s \\ t \\ s \end{bmatrix} = \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} + \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$$

Hence the two eigenvectors associated with this eigenvalue are

$$\left(\begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}\right)$$

Considering the eigenvalue $\lambda_2 = 2$

We need to solve $A\vec{v} = \lambda\vec{v}$ or $(A - \lambda I)\vec{v} = \vec{0}$ which becomes

$$\left(\begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix} - (2)\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\right)\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} -2 & 1 & 1 \\ 1 & -2 & 1 \\ 1 & 1 & -2 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Now forward elimination is applied to solve for the eigenvector $\vec{v}$. The augmented matrix is

$$\left[\begin{array}{ccc|c} -2 & 1 & 1 & 0 \\ 1 & -2 & 1 & 0 \\ 1 & 1 & -2 & 0 \end{array}\right]$$

$$R_2 = R_2 + \frac{R_1}{2} \Longrightarrow \left[\begin{array}{ccc|c} -2 & 1 & 1 & 0 \\ 0 & -\frac{3}{2} & \frac{3}{2} & 0 \\ 1 & 1 & -2 & 0 \end{array}\right]$$

$$R_3 = R_3 + \frac{R_1}{2} \Longrightarrow \left[\begin{array}{ccc|c} -2 & 1 & 1 & 0 \\ 0 & -\frac{3}{2} & \frac{3}{2} & 0 \\ 0 & \frac{3}{2} & -\frac{3}{2} & 0 \end{array}\right]$$

$$R_3 = R_3 + R_2 \Longrightarrow \left[\begin{array}{ccc|c} -2 & 1 & 1 & 0 \\ 0 & -\frac{3}{2} & \frac{3}{2} & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

Therefore the system in Echelon form is

$$\begin{bmatrix} -2 & 1 & 1 \\ 0 & -\frac{3}{2} & \frac{3}{2} \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

The free variables are $\{v_3\}$ and the leading variables are $\{v_1, v_2\}$. Let $v_3 = t$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\{v_1 = t, v_2 = t\}$

Hence the solution is

$$\begin{bmatrix} t \\ t \\ t \end{bmatrix} = \begin{bmatrix} t \\ t \\ t \end{bmatrix}$$

Since there is one free Variable, we have found one eigenvector associated with this eigenvalue. The above can be written as

$$\begin{bmatrix} t \\ t \\ t \end{bmatrix} = t \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

Let $t = 1$ the eigenvector becomes

$$\begin{bmatrix} t \\ t \\ t \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

The following table gives a summary of this result. It shows for each eigenvalue the algebraic multiplicity m, and its geometric multiplicity k and the eigenvectors associated with the eigenvalue. If $m > k$ then the eigenvalue is defective which means the number of normal linearly independent eigenvectors associated with this eigenvalue (called the geometric multiplicity k) does not equal the algebraic multiplicity m, and we need to determine an additional $m - k$ generalized eigenvectors for this eigenvalue.

eigenvalue	multiplicity		defective?	eigenvectors
	algebraic m	geometric k		
2	1	1	No	$\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$
-1	2	2	No	$\begin{bmatrix} -1 & -1 \\ 0 & 1 \\ 1 & 0 \end{bmatrix}$

Now that we found the eigenvalues and associated eigenvectors, we will go over each eigenvalue and generate the solution basis. The only problem we need to take care of is if the eigenvalue is defective. Since eigenvalue 2 is real and distinct then the corresponding eigenvector solution is

$$
\begin{aligned}
\vec{x}_1(t) &= \vec{v}_1 e^{2t} \\
&= \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} e^{2t}
\end{aligned}
$$

eigenvalue -1 is real and repated eigenvalue of multiplicity 2.There are two possible cases that can happen. This is illustrated in this diagram

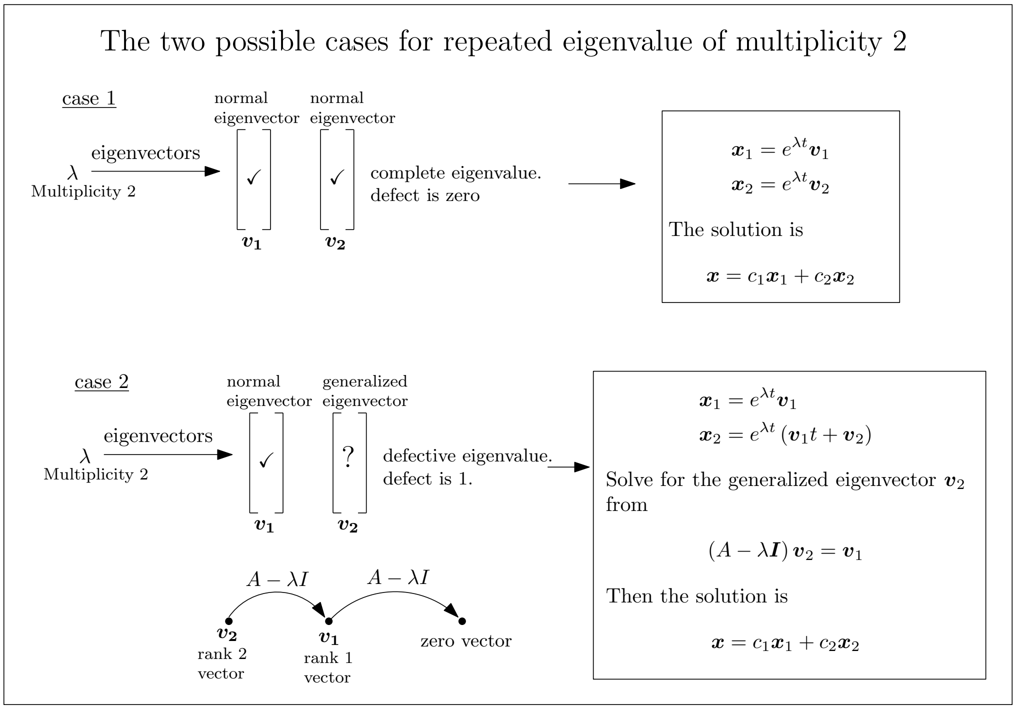


Figure 527: Possible case for repeated λ of multiplicity 2

This eigenvalue has algebraic multiplicity of 2 which is the same as its geometric multiplicity 2, then it is complete eigenvalue and this falls into case 1 shown above. Hence the corresponding eigenvector basis are

$$
\begin{aligned}
\vec{x}_2(t) &= \vec{v}_2 e^{-t} \\
&= \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} e^{-t}
\end{aligned}
$$

$$
\begin{aligned}
\vec{x}_3(t) &= \vec{v}_3 e^{-t} \\
&= \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} e^{-t}
\end{aligned}
$$

Therefore the final solution is

$$
\vec{x}_h(t) = c_1 \vec{x}_1(t) + c_2 \vec{x}_2(t) + c_3 \vec{x}_3(t)
$$

Which is written as

$$\begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{bmatrix} = c_1 \begin{bmatrix} e^{2t} \\ e^{2t} \\ e^{2t} \end{bmatrix} + c_2 \begin{bmatrix} -e^{-t} \\ 0 \\ e^{-t} \end{bmatrix} + c_3 \begin{bmatrix} -e^{-t} \\ e^{-t} \\ 0 \end{bmatrix}$$

Which becomes

$$\begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{bmatrix} = \begin{bmatrix} (-c_2 - c_3)\, e^{-t} + c_1 e^{2t} \\ c_1 e^{2t} + c_3 e^{-t} \\ c_1 e^{2t} + c_2 e^{-t} \end{bmatrix}$$

17.6.3 Maple step by step solution

Let's solve

$[x_1'(t) = x_2(t) + x_3(t)\,, x_2'(t) = x_1(t) + x_3(t)\,, x_3'(t) = x_1(t) + x_2(t)]$

- Define vector

$$\underline{x}^{\rightarrow}(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{bmatrix}$$

- Convert system into a vector equation

$$\underline{x}^{\rightarrow\prime}(t) = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix} \cdot \underline{x}^{\rightarrow}(t) + \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

- System to solve

$$\underline{x}^{\rightarrow\prime}(t) = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix} \cdot \underline{x}^{\rightarrow}(t)$$

- Define the coefficient matrix

$$A = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}$$

- Rewrite the system as

  $\underline{x}^{\rightarrow}{}'(t) = A \cdot \underline{x}^{\rightarrow}(t)$

- To solve the system, find the eigenvalues and eigenvectors of A
- Eigenpairs of A

  $$\left[\left[-1, \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}\right], \left[-1, \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}\right], \left[2, \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}\right]\right]$$

- Consider eigenpair, with eigenvalue of algebraic multiplicity 2

  $$\left[-1, \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}\right]$$

- First solution from eigenvalue -1

  $$\underline{x}^{\rightarrow}{}_1(t) = e^{-t} \cdot \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$$

- Form of the 2nd homogeneous solution where $\vec{p}$ is to be solved for, $\lambda = -1$ is the eigenvalue, an

  $\underline{x}^{\rightarrow}{}_2(t) = e^{\lambda t}\left(t\vec{v} + \vec{p}\right)$

- Note that the t multiplying $\vec{v}$ makes this solution linearly independent to the 1st solution obtai
- Substitute $\underline{x}^{\rightarrow}{}_2(t)$ into the homogeneous system

  $\lambda e^{\lambda t}\left(t\vec{v} + \vec{p}\right) + e^{\lambda t}\vec{v} = \left(e^{\lambda t}A\right) \cdot \left(t\vec{v} + \vec{p}\right)$

- Use the fact that $\vec{v}$ is an eigenvector of A

  $\lambda e^{\lambda t}\left(t\vec{v} + \vec{p}\right) + e^{\lambda t}\vec{v} = e^{\lambda t}\left(\lambda t\vec{v} + A \cdot \vec{p}\right)$

- Simplify equation

  $\lambda\vec{p} + \vec{v} = A \cdot \vec{p}$

- Make use of the identity matrix I

  $(\lambda \cdot I) \cdot \vec{p} + \vec{v} = A \cdot \vec{p}$

- Condition $\vec{p}$ must meet for $\underline{x}^{\rightarrow}{}_2(t)$ to be a solution to the homogeneous system

  $(A - \lambda \cdot I) \cdot \vec{p} = \vec{v}$

- Choose $\vec{p}$ to use in the second solution to the homogeneous system from eigenvalue -1

$$\left(\begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix} - (-1) \cdot \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\right) \cdot \vec{p} = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$$

- Choice of $\vec{p}$

$$\vec{p} = \begin{bmatrix} -1 \\ 0 \\ 0 \end{bmatrix}$$

- Second solution from eigenvalue -1

$$\underline{x}^{\rightarrow}{}_2(t) = \mathrm{e}^{-t} \cdot \left(t \cdot \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} + \begin{bmatrix} -1 \\ 0 \\ 0 \end{bmatrix}\right)$$

- Consider eigenpair

$$\left[2, \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}\right]$$

- Solution to homogeneous system from eigenpair

$$\underline{x}^{\rightarrow}{}_3 = \mathrm{e}^{2t} \cdot \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

- General solution to the system of ODEs

  $\underline{x}^{\rightarrow} = c_1 \underline{x}^{\rightarrow}{}_1(t) + c_2 \underline{x}^{\rightarrow}{}_2(t) + c_3 \underline{x}^{\rightarrow}{}_3$

- Substitute solutions into the general solution

$$\underline{x^{\rightarrow}} = c_1 e^{-t} \cdot \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} + c_2 e^{-t} \cdot \left(t \cdot \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} + \begin{bmatrix} -1 \\ 0 \\ 0 \end{bmatrix} \right) + c_3 e^{2t} \cdot \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

- Substitute in vector of dependent variables

$$\begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{bmatrix} = \begin{bmatrix} ((-t-1)\, c_2 - c_1)\, e^{-t} + c_3 e^{2t} \\ c_3 e^{2t} \\ (c_2 t + c_1)\, e^{-t} + c_3 e^{2t} \end{bmatrix}$$

- Solution to the system of ODEs

$$\{x_1(t) = ((-t-1)\, c_2 - c_1)\, e^{-t} + c_3 e^{2t}, x_2(t) = c_3 e^{2t}, x_3(t) = (c_2 t + c_1)\, e^{-t} + c_3 e^{2t}\}$$

✓ Solution by Maple

Time used: 0.031 (sec). Leaf size: 64

```
dsolve([diff(x__1(t),t)=0*x__1(t)+1*x__2(t)+1*x__3(t),diff(x__2(t),t)=1*x__1(t)+0*x__2(t)+1*
```

$$x_1(t) = c_2 e^{-t} + c_3 e^{2t}$$
$$x_2(t) = c_2 e^{-t} + c_3 e^{2t} + e^{-t} c_1$$
$$x_3(t) = -2 c_2 e^{-t} + c_3 e^{2t} - e^{-t} c_1$$

✓ Solution by Mathematica

Time used: 0.005 (sec). Leaf size: 124

```
DSolve[{x1'[t]==0*x1[t]+1*x2[t]+1*x3[t],x2'[t]==1*x1[t]+0*x2[t]+1*x3[t],x3'[t]==1*x1[t]+1*x2
```

$$\text{x1}(t) \to \frac{1}{3} e^{-t} \left(c_1 \left(e^{3t} + 2 \right) + (c_2 + c_3) \left(e^{3t} - 1 \right) \right)$$
$$\text{x2}(t) \to \frac{1}{3} e^{-t} \left(c_1 \left(e^{3t} - 1 \right) + c_2 \left(e^{3t} + 2 \right) + c_3 \left(e^{3t} - 1 \right) \right)$$
$$\text{x3}(t) \to \frac{1}{3} e^{-t} \left(c_1 \left(e^{3t} - 1 \right) + c_2 \left(e^{3t} - 1 \right) + c_3 \left(e^{3t} + 2 \right) \right)$$

17.7 problem 7

Internal problem ID [772]
Internal file name [OUTPUT/772_Sunday_June_05_2022_01_49_17_AM_55713821/index.tex]

Book: Elementary differential equations and boundary value problems, 10th ed., Boyce and DiPrima
Section: Chapter 7.8, Repeated Eigenvalues. page 436
Problem number: 7.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"system of linear ODEs"**

Solve

$$
\begin{aligned}
x_1'(t) &= x_1(t) - 4x_2(t) \\
x_2'(t) &= 4x_1(t) - 7x_2(t)
\end{aligned}
$$

With initial conditions

$$[x_1(0) = 3, x_2(0) = 2]$$

17.7.1 Solution using Matrix exponential method

In this method, we will assume we have found the matrix exponential e^{At} allready. There are different methods to determine this but will not be shown here. This is a system of linear ODE's given as

$$\vec{x}'(t) = A\,\vec{x}(t)$$

Or

$$\begin{bmatrix} x_1'(t) \\ x_2'(t) \end{bmatrix} = \begin{bmatrix} 1 & -4 \\ 4 & -7 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}$$

For the above matrix A, the matrix exponential can be found to be

$$e^{At} = \begin{bmatrix} e^{-3t}(4t+1) & -4t\,e^{-3t} \\ 4t\,e^{-3t} & e^{-3t}(1-4t) \end{bmatrix}$$

Therefore the homogeneous solution is

$$\begin{aligned}\vec{x}_h(t) &= e^{At}\vec{x}_0 \\ &= \begin{bmatrix} e^{-3t}(4t+1) & -4t\,e^{-3t} \\ 4t\,e^{-3t} & e^{-3t}(1-4t) \end{bmatrix} \begin{bmatrix} 3 \\ 2 \end{bmatrix} \\ &= \begin{bmatrix} 3\,e^{-3t}(4t+1) - 8t\,e^{-3t} \\ 12t\,e^{-3t} + 2\,e^{-3t}(1-4t) \end{bmatrix} \\ &= \begin{bmatrix} e^{-3t}(4t+3) \\ (4t+2)\,e^{-3t} \end{bmatrix}\end{aligned}$$

Since no forcing function is given, then the final solution is $\vec{x}_h(t)$ above.

17.7.2 Solution using explicit Eigenvalue and Eigenvector method

This is a system of linear ODE's given as

$$\vec{x}'(t) = A\,\vec{x}(t)$$

Or

$$\begin{bmatrix} x_1'(t) \\ x_2'(t) \end{bmatrix} = \begin{bmatrix} 1 & -4 \\ 4 & -7 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}$$

The first step is find the homogeneous solution. We start by finding the eigenvalues of A. This is done by solving the following equation for the eigenvalues λ

$$\det(A - \lambda I) = 0$$

Expanding gives

$$\det\left(\begin{bmatrix} 1 & -4 \\ 4 & -7 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right) = 0$$

Therefore

$$\det\left(\begin{bmatrix} 1-\lambda & -4 \\ 4 & -7-\lambda \end{bmatrix}\right) = 0$$

Which gives the characteristic equation

$$\lambda^2 + 6\lambda + 9 = 0$$

The roots of the above are the eigenvalues.

$$\lambda_1 = -3$$

This table summarises the above result

eigenvalue	algebraic multiplicity	type of eigenvalue
-3	1	real eigenvalue

Now the eigenvector for each eigenvalue are found.

Considering the eigenvalue $\lambda_1 = -3$

We need to solve $A\vec{v} = \lambda\vec{v}$ or $(A - \lambda I)\vec{v} = \vec{0}$ which becomes

$$\left(\begin{bmatrix} 1 & -4 \\ 4 & -7 \end{bmatrix} - (-3)\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right)\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 4 & -4 \\ 4 & -4 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Now forward elimination is applied to solve for the eigenvector $\vec{v}$. The augmented matrix is

$$\left[\begin{array}{cc|c} 4 & -4 & 0 \\ 4 & -4 & 0 \end{array}\right]$$

$$R_2 = R_2 - R_1 \Longrightarrow \left[\begin{array}{cc|c} 4 & -4 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

Therefore the system in Echelon form is

$$\begin{bmatrix} 4 & -4 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

The free variables are $\{v_2\}$ and the leading variables are $\{v_1\}$. Let $v_2 = t$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\{v_1 = t\}$

Hence the solution is

$$\begin{bmatrix} t \\ t \end{bmatrix} = \begin{bmatrix} t \\ t \end{bmatrix}$$

Since there is one free Variable, we have found one eigenvector associated with this eigenvalue. The above can be written as

$$\begin{bmatrix} t \\ t \end{bmatrix} = t \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

Let $t = 1$ the eigenvector becomes

$$\begin{bmatrix} t \\ t \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

The following table gives a summary of this result. It shows for each eigenvalue the algebraic multiplicity m, and its geometric multiplicity k and the eigenvectors associated with the eigenvalue. If $m > k$ then the eigenvalue is defective which means the number of normal linearly independent eigenvectors associated with this eigenvalue (called the geometric multiplicity k) does not equal the algebraic multiplicity m, and we need to determine an additional $m - k$ generalized eigenvectors for this eigenvalue.

	multiplicity			
eigenvalue	algebraic m	geometric k	defective?	eigenvectors
-3	2	1	Yes	$\begin{bmatrix} 1 \\ 1 \end{bmatrix}$

Now that we found the eigenvalues and associated eigenvectors, we will go over each eigenvalue and generate the solution basis. The only problem we need to take care of is if the eigenvalue is defective. eigenvalue -3 is real and repated eigenvalue of multiplicity 2.There are two possible cases that can happen. This is illustrated in this diagram

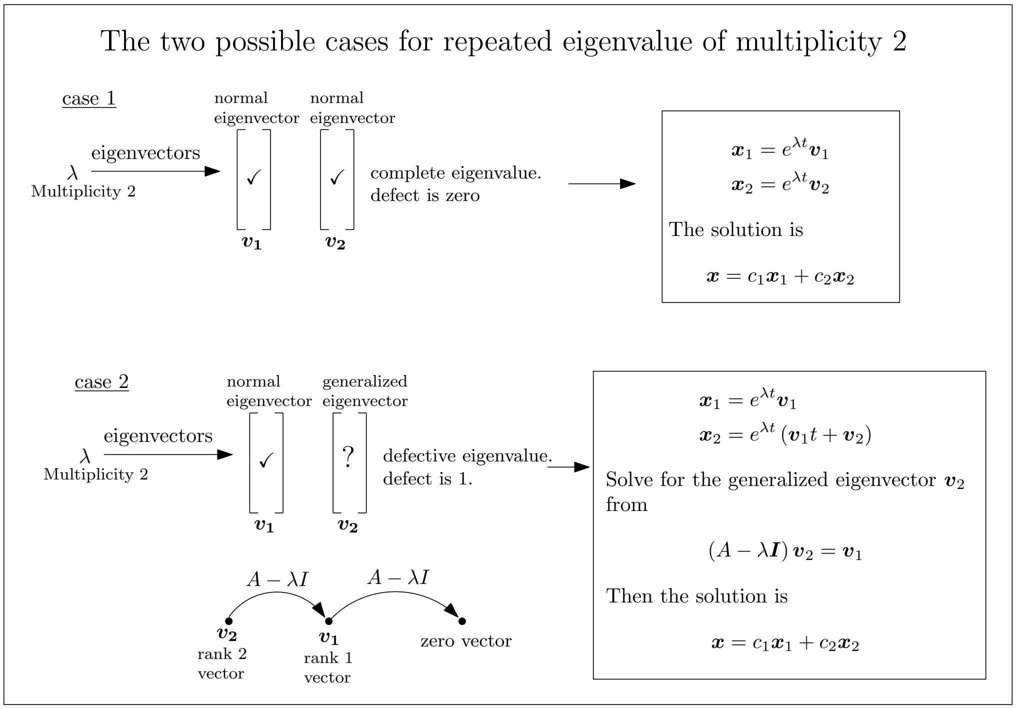


Figure 528: Possible case for repeated λ of multiplicity 2

This eigenvalue has algebraic multiplicity of 2, and geometric multiplicity 1, therefore this is defective eigenvalue. The defect is 1. This falls into case 2 shown above. We need to generate the missing additonal generalized eigevector $\vec{v}_2$ by solving

$$(A - \lambda I)\,\vec{v}_2 = \vec{v}_1$$

Where $\vec{v}_1$ is the normal (rank 1) eigenvector found above. Hence we need to solve

$$\left(\begin{bmatrix} 1 & -4 \\ 4 & -7 \end{bmatrix} - (-3)\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right)\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$
$$\begin{bmatrix} 4 & -4 \\ 4 & -4 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

Solving for $\vec{v}_2$ gives

$$\vec{v}_2 = \begin{bmatrix} \frac{5}{4} \\ 1 \end{bmatrix}$$

We have found two generalized eigenvectors for eigenvalue -3. Therefore the two basis solution associated with this eigenvalue are

$$
\begin{aligned}
\vec{x}_1(t) &= \vec{v}_1 e^{\lambda t} \\
&= \begin{bmatrix} 1 \\ 1 \end{bmatrix} \mathrm{e}^{-3t} \\
&= \begin{bmatrix} \mathrm{e}^{-3t} \\ \mathrm{e}^{-3t} \end{bmatrix}
\end{aligned}
$$

And

$$
\begin{aligned}
\vec{x}_2(t) &= (\vec{v}_1 t + \vec{v}_2)\, e^{\lambda t} \\
&= \left(\begin{bmatrix} 1 \\ 1 \end{bmatrix} t + \begin{bmatrix} \frac{5}{4} \\ 1 \end{bmatrix} \right) \mathrm{e}^{-3t} \\
&= \begin{bmatrix} \frac{\mathrm{e}^{-3t}(4t+5)}{4} \\ \mathrm{e}^{-3t}(t+1) \end{bmatrix}
\end{aligned}
$$

Therefore the final solution is

$$
\vec{x}_h(t) = c_1 \vec{x}_1(t) + c_2 \vec{x}_2(t)
$$

Which is written as

$$
\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = c_1 \begin{bmatrix} \mathrm{e}^{-3t} \\ \mathrm{e}^{-3t} \end{bmatrix} + c_2 \begin{bmatrix} \mathrm{e}^{-3t}\left(t + \frac{5}{4}\right) \\ \mathrm{e}^{-3t}(t+1) \end{bmatrix}
$$

Which becomes

$$
\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = \begin{bmatrix} \mathrm{e}^{-3t}\left(c_1 + c_2 t + \frac{5}{4} c_2\right) \\ \mathrm{e}^{-3t}(c_2 t + c_1 + c_2) \end{bmatrix}
$$

Since initial conditions are given, the solution above needs to be updated by solving for the constants of integrations using the given initial conditions

$$
\begin{bmatrix} x_1(0) = 3 \\ x_2(0) = 2 \end{bmatrix} \tag{1}
$$

Substituting initial conditions into the above solution at $t = 0$ gives

$$\begin{bmatrix} 3 \\ 2 \end{bmatrix} = \begin{bmatrix} c_1 + \frac{5c_2}{4} \\ c_1 + c_2 \end{bmatrix}$$

Solving for the constants of integrations gives

$$\begin{bmatrix} c_1 = -2 \\ c_2 = 4 \end{bmatrix}$$

Substituting these constants back in original solution in Eq. (1) gives

$$\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = \begin{bmatrix} \mathrm{e}^{-3t}(4t+3) \\ (4t+2)\,\mathrm{e}^{-3t} \end{bmatrix}$$

The following is the phase plot of the system.

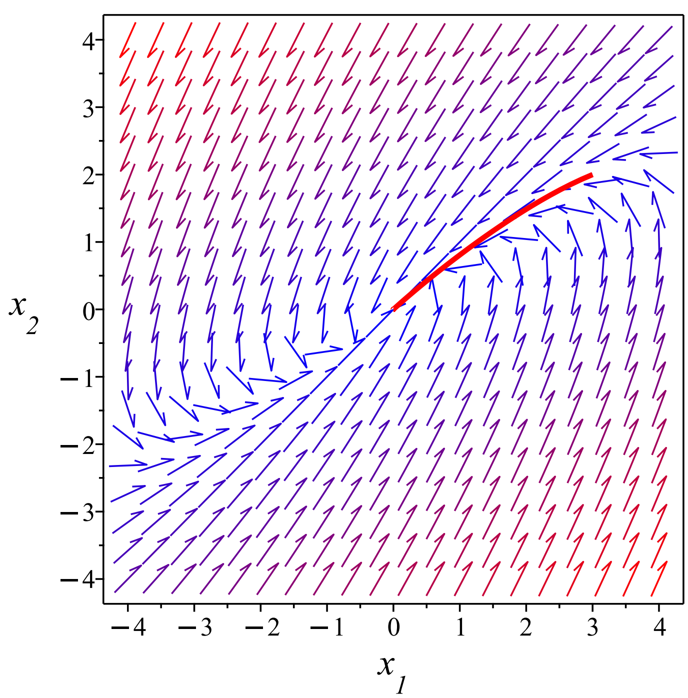
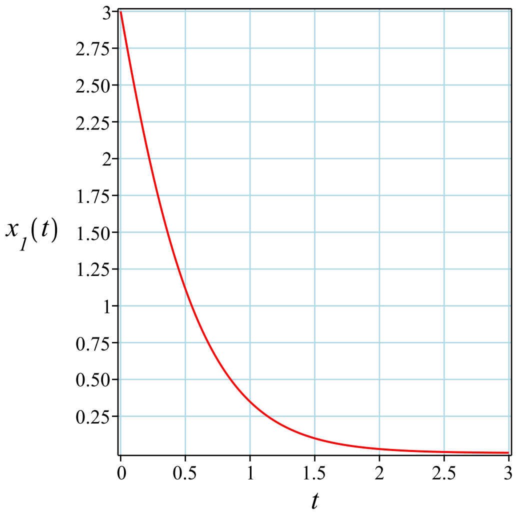
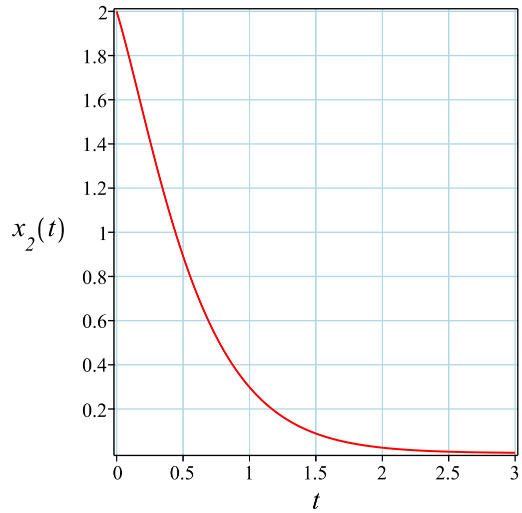


Figure 529: Phase plot

The following are plots of each solution.





✓ Solution by Maple

Time used: 0.015 (sec). Leaf size: 29

```
dsolve([diff(x__1(t),t) = x__1(t)-4*x__2(t), diff(x__2(t),t) = 4*x__1(t)-7*x__2(t), x__1(0)
```

$$x_1(t) = \mathrm{e}^{-3t}(4t+3)$$
$$x_2(t) = \frac{\mathrm{e}^{-3t}(16t+8)}{4}$$

✓ Solution by Mathematica

Time used: 0.005 (sec). Leaf size: 34

```
DSolve[{x1'[t]==1*x1[t]-4*x2[t],x2'[t]==1*x1[t]-4*x2[t]},{x1[0]==3,x2[0]==2},{x1[t],x2[t]},t
```

$$\mathrm{x1}(t) \to \frac{5e^{-3t}}{3} + \frac{4}{3}$$
$$\mathrm{x2}(t) \to \frac{5e^{-3t}}{3} + \frac{1}{3}$$

17.8 problem 8

Internal problem ID [773]
Internal file name [`OUTPUT/773_Sunday_June_05_2022_01_49_18_AM_81518641/index.tex`]

Book: Elementary differential equations and boundary value problems, 10th ed., Boyce and DiPrima
Section: Chapter 7.8, Repeated Eigenvalues. page 436
Problem number: 8.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"system of linear ODEs"**

Solve

$$
\begin{aligned}
x_1'(t) &= -\frac{5x_1(t)}{2} + \frac{3x_2(t)}{2} \\
x_2'(t) &= -\frac{3x_1(t)}{2} + \frac{x_2(t)}{2}
\end{aligned}
$$

With initial conditions

$$[x_1(0) = 3, x_2(0) = -1]$$

17.8.1 Solution using Matrix exponential method

In this method, we will assume we have found the matrix exponential e^{At} allready. There are different methods to determine this but will not be shown here. This is a system of linear ODE's given as

$$\vec{x}'(t) = A\,\vec{x}(t)$$

Or

$$\begin{bmatrix} x_1'(t) \\ x_2'(t) \end{bmatrix} = \begin{bmatrix} -\frac{5}{2} & \frac{3}{2} \\ -\frac{3}{2} & \frac{1}{2} \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}$$

For the above matrix A, the matrix exponential can be found to be

$$e^{At} = \begin{bmatrix} \mathrm{e}^{-t}\left(1 - \frac{3t}{2}\right) & \frac{3t\,\mathrm{e}^{-t}}{2} \\ -\frac{3t\,\mathrm{e}^{-t}}{2} & \mathrm{e}^{-t}\left(1 + \frac{3t}{2}\right) \end{bmatrix}$$

Therefore the homogeneous solution is

$$
\begin{aligned}
\vec{x}_h(t) &= e^{At}\vec{x}_0 \\
&= \begin{bmatrix} e^{-t}\left(1-\frac{3t}{2}\right) & \frac{3t\,e^{-t}}{2} \\ -\frac{3t\,e^{-t}}{2} & e^{-t}\left(1+\frac{3t}{2}\right) \end{bmatrix} \begin{bmatrix} 3 \\ -1 \end{bmatrix} \\
&= \begin{bmatrix} 3\,e^{-t}\left(1-\frac{3t}{2}\right) - \frac{3t\,e^{-t}}{2} \\ -\frac{9t\,e^{-t}}{2} - e^{-t}\left(1+\frac{3t}{2}\right) \end{bmatrix} \\
&= \begin{bmatrix} (-6t+3)\,e^{-t} \\ (-6t-1)\,e^{-t} \end{bmatrix}
\end{aligned}
$$

Since no forcing function is given, then the final solution is $\vec{x}_h(t)$ above.

17.8.2 Solution using explicit Eigenvalue and Eigenvector method

This is a system of linear ODE's given as

$$
\vec{x}'(t) = A\,\vec{x}(t)
$$

Or

$$
\begin{bmatrix} x_1'(t) \\ x_2'(t) \end{bmatrix} = \begin{bmatrix} -\frac{5}{2} & \frac{3}{2} \\ -\frac{3}{2} & \frac{1}{2} \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}
$$

The first step is find the homogeneous solution. We start by finding the eigenvalues of A. This is done by solving the following equation for the eigenvalues λ

$$
\det(A - \lambda I) = 0
$$

Expanding gives

$$
\det\left(\begin{bmatrix} -\frac{5}{2} & \frac{3}{2} \\ -\frac{3}{2} & \frac{1}{2} \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right) = 0
$$

Therefore

$$
\det\left(\begin{bmatrix} -\frac{5}{2} - \lambda & \frac{3}{2} \\ -\frac{3}{2} & \frac{1}{2} - \lambda \end{bmatrix}\right) = 0
$$

Which gives the characteristic equation

$$\lambda^2 + 2\lambda + 1 = 0$$

The roots of the above are the eigenvalues.

$$\lambda_1 = -1$$

This table summarises the above result

eigenvalue	algebraic multiplicity	type of eigenvalue
-1	1	real eigenvalue

Now the eigenvector for each eigenvalue are found.

Considering the eigenvalue $\lambda_1 = -1$

We need to solve $A\vec{v} = \lambda\vec{v}$ or $(A - \lambda I)\vec{v} = \vec{0}$ which becomes

$$\left(\begin{bmatrix} -\frac{5}{2} & \frac{3}{2} \\ -\frac{3}{2} & \frac{1}{2} \end{bmatrix} - (-1)\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right)\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} -\frac{3}{2} & \frac{3}{2} \\ -\frac{3}{2} & \frac{3}{2} \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Now forward elimination is applied to solve for the eigenvector $\vec{v}$. The augmented matrix is

$$\left[\begin{array}{cc|c} -\frac{3}{2} & \frac{3}{2} & 0 \\ -\frac{3}{2} & \frac{3}{2} & 0 \end{array}\right]$$

$$R_2 = R_2 - R_1 \Longrightarrow \left[\begin{array}{cc|c} -\frac{3}{2} & \frac{3}{2} & 0 \\ 0 & 0 & 0 \end{array}\right]$$

Therefore the system in Echelon form is

$$\begin{bmatrix} -\frac{3}{2} & \frac{3}{2} \\ 0 & 0 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

The free variables are $\{v_2\}$ and the leading variables are $\{v_1\}$. Let $v_2 = t$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\{v_1 = t\}$

Hence the solution is

$$\begin{bmatrix} t \\ t \end{bmatrix} = \begin{bmatrix} t \\ t \end{bmatrix}$$

Since there is one free Variable, we have found one eigenvector associated with this eigenvalue. The above can be written as

$$\begin{bmatrix} t \\ t \end{bmatrix} = t \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

Let $t = 1$ the eigenvector becomes

$$\begin{bmatrix} t \\ t \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

The following table gives a summary of this result. It shows for each eigenvalue the algebraic multiplicity m, and its geometric multiplicity k and the eigenvectors associated with the eigenvalue. If $m > k$ then the eigenvalue is defective which means the number of normal linearly independent eigenvectors associated with this eigenvalue (called the geometric multiplicity k) does not equal the algebraic multiplicity m, and we need to determine an additional $m - k$ generalized eigenvectors for this eigenvalue.

eigenvalue	multiplicity		defective?	eigenvectors
	algebraic m	geometric k		
-1	2	1	Yes	$\begin{bmatrix} 1 \\ 1 \end{bmatrix}$

Now that we found the eigenvalues and associated eigenvectors, we will go over each eigenvalue and generate the solution basis. The only problem we need to take care of is if the eigenvalue is defective. eigenvalue -1 is real and repated eigenvalue of multiplicity 2.There are two possible cases that can happen. This is illustrated in this diagram

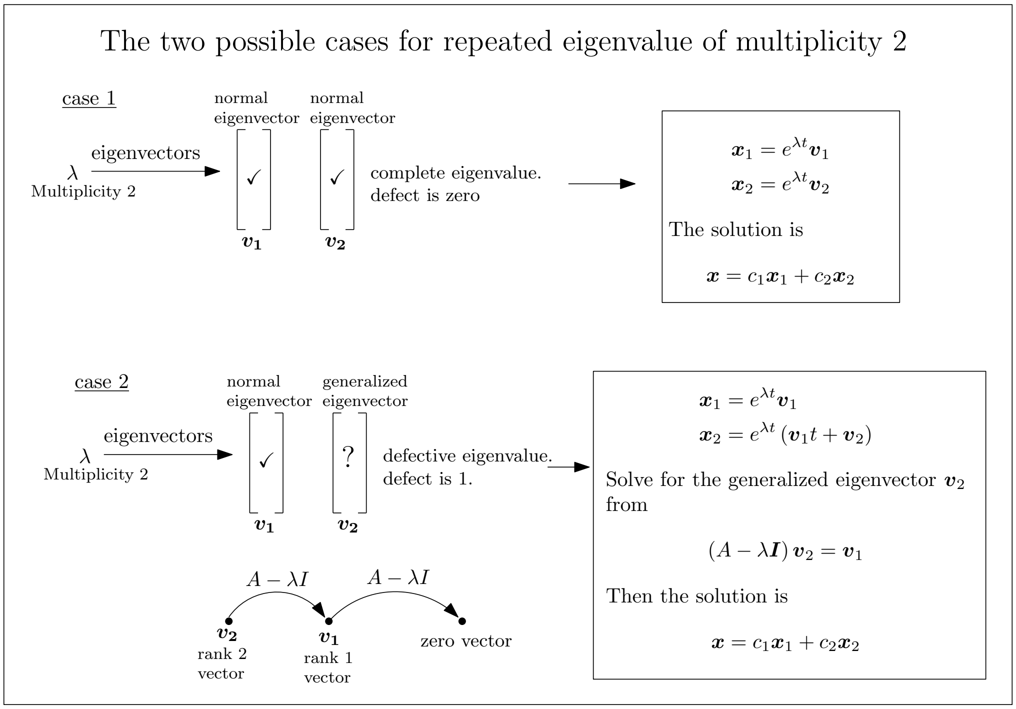


Figure 530: Possible case for repeated λ of multiplicity 2

This eigenvalue has algebraic multiplicity of 2, and geometric multiplicity 1, therefore this is defective eigenvalue. The defect is 1. This falls into case 2 shown above. We need to generate the missing additonal generalized eigevector $\vec{v}_2$ by solving

$$(A - \lambda I)\,\vec{v}_2 = \vec{v}_1$$

Where $\vec{v}_1$ is the normal (rank 1) eigenvector found above. Hence we need to solve

$$\left(\begin{bmatrix} -\frac{5}{2} & \frac{3}{2} \\ -\frac{3}{2} & \frac{1}{2} \end{bmatrix} - (-1)\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right)\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$\begin{bmatrix} -\frac{3}{2} & \frac{3}{2} \\ -\frac{3}{2} & \frac{3}{2} \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

Solving for $\vec{v}_2$ gives

$$\vec{v}_2 = \begin{bmatrix} \frac{1}{3} \\ 1 \end{bmatrix}$$

We have found two generalized eigenvectors for eigenvalue -1. Therefore the two basis solution associated with this eigenvalue are

$$\begin{aligned}\vec{x}_1(t) &= \vec{v}_1 e^{\lambda t} \\ &= \begin{bmatrix} 1 \\ 1 \end{bmatrix} \mathrm{e}^{-t} \\ &= \begin{bmatrix} \mathrm{e}^{-t} \\ \mathrm{e}^{-t} \end{bmatrix}\end{aligned}$$

And

$$\begin{aligned}\vec{x}_2(t) &= (\vec{v}_1 t + \vec{v}_2)\, e^{\lambda t} \\ &= \left(\begin{bmatrix} 1 \\ 1 \end{bmatrix} t + \begin{bmatrix} \frac{1}{3} \\ 1 \end{bmatrix} \right) \mathrm{e}^{-t} \\ &= \begin{bmatrix} \frac{(1+3t)\mathrm{e}^{-t}}{3} \\ \mathrm{e}^{-t}(t+1) \end{bmatrix}\end{aligned}$$

Therefore the final solution is

$$\vec{x}_h(t) = c_1 \vec{x}_1(t) + c_2 \vec{x}_2(t)$$

Which is written as

$$\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = c_1 \begin{bmatrix} \mathrm{e}^{-t} \\ \mathrm{e}^{-t} \end{bmatrix} + c_2 \begin{bmatrix} \mathrm{e}^{-t}\left(t + \frac{1}{3}\right) \\ \mathrm{e}^{-t}(t+1) \end{bmatrix}$$

Which becomes

$$\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = \begin{bmatrix} \mathrm{e}^{-t}\left(c_1 + c_2 t + \frac{1}{3}c_2\right) \\ \mathrm{e}^{-t}(c_2 t + c_1 + c_2) \end{bmatrix}$$

Since initial conditions are given, the solution above needs to be updated by solving for the constants of integrations using the given initial conditions

$$\begin{bmatrix} x_1(0) = 3 \\ x_2(0) = -1 \end{bmatrix} \tag{1}$$

Substituting initial conditions into the above solution at $t = 0$ gives

$$\begin{bmatrix} 3 \\ -1 \end{bmatrix} = \begin{bmatrix} c_1 + \frac{c_2}{3} \\ c_1 + c_2 \end{bmatrix}$$

Solving for the constants of integrations gives

$$\begin{bmatrix} c_1 = 5 \\ c_2 = -6 \end{bmatrix}$$

Substituting these constants back in original solution in Eq. (1) gives

$$\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = \begin{bmatrix} (-6t + 3)\,\mathrm{e}^{-t} \\ (-6t - 1)\,\mathrm{e}^{-t} \end{bmatrix}$$

The following is the phase plot of the system.

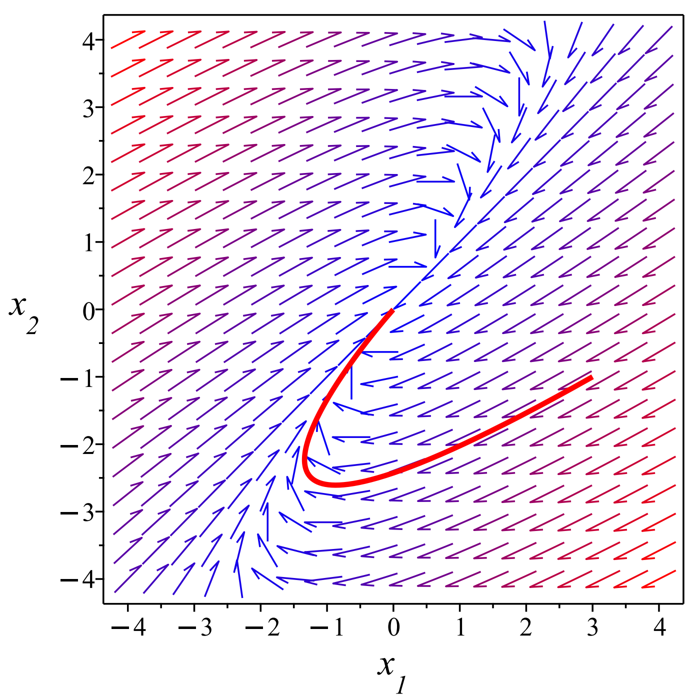
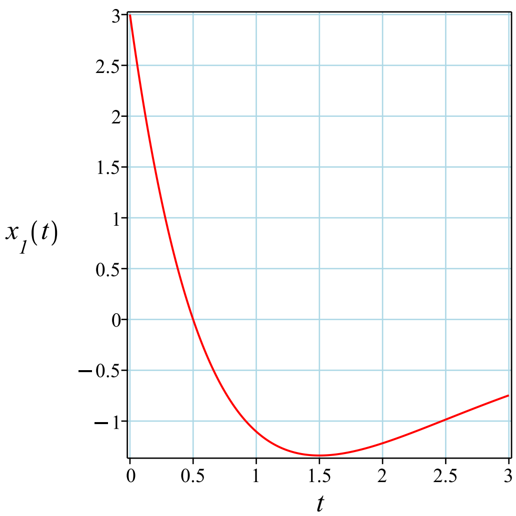
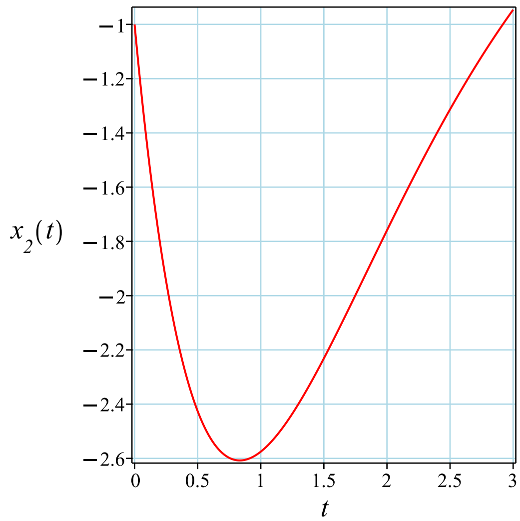


Figure 531: Phase plot

The following are plots of each solution.





✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 29

```
dsolve([diff(x__1(t),t) = -5/2*x__1(t)+3/2*x__2(t), diff(x__2(t),t) = -3/2*x__1(t)+1/2*x__2(
```

$$x_1(t) = \mathrm{e}^{-t}(-6t + 3)$$
$$x_2(t) = \frac{\mathrm{e}^{-t}(-18t - 3)}{3}$$

✓ Solution by Mathematica

Time used: 0.003 (sec). Leaf size: 31

```
DSolve[{x1'[t]==-5/2*x1[t]+3/2*x2[t],x2'[t]==-3/2*x1[t]+1/2*x2[t]},{x1[0]==3,x2[0]==-1},{x1[
```

$$\mathrm{x1}(t) \to e^{-t}(3 - 6t)$$
$$\mathrm{x2}(t) \to -e^{-t}(6t + 1)$$

17.9 problem 9

Internal problem ID [774]
Internal file name [`OUTPUT/774_Sunday_June_05_2022_01_49_19_AM_6565187/index.tex`]

Book: Elementary differential equations and boundary value problems, 10th ed., Boyce and DiPrima
Section: Chapter 7.8, Repeated Eigenvalues. page 436
Problem number: 9.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"system of linear ODEs"**

Solve

$$x_1'(t) = 2x_1(t) + \frac{3x_2(t)}{2}$$
$$x_2'(t) = -\frac{3x_1(t)}{2} - x_2(t)$$

With initial conditions

$$[x_1(0) = 3, x_2(0) = -2]$$

17.9.1 Solution using Matrix exponential method

In this method, we will assume we have found the matrix exponential e^{At} allready. There are different methods to determine this but will not be shown here. This is a system of linear ODE's given as

$$\vec{x}'(t) = A\,\vec{x}(t)$$

Or

$$\begin{bmatrix} x_1'(t) \\ x_2'(t) \end{bmatrix} = \begin{bmatrix} 2 & \frac{3}{2} \\ -\frac{3}{2} & -1 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}$$

For the above matrix A, the matrix exponential can be found to be

$$e^{At} = \begin{bmatrix} e^{\frac{t}{2}}\left(1 + \frac{3t}{2}\right) & \frac{3t\,e^{\frac{t}{2}}}{2} \\ -\frac{3t\,e^{\frac{t}{2}}}{2} & e^{\frac{t}{2}}\left(1 - \frac{3t}{2}\right) \end{bmatrix}$$

Therefore the homogeneous solution is

$$
\begin{aligned}
\vec{x}_h(t) &= e^{At}\vec{x}_0 \\
&= \begin{bmatrix} e^{\frac{t}{2}}\left(1+\frac{3t}{2}\right) & \frac{3t\,e^{\frac{t}{2}}}{2} \\ -\frac{3t\,e^{\frac{t}{2}}}{2} & e^{\frac{t}{2}}\left(1-\frac{3t}{2}\right) \end{bmatrix} \begin{bmatrix} 3 \\ -2 \end{bmatrix} \\
&= \begin{bmatrix} 3\,e^{\frac{t}{2}}\left(1+\frac{3t}{2}\right) - 3t\,e^{\frac{t}{2}} \\ -\frac{9t\,e^{\frac{t}{2}}}{2} - 2\,e^{\frac{t}{2}}\left(1-\frac{3t}{2}\right) \end{bmatrix} \\
&= \begin{bmatrix} \frac{3\,e^{\frac{t}{2}}(2+t)}{2} \\ e^{\frac{t}{2}}\left(-2-\frac{3t}{2}\right) \end{bmatrix}
\end{aligned}
$$

Since no forcing function is given, then the final solution is $\vec{x}_h(t)$ above.

17.9.2 Solution using explicit Eigenvalue and Eigenvector method

This is a system of linear ODE's given as

$$
\vec{x}'(t) = A\,\vec{x}(t)
$$

Or

$$
\begin{bmatrix} x_1'(t) \\ x_2'(t) \end{bmatrix} = \begin{bmatrix} 2 & \frac{3}{2} \\ -\frac{3}{2} & -1 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}
$$

The first step is find the homogeneous solution. We start by finding the eigenvalues of A. This is done by solving the following equation for the eigenvalues λ

$$
\det\left(A - \lambda I\right) = 0
$$

Expanding gives

$$
\det\left(\begin{bmatrix} 2 & \frac{3}{2} \\ -\frac{3}{2} & -1 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right) = 0
$$

Therefore

$$
\det\left(\begin{bmatrix} 2-\lambda & \frac{3}{2} \\ -\frac{3}{2} & -1-\lambda \end{bmatrix}\right) = 0
$$

Which gives the characteristic equation

$$\lambda^2 - \lambda + \frac{1}{4} = 0$$

The roots of the above are the eigenvalues.

$$\lambda_1 = \frac{1}{2}$$

This table summarises the above result

eigenvalue	algebraic multiplicity	type of eigenvalue
$\frac{1}{2}$	1	real eigenvalue

Now the eigenvector for each eigenvalue are found.

Considering the eigenvalue $\lambda_1 = \frac{1}{2}$

We need to solve $A\vec{v} = \lambda\vec{v}$ or $(A - \lambda I)\vec{v} = \vec{0}$ which becomes

$$\left(\begin{bmatrix} 2 & \frac{3}{2} \\ -\frac{3}{2} & -1 \end{bmatrix} - \left(\frac{1}{2}\right)\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right)\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} \frac{3}{2} & \frac{3}{2} \\ -\frac{3}{2} & -\frac{3}{2} \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Now forward elimination is applied to solve for the eigenvector $\vec{v}$. The augmented matrix is

$$\left[\begin{array}{cc|c} \frac{3}{2} & \frac{3}{2} & 0 \\ -\frac{3}{2} & -\frac{3}{2} & 0 \end{array}\right]$$

$$R_2 = R_2 + R_1 \Longrightarrow \left[\begin{array}{cc|c} \frac{3}{2} & \frac{3}{2} & 0 \\ 0 & 0 & 0 \end{array}\right]$$

Therefore the system in Echelon form is

$$\begin{bmatrix} \frac{3}{2} & \frac{3}{2} \\ 0 & 0 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

The free variables are $\{v_2\}$ and the leading variables are $\{v_1\}$. Let $v_2 = t$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\{v_1 = -t\}$

Hence the solution is

$$\begin{bmatrix} -t \\ t \end{bmatrix} = \begin{bmatrix} -t \\ t \end{bmatrix}$$

Since there is one free Variable, we have found one eigenvector associated with this eigenvalue. The above can be written as

$$\begin{bmatrix} -t \\ t \end{bmatrix} = t \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

Let $t = 1$ the eigenvector becomes

$$\begin{bmatrix} -t \\ t \end{bmatrix} = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

The following table gives a summary of this result. It shows for each eigenvalue the algebraic multiplicity m, and its geometric multiplicity k and the eigenvectors associated with the eigenvalue. If $m > k$ then the eigenvalue is defective which means the number of normal linearly independent eigenvectors associated with this eigenvalue (called the geometric multiplicity k) does not equal the algebraic multiplicity m, and we need to determine an additional $m - k$ generalized eigenvectors for this eigenvalue.

	multiplicity			
eigenvalue	algebraic m	geometric k	defective?	eigenvectors
$\frac{1}{2}$	2	1	Yes	$\begin{bmatrix} -1 \\ 1 \end{bmatrix}$

Now that we found the eigenvalues and associated eigenvectors, we will go over each eigenvalue and generate the solution basis. The only problem we need to take care of is if the eigenvalue is defective. eigenvalue $\frac{1}{2}$ is real and repated eigenvalue of multiplicity 2.There are two possible cases that can happen. This is illustrated in this diagram

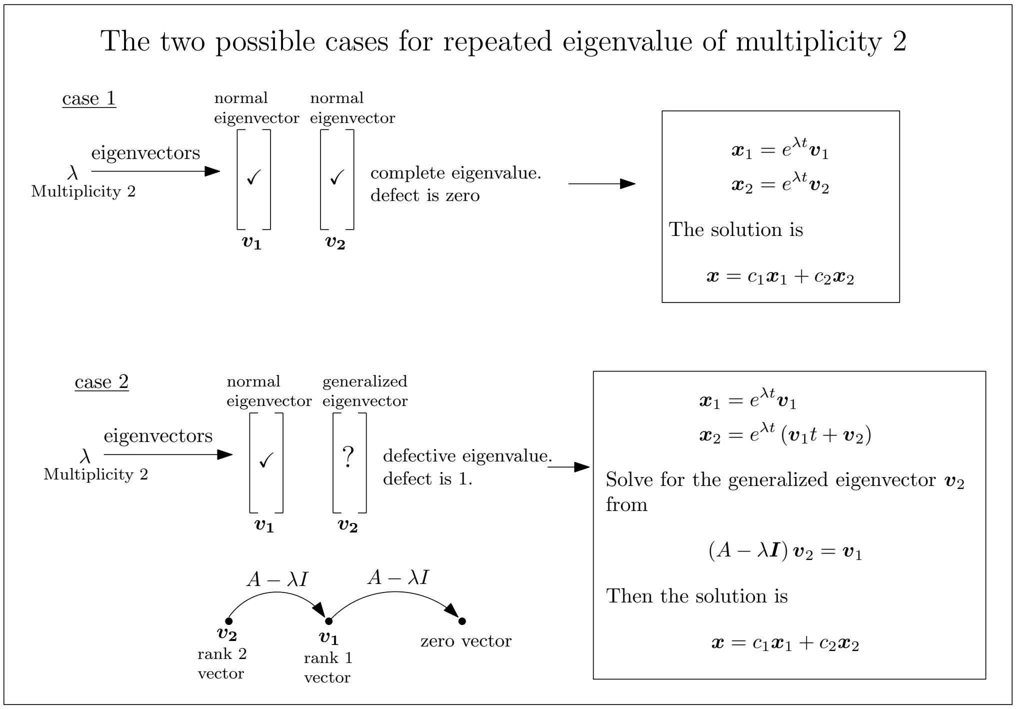


Figure 532: Possible case for repeated λ of multiplicity 2

This eigenvalue has algebraic multiplicity of 2, and geometric multiplicity 1, therefore this is defective eigenvalue. The defect is 1. This falls into case 2 shown above. We need to generate the missing additonal generalized eigevector $\vec{v}_2$ by solving

$$(A - \lambda I)\,\vec{v}_2 = \vec{v}_1$$

Where $\vec{v}_1$ is the normal (rank 1) eigenvector found above. Hence we need to solve

$$\left(\begin{bmatrix} 2 & \frac{3}{2} \\ -\frac{3}{2} & -1 \end{bmatrix} - \left(\frac{1}{2}\right) \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right) \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

$$\begin{bmatrix} \frac{3}{2} & \frac{3}{2} \\ -\frac{3}{2} & -\frac{3}{2} \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

Solving for $\vec{v}_2$ gives

$$\vec{v}_2 = \begin{bmatrix} -\frac{5}{3} \\ 1 \end{bmatrix}$$

We have found two generalized eigenvectors for eigenvalue $\frac{1}{2}$. Therefore the two basis solution associated with this eigenvalue are

$$\begin{aligned}\vec{x}_1(t) &= \vec{v}_1 e^{\lambda t} \\ &= \begin{bmatrix} -1 \\ 1 \end{bmatrix} \mathrm{e}^{\frac{t}{2}} \\ &= \begin{bmatrix} -\mathrm{e}^{\frac{t}{2}} \\ \mathrm{e}^{\frac{t}{2}} \end{bmatrix}\end{aligned}$$

And

$$\begin{aligned}\vec{x}_2(t) &= (\vec{v}_1 t + \vec{v}_2)\, e^{\lambda t} \\ &= \left(\begin{bmatrix} -1 \\ 1 \end{bmatrix} t + \begin{bmatrix} -\frac{5}{3} \\ 1 \end{bmatrix}\right) \mathrm{e}^{\frac{t}{2}} \\ &= \begin{bmatrix} -\frac{\mathrm{e}^{\frac{t}{2}}(3t+5)}{3} \\ \mathrm{e}^{\frac{t}{2}}(t+1) \end{bmatrix}\end{aligned}$$

Therefore the final solution is

$$\vec{x}_h(t) = c_1 \vec{x}_1(t) + c_2 \vec{x}_2(t)$$

Which is written as

$$\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = c_1 \begin{bmatrix} -\mathrm{e}^{\frac{t}{2}} \\ \mathrm{e}^{\frac{t}{2}} \end{bmatrix} + c_2 \begin{bmatrix} \mathrm{e}^{\frac{t}{2}}\left(-t - \frac{5}{3}\right) \\ \mathrm{e}^{\frac{t}{2}}(t+1) \end{bmatrix}$$

Which becomes

$$\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = \begin{bmatrix} \mathrm{e}^{\frac{t}{2}}\left(-c_1 - c_2 t - \frac{5}{3}c_2\right) \\ \mathrm{e}^{\frac{t}{2}}(c_2 t + c_1 + c_2) \end{bmatrix}$$

Since initial conditions are given, the solution above needs to be updated by solving for the constants of integrations using the given initial conditions

$$\begin{bmatrix} x_1(0) = 3 \\ x_2(0) = -2 \end{bmatrix} \tag{1}$$

Substituting initial conditions into the above solution at $t = 0$ gives

$$\begin{bmatrix} 3 \\ -2 \end{bmatrix} = \begin{bmatrix} -c_1 - \frac{5c_2}{3} \\ c_1 + c_2 \end{bmatrix}$$

Solving for the constants of integrations gives

$$\begin{bmatrix} c_1 = -\frac{1}{2} \\ c_2 = -\frac{3}{2} \end{bmatrix}$$

Substituting these constants back in original solution in Eq. (1) gives

$$\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = \begin{bmatrix} e^{\frac{t}{2}}\left(3 + \frac{3t}{2}\right) \\ e^{\frac{t}{2}}\left(-2 - \frac{3t}{2}\right) \end{bmatrix}$$

The following is the phase plot of the system.

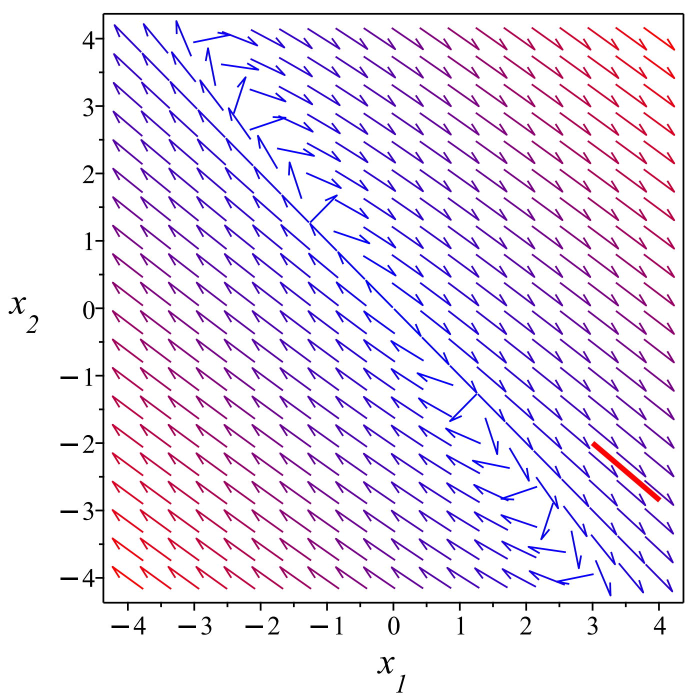
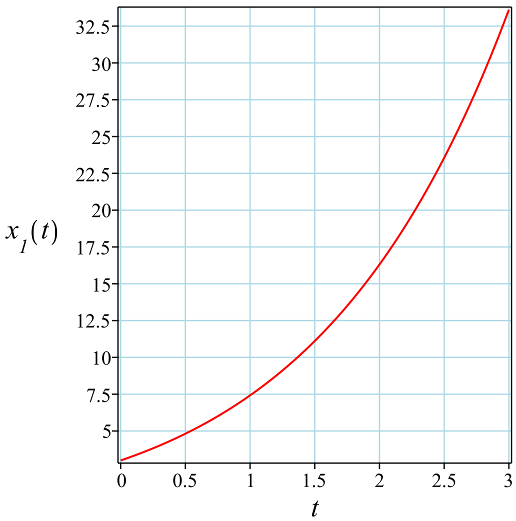
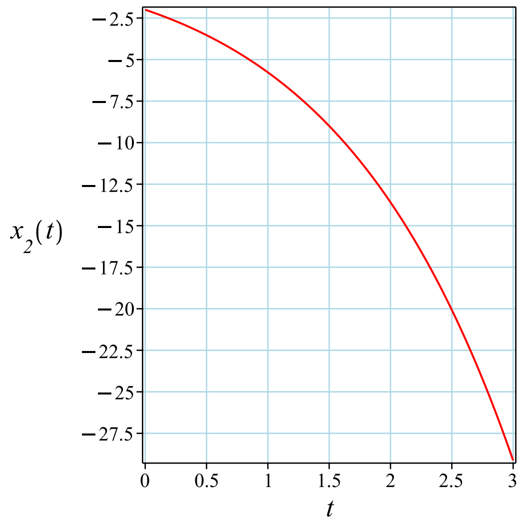


Figure 533: Phase plot

The following are plots of each solution.





✓ Solution by Maple

Time used: 0.015 (sec). Leaf size: 29

```
dsolve([diff(x__1(t),t) = 2*x__1(t)+3/2*x__2(t), diff(x__2(t),t) = -3/2*x__1(t)-x__2(t), x__
```

$$x_1(t) = e^{\frac{t}{2}}\left(\frac{3t}{2}+3\right)$$
$$x_2(t) = -\frac{e^{\frac{t}{2}}\left(\frac{9t}{2}+6\right)}{3}$$

✓ Solution by Mathematica

Time used: 0.003 (sec). Leaf size: 38

```
DSolve[{x1'[t]==2*x1[t]+3/2*x2[t],x2'[t]==-3/2*x1[t]-1*x2[t]},{x1[0]==3,x2[0]==-2},{x1[t],x2
```

$$\text{x1}(t) \to \frac{3}{2}e^{t/2}(t+2)$$
$$\text{x2}(t) \to -\frac{1}{2}e^{t/2}(3t+4)$$

17.10 problem 10

Internal problem ID [775]
Internal file name [OUTPUT/775_Sunday_June_05_2022_01_49_21_AM_24868463/index.tex]

Book: Elementary differential equations and boundary value problems, 10th ed., Boyce and DiPrima
Section: Chapter 7.8, Repeated Eigenvalues. page 436
Problem number: 10.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"system of linear ODEs"**

Solve

$$
\begin{aligned}
x_1'(t) &= 3x_1(t) + 9x_2(t) \\
x_2'(t) &= -x_1(t) - 3x_2(t)
\end{aligned}
$$

With initial conditions

$$
[x_1(0) = 2, x_2(0) = 4]
$$

17.10.1 Solution using Matrix exponential method

In this method, we will assume we have found the matrix exponential e^{At} allready. There are different methods to determine this but will not be shown here. This is a system of linear ODE's given as

$$
\vec{x}'(t) = A\,\vec{x}(t)
$$

Or

$$
\begin{bmatrix} x_1'(t) \\ x_2'(t) \end{bmatrix} = \begin{bmatrix} 3 & 9 \\ -1 & -3 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}
$$

For the above matrix A, the matrix exponential can be found to be

$$
e^{At} = \begin{bmatrix} 1+3t & 9t \\ -t & 1-3t \end{bmatrix}
$$

Therefore the homogeneous solution is

$$\vec{x}_h(t) = e^{At}\vec{x}_0$$
$$= \begin{bmatrix} 1+3t & 9t \\ -t & 1-3t \end{bmatrix} \begin{bmatrix} 2 \\ 4 \end{bmatrix}$$
$$= \begin{bmatrix} 2+42t \\ -14t+4 \end{bmatrix}$$

Since no forcing function is given, then the final solution is $\vec{x}_h(t)$ above.

17.10.2 Solution using explicit Eigenvalue and Eigenvector method

This is a system of linear ODE's given as

$$\vec{x}'(t) = A\,\vec{x}(t)$$

Or

$$\begin{bmatrix} x_1'(t) \\ x_2'(t) \end{bmatrix} = \begin{bmatrix} 3 & 9 \\ -1 & -3 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}$$

The first step is find the homogeneous solution. We start by finding the eigenvalues of A. This is done by solving the following equation for the eigenvalues λ

$$\det(A - \lambda I) = 0$$

Expanding gives

$$\det\left(\begin{bmatrix} 3 & 9 \\ -1 & -3 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right) = 0$$

Therefore

$$\det\left(\begin{bmatrix} 3-\lambda & 9 \\ -1 & -3-\lambda \end{bmatrix}\right) = 0$$

Which gives the characteristic equation

$$\lambda^2 = 0$$

The roots of the above are the eigenvalues.

$$\lambda_1 = 0$$

This table summarises the above result

eigenvalue	algebraic multiplicity	type of eigenvalue
0	1	real eigenvalue

Now the eigenvector for each eigenvalue are found.

Considering the eigenvalue $\lambda_1 = 0$

We need to solve $A\vec{v} = \lambda\vec{v}$ or $(A - \lambda I)\vec{v} = \vec{0}$ which becomes

$$\left(\begin{bmatrix} 3 & 9 \\ -1 & -3 \end{bmatrix} - (0) \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right) \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 3 & 9 \\ -1 & -3 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Now forward elimination is applied to solve for the eigenvector $\vec{v}$. The augmented matrix is

$$\left[\begin{array}{cc|c} 3 & 9 & 0 \\ -1 & -3 & 0 \end{array}\right]$$

$$R_2 = R_2 + \frac{R_1}{3} \Longrightarrow \left[\begin{array}{cc|c} 3 & 9 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

Therefore the system in Echelon form is

$$\begin{bmatrix} 3 & 9 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

The free variables are $\{v_2\}$ and the leading variables are $\{v_1\}$. Let $v_2 = t$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\{v_1 = -3t\}$

Hence the solution is

$$\begin{bmatrix} -3t \\ t \end{bmatrix} = \begin{bmatrix} -3t \\ t \end{bmatrix}$$

Since there is one free Variable, we have found one eigenvector associated with this eigenvalue. The above can be written as

$$\begin{bmatrix} -3t \\ t \end{bmatrix} = t \begin{bmatrix} -3 \\ 1 \end{bmatrix}$$

Let $t = 1$ the eigenvector becomes

$$\begin{bmatrix} -3t \\ t \end{bmatrix} = \begin{bmatrix} -3 \\ 1 \end{bmatrix}$$

The following table gives a summary of this result. It shows for each eigenvalue the algebraic multiplicity m, and its geometric multiplicity k and the eigenvectors associated with the eigenvalue. If $m > k$ then the eigenvalue is defective which means the number of normal linearly independent eigenvectors associated with this eigenvalue (called the geometric multiplicity k) does not equal the algebraic multiplicity m, and we need to determine an additional $m - k$ generalized eigenvectors for this eigenvalue.

eigenvalue	multiplicity		defective?	eigenvectors
	algebraic m	geometric k		
0	2	1	Yes	$\begin{bmatrix} -3 \\ 1 \end{bmatrix}$

Now that we found the eigenvalues and associated eigenvectors, we will go over each eigenvalue and generate the solution basis. The only problem we need to take care of is if the eigenvalue is defective. eigenvalue 0 is real and repated eigenvalue of multiplicity 2.There are two possible cases that can happen. This is illustrated in this diagram

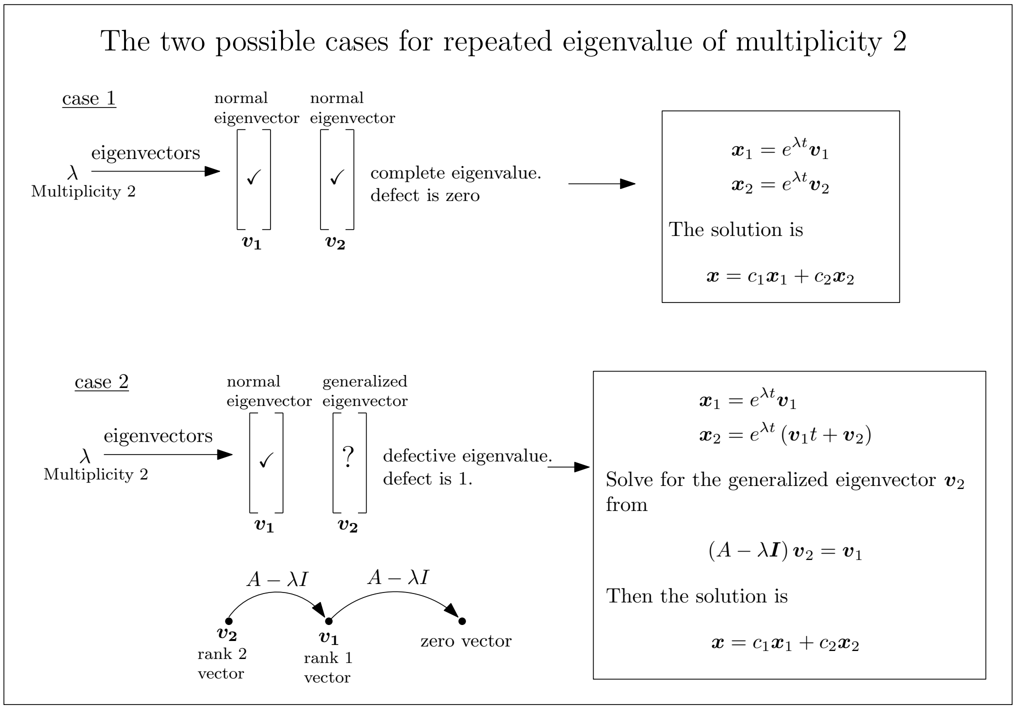


Figure 534: Possible case for repeated λ of multiplicity 2

This eigenvalue has algebraic multiplicity of 2, and geometric multiplicity 1, therefore this is defective eigenvalue. The defect is 1. This falls into case 2 shown above. We need to generate the missing additonal generalized eigevector $\vec{v}_2$ by solving

$$(A - \lambda I)\vec{v}_2 = \vec{v}_1$$

Where $\vec{v}_1$ is the normal (rank 1) eigenvector found above. Hence we need to solve

$$\left(\begin{bmatrix} 3 & 9 \\ -1 & -3 \end{bmatrix} - (0)\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right)\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} -3 \\ 1 \end{bmatrix}$$
$$\begin{bmatrix} 3 & 9 \\ -1 & -3 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} -3 \\ 1 \end{bmatrix}$$

Solving for $\vec{v}_2$ gives

$$\vec{v}_2 = \begin{bmatrix} -4 \\ 1 \end{bmatrix}$$

We have found two generalized eigenvectors for eigenvalue 0. Therefore the two basis solution associated with this eigenvalue are

$$\begin{aligned}\vec{x}_1(t) &= \vec{v}_1 e^{\lambda t} \\ &= \begin{bmatrix} -3 \\ 1 \end{bmatrix} 1 \\ &= \begin{bmatrix} -3 \\ 1 \end{bmatrix}\end{aligned}$$

And

$$\begin{aligned}\vec{x}_2(t) &= (\vec{v}_1 t + \vec{v}_2)\, e^{\lambda t} \\ &= \left(\begin{bmatrix} -3 \\ 1 \end{bmatrix} t + \begin{bmatrix} -4 \\ 1 \end{bmatrix} \right) 1 \\ &= \begin{bmatrix} -4 - 3t \\ t + 1 \end{bmatrix}\end{aligned}$$

Therefore the final solution is

$$\vec{x}_h(t) = c_1 \vec{x}_1(t) + c_2 \vec{x}_2(t)$$

Which is written as

$$\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = c_1 \begin{bmatrix} -3 \\ 1 \end{bmatrix} + c_2 \begin{bmatrix} -4 - 3t \\ t + 1 \end{bmatrix}$$

Which becomes

$$\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = \begin{bmatrix} -3c_1 + c_2(-4 - 3t) \\ c_2 t + c_1 + c_2 \end{bmatrix}$$

Since initial conditions are given, the solution above needs to be updated by solving for the constants of integrations using the given initial conditions

$$\begin{bmatrix} x_1(0) = 2 \\ x_2(0) = 4 \end{bmatrix} \tag{1}$$

Substituting initial conditions into the above solution at $t = 0$ gives

$$\begin{bmatrix} 2 \\ 4 \end{bmatrix} = \begin{bmatrix} -3c_1 - 4c_2 \\ c_1 + c_2 \end{bmatrix}$$

Solving for the constants of integrations gives

$$\begin{bmatrix} c_1 = 18 \\ c_2 = -14 \end{bmatrix}$$

Substituting these constants back in original solution in Eq. (1) gives

$$\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = \begin{bmatrix} 2 + 42t \\ -14t + 4 \end{bmatrix}$$

The following is the phase plot of the system.

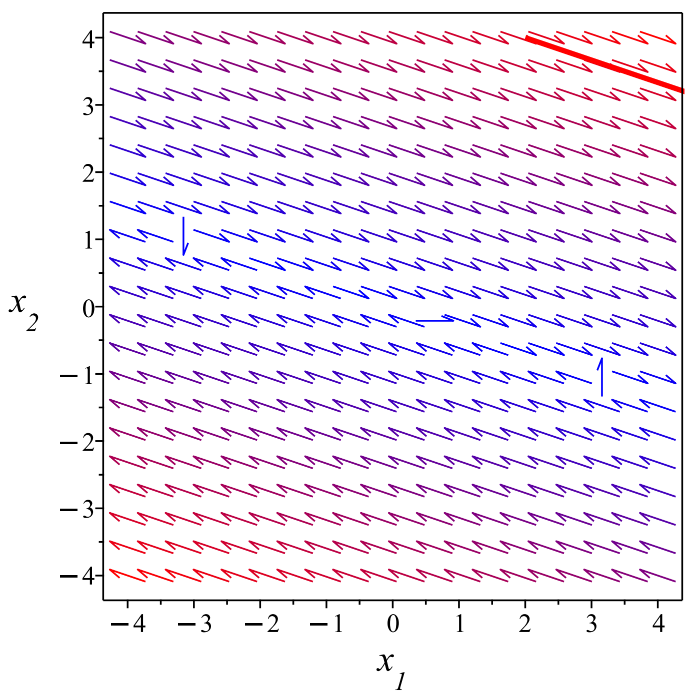
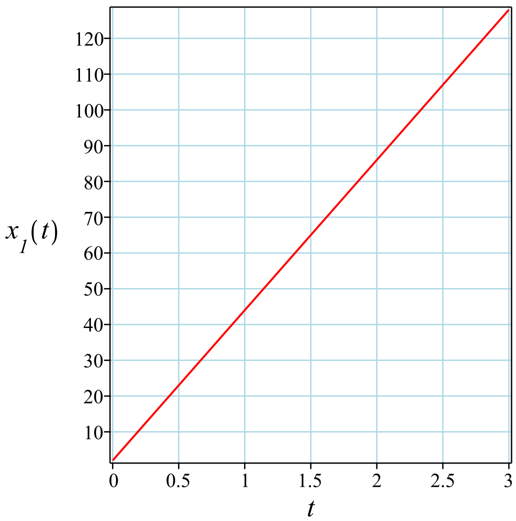
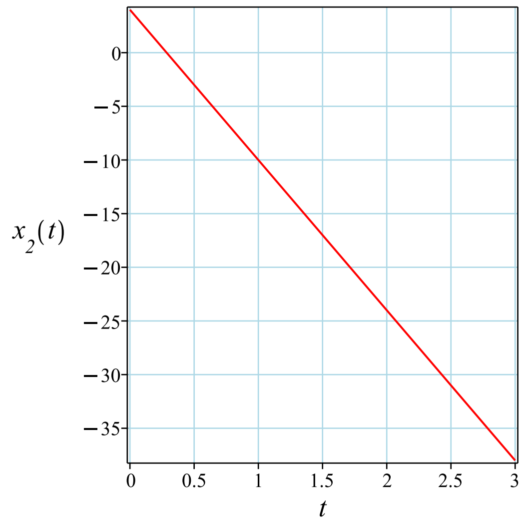


Figure 535: Phase plot

The following are plots of each solution.





✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 18

```
dsolve([diff(x__1(t),t) = 3*x__1(t)+9*x__2(t), diff(x__2(t),t) = -x__1(t)-3*x__2(t), x__1(0)
```

$$x_1(t) = 42t + 2$$
$$x_2(t) = 4 - 14t$$

✓ Solution by Mathematica

Time used: 0.003 (sec). Leaf size: 18

```
DSolve[{x1'[t]==3*x1[t]+9*x2[t],x2'[t]==-1*x1[t]-3*x2[t]},{x1[0]==2,x2[0]==4},{x1[t],x2[t]},
```

$$\text{x1}(t) \to 42t + 2$$
$$\text{x2}(t) \to 4 - 14t$$

17.11 problem 11

Internal problem ID [776]
Internal file name [`OUTPUT/776_Sunday_June_05_2022_01_49_22_AM_63754416/index.tex`]

Book: Elementary differential equations and boundary value problems, 10th ed., Boyce and DiPrima
Section: Chapter 7.8, Repeated Eigenvalues. page 436
Problem number: 11.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"system of linear ODEs"**

Solve

$$
\begin{aligned}
x_1'(t) &= x_1(t) \\
x_2'(t) &= -4x_1(t) + x_2(t) \\
x_3'(t) &= 3x_1(t) + 6x_2(t) + 2x_3(t)
\end{aligned}
$$

With initial conditions

$$
[x_1(0) = -1, x_2(0) = 2, x_3(0) = -30]
$$

17.11.1 Solution using Matrix exponential method

In this method, we will assume we have found the matrix exponential e^{At} allready. There are different methods to determine this but will not be shown here. This is a system of linear ODE's given as

$$
\vec{x}'(t) = A\,\vec{x}(t)
$$

Or

$$
\begin{bmatrix} x_1'(t) \\ x_2'(t) \\ x_3'(t) \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ -4 & 1 & 0 \\ 3 & 6 & 2 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{bmatrix}
$$

For the above matrix A, the matrix exponential can be found to be

$$e^{At} = \begin{bmatrix} e^{t} & 0 & 0 \\ -4t\,e^{t} & e^{t} & 0 \\ -21\,e^{2t} + (24t+21)\,e^{t} & 6\,e^{2t} - 6\,e^{t} & e^{2t} \end{bmatrix}$$

Therefore the homogeneous solution is

$$\begin{aligned} \vec{x}_h(t) &= e^{At}\vec{x}_0 \\ &= \begin{bmatrix} e^{t} & 0 & 0 \\ -4t\,e^{t} & e^{t} & 0 \\ -21\,e^{2t} + (24t+21)\,e^{t} & 6\,e^{2t} - 6\,e^{t} & e^{2t} \end{bmatrix} \begin{bmatrix} -1 \\ 2 \\ -30 \end{bmatrix} \\ &= \begin{bmatrix} -e^{t} \\ 4t\,e^{t} + 2\,e^{t} \\ 3\,e^{2t} - (24t+21)\,e^{t} - 12\,e^{t} \end{bmatrix} \\ &= \begin{bmatrix} -e^{t} \\ (4t+2)\,e^{t} \\ 3\,e^{2t} + (-24t-33)\,e^{t} \end{bmatrix} \end{aligned}$$

Since no forcing function is given, then the final solution is $\vec{x}_h(t)$ above.

17.11.2 Solution using explicit Eigenvalue and Eigenvector method

This is a system of linear ODE's given as

$$\vec{x}'(t) = A\,\vec{x}(t)$$

Or

$$\begin{bmatrix} x_1'(t) \\ x_2'(t) \\ x_3'(t) \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ -4 & 1 & 0 \\ 3 & 6 & 2 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{bmatrix}$$

The first step is find the homogeneous solution. We start by finding the eigenvalues of A. This is done by solving the following equation for the eigenvalues λ

$$\det(A - \lambda I) = 0$$

Expanding gives

$$\det\left(\begin{bmatrix}1 & 0 & 0\\ -4 & 1 & 0\\ 3 & 6 & 2\end{bmatrix} - \lambda\begin{bmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{bmatrix}\right) = 0$$

Therefore

$$\det\left(\begin{bmatrix}1-\lambda & 0 & 0\\ -4 & 1-\lambda & 0\\ 3 & 6 & 2-\lambda\end{bmatrix}\right) = 0$$

Since the matrix A is triangular matrix, then the determinant is the product of the elements along the diagonal. Therefore the above becomes

$$(1-\lambda)(1-\lambda)(2-\lambda) = 0$$

The roots of the above are the eigenvalues.

$$\lambda_1 = 2$$
$$\lambda_2 = 1$$

This table summarises the above result

eigenvalue	algebraic multiplicity	type of eigenvalue
1	1	real eigenvalue
2	1	real eigenvalue

Now the eigenvector for each eigenvalue are found.

Considering the eigenvalue $\lambda_1 = 1$

We need to solve $A\vec{v} = \lambda\vec{v}$ or $(A - \lambda I)\vec{v} = \vec{0}$ which becomes

$$\left(\begin{bmatrix}1 & 0 & 0\\ -4 & 1 & 0\\ 3 & 6 & 2\end{bmatrix} - (1)\begin{bmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{bmatrix}\right)\begin{bmatrix}v_1\\ v_2\\ v_3\end{bmatrix} = \begin{bmatrix}0\\ 0\\ 0\end{bmatrix}$$

$$\begin{bmatrix}0 & 0 & 0\\ -4 & 0 & 0\\ 3 & 6 & 1\end{bmatrix}\begin{bmatrix}v_1\\ v_2\\ v_3\end{bmatrix} = \begin{bmatrix}0\\ 0\\ 0\end{bmatrix}$$

Now forward elimination is applied to solve for the eigenvector $\vec{v}$. The augmented matrix is

$$\left[\begin{array}{ccc|c} 0 & 0 & 0 & 0 \\ -4 & 0 & 0 & 0 \\ 3 & 6 & 1 & 0 \end{array}\right]$$

Since the current pivot $A(1,1)$ is zero, then the current pivot row is replaced with a row with a non-zero pivot. Swapping row 1 and row 2 gives

$$\left[\begin{array}{ccc|c} -4 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 3 & 6 & 1 & 0 \end{array}\right]$$

$$R_3 = R_3 + \frac{3R_1}{4} \Longrightarrow \left[\begin{array}{ccc|c} -4 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 6 & 1 & 0 \end{array}\right]$$

Since the current pivot $A(2,2)$ is zero, then the current pivot row is replaced with a row with a non-zero pivot. Swapping row 2 and row 3 gives

$$\left[\begin{array}{ccc|c} -4 & 0 & 0 & 0 \\ 0 & 6 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

Therefore the system in Echelon form is

$$\begin{bmatrix} -4 & 0 & 0 \\ 0 & 6 & 1 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

The free variables are $\{v_3\}$ and the leading variables are $\{v_1, v_2\}$. Let $v_3 = t$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\left\{v_1 = 0, v_2 = -\frac{t}{6}\right\}$

Hence the solution is

$$\begin{bmatrix} 0 \\ -\frac{t}{6} \\ t \end{bmatrix} = \begin{bmatrix} 0 \\ -\frac{t}{6} \\ t \end{bmatrix}$$

Since there is one free Variable, we have found one eigenvector associated with this eigenvalue. The above can be written as

$$\begin{bmatrix} 0 \\ -\frac{t}{6} \\ t \end{bmatrix} = t \begin{bmatrix} 0 \\ -\frac{1}{6} \\ 1 \end{bmatrix}$$

Let $t = 1$ the eigenvector becomes

$$\begin{bmatrix} 0 \\ -\frac{t}{6} \\ t \end{bmatrix} = \begin{bmatrix} 0 \\ -\frac{1}{6} \\ 1 \end{bmatrix}$$

Which is normalized to

$$\begin{bmatrix} 0 \\ -\frac{t}{6} \\ t \end{bmatrix} = \begin{bmatrix} 0 \\ -1 \\ 6 \end{bmatrix}$$

Considering the eigenvalue $\lambda_2 = 2$

We need to solve $A\vec{v} = \lambda\vec{v}$ or $(A - \lambda I)\vec{v} = \vec{0}$ which becomes

$$\left(\begin{bmatrix} 1 & 0 & 0 \\ -4 & 1 & 0 \\ 3 & 6 & 2 \end{bmatrix} - (2) \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \right) \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} -1 & 0 & 0 \\ -4 & -1 & 0 \\ 3 & 6 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Now forward elimination is applied to solve for the eigenvector $\vec{v}$. The augmented matrix is

$$\left[\begin{array}{ccc|c} -1 & 0 & 0 & 0 \\ -4 & -1 & 0 & 0 \\ 3 & 6 & 0 & 0 \end{array}\right]$$

$$R_2 = R_2 - 4R_1 \Longrightarrow \left[\begin{array}{ccc|c} -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 3 & 6 & 0 & 0 \end{array}\right]$$

$$R_3 = R_3 + 3R_1 \Longrightarrow \left[\begin{array}{ccc|c} -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 6 & 0 & 0 \end{array}\right]$$

$$R_3 = R_3 + 6R_2 \Longrightarrow \left[\begin{array}{ccc|c} -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

Therefore the system in Echelon form is

$$\begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

The free variables are $\{v_3\}$ and the leading variables are $\{v_1, v_2\}$. Let $v_3 = t$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\{v_1 = 0, v_2 = 0\}$

Hence the solution is

$$\begin{bmatrix} 0 \\ 0 \\ t \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ t \end{bmatrix}$$

Since there is one free Variable, we have found one eigenvector associated with this eigenvalue. The above can be written as

$$\begin{bmatrix} 0 \\ 0 \\ t \end{bmatrix} = t \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

Let $t = 1$ the eigenvector becomes

$$\begin{bmatrix} 0 \\ 0 \\ t \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

The following table gives a summary of this result. It shows for each eigenvalue the algebraic multiplicity m, and its geometric multiplicity k and the eigenvectors associated with the eigenvalue. If $m > k$ then the eigenvalue is defective which means the number of normal linearly independent eigenvectors associated with this eigenvalue (called the geometric multiplicity k) does not equal the algebraic multiplicity m, and we need to determine an additional $m-k$ generalized eigenvectors for this eigenvalue.

eigenvalue	multiplicity		defective?	eigenvectors
	algebraic m	geometric k		
2	1	1	No	$\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$
1	2	1	Yes	$\begin{bmatrix} 0 \\ -\frac{1}{6} \\ 1 \end{bmatrix}$

Now that we found the eigenvalues and associated eigenvectors, we will go over each eigenvalue and generate the solution basis. The only problem we need to take care of is if the eigenvalue is defective. Since eigenvalue 2 is real and distinct then the corresponding eigenvector solution is

$$\begin{aligned} \vec{x}_1(t) &= \vec{v}_1 e^{2t} \\ &= \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} e^{2t} \end{aligned}$$

eigenvalue 1 is real and repated eigenvalue of multiplicity 2.There are two possible cases that can happen. This is illustrated in this diagram

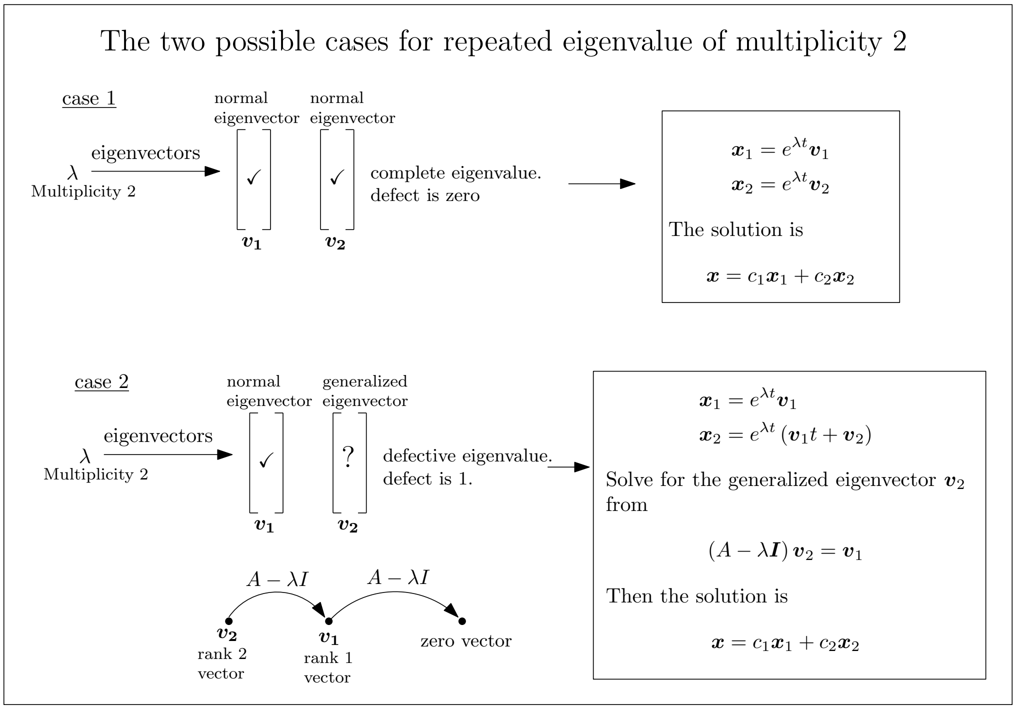


Figure 536: Possible case for repeated λ of multiplicity 2

This eigenvalue has algebraic multiplicity of 2, and geometric multiplicity 1, therefore this is defective eigenvalue. The defect is 1. This falls into case 2 shown above. We need to generate the missing additonal generalized eigevector $\vec{v}_2$ by solving

$$(A - \lambda I)\,\vec{v}_2 = \vec{v}_1$$

Where $\vec{v}_1$ is the normal (rank 1) eigenvector found above. Hence we need to solve

$$\left(\begin{bmatrix} 1 & 0 & 0 \\ -4 & 1 & 0 \\ 3 & 6 & 2 \end{bmatrix} - (1)\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\right)\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ -\frac{1}{6} \\ 1 \end{bmatrix}$$

$$\begin{bmatrix} 0 & 0 & 0 \\ -4 & 0 & 0 \\ 3 & 6 & 1 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ -\frac{1}{6} \\ 1 \end{bmatrix}$$

Solving for $\vec{v}_2$ gives

$$\vec{v}_2 = \begin{bmatrix} \frac{1}{24} \\ 1 \\ -\frac{41}{8} \end{bmatrix}$$

We have found two generalized eigenvectors for eigenvalue 1. Therefore the two basis solution associated with this eigenvalue are

$$\begin{aligned} \vec{x}_2(t) &= \vec{v}_1 e^{\lambda t} \\ &= \begin{bmatrix} 0 \\ -\frac{1}{6} \\ 1 \end{bmatrix} \mathrm{e}^{t} \\ &= \begin{bmatrix} 0 \\ -\frac{\mathrm{e}^{t}}{6} \\ \mathrm{e}^{t} \end{bmatrix} \end{aligned}$$

And

$$\begin{aligned} \vec{x}_3(t) &= (\vec{v}_1 t + \vec{v}_2)\, e^{\lambda t} \\ &= \left(\begin{bmatrix} 0 \\ -\frac{1}{6} \\ 1 \end{bmatrix} t + \begin{bmatrix} \frac{1}{24} \\ 1 \\ -\frac{41}{8} \end{bmatrix} \right) \mathrm{e}^{t} \\ &= \begin{bmatrix} \frac{\mathrm{e}^{t}}{24} \\ -\frac{\mathrm{e}^{t}(t-6)}{6} \\ \frac{\mathrm{e}^{t}(8t-41)}{8} \end{bmatrix} \end{aligned}$$

Therefore the final solution is

$$\vec{x}_h(t) = c_1 \vec{x}_1(t) + c_2 \vec{x}_2(t) + c_3 \vec{x}_3(t)$$

Which is written as

$$\begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{bmatrix} = c_1 \begin{bmatrix} 0 \\ 0 \\ \mathrm{e}^{2t} \end{bmatrix} + c_2 \begin{bmatrix} 0 \\ -\frac{\mathrm{e}^{t}}{6} \\ \mathrm{e}^{t} \end{bmatrix} + c_3 \begin{bmatrix} \frac{\mathrm{e}^{t}}{24} \\ \mathrm{e}^{t}\left(-\frac{t}{6} + 1\right) \\ \mathrm{e}^{t}\left(t - \frac{41}{8}\right) \end{bmatrix}$$

Which becomes

$$\begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{bmatrix} = \begin{bmatrix} \frac{c_3 e^t}{24} \\ -\frac{((t-6)c_3+c_2)e^t}{6} \\ c_1 e^{2t} + \left(\left(t-\frac{41}{8}\right)c_3 + c_2\right)e^t \end{bmatrix}$$

Since initial conditions are given, the solution above needs to be updated by solving for the constants of integrations using the given initial conditions

$$\begin{bmatrix} x_1(0) = -1 \\ x_2(0) = 2 \\ x_3(0) = -30 \end{bmatrix} \tag{1}$$

Substituting initial conditions into the above solution at $t = 0$ gives

$$\begin{bmatrix} -1 \\ 2 \\ -30 \end{bmatrix} = \begin{bmatrix} \frac{c_3}{24} \\ c_3 - \frac{c_2}{6} \\ c_1 - \frac{41c_3}{8} + c_2 \end{bmatrix}$$

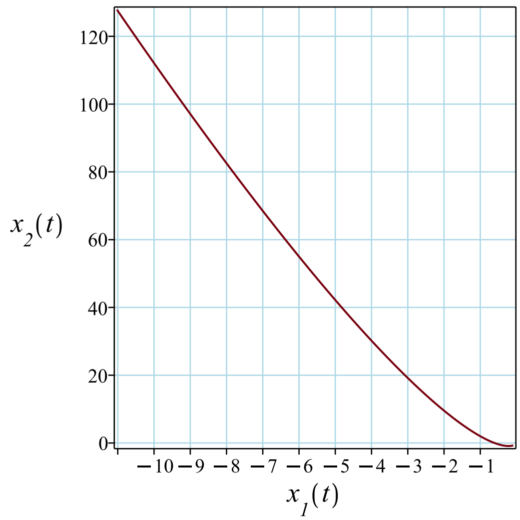
Solving for the constants of integrations gives

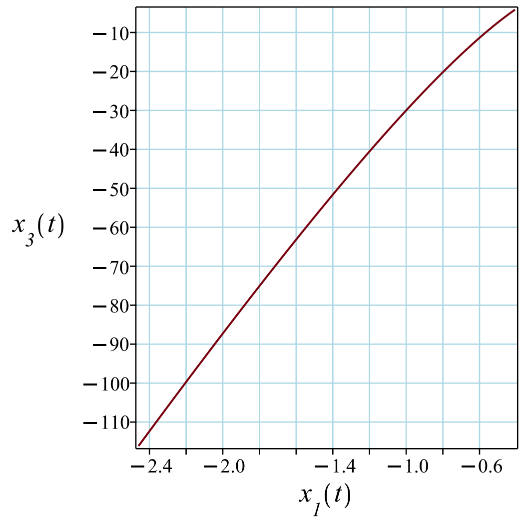
$$\begin{bmatrix} c_1 = 3 \\ c_2 = -156 \\ c_3 = -24 \end{bmatrix}$$

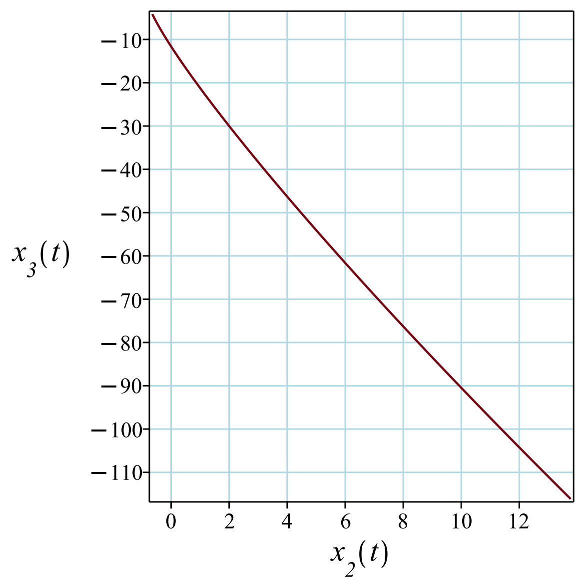
Substituting these constants back in original solution in Eq. (1) gives

$$\begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{bmatrix} = \begin{bmatrix} -e^t \\ -\frac{(-24t-12)e^t}{6} \\ 3\,e^{2t} + (-24t-33)\,e^t \end{bmatrix}$$

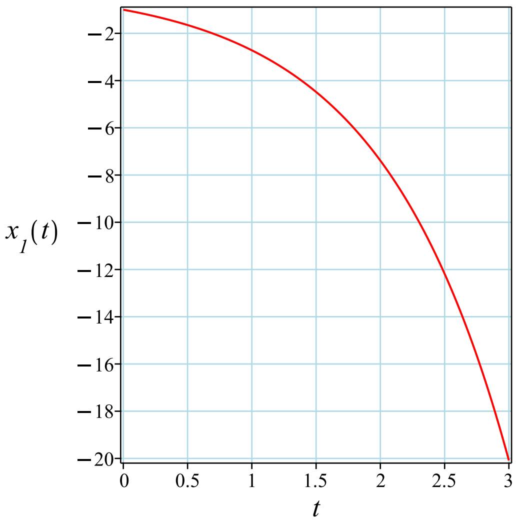
The following are plots of each solution against another.

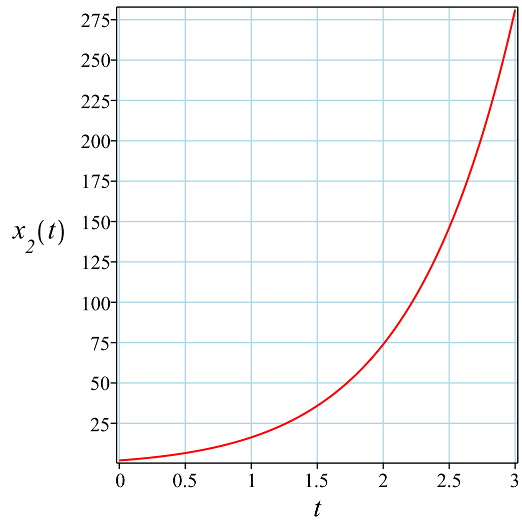


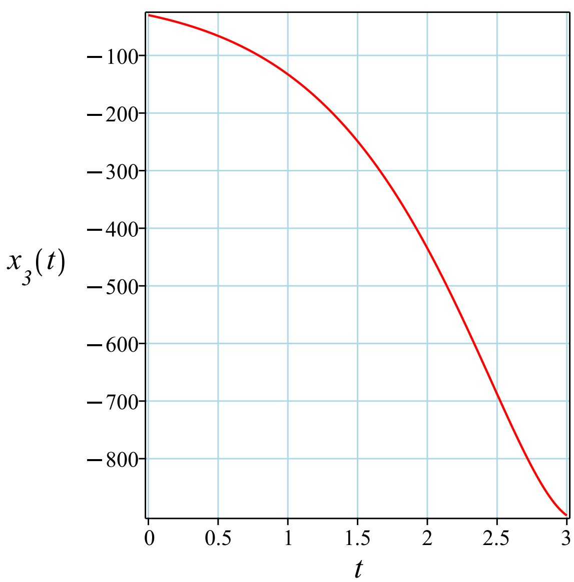




The following are plots of each solution.







✓ Solution by Maple

Time used: 0.032 (sec). Leaf size: 39

```
dsolve([diff(x__1(t),t) = x__1(t), diff(x__2(t),t) = -4*x__1(t)+x__2(t), diff(x__3(t),t) = 3
```

$$x_1(t) = -e^t$$
$$x_2(t) = (4t + 2)\,e^t$$
$$x_3(t) = -24\,e^t t - 33\,e^t + 3\,e^{2t}$$

✓ Solution by Mathematica

Time used: 0.007 (sec). Leaf size: 39

```
DSolve[{x1'[t]==1*x1[t]+0*x2[t]+0*x3[t],x2'[t]==-4*x1[t]+1*x2[t]+0*x3[t],x3'[t]==3*x1[t]+6*x
```

$$
\begin{aligned}
\text{x1}(t) &\to -e^t \\
\text{x2}(t) &\to 2e^t(2t+1) \\
\text{x3}(t) &\to 3e^t\left(-8t+e^t-11\right)
\end{aligned}
$$

17.12 problem 12

Internal problem ID [777]
Internal file name [OUTPUT/777_Sunday_June_05_2022_01_49_24_AM_20783531/index.tex]

Book: Elementary differential equations and boundary value problems, 10th ed., Boyce and DiPrima
Section: Chapter 7.8, Repeated Eigenvalues. page 436
Problem number: 12.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"system of linear ODEs"**

Solve

$$
\begin{aligned}
x_1'(t) &= -\frac{5x_1(t)}{2} + x_2(t) + x_3(t) \\
x_2'(t) &= x_1(t) - \frac{5x_2(t)}{2} + x_3(t) \\
x_3'(t) &= x_1(t) + x_2(t) - \frac{5x_3(t)}{2}
\end{aligned}
$$

With initial conditions

$$
[x_1(0) = 2, x_2(0) = 3, x_3(0) = -1]
$$

17.12.1 Solution using Matrix exponential method

In this method, we will assume we have found the matrix exponential e^{At} allready. There are different methods to determine this but will not be shown here. This is a system of linear ODE's given as

$$
\vec{x}'(t) = A\,\vec{x}(t)
$$

Or

$$
\begin{bmatrix} x_1'(t) \\ x_2'(t) \\ x_3'(t) \end{bmatrix} = \begin{bmatrix} -\frac{5}{2} & 1 & 1 \\ 1 & -\frac{5}{2} & 1 \\ 1 & 1 & -\frac{5}{2} \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{bmatrix}
$$

For the above matrix A, the matrix exponential can be found to be

$$e^{At} = \begin{bmatrix} \frac{2\,e^{-\frac{7t}{2}}}{3}+\frac{e^{-\frac{t}{2}}}{3} & \frac{e^{-\frac{t}{2}}}{3}-\frac{e^{-\frac{7t}{2}}}{3} & \frac{e^{-\frac{t}{2}}}{3}-\frac{e^{-\frac{7t}{2}}}{3} \\ \frac{e^{-\frac{t}{2}}}{3}-\frac{e^{-\frac{7t}{2}}}{3} & \frac{2\,e^{-\frac{7t}{2}}}{3}+\frac{e^{-\frac{t}{2}}}{3} & \frac{e^{-\frac{t}{2}}}{3}-\frac{e^{-\frac{7t}{2}}}{3} \\ \frac{e^{-\frac{t}{2}}}{3}-\frac{e^{-\frac{7t}{2}}}{3} & \frac{e^{-\frac{t}{2}}}{3}-\frac{e^{-\frac{7t}{2}}}{3} & \frac{2\,e^{-\frac{7t}{2}}}{3}+\frac{e^{-\frac{t}{2}}}{3} \end{bmatrix}$$

Therefore the homogeneous solution is

$$\begin{aligned} \vec{x}_h(t) &= e^{At}\vec{x}_0 \\ &= \begin{bmatrix} \frac{2\,e^{-\frac{7t}{2}}}{3}+\frac{e^{-\frac{t}{2}}}{3} & \frac{e^{-\frac{t}{2}}}{3}-\frac{e^{-\frac{7t}{2}}}{3} & \frac{e^{-\frac{t}{2}}}{3}-\frac{e^{-\frac{7t}{2}}}{3} \\ \frac{e^{-\frac{t}{2}}}{3}-\frac{e^{-\frac{7t}{2}}}{3} & \frac{2\,e^{-\frac{7t}{2}}}{3}+\frac{e^{-\frac{t}{2}}}{3} & \frac{e^{-\frac{t}{2}}}{3}-\frac{e^{-\frac{7t}{2}}}{3} \\ \frac{e^{-\frac{t}{2}}}{3}-\frac{e^{-\frac{7t}{2}}}{3} & \frac{e^{-\frac{t}{2}}}{3}-\frac{e^{-\frac{7t}{2}}}{3} & \frac{2\,e^{-\frac{7t}{2}}}{3}+\frac{e^{-\frac{t}{2}}}{3} \end{bmatrix} \begin{bmatrix} 2 \\ 3 \\ -1 \end{bmatrix} \\ &= \begin{bmatrix} \frac{2\,e^{-\frac{7t}{2}}}{3}+\frac{4\,e^{-\frac{t}{2}}}{3} \\ \frac{4\,e^{-\frac{t}{2}}}{3}+\frac{5\,e^{-\frac{7t}{2}}}{3} \\ \frac{4\,e^{-\frac{t}{2}}}{3}-\frac{7\,e^{-\frac{7t}{2}}}{3} \end{bmatrix} \end{aligned}$$

Since no forcing function is given, then the final solution is $\vec{x}_h(t)$ above.

17.12.2 Solution using explicit Eigenvalue and Eigenvector method

This is a system of linear ODE's given as

$$\vec{x}'(t) = A\,\vec{x}(t)$$

Or

$$\begin{bmatrix} x_1'(t) \\ x_2'(t) \\ x_3'(t) \end{bmatrix} = \begin{bmatrix} -\frac{5}{2} & 1 & 1 \\ 1 & -\frac{5}{2} & 1 \\ 1 & 1 & -\frac{5}{2} \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{bmatrix}$$

The first step is find the homogeneous solution. We start by finding the eigenvalues of A. This is done by solving the following equation for the eigenvalues λ

$$\det(A - \lambda I) = 0$$

Expanding gives

$$\det\left(\begin{bmatrix}-\frac{5}{2} & 1 & 1\\ 1 & -\frac{5}{2} & 1\\ 1 & 1 & -\frac{5}{2}\end{bmatrix} - \lambda\begin{bmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{bmatrix}\right) = 0$$

Therefore

$$\det\left(\begin{bmatrix}-\frac{5}{2}-\lambda & 1 & 1\\ 1 & -\frac{5}{2}-\lambda & 1\\ 1 & 1 & -\frac{5}{2}-\lambda\end{bmatrix}\right) = 0$$

Which gives the characteristic equation

$$\lambda^3 + \frac{15}{2}\lambda^2 + \frac{63}{4}\lambda + \frac{49}{8} = 0$$

The roots of the above are the eigenvalues.

$$\lambda_1 = -\frac{1}{2}$$
$$\lambda_2 = -\frac{7}{2}$$

This table summarises the above result

eigenvalue	algebraic multiplicity	type of eigenvalue
$-\frac{1}{2}$	1	real eigenvalue
$-\frac{7}{2}$	1	real eigenvalue

Now the eigenvector for each eigenvalue are found.

Considering the eigenvalue $\lambda_1 = -\frac{7}{2}$

We need to solve $A\vec{v} = \lambda\vec{v}$ or $(A - \lambda I)\vec{v} = \vec{0}$ which becomes

$$\left(\begin{bmatrix}-\frac{5}{2} & 1 & 1\\ 1 & -\frac{5}{2} & 1\\ 1 & 1 & -\frac{5}{2}\end{bmatrix} - \left(-\frac{7}{2}\right)\begin{bmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{bmatrix}\right)\begin{bmatrix}v_1\\ v_2\\ v_3\end{bmatrix} = \begin{bmatrix}0\\ 0\\ 0\end{bmatrix}$$

$$\begin{bmatrix}1 & 1 & 1\\ 1 & 1 & 1\\ 1 & 1 & 1\end{bmatrix}\begin{bmatrix}v_1\\ v_2\\ v_3\end{bmatrix} = \begin{bmatrix}0\\ 0\\ 0\end{bmatrix}$$

Now forward elimination is applied to solve for the eigenvector $\vec{v}$. The augmented matrix is

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 0 \end{array}\right]$$

$$R_2 = R_2 - R_1 \Longrightarrow \left[\begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 \end{array}\right]$$

$$R_3 = R_3 - R_1 \Longrightarrow \left[\begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

Therefore the system in Echelon form is

$$\begin{bmatrix} 1 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

The free variables are $\{v_2, v_3\}$ and the leading variables are $\{v_1\}$. Let $v_2 = t$. Let $v_3 = s$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\{v_1 = -t - s\}$

Hence the solution is

$$\begin{bmatrix} -t - s \\ t \\ s \end{bmatrix} = \begin{bmatrix} -t - s \\ t \\ s \end{bmatrix}$$

Since there are two free Variable, we have found two eigenvectors associated with this eigenvalue. The above can be written as

$$\begin{aligned} \begin{bmatrix} -t - s \\ t \\ s \end{bmatrix} &= \begin{bmatrix} -t \\ t \\ 0 \end{bmatrix} + \begin{bmatrix} -s \\ 0 \\ s \end{bmatrix} \\ &= t \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} + s \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} \end{aligned}$$

By letting $t=1$ and $s=1$ then the above becomes

$$\begin{bmatrix} -t-s \\ t \\ s \end{bmatrix} = \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} + \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$$

Hence the two eigenvectors associated with this eigenvalue are

$$\left(\begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} \right)$$

Considering the eigenvalue $\lambda_2 = -\frac{1}{2}$

We need to solve $A\vec{v} = \lambda\vec{v}$ or $(A-\lambda I)\vec{v} = \vec{0}$ which becomes

$$\left(\begin{bmatrix} -\frac{5}{2} & 1 & 1 \\ 1 & -\frac{5}{2} & 1 \\ 1 & 1 & -\frac{5}{2} \end{bmatrix} - \left(-\frac{1}{2}\right) \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \right) \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} -2 & 1 & 1 \\ 1 & -2 & 1 \\ 1 & 1 & -2 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Now forward elimination is applied to solve for the eigenvector $\vec{v}$. The augmented matrix is

$$\left[\begin{array}{ccc|c} -2 & 1 & 1 & 0 \\ 1 & -2 & 1 & 0 \\ 1 & 1 & -2 & 0 \end{array}\right]$$

$$R_2 = R_2 + \frac{R_1}{2} \Longrightarrow \left[\begin{array}{ccc|c} -2 & 1 & 1 & 0 \\ 0 & -\frac{3}{2} & \frac{3}{2} & 0 \\ 1 & 1 & -2 & 0 \end{array}\right]$$

$$R_3 = R_3 + \frac{R_1}{2} \Longrightarrow \left[\begin{array}{ccc|c} -2 & 1 & 1 & 0 \\ 0 & -\frac{3}{2} & \frac{3}{2} & 0 \\ 0 & \frac{3}{2} & -\frac{3}{2} & 0 \end{array}\right]$$

$$R_3 = R_3 + R_2 \Longrightarrow \left[\begin{array}{ccc|c} -2 & 1 & 1 & 0 \\ 0 & -\frac{3}{2} & \frac{3}{2} & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

Therefore the system in Echelon form is

$$\begin{bmatrix} -2 & 1 & 1 \\ 0 & -\frac{3}{2} & \frac{3}{2} \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

The free variables are $\{v_3\}$ and the leading variables are $\{v_1, v_2\}$. Let $v_3 = t$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\{v_1 = t, v_2 = t\}$

Hence the solution is

$$\begin{bmatrix} t \\ t \\ t \end{bmatrix} = \begin{bmatrix} t \\ t \\ t \end{bmatrix}$$

Since there is one free Variable, we have found one eigenvector associated with this eigenvalue. The above can be written as

$$\begin{bmatrix} t \\ t \\ t \end{bmatrix} = t \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

Let $t = 1$ the eigenvector becomes

$$\begin{bmatrix} t \\ t \\ t \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

The following table gives a summary of this result. It shows for each eigenvalue the algebraic multiplicity m, and its geometric multiplicity k and the eigenvectors associated with the eigenvalue. If $m > k$ then the eigenvalue is defective which means the number of normal linearly independent eigenvectors associated with this eigenvalue (called the geometric multiplicity k) does not equal the algebraic multiplicity m, and we need to determine an additional $m - k$ generalized eigenvectors for this eigenvalue.

eigenvalue	multiplicity		defective?	eigenvectors
	algebraic m	geometric k		
$-\frac{1}{2}$	1	1	No	$\left[\begin{array}{c}1\\1\\1\end{array}\right]$
$-\frac{7}{2}$	2	2	No	$\left[\begin{array}{cc}-1 & -1\\0 & 1\\1 & 0\end{array}\right]$

Now that we found the eigenvalues and associated eigenvectors, we will go over each eigenvalue and generate the solution basis. The only problem we need to take care of is if the eigenvalue is defective. Since eigenvalue $-\frac{1}{2}$ is real and distinct then the corresponding eigenvector solution is

$$
\begin{aligned}
\vec{x}_1(t) &= \vec{v}_1 e^{-\frac{t}{2}} \\
&= \left[\begin{array}{c}1\\1\\1\end{array}\right] e^{-\frac{t}{2}}
\end{aligned}
$$

eigenvalue $-\frac{7}{2}$ is real and repated eigenvalue of multiplicity 2.There are two possible cases that can happen. This is illustrated in this diagram

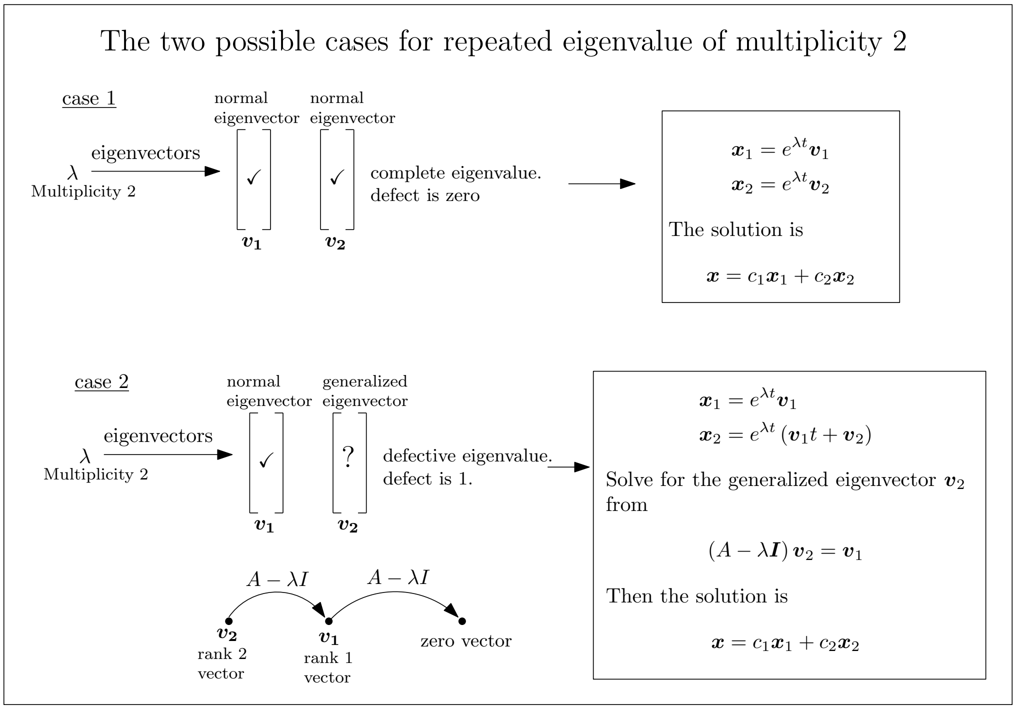


Figure 537: Possible case for repeated λ of multiplicity 2

This eigenvalue has algebraic multiplicity of 2 which is the same as its geometric multiplicity 2, then it is complete eigenvalue and this falls into case 1 shown above. Hence the corresponding eigenvector basis are

$$
\begin{aligned}
\vec{x}_2(t) &= \vec{v}_2 e^{-\frac{7t}{2}} \\
&= \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} e^{-\frac{7t}{2}}
\end{aligned}
$$

$$
\begin{aligned}
\vec{x}_3(t) &= \vec{v}_3 e^{-\frac{7t}{2}} \\
&= \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} e^{-\frac{7t}{2}}
\end{aligned}
$$

Therefore the final solution is

$$
\vec{x}_h(t) = c_1 \vec{x}_1(t) + c_2 \vec{x}_2(t) + c_3 \vec{x}_3(t)
$$

Which is written as

$$\begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{bmatrix} = c_1 \begin{bmatrix} e^{-\frac{t}{2}} \\ e^{-\frac{t}{2}} \\ e^{-\frac{t}{2}} \end{bmatrix} + c_2 \begin{bmatrix} -e^{-\frac{7t}{2}} \\ 0 \\ e^{-\frac{7t}{2}} \end{bmatrix} + c_3 \begin{bmatrix} -e^{-\frac{7t}{2}} \\ e^{-\frac{7t}{2}} \\ 0 \end{bmatrix}$$

Which becomes

$$\begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{bmatrix} = \begin{bmatrix} (-c_2 - c_3)\, e^{-\frac{7t}{2}} + c_1 e^{-\frac{t}{2}} \\ c_1 e^{-\frac{t}{2}} + c_3 e^{-\frac{7t}{2}} \\ c_1 e^{-\frac{t}{2}} + c_2 e^{-\frac{7t}{2}} \end{bmatrix}$$

Since initial conditions are given, the solution above needs to be updated by solving for the constants of integrations using the given initial conditions

$$\begin{bmatrix} x_1(0) = 2 \\ x_2(0) = 3 \\ x_3(0) = -1 \end{bmatrix} \tag{1}$$

Substituting initial conditions into the above solution at $t = 0$ gives

$$\begin{bmatrix} 2 \\ 3 \\ -1 \end{bmatrix} = \begin{bmatrix} -c_2 - c_3 + c_1 \\ c_1 + c_3 \\ c_1 + c_2 \end{bmatrix}$$

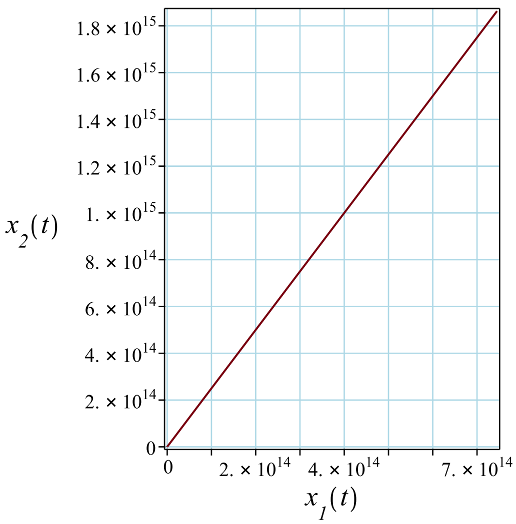
Solving for the constants of integrations gives

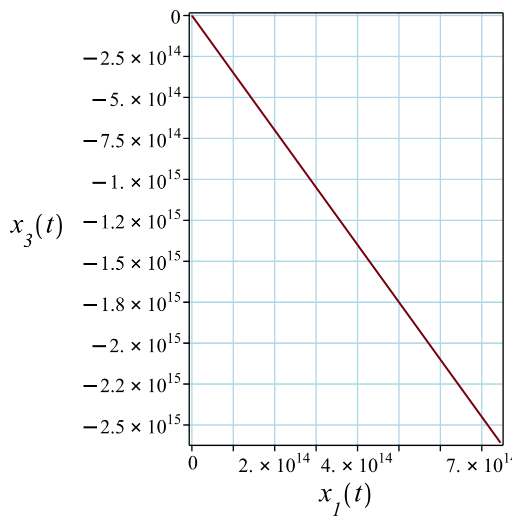
$$\begin{bmatrix} c_1 = \frac{4}{3} \\ c_2 = -\frac{7}{3} \\ c_3 = \frac{5}{3} \end{bmatrix}$$

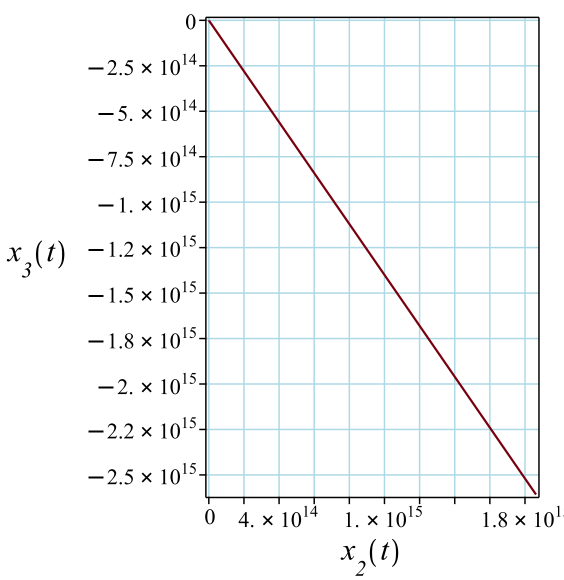
Substituting these constants back in original solution in Eq. (1) gives

$$\begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{bmatrix} = \begin{bmatrix} \frac{2\,e^{-\frac{7t}{2}}}{3} + \frac{4\,e^{-\frac{t}{2}}}{3} \\ \frac{4\,e^{-\frac{t}{2}}}{3} + \frac{5\,e^{-\frac{7t}{2}}}{3} \\ \frac{4\,e^{-\frac{t}{2}}}{3} - \frac{7\,e^{-\frac{7t}{2}}}{3} \end{bmatrix}$$

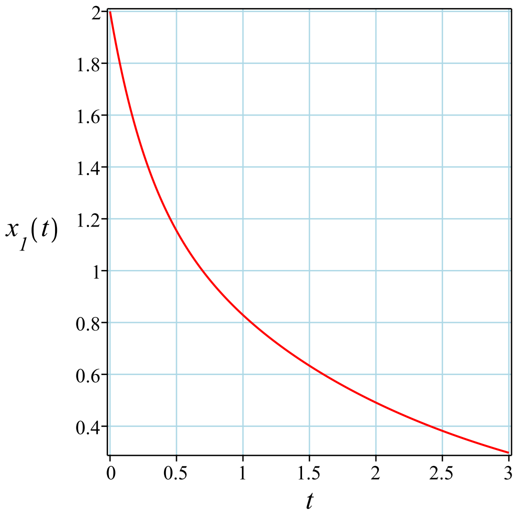
The following are plots of each solution against another.

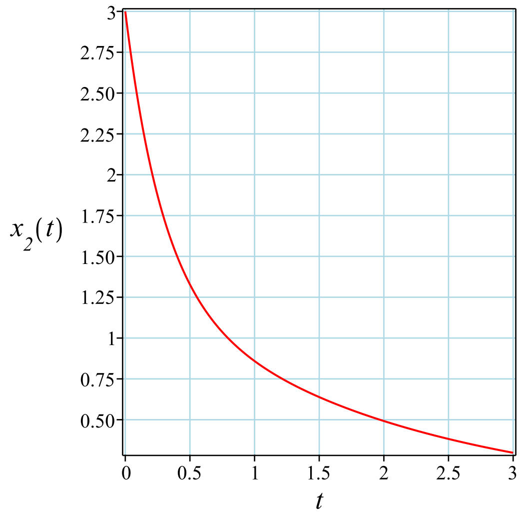


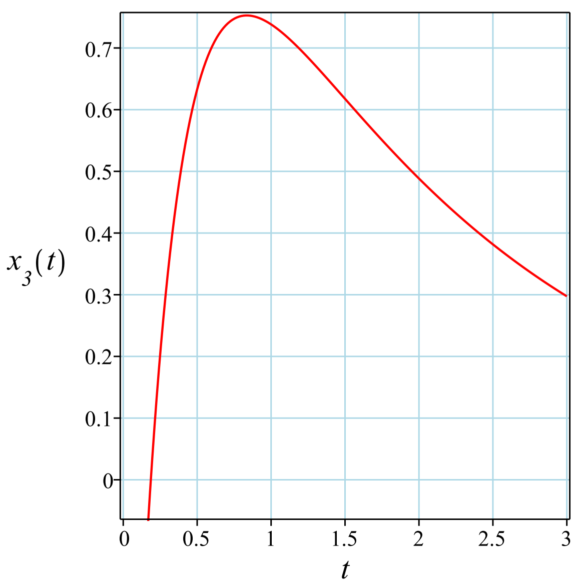




The following are plots of each solution.


2
1.8
1.6
1.4
1.2
1
0.8
0.6
0.4
$x_1(t)$
0
0.5
1
1.5
2
2.5
3
t


3
2.75
2.50
2.25
2
1.75
1.50
1.25
1
0.75
0.50
$x_2(t)$
0
0.5
1
1.5
2
2.5
3
t


0.7
0.6
0.5
0.4
0.3
0.2
0.1
0
$x_3(t)$
0
0.5
1
1.5
2
2.5
3
t

✓ Solution by Maple

Time used: 0.031 (sec). Leaf size: 50

```
dsolve([diff(x__1(t),t) = -5/2*x__1(t)+x__2(t)+x__3(t), diff(x__2(t),t) = x__1(t)-5/2*x__2(t
```

$$x_1(t) = \frac{2\,\mathrm{e}^{-\frac{7t}{2}}}{3} + \frac{4\,\mathrm{e}^{-\frac{t}{2}}}{3}$$
$$x_2(t) = \frac{5\,\mathrm{e}^{-\frac{7t}{2}}}{3} + \frac{4\,\mathrm{e}^{-\frac{t}{2}}}{3}$$
$$x_3(t) = -\frac{7\,\mathrm{e}^{-\frac{7t}{2}}}{3} + \frac{4\,\mathrm{e}^{-\frac{t}{2}}}{3}$$

✓ Solution by Mathematica

Time used: 0.008 (sec). Leaf size: 71

```
DSolve[{x1'[t]==-5/2*x1[t]+1*x2[t]+1*x3[t],x2'[t]==1*x1[t]-5/2*x2[t]+1*x3[t],x3'[t]==1*x1[t]
```

$$\mathrm{x1}(t) \to \frac{2}{3}e^{-7t/2}\left(2e^{3t}+1\right)$$
$$\mathrm{x2}(t) \to \frac{1}{3}e^{-7t/2}\left(4e^{3t}+5\right)$$
$$\mathrm{x3}(t) \to \frac{1}{3}e^{-7t/2}\left(4e^{3t}-7\right)$$

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18.1 problem 1

Internal problem ID [778]
Internal file name [`OUTPUT/778_Sunday_June_05_2022_01_49_25_AM_67064786/index.tex`]

Book: Elementary differential equations and boundary value problems, 10th ed., Boyce and DiPrima
Section: Chapter 7.9, Nonhomogeneous Linear Systems. page 447
Problem number: 1.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"system of linear ODEs"**

Solve

$$x_1'(t) = 2x_1(t) - x_2(t) + \mathrm{e}^t$$
$$x_2'(t) = 3x_1(t) - 2x_2(t) + t$$

18.1.1 Solution using Matrix exponential method

In this method, we will assume we have found the matrix exponential e^{At} allready. There are different methods to determine this but will not be shown here. This is a system of linear ODE's given as

$$\vec{x}'(t) = A\,\vec{x}(t) + \vec{G}(t)$$

Or

$$\begin{bmatrix} x_1'(t) \\ x_2'(t) \end{bmatrix} = \begin{bmatrix} 2 & -1 \\ 3 & -2 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} + \begin{bmatrix} \mathrm{e}^t \\ t \end{bmatrix}$$

Since the system is nonhomogeneous, then the solution is given by

$$\vec{x}(t) = \vec{x}_h(t) + \vec{x}_p(t)$$

Where $\vec{x}_h(t)$ is the homogeneous solution to $\vec{x}'(t) = A\,\vec{x}(t)$ and $\vec{x}_p(t)$ is a particular solution to $\vec{x}'(t) = A\,\vec{x}(t) + \vec{G}(t)$. The particular solution will be found using variation

of parameters method applied to the fundamental matrix. For the above matrix A, the matrix exponential can be found to be

$$e^{At} = \begin{bmatrix} -\frac{e^{-t}}{2} + \frac{3e^{t}}{2} & -\frac{e^{t}}{2} + \frac{e^{-t}}{2} \\ \frac{3e^{t}}{2} - \frac{3e^{-t}}{2} & \frac{3e^{-t}}{2} - \frac{e^{t}}{2} \end{bmatrix}$$

Therefore the homogeneous solution is

$$\begin{aligned} \vec{x}_h(t) &= e^{At}\vec{c} \\ &= \begin{bmatrix} -\frac{e^{-t}}{2} + \frac{3e^{t}}{2} & -\frac{e^{t}}{2} + \frac{e^{-t}}{2} \\ \frac{3e^{t}}{2} - \frac{3e^{-t}}{2} & \frac{3e^{-t}}{2} - \frac{e^{t}}{2} \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} \\ &= \begin{bmatrix} \left(-\frac{e^{-t}}{2} + \frac{3e^{t}}{2}\right) c_1 + \left(-\frac{e^{t}}{2} + \frac{e^{-t}}{2}\right) c_2 \\ \left(\frac{3e^{t}}{2} - \frac{3e^{-t}}{2}\right) c_1 + \left(\frac{3e^{-t}}{2} - \frac{e^{t}}{2}\right) c_2 \end{bmatrix} \\ &= \begin{bmatrix} \frac{(c_2-c_1)e^{-t}}{2} + \frac{3\left(-\frac{c_2}{3}+c_1\right)e^{t}}{2} \\ \frac{(-3c_1+3c_2)e^{-t}}{2} + \frac{3\left(-\frac{c_2}{3}+c_1\right)e^{t}}{2} \end{bmatrix} \end{aligned}$$

The particular solution given by

$$\vec{x}_p(t) = e^{At} \int e^{-At}\vec{G}(t)\,dt$$

But

$$\begin{aligned} e^{-At} &= \left(e^{At}\right)^{-1} \\ &= \begin{bmatrix} \frac{3e^{-t}}{2} - \frac{e^{t}}{2} & \frac{e^{t}}{2} - \frac{e^{-t}}{2} \\ -\frac{3e^{t}}{2} + \frac{3e^{-t}}{2} & -\frac{e^{-t}}{2} + \frac{3e^{t}}{2} \end{bmatrix} \end{aligned}$$

Hence

$$\begin{aligned} \vec{x}_p(t) &= \begin{bmatrix} -\frac{e^{-t}}{2} + \frac{3e^{t}}{2} & -\frac{e^{t}}{2} + \frac{e^{-t}}{2} \\ \frac{3e^{t}}{2} - \frac{3e^{-t}}{2} & \frac{3e^{-t}}{2} - \frac{e^{t}}{2} \end{bmatrix} \int \begin{bmatrix} \frac{3e^{-t}}{2} - \frac{e^{t}}{2} & \frac{e^{t}}{2} - \frac{e^{-t}}{2} \\ -\frac{3e^{t}}{2} + \frac{3e^{-t}}{2} & -\frac{e^{-t}}{2} + \frac{3e^{t}}{2} \end{bmatrix} \begin{bmatrix} e^{t} \\ t \end{bmatrix} dt \\ &= \begin{bmatrix} -\frac{e^{-t}}{2} + \frac{3e^{t}}{2} & -\frac{e^{t}}{2} + \frac{e^{-t}}{2} \\ \frac{3e^{t}}{2} - \frac{3e^{-t}}{2} & \frac{3e^{-t}}{2} - \frac{e^{t}}{2} \end{bmatrix} \begin{bmatrix} \frac{(2t+2)e^{-t}}{4} - \frac{e^{2t}}{4} + \frac{(2t-2)e^{t}}{4} + \frac{3t}{2} \\ \frac{(2t+2)e^{-t}}{4} - \frac{3e^{2t}}{4} + \frac{(6t-6)e^{t}}{4} + \frac{3t}{2} \end{bmatrix} \\ &= \begin{bmatrix} \frac{(6t-1)e^{t}}{4} + t \\ \frac{(6t-3)e^{t}}{4} + 2t - 1 \end{bmatrix} \end{aligned}$$

Hence the complete solution is

$$\vec{x}(t) = \vec{x}_h(t) + \vec{x}_p(t)$$

$$= \begin{bmatrix} \frac{(-2c_1+2c_2)e^{-t}}{4} + \frac{(6t+6c_1-2c_2-1)e^t}{4} + t \\ \frac{(6c_2-6c_1)e^{-t}}{4} + \frac{(6t+6c_1-2c_2-3)e^t}{4} + 2t - 1 \end{bmatrix}$$

18.1.2 Solution using explicit Eigenvalue and Eigenvector method

This is a system of linear ODE's given as

$$\vec{x}'(t) = A\,\vec{x}(t) + \vec{G}(t)$$

Or

$$\begin{bmatrix} x_1'(t) \\ x_2'(t) \end{bmatrix} = \begin{bmatrix} 2 & -1 \\ 3 & -2 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} + \begin{bmatrix} e^t \\ t \end{bmatrix}$$

Since the system is nonhomogeneous, then the solution is given by

$$\vec{x}(t) = \vec{x}_h(t) + \vec{x}_p(t)$$

Where $\vec{x}_h(t)$ is the homogeneous solution to $\vec{x}'(t) = A\,\vec{x}(t)$ and $\vec{x}_p(t)$ is a particular solution to $\vec{x}'(t) = A\,\vec{x}(t) + \vec{G}(t)$. The particular solution will be found using variation of parameters method applied to the fundamental matrix.

The first step is find the homogeneous solution. We start by finding the eigenvalues of A. This is done by solving the following equation for the eigenvalues λ

$$\det(A - \lambda I) = 0$$

Expanding gives

$$\det\left(\begin{bmatrix} 2 & -1 \\ 3 & -2 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right) = 0$$

Therefore

$$\det\left(\begin{bmatrix} 2-\lambda & -1 \\ 3 & -2-\lambda \end{bmatrix}\right) = 0$$

Which gives the characteristic equation

$$\lambda^2 - 1 = 0$$

The roots of the above are the eigenvalues.

$$\lambda_1 = 1$$
$$\lambda_2 = -1$$

This table summarises the above result

eigenvalue	algebraic multiplicity	type of eigenvalue
-1	1	real eigenvalue
1	1	real eigenvalue

Now the eigenvector for each eigenvalue are found.

Considering the eigenvalue $\lambda_1 = -1$

We need to solve $A\vec{v} = \lambda\vec{v}$ or $(A - \lambda I)\vec{v} = \vec{0}$ which becomes

$$\left(\begin{bmatrix} 2 & -1 \\ 3 & -2 \end{bmatrix} - (-1)\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right)\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$
$$\begin{bmatrix} 3 & -1 \\ 3 & -1 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Now forward elimination is applied to solve for the eigenvector $\vec{v}$. The augmented matrix is

$$\left[\begin{array}{cc|c} 3 & -1 & 0 \\ 3 & -1 & 0 \end{array}\right]$$

$$R_2 = R_2 - R_1 \Longrightarrow \left[\begin{array}{cc|c} 3 & -1 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

Therefore the system in Echelon form is

$$\begin{bmatrix} 3 & -1 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

The free variables are $\{v_2\}$ and the leading variables are $\{v_1\}$. Let $v_2 = t$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\left\{v_1 = \frac{t}{3}\right\}$

Hence the solution is

$$\begin{bmatrix} \frac{t}{3} \\ t \end{bmatrix} = \begin{bmatrix} \frac{t}{3} \\ t \end{bmatrix}$$

Since there is one free Variable, we have found one eigenvector associated with this eigenvalue. The above can be written as

$$\begin{bmatrix} \frac{t}{3} \\ t \end{bmatrix} = t \begin{bmatrix} \frac{1}{3} \\ 1 \end{bmatrix}$$

Let $t = 1$ the eigenvector becomes

$$\begin{bmatrix} \frac{t}{3} \\ t \end{bmatrix} = \begin{bmatrix} \frac{1}{3} \\ 1 \end{bmatrix}$$

Which is normalized to

$$\begin{bmatrix} \frac{t}{3} \\ t \end{bmatrix} = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$$

Considering the eigenvalue $\lambda_2 = 1$

We need to solve $A\vec{v} = \lambda\vec{v}$ or $(A - \lambda I)\vec{v} = \vec{0}$ which becomes

$$\left(\begin{bmatrix} 2 & -1 \\ 3 & -2 \end{bmatrix} - (1) \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right) \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & -1 \\ 3 & -3 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Now forward elimination is applied to solve for the eigenvector $\vec{v}$. The augmented matrix is

$$\left[\begin{array}{cc|c} 1 & -1 & 0 \\ 3 & -3 & 0 \end{array}\right]$$

$$R_2 = R_2 - 3R_1 \Longrightarrow \left[\begin{array}{cc|c} 1 & -1 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

Therefore the system in Echelon form is

$$\begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

The free variables are $\{v_2\}$ and the leading variables are $\{v_1\}$. Let $v_2 = t$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\{v_1 = t\}$

Hence the solution is

$$\begin{bmatrix} t \\ t \end{bmatrix} = \begin{bmatrix} t \\ t \end{bmatrix}$$

Since there is one free Variable, we have found one eigenvector associated with this eigenvalue. The above can be written as

$$\begin{bmatrix} t \\ t \end{bmatrix} = t \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

Let $t = 1$ the eigenvector becomes

$$\begin{bmatrix} t \\ t \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

The following table gives a summary of this result. It shows for each eigenvalue the algebraic multiplicity m, and its geometric multiplicity k and the eigenvectors associated with the eigenvalue. If $m > k$ then the eigenvalue is defective which means the number of normal linearly independent eigenvectors associated with this eigenvalue (called the geometric multiplicity k) does not equal the algebraic multiplicity m, and we need to determine an additional $m - k$ generalized eigenvectors for this eigenvalue.

	multiplicity			
eigenvalue	algebraic m	geometric k	defective?	eigenvectors
1	1	1	No	$\begin{bmatrix} 1 \\ 1 \end{bmatrix}$
-1	1	1	No	$\begin{bmatrix} \frac{1}{3} \\ 1 \end{bmatrix}$

Now that we found the eigenvalues and associated eigenvectors, we will go over each eigenvalue and generate the solution basis. The only problem we need to take care of is if the eigenvalue is defective. Since eigenvalue 1 is real and distinct then the

corresponding eigenvector solution is

$$\vec{x}_1(t) = \vec{v}_1 e^t$$
$$= \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^t$$

Since eigenvalue -1 is real and distinct then the corresponding eigenvector solution is

$$\vec{x}_2(t) = \vec{v}_2 e^{-t}$$
$$= \begin{bmatrix} \frac{1}{3} \\ 1 \end{bmatrix} e^{-t}$$

Therefore the homogeneous solution is

$$\vec{x}_h(t) = c_1 \vec{x}_1(t) + c_2 \vec{x}_2(t)$$

Which is written as

$$\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = c_1 \begin{bmatrix} e^t \\ e^t \end{bmatrix} + c_2 \begin{bmatrix} \frac{e^{-t}}{3} \\ e^{-t} \end{bmatrix}$$

Now that we found homogeneous solution above, we need to find a particular solution $\vec{x}_p(t)$. We will use Variation of parameters. The fundamental matrix is

$$\Phi = \begin{bmatrix} \vec{x}_1 & \vec{x}_2 & \cdots \end{bmatrix}$$

Where $\vec{x}_i$ are the solution basis found above. Therefore the fundamental matrix is

$$\Phi(t) = \begin{bmatrix} e^t & \frac{e^{-t}}{3} \\ e^t & e^{-t} \end{bmatrix}$$

The particular solution is then given by

$$\vec{x}_p(t) = \Phi \int \Phi^{-1} \vec{G}(t) \, dt$$

But

$$\Phi^{-1} = \begin{bmatrix} \frac{3 e^{-t}}{2} & -\frac{e^{-t}}{2} \\ -\frac{3 e^t}{2} & \frac{3 e^t}{2} \end{bmatrix}$$

Hence

$$\vec{x}_p(t) = \begin{bmatrix} e^t & \frac{e^{-t}}{3} \\ e^t & e^{-t} \end{bmatrix} \int \begin{bmatrix} \frac{3\,e^{-t}}{2} & -\frac{e^{-t}}{2} \\ -\frac{3\,e^t}{2} & \frac{3\,e^t}{2} \end{bmatrix} \begin{bmatrix} e^t \\ t \end{bmatrix} dt$$

$$= \begin{bmatrix} e^t & \frac{e^{-t}}{3} \\ e^t & e^{-t} \end{bmatrix} \int \begin{bmatrix} \frac{3}{2} - \frac{t\,e^{-t}}{2} \\ -\frac{3\,e^{2t}}{2} + \frac{3t\,e^t}{2} \end{bmatrix} dt$$

$$= \begin{bmatrix} e^t & \frac{e^{-t}}{3} \\ e^t & e^{-t} \end{bmatrix} \begin{bmatrix} \frac{e^{-t}(t+1)}{2} + \frac{3t}{2} \\ -\frac{3\,e^{2t}}{4} + \frac{(6t-6)e^t}{4} \end{bmatrix}$$

$$= \begin{bmatrix} \frac{(6t-1)e^t}{4} + t \\ \frac{(6t-3)e^t}{4} + 2t - 1 \end{bmatrix}$$

Now that we found particular solution, the final solution is

$$\vec{x}(t) = \vec{x}_h(t) + \vec{x}_p(t)$$

$$\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = \begin{bmatrix} c_1 e^t \\ c_1 e^t \end{bmatrix} + \begin{bmatrix} \frac{c_2 e^{-t}}{3} \\ c_2 e^{-t} \end{bmatrix} + \begin{bmatrix} \frac{(6t-1)e^t}{4} + t \\ \frac{(6t-3)e^t}{4} + 2t - 1 \end{bmatrix}$$

Which becomes

$$\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = \begin{bmatrix} \frac{c_2 e^{-t}}{3} + \frac{(-1+6t+4c_1)e^t}{4} + t \\ c_2 e^{-t} + \frac{(6t+4c_1-3)e^t}{4} + 2t - 1 \end{bmatrix}$$

18.1.3 Maple step by step solution

Let's solve

$[x_1'(t) = 2x_1(t) - x_2(t) + e^t, x_2'(t) = 3x_1(t) - 2x_2(t) + t]$

- Define vector

$$\underline{x}^{\rightarrow}(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}$$

- Convert system into a vector equation

$$\underline{x}^{\rightarrow}{}'(t) = \begin{bmatrix} 2 & -1 \\ 3 & -2 \end{bmatrix} \cdot \underline{x}^{\rightarrow}(t) + \begin{bmatrix} e^t \\ t \end{bmatrix}$$

- System to solve

$$\underline{\vec{x}}'(t) = \begin{bmatrix} 2 & -1 \\ 3 & -2 \end{bmatrix} \cdot \underline{\vec{x}}(t) + \begin{bmatrix} e^t \\ t \end{bmatrix}$$

- Define the forcing function

$$\vec{f}(t) = \begin{bmatrix} e^t \\ t \end{bmatrix}$$

- Define the coefficient matrix

$$A = \begin{bmatrix} 2 & -1 \\ 3 & -2 \end{bmatrix}$$

- Rewrite the system as

$$\underline{\vec{x}}'(t) = A \cdot \underline{\vec{x}}(t) + \vec{f}$$

- To solve the system, find the eigenvalues and eigenvectors of A
- Eigenpairs of A

$$\left[\left[-1, \begin{bmatrix} \frac{1}{3} \\ 1 \end{bmatrix}\right], \left[1, \begin{bmatrix} 1 \\ 1 \end{bmatrix}\right]\right]$$

- Consider eigenpair

$$\left[-1, \begin{bmatrix} \frac{1}{3} \\ 1 \end{bmatrix}\right]$$

- Solution to homogeneous system from eigenpair

$$\underline{\vec{x}}_1 = e^{-t} \cdot \begin{bmatrix} \frac{1}{3} \\ 1 \end{bmatrix}$$

- Consider eigenpair

$$\left[1, \begin{bmatrix} 1 \\ 1 \end{bmatrix}\right]$$

- Solution to homogeneous system from eigenpair

$$\underline{\vec{x}}_2 = e^t \cdot \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

- General solution of the system of ODEs can be written in terms of the particular solution $\underline{\vec{x}}$

  $\underline{\vec{x}}(t)=c_1\underline{\vec{x}}_1+c_2\underline{\vec{x}}_2+\underline{\vec{x}}_p(t)$

☐ Fundamental matrix

- Let $\phi(t)$ be the matrix whose columns are the independent solutions of the homogeneous syst

  $$\phi(t)=\begin{bmatrix}\frac{e^{-t}}{3} & e^{t}\\ e^{-t} & e^{t}\end{bmatrix}$$

- The fundamental matrix, $\Phi(t)$ is a normalized version of $\phi(t)$ satisfying $\Phi(0)=I$ where I is th

  $\Phi(t)=\phi(t)\cdot\frac{1}{\phi(0)}$

- Substitute the value of $\phi(t)$ and $\phi(0)$

  $$\Phi(t)=\begin{bmatrix}\frac{e^{-t}}{3} & e^{t}\\ e^{-t} & e^{t}\end{bmatrix}\cdot\frac{1}{\begin{bmatrix}\frac{1}{3} & 1\\ 1 & 1\end{bmatrix}}$$

- Evaluate and simplify to get the fundamental matrix

  $$\Phi(t)=\begin{bmatrix}-\frac{e^{-t}}{2}+\frac{3e^{t}}{2} & -\frac{e^{t}}{2}+\frac{e^{-t}}{2}\\ \frac{3e^{t}}{2}-\frac{3e^{-t}}{2} & \frac{3e^{-t}}{2}-\frac{e^{t}}{2}\end{bmatrix}$$

☐ Find a particular solution of the system of ODEs using variation of parameters

- Let the particular solution be the fundamental matrix multiplied by $\vec{v}(t)$ and solve for $\vec{v}(t)$

  $\underline{\vec{x}}_p(t)=\Phi(t)\cdot\vec{v}(t)$

- Take the derivative of the particular solution

  $\underline{\vec{x}}_p'(t)=\Phi'(t)\cdot\vec{v}(t)+\Phi(t)\cdot\vec{v}'(t)$

- Substitute particular solution and its derivative into the system of ODEs

  $\Phi'(t)\cdot\vec{v}(t)+\Phi(t)\cdot\vec{v}'(t)=A\cdot\Phi(t)\cdot\vec{v}(t)+\vec{f}(t)$

- The fundamental matrix has columns that are solutions to the homogeneous system so its der

  $A\cdot\Phi(t)\cdot\vec{v}(t)+\Phi(t)\cdot\vec{v}'(t)=A\cdot\Phi(t)\cdot\vec{v}(t)+\vec{f}(t)$

- Cancel like terms

  $\Phi(t)\cdot\vec{v}'(t)=\vec{f}(t)$

- Multiply by the inverse of the fundamental matrix

$$\vec{v}'(t) = \frac{1}{\Phi(t)} \cdot \vec{f}(t)$$

- Integrate to solve for $\vec{v}(t)$

$$\vec{v}(t) = \int_0^t \frac{1}{\Phi(s)} \cdot \vec{f}(s)\, ds$$

- Plug $\vec{v}(t)$ into the equation for the particular solution

$$\underline{\vec{x}}_p(t) = \Phi(t) \cdot \left(\int_0^t \frac{1}{\Phi(s)} \cdot \vec{f}(s)\, ds\right)$$

- Plug in the fundamental matrix and the forcing function and compute

$$\underline{\vec{x}}_p(t) = \begin{bmatrix} \frac{3\,\mathrm{e}^{-t}}{4} + \frac{(6t-3)\mathrm{e}^{t}}{4} + t \\ \frac{9\,\mathrm{e}^{-t}}{4} + \frac{(-5+6t)\mathrm{e}^{t}}{4} + 2t - 1 \end{bmatrix}$$

- Plug particular solution back into general solution

$$\underline{\vec{x}}(t) = c_1\underline{\vec{x}}_1 + c_2\underline{\vec{x}}_2 + \begin{bmatrix} \frac{3\,\mathrm{e}^{-t}}{4} + \frac{(6t-3)\mathrm{e}^{t}}{4} + t \\ \frac{9\,\mathrm{e}^{-t}}{4} + \frac{(-5+6t)\mathrm{e}^{t}}{4} + 2t - 1 \end{bmatrix}$$

- Substitute in vector of dependent variables

$$\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = \begin{bmatrix} \frac{(9+4c_1)\mathrm{e}^{-t}}{12} + \frac{(-3+6t+4c_2)\mathrm{e}^{t}}{4} + t \\ \frac{(9+4c_1)\mathrm{e}^{-t}}{4} + \frac{(6t+4c_2-5)\mathrm{e}^{t}}{4} + 2t - 1 \end{bmatrix}$$

- Solution to the system of ODEs

$$\left\{x_1(t) = \frac{(9+4c_1)\mathrm{e}^{-t}}{12} + \frac{(-3+6t+4c_2)\mathrm{e}^{t}}{4} + t, x_2(t) = \frac{(9+4c_1)\mathrm{e}^{-t}}{4} + \frac{(6t+4c_2-5)\mathrm{e}^{t}}{4} + 2t - 1\right\}$$

✓ Solution by Maple

Time used: 0.031 (sec). Leaf size: 54

```
dsolve([diff(x__1(t),t)=2*x__1(t)-1*x__2(t)+exp(t),diff(x__2(t),t)=3*x__1(t)-2*x__2(t)+t],si
```

$$x_1(t) = \frac{\mathrm{e}^{-t}c_1}{3} + c_2\mathrm{e}^{t} + \frac{3\,\mathrm{e}^{t}t}{2} - \frac{\mathrm{e}^{t}}{4} + t$$
$$x_2(t) = c_2\mathrm{e}^{t} + \mathrm{e}^{-t}c_1 + \frac{3\,\mathrm{e}^{t}t}{2} - \frac{3\,\mathrm{e}^{t}}{4} + 2t - 1$$

✓ Solution by Mathematica

Time used: 0.123 (sec). Leaf size: 97

```
DSolve[{x1'[t]==2*x1[t]-1*x2[t]+Exp[t],x2'[t]==3*x1[t]-2*x2[t]+t},{x1[t],x2[t]},t,IncludeSin
```

$$\text{x1}(t) \to \frac{1}{4}e^{-t}\left(4e^t t + e^{2t}(6t - 1 + 6c_1 - 2c_2) - 2c_1 + 2c_2\right)$$
$$\text{x2}(t) \to \frac{1}{4}e^{-t}\left(e^t(8t - 4) + e^{2t}(6t - 3 + 6c_1 - 2c_2) - 6c_1 + 6c_2\right)$$

18.2 problem 2

Internal problem ID [779]
Internal file name [OUTPUT/779_Sunday_June_05_2022_01_49_27_AM_63990030/index.tex]

Book: Elementary differential equations and boundary value problems, 10th ed., Boyce and DiPrima
Section: Chapter 7.9, Nonhomogeneous Linear Systems. page 447
Problem number: 2.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"system of linear ODEs"**

Solve

$$x_1'(t) = x_1(t) + \sqrt{3}\, x_2(t) + \mathrm{e}^t$$
$$x_2'(t) = \sqrt{3}\, x_1(t) - x_2(t) + \sqrt{3}\, \mathrm{e}^{-t}$$

18.2.1 Solution using Matrix exponential method

In this method, we will assume we have found the matrix exponential e^{At} allready. There are different methods to determine this but will not be shown here. This is a system of linear ODE's given as

$$\vec{x}'(t) = A\,\vec{x}(t) + \vec{G}(t)$$

Or

$$\begin{bmatrix} x_1'(t) \\ x_2'(t) \end{bmatrix} = \begin{bmatrix} 1 & \sqrt{3} \\ \sqrt{3} & -1 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} + \begin{bmatrix} \mathrm{e}^t \\ \sqrt{3}\,\mathrm{e}^{-t} \end{bmatrix}$$

Since the system is nonhomogeneous, then the solution is given by

$$\vec{x}(t) = \vec{x}_h(t) + \vec{x}_p(t)$$

Where $\vec{x}_h(t)$ is the homogeneous solution to $\vec{x}'(t) = A\,\vec{x}(t)$ and $\vec{x}_p(t)$ is a particular solution to $\vec{x}'(t) = A\,\vec{x}(t) + \vec{G}(t)$. The particular solution will be found using variation

of parameters method applied to the fundamental matrix. For the above matrix A, the matrix exponential can be found to be

$$
e^{At} = \begin{bmatrix} \frac{e^{-2t}}{4}+\frac{3\,e^{2t}}{4} & -\frac{\sqrt{3}\,(e^{-2t}-e^{2t})}{4} \\ -\frac{\sqrt{3}\,(e^{-2t}-e^{2t})}{4} & \frac{3\,e^{-2t}}{4}+\frac{e^{2t}}{4} \end{bmatrix}
$$

Therefore the homogeneous solution is

$$
\begin{aligned}
\vec{x}_h(t) &= e^{At}\vec{c} \\
&= \begin{bmatrix} \frac{e^{-2t}}{4}+\frac{3\,e^{2t}}{4} & -\frac{\sqrt{3}\,(e^{-2t}-e^{2t})}{4} \\ -\frac{\sqrt{3}\,(e^{-2t}-e^{2t})}{4} & \frac{3\,e^{-2t}}{4}+\frac{e^{2t}}{4} \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} \\
&= \begin{bmatrix} \left(\frac{e^{-2t}}{4}+\frac{3\,e^{2t}}{4}\right)c_1 - \frac{\sqrt{3}\,(e^{-2t}-e^{2t})c_2}{4} \\ -\frac{\sqrt{3}\,(e^{-2t}-e^{2t})c_1}{4} + \left(\frac{3\,e^{-2t}}{4}+\frac{e^{2t}}{4}\right)c_2 \end{bmatrix}
\end{aligned}
$$

The particular solution given by

$$
\vec{x}_p(t) = e^{At}\int e^{-At}\vec{G}(t)\,dt
$$

But

$$
\begin{aligned}
e^{-At} &= (e^{At})^{-1} \\
&= \begin{bmatrix} \frac{3\,e^{-2t}}{4}+\frac{e^{2t}}{4} & \frac{\sqrt{3}\,(e^{-2t}-e^{2t})}{4} \\ \frac{\sqrt{3}\,(e^{-2t}-e^{2t})}{4} & \frac{e^{-2t}}{4}+\frac{3\,e^{2t}}{4} \end{bmatrix}
\end{aligned}
$$

Hence

$$
\begin{aligned}
\vec{x}_p(t) &= \begin{bmatrix} \frac{e^{-2t}}{4}+\frac{3\,e^{2t}}{4} & -\frac{\sqrt{3}\,(e^{-2t}-e^{2t})}{4} \\ -\frac{\sqrt{3}\,(e^{-2t}-e^{2t})}{4} & \frac{3\,e^{-2t}}{4}+\frac{e^{2t}}{4} \end{bmatrix} \int \begin{bmatrix} \frac{3\,e^{-2t}}{4}+\frac{e^{2t}}{4} & \frac{\sqrt{3}\,(e^{-2t}-e^{2t})}{4} \\ \frac{\sqrt{3}\,(e^{-2t}-e^{2t})}{4} & \frac{e^{-2t}}{4}+\frac{3\,e^{2t}}{4} \end{bmatrix} \begin{bmatrix} e^{t} \\ \sqrt{3}\,e^{-t} \end{bmatrix} dt \\
&= \begin{bmatrix} \frac{e^{-2t}}{4}+\frac{3\,e^{2t}}{4} & -\frac{\sqrt{3}\,(e^{-2t}-e^{2t})}{4} \\ -\frac{\sqrt{3}\,(e^{-2t}-e^{2t})}{4} & \frac{3\,e^{-2t}}{4}+\frac{e^{2t}}{4} \end{bmatrix} \begin{bmatrix} \frac{(e^{6t}-9\,e^{4t}-9\,e^{2t}-3)e^{-3t}}{12} \\ -\frac{\sqrt{3}\,(e^{6t}-9\,e^{4t}+3\,e^{2t}+1)e^{-3t}}{12} \end{bmatrix} \\
&= \begin{bmatrix} -\frac{2\,e^{t}}{3} - e^{-t} \\ -\frac{\sqrt{3}\,(e^{t}-2\,e^{-t})}{3} \end{bmatrix}
\end{aligned}
$$

Hence the complete solution is

$$\vec{x}(t) = \vec{x}_h(t) + \vec{x}_p(t)$$

$$= \begin{bmatrix} \frac{e^{-2t}\left(\left(c_2\sqrt{3}+3c_1\right)e^{4t}-c_2\sqrt{3}+c_1-4\,e^{t}-\frac{8\,e^{3t}}{3}\right)}{4} \\ \frac{\left(\left(c_1\sqrt{3}+c_2\right)e^{4t}-\frac{4\sqrt{3}\,e^{3t}}{3}+\left(-c_1+\frac{8\,e^{t}}{3}\right)\sqrt{3}+3c_2\right)e^{-2t}}{4} \end{bmatrix}$$

18.2.2 Solution using explicit Eigenvalue and Eigenvector method

This is a system of linear ODE's given as

$$\vec{x}'(t) = A\,\vec{x}(t) + \vec{G}(t)$$

Or

$$\begin{bmatrix} x_1'(t) \\ x_2'(t) \end{bmatrix} = \begin{bmatrix} 1 & \sqrt{3} \\ \sqrt{3} & -1 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} + \begin{bmatrix} e^{t} \\ \sqrt{3}\,e^{-t} \end{bmatrix}$$

Since the system is nonhomogeneous, then the solution is given by

$$\vec{x}(t) = \vec{x}_h(t) + \vec{x}_p(t)$$

Where $\vec{x}_h(t)$ is the homogeneous solution to $\vec{x}'(t) = A\,\vec{x}(t)$ and $\vec{x}_p(t)$ is a particular solution to $\vec{x}'(t) = A\,\vec{x}(t) + \vec{G}(t)$. The particular solution will be found using variation of parameters method applied to the fundamental matrix.

The first step is find the homogeneous solution. We start by finding the eigenvalues of A. This is done by solving the following equation for the eigenvalues λ

$$\det\left(A - \lambda I\right) = 0$$

Expanding gives

$$\det\left(\begin{bmatrix} 1 & \sqrt{3} \\ \sqrt{3} & -1 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right) = 0$$

Therefore

$$\det\left(\begin{bmatrix} 1-\lambda & \sqrt{3} \\ \sqrt{3} & -1-\lambda \end{bmatrix}\right) = 0$$

Which gives the characteristic equation

$$\lambda^2 - 4 = 0$$

The roots of the above are the eigenvalues.

$$\lambda_1 = -2$$
$$\lambda_2 = 2$$

This table summarises the above result

eigenvalue	algebraic multiplicity	type of eigenvalue
-2	1	real eigenvalue
2	1	real eigenvalue

Now the eigenvector for each eigenvalue are found.

Considering the eigenvalue $\lambda_1 = -2$

We need to solve $A\vec{v} = \lambda \vec{v}$ or $(A - \lambda I)\vec{v} = \vec{0}$ which becomes

$$\left(\begin{bmatrix} 1 & \sqrt{3} \\ \sqrt{3} & -1 \end{bmatrix} - (-2) \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right) \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$
$$\begin{bmatrix} 3 & \sqrt{3} \\ \sqrt{3} & 1 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Now forward elimination is applied to solve for the eigenvector $\vec{v}$. The augmented matrix is

$$\left[\begin{array}{cc|c} 3 & \sqrt{3} & 0 \\ \sqrt{3} & 1 & 0 \end{array}\right]$$

$$R_2 = R_2 - \frac{\sqrt{3}\, R_1}{3} \Longrightarrow \left[\begin{array}{cc|c} 3 & \sqrt{3} & 0 \\ 0 & 0 & 0 \end{array}\right]$$

Therefore the system in Echelon form is

$$\begin{bmatrix} 3 & \sqrt{3} \\ 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

The free variables are $\{v_2\}$ and the leading variables are $\{v_1\}$. Let $v_2 = t$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\left\{v_1 = -\frac{\sqrt{3}\,t}{3}\right\}$

Hence the solution is

$$\begin{bmatrix} -\frac{\sqrt{3}\,t}{3} \\ t \end{bmatrix} = \begin{bmatrix} -\frac{\sqrt{3}\,t}{3} \\ t \end{bmatrix}$$

Since there is one free Variable, we have found one eigenvector associated with this eigenvalue. The above can be written as

$$\begin{bmatrix} -\frac{\sqrt{3}\,t}{3} \\ t \end{bmatrix} = t \begin{bmatrix} -\frac{\sqrt{3}}{3} \\ 1 \end{bmatrix}$$

Let $t = 1$ the eigenvector becomes

$$\begin{bmatrix} -\frac{\sqrt{3}\,t}{3} \\ t \end{bmatrix} = \begin{bmatrix} -\frac{\sqrt{3}}{3} \\ 1 \end{bmatrix}$$

Which is normalized to

$$\begin{bmatrix} -\frac{\sqrt{3}\,t}{3} \\ t \end{bmatrix} = \begin{bmatrix} -\sqrt{3} \\ 3 \end{bmatrix}$$

Considering the eigenvalue $\lambda_2 = 2$

We need to solve $A\vec{v} = \lambda\vec{v}$ or $(A - \lambda I)\vec{v} = \vec{0}$ which becomes

$$\left(\begin{bmatrix} 1 & \sqrt{3} \\ \sqrt{3} & -1 \end{bmatrix} - (2) \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right) \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} -1 & \sqrt{3} \\ \sqrt{3} & -3 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Now forward elimination is applied to solve for the eigenvector $\vec{v}$. The augmented matrix is

$$\left[\begin{array}{cc|c} -1 & \sqrt{3} & 0 \\ \sqrt{3} & -3 & 0 \end{array}\right]$$

$$R_2 = R_2 + \sqrt{3}\,R_1 \Longrightarrow \left[\begin{array}{cc|c} -1 & \sqrt{3} & 0 \\ 0 & 0 & 0 \end{array}\right]$$

Therefore the system in Echelon form is

$$\begin{bmatrix} -1 & \sqrt{3} \\ 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

The free variables are $\{v_2\}$ and the leading variables are $\{v_1\}$. Let $v_2 = t$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\{v_1 = \sqrt{3}\,t\}$

Hence the solution is

$$\begin{bmatrix} \sqrt{3}\,t \\ t \end{bmatrix} = \begin{bmatrix} \sqrt{3}\,t \\ t \end{bmatrix}$$

Since there is one free Variable, we have found one eigenvector associated with this eigenvalue. The above can be written as

$$\begin{bmatrix} \sqrt{3}\,t \\ t \end{bmatrix} = t \begin{bmatrix} \sqrt{3} \\ 1 \end{bmatrix}$$

Let $t = 1$ the eigenvector becomes

$$\begin{bmatrix} \sqrt{3}\,t \\ t \end{bmatrix} = \begin{bmatrix} \sqrt{3} \\ 1 \end{bmatrix}$$

The following table gives a summary of this result. It shows for each eigenvalue the algebraic multiplicity m, and its geometric multiplicity k and the eigenvectors associated with the eigenvalue. If $m > k$ then the eigenvalue is defective which means the number of normal linearly independent eigenvectors associated with this eigenvalue (called the geometric multiplicity k) does not equal the algebraic multiplicity m, and we need to determine an additional $m - k$ generalized eigenvectors for this eigenvalue.

	multiplicity			
eigenvalue	algebraic m	geometric k	defective?	eigenvectors
-2	1	1	No	$\begin{bmatrix} -\frac{\sqrt{3}}{3} \\ 1 \end{bmatrix}$
2	1	1	No	$\begin{bmatrix} \sqrt{3} \\ 1 \end{bmatrix}$

Now that we found the eigenvalues and associated eigenvectors, we will go over each eigenvalue and generate the solution basis. The only problem we need to take care of is if the eigenvalue is defective. Since eigenvalue -2 is real and distinct then the corresponding eigenvector solution is

$$
\begin{aligned}
\vec{x}_1(t) &= \vec{v}_1 e^{-2t} \\
&= \begin{bmatrix} -\frac{\sqrt{3}}{3} \\ 1 \end{bmatrix} e^{-2t}
\end{aligned}
$$

Since eigenvalue 2 is real and distinct then the corresponding eigenvector solution is

$$
\begin{aligned}
\vec{x}_2(t) &= \vec{v}_2 e^{2t} \\
&= \begin{bmatrix} \sqrt{3} \\ 1 \end{bmatrix} e^{2t}
\end{aligned}
$$

Therefore the homogeneous solution is

$$
\vec{x}_h(t) = c_1 \vec{x}_1(t) + c_2 \vec{x}_2(t)
$$

Which is written as

$$
\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = c_1 \begin{bmatrix} -\frac{\sqrt{3}\,\mathrm{e}^{-2t}}{3} \\ \mathrm{e}^{-2t} \end{bmatrix} + c_2 \begin{bmatrix} \sqrt{3}\,\mathrm{e}^{2t} \\ \mathrm{e}^{2t} \end{bmatrix}
$$

Now that we found homogeneous solution above, we need to find a particular solution $\vec{x}_p(t)$. We will use Variation of parameters. The fundamental matrix is

$$
\Phi = \begin{bmatrix} \vec{x}_1 & \vec{x}_2 & \cdots \end{bmatrix}
$$

Where $\vec{x}_i$ are the solution basis found above. Therefore the fundamental matrix is

$$
\Phi(t) = \begin{bmatrix} -\frac{\sqrt{3}\,\mathrm{e}^{-2t}}{3} & \sqrt{3}\,\mathrm{e}^{2t} \\ \mathrm{e}^{-2t} & \mathrm{e}^{2t} \end{bmatrix}
$$

The particular solution is then given by

$$
\vec{x}_p(t) = \Phi \int \Phi^{-1} \vec{G}(t)\, dt
$$

But

$$\Phi^{-1} = \begin{bmatrix} -\frac{\sqrt{3}\,e^{2t}}{4} & \frac{3\,e^{2t}}{4} \\ \frac{\sqrt{3}\,e^{-2t}}{4} & \frac{e^{-2t}}{4} \end{bmatrix}$$

Hence

$$\begin{aligned}
\vec{x}_p(t) &= \begin{bmatrix} -\frac{\sqrt{3}\,e^{-2t}}{3} & \sqrt{3}\,e^{2t} \\ e^{-2t} & e^{2t} \end{bmatrix} \int \begin{bmatrix} -\frac{\sqrt{3}\,e^{2t}}{4} & \frac{3\,e^{2t}}{4} \\ \frac{\sqrt{3}\,e^{-2t}}{4} & \frac{e^{-2t}}{4} \end{bmatrix} \begin{bmatrix} e^{t} \\ \sqrt{3}\,e^{-t} \end{bmatrix} dt \\
&= \begin{bmatrix} -\frac{\sqrt{3}\,e^{-2t}}{3} & \sqrt{3}\,e^{2t} \\ e^{-2t} & e^{2t} \end{bmatrix} \int \begin{bmatrix} -\frac{\sqrt{3}\,e^{t}(e^{2t}-3)}{4} \\ \frac{\sqrt{3}\,e^{-3t}(e^{2t}+1)}{4} \end{bmatrix} dt \\
&= \begin{bmatrix} -\frac{\sqrt{3}\,e^{-2t}}{3} & \sqrt{3}\,e^{2t} \\ e^{-2t} & e^{2t} \end{bmatrix} \begin{bmatrix} \frac{(-e^{3t}+9\,e^{t})\sqrt{3}}{12} \\ -\frac{\sqrt{3}\,(3\,e^{2t}+1)e^{-3t}}{12} \end{bmatrix} \\
&= \begin{bmatrix} -\frac{2\,e^{t}}{3} - e^{-t} \\ -\frac{\sqrt{3}\,(e^{t}-2\,e^{-t})}{3} \end{bmatrix}
\end{aligned}$$

Now that we found particular solution, the final solution is

$$\vec{x}(t) = \vec{x}_h(t) + \vec{x}_p(t)$$

$$\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = \begin{bmatrix} -\frac{c_1\sqrt{3}\,e^{-2t}}{3} \\ c_1 e^{-2t} \end{bmatrix} + \begin{bmatrix} c_2\sqrt{3}\,e^{2t} \\ c_2 e^{2t} \end{bmatrix} + \begin{bmatrix} -\frac{2\,e^{t}}{3} - e^{-t} \\ -\frac{\sqrt{3}\,(e^{t}-2\,e^{-t})}{3} \end{bmatrix}$$

Which becomes

$$\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = \begin{bmatrix} -\frac{\left(-3c_2\sqrt{3}\,e^{4t}+\sqrt{3}\,c_1+3\,e^{t}+2\,e^{3t}\right)e^{-2t}}{3} \\ \frac{\left(3\,e^{4t}c_2-\sqrt{3}\,e^{3t}+2\sqrt{3}\,e^{t}+3c_1\right)e^{-2t}}{3} \end{bmatrix}$$

✓ Solution by Maple

Time used: 0.141 (sec). Leaf size: 71

```
dsolve([diff(x__1(t),t)=1*x__1(t)+sqrt(3)*x__2(t)+exp(t),diff(x__2(t),t)=sqrt(3)*x__1(t)-1*x
```

$$x_1(t) = \sinh(2t)\,c_2 + \cosh(2t)\,c_1 - \frac{5\cosh(t)}{3} + \frac{\sinh(t)}{3}$$

$$x_2(t) = -\frac{\sqrt{3}\,(\cosh(2t)\,c_1 - 2\cosh(2t)\,c_2 - 2\sinh(2t)\,c_1 + \sinh(2t)\,c_2 + e^{t} + 2\sinh(t) - 2\cosh(t))}{3}$$

✓ Solution by Mathematica

Time used: 2.501 (sec). Leaf size: 313

```
DSolve[{x1'[t]==1*x1[t]+Sqrt[4]*x2[t]+Exp[t],x2'[t]==Sqrt[3]*x1[t]-1*x2[t]+Sqrt[3]*Exp[-t]},
```

$$x1(t) \to \frac{1}{6}\left(-6e^{-t} - \frac{2(6+\sqrt{3})\,e^{t}}{1+2\sqrt{3}} + \frac{\left(3\left(\sqrt{1+2\sqrt{3}}-1\right)c_1 - 6c_2\right)e^{-\sqrt{1+2\sqrt{3}}t}}{\sqrt{1+2\sqrt{3}}} + \frac{3\left(\left(1+\sqrt{1+2\sqrt{3}}\right)c_1 + 2c_2\right)e^{\sqrt{1+2\sqrt{3}}t}}{\sqrt{1+2\sqrt{3}}}\right)$$

$$x2(t) \to \frac{1}{4}\left(4e^{-t} - 2e^{t} + \frac{2\left((6+\sqrt{3})\,c_1 + (1+2\sqrt{3})\left(\sqrt{1+2\sqrt{3}}-1\right)c_2\right)e^{\sqrt{1+2\sqrt{3}}t}}{\left(1+2\sqrt{3}\right)^{3/2}} + \frac{\left(2(1+2\sqrt{3})\left(1+\sqrt{1+2\sqrt{3}}\right)c_2 - 2(6+\sqrt{3})\,c_1\right)e^{-\sqrt{1+2\sqrt{3}}t}}{\left(1+2\sqrt{3}\right)^{3/2}}\right)$$

18.3 problem 3

Internal problem ID [780]
Internal file name [`OUTPUT/780_Sunday_June_05_2022_01_49_29_AM_86783349/index.tex`]

Book: Elementary differential equations and boundary value problems, 10th ed., Boyce and DiPrima
Section: Chapter 7.9, Nonhomogeneous Linear Systems. page 447
Problem number: 3.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"system of linear ODEs"**

Solve

$$x_1'(t) = 2x_1(t) - 5x_2(t) - \cos(t)$$
$$x_2'(t) = x_1(t) - 2x_2(t) + \sin(t)$$

18.3.1 Solution using Matrix exponential method

In this method, we will assume we have found the matrix exponential e^{At} allready. There are different methods to determine this but will not be shown here. This is a system of linear ODE's given as

$$\vec{x}'(t) = A\,\vec{x}(t) + \vec{G}(t)$$

Or

$$\begin{bmatrix} x_1'(t) \\ x_2'(t) \end{bmatrix} = \begin{bmatrix} 2 & -5 \\ 1 & -2 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} + \begin{bmatrix} -\cos(t) \\ \sin(t) \end{bmatrix}$$

Since the system is nonhomogeneous, then the solution is given by

$$\vec{x}(t) = \vec{x}_h(t) + \vec{x}_p(t)$$

Where $\vec{x}_h(t)$ is the homogeneous solution to $\vec{x}'(t) = A\,\vec{x}(t)$ and $\vec{x}_p(t)$ is a particular solution to $\vec{x}'(t) = A\,\vec{x}(t) + \vec{G}(t)$. The particular solution will be found using variation

of parameters method applied to the fundamental matrix. For the above matrix A, the matrix exponential can be found to be

$$e^{At} = \begin{bmatrix} \cos(t) + 2\sin(t) & -5\sin(t) \\ \sin(t) & \cos(t) - 2\sin(t) \end{bmatrix}$$

Therefore the homogeneous solution is

$$\begin{aligned} \vec{x}_h(t) &= e^{At}\vec{c} \\ &= \begin{bmatrix} \cos(t) + 2\sin(t) & -5\sin(t) \\ \sin(t) & \cos(t) - 2\sin(t) \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} \\ &= \begin{bmatrix} (\cos(t) + 2\sin(t))\, c_1 - 5\sin(t)\, c_2 \\ \sin(t)\, c_1 + (\cos(t) - 2\sin(t))\, c_2 \end{bmatrix} \\ &= \begin{bmatrix} (2c_1 - 5c_2)\sin(t) + c_1\cos(t) \\ (c_1 - 2c_2)\sin(t) + c_2\cos(t) \end{bmatrix} \end{aligned}$$

The particular solution given by

$$\vec{x}_p(t) = e^{At} \int e^{-At}\vec{G}(t)\, dt$$

But

$$\begin{aligned} e^{-At} &= (e^{At})^{-1} \\ &= \begin{bmatrix} \cos(t) - 2\sin(t) & 5\sin(t) \\ -\sin(t) & \cos(t) + 2\sin(t) \end{bmatrix} \end{aligned}$$

Hence

$$\begin{aligned} \vec{x}_p(t) &= \begin{bmatrix} \cos(t) + 2\sin(t) & -5\sin(t) \\ \sin(t) & \cos(t) - 2\sin(t) \end{bmatrix} \int \begin{bmatrix} \cos(t) - 2\sin(t) & 5\sin(t) \\ -\sin(t) & \cos(t) + 2\sin(t) \end{bmatrix} \begin{bmatrix} -\cos(t) \\ \sin(t) \end{bmatrix} \\ &= \begin{bmatrix} \cos(t) + 2\sin(t) & -5\sin(t) \\ \sin(t) & \cos(t) - 2\sin(t) \end{bmatrix} \begin{bmatrix} -3\sin(t)\cos(t) + 2t - \cos(t)^2 \\ -\cos(t)^2 - \sin(t)\cos(t) + t + \frac{1}{2} \end{bmatrix} \\ &= \begin{bmatrix} \cos(t)(2t - 1) + \frac{(-2t-5)\sin(t)}{2} \\ -\frac{\cos(t)}{2} - \sin(t) + t\cos(t) \end{bmatrix} \end{aligned}$$

Hence the complete solution is

$$\vec{x}(t) = \vec{x}_h(t) + \vec{x}_p(t)$$

$$= \begin{bmatrix} \frac{(-2t+4c_1-10c_2-5)\sin(t)}{2} + 2\left(t + \frac{c_1}{2} - \frac{1}{2}\right)\cos(t) \\ \frac{(2t+2c_2-1)\cos(t)}{2} + \sin(t)(c_1 - 2c_2 - 1) \end{bmatrix}$$

18.3.2 Solution using explicit Eigenvalue and Eigenvector method

This is a system of linear ODE's given as

$$\vec{x}'(t) = A\,\vec{x}(t) + \vec{G}(t)$$

Or

$$\begin{bmatrix} x_1'(t) \\ x_2'(t) \end{bmatrix} = \begin{bmatrix} 2 & -5 \\ 1 & -2 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} + \begin{bmatrix} -\cos(t) \\ \sin(t) \end{bmatrix}$$

Since the system is nonhomogeneous, then the solution is given by

$$\vec{x}(t) = \vec{x}_h(t) + \vec{x}_p(t)$$

Where $\vec{x}_h(t)$ is the homogeneous solution to $\vec{x}'(t) = A\,\vec{x}(t)$ and $\vec{x}_p(t)$ is a particular solution to $\vec{x}'(t) = A\,\vec{x}(t) + \vec{G}(t)$. The particular solution will be found using variation of parameters method applied to the fundamental matrix.

The first step is find the homogeneous solution. We start by finding the eigenvalues of A. This is done by solving the following equation for the eigenvalues λ

$$\det(A - \lambda I) = 0$$

Expanding gives

$$\det\left(\begin{bmatrix} 2 & -5 \\ 1 & -2 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right) = 0$$

Therefore

$$\det\left(\begin{bmatrix} 2-\lambda & -5 \\ 1 & -2-\lambda \end{bmatrix}\right) = 0$$

Which gives the characteristic equation

$$\lambda^2 + 1 = 0$$

The roots of the above are the eigenvalues.

$$\lambda_1 = i$$
$$\lambda_2 = -i$$

This table summarises the above result

eigenvalue	algebraic multiplicity	type of eigenvalue
$-i$	1	complex eigenvalue
i	1	complex eigenvalue

Now the eigenvector for each eigenvalue are found.

Considering the eigenvalue $\lambda_1 = -i$

We need to solve $A\vec{v} = \lambda\vec{v}$ or $(A - \lambda I)\vec{v} = \vec{0}$ which becomes

$$\left(\begin{bmatrix} 2 & -5 \\ 1 & -2 \end{bmatrix} - (-i)\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right)\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$
$$\begin{bmatrix} 2+i & -5 \\ 1 & -2+i \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Now forward elimination is applied to solve for the eigenvector $\vec{v}$. The augmented matrix is

$$\left[\begin{array}{cc|c} 2+i & -5 & 0 \\ 1 & -2+i & 0 \end{array}\right]$$

$$R_2 = R_2 + \left(-\frac{2}{5} + \frac{i}{5}\right) R_1 \Longrightarrow \left[\begin{array}{cc|c} 2+i & -5 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

Therefore the system in Echelon form is

$$\begin{bmatrix} 2+i & -5 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

The free variables are $\{v_2\}$ and the leading variables are $\{v_1\}$. Let $v_2 = t$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\{v_1 = (2-i)\,t\}$

Hence the solution is

$$\begin{bmatrix} (2-\mathrm{I})\,t \\ t \end{bmatrix} = \begin{bmatrix} (2-i)\,t \\ t \end{bmatrix}$$

Since there is one free Variable, we have found one eigenvector associated with this eigenvalue. The above can be written as

$$\begin{bmatrix} (2-\mathrm{I})\,t \\ t \end{bmatrix} = t \begin{bmatrix} 2-i \\ 1 \end{bmatrix}$$

Let $t = 1$ the eigenvector becomes

$$\begin{bmatrix} (2-\mathrm{I})\,t \\ t \end{bmatrix} = \begin{bmatrix} 2-i \\ 1 \end{bmatrix}$$

Considering the eigenvalue $\lambda_2 = i$

We need to solve $A\vec{v} = \lambda\vec{v}$ or $(A - \lambda I)\vec{v} = \vec{0}$ which becomes

$$\left(\begin{bmatrix} 2 & -5 \\ 1 & -2 \end{bmatrix} - (i) \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right) \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 2-i & -5 \\ 1 & -2-i \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Now forward elimination is applied to solve for the eigenvector $\vec{v}$. The augmented matrix is

$$\left[\begin{array}{cc|c} 2-i & -5 & 0 \\ 1 & -2-i & 0 \end{array}\right]$$

$$R_2 = R_2 + \left(-\frac{2}{5} - \frac{i}{5}\right) R_1 \Longrightarrow \left[\begin{array}{cc|c} 2-i & -5 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

Therefore the system in Echelon form is

$$\begin{bmatrix} 2-i & -5 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

The free variables are $\{v_2\}$ and the leading variables are $\{v_1\}$. Let $v_2 = t$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\{v_1 = (2+i)\,t\}$

Hence the solution is

$$\begin{bmatrix} (2+\mathrm{I})\,t \\ t \end{bmatrix} = \begin{bmatrix} (2+i)\,t \\ t \end{bmatrix}$$

Since there is one free Variable, we have found one eigenvector associated with this eigenvalue. The above can be written as

$$\begin{bmatrix} (2+\mathrm{I})\,t \\ t \end{bmatrix} = t \begin{bmatrix} 2+i \\ 1 \end{bmatrix}$$

Let $t = 1$ the eigenvector becomes

$$\begin{bmatrix} (2+\mathrm{I})\,t \\ t \end{bmatrix} = \begin{bmatrix} 2+i \\ 1 \end{bmatrix}$$

The following table gives a summary of this result. It shows for each eigenvalue the algebraic multiplicity m, and its geometric multiplicity k and the eigenvectors associated with the eigenvalue. If $m > k$ then the eigenvalue is defective which means the number of normal linearly independent eigenvectors associated with this eigenvalue (called the geometric multiplicity k) does not equal the algebraic multiplicity m, and we need to determine an additional $m - k$ generalized eigenvectors for this eigenvalue.

	multiplicity			
eigenvalue	algebraic m	geometric k	defective?	eigenvectors
i	1	1	No	$\begin{bmatrix} 2+i \\ 1 \end{bmatrix}$
$-i$	1	1	No	$\begin{bmatrix} 2-i \\ 1 \end{bmatrix}$

Now that we found the eigenvalues and associated eigenvectors, we will go over each eigenvalue and generate the solution basis. The only problem we need to take care of is if the eigenvalue is defective. Therefore the homogeneous solution is

$$\vec{x}_h(t) = c_1 \vec{x}_1(t) + c_2 \vec{x}_2(t)$$

Which is written as

$$\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = c_1 \begin{bmatrix} (2+i)\,\mathrm{e}^{it} \\ \mathrm{e}^{it} \end{bmatrix} + c_2 \begin{bmatrix} (2-i)\,\mathrm{e}^{-it} \\ \mathrm{e}^{-it} \end{bmatrix}$$

Now that we found homogeneous solution above, we need to find a particular solution $\vec{x}_p(t)$. We will use Variation of parameters. The fundamental matrix is

$$\Phi = \begin{bmatrix} \vec{x}_1 & \vec{x}_2 & \cdots \end{bmatrix}$$

Where $\vec{x}_i$ are the solution basis found above. Therefore the fundamental matrix is

$$\Phi(t) = \begin{bmatrix} (2+i)\,\mathrm{e}^{it} & (2-i)\,\mathrm{e}^{-it} \\ \mathrm{e}^{it} & \mathrm{e}^{-it} \end{bmatrix}$$

The particular solution is then given by

$$\vec{x}_p(t) = \Phi \int \Phi^{-1}\vec{G}(t)\,dt$$

But

$$\Phi^{-1} = \begin{bmatrix} -\frac{i\mathrm{e}^{-it}}{2} & \left(\frac{1}{2}+i\right)\mathrm{e}^{-it} \\ \frac{i\mathrm{e}^{it}}{2} & \left(\frac{1}{2}-i\right)\mathrm{e}^{it} \end{bmatrix}$$

Hence

$$\begin{aligned}
\vec{x}_p(t) &= \begin{bmatrix} (2+i)\,\mathrm{e}^{it} & (2-i)\,\mathrm{e}^{-it} \\ \mathrm{e}^{it} & \mathrm{e}^{-it} \end{bmatrix} \int \begin{bmatrix} -\frac{i\mathrm{e}^{-it}}{2} & \left(\frac{1}{2}+i\right)\mathrm{e}^{-it} \\ \frac{i\mathrm{e}^{it}}{2} & \left(\frac{1}{2}-i\right)\mathrm{e}^{it} \end{bmatrix} \begin{bmatrix} -\cos(t) \\ \sin(t) \end{bmatrix} dt \\
&= \begin{bmatrix} (2+i)\,\mathrm{e}^{it} & (2-i)\,\mathrm{e}^{-it} \\ \mathrm{e}^{it} & \mathrm{e}^{-it} \end{bmatrix} \int \begin{bmatrix} \frac{((1+2i)\sin(t)+i\cos(t))\mathrm{e}^{-it}}{2} \\ -\frac{((-1+2i)\sin(t)+i\cos(t))\mathrm{e}^{it}}{2} \end{bmatrix} dt \\
&= \begin{bmatrix} (2+i)\,\mathrm{e}^{it} & (2-i)\,\mathrm{e}^{-it} \\ \mathrm{e}^{it} & \mathrm{e}^{-it} \end{bmatrix} \begin{bmatrix} \frac{((it+i-1)\sin(t)+t\cos(t))\mathrm{e}^{-it}}{2} \\ -\frac{((it+i+1)\sin(t)-t\cos(t))\mathrm{e}^{it}}{2} \end{bmatrix} \\
&= \begin{bmatrix} (-t-3)\sin(t)+2t\cos(t) \\ t\cos(t)-\sin(t) \end{bmatrix}
\end{aligned}$$

Now that we found particular solution, the final solution is

$$\vec{x}(t) = \vec{x}_h(t) + \vec{x}_p(t)$$

$$\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = \begin{bmatrix} (2+i)\,c_1\mathrm{e}^{it} \\ c_1\mathrm{e}^{it} \end{bmatrix} + \begin{bmatrix} (2-i)\,c_2\mathrm{e}^{-it} \\ c_2\mathrm{e}^{-it} \end{bmatrix} + \begin{bmatrix} (-t-3)\sin(t)+2t\cos(t) \\ t\cos(t)-\sin(t) \end{bmatrix}$$

Which becomes

$$\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = \begin{bmatrix} (2+i)\,c_1\mathrm{e}^{it} + (2-i)\,c_2\mathrm{e}^{-it} + (-t-3)\sin(t) + 2t\cos(t) \\ c_1\mathrm{e}^{it} + c_2\mathrm{e}^{-it} + t\cos(t) - \sin(t) \end{bmatrix}$$

18.3.3 Maple step by step solution

Let's solve

$[x_1'(t) = 2x_1(t) - 5x_2(t) - \cos(t), x_2'(t) = x_1(t) - 2x_2(t) + \sin(t)]$

- Define vector

$\underline{x}^{\rightarrow}(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}$

- Convert system into a vector equation

$\underline{x}^{\rightarrow\prime}(t) = \begin{bmatrix} 2 & -5 \\ 1 & -2 \end{bmatrix} \cdot \underline{x}^{\rightarrow}(t) + \begin{bmatrix} -\cos(t) \\ \sin(t) \end{bmatrix}$

- System to solve

$\underline{x}^{\rightarrow\prime}(t) = \begin{bmatrix} 2 & -5 \\ 1 & -2 \end{bmatrix} \cdot \underline{x}^{\rightarrow}(t) + \begin{bmatrix} -\cos(t) \\ \sin(t) \end{bmatrix}$

- Define the forcing function

$\vec{f}(t) = \begin{bmatrix} -\cos(t) \\ \sin(t) \end{bmatrix}$

- Define the coefficient matrix

$A = \begin{bmatrix} 2 & -5 \\ 1 & -2 \end{bmatrix}$

- Rewrite the system as

$\underline{x}^{\rightarrow\prime}(t) = A \cdot \underline{x}^{\rightarrow}(t) + \vec{f}$

- To solve the system, find the eigenvalues and eigenvectors of A
- Eigenpairs of A

$\left[\left[-\mathrm{I}, \begin{bmatrix} 2-\mathrm{I} \\ 1 \end{bmatrix}\right], \left[\mathrm{I}, \begin{bmatrix} 2+\mathrm{I} \\ 1 \end{bmatrix}\right]\right]$

- Consider complex eigenpair, complex conjugate eigenvalue can be ignored

$\left[-\mathrm{I}, \begin{bmatrix} 2-\mathrm{I} \\ 1 \end{bmatrix}\right]$

- Solution from eigenpair

$$e^{-It} \cdot \begin{bmatrix} 2-I \\ 1 \end{bmatrix}$$

- Use Euler identity to write solution in terms of sin and cos

$$(\cos(t) - I\sin(t)) \cdot \begin{bmatrix} 2-I \\ 1 \end{bmatrix}$$

- Simplify expression

$$\begin{bmatrix} (2-I)(\cos(t) - I\sin(t)) \\ \cos(t) - I\sin(t) \end{bmatrix}$$

- Both real and imaginary parts are solutions to the homogeneous system

$$\left[\underline{\vec{x}}_1(t) = \begin{bmatrix} 2\cos(t) - \sin(t) \\ \cos(t) \end{bmatrix}, \underline{\vec{x}}_2(t) = \begin{bmatrix} -\cos(t) - 2\sin(t) \\ -\sin(t) \end{bmatrix}\right]$$

- General solution of the system of ODEs can be written in terms of the particular solution $\underline{\vec{x}}$

$$\underline{\vec{x}}(t) = c_1\underline{\vec{x}}_1(t) + c_2\underline{\vec{x}}_2(t) + \underline{\vec{x}}_p(t)$$

☐ Fundamental matrix

  - Let $\phi(t)$ be the matrix whose columns are the independent solutions of the homogeneous syst

$$\phi(t) = \begin{bmatrix} 2\cos(t) - \sin(t) & -\cos(t) - 2\sin(t) \\ \cos(t) & -\sin(t) \end{bmatrix}$$

  - The fundamental matrix, $\Phi(t)$ is a normalized version of $\phi(t)$ satisfying $\Phi(0) = I$ where I is th

$$\Phi(t) = \phi(t) \cdot \frac{1}{\phi(0)}$$

  - Substitute the value of $\phi(t)$ and $\phi(0)$

$$\Phi(t) = \begin{bmatrix} 2\cos(t) - \sin(t) & -\cos(t) - 2\sin(t) \\ \cos(t) & -\sin(t) \end{bmatrix} \cdot \frac{1}{\begin{bmatrix} 2 & -1 \\ 1 & 0 \end{bmatrix}}$$

  - Evaluate and simplify to get the fundamental matrix

$$\Phi(t) = \begin{bmatrix} \cos(t) + 2\sin(t) & -5\sin(t) \\ \sin(t) & \cos(t) - 2\sin(t) \end{bmatrix}$$

- ☐ Find a particular solution of the system of ODEs using variation of parameters
  - Let the particular solution be the fundamental matrix multiplied by $\vec{v}(t)$ and solve for $\vec{v}(t)$

    $\overrightarrow{x}_p(t) = \Phi(t) \cdot \vec{v}(t)$
  - Take the derivative of the particular solution

    $\overrightarrow{x}_p'(t) = \Phi'(t) \cdot \vec{v}(t) + \Phi(t) \cdot \vec{v}'(t)$
  - Substitute particular solution and its derivative into the system of ODEs

    $\Phi'(t) \cdot \vec{v}(t) + \Phi(t) \cdot \vec{v}'(t) = A \cdot \Phi(t) \cdot \vec{v}(t) + \vec{f}(t)$
  - The fundamental matrix has columns that are solutions to the homogeneous system so its der

    $A \cdot \Phi(t) \cdot \vec{v}(t) + \Phi(t) \cdot \vec{v}'(t) = A \cdot \Phi(t) \cdot \vec{v}(t) + \vec{f}(t)$
  - Cancel like terms

    $\Phi(t) \cdot \vec{v}'(t) = \vec{f}(t)$
  - Multiply by the inverse of the fundamental matrix

    $\vec{v}'(t) = \frac{1}{\Phi(t)} \cdot \vec{f}(t)$
  - Integrate to solve for $\vec{v}(t)$

    $\vec{v}(t) = \int_0^t \frac{1}{\Phi(s)} \cdot \vec{f}(s)\, ds$
  - Plug $\vec{v}(t)$ into the equation for the particular solution

    $\overrightarrow{x}_p(t) = \Phi(t) \cdot \left(\int_0^t \frac{1}{\Phi(s)} \cdot \vec{f}(s)\, ds\right)$
  - Plug in the fundamental matrix and the forcing function and compute

    $$\overrightarrow{x}_p(t) = \begin{bmatrix} (-t-3)\sin(t) + 2t\cos(t) \\ t\cos(t) - \sin(t) \end{bmatrix}$$
- Plug particular solution back into general solution

  $$\overrightarrow{x}(t) = c_1 \overrightarrow{x}_1(t) + c_2 \overrightarrow{x}_2(t) + \begin{bmatrix} (-t-3)\sin(t) + 2t\cos(t) \\ t\cos(t) - \sin(t) \end{bmatrix}$$
- Substitute in vector of dependent variables

  $$\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = \begin{bmatrix} (-c_1 - 2c_2 - t - 3)\sin(t) + 2\left(t + c_1 - \frac{c_2}{2}\right)\cos(t) \\ (t + c_1)\cos(t) - \sin(t)(c_2 + 1) \end{bmatrix}$$
- Solution to the system of ODEs

$$\left\{x_1(t) = (-c_1 - 2c_2 - t - 3)\sin(t) + 2\left(t + c_1 - \frac{c_2}{2}\right)\cos(t), x_2(t) = (t + c_1)\cos(t) - \sin(t)(c_2 +$$

✓ Solution by Maple

Time used: 0.063 (sec). Leaf size: 60

```
dsolve([diff(x__1(t),t)=2*x__1(t)-5*x__2(t)-cos(t),diff(x__2(t),t)=1*x__1(t)-2*x__2(t)+sin(t
```

$$x_1(t) = c_2 \sin(t) - \sin(t)\, t + c_1 \cos(t) + 2\cos(t)\, t - \cos(t)$$
$$x_2(t) = \frac{c_1 \sin(t)}{5} + \frac{2c_2 \sin(t)}{5} + \frac{2c_1 \cos(t)}{5} - \frac{c_2 \cos(t)}{5} + \cos(t)\, t - \cos(t)$$

✓ Solution by Mathematica

Time used: 0.069 (sec). Leaf size: 61

```
DSolve[{x1'[t]==2*x1[t]-5*x2[t]-Cos[t],x2'[t]==1*x1[t]-2*x2[t]+Sin[t]},{x1[t],x2[t]},t,Inclu
```

$$\text{x1}(t) \to \left(2t - \frac{1}{2} + c_1\right)\cos(t) - (t - 1 - 2c_1 + 5c_2)\sin(t)$$
$$\text{x2}(t) \to (t - 1 + c_2)\cos(t) + \frac{1}{2}(1 + 2c_1 - 4c_2)\sin(t)$$

18.4 problem 4

Internal problem ID [781]
Internal file name [`OUTPUT/781_Sunday_June_05_2022_01_49_32_AM_79929771/index.tex`]

Book: Elementary differential equations and boundary value problems, 10th ed., Boyce and DiPrima
Section: Chapter 7.9, Nonhomogeneous Linear Systems. page 447
Problem number: 4.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"system of linear ODEs"**

Solve

$$
\begin{aligned}
x_1'(t) &= x_1(t) + x_2(t) + \mathrm{e}^{-2t} \\
x_2'(t) &= 4x_1(t) - 2x_2(t) - 2\,\mathrm{e}^{t}
\end{aligned}
$$

18.4.1 Solution using Matrix exponential method

In this method, we will assume we have found the matrix exponential e^{At} allready. There are different methods to determine this but will not be shown here. This is a system of linear ODE's given as

$$\vec{x}'(t) = A\,\vec{x}(t) + \vec{G}(t)$$

Or

$$\begin{bmatrix} x_1'(t) \\ x_2'(t) \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 4 & -2 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} + \begin{bmatrix} \mathrm{e}^{-2t} \\ -2\,\mathrm{e}^{t} \end{bmatrix}$$

Since the system is nonhomogeneous, then the solution is given by

$$\vec{x}(t) = \vec{x}_h(t) + \vec{x}_p(t)$$

Where $\vec{x}_h(t)$ is the homogeneous solution to $\vec{x}'(t) = A\,\vec{x}(t)$ and $\vec{x}_p(t)$ is a particular solution to $\vec{x}'(t) = A\,\vec{x}(t) + \vec{G}(t)$. The particular solution will be found using variation

of parameters method applied to the fundamental matrix. For the above matrix A, the matrix exponential can be found to be

$$e^{At} = \begin{bmatrix} \frac{(4\,e^{5t}+1)\,e^{-3t}}{5} & \frac{(e^{5t}-1)\,e^{-3t}}{5} \\ \frac{4(e^{5t}-1)\,e^{-3t}}{5} & \frac{(e^{5t}+4)\,e^{-3t}}{5} \end{bmatrix}$$

Therefore the homogeneous solution is

$$\begin{aligned}
\vec{x}_h(t) &= e^{At}\vec{c} \\
&= \begin{bmatrix} \frac{(4\,e^{5t}+1)\,e^{-3t}}{5} & \frac{(e^{5t}-1)\,e^{-3t}}{5} \\ \frac{4(e^{5t}-1)\,e^{-3t}}{5} & \frac{(e^{5t}+4)\,e^{-3t}}{5} \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} \\
&= \begin{bmatrix} \frac{(4\,e^{5t}+1)\,e^{-3t}c_1}{5} + \frac{(e^{5t}-1)\,e^{-3t}c_2}{5} \\ \frac{4(e^{5t}-1)\,e^{-3t}c_1}{5} + \frac{(e^{5t}+4)\,e^{-3t}c_2}{5} \end{bmatrix} \\
&= \begin{bmatrix} \frac{e^{-3t}((4c_1+c_2)\,e^{5t}-c_2+c_1)}{5} \\ \frac{4\left(\left(c_1+\frac{c_2}{4}\right)e^{5t}+c_2-c_1\right)e^{-3t}}{5} \end{bmatrix}
\end{aligned}$$

The particular solution given by

$$\vec{x}_p(t) = e^{At} \int e^{-At}\vec{G}(t)\,dt$$

But

$$\begin{aligned}
e^{-At} &= (e^{At})^{-1} \\
&= \begin{bmatrix} \frac{(e^{5t}+4)\,e^{-2t}}{5} & -\frac{(e^{5t}-1)\,e^{-2t}}{5} \\ -\frac{4(e^{5t}-1)\,e^{-2t}}{5} & \frac{(4\,e^{5t}+1)\,e^{-2t}}{5} \end{bmatrix}
\end{aligned}$$

Hence

$$\begin{aligned}
\vec{x}_p(t) &= \begin{bmatrix} \frac{(4\,e^{5t}+1)\,e^{-3t}}{5} & \frac{(e^{5t}-1)\,e^{-3t}}{5} \\ \frac{4(e^{5t}-1)\,e^{-3t}}{5} & \frac{(e^{5t}+4)\,e^{-3t}}{5} \end{bmatrix} \int \begin{bmatrix} \frac{(e^{5t}+4)\,e^{-2t}}{5} & -\frac{(e^{5t}-1)\,e^{-2t}}{5} \\ -\frac{4(e^{5t}-1)\,e^{-2t}}{5} & \frac{(4\,e^{5t}+1)\,e^{-2t}}{5} \end{bmatrix} \begin{bmatrix} e^{-2t} \\ -2\,e^{t} \end{bmatrix} dt \\
&= \begin{bmatrix} \frac{(4\,e^{5t}+1)\,e^{-3t}}{5} & \frac{(e^{5t}-1)\,e^{-3t}}{5} \\ \frac{4(e^{5t}-1)\,e^{-3t}}{5} & \frac{(e^{5t}+4)\,e^{-3t}}{5} \end{bmatrix} \begin{bmatrix} \frac{(e^{8t}+2\,e^{5t}+4\,e^{3t}-2)\,e^{-4t}}{10} \\ -\frac{(2\,e^{8t}+4\,e^{5t}-2\,e^{3t}+1)\,e^{-4t}}{5} \end{bmatrix} \\
&= \begin{bmatrix} \frac{e^{t}}{2} \\ -e^{-2t} \end{bmatrix}
\end{aligned}$$

Hence the complete solution is

$$\vec{x}(t) = \vec{x}_h(t) + \vec{x}_p(t)$$
$$= \begin{bmatrix} \frac{e^{-3t}\left((4c_1+c_2)e^{5t}+c_1-c_2+\frac{5\,e^{4t}}{2}\right)}{5} \\ \frac{4\left((c_1+\frac{c_2}{4})e^{5t}-c_1+c_2-\frac{5\,e^{t}}{4}\right)e^{-3t}}{5} \end{bmatrix}$$

18.4.2 Solution using explicit Eigenvalue and Eigenvector method

This is a system of linear ODE's given as

$$\vec{x}'(t) = A\,\vec{x}(t) + \vec{G}(t)$$

Or

$$\begin{bmatrix} x_1'(t) \\ x_2'(t) \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 4 & -2 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} + \begin{bmatrix} e^{-2t} \\ -2\,e^{t} \end{bmatrix}$$

Since the system is nonhomogeneous, then the solution is given by

$$\vec{x}(t) = \vec{x}_h(t) + \vec{x}_p(t)$$

Where $\vec{x}_h(t)$ is the homogeneous solution to $\vec{x}'(t) = A\,\vec{x}(t)$ and $\vec{x}_p(t)$ is a particular solution to $\vec{x}'(t) = A\,\vec{x}(t) + \vec{G}(t)$. The particular solution will be found using variation of parameters method applied to the fundamental matrix.

The first step is find the homogeneous solution. We start by finding the eigenvalues of A. This is done by solving the following equation for the eigenvalues λ

$$\det\left(A - \lambda I\right) = 0$$

Expanding gives

$$\det\left(\begin{bmatrix} 1 & 1 \\ 4 & -2 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right) = 0$$

Therefore

$$\det\left(\begin{bmatrix} 1-\lambda & 1 \\ 4 & -2-\lambda \end{bmatrix}\right) = 0$$

Which gives the characteristic equation

$$\lambda^2 + \lambda - 6 = 0$$

The roots of the above are the eigenvalues.

$$\lambda_1 = 2$$
$$\lambda_2 = -3$$

This table summarises the above result

eigenvalue	algebraic multiplicity	type of eigenvalue
-3	1	real eigenvalue
2	1	real eigenvalue

Now the eigenvector for each eigenvalue are found.

Considering the eigenvalue $\lambda_1 = -3$

We need to solve $A\vec{v} = \lambda\vec{v}$ or $(A - \lambda I)\vec{v} = \vec{0}$ which becomes

$$\left(\begin{bmatrix} 1 & 1 \\ 4 & -2 \end{bmatrix} - (-3)\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right)\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$
$$\begin{bmatrix} 4 & 1 \\ 4 & 1 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Now forward elimination is applied to solve for the eigenvector $\vec{v}$. The augmented matrix is

$$\left[\begin{array}{cc|c} 4 & 1 & 0 \\ 4 & 1 & 0 \end{array}\right]$$

$$R_2 = R_2 - R_1 \Longrightarrow \left[\begin{array}{cc|c} 4 & 1 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

Therefore the system in Echelon form is

$$\begin{bmatrix} 4 & 1 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

The free variables are $\{v_2\}$ and the leading variables are $\{v_1\}$. Let $v_2 = t$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\{v_1 = -\frac{t}{4}\}$

Hence the solution is

$$\begin{bmatrix} -\frac{t}{4} \\ t \end{bmatrix} = \begin{bmatrix} -\frac{t}{4} \\ t \end{bmatrix}$$

Since there is one free Variable, we have found one eigenvector associated with this eigenvalue. The above can be written as

$$\begin{bmatrix} -\frac{t}{4} \\ t \end{bmatrix} = t \begin{bmatrix} -\frac{1}{4} \\ 1 \end{bmatrix}$$

Let $t = 1$ the eigenvector becomes

$$\begin{bmatrix} -\frac{t}{4} \\ t \end{bmatrix} = \begin{bmatrix} -\frac{1}{4} \\ 1 \end{bmatrix}$$

Which is normalized to

$$\begin{bmatrix} -\frac{t}{4} \\ t \end{bmatrix} = \begin{bmatrix} -1 \\ 4 \end{bmatrix}$$

Considering the eigenvalue $\lambda_2 = 2$

We need to solve $A\vec{v} = \lambda\vec{v}$ or $(A - \lambda I)\vec{v} = \vec{0}$ which becomes

$$\left(\begin{bmatrix} 1 & 1 \\ 4 & -2 \end{bmatrix} - (2) \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right) \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} -1 & 1 \\ 4 & -4 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Now forward elimination is applied to solve for the eigenvector $\vec{v}$. The augmented matrix is

$$\left[\begin{array}{cc|c} -1 & 1 & 0 \\ 4 & -4 & 0 \end{array}\right]$$

$$R_2 = R_2 + 4R_1 \Longrightarrow \left[\begin{array}{cc|c} -1 & 1 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

Therefore the system in Echelon form is

$$\begin{bmatrix} -1 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

The free variables are $\{v_2\}$ and the leading variables are $\{v_1\}$. Let $v_2 = t$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\{v_1 = t\}$

Hence the solution is

$$\begin{bmatrix} t \\ t \end{bmatrix} = \begin{bmatrix} t \\ t \end{bmatrix}$$

Since there is one free Variable, we have found one eigenvector associated with this eigenvalue. The above can be written as

$$\begin{bmatrix} t \\ t \end{bmatrix} = t \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

Let $t = 1$ the eigenvector becomes

$$\begin{bmatrix} t \\ t \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

The following table gives a summary of this result. It shows for each eigenvalue the algebraic multiplicity m, and its geometric multiplicity k and the eigenvectors associated with the eigenvalue. If $m > k$ then the eigenvalue is defective which means the number of normal linearly independent eigenvectors associated with this eigenvalue (called the geometric multiplicity k) does not equal the algebraic multiplicity m, and we need to determine an additional $m - k$ generalized eigenvectors for this eigenvalue.

	multiplicity			
eigenvalue	algebraic m	geometric k	defective?	eigenvectors
2	1	1	No	$\begin{bmatrix} 1 \\ 1 \end{bmatrix}$
-3	1	1	No	$\begin{bmatrix} -\frac{1}{4} \\ 1 \end{bmatrix}$

Now that we found the eigenvalues and associated eigenvectors, we will go over each eigenvalue and generate the solution basis. The only problem we need to take care of is if the eigenvalue is defective. Since eigenvalue 2 is real and distinct then the corresponding eigenvector solution is

$$\begin{aligned}\vec{x}_1(t) &= \vec{v}_1 e^{2t} \\ &= \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{2t}\end{aligned}$$

Since eigenvalue -3 is real and distinct then the corresponding eigenvector solution is

$$\begin{aligned}\vec{x}_2(t) &= \vec{v}_2 e^{-3t} \\ &= \begin{bmatrix} -\frac{1}{4} \\ 1 \end{bmatrix} e^{-3t}\end{aligned}$$

Therefore the homogeneous solution is

$$\vec{x}_h(t) = c_1 \vec{x}_1(t) + c_2 \vec{x}_2(t)$$

Which is written as

$$\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = c_1 \begin{bmatrix} \mathrm{e}^{2t} \\ \mathrm{e}^{2t} \end{bmatrix} + c_2 \begin{bmatrix} -\frac{\mathrm{e}^{-3t}}{4} \\ \mathrm{e}^{-3t} \end{bmatrix}$$

Now that we found homogeneous solution above, we need to find a particular solution $\vec{x}_p(t)$. We will use Variation of parameters. The fundamental matrix is

$$\Phi = \begin{bmatrix} \vec{x}_1 & \vec{x}_2 & \cdots \end{bmatrix}$$

Where $\vec{x}_i$ are the solution basis found above. Therefore the fundamental matrix is

$$\Phi(t) = \begin{bmatrix} \mathrm{e}^{2t} & -\frac{\mathrm{e}^{-3t}}{4} \\ \mathrm{e}^{2t} & \mathrm{e}^{-3t} \end{bmatrix}$$

The particular solution is then given by

$$\vec{x}_p(t) = \Phi \int \Phi^{-1} \vec{G}(t)\, dt$$

But

$$\Phi^{-1} = \begin{bmatrix} \frac{4\,e^{-2t}}{5} & \frac{e^{-2t}}{5} \\ -\frac{4\,e^{3t}}{5} & \frac{4\,e^{3t}}{5} \end{bmatrix}$$

Hence

$$\begin{aligned}
\vec{x}_p(t) &= \begin{bmatrix} e^{2t} & -\frac{e^{-3t}}{4} \\ e^{2t} & e^{-3t} \end{bmatrix} \int \begin{bmatrix} \frac{4\,e^{-2t}}{5} & \frac{e^{-2t}}{5} \\ -\frac{4\,e^{3t}}{5} & \frac{4\,e^{3t}}{5} \end{bmatrix} \begin{bmatrix} e^{-2t} \\ -2\,e^{t} \end{bmatrix} dt \\
&= \begin{bmatrix} e^{2t} & -\frac{e^{-3t}}{4} \\ e^{2t} & e^{-3t} \end{bmatrix} \int \begin{bmatrix} \frac{4\,e^{-4t}}{5} - \frac{2\,e^{-t}}{5} \\ -\frac{4\,e^{t}}{5} - \frac{8\,e^{4t}}{5} \end{bmatrix} dt \\
&= \begin{bmatrix} e^{2t} & -\frac{e^{-3t}}{4} \\ e^{2t} & e^{-3t} \end{bmatrix} \begin{bmatrix} -\frac{e^{-4t}}{5} + \frac{2\,e^{-t}}{5} \\ -\frac{4\,e^{t}}{5} - \frac{2\,e^{4t}}{5} \end{bmatrix} \\
&= \begin{bmatrix} \frac{e^{t}}{2} \\ -e^{-2t} \end{bmatrix}
\end{aligned}$$

Now that we found particular solution, the final solution is

$$\vec{x}(t) = \vec{x}_h(t) + \vec{x}_p(t)$$

$$\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = \begin{bmatrix} c_1 e^{2t} \\ c_1 e^{2t} \end{bmatrix} + \begin{bmatrix} -\frac{c_2 e^{-3t}}{4} \\ c_2 e^{-3t} \end{bmatrix} + \begin{bmatrix} \frac{e^{t}}{2} \\ -e^{-2t} \end{bmatrix}$$

Which becomes

$$\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = \begin{bmatrix} \frac{(4c_1 e^{5t} + 2\,e^{4t} - c_2)\,e^{-3t}}{4} \\ (c_1 e^{5t} - e^{t} + c_2)\,e^{-3t} \end{bmatrix}$$

18.4.3 Maple step by step solution

Let's solve

$$\left[x_1'(t) = x_1(t) + x_2(t) + \frac{1}{(e^{t})^2}, x_2'(t) = 4x_1(t) - 2x_2(t) - 2\,e^{t}\right]$$

- Define vector

$$\underline{\vec{x}}(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}$$

- Convert system into a vector equation

$$\underline{\vec{x}}'(t) = \begin{bmatrix} 1 & 1 \\ 4 & -2 \end{bmatrix} \cdot \underline{\vec{x}}(t) + \begin{bmatrix} \frac{x_1(t)(e^t)^2 + x_2(t)(e^t)^2 + 1}{(e^t)^2} - x_1(t) - x_2(t) \\ -2\,e^t \end{bmatrix}$$

- System to solve

$$\underline{\vec{x}}'(t) = \begin{bmatrix} 1 & 1 \\ 4 & -2 \end{bmatrix} \cdot \underline{\vec{x}}(t) + \begin{bmatrix} 0 \\ -2\,e^t \end{bmatrix}$$

- Define the forcing function

$$\vec{f}(t) = \begin{bmatrix} 0 \\ -2\,e^t \end{bmatrix}$$

- Define the coefficient matrix

$$A = \begin{bmatrix} 1 & 1 \\ 4 & -2 \end{bmatrix}$$

- Rewrite the system as

$$\underline{\vec{x}}'(t) = A \cdot \underline{\vec{x}}(t) + \vec{f}$$

- To solve the system, find the eigenvalues and eigenvectors of A
- Eigenpairs of A

$$\left[\left[-3, \begin{bmatrix} -\frac{1}{4} \\ 1 \end{bmatrix}\right], \left[2, \begin{bmatrix} 1 \\ 1 \end{bmatrix}\right]\right]$$

- Consider eigenpair

$$\left[-3, \begin{bmatrix} -\frac{1}{4} \\ 1 \end{bmatrix}\right]$$

- Solution to homogeneous system from eigenpair

$$\underline{\vec{x}}_1 = e^{-3t} \cdot \begin{bmatrix} -\frac{1}{4} \\ 1 \end{bmatrix}$$

- Consider eigenpair

$$\left[2, \begin{bmatrix}1\\1\end{bmatrix}\right]$$

- Solution to homogeneous system from eigenpair

$$\underline{\vec{x}}_{2} = e^{2t} \cdot \begin{bmatrix}1\\1\end{bmatrix}$$

- General solution of the system of ODEs can be written in terms of the particular solution $\underline{\vec{x}}$

$$\underline{\vec{x}}(t) = c_1 \underline{\vec{x}}_{1} + c_2 \underline{\vec{x}}_{2} + \underline{\vec{x}}_{p}(t)$$

- ☐ Fundamental matrix
  - Let $\phi(t)$ be the matrix whose columns are the independent solutions of the homogeneous syste

$$\phi(t) = \begin{bmatrix} -\frac{e^{-3t}}{4} & e^{2t} \\ e^{-3t} & e^{2t} \end{bmatrix}$$

  - The fundamental matrix, $\Phi(t)$ is a normalized version of $\phi(t)$ satisfying $\Phi(0) = I$ where I is th

$$\Phi(t) = \phi(t) \cdot \frac{1}{\phi(0)}$$

  - Substitute the value of $\phi(t)$ and $\phi(0)$

$$\Phi(t) = \begin{bmatrix} -\frac{e^{-3t}}{4} & e^{2t} \\ e^{-3t} & e^{2t} \end{bmatrix} \cdot \frac{1}{\begin{bmatrix} -\frac{1}{4} & 1 \\ 1 & 1 \end{bmatrix}}$$

  - Evaluate and simplify to get the fundamental matrix

$$\Phi(t) = \begin{bmatrix} \frac{(4\,e^{5t}+1)e^{-3t}}{5} & \frac{(e^{5t}-1)e^{-3t}}{5} \\ \frac{4(e^{5t}-1)e^{-3t}}{5} & \frac{(e^{5t}+4)e^{-3t}}{5} \end{bmatrix}$$

- ☐ Find a particular solution of the system of ODEs using variation of parameters
  - Let the particular solution be the fundamental matrix multiplied by $\vec{v}(t)$ and solve for $\vec{v}(t)$

$$\underline{\vec{x}}_{p}(t) = \Phi(t) \cdot \vec{v}(t)$$

  - Take the derivative of the particular solution

$$\underline{\vec{x}}'_{p}(t) = \Phi'(t) \cdot \vec{v}(t) + \Phi(t) \cdot \vec{v}'(t)$$

  - Substitute particular solution and its derivative into the system of ODEs

$\Phi'(t) \cdot \vec{v}(t) + \Phi(t) \cdot \vec{v}'(t) = A \cdot \Phi(t) \cdot \vec{v}(t) + \vec{f}(t)$

- The fundamental matrix has columns that are solutions to the homogeneous system so its der

  $A \cdot \Phi(t) \cdot \vec{v}(t) + \Phi(t) \cdot \vec{v}'(t) = A \cdot \Phi(t) \cdot \vec{v}(t) + \vec{f}(t)$

- Cancel like terms

  $\Phi(t) \cdot \vec{v}'(t) = \vec{f}(t)$

- Multiply by the inverse of the fundamental matrix

  $\vec{v}'(t) = \frac{1}{\Phi(t)} \cdot \vec{f}(t)$

- Integrate to solve for $\vec{v}(t)$

  $\vec{v}(t) = \int_0^t \frac{1}{\Phi(s)} \cdot \vec{f}(s)\, ds$

- Plug $\vec{v}(t)$ into the equation for the particular solution

  $\underline{\vec{x}}_p(t) = \Phi(t) \cdot \left(\int_0^t \frac{1}{\Phi(s)} \cdot \vec{f}(s)\, ds\right)$

- Plug in the fundamental matrix and the forcing function and compute

  $\underline{\vec{x}}_p(t) = \left[\begin{array}{c} -\frac{(4\,\mathrm{e}^{5t}-5\,\mathrm{e}^{4t}+1)\mathrm{e}^{-3t}}{10} \\ -\frac{2(\mathrm{e}^{5t}-1)\mathrm{e}^{-3t}}{5} \end{array}\right]$

- Plug particular solution back into general solution

  $\underline{\vec{x}}(t) = c_1 \underline{\vec{x}}_1 + c_2 \underline{\vec{x}}_2 + \left[\begin{array}{c} -\frac{(4\,\mathrm{e}^{5t}-5\,\mathrm{e}^{4t}+1)\mathrm{e}^{-3t}}{10} \\ -\frac{2(\mathrm{e}^{5t}-1)\mathrm{e}^{-3t}}{5} \end{array}\right]$

- Substitute in vector of dependent variables

  $\left[\begin{array}{c} x_1(t) \\ x_2(t) \end{array}\right] = \left[\begin{array}{c} \frac{(20c_2\mathrm{e}^{5t}-8\,\mathrm{e}^{5t}+10\,\mathrm{e}^{4t}-5c_1-2)\mathrm{e}^{-3t}}{20} \\ \frac{(5c_2\mathrm{e}^{5t}-2\,\mathrm{e}^{5t}+5c_1+2)\mathrm{e}^{-3t}}{5} \end{array}\right]$

- Solution to the system of ODEs

  $\left\{x_1(t) = \frac{(20c_2\mathrm{e}^{5t}-8\,\mathrm{e}^{5t}+10\,\mathrm{e}^{4t}-5c_1-2)\mathrm{e}^{-3t}}{20}, x_2(t) = \frac{(5c_2\mathrm{e}^{5t}-2\,\mathrm{e}^{5t}+5c_1+2)\mathrm{e}^{-3t}}{5}\right\}$

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 45

```
dsolve([diff(x__1(t),t)=1*x__1(t)+1*x__2(t)+exp(-2*t),diff(x__2(t),t)=4*x__1(t)-2*x__2(t)-2*
```

$$x_1(t) = c_2 e^{2t} - \frac{c_1 e^{-3t}}{4} + \frac{e^t}{2}$$
$$x_2(t) = c_2 e^{2t} + c_1 e^{-3t} - e^{-2t}$$

✓ Solution by Mathematica

Time used: 0.585 (sec). Leaf size: 84

```
DSolve[{x1'[t]==1*x1[t]+1*x2[t]+Exp[-2*t],x2'[t]==4*x1[t]-2*x2[t]-2*Exp[t]},{x1[t],x2[t]},t,
```

$$\text{x1}(t) \to \frac{e^t}{2} + \frac{1}{5}(c_1 - c_2)e^{-3t} + \frac{1}{5}(4c_1 + c_2)e^{2t}$$
$$\text{x2}(t) \to \frac{1}{5}e^{-3t}\left(-5e^t + (4c_1 + c_2)e^{5t} - 4c_1 + 4c_2\right)$$

18.5 problem 5

Internal problem ID [782]
Internal file name [`OUTPUT/782_Sunday_June_05_2022_01_49_34_AM_87329820/index.tex`]

Book: Elementary differential equations and boundary value problems, 10th ed., Boyce and DiPrima
Section: Chapter 7.9, Nonhomogeneous Linear Systems. page 447
Problem number: 5.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"system of linear ODEs"**

Solve

$$x_1'(t) = 4x_1(t) - 2x_2(t) + \frac{1}{t^3}$$
$$x_2'(t) = 8x_1(t) - 4x_2(t) - \frac{1}{t^2}$$

18.5.1 Solution using Matrix exponential method

In this method, we will assume we have found the matrix exponential e^{At} allready. There are different methods to determine this but will not be shown here. This is a system of linear ODE's given as

$$\vec{x}'(t) = A\,\vec{x}(t) + \vec{G}(t)$$

Or

$$\begin{bmatrix} x_1'(t) \\ x_2'(t) \end{bmatrix} = \begin{bmatrix} 4 & -2 \\ 8 & -4 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} + \begin{bmatrix} \frac{1}{t^3} \\ -\frac{1}{t^2} \end{bmatrix}$$

Since the system is nonhomogeneous, then the solution is given by

$$\vec{x}(t) = \vec{x}_h(t) + \vec{x}_p(t)$$

Where $\vec{x}_h(t)$ is the homogeneous solution to $\vec{x}'(t) = A\,\vec{x}(t)$ and $\vec{x}_p(t)$ is a particular solution to $\vec{x}'(t) = A\,\vec{x}(t) + \vec{G}(t)$. The particular solution will be found using variation

of parameters method applied to the fundamental matrix. For the above matrix A, the matrix exponential can be found to be

$$e^{At} = \begin{bmatrix} 4t+1 & -2t \\ 8t & 1-4t \end{bmatrix}$$

Therefore the homogeneous solution is

$$\begin{aligned}
\vec{x}_h(t) &= e^{At}\vec{c} \\
&= \begin{bmatrix} 4t+1 & -2t \\ 8t & 1-4t \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} \\
&= \begin{bmatrix} (4t+1)\,c_1 - 2tc_2 \\ 8tc_1 + (1-4t)\,c_2 \end{bmatrix} \\
&= \begin{bmatrix} (4c_1 - 2c_2)\,t + c_1 \\ (8c_1 - 4c_2)\,t + c_2 \end{bmatrix}
\end{aligned}$$

The particular solution given by

$$\vec{x}_p(t) = e^{At} \int e^{-At} \vec{G}(t)\, dt$$

But

$$\begin{aligned}
e^{-At} &= (e^{At})^{-1} \\
&= \begin{bmatrix} 1-4t & 2t \\ -8t & 4t+1 \end{bmatrix}
\end{aligned}$$

Hence

$$\begin{aligned}
\vec{x}_p(t) &= \begin{bmatrix} 4t+1 & -2t \\ 8t & 1-4t \end{bmatrix} \int \begin{bmatrix} 1-4t & 2t \\ -8t & 4t+1 \end{bmatrix} \begin{bmatrix} \frac{1}{t^3} \\ -\frac{1}{t^2} \end{bmatrix} dt \\
&= \begin{bmatrix} 4t+1 & -2t \\ 8t & 1-4t \end{bmatrix} \begin{bmatrix} \frac{4}{t} - \frac{1}{2t^2} - 2\ln(t) \\ \frac{9}{t} - 4\ln(t) \end{bmatrix} \\
&= \begin{bmatrix} \frac{-2t^2 \ln(t) - 2\left(t-\frac{1}{2}\right)^2}{t^2} \\ \frac{-4\ln(t)t - 4t + 5}{t} \end{bmatrix}
\end{aligned}$$

Hence the complete solution is

$$\vec{x}(t) = \vec{x}_h(t) + \vec{x}_p(t)$$
$$= \begin{bmatrix} (4c_1 - 2c_2)\, t + c_1 + \frac{-2t^2 \ln(t) - 2\left(t - \frac{1}{2}\right)^2}{t^2} \\ (8c_1 - 4c_2)\, t + c_2 + \frac{-4\ln(t)t - 4t + 5}{t} \end{bmatrix}$$

18.5.2 Solution using explicit Eigenvalue and Eigenvector method

This is a system of linear ODE's given as

$$\vec{x}'(t) = A\,\vec{x}(t) + \vec{G}(t)$$

Or

$$\begin{bmatrix} x_1'(t) \\ x_2'(t) \end{bmatrix} = \begin{bmatrix} 4 & -2 \\ 8 & -4 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} + \begin{bmatrix} \frac{1}{t^3} \\ -\frac{1}{t^2} \end{bmatrix}$$

Since the system is nonhomogeneous, then the solution is given by

$$\vec{x}(t) = \vec{x}_h(t) + \vec{x}_p(t)$$

Where $\vec{x}_h(t)$ is the homogeneous solution to $\vec{x}'(t) = A\,\vec{x}(t)$ and $\vec{x}_p(t)$ is a particular solution to $\vec{x}'(t) = A\,\vec{x}(t) + \vec{G}(t)$. The particular solution will be found using variation of parameters method applied to the fundamental matrix.

The first step is find the homogeneous solution. We start by finding the eigenvalues of A. This is done by solving the following equation for the eigenvalues λ

$$\det\left(A - \lambda I\right) = 0$$

Expanding gives

$$\det\left(\begin{bmatrix} 4 & -2 \\ 8 & -4 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right) = 0$$

Therefore

$$\det\left(\begin{bmatrix} 4-\lambda & -2 \\ 8 & -4-\lambda \end{bmatrix}\right) = 0$$

Which gives the characteristic equation

$$\lambda^2 = 0$$

The roots of the above are the eigenvalues.

$$\lambda_1 = 0$$

This table summarises the above result

eigenvalue	algebraic multiplicity	type of eigenvalue
0	1	real eigenvalue

Now the eigenvector for each eigenvalue are found.

Considering the eigenvalue $\lambda_1 = 0$

We need to solve $A\vec{v} = \lambda\vec{v}$ or $(A - \lambda I)\vec{v} = \vec{0}$ which becomes

$$\left(\begin{bmatrix} 4 & -2 \\ 8 & -4 \end{bmatrix} - (0)\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right)\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 4 & -2 \\ 8 & -4 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Now forward elimination is applied to solve for the eigenvector $\vec{v}$. The augmented matrix is

$$\left[\begin{array}{cc|c} 4 & -2 & 0 \\ 8 & -4 & 0 \end{array}\right]$$

$$R_2 = R_2 - 2R_1 \Longrightarrow \left[\begin{array}{cc|c} 4 & -2 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

Therefore the system in Echelon form is

$$\begin{bmatrix} 4 & -2 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

The free variables are $\{v_2\}$ and the leading variables are $\{v_1\}$. Let $v_2 = t$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\{v_1 = \frac{t}{2}\}$

Hence the solution is

$$\begin{bmatrix} \frac{t}{2} \\ t \end{bmatrix} = \begin{bmatrix} \frac{t}{2} \\ t \end{bmatrix}$$

Since there is one free Variable, we have found one eigenvector associated with this eigenvalue. The above can be written as

$$\begin{bmatrix} \frac{t}{2} \\ t \end{bmatrix} = t \begin{bmatrix} \frac{1}{2} \\ 1 \end{bmatrix}$$

Let $t = 1$ the eigenvector becomes

$$\begin{bmatrix} \frac{t}{2} \\ t \end{bmatrix} = \begin{bmatrix} \frac{1}{2} \\ 1 \end{bmatrix}$$

Which is normalized to

$$\begin{bmatrix} \frac{t}{2} \\ t \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

The following table gives a summary of this result. It shows for each eigenvalue the algebraic multiplicity m, and its geometric multiplicity k and the eigenvectors associated with the eigenvalue. If $m > k$ then the eigenvalue is defective which means the number of normal linearly independent eigenvectors associated with this eigenvalue (called the geometric multiplicity k) does not equal the algebraic multiplicity m, and we need to determine an additional $m - k$ generalized eigenvectors for this eigenvalue.

	multiplicity			
eigenvalue	algebraic m	geometric k	defective?	eigenvectors
0	2	1	Yes	$\begin{bmatrix} \frac{1}{2} \\ 1 \end{bmatrix}$

Now that we found the eigenvalues and associated eigenvectors, we will go over each eigenvalue and generate the solution basis. The only problem we need to take care of is if the eigenvalue is defective. eigenvalue 0 is real and repated eigenvalue of multiplicity 2.There are two possible cases that can happen. This is illustrated in this diagram

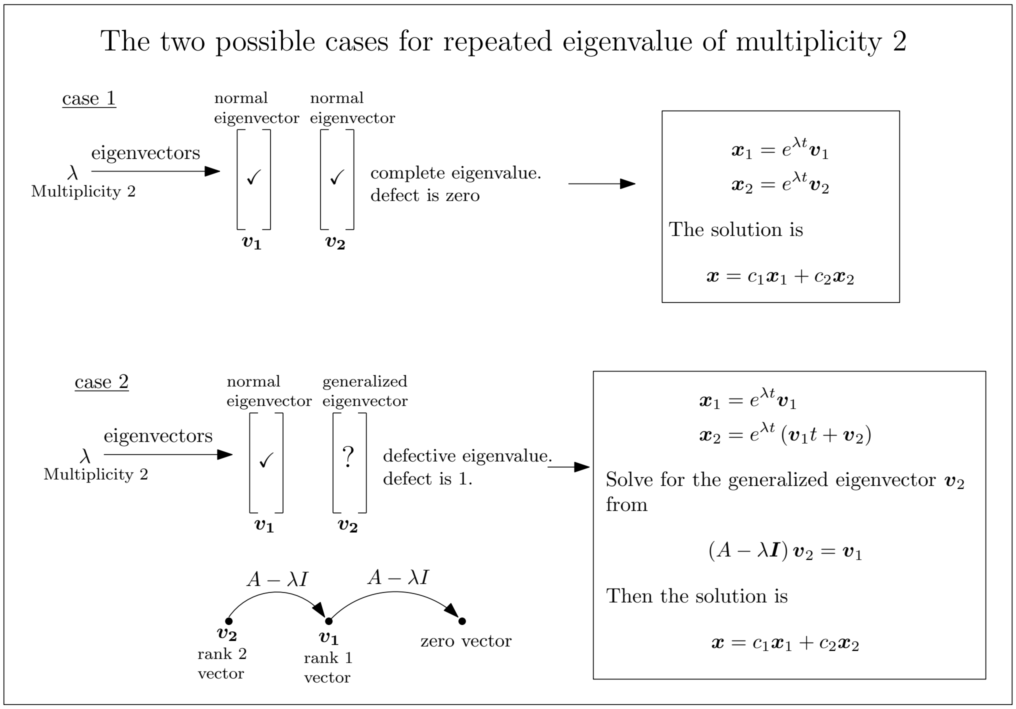


Figure 538: Possible case for repeated λ of multiplicity 2

This eigenvalue has algebraic multiplicity of 2, and geometric multiplicity 1, therefore this is defective eigenvalue. The defect is 1. This falls into case 2 shown above. We need to generate the missing additonal generalized eigevector $\vec{v}_2$ by solving

$$(A - \lambda I)\,\vec{v}_2 = \vec{v}_1$$

Where $\vec{v}_1$ is the normal (rank 1) eigenvector found above. Hence we need to solve

$$\left(\begin{bmatrix} 4 & -2 \\ 8 & -4 \end{bmatrix} - (0)\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right)\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} \frac{1}{2} \\ 1 \end{bmatrix}$$
$$\begin{bmatrix} 4 & -2 \\ 8 & -4 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} \frac{1}{2} \\ 1 \end{bmatrix}$$

Solving for $\vec{v}_2$ gives

$$\vec{v}_2 = \begin{bmatrix} 1 \\ \frac{7}{4} \end{bmatrix}$$

We have found two generalized eigenvectors for eigenvalue 0. Therefore the two basis solution associated with this eigenvalue are

$$\begin{aligned}\vec{x}_1(t) &= \vec{v}_1 e^{\lambda t}\\ &= \begin{bmatrix}\frac{1}{2}\\ 1\end{bmatrix} 1\\ &= \begin{bmatrix}\frac{1}{2}\\ 1\end{bmatrix}\end{aligned}$$

And

$$\begin{aligned}\vec{x}_2(t) &= (\vec{v}_1 t + \vec{v}_2)\, e^{\lambda t}\\ &= \left(\begin{bmatrix}\frac{1}{2}\\ 1\end{bmatrix} t + \begin{bmatrix}1\\ \frac{7}{4}\end{bmatrix}\right) 1\\ &= \begin{bmatrix}\frac{t}{2}+1\\ t+\frac{7}{4}\end{bmatrix}\end{aligned}$$

Therefore the homogeneous solution is

$$\vec{x}_h(t) = c_1 \vec{x}_1(t) + c_2 \vec{x}_2(t)$$

Which is written as

$$\begin{bmatrix}x_1(t)\\ x_2(t)\end{bmatrix} = c_1 \begin{bmatrix}\frac{1}{2}\\ 1\end{bmatrix} + c_2 \begin{bmatrix}\frac{t}{2}+1\\ t+\frac{7}{4}\end{bmatrix}$$

Now that we found homogeneous solution above, we need to find a particular solution $\vec{x}_p(t)$. We will use Variation of parameters. The fundamental matrix is

$$\Phi = \begin{bmatrix}\vec{x}_1 & \vec{x}_2 & \cdots\end{bmatrix}$$

Where $\vec{x}_i$ are the solution basis found above. Therefore the fundamental matrix is

$$\Phi(t) = \begin{bmatrix}\frac{1}{2} & \frac{t}{2}+1\\ 1 & t+\frac{7}{4}\end{bmatrix}$$

The particular solution is then given by

$$\vec{x}_p(t) = \Phi \int \Phi^{-1} \vec{G}(t)\, dt$$

But

$$\Phi^{-1} = \begin{bmatrix} -8t-14 & 8+4t \\ 8 & -4 \end{bmatrix}$$

Hence

$$\begin{aligned}
\vec{x}_p(t) &= \begin{bmatrix} \frac{1}{2} & \frac{t}{2}+1 \\ 1 & t+\frac{7}{4} \end{bmatrix} \int \begin{bmatrix} -8t-14 & 8+4t \\ 8 & -4 \end{bmatrix} \begin{bmatrix} \frac{1}{t^3} \\ -\frac{1}{t^2} \end{bmatrix} dt \\
&= \begin{bmatrix} \frac{1}{2} & \frac{t}{2}+1 \\ 1 & t+\frac{7}{4} \end{bmatrix} \int \begin{bmatrix} \frac{-4t^2-16t-14}{t^3} \\ \frac{8+4t}{t^3} \end{bmatrix} dt \\
&= \begin{bmatrix} \frac{1}{2} & \frac{t}{2}+1 \\ 1 & t+\frac{7}{4} \end{bmatrix} \begin{bmatrix} \frac{16}{t}+\frac{7}{t^2}-4\ln(t) \\ \frac{-4t-4}{t^2} \end{bmatrix} \\
&= \begin{bmatrix} \frac{-2t^2\ln(t)-2\left(t-\frac{1}{2}\right)^2}{t^2} \\ \frac{-4\ln(t)t-4t+5}{t} \end{bmatrix}
\end{aligned}$$

Now that we found particular solution, the final solution is

$$\vec{x}(t) = \vec{x}_h(t) + \vec{x}_p(t)$$

$$\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = \begin{bmatrix} \frac{c_1}{2} \\ c_1 \end{bmatrix} + \begin{bmatrix} c_2\left(\frac{t}{2}+1\right) \\ c_2\left(t+\frac{7}{4}\right) \end{bmatrix} + \begin{bmatrix} \frac{-2t^2\ln(t)-2\left(t-\frac{1}{2}\right)^2}{t^2} \\ \frac{-4\ln(t)t-4t+5}{t} \end{bmatrix}$$

Which becomes

$$\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = \begin{bmatrix} \frac{-4t^2\ln(t)-1+c_2t^3+(c_1+2c_2-4)t^2+4t}{2t^2} \\ c_1+c_2\left(t+\frac{7}{4}\right)+\frac{-4\ln(t)t-4t+5}{t} \end{bmatrix}$$

18.5.3 Maple step by step solution

Let's solve

$\left[x_1'(t) = 4x_1(t) - 2x_2(t) + \frac{1}{t^3}, x_2'(t) = 8x_1(t) - 4x_2(t) - \frac{1}{t^2}\right]$

- Define vector

$$\underline{x}^{\rightarrow}(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}$$

- Convert system into a vector equation

$$\underline{x}^{\rightarrow\prime}(t) = \begin{bmatrix} 4 & -2 \\ 8 & -4 \end{bmatrix} \cdot \underline{x}^{\rightarrow}(t) + \begin{bmatrix} \frac{4x_1(t)t^3-2x_2(t)t^3+1}{t^3} - 4x_1(t) + 2x_2(t) \\ \frac{8x_1(t)t^2-4x_2(t)t^2-1}{t^2} - 8x_1(t) + 4x_2(t) \end{bmatrix}$$

- System to solve

$$\underline{x}^{\rightarrow\prime}(t) = \begin{bmatrix} 4 & -2 \\ 8 & -4 \end{bmatrix} \cdot \underline{x}^{\rightarrow}(t)$$

- Define the coefficient matrix

$$A = \begin{bmatrix} 4 & -2 \\ 8 & -4 \end{bmatrix}$$

- Rewrite the system as

$$\underline{x}^{\rightarrow\prime}(t) = A \cdot \underline{x}^{\rightarrow}(t)$$

- To solve the system, find the eigenvalues and eigenvectors of A
- Eigenpairs of A

$$\left[\left[0, \begin{bmatrix} \frac{1}{2} \\ 1 \end{bmatrix}\right], \left[0, \begin{bmatrix} 0 \\ 0 \end{bmatrix}\right]\right]$$

- Consider eigenpair

$$\left[0, \begin{bmatrix} \frac{1}{2} \\ 1 \end{bmatrix}\right]$$

- Solution to homogeneous system from eigenpair

$$\underline{x}^{\rightarrow}_{1} = \begin{bmatrix} \frac{1}{2} \\ 1 \end{bmatrix}$$

- Consider eigenpair

$$\left[0, \begin{bmatrix} 0 \\ 0 \end{bmatrix}\right]$$

- Solution to homogeneous system from eigenpair

$$\underline{x}^{\rightarrow}{}_{2} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

- General solution to the system of ODEs

$$\underline{x}^{\rightarrow} = c_1 \underline{x}^{\rightarrow}{}_{1} + c_2 \underline{x}^{\rightarrow}{}_{2}$$

- Substitute solutions into the general solution

$$\underline{x}^{\rightarrow} = \begin{bmatrix} \frac{c_1}{2} \\ c_1 \end{bmatrix}$$

- Substitute in vector of dependent variables

$$\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = \begin{bmatrix} \frac{c_1}{2} \\ c_1 \end{bmatrix}$$

- Solution to the system of ODEs

$$\left\{x_1(t) = \frac{c_1}{2}, x_2(t) = c_1\right\}$$

✓ Solution by Maple

Time used: 0.031 (sec). Leaf size: 47

```
dsolve([diff(x__1(t),t)=4*x__1(t)-2*x__2(t)+1/(t^3),diff(x__2(t),t)=8*x__1(t)-4*x__2(t)-1/(t
```

$$x_1(t) = -\frac{1}{2t^2} + \frac{2}{t} - 2\ln(t) + c_1 t + c_2$$

$$x_2(t) = 2c_1 t - 4\ln(t) - \frac{c_1}{2} + 2c_2 + \frac{5}{t}$$

✓ Solution by Mathematica

Time used: 0.006 (sec). Leaf size: 61

```
DSolve[{x1'[t]==4*x1[t]-2*x2[t]+1/(t^3),x2'[t]==8*x1[t]-4*x2[t]-1/(t^2)},{x1[t],x2[t]},t,Inc
```

$$\text{x1}(t) \to -\frac{1}{2t^2} + \frac{2}{t} - 2\log(t) + 4c_1 t - 2c_2 t - 2 + c_1$$

$$\text{x2}(t) \to \frac{5}{t} - 4\log(t) + 8c_1 t - 4c_2 t - 4 + c_2$$

18.6 problem 6

Internal problem ID [783]
Internal file name [`OUTPUT/783_Sunday_June_05_2022_01_49_35_AM_71046668/index.tex`]

Book: Elementary differential equations and boundary value problems, 10th ed., Boyce and DiPrima
Section: Chapter 7.9, Nonhomogeneous Linear Systems. page 447
Problem number: 6.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"system of linear ODEs"**

Solve

$$
\begin{aligned}
x_1'(t) &= -4x_1(t) + 2x_2(t) + \frac{1}{t} \\
x_2'(t) &= 2x_1(t) - x_2(t) + \frac{2}{t} + 4
\end{aligned}
$$

18.6.1 Solution using Matrix exponential method

In this method, we will assume we have found the matrix exponential e^{At} allready. There are different methods to determine this but will not be shown here. This is a system of linear ODE's given as

$$\vec{x}'(t) = A\,\vec{x}(t) + \vec{G}(t)$$

Or

$$\begin{bmatrix} x_1'(t) \\ x_2'(t) \end{bmatrix} = \begin{bmatrix} -4 & 2 \\ 2 & -1 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} + \begin{bmatrix} \frac{1}{t} \\ \frac{2}{t} + 4 \end{bmatrix}$$

Since the system is nonhomogeneous, then the solution is given by

$$\vec{x}(t) = \vec{x}_h(t) + \vec{x}_p(t)$$

Where $\vec{x}_h(t)$ is the homogeneous solution to $\vec{x}'(t) = A\,\vec{x}(t)$ and $\vec{x}_p(t)$ is a particular solution to $\vec{x}'(t) = A\,\vec{x}(t) + \vec{G}(t)$. The particular solution will be found using variation

of parameters method applied to the fundamental matrix. For the above matrix A, the matrix exponential can be found to be

$$e^{At} = \begin{bmatrix} \frac{4\,e^{-5t}}{5}+\frac{1}{5} & \frac{2}{5}-\frac{2\,e^{-5t}}{5} \\ \frac{2}{5}-\frac{2\,e^{-5t}}{5} & \frac{e^{-5t}}{5}+\frac{4}{5} \end{bmatrix}$$

Therefore the homogeneous solution is

$$\begin{aligned}
\vec{x}_h(t) &= e^{At}\vec{c} \\
&= \begin{bmatrix} \frac{4\,e^{-5t}}{5}+\frac{1}{5} & \frac{2}{5}-\frac{2\,e^{-5t}}{5} \\ \frac{2}{5}-\frac{2\,e^{-5t}}{5} & \frac{e^{-5t}}{5}+\frac{4}{5} \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} \\
&= \begin{bmatrix} \left(\frac{4\,e^{-5t}}{5}+\frac{1}{5}\right)c_1+\left(\frac{2}{5}-\frac{2\,e^{-5t}}{5}\right)c_2 \\ \left(\frac{2}{5}-\frac{2\,e^{-5t}}{5}\right)c_1+\left(\frac{e^{-5t}}{5}+\frac{4}{5}\right)c_2 \end{bmatrix} \\
&= \begin{bmatrix} \frac{(4c_1-2c_2)e^{-5t}}{5}+\frac{c_1}{5}+\frac{2c_2}{5} \\ \frac{(-2c_1+c_2)e^{-5t}}{5}+\frac{2c_1}{5}+\frac{4c_2}{5} \end{bmatrix}
\end{aligned}$$

The particular solution given by

$$\vec{x}_p(t) = e^{At}\int e^{-At}\vec{G}(t)\,dt$$

But

$$\begin{aligned}
e^{-At} &= (e^{At})^{-1} \\
&= \begin{bmatrix} \frac{4\,e^{5t}}{5}+\frac{1}{5} & -\frac{2\,e^{5t}}{5}+\frac{2}{5} \\ -\frac{2\,e^{5t}}{5}+\frac{2}{5} & \frac{e^{5t}}{5}+\frac{4}{5} \end{bmatrix}
\end{aligned}$$

Hence

$$\begin{aligned}
\vec{x}_p(t) &= \begin{bmatrix} \frac{4\,e^{-5t}}{5}+\frac{1}{5} & \frac{2}{5}-\frac{2\,e^{-5t}}{5} \\ \frac{2}{5}-\frac{2\,e^{-5t}}{5} & \frac{e^{-5t}}{5}+\frac{4}{5} \end{bmatrix} \int \begin{bmatrix} \frac{4\,e^{5t}}{5}+\frac{1}{5} & -\frac{2\,e^{5t}}{5}+\frac{2}{5} \\ -\frac{2\,e^{5t}}{5}+\frac{2}{5} & \frac{e^{5t}}{5}+\frac{4}{5} \end{bmatrix} \begin{bmatrix} \frac{1}{t} \\ \frac{2}{t}+4 \end{bmatrix} dt \\
&= \begin{bmatrix} \frac{4\,e^{-5t}}{5}+\frac{1}{5} & \frac{2}{5}-\frac{2\,e^{-5t}}{5} \\ \frac{2}{5}-\frac{2\,e^{-5t}}{5} & \frac{e^{-5t}}{5}+\frac{4}{5} \end{bmatrix} \begin{bmatrix} \frac{8t}{5}+\ln(5)+\ln(t)-\frac{8\,e^{5t}}{25} \\ 2\ln(5)+2\ln(t)+\frac{16t}{5}+\frac{4\,e^{5t}}{25} \end{bmatrix} \\
&= \begin{bmatrix} -\frac{8}{25}+\frac{8t}{5}+\ln(5)+\ln(t) \\ \frac{16t}{5}+2\ln(5)+2\ln(t)+\frac{4}{25} \end{bmatrix}
\end{aligned}$$

Hence the complete solution is

$$\vec{x}(t) = \vec{x}_h(t) + \vec{x}_p(t)$$
$$= \begin{bmatrix} -\frac{8}{25} + \frac{2(2c_1-c_2)e^{-5t}}{5} + \frac{8t}{5} + \frac{c_1}{5} + \frac{2c_2}{5} + \ln(5) + \ln(t) \\ \frac{(-2c_1+c_2)e^{-5t}}{5} + \frac{2c_1}{5} + \frac{4c_2}{5} + \frac{16t}{5} + 2\ln(5) + 2\ln(t) + \frac{4}{25} \end{bmatrix}$$

18.6.2 Solution using explicit Eigenvalue and Eigenvector method

This is a system of linear ODE's given as

$$\vec{x}'(t) = A\,\vec{x}(t) + \vec{G}(t)$$

Or

$$\begin{bmatrix} x_1'(t) \\ x_2'(t) \end{bmatrix} = \begin{bmatrix} -4 & 2 \\ 2 & -1 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} + \begin{bmatrix} \frac{1}{t} \\ \frac{2}{t} + 4 \end{bmatrix}$$

Since the system is nonhomogeneous, then the solution is given by

$$\vec{x}(t) = \vec{x}_h(t) + \vec{x}_p(t)$$

Where $\vec{x}_h(t)$ is the homogeneous solution to $\vec{x}'(t) = A\,\vec{x}(t)$ and $\vec{x}_p(t)$ is a particular solution to $\vec{x}'(t) = A\,\vec{x}(t) + \vec{G}(t)$. The particular solution will be found using variation of parameters method applied to the fundamental matrix.

The first step is find the homogeneous solution. We start by finding the eigenvalues of A. This is done by solving the following equation for the eigenvalues λ

$$\det(A - \lambda I) = 0$$

Expanding gives

$$\det\left(\begin{bmatrix} -4 & 2 \\ 2 & -1 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right) = 0$$

Therefore

$$\det\left(\begin{bmatrix} -4-\lambda & 2 \\ 2 & -1-\lambda \end{bmatrix}\right) = 0$$

Which gives the characteristic equation

$$\lambda^2 + 5\lambda = 0$$

The roots of the above are the eigenvalues.

$$\lambda_1 = -5$$
$$\lambda_2 = 0$$

This table summarises the above result

eigenvalue	algebraic multiplicity	type of eigenvalue
0	1	real eigenvalue
-5	1	real eigenvalue

Now the eigenvector for each eigenvalue are found.

Considering the eigenvalue $\lambda_1 = -5$

We need to solve $A\vec{v} = \lambda\vec{v}$ or $(A - \lambda I)\vec{v} = \vec{0}$ which becomes

$$\left(\begin{bmatrix} -4 & 2 \\ 2 & -1 \end{bmatrix} - (-5)\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right)\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$
$$\begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Now forward elimination is applied to solve for the eigenvector $\vec{v}$. The augmented matrix is

$$\left[\begin{array}{cc|c} 1 & 2 & 0 \\ 2 & 4 & 0 \end{array}\right]$$

$$R_2 = R_2 - 2R_1 \Longrightarrow \left[\begin{array}{cc|c} 1 & 2 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

Therefore the system in Echelon form is

$$\begin{bmatrix} 1 & 2 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

The free variables are $\{v_2\}$ and the leading variables are $\{v_1\}$. Let $v_2 = t$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\{v_1 = -2t\}$

Hence the solution is

$$\begin{bmatrix} -2t \\ t \end{bmatrix} = \begin{bmatrix} -2t \\ t \end{bmatrix}$$

Since there is one free Variable, we have found one eigenvector associated with this eigenvalue. The above can be written as

$$\begin{bmatrix} -2t \\ t \end{bmatrix} = t \begin{bmatrix} -2 \\ 1 \end{bmatrix}$$

Let $t = 1$ the eigenvector becomes

$$\begin{bmatrix} -2t \\ t \end{bmatrix} = \begin{bmatrix} -2 \\ 1 \end{bmatrix}$$

Considering the eigenvalue $\lambda_2 = 0$

We need to solve $A\vec{v} = \lambda\vec{v}$ or $(A - \lambda I)\vec{v} = \vec{0}$ which becomes

$$\left(\begin{bmatrix} -4 & 2 \\ 2 & -1 \end{bmatrix} - (0) \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right) \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} -4 & 2 \\ 2 & -1 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Now forward elimination is applied to solve for the eigenvector $\vec{v}$. The augmented matrix is

$$\left[\begin{array}{cc|c} -4 & 2 & 0 \\ 2 & -1 & 0 \end{array}\right]$$

$$R_2 = R_2 + \frac{R_1}{2} \implies \left[\begin{array}{cc|c} -4 & 2 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

Therefore the system in Echelon form is

$$\begin{bmatrix} -4 & 2 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

The free variables are $\{v_2\}$ and the leading variables are $\{v_1\}$. Let $v_2 = t$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\left\{v_1 = \frac{t}{2}\right\}$

Hence the solution is

$$\begin{bmatrix} \frac{t}{2} \\ t \end{bmatrix} = \begin{bmatrix} \frac{t}{2} \\ t \end{bmatrix}$$

Since there is one free Variable, we have found one eigenvector associated with this eigenvalue. The above can be written as

$$\begin{bmatrix} \frac{t}{2} \\ t \end{bmatrix} = t \begin{bmatrix} \frac{1}{2} \\ 1 \end{bmatrix}$$

Let $t = 1$ the eigenvector becomes

$$\begin{bmatrix} \frac{t}{2} \\ t \end{bmatrix} = \begin{bmatrix} \frac{1}{2} \\ 1 \end{bmatrix}$$

Which is normalized to

$$\begin{bmatrix} \frac{t}{2} \\ t \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

The following table gives a summary of this result. It shows for each eigenvalue the algebraic multiplicity m, and its geometric multiplicity k and the eigenvectors associated with the eigenvalue. If $m > k$ then the eigenvalue is defective which means the number of normal linearly independent eigenvectors associated with this eigenvalue (called the geometric multiplicity k) does not equal the algebraic multiplicity m, and we need to determine an additional $m - k$ generalized eigenvectors for this eigenvalue.

	multiplicity			
eigenvalue	algebraic m	geometric k	defective?	eigenvectors
-5	1	1	No	$\begin{bmatrix} -2 \\ 1 \end{bmatrix}$
0	1	1	No	$\begin{bmatrix} \frac{1}{2} \\ 1 \end{bmatrix}$

Now that we found the eigenvalues and associated eigenvectors, we will go over each eigenvalue and generate the solution basis. The only problem we need to take care of is if the eigenvalue is defective. Since eigenvalue -5 is real and distinct then the

corresponding eigenvector solution is

$$\vec{x}_1(t) = \vec{v}_1 e^{-5t} = \begin{bmatrix} -2 \\ 1 \end{bmatrix} e^{-5t}$$

Since eigenvalue 0 is real and distinct then the corresponding eigenvector solution is

$$\vec{x}_2(t) = \vec{v}_2 e^{0} = \begin{bmatrix} \frac{1}{2} \\ 1 \end{bmatrix} e^{0}$$

Therefore the homogeneous solution is

$$\vec{x}_h(t) = c_1 \vec{x}_1(t) + c_2 \vec{x}_2(t)$$

Which is written as

$$\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = c_1 \begin{bmatrix} -2\,\mathrm{e}^{-5t} \\ \mathrm{e}^{-5t} \end{bmatrix} + c_2 \begin{bmatrix} \frac{1}{2} \\ 1 \end{bmatrix}$$

Now that we found homogeneous solution above, we need to find a particular solution $\vec{x}_p(t)$. We will use Variation of parameters. The fundamental matrix is

$$\Phi = \begin{bmatrix} \vec{x}_1 & \vec{x}_2 & \cdots \end{bmatrix}$$

Where $\vec{x}_i$ are the solution basis found above. Therefore the fundamental matrix is

$$\Phi(t) = \begin{bmatrix} -2\,\mathrm{e}^{-5t} & \frac{1}{2} \\ \mathrm{e}^{-5t} & 1 \end{bmatrix}$$

The particular solution is then given by

$$\vec{x}_p(t) = \Phi \int \Phi^{-1} \vec{G}(t)\,dt$$

But

$$\Phi^{-1} = \begin{bmatrix} -\frac{2\,\mathrm{e}^{5t}}{5} & \frac{\mathrm{e}^{5t}}{5} \\ \frac{2}{5} & \frac{4}{5} \end{bmatrix}$$

Hence

$$\vec{x}_p(t) = \begin{bmatrix} -2\,\mathrm{e}^{-5t} & \frac{1}{2} \\ \mathrm{e}^{-5t} & 1 \end{bmatrix} \int \begin{bmatrix} -\frac{2\,\mathrm{e}^{5t}}{5} & \frac{\mathrm{e}^{5t}}{5} \\ \frac{2}{5} & \frac{4}{5} \end{bmatrix} \begin{bmatrix} \frac{1}{t} \\ \frac{2}{t}+4 \end{bmatrix} dt$$

$$= \begin{bmatrix} -2\,\mathrm{e}^{-5t} & \frac{1}{2} \\ \mathrm{e}^{-5t} & 1 \end{bmatrix} \int \begin{bmatrix} \frac{4\,\mathrm{e}^{5t}}{5} \\ \frac{2}{t}+\frac{16}{5} \end{bmatrix} dt$$

$$= \begin{bmatrix} -2\,\mathrm{e}^{-5t} & \frac{1}{2} \\ \mathrm{e}^{-5t} & 1 \end{bmatrix} \begin{bmatrix} \frac{4\,\mathrm{e}^{5t}}{25} \\ \frac{16t}{5}+2\ln(t) \end{bmatrix}$$

$$= \begin{bmatrix} -\frac{8}{25}+\frac{8t}{5}+\ln(t) \\ \frac{4}{25}+\frac{16t}{5}+2\ln(t) \end{bmatrix}$$

Now that we found particular solution, the final solution is

$$\vec{x}(t) = \vec{x}_h(t) + \vec{x}_p(t)$$

$$\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = \begin{bmatrix} -2c_1\mathrm{e}^{-5t} \\ c_1\mathrm{e}^{-5t} \end{bmatrix} + \begin{bmatrix} \frac{c_2}{2} \\ c_2 \end{bmatrix} + \begin{bmatrix} -\frac{8}{25}+\frac{8t}{5}+\ln(t) \\ \frac{4}{25}+\frac{16t}{5}+2\ln(t) \end{bmatrix}$$

Which becomes

$$\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = \begin{bmatrix} -2c_1\mathrm{e}^{-5t}+\frac{c_2}{2}-\frac{8}{25}+\frac{8t}{5}+\ln(t) \\ c_1\mathrm{e}^{-5t}+c_2+\frac{4}{25}+\frac{16t}{5}+2\ln(t) \end{bmatrix}$$

18.6.3 Maple step by step solution

Let's solve

$\left[x_1'(t) = -4x_1(t)+2x_2(t)+\frac{1}{t}, x_2'(t) = 2x_1(t)-x_2(t)+\frac{2}{t}+4\right]$

- Define vector

$$\underline{x}^{\rightarrow}(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}$$

- Convert system into a vector equation

$$\underline{x}^{\rightarrow}{}'(t) = \begin{bmatrix} -4 & 2 \\ 2 & -1 \end{bmatrix} \cdot \underline{x}^{\rightarrow}(t) + \begin{bmatrix} -\frac{4x_1(t)t-2x_2(t)t-1}{t}+4x_1(t)-2x_2(t) \\ \frac{2x_1(t)t-x_2(t)t+4t+2}{t}-2x_1(t)+x_2(t) \end{bmatrix}$$

- System to solve

$$\underline{\vec{x}}'(t) = \begin{bmatrix} -4 & 2 \\ 2 & -1 \end{bmatrix} \cdot \underline{\vec{x}}(t)$$

- Define the coefficient matrix

$$A = \begin{bmatrix} -4 & 2 \\ 2 & -1 \end{bmatrix}$$

- Rewrite the system as

$$\underline{\vec{x}}'(t) = A \cdot \underline{\vec{x}}(t)$$

- To solve the system, find the eigenvalues and eigenvectors of A
- Eigenpairs of A

$$\left[\left[-5, \begin{bmatrix} -2 \\ 1 \end{bmatrix}\right], \left[0, \begin{bmatrix} \frac{1}{2} \\ 1 \end{bmatrix}\right]\right]$$

- Consider eigenpair

$$\left[-5, \begin{bmatrix} -2 \\ 1 \end{bmatrix}\right]$$

- Solution to homogeneous system from eigenpair

$$\underline{\vec{x}}_1 = \mathrm{e}^{-5t} \cdot \begin{bmatrix} -2 \\ 1 \end{bmatrix}$$

- Consider eigenpair

$$\left[0, \begin{bmatrix} \frac{1}{2} \\ 1 \end{bmatrix}\right]$$

- Solution to homogeneous system from eigenpair

$$\underline{\vec{x}}_2 = \begin{bmatrix} \frac{1}{2} \\ 1 \end{bmatrix}$$

- General solution to the system of ODEs

$$\underline{\vec{x}} = c_1 \underline{\vec{x}}_1 + c_2 \underline{\vec{x}}_2$$

- Substitute solutions into the general solution

$$\underline{x^{\rightarrow}} = c_1 \mathrm{e}^{-5t} \cdot \begin{bmatrix} -2 \\ 1 \end{bmatrix} + \begin{bmatrix} \frac{c_2}{2} \\ c_2 \end{bmatrix}$$

- Substitute in vector of dependent variables

$$\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = \begin{bmatrix} -2c_1 \mathrm{e}^{-5t} + \frac{c_2}{2} \\ c_1 \mathrm{e}^{-5t} + c_2 \end{bmatrix}$$

- Solution to the system of ODEs

$$\left\{x_1(t) = -2c_1 \mathrm{e}^{-5t} + \frac{c_2}{2}, x_2(t) = c_1 \mathrm{e}^{-5t} + c_2\right\}$$

✓ Solution by Maple

Time used: 0.031 (sec). Leaf size: 45

```
dsolve([diff(x__1(t),t)=-4*x__1(t)+2*x__2(t)+1/t,diff(x__2(t),t)=2*x__1(t)-1*x__2(t)+2/t+4],
```

$$x_1(t) = \ln(-5t) - \frac{c_1 \mathrm{e}^{-5t}}{5} + \frac{8t}{5} + c_2$$

$$x_2(t) = \frac{c_1 \mathrm{e}^{-5t}}{10} + 2\ln(-5t) + 2c_2 + \frac{16t}{5} + \frac{4}{5}$$

✓ Solution by Mathematica

Time used: 0.02 (sec). Leaf size: 86

```
DSolve[{x1'[t]==-4*x1[t]+2*x2[t]+1/t,x2'[t]==2*x1[t]-1*x2[t]+2/t+4},{x1[t],x2[t]},t,IncludeS
```

$$\mathrm{x1}(t) \rightarrow \frac{1}{25}\left(40t + 25\log(t) + 20c_1 e^{-5t} - 10c_2 e^{-5t} - 8 + 5c_1 + 10c_2\right)$$

$$\mathrm{x2}(t) \rightarrow \frac{1}{25}\left(80t + 50\log(t) - 10c_1 e^{-5t} + 5c_2 e^{-5t} + 4 + 10c_1 + 20c_2\right)$$

18.7 problem 7

Internal problem ID [784]
Internal file name [`OUTPUT/784_Sunday_June_05_2022_01_49_37_AM_33602493/index.tex`]

Book: Elementary differential equations and boundary value problems, 10th ed., Boyce and DiPrima
Section: Chapter 7.9, Nonhomogeneous Linear Systems. page 447
Problem number: 7.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"system of linear ODEs"**

Solve

$$x_1'(t) = x_1(t) + x_2(t) + 2\,\mathrm{e}^t$$
$$x_2'(t) = 4x_1(t) + x_2(t) - \mathrm{e}^t$$

18.7.1 Solution using Matrix exponential method

In this method, we will assume we have found the matrix exponential e^{At} allready. There are different methods to determine this but will not be shown here. This is a system of linear ODE's given as

$$\vec{x}'(t) = A\,\vec{x}(t) + \vec{G}(t)$$

Or

$$\begin{bmatrix} x_1'(t) \\ x_2'(t) \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 4 & 1 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} + \begin{bmatrix} 2\,\mathrm{e}^t \\ -\mathrm{e}^t \end{bmatrix}$$

Since the system is nonhomogeneous, then the solution is given by

$$\vec{x}(t) = \vec{x}_h(t) + \vec{x}_p(t)$$

Where $\vec{x}_h(t)$ is the homogeneous solution to $\vec{x}'(t) = A\,\vec{x}(t)$ and $\vec{x}_p(t)$ is a particular solution to $\vec{x}'(t) = A\,\vec{x}(t) + \vec{G}(t)$. The particular solution will be found using variation

of parameters method applied to the fundamental matrix. For the above matrix A, the matrix exponential can be found to be

$$e^{At} = \begin{bmatrix} \frac{e^{-t}}{2} + \frac{e^{3t}}{2} & \frac{e^{3t}}{4} - \frac{e^{-t}}{4} \\ e^{3t} - e^{-t} & \frac{e^{-t}}{2} + \frac{e^{3t}}{2} \end{bmatrix}$$

Therefore the homogeneous solution is

$$\begin{aligned} \vec{x}_h(t) &= e^{At}\vec{c} \\ &= \begin{bmatrix} \frac{e^{-t}}{2} + \frac{e^{3t}}{2} & \frac{e^{3t}}{4} - \frac{e^{-t}}{4} \\ e^{3t} - e^{-t} & \frac{e^{-t}}{2} + \frac{e^{3t}}{2} \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} \\ &= \begin{bmatrix} \left(\frac{e^{-t}}{2} + \frac{e^{3t}}{2}\right) c_1 + \left(\frac{e^{3t}}{4} - \frac{e^{-t}}{4}\right) c_2 \\ \left(e^{3t} - e^{-t}\right) c_1 + \left(\frac{e^{-t}}{2} + \frac{e^{3t}}{2}\right) c_2 \end{bmatrix} \\ &= \begin{bmatrix} \frac{(2c_1 - c_2)e^{-t}}{4} + \frac{\left(c_1 + \frac{c_2}{2}\right)e^{3t}}{2} \\ \frac{(-2c_1 + c_2)e^{-t}}{2} + \left(c_1 + \frac{c_2}{2}\right) e^{3t} \end{bmatrix} \end{aligned}$$

The particular solution given by

$$\vec{x}_p(t) = e^{At} \int e^{-At} \vec{G}(t)\, dt$$

But

$$\begin{aligned} e^{-At} &= \left(e^{At}\right)^{-1} \\ &= \begin{bmatrix} \frac{(e^{4t}+1)e^{-3t}}{2} & -\frac{(e^{4t}-1)e^{-3t}}{4} \\ -(e^{4t} - 1)\, e^{-3t} & \frac{(e^{4t}+1)e^{-3t}}{2} \end{bmatrix} \end{aligned}$$

Hence

$$\begin{aligned} \vec{x}_p(t) &= \begin{bmatrix} \frac{e^{-t}}{2} + \frac{e^{3t}}{2} & \frac{e^{3t}}{4} - \frac{e^{-t}}{4} \\ e^{3t} - e^{-t} & \frac{e^{-t}}{2} + \frac{e^{3t}}{2} \end{bmatrix} \int \begin{bmatrix} \frac{(e^{4t}+1)e^{-3t}}{2} & -\frac{(e^{4t}-1)e^{-3t}}{4} \\ -(e^{4t} - 1)\, e^{-3t} & \frac{(e^{4t}+1)e^{-3t}}{2} \end{bmatrix} \begin{bmatrix} 2\, e^{t} \\ -e^{t} \end{bmatrix} dt \\ &= \begin{bmatrix} \frac{e^{-t}}{2} + \frac{e^{3t}}{2} & \frac{e^{3t}}{4} - \frac{e^{-t}}{4} \\ e^{3t} - e^{-t} & \frac{e^{-t}}{2} + \frac{e^{3t}}{2} \end{bmatrix} \begin{bmatrix} \frac{5\, e^{2t}}{8} - \frac{3\, e^{-2t}}{8} \\ -\frac{5\, e^{2t}}{4} - \frac{3\, e^{-2t}}{4} \end{bmatrix} \\ &= \begin{bmatrix} \frac{e^{t}}{4} \\ -2\, e^{t} \end{bmatrix} \end{aligned}$$

Hence the complete solution is

$$\vec{x}(t) = \vec{x}_h(t) + \vec{x}_p(t)$$
$$= \begin{bmatrix} \frac{(2c_1-c_2)\mathrm{e}^{-t}}{4} + \frac{(2c_1+c_2)\mathrm{e}^{3t}}{4} + \frac{\mathrm{e}^t}{4} \\ \frac{(-2c_1+c_2)\mathrm{e}^{-t}}{2} + \frac{(2c_1+c_2)\mathrm{e}^{3t}}{2} - 2\,\mathrm{e}^t \end{bmatrix}$$

18.7.2 Solution using explicit Eigenvalue and Eigenvector method

This is a system of linear ODE's given as

$$\vec{x}'(t) = A\,\vec{x}(t) + \vec{G}(t)$$

Or

$$\begin{bmatrix} x_1'(t) \\ x_2'(t) \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 4 & 1 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} + \begin{bmatrix} 2\,\mathrm{e}^t \\ -\mathrm{e}^t \end{bmatrix}$$

Since the system is nonhomogeneous, then the solution is given by

$$\vec{x}(t) = \vec{x}_h(t) + \vec{x}_p(t)$$

Where $\vec{x}_h(t)$ is the homogeneous solution to $\vec{x}'(t) = A\,\vec{x}(t)$ and $\vec{x}_p(t)$ is a particular solution to $\vec{x}'(t) = A\,\vec{x}(t) + \vec{G}(t)$. The particular solution will be found using variation of parameters method applied to the fundamental matrix.

The first step is find the homogeneous solution. We start by finding the eigenvalues of A. This is done by solving the following equation for the eigenvalues λ

$$\det(A - \lambda I) = 0$$

Expanding gives

$$\det\left(\begin{bmatrix} 1 & 1 \\ 4 & 1 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right) = 0$$

Therefore

$$\det\left(\begin{bmatrix} 1-\lambda & 1 \\ 4 & 1-\lambda \end{bmatrix}\right) = 0$$

Which gives the characteristic equation

$$\lambda^2 - 2\lambda - 3 = 0$$

The roots of the above are the eigenvalues.

$$
\begin{aligned}
\lambda_1 &= 3 \\
\lambda_2 &= -1
\end{aligned}
$$

This table summarises the above result

eigenvalue	algebraic multiplicity	type of eigenvalue
-1	1	real eigenvalue
3	1	real eigenvalue

Now the eigenvector for each eigenvalue are found.

Considering the eigenvalue $\lambda_1 = -1$

We need to solve $A\vec{v} = \lambda\vec{v}$ or $(A - \lambda I)\vec{v} = \vec{0}$ which becomes

$$
\begin{aligned}
\left(\begin{bmatrix} 1 & 1 \\ 4 & 1 \end{bmatrix} - (-1)\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right)\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} &= \begin{bmatrix} 0 \\ 0 \end{bmatrix} \\
\begin{bmatrix} 2 & 1 \\ 4 & 2 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} &= \begin{bmatrix} 0 \\ 0 \end{bmatrix}
\end{aligned}
$$

Now forward elimination is applied to solve for the eigenvector $\vec{v}$. The augmented matrix is

$$
\left[\begin{array}{cc|c} 2 & 1 & 0 \\ 4 & 2 & 0 \end{array}\right]
$$

$$
R_2 = R_2 - 2R_1 \Longrightarrow \left[\begin{array}{cc|c} 2 & 1 & 0 \\ 0 & 0 & 0 \end{array}\right]
$$

Therefore the system in Echelon form is

$$
\begin{bmatrix} 2 & 1 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}
$$

The free variables are $\{v_2\}$ and the leading variables are $\{v_1\}$. Let $v_2 = t$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\left\{v_1 = -\frac{t}{2}\right\}$

Hence the solution is

$$\begin{bmatrix} -\frac{t}{2} \\ t \end{bmatrix} = \begin{bmatrix} -\frac{t}{2} \\ t \end{bmatrix}$$

Since there is one free Variable, we have found one eigenvector associated with this eigenvalue. The above can be written as

$$\begin{bmatrix} -\frac{t}{2} \\ t \end{bmatrix} = t \begin{bmatrix} -\frac{1}{2} \\ 1 \end{bmatrix}$$

Let $t = 1$ the eigenvector becomes

$$\begin{bmatrix} -\frac{t}{2} \\ t \end{bmatrix} = \begin{bmatrix} -\frac{1}{2} \\ 1 \end{bmatrix}$$

Which is normalized to

$$\begin{bmatrix} -\frac{t}{2} \\ t \end{bmatrix} = \begin{bmatrix} -1 \\ 2 \end{bmatrix}$$

Considering the eigenvalue $\lambda_2 = 3$

We need to solve $A\vec{v} = \lambda\vec{v}$ or $(A - \lambda I)\vec{v} = \vec{0}$ which becomes

$$\left(\begin{bmatrix} 1 & 1 \\ 4 & 1 \end{bmatrix} - (3) \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right) \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} -2 & 1 \\ 4 & -2 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Now forward elimination is applied to solve for the eigenvector $\vec{v}$. The augmented matrix is

$$\left[\begin{array}{cc|c} -2 & 1 & 0 \\ 4 & -2 & 0 \end{array}\right]$$

$$R_2 = R_2 + 2R_1 \Longrightarrow \left[\begin{array}{cc|c} -2 & 1 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

Therefore the system in Echelon form is

$$\begin{bmatrix} -2 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

The free variables are $\{v_2\}$ and the leading variables are $\{v_1\}$. Let $v_2 = t$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\{v_1 = \frac{t}{2}\}$

Hence the solution is

$$\begin{bmatrix} \frac{t}{2} \\ t \end{bmatrix} = \begin{bmatrix} \frac{t}{2} \\ t \end{bmatrix}$$

Since there is one free Variable, we have found one eigenvector associated with this eigenvalue. The above can be written as

$$\begin{bmatrix} \frac{t}{2} \\ t \end{bmatrix} = t \begin{bmatrix} \frac{1}{2} \\ 1 \end{bmatrix}$$

Let $t = 1$ the eigenvector becomes

$$\begin{bmatrix} \frac{t}{2} \\ t \end{bmatrix} = \begin{bmatrix} \frac{1}{2} \\ 1 \end{bmatrix}$$

Which is normalized to

$$\begin{bmatrix} \frac{t}{2} \\ t \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

The following table gives a summary of this result. It shows for each eigenvalue the algebraic multiplicity m, and its geometric multiplicity k and the eigenvectors associated with the eigenvalue. If $m > k$ then the eigenvalue is defective which means the number of normal linearly independent eigenvectors associated with this eigenvalue (called the geometric multiplicity k) does not equal the algebraic multiplicity m, and we need to determine an additional $m - k$ generalized eigenvectors for this eigenvalue.

eigenvalue	multiplicity		defective?	eigenvectors
	algebraic m	geometric k		
3	1	1	No	$\begin{bmatrix} \frac{1}{2} \\ 1 \end{bmatrix}$
-1	1	1	No	$\begin{bmatrix} -\frac{1}{2} \\ 1 \end{bmatrix}$

Now that we found the eigenvalues and associated eigenvectors, we will go over each eigenvalue and generate the solution basis. The only problem we need to take care of is if the eigenvalue is defective. Since eigenvalue 3 is real and distinct then the corresponding eigenvector solution is

$$
\begin{aligned}
\vec{x}_1(t) &= \vec{v}_1 e^{3t} \\
&= \begin{bmatrix} \frac{1}{2} \\ 1 \end{bmatrix} e^{3t}
\end{aligned}
$$

Since eigenvalue -1 is real and distinct then the corresponding eigenvector solution is

$$
\begin{aligned}
\vec{x}_2(t) &= \vec{v}_2 e^{-t} \\
&= \begin{bmatrix} -\frac{1}{2} \\ 1 \end{bmatrix} e^{-t}
\end{aligned}
$$

Therefore the homogeneous solution is

$$
\vec{x}_h(t) = c_1 \vec{x}_1(t) + c_2 \vec{x}_2(t)
$$

Which is written as

$$
\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = c_1 \begin{bmatrix} \frac{e^{3t}}{2} \\ e^{3t} \end{bmatrix} + c_2 \begin{bmatrix} -\frac{e^{-t}}{2} \\ e^{-t} \end{bmatrix}
$$

Now that we found homogeneous solution above, we need to find a particular solution $\vec{x}_p(t)$. We will use Variation of parameters. The fundamental matrix is

$$
\Phi = \begin{bmatrix} \vec{x}_1 & \vec{x}_2 & \cdots \end{bmatrix}
$$

Where $\vec{x}_i$ are the solution basis found above. Therefore the fundamental matrix is

$$
\Phi(t) = \begin{bmatrix} \frac{e^{3t}}{2} & -\frac{e^{-t}}{2} \\ e^{3t} & e^{-t} \end{bmatrix}
$$

The particular solution is then given by

$$
\vec{x}_p(t) = \Phi \int \Phi^{-1} \vec{G}(t)\, dt
$$

But

$$\Phi^{-1} = \begin{bmatrix} e^{-3t} & \frac{e^{-3t}}{2} \\ -e^{t} & \frac{e^{t}}{2} \end{bmatrix}$$

Hence

$$\begin{aligned}
\vec{x}_p(t) &= \begin{bmatrix} \frac{e^{3t}}{2} & -\frac{e^{-t}}{2} \\ e^{3t} & e^{-t} \end{bmatrix} \int \begin{bmatrix} e^{-3t} & \frac{e^{-3t}}{2} \\ -e^{t} & \frac{e^{t}}{2} \end{bmatrix} \begin{bmatrix} 2\,e^{t} \\ -e^{t} \end{bmatrix} dt \\
&= \begin{bmatrix} \frac{e^{3t}}{2} & -\frac{e^{-t}}{2} \\ e^{3t} & e^{-t} \end{bmatrix} \int \begin{bmatrix} \frac{3\,e^{-2t}}{2} \\ -\frac{5\,e^{2t}}{2} \end{bmatrix} dt \\
&= \begin{bmatrix} \frac{e^{3t}}{2} & -\frac{e^{-t}}{2} \\ e^{3t} & e^{-t} \end{bmatrix} \begin{bmatrix} -\frac{3\,e^{-2t}}{4} \\ -\frac{5\,e^{2t}}{4} \end{bmatrix} \\
&= \begin{bmatrix} \frac{e^{t}}{4} \\ -2\,e^{t} \end{bmatrix}
\end{aligned}$$

Now that we found particular solution, the final solution is

$$\vec{x}(t) = \vec{x}_h(t) + \vec{x}_p(t)$$

$$\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = \begin{bmatrix} \frac{c_1 e^{3t}}{2} \\ c_1 e^{3t} \end{bmatrix} + \begin{bmatrix} -\frac{c_2 e^{-t}}{2} \\ c_2 e^{-t} \end{bmatrix} + \begin{bmatrix} \frac{e^{t}}{4} \\ -2\,e^{t} \end{bmatrix}$$

Which becomes

$$\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = \begin{bmatrix} \frac{c_1 e^{3t}}{2} - \frac{c_2 e^{-t}}{2} + \frac{e^{t}}{4} \\ c_1 e^{3t} + c_2 e^{-t} - 2\,e^{t} \end{bmatrix}$$

18.7.3 Maple step by step solution

Let's solve

$[x_1'(t) = x_1(t) + x_2(t) + 2\,e^{t}, x_2'(t) = 4x_1(t) + x_2(t) - e^{t}]$

- Define vector

$$\underline{x^{\rightarrow}}(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}$$

- Convert system into a vector equation

$$\underline{x^{\rightarrow}}'(t) = \begin{bmatrix} 1 & 1 \\ 4 & 1 \end{bmatrix} \cdot \underline{x^{\rightarrow}}(t) + \begin{bmatrix} 2\,\mathrm{e}^{t} \\ -\mathrm{e}^{t} \end{bmatrix}$$

- System to solve

$$\underline{x^{\rightarrow}}'(t) = \begin{bmatrix} 1 & 1 \\ 4 & 1 \end{bmatrix} \cdot \underline{x^{\rightarrow}}(t) + \begin{bmatrix} 2\,\mathrm{e}^{t} \\ -\mathrm{e}^{t} \end{bmatrix}$$

- Define the forcing function

$$\vec{f}(t) = \begin{bmatrix} 2\,\mathrm{e}^{t} \\ -\mathrm{e}^{t} \end{bmatrix}$$

- Define the coefficient matrix

$$A = \begin{bmatrix} 1 & 1 \\ 4 & 1 \end{bmatrix}$$

- Rewrite the system as

$$\underline{x^{\rightarrow}}'(t) = A \cdot \underline{x^{\rightarrow}}(t) + \vec{f}$$

- To solve the system, find the eigenvalues and eigenvectors of A
- Eigenpairs of A

$$\left[\left[-1, \begin{bmatrix} -\frac{1}{2} \\ 1 \end{bmatrix}\right], \left[3, \begin{bmatrix} \frac{1}{2} \\ 1 \end{bmatrix}\right]\right]$$

- Consider eigenpair

$$\left[-1, \begin{bmatrix} -\frac{1}{2} \\ 1 \end{bmatrix}\right]$$

- Solution to homogeneous system from eigenpair

$$\underline{x^{\rightarrow}}_{1} = \mathrm{e}^{-t} \cdot \begin{bmatrix} -\frac{1}{2} \\ 1 \end{bmatrix}$$

- Consider eigenpair

$$\left[3, \begin{bmatrix} \frac{1}{2} \\ 1 \end{bmatrix}\right]$$

- Solution to homogeneous system from eigenpair

$\underline{x}^{\rightarrow}_{2} = \mathrm{e}^{3t} \cdot \begin{bmatrix} \frac{1}{2} \\ 1 \end{bmatrix}$

- General solution of the system of ODEs can be written in terms of the particular solution $\underline{x}^{\rightarrow}$

$\underline{x}^{\rightarrow}(t) = c_1 \underline{x}^{\rightarrow}_{1} + c_2 \underline{x}^{\rightarrow}_{2} + \underline{x}^{\rightarrow}_{p}(t)$

☐ Fundamental matrix

  - Let $\phi(t)$ be the matrix whose columns are the independent solutions of the homogeneous syst

    $\phi(t) = \begin{bmatrix} -\frac{\mathrm{e}^{-t}}{2} & \frac{\mathrm{e}^{3t}}{2} \\ \mathrm{e}^{-t} & \mathrm{e}^{3t} \end{bmatrix}$

  - The fundamental matrix, $\Phi(t)$ is a normalized version of $\phi(t)$ satisfying $\Phi(0) = I$ where I is th

    $\Phi(t) = \phi(t) \cdot \frac{1}{\phi(0)}$

  - Substitute the value of $\phi(t)$ and $\phi(0)$

    $\Phi(t) = \begin{bmatrix} -\frac{\mathrm{e}^{-t}}{2} & \frac{\mathrm{e}^{3t}}{2} \\ \mathrm{e}^{-t} & \mathrm{e}^{3t} \end{bmatrix} \cdot \frac{1}{\begin{bmatrix} -\frac{1}{2} & \frac{1}{2} \\ 1 & 1 \end{bmatrix}}$

  - Evaluate and simplify to get the fundamental matrix

    $\Phi(t) = \begin{bmatrix} \frac{\mathrm{e}^{-t}}{2} + \frac{\mathrm{e}^{3t}}{2} & \frac{\mathrm{e}^{3t}}{4} - \frac{\mathrm{e}^{-t}}{4} \\ \mathrm{e}^{3t} - \mathrm{e}^{-t} & \frac{\mathrm{e}^{-t}}{2} + \frac{\mathrm{e}^{3t}}{2} \end{bmatrix}$

☐ Find a particular solution of the system of ODEs using variation of parameters

  - Let the particular solution be the fundamental matrix multiplied by $\vec{v}(t)$ and solve for $\vec{v}(t)$

    $\underline{x}^{\rightarrow}_{p}(t) = \Phi(t) \cdot \vec{v}(t)$

  - Take the derivative of the particular solution

    $\underline{x}^{\rightarrow}{}'_{p}(t) = \Phi'(t) \cdot \vec{v}(t) + \Phi(t) \cdot \vec{v}'(t)$

  - Substitute particular solution and its derivative into the system of ODEs

    $\Phi'(t) \cdot \vec{v}(t) + \Phi(t) \cdot \vec{v}'(t) = A \cdot \Phi(t) \cdot \vec{v}(t) + \vec{f}(t)$

  - The fundamental matrix has columns that are solutions to the homogeneous system so its der

    $A \cdot \Phi(t) \cdot \vec{v}(t) + \Phi(t) \cdot \vec{v}'(t) = A \cdot \Phi(t) \cdot \vec{v}(t) + \vec{f}(t)$

- Cancel like terms

  $\Phi(t)\cdot \vec{v}'(t)=\vec{f}(t)$

- Multiply by the inverse of the fundamental matrix

  $\vec{v}'(t)=\frac{1}{\Phi(t)}\cdot \vec{f}(t)$

- Integrate to solve for $\vec{v}(t)$

  $\vec{v}(t)=\int_0^t \frac{1}{\Phi(s)}\cdot \vec{f}(s)\,ds$

- Plug $\vec{v}(t)$ into the equation for the particular solution

  $\vec{x}_p(t)=\Phi(t)\cdot\left(\int_0^t \frac{1}{\Phi(s)}\cdot \vec{f}(s)\,ds\right)$

- Plug in the fundamental matrix and the forcing function and compute

  $\vec{x}_p(t)=\left[\begin{array}{c}\frac{e^t}{4}-\frac{5\,e^{-t}}{8}+\frac{3\,e^{3t}}{8}\\ \frac{3\,e^{3t}}{4}+\frac{5\,e^{-t}}{4}-2\,e^t\end{array}\right]$

- Plug particular solution back into general solution

  $\vec{x}(t)=c_1\vec{x}_1+c_2\vec{x}_2+\left[\begin{array}{c}\frac{e^t}{4}-\frac{5\,e^{-t}}{8}+\frac{3\,e^{3t}}{8}\\ \frac{3\,e^{3t}}{4}+\frac{5\,e^{-t}}{4}-2\,e^t\end{array}\right]$

- Substitute in vector of dependent variables

  $\left[\begin{array}{c}x_1(t)\\ x_2(t)\end{array}\right]=\left[\begin{array}{c}\frac{(-4c_1-5)e^{-t}}{8}+\frac{(4c_2+3)e^{3t}}{8}+\frac{e^t}{4}\\ \frac{(5+4c_1)e^{-t}}{4}+\frac{(4c_2+3)e^{3t}}{4}-2\,e^t\end{array}\right]$

- Solution to the system of ODEs

  $\left\{x_1(t)=\frac{(-4c_1-5)e^{-t}}{8}+\frac{(4c_2+3)e^{3t}}{8}+\frac{e^t}{4}, x_2(t)=\frac{(5+4c_1)e^{-t}}{4}+\frac{(4c_2+3)e^{3t}}{4}-2\,e^t\right\}$

✓ Solution by Maple

Time used: 0.015 (sec). Leaf size: 44

```
dsolve([diff(x__1(t),t)=1*x__1(t)+1*x__2(t)+2*exp(t),diff(x__2(t),t)=4*x__1(t)+1*x__2(t)-exp
```

$$x_1(t)=c_2e^{3t}+e^{-t}c_1+\frac{e^t}{4}$$
$$x_2(t)=2c_2e^{3t}-2\,e^{-t}c_1-2\,e^t$$

✓ Solution by Mathematica

Time used: 0.004 (sec). Leaf size: 80

```
DSolve[{x1'[t]==1*x1[t]+1*x2[t]+2*Exp[t],x2'[t]==4*x1[t]+1*x2[t]-Exp[t]},{x1[t],x2[t]},t,Inc
```

$$x1(t) \to \frac{1}{4}e^{-t}\left(e^{2t} + (2c_1 + c_2)e^{4t} + 2c_1 - c_2\right)$$
$$x2(t) \to \frac{1}{2}e^{-t}\left(-4e^{2t} + (2c_1 + c_2)e^{4t} - 2c_1 + c_2\right)$$

18.8 problem 8

Internal problem ID [785]
Internal file name [`OUTPUT/785_Sunday_June_05_2022_01_49_39_AM_83821617/index.tex`]

Book: Elementary differential equations and boundary value problems, 10th ed., Boyce and DiPrima
Section: Chapter 7.9, Nonhomogeneous Linear Systems. page 447
Problem number: 8.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"system of linear ODEs"**

Solve

$$x_1'(t) = 2x_1(t) - x_2(t) + \mathrm{e}^t$$
$$x_2'(t) = 3x_1(t) - 2x_2(t) - \mathrm{e}^t$$

18.8.1 Solution using Matrix exponential method

In this method, we will assume we have found the matrix exponential e^{At} allready. There are different methods to determine this but will not be shown here. This is a system of linear ODE's given as

$$\vec{x}'(t) = A\,\vec{x}(t) + \vec{G}(t)$$

Or

$$\begin{bmatrix} x_1'(t) \\ x_2'(t) \end{bmatrix} = \begin{bmatrix} 2 & -1 \\ 3 & -2 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} + \begin{bmatrix} \mathrm{e}^t \\ -\mathrm{e}^t \end{bmatrix}$$

Since the system is nonhomogeneous, then the solution is given by

$$\vec{x}(t) = \vec{x}_h(t) + \vec{x}_p(t)$$

Where $\vec{x}_h(t)$ is the homogeneous solution to $\vec{x}'(t) = A\,\vec{x}(t)$ and $\vec{x}_p(t)$ is a particular solution to $\vec{x}'(t) = A\,\vec{x}(t) + \vec{G}(t)$. The particular solution will be found using variation

of parameters method applied to the fundamental matrix. For the above matrix A, the matrix exponential can be found to be

$$e^{At} = \begin{bmatrix} -\frac{e^{-t}}{2} + \frac{3\,e^{t}}{2} & -\frac{e^{t}}{2} + \frac{e^{-t}}{2} \\ \frac{3\,e^{t}}{2} - \frac{3\,e^{-t}}{2} & \frac{3\,e^{-t}}{2} - \frac{e^{t}}{2} \end{bmatrix}$$

Therefore the homogeneous solution is

$$\begin{aligned}
\vec{x}_h(t) &= e^{At}\vec{c} \\
&= \begin{bmatrix} -\frac{e^{-t}}{2} + \frac{3\,e^{t}}{2} & -\frac{e^{t}}{2} + \frac{e^{-t}}{2} \\ \frac{3\,e^{t}}{2} - \frac{3\,e^{-t}}{2} & \frac{3\,e^{-t}}{2} - \frac{e^{t}}{2} \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} \\
&= \begin{bmatrix} \left(-\frac{e^{-t}}{2} + \frac{3\,e^{t}}{2}\right) c_1 + \left(-\frac{e^{t}}{2} + \frac{e^{-t}}{2}\right) c_2 \\ \left(\frac{3\,e^{t}}{2} - \frac{3\,e^{-t}}{2}\right) c_1 + \left(\frac{3\,e^{-t}}{2} - \frac{e^{t}}{2}\right) c_2 \end{bmatrix} \\
&= \begin{bmatrix} \frac{(c_2 - c_1)e^{-t}}{2} + \frac{3\left(-\frac{c_2}{3} + c_1\right)e^{t}}{2} \\ \frac{(-3c_1 + 3c_2)e^{-t}}{2} + \frac{3\left(-\frac{c_2}{3} + c_1\right)e^{t}}{2} \end{bmatrix}
\end{aligned}$$

The particular solution given by

$$\vec{x}_p(t) = e^{At} \int e^{-At} \vec{G}(t)\, dt$$

But

$$\begin{aligned}
e^{-At} &= \left(e^{At}\right)^{-1} \\
&= \begin{bmatrix} \frac{3\,e^{-t}}{2} - \frac{e^{t}}{2} & \frac{e^{t}}{2} - \frac{e^{-t}}{2} \\ -\frac{3\,e^{t}}{2} + \frac{3\,e^{-t}}{2} & -\frac{e^{-t}}{2} + \frac{3\,e^{t}}{2} \end{bmatrix}
\end{aligned}$$

Hence

$$\begin{aligned}
\vec{x}_p(t) &= \begin{bmatrix} -\frac{e^{-t}}{2} + \frac{3\,e^{t}}{2} & -\frac{e^{t}}{2} + \frac{e^{-t}}{2} \\ \frac{3\,e^{t}}{2} - \frac{3\,e^{-t}}{2} & \frac{3\,e^{-t}}{2} - \frac{e^{t}}{2} \end{bmatrix} \int \begin{bmatrix} \frac{3\,e^{-t}}{2} - \frac{e^{t}}{2} & \frac{e^{t}}{2} - \frac{e^{-t}}{2} \\ -\frac{3\,e^{t}}{2} + \frac{3\,e^{-t}}{2} & -\frac{e^{-t}}{2} + \frac{3\,e^{t}}{2} \end{bmatrix} \begin{bmatrix} e^{t} \\ -e^{t} \end{bmatrix} dt \\
&= \begin{bmatrix} -\frac{e^{-t}}{2} + \frac{3\,e^{t}}{2} & -\frac{e^{t}}{2} + \frac{e^{-t}}{2} \\ \frac{3\,e^{t}}{2} - \frac{3\,e^{-t}}{2} & \frac{3\,e^{-t}}{2} - \frac{e^{t}}{2} \end{bmatrix} \begin{bmatrix} 2t - \frac{e^{2t}}{2} \\ 2t - \frac{3\,e^{2t}}{2} \end{bmatrix} \\
&= \begin{bmatrix} e^{t}\left(-\frac{1}{2} + 2t\right) \\ e^{t}\left(-\frac{3}{2} + 2t\right) \end{bmatrix}
\end{aligned}$$

Hence the complete solution is

$$\vec{x}(t) = \vec{x}_h(t) + \vec{x}_p(t)$$
$$= \begin{bmatrix} \frac{(c_2-c_1)\mathrm{e}^{-t}}{2} + 2\left(t + \frac{3c_1}{4} - \frac{c_2}{4} - \frac{1}{4}\right)\mathrm{e}^t \\ \frac{(-3c_1+3c_2)\mathrm{e}^{-t}}{2} + 2\left(t + \frac{3c_1}{4} - \frac{c_2}{4} - \frac{3}{4}\right)\mathrm{e}^t \end{bmatrix}$$

18.8.2 Solution using explicit Eigenvalue and Eigenvector method

This is a system of linear ODE's given as

$$\vec{x}'(t) = A\,\vec{x}(t) + \vec{G}(t)$$

Or

$$\begin{bmatrix} x_1'(t) \\ x_2'(t) \end{bmatrix} = \begin{bmatrix} 2 & -1 \\ 3 & -2 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} + \begin{bmatrix} \mathrm{e}^t \\ -\mathrm{e}^t \end{bmatrix}$$

Since the system is nonhomogeneous, then the solution is given by

$$\vec{x}(t) = \vec{x}_h(t) + \vec{x}_p(t)$$

Where $\vec{x}_h(t)$ is the homogeneous solution to $\vec{x}'(t) = A\,\vec{x}(t)$ and $\vec{x}_p(t)$ is a particular solution to $\vec{x}'(t) = A\,\vec{x}(t) + \vec{G}(t)$. The particular solution will be found using variation of parameters method applied to the fundamental matrix.

The first step is find the homogeneous solution. We start by finding the eigenvalues of A. This is done by solving the following equation for the eigenvalues λ

$$\det(A - \lambda I) = 0$$

Expanding gives

$$\det\left(\begin{bmatrix} 2 & -1 \\ 3 & -2 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right) = 0$$

Therefore

$$\det\left(\begin{bmatrix} 2-\lambda & -1 \\ 3 & -2-\lambda \end{bmatrix}\right) = 0$$

Which gives the characteristic equation

$$\lambda^2 - 1 = 0$$

The roots of the above are the eigenvalues.

$$\lambda_1 = -1$$
$$\lambda_2 = 1$$

This table summarises the above result

eigenvalue	algebraic multiplicity	type of eigenvalue
-1	1	real eigenvalue
1	1	real eigenvalue

Now the eigenvector for each eigenvalue are found.

Considering the eigenvalue $\lambda_1 = -1$

We need to solve $A\vec{v} = \lambda\vec{v}$ or $(A - \lambda I)\vec{v} = \vec{0}$ which becomes

$$\left(\begin{bmatrix} 2 & -1 \\ 3 & -2 \end{bmatrix} - (-1)\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right)\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$
$$\begin{bmatrix} 3 & -1 \\ 3 & -1 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Now forward elimination is applied to solve for the eigenvector $\vec{v}$. The augmented matrix is

$$\left[\begin{array}{cc|c} 3 & -1 & 0 \\ 3 & -1 & 0 \end{array}\right]$$

$$R_2 = R_2 - R_1 \Longrightarrow \left[\begin{array}{cc|c} 3 & -1 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

Therefore the system in Echelon form is

$$\begin{bmatrix} 3 & -1 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

The free variables are $\{v_2\}$ and the leading variables are $\{v_1\}$. Let $v_2 = t$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\left\{v_1 = \frac{t}{3}\right\}$

Hence the solution is

$$\begin{bmatrix} \frac{t}{3} \\ t \end{bmatrix} = \begin{bmatrix} \frac{t}{3} \\ t \end{bmatrix}$$

Since there is one free Variable, we have found one eigenvector associated with this eigenvalue. The above can be written as

$$\begin{bmatrix} \frac{t}{3} \\ t \end{bmatrix} = t \begin{bmatrix} \frac{1}{3} \\ 1 \end{bmatrix}$$

Let $t = 1$ the eigenvector becomes

$$\begin{bmatrix} \frac{t}{3} \\ t \end{bmatrix} = \begin{bmatrix} \frac{1}{3} \\ 1 \end{bmatrix}$$

Which is normalized to

$$\begin{bmatrix} \frac{t}{3} \\ t \end{bmatrix} = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$$

Considering the eigenvalue $\lambda_2 = 1$

We need to solve $A\vec{v} = \lambda\vec{v}$ or $(A - \lambda I)\vec{v} = \vec{0}$ which becomes

$$\left(\begin{bmatrix} 2 & -1 \\ 3 & -2 \end{bmatrix} - (1) \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right) \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & -1 \\ 3 & -3 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Now forward elimination is applied to solve for the eigenvector $\vec{v}$. The augmented matrix is

$$\left[\begin{array}{cc|c} 1 & -1 & 0 \\ 3 & -3 & 0 \end{array}\right]$$

$$R_2 = R_2 - 3R_1 \Longrightarrow \left[\begin{array}{cc|c} 1 & -1 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

Therefore the system in Echelon form is

$$\begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

The free variables are $\{v_2\}$ and the leading variables are $\{v_1\}$. Let $v_2 = t$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\{v_1 = t\}$

Hence the solution is

$$\begin{bmatrix} t \\ t \end{bmatrix} = \begin{bmatrix} t \\ t \end{bmatrix}$$

Since there is one free Variable, we have found one eigenvector associated with this eigenvalue. The above can be written as

$$\begin{bmatrix} t \\ t \end{bmatrix} = t \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

Let $t = 1$ the eigenvector becomes

$$\begin{bmatrix} t \\ t \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

The following table gives a summary of this result. It shows for each eigenvalue the algebraic multiplicity m, and its geometric multiplicity k and the eigenvectors associated with the eigenvalue. If $m > k$ then the eigenvalue is defective which means the number of normal linearly independent eigenvectors associated with this eigenvalue (called the geometric multiplicity k) does not equal the algebraic multiplicity m, and we need to determine an additional $m - k$ generalized eigenvectors for this eigenvalue.

	multiplicity			
eigenvalue	algebraic m	geometric k	defective?	eigenvectors
-1	1	1	No	$\begin{bmatrix} \frac{1}{3} \\ 1 \end{bmatrix}$
1	1	1	No	$\begin{bmatrix} 1 \\ 1 \end{bmatrix}$

Now that we found the eigenvalues and associated eigenvectors, we will go over each eigenvalue and generate the solution basis. The only problem we need to take care of is if the eigenvalue is defective. Since eigenvalue -1 is real and distinct then the

corresponding eigenvector solution is

$$\vec{x}_1(t) = \vec{v}_1 e^{-t}$$
$$= \begin{bmatrix} \frac{1}{3} \\ 1 \end{bmatrix} e^{-t}$$

Since eigenvalue 1 is real and distinct then the corresponding eigenvector solution is

$$\vec{x}_2(t) = \vec{v}_2 e^{t}$$
$$= \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{t}$$

Therefore the homogeneous solution is

$$\vec{x}_h(t) = c_1 \vec{x}_1(t) + c_2 \vec{x}_2(t)$$

Which is written as

$$\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = c_1 \begin{bmatrix} \frac{e^{-t}}{3} \\ e^{-t} \end{bmatrix} + c_2 \begin{bmatrix} e^{t} \\ e^{t} \end{bmatrix}$$

Now that we found homogeneous solution above, we need to find a particular solution $\vec{x}_p(t)$. We will use Variation of parameters. The fundamental matrix is

$$\Phi = \begin{bmatrix} \vec{x}_1 & \vec{x}_2 & \cdots \end{bmatrix}$$

Where $\vec{x}_i$ are the solution basis found above. Therefore the fundamental matrix is

$$\Phi(t) = \begin{bmatrix} \frac{e^{-t}}{3} & e^{t} \\ e^{-t} & e^{t} \end{bmatrix}$$

The particular solution is then given by

$$\vec{x}_p(t) = \Phi \int \Phi^{-1} \vec{G}(t)\, dt$$

But

$$\Phi^{-1} = \begin{bmatrix} -\frac{3\,e^{t}}{2} & \frac{3\,e^{t}}{2} \\ \frac{3\,e^{-t}}{2} & -\frac{e^{-t}}{2} \end{bmatrix}$$

Hence

$$\vec{x}_p(t) = \begin{bmatrix} \frac{e^{-t}}{3} & e^{t} \\ e^{-t} & e^{t} \end{bmatrix} \int \begin{bmatrix} -\frac{3e^{t}}{2} & \frac{3e^{t}}{2} \\ \frac{3e^{-t}}{2} & -\frac{e^{-t}}{2} \end{bmatrix} \begin{bmatrix} e^{t} \\ -e^{t} \end{bmatrix} dt$$
$$= \begin{bmatrix} \frac{e^{-t}}{3} & e^{t} \\ e^{-t} & e^{t} \end{bmatrix} \int \begin{bmatrix} -3e^{2t} \\ 2 \end{bmatrix} dt$$
$$= \begin{bmatrix} \frac{e^{-t}}{3} & e^{t} \\ e^{-t} & e^{t} \end{bmatrix} \begin{bmatrix} -\frac{3e^{2t}}{2} \\ 2t \end{bmatrix}$$
$$= \begin{bmatrix} e^{t}\left(-\frac{1}{2} + 2t\right) \\ e^{t}\left(-\frac{3}{2} + 2t\right) \end{bmatrix}$$

Now that we found particular solution, the final solution is

$$\vec{x}(t) = \vec{x}_h(t) + \vec{x}_p(t)$$
$$\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = \begin{bmatrix} \frac{c_1 e^{-t}}{3} \\ c_1 e^{-t} \end{bmatrix} + \begin{bmatrix} c_2 e^{t} \\ c_2 e^{t} \end{bmatrix} + \begin{bmatrix} e^{t}\left(-\frac{1}{2} + 2t\right) \\ e^{t}\left(-\frac{3}{2} + 2t\right) \end{bmatrix}$$

Which becomes

$$\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = \begin{bmatrix} \frac{c_1 e^{-t}}{3} + 2\left(t + \frac{c_2}{2} - \frac{1}{4}\right) e^{t} \\ c_1 e^{-t} + 2\left(t + \frac{c_2}{2} - \frac{3}{4}\right) e^{t} \end{bmatrix}$$

18.8.3 Maple step by step solution

Let's solve

$[x_1'(t) = 2x_1(t) - x_2(t) + e^{t}, x_2'(t) = 3x_1(t) - 2x_2(t) - e^{t}]$

- Define vector

$$\underline{x}^{\rightarrow}(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}$$

- Convert system into a vector equation

$$\underline{x}^{\rightarrow\prime}(t) = \begin{bmatrix} 2 & -1 \\ 3 & -2 \end{bmatrix} \cdot \underline{x}^{\rightarrow}(t) + \begin{bmatrix} e^{t} \\ -e^{t} \end{bmatrix}$$

- System to solve

$$\underline{\vec{x}}'(t)=\begin{bmatrix}2 & -1\\3 & -2\end{bmatrix}\cdot\underline{\vec{x}}(t)+\begin{bmatrix}e^{t}\\-e^{t}\end{bmatrix}$$

- Define the forcing function

$$\vec{f}(t)=\begin{bmatrix}e^{t}\\-e^{t}\end{bmatrix}$$

- Define the coefficient matrix

$$A=\begin{bmatrix}2 & -1\\3 & -2\end{bmatrix}$$

- Rewrite the system as

$$\underline{\vec{x}}'(t)=A\cdot\underline{\vec{x}}(t)+\vec{f}$$

- To solve the system, find the eigenvalues and eigenvectors of A
- Eigenpairs of A

$$\left[\left[-1,\begin{bmatrix}\frac{1}{3}\\1\end{bmatrix}\right],\left[1,\begin{bmatrix}1\\1\end{bmatrix}\right]\right]$$

- Consider eigenpair

$$\left[-1,\begin{bmatrix}\frac{1}{3}\\1\end{bmatrix}\right]$$

- Solution to homogeneous system from eigenpair

$$\underline{\vec{x}}_{1}=e^{-t}\cdot\begin{bmatrix}\frac{1}{3}\\1\end{bmatrix}$$

- Consider eigenpair

$$\left[1,\begin{bmatrix}1\\1\end{bmatrix}\right]$$

- Solution to homogeneous system from eigenpair

$$\underline{\vec{x}}_{2}=e^{t}\cdot\begin{bmatrix}1\\1\end{bmatrix}$$

- General solution of the system of ODEs can be written in terms of the particular solution $\underline{\vec{x}}$

  $\underline{\vec{x}}(t) = c_1\underline{\vec{x}}_1 + c_2\underline{\vec{x}}_2 + \underline{\vec{x}}_p(t)$

☐ Fundamental matrix

  - Let $\phi(t)$ be the matrix whose columns are the independent solutions of the homogeneous syst

    $$\phi(t) = \begin{bmatrix} \frac{e^{-t}}{3} & e^{t} \\ e^{-t} & e^{t} \end{bmatrix}$$

  - The fundamental matrix, $\Phi(t)$ is a normalized version of $\phi(t)$ satisfying $\Phi(0) = I$ where I is th

    $\Phi(t) = \phi(t) \cdot \frac{1}{\phi(0)}$

  - Substitute the value of $\phi(t)$ and $\phi(0)$

    $$\Phi(t) = \begin{bmatrix} \frac{e^{-t}}{3} & e^{t} \\ e^{-t} & e^{t} \end{bmatrix} \cdot \frac{1}{\begin{bmatrix} \frac{1}{3} & 1 \\ 1 & 1 \end{bmatrix}}$$

  - Evaluate and simplify to get the fundamental matrix

    $$\Phi(t) = \begin{bmatrix} -\frac{e^{-t}}{2} + \frac{3\,e^{t}}{2} & -\frac{e^{t}}{2} + \frac{e^{-t}}{2} \\ \frac{3\,e^{t}}{2} - \frac{3\,e^{-t}}{2} & \frac{3\,e^{-t}}{2} - \frac{e^{t}}{2} \end{bmatrix}$$

☐ Find a particular solution of the system of ODEs using variation of parameters

  - Let the particular solution be the fundamental matrix multiplied by $\vec{v}(t)$ and solve for $\vec{v}(t)$

    $\underline{\vec{x}}_p(t) = \Phi(t) \cdot \vec{v}(t)$

  - Take the derivative of the particular solution

    $\underline{\vec{x}}'_p(t) = \Phi'(t) \cdot \vec{v}(t) + \Phi(t) \cdot \vec{v}'(t)$

  - Substitute particular solution and its derivative into the system of ODEs

    $\Phi'(t) \cdot \vec{v}(t) + \Phi(t) \cdot \vec{v}'(t) = A \cdot \Phi(t) \cdot \vec{v}(t) + \vec{f}(t)$

  - The fundamental matrix has columns that are solutions to the homogeneous system so its der

    $A \cdot \Phi(t) \cdot \vec{v}(t) + \Phi(t) \cdot \vec{v}'(t) = A \cdot \Phi(t) \cdot \vec{v}(t) + \vec{f}(t)$

  - Cancel like terms

    $\Phi(t) \cdot \vec{v}'(t) = \vec{f}(t)$

  - Multiply by the inverse of the fundamental matrix

$\overrightarrow{v}'(t) = \frac{1}{\Phi(t)} \cdot \overrightarrow{f}(t)$

- Integrate to solve for $\overrightarrow{v}(t)$

  $\overrightarrow{v}(t) = \int_0^t \frac{1}{\Phi(s)} \cdot \overrightarrow{f}(s)\, ds$

- Plug $\overrightarrow{v}(t)$ into the equation for the particular solution

  $\underline{x}^{\rightarrow}{}_p(t) = \Phi(t) \cdot \left(\int_0^t \frac{1}{\Phi(s)} \cdot \overrightarrow{f}(s)\, ds\right)$

- Plug in the fundamental matrix and the forcing function and compute

  $\underline{x}^{\rightarrow}{}_p(t) = \begin{bmatrix} \frac{e^{-t}}{2} - \frac{e^t}{2} + 2t\, e^t \\ 2t\, e^t + \frac{3\, e^{-t}}{2} - \frac{3\, e^t}{2} \end{bmatrix}$

- Plug particular solution back into general solution

  $\underline{x}^{\rightarrow}(t) = c_1 \underline{x}^{\rightarrow}{}_1 + c_2 \underline{x}^{\rightarrow}{}_2 + \begin{bmatrix} \frac{e^{-t}}{2} - \frac{e^t}{2} + 2t\, e^t \\ 2t\, e^t + \frac{3\, e^{-t}}{2} - \frac{3\, e^t}{2} \end{bmatrix}$

- Substitute in vector of dependent variables

  $\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = \begin{bmatrix} \frac{(2c_1+3)e^{-t}}{6} + 2\left(t + \frac{c_2}{2} - \frac{1}{4}\right) e^t \\ \frac{(2c_1+3)e^{-t}}{2} + 2\left(t + \frac{c_2}{2} - \frac{3}{4}\right) e^t \end{bmatrix}$

- Solution to the system of ODEs

  $\left\{x_1(t) = \frac{(2c_1+3)e^{-t}}{6} + 2\left(t + \frac{c_2}{2} - \frac{1}{4}\right) e^t, x_2(t) = \frac{(2c_1+3)e^{-t}}{2} + 2\left(t + \frac{c_2}{2} - \frac{3}{4}\right) e^t\right\}$

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 45

```
dsolve([diff(x__1(t),t)=2*x__1(t)-1*x__2(t)+exp(t),diff(x__2(t),t)=3*x__1(t)-2*x__2(t)-exp(t
```

$$x_1(t) = c_2 e^t + e^{-t} c_1 + 2\, e^t t$$
$$x_2(t) = c_2 e^t + 3\, e^{-t} c_1 + 2\, e^t t - e^t$$

✓ Solution by Mathematica

Time used: 0.009 (sec). Leaf size: 80

```
DSolve[{x1'[t]==2*x1[t]-1*x2[t]+Exp[t],x2'[t]==3*x1[t]-2*x2[t]-Exp[t]},{x1[t],x2[t]},t,Inclu
```

$$\begin{aligned} \text{x1}(t) &\to \frac{1}{2}e^{-t}\left(e^{2t}(4t-1+3c_1-c_2)-c_1+c_2\right) \\ \text{x2}(t) &\to \frac{1}{2}e^{-t}\left(e^{2t}(4t-3+3c_1-c_2)-3c_1+3c_2\right) \end{aligned}$$

18.9 problem 9

Internal problem ID [786]
Internal file name [`OUTPUT/786_Sunday_June_05_2022_01_49_41_AM_45101061/index.tex`]

Book: Elementary differential equations and boundary value problems, 10th ed., Boyce and DiPrima
Section: Chapter 7.9, Nonhomogeneous Linear Systems. page 447
Problem number: 9.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"system of linear ODEs"**

Solve

$$
\begin{aligned}
x_1'(t) &= -\frac{5x_1(t)}{4} + \frac{3x_2(t)}{4} + 2t \\
x_2'(t) &= \frac{3x_1(t)}{4} - \frac{5x_2(t)}{4} + \mathrm{e}^t
\end{aligned}
$$

18.9.1 Solution using Matrix exponential method

In this method, we will assume we have found the matrix exponential e^{At} allready. There are different methods to determine this but will not be shown here. This is a system of linear ODE's given as

$$
\vec{x}'(t) = A\,\vec{x}(t) + \vec{G}(t)
$$

Or

$$
\begin{bmatrix} x_1'(t) \\ x_2'(t) \end{bmatrix} = \begin{bmatrix} -\frac{5}{4} & \frac{3}{4} \\ \frac{3}{4} & -\frac{5}{4} \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} + \begin{bmatrix} 2t \\ \mathrm{e}^t \end{bmatrix}
$$

Since the system is nonhomogeneous, then the solution is given by

$$
\vec{x}(t) = \vec{x}_h(t) + \vec{x}_p(t)
$$

Where $\vec{x}_h(t)$ is the homogeneous solution to $\vec{x}'(t) = A\,\vec{x}(t)$ and $\vec{x}_p(t)$ is a particular solution to $\vec{x}'(t) = A\,\vec{x}(t) + \vec{G}(t)$. The particular solution will be found using variation

of parameters method applied to the fundamental matrix. For the above matrix A, the matrix exponential can be found to be

$$e^{At} = \begin{bmatrix} \frac{e^{-2t}}{2} + \frac{e^{-\frac{t}{2}}}{2} & \frac{e^{-\frac{t}{2}}}{2} - \frac{e^{-2t}}{2} \\ \frac{e^{-\frac{t}{2}}}{2} - \frac{e^{-2t}}{2} & \frac{e^{-2t}}{2} + \frac{e^{-\frac{t}{2}}}{2} \end{bmatrix}$$

Therefore the homogeneous solution is

$$\begin{aligned} \vec{x}_h(t) &= e^{At}\vec{c} \\ &= \begin{bmatrix} \frac{e^{-2t}}{2} + \frac{e^{-\frac{t}{2}}}{2} & \frac{e^{-\frac{t}{2}}}{2} - \frac{e^{-2t}}{2} \\ \frac{e^{-\frac{t}{2}}}{2} - \frac{e^{-2t}}{2} & \frac{e^{-2t}}{2} + \frac{e^{-\frac{t}{2}}}{2} \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} \\ &= \begin{bmatrix} \left(\frac{e^{-2t}}{2} + \frac{e^{-\frac{t}{2}}}{2}\right) c_1 + \left(\frac{e^{-\frac{t}{2}}}{2} - \frac{e^{-2t}}{2}\right) c_2 \\ \left(\frac{e^{-\frac{t}{2}}}{2} - \frac{e^{-2t}}{2}\right) c_1 + \left(\frac{e^{-2t}}{2} + \frac{e^{-\frac{t}{2}}}{2}\right) c_2 \end{bmatrix} \\ &= \begin{bmatrix} \frac{(c_1+c_2)e^{-\frac{t}{2}}}{2} + \frac{(-c_2+c_1)e^{-2t}}{2} \\ \frac{(c_1+c_2)e^{-\frac{t}{2}}}{2} - \frac{(-c_2+c_1)e^{-2t}}{2} \end{bmatrix} \end{aligned}$$

The particular solution given by

$$\vec{x}_p(t) = e^{At} \int e^{-At} \vec{G}(t)\, dt$$

But

$$\begin{aligned} e^{-At} &= (e^{At})^{-1} \\ &= \begin{bmatrix} \frac{\left(e^{\frac{3t}{2}}+1\right)e^{\frac{t}{2}}}{2} & -\frac{\left(e^{\frac{3t}{2}}-1\right)e^{\frac{t}{2}}}{2} \\ -\frac{\left(e^{\frac{3t}{2}}-1\right)e^{\frac{t}{2}}}{2} & \frac{\left(e^{\frac{3t}{2}}+1\right)e^{\frac{t}{2}}}{2} \end{bmatrix} \end{aligned}$$

Hence

$$\vec{x}_p(t) = \begin{bmatrix} \frac{e^{-2t}}{2}+\frac{e^{-\frac{t}{2}}}{2} & \frac{e^{-\frac{t}{2}}}{2}-\frac{e^{-2t}}{2} \\ \frac{e^{-\frac{t}{2}}}{2}-\frac{e^{-2t}}{2} & \frac{e^{-2t}}{2}+\frac{e^{-\frac{t}{2}}}{2} \end{bmatrix} \int \begin{bmatrix} \frac{\left(e^{\frac{3t}{2}}+1\right)e^{\frac{t}{2}}}{2} & -\frac{\left(e^{\frac{3t}{2}}-1\right)e^{\frac{t}{2}}}{2} \\ -\frac{\left(e^{\frac{3t}{2}}-1\right)e^{\frac{t}{2}}}{2} & \frac{\left(e^{\frac{3t}{2}}+1\right)e^{\frac{t}{2}}}{2} \end{bmatrix} \begin{bmatrix} 2t \\ e^{t} \end{bmatrix} dt$$

$$= \begin{bmatrix} \frac{e^{-2t}}{2}+\frac{e^{-\frac{t}{2}}}{2} & \frac{e^{-\frac{t}{2}}}{2}-\frac{e^{-2t}}{2} \\ \frac{e^{-\frac{t}{2}}}{2}-\frac{e^{-2t}}{2} & \frac{e^{-2t}}{2}+\frac{e^{-\frac{t}{2}}}{2} \end{bmatrix} \begin{bmatrix} 2(t-2)\,e^{\frac{t}{2}}+\frac{e^{\frac{3t}{2}}}{3}+\frac{(2t-1)e^{2t}}{4}-\frac{e^{3t}}{6} \\ 2(t-2)\,e^{\frac{t}{2}}+\frac{e^{\frac{3t}{2}}}{3}+\frac{(1-2t)e^{2t}}{4}+\frac{e^{3t}}{6} \end{bmatrix}$$

$$= \begin{bmatrix} \frac{e^{t}}{6}+\frac{5t}{2}-\frac{17}{4} \\ \frac{e^{t}}{2}+\frac{3t}{2}-\frac{15}{4} \end{bmatrix}$$

Hence the complete solution is

$$\begin{aligned} \vec{x}(t) &= \vec{x}_h(t)+\vec{x}_p(t) \\ &= \begin{bmatrix} \frac{e^{-2t}\left((c_1+c_2)e^{\frac{3t}{2}}+5\,e^{2t}t+c_1-c_2-\frac{17\,e^{2t}}{2}+\frac{e^{3t}}{3}\right)}{2} \\ \frac{e^{-2t}\left((c_1+c_2)e^{\frac{3t}{2}}+3\,e^{2t}t-c_1+c_2-\frac{15\,e^{2t}}{2}+e^{3t}\right)}{2} \end{bmatrix} \end{aligned}$$

18.9.2 Solution using explicit Eigenvalue and Eigenvector method

This is a system of linear ODE's given as

$$\vec{x}'(t) = A\,\vec{x}(t)+\vec{G}(t)$$

Or

$$\begin{bmatrix} x_1'(t) \\ x_2'(t) \end{bmatrix} = \begin{bmatrix} -\frac{5}{4} & \frac{3}{4} \\ \frac{3}{4} & -\frac{5}{4} \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} + \begin{bmatrix} 2t \\ e^{t} \end{bmatrix}$$

Since the system is nonhomogeneous, then the solution is given by

$$\vec{x}(t) = \vec{x}_h(t)+\vec{x}_p(t)$$

Where $\vec{x}_h(t)$ is the homogeneous solution to $\vec{x}'(t) = A\,\vec{x}(t)$ and $\vec{x}_p(t)$ is a particular solution to $\vec{x}'(t) = A\,\vec{x}(t)+\vec{G}(t)$. The particular solution will be found using variation of parameters method applied to the fundamental matrix.

The first step is find the homogeneous solution. We start by finding the eigenvalues of A. This is done by solving the following equation for the eigenvalues λ

$$\det\left(A-\lambda I\right) = 0$$

Expanding gives

$$\det\left(\begin{bmatrix}-\frac{5}{4} & \frac{3}{4}\\ \frac{3}{4} & -\frac{5}{4}\end{bmatrix} - \lambda\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}\right) = 0$$

Therefore

$$\det\left(\begin{bmatrix}-\frac{5}{4}-\lambda & \frac{3}{4}\\ \frac{3}{4} & -\frac{5}{4}-\lambda\end{bmatrix}\right) = 0$$

Which gives the characteristic equation

$$\lambda^2 + \frac{5}{2}\lambda + 1 = 0$$

The roots of the above are the eigenvalues.

$$\lambda_1 = -2$$
$$\lambda_2 = -\frac{1}{2}$$

This table summarises the above result

eigenvalue	algebraic multiplicity	type of eigenvalue
-2	1	real eigenvalue
$-\frac{1}{2}$	1	real eigenvalue

Now the eigenvector for each eigenvalue are found.

Considering the eigenvalue $\lambda_1 = -2$

We need to solve $A\vec{v} = \lambda\vec{v}$ or $(A - \lambda I)\vec{v} = \vec{0}$ which becomes

$$\left(\begin{bmatrix}-\frac{5}{4} & \frac{3}{4}\\ \frac{3}{4} & -\frac{5}{4}\end{bmatrix} - (-2)\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}\right)\begin{bmatrix}v_1\\ v_2\end{bmatrix} = \begin{bmatrix}0\\ 0\end{bmatrix}$$
$$\begin{bmatrix}\frac{3}{4} & \frac{3}{4}\\ \frac{3}{4} & \frac{3}{4}\end{bmatrix}\begin{bmatrix}v_1\\ v_2\end{bmatrix} = \begin{bmatrix}0\\ 0\end{bmatrix}$$

Now forward elimination is applied to solve for the eigenvector $\vec{v}$. The augmented matrix is

$$\left[\begin{array}{cc|c}\frac{3}{4} & \frac{3}{4} & 0\\ \frac{3}{4} & \frac{3}{4} & 0\end{array}\right]$$

$$R_2 = R_2 - R_1 \Longrightarrow \left[\begin{array}{cc|c} \frac{3}{4} & \frac{3}{4} & 0 \\ 0 & 0 & 0 \end{array}\right]$$

Therefore the system in Echelon form is

$$\begin{bmatrix} \frac{3}{4} & \frac{3}{4} \\ 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

The free variables are $\{v_2\}$ and the leading variables are $\{v_1\}$. Let $v_2 = t$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\{v_1 = -t\}$

Hence the solution is

$$\begin{bmatrix} -t \\ t \end{bmatrix} = \begin{bmatrix} -t \\ t \end{bmatrix}$$

Since there is one free Variable, we have found one eigenvector associated with this eigenvalue. The above can be written as

$$\begin{bmatrix} -t \\ t \end{bmatrix} = t \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

Let $t = 1$ the eigenvector becomes

$$\begin{bmatrix} -t \\ t \end{bmatrix} = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

Considering the eigenvalue $\lambda_2 = -\frac{1}{2}$

We need to solve $A\vec{v} = \lambda\vec{v}$ or $(A - \lambda I)\vec{v} = \vec{0}$ which becomes

$$\left(\begin{bmatrix} -\frac{5}{4} & \frac{3}{4} \\ \frac{3}{4} & -\frac{5}{4} \end{bmatrix} - \left(-\frac{1}{2}\right) \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right) \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} -\frac{3}{4} & \frac{3}{4} \\ \frac{3}{4} & -\frac{3}{4} \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Now forward elimination is applied to solve for the eigenvector $\vec{v}$. The augmented matrix is

$$\left[\begin{array}{cc|c} -\frac{3}{4} & \frac{3}{4} & 0 \\ \frac{3}{4} & -\frac{3}{4} & 0 \end{array}\right]$$

$$
R_2 = R_2 + R_1 \Longrightarrow \left[\begin{array}{cc|c} -\frac{3}{4} & \frac{3}{4} & 0 \\ 0 & 0 & 0 \end{array}\right]
$$

Therefore the system in Echelon form is

$$
\begin{bmatrix} -\frac{3}{4} & \frac{3}{4} \\ 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}
$$

The free variables are $\{v_2\}$ and the leading variables are $\{v_1\}$. Let $v_2 = t$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\{v_1 = t\}$

Hence the solution is

$$
\begin{bmatrix} t \\ t \end{bmatrix} = \begin{bmatrix} t \\ t \end{bmatrix}
$$

Since there is one free Variable, we have found one eigenvector associated with this eigenvalue. The above can be written as

$$
\begin{bmatrix} t \\ t \end{bmatrix} = t \begin{bmatrix} 1 \\ 1 \end{bmatrix}
$$

Let $t = 1$ the eigenvector becomes

$$
\begin{bmatrix} t \\ t \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}
$$

The following table gives a summary of this result. It shows for each eigenvalue the algebraic multiplicity m, and its geometric multiplicity k and the eigenvectors associated with the eigenvalue. If $m > k$ then the eigenvalue is defective which means the number of normal linearly independent eigenvectors associated with this eigenvalue (called the geometric multiplicity k) does not equal the algebraic multiplicity m, and we need to determine an additional $m - k$ generalized eigenvectors for this eigenvalue.

	multiplicity			
eigenvalue	algebraic m	geometric k	defective?	eigenvectors
-2	1	1	No	$\begin{bmatrix} -1 \\ 1 \end{bmatrix}$
$-\frac{1}{2}$	1	1	No	$\begin{bmatrix} 1 \\ 1 \end{bmatrix}$

Now that we found the eigenvalues and associated eigenvectors, we will go over each eigenvalue and generate the solution basis. The only problem we need to take care of is if the eigenvalue is defective. Since eigenvalue -2 is real and distinct then the corresponding eigenvector solution is

$$\begin{aligned}\vec{x}_1(t) &= \vec{v}_1 e^{-2t} \\ &= \begin{bmatrix} -1 \\ 1 \end{bmatrix} e^{-2t}\end{aligned}$$

Since eigenvalue $-\frac{1}{2}$ is real and distinct then the corresponding eigenvector solution is

$$\begin{aligned}\vec{x}_2(t) &= \vec{v}_2 e^{-\frac{t}{2}} \\ &= \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{-\frac{t}{2}}\end{aligned}$$

Therefore the homogeneous solution is

$$\vec{x}_h(t) = c_1 \vec{x}_1(t) + c_2 \vec{x}_2(t)$$

Which is written as

$$\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = c_1 \begin{bmatrix} -\mathrm{e}^{-2t} \\ \mathrm{e}^{-2t} \end{bmatrix} + c_2 \begin{bmatrix} \mathrm{e}^{-\frac{t}{2}} \\ \mathrm{e}^{-\frac{t}{2}} \end{bmatrix}$$

Now that we found homogeneous solution above, we need to find a particular solution $\vec{x}_p(t)$. We will use Variation of parameters. The fundamental matrix is

$$\Phi = \begin{bmatrix} \vec{x}_1 & \vec{x}_2 & \cdots \end{bmatrix}$$

Where $\vec{x}_i$ are the solution basis found above. Therefore the fundamental matrix is

$$\Phi(t) = \begin{bmatrix} -\mathrm{e}^{-2t} & \mathrm{e}^{-\frac{t}{2}} \\ \mathrm{e}^{-2t} & \mathrm{e}^{-\frac{t}{2}} \end{bmatrix}$$

The particular solution is then given by

$$\vec{x}_p(t) = \Phi \int \Phi^{-1} \vec{G}(t)\, dt$$

But

$$\Phi^{-1} = \begin{bmatrix} -\frac{e^{2t}}{2} & \frac{e^{2t}}{2} \\ \frac{e^{\frac{t}{2}}}{2} & \frac{e^{\frac{t}{2}}}{2} \end{bmatrix}$$

Hence

$$\begin{aligned}
\vec{x}_p(t) &= \begin{bmatrix} -e^{-2t} & e^{-\frac{t}{2}} \\ e^{-2t} & e^{-\frac{t}{2}} \end{bmatrix} \int \begin{bmatrix} -\frac{e^{2t}}{2} & \frac{e^{2t}}{2} \\ \frac{e^{\frac{t}{2}}}{2} & \frac{e^{\frac{t}{2}}}{2} \end{bmatrix} \begin{bmatrix} 2t \\ e^t \end{bmatrix} dt \\
&= \begin{bmatrix} -e^{-2t} & e^{-\frac{t}{2}} \\ e^{-2t} & e^{-\frac{t}{2}} \end{bmatrix} \int \begin{bmatrix} -e^{2t}t + \frac{e^{3t}}{2} \\ t\,e^{\frac{t}{2}} + \frac{e^{\frac{3t}{2}}}{2} \end{bmatrix} dt \\
&= \begin{bmatrix} -e^{-2t} & e^{-\frac{t}{2}} \\ e^{-2t} & e^{-\frac{t}{2}} \end{bmatrix} \begin{bmatrix} \frac{(1-2t)e^{2t}}{4} + \frac{e^{3t}}{6} \\ 2(t-2)\,e^{\frac{t}{2}} + \frac{e^{\frac{3t}{2}}}{3} \end{bmatrix} \\
&= \begin{bmatrix} \frac{e^t}{6} + \frac{5t}{2} - \frac{17}{4} \\ \frac{e^t}{2} + \frac{3t}{2} - \frac{15}{4} \end{bmatrix}
\end{aligned}$$

Now that we found particular solution, the final solution is

$$\begin{aligned}
\vec{x}(t) &= \vec{x}_h(t) + \vec{x}_p(t) \\
\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} &= \begin{bmatrix} -c_1 e^{-2t} \\ c_1 e^{-2t} \end{bmatrix} + \begin{bmatrix} c_2 e^{-\frac{t}{2}} \\ c_2 e^{-\frac{t}{2}} \end{bmatrix} + \begin{bmatrix} \frac{e^t}{6} + \frac{5t}{2} - \frac{17}{4} \\ \frac{e^t}{2} + \frac{3t}{2} - \frac{15}{4} \end{bmatrix}
\end{aligned}$$

Which becomes

$$\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = \begin{bmatrix} \frac{\left(2\,e^{3t} + 30\,e^{2t}t - 51\,e^{2t} + 12c_2 e^{\frac{3t}{2}} - 12c_1\right)e^{-2t}}{12} \\ \frac{\left(2\,e^{3t} + 6\,e^{2t}t - 15\,e^{2t} + 4c_2 e^{\frac{3t}{2}} + 4c_1\right)e^{-2t}}{4} \end{bmatrix}$$

18.9.3 Maple step by step solution

Let's solve

$\left[x_1'(t) = -\frac{5x_1(t)}{4} + \frac{3x_2(t)}{4} + 2t, x_2'(t) = \frac{3x_1(t)}{4} - \frac{5x_2(t)}{4} + e^t\right]$

- Define vector

$$\underline{\vec{x}}(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}$$

- Convert system into a vector equation

$$\underline{\vec{x}}'(t) = \begin{bmatrix} -\frac{5}{4} & \frac{3}{4} \\ \frac{3}{4} & -\frac{5}{4} \end{bmatrix} \cdot \underline{\vec{x}}(t) + \begin{bmatrix} 2t \\ \mathrm{e}^t \end{bmatrix}$$

- System to solve

$$\underline{\vec{x}}'(t) = \begin{bmatrix} -\frac{5}{4} & \frac{3}{4} \\ \frac{3}{4} & -\frac{5}{4} \end{bmatrix} \cdot \underline{\vec{x}}(t) + \begin{bmatrix} 2t \\ \mathrm{e}^t \end{bmatrix}$$

- Define the forcing function

$$\vec{f}(t) = \begin{bmatrix} 2t \\ \mathrm{e}^t \end{bmatrix}$$

- Define the coefficient matrix

$$A = \begin{bmatrix} -\frac{5}{4} & \frac{3}{4} \\ \frac{3}{4} & -\frac{5}{4} \end{bmatrix}$$

- Rewrite the system as

$$\underline{\vec{x}}'(t) = A \cdot \underline{\vec{x}}(t) + \vec{f}$$

- To solve the system, find the eigenvalues and eigenvectors of A
- Eigenpairs of A

$$\left[\left[-2, \begin{bmatrix} -1 \\ 1 \end{bmatrix}\right], \left[-\frac{1}{2}, \begin{bmatrix} 1 \\ 1 \end{bmatrix}\right]\right]$$

- Consider eigenpair

$$\left[-2, \begin{bmatrix} -1 \\ 1 \end{bmatrix}\right]$$

- Solution to homogeneous system from eigenpair

$$\underline{\vec{x}}_1 = \mathrm{e}^{-2t} \cdot \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

- Consider eigenpair

$\left[-\frac{1}{2}, \begin{bmatrix} 1 \\ 1 \end{bmatrix}\right]$

- Solution to homogeneous system from eigenpair

$\overrightarrow{x}_2 = e^{-\frac{t}{2}} \cdot \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

- General solution of the system of ODEs can be written in terms of the particular solution $\overrightarrow{x}$

$\overrightarrow{x}(t) = c_1\overrightarrow{x}_1 + c_2\overrightarrow{x}_2 + \overrightarrow{x}_p(t)$

☐ Fundamental matrix

- Let $\phi(t)$ be the matrix whose columns are the independent solutions of the homogeneous syst

$\phi(t) = \begin{bmatrix} -e^{-2t} & e^{-\frac{t}{2}} \\ e^{-2t} & e^{-\frac{t}{2}} \end{bmatrix}$

- The fundamental matrix, $\Phi(t)$ is a normalized version of $\phi(t)$ satisfying $\Phi(0) = I$ where I is th

$\Phi(t) = \phi(t) \cdot \frac{1}{\phi(0)}$

- Substitute the value of $\phi(t)$ and $\phi(0)$

$\Phi(t) = \begin{bmatrix} -e^{-2t} & e^{-\frac{t}{2}} \\ e^{-2t} & e^{-\frac{t}{2}} \end{bmatrix} \cdot \frac{1}{\begin{bmatrix} -1 & 1 \\ 1 & 1 \end{bmatrix}}$

- Evaluate and simplify to get the fundamental matrix

$\Phi(t) = \begin{bmatrix} \frac{e^{-2t}}{2} + \frac{e^{-\frac{t}{2}}}{2} & \frac{e^{-\frac{t}{2}}}{2} - \frac{e^{-2t}}{2} \\ \frac{e^{-\frac{t}{2}}}{2} - \frac{e^{-2t}}{2} & \frac{e^{-2t}}{2} + \frac{e^{-\frac{t}{2}}}{2} \end{bmatrix}$

☐ Find a particular solution of the system of ODEs using variation of parameters

- Let the particular solution be the fundamental matrix multiplied by $\overrightarrow{v}(t)$ and solve for $\overrightarrow{v}(t)$

$\overrightarrow{x}_p(t) = \Phi(t) \cdot \overrightarrow{v}(t)$

- Take the derivative of the particular solution

$\overrightarrow{x}'_p(t) = \Phi'(t) \cdot \overrightarrow{v}(t) + \Phi(t) \cdot \overrightarrow{v}'(t)$

- Substitute particular solution and its derivative into the system of ODEs

$\Phi'(t) \cdot \overrightarrow{v}(t) + \Phi(t) \cdot \overrightarrow{v}'(t) = A \cdot \Phi(t) \cdot \overrightarrow{v}(t) + \overrightarrow{f}(t)$

- The fundamental matrix has columns that are solutions to the homogeneous system so its der

  $A\cdot\Phi(t)\cdot\vec{v}(t)+\Phi(t)\cdot\vec{v}'(t)=A\cdot\Phi(t)\cdot\vec{v}(t)+\vec{f}(t)$

- Cancel like terms

  $\Phi(t)\cdot\vec{v}'(t)=\vec{f}(t)$

- Multiply by the inverse of the fundamental matrix

  $\vec{v}'(t)=\frac{1}{\Phi(t)}\cdot\vec{f}(t)$

- Integrate to solve for $\vec{v}(t)$

  $\vec{v}(t)=\int_0^t\frac{1}{\Phi(s)}\cdot\vec{f}(s)\,ds$

- Plug $\vec{v}(t)$ into the equation for the particular solution

  $\underline{\vec{x}}_p(t)=\Phi(t)\cdot\left(\int_0^t\frac{1}{\Phi(s)}\cdot\vec{f}(s)\,ds\right)$

- Plug in the fundamental matrix and the forcing function and compute

  $$\underline{\vec{x}}_p(t)=\left[\begin{array}{c}\frac{\left(2\,e^{3t}+30\,e^{2t}t-51\,e^{2t}+44\,e^{\frac{3t}{2}}+5\right)e^{-2t}}{12}\\ \frac{\left(6\,e^{3t}+18\,e^{2t}t-45\,e^{2t}+44\,e^{\frac{3t}{2}}-5\right)e^{-2t}}{12}\end{array}\right]$$

- Plug particular solution back into general solution

  $$\underline{\vec{x}}(t)=c_1\underline{\vec{x}}_1+c_2\underline{\vec{x}}_2+\left[\begin{array}{c}\frac{\left(2\,e^{3t}+30\,e^{2t}t-51\,e^{2t}+44\,e^{\frac{3t}{2}}+5\right)e^{-2t}}{12}\\ \frac{\left(6\,e^{3t}+18\,e^{2t}t-45\,e^{2t}+44\,e^{\frac{3t}{2}}-5\right)e^{-2t}}{12}\end{array}\right]$$

- Substitute in vector of dependent variables

  $$\left[\begin{array}{c}x_1(t)\\ x_2(t)\end{array}\right]=\left[\begin{array}{c}\frac{\left(2\,e^{3t}+30\,e^{2t}t-51\,e^{2t}+12c_2e^{\frac{3t}{2}}+44\,e^{\frac{3t}{2}}-12c_1+5\right)e^{-2t}}{12}\\ \frac{\left(6\,e^{3t}+18\,e^{2t}t-45\,e^{2t}+12c_2e^{\frac{3t}{2}}+44\,e^{\frac{3t}{2}}+12c_1-5\right)e^{-2t}}{12}\end{array}\right]$$

- Solution to the system of ODEs

  $$\left\{x_1(t)=\frac{\left(2\,e^{3t}+30\,e^{2t}t-51\,e^{2t}+12c_2e^{\frac{3t}{2}}+44\,e^{\frac{3t}{2}}-12c_1+5\right)e^{-2t}}{12},x_2(t)=\frac{\left(6\,e^{3t}+18\,e^{2t}t-45\,e^{2t}+12c_2e^{\frac{3t}{2}}+44\,e^{\frac{3t}{2}}+12c_1-5\right)}{12}\right.$$

✓ Solution by Maple

Time used: 0.031 (sec). Leaf size: 51

```
dsolve([diff(x__1(t),t)=-5/4*x__1(t)+3/4*x__2(t)+2*t,diff(x__2(t),t)=3/4*x__1(t)-5/4*x__2(t)
```

$$x_1(t) = c_2\mathrm{e}^{-2t} + c_1\mathrm{e}^{-\frac{t}{2}} - \frac{17}{4} + \frac{\mathrm{e}^t}{6} + \frac{5t}{2}$$
$$x_2(t) = -c_2\mathrm{e}^{-2t} + c_1\mathrm{e}^{-\frac{t}{2}} + \frac{\mathrm{e}^t}{2} - \frac{15}{4} + \frac{3t}{2}$$

✓ Solution by Mathematica

Time used: 0.349 (sec). Leaf size: 101

```
DSolve[{x1'[t]==-5/4*x1[t]+3/4*x2[t]+2*t,x2'[t]==3/4*x1[t]-5/4*x2[t]+Exp[t]},{x1[t],x2[t]},t
```

$$\mathrm{x1}(t) \to \frac{1}{12}\left(30t + 2e^t + 6(c_1 - c_2)e^{-2t} + 6(c_1 + c_2)e^{-t/2} - 51\right)$$
$$\mathrm{x2}(t) \to \frac{1}{4}e^{-2t}\left(3e^{2t}(2t - 5) + 2e^{3t} + 2(c_1 + c_2)e^{3t/2} - 2c_1 + 2c_2\right)$$

18.10 problem 10

Internal problem ID [787]
Internal file name [`OUTPUT/787_Sunday_June_05_2022_01_49_43_AM_10721479/index.tex`]

Book: Elementary differential equations and boundary value problems, 10th ed., Boyce and DiPrima
Section: Chapter 7.9, Nonhomogeneous Linear Systems. page 447
Problem number: 10.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"system of linear ODEs"**

Solve

$$
\begin{aligned}
x_1'(t) &= -3x_1(t) + \sqrt{2}\, x_2(t) + \mathrm{e}^{-t} \\
x_2'(t) &= \sqrt{2}\, x_1(t) - 2x_2(t) - \mathrm{e}^{-t}
\end{aligned}
$$

18.10.1 Solution using Matrix exponential method

In this method, we will assume we have found the matrix exponential e^{At} allready. There are different methods to determine this but will not be shown here. This is a system of linear ODE's given as

$$\vec{x}'(t) = A\,\vec{x}(t) + \vec{G}(t)$$

Or

$$\begin{bmatrix} x_1'(t) \\ x_2'(t) \end{bmatrix} = \begin{bmatrix} -3 & \sqrt{2} \\ \sqrt{2} & -2 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} + \begin{bmatrix} \mathrm{e}^{-t} \\ -\mathrm{e}^{-t} \end{bmatrix}$$

Since the system is nonhomogeneous, then the solution is given by

$$\vec{x}(t) = \vec{x}_h(t) + \vec{x}_p(t)$$

Where $\vec{x}_h(t)$ is the homogeneous solution to $\vec{x}'(t) = A\,\vec{x}(t)$ and $\vec{x}_p(t)$ is a particular solution to $\vec{x}'(t) = A\,\vec{x}(t) + \vec{G}(t)$. The particular solution will be found using variation

of parameters method applied to the fundamental matrix. For the above matrix A, the matrix exponential can be found to be

$$e^{At} = \begin{bmatrix} \frac{2\,e^{-4t}}{3}+\frac{e^{-t}}{3} & \frac{\sqrt{2}\,(e^{-t}-e^{-4t})}{3} \\ \frac{\sqrt{2}\,(e^{-t}-e^{-4t})}{3} & \frac{e^{-4t}}{3}+\frac{2\,e^{-t}}{3} \end{bmatrix}$$

Therefore the homogeneous solution is

$$\begin{aligned} \vec{x}_h(t) &= e^{At}\vec{c} \\ &= \begin{bmatrix} \frac{2\,e^{-4t}}{3}+\frac{e^{-t}}{3} & \frac{\sqrt{2}\,(e^{-t}-e^{-4t})}{3} \\ \frac{\sqrt{2}\,(e^{-t}-e^{-4t})}{3} & \frac{e^{-4t}}{3}+\frac{2\,e^{-t}}{3} \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} \\ &= \begin{bmatrix} \left(\frac{2\,e^{-4t}}{3}+\frac{e^{-t}}{3}\right)c_1 + \frac{\sqrt{2}\,(e^{-t}-e^{-4t})c_2}{3} \\ \frac{\sqrt{2}\,(e^{-t}-e^{-4t})c_1}{3} + \left(\frac{e^{-4t}}{3}+\frac{2\,e^{-t}}{3}\right)c_2 \end{bmatrix} \end{aligned}$$

The particular solution given by

$$\vec{x}_p(t) = e^{At}\int e^{-At}\vec{G}(t)\,dt$$

But

$$\begin{aligned} e^{-At} &= (e^{At})^{-1} \\ &= \begin{bmatrix} \frac{(2\,e^{3t}+1)e^{t}}{3} & -\frac{\sqrt{2}\,e^{t}\,(e^{3t}-1)}{3} \\ -\frac{\sqrt{2}\,e^{t}\,(e^{3t}-1)}{3} & \frac{(e^{3t}+2)e^{t}}{3} \end{bmatrix} \end{aligned}$$

Hence

$$\begin{aligned} \vec{x}_p(t) &= \begin{bmatrix} \frac{2\,e^{-4t}}{3}+\frac{e^{-t}}{3} & \frac{\sqrt{2}\,(e^{-t}-e^{-4t})}{3} \\ \frac{\sqrt{2}\,(e^{-t}-e^{-4t})}{3} & \frac{e^{-4t}}{3}+\frac{2\,e^{-t}}{3} \end{bmatrix} \int \begin{bmatrix} \frac{(2\,e^{3t}+1)e^{t}}{3} & -\frac{\sqrt{2}\,e^{t}\,(e^{3t}-1)}{3} \\ -\frac{\sqrt{2}\,e^{t}\,(e^{3t}-1)}{3} & \frac{(e^{3t}+2)e^{t}}{3} \end{bmatrix} \begin{bmatrix} e^{-t} \\ -e^{-t} \end{bmatrix} dt \\ &= \begin{bmatrix} \frac{2\,e^{-4t}}{3}+\frac{e^{-t}}{3} & \frac{\sqrt{2}\,(e^{-t}-e^{-4t})}{3} \\ \frac{\sqrt{2}\,(e^{-t}-e^{-4t})}{3} & \frac{e^{-4t}}{3}+\frac{2\,e^{-t}}{3} \end{bmatrix} \begin{bmatrix} \frac{\left(2+\sqrt{2}\right)e^{3t}}{9} - \frac{t\left(\sqrt{2}-1\right)}{3} \\ \frac{\left(-1-\sqrt{2}\right)e^{3t}}{9} + \frac{t\left(-2+\sqrt{2}\right)}{3} \end{bmatrix} \\ &= \begin{bmatrix} \frac{\left((1-3t)\sqrt{2}+3t+2\right)e^{-t}}{9} \\ \frac{\left((-1+3t)\sqrt{2}-6t-1\right)e^{-t}}{9} \end{bmatrix} \end{aligned}$$

Hence the complete solution is

$$\vec{x}(t) = \vec{x}_h(t) + \vec{x}_p(t)$$
$$= \left[\begin{array}{c} \frac{\left((-3t+3c_2+1)\sqrt{2}+3t+3c_1+2\right)e^{-t}}{9} + \frac{2\left(-\frac{c_2\sqrt{2}}{2}+c_1\right)e^{-4t}}{3} \\ \frac{\left((3t+3c_1-1)\sqrt{2}-6t+6c_2-1\right)e^{-t}}{9} - \frac{e^{-4t}\left(c_1\sqrt{2}-c_2\right)}{3} \end{array}\right]$$

18.10.2 Solution using explicit Eigenvalue and Eigenvector method

This is a system of linear ODE's given as

$$\vec{x}'(t) = A\,\vec{x}(t) + \vec{G}(t)$$

Or

$$\left[\begin{array}{c} x_1'(t) \\ x_2'(t) \end{array}\right] = \left[\begin{array}{cc} -3 & \sqrt{2} \\ \sqrt{2} & -2 \end{array}\right] \left[\begin{array}{c} x_1(t) \\ x_2(t) \end{array}\right] + \left[\begin{array}{c} e^{-t} \\ -e^{-t} \end{array}\right]$$

Since the system is nonhomogeneous, then the solution is given by

$$\vec{x}(t) = \vec{x}_h(t) + \vec{x}_p(t)$$

Where $\vec{x}_h(t)$ is the homogeneous solution to $\vec{x}'(t) = A\,\vec{x}(t)$ and $\vec{x}_p(t)$ is a particular solution to $\vec{x}'(t) = A\,\vec{x}(t) + \vec{G}(t)$. The particular solution will be found using variation of parameters method applied to the fundamental matrix.

The first step is find the homogeneous solution. We start by finding the eigenvalues of A. This is done by solving the following equation for the eigenvalues λ

$$\det\left(A - \lambda I\right) = 0$$

Expanding gives

$$\det\left(\left[\begin{array}{cc} -3 & \sqrt{2} \\ \sqrt{2} & -2 \end{array}\right] - \lambda \left[\begin{array}{cc} 1 & 0 \\ 0 & 1 \end{array}\right]\right) = 0$$

Therefore

$$\det\left(\left[\begin{array}{cc} -3-\lambda & \sqrt{2} \\ \sqrt{2} & -2-\lambda \end{array}\right]\right) = 0$$

Which gives the characteristic equation

$$\lambda^2 + 5\lambda + 4 = 0$$

The roots of the above are the eigenvalues.

$$\lambda_1 = -4$$
$$\lambda_2 = -1$$

This table summarises the above result

eigenvalue	algebraic multiplicity	type of eigenvalue
-1	1	real eigenvalue
-4	1	real eigenvalue

Now the eigenvector for each eigenvalue are found.

Considering the eigenvalue $\lambda_1 = -4$

We need to solve $A\vec{v} = \lambda\vec{v}$ or $(A - \lambda I)\vec{v} = \vec{0}$ which becomes

$$\left(\begin{bmatrix} -3 & \sqrt{2} \\ \sqrt{2} & -2 \end{bmatrix} - (-4)\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right)\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$
$$\begin{bmatrix} 1 & \sqrt{2} \\ \sqrt{2} & 2 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Now forward elimination is applied to solve for the eigenvector $\vec{v}$. The augmented matrix is

$$\left[\begin{array}{cc|c} 1 & \sqrt{2} & 0 \\ \sqrt{2} & 2 & 0 \end{array}\right]$$

$$R_2 = R_2 - \sqrt{2}\,R_1 \Longrightarrow \left[\begin{array}{cc|c} 1 & \sqrt{2} & 0 \\ 0 & 0 & 0 \end{array}\right]$$

Therefore the system in Echelon form is

$$\begin{bmatrix} 1 & \sqrt{2} \\ 0 & 0 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

The free variables are $\{v_2\}$ and the leading variables are $\{v_1\}$. Let $v_2 = t$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\{v_1 = -\sqrt{2}\,t\}$

Hence the solution is

$$\begin{bmatrix} -\sqrt{2}\,t \\ t \end{bmatrix} = \begin{bmatrix} -\sqrt{2}\,t \\ t \end{bmatrix}$$

Since there is one free Variable, we have found one eigenvector associated with this eigenvalue. The above can be written as

$$\begin{bmatrix} -\sqrt{2}\,t \\ t \end{bmatrix} = t \begin{bmatrix} -\sqrt{2} \\ 1 \end{bmatrix}$$

Let $t = 1$ the eigenvector becomes

$$\begin{bmatrix} -\sqrt{2}\,t \\ t \end{bmatrix} = \begin{bmatrix} -\sqrt{2} \\ 1 \end{bmatrix}$$

Considering the eigenvalue $\lambda_2 = -1$

We need to solve $A\vec{v} = \lambda \vec{v}$ or $(A - \lambda I)\vec{v} = \vec{0}$ which becomes

$$\left(\begin{bmatrix} -3 & \sqrt{2} \\ \sqrt{2} & -2 \end{bmatrix} - (-1) \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right) \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$
$$\begin{bmatrix} -2 & \sqrt{2} \\ \sqrt{2} & -1 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Now forward elimination is applied to solve for the eigenvector $\vec{v}$. The augmented matrix is

$$\left[\begin{array}{cc|c} -2 & \sqrt{2} & 0 \\ \sqrt{2} & -1 & 0 \end{array}\right]$$

$$R_2 = R_2 + \frac{\sqrt{2}\,R_1}{2} \Longrightarrow \left[\begin{array}{cc|c} -2 & \sqrt{2} & 0 \\ 0 & 0 & 0 \end{array}\right]$$

Therefore the system in Echelon form is

$$\begin{bmatrix} -2 & \sqrt{2} \\ 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

The free variables are $\{v_2\}$ and the leading variables are $\{v_1\}$. Let $v_2 = t$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\left\{v_1 = \frac{\sqrt{2}\, t}{2}\right\}$

Hence the solution is

$$\begin{bmatrix} \frac{\sqrt{2}\, t}{2} \\ t \end{bmatrix} = \begin{bmatrix} \frac{\sqrt{2}\, t}{2} \\ t \end{bmatrix}$$

Since there is one free Variable, we have found one eigenvector associated with this eigenvalue. The above can be written as

$$\begin{bmatrix} \frac{\sqrt{2}\, t}{2} \\ t \end{bmatrix} = t \begin{bmatrix} \frac{\sqrt{2}}{2} \\ 1 \end{bmatrix}$$

Let $t = 1$ the eigenvector becomes

$$\begin{bmatrix} \frac{\sqrt{2}\, t}{2} \\ t \end{bmatrix} = \begin{bmatrix} \frac{\sqrt{2}}{2} \\ 1 \end{bmatrix}$$

Which is normalized to

$$\begin{bmatrix} \frac{\sqrt{2}\, t}{2} \\ t \end{bmatrix} = \begin{bmatrix} \sqrt{2} \\ 2 \end{bmatrix}$$

The following table gives a summary of this result. It shows for each eigenvalue the algebraic multiplicity m, and its geometric multiplicity k and the eigenvectors associated with the eigenvalue. If $m > k$ then the eigenvalue is defective which means the number of normal linearly independent eigenvectors associated with this eigenvalue (called the geometric multiplicity k) does not equal the algebraic multiplicity m, and we need to determine an additional $m - k$ generalized eigenvectors for this eigenvalue.

	multiplicity			
eigenvalue	algebraic m	geometric k	defective?	eigenvectors
-4	1	1	No	$\begin{bmatrix} -\sqrt{2} \\ 1 \end{bmatrix}$
-1	1	1	No	$\begin{bmatrix} \frac{\sqrt{2}}{2} \\ 1 \end{bmatrix}$

Now that we found the eigenvalues and associated eigenvectors, we will go over each eigenvalue and generate the solution basis. The only problem we need to take care of is if the eigenvalue is defective. Since eigenvalue -4 is real and distinct then the corresponding eigenvector solution is

$$\begin{aligned}\vec{x}_1(t) &= \vec{v}_1 e^{-4t} \\ &= \begin{bmatrix} -\sqrt{2} \\ 1 \end{bmatrix} e^{-4t}\end{aligned}$$

Since eigenvalue -1 is real and distinct then the corresponding eigenvector solution is

$$\begin{aligned}\vec{x}_2(t) &= \vec{v}_2 e^{-t} \\ &= \begin{bmatrix} \frac{\sqrt{2}}{2} \\ 1 \end{bmatrix} e^{-t}\end{aligned}$$

Therefore the homogeneous solution is

$$\vec{x}_h(t) = c_1 \vec{x}_1(t) + c_2 \vec{x}_2(t)$$

Which is written as

$$\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = c_1 \begin{bmatrix} -\sqrt{2}\, \mathrm{e}^{-4t} \\ \mathrm{e}^{-4t} \end{bmatrix} + c_2 \begin{bmatrix} \frac{\sqrt{2}\, \mathrm{e}^{-t}}{2} \\ \mathrm{e}^{-t} \end{bmatrix}$$

Now that we found homogeneous solution above, we need to find a particular solution $\vec{x}_p(t)$. We will use Variation of parameters. The fundamental matrix is

$$\Phi = \begin{bmatrix} \vec{x}_1 & \vec{x}_2 & \cdots \end{bmatrix}$$

Where $\vec{x}_i$ are the solution basis found above. Therefore the fundamental matrix is

$$\Phi(t) = \begin{bmatrix} -\sqrt{2}\, \mathrm{e}^{-4t} & \frac{\sqrt{2}\, \mathrm{e}^{-t}}{2} \\ \mathrm{e}^{-4t} & \mathrm{e}^{-t} \end{bmatrix}$$

The particular solution is then given by

$$\vec{x}_p(t) = \Phi \int \Phi^{-1} \vec{G}(t)\, dt$$

But

$$\Phi^{-1} = \begin{bmatrix} -\frac{\sqrt{2}\,e^{4t}}{3} & \frac{e^{4t}}{3} \\ \frac{\sqrt{2}\,e^{t}}{3} & \frac{2\,e^{t}}{3} \end{bmatrix}$$

Hence

$$\begin{aligned}
\vec{x}_p(t) &= \begin{bmatrix} -\sqrt{2}\,e^{-4t} & \frac{\sqrt{2}\,e^{-t}}{2} \\ e^{-4t} & e^{-t} \end{bmatrix} \int \begin{bmatrix} -\frac{\sqrt{2}\,e^{4t}}{3} & \frac{e^{4t}}{3} \\ \frac{\sqrt{2}\,e^{t}}{3} & \frac{2\,e^{t}}{3} \end{bmatrix} \begin{bmatrix} e^{-t} \\ -e^{-t} \end{bmatrix} dt \\
&= \begin{bmatrix} -\sqrt{2}\,e^{-4t} & \frac{\sqrt{2}\,e^{-t}}{2} \\ e^{-4t} & e^{-t} \end{bmatrix} \int \begin{bmatrix} -\frac{e^{3t}\left(1+\sqrt{2}\right)}{3} \\ \frac{\sqrt{2}}{3} - \frac{2}{3} \end{bmatrix} dt \\
&= \begin{bmatrix} -\sqrt{2}\,e^{-4t} & \frac{\sqrt{2}\,e^{-t}}{2} \\ e^{-4t} & e^{-t} \end{bmatrix} \begin{bmatrix} -\frac{e^{3t}\left(1+\sqrt{2}\right)}{9} \\ \frac{t\left(-2+\sqrt{2}\right)}{3} \end{bmatrix} \\
&= \begin{bmatrix} \frac{\left((1-3t)\sqrt{2}+3t+2\right)e^{-t}}{9} \\ \frac{\left((-1+3t)\sqrt{2}-6t-1\right)e^{-t}}{9} \end{bmatrix}
\end{aligned}$$

Now that we found particular solution, the final solution is

$$\vec{x}(t) = \vec{x}_h(t) + \vec{x}_p(t)$$

$$\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = \begin{bmatrix} -c_1\sqrt{2}\,e^{-4t} \\ c_1 e^{-4t} \end{bmatrix} + \begin{bmatrix} \frac{c_2\sqrt{2}\,e^{-t}}{2} \\ c_2 e^{-t} \end{bmatrix} + \begin{bmatrix} \frac{\left((1-3t)\sqrt{2}+3t+2\right)e^{-t}}{9} \\ \frac{\left((-1+3t)\sqrt{2}-6t-1\right)e^{-t}}{9} \end{bmatrix}$$

Which becomes

$$\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = \begin{bmatrix} \frac{\left((-6t+9c_2+2)\sqrt{2}+6t+4\right)e^{-t}}{18} - c_1\sqrt{2}\,e^{-4t} \\ \frac{\left((-1+3t)\sqrt{2}-6t+9c_2-1\right)e^{-t}}{9} + c_1 e^{-4t} \end{bmatrix}$$

✓ Solution by Maple

Time used: 0.063 (sec). Leaf size: 90

```
dsolve([diff(x__1(t),t)=-3*x__1(t)+sqrt(2)*x__2(t)+exp(-t),diff(x__2(t),t)=sqrt(2)*x__1(t)-2
```

$$x_1(t) = c_2 e^{-t} + e^{-4t} c_1 - \frac{t\,e^{-t}\sqrt{2}}{3} + \frac{t\,e^{-t}}{3}$$
$$x_2(t) = -\frac{2t\,e^{-t}}{3} - \frac{e^{-t}}{3} + \sqrt{2}\,e^{-t} c_2 + \frac{t\,e^{-t}\sqrt{2}}{3} - \frac{\sqrt{2}\,e^{-4t} c_1}{2} - \frac{\sqrt{2}\,e^{-t}}{3}$$

✓ Solution by Mathematica

Time used: 0.035 (sec). Leaf size: 128

```
DSolve[{x1'[t]==-3*x1[t]+Sqrt[2]*x2[t]+Exp[-t],x2'[t]==Sqrt[2]*x1[t]-2*x2[t]-Exp[-t]},{x1[t]
```

$$\text{x1}(t) \to \frac{1}{9}e^{-4t}\left(e^{3t}\left(-3\left(\sqrt{2}-1\right)t + \sqrt{2} + 2 + 3c_1 + 3\sqrt{2}c_2\right) + 6c_1 - 3\sqrt{2}c_2\right)$$
$$\text{x2}(t) \to \frac{1}{9}e^{-4t}\left(e^{3t}\left(3\left(\sqrt{2}-2\right)t - \sqrt{2} - 1 + 3\sqrt{2}c_1 + 6c_2\right) - 3\sqrt{2}c_1 + 3c_2\right)$$

18.11 problem 11

Internal problem ID [788]
Internal file name [`OUTPUT/788_Sunday_June_05_2022_01_49_46_AM_47915974/index.tex`]

Book: Elementary differential equations and boundary value problems, 10th ed., Boyce and DiPrima
Section: Chapter 7.9, Nonhomogeneous Linear Systems. page 447
Problem number: 11.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"system of linear ODEs"**

Solve

$$
\begin{aligned}
x_1'(t) &= 2x_1(t) - 5x_2(t) \\
x_2'(t) &= x_1(t) - 2x_2(t) + \cos(t)
\end{aligned}
$$

18.11.1 Solution using Matrix exponential method

In this method, we will assume we have found the matrix exponential e^{At} allready. There are different methods to determine this but will not be shown here. This is a system of linear ODE's given as

$$\vec{x}'(t) = A\,\vec{x}(t) + \vec{G}(t)$$

Or

$$\begin{bmatrix} x_1'(t) \\ x_2'(t) \end{bmatrix} = \begin{bmatrix} 2 & -5 \\ 1 & -2 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} + \begin{bmatrix} 0 \\ \cos(t) \end{bmatrix}$$

Since the system is nonhomogeneous, then the solution is given by

$$\vec{x}(t) = \vec{x}_h(t) + \vec{x}_p(t)$$

Where $\vec{x}_h(t)$ is the homogeneous solution to $\vec{x}'(t) = A\,\vec{x}(t)$ and $\vec{x}_p(t)$ is a particular solution to $\vec{x}'(t) = A\,\vec{x}(t) + \vec{G}(t)$. The particular solution will be found using variation

of parameters method applied to the fundamental matrix. For the above matrix A, the matrix exponential can be found to be

$$e^{At} = \begin{bmatrix} \cos(t) + 2\sin(t) & -5\sin(t) \\ \sin(t) & \cos(t) - 2\sin(t) \end{bmatrix}$$

Therefore the homogeneous solution is

$$\begin{aligned}
\vec{x}_h(t) &= e^{At}\vec{c} \\
&= \begin{bmatrix} \cos(t) + 2\sin(t) & -5\sin(t) \\ \sin(t) & \cos(t) - 2\sin(t) \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} \\
&= \begin{bmatrix} (\cos(t) + 2\sin(t))\, c_1 - 5\sin(t)\, c_2 \\ \sin(t)\, c_1 + (\cos(t) - 2\sin(t))\, c_2 \end{bmatrix} \\
&= \begin{bmatrix} (2c_1 - 5c_2)\sin(t) + c_1\cos(t) \\ (c_1 - 2c_2)\sin(t) + c_2\cos(t) \end{bmatrix}
\end{aligned}$$

The particular solution given by

$$\vec{x}_p(t) = e^{At} \int e^{-At}\vec{G}(t)\, dt$$

But

$$\begin{aligned}
e^{-At} &= (e^{At})^{-1} \\
&= \begin{bmatrix} \cos(t) - 2\sin(t) & 5\sin(t) \\ -\sin(t) & \cos(t) + 2\sin(t) \end{bmatrix}
\end{aligned}$$

Hence

$$\begin{aligned}
\vec{x}_p(t) &= \begin{bmatrix} \cos(t) + 2\sin(t) & -5\sin(t) \\ \sin(t) & \cos(t) - 2\sin(t) \end{bmatrix} \int \begin{bmatrix} \cos(t) - 2\sin(t) & 5\sin(t) \\ -\sin(t) & \cos(t) + 2\sin(t) \end{bmatrix} \begin{bmatrix} 0 \\ \cos(t) \end{bmatrix} \\
&= \begin{bmatrix} \cos(t) + 2\sin(t) & -5\sin(t) \\ \sin(t) & \cos(t) - 2\sin(t) \end{bmatrix} \begin{bmatrix} \frac{5\sin(t)^2}{2} \\ \frac{\sin(t)\cos(t)}{2} + \frac{t}{2} - \cos(t)^2 \end{bmatrix} \\
&= \begin{bmatrix} -\frac{5\sin(t)(t-2)}{2} \\ \frac{(t-2)\cos(t)}{2} + \frac{(-2t+5)\sin(t)}{2} \end{bmatrix}
\end{aligned}$$

Hence the complete solution is

$$\vec{x}(t) = \vec{x}_h(t) + \vec{x}_p(t)$$

$$= \begin{bmatrix} \frac{(-5t+4c_1-10c_2+10)\sin(t)}{2} + c_1\cos(t) \\ \frac{(-2t+2c_1-4c_2+5)\sin(t)}{2} + \frac{\cos(t)(t+2c_2-2)}{2} \end{bmatrix}$$

18.11.2 Solution using explicit Eigenvalue and Eigenvector method

This is a system of linear ODE's given as

$$\vec{x}'(t) = A\,\vec{x}(t) + \vec{G}(t)$$

Or

$$\begin{bmatrix} x_1'(t) \\ x_2'(t) \end{bmatrix} = \begin{bmatrix} 2 & -5 \\ 1 & -2 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} + \begin{bmatrix} 0 \\ \cos(t) \end{bmatrix}$$

Since the system is nonhomogeneous, then the solution is given by

$$\vec{x}(t) = \vec{x}_h(t) + \vec{x}_p(t)$$

Where $\vec{x}_h(t)$ is the homogeneous solution to $\vec{x}'(t) = A\,\vec{x}(t)$ and $\vec{x}_p(t)$ is a particular solution to $\vec{x}'(t) = A\,\vec{x}(t) + \vec{G}(t)$. The particular solution will be found using variation of parameters method applied to the fundamental matrix.

The first step is find the homogeneous solution. We start by finding the eigenvalues of A. This is done by solving the following equation for the eigenvalues λ

$$\det(A - \lambda I) = 0$$

Expanding gives

$$\det\left(\begin{bmatrix} 2 & -5 \\ 1 & -2 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right) = 0$$

Therefore

$$\det\left(\begin{bmatrix} 2-\lambda & -5 \\ 1 & -2-\lambda \end{bmatrix}\right) = 0$$

Which gives the characteristic equation

$$\lambda^2 + 1 = 0$$

The roots of the above are the eigenvalues.

$$\lambda_1 = i$$
$$\lambda_2 = -i$$

This table summarises the above result

eigenvalue	algebraic multiplicity	type of eigenvalue
$-i$	1	complex eigenvalue
i	1	complex eigenvalue

Now the eigenvector for each eigenvalue are found.

Considering the eigenvalue $\lambda_1 = -i$

We need to solve $A\vec{v} = \lambda\vec{v}$ or $(A - \lambda I)\vec{v} = \vec{0}$ which becomes

$$\left(\begin{bmatrix} 2 & -5 \\ 1 & -2 \end{bmatrix} - (-i)\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right)\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$
$$\begin{bmatrix} 2+i & -5 \\ 1 & -2+i \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Now forward elimination is applied to solve for the eigenvector $\vec{v}$. The augmented matrix is

$$\left[\begin{array}{cc|c} 2+i & -5 & 0 \\ 1 & -2+i & 0 \end{array}\right]$$

$$R_2 = R_2 + \left(-\frac{2}{5} + \frac{i}{5}\right) R_1 \Longrightarrow \left[\begin{array}{cc|c} 2+i & -5 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

Therefore the system in Echelon form is

$$\begin{bmatrix} 2+i & -5 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

The free variables are $\{v_2\}$ and the leading variables are $\{v_1\}$. Let $v_2 = t$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\{v_1 = (2-i)\,t\}$

Hence the solution is

$$\begin{bmatrix}(2-I)\,t\\ t\end{bmatrix} = \begin{bmatrix}(2-i)\,t\\ t\end{bmatrix}$$

Since there is one free Variable, we have found one eigenvector associated with this eigenvalue. The above can be written as

$$\begin{bmatrix}(2-I)\,t\\ t\end{bmatrix} = t\begin{bmatrix}2-i\\ 1\end{bmatrix}$$

Let $t=1$ the eigenvector becomes

$$\begin{bmatrix}(2-I)\,t\\ t\end{bmatrix} = \begin{bmatrix}2-i\\ 1\end{bmatrix}$$

Considering the eigenvalue $\lambda_2 = i$

We need to solve $A\vec{v} = \lambda\vec{v}$ or $(A-\lambda I)\vec{v} = \vec{0}$ which becomes

$$\left(\begin{bmatrix}2 & -5\\ 1 & -2\end{bmatrix} - (i)\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}\right)\begin{bmatrix}v_1\\ v_2\end{bmatrix} = \begin{bmatrix}0\\ 0\end{bmatrix}$$

$$\begin{bmatrix}2-i & -5\\ 1 & -2-i\end{bmatrix}\begin{bmatrix}v_1\\ v_2\end{bmatrix} = \begin{bmatrix}0\\ 0\end{bmatrix}$$

Now forward elimination is applied to solve for the eigenvector $\vec{v}$. The augmented matrix is

$$\left[\begin{array}{cc|c}2-i & -5 & 0\\ 1 & -2-i & 0\end{array}\right]$$

$$R_2 = R_2 + \left(-\frac{2}{5} - \frac{i}{5}\right) R_1 \Longrightarrow \left[\begin{array}{cc|c}2-i & -5 & 0\\ 0 & 0 & 0\end{array}\right]$$

Therefore the system in Echelon form is

$$\begin{bmatrix}2-i & -5\\ 0 & 0\end{bmatrix}\begin{bmatrix}v_1\\ v_2\end{bmatrix} = \begin{bmatrix}0\\ 0\end{bmatrix}$$

The free variables are $\{v_2\}$ and the leading variables are $\{v_1\}$. Let $v_2 = t$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\{v_1 = (2+i)\,t\}$

Hence the solution is

$$\begin{bmatrix} (2+I)\,t \\ t \end{bmatrix} = \begin{bmatrix} (2+i)\,t \\ t \end{bmatrix}$$

Since there is one free Variable, we have found one eigenvector associated with this eigenvalue. The above can be written as

$$\begin{bmatrix} (2+I)\,t \\ t \end{bmatrix} = t \begin{bmatrix} 2+i \\ 1 \end{bmatrix}$$

Let $t = 1$ the eigenvector becomes

$$\begin{bmatrix} (2+I)\,t \\ t \end{bmatrix} = \begin{bmatrix} 2+i \\ 1 \end{bmatrix}$$

The following table gives a summary of this result. It shows for each eigenvalue the algebraic multiplicity m, and its geometric multiplicity k and the eigenvectors associated with the eigenvalue. If $m > k$ then the eigenvalue is defective which means the number of normal linearly independent eigenvectors associated with this eigenvalue (called the geometric multiplicity k) does not equal the algebraic multiplicity m, and we need to determine an additional $m - k$ generalized eigenvectors for this eigenvalue.

	multiplicity			
eigenvalue	algebraic m	geometric k	defective?	eigenvectors
i	1	1	No	$\begin{bmatrix} 2+i \\ 1 \end{bmatrix}$
$-i$	1	1	No	$\begin{bmatrix} 2-i \\ 1 \end{bmatrix}$

Now that we found the eigenvalues and associated eigenvectors, we will go over each eigenvalue and generate the solution basis. The only problem we need to take care of is if the eigenvalue is defective. Therefore the homogeneous solution is

$$\vec{x}_h(t) = c_1 \vec{x}_1(t) + c_2 \vec{x}_2(t)$$

Which is written as

$$\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = c_1 \begin{bmatrix} (2+i)\,\mathrm{e}^{it} \\ \mathrm{e}^{it} \end{bmatrix} + c_2 \begin{bmatrix} (2-i)\,\mathrm{e}^{-it} \\ \mathrm{e}^{-it} \end{bmatrix}$$

Now that we found homogeneous solution above, we need to find a particular solution $\vec{x}_p(t)$. We will use Variation of parameters. The fundamental matrix is

$$\Phi = \begin{bmatrix} \vec{x}_1 & \vec{x}_2 & \cdots \end{bmatrix}$$

Where $\vec{x}_i$ are the solution basis found above. Therefore the fundamental matrix is

$$\Phi(t) = \begin{bmatrix} (2+i)\,e^{it} & (2-i)\,e^{-it} \\ e^{it} & e^{-it} \end{bmatrix}$$

The particular solution is then given by

$$\vec{x}_p(t) = \Phi \int \Phi^{-1}\vec{G}(t)\,dt$$

But

$$\Phi^{-1} = \begin{bmatrix} -\frac{i e^{-it}}{2} & \left(\frac{1}{2}+i\right)e^{-it} \\ \frac{i e^{it}}{2} & \left(\frac{1}{2}-i\right)e^{it} \end{bmatrix}$$

Hence

$$\begin{aligned}
\vec{x}_p(t) &= \begin{bmatrix} (2+i)\,e^{it} & (2-i)\,e^{-it} \\ e^{it} & e^{-it} \end{bmatrix} \int \begin{bmatrix} -\frac{i e^{-it}}{2} & \left(\frac{1}{2}+i\right)e^{-it} \\ \frac{i e^{it}}{2} & \left(\frac{1}{2}-i\right)e^{it} \end{bmatrix} \begin{bmatrix} 0 \\ \cos(t) \end{bmatrix} dt \\
&= \begin{bmatrix} (2+i)\,e^{it} & (2-i)\,e^{-it} \\ e^{it} & e^{-it} \end{bmatrix} \int \begin{bmatrix} \left(\frac{1}{2}+i\right)e^{-it}\cos(t) \\ \left(\frac{1}{2}-i\right)e^{it}\cos(t) \end{bmatrix} dt \\
&= \begin{bmatrix} (2+i)\,e^{it} & (2-i)\,e^{-it} \\ e^{it} & e^{-it} \end{bmatrix} \begin{bmatrix} \left(\frac{1}{4}+\frac{i}{2}\right)(it\sin(t) + t\cos(t) + \sin(t))\,e^{-it} \\ \left(\frac{1}{4}-\frac{i}{2}\right)t + \left(-\frac{1}{4}-\frac{i}{8}\right)e^{2it} \end{bmatrix} \\
&= \begin{bmatrix} -\frac{5\cos(t)}{8} + \frac{5(i-4t)\sin(t)}{8} \\ \frac{\cos(t)(-2-i+4t)}{8} + \frac{\left(\frac{3}{2}+i-4t\right)\sin(t)}{4} \end{bmatrix}
\end{aligned}$$

Now that we found particular solution, the final solution is

$$\vec{x}(t) = \vec{x}_h(t) + \vec{x}_p(t)$$

$$\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = \begin{bmatrix} (2+i)\,c_1 e^{it} \\ c_1 e^{it} \end{bmatrix} + \begin{bmatrix} (2-i)\,c_2 e^{-it} \\ c_2 e^{-it} \end{bmatrix} + \begin{bmatrix} -\frac{5\cos(t)}{8} + \frac{5(i-4t)\sin(t)}{8} \\ \frac{\cos(t)(-2-i+4t)}{8} + \frac{\left(\frac{3}{2}+i-4t\right)\sin(t)}{4} \end{bmatrix}$$

Which becomes

$$\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = \begin{bmatrix} \frac{((16-8i)c_2-10it-5)\mathrm{e}^{-it}}{8} + \frac{5\left(\left(\frac{8}{5}+\frac{4i}{5}\right)c_1+it\right)\mathrm{e}^{it}}{4} \\ \frac{(8c_1-2-i+4t+8c_2)\cos(t)}{8} + \left(ic_1 - t + \frac{3}{8} + \frac{1}{4}i - ic_2\right)\sin(t) \end{bmatrix}$$

18.11.3 Maple step by step solution

Let's solve

$[x_1'(t) = 2x_1(t) - 5x_2(t), x_2'(t) = x_1(t) - 2x_2(t) + \cos(t)]$

- Define vector

$\underline{x}^{\rightarrow}(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}$

- Convert system into a vector equation

$\underline{x}^{\rightarrow}{}'(t) = \begin{bmatrix} 2 & -5 \\ 1 & -2 \end{bmatrix} \cdot \underline{x}^{\rightarrow}(t) + \begin{bmatrix} 0 \\ \cos(t) \end{bmatrix}$

- System to solve

$\underline{x}^{\rightarrow}{}'(t) = \begin{bmatrix} 2 & -5 \\ 1 & -2 \end{bmatrix} \cdot \underline{x}^{\rightarrow}(t) + \begin{bmatrix} 0 \\ \cos(t) \end{bmatrix}$

- Define the forcing function

$\vec{f}(t) = \begin{bmatrix} 0 \\ \cos(t) \end{bmatrix}$

- Define the coefficient matrix

$A = \begin{bmatrix} 2 & -5 \\ 1 & -2 \end{bmatrix}$

- Rewrite the system as

$\underline{x}^{\rightarrow}{}'(t) = A \cdot \underline{x}^{\rightarrow}(t) + \vec{f}$

- To solve the system, find the eigenvalues and eigenvectors of A
- Eigenpairs of A

$$\left[\left[-I, \begin{bmatrix} 2-I \\ 1 \end{bmatrix}\right], \left[I, \begin{bmatrix} 2+I \\ 1 \end{bmatrix}\right]\right]$$

- Consider complex eigenpair, complex conjugate eigenvalue can be ignored

$$\left[-I, \begin{bmatrix} 2-I \\ 1 \end{bmatrix}\right]$$

- Solution from eigenpair

$$e^{-It} \cdot \begin{bmatrix} 2-I \\ 1 \end{bmatrix}$$

- Use Euler identity to write solution in terms of sin and cos

$$(\cos(t) - I\sin(t)) \cdot \begin{bmatrix} 2-I \\ 1 \end{bmatrix}$$

- Simplify expression

$$\begin{bmatrix} (2-I)(\cos(t) - I\sin(t)) \\ \cos(t) - I\sin(t) \end{bmatrix}$$

- Both real and imaginary parts are solutions to the homogeneous system

$$\left[\vec{x}_1(t) = \begin{bmatrix} 2\cos(t) - \sin(t) \\ \cos(t) \end{bmatrix}, \vec{x}_2(t) = \begin{bmatrix} -\cos(t) - 2\sin(t) \\ -\sin(t) \end{bmatrix}\right]$$

- General solution of the system of ODEs can be written in terms of the particular solution $\vec{x}$

$$\vec{x}(t) = c_1\vec{x}_1(t) + c_2\vec{x}_2(t) + \vec{x}_p(t)$$

☐ Fundamental matrix

- Let $\phi(t)$ be the matrix whose columns are the independent solutions of the homogeneous syst

$$\phi(t) = \begin{bmatrix} 2\cos(t) - \sin(t) & -\cos(t) - 2\sin(t) \\ \cos(t) & -\sin(t) \end{bmatrix}$$

- The fundamental matrix, $\Phi(t)$ is a normalized version of $\phi(t)$ satisfying $\Phi(0) = I$ where I is th

$$\Phi(t) = \phi(t) \cdot \frac{1}{\phi(0)}$$

- Substitute the value of $\phi(t)$ and $\phi(0)$

$$\Phi(t) = \begin{bmatrix} 2\cos(t) - \sin(t) & -\cos(t) - 2\sin(t) \\ \cos(t) & -\sin(t) \end{bmatrix} \cdot \frac{1}{\begin{bmatrix} 2 & -1 \\ 1 & 0 \end{bmatrix}}$$

- Evaluate and simplify to get the fundamental matrix

$$\Phi(t) = \begin{bmatrix} \cos(t) + 2\sin(t) & -5\sin(t) \\ \sin(t) & \cos(t) - 2\sin(t) \end{bmatrix}$$

☐ Find a particular solution of the system of ODEs using variation of parameters

- Let the particular solution be the fundamental matrix multiplied by $\vec{v}(t)$ and solve for $\vec{v}(t)$

$$\vec{x}_p(t) = \Phi(t) \cdot \vec{v}(t)$$

- Take the derivative of the particular solution

$$\vec{x}_p{}'(t) = \Phi'(t) \cdot \vec{v}(t) + \Phi(t) \cdot \vec{v}'(t)$$

- Substitute particular solution and its derivative into the system of ODEs

$$\Phi'(t) \cdot \vec{v}(t) + \Phi(t) \cdot \vec{v}'(t) = A \cdot \Phi(t) \cdot \vec{v}(t) + \vec{f}(t)$$

- The fundamental matrix has columns that are solutions to the homogeneous system so its der

$$A \cdot \Phi(t) \cdot \vec{v}(t) + \Phi(t) \cdot \vec{v}'(t) = A \cdot \Phi(t) \cdot \vec{v}(t) + \vec{f}(t)$$

- Cancel like terms

$$\Phi(t) \cdot \vec{v}'(t) = \vec{f}(t)$$

- Multiply by the inverse of the fundamental matrix

$$\vec{v}'(t) = \frac{1}{\Phi(t)} \cdot \vec{f}(t)$$

- Integrate to solve for $\vec{v}(t)$

$$\vec{v}(t) = \int_0^t \frac{1}{\Phi(s)} \cdot \vec{f}(s)\, ds$$

- Plug $\vec{v}(t)$ into the equation for the particular solution

$$\vec{x}_p(t) = \Phi(t) \cdot \left(\int_0^t \frac{1}{\Phi(s)} \cdot \vec{f}(s)\, ds\right)$$

- Plug in the fundamental matrix and the forcing function and compute

$$\vec{x}_p(t) = \begin{bmatrix} -\frac{5t\sin(t)}{2} \\ -t\sin(t) + \frac{\sin(t)}{2} + \frac{t\cos(t)}{2} \end{bmatrix}$$

- Plug particular solution back into general solution

$$\overrightarrow{x}(t) = c_1 \overrightarrow{x}_1(t) + c_2 \overrightarrow{x}_2(t) + \begin{bmatrix} -\frac{5t\sin(t)}{2} \\ -t\sin(t) + \frac{\sin(t)}{2} + \frac{t\cos(t)}{2} \end{bmatrix}$$

- Substitute in vector of dependent variables

$$\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = \begin{bmatrix} \frac{(-5t-2c_1-4c_2)\sin(t)}{2} + 2\left(c_1 - \frac{c_2}{2}\right)\cos(t) \\ \frac{(-2t-2c_2+1)\sin(t)}{2} + \frac{\cos(t)(2c_1+t)}{2} \end{bmatrix}$$

- Solution to the system of ODEs

$$\left\{x_1(t) = \frac{(-5t-2c_1-4c_2)\sin(t)}{2} + 2\left(c_1 - \frac{c_2}{2}\right)\cos(t), x_2(t) = \frac{(-2t-2c_2+1)\sin(t)}{2} + \frac{\cos(t)(2c_1+t)}{2}\right\}$$

✓ Solution by Maple

Time used: 0.031 (sec). Leaf size: 57

```
dsolve([diff(x__1(t),t)=2*x__1(t)-5*x__2(t)+0,diff(x__2(t),t)=1*x__1(t)-2*x__2(t)+cos(t)],si
```

$$x_1(t) = c_2 \sin(t) + c_1 \cos(t) - \frac{5\sin(t)\,t}{2}$$
$$x_2(t) = -\frac{c_2 \cos(t)}{5} + \frac{c_1 \sin(t)}{5} + \frac{\cos(t)\,t}{2} + \frac{\sin(t)}{2} + \frac{2c_2 \sin(t)}{5} + \frac{2c_1 \cos(t)}{5} - \sin(t)\,t$$

✓ Solution by Mathematica

Time used: 0.024 (sec). Leaf size: 60

```
DSolve[{x1'[t]==2*x1[t]-5*x2[t]+0,x2'[t]==1*x1[t]-2*x2[t]-Cos[t]},{x1[t],x2[t]},t,IncludeSin
```

$$\text{x1}(t) \to \left(\frac{5}{2} + c_1\right)\cos(t) + \frac{1}{2}(5t + 4c_1 - 10c_2)\sin(t)$$
$$\text{x2}(t) \to \left(-\frac{t}{2} + 1 + c_2\right)\cos(t) + (t + c_1 - 2c_2)\sin(t)$$

18.12 problem 12

Internal problem ID [789]
Internal file name [`OUTPUT/789_Sunday_June_05_2022_01_49_48_AM_41360408/index.tex`]

Book: Elementary differential equations and boundary value problems, 10th ed., Boyce and DiPrima
Section: Chapter 7.9, Nonhomogeneous Linear Systems. page 447
Problem number: 12.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"system of linear ODEs"**

Solve

$$
\begin{aligned}
x_1'(t) &= 2x_1(t) - 5x_2(t) + \csc(t) \\
x_2'(t) &= x_1(t) - 2x_2(t) + \sec(t)
\end{aligned}
$$

18.12.1 Solution using Matrix exponential method

In this method, we will assume we have found the matrix exponential e^{At} allready. There are different methods to determine this but will not be shown here. This is a system of linear ODE's given as

$$\vec{x}'(t) = A\,\vec{x}(t) + \vec{G}(t)$$

Or

$$\begin{bmatrix} x_1'(t) \\ x_2'(t) \end{bmatrix} = \begin{bmatrix} 2 & -5 \\ 1 & -2 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} + \begin{bmatrix} \csc(t) \\ \sec(t) \end{bmatrix}$$

Since the system is nonhomogeneous, then the solution is given by

$$\vec{x}(t) = \vec{x}_h(t) + \vec{x}_p(t)$$

Where $\vec{x}_h(t)$ is the homogeneous solution to $\vec{x}'(t) = A\,\vec{x}(t)$ and $\vec{x}_p(t)$ is a particular solution to $\vec{x}'(t) = A\,\vec{x}(t) + \vec{G}(t)$. The particular solution will be found using variation

of parameters method applied to the fundamental matrix. For the above matrix A, the matrix exponential can be found to be

$$e^{At} = \begin{bmatrix} \cos(t)+2\sin(t) & -5\sin(t) \\ \sin(t) & \cos(t)-2\sin(t) \end{bmatrix}$$

Therefore the homogeneous solution is

$$\begin{aligned} \vec{x}_h(t) &= e^{At}\vec{c} \\ &= \begin{bmatrix} \cos(t)+2\sin(t) & -5\sin(t) \\ \sin(t) & \cos(t)-2\sin(t) \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} \\ &= \begin{bmatrix} (\cos(t)+2\sin(t))\,c_1-5\sin(t)\,c_2 \\ \sin(t)\,c_1+(\cos(t)-2\sin(t))\,c_2 \end{bmatrix} \\ &= \begin{bmatrix} (2c_1-5c_2)\sin(t)+c_1\cos(t) \\ (c_1-2c_2)\sin(t)+c_2\cos(t) \end{bmatrix} \end{aligned}$$

The particular solution given by

$$\vec{x}_p(t) = e^{At}\int e^{-At}\vec{G}(t)\,dt$$

But

$$\begin{aligned} e^{-At} &= (e^{At})^{-1} \\ &= \begin{bmatrix} \cos(t)-2\sin(t) & 5\sin(t) \\ -\sin(t) & \cos(t)+2\sin(t) \end{bmatrix} \end{aligned}$$

Hence

$$\begin{aligned} \vec{x}_p(t) &= \begin{bmatrix} \cos(t)+2\sin(t) & -5\sin(t) \\ \sin(t) & \cos(t)-2\sin(t) \end{bmatrix} \int \begin{bmatrix} \cos(t)-2\sin(t) & 5\sin(t) \\ -\sin(t) & \cos(t)+2\sin(t) \end{bmatrix} \begin{bmatrix} \csc(t) \\ \sec(t) \end{bmatrix} d \\ &= \begin{bmatrix} \cos(t)+2\sin(t) & -5\sin(t) \\ \sin(t) & \cos(t)-2\sin(t) \end{bmatrix} \begin{bmatrix} -2t+\ln(\sin(t))+5\ln(\sec(t)) \\ -2\ln(\cos(t)) \end{bmatrix} \\ &= \begin{bmatrix} (\cos(t)+2\sin(t))\,(-2t+\ln(\sin(t))+5\ln(\sec(t)))+10\sin(t)\ln(\cos(t)) \\ (-2\cos(t)+4\sin(t))\ln(\cos(t))-2\left(t-\frac{5\ln(\sec(t))}{2}-\frac{\ln(\sin(t))}{2}\right)\sin(t) \end{bmatrix} \end{aligned}$$

Hence the complete solution is

$$\vec{x}(t) = \vec{x}_h(t) + \vec{x}_p(t)$$

$$= \left[\begin{array}{c} (2c_1 - 5c_2)\sin(t) + c_1\cos(t) + (\cos(t) + 2\sin(t))(-2t + \ln(\sin(t)) + 5\ln(\sec(t))) + 10\sin(\\ (-2\cos(t) + 4\sin(t))\ln(\cos(t)) + 5\sin(t)\ln(\sec(t)) + \sin(t)\ln(\sin(t)) + (-2t + c_1 - 2c_2)\sin \end{array}\right.$$

18.12.2 Solution using explicit Eigenvalue and Eigenvector method

This is a system of linear ODE's given as

$$\vec{x}'(t) = A\,\vec{x}(t) + \vec{G}(t)$$

Or

$$\begin{bmatrix} x_1'(t) \\ x_2'(t) \end{bmatrix} = \begin{bmatrix} 2 & -5 \\ 1 & -2 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} + \begin{bmatrix} \csc(t) \\ \sec(t) \end{bmatrix}$$

Since the system is nonhomogeneous, then the solution is given by

$$\vec{x}(t) = \vec{x}_h(t) + \vec{x}_p(t)$$

Where $\vec{x}_h(t)$ is the homogeneous solution to $\vec{x}'(t) = A\,\vec{x}(t)$ and $\vec{x}_p(t)$ is a particular solution to $\vec{x}'(t) = A\,\vec{x}(t) + \vec{G}(t)$. The particular solution will be found using variation of parameters method applied to the fundamental matrix.

The first step is find the homogeneous solution. We start by finding the eigenvalues of A. This is done by solving the following equation for the eigenvalues λ

$$\det(A - \lambda I) = 0$$

Expanding gives

$$\det\left(\begin{bmatrix} 2 & -5 \\ 1 & -2 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right) = 0$$

Therefore

$$\det\left(\begin{bmatrix} 2-\lambda & -5 \\ 1 & -2-\lambda \end{bmatrix}\right) = 0$$

Which gives the characteristic equation

$$\lambda^2 + 1 = 0$$

The roots of the above are the eigenvalues.

$$\lambda_1 = i$$
$$\lambda_2 = -i$$

This table summarises the above result

eigenvalue	algebraic multiplicity	type of eigenvalue
$-i$	1	complex eigenvalue
i	1	complex eigenvalue

Now the eigenvector for each eigenvalue are found.

Considering the eigenvalue $\lambda_1 = -i$

We need to solve $A\vec{v} = \lambda\vec{v}$ or $(A - \lambda I)\vec{v} = \vec{0}$ which becomes

$$\left(\begin{bmatrix} 2 & -5 \\ 1 & -2 \end{bmatrix} - (-i)\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right)\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$
$$\begin{bmatrix} 2+i & -5 \\ 1 & -2+i \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Now forward elimination is applied to solve for the eigenvector $\vec{v}$. The augmented matrix is

$$\left[\begin{array}{cc|c} 2+i & -5 & 0 \\ 1 & -2+i & 0 \end{array}\right]$$

$$R_2 = R_2 + \left(-\frac{2}{5} + \frac{i}{5}\right) R_1 \Longrightarrow \left[\begin{array}{cc|c} 2+i & -5 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

Therefore the system in Echelon form is

$$\begin{bmatrix} 2+i & -5 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

The free variables are $\{v_2\}$ and the leading variables are $\{v_1\}$. Let $v_2 = t$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\{v_1 = (2 - i)\, t\}$

Hence the solution is

$$\begin{bmatrix}(2-\mathrm{I})\,t\\ t\end{bmatrix}=\begin{bmatrix}(2-i)\,t\\ t\end{bmatrix}$$

Since there is one free Variable, we have found one eigenvector associated with this eigenvalue. The above can be written as

$$\begin{bmatrix}(2-\mathrm{I})\,t\\ t\end{bmatrix}=t\begin{bmatrix}2-i\\ 1\end{bmatrix}$$

Let $t=1$ the eigenvector becomes

$$\begin{bmatrix}(2-\mathrm{I})\,t\\ t\end{bmatrix}=\begin{bmatrix}2-i\\ 1\end{bmatrix}$$

Considering the eigenvalue $\lambda_2=i$

We need to solve $A\vec{v}=\lambda\vec{v}$ or $(A-\lambda I)\vec{v}=\vec{0}$ which becomes

$$\left(\begin{bmatrix}2&-5\\1&-2\end{bmatrix}-(i)\begin{bmatrix}1&0\\0&1\end{bmatrix}\right)\begin{bmatrix}v_1\\v_2\end{bmatrix}=\begin{bmatrix}0\\0\end{bmatrix}$$
$$\begin{bmatrix}2-i&-5\\1&-2-i\end{bmatrix}\begin{bmatrix}v_1\\v_2\end{bmatrix}=\begin{bmatrix}0\\0\end{bmatrix}$$

Now forward elimination is applied to solve for the eigenvector $\vec{v}$. The augmented matrix is

$$\left[\begin{array}{cc|c}2-i&-5&0\\1&-2-i&0\end{array}\right]$$

$$R_2=R_2+\left(-\frac{2}{5}-\frac{i}{5}\right)R_1\Longrightarrow\left[\begin{array}{cc|c}2-i&-5&0\\0&0&0\end{array}\right]$$

Therefore the system in Echelon form is

$$\begin{bmatrix}2-i&-5\\0&0\end{bmatrix}\begin{bmatrix}v_1\\v_2\end{bmatrix}=\begin{bmatrix}0\\0\end{bmatrix}$$

The free variables are $\{v_2\}$ and the leading variables are $\{v_1\}$. Let $v_2=t$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\{v_1=(2+i)\,t\}$

Hence the solution is

$$\begin{bmatrix} (2+I)\,t \\ t \end{bmatrix} = \begin{bmatrix} (2+i)\,t \\ t \end{bmatrix}$$

Since there is one free Variable, we have found one eigenvector associated with this eigenvalue. The above can be written as

$$\begin{bmatrix} (2+I)\,t \\ t \end{bmatrix} = t \begin{bmatrix} 2+i \\ 1 \end{bmatrix}$$

Let $t = 1$ the eigenvector becomes

$$\begin{bmatrix} (2+I)\,t \\ t \end{bmatrix} = \begin{bmatrix} 2+i \\ 1 \end{bmatrix}$$

The following table gives a summary of this result. It shows for each eigenvalue the algebraic multiplicity m, and its geometric multiplicity k and the eigenvectors associated with the eigenvalue. If $m > k$ then the eigenvalue is defective which means the number of normal linearly independent eigenvectors associated with this eigenvalue (called the geometric multiplicity k) does not equal the algebraic multiplicity m, and we need to determine an additional $m - k$ generalized eigenvectors for this eigenvalue.

	multiplicity			
eigenvalue	algebraic m	geometric k	defective?	eigenvectors
i	1	1	No	$\begin{bmatrix} 2+i \\ 1 \end{bmatrix}$
$-i$	1	1	No	$\begin{bmatrix} 2-i \\ 1 \end{bmatrix}$

Now that we found the eigenvalues and associated eigenvectors, we will go over each eigenvalue and generate the solution basis. The only problem we need to take care of is if the eigenvalue is defective. Therefore the homogeneous solution is

$$\vec{x}_h(t) = c_1 \vec{x}_1(t) + c_2 \vec{x}_2(t)$$

Which is written as

$$\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = c_1 \begin{bmatrix} (2+i)\,\mathrm{e}^{it} \\ \mathrm{e}^{it} \end{bmatrix} + c_2 \begin{bmatrix} (2-i)\,\mathrm{e}^{-it} \\ \mathrm{e}^{-it} \end{bmatrix}$$

Now that we found homogeneous solution above, we need to find a particular solution $\vec{x}_p(t)$. We will use Variation of parameters. The fundamental matrix is

$$\Phi = \begin{bmatrix} \vec{x}_1 & \vec{x}_2 & \cdots \end{bmatrix}$$

Where $\vec{x}_i$ are the solution basis found above. Therefore the fundamental matrix is

$$\Phi(t) = \begin{bmatrix} (2+i)\,\mathrm{e}^{it} & (2-i)\,\mathrm{e}^{-it} \\ \mathrm{e}^{it} & \mathrm{e}^{-it} \end{bmatrix}$$

The particular solution is then given by

$$\vec{x}_p(t) = \Phi \int \Phi^{-1}\vec{G}(t)\,dt$$

But

$$\Phi^{-1} = \begin{bmatrix} -\frac{i\mathrm{e}^{-it}}{2} & \left(\frac{1}{2}+i\right)\mathrm{e}^{-it} \\ \frac{i\mathrm{e}^{it}}{2} & \left(\frac{1}{2}-i\right)\mathrm{e}^{it} \end{bmatrix}$$

Hence

$$\begin{aligned}
\vec{x}_p(t) &= \begin{bmatrix} (2+i)\,\mathrm{e}^{it} & (2-i)\,\mathrm{e}^{-it} \\ \mathrm{e}^{it} & \mathrm{e}^{-it} \end{bmatrix} \int \begin{bmatrix} -\frac{i\mathrm{e}^{-it}}{2} & \left(\frac{1}{2}+i\right)\mathrm{e}^{-it} \\ \frac{i\mathrm{e}^{it}}{2} & \left(\frac{1}{2}-i\right)\mathrm{e}^{it} \end{bmatrix} \begin{bmatrix} \csc(t) \\ \sec(t) \end{bmatrix} dt \\
&= \begin{bmatrix} (2+i)\,\mathrm{e}^{it} & (2-i)\,\mathrm{e}^{-it} \\ \mathrm{e}^{it} & \mathrm{e}^{-it} \end{bmatrix} \int \begin{bmatrix} -\frac{((-1-2i)\sec(t)+i\csc(t))\mathrm{e}^{-it}}{2} \\ \frac{((1-2i)\sec(t)+i\csc(t))\mathrm{e}^{it}}{2} \end{bmatrix} dt \\
&= \begin{bmatrix} (2+i)\,\mathrm{e}^{it} & (2-i)\,\mathrm{e}^{-it} \\ \mathrm{e}^{it} & \mathrm{e}^{-it} \end{bmatrix} \begin{bmatrix} \left(-1+\frac{i}{2}\right)\ln\left(\mathrm{e}^{2it}+1\right) - \frac{i\ln(\mathrm{e}^{it}-1)}{2} - \frac{i\ln(\mathrm{e}^{it}+1)}{2} + 2\ln\left(\mathrm{e}^{it}\right) \\ \frac{i\ln(\mathrm{e}^{2it}-1)}{2} + \left(-1-\frac{i}{2}\right)\ln\left(\mathrm{e}^{2it}+1\right) \end{bmatrix} \\
&= \begin{bmatrix} \left(\frac{1}{2}+i\right)\mathrm{e}^{-it}\ln\left(\mathrm{e}^{2it}-1\right) + \frac{(-5\,\mathrm{e}^{-it}-5\,\mathrm{e}^{it})\ln(\mathrm{e}^{2it}+1)}{2} + 2\left(\left(\frac{1}{4}-\frac{i}{2}\right)\ln\left(\mathrm{e}^{it}-1\right) + \left(\frac{1}{4}-\frac{i}{2}\right)\ln\left(\mathrm{e}^{it}+1\right) + \right. \\ \frac{((-2-i)\mathrm{e}^{-it}+(-2+i)\mathrm{e}^{it})\ln(\mathrm{e}^{2it}+1)}{2} + \frac{i\mathrm{e}^{-it}\ln(\mathrm{e}^{2it}-1)}{2} - \frac{(i\ln(\mathrm{e}^{it}-1)+i\ln(\mathrm{e}^{it}+1)-4\ln(\mathrm{e}^{it}))\mathrm{e}^{it}}{2} \end{bmatrix}
\end{aligned}$$

Now that we found particular solution, the final solution is

$$\vec{x}(t) = \vec{x}_h(t) + \vec{x}_p(t)$$

$$\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = \begin{bmatrix} (2+i)\,c_1\mathrm{e}^{it} \\ c_1\mathrm{e}^{it} \end{bmatrix} + \begin{bmatrix} (2-i)\,c_2\mathrm{e}^{-it} \\ c_2\mathrm{e}^{-it} \end{bmatrix} + \begin{bmatrix} \left(\frac{1}{2}+i\right)\mathrm{e}^{-it}\ln\left(\mathrm{e}^{2it}-1\right) + \frac{(-5\,\mathrm{e}^{-it}-5\,\mathrm{e}^{it})\ln(\mathrm{e}^{2it}+1)}{2} + \\ \frac{((-2-i)\mathrm{e}^{-it}+(-2+i)\mathrm{e}^{it})\ln(\mathrm{e}^{2it}+1)}{2} + \end{bmatrix}$$

Which becomes

$$\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = \begin{bmatrix} \left(\frac{1}{2}+i\right) e^{-it} \ln\left(e^{2it}-1\right) + \frac{\left(-5\, e^{-it}-5\, e^{it}\right) \ln\left(e^{2it}+1\right)}{2} + \left(\frac{1}{2}-i\right) e^{it} \ln\left(e^{it}-1\right) + \left(\frac{1}{2}-i\right) e^{it} \ln\left(e^{it} \\ \frac{\left((-2-i)e^{-it}+(-2+i)e^{it}\right) \ln\left(e^{2it}+1\right)}{2} + \frac{i e^{-it} \ln\left(e^{2it}-1\right)}{2} - \frac{i e^{it} \ln\left(e^{it}-1\right)}{2} - \frac{i e^{it} \ln\left(e^{it}+1\right)}{2} \end{bmatrix}$$

18.12.3 Maple step by step solution

Let's solve

$[x_1'(t) = 2x_1(t) - 5x_2(t) + \csc(t), x_2'(t) = x_1(t) - 2x_2(t) + \sec(t)]$

- Define vector

$$\underline{\vec{x}}(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}$$

- Convert system into a vector equation

$$\underline{\vec{x}}'(t) = \begin{bmatrix} 2 & -5 \\ 1 & -2 \end{bmatrix} \cdot \underline{\vec{x}}(t) + \begin{bmatrix} \csc(t) \\ \sec(t) \end{bmatrix}$$

- System to solve

$$\underline{\vec{x}}'(t) = \begin{bmatrix} 2 & -5 \\ 1 & -2 \end{bmatrix} \cdot \underline{\vec{x}}(t) + \begin{bmatrix} \csc(t) \\ \sec(t) \end{bmatrix}$$

- Define the forcing function

$$\vec{f}(t) = \begin{bmatrix} \csc(t) \\ \sec(t) \end{bmatrix}$$

- Define the coefficient matrix

$$A = \begin{bmatrix} 2 & -5 \\ 1 & -2 \end{bmatrix}$$

- Rewrite the system as

$$\underline{\vec{x}}'(t) = A \cdot \underline{\vec{x}}(t) + \vec{f}$$

- To solve the system, find the eigenvalues and eigenvectors of A
- Eigenpairs of A

$$\left[\left[-I, \left[\begin{array}{c} 2-I \\ 1 \end{array}\right]\right], \left[I, \left[\begin{array}{c} 2+I \\ 1 \end{array}\right]\right]\right]$$

- Consider complex eigenpair, complex conjugate eigenvalue can be ignored

$$\left[-I, \left[\begin{array}{c} 2-I \\ 1 \end{array}\right]\right]$$

- Solution from eigenpair

$$e^{-It} \cdot \left[\begin{array}{c} 2-I \\ 1 \end{array}\right]$$

- Use Euler identity to write solution in terms of sin and cos

$$(\cos(t) - I\sin(t)) \cdot \left[\begin{array}{c} 2-I \\ 1 \end{array}\right]$$

- Simplify expression

$$\left[\begin{array}{c} (2-I)(\cos(t) - I\sin(t)) \\ \cos(t) - I\sin(t) \end{array}\right]$$

- Both real and imaginary parts are solutions to the homogeneous system

$$\left[\vec{x}_{1}(t) = \left[\begin{array}{c} 2\cos(t) - \sin(t) \\ \cos(t) \end{array}\right], \vec{x}_{2}(t) = \left[\begin{array}{c} -\cos(t) - 2\sin(t) \\ -\sin(t) \end{array}\right]\right]$$

- General solution of the system of ODEs can be written in terms of the particular solution $\vec{x}$

$$\vec{x}(t) = c_1\vec{x}_{1}(t) + c_2\vec{x}_{2}(t) + \vec{x}_{p}(t)$$

□ Fundamental matrix

  - Let $\phi(t)$ be the matrix whose columns are the independent solutions of the homogeneous syst

$$\phi(t) = \left[\begin{array}{cc} 2\cos(t) - \sin(t) & -\cos(t) - 2\sin(t) \\ \cos(t) & -\sin(t) \end{array}\right]$$

  - The fundamental matrix, $\Phi(t)$ is a normalized version of $\phi(t)$ satisfying $\Phi(0) = I$ where I is th

$$\Phi(t) = \phi(t) \cdot \frac{1}{\phi(0)}$$

  - Substitute the value of $\phi(t)$ and $\phi(0)$

$$\Phi(t) = \begin{bmatrix} 2\cos(t) - \sin(t) & -\cos(t) - 2\sin(t) \\ \cos(t) & -\sin(t) \end{bmatrix} \cdot \frac{1}{\begin{bmatrix} 2 & -1 \\ 1 & 0 \end{bmatrix}}$$

- Evaluate and simplify to get the fundamental matrix

  $$\Phi(t) = \begin{bmatrix} \cos(t) + 2\sin(t) & -5\sin(t) \\ \sin(t) & \cos(t) - 2\sin(t) \end{bmatrix}$$

☐ Find a particular solution of the system of ODEs using variation of parameters

- Let the particular solution be the fundamental matrix multiplied by $\vec{v}(t)$ and solve for $\vec{v}(t)$

  $\vec{x}_p(t) = \Phi(t) \cdot \vec{v}(t)$

- Take the derivative of the particular solution

  $\vec{x}_p'(t) = \Phi'(t) \cdot \vec{v}(t) + \Phi(t) \cdot \vec{v}'(t)$

- Substitute particular solution and its derivative into the system of ODEs

  $\Phi'(t) \cdot \vec{v}(t) + \Phi(t) \cdot \vec{v}'(t) = A \cdot \Phi(t) \cdot \vec{v}(t) + \vec{f}(t)$

- The fundamental matrix has columns that are solutions to the homogeneous system so its der

  $A \cdot \Phi(t) \cdot \vec{v}(t) + \Phi(t) \cdot \vec{v}'(t) = A \cdot \Phi(t) \cdot \vec{v}(t) + \vec{f}(t)$

- Cancel like terms

  $\Phi(t) \cdot \vec{v}'(t) = \vec{f}(t)$

- Multiply by the inverse of the fundamental matrix

  $\vec{v}'(t) = \frac{1}{\Phi(t)} \cdot \vec{f}(t)$

- Integrate to solve for $\vec{v}(t)$

  $\vec{v}(t) = \int_0^t \frac{1}{\Phi(s)} \cdot \vec{f}(s)\, ds$

- Plug $\vec{v}(t)$ into the equation for the particular solution

  $\vec{x}_p(t) = \Phi(t) \cdot \left(\int_0^t \frac{1}{\Phi(s)} \cdot \vec{f}(s)\, ds\right)$

- Plug in the fundamental matrix and the forcing function and compute

  $$\vec{x}_p(t) = \begin{bmatrix} (\cos(t) + 2\sin(t)) \left(\int_0^t (-4\cot(s) - 2 + 5\sec(s)\csc(s))\, ds\right) - 10\sin(t) \left(\int_0^t \tan(s \\ \left(\int_0^t (-4\cot(s) - 2 + 5\sec(s)\csc(s))\, ds\right) \sin(t) + 2(\cos(t) - 2\sin(t)) \left(\int_0^t \tan(s)\right. \end{bmatrix}$$

- Plug particular solution back into general solution

$$\underline{x^{\rightarrow}}(t) = c_1\underline{x^{\rightarrow}}_1(t) + c_2\underline{x^{\rightarrow}}_2(t) + \left[\begin{array}{c} (\cos(t) + 2\sin(t))\left(\int_0^t (-4\cot(s) - 2 + 5\sec(s)\csc(s))\, d \right. \\ \left(\int_0^t (-4\cot(s) - 2 + 5\sec(s)\csc(s))\, ds\right)\sin(t) + 2(\cos \end{array}\right.$$

- Substitute in vector of dependent variables

$$\left[\begin{array}{c} x_1(t) \\ x_2(t) \end{array}\right] = \left[\begin{array}{c} (\cos(t) + 2\sin(t))\left(\int_0^t (-4\cot(s) - 2 + 5\sec(s)\csc(s))\, ds\right) - 10\sin(t)\left(\int_0^t \tan \right. \\ \left(\int_0^t (-4\cot(s) - 2 + 5\sec(s)\csc(s))\, ds\right)\sin(t) + (2\cos(t) - 4\sin(t)) \end{array}\right.$$

- Solution to the system of ODEs

$$\left\{x_1(t) = (\cos(t) + 2\sin(t))\left(\int_0^t (-4\cot(s) - 2 + 5\sec(s)\csc(s))\, ds\right) - 10\sin(t)\left(\int_0^t \tan(s)\, ds\right)\right.$$

✓ Solution by Maple

Time used: 0.235 (sec). Leaf size: 113

```
dsolve([diff(x__1(t),t)=2*x__1(t)-5*x__2(t)+csc(t),diff(x__2(t),t)=1*x__1(t)-2*x__2(t)+sec(t
```

$$x_1(t) = \ln(\sin(t))\cos(t) - 5\cos(t)\ln(\cos(t)) + c_1\cos(t) - 2\cos(t)t + 2\ln(\sin(t))\sin(t) + c_2\sin(t) - 4\sin(t)t + \cos(t)$$

$$x_2(t) = -2\cos(t)\ln(\cos(t)) + \frac{2c_1\cos(t)}{5} - \frac{c_2\cos(t)}{5} + \ln(\sin(t))\sin(t) - \sin(t)\ln(\cos(t)) + \frac{c_1\sin(t)}{5} + \frac{2c_2\sin(t)}{5} - 2\sin(t)t - \frac{\cos(t)^2}{5\sin(t)} + \frac{2\cos(t)}{5} + \frac{\csc(t)}{5}$$

✓ Solution by Mathematica

Time used: 0.018 (sec). Leaf size: 79

```
DSolve[{x1'[t]==2*x1[t]-5*x2[t]+Csc[t],x2'[t]==1*x1[t]-2*x2[t]+Sec[t]},{x1[t],x2[t]},t,Inclu
```

$$\text{x1}(t) \to \sin(t)(-4t + 2\log(\sin(t)) + 2c_1 - 5c_2) + \cos(t)(-2t + \log(\sin(t)) - 5\log(\cos(t)) + c_1)$$

$$\text{x2}(t) \to \cos(t)(-2\log(\cos(t)) + c_2) + \sin(t)(-2t + \log(\sin(t)) - \log(\cos(t)) + c_1 - 2c_2)$$

18.13 problem 13

Internal problem ID [790]
Internal file name [OUTPUT/790_Sunday_June_05_2022_01_49_50_AM_26957802/index.tex]

Book: Elementary differential equations and boundary value problems, 10th ed., Boyce and DiPrima
Section: Chapter 7.9, Nonhomogeneous Linear Systems. page 447
Problem number: 13.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"system of linear ODEs"**

Solve

$$x_1'(t) = -\frac{x_1(t)}{2} - \frac{x_2(t)}{8} + \frac{e^{-\frac{t}{2}}}{2}$$
$$x_2'(t) = 2x_1(t) - \frac{x_2(t)}{2}$$

18.13.1 Solution using Matrix exponential method

In this method, we will assume we have found the matrix exponential e^{At} allready. There are different methods to determine this but will not be shown here. This is a system of linear ODE's given as

$$\vec{x}'(t) = A\,\vec{x}(t) + \vec{G}(t)$$

Or

$$\begin{bmatrix} x_1'(t) \\ x_2'(t) \end{bmatrix} = \begin{bmatrix} -\frac{1}{2} & -\frac{1}{8} \\ 2 & -\frac{1}{2} \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} + \begin{bmatrix} \frac{e^{-\frac{t}{2}}}{2} \\ 0 \end{bmatrix}$$

Since the system is nonhomogeneous, then the solution is given by

$$\vec{x}(t) = \vec{x}_h(t) + \vec{x}_p(t)$$

Where $\vec{x}_h(t)$ is the homogeneous solution to $\vec{x}'(t) = A\,\vec{x}(t)$ and $\vec{x}_p(t)$ is a particular solution to $\vec{x}'(t) = A\,\vec{x}(t) + \vec{G}(t)$. The particular solution will be found using variation

of parameters method applied to the fundamental matrix. For the above matrix A, the matrix exponential can be found to be

$$e^{At} = \begin{bmatrix} e^{-\frac{t}{2}} \cos\left(\frac{t}{2}\right) & -\frac{e^{-\frac{t}{2}} \sin\left(\frac{t}{2}\right)}{4} \\ 4\,e^{-\frac{t}{2}} \sin\left(\frac{t}{2}\right) & e^{-\frac{t}{2}} \cos\left(\frac{t}{2}\right) \end{bmatrix}$$

Therefore the homogeneous solution is

$$\begin{aligned}
\vec{x}_h(t) &= e^{At}\vec{c} \\
&= \begin{bmatrix} e^{-\frac{t}{2}} \cos\left(\frac{t}{2}\right) & -\frac{e^{-\frac{t}{2}} \sin\left(\frac{t}{2}\right)}{4} \\ 4\,e^{-\frac{t}{2}} \sin\left(\frac{t}{2}\right) & e^{-\frac{t}{2}} \cos\left(\frac{t}{2}\right) \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} \\
&= \begin{bmatrix} e^{-\frac{t}{2}} \cos\left(\frac{t}{2}\right) c_1 - \frac{e^{-\frac{t}{2}} \sin\left(\frac{t}{2}\right) c_2}{4} \\ 4\,e^{-\frac{t}{2}} \sin\left(\frac{t}{2}\right) c_1 + e^{-\frac{t}{2}} \cos\left(\frac{t}{2}\right) c_2 \end{bmatrix} \\
&= \begin{bmatrix} e^{-\frac{t}{2}} \cos\left(\frac{t}{2}\right) c_1 - \frac{e^{-\frac{t}{2}} \sin\left(\frac{t}{2}\right) c_2}{4} \\ e^{-\frac{t}{2}} \left(4 \sin\left(\frac{t}{2}\right) c_1 + \cos\left(\frac{t}{2}\right) c_2\right) \end{bmatrix}
\end{aligned}$$

The particular solution given by

$$\vec{x}_p(t) = e^{At} \int e^{-At} \vec{G}(t)\, dt$$

But

$$\begin{aligned}
e^{-At} &= \left(e^{At}\right)^{-1} \\
&= \begin{bmatrix} \cos\left(\frac{t}{2}\right) e^{\frac{t}{2}} & \frac{\sin\left(\frac{t}{2}\right) e^{\frac{t}{2}}}{4} \\ -4 \sin\left(\frac{t}{2}\right) e^{\frac{t}{2}} & \cos\left(\frac{t}{2}\right) e^{\frac{t}{2}} \end{bmatrix}
\end{aligned}$$

Hence

$$\begin{aligned}
\vec{x}_p(t) &= \begin{bmatrix} e^{-\frac{t}{2}} \cos\left(\frac{t}{2}\right) & -\frac{e^{-\frac{t}{2}} \sin\left(\frac{t}{2}\right)}{4} \\ 4\,e^{-\frac{t}{2}} \sin\left(\frac{t}{2}\right) & e^{-\frac{t}{2}} \cos\left(\frac{t}{2}\right) \end{bmatrix} \int \begin{bmatrix} \cos\left(\frac{t}{2}\right) e^{\frac{t}{2}} & \frac{\sin\left(\frac{t}{2}\right) e^{\frac{t}{2}}}{4} \\ -4 \sin\left(\frac{t}{2}\right) e^{\frac{t}{2}} & \cos\left(\frac{t}{2}\right) e^{\frac{t}{2}} \end{bmatrix} \begin{bmatrix} \frac{e^{-\frac{t}{2}}}{2} \\ 0 \end{bmatrix} dt \\
&= \begin{bmatrix} e^{-\frac{t}{2}} \cos\left(\frac{t}{2}\right) & -\frac{e^{-\frac{t}{2}} \sin\left(\frac{t}{2}\right)}{4} \\ 4\,e^{-\frac{t}{2}} \sin\left(\frac{t}{2}\right) & e^{-\frac{t}{2}} \cos\left(\frac{t}{2}\right) \end{bmatrix} \begin{bmatrix} \sin\left(\frac{t}{2}\right) \\ 4 \cos\left(\frac{t}{2}\right) \end{bmatrix} \\
&= \begin{bmatrix} 0 \\ 4\,e^{-\frac{t}{2}} \end{bmatrix}
\end{aligned}$$

Hence the complete solution is

$$
\begin{aligned}
\vec{x}(t) &= \vec{x}_h(t) + \vec{x}_p(t) \\
&= \left[\begin{array}{c} e^{-\frac{t}{2}} \cos\left(\frac{t}{2}\right) c_1 - \frac{e^{-\frac{t}{2}} \sin\left(\frac{t}{2}\right) c_2}{4} \\ e^{-\frac{t}{2}} \left(4 + 4 \sin\left(\frac{t}{2}\right) c_1 + \cos\left(\frac{t}{2}\right) c_2\right) \end{array}\right]
\end{aligned}
$$

18.13.2 Solution using explicit Eigenvalue and Eigenvector method

This is a system of linear ODE's given as

$$
\vec{x}'(t) = A\,\vec{x}(t) + \vec{G}(t)
$$

Or

$$
\left[\begin{array}{c} x_1'(t) \\ x_2'(t) \end{array}\right] = \left[\begin{array}{cc} -\frac{1}{2} & -\frac{1}{8} \\ 2 & -\frac{1}{2} \end{array}\right] \left[\begin{array}{c} x_1(t) \\ x_2(t) \end{array}\right] + \left[\begin{array}{c} \frac{e^{-\frac{t}{2}}}{2} \\ 0 \end{array}\right]
$$

Since the system is nonhomogeneous, then the solution is given by

$$
\vec{x}(t) = \vec{x}_h(t) + \vec{x}_p(t)
$$

Where $\vec{x}_h(t)$ is the homogeneous solution to $\vec{x}'(t) = A\,\vec{x}(t)$ and $\vec{x}_p(t)$ is a particular solution to $\vec{x}'(t) = A\,\vec{x}(t) + \vec{G}(t)$. The particular solution will be found using variation of parameters method applied to the fundamental matrix.

The first step is find the homogeneous solution. We start by finding the eigenvalues of A. This is done by solving the following equation for the eigenvalues λ

$$
\det\left(A - \lambda I\right) = 0
$$

Expanding gives

$$
\det\left(\left[\begin{array}{cc} -\frac{1}{2} & -\frac{1}{8} \\ 2 & -\frac{1}{2} \end{array}\right] - \lambda \left[\begin{array}{cc} 1 & 0 \\ 0 & 1 \end{array}\right]\right) = 0
$$

Therefore

$$
\det\left(\left[\begin{array}{cc} -\frac{1}{2} - \lambda & -\frac{1}{8} \\ 2 & -\frac{1}{2} - \lambda \end{array}\right]\right) = 0
$$

Which gives the characteristic equation

$$\lambda^2 + \lambda + \frac{1}{2} = 0$$

The roots of the above are the eigenvalues.

$$\lambda_1 = -\frac{1}{2} + \frac{i}{2}$$
$$\lambda_2 = -\frac{1}{2} - \frac{i}{2}$$

This table summarises the above result

eigenvalue	algebraic multiplicity	type of eigenvalue
$-\frac{1}{2} + \frac{i}{2}$	1	complex eigenvalue
$-\frac{1}{2} - \frac{i}{2}$	1	complex eigenvalue

Now the eigenvector for each eigenvalue are found.

Considering the eigenvalue $\lambda_1 = -\frac{1}{2} - \frac{i}{2}$

We need to solve $A\vec{v} = \lambda\vec{v}$ or $(A - \lambda I)\vec{v} = \vec{0}$ which becomes

$$\left(\begin{bmatrix} -\frac{1}{2} & -\frac{1}{8} \\ 2 & -\frac{1}{2} \end{bmatrix} - \left(-\frac{1}{2} - \frac{i}{2}\right)\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right)\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$
$$\begin{bmatrix} \frac{i}{2} & -\frac{1}{8} \\ 2 & \frac{i}{2} \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Now forward elimination is applied to solve for the eigenvector $\vec{v}$. The augmented matrix is

$$\left[\begin{array}{cc|c} \frac{i}{2} & -\frac{1}{8} & 0 \\ 2 & \frac{i}{2} & 0 \end{array}\right]$$

$$R_2 = 4iR_1 + R_2 \Longrightarrow \left[\begin{array}{cc|c} \frac{i}{2} & -\frac{1}{8} & 0 \\ 0 & 0 & 0 \end{array}\right]$$

Therefore the system in Echelon form is

$$\begin{bmatrix} \frac{i}{2} & -\frac{1}{8} \\ 0 & 0 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

The free variables are $\{v_2\}$ and the leading variables are $\{v_1\}$. Let $v_2 = t$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\left\{v_1 = -\frac{it}{4}\right\}$

Hence the solution is

$$\begin{bmatrix} -\frac{I}{4}t \\ t \end{bmatrix} = \begin{bmatrix} -\frac{it}{4} \\ t \end{bmatrix}$$

Since there is one free Variable, we have found one eigenvector associated with this eigenvalue. The above can be written as

$$\begin{bmatrix} -\frac{I}{4}t \\ t \end{bmatrix} = t \begin{bmatrix} -\frac{i}{4} \\ 1 \end{bmatrix}$$

Let $t = 1$ the eigenvector becomes

$$\begin{bmatrix} -\frac{I}{4}t \\ t \end{bmatrix} = \begin{bmatrix} -\frac{i}{4} \\ 1 \end{bmatrix}$$

Which is normalized to

$$\begin{bmatrix} -\frac{I}{4}t \\ t \end{bmatrix} = \begin{bmatrix} -i \\ 4 \end{bmatrix}$$

Considering the eigenvalue $\lambda_2 = -\frac{1}{2} + \frac{i}{2}$

We need to solve $A\vec{v} = \lambda\vec{v}$ or $(A - \lambda I)\vec{v} = \vec{0}$ which becomes

$$\left(\begin{bmatrix} -\frac{1}{2} & -\frac{1}{8} \\ 2 & -\frac{1}{2} \end{bmatrix} - \left(-\frac{1}{2} + \frac{i}{2}\right)\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right)\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} -\frac{i}{2} & -\frac{1}{8} \\ 2 & -\frac{i}{2} \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Now forward elimination is applied to solve for the eigenvector $\vec{v}$. The augmented matrix is

$$\left[\begin{array}{cc|c} -\frac{i}{2} & -\frac{1}{8} & 0 \\ 2 & -\frac{i}{2} & 0 \end{array}\right]$$

$$R_2 = -4iR_1 + R_2 \Longrightarrow \left[\begin{array}{cc|c} -\frac{i}{2} & -\frac{1}{8} & 0 \\ 0 & 0 & 0 \end{array}\right]$$

Therefore the system in Echelon form is

$$\begin{bmatrix} -\frac{i}{2} & -\frac{1}{8} \\ 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

The free variables are $\{v_2\}$ and the leading variables are $\{v_1\}$. Let $v_2 = t$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\left\{v_1 = \frac{it}{4}\right\}$

Hence the solution is

$$\begin{bmatrix} \frac{\mathrm{I}}{4}t \\ t \end{bmatrix} = \begin{bmatrix} \frac{it}{4} \\ t \end{bmatrix}$$

Since there is one free Variable, we have found one eigenvector associated with this eigenvalue. The above can be written as

$$\begin{bmatrix} \frac{\mathrm{I}}{4}t \\ t \end{bmatrix} = t \begin{bmatrix} \frac{i}{4} \\ 1 \end{bmatrix}$$

Let $t = 1$ the eigenvector becomes

$$\begin{bmatrix} \frac{\mathrm{I}}{4}t \\ t \end{bmatrix} = \begin{bmatrix} \frac{i}{4} \\ 1 \end{bmatrix}$$

Which is normalized to

$$\begin{bmatrix} \frac{\mathrm{I}}{4}t \\ t \end{bmatrix} = \begin{bmatrix} i \\ 4 \end{bmatrix}$$

The following table gives a summary of this result. It shows for each eigenvalue the algebraic multiplicity m, and its geometric multiplicity k and the eigenvectors associated with the eigenvalue. If $m > k$ then the eigenvalue is defective which means the number of normal linearly independent eigenvectors associated with this eigenvalue (called the geometric multiplicity k) does not equal the algebraic multiplicity m, and we need to determine an additional $m - k$ generalized eigenvectors for this eigenvalue.

eigenvalue	multiplicity		defective?	eigenvectors
	algebraic m	geometric k		
$-\frac{1}{2}+\frac{i}{2}$	1	1	No	$\begin{bmatrix} \frac{i}{4} \\ 1 \end{bmatrix}$
$-\frac{1}{2}-\frac{i}{2}$	1	1	No	$\begin{bmatrix} -\frac{i}{4} \\ 1 \end{bmatrix}$

Now that we found the eigenvalues and associated eigenvectors, we will go over each eigenvalue and generate the solution basis. The only problem we need to take care of is if the eigenvalue is defective. Therefore the homogeneous solution is

$$\vec{x}_h(t) = c_1\vec{x}_1(t) + c_2\vec{x}_2(t)$$

Which is written as

$$\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = c_1 \begin{bmatrix} \frac{i\,e^{\left(-\frac{1}{2}+\frac{i}{2}\right)t}}{4} \\ e^{\left(-\frac{1}{2}+\frac{i}{2}\right)t} \end{bmatrix} + c_2 \begin{bmatrix} -\frac{i\,e^{\left(-\frac{1}{2}-\frac{i}{2}\right)t}}{4} \\ e^{\left(-\frac{1}{2}-\frac{i}{2}\right)t} \end{bmatrix}$$

Now that we found homogeneous solution above, we need to find a particular solution $\vec{x}_p(t)$. We will use Variation of parameters. The fundamental matrix is

$$\Phi = \begin{bmatrix} \vec{x}_1 & \vec{x}_2 & \cdots \end{bmatrix}$$

Where $\vec{x}_i$ are the solution basis found above. Therefore the fundamental matrix is

$$\Phi(t) = \begin{bmatrix} \frac{i\,e^{\left(-\frac{1}{2}+\frac{i}{2}\right)t}}{4} & -\frac{i\,e^{\left(-\frac{1}{2}-\frac{i}{2}\right)t}}{4} \\ e^{\left(-\frac{1}{2}+\frac{i}{2}\right)t} & e^{\left(-\frac{1}{2}-\frac{i}{2}\right)t} \end{bmatrix}$$

The particular solution is then given by

$$\vec{x}_p(t) = \Phi \int \Phi^{-1}\vec{G}(t)\,dt$$

But

$$\Phi^{-1} = \begin{bmatrix} -2i\,e^{\left(\frac{1}{2}-\frac{i}{2}\right)t} & \frac{e^{\left(\frac{1}{2}-\frac{i}{2}\right)t}}{2} \\ 2i\,e^{\left(\frac{1}{2}+\frac{i}{2}\right)t} & \frac{e^{\left(\frac{1}{2}+\frac{i}{2}\right)t}}{2} \end{bmatrix}$$

Hence

$$\vec{x}_p(t) = \begin{bmatrix} \frac{i e^{\left(-\frac{1}{2}+\frac{i}{2}\right)t}}{4} & -\frac{i e^{\left(-\frac{1}{2}-\frac{i}{2}\right)t}}{4} \\ e^{\left(-\frac{1}{2}+\frac{i}{2}\right)t} & e^{\left(-\frac{1}{2}-\frac{i}{2}\right)t} \end{bmatrix} \int \begin{bmatrix} -2i e^{\left(\frac{1}{2}-\frac{i}{2}\right)t} & \frac{e^{\left(\frac{1}{2}-\frac{i}{2}\right)t}}{2} \\ 2i e^{\left(\frac{1}{2}+\frac{i}{2}\right)t} & \frac{e^{\left(\frac{1}{2}+\frac{i}{2}\right)t}}{2} \end{bmatrix} \begin{bmatrix} \frac{e^{-\frac{t}{2}}}{2} \\ 0 \end{bmatrix} dt$$

$$= \begin{bmatrix} \frac{i e^{\left(-\frac{1}{2}+\frac{i}{2}\right)t}}{4} & -\frac{i e^{\left(-\frac{1}{2}-\frac{i}{2}\right)t}}{4} \\ e^{\left(-\frac{1}{2}+\frac{i}{2}\right)t} & e^{\left(-\frac{1}{2}-\frac{i}{2}\right)t} \end{bmatrix} \int \begin{bmatrix} -i e^{-\frac{it}{2}} \\ i e^{\frac{it}{2}} \end{bmatrix} dt$$

$$= \begin{bmatrix} \frac{i e^{\left(-\frac{1}{2}+\frac{i}{2}\right)t}}{4} & -\frac{i e^{\left(-\frac{1}{2}-\frac{i}{2}\right)t}}{4} \\ e^{\left(-\frac{1}{2}+\frac{i}{2}\right)t} & e^{\left(-\frac{1}{2}-\frac{i}{2}\right)t} \end{bmatrix} \begin{bmatrix} 2\,e^{-\frac{it}{2}} \\ 2\,e^{\frac{it}{2}} \end{bmatrix}$$

$$= \begin{bmatrix} 0 \\ 4\,e^{-\frac{t}{2}} \end{bmatrix}$$

Now that we found particular solution, the final solution is

$$\vec{x}(t) = \vec{x}_h(t) + \vec{x}_p(t)$$

$$\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = \begin{bmatrix} \frac{i c_1 e^{\left(-\frac{1}{2}+\frac{i}{2}\right)t}}{4} \\ c_1 e^{\left(-\frac{1}{2}+\frac{i}{2}\right)t} \end{bmatrix} + \begin{bmatrix} -\frac{i c_2 e^{\left(-\frac{1}{2}-\frac{i}{2}\right)t}}{4} \\ c_2 e^{\left(-\frac{1}{2}-\frac{i}{2}\right)t} \end{bmatrix} + \begin{bmatrix} 0 \\ 4\,e^{-\frac{t}{2}} \end{bmatrix}$$

Which becomes

$$\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = \begin{bmatrix} \frac{i\left(c_1 e^{\left(-\frac{1}{2}+\frac{i}{2}\right)t} - c_2 e^{\left(-\frac{1}{2}-\frac{i}{2}\right)t}\right)}{4} \\ c_1 e^{\left(-\frac{1}{2}+\frac{i}{2}\right)t} + c_2 e^{\left(-\frac{1}{2}-\frac{i}{2}\right)t} + 4\,e^{-\frac{t}{2}} \end{bmatrix}$$

18.13.3 Maple step by step solution

Let's solve

$$\left[x_1'(t) = -\frac{x_1(t)}{2} - \frac{x_2(t)}{8} + \frac{e^{-\frac{t}{2}}}{2}, x_2'(t) = 2x_1(t) - \frac{x_2(t)}{2}\right]$$

- Define vector

$$\underline{x}^{\rightarrow}(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}$$

- Convert system into a vector equation

$$\underline{x^{\rightarrow}}'(t) = \begin{bmatrix} -\frac{1}{2} & -\frac{1}{8} \\ 2 & -\frac{1}{2} \end{bmatrix} \cdot \underline{x^{\rightarrow}}(t) + \begin{bmatrix} \frac{e^{-\frac{t}{2}}}{2} \\ 0 \end{bmatrix}$$

- System to solve

$$\underline{x^{\rightarrow}}'(t) = \begin{bmatrix} -\frac{1}{2} & -\frac{1}{8} \\ 2 & -\frac{1}{2} \end{bmatrix} \cdot \underline{x^{\rightarrow}}(t) + \begin{bmatrix} \frac{e^{-\frac{t}{2}}}{2} \\ 0 \end{bmatrix}$$

- Define the forcing function

$$\vec{f}(t) = \begin{bmatrix} \frac{e^{-\frac{t}{2}}}{2} \\ 0 \end{bmatrix}$$

- Define the coefficient matrix

$$A = \begin{bmatrix} -\frac{1}{2} & -\frac{1}{8} \\ 2 & -\frac{1}{2} \end{bmatrix}$$

- Rewrite the system as

$$\underline{x^{\rightarrow}}'(t) = A \cdot \underline{x^{\rightarrow}}(t) + \vec{f}$$

- To solve the system, find the eigenvalues and eigenvectors of A
- Eigenpairs of A

$$\left[\left[-\frac{1}{2} - \frac{I}{2}, \begin{bmatrix} -\frac{I}{4} \\ 1 \end{bmatrix}\right], \left[-\frac{1}{2} + \frac{I}{2}, \begin{bmatrix} \frac{I}{4} \\ 1 \end{bmatrix}\right]\right]$$

- Consider complex eigenpair, complex conjugate eigenvalue can be ignored

$$\left[-\frac{1}{2} - \frac{I}{2}, \begin{bmatrix} -\frac{I}{4} \\ 1 \end{bmatrix}\right]$$

- Solution from eigenpair

$$e^{\left(-\frac{1}{2}-\frac{I}{2}\right)t} \cdot \begin{bmatrix} -\frac{I}{4} \\ 1 \end{bmatrix}$$

- Use Euler identity to write solution in terms of $\sin$ and $\cos$

$$e^{-\frac{t}{2}} \cdot \left(\cos\left(\frac{t}{2}\right) - I\sin\left(\frac{t}{2}\right)\right) \cdot \begin{bmatrix} -\frac{I}{4} \\ 1 \end{bmatrix}$$

- Simplify expression

$$e^{-\frac{t}{2}} \cdot \begin{bmatrix} -\frac{I}{4}\left(\cos\left(\frac{t}{2}\right) - I\sin\left(\frac{t}{2}\right)\right) \\ \cos\left(\frac{t}{2}\right) - I\sin\left(\frac{t}{2}\right) \end{bmatrix}$$

- Both real and imaginary parts are solutions to the homogeneous system

$$\left[\underline{x}^{\rightarrow}{}_1(t) = e^{-\frac{t}{2}} \cdot \begin{bmatrix} -\frac{\sin\left(\frac{t}{2}\right)}{4} \\ \cos\left(\frac{t}{2}\right) \end{bmatrix}, \underline{x}^{\rightarrow}{}_2(t) = e^{-\frac{t}{2}} \cdot \begin{bmatrix} -\frac{\cos\left(\frac{t}{2}\right)}{4} \\ -\sin\left(\frac{t}{2}\right) \end{bmatrix}\right]$$

- General solution of the system of ODEs can be written in terms of the particular solution $\underline{x}^{\rightarrow}$

$\underline{x}^{\rightarrow}(t) = c_1\underline{x}^{\rightarrow}{}_1(t) + c_2\underline{x}^{\rightarrow}{}_2(t) + \underline{x}^{\rightarrow}{}_p(t)$

☐ Fundamental matrix

- ○ Let $\phi(t)$ be the matrix whose columns are the independent solutions of the homogeneous syst

$$\phi(t) = \begin{bmatrix} -\frac{e^{-\frac{t}{2}}\sin\left(\frac{t}{2}\right)}{4} & -\frac{e^{-\frac{t}{2}}\cos\left(\frac{t}{2}\right)}{4} \\ e^{-\frac{t}{2}}\cos\left(\frac{t}{2}\right) & -e^{-\frac{t}{2}}\sin\left(\frac{t}{2}\right) \end{bmatrix}$$

- ○ The fundamental matrix, $\Phi(t)$ is a normalized version of $\phi(t)$ satisfying $\Phi(0) = I$ where I is th

$\Phi(t) = \phi(t) \cdot \frac{1}{\phi(0)}$

- ○ Substitute the value of $\phi(t)$ and $\phi(0)$

$$\Phi(t) = \begin{bmatrix} -\frac{e^{-\frac{t}{2}}\sin\left(\frac{t}{2}\right)}{4} & -\frac{e^{-\frac{t}{2}}\cos\left(\frac{t}{2}\right)}{4} \\ e^{-\frac{t}{2}}\cos\left(\frac{t}{2}\right) & -e^{-\frac{t}{2}}\sin\left(\frac{t}{2}\right) \end{bmatrix} \cdot \frac{1}{\begin{bmatrix} 0 & -\frac{1}{4} \\ 1 & 0 \end{bmatrix}}$$

- ○ Evaluate and simplify to get the fundamental matrix

$$\Phi(t) = \begin{bmatrix} e^{-\frac{t}{2}}\cos\left(\frac{t}{2}\right) & -\frac{e^{-\frac{t}{2}}\sin\left(\frac{t}{2}\right)}{4} \\ 4e^{-\frac{t}{2}}\sin\left(\frac{t}{2}\right) & e^{-\frac{t}{2}}\cos\left(\frac{t}{2}\right) \end{bmatrix}$$

☐ Find a particular solution of the system of ODEs using variation of parameters

- ○ Let the particular solution be the fundamental matrix multiplied by $\vec{v}(t)$ and solve for $\vec{v}(t)$

$\underline{x}^{\rightarrow}{}_p(t) = \Phi(t) \cdot \vec{v}(t)$

- ○ Take the derivative of the particular solution

$\underline{x}^{\rightarrow}{}'_p(t) = \Phi'(t) \cdot \vec{v}(t) + \Phi(t) \cdot \vec{v}'(t)$

- Substitute particular solution and its derivative into the system of ODEs

  $\Phi'(t) \cdot \vec{v}(t) + \Phi(t) \cdot \vec{v}'(t) = A \cdot \Phi(t) \cdot \vec{v}(t) + \vec{f}(t)$

- The fundamental matrix has columns that are solutions to the homogeneous system so its der

  $A \cdot \Phi(t) \cdot \vec{v}(t) + \Phi(t) \cdot \vec{v}'(t) = A \cdot \Phi(t) \cdot \vec{v}(t) + \vec{f}(t)$

- Cancel like terms

  $\Phi(t) \cdot \vec{v}'(t) = \vec{f}(t)$

- Multiply by the inverse of the fundamental matrix

  $\vec{v}'(t) = \frac{1}{\Phi(t)} \cdot \vec{f}(t)$

- Integrate to solve for $\vec{v}(t)$

  $\vec{v}(t) = \int_0^t \frac{1}{\Phi(s)} \cdot \vec{f}(s)\, ds$

- Plug $\vec{v}(t)$ into the equation for the particular solution

  $\underline{\vec{x}}_p(t) = \Phi(t) \cdot \left(\int_0^t \frac{1}{\Phi(s)} \cdot \vec{f}(s)\, ds\right)$

- Plug in the fundamental matrix and the forcing function and compute

  $$\underline{\vec{x}}_p(t) = \left[\begin{array}{c} \mathrm{e}^{-\frac{t}{2}} \sin\left(\frac{t}{2}\right) \\ -4\,\mathrm{e}^{-\frac{t}{2}}\left(-1+\cos\left(\frac{t}{2}\right)\right) \end{array}\right]$$

- Plug particular solution back into general solution

  $$\underline{\vec{x}}(t) = c_1 \underline{\vec{x}}_1(t) + c_2 \underline{\vec{x}}_2(t) + \left[\begin{array}{c} \mathrm{e}^{-\frac{t}{2}} \sin\left(\frac{t}{2}\right) \\ -4\,\mathrm{e}^{-\frac{t}{2}}\left(-1+\cos\left(\frac{t}{2}\right)\right) \end{array}\right]$$

- Substitute in vector of dependent variables

  $$\left[\begin{array}{c} x_1(t) \\ x_2(t) \end{array}\right] = \left[\begin{array}{c} -\frac{\left(\left(-4+c_1\right)\sin\left(\frac{t}{2}\right)+c_2\cos\left(\frac{t}{2}\right)\right)\mathrm{e}^{-\frac{t}{2}}}{4} \\ \left(\left(-4+c_1\right)\cos\left(\frac{t}{2}\right)-c_2\sin\left(\frac{t}{2}\right)+4\right)\mathrm{e}^{-\frac{t}{2}} \end{array}\right]$$

- Solution to the system of ODEs

  $$\left\{x_1(t) = -\frac{\left(\left(-4+c_1\right)\sin\left(\frac{t}{2}\right)+c_2\cos\left(\frac{t}{2}\right)\right)\mathrm{e}^{-\frac{t}{2}}}{4}, x_2(t) = \left(\left(-4+c_1\right)\cos\left(\frac{t}{2}\right)-c_2\sin\left(\frac{t}{2}\right)+4\right)\mathrm{e}^{-\frac{t}{2}}\right\}$$

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 47

```
dsolve([diff(x__1(t),t)=-1/2*x__1(t)-1/8*x__2(t)+1/2*exp(-t/2),diff(x__2(t),t)=2*x__1(t)-1/2
```

$$x_1(t) = \frac{e^{-\frac{t}{2}}\left(c_2\cos\left(\frac{t}{2}\right) - c_1\sin\left(\frac{t}{2}\right)\right)}{4}$$
$$x_2(t) = e^{-\frac{t}{2}}\left(4 + \cos\left(\frac{t}{2}\right)c_1 + \sin\left(\frac{t}{2}\right)c_2\right)$$

✓ Solution by Mathematica

Time used: 0.022 (sec). Leaf size: 69

```
DSolve[{x1'[t]==-1/2*x1[t]-1/8*x2[t]+1/2*Exp[-t/2],x2'[t]==2*x1[t]-1/2*x2[t]+0},{x1[t],x2[t]
```

$$\text{x1}(t) \to \frac{1}{4}e^{-t/2}\left(4c_1\cos\left(\frac{t}{2}\right) - c_2\sin\left(\frac{t}{2}\right)\right)$$
$$\text{x2}(t) \to e^{-t/2}\left(c_2\cos\left(\frac{t}{2}\right) + 4c_1\sin\left(\frac{t}{2}\right) + 4\right)$$

18.14 problem 18

Internal problem ID [791]
Internal file name [OUTPUT/791_Sunday_June_05_2022_01_49_52_AM_56180441/index.tex]

Book: Elementary differential equations and boundary value problems, 10th ed., Boyce and DiPrima
Section: Chapter 7.9, Nonhomogeneous Linear Systems. page 447
Problem number: 18.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"system of linear ODEs"**

Solve

$$x_1'(t) = -2x_1(t) + x_2(t) + 2\,\mathrm{e}^{-t}$$
$$x_2'(t) = x_1(t) - 2x_2(t) + 3t$$

With initial conditions

$$[x_1(0) = \alpha_1, x_2(0) = \alpha_2]$$

18.14.1 Solution using Matrix exponential method

In this method, we will assume we have found the matrix exponential e^{At} allready. There are different methods to determine this but will not be shown here. This is a system of linear ODE's given as

$$\vec{x}'(t) = A\,\vec{x}(t) + \vec{G}(t)$$

Or

$$\begin{bmatrix} x_1'(t) \\ x_2'(t) \end{bmatrix} = \begin{bmatrix} -2 & 1 \\ 1 & -2 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} + \begin{bmatrix} 2\,\mathrm{e}^{-t} \\ 3t \end{bmatrix}$$

Since the system is nonhomogeneous, then the solution is given by

$$\vec{x}(t) = \vec{x}_h(t) + \vec{x}_p(t)$$

Where $\vec{x}_h(t)$ is the homogeneous solution to $\vec{x}'(t) = A\,\vec{x}(t)$ and $\vec{x}_p(t)$ is a particular solution to $\vec{x}'(t) = A\,\vec{x}(t) + \vec{G}(t)$. The particular solution will be found using variation

of parameters method applied to the fundamental matrix. For the above matrix A, the matrix exponential can be found to be

$$e^{At} = \begin{bmatrix} \frac{e^{-3t}}{2} + \frac{e^{-t}}{2} & \frac{e^{-t}}{2} - \frac{e^{-3t}}{2} \\ \frac{e^{-t}}{2} - \frac{e^{-3t}}{2} & \frac{e^{-3t}}{2} + \frac{e^{-t}}{2} \end{bmatrix}$$

Therefore the homogeneous solution is

$$\begin{aligned} \vec{x}_h(t) &= e^{At}\vec{x}_0 \\ &= \begin{bmatrix} \frac{e^{-3t}}{2} + \frac{e^{-t}}{2} & \frac{e^{-t}}{2} - \frac{e^{-3t}}{2} \\ \frac{e^{-t}}{2} - \frac{e^{-3t}}{2} & \frac{e^{-3t}}{2} + \frac{e^{-t}}{2} \end{bmatrix} \begin{bmatrix} \alpha_1 \\ \alpha_2 \end{bmatrix} \\ &= \begin{bmatrix} \left(\frac{e^{-3t}}{2} + \frac{e^{-t}}{2}\right)\alpha_1 + \left(\frac{e^{-t}}{2} - \frac{e^{-3t}}{2}\right)\alpha_2 \\ \left(\frac{e^{-t}}{2} - \frac{e^{-3t}}{2}\right)\alpha_1 + \left(\frac{e^{-3t}}{2} + \frac{e^{-t}}{2}\right)\alpha_2 \end{bmatrix} \\ &= \begin{bmatrix} \frac{(-\alpha_2+\alpha_1)e^{-3t}}{2} + \frac{e^{-t}(\alpha_1+\alpha_2)}{2} \\ \frac{(\alpha_2-\alpha_1)e^{-3t}}{2} + \frac{e^{-t}(\alpha_1+\alpha_2)}{2} \end{bmatrix} \end{aligned}$$

The particular solution given by

$$\vec{x}_p(t) = e^{At} \int e^{-At}\vec{G}(t)\, dt$$

But

$$\begin{aligned} e^{-At} &= (e^{At})^{-1} \\ &= \begin{bmatrix} \frac{(e^{2t}+1)e^{t}}{2} & -\frac{(e^{2t}-1)e^{t}}{2} \\ -\frac{(e^{2t}-1)e^{t}}{2} & \frac{(e^{2t}+1)e^{t}}{2} \end{bmatrix} \end{aligned}$$

Hence

$$\begin{aligned} \vec{x}_p(t) &= \begin{bmatrix} \frac{e^{-3t}}{2} + \frac{e^{-t}}{2} & \frac{e^{-t}}{2} - \frac{e^{-3t}}{2} \\ \frac{e^{-t}}{2} - \frac{e^{-3t}}{2} & \frac{e^{-3t}}{2} + \frac{e^{-t}}{2} \end{bmatrix} \int \begin{bmatrix} \frac{(e^{2t}+1)e^{t}}{2} & -\frac{(e^{2t}-1)e^{t}}{2} \\ -\frac{(e^{2t}-1)e^{t}}{2} & \frac{(e^{2t}+1)e^{t}}{2} \end{bmatrix} \begin{bmatrix} 2\,e^{-t} \\ 3t \end{bmatrix} dt \\ &= \begin{bmatrix} \frac{e^{-3t}}{2} + \frac{e^{-t}}{2} & \frac{e^{-t}}{2} - \frac{e^{-3t}}{2} \\ \frac{e^{-t}}{2} - \frac{e^{-3t}}{2} & \frac{e^{-3t}}{2} + \frac{e^{-t}}{2} \end{bmatrix} \begin{bmatrix} \frac{(1-3t)e^{3t}}{6} + \frac{e^{2t}}{2} + \frac{(9t-9)e^{t}}{6} + t \\ \frac{(-1+3t)e^{3t}}{6} - \frac{e^{2t}}{2} + \frac{(9t-9)e^{t}}{6} + t \end{bmatrix} \\ &= \begin{bmatrix} t - \frac{4}{3} + \frac{e^{-t}}{2} + t\,e^{-t} \\ t\,e^{-t} + 2t - \frac{5}{3} - \frac{e^{-t}}{2} \end{bmatrix} \end{aligned}$$

Hence the complete solution is

$$
\begin{aligned}
\vec{x}(t) &= \vec{x}_h(t) + \vec{x}_p(t) \\
&= \begin{bmatrix} \frac{(6t+3\alpha_1+3\alpha_2+3)\mathrm{e}^{-t}}{6} + \frac{(3\alpha_1-3\alpha_2)\mathrm{e}^{-3t}}{6} + t - \frac{4}{3} \\ \frac{(6t+3\alpha_1+3\alpha_2-3)\mathrm{e}^{-t}}{6} + \frac{(-3\alpha_1+3\alpha_2)\mathrm{e}^{-3t}}{6} + 2t - \frac{5}{3} \end{bmatrix}
\end{aligned}
$$

18.14.2 Solution using explicit Eigenvalue and Eigenvector method

This is a system of linear ODE's given as

$$\vec{x}'(t) = A\,\vec{x}(t) + \vec{G}(t)$$

Or

$$\begin{bmatrix} x_1'(t) \\ x_2'(t) \end{bmatrix} = \begin{bmatrix} -2 & 1 \\ 1 & -2 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} + \begin{bmatrix} 2\,\mathrm{e}^{-t} \\ 3t \end{bmatrix}$$

Since the system is nonhomogeneous, then the solution is given by

$$\vec{x}(t) = \vec{x}_h(t) + \vec{x}_p(t)$$

Where $\vec{x}_h(t)$ is the homogeneous solution to $\vec{x}'(t) = A\,\vec{x}(t)$ and $\vec{x}_p(t)$ is a particular solution to $\vec{x}'(t) = A\,\vec{x}(t) + \vec{G}(t)$. The particular solution will be found using variation of parameters method applied to the fundamental matrix.

The first step is find the homogeneous solution. We start by finding the eigenvalues of A. This is done by solving the following equation for the eigenvalues λ

$$\det(A - \lambda I) = 0$$

Expanding gives

$$\det\left(\begin{bmatrix} -2 & 1 \\ 1 & -2 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right) = 0$$

Therefore

$$\det\left(\begin{bmatrix} -2-\lambda & 1 \\ 1 & -2-\lambda \end{bmatrix}\right) = 0$$

Which gives the characteristic equation

$$\lambda^2 + 4\lambda + 3 = 0$$

The roots of the above are the eigenvalues.

$$\lambda_1 = -1$$
$$\lambda_2 = -3$$

This table summarises the above result

eigenvalue	algebraic multiplicity	type of eigenvalue
-1	1	real eigenvalue
-3	1	real eigenvalue

Now the eigenvector for each eigenvalue are found.

Considering the eigenvalue $\lambda_1 = -3$

We need to solve $A\vec{v} = \lambda\vec{v}$ or $(A - \lambda I)\vec{v} = \vec{0}$ which becomes

$$\left(\begin{bmatrix} -2 & 1 \\ 1 & -2 \end{bmatrix} - (-3)\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right)\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$
$$\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Now forward elimination is applied to solve for the eigenvector $\vec{v}$. The augmented matrix is

$$\left[\begin{array}{cc|c} 1 & 1 & 0 \\ 1 & 1 & 0 \end{array}\right]$$

$$R_2 = R_2 - R_1 \Longrightarrow \left[\begin{array}{cc|c} 1 & 1 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

Therefore the system in Echelon form is

$$\begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

The free variables are $\{v_2\}$ and the leading variables are $\{v_1\}$. Let $v_2 = t$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\{v_1 = -t\}$

Hence the solution is

$$\begin{bmatrix} -t \\ t \end{bmatrix} = \begin{bmatrix} -t \\ t \end{bmatrix}$$

Since there is one free Variable, we have found one eigenvector associated with this eigenvalue. The above can be written as

$$\begin{bmatrix} -t \\ t \end{bmatrix} = t \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

Let $t = 1$ the eigenvector becomes

$$\begin{bmatrix} -t \\ t \end{bmatrix} = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

Considering the eigenvalue $\lambda_2 = -1$

We need to solve $A\vec{v} = \lambda\vec{v}$ or $(A - \lambda I)\vec{v} = \vec{0}$ which becomes

$$\left(\begin{bmatrix} -2 & 1 \\ 1 & -2 \end{bmatrix} - (-1) \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right) \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Now forward elimination is applied to solve for the eigenvector $\vec{v}$. The augmented matrix is

$$\left[\begin{array}{cc|c} -1 & 1 & 0 \\ 1 & -1 & 0 \end{array}\right]$$

$$R_2 = R_2 + R_1 \Longrightarrow \left[\begin{array}{cc|c} -1 & 1 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

Therefore the system in Echelon form is

$$\begin{bmatrix} -1 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

The free variables are $\{v_2\}$ and the leading variables are $\{v_1\}$. Let $v_2 = t$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\{v_1 = t\}$

Hence the solution is

$$\begin{bmatrix} t \\ t \end{bmatrix} = \begin{bmatrix} t \\ t \end{bmatrix}$$

Since there is one free Variable, we have found one eigenvector associated with this eigenvalue. The above can be written as

$$\begin{bmatrix} t \\ t \end{bmatrix} = t \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

Let $t = 1$ the eigenvector becomes

$$\begin{bmatrix} t \\ t \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

The following table gives a summary of this result. It shows for each eigenvalue the algebraic multiplicity m, and its geometric multiplicity k and the eigenvectors associated with the eigenvalue. If $m > k$ then the eigenvalue is defective which means the number of normal linearly independent eigenvectors associated with this eigenvalue (called the geometric multiplicity k) does not equal the algebraic multiplicity m, and we need to determine an additional $m - k$ generalized eigenvectors for this eigenvalue.

	multiplicity			
eigenvalue	algebraic m	geometric k	defective?	eigenvectors
-1	1	1	No	$\begin{bmatrix} 1 \\ 1 \end{bmatrix}$
-3	1	1	No	$\begin{bmatrix} -1 \\ 1 \end{bmatrix}$

Now that we found the eigenvalues and associated eigenvectors, we will go over each eigenvalue and generate the solution basis. The only problem we need to take care of is if the eigenvalue is defective. Since eigenvalue -1 is real and distinct then the corresponding eigenvector solution is

$$\begin{aligned} \vec{x}_1(t) &= \vec{v}_1 e^{-t} \\ &= \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{-t} \end{aligned}$$

Since eigenvalue -3 is real and distinct then the corresponding eigenvector solution is

$$\vec{x}_2(t) = \vec{v}_2 e^{-3t}$$
$$= \begin{bmatrix} -1 \\ 1 \end{bmatrix} e^{-3t}$$

Therefore the homogeneous solution is

$$\vec{x}_h(t) = c_1 \vec{x}_1(t) + c_2 \vec{x}_2(t)$$

Which is written as

$$\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = c_1 \begin{bmatrix} \mathrm{e}^{-t} \\ \mathrm{e}^{-t} \end{bmatrix} + c_2 \begin{bmatrix} -\mathrm{e}^{-3t} \\ \mathrm{e}^{-3t} \end{bmatrix}$$

Now that we found homogeneous solution above, we need to find a particular solution $\vec{x}_p(t)$. We will use Variation of parameters. The fundamental matrix is

$$\Phi = \begin{bmatrix} \vec{x}_1 & \vec{x}_2 & \cdots \end{bmatrix}$$

Where $\vec{x}_i$ are the solution basis found above. Therefore the fundamental matrix is

$$\Phi(t) = \begin{bmatrix} \mathrm{e}^{-t} & -\mathrm{e}^{-3t} \\ \mathrm{e}^{-t} & \mathrm{e}^{-3t} \end{bmatrix}$$

The particular solution is then given by

$$\vec{x}_p(t) = \Phi \int \Phi^{-1} \vec{G}(t) \, dt$$

But

$$\Phi^{-1} = \begin{bmatrix} \frac{\mathrm{e}^{t}}{2} & \frac{\mathrm{e}^{t}}{2} \\ -\frac{\mathrm{e}^{3t}}{2} & \frac{\mathrm{e}^{3t}}{2} \end{bmatrix}$$

Hence

$$\vec{x}_p(t) = \begin{bmatrix} e^{-t} & -e^{-3t} \\ e^{-t} & e^{-3t} \end{bmatrix} \int \begin{bmatrix} \frac{e^{t}}{2} & \frac{e^{t}}{2} \\ -\frac{e^{3t}}{2} & \frac{e^{3t}}{2} \end{bmatrix} \begin{bmatrix} 2\,e^{-t} \\ 3t \end{bmatrix} dt$$

$$= \begin{bmatrix} e^{-t} & -e^{-3t} \\ e^{-t} & e^{-3t} \end{bmatrix} \int \begin{bmatrix} 1+\frac{3t\,e^{t}}{2} \\ -e^{2t}+\frac{3\,e^{3t}t}{2} \end{bmatrix} dt$$

$$= \begin{bmatrix} e^{-t} & -e^{-3t} \\ e^{-t} & e^{-3t} \end{bmatrix} \begin{bmatrix} \frac{(3t-3)e^{t}}{2}+t \\ \frac{(-1+3t)e^{3t}}{6}-\frac{e^{2t}}{2} \end{bmatrix}$$

$$= \begin{bmatrix} t-\frac{4}{3}+\frac{e^{-t}}{2}+t\,e^{-t} \\ t\,e^{-t}+2t-\frac{5}{3}-\frac{e^{-t}}{2} \end{bmatrix}$$

Now that we found particular solution, the final solution is

$$\vec{x}(t) = \vec{x}_h(t) + \vec{x}_p(t)$$

$$\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = \begin{bmatrix} c_1 e^{-t} \\ c_1 e^{-t} \end{bmatrix} + \begin{bmatrix} -c_2 e^{-3t} \\ c_2 e^{-3t} \end{bmatrix} + \begin{bmatrix} t-\frac{4}{3}+\frac{e^{-t}}{2}+t\,e^{-t} \\ t\,e^{-t}+2t-\frac{5}{3}-\frac{e^{-t}}{2} \end{bmatrix}$$

Which becomes

$$\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = \begin{bmatrix} \frac{(6t+6c_1+3)e^{-t}}{6} - c_2 e^{-3t} + t - \frac{4}{3} \\ \frac{(6t+6c_1-3)e^{-t}}{6} + c_2 e^{-3t} + 2t - \frac{5}{3} \end{bmatrix}$$

Since initial conditions are given, the solution above needs to be updated by solving for the constants of integrations using the given initial conditions

$$\begin{bmatrix} x_1(0) = \alpha_1 \\ x_2(0) = \alpha_2 \end{bmatrix} \tag{1}$$

Substituting initial conditions into the above solution at $t = 0$ gives

$$\begin{bmatrix} \alpha_1 \\ \alpha_2 \end{bmatrix} = \begin{bmatrix} -\frac{5}{6}+c_1-c_2 \\ -\frac{13}{6}+c_1+c_2 \end{bmatrix}$$

Solving for the constants of integrations gives

$$\begin{bmatrix} c_1 = \frac{\alpha_1}{2}+\frac{3}{2}+\frac{\alpha_2}{2} \\ c_2 = \frac{\alpha_2}{2}+\frac{2}{3}-\frac{\alpha_1}{2} \end{bmatrix}$$

Substituting these constants back in original solution in Eq. (1) gives

$$\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = \begin{bmatrix} \frac{(6t+3\alpha_1+12+3\alpha_2)\mathrm{e}^{-t}}{6} - \left(\frac{\alpha_2}{2}+\frac{2}{3}-\frac{\alpha_1}{2}\right)\mathrm{e}^{-3t}+t-\frac{4}{3} \\ \frac{(6t+3\alpha_1+6+3\alpha_2)\mathrm{e}^{-t}}{6} + \left(\frac{\alpha_2}{2}+\frac{2}{3}-\frac{\alpha_1}{2}\right)\mathrm{e}^{-3t}+2t-\frac{5}{3} \end{bmatrix}$$

The following are plots of each solution.

✓ Solution by Maple

Time used: 0.031 (sec). Leaf size: 93

```
dsolve([diff(x__1(t),t) = -2*x__1(t)+x__2(t)+2*exp(-t), diff(x__2(t),t) = x__1(t)-2*x__2(t)+
```

$$x_1(t) = \left(\frac{3}{2}+\frac{\alpha_2}{2}+\frac{\alpha_1}{2}\right)\mathrm{e}^{-t} - \left(\frac{2}{3}+\frac{\alpha_2}{2}-\frac{\alpha_1}{2}\right)\mathrm{e}^{-3t} + \frac{\mathrm{e}^{-t}}{2} + t\,\mathrm{e}^{-t} - \frac{4}{3} + t$$

$$x_2(t) = \left(\frac{3}{2}+\frac{\alpha_2}{2}+\frac{\alpha_1}{2}\right)\mathrm{e}^{-t} + \left(\frac{2}{3}+\frac{\alpha_2}{2}-\frac{\alpha_1}{2}\right)\mathrm{e}^{-3t} + t\,\mathrm{e}^{-t} + 2t - \frac{5}{3} - \frac{\mathrm{e}^{-t}}{2}$$

✓ Solution by Mathematica

Time used: 0.061 (sec). Leaf size: 122

```
DSolve[{x1'[t]==-2*x1[t]+1*x2[t]+2*Exp[-t],x2'[t]==1*x1[t]-2*x2[t]+3*t},{x1[0]==a1,x2[0]==a2
```

$$\mathrm{x1}(t) \to \frac{1}{6}e^{-3t}\left(3\mathrm{a1}\left(e^{2t}+1\right)+3\mathrm{a2}\left(e^{2t}-1\right)+12e^{2t}-8e^{3t}+6e^{2t}t+6e^{3t}t-4\right)$$

$$\mathrm{x2}(t) \to \frac{1}{6}e^{-3t}\left(3\mathrm{a1}\left(e^{2t}-1\right)+3\mathrm{a2}\left(e^{2t}+1\right)+6e^{2t}(t+1)+2e^{3t}(6t-5)+4\right)$$

19 Chapter 9.1, The Phase Plane: Linear Systems. page 505

19.1 problem 1

Internal problem ID [792]
Internal file name [`OUTPUT/792_Sunday_June_05_2022_01_49_55_AM_22760055/index.tex`]

Book: Elementary differential equations and boundary value problems, 10th ed., Boyce and DiPrima
Section: Chapter 9.1, The Phase Plane: Linear Systems. page 505
Problem number: 1.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"system of linear ODEs"**

Solve

$$x_1'(t) = 3x_1(t) - 2x_2(t)$$
$$x_2'(t) = 2x_1(t) - 2x_2(t)$$

19.1.1 Solution using Matrix exponential method

In this method, we will assume we have found the matrix exponential e^{At} allready. There are different methods to determine this but will not be shown here. This is a system of linear ODE's given as

$$\vec{x}'(t) = A\,\vec{x}(t)$$

Or

$$\begin{bmatrix} x_1'(t) \\ x_2'(t) \end{bmatrix} = \begin{bmatrix} 3 & -2 \\ 2 & -2 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}$$

For the above matrix A, the matrix exponential can be found to be

$$e^{At} = \begin{bmatrix} -\frac{e^{-t}}{3} + \frac{4\,e^{2t}}{3} & -\frac{2\,e^{2t}}{3} + \frac{2\,e^{-t}}{3} \\ \frac{2\,e^{2t}}{3} - \frac{2\,e^{-t}}{3} & \frac{4\,e^{-t}}{3} - \frac{e^{2t}}{3} \end{bmatrix}$$

Therefore the homogeneous solution is

$$
\begin{aligned}
\vec{x}_h(t) &= e^{At}\vec{c} \\
&= \begin{bmatrix} -\frac{e^{-t}}{3}+\frac{4\,e^{2t}}{3} & -\frac{2\,e^{2t}}{3}+\frac{2\,e^{-t}}{3} \\ \frac{2\,e^{2t}}{3}-\frac{2\,e^{-t}}{3} & \frac{4\,e^{-t}}{3}-\frac{e^{2t}}{3} \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} \\
&= \begin{bmatrix} \left(-\frac{e^{-t}}{3}+\frac{4\,e^{2t}}{3}\right)c_1+\left(-\frac{2\,e^{2t}}{3}+\frac{2\,e^{-t}}{3}\right)c_2 \\ \left(\frac{2\,e^{2t}}{3}-\frac{2\,e^{-t}}{3}\right)c_1+\left(\frac{4\,e^{-t}}{3}-\frac{e^{2t}}{3}\right)c_2 \end{bmatrix} \\
&= \begin{bmatrix} \frac{(-c_1+2c_2)e^{-t}}{3}+\frac{4\left(c_1-\frac{c_2}{2}\right)e^{2t}}{3} \\ \frac{(-2c_1+4c_2)e^{-t}}{3}+\frac{2\left(c_1-\frac{c_2}{2}\right)e^{2t}}{3} \end{bmatrix}
\end{aligned}
$$

Since no forcing function is given, then the final solution is $\vec{x}_h(t)$ above.

19.1.2 Solution using explicit Eigenvalue and Eigenvector method

This is a system of linear ODE's given as

$$\vec{x}'(t) = A\,\vec{x}(t)$$

Or

$$\begin{bmatrix} x_1'(t) \\ x_2'(t) \end{bmatrix} = \begin{bmatrix} 3 & -2 \\ 2 & -2 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}$$

The first step is find the homogeneous solution. We start by finding the eigenvalues of A. This is done by solving the following equation for the eigenvalues λ

$$\det(A-\lambda I) = 0$$

Expanding gives

$$\det\left(\begin{bmatrix} 3 & -2 \\ 2 & -2 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right) = 0$$

Therefore

$$\det\left(\begin{bmatrix} 3-\lambda & -2 \\ 2 & -2-\lambda \end{bmatrix}\right) = 0$$

Which gives the characteristic equation

$$\lambda^2 - \lambda - 2 = 0$$

The roots of the above are the eigenvalues.

$$\lambda_1 = 2$$
$$\lambda_2 = -1$$

This table summarises the above result

eigenvalue	algebraic multiplicity	type of eigenvalue
-1	1	real eigenvalue
2	1	real eigenvalue

Now the eigenvector for each eigenvalue are found.

Considering the eigenvalue $\lambda_1 = -1$

We need to solve $A\vec{v} = \lambda\vec{v}$ or $(A - \lambda I)\vec{v} = \vec{0}$ which becomes

$$\left(\begin{bmatrix} 3 & -2 \\ 2 & -2 \end{bmatrix} - (-1)\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right)\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$
$$\begin{bmatrix} 4 & -2 \\ 2 & -1 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Now forward elimination is applied to solve for the eigenvector $\vec{v}$. The augmented matrix is

$$\left[\begin{array}{cc|c} 4 & -2 & 0 \\ 2 & -1 & 0 \end{array}\right]$$

$$R_2 = R_2 - \frac{R_1}{2} \Longrightarrow \left[\begin{array}{cc|c} 4 & -2 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

Therefore the system in Echelon form is

$$\begin{bmatrix} 4 & -2 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

The free variables are $\{v_2\}$ and the leading variables are $\{v_1\}$. Let $v_2 = t$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\left\{v_1 = \frac{t}{2}\right\}$

Hence the solution is

$$\begin{bmatrix} \frac{t}{2} \\ t \end{bmatrix} = \begin{bmatrix} \frac{t}{2} \\ t \end{bmatrix}$$

Since there is one free Variable, we have found one eigenvector associated with this eigenvalue. The above can be written as

$$\begin{bmatrix} \frac{t}{2} \\ t \end{bmatrix} = t \begin{bmatrix} \frac{1}{2} \\ 1 \end{bmatrix}$$

Let $t = 1$ the eigenvector becomes

$$\begin{bmatrix} \frac{t}{2} \\ t \end{bmatrix} = \begin{bmatrix} \frac{1}{2} \\ 1 \end{bmatrix}$$

Which is normalized to

$$\begin{bmatrix} \frac{t}{2} \\ t \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

Considering the eigenvalue $\lambda_2 = 2$

We need to solve $A\vec{v} = \lambda\vec{v}$ or $(A - \lambda I)\vec{v} = \vec{0}$ which becomes

$$\left(\begin{bmatrix} 3 & -2 \\ 2 & -2 \end{bmatrix} - (2) \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right) \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & -2 \\ 2 & -4 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Now forward elimination is applied to solve for the eigenvector $\vec{v}$. The augmented matrix is

$$\left[\begin{array}{cc|c} 1 & -2 & 0 \\ 2 & -4 & 0 \end{array}\right]$$

$$R_2 = R_2 - 2R_1 \Longrightarrow \left[\begin{array}{cc|c} 1 & -2 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

Therefore the system in Echelon form is

$$\begin{bmatrix} 1 & -2 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

The free variables are $\{v_2\}$ and the leading variables are $\{v_1\}$. Let $v_2 = t$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\{v_1 = 2t\}$

Hence the solution is

$$\begin{bmatrix} 2t \\ t \end{bmatrix} = \begin{bmatrix} 2t \\ t \end{bmatrix}$$

Since there is one free Variable, we have found one eigenvector associated with this eigenvalue. The above can be written as

$$\begin{bmatrix} 2t \\ t \end{bmatrix} = t \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

Let $t = 1$ the eigenvector becomes

$$\begin{bmatrix} 2t \\ t \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

The following table gives a summary of this result. It shows for each eigenvalue the algebraic multiplicity m, and its geometric multiplicity k and the eigenvectors associated with the eigenvalue. If $m > k$ then the eigenvalue is defective which means the number of normal linearly independent eigenvectors associated with this eigenvalue (called the geometric multiplicity k) does not equal the algebraic multiplicity m, and we need to determine an additional $m - k$ generalized eigenvectors for this eigenvalue.

	multiplicity			
eigenvalue	algebraic m	geometric k	defective?	eigenvectors
2	1	1	No	$\begin{bmatrix} 2 \\ 1 \end{bmatrix}$
-1	1	1	No	$\begin{bmatrix} \frac{1}{2} \\ 1 \end{bmatrix}$

Now that we found the eigenvalues and associated eigenvectors, we will go over each eigenvalue and generate the solution basis. The only problem we need to take care of is if the eigenvalue is defective. Since eigenvalue 2 is real and distinct then the corresponding eigenvector solution is

$$\begin{aligned}\vec{x}_1(t) &= \vec{v}_1 e^{2t} \\ &= \begin{bmatrix} 2 \\ 1 \end{bmatrix} e^{2t}\end{aligned}$$

Since eigenvalue -1 is real and distinct then the corresponding eigenvector solution is

$$\begin{aligned}\vec{x}_2(t) &= \vec{v}_2 e^{-t} \\ &= \begin{bmatrix} \frac{1}{2} \\ 1 \end{bmatrix} e^{-t}\end{aligned}$$

Therefore the final solution is

$$\vec{x}_h(t) = c_1 \vec{x}_1(t) + c_2 \vec{x}_2(t)$$

Which is written as

$$\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = c_1 \begin{bmatrix} 2\,\mathrm{e}^{2t} \\ \mathrm{e}^{2t} \end{bmatrix} + c_2 \begin{bmatrix} \frac{\mathrm{e}^{-t}}{2} \\ \mathrm{e}^{-t} \end{bmatrix}$$

Which becomes

$$\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = \begin{bmatrix} 2c_1 \mathrm{e}^{2t} + \frac{c_2 \mathrm{e}^{-t}}{2} \\ c_1 \mathrm{e}^{2t} + c_2 \mathrm{e}^{-t} \end{bmatrix}$$

The following is the phase plot of the system.

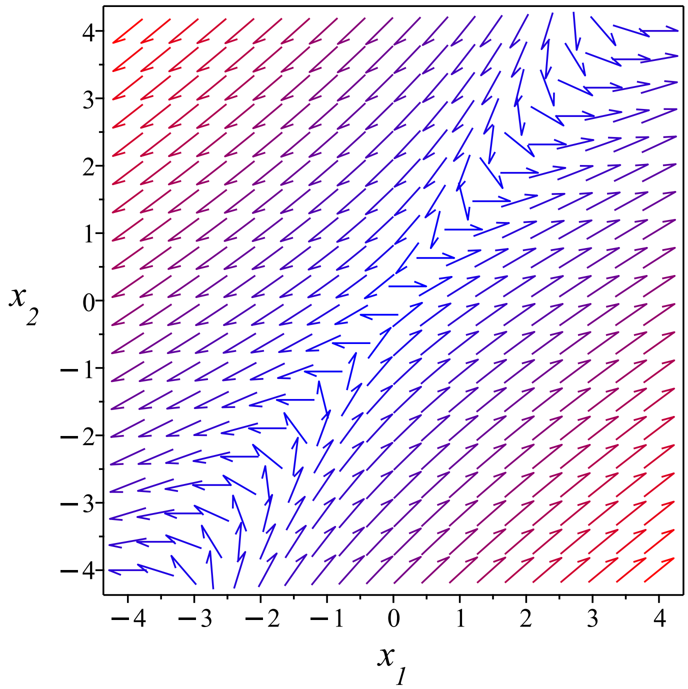


Figure 539: Phase plot

19.1.3 Maple step by step solution

Let's solve

$[x_1'(t) = 3x_1(t) - 2x_2(t), x_2'(t) = 2x_1(t) - 2x_2(t)]$

- Define vector

$$\underline{x}^{\rightarrow}(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}$$

- Convert system into a vector equation

$$\underline{x}^{\rightarrow\prime}(t) = \begin{bmatrix} 3 & -2 \\ 2 & -2 \end{bmatrix} \cdot \underline{x}^{\rightarrow}(t) + \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

- System to solve

$$\underline{x}^{\rightarrow\prime}(t) = \begin{bmatrix} 3 & -2 \\ 2 & -2 \end{bmatrix} \cdot \underline{x}^{\rightarrow}(t)$$

- Define the coefficient matrix

$$A = \begin{bmatrix} 3 & -2 \\ 2 & -2 \end{bmatrix}$$

- Rewrite the system as

$$\underline{\vec{x}}'(t) = A \cdot \underline{\vec{x}}(t)$$

- To solve the system, find the eigenvalues and eigenvectors of A
- Eigenpairs of A

$$\left[\left[-1, \begin{bmatrix} \frac{1}{2} \\ 1 \end{bmatrix}\right], \left[2, \begin{bmatrix} 2 \\ 1 \end{bmatrix}\right]\right]$$

- Consider eigenpair

$$\left[-1, \begin{bmatrix} \frac{1}{2} \\ 1 \end{bmatrix}\right]$$

- Solution to homogeneous system from eigenpair

$$\underline{\vec{x}}_1 = \mathrm{e}^{-t} \cdot \begin{bmatrix} \frac{1}{2} \\ 1 \end{bmatrix}$$

- Consider eigenpair

$$\left[2, \begin{bmatrix} 2 \\ 1 \end{bmatrix}\right]$$

- Solution to homogeneous system from eigenpair

$$\underline{\vec{x}}_2 = \mathrm{e}^{2t} \cdot \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

- General solution to the system of ODEs

$$\underline{\vec{x}} = c_1 \underline{\vec{x}}_1 + c_2 \underline{\vec{x}}_2$$

- Substitute solutions into the general solution

$$\underline{\vec{x}} = c_1 \mathrm{e}^{-t} \cdot \begin{bmatrix} \frac{1}{2} \\ 1 \end{bmatrix} + c_2 \mathrm{e}^{2t} \cdot \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

- Substitute in vector of dependent variables

$$\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = \begin{bmatrix} \frac{c_1 \mathrm{e}^{-t}}{2} + 2c_2 \mathrm{e}^{2t} \\ c_1 \mathrm{e}^{-t} + c_2 \mathrm{e}^{2t} \end{bmatrix}$$

- Solution to the system of ODEs

  $$\left\{x_1(t) = \frac{c_1 \mathrm{e}^{-t}}{2} + 2c_2 \mathrm{e}^{2t}, x_2(t) = c_1 \mathrm{e}^{-t} + c_2 \mathrm{e}^{2t}\right\}$$

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 36

```
dsolve([diff(x__1(t),t)=3*x__1(t)-2*x__2(t),diff(x__2(t),t)=2*x__1(t)-2*x__2(t)],singsol=all
```

$$x_1(t) = \mathrm{e}^{-t} c_1 + c_2 \mathrm{e}^{2t}$$
$$x_2(t) = 2\,\mathrm{e}^{-t} c_1 + \frac{c_2 \mathrm{e}^{2t}}{2}$$

✓ Solution by Mathematica

Time used: 0.003 (sec). Leaf size: 73

```
DSolve[{x1'[t]==3*x1[t]-2*x2[t],x2'[t]==2*x1[t]-2*x2[t]},{x1[t],x2[t]},t,IncludeSingularSolu
```

$$\text{x1}(t) \to \frac{1}{3} e^{-t} \left(c_1 \left(4e^{3t} - 1\right) - 2c_2 \left(e^{3t} - 1\right)\right)$$
$$\text{x2}(t) \to \frac{1}{3} e^{-t} \left(2c_1 \left(e^{3t} - 1\right) - c_2 \left(e^{3t} - 4\right)\right)$$

19.2 problem 2

Internal problem ID [793]
Internal file name [`OUTPUT/793_Sunday_June_05_2022_01_49_56_AM_69247619/index.tex`]

Book: Elementary differential equations and boundary value problems, 10th ed., Boyce and DiPrima
Section: Chapter 9.1, The Phase Plane: Linear Systems. page 505
Problem number: 2.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"system of linear ODEs"**

Solve

$$x_1'(t) = 5x_1(t) - x_2(t)$$
$$x_2'(t) = 3x_1(t) + x_2(t)$$

19.2.1 Solution using Matrix exponential method

In this method, we will assume we have found the matrix exponential e^{At} allready. There are different methods to determine this but will not be shown here. This is a system of linear ODE's given as

$$\vec{x}'(t) = A\,\vec{x}(t)$$

Or

$$\begin{bmatrix} x_1'(t) \\ x_2'(t) \end{bmatrix} = \begin{bmatrix} 5 & -1 \\ 3 & 1 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}$$

For the above matrix A, the matrix exponential can be found to be

$$e^{At} = \begin{bmatrix} -\frac{e^{2t}}{2} + \frac{3\,e^{4t}}{2} & -\frac{e^{4t}}{2} + \frac{e^{2t}}{2} \\ \frac{3\,e^{4t}}{2} - \frac{3\,e^{2t}}{2} & \frac{3\,e^{2t}}{2} - \frac{e^{4t}}{2} \end{bmatrix}$$

Therefore the homogeneous solution is

$$
\begin{aligned}
\vec{x}_h(t) &= e^{At}\vec{c} \\
&= \begin{bmatrix} -\frac{e^{2t}}{2} + \frac{3e^{4t}}{2} & -\frac{e^{4t}}{2} + \frac{e^{2t}}{2} \\ \frac{3e^{4t}}{2} - \frac{3e^{2t}}{2} & \frac{3e^{2t}}{2} - \frac{e^{4t}}{2} \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} \\
&= \begin{bmatrix} \left(-\frac{e^{2t}}{2} + \frac{3e^{4t}}{2}\right) c_1 + \left(-\frac{e^{4t}}{2} + \frac{e^{2t}}{2}\right) c_2 \\ \left(\frac{3e^{4t}}{2} - \frac{3e^{2t}}{2}\right) c_1 + \left(\frac{3e^{2t}}{2} - \frac{e^{4t}}{2}\right) c_2 \end{bmatrix} \\
&= \begin{bmatrix} \frac{(c_2-c_1)e^{2t}}{2} + \frac{3\left(c_1-\frac{c_2}{3}\right)e^{4t}}{2} \\ \frac{(-3c_1+3c_2)e^{2t}}{2} + \frac{3\left(c_1-\frac{c_2}{3}\right)e^{4t}}{2} \end{bmatrix}
\end{aligned}
$$

Since no forcing function is given, then the final solution is $\vec{x}_h(t)$ above.

19.2.2 Solution using explicit Eigenvalue and Eigenvector method

This is a system of linear ODE's given as

$$\vec{x}'(t) = A\,\vec{x}(t)$$

Or

$$\begin{bmatrix} x_1'(t) \\ x_2'(t) \end{bmatrix} = \begin{bmatrix} 5 & -1 \\ 3 & 1 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}$$

The first step is find the homogeneous solution. We start by finding the eigenvalues of A. This is done by solving the following equation for the eigenvalues λ

$$\det(A - \lambda I) = 0$$

Expanding gives

$$\det\left(\begin{bmatrix} 5 & -1 \\ 3 & 1 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right) = 0$$

Therefore

$$\det\left(\begin{bmatrix} 5-\lambda & -1 \\ 3 & 1-\lambda \end{bmatrix}\right) = 0$$

Which gives the characteristic equation

$$\lambda^2 - 6\lambda + 8 = 0$$

The roots of the above are the eigenvalues.

$$\lambda_1 = 2$$
$$\lambda_2 = 4$$

This table summarises the above result

eigenvalue	algebraic multiplicity	type of eigenvalue
2	1	real eigenvalue
4	1	real eigenvalue

Now the eigenvector for each eigenvalue are found.

Considering the eigenvalue $\lambda_1 = 2$

We need to solve $A\vec{v} = \lambda\vec{v}$ or $(A - \lambda I)\vec{v} = \vec{0}$ which becomes

$$\left(\begin{bmatrix} 5 & -1 \\ 3 & 1 \end{bmatrix} - (2)\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right)\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$
$$\begin{bmatrix} 3 & -1 \\ 3 & -1 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Now forward elimination is applied to solve for the eigenvector $\vec{v}$. The augmented matrix is

$$\left[\begin{array}{cc|c} 3 & -1 & 0 \\ 3 & -1 & 0 \end{array}\right]$$

$$R_2 = R_2 - R_1 \Longrightarrow \left[\begin{array}{cc|c} 3 & -1 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

Therefore the system in Echelon form is

$$\begin{bmatrix} 3 & -1 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

The free variables are $\{v_2\}$ and the leading variables are $\{v_1\}$. Let $v_2 = t$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\{v_1 = \frac{t}{3}\}$

Hence the solution is

$$\begin{bmatrix} \frac{t}{3} \\ t \end{bmatrix} = \begin{bmatrix} \frac{t}{3} \\ t \end{bmatrix}$$

Since there is one free Variable, we have found one eigenvector associated with this eigenvalue. The above can be written as

$$\begin{bmatrix} \frac{t}{3} \\ t \end{bmatrix} = t \begin{bmatrix} \frac{1}{3} \\ 1 \end{bmatrix}$$

Let $t = 1$ the eigenvector becomes

$$\begin{bmatrix} \frac{t}{3} \\ t \end{bmatrix} = \begin{bmatrix} \frac{1}{3} \\ 1 \end{bmatrix}$$

Which is normalized to

$$\begin{bmatrix} \frac{t}{3} \\ t \end{bmatrix} = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$$

Considering the eigenvalue $\lambda_2 = 4$

We need to solve $A\vec{v} = \lambda\vec{v}$ or $(A - \lambda I)\vec{v} = \vec{0}$ which becomes

$$\left(\begin{bmatrix} 5 & -1 \\ 3 & 1 \end{bmatrix} - (4) \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right) \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & -1 \\ 3 & -3 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Now forward elimination is applied to solve for the eigenvector $\vec{v}$. The augmented matrix is

$$\left[\begin{array}{cc|c} 1 & -1 & 0 \\ 3 & -3 & 0 \end{array}\right]$$

$$R_2 = R_2 - 3R_1 \Longrightarrow \left[\begin{array}{cc|c} 1 & -1 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

Therefore the system in Echelon form is

$$\begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

The free variables are $\{v_2\}$ and the leading variables are $\{v_1\}$. Let $v_2 = t$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\{v_1 = t\}$

Hence the solution is

$$\begin{bmatrix} t \\ t \end{bmatrix} = \begin{bmatrix} t \\ t \end{bmatrix}$$

Since there is one free Variable, we have found one eigenvector associated with this eigenvalue. The above can be written as

$$\begin{bmatrix} t \\ t \end{bmatrix} = t \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

Let $t = 1$ the eigenvector becomes

$$\begin{bmatrix} t \\ t \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

The following table gives a summary of this result. It shows for each eigenvalue the algebraic multiplicity m, and its geometric multiplicity k and the eigenvectors associated with the eigenvalue. If $m > k$ then the eigenvalue is defective which means the number of normal linearly independent eigenvectors associated with this eigenvalue (called the geometric multiplicity k) does not equal the algebraic multiplicity m, and we need to determine an additional $m - k$ generalized eigenvectors for this eigenvalue.

	multiplicity			
eigenvalue	algebraic m	geometric k	defective?	eigenvectors
2	1	1	No	$\begin{bmatrix} \frac{1}{3} \\ 1 \end{bmatrix}$
4	1	1	No	$\begin{bmatrix} 1 \\ 1 \end{bmatrix}$

Now that we found the eigenvalues and associated eigenvectors, we will go over each eigenvalue and generate the solution basis. The only problem we need to take care of is if the eigenvalue is defective. Since eigenvalue 2 is real and distinct then the corresponding eigenvector solution is

$$
\begin{aligned}
\vec{x}_1(t) &= \vec{v}_1 e^{2t} \\
&= \begin{bmatrix} \frac{1}{3} \\ 1 \end{bmatrix} e^{2t}
\end{aligned}
$$

Since eigenvalue 4 is real and distinct then the corresponding eigenvector solution is

$$
\begin{aligned}
\vec{x}_2(t) &= \vec{v}_2 e^{4t} \\
&= \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{4t}
\end{aligned}
$$

Therefore the final solution is

$$
\vec{x}_h(t) = c_1 \vec{x}_1(t) + c_2 \vec{x}_2(t)
$$

Which is written as

$$
\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = c_1 \begin{bmatrix} \frac{e^{2t}}{3} \\ e^{2t} \end{bmatrix} + c_2 \begin{bmatrix} e^{4t} \\ e^{4t} \end{bmatrix}
$$

Which becomes

$$
\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = \begin{bmatrix} \frac{c_1 e^{2t}}{3} + c_2 e^{4t} \\ c_1 e^{2t} + c_2 e^{4t} \end{bmatrix}
$$

The following is the phase plot of the system.

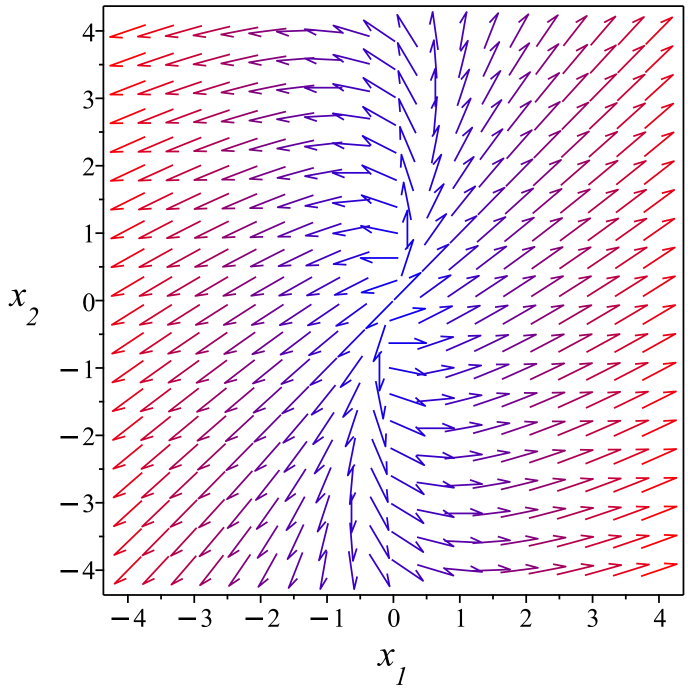


Figure 540: Phase plot

19.2.3 Maple step by step solution

Let's solve

$[x_1'(t) = 5x_1(t) - x_2(t), x_2'(t) = 3x_1(t) + x_2(t)]$

- Define vector

$$\underline{x}^{\rightarrow}(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}$$

- Convert system into a vector equation

$$\underline{x}^{\rightarrow}{}'(t) = \begin{bmatrix} 5 & -1 \\ 3 & 1 \end{bmatrix} \cdot \underline{x}^{\rightarrow}(t) + \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

- System to solve

$$\underline{x}^{\rightarrow}{}'(t) = \begin{bmatrix} 5 & -1 \\ 3 & 1 \end{bmatrix} \cdot \underline{x}^{\rightarrow}(t)$$

- Define the coefficient matrix

$$A = \begin{bmatrix} 5 & -1 \\ 3 & 1 \end{bmatrix}$$

- Rewrite the system as

$$\underline{\overrightarrow{x}}'(t) = A \cdot \underline{\overrightarrow{x}}(t)$$

- To solve the system, find the eigenvalues and eigenvectors of A
- Eigenpairs of A

$$\left[\left[2, \begin{bmatrix} \frac{1}{3} \\ 1 \end{bmatrix}\right], \left[4, \begin{bmatrix} 1 \\ 1 \end{bmatrix}\right]\right]$$

- Consider eigenpair

$$\left[2, \begin{bmatrix} \frac{1}{3} \\ 1 \end{bmatrix}\right]$$

- Solution to homogeneous system from eigenpair

$$\underline{\overrightarrow{x}}_1 = e^{2t} \cdot \begin{bmatrix} \frac{1}{3} \\ 1 \end{bmatrix}$$

- Consider eigenpair

$$\left[4, \begin{bmatrix} 1 \\ 1 \end{bmatrix}\right]$$

- Solution to homogeneous system from eigenpair

$$\underline{\overrightarrow{x}}_2 = e^{4t} \cdot \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

- General solution to the system of ODEs

$$\underline{\overrightarrow{x}} = c_1 \underline{\overrightarrow{x}}_1 + c_2 \underline{\overrightarrow{x}}_2$$

- Substitute solutions into the general solution

$$\underline{\overrightarrow{x}} = c_1 e^{2t} \cdot \begin{bmatrix} \frac{1}{3} \\ 1 \end{bmatrix} + c_2 e^{4t} \cdot \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

- Substitute in vector of dependent variables

$$\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = \begin{bmatrix} \frac{c_1 e^{2t}}{3} + c_2 e^{4t} \\ c_1 e^{2t} + c_2 e^{4t} \end{bmatrix}$$

- Solution to the system of ODEs

  $\left\{x_1(t) = \frac{c_1 e^{2t}}{3} + c_2 e^{4t}, x_2(t) = c_1 e^{2t} + c_2 e^{4t}\right\}$

✓ Solution by Maple

Time used: 0.015 (sec). Leaf size: 35

```
dsolve([diff(x__1(t),t)=5*x__1(t)-1*x__2(t),diff(x__2(t),t)=3*x__1(t)+1*x__2(t)],singsol=all
```

$$x_1(t) = c_1 e^{4t} + c_2 e^{2t}$$
$$x_2(t) = c_1 e^{4t} + 3c_2 e^{2t}$$

✓ Solution by Mathematica

Time used: 0.003 (sec). Leaf size: 73

```
DSolve[{x1'[t]==5*x1[t]-1*x2[t],x2'[t]==3*x1[t]+1*x2[t]},{x1[t],x2[t]},t,IncludeSingularSolu
```

$$\text{x1}(t) \to \frac{1}{2} e^{2t} \left(c_1 \left(3e^{2t} - 1\right) - c_2 \left(e^{2t} - 1\right)\right)$$
$$\text{x2}(t) \to \frac{1}{2} e^{2t} \left(3c_1 \left(e^{2t} - 1\right) - c_2 \left(e^{2t} - 3\right)\right)$$

19.3 problem 3

Internal problem ID [794]
Internal file name [`OUTPUT/794_Sunday_June_05_2022_01_49_57_AM_46336794/index.tex`]

Book: Elementary differential equations and boundary value problems, 10th ed., Boyce and DiPrima
Section: Chapter 9.1, The Phase Plane: Linear Systems. page 505
Problem number: 3.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"system of linear ODEs"**

Solve

$$x_1'(t) = 2x_1(t) - x_2(t)$$
$$x_2'(t) = 3x_1(t) - 2x_2(t)$$

19.3.1 Solution using Matrix exponential method

In this method, we will assume we have found the matrix exponential e^{At} allready. There are different methods to determine this but will not be shown here. This is a system of linear ODE's given as

$$\vec{x}'(t) = A\,\vec{x}(t)$$

Or

$$\begin{bmatrix} x_1'(t) \\ x_2'(t) \end{bmatrix} = \begin{bmatrix} 2 & -1 \\ 3 & -2 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}$$

For the above matrix A, the matrix exponential can be found to be

$$e^{At} = \begin{bmatrix} -\frac{e^{-t}}{2} + \frac{3\,e^{t}}{2} & -\frac{e^{t}}{2} + \frac{e^{-t}}{2} \\ \frac{3\,e^{t}}{2} - \frac{3\,e^{-t}}{2} & \frac{3\,e^{-t}}{2} - \frac{e^{t}}{2} \end{bmatrix}$$

Therefore the homogeneous solution is

$$
\begin{aligned}
\vec{x}_h(t) &= e^{At}\vec{c} \\
&= \begin{bmatrix} -\frac{e^{-t}}{2} + \frac{3\,e^{t}}{2} & -\frac{e^{t}}{2} + \frac{e^{-t}}{2} \\ \frac{3\,e^{t}}{2} - \frac{3\,e^{-t}}{2} & \frac{3\,e^{-t}}{2} - \frac{e^{t}}{2} \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} \\
&= \begin{bmatrix} \left(-\frac{e^{-t}}{2} + \frac{3\,e^{t}}{2}\right) c_1 + \left(-\frac{e^{t}}{2} + \frac{e^{-t}}{2}\right) c_2 \\ \left(\frac{3\,e^{t}}{2} - \frac{3\,e^{-t}}{2}\right) c_1 + \left(\frac{3\,e^{-t}}{2} - \frac{e^{t}}{2}\right) c_2 \end{bmatrix} \\
&= \begin{bmatrix} \frac{(c_2-c_1)e^{-t}}{2} + \frac{3\left(c_1-\frac{c_2}{3}\right)e^{t}}{2} \\ \frac{(-3c_1+3c_2)e^{-t}}{2} + \frac{3\left(c_1-\frac{c_2}{3}\right)e^{t}}{2} \end{bmatrix}
\end{aligned}
$$

Since no forcing function is given, then the final solution is $\vec{x}_h(t)$ above.

19.3.2 Solution using explicit Eigenvalue and Eigenvector method

This is a system of linear ODE's given as

$$
\vec{x}'(t) = A\,\vec{x}(t)
$$

Or

$$
\begin{bmatrix} x_1'(t) \\ x_2'(t) \end{bmatrix} = \begin{bmatrix} 2 & -1 \\ 3 & -2 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}
$$

The first step is find the homogeneous solution. We start by finding the eigenvalues of A. This is done by solving the following equation for the eigenvalues λ

$$
\det\left(A - \lambda I\right) = 0
$$

Expanding gives

$$
\det\left(\begin{bmatrix} 2 & -1 \\ 3 & -2 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right) = 0
$$

Therefore

$$
\det\left(\begin{bmatrix} 2-\lambda & -1 \\ 3 & -2-\lambda \end{bmatrix}\right) = 0
$$

Which gives the characteristic equation

$$\lambda^2 - 1 = 0$$

The roots of the above are the eigenvalues.

$$\lambda_1 = -1$$
$$\lambda_2 = 1$$

This table summarises the above result

eigenvalue	algebraic multiplicity	type of eigenvalue
-1	1	real eigenvalue
1	1	real eigenvalue

Now the eigenvector for each eigenvalue are found.

Considering the eigenvalue $\lambda_1 = -1$

We need to solve $A\vec{v} = \lambda\vec{v}$ or $(A - \lambda I)\vec{v} = \vec{0}$ which becomes

$$\left(\begin{bmatrix} 2 & -1 \\ 3 & -2 \end{bmatrix} - (-1)\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right)\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$
$$\begin{bmatrix} 3 & -1 \\ 3 & -1 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Now forward elimination is applied to solve for the eigenvector $\vec{v}$. The augmented matrix is

$$\left[\begin{array}{cc|c} 3 & -1 & 0 \\ 3 & -1 & 0 \end{array}\right]$$

$$R_2 = R_2 - R_1 \Longrightarrow \left[\begin{array}{cc|c} 3 & -1 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

Therefore the system in Echelon form is

$$\begin{bmatrix} 3 & -1 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

The free variables are $\{v_2\}$ and the leading variables are $\{v_1\}$. Let $v_2 = t$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\left\{v_1 = \frac{t}{3}\right\}$

Hence the solution is

$$\begin{bmatrix} \frac{t}{3} \\ t \end{bmatrix} = \begin{bmatrix} \frac{t}{3} \\ t \end{bmatrix}$$

Since there is one free Variable, we have found one eigenvector associated with this eigenvalue. The above can be written as

$$\begin{bmatrix} \frac{t}{3} \\ t \end{bmatrix} = t \begin{bmatrix} \frac{1}{3} \\ 1 \end{bmatrix}$$

Let $t = 1$ the eigenvector becomes

$$\begin{bmatrix} \frac{t}{3} \\ t \end{bmatrix} = \begin{bmatrix} \frac{1}{3} \\ 1 \end{bmatrix}$$

Which is normalized to

$$\begin{bmatrix} \frac{t}{3} \\ t \end{bmatrix} = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$$

Considering the eigenvalue $\lambda_2 = 1$

We need to solve $A\vec{v} = \lambda\vec{v}$ or $(A - \lambda I)\vec{v} = \vec{0}$ which becomes

$$\left(\begin{bmatrix} 2 & -1 \\ 3 & -2 \end{bmatrix} - (1) \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right) \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & -1 \\ 3 & -3 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Now forward elimination is applied to solve for the eigenvector $\vec{v}$. The augmented matrix is

$$\left[\begin{array}{cc|c} 1 & -1 & 0 \\ 3 & -3 & 0 \end{array}\right]$$

$$R_2 = R_2 - 3R_1 \Longrightarrow \left[\begin{array}{cc|c} 1 & -1 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

Therefore the system in Echelon form is

$$\begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

The free variables are $\{v_2\}$ and the leading variables are $\{v_1\}$. Let $v_2 = t$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\{v_1 = t\}$

Hence the solution is

$$\begin{bmatrix} t \\ t \end{bmatrix} = \begin{bmatrix} t \\ t \end{bmatrix}$$

Since there is one free Variable, we have found one eigenvector associated with this eigenvalue. The above can be written as

$$\begin{bmatrix} t \\ t \end{bmatrix} = t \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

Let $t = 1$ the eigenvector becomes

$$\begin{bmatrix} t \\ t \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

The following table gives a summary of this result. It shows for each eigenvalue the algebraic multiplicity m, and its geometric multiplicity k and the eigenvectors associated with the eigenvalue. If $m > k$ then the eigenvalue is defective which means the number of normal linearly independent eigenvectors associated with this eigenvalue (called the geometric multiplicity k) does not equal the algebraic multiplicity m, and we need to determine an additional $m - k$ generalized eigenvectors for this eigenvalue.

	multiplicity			
eigenvalue	algebraic m	geometric k	defective?	eigenvectors
-1	1	1	No	$\begin{bmatrix} \frac{1}{3} \\ 1 \end{bmatrix}$
1	1	1	No	$\begin{bmatrix} 1 \\ 1 \end{bmatrix}$

Now that we found the eigenvalues and associated eigenvectors, we will go over each eigenvalue and generate the solution basis. The only problem we need to take care of is if the eigenvalue is defective. Since eigenvalue -1 is real and distinct then the corresponding eigenvector solution is

$$
\begin{aligned}
\vec{x}_1(t) &= \vec{v}_1 e^{-t} \\
&= \begin{bmatrix} \frac{1}{3} \\ 1 \end{bmatrix} e^{-t}
\end{aligned}
$$

Since eigenvalue 1 is real and distinct then the corresponding eigenvector solution is

$$
\begin{aligned}
\vec{x}_2(t) &= \vec{v}_2 e^{t} \\
&= \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{t}
\end{aligned}
$$

Therefore the final solution is

$$
\vec{x}_h(t) = c_1 \vec{x}_1(t) + c_2 \vec{x}_2(t)
$$

Which is written as

$$
\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = c_1 \begin{bmatrix} \frac{\mathrm{e}^{-t}}{3} \\ \mathrm{e}^{-t} \end{bmatrix} + c_2 \begin{bmatrix} \mathrm{e}^{t} \\ \mathrm{e}^{t} \end{bmatrix}
$$

Which becomes

$$
\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = \begin{bmatrix} \frac{c_1 \mathrm{e}^{-t}}{3} + c_2 \mathrm{e}^{t} \\ c_1 \mathrm{e}^{-t} + c_2 \mathrm{e}^{t} \end{bmatrix}
$$

The following is the phase plot of the system.

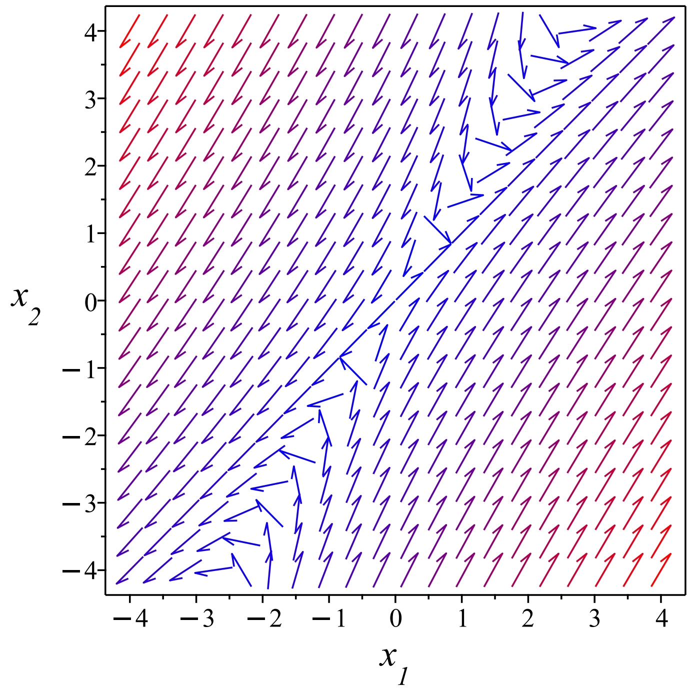


Figure 541: Phase plot

19.3.3 Maple step by step solution

Let's solve

$[x_1'(t) = 2x_1(t) - x_2(t), x_2'(t) = 3x_1(t) - 2x_2(t)]$

- Define vector

$$\underline{x}^{\rightarrow}(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}$$

- Convert system into a vector equation

$$\underline{x}^{\rightarrow\prime}(t) = \begin{bmatrix} 2 & -1 \\ 3 & -2 \end{bmatrix} \cdot \underline{x}^{\rightarrow}(t) + \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

- System to solve

$$\underline{x}^{\rightarrow\prime}(t) = \begin{bmatrix} 2 & -1 \\ 3 & -2 \end{bmatrix} \cdot \underline{x}^{\rightarrow}(t)$$

- Define the coefficient matrix

$$A = \begin{bmatrix} 2 & -1 \\ 3 & -2 \end{bmatrix}$$

- Rewrite the system as

$$\underline{\vec{x}}'(t) = A \cdot \underline{\vec{x}}(t)$$

- To solve the system, find the eigenvalues and eigenvectors of A
- Eigenpairs of A

$$\left[\left[-1, \begin{bmatrix} \frac{1}{3} \\ 1 \end{bmatrix}\right], \left[1, \begin{bmatrix} 1 \\ 1 \end{bmatrix}\right]\right]$$

- Consider eigenpair

$$\left[-1, \begin{bmatrix} \frac{1}{3} \\ 1 \end{bmatrix}\right]$$

- Solution to homogeneous system from eigenpair

$$\underline{\vec{x}}_1 = e^{-t} \cdot \begin{bmatrix} \frac{1}{3} \\ 1 \end{bmatrix}$$

- Consider eigenpair

$$\left[1, \begin{bmatrix} 1 \\ 1 \end{bmatrix}\right]$$

- Solution to homogeneous system from eigenpair

$$\underline{\vec{x}}_2 = e^{t} \cdot \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

- General solution to the system of ODEs

$$\underline{\vec{x}} = c_1 \underline{\vec{x}}_1 + c_2 \underline{\vec{x}}_2$$

- Substitute solutions into the general solution

$$\underline{\vec{x}} = c_1 e^{-t} \cdot \begin{bmatrix} \frac{1}{3} \\ 1 \end{bmatrix} + c_2 e^{t} \cdot \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

- Substitute in vector of dependent variables

$$\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = \begin{bmatrix} \frac{c_1 \mathrm{e}^{-t}}{3} + c_2 \mathrm{e}^{t} \\ c_1 \mathrm{e}^{-t} + c_2 \mathrm{e}^{t} \end{bmatrix}$$

- Solution to the system of ODEs

  $\left\{x_1(t) = \frac{c_1 \mathrm{e}^{-t}}{3} + c_2 \mathrm{e}^{t}, x_2(t) = c_1 \mathrm{e}^{-t} + c_2 \mathrm{e}^{t}\right\}$

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 31

```
dsolve([diff(x__1(t),t)=2*x__1(t)-1*x__2(t),diff(x__2(t),t)=3*x__1(t)-2*x__2(t)],singsol=all
```

$$x_1(t) = c_1 \mathrm{e}^{t} + c_2 \mathrm{e}^{-t}$$
$$x_2(t) = c_1 \mathrm{e}^{t} + 3c_2 \mathrm{e}^{-t}$$

✓ Solution by Mathematica

Time used: 0.003 (sec). Leaf size: 73

```
DSolve[{x1'[t]==2*x1[t]-1*x2[t],x2'[t]==3*x1[t]-2*x2[t]},{x1[t],x2[t]},t,IncludeSingularSolu
```

$$\text{x1}(t) \to \frac{1}{2} e^{-t} \left(c_1 \left(3e^{2t} - 1\right) - c_2 \left(e^{2t} - 1\right)\right)$$
$$\text{x2}(t) \to \frac{1}{2} e^{-t} \left(3c_1 \left(e^{2t} - 1\right) - c_2 \left(e^{2t} - 3\right)\right)$$

19.4 problem 4

Internal problem ID [795]
Internal file name [`OUTPUT/795_Sunday_June_05_2022_01_49_58_AM_52350690/index.tex`]

Book: Elementary differential equations and boundary value problems, 10th ed., Boyce and DiPrima
Section: Chapter 9.1, The Phase Plane: Linear Systems. page 505
Problem number: 4.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"system of linear ODEs"**

Solve

$$x_1'(t) = x_1(t) - 4x_2(t)$$
$$x_2'(t) = 4x_1(t) - 7x_2(t)$$

19.4.1 Solution using Matrix exponential method

In this method, we will assume we have found the matrix exponential e^{At} allready. There are different methods to determine this but will not be shown here. This is a system of linear ODE's given as

$$\vec{x}'(t) = A\,\vec{x}(t)$$

Or

$$\begin{bmatrix} x_1'(t) \\ x_2'(t) \end{bmatrix} = \begin{bmatrix} 1 & -4 \\ 4 & -7 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}$$

For the above matrix A, the matrix exponential can be found to be

$$e^{At} = \begin{bmatrix} \mathrm{e}^{-3t}(4t+1) & -4t\,\mathrm{e}^{-3t} \\ 4t\,\mathrm{e}^{-3t} & \mathrm{e}^{-3t}(1-4t) \end{bmatrix}$$

Therefore the homogeneous solution is

$$
\begin{aligned}
\vec{x}_h(t) &= e^{At}\vec{c} \\
&= \begin{bmatrix} \mathrm{e}^{-3t}(4t+1) & -4t\,\mathrm{e}^{-3t} \\ 4t\,\mathrm{e}^{-3t} & \mathrm{e}^{-3t}(1-4t) \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} \\
&= \begin{bmatrix} \mathrm{e}^{-3t}(4t+1)\,c_1 - 4t\,\mathrm{e}^{-3t}c_2 \\ 4t\,\mathrm{e}^{-3t}c_1 + \mathrm{e}^{-3t}(1-4t)\,c_2 \end{bmatrix} \\
&= \begin{bmatrix} \mathrm{e}^{-3t}(4tc_1 - 4c_2t + c_1) \\ \mathrm{e}^{-3t}(4tc_1 - 4c_2t + c_2) \end{bmatrix}
\end{aligned}
$$

Since no forcing function is given, then the final solution is $\vec{x}_h(t)$ above.

19.4.2 Solution using explicit Eigenvalue and Eigenvector method

This is a system of linear ODE's given as

$$\vec{x}'(t) = A\,\vec{x}(t)$$

Or

$$\begin{bmatrix} x_1'(t) \\ x_2'(t) \end{bmatrix} = \begin{bmatrix} 1 & -4 \\ 4 & -7 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}$$

The first step is find the homogeneous solution. We start by finding the eigenvalues of A. This is done by solving the following equation for the eigenvalues λ

$$\det(A - \lambda I) = 0$$

Expanding gives

$$\det\left(\begin{bmatrix} 1 & -4 \\ 4 & -7 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right) = 0$$

Therefore

$$\det\left(\begin{bmatrix} 1-\lambda & -4 \\ 4 & -7-\lambda \end{bmatrix}\right) = 0$$

Which gives the characteristic equation

$$\lambda^2 + 6\lambda + 9 = 0$$

The roots of the above are the eigenvalues.

$$\lambda_1 = -3$$

This table summarises the above result

eigenvalue	algebraic multiplicity	type of eigenvalue
-3	1	real eigenvalue

Now the eigenvector for each eigenvalue are found.

Considering the eigenvalue $\lambda_1 = -3$

We need to solve $A\vec{v} = \lambda\vec{v}$ or $(A - \lambda I)\vec{v} = \vec{0}$ which becomes

$$\left(\begin{bmatrix} 1 & -4 \\ 4 & -7 \end{bmatrix} - (-3) \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right) \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 4 & -4 \\ 4 & -4 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Now forward elimination is applied to solve for the eigenvector $\vec{v}$. The augmented matrix is

$$\left[\begin{array}{cc|c} 4 & -4 & 0 \\ 4 & -4 & 0 \end{array}\right]$$

$$R_2 = R_2 - R_1 \Longrightarrow \left[\begin{array}{cc|c} 4 & -4 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

Therefore the system in Echelon form is

$$\begin{bmatrix} 4 & -4 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

The free variables are $\{v_2\}$ and the leading variables are $\{v_1\}$. Let $v_2 = t$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\{v_1 = t\}$

Hence the solution is

$$\begin{bmatrix} t \\ t \end{bmatrix} = \begin{bmatrix} t \\ t \end{bmatrix}$$

Since there is one free Variable, we have found one eigenvector associated with this eigenvalue. The above can be written as

$$\begin{bmatrix} t \\ t \end{bmatrix} = t \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

Let $t = 1$ the eigenvector becomes

$$\begin{bmatrix} t \\ t \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

The following table gives a summary of this result. It shows for each eigenvalue the algebraic multiplicity m, and its geometric multiplicity k and the eigenvectors associated with the eigenvalue. If $m > k$ then the eigenvalue is defective which means the number of normal linearly independent eigenvectors associated with this eigenvalue (called the geometric multiplicity k) does not equal the algebraic multiplicity m, and we need to determine an additional $m - k$ generalized eigenvectors for this eigenvalue.

	multiplicity			
eigenvalue	algebraic m	geometric k	defective?	eigenvectors
-3	2	1	Yes	$\begin{bmatrix} 1 \\ 1 \end{bmatrix}$

Now that we found the eigenvalues and associated eigenvectors, we will go over each eigenvalue and generate the solution basis. The only problem we need to take care of is if the eigenvalue is defective. eigenvalue -3 is real and repated eigenvalue of multiplicity 2.There are two possible cases that can happen. This is illustrated in this diagram

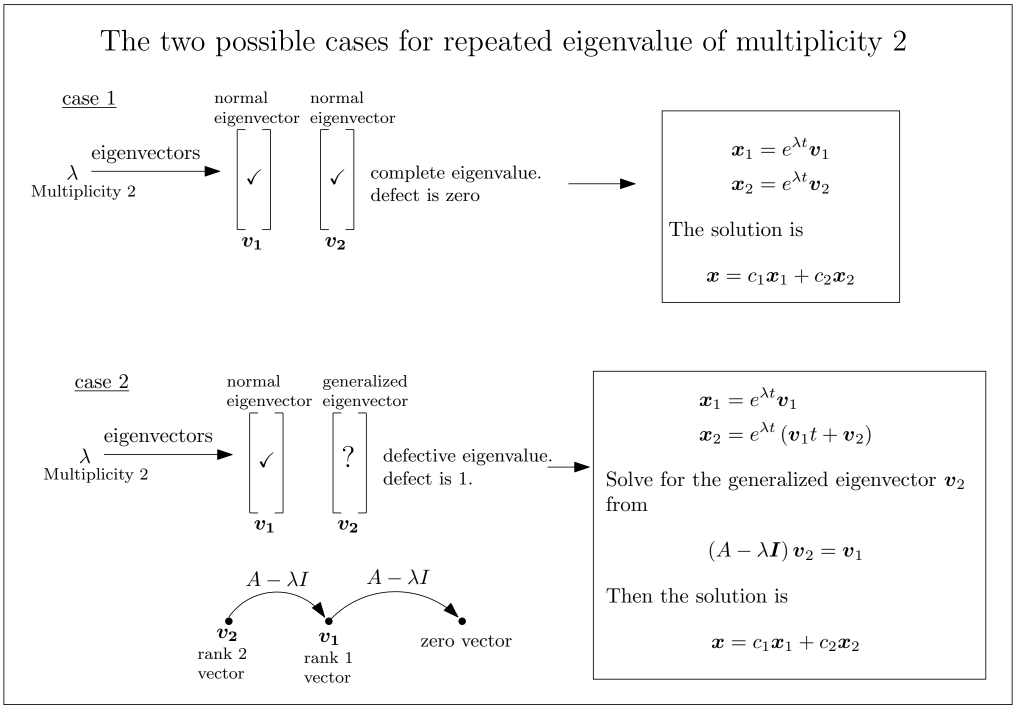


Figure 542: Possible case for repeated λ of multiplicity 2

This eigenvalue has algebraic multiplicity of 2, and geometric multiplicity 1, therefore this is defective eigenvalue. The defect is 1. This falls into case 2 shown above. We need to generate the missing additonal generalized eigevector $\vec{v}_2$ by solving

$$(A-\lambda I)\,\vec{v}_2=\vec{v}_1$$

Where $\vec{v}_1$ is the normal (rank 1) eigenvector found above. Hence we need to solve

$$\left(\begin{bmatrix}1 & -4\\ 4 & -7\end{bmatrix}-(-3)\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}\right)\begin{bmatrix}v_1\\ v_2\end{bmatrix}=\begin{bmatrix}1\\ 1\end{bmatrix}$$

$$\begin{bmatrix}4 & -4\\ 4 & -4\end{bmatrix}\begin{bmatrix}v_1\\ v_2\end{bmatrix}=\begin{bmatrix}1\\ 1\end{bmatrix}$$

Solving for $\vec{v}_2$ gives

$$\vec{v}_2=\begin{bmatrix}\frac{5}{4}\\ 1\end{bmatrix}$$

We have found two generalized eigenvectors for eigenvalue -3. Therefore the two basis solution associated with this eigenvalue are

$$\begin{aligned}\vec{x}_1(t) &= \vec{v}_1 e^{\lambda t} \\ &= \begin{bmatrix} 1 \\ 1 \end{bmatrix} \mathrm{e}^{-3t} \\ &= \begin{bmatrix} \mathrm{e}^{-3t} \\ \mathrm{e}^{-3t} \end{bmatrix}\end{aligned}$$

And

$$\begin{aligned}\vec{x}_2(t) &= (\vec{v}_1 t + \vec{v}_2)\, e^{\lambda t} \\ &= \left(\begin{bmatrix} 1 \\ 1 \end{bmatrix} t + \begin{bmatrix} \frac{5}{4} \\ 1 \end{bmatrix} \right) \mathrm{e}^{-3t} \\ &= \begin{bmatrix} \frac{\mathrm{e}^{-3t}(4t+5)}{4} \\ \mathrm{e}^{-3t}(t+1) \end{bmatrix}\end{aligned}$$

Therefore the final solution is

$$\vec{x}_h(t) = c_1 \vec{x}_1(t) + c_2 \vec{x}_2(t)$$

Which is written as

$$\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = c_1 \begin{bmatrix} \mathrm{e}^{-3t} \\ \mathrm{e}^{-3t} \end{bmatrix} + c_2 \begin{bmatrix} \mathrm{e}^{-3t}\left(t + \frac{5}{4}\right) \\ \mathrm{e}^{-3t}(t+1) \end{bmatrix}$$

Which becomes

$$\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = \begin{bmatrix} \mathrm{e}^{-3t}\left(c_1 + c_2 t + \frac{5}{4} c_2\right) \\ \mathrm{e}^{-3t}(c_2 t + c_1 + c_2) \end{bmatrix}$$

The following is the phase plot of the system.

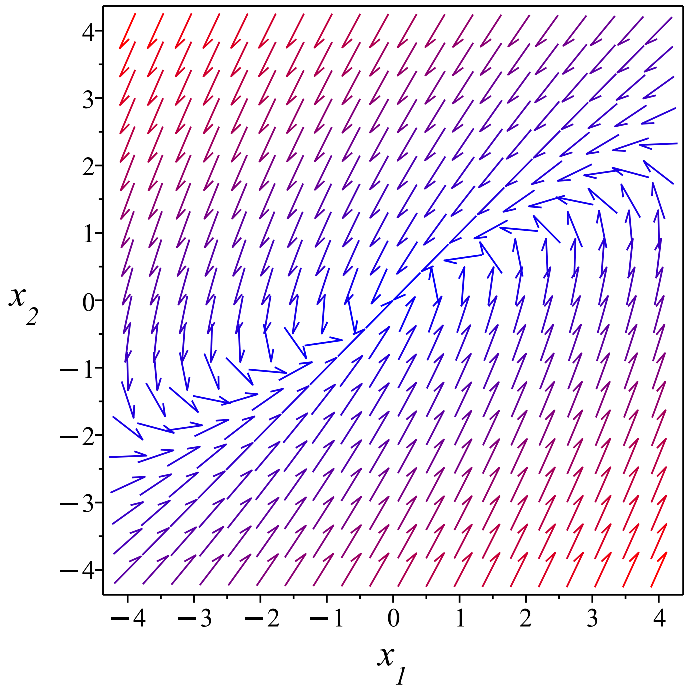


Figure 543: Phase plot

19.4.3 Maple step by step solution

Let's solve

$[x_1'(t) = x_1(t) - 4x_2(t), x_2'(t) = 4x_1(t) - 7x_2(t)]$

- Define vector

$$\underline{x^{\rightarrow}}(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}$$

- Convert system into a vector equation

$$\underline{x^{\rightarrow}}'(t) = \begin{bmatrix} 1 & -4 \\ 4 & -7 \end{bmatrix} \cdot \underline{x^{\rightarrow}}(t) + \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

- System to solve

$$\underline{x^{\rightarrow}}'(t) = \begin{bmatrix} 1 & -4 \\ 4 & -7 \end{bmatrix} \cdot \underline{x^{\rightarrow}}(t)$$

- Define the coefficient matrix

$$A = \begin{bmatrix} 1 & -4 \\ 4 & -7 \end{bmatrix}$$

- Rewrite the system as

  $\underline{\vec{x}}'(t) = A \cdot \underline{\vec{x}}(t)$

- To solve the system, find the eigenvalues and eigenvectors of A
- Eigenpairs of A

  $\left[\left[-3, \begin{bmatrix} 1 \\ 1 \end{bmatrix}\right], \left[-3, \begin{bmatrix} 0 \\ 0 \end{bmatrix}\right]\right]$

- Consider eigenpair, with eigenvalue of algebraic multiplicity 2

  $\left[-3, \begin{bmatrix} 1 \\ 1 \end{bmatrix}\right]$

- First solution from eigenvalue -3

  $\underline{\vec{x}}_1(t) = e^{-3t} \cdot \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

- Form of the 2nd homogeneous solution where $\vec{p}$ is to be solved for, $\lambda = -3$ is the eigenvalue, an

  $\underline{\vec{x}}_2(t) = e^{\lambda t}\left(t\vec{v} + \vec{p}\right)$

- Note that the t multiplying $\vec{v}$ makes this solution linearly independent to the 1st solution obtai
- Substitute $\underline{\vec{x}}_2(t)$ into the homogeneous system

  $\lambda e^{\lambda t}\left(t\vec{v} + \vec{p}\right) + e^{\lambda t}\vec{v} = \left(e^{\lambda t}A\right) \cdot \left(t\vec{v} + \vec{p}\right)$

- Use the fact that $\vec{v}$ is an eigenvector of A

  $\lambda e^{\lambda t}\left(t\vec{v} + \vec{p}\right) + e^{\lambda t}\vec{v} = e^{\lambda t}\left(\lambda t\vec{v} + A \cdot \vec{p}\right)$

- Simplify equation

  $\lambda\vec{p} + \vec{v} = A \cdot \vec{p}$

- Make use of the identity matrix I

  $(\lambda \cdot I) \cdot \vec{p} + \vec{v} = A \cdot \vec{p}$

- Condition $\vec{p}$ must meet for $\underline{\vec{x}}_2(t)$ to be a solution to the homogeneous system

  $(A - \lambda \cdot I) \cdot \vec{p} = \vec{v}$

- Choose $\vec{p}$ to use in the second solution to the homogeneous system from eigenvalue -3

$$\left(\begin{bmatrix} 1 & -4 \\ 4 & -7 \end{bmatrix} - (-3)\cdot \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right)\cdot \vec{p} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

- Choice of $\vec{p}$

$$\vec{p} = \begin{bmatrix} \frac{1}{4} \\ 0 \end{bmatrix}$$

- Second solution from eigenvalue -3

$$\vec{x\_\_}_2(t) = \mathrm{e}^{-3t}\cdot \left(t\cdot \begin{bmatrix} 1 \\ 1 \end{bmatrix} + \begin{bmatrix} \frac{1}{4} \\ 0 \end{bmatrix}\right)$$

- General solution to the system of ODEs

$$\vec{x\_\_} = c_1 \vec{x\_\_}_1(t) + c_2 \vec{x\_\_}_2(t)$$

- Substitute solutions into the general solution

$$\vec{x\_\_} = c_1 \mathrm{e}^{-3t}\cdot \begin{bmatrix} 1 \\ 1 \end{bmatrix} + \mathrm{e}^{-3t} c_2 \cdot \left(t\cdot \begin{bmatrix} 1 \\ 1 \end{bmatrix} + \begin{bmatrix} \frac{1}{4} \\ 0 \end{bmatrix}\right)$$

- Substitute in vector of dependent variables

$$\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = \begin{bmatrix} \mathrm{e}^{-3t}\left(c_1 + c_2 t + \frac{1}{4}c_2\right) \\ \mathrm{e}^{-3t}(c_2 t + c_1) \end{bmatrix}$$

- Solution to the system of ODEs

$$\left\{x_1(t) = \mathrm{e}^{-3t}\left(c_1 + c_2 t + \frac{1}{4}c_2\right), x_2(t) = \mathrm{e}^{-3t}(c_2 t + c_1)\right\}$$

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 35

```
dsolve([diff(x__1(t),t)=1*x__1(t)-4*x__2(t),diff(x__2(t),t)=4*x__1(t)-7*x__2(t)],singsol=all
```

$$x_1(t) = \mathrm{e}^{-3t}(c_2 t + c_1)$$
$$x_2(t) = \frac{\mathrm{e}^{-3t}(4c_2 t + 4c_1 - c_2)}{4}$$

✓ Solution by Mathematica

Time used: 0.002 (sec). Leaf size: 46

```
DSolve[{x1'[t]==1*x1[t]-4*x2[t],x2'[t]==4*x1[t]-7*x2[t]},{x1[t],x2[t]},t,IncludeSingularSolu
```

$$\begin{aligned} \mathrm{x1}(t) &\to e^{-3t}(4c_1t - 4c_2t + c_1) \\ \mathrm{x2}(t) &\to e^{-3t}(4(c_1 - c_2)t + c_2) \end{aligned}$$

19.5 problem 5

Internal problem ID [796]
Internal file name [`OUTPUT/796_Sunday_June_05_2022_01_49_59_AM_55876202/index.tex`]

Book: Elementary differential equations and boundary value problems, 10th ed., Boyce and DiPrima
Section: Chapter 9.1, The Phase Plane: Linear Systems. page 505
Problem number: 5.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"system of linear ODEs"**

Solve

$$x_1'(t) = x_1(t) - 5x_2(t)$$
$$x_2'(t) = x_1(t) - 3x_2(t)$$

19.5.1 Solution using Matrix exponential method

In this method, we will assume we have found the matrix exponential e^{At} allready. There are different methods to determine this but will not be shown here. This is a system of linear ODE's given as

$$\vec{x}'(t) = A\,\vec{x}(t)$$

Or

$$\begin{bmatrix} x_1'(t) \\ x_2'(t) \end{bmatrix} = \begin{bmatrix} 1 & -5 \\ 1 & -3 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}$$

For the above matrix A, the matrix exponential can be found to be

$$e^{At} = \begin{bmatrix} e^{-t}\cos(t) + 2\,e^{-t}\sin(t) & -5\,e^{-t}\sin(t) \\ e^{-t}\sin(t) & e^{-t}\cos(t) - 2\,e^{-t}\sin(t) \end{bmatrix}$$
$$= \begin{bmatrix} e^{-t}(\cos(t) + 2\sin(t)) & -5\,e^{-t}\sin(t) \\ e^{-t}\sin(t) & e^{-t}(\cos(t) - 2\sin(t)) \end{bmatrix}$$

Therefore the homogeneous solution is

$$
\begin{aligned}
\vec{x}_h(t) &= e^{At}\vec{c} \\
&= \begin{bmatrix} e^{-t}(\cos(t)+2\sin(t)) & -5\,e^{-t}\sin(t) \\ e^{-t}\sin(t) & e^{-t}(\cos(t)-2\sin(t)) \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} \\
&= \begin{bmatrix} e^{-t}(\cos(t)+2\sin(t))\,c_1 - 5\,e^{-t}\sin(t)\,c_2 \\ e^{-t}\sin(t)\,c_1 + e^{-t}(\cos(t)-2\sin(t))\,c_2 \end{bmatrix} \\
&= \begin{bmatrix} ((2c_1-5c_2)\sin(t)+c_1\cos(t))\,e^{-t} \\ ((c_1-2c_2)\sin(t)+c_2\cos(t))\,e^{-t} \end{bmatrix}
\end{aligned}
$$

Since no forcing function is given, then the final solution is $\vec{x}_h(t)$ above.

19.5.2 Solution using explicit Eigenvalue and Eigenvector method

This is a system of linear ODE's given as

$$\vec{x}'(t) = A\,\vec{x}(t)$$

Or

$$\begin{bmatrix} x_1'(t) \\ x_2'(t) \end{bmatrix} = \begin{bmatrix} 1 & -5 \\ 1 & -3 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}$$

The first step is find the homogeneous solution. We start by finding the eigenvalues of A. This is done by solving the following equation for the eigenvalues λ

$$\det(A-\lambda I) = 0$$

Expanding gives

$$\det\left(\begin{bmatrix} 1 & -5 \\ 1 & -3 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right) = 0$$

Therefore

$$\det\left(\begin{bmatrix} 1-\lambda & -5 \\ 1 & -3-\lambda \end{bmatrix}\right) = 0$$

Which gives the characteristic equation

$$\lambda^2 + 2\lambda + 2 = 0$$

The roots of the above are the eigenvalues.

$$\lambda_1 = -1 + i$$
$$\lambda_2 = -1 - i$$

This table summarises the above result

eigenvalue	algebraic multiplicity	type of eigenvalue
$-1 - i$	1	complex eigenvalue
$-1 + i$	1	complex eigenvalue

Now the eigenvector for each eigenvalue are found.

Considering the eigenvalue $\lambda_1 = -1 - i$

We need to solve $A\vec{v} = \lambda\vec{v}$ or $(A - \lambda I)\vec{v} = \vec{0}$ which becomes

$$\left(\begin{bmatrix} 1 & -5 \\ 1 & -3 \end{bmatrix} - (-1 - i)\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right)\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$
$$\begin{bmatrix} 2+i & -5 \\ 1 & -2+i \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Now forward elimination is applied to solve for the eigenvector $\vec{v}$. The augmented matrix is

$$\left[\begin{array}{cc|c} 2+i & -5 & 0 \\ 1 & -2+i & 0 \end{array}\right]$$

$$R_2 = R_2 + \left(-\frac{2}{5} + \frac{i}{5}\right)R_1 \Longrightarrow \left[\begin{array}{cc|c} 2+i & -5 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

Therefore the system in Echelon form is

$$\begin{bmatrix} 2+i & -5 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

The free variables are $\{v_2\}$ and the leading variables are $\{v_1\}$. Let $v_2 = t$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\{v_1 = (2 - i)\, t\}$

Hence the solution is

$$\begin{bmatrix} (2 - \mathrm{I})\, t \\ t \end{bmatrix} = \begin{bmatrix} (2 - i)\, t \\ t \end{bmatrix}$$

Since there is one free Variable, we have found one eigenvector associated with this eigenvalue. The above can be written as

$$\begin{bmatrix} (2 - \mathrm{I})\, t \\ t \end{bmatrix} = t \begin{bmatrix} 2 - i \\ 1 \end{bmatrix}$$

Let $t = 1$ the eigenvector becomes

$$\begin{bmatrix} (2 - \mathrm{I})\, t \\ t \end{bmatrix} = \begin{bmatrix} 2 - i \\ 1 \end{bmatrix}$$

Considering the eigenvalue $\lambda_2 = -1 + i$

We need to solve $A\vec{v} = \lambda\vec{v}$ or $(A - \lambda I)\vec{v} = \vec{0}$ which becomes

$$\left(\begin{bmatrix} 1 & -5 \\ 1 & -3 \end{bmatrix} - (-1 + i) \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right) \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 2 - i & -5 \\ 1 & -2 - i \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Now forward elimination is applied to solve for the eigenvector $\vec{v}$. The augmented matrix is

$$\left[\begin{array}{cc|c} 2 - i & -5 & 0 \\ 1 & -2 - i & 0 \end{array}\right]$$

$$R_2 = R_2 + \left(-\frac{2}{5} - \frac{i}{5}\right) R_1 \Longrightarrow \left[\begin{array}{cc|c} 2 - i & -5 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

Therefore the system in Echelon form is

$$\begin{bmatrix} 2 - i & -5 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

The free variables are $\{v_2\}$ and the leading variables are $\{v_1\}$. Let $v_2 = t$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\{v_1 = (2+i)\,t\}$

Hence the solution is

$$\begin{bmatrix} (2+\mathrm{I})\,t \\ t \end{bmatrix} = \begin{bmatrix} (2+i)\,t \\ t \end{bmatrix}$$

Since there is one free Variable, we have found one eigenvector associated with this eigenvalue. The above can be written as

$$\begin{bmatrix} (2+\mathrm{I})\,t \\ t \end{bmatrix} = t \begin{bmatrix} 2+i \\ 1 \end{bmatrix}$$

Let $t = 1$ the eigenvector becomes

$$\begin{bmatrix} (2+\mathrm{I})\,t \\ t \end{bmatrix} = \begin{bmatrix} 2+i \\ 1 \end{bmatrix}$$

The following table gives a summary of this result. It shows for each eigenvalue the algebraic multiplicity m, and its geometric multiplicity k and the eigenvectors associated with the eigenvalue. If $m > k$ then the eigenvalue is defective which means the number of normal linearly independent eigenvectors associated with this eigenvalue (called the geometric multiplicity k) does not equal the algebraic multiplicity m, and we need to determine an additional $m - k$ generalized eigenvectors for this eigenvalue.

	multiplicity			
eigenvalue	algebraic m	geometric k	defective?	eigenvectors
$-1+i$	1	1	No	$\begin{bmatrix} 2+i \\ 1 \end{bmatrix}$
$-1-i$	1	1	No	$\begin{bmatrix} 2-i \\ 1 \end{bmatrix}$

Now that we found the eigenvalues and associated eigenvectors, we will go over each eigenvalue and generate the solution basis. The only problem we need to take care of is if the eigenvalue is defective. Therefore the final solution is

$$\vec{x}_h(t) = c_1 \vec{x}_1(t) + c_2 \vec{x}_2(t)$$

Which is written as

$$\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = c_1 \begin{bmatrix} (2+i)\, \mathrm{e}^{(-1+i)t} \\ \mathrm{e}^{(-1+i)t} \end{bmatrix} + c_2 \begin{bmatrix} (2-i)\, \mathrm{e}^{(-1-i)t} \\ \mathrm{e}^{(-1-i)t} \end{bmatrix}$$

Which becomes

$$\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = \begin{bmatrix} (2+i)\, c_1 \mathrm{e}^{(-1+i)t} + (2-i)\, c_2 \mathrm{e}^{(-1-i)t} \\ c_1 \mathrm{e}^{(-1+i)t} + c_2 \mathrm{e}^{(-1-i)t} \end{bmatrix}$$

The following is the phase plot of the system.

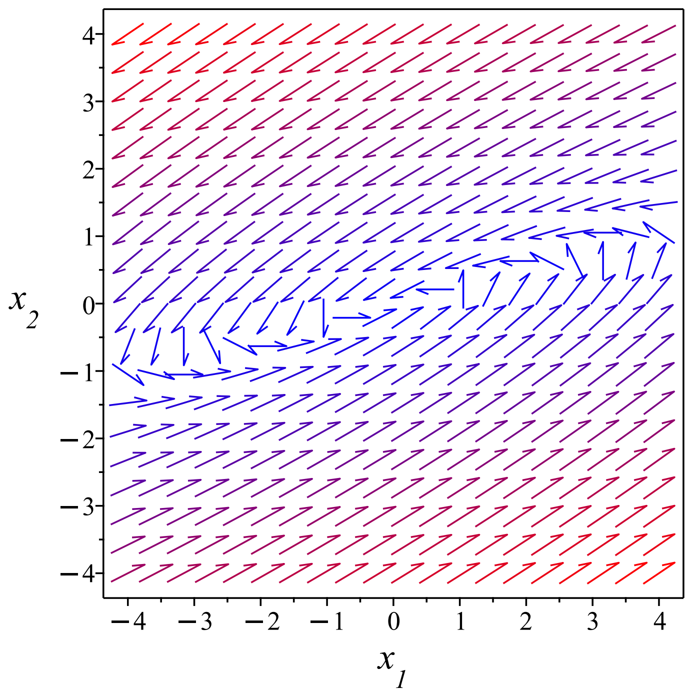


Figure 544: Phase plot

19.5.3 Maple step by step solution

Let's solve

$[x_1'(t) = x_1(t) - 5x_2(t)\,, x_2'(t) = x_1(t) - 3x_2(t)]$

- Define vector

$$\underline{\vec{x}}(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}$$

- Convert system into a vector equation

$$\underline{\vec{x}}'(t) = \begin{bmatrix} 1 & -5 \\ 1 & -3 \end{bmatrix} \cdot \underline{\vec{x}}(t) + \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

- System to solve

$$\underline{\vec{x}}'(t) = \begin{bmatrix} 1 & -5 \\ 1 & -3 \end{bmatrix} \cdot \underline{\vec{x}}(t)$$

- Define the coefficient matrix

$$A = \begin{bmatrix} 1 & -5 \\ 1 & -3 \end{bmatrix}$$

- Rewrite the system as

$$\underline{\vec{x}}'(t) = A \cdot \underline{\vec{x}}(t)$$

- To solve the system, find the eigenvalues and eigenvectors of A
- Eigenpairs of A

$$\left[\left[-1 - \mathrm{I}, \begin{bmatrix} 2 - \mathrm{I} \\ 1 \end{bmatrix}\right], \left[-1 + \mathrm{I}, \begin{bmatrix} 2 + \mathrm{I} \\ 1 \end{bmatrix}\right]\right]$$

- Consider complex eigenpair, complex conjugate eigenvalue can be ignored

$$\left[-1 - \mathrm{I}, \begin{bmatrix} 2 - \mathrm{I} \\ 1 \end{bmatrix}\right]$$

- Solution from eigenpair

$$\mathrm{e}^{(-1-\mathrm{I})t} \cdot \begin{bmatrix} 2 - \mathrm{I} \\ 1 \end{bmatrix}$$

- Use Euler identity to write solution in terms of sin and cos

$$\mathrm{e}^{-t} \cdot (\cos(t) - \mathrm{I}\sin(t)) \cdot \begin{bmatrix} 2 - \mathrm{I} \\ 1 \end{bmatrix}$$

- Simplify expression

$$e^{-t} \cdot \begin{bmatrix} (2-I)(\cos(t) - I\sin(t)) \\ \cos(t) - I\sin(t) \end{bmatrix}$$

- Both real and imaginary parts are solutions to the homogeneous system

$$\left[\underline{x}^{\rightarrow}{}_1(t) = e^{-t} \cdot \begin{bmatrix} 2\cos(t) - \sin(t) \\ \cos(t) \end{bmatrix}, \underline{x}^{\rightarrow}{}_2(t) = e^{-t} \cdot \begin{bmatrix} -\cos(t) - 2\sin(t) \\ -\sin(t) \end{bmatrix}\right]$$

- General solution to the system of ODEs

$$\underline{x}^{\rightarrow} = c_1 \underline{x}^{\rightarrow}{}_1(t) + c_2 \underline{x}^{\rightarrow}{}_2(t)$$

- Substitute solutions into the general solution

$$\underline{x}^{\rightarrow} = c_1 e^{-t} \cdot \begin{bmatrix} 2\cos(t) - \sin(t) \\ \cos(t) \end{bmatrix} + c_2 e^{-t} \cdot \begin{bmatrix} -\cos(t) - 2\sin(t) \\ -\sin(t) \end{bmatrix}$$

- Substitute in vector of dependent variables

$$\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = \begin{bmatrix} 2\left(\left(c_1 - \frac{c_2}{2}\right)\cos(t) - \frac{\sin(t)(c_1+2c_2)}{2}\right) e^{-t} \\ e^{-t}(-c_2\sin(t) + c_1\cos(t)) \end{bmatrix}$$

- Solution to the system of ODEs

$$\left\{x_1(t) = 2\left(\left(c_1 - \frac{c_2}{2}\right)\cos(t) - \frac{\sin(t)(c_1+2c_2)}{2}\right) e^{-t}, x_2(t) = e^{-t}(-c_2\sin(t) + c_1\cos(t))\right\}$$

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 48

```
dsolve([diff(x__1(t),t)=1*x__1(t)-5*x__2(t),diff(x__2(t),t)=1*x__1(t)-3*x__2(t)],singsol=all
```

$$x_1(t) = e^{-t}(c_1\sin(t) + c_2\cos(t))$$
$$x_2(t) = \frac{e^{-t}(-c_1\cos(t) + c_2\sin(t) + 2c_1\sin(t) + 2c_2\cos(t))}{5}$$

✓ Solution by Mathematica

Time used: 0.005 (sec). Leaf size: 54

```
DSolve[{x1'[t]==1*x1[t]-5*x2[t],x2'[t]==1*x1[t]-3*x2[t]},{x1[t],x2[t]},t,IncludeSingularSolu
```

$$\begin{aligned} \text{x1}(t) &\to e^{-t}(c_1\cos(t) + (2c_1 - 5c_2)\sin(t)) \\ \text{x2}(t) &\to e^{-t}(c_2\cos(t) + (c_1 - 2c_2)\sin(t)) \end{aligned}$$

19.6 problem 6

Internal problem ID [797]
Internal file name [`OUTPUT/797_Sunday_June_05_2022_01_50_01_AM_59153892/index.tex`]

Book: Elementary differential equations and boundary value problems, 10th ed., Boyce and DiPrima
Section: Chapter 9.1, The Phase Plane: Linear Systems. page 505
Problem number: 6.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"system of linear ODEs"**

Solve

$$
\begin{aligned}
x_1'(t) &= 2x_1(t) - 5x_2(t) \\
x_2'(t) &= x_1(t) - 2x_2(t)
\end{aligned}
$$

19.6.1 Solution using Matrix exponential method

In this method, we will assume we have found the matrix exponential e^{At} allready. There are different methods to determine this but will not be shown here. This is a system of linear ODE's given as

$$\vec{x}'(t) = A\,\vec{x}(t)$$

Or

$$\begin{bmatrix} x_1'(t) \\ x_2'(t) \end{bmatrix} = \begin{bmatrix} 2 & -5 \\ 1 & -2 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}$$

For the above matrix A, the matrix exponential can be found to be

$$e^{At} = \begin{bmatrix} \cos(t) + 2\sin(t) & -5\sin(t) \\ \sin(t) & \cos(t) - 2\sin(t) \end{bmatrix}$$

Therefore the homogeneous solution is

$$
\begin{aligned}
\vec{x}_h(t) &= e^{At}\vec{c} \\
&= \begin{bmatrix} \cos(t) + 2\sin(t) & -5\sin(t) \\ \sin(t) & \cos(t) - 2\sin(t) \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} \\
&= \begin{bmatrix} (\cos(t) + 2\sin(t))\, c_1 - 5\sin(t)\, c_2 \\ \sin(t)\, c_1 + (\cos(t) - 2\sin(t))\, c_2 \end{bmatrix} \\
&= \begin{bmatrix} (2c_1 - 5c_2)\sin(t) + c_1\cos(t) \\ (c_1 - 2c_2)\sin(t) + c_2\cos(t) \end{bmatrix}
\end{aligned}
$$

Since no forcing function is given, then the final solution is $\vec{x}_h(t)$ above.

19.6.2 Solution using explicit Eigenvalue and Eigenvector method

This is a system of linear ODE's given as

$$
\vec{x}'(t) = A\,\vec{x}(t)
$$

Or

$$
\begin{bmatrix} x_1'(t) \\ x_2'(t) \end{bmatrix} = \begin{bmatrix} 2 & -5 \\ 1 & -2 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}
$$

The first step is find the homogeneous solution. We start by finding the eigenvalues of A. This is done by solving the following equation for the eigenvalues λ

$$
\det(A - \lambda I) = 0
$$

Expanding gives

$$
\det\left(\begin{bmatrix} 2 & -5 \\ 1 & -2 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right) = 0
$$

Therefore

$$
\det\left(\begin{bmatrix} 2-\lambda & -5 \\ 1 & -2-\lambda \end{bmatrix}\right) = 0
$$

Which gives the characteristic equation

$$\lambda^2 + 1 = 0$$

The roots of the above are the eigenvalues.

$$\lambda_1 = i$$
$$\lambda_2 = -i$$

This table summarises the above result

eigenvalue	algebraic multiplicity	type of eigenvalue
$-i$	1	complex eigenvalue
i	1	complex eigenvalue

Now the eigenvector for each eigenvalue are found.

Considering the eigenvalue $\lambda_1 = -i$

We need to solve $A\vec{v} = \lambda\vec{v}$ or $(A - \lambda I)\vec{v} = \vec{0}$ which becomes

$$\left(\begin{bmatrix} 2 & -5 \\ 1 & -2 \end{bmatrix} - (-i)\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right)\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$
$$\begin{bmatrix} 2+i & -5 \\ 1 & -2+i \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Now forward elimination is applied to solve for the eigenvector $\vec{v}$. The augmented matrix is

$$\left[\begin{array}{cc|c} 2+i & -5 & 0 \\ 1 & -2+i & 0 \end{array}\right]$$

$$R_2 = R_2 + \left(-\frac{2}{5} + \frac{i}{5}\right)R_1 \Longrightarrow \left[\begin{array}{cc|c} 2+i & -5 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

Therefore the system in Echelon form is

$$\begin{bmatrix} 2+i & -5 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

The free variables are $\{v_2\}$ and the leading variables are $\{v_1\}$. Let $v_2 = t$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\{v_1 = (2-i)\,t\}$

Hence the solution is

$$\begin{bmatrix} (2-\mathrm{I})\,t \\ t \end{bmatrix} = \begin{bmatrix} (2-i)\,t \\ t \end{bmatrix}$$

Since there is one free Variable, we have found one eigenvector associated with this eigenvalue. The above can be written as

$$\begin{bmatrix} (2-\mathrm{I})\,t \\ t \end{bmatrix} = t \begin{bmatrix} 2-i \\ 1 \end{bmatrix}$$

Let $t = 1$ the eigenvector becomes

$$\begin{bmatrix} (2-\mathrm{I})\,t \\ t \end{bmatrix} = \begin{bmatrix} 2-i \\ 1 \end{bmatrix}$$

Considering the eigenvalue $\lambda_2 = i$

We need to solve $A\vec{v} = \lambda\vec{v}$ or $(A - \lambda I)\vec{v} = \vec{0}$ which becomes

$$\left(\begin{bmatrix} 2 & -5 \\ 1 & -2 \end{bmatrix} - (i) \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right) \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 2-i & -5 \\ 1 & -2-i \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Now forward elimination is applied to solve for the eigenvector $\vec{v}$. The augmented matrix is

$$\left[\begin{array}{cc|c} 2-i & -5 & 0 \\ 1 & -2-i & 0 \end{array}\right]$$

$$R_2 = R_2 + \left(-\frac{2}{5} - \frac{i}{5}\right) R_1 \Longrightarrow \left[\begin{array}{cc|c} 2-i & -5 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

Therefore the system in Echelon form is

$$\begin{bmatrix} 2-i & -5 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

The free variables are $\{v_2\}$ and the leading variables are $\{v_1\}$. Let $v_2 = t$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\{v_1 = (2+i)\,t\}$

Hence the solution is

$$\begin{bmatrix} (2+\mathrm{I})\,t \\ t \end{bmatrix} = \begin{bmatrix} (2+i)\,t \\ t \end{bmatrix}$$

Since there is one free Variable, we have found one eigenvector associated with this eigenvalue. The above can be written as

$$\begin{bmatrix} (2+\mathrm{I})\,t \\ t \end{bmatrix} = t \begin{bmatrix} 2+i \\ 1 \end{bmatrix}$$

Let $t = 1$ the eigenvector becomes

$$\begin{bmatrix} (2+\mathrm{I})\,t \\ t \end{bmatrix} = \begin{bmatrix} 2+i \\ 1 \end{bmatrix}$$

The following table gives a summary of this result. It shows for each eigenvalue the algebraic multiplicity m, and its geometric multiplicity k and the eigenvectors associated with the eigenvalue. If $m > k$ then the eigenvalue is defective which means the number of normal linearly independent eigenvectors associated with this eigenvalue (called the geometric multiplicity k) does not equal the algebraic multiplicity m, and we need to determine an additional $m - k$ generalized eigenvectors for this eigenvalue.

	multiplicity			
eigenvalue	algebraic m	geometric k	defective?	eigenvectors
i	1	1	No	$\begin{bmatrix} 2+i \\ 1 \end{bmatrix}$
$-i$	1	1	No	$\begin{bmatrix} 2-i \\ 1 \end{bmatrix}$

Now that we found the eigenvalues and associated eigenvectors, we will go over each eigenvalue and generate the solution basis. The only problem we need to take care of is if the eigenvalue is defective. Therefore the final solution is

$$\vec{x}_h(t) = c_1\vec{x}_1(t) + c_2\vec{x}_2(t)$$

Which is written as

$$\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = c_1 \begin{bmatrix} (2+i)\,\mathrm{e}^{it} \\ \mathrm{e}^{it} \end{bmatrix} + c_2 \begin{bmatrix} (2-i)\,\mathrm{e}^{-it} \\ \mathrm{e}^{-it} \end{bmatrix}$$

Which becomes

$$\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = \begin{bmatrix} (2+i)\,c_1\mathrm{e}^{it} + (2-i)\,c_2\mathrm{e}^{-it} \\ c_1\mathrm{e}^{it} + c_2\mathrm{e}^{-it} \end{bmatrix}$$

The following is the phase plot of the system.

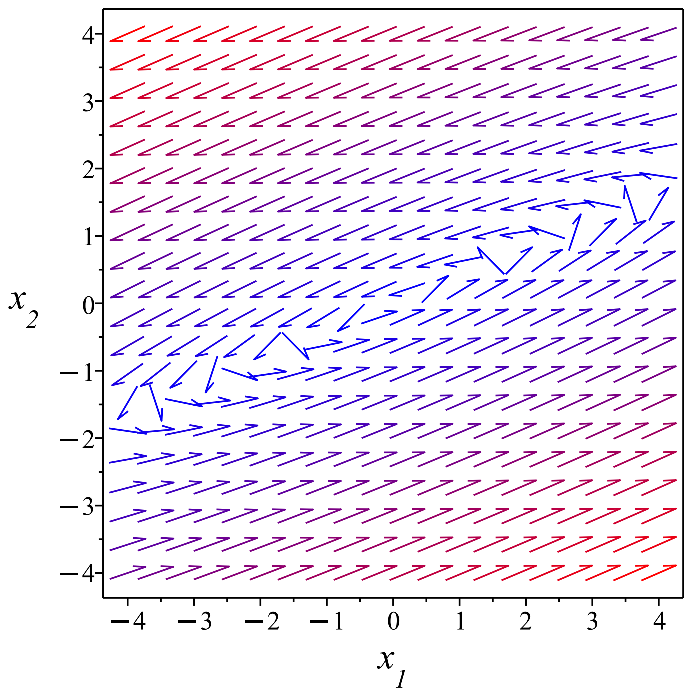


Figure 545: Phase plot

19.6.3 Maple step by step solution

Let's solve

$[x_1'(t) = 2x_1(t) - 5x_2(t)\,, x_2'(t) = x_1(t) - 2x_2(t)]$

- Define vector

$$\underline{\vec{x}}(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}$$

- Convert system into a vector equation

$$\underline{\vec{x}}'(t) = \begin{bmatrix} 2 & -5 \\ 1 & -2 \end{bmatrix} \cdot \underline{\vec{x}}(t) + \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

- System to solve

$$\underline{\vec{x}}'(t) = \begin{bmatrix} 2 & -5 \\ 1 & -2 \end{bmatrix} \cdot \underline{\vec{x}}(t)$$

- Define the coefficient matrix

$$A = \begin{bmatrix} 2 & -5 \\ 1 & -2 \end{bmatrix}$$

- Rewrite the system as

$$\underline{\vec{x}}'(t) = A \cdot \underline{\vec{x}}(t)$$

- To solve the system, find the eigenvalues and eigenvectors of A
- Eigenpairs of A

$$\left[\left[-\mathrm{I}, \begin{bmatrix} 2-\mathrm{I} \\ 1 \end{bmatrix}\right], \left[\mathrm{I}, \begin{bmatrix} 2+\mathrm{I} \\ 1 \end{bmatrix}\right]\right]$$

- Consider complex eigenpair, complex conjugate eigenvalue can be ignored

$$\left[-\mathrm{I}, \begin{bmatrix} 2-\mathrm{I} \\ 1 \end{bmatrix}\right]$$

- Solution from eigenpair

$$\mathrm{e}^{-\mathrm{I}t} \cdot \begin{bmatrix} 2-\mathrm{I} \\ 1 \end{bmatrix}$$

- Use Euler identity to write solution in terms of sin and cos

$$(\cos(t) - \mathrm{I}\sin(t)) \cdot \begin{bmatrix} 2-\mathrm{I} \\ 1 \end{bmatrix}$$

- Simplify expression

$$\left[\begin{array}{c}(2-\mathrm{I})(\cos(t)-\mathrm{I}\sin(t)) \\ \cos(t)-\mathrm{I}\sin(t)\end{array}\right]$$

- Both real and imaginary parts are solutions to the homogeneous system

$$\left[\underline{\vec{x}}_1(t)=\left[\begin{array}{c}2\cos(t)-\sin(t) \\ \cos(t)\end{array}\right], \underline{\vec{x}}_2(t)=\left[\begin{array}{c}-\cos(t)-2\sin(t) \\ -\sin(t)\end{array}\right]\right]$$

- General solution to the system of ODEs

$$\underline{\vec{x}}=c_1\underline{\vec{x}}_1(t)+c_2\underline{\vec{x}}_2(t)$$

- Substitute solutions into the general solution

$$\underline{\vec{x}}=\left[\begin{array}{c}c_2(-\cos(t)-2\sin(t))+c_1(2\cos(t)-\sin(t)) \\ -c_2\sin(t)+c_1\cos(t)\end{array}\right]$$

- Substitute in vector of dependent variables

$$\left[\begin{array}{c}x_1(t) \\ x_2(t)\end{array}\right]=\left[\begin{array}{c}\cos(t)(2c_1-c_2)-\sin(t)(c_1+2c_2) \\ -c_2\sin(t)+c_1\cos(t)\end{array}\right]$$

- Solution to the system of ODEs

$$\{x_1(t)=\cos(t)(2c_1-c_2)-\sin(t)(c_1+2c_2), x_2(t)=-c_2\sin(t)+c_1\cos(t)\}$$

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 38

```
dsolve([diff(x__1(t),t)=2*x__1(t)-5*x__2(t),diff(x__2(t),t)=1*x__1(t)-2*x__2(t)],singsol=all
```

$$x_1(t)=c_1\sin(t)+c_2\cos(t)$$
$$x_2(t)=-\frac{c_1\cos(t)}{5}+\frac{c_2\sin(t)}{5}+\frac{2c_1\sin(t)}{5}+\frac{2c_2\cos(t)}{5}$$

✓ Solution by Mathematica

Time used: 0.004 (sec). Leaf size: 41

```
DSolve[{x1'[t]==2*x1[t]-5*x2[t],x2'[t]==1*x1[t]-2*x2[t]},{x1[t],x2[t]},t,IncludeSingularSolu
```

$$\mathrm{x1}(t)\to c_1(2\sin(t)+\cos(t))-5c_2\sin(t)$$
$$\mathrm{x2}(t)\to c_2\cos(t)+(c_1-2c_2)\sin(t)$$

19.7 problem 7

Internal problem ID [798]
Internal file name [`OUTPUT/798_Sunday_June_05_2022_01_50_03_AM_80643564/index.tex`]

Book: Elementary differential equations and boundary value problems, 10th ed., Boyce and DiPrima
Section: Chapter 9.1, The Phase Plane: Linear Systems. page 505
Problem number: 7.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"system of linear ODEs"**

Solve

$$
\begin{aligned}
x_1'(t) &= 3x_1(t) - 2x_2(t) \\
x_2'(t) &= 4x_1(t) - x_2(t)
\end{aligned}
$$

19.7.1 Solution using Matrix exponential method

In this method, we will assume we have found the matrix exponential e^{At} allready. There are different methods to determine this but will not be shown here. This is a system of linear ODE's given as

$$\vec{x}'(t) = A\,\vec{x}(t)$$

Or

$$\begin{bmatrix} x_1'(t) \\ x_2'(t) \end{bmatrix} = \begin{bmatrix} 3 & -2 \\ 4 & -1 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}$$

For the above matrix A, the matrix exponential can be found to be

$$
\begin{aligned}
e^{At} &= \begin{bmatrix} e^t \cos(2t) + e^t \sin(2t) & -e^t \sin(2t) \\ 2\,e^t \sin(2t) & e^t \cos(2t) - e^t \sin(2t) \end{bmatrix} \\
&= \begin{bmatrix} e^t (\cos(2t) + \sin(2t)) & -e^t \sin(2t) \\ 2\,e^t \sin(2t) & e^t (\cos(2t) - \sin(2t)) \end{bmatrix}
\end{aligned}
$$

Therefore the homogeneous solution is

$$
\begin{aligned}
\vec{x}_h(t) &= e^{At}\vec{c} \\
&= \begin{bmatrix} e^t(\cos(2t)+\sin(2t)) & -e^t\sin(2t) \\ 2e^t\sin(2t) & e^t(\cos(2t)-\sin(2t)) \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} \\
&= \begin{bmatrix} e^t(\cos(2t)+\sin(2t))\, c_1 - e^t\sin(2t)\, c_2 \\ 2e^t\sin(2t)\, c_1 + e^t(\cos(2t)-\sin(2t))\, c_2 \end{bmatrix} \\
&= \begin{bmatrix} e^t((-c_2+c_1)\sin(2t)+c_1\cos(2t)) \\ e^t(2c_1-c_2)\sin(2t)+e^t\cos(2t)\, c_2 \end{bmatrix}
\end{aligned}
$$

Since no forcing function is given, then the final solution is $\vec{x}_h(t)$ above.

19.7.2 Solution using explicit Eigenvalue and Eigenvector method

This is a system of linear ODE's given as

$$\vec{x}'(t) = A\,\vec{x}(t)$$

Or

$$\begin{bmatrix} x_1'(t) \\ x_2'(t) \end{bmatrix} = \begin{bmatrix} 3 & -2 \\ 4 & -1 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}$$

The first step is find the homogeneous solution. We start by finding the eigenvalues of A. This is done by solving the following equation for the eigenvalues λ

$$\det(A - \lambda I) = 0$$

Expanding gives

$$\det\left(\begin{bmatrix} 3 & -2 \\ 4 & -1 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right) = 0$$

Therefore

$$\det\left(\begin{bmatrix} 3-\lambda & -2 \\ 4 & -1-\lambda \end{bmatrix}\right) = 0$$

Which gives the characteristic equation

$$\lambda^2 - 2\lambda + 5 = 0$$

The roots of the above are the eigenvalues.

$$\lambda_1 = 1 + 2i$$
$$\lambda_2 = 1 - 2i$$

This table summarises the above result

eigenvalue	algebraic multiplicity	type of eigenvalue
$1 + 2i$	1	complex eigenvalue
$1 - 2i$	1	complex eigenvalue

Now the eigenvector for each eigenvalue are found.

Considering the eigenvalue $\lambda_1 = 1 - 2i$

We need to solve $A\vec{v} = \lambda\vec{v}$ or $(A - \lambda I)\vec{v} = \vec{0}$ which becomes

$$\left(\begin{bmatrix} 3 & -2 \\ 4 & -1 \end{bmatrix} - (1 - 2i)\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right)\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$
$$\begin{bmatrix} 2 + 2i & -2 \\ 4 & -2 + 2i \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Now forward elimination is applied to solve for the eigenvector $\vec{v}$. The augmented matrix is

$$\left[\begin{array}{cc|c} 2 + 2i & -2 & 0 \\ 4 & -2 + 2i & 0 \end{array}\right]$$

$$R_2 = R_2 + (-1 + i)\, R_1 \Longrightarrow \left[\begin{array}{cc|c} 2 + 2i & -2 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

Therefore the system in Echelon form is

$$\begin{bmatrix} 2 + 2i & -2 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

The free variables are $\{v_2\}$ and the leading variables are $\{v_1\}$. Let $v_2 = t$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\left\{v_1 = \left(\frac{1}{2} - \frac{i}{2}\right) t\right\}$

Hence the solution is

$$\begin{bmatrix} \left(\frac{1}{2} - \frac{I}{2}\right) t \\ t \end{bmatrix} = \begin{bmatrix} \left(\frac{1}{2} - \frac{i}{2}\right) t \\ t \end{bmatrix}$$

Since there is one free Variable, we have found one eigenvector associated with this eigenvalue. The above can be written as

$$\begin{bmatrix} \left(\frac{1}{2} - \frac{I}{2}\right) t \\ t \end{bmatrix} = t \begin{bmatrix} \frac{1}{2} - \frac{i}{2} \\ 1 \end{bmatrix}$$

Let $t = 1$ the eigenvector becomes

$$\begin{bmatrix} \left(\frac{1}{2} - \frac{I}{2}\right) t \\ t \end{bmatrix} = \begin{bmatrix} \frac{1}{2} - \frac{i}{2} \\ 1 \end{bmatrix}$$

Which is normalized to

$$\begin{bmatrix} \left(\frac{1}{2} - \frac{I}{2}\right) t \\ t \end{bmatrix} = \begin{bmatrix} 1 - i \\ 2 \end{bmatrix}$$

Considering the eigenvalue $\lambda_2 = 1 + 2i$

We need to solve $A\vec{v} = \lambda\vec{v}$ or $(A - \lambda I)\vec{v} = \vec{0}$ which becomes

$$\left(\begin{bmatrix} 3 & -2 \\ 4 & -1 \end{bmatrix} - (1 + 2i) \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right) \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 2 - 2i & -2 \\ 4 & -2 - 2i \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Now forward elimination is applied to solve for the eigenvector $\vec{v}$. The augmented matrix is

$$\left[\begin{array}{cc|c} 2 - 2i & -2 & 0 \\ 4 & -2 - 2i & 0 \end{array}\right]$$

$$R_2 = R_2 + (-1 - i) R_1 \Longrightarrow \left[\begin{array}{cc|c} 2 - 2i & -2 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

Therefore the system in Echelon form is

$$\begin{bmatrix} 2-2i & -2 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

The free variables are $\{v_2\}$ and the leading variables are $\{v_1\}$. Let $v_2 = t$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\left\{v_1 = \left(\frac{1}{2} + \frac{i}{2}\right) t\right\}$

Hence the solution is

$$\begin{bmatrix} \left(\frac{1}{2} + \frac{I}{2}\right) t \\ t \end{bmatrix} = \begin{bmatrix} \left(\frac{1}{2} + \frac{i}{2}\right) t \\ t \end{bmatrix}$$

Since there is one free Variable, we have found one eigenvector associated with this eigenvalue. The above can be written as

$$\begin{bmatrix} \left(\frac{1}{2} + \frac{I}{2}\right) t \\ t \end{bmatrix} = t \begin{bmatrix} \frac{1}{2} + \frac{i}{2} \\ 1 \end{bmatrix}$$

Let $t = 1$ the eigenvector becomes

$$\begin{bmatrix} \left(\frac{1}{2} + \frac{I}{2}\right) t \\ t \end{bmatrix} = \begin{bmatrix} \frac{1}{2} + \frac{i}{2} \\ 1 \end{bmatrix}$$

Which is normalized to

$$\begin{bmatrix} \left(\frac{1}{2} + \frac{I}{2}\right) t \\ t \end{bmatrix} = \begin{bmatrix} 1+i \\ 2 \end{bmatrix}$$

The following table gives a summary of this result. It shows for each eigenvalue the algebraic multiplicity m, and its geometric multiplicity k and the eigenvectors associated with the eigenvalue. If $m > k$ then the eigenvalue is defective which means the number of normal linearly independent eigenvectors associated with this eigenvalue (called the geometric multiplicity k) does not equal the algebraic multiplicity m, and we need to determine an additional $m - k$ generalized eigenvectors for this eigenvalue.

eigenvalue	multiplicity algebraic m	multiplicity geometric k	defective?	eigenvectors
$1+2i$	1	1	No	$\left[\begin{array}{c} \frac{1}{2}+\frac{i}{2} \\ 1 \end{array}\right]$
$1-2i$	1	1	No	$\left[\begin{array}{c} \frac{1}{2}-\frac{i}{2} \\ 1 \end{array}\right]$

Now that we found the eigenvalues and associated eigenvectors, we will go over each eigenvalue and generate the solution basis. The only problem we need to take care of is if the eigenvalue is defective. Therefore the final solution is

$$\vec{x}_h(t) = c_1 \vec{x}_1(t) + c_2 \vec{x}_2(t)$$

Which is written as

$$\left[\begin{array}{c} x_1(t) \\ x_2(t) \end{array}\right] = c_1 \left[\begin{array}{c} \left(\frac{1}{2}+\frac{i}{2}\right) \mathrm{e}^{(1+2i)t} \\ \mathrm{e}^{(1+2i)t} \end{array}\right] + c_2 \left[\begin{array}{c} \left(\frac{1}{2}-\frac{i}{2}\right) \mathrm{e}^{(1-2i)t} \\ \mathrm{e}^{(1-2i)t} \end{array}\right]$$

Which becomes

$$\left[\begin{array}{c} x_1(t) \\ x_2(t) \end{array}\right] = \left[\begin{array}{c} \left(\frac{1}{2}+\frac{i}{2}\right) c_1 \mathrm{e}^{(1+2i)t} + \left(\frac{1}{2}-\frac{i}{2}\right) c_2 \mathrm{e}^{(1-2i)t} \\ c_1 \mathrm{e}^{(1+2i)t} + c_2 \mathrm{e}^{(1-2i)t} \end{array}\right]$$

The following is the phase plot of the system.

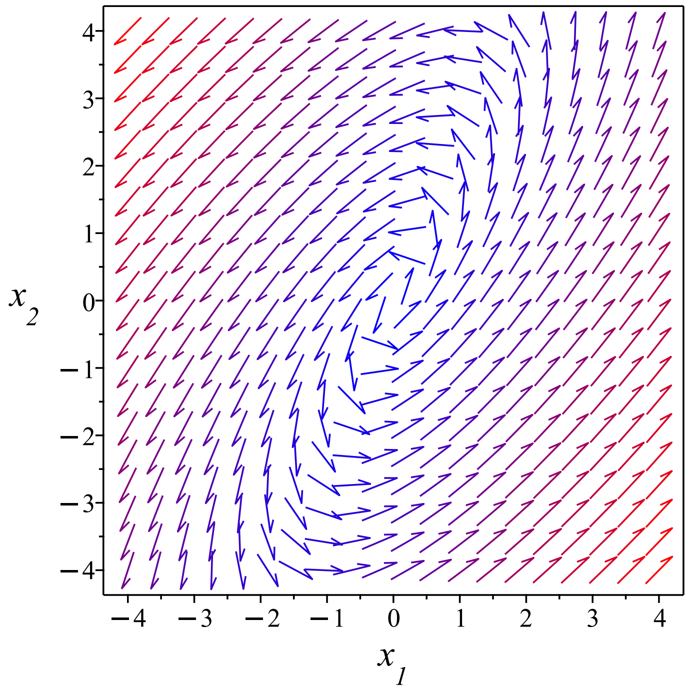


Figure 546: Phase plot

19.7.3 Maple step by step solution

Let's solve

$[x_1'(t) = 3x_1(t) - 2x_2(t), x_2'(t) = 4x_1(t) - x_2(t)]$

- Define vector

$$\underline{x}^{\rightarrow}(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}$$

- Convert system into a vector equation

$$\underline{x}^{\rightarrow}{}'(t) = \begin{bmatrix} 3 & -2 \\ 4 & -1 \end{bmatrix} \cdot \underline{x}^{\rightarrow}(t) + \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

- System to solve

$$\underline{x}^{\rightarrow}{}'(t) = \begin{bmatrix} 3 & -2 \\ 4 & -1 \end{bmatrix} \cdot \underline{x}^{\rightarrow}(t)$$

- Define the coefficient matrix

$$A = \begin{bmatrix} 3 & -2 \\ 4 & -1 \end{bmatrix}$$

- Rewrite the system as

$$\underline{x^{\rightarrow}}'(t) = A \cdot \underline{x^{\rightarrow}}(t)$$

- To solve the system, find the eigenvalues and eigenvectors of A
- Eigenpairs of A

$$\left[\left[1-2\,\mathrm{I}, \begin{bmatrix} \frac{1}{2}-\frac{\mathrm{I}}{2} \\ 1 \end{bmatrix}\right], \left[1+2\,\mathrm{I}, \begin{bmatrix} \frac{1}{2}+\frac{\mathrm{I}}{2} \\ 1 \end{bmatrix}\right]\right]$$

- Consider complex eigenpair, complex conjugate eigenvalue can be ignored

$$\left[1-2\,\mathrm{I}, \begin{bmatrix} \frac{1}{2}-\frac{\mathrm{I}}{2} \\ 1 \end{bmatrix}\right]$$

- Solution from eigenpair

$$\mathrm{e}^{(1-2\,\mathrm{I})t} \cdot \begin{bmatrix} \frac{1}{2}-\frac{\mathrm{I}}{2} \\ 1 \end{bmatrix}$$

- Use Euler identity to write solution in terms of $\sin$ and $\cos$

$$\mathrm{e}^{t} \cdot (\cos(2t) - \mathrm{I}\sin(2t)) \cdot \begin{bmatrix} \frac{1}{2}-\frac{\mathrm{I}}{2} \\ 1 \end{bmatrix}$$

- Simplify expression

$$\mathrm{e}^{t} \cdot \begin{bmatrix} \left(\frac{1}{2}-\frac{\mathrm{I}}{2}\right)(\cos(2t) - \mathrm{I}\sin(2t)) \\ \cos(2t) - \mathrm{I}\sin(2t) \end{bmatrix}$$

- Both real and imaginary parts are solutions to the homogeneous system

$$\left[\underline{x^{\rightarrow}}_1(t) = \mathrm{e}^{t} \cdot \begin{bmatrix} \frac{\cos(2t)}{2} - \frac{\sin(2t)}{2} \\ \cos(2t) \end{bmatrix}, \underline{x^{\rightarrow}}_2(t) = \mathrm{e}^{t} \cdot \begin{bmatrix} -\frac{\sin(2t)}{2} - \frac{\cos(2t)}{2} \\ -\sin(2t) \end{bmatrix}\right]$$

- General solution to the system of ODEs

$$\underline{x^{\rightarrow}} = c_1 \underline{x^{\rightarrow}}_1(t) + c_2 \underline{x^{\rightarrow}}_2(t)$$

- Substitute solutions into the general solution

$$\underline{x^{\rightarrow}} = c_1 e^t \cdot \begin{bmatrix} \frac{\cos(2t)}{2} - \frac{\sin(2t)}{2} \\ \cos(2t) \end{bmatrix} + c_2 e^t \cdot \begin{bmatrix} -\frac{\sin(2t)}{2} - \frac{\cos(2t)}{2} \\ -\sin(2t) \end{bmatrix}$$

- Substitute in vector of dependent variables

$$\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = \begin{bmatrix} \frac{e^t((c_1-c_2)\cos(2t)-\sin(2t)(c_1+c_2))}{2} \\ e^t(-c_2\sin(2t) + c_1\cos(2t)) \end{bmatrix}$$

- Solution to the system of ODEs

$$\left\{x_1(t) = \frac{e^t((c_1-c_2)\cos(2t)-\sin(2t)(c_1+c_2))}{2}, x_2(t) = e^t(-c_2\sin(2t) + c_1\cos(2t))\right\}$$

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 56

```
dsolve([diff(x__1(t),t)=3*x__1(t)-2*x__2(t),diff(x__2(t),t)=4*x__1(t)-1*x__2(t)],singsol=all
```

$$x_1(t) = e^t(c_1\sin(2t) + c_2\cos(2t))$$
$$x_2(t) = -e^t(c_1\cos(2t) - c_2\cos(2t) - c_1\sin(2t) - c_2\sin(2t))$$

✓ Solution by Mathematica

Time used: 0.005 (sec). Leaf size: 58

```
DSolve[{x1'[t]==3*x1[t]-2*x2[t],x2'[t]==4*x1[t]-1*x2[t]},{x1[t],x2[t]},t,IncludeSingularSolu
```

$$\text{x1}(t) \rightarrow e^t(c_1\cos(2t) + (c_1 - c_2)\sin(2t))$$
$$\text{x2}(t) \rightarrow e^t(c_2\cos(2t) + (2c_1 - c_2)\sin(2t))$$

19.8 problem 8

Internal problem ID [799]
Internal file name [`OUTPUT/799_Sunday_June_05_2022_01_50_04_AM_7663506/index.tex`]

Book: Elementary differential equations and boundary value problems, 10th ed., Boyce and DiPrima
Section: Chapter 9.1, The Phase Plane: Linear Systems. page 505
Problem number: 8.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"system of linear ODEs"**

Solve

$$x_1'(t) = -x_1(t) - x_2(t)$$
$$x_2'(t) = -\frac{5x_2(t)}{2}$$

19.8.1 Solution using Matrix exponential method

In this method, we will assume we have found the matrix exponential e^{At} allready. There are different methods to determine this but will not be shown here. This is a system of linear ODE's given as

$$\vec{x}'(t) = A\,\vec{x}(t)$$

Or

$$\begin{bmatrix} x_1'(t) \\ x_2'(t) \end{bmatrix} = \begin{bmatrix} -1 & -1 \\ 0 & -\frac{5}{2} \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}$$

For the above matrix A, the matrix exponential can be found to be

$$e^{At} = \begin{bmatrix} e^{-t} & -\frac{2\,e^{-t}}{3} + \frac{2\,e^{-\frac{5t}{2}}}{3} \\ 0 & e^{-\frac{5t}{2}} \end{bmatrix}$$

Therefore the homogeneous solution is

$$
\begin{aligned}
\vec{x}_h(t) &= e^{At}\vec{c} \\
&= \begin{bmatrix} \mathrm{e}^{-t} & -\frac{2\,\mathrm{e}^{-t}}{3}+\frac{2\,\mathrm{e}^{-\frac{5t}{2}}}{3} \\ 0 & \mathrm{e}^{-\frac{5t}{2}} \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} \\
&= \begin{bmatrix} \mathrm{e}^{-t}c_1 + \left(-\frac{2\,\mathrm{e}^{-t}}{3}+\frac{2\,\mathrm{e}^{-\frac{5t}{2}}}{3}\right)c_2 \\ \mathrm{e}^{-\frac{5t}{2}}c_2 \end{bmatrix} \\
&= \begin{bmatrix} \frac{2\,\mathrm{e}^{-\frac{5t}{2}}c_2}{3} + \left(c_1 - \frac{2c_2}{3}\right)\mathrm{e}^{-t} \\ \mathrm{e}^{-\frac{5t}{2}}c_2 \end{bmatrix}
\end{aligned}
$$

Since no forcing function is given, then the final solution is $\vec{x}_h(t)$ above.

19.8.2 Solution using explicit Eigenvalue and Eigenvector method

This is a system of linear ODE's given as

$$
\vec{x}'(t) = A\,\vec{x}(t)
$$

Or

$$
\begin{bmatrix} x_1'(t) \\ x_2'(t) \end{bmatrix} = \begin{bmatrix} -1 & -1 \\ 0 & -\frac{5}{2} \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}
$$

The first step is find the homogeneous solution. We start by finding the eigenvalues of A. This is done by solving the following equation for the eigenvalues λ

$$
\det(A - \lambda I) = 0
$$

Expanding gives

$$
\det\left(\begin{bmatrix} -1 & -1 \\ 0 & -\frac{5}{2} \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right) = 0
$$

Therefore

$$
\det\left(\begin{bmatrix} -1-\lambda & -1 \\ 0 & -\frac{5}{2}-\lambda \end{bmatrix}\right) = 0
$$

Since the matrix A is triangular matrix, then the determinant is the product of the elements along the diagonal. Therefore the above becomes

$$(-1-\lambda)(-\frac{5}{2}-\lambda)=0$$

The roots of the above are the eigenvalues.

$$\lambda_1=-\frac{5}{2}$$
$$\lambda_2=-1$$

This table summarises the above result

eigenvalue	algebraic multiplicity	type of eigenvalue
-1	1	real eigenvalue
$-\frac{5}{2}$	1	real eigenvalue

Now the eigenvector for each eigenvalue are found.

Considering the eigenvalue $\lambda_1=-1$

We need to solve $A\vec{v}=\lambda\vec{v}$ or $(A-\lambda I)\vec{v}=\vec{0}$ which becomes

$$\left(\begin{bmatrix}-1 & -1\\ 0 & -\frac{5}{2}\end{bmatrix}-(-1)\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}\right)\begin{bmatrix}v_1\\ v_2\end{bmatrix}=\begin{bmatrix}0\\ 0\end{bmatrix}$$
$$\begin{bmatrix}0 & -1\\ 0 & -\frac{3}{2}\end{bmatrix}\begin{bmatrix}v_1\\ v_2\end{bmatrix}=\begin{bmatrix}0\\ 0\end{bmatrix}$$

Now forward elimination is applied to solve for the eigenvector $\vec{v}$. The augmented matrix is

$$\left[\begin{array}{cc|c}0 & -1 & 0\\ 0 & -\frac{3}{2} & 0\end{array}\right]$$

$$R_2=R_2-\frac{3R_1}{2}\Longrightarrow\left[\begin{array}{cc|c}0 & -1 & 0\\ 0 & 0 & 0\end{array}\right]$$

Therefore the system in Echelon form is

$$\begin{bmatrix}0 & -1\\ 0 & 0\end{bmatrix}\begin{bmatrix}v_1\\ v_2\end{bmatrix}=\begin{bmatrix}0\\ 0\end{bmatrix}$$

The free variables are $\{v_1\}$ and the leading variables are $\{v_2\}$. Let $v_1 = t$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\{v_2 = 0\}$

Hence the solution is

$$\begin{bmatrix} t \\ 0 \end{bmatrix} = \begin{bmatrix} t \\ 0 \end{bmatrix}$$

Since there is one free Variable, we have found one eigenvector associated with this eigenvalue. The above can be written as

$$\begin{bmatrix} t \\ 0 \end{bmatrix} = t \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

Let $t = 1$ the eigenvector becomes

$$\begin{bmatrix} t \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

Considering the eigenvalue $\lambda_2 = -\frac{5}{2}$

We need to solve $A\vec{v} = \lambda\vec{v}$ or $(A - \lambda I)\vec{v} = \vec{0}$ which becomes

$$\left(\begin{bmatrix} -1 & -1 \\ 0 & -\frac{5}{2} \end{bmatrix} - \left(-\frac{5}{2}\right) \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right) \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} \frac{3}{2} & -1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Now forward elimination is applied to solve for the eigenvector $\vec{v}$. The augmented matrix is

$$\left[\begin{array}{cc|c} \frac{3}{2} & -1 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

Therefore the system in Echelon form is

$$\begin{bmatrix} \frac{3}{2} & -1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

The free variables are $\{v_2\}$ and the leading variables are $\{v_1\}$. Let $v_2 = t$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\left\{v_1 = \frac{2t}{3}\right\}$

Hence the solution is

$$\begin{bmatrix} \frac{2t}{3} \\ t \end{bmatrix} = \begin{bmatrix} \frac{2t}{3} \\ t \end{bmatrix}$$

Since there is one free Variable, we have found one eigenvector associated with this eigenvalue. The above can be written as

$$\begin{bmatrix} \frac{2t}{3} \\ t \end{bmatrix} = t \begin{bmatrix} \frac{2}{3} \\ 1 \end{bmatrix}$$

Let $t = 1$ the eigenvector becomes

$$\begin{bmatrix} \frac{2t}{3} \\ t \end{bmatrix} = \begin{bmatrix} \frac{2}{3} \\ 1 \end{bmatrix}$$

Which is normalized to

$$\begin{bmatrix} \frac{2t}{3} \\ t \end{bmatrix} = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$$

The following table gives a summary of this result. It shows for each eigenvalue the algebraic multiplicity m, and its geometric multiplicity k and the eigenvectors associated with the eigenvalue. If $m > k$ then the eigenvalue is defective which means the number of normal linearly independent eigenvectors associated with this eigenvalue (called the geometric multiplicity k) does not equal the algebraic multiplicity m, and we need to determine an additional $m - k$ generalized eigenvectors for this eigenvalue.

	multiplicity			
eigenvalue	algebraic m	geometric k	defective?	eigenvectors
$-\frac{5}{2}$	1	1	No	$\begin{bmatrix} \frac{2}{3} \\ 1 \end{bmatrix}$
-1	1	1	No	$\begin{bmatrix} 1 \\ 0 \end{bmatrix}$

Now that we found the eigenvalues and associated eigenvectors, we will go over each eigenvalue and generate the solution basis. The only problem we need to take care of is if the eigenvalue is defective. Since eigenvalue $-\frac{5}{2}$ is real and distinct then the

corresponding eigenvector solution is

$$\vec{x}_1(t) = \vec{v}_1 e^{-\frac{5t}{2}}$$
$$= \begin{bmatrix} \frac{2}{3} \\ 1 \end{bmatrix} e^{-\frac{5t}{2}}$$

Since eigenvalue -1 is real and distinct then the corresponding eigenvector solution is

$$\vec{x}_2(t) = \vec{v}_2 e^{-t}$$
$$= \begin{bmatrix} 1 \\ 0 \end{bmatrix} e^{-t}$$

Therefore the final solution is

$$\vec{x}_h(t) = c_1 \vec{x}_1(t) + c_2 \vec{x}_2(t)$$

Which is written as

$$\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = c_1 \begin{bmatrix} \frac{2\,\mathrm{e}^{-\frac{5t}{2}}}{3} \\ \mathrm{e}^{-\frac{5t}{2}} \end{bmatrix} + c_2 \begin{bmatrix} \mathrm{e}^{-t} \\ 0 \end{bmatrix}$$

Which becomes

$$\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = \begin{bmatrix} \frac{2c_1\mathrm{e}^{-\frac{5t}{2}}}{3} + c_2\mathrm{e}^{-t} \\ c_1\mathrm{e}^{-\frac{5t}{2}} \end{bmatrix}$$

The following is the phase plot of the system.

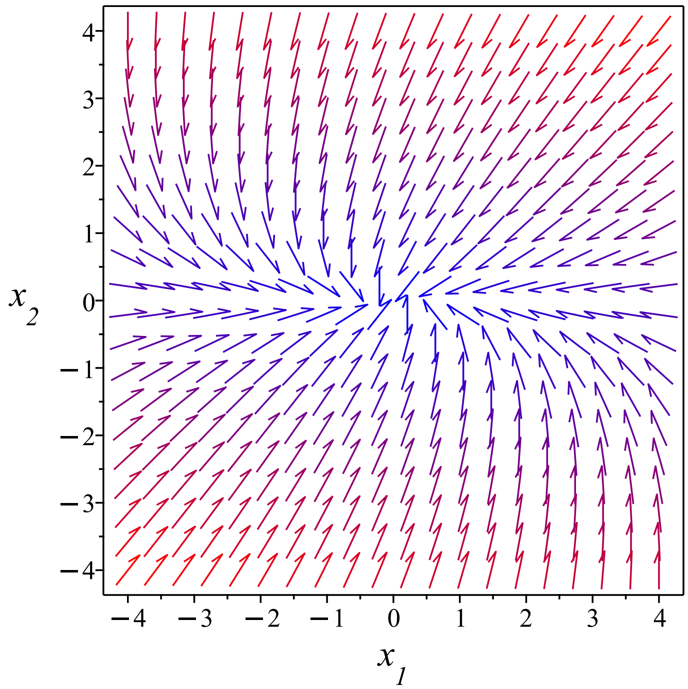


Figure 547: Phase plot

19.8.3 Maple step by step solution

Let's solve

$\left[x_1'(t) = -x_1(t) - x_2(t), x_2'(t) = -\frac{5x_2(t)}{2}\right]$

- Define vector

$$\underline{x}^{\rightarrow}(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}$$

- Convert system into a vector equation

$$\underline{x}^{\rightarrow}{}'(t) = \begin{bmatrix} -1 & -1 \\ 0 & -\frac{5}{2} \end{bmatrix} \cdot \underline{x}^{\rightarrow}(t) + \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

- System to solve

$$\underline{x}^{\rightarrow}{}'(t) = \begin{bmatrix} -1 & -1 \\ 0 & -\frac{5}{2} \end{bmatrix} \cdot \underline{x}^{\rightarrow}(t)$$

- Define the coefficient matrix

$$A = \begin{bmatrix} -1 & -1 \\ 0 & -\frac{5}{2} \end{bmatrix}$$

- Rewrite the system as

$$\underline{x}^{\rightarrow\prime}(t) = A \cdot \underline{x}^{\rightarrow}(t)$$

- To solve the system, find the eigenvalues and eigenvectors of A
- Eigenpairs of A

$$\left[\left[-\frac{5}{2}, \begin{bmatrix} \frac{2}{3} \\ 1 \end{bmatrix}\right], \left[-1, \begin{bmatrix} 1 \\ 0 \end{bmatrix}\right]\right]$$

- Consider eigenpair

$$\left[-\frac{5}{2}, \begin{bmatrix} \frac{2}{3} \\ 1 \end{bmatrix}\right]$$

- Solution to homogeneous system from eigenpair

$$\underline{x}^{\rightarrow}_{1} = e^{-\frac{5t}{2}} \cdot \begin{bmatrix} \frac{2}{3} \\ 1 \end{bmatrix}$$

- Consider eigenpair

$$\left[-1, \begin{bmatrix} 1 \\ 0 \end{bmatrix}\right]$$

- Solution to homogeneous system from eigenpair

$$\underline{x}^{\rightarrow}_{2} = e^{-t} \cdot \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

- General solution to the system of ODEs

$$\underline{x}^{\rightarrow} = c_1 \underline{x}^{\rightarrow}_{1} + c_2 \underline{x}^{\rightarrow}_{2}$$

- Substitute solutions into the general solution

$$\underline{x}^{\rightarrow} = c_1 e^{-\frac{5t}{2}} \cdot \begin{bmatrix} \frac{2}{3} \\ 1 \end{bmatrix} + c_2 e^{-t} \cdot \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

- Substitute in vector of dependent variables

$$\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = \begin{bmatrix} \frac{2c_1 e^{-\frac{5t}{2}}}{3} + c_2 e^{-t} \\ c_1 e^{-\frac{5t}{2}} \end{bmatrix}$$

- Solution to the system of ODEs

  $$\left\{x_1(t) = \frac{2c_1 e^{-\frac{5t}{2}}}{3} + c_2 e^{-t}, x_2(t) = c_1 e^{-\frac{5t}{2}}\right\}$$

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 28

```
dsolve([diff(x__1(t),t)=-1*x__1(t)-1*x__2(t),diff(x__2(t),t)=0*x__1(t)-25/10*x__2(t)],singso
```

$$x_1(t) = \frac{2c_2 e^{-\frac{5t}{2}}}{3} + e^{-t} c_1$$
$$x_2(t) = c_2 e^{-\frac{5t}{2}}$$

✓ Solution by Mathematica

Time used: 0.006 (sec). Leaf size: 47

```
DSolve[{x1'[t]==-1*x1[t]-1*x2[t],x2'[t]==0*x1[t]-25/10*x2[t]},{x1[t],x2[t]},t,IncludeSingula
```

$$\text{x1}(t) \to \left(c_1 - \frac{2c_2}{3}\right) e^{-t} + \frac{2}{3} c_2 e^{-5t/2}$$
$$\text{x2}(t) \to c_2 e^{-5t/2}$$

19.9 problem 9

Internal problem ID [800]
Internal file name [`OUTPUT/800_Sunday_June_05_2022_01_50_05_AM_83117480/index.tex`]

Book: Elementary differential equations and boundary value problems, 10th ed., Boyce and DiPrima
Section: Chapter 9.1, The Phase Plane: Linear Systems. page 505
Problem number: 9.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"system of linear ODEs"**

Solve

$$x_1'(t) = 3x_1(t) - 4x_2(t)$$
$$x_2'(t) = x_1(t) - x_2(t)$$

19.9.1 Solution using Matrix exponential method

In this method, we will assume we have found the matrix exponential e^{At} allready. There are different methods to determine this but will not be shown here. This is a system of linear ODE's given as

$$\vec{x}'(t) = A\,\vec{x}(t)$$

Or

$$\begin{bmatrix} x_1'(t) \\ x_2'(t) \end{bmatrix} = \begin{bmatrix} 3 & -4 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}$$

For the above matrix A, the matrix exponential can be found to be

$$e^{At} = \begin{bmatrix} e^t(1+2t) & -4t\,e^t \\ t\,e^t & e^t(1-2t) \end{bmatrix}$$

Therefore the homogeneous solution is

$$
\begin{aligned}
\vec{x}_h(t) &= e^{At}\vec{c} \\
&= \begin{bmatrix} e^t(1+2t) & -4t\,e^t \\ t\,e^t & e^t(1-2t) \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} \\
&= \begin{bmatrix} e^t(1+2t)\,c_1 - 4t\,e^t c_2 \\ t\,e^t c_1 + e^t(1-2t)\,c_2 \end{bmatrix} \\
&= \begin{bmatrix} e^t(2tc_1 - 4c_2t + c_1) \\ e^t(tc_1 - 2c_2t + c_2) \end{bmatrix}
\end{aligned}
$$

Since no forcing function is given, then the final solution is $\vec{x}_h(t)$ above.

19.9.2 Solution using explicit Eigenvalue and Eigenvector method

This is a system of linear ODE's given as

$$
\vec{x}'(t) = A\,\vec{x}(t)
$$

Or

$$
\begin{bmatrix} x_1'(t) \\ x_2'(t) \end{bmatrix} = \begin{bmatrix} 3 & -4 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}
$$

The first step is find the homogeneous solution. We start by finding the eigenvalues of A. This is done by solving the following equation for the eigenvalues λ

$$
\det(A - \lambda I) = 0
$$

Expanding gives

$$
\det\left(\begin{bmatrix} 3 & -4 \\ 1 & -1 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right) = 0
$$

Therefore

$$
\det\left(\begin{bmatrix} 3-\lambda & -4 \\ 1 & -1-\lambda \end{bmatrix}\right) = 0
$$

Which gives the characteristic equation

$$\lambda^2 - 2\lambda + 1 = 0$$

The roots of the above are the eigenvalues.

$$\lambda_1 = 1$$

This table summarises the above result

eigenvalue	algebraic multiplicity	type of eigenvalue
1	1	real eigenvalue

Now the eigenvector for each eigenvalue are found.

Considering the eigenvalue $\lambda_1 = 1$

We need to solve $A\vec{v} = \lambda\vec{v}$ or $(A - \lambda I)\vec{v} = \vec{0}$ which becomes

$$\left(\begin{bmatrix} 3 & -4 \\ 1 & -1 \end{bmatrix} - (1)\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right)\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$
$$\begin{bmatrix} 2 & -4 \\ 1 & -2 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Now forward elimination is applied to solve for the eigenvector $\vec{v}$. The augmented matrix is

$$\left[\begin{array}{cc|c} 2 & -4 & 0 \\ 1 & -2 & 0 \end{array}\right]$$

$$R_2 = R_2 - \frac{R_1}{2} \Longrightarrow \left[\begin{array}{cc|c} 2 & -4 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

Therefore the system in Echelon form is

$$\begin{bmatrix} 2 & -4 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

The free variables are $\{v_2\}$ and the leading variables are $\{v_1\}$. Let $v_2 = t$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\{v_1 = 2t\}$

Hence the solution is

$$\begin{bmatrix} 2t \\ t \end{bmatrix} = \begin{bmatrix} 2t \\ t \end{bmatrix}$$

Since there is one free Variable, we have found one eigenvector associated with this eigenvalue. The above can be written as

$$\begin{bmatrix} 2t \\ t \end{bmatrix} = t \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

Let $t = 1$ the eigenvector becomes

$$\begin{bmatrix} 2t \\ t \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

The following table gives a summary of this result. It shows for each eigenvalue the algebraic multiplicity m, and its geometric multiplicity k and the eigenvectors associated with the eigenvalue. If $m > k$ then the eigenvalue is defective which means the number of normal linearly independent eigenvectors associated with this eigenvalue (called the geometric multiplicity k) does not equal the algebraic multiplicity m, and we need to determine an additional $m - k$ generalized eigenvectors for this eigenvalue.

	multiplicity			
eigenvalue	algebraic m	geometric k	defective?	eigenvectors
1	2	1	Yes	$\begin{bmatrix} 2 \\ 1 \end{bmatrix}$

Now that we found the eigenvalues and associated eigenvectors, we will go over each eigenvalue and generate the solution basis. The only problem we need to take care of is if the eigenvalue is defective. eigenvalue 1 is real and repated eigenvalue of multiplicity 2.There are two possible cases that can happen. This is illustrated in this diagram

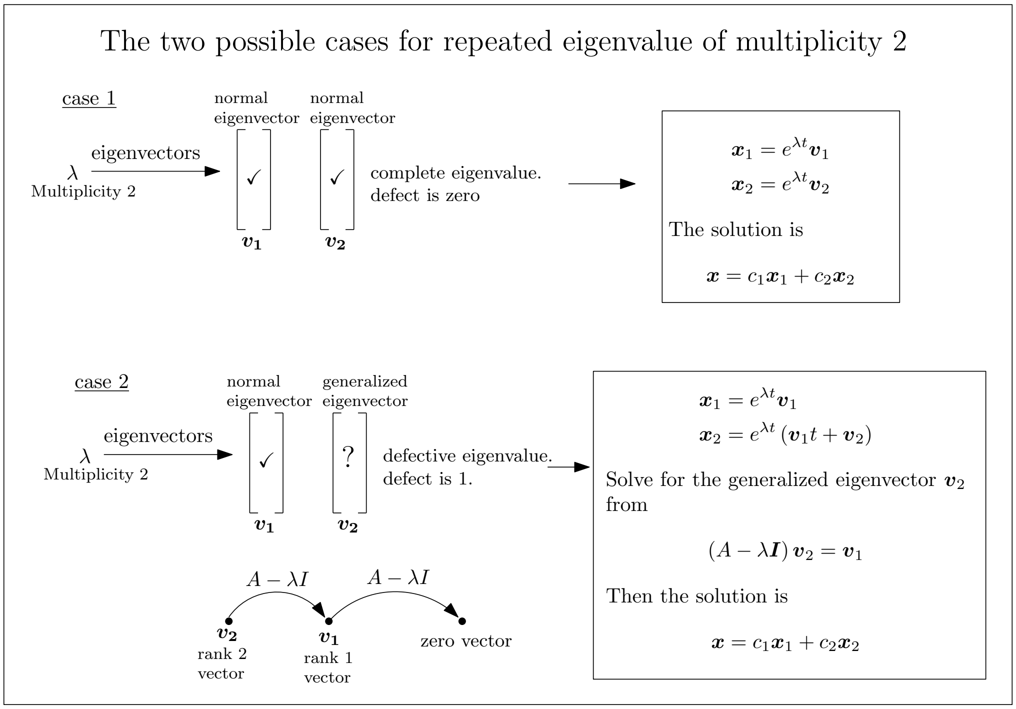


Figure 548: Possible case for repeated λ of multiplicity 2

This eigenvalue has algebraic multiplicity of 2, and geometric multiplicity 1, therefore this is defective eigenvalue. The defect is 1. This falls into case 2 shown above. We need to generate the missing additonal generalized eigevector $\vec{v}_2$ by solving

$$(A - \lambda I)\,\vec{v}_2 = \vec{v}_1$$

Where $\vec{v}_1$ is the normal (rank 1) eigenvector found above. Hence we need to solve

$$\left(\begin{bmatrix} 3 & -4 \\ 1 & -1 \end{bmatrix} - (1)\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right)\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

$$\begin{bmatrix} 2 & -4 \\ 1 & -2 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

Solving for $\vec{v}_2$ gives

$$\vec{v}_2 = \begin{bmatrix} 3 \\ 1 \end{bmatrix}$$

We have found two generalized eigenvectors for eigenvalue 1. Therefore the two basis solution associated with this eigenvalue are

$$\begin{aligned}\vec{x}_1(t) &= \vec{v}_1 e^{\lambda t} \\ &= \begin{bmatrix} 2 \\ 1 \end{bmatrix} \mathrm{e}^{t} \\ &= \begin{bmatrix} 2\,\mathrm{e}^{t} \\ \mathrm{e}^{t} \end{bmatrix}\end{aligned}$$

And

$$\begin{aligned}\vec{x}_2(t) &= (\vec{v}_1 t + \vec{v}_2)\, e^{\lambda t} \\ &= \left(\begin{bmatrix} 2 \\ 1 \end{bmatrix} t + \begin{bmatrix} 3 \\ 1 \end{bmatrix}\right) \mathrm{e}^{t} \\ &= \begin{bmatrix} \mathrm{e}^{t}(2t+3) \\ \mathrm{e}^{t}(t+1) \end{bmatrix}\end{aligned}$$

Therefore the final solution is

$$\vec{x}_h(t) = c_1 \vec{x}_1(t) + c_2 \vec{x}_2(t)$$

Which is written as

$$\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = c_1 \begin{bmatrix} 2\,\mathrm{e}^{t} \\ \mathrm{e}^{t} \end{bmatrix} + c_2 \begin{bmatrix} \mathrm{e}^{t}(2t+3) \\ \mathrm{e}^{t}(t+1) \end{bmatrix}$$

Which becomes

$$\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = \begin{bmatrix} ((2t+3)\,c_2 + 2c_1)\,\mathrm{e}^{t} \\ \mathrm{e}^{t}(c_2 t + c_1 + c_2) \end{bmatrix}$$

The following is the phase plot of the system.

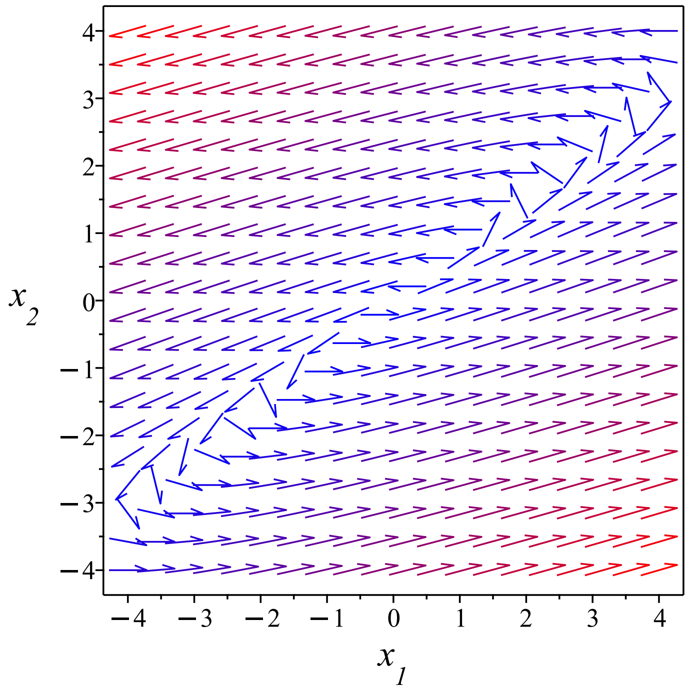


Figure 549: Phase plot

19.9.3 Maple step by step solution

Let's solve

$[x_1'(t) = 3x_1(t) - 4x_2(t), x_2'(t) = x_1(t) - x_2(t)]$

- Define vector

$$\underline{x}^{\rightarrow}(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}$$

- Convert system into a vector equation

$$\underline{x}^{\rightarrow\prime}(t) = \begin{bmatrix} 3 & -4 \\ 1 & -1 \end{bmatrix} \cdot \underline{x}^{\rightarrow}(t) + \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

- System to solve

$$\underline{x}^{\rightarrow\prime}(t) = \begin{bmatrix} 3 & -4 \\ 1 & -1 \end{bmatrix} \cdot \underline{x}^{\rightarrow}(t)$$

- Define the coefficient matrix

$A = \begin{bmatrix} 3 & -4 \\ 1 & -1 \end{bmatrix}$

- Rewrite the system as

  $\underline{x^{\rightarrow}}'(t) = A \cdot \underline{x^{\rightarrow}}(t)$

- To solve the system, find the eigenvalues and eigenvectors of A
- Eigenpairs of A

  $\left[\left[1, \begin{bmatrix} 2 \\ 1 \end{bmatrix}\right], \left[1, \begin{bmatrix} 0 \\ 0 \end{bmatrix}\right]\right]$

- Consider eigenpair, with eigenvalue of algebraic multiplicity 2

  $\left[1, \begin{bmatrix} 2 \\ 1 \end{bmatrix}\right]$

- First solution from eigenvalue 1

  $\underline{x^{\rightarrow}}_1(t) = \mathrm{e}^t \cdot \begin{bmatrix} 2 \\ 1 \end{bmatrix}$

- Form of the 2nd homogeneous solution where $\vec{p}$ is to be solved for, $\lambda = 1$ is the eigenvalue, and

  $\underline{x^{\rightarrow}}_2(t) = \mathrm{e}^{\lambda t}\left(t\vec{v} + \vec{p}\right)$

- Note that the t multiplying $\vec{v}$ makes this solution linearly independent to the 1st solution obtai
- Substitute $\underline{x^{\rightarrow}}_2(t)$ into the homogeneous system

  $\lambda\,\mathrm{e}^{\lambda t}\left(t\vec{v} + \vec{p}\right) + \mathrm{e}^{\lambda t}\vec{v} = \left(\mathrm{e}^{\lambda t} A\right) \cdot \left(t\vec{v} + \vec{p}\right)$

- Use the fact that $\vec{v}$ is an eigenvector of A

  $\lambda\,\mathrm{e}^{\lambda t}\left(t\vec{v} + \vec{p}\right) + \mathrm{e}^{\lambda t}\vec{v} = \mathrm{e}^{\lambda t}\left(\lambda t\vec{v} + A \cdot \vec{p}\right)$

- Simplify equation

  $\lambda\vec{p} + \vec{v} = A \cdot \vec{p}$

- Make use of the identity matrix I

  $(\lambda \cdot I) \cdot \vec{p} + \vec{v} = A \cdot \vec{p}$

- Condition $\vec{p}$ must meet for $\underline{x^{\rightarrow}}_2(t)$ to be a solution to the homogeneous system

  $(A - \lambda \cdot I) \cdot \vec{p} = \vec{v}$

- Choose $\vec{p}$ to use in the second solution to the homogeneous system from eigenvalue 1

$$\left(\begin{bmatrix} 3 & -4 \\ 1 & -1 \end{bmatrix} - 1 \cdot \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right) \cdot \vec{p} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

- Choice of $\vec{p}$

$$\vec{p} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

- Second solution from eigenvalue 1

$$\underline{\vec{x}}_{2}(t) = \mathrm{e}^{t} \cdot \left(t \cdot \begin{bmatrix} 2 \\ 1 \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \end{bmatrix}\right)$$

- General solution to the system of ODEs

$$\underline{\vec{x}} = c_1 \underline{\vec{x}}_{1}(t) + c_2 \underline{\vec{x}}_{2}(t)$$

- Substitute solutions into the general solution

$$\underline{\vec{x}} = c_1 \mathrm{e}^{t} \cdot \begin{bmatrix} 2 \\ 1 \end{bmatrix} + c_2 \mathrm{e}^{t} \cdot \left(t \cdot \begin{bmatrix} 2 \\ 1 \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \end{bmatrix}\right)$$

- Substitute in vector of dependent variables

$$\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = \begin{bmatrix} \mathrm{e}^{t}(2c_2 t + 2c_1 + c_2) \\ \mathrm{e}^{t}(c_2 t + c_1) \end{bmatrix}$$

- Solution to the system of ODEs

$$\{x_1(t) = \mathrm{e}^{t}(2c_2 t + 2c_1 + c_2), x_2(t) = \mathrm{e}^{t}(c_2 t + c_1)\}$$

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 31

```
dsolve([diff(x__1(t),t)=3*x__1(t)-4*x__2(t),diff(x__2(t),t)=1*x__1(t)-1*x__2(t)],singsol=all
```

$$x_1(t) = \mathrm{e}^{t}(c_2 t + c_1)$$
$$x_2(t) = \frac{\mathrm{e}^{t}(2c_2 t + 2c_1 - c_2)}{4}$$

✓ Solution by Mathematica

Time used: 0.002 (sec). Leaf size: 41

```
DSolve[{x1'[t]==3*x1[t]-4*x2[t],x2'[t]==1*x1[t]-1*x2[t]},{x1[t],x2[t]},t,IncludeSingularSolu
```

$$\text{x1}(t) \to e^t(2c_1t - 4c_2t + c_1)$$
$$\text{x2}(t) \to e^t((c_1 - 2c_2)t + c_2)$$

19.10 problem 10

Internal problem ID [801]
Internal file name [`OUTPUT/801_Sunday_June_05_2022_01_50_06_AM_94844149/index.tex`]

Book: Elementary differential equations and boundary value problems, 10th ed., Boyce and DiPrima
Section: Chapter 9.1, The Phase Plane: Linear Systems. page 505
Problem number: 10.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"system of linear ODEs"**

Solve

$$
\begin{aligned}
x_1'(t) &= x_1(t) + 2x_2(t) \\
x_2'(t) &= -5x_1(t)
\end{aligned}
$$

19.10.1 Solution using Matrix exponential method

In this method, we will assume we have found the matrix exponential e^{At} allready. There are different methods to determine this but will not be shown here. This is a system of linear ODE's given as

$$\vec{x}'(t) = A\,\vec{x}(t)$$

Or

$$\begin{bmatrix} x_1'(t) \\ x_2'(t) \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ -5 & 0 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}$$

For the above matrix A, the matrix exponential can be found to be

$$e^{At} = \begin{bmatrix} e^{\frac{t}{2}}\cos\left(\frac{\sqrt{39}\,t}{2}\right) + \frac{\sqrt{39}\,e^{\frac{t}{2}}\sin\left(\frac{\sqrt{39}\,t}{2}\right)}{39} & \frac{4\sqrt{39}\,e^{\frac{t}{2}}\sin\left(\frac{\sqrt{39}\,t}{2}\right)}{39} \\ -\frac{10\sqrt{39}\,e^{\frac{t}{2}}\sin\left(\frac{\sqrt{39}\,t}{2}\right)}{39} & e^{\frac{t}{2}}\cos\left(\frac{\sqrt{39}\,t}{2}\right) - \frac{\sqrt{39}\,e^{\frac{t}{2}}\sin\left(\frac{\sqrt{39}\,t}{2}\right)}{39} \end{bmatrix}$$

$$= \begin{bmatrix} \frac{e^{\frac{t}{2}}\left(\sqrt{39}\sin\left(\frac{\sqrt{39}\,t}{2}\right)+39\cos\left(\frac{\sqrt{39}\,t}{2}\right)\right)}{39} & \frac{4\sqrt{39}\,e^{\frac{t}{2}}\sin\left(\frac{\sqrt{39}\,t}{2}\right)}{39} \\ -\frac{10\sqrt{39}\,e^{\frac{t}{2}}\sin\left(\frac{\sqrt{39}\,t}{2}\right)}{39} & -\frac{e^{\frac{t}{2}}\left(\sqrt{39}\sin\left(\frac{\sqrt{39}\,t}{2}\right)-39\cos\left(\frac{\sqrt{39}\,t}{2}\right)\right)}{39} \end{bmatrix}$$

Therefore the homogeneous solution is

$$\vec{x}_h(t) = e^{At}\vec{c}$$

$$= \begin{bmatrix} \frac{e^{\frac{t}{2}}\left(\sqrt{39}\sin\left(\frac{\sqrt{39}\,t}{2}\right)+39\cos\left(\frac{\sqrt{39}\,t}{2}\right)\right)}{39} & \frac{4\sqrt{39}\,e^{\frac{t}{2}}\sin\left(\frac{\sqrt{39}\,t}{2}\right)}{39} \\ -\frac{10\sqrt{39}\,e^{\frac{t}{2}}\sin\left(\frac{\sqrt{39}\,t}{2}\right)}{39} & -\frac{e^{\frac{t}{2}}\left(\sqrt{39}\sin\left(\frac{\sqrt{39}\,t}{2}\right)-39\cos\left(\frac{\sqrt{39}\,t}{2}\right)\right)}{39} \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix}$$

$$= \begin{bmatrix} \frac{e^{\frac{t}{2}}\left(\sqrt{39}\sin\left(\frac{\sqrt{39}\,t}{2}\right)+39\cos\left(\frac{\sqrt{39}\,t}{2}\right)\right)c_1}{39} + \frac{4\sqrt{39}\,e^{\frac{t}{2}}\sin\left(\frac{\sqrt{39}\,t}{2}\right)c_2}{39} \\ -\frac{10\sqrt{39}\,e^{\frac{t}{2}}\sin\left(\frac{\sqrt{39}\,t}{2}\right)c_1}{39} - \frac{e^{\frac{t}{2}}\left(\sqrt{39}\sin\left(\frac{\sqrt{39}\,t}{2}\right)-39\cos\left(\frac{\sqrt{39}\,t}{2}\right)\right)c_2}{39} \end{bmatrix}$$

$$= \begin{bmatrix} \frac{\left(\sqrt{39}\,(c_1+4c_2)\sin\left(\frac{\sqrt{39}\,t}{2}\right)+39\cos\left(\frac{\sqrt{39}\,t}{2}\right)c_1\right)e^{\frac{t}{2}}}{39} \\ -\frac{10\,e^{\frac{t}{2}}\left(\sqrt{39}\,\left(c_1+\frac{c_2}{10}\right)\sin\left(\frac{\sqrt{39}\,t}{2}\right)-\frac{39\cos\left(\frac{\sqrt{39}\,t}{2}\right)c_2}{10}\right)}{39} \end{bmatrix}$$

Since no forcing function is given, then the final solution is $\vec{x}_h(t)$ above.

19.10.2 Solution using explicit Eigenvalue and Eigenvector method

This is a system of linear ODE's given as

$$\vec{x}'(t) = A\,\vec{x}(t)$$

Or

$$\begin{bmatrix} x_1'(t) \\ x_2'(t) \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ -5 & 0 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}$$

The first step is find the homogeneous solution. We start by finding the eigenvalues of A. This is done by solving the following equation for the eigenvalues λ

$$\det(A - \lambda I) = 0$$

Expanding gives

$$\det\left(\begin{bmatrix}1 & 2\\ -5 & 0\end{bmatrix} - \lambda\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}\right) = 0$$

Therefore

$$\det\left(\begin{bmatrix}1-\lambda & 2\\ -5 & -\lambda\end{bmatrix}\right) = 0$$

Which gives the characteristic equation

$$\lambda^2 - \lambda + 10 = 0$$

The roots of the above are the eigenvalues.

$$\lambda_1 = \frac{1}{2} + \frac{i\sqrt{39}}{2}$$
$$\lambda_2 = \frac{1}{2} - \frac{i\sqrt{39}}{2}$$

This table summarises the above result

eigenvalue	algebraic multiplicity	type of eigenvalue
$\frac{1}{2} - \frac{i\sqrt{39}}{2}$	1	complex eigenvalue
$\frac{1}{2} + \frac{i\sqrt{39}}{2}$	1	complex eigenvalue

Now the eigenvector for each eigenvalue are found.

Considering the eigenvalue $\lambda_1 = \frac{1}{2} - \frac{i\sqrt{39}}{2}$

We need to solve $A\vec{v} = \lambda\vec{v}$ or $(A - \lambda I)\vec{v} = \vec{0}$ which becomes

$$\left(\begin{bmatrix}1 & 2\\ -5 & 0\end{bmatrix} - \left(\frac{1}{2} - \frac{i\sqrt{39}}{2}\right)\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}\right)\begin{bmatrix}v_1\\ v_2\end{bmatrix} = \begin{bmatrix}0\\ 0\end{bmatrix}$$
$$\begin{bmatrix}\frac{1}{2} + \frac{i\sqrt{39}}{2} & 2\\ -5 & -\frac{1}{2} + \frac{i\sqrt{39}}{2}\end{bmatrix}\begin{bmatrix}v_1\\ v_2\end{bmatrix} = \begin{bmatrix}0\\ 0\end{bmatrix}$$

Now forward elimination is applied to solve for the eigenvector $\vec{v}$. The augmented matrix is

$$\left[\begin{array}{cc|c}\frac{1}{2} + \frac{i\sqrt{39}}{2} & 2 & 0\\ -5 & -\frac{1}{2} + \frac{i\sqrt{39}}{2} & 0\end{array}\right]$$

$$R_2 = R_2 + \frac{5R_1}{\frac{1}{2} + \frac{i\sqrt{39}}{2}} \Longrightarrow \left[\begin{array}{cc|c} \frac{1}{2} + \frac{i\sqrt{39}}{2} & 2 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

Therefore the system in Echelon form is

$$\begin{bmatrix} \frac{1}{2} + \frac{i\sqrt{39}}{2} & 2 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

The free variables are $\{v_2\}$ and the leading variables are $\{v_1\}$. Let $v_2 = t$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\left\{v_1 = -\frac{4t}{1+i\sqrt{39}}\right\}$

Hence the solution is

$$\begin{bmatrix} -\frac{4t}{1+\mathrm{I}\sqrt{39}} \\ t \end{bmatrix} = \begin{bmatrix} -\frac{4t}{1+i\sqrt{39}} \\ t \end{bmatrix}$$

Since there is one free Variable, we have found one eigenvector associated with this eigenvalue. The above can be written as

$$\begin{bmatrix} -\frac{4t}{1+\mathrm{I}\sqrt{39}} \\ t \end{bmatrix} = t \begin{bmatrix} -\frac{4}{1+i\sqrt{39}} \\ 1 \end{bmatrix}$$

Let $t = 1$ the eigenvector becomes

$$\begin{bmatrix} -\frac{4t}{1+\mathrm{I}\sqrt{39}} \\ t \end{bmatrix} = \begin{bmatrix} -\frac{4}{1+i\sqrt{39}} \\ 1 \end{bmatrix}$$

Which is normalized to

$$\begin{bmatrix} -\frac{4t}{1+\mathrm{I}\sqrt{39}} \\ t \end{bmatrix} = \begin{bmatrix} -\frac{4}{1+i\sqrt{39}} \\ 1 \end{bmatrix}$$

Considering the eigenvalue $\lambda_2 = \frac{1}{2} + \frac{i\sqrt{39}}{2}$

We need to solve $A\vec{v} = \lambda\vec{v}$ or $(A - \lambda I)\vec{v} = \vec{0}$ which becomes

$$\left(\begin{bmatrix} 1 & 2 \\ -5 & 0 \end{bmatrix} - \left(\frac{1}{2} + \frac{i\sqrt{39}}{2}\right)\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right)\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} \frac{1}{2} - \frac{i\sqrt{39}}{2} & 2 \\ -5 & -\frac{1}{2} - \frac{i\sqrt{39}}{2} \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Now forward elimination is applied to solve for the eigenvector $\vec{v}$. The augmented matrix is

$$\left[\begin{array}{cc|c} \frac{1}{2}-\frac{i\sqrt{39}}{2} & 2 & 0 \\ -5 & -\frac{1}{2}-\frac{i\sqrt{39}}{2} & 0 \end{array}\right]$$

$$R_2 = R_2 + \frac{5R_1}{\frac{1}{2}-\frac{i\sqrt{39}}{2}} \Longrightarrow \left[\begin{array}{cc|c} \frac{1}{2}-\frac{i\sqrt{39}}{2} & 2 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

Therefore the system in Echelon form is

$$\begin{bmatrix} \frac{1}{2}-\frac{i\sqrt{39}}{2} & 2 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

The free variables are $\{v_2\}$ and the leading variables are $\{v_1\}$. Let $v_2 = t$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\left\{v_1 = \frac{4t}{i\sqrt{39}-1}\right\}$

Hence the solution is

$$\begin{bmatrix} \frac{4t}{\mathrm{I}\sqrt{39}-1} \\ t \end{bmatrix} = \begin{bmatrix} \frac{4t}{i\sqrt{39}-1} \\ t \end{bmatrix}$$

Since there is one free Variable, we have found one eigenvector associated with this eigenvalue. The above can be written as

$$\begin{bmatrix} \frac{4t}{\mathrm{I}\sqrt{39}-1} \\ t \end{bmatrix} = t \begin{bmatrix} \frac{4}{i\sqrt{39}-1} \\ 1 \end{bmatrix}$$

Let $t = 1$ the eigenvector becomes

$$\begin{bmatrix} \frac{4t}{\mathrm{I}\sqrt{39}-1} \\ t \end{bmatrix} = \begin{bmatrix} \frac{4}{i\sqrt{39}-1} \\ 1 \end{bmatrix}$$

Which is normalized to

$$\begin{bmatrix} \frac{4t}{\mathrm{I}\sqrt{39}-1} \\ t \end{bmatrix} = \begin{bmatrix} \frac{4}{i\sqrt{39}-1} \\ 1 \end{bmatrix}$$

The following table gives a summary of this result. It shows for each eigenvalue the algebraic multiplicity m, and its geometric multiplicity k and the eigenvectors associated with the eigenvalue. If $m > k$ then the eigenvalue is defective which means the number of normal linearly independent eigenvectors associated with this eigenvalue (called the geometric multiplicity k) does not equal the algebraic multiplicity m, and we need to determine an additional $m-k$ generalized eigenvectors for this eigenvalue.

eigenvalue	multiplicity		defective?	eigenvectors
	algebraic m	geometric k		
$\frac{1}{2}+\frac{i\sqrt{39}}{2}$	1	1	No	$\left[\begin{array}{c}\frac{2}{-\frac{1}{2}+\frac{i\sqrt{39}}{2}} \\ 1\end{array}\right]$
$\frac{1}{2}-\frac{i\sqrt{39}}{2}$	1	1	No	$\left[\begin{array}{c}\frac{2}{-\frac{1}{2}-\frac{i\sqrt{39}}{2}} \\ 1\end{array}\right]$

Now that we found the eigenvalues and associated eigenvectors, we will go over each eigenvalue and generate the solution basis. The only problem we need to take care of is if the eigenvalue is defective. Therefore the final solution is

$$\vec{x}_h(t) = c_1\vec{x}_1(t) + c_2\vec{x}_2(t)$$

Which is written as

$$\left[\begin{array}{c}x_1(t) \\ x_2(t)\end{array}\right] = c_1\left[\begin{array}{c}\frac{2\,\mathrm{e}^{\left(\frac{1}{2}+\frac{i\sqrt{39}}{2}\right)t}}{-\frac{1}{2}+\frac{i\sqrt{39}}{2}} \\ \mathrm{e}^{\left(\frac{1}{2}+\frac{i\sqrt{39}}{2}\right)t}\end{array}\right] + c_2\left[\begin{array}{c}\frac{2\,\mathrm{e}^{\left(\frac{1}{2}-\frac{i\sqrt{39}}{2}\right)t}}{-\frac{1}{2}-\frac{i\sqrt{39}}{2}} \\ \mathrm{e}^{\left(\frac{1}{2}-\frac{i\sqrt{39}}{2}\right)t}\end{array}\right]$$

Which becomes

$$\left[\begin{array}{c}x_1(t) \\ x_2(t)\end{array}\right] = \left[\begin{array}{c}\frac{i\left(i-\sqrt{39}\right)c_1\mathrm{e}^{\frac{\left(1+i\sqrt{39}\right)t}{2}}}{10} + \frac{i\left(\sqrt{39}+i\right)c_2\mathrm{e}^{-\frac{\left(i\sqrt{39}-1\right)t}{2}}}{10} \\ c_1\mathrm{e}^{\frac{\left(1+i\sqrt{39}\right)t}{2}} + c_2\mathrm{e}^{-\frac{\left(i\sqrt{39}-1\right)t}{2}}\end{array}\right]$$

The following is the phase plot of the system.

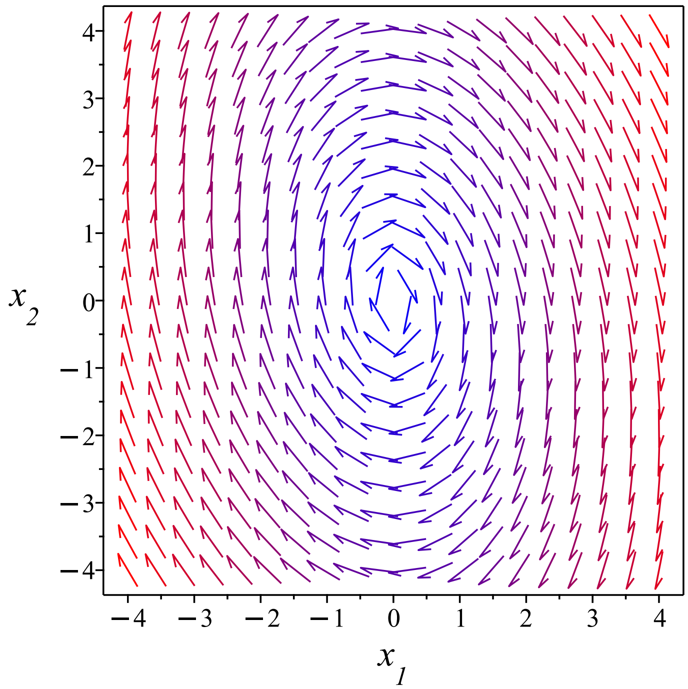


Figure 550: Phase plot

19.10.3 Maple step by step solution

Let's solve

$[x_1'(t) = x_1(t) + 2x_2(t), x_2'(t) = -5x_1(t)]$

- Define vector

$$\underline{x}^{\rightarrow}(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}$$

- Convert system into a vector equation

$$\underline{x}^{\rightarrow\prime}(t) = \begin{bmatrix} 1 & 2 \\ -5 & 0 \end{bmatrix} \cdot \underline{x}^{\rightarrow}(t) + \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

- System to solve

$$\underline{x}^{\rightarrow\prime}(t) = \begin{bmatrix} 1 & 2 \\ -5 & 0 \end{bmatrix} \cdot \underline{x}^{\rightarrow}(t)$$

- Define the coefficient matrix

$$A = \begin{bmatrix} 1 & 2 \\ -5 & 0 \end{bmatrix}$$

- Rewrite the system as

$$\underline{\vec{x}}'(t) = A \cdot \underline{\vec{x}}(t)$$

- To solve the system, find the eigenvalues and eigenvectors of A
- Eigenpairs of A

$$\left[\left[\frac{1}{2} - \frac{\mathrm{I}\sqrt{39}}{2}, \begin{bmatrix} \frac{2}{-\frac{1}{2}-\frac{\mathrm{I}\sqrt{39}}{2}} \\ 1 \end{bmatrix}\right], \left[\frac{1}{2} + \frac{\mathrm{I}\sqrt{39}}{2}, \begin{bmatrix} \frac{2}{-\frac{1}{2}+\frac{\mathrm{I}\sqrt{39}}{2}} \\ 1 \end{bmatrix}\right]\right]$$

- Consider complex eigenpair, complex conjugate eigenvalue can be ignored

$$\left[\frac{1}{2} - \frac{\mathrm{I}\sqrt{39}}{2}, \begin{bmatrix} \frac{2}{-\frac{1}{2}-\frac{\mathrm{I}\sqrt{39}}{2}} \\ 1 \end{bmatrix}\right]$$

- Solution from eigenpair

$$\mathrm{e}^{\left(\frac{1}{2}-\frac{\mathrm{I}\sqrt{39}}{2}\right)t} \cdot \begin{bmatrix} \frac{2}{-\frac{1}{2}-\frac{\mathrm{I}\sqrt{39}}{2}} \\ 1 \end{bmatrix}$$

- Use Euler identity to write solution in terms of sin and cos

$$\mathrm{e}^{\frac{t}{2}} \cdot \left(\cos\left(\frac{\sqrt{39}\,t}{2}\right) - \mathrm{I}\sin\left(\frac{\sqrt{39}\,t}{2}\right)\right) \cdot \begin{bmatrix} \frac{2}{-\frac{1}{2}-\frac{\mathrm{I}\sqrt{39}}{2}} \\ 1 \end{bmatrix}$$

- Simplify expression

$$\mathrm{e}^{\frac{t}{2}} \cdot \begin{bmatrix} \frac{2\left(\cos\left(\frac{\sqrt{39}\,t}{2}\right)-\mathrm{I}\sin\left(\frac{\sqrt{39}\,t}{2}\right)\right)}{-\frac{1}{2}-\frac{\mathrm{I}\sqrt{39}}{2}} \\ \cos\left(\frac{\sqrt{39}\,t}{2}\right) - \mathrm{I}\sin\left(\frac{\sqrt{39}\,t}{2}\right) \end{bmatrix}$$

- Both real and imaginary parts are solutions to the homogeneous system

$$\left[\underline{\vec{x}}_1(t) = \mathrm{e}^{\frac{t}{2}} \cdot \begin{bmatrix} -\frac{\cos\left(\frac{\sqrt{39}\,t}{2}\right)}{10} + \frac{\sqrt{39}\,\sin\left(\frac{\sqrt{39}\,t}{2}\right)}{10} \\ \cos\left(\frac{\sqrt{39}\,t}{2}\right) \end{bmatrix}, \underline{\vec{x}}_2(t) = \mathrm{e}^{\frac{t}{2}} \cdot \begin{bmatrix} \frac{\cos\left(\frac{\sqrt{39}\,t}{2}\right)\sqrt{39}}{10} + \frac{\sin\left(\frac{\sqrt{39}\,t}{2}\right)}{10} \\ -\sin\left(\frac{\sqrt{39}\,t}{2}\right) \end{bmatrix}\right]$$

- General solution to the system of ODEs

$$\underline{\vec{x}} = c_1\underline{\vec{x}}_1(t) + c_2\underline{\vec{x}}_2(t)$$

- Substitute solutions into the general solution

$$\underline{x^{\rightarrow}} = c_1 e^{\frac{t}{2}} \cdot \begin{bmatrix} -\frac{\cos\left(\frac{\sqrt{39}\,t}{2}\right)}{10} + \frac{\sqrt{39}\,\sin\left(\frac{\sqrt{39}\,t}{2}\right)}{10} \\ \cos\left(\frac{\sqrt{39}\,t}{2}\right) \end{bmatrix} + c_2 e^{\frac{t}{2}} \cdot \begin{bmatrix} \frac{\cos\left(\frac{\sqrt{39}\,t}{2}\right)\sqrt{39}}{10} + \frac{\sin\left(\frac{\sqrt{39}\,t}{2}\right)}{10} \\ -\sin\left(\frac{\sqrt{39}\,t}{2}\right) \end{bmatrix}$$

- Substitute in vector of dependent variables

$$\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = \begin{bmatrix} \frac{\left(\left(\sqrt{39}\,c_2 - c_1\right)\cos\left(\frac{\sqrt{39}\,t}{2}\right) + \sin\left(\frac{\sqrt{39}\,t}{2}\right)\left(c_1\sqrt{39} + c_2\right)\right) e^{\frac{t}{2}}}{10} \\ e^{\frac{t}{2}}\left(\cos\left(\frac{\sqrt{39}\,t}{2}\right) c_1 - \sin\left(\frac{\sqrt{39}\,t}{2}\right) c_2\right) \end{bmatrix}$$

- Solution to the system of ODEs

$$\left\{x_1(t) = \frac{\left(\left(\sqrt{39}\,c_2 - c_1\right)\cos\left(\frac{\sqrt{39}\,t}{2}\right) + \sin\left(\frac{\sqrt{39}\,t}{2}\right)\left(c_1\sqrt{39} + c_2\right)\right) e^{\frac{t}{2}}}{10}, x_2(t) = e^{\frac{t}{2}}\left(\cos\left(\frac{\sqrt{39}\,t}{2}\right) c_1 - \sin\left(\frac{\sqrt{39}\,t}{2}\right) c_2\right)\right\}$$

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 84

```
dsolve([diff(x__1(t),t)=1*x__1(t)+2*x__2(t),diff(x__2(t),t)=-5*x__1(t)-0*x__2(t)],singsol=al
```

$$x_1(t) = \frac{e^{\frac{t}{2}}\left(\sin\left(\frac{\sqrt{39}\,t}{2}\right)\sqrt{39}\,c_2 - \cos\left(\frac{\sqrt{39}\,t}{2}\right)\sqrt{39}\,c_1 - \sin\left(\frac{\sqrt{39}\,t}{2}\right) c_1 - \cos\left(\frac{\sqrt{39}\,t}{2}\right) c_2\right)}{10}$$

$$x_2(t) = e^{\frac{t}{2}}\left(\sin\left(\frac{\sqrt{39}\,t}{2}\right) c_1 + \cos\left(\frac{\sqrt{39}\,t}{2}\right) c_2\right)$$

✓ Solution by Mathematica

Time used: 0.007 (sec). Leaf size: 54

```
DSolve[{x1'[t]==1*x1[t]+2*x2[t],x2'[t]==-5*x1[t]-1*x2[t]},{x1[t],x2[t]},t,IncludeSingularSol
```

$$\text{x1}(t) \rightarrow c_1 \cos(3t) + \frac{1}{3}(c_1 + 2c_2)\sin(3t)$$
$$\text{x2}(t) \rightarrow c_2 \cos(3t) - \frac{1}{3}(5c_1 + c_2)\sin(3t)$$

19.11 problem 11

Internal problem ID [802]
Internal file name [`OUTPUT/802_Sunday_June_05_2022_01_50_08_AM_89376539/index.tex`]

Book: Elementary differential equations and boundary value problems, 10th ed., Boyce and DiPrima
Section: Chapter 9.1, The Phase Plane: Linear Systems. page 505
Problem number: 11.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"system of linear ODEs"**

Solve

$$x_1'(t) = -x_1(t)$$
$$x_2'(t) = -x_2(t)$$

19.11.1 Solution using Matrix exponential method

In this method, we will assume we have found the matrix exponential e^{At} allready. There are different methods to determine this but will not be shown here. This is a system of linear ODE's given as

$$\vec{x}'(t) = A\,\vec{x}(t)$$

Or

$$\begin{bmatrix} x_1'(t) \\ x_2'(t) \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}$$

For the above matrix A, the matrix exponential can be found to be

$$e^{At} = \begin{bmatrix} \mathrm{e}^{-t} & 0 \\ 0 & \mathrm{e}^{-t} \end{bmatrix}$$

Therefore the homogeneous solution is

$$
\begin{aligned}
\vec{x}_h(t) &= e^{At}\vec{c} \\
&= \begin{bmatrix} e^{-t} & 0 \\ 0 & e^{-t} \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} \\
&= \begin{bmatrix} e^{-t}c_1 \\ e^{-t}c_2 \end{bmatrix}
\end{aligned}
$$

Since no forcing function is given, then the final solution is $\vec{x}_h(t)$ above.

19.11.2 Solution using explicit Eigenvalue and Eigenvector method

This is a system of linear ODE's given as

$$\vec{x}'(t) = A\,\vec{x}(t)$$

Or

$$\begin{bmatrix} x_1'(t) \\ x_2'(t) \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}$$

The first step is find the homogeneous solution. We start by finding the eigenvalues of A. This is done by solving the following equation for the eigenvalues λ

$$\det(A - \lambda I) = 0$$

Expanding gives

$$\det\left(\begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right) = 0$$

Therefore

$$\det\left(\begin{bmatrix} -1-\lambda & 0 \\ 0 & -1-\lambda \end{bmatrix}\right) = 0$$

Since the matrix A is triangular matrix, then the determinant is the product of the elements along the diagonal. Therefore the above becomes

$$(-1-\lambda)(-1-\lambda) = 0$$

The roots of the above are the eigenvalues.

$$\lambda_1 = -1$$

This table summarises the above result

eigenvalue	algebraic multiplicity	type of eigenvalue
-1	1	real eigenvalue

Now the eigenvector for each eigenvalue are found.

Considering the eigenvalue $\lambda_1 = -1$

We need to solve $A\vec{v} = \lambda\vec{v}$ or $(A - \lambda I)\vec{v} = \vec{0}$ which becomes

$$\left(\begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} - (-1)\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right)\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$
$$\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Now forward elimination is applied to solve for the eigenvector $\vec{v}$. The augmented matrix is

$$\left[\begin{array}{cc|c} 0 & 0 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

Therefore the system in Echelon form is

$$\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

The free variables are $\{v_1, v_2\}$ and there are no leading variables. Let $v_1 = t$. Let $v_2 = s$. Since there are two free Variable, we have found two eigenvectors associated with this eigenvalue. The above can be written as

$$\begin{bmatrix} t \\ s \end{bmatrix} = \begin{bmatrix} t \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ s \end{bmatrix}$$
$$= t\begin{bmatrix} 1 \\ 0 \end{bmatrix} + s\begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

By letting $t = 1$ and $s = 1$ then the above becomes

$$\begin{bmatrix} t \\ s \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

Hence the two eigenvectors associated with this eigenvalue are

$$\left(\begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right)$$

The following table gives a summary of this result. It shows for each eigenvalue the algebraic multiplicity m, and its geometric multiplicity k and the eigenvectors associated with the eigenvalue. If $m > k$ then the eigenvalue is defective which means the number of normal linearly independent eigenvectors associated with this eigenvalue (called the geometric multiplicity k) does not equal the algebraic multiplicity m, and we need to determine an additional $m - k$ generalized eigenvectors for this eigenvalue.

	multiplicity			
eigenvalue	algebraic m	geometric k	defective?	eigenvectors
-1	2	2	No	$\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$

Now that we found the eigenvalues and associated eigenvectors, we will go over each eigenvalue and generate the solution basis. The only problem we need to take care of is if the eigenvalue is defective. eigenvalue -1 is real and repated eigenvalue of multiplicity 2.There are two possible cases that can happen. This is illustrated in this diagram

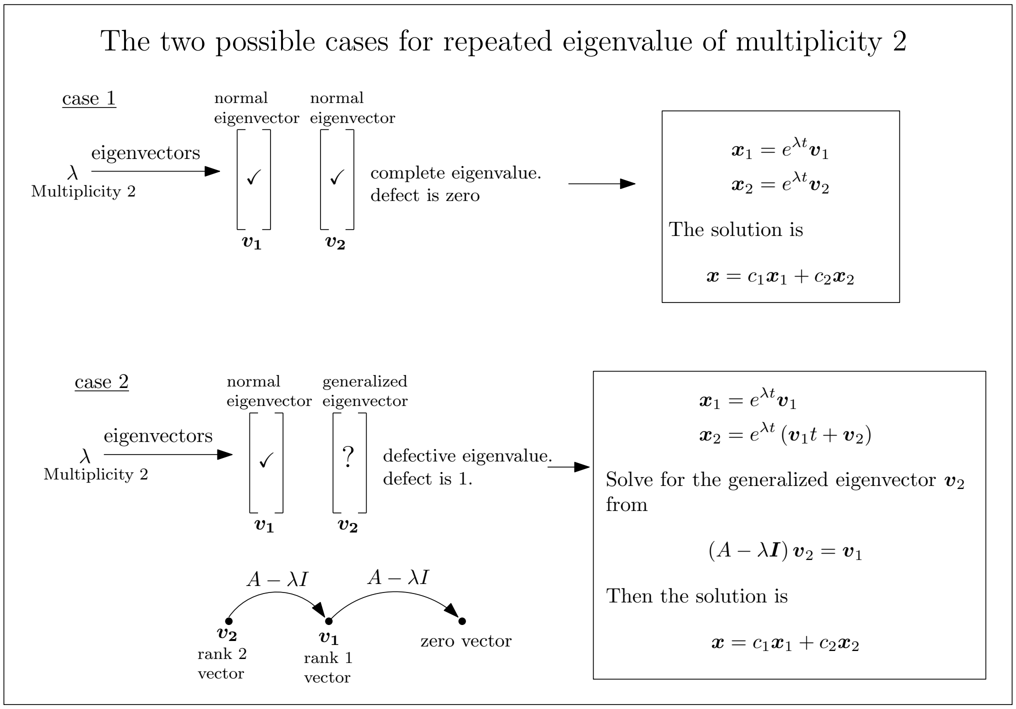


Figure 551: Possible case for repeated λ of multiplicity 2

This eigenvalue has algebraic multiplicity of 2 which is the same as its geometric multiplicity 2, then it is complete eigenvalue and this falls into case 1 shown above. Hence the corresponding eigenvector basis are

$$
\begin{aligned}
\vec{x}_1(t) &= \vec{v}_1 e^{-t} \\
&= \begin{bmatrix} 0 \\ 1 \end{bmatrix} e^{-t}
\end{aligned}
$$

$$
\begin{aligned}
\vec{x}_2(t) &= \vec{v}_2 e^{-t} \\
&= \begin{bmatrix} 1 \\ 0 \end{bmatrix} e^{-t}
\end{aligned}
$$

Therefore the final solution is

$$
\vec{x}_h(t) = c_1 \vec{x}_1(t) + c_2 \vec{x}_2(t)
$$

Which is written as

$$\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = c_1 \begin{bmatrix} 0 \\ e^{-t} \end{bmatrix} + c_2 \begin{bmatrix} e^{-t} \\ 0 \end{bmatrix}$$

Which becomes

$$\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = \begin{bmatrix} c_2 e^{-t} \\ c_1 e^{-t} \end{bmatrix}$$

The following is the phase plot of the system.

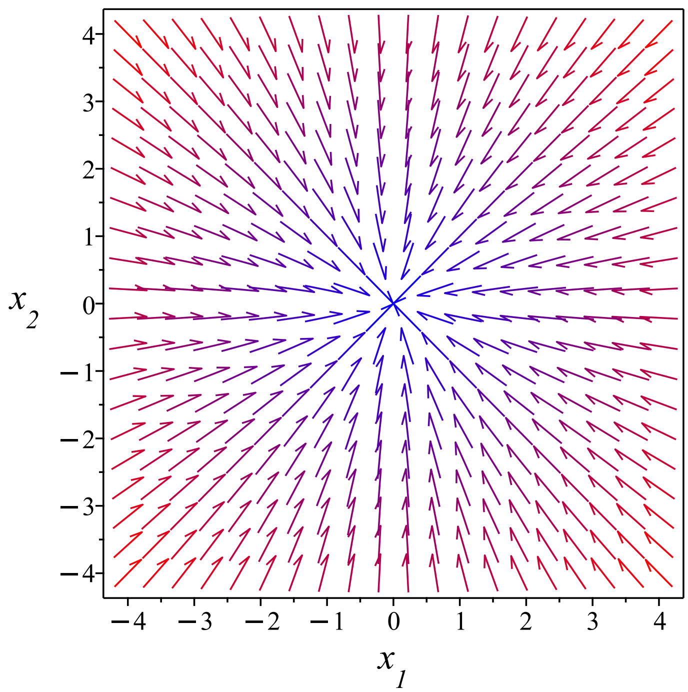


Figure 552: Phase plot

19.11.3 Maple step by step solution

Let's solve

$[x_1'(t) = -x_1(t), x_2'(t) = -x_2(t)]$

- Define vector

$$\underline{x}^{\rightarrow}(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}$$

- Convert system into a vector equation

$$\underline{x}^{\rightarrow}{}'(t) = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} \cdot \underline{x}^{\rightarrow}(t) + \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

- System to solve

$$\underline{x}^{\rightarrow}{}'(t) = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} \cdot \underline{x}^{\rightarrow}(t)$$

- Define the coefficient matrix

$$A = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$$

- Rewrite the system as

$$\underline{x}^{\rightarrow}{}'(t) = A \cdot \underline{x}^{\rightarrow}(t)$$

- To solve the system, find the eigenvalues and eigenvectors of A
- Eigenpairs of A

$$\left[\left[-1, \begin{bmatrix} 0 \\ 1 \end{bmatrix}\right], \left[-1, \begin{bmatrix} 1 \\ 0 \end{bmatrix}\right]\right]$$

- Consider eigenpair, with eigenvalue of algebraic multiplicity 2

$$\left[-1, \begin{bmatrix} 0 \\ 1 \end{bmatrix}\right]$$

- First solution from eigenvalue -1

$$\underline{x}^{\rightarrow}{}_1(t) = \mathrm{e}^{-t} \cdot \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

- Form of the 2nd homogeneous solution where $\vec{p}$ is to be solved for, $\lambda = -1$ is the eigenvalue, an

$$\underline{x}^{\rightarrow}{}_2(t) = \mathrm{e}^{\lambda t}\left(t\vec{v} + \vec{p}\right)$$

- Note that the t multiplying $\vec{v}$ makes this solution linearly independent to the 1st solution obtai
- Substitute $\underline{x}^{\rightarrow}{}_2(t)$ into the homogeneous system

$$\lambda e^{\lambda t}\left(t\vec{v}+\vec{p}\right)+e^{\lambda t}\vec{v}=\left(e^{\lambda t}A\right)\cdot\left(t\vec{v}+\vec{p}\right)$$

- Use the fact that $\vec{v}$ is an eigenvector of A

$$\lambda e^{\lambda t}\left(t\vec{v}+\vec{p}\right)+e^{\lambda t}\vec{v}=e^{\lambda t}\left(\lambda t\vec{v}+A\cdot\vec{p}\right)$$

- Simplify equation

$$\lambda\vec{p}+\vec{v}=A\cdot\vec{p}$$

- Make use of the identity matrix I

$$(\lambda\cdot I)\cdot\vec{p}+\vec{v}=A\cdot\vec{p}$$

- Condition $\vec{p}$ must meet for $\underline{\vec{x}}_2(t)$ to be a solution to the homogeneous system

$$(A-\lambda\cdot I)\cdot\vec{p}=\vec{v}$$

- Choose $\vec{p}$ to use in the second solution to the homogeneous system from eigenvalue -1

$$\left(\begin{bmatrix}-1 & 0\\ 0 & -1\end{bmatrix}-(-1)\cdot\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}\right)\cdot\vec{p}=\begin{bmatrix}0\\ 1\end{bmatrix}$$

- Choice of $\vec{p}$

$$\vec{p}=\begin{bmatrix}0\\ 0\end{bmatrix}$$

- Second solution from eigenvalue -1

$$\underline{\vec{x}}_2(t)=e^{-t}\cdot\left(t\cdot\begin{bmatrix}0\\ 1\end{bmatrix}+\begin{bmatrix}0\\ 0\end{bmatrix}\right)$$

- General solution to the system of ODEs

$$\underline{\vec{x}}=c_1\underline{\vec{x}}_1(t)+c_2\underline{\vec{x}}_2(t)$$

- Substitute solutions into the general solution

$$\underline{\vec{x}}=c_1e^{-t}\cdot\begin{bmatrix}0\\ 1\end{bmatrix}+c_2e^{-t}\cdot\left(t\cdot\begin{bmatrix}0\\ 1\end{bmatrix}+\begin{bmatrix}0\\ 0\end{bmatrix}\right)$$

- Substitute in vector of dependent variables

$$\begin{bmatrix}x_1(t)\\ x_2(t)\end{bmatrix}=\begin{bmatrix}0\\ (c_2t+c_1)\,e^{-t}\end{bmatrix}$$

- Solution to the system of ODEs

$$\{x_1(t) = 0, x_2(t) = (c_2 t + c_1)\,\mathrm{e}^{-t}\}$$

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 20

```
dsolve([diff(x__1(t),t)=-1*x__1(t)-0*x__2(t),diff(x__2(t),t)=0*x__1(t)-1*x__2(t)],singsol=al
```

$$x_1(t) = c_2 \mathrm{e}^{-t}$$
$$x_2(t) = \mathrm{e}^{-t} c_1$$

✓ Solution by Mathematica

Time used: 0.034 (sec). Leaf size: 65

```
DSolve[{x1'[t]==-1*x1[t]-0*x2[t],x2'[t]==0*x1[t]-1*x2[t]},{x1[t],x2[t]},t,IncludeSingularSol
```

$$\mathrm{x1}(t) \to c_1 e^{-t}$$
$$\mathrm{x2}(t) \to c_2 e^{-t}$$
$$\mathrm{x1}(t) \to c_1 e^{-t}$$
$$\mathrm{x2}(t) \to 0$$
$$\mathrm{x1}(t) \to 0$$
$$\mathrm{x2}(t) \to c_2 e^{-t}$$
$$\mathrm{x1}(t) \to 0$$
$$\mathrm{x2}(t) \to 0$$

19.12 problem 12

Internal problem ID [803]
Internal file name [`OUTPUT/803_Sunday_June_05_2022_01_50_09_AM_89623807/index.tex`]

Book: Elementary differential equations and boundary value problems, 10th ed., Boyce and DiPrima
Section: Chapter 9.1, The Phase Plane: Linear Systems. page 505
Problem number: 12.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"system of linear ODEs"**

Solve

$$x_1'(t) = 2x_1(t) - \frac{5x_2(t)}{2}$$
$$x_2'(t) = \frac{9x_1(t)}{5} - x_2(t)$$

19.12.1 Solution using Matrix exponential method

In this method, we will assume we have found the matrix exponential e^{At} allready. There are different methods to determine this but will not be shown here. This is a system of linear ODE's given as

$$\vec{x}'(t) = A\,\vec{x}(t)$$

Or

$$\begin{bmatrix} x_1'(t) \\ x_2'(t) \end{bmatrix} = \begin{bmatrix} 2 & -\frac{5}{2} \\ \frac{9}{5} & -1 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}$$

For the above matrix A, the matrix exponential can be found to be

$$e^{At} = \begin{bmatrix} e^{\frac{t}{2}}\cos\left(\frac{3t}{2}\right)+e^{\frac{t}{2}}\sin\left(\frac{3t}{2}\right) & -\frac{5\,e^{\frac{t}{2}}\sin\left(\frac{3t}{2}\right)}{3} \\ \frac{6\,e^{\frac{t}{2}}\sin\left(\frac{3t}{2}\right)}{5} & e^{\frac{t}{2}}\cos\left(\frac{3t}{2}\right)-e^{\frac{t}{2}}\sin\left(\frac{3t}{2}\right) \end{bmatrix}$$
$$= \begin{bmatrix} e^{\frac{t}{2}}\left(\cos\left(\frac{3t}{2}\right)+\sin\left(\frac{3t}{2}\right)\right) & -\frac{5\,e^{\frac{t}{2}}\sin\left(\frac{3t}{2}\right)}{3} \\ \frac{6\,e^{\frac{t}{2}}\sin\left(\frac{3t}{2}\right)}{5} & e^{\frac{t}{2}}\left(\cos\left(\frac{3t}{2}\right)-\sin\left(\frac{3t}{2}\right)\right) \end{bmatrix}$$

Therefore the homogeneous solution is

$$\begin{aligned}
\vec{x}_h(t) &= e^{At}\vec{c} \\
&= \begin{bmatrix} e^{\frac{t}{2}}\left(\cos\left(\frac{3t}{2}\right)+\sin\left(\frac{3t}{2}\right)\right) & -\frac{5\,e^{\frac{t}{2}}\sin\left(\frac{3t}{2}\right)}{3} \\ \frac{6\,e^{\frac{t}{2}}\sin\left(\frac{3t}{2}\right)}{5} & e^{\frac{t}{2}}\left(\cos\left(\frac{3t}{2}\right)-\sin\left(\frac{3t}{2}\right)\right) \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} \\
&= \begin{bmatrix} e^{\frac{t}{2}}\left(\cos\left(\frac{3t}{2}\right)+\sin\left(\frac{3t}{2}\right)\right)c_1 - \frac{5\,e^{\frac{t}{2}}\sin\left(\frac{3t}{2}\right)c_2}{3} \\ \frac{6\,e^{\frac{t}{2}}\sin\left(\frac{3t}{2}\right)c_1}{5} + e^{\frac{t}{2}}\left(\cos\left(\frac{3t}{2}\right)-\sin\left(\frac{3t}{2}\right)\right)c_2 \end{bmatrix} \\
&= \begin{bmatrix} \left(\left(c_1-\frac{5c_2}{3}\right)\sin\left(\frac{3t}{2}\right)+c_1\cos\left(\frac{3t}{2}\right)\right)e^{\frac{t}{2}} \\ \frac{e^{\frac{t}{2}}(6c_1-5c_2)\sin\left(\frac{3t}{2}\right)}{5} + e^{\frac{t}{2}}\cos\left(\frac{3t}{2}\right)c_2 \end{bmatrix}
\end{aligned}$$

Since no forcing function is given, then the final solution is $\vec{x}_h(t)$ above.

19.12.2 Solution using explicit Eigenvalue and Eigenvector method

This is a system of linear ODE's given as

$$\vec{x}'(t) = A\,\vec{x}(t)$$

Or

$$\begin{bmatrix} x_1'(t) \\ x_2'(t) \end{bmatrix} = \begin{bmatrix} 2 & -\frac{5}{2} \\ \frac{9}{5} & -1 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}$$

The first step is find the homogeneous solution. We start by finding the eigenvalues of A. This is done by solving the following equation for the eigenvalues λ

$$\det(A-\lambda I)=0$$

Expanding gives

$$\det\left(\begin{bmatrix}2 & -\frac{5}{2}\\ \frac{9}{5} & -1\end{bmatrix} - \lambda\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}\right) = 0$$

Therefore

$$\det\left(\begin{bmatrix}2-\lambda & -\frac{5}{2}\\ \frac{9}{5} & -1-\lambda\end{bmatrix}\right) = 0$$

Which gives the characteristic equation

$$\lambda^2 - \lambda + \frac{5}{2} = 0$$

The roots of the above are the eigenvalues.

$$\lambda_1 = \frac{1}{2} + \frac{3i}{2}$$
$$\lambda_2 = \frac{1}{2} - \frac{3i}{2}$$

This table summarises the above result

eigenvalue	algebraic multiplicity	type of eigenvalue
$\frac{1}{2} - \frac{3i}{2}$	1	complex eigenvalue
$\frac{1}{2} + \frac{3i}{2}$	1	complex eigenvalue

Now the eigenvector for each eigenvalue are found.

Considering the eigenvalue $\lambda_1 = \frac{1}{2} - \frac{3i}{2}$

We need to solve $A\vec{v} = \lambda\vec{v}$ or $(A - \lambda I)\vec{v} = \vec{0}$ which becomes

$$\left(\begin{bmatrix}2 & -\frac{5}{2}\\ \frac{9}{5} & -1\end{bmatrix} - \left(\frac{1}{2} - \frac{3i}{2}\right)\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}\right)\begin{bmatrix}v_1\\ v_2\end{bmatrix} = \begin{bmatrix}0\\ 0\end{bmatrix}$$
$$\begin{bmatrix}\frac{3}{2} + \frac{3i}{2} & -\frac{5}{2}\\ \frac{9}{5} & -\frac{3}{2} + \frac{3i}{2}\end{bmatrix}\begin{bmatrix}v_1\\ v_2\end{bmatrix} = \begin{bmatrix}0\\ 0\end{bmatrix}$$

Now forward elimination is applied to solve for the eigenvector $\vec{v}$. The augmented matrix is

$$\left[\begin{array}{cc|c}\frac{3}{2} + \frac{3i}{2} & -\frac{5}{2} & 0\\ \frac{9}{5} & -\frac{3}{2} + \frac{3i}{2} & 0\end{array}\right]$$

$$R_2 = R_2 + \left(-\frac{3}{5} + \frac{3i}{5}\right) R_1 \Longrightarrow \left[\begin{array}{cc|c} \frac{3}{2} + \frac{3i}{2} & -\frac{5}{2} & 0 \\ 0 & 0 & 0 \end{array}\right]$$

Therefore the system in Echelon form is

$$\begin{bmatrix} \frac{3}{2} + \frac{3i}{2} & -\frac{5}{2} \\ 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

The free variables are $\{v_2\}$ and the leading variables are $\{v_1\}$. Let $v_2 = t$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\left\{v_1 = \left(\frac{5}{6} - \frac{5i}{6}\right) t\right\}$

Hence the solution is

$$\begin{bmatrix} \left(\frac{5}{6} - \frac{5\,\mathrm{I}}{6}\right) t \\ t \end{bmatrix} = \begin{bmatrix} \left(\frac{5}{6} - \frac{5i}{6}\right) t \\ t \end{bmatrix}$$

Since there is one free Variable, we have found one eigenvector associated with this eigenvalue. The above can be written as

$$\begin{bmatrix} \left(\frac{5}{6} - \frac{5\,\mathrm{I}}{6}\right) t \\ t \end{bmatrix} = t \begin{bmatrix} \frac{5}{6} - \frac{5i}{6} \\ 1 \end{bmatrix}$$

Let $t = 1$ the eigenvector becomes

$$\begin{bmatrix} \left(\frac{5}{6} - \frac{5\,\mathrm{I}}{6}\right) t \\ t \end{bmatrix} = \begin{bmatrix} \frac{5}{6} - \frac{5i}{6} \\ 1 \end{bmatrix}$$

Which is normalized to

$$\begin{bmatrix} \left(\frac{5}{6} - \frac{5\,\mathrm{I}}{6}\right) t \\ t \end{bmatrix} = \begin{bmatrix} 5 - 5i \\ 6 \end{bmatrix}$$

Considering the eigenvalue $\lambda_2 = \frac{1}{2} + \frac{3i}{2}$

We need to solve $A\vec{v} = \lambda\vec{v}$ or $(A - \lambda I)\vec{v} = \vec{0}$ which becomes

$$\left(\begin{bmatrix} 2 & -\frac{5}{2} \\ \frac{9}{5} & -1 \end{bmatrix} - \left(\frac{1}{2} + \frac{3i}{2}\right) \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right) \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} \frac{3}{2} - \frac{3i}{2} & -\frac{5}{2} \\ \frac{9}{5} & -\frac{3}{2} - \frac{3i}{2} \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Now forward elimination is applied to solve for the eigenvector $\vec{v}$. The augmented matrix is

$$\left[\begin{array}{cc|c} \frac{3}{2}-\frac{3i}{2} & -\frac{5}{2} & 0 \\ \frac{9}{5} & -\frac{3}{2}-\frac{3i}{2} & 0 \end{array}\right]$$

$$R_2 = R_2 + \left(-\frac{3}{5}-\frac{3i}{5}\right) R_1 \Longrightarrow \left[\begin{array}{cc|c} \frac{3}{2}-\frac{3i}{2} & -\frac{5}{2} & 0 \\ 0 & 0 & 0 \end{array}\right]$$

Therefore the system in Echelon form is

$$\begin{bmatrix} \frac{3}{2}-\frac{3i}{2} & -\frac{5}{2} \\ 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

The free variables are $\{v_2\}$ and the leading variables are $\{v_1\}$. Let $v_2 = t$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\left\{v_1 = \left(\frac{5}{6}+\frac{5i}{6}\right)t\right\}$

Hence the solution is

$$\begin{bmatrix} \left(\frac{5}{6}+\frac{5\,\mathrm{I}}{6}\right)t \\ t \end{bmatrix} = \begin{bmatrix} \left(\frac{5}{6}+\frac{5i}{6}\right)t \\ t \end{bmatrix}$$

Since there is one free Variable, we have found one eigenvector associated with this eigenvalue. The above can be written as

$$\begin{bmatrix} \left(\frac{5}{6}+\frac{5\,\mathrm{I}}{6}\right)t \\ t \end{bmatrix} = t \begin{bmatrix} \frac{5}{6}+\frac{5i}{6} \\ 1 \end{bmatrix}$$

Let $t = 1$ the eigenvector becomes

$$\begin{bmatrix} \left(\frac{5}{6}+\frac{5\,\mathrm{I}}{6}\right)t \\ t \end{bmatrix} = \begin{bmatrix} \frac{5}{6}+\frac{5i}{6} \\ 1 \end{bmatrix}$$

Which is normalized to

$$\begin{bmatrix} \left(\frac{5}{6}+\frac{5\,\mathrm{I}}{6}\right)t \\ t \end{bmatrix} = \begin{bmatrix} 5+5i \\ 6 \end{bmatrix}$$

The following table gives a summary of this result. It shows for each eigenvalue the algebraic multiplicity m, and its geometric multiplicity k and the eigenvectors associated with the eigenvalue. If $m > k$ then the eigenvalue is defective which means the number of normal linearly independent eigenvectors associated with this eigenvalue (called the geometric multiplicity k) does not equal the algebraic multiplicity m, and we need to determine an additional $m-k$ generalized eigenvectors for this eigenvalue.

	multiplicity			
eigenvalue	algebraic m	geometric k	defective?	eigenvectors
$\frac{1}{2}+\frac{3i}{2}$	1	1	No	$\begin{bmatrix}\frac{5}{6}+\frac{5i}{6}\\1\end{bmatrix}$
$\frac{1}{2}-\frac{3i}{2}$	1	1	No	$\begin{bmatrix}\frac{5}{6}-\frac{5i}{6}\\1\end{bmatrix}$

Now that we found the eigenvalues and associated eigenvectors, we will go over each eigenvalue and generate the solution basis. The only problem we need to take care of is if the eigenvalue is defective. Therefore the final solution is

$$\vec{x}_h(t) = c_1\vec{x}_1(t) + c_2\vec{x}_2(t)$$

Which is written as

$$\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = c_1 \begin{bmatrix} \left(\frac{5}{6}+\frac{5i}{6}\right) e^{\left(\frac{1}{2}+\frac{3i}{2}\right)t} \\ e^{\left(\frac{1}{2}+\frac{3i}{2}\right)t} \end{bmatrix} + c_2 \begin{bmatrix} \left(\frac{5}{6}-\frac{5i}{6}\right) e^{\left(\frac{1}{2}-\frac{3i}{2}\right)t} \\ e^{\left(\frac{1}{2}-\frac{3i}{2}\right)t} \end{bmatrix}$$

Which becomes

$$\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = \begin{bmatrix} \left(\frac{5}{6}+\frac{5i}{6}\right) c_1 e^{\left(\frac{1}{2}+\frac{3i}{2}\right)t} + \left(\frac{5}{6}-\frac{5i}{6}\right) c_2 e^{\left(\frac{1}{2}-\frac{3i}{2}\right)t} \\ c_1 e^{\left(\frac{1}{2}+\frac{3i}{2}\right)t} + c_2 e^{\left(\frac{1}{2}-\frac{3i}{2}\right)t} \end{bmatrix}$$

The following is the phase plot of the system.

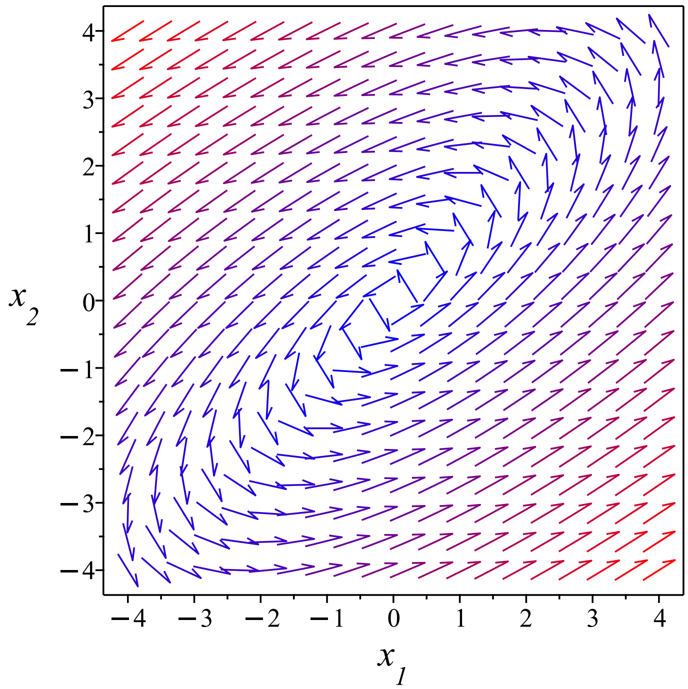


Figure 553: Phase plot

19.12.3 Maple step by step solution

Let's solve

$$\left[x_1'(t) = 2x_1(t) - \frac{5x_2(t)}{2}, x_2'(t) = \frac{9x_1(t)}{5} - x_2(t)\right]$$

- Define vector

$$\underline{x}^{\rightarrow}(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}$$

- Convert system into a vector equation

$$\underline{x}^{\rightarrow\prime}(t) = \begin{bmatrix} 2 & -\frac{5}{2} \\ \frac{9}{5} & -1 \end{bmatrix} \cdot \underline{x}^{\rightarrow}(t) + \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

- System to solve

$$\underline{x}^{\rightarrow\prime}(t) = \begin{bmatrix} 2 & -\frac{5}{2} \\ \frac{9}{5} & -1 \end{bmatrix} \cdot \underline{x}^{\rightarrow}(t)$$

- Define the coefficient matrix

$$A = \begin{bmatrix} 2 & -\frac{5}{2} \\ \frac{9}{5} & -1 \end{bmatrix}$$

- Rewrite the system as

$$\underline{x}^{\rightarrow}{}'(t) = A \cdot \underline{x}^{\rightarrow}(t)$$

- To solve the system, find the eigenvalues and eigenvectors of A
- Eigenpairs of A

$$\left[\left[\frac{1}{2}-\frac{3\,\mathrm{I}}{2}, \begin{bmatrix} \frac{5}{6}-\frac{5\,\mathrm{I}}{6} \\ 1 \end{bmatrix}\right], \left[\frac{1}{2}+\frac{3\,\mathrm{I}}{2}, \begin{bmatrix} \frac{5}{6}+\frac{5\,\mathrm{I}}{6} \\ 1 \end{bmatrix}\right]\right]$$

- Consider complex eigenpair, complex conjugate eigenvalue can be ignored

$$\left[\frac{1}{2}-\frac{3\,\mathrm{I}}{2}, \begin{bmatrix} \frac{5}{6}-\frac{5\,\mathrm{I}}{6} \\ 1 \end{bmatrix}\right]$$

- Solution from eigenpair

$$\mathrm{e}^{\left(\frac{1}{2}-\frac{3\,\mathrm{I}}{2}\right)t} \cdot \begin{bmatrix} \frac{5}{6}-\frac{5\,\mathrm{I}}{6} \\ 1 \end{bmatrix}$$

- Use Euler identity to write solution in terms of $\sin$ and $\cos$

$$\mathrm{e}^{\frac{t}{2}} \cdot \left(\cos\left(\frac{3t}{2}\right) - \mathrm{I}\sin\left(\frac{3t}{2}\right)\right) \cdot \begin{bmatrix} \frac{5}{6}-\frac{5\,\mathrm{I}}{6} \\ 1 \end{bmatrix}$$

- Simplify expression

$$\mathrm{e}^{\frac{t}{2}} \cdot \begin{bmatrix} \left(\frac{5}{6}-\frac{5\,\mathrm{I}}{6}\right)\left(\cos\left(\frac{3t}{2}\right) - \mathrm{I}\sin\left(\frac{3t}{2}\right)\right) \\ \cos\left(\frac{3t}{2}\right) - \mathrm{I}\sin\left(\frac{3t}{2}\right) \end{bmatrix}$$

- Both real and imaginary parts are solutions to the homogeneous system

$$\left[\underline{x}^{\rightarrow}{}_{1}(t) = \mathrm{e}^{\frac{t}{2}} \cdot \begin{bmatrix} \frac{5\cos\left(\frac{3t}{2}\right)}{6} - \frac{5\sin\left(\frac{3t}{2}\right)}{6} \\ \cos\left(\frac{3t}{2}\right) \end{bmatrix}, \underline{x}^{\rightarrow}{}_{2}(t) = \mathrm{e}^{\frac{t}{2}} \cdot \begin{bmatrix} -\frac{5\sin\left(\frac{3t}{2}\right)}{6} - \frac{5\cos\left(\frac{3t}{2}\right)}{6} \\ -\sin\left(\frac{3t}{2}\right) \end{bmatrix}\right]$$

- General solution to the system of ODEs

$$\underline{x}^{\rightarrow} = c_1 \underline{x}^{\rightarrow}{}_{1}(t) + c_2 \underline{x}^{\rightarrow}{}_{2}(t)$$

- Substitute solutions into the general solution

$$\underline{x^{\rightarrow}} = c_1 e^{\frac{t}{2}} \cdot \begin{bmatrix} \frac{5\cos\left(\frac{3t}{2}\right)}{6} - \frac{5\sin\left(\frac{3t}{2}\right)}{6} \\ \cos\left(\frac{3t}{2}\right) \end{bmatrix} + c_2 e^{\frac{t}{2}} \cdot \begin{bmatrix} -\frac{5\sin\left(\frac{3t}{2}\right)}{6} - \frac{5\cos\left(\frac{3t}{2}\right)}{6} \\ -\sin\left(\frac{3t}{2}\right) \end{bmatrix}$$

- Substitute in vector of dependent variables

$$\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = \begin{bmatrix} \frac{5\left((c_1-c_2)\cos\left(\frac{3t}{2}\right)-\sin\left(\frac{3t}{2}\right)(c_1+c_2)\right)e^{\frac{t}{2}}}{6} \\ e^{\frac{t}{2}}\left(c_1\cos\left(\frac{3t}{2}\right) - c_2\sin\left(\frac{3t}{2}\right)\right) \end{bmatrix}$$

- Solution to the system of ODEs

$$\left\{x_1(t) = \frac{5\left((c_1-c_2)\cos\left(\frac{3t}{2}\right)-\sin\left(\frac{3t}{2}\right)(c_1+c_2)\right)e^{\frac{t}{2}}}{6}, x_2(t) = e^{\frac{t}{2}}\left(c_1\cos\left(\frac{3t}{2}\right) - c_2\sin\left(\frac{3t}{2}\right)\right)\right\}$$

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 58

```
dsolve([diff(x__1(t),t)=2*x__1(t)-5/2*x__2(t),diff(x__2(t),t)=9/5*x__1(t)-1*x__2(t)],singsol
```

$$x_1(t) = e^{\frac{t}{2}}\left(\sin\left(\frac{3t}{2}\right)c_1 + \cos\left(\frac{3t}{2}\right)c_2\right)$$

$$x_2(t) = \frac{3\,e^{\frac{t}{2}}\left(\sin\left(\frac{3t}{2}\right)c_1 + \sin\left(\frac{3t}{2}\right)c_2 - \cos\left(\frac{3t}{2}\right)c_1 + \cos\left(\frac{3t}{2}\right)c_2\right)}{5}$$

✓ Solution by Mathematica

Time used: 0.005 (sec). Leaf size: 84

```
DSolve[{x1'[t]==2*x1[t]-5/2*x2[t],x2'[t]==9/5*x1[t]-1*x2[t]},{x1[t],x2[t]},t,IncludeSingular
```

$$\text{x1}(t) \to \frac{1}{3}e^{t/2}\left(3c_1\cos\left(\frac{3t}{2}\right) + (3c_1 - 5c_2)\sin\left(\frac{3t}{2}\right)\right)$$

$$\text{x2}(t) \to \frac{1}{5}e^{t/2}\left(5c_2\cos\left(\frac{3t}{2}\right) + (6c_1 - 5c_2)\sin\left(\frac{3t}{2}\right)\right)$$

19.13 problem 13

Internal problem ID [804]
Internal file name [`OUTPUT/804_Sunday_June_05_2022_01_50_11_AM_95295454/index.tex`]

Book: Elementary differential equations and boundary value problems, 10th ed., Boyce and DiPrima
Section: Chapter 9.1, The Phase Plane: Linear Systems. page 505
Problem number: 13.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"system of linear ODEs"**

Solve

$$x_1'(t) = x_1(t) + x_2(t) - 2$$
$$x_2'(t) = x_1(t) - x_2(t)$$

19.13.1 Solution using Matrix exponential method

In this method, we will assume we have found the matrix exponential e^{At} allready. There are different methods to determine this but will not be shown here. This is a system of linear ODE's given as

$$\vec{x}'(t) = A\,\vec{x}(t) + \vec{G}(t)$$

Or

$$\begin{bmatrix} x_1'(t) \\ x_2'(t) \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} + \begin{bmatrix} -2 \\ 0 \end{bmatrix}$$

Since the system is nonhomogeneous, then the solution is given by

$$\vec{x}(t) = \vec{x}_h(t) + \vec{x}_p(t)$$

Where $\vec{x}_h(t)$ is the homogeneous solution to $\vec{x}'(t) = A\,\vec{x}(t)$ and $\vec{x}_p(t)$ is a particular solution to $\vec{x}'(t) = A\,\vec{x}(t) + \vec{G}(t)$. The particular solution will be found using variation

of parameters method applied to the fundamental matrix. For the above matrix A, the matrix exponential can be found to be

$$e^{At} = \begin{bmatrix} \frac{(2-\sqrt{2})e^{-\sqrt{2}\,t}}{4} + \frac{e^{\sqrt{2}\,t}(2+\sqrt{2})}{4} & \frac{(-e^{-\sqrt{2}\,t}+e^{\sqrt{2}\,t})\sqrt{2}}{4} \\ \frac{(-e^{-\sqrt{2}\,t}+e^{\sqrt{2}\,t})\sqrt{2}}{4} & \frac{(2+\sqrt{2})e^{-\sqrt{2}\,t}}{4} - \frac{e^{\sqrt{2}\,t}(-2+\sqrt{2})}{4} \end{bmatrix}$$

Therefore the homogeneous solution is

$$\begin{aligned} \vec{x}_h(t) &= e^{At}\vec{c} \\ &= \begin{bmatrix} \frac{(2-\sqrt{2})e^{-\sqrt{2}\,t}}{4} + \frac{e^{\sqrt{2}\,t}(2+\sqrt{2})}{4} & \frac{(-e^{-\sqrt{2}\,t}+e^{\sqrt{2}\,t})\sqrt{2}}{4} \\ \frac{(-e^{-\sqrt{2}\,t}+e^{\sqrt{2}\,t})\sqrt{2}}{4} & \frac{(2+\sqrt{2})e^{-\sqrt{2}\,t}}{4} - \frac{e^{\sqrt{2}\,t}(-2+\sqrt{2})}{4} \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} \\ &= \begin{bmatrix} \left(\frac{(2-\sqrt{2})e^{-\sqrt{2}\,t}}{4} + \frac{e^{\sqrt{2}\,t}(2+\sqrt{2})}{4}\right) c_1 + \frac{(-e^{-\sqrt{2}\,t}+e^{\sqrt{2}\,t})\sqrt{2}\,c_2}{4} \\ \frac{(-e^{-\sqrt{2}\,t}+e^{\sqrt{2}\,t})\sqrt{2}\,c_1}{4} + \left(\frac{(2+\sqrt{2})e^{-\sqrt{2}\,t}}{4} - \frac{e^{\sqrt{2}\,t}(-2+\sqrt{2})}{4}\right) c_2 \end{bmatrix} \\ &= \begin{bmatrix} \frac{((-c_1-c_2)\sqrt{2}+2c_1)e^{-\sqrt{2}\,t}}{4} + \frac{((c_1+c_2)\sqrt{2}+2c_1)e^{\sqrt{2}\,t}}{4} \\ \frac{((c_2-c_1)\sqrt{2}+2c_2)e^{-\sqrt{2}\,t}}{4} + \frac{e^{\sqrt{2}\,t}((-c_2+c_1)\sqrt{2}+2c_2)}{4} \end{bmatrix} \end{aligned}$$

The particular solution given by

$$\vec{x}_p(t) = e^{At} \int e^{-At}\vec{G}(t)\,dt$$

But

$$\begin{aligned} e^{-At} &= (e^{At})^{-1} \\ &= \begin{bmatrix} \frac{(2+\sqrt{2})e^{-\sqrt{2}\,t}}{4} - \frac{e^{\sqrt{2}\,t}(-2+\sqrt{2})}{4} & -\frac{(-e^{-\sqrt{2}\,t}+e^{\sqrt{2}\,t})\sqrt{2}}{4} \\ -\frac{(-e^{-\sqrt{2}\,t}+e^{\sqrt{2}\,t})\sqrt{2}}{4} & \frac{(2-\sqrt{2})e^{-\sqrt{2}\,t}}{4} + \frac{e^{\sqrt{2}\,t}(2+\sqrt{2})}{4} \end{bmatrix} \end{aligned}$$

Hence

$$
\begin{aligned}
\vec{x}_p(t) &= \left[\begin{array}{cc} \frac{\left(2-\sqrt{2}\right)\mathrm{e}^{-\sqrt{2}\, t}}{4}+\frac{\mathrm{e}^{\sqrt{2}\, t}\left(2+\sqrt{2}\right)}{4} & \frac{\left(-\mathrm{e}^{-\sqrt{2}\, t}+\mathrm{e}^{\sqrt{2}\, t}\right)\sqrt{2}}{4} \\ \frac{\left(-\mathrm{e}^{-\sqrt{2}\, t}+\mathrm{e}^{\sqrt{2}\, t}\right)\sqrt{2}}{4} & \frac{\left(2+\sqrt{2}\right)\mathrm{e}^{-\sqrt{2}\, t}}{4}-\frac{\mathrm{e}^{\sqrt{2}\, t}\left(-2+\sqrt{2}\right)}{4} \end{array}\right] \int \left[\begin{array}{cc} \frac{\left(2+\sqrt{2}\right)\mathrm{e}^{-\sqrt{2}\, t}}{4}-\frac{\mathrm{e}^{\sqrt{2}\, t}\left(-2+\sqrt{2}\right)}{4} & \\ -\frac{\left(-\mathrm{e}^{-\sqrt{2}\, t}+\mathrm{e}^{\sqrt{2}\, t}\right)\sqrt{2}}{4} & \left(2-\right. \end{array}\right. \\
&= \left[\begin{array}{cc} \frac{\left(2-\sqrt{2}\right)\mathrm{e}^{-\sqrt{2}\, t}}{4}+\frac{\mathrm{e}^{\sqrt{2}\, t}\left(2+\sqrt{2}\right)}{4} & \frac{\left(-\mathrm{e}^{-\sqrt{2}\, t}+\mathrm{e}^{\sqrt{2}\, t}\right)\sqrt{2}}{4} \\ \frac{\left(-\mathrm{e}^{-\sqrt{2}\, t}+\mathrm{e}^{\sqrt{2}\, t}\right)\sqrt{2}}{4} & \frac{\left(2+\sqrt{2}\right)\mathrm{e}^{-\sqrt{2}\, t}}{4}-\frac{\mathrm{e}^{\sqrt{2}\, t}\left(-2+\sqrt{2}\right)}{4} \end{array}\right] \left[\begin{array}{c} \frac{\left(1+\sqrt{2}\right)\mathrm{e}^{-\sqrt{2}\, t}}{2}-\frac{\mathrm{e}^{\sqrt{2}\, t}\left(\sqrt{2}-1\right)}{2} \\ \frac{\mathrm{e}^{\sqrt{2}\, t}}{2}+\frac{\mathrm{e}^{-\sqrt{2}\, t}}{2} \end{array}\right] \\
&= \left[\begin{array}{c} 1 \\ 1 \end{array}\right]
\end{aligned}
$$

Hence the complete solution is

$$
\begin{aligned}
\vec{x}(t) &= \vec{x}_h(t) + \vec{x}_p(t) \\
&= \left[\begin{array}{c} \frac{\left(\left(-c_1-c_2\right)\sqrt{2}+2c_1\right)\mathrm{e}^{-\sqrt{2}\, t}}{4}+\frac{\left(\left(c_1+c_2\right)\sqrt{2}+2c_1\right)\mathrm{e}^{\sqrt{2}\, t}}{4}+1 \\ \frac{\left(\left(c_2-c_1\right)\sqrt{2}+2c_2\right)\mathrm{e}^{-\sqrt{2}\, t}}{4}+\frac{\mathrm{e}^{\sqrt{2}\, t}\left(\left(-c_2+c_1\right)\sqrt{2}+2c_2\right)}{4}+1 \end{array}\right]
\end{aligned}
$$

19.13.2 Solution using explicit Eigenvalue and Eigenvector method

This is a system of linear ODE's given as

$$\vec{x}'(t) = A\,\vec{x}(t) + \vec{G}(t)$$

Or

$$\left[\begin{array}{c} x_1'(t) \\ x_2'(t) \end{array}\right] = \left[\begin{array}{cc} 1 & 1 \\ 1 & -1 \end{array}\right] \left[\begin{array}{c} x_1(t) \\ x_2(t) \end{array}\right] + \left[\begin{array}{c} -2 \\ 0 \end{array}\right]$$

Since the system is nonhomogeneous, then the solution is given by

$$\vec{x}(t) = \vec{x}_h(t) + \vec{x}_p(t)$$

Where $\vec{x}_h(t)$ is the homogeneous solution to $\vec{x}'(t) = A\,\vec{x}(t)$ and $\vec{x}_p(t)$ is a particular solution to $\vec{x}'(t) = A\,\vec{x}(t) + \vec{G}(t)$. The particular solution will be found using variation of parameters method applied to the fundamental matrix.

The first step is find the homogeneous solution. We start by finding the eigenvalues of A. This is done by solving the following equation for the eigenvalues λ

$$\det\left(A - \lambda I\right) = 0$$

Expanding gives

$$\det\left(\begin{bmatrix}1 & 1\\ 1 & -1\end{bmatrix} - \lambda\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}\right) = 0$$

Therefore

$$\det\left(\begin{bmatrix}1-\lambda & 1\\ 1 & -1-\lambda\end{bmatrix}\right) = 0$$

Which gives the characteristic equation

$$\lambda^2 - 2 = 0$$

The roots of the above are the eigenvalues.

$$\lambda_1 = \sqrt{2}$$
$$\lambda_2 = -\sqrt{2}$$

This table summarises the above result

eigenvalue	algebraic multiplicity	type of eigenvalue
$\sqrt{2}$	1	real eigenvalue
$-\sqrt{2}$	1	real eigenvalue

Now the eigenvector for each eigenvalue are found.

Considering the eigenvalue $\lambda_1 = \sqrt{2}$

We need to solve $A\vec{v} = \lambda\vec{v}$ or $(A - \lambda I)\vec{v} = \vec{0}$ which becomes

$$\left(\begin{bmatrix}1 & 1\\ 1 & -1\end{bmatrix} - \left(\sqrt{2}\right)\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}\right)\begin{bmatrix}v_1\\ v_2\end{bmatrix} = \begin{bmatrix}0\\ 0\end{bmatrix}$$
$$\begin{bmatrix}1-\sqrt{2} & 1\\ 1 & -1-\sqrt{2}\end{bmatrix}\begin{bmatrix}v_1\\ v_2\end{bmatrix} = \begin{bmatrix}0\\ 0\end{bmatrix}$$

Now forward elimination is applied to solve for the eigenvector $\vec{v}$. The augmented matrix is

$$\left[\begin{array}{cc|c}1-\sqrt{2} & 1 & 0\\ 1 & -1-\sqrt{2} & 0\end{array}\right]$$

$$R_2 = R_2 - \frac{R_1}{1-\sqrt{2}} \Longrightarrow \left[\begin{array}{cc|c} 1-\sqrt{2} & 1 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

Therefore the system in Echelon form is

$$\begin{bmatrix} 1-\sqrt{2} & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

The free variables are $\{v_2\}$ and the leading variables are $\{v_1\}$. Let $v_2 = t$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\left\{v_1 = \frac{t}{\sqrt{2}-1}\right\}$

Hence the solution is

$$\begin{bmatrix} \frac{t}{\sqrt{2}-1} \\ t \end{bmatrix} = \begin{bmatrix} \frac{t}{\sqrt{2}-1} \\ t \end{bmatrix}$$

Since there is one free Variable, we have found one eigenvector associated with this eigenvalue. The above can be written as

$$\begin{bmatrix} \frac{t}{\sqrt{2}-1} \\ t \end{bmatrix} = t \begin{bmatrix} \frac{1}{\sqrt{2}-1} \\ 1 \end{bmatrix}$$

Let $t = 1$ the eigenvector becomes

$$\begin{bmatrix} \frac{t}{\sqrt{2}-1} \\ t \end{bmatrix} = \begin{bmatrix} \frac{1}{\sqrt{2}-1} \\ 1 \end{bmatrix}$$

Considering the eigenvalue $\lambda_2 = -\sqrt{2}$

We need to solve $A\vec{v} = \lambda\vec{v}$ or $(A - \lambda I)\vec{v} = \vec{0}$ which becomes

$$\left(\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} - \left(-\sqrt{2}\right) \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right) \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 1+\sqrt{2} & 1 \\ 1 & \sqrt{2}-1 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Now forward elimination is applied to solve for the eigenvector $\vec{v}$. The augmented matrix is

$$\left[\begin{array}{cc|c} 1+\sqrt{2} & 1 & 0 \\ 1 & \sqrt{2}-1 & 0 \end{array}\right]$$

$$R_2 = R_2 - \frac{R_1}{1+\sqrt{2}} \Longrightarrow \left[\begin{array}{cc|c} 1+\sqrt{2} & 1 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

Therefore the system in Echelon form is

$$\begin{bmatrix} 1+\sqrt{2} & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

The free variables are $\{v_2\}$ and the leading variables are $\{v_1\}$. Let $v_2 = t$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\left\{v_1 = -\frac{t}{1+\sqrt{2}}\right\}$

Hence the solution is

$$\begin{bmatrix} -\frac{t}{1+\sqrt{2}} \\ t \end{bmatrix} = \begin{bmatrix} -\frac{t}{1+\sqrt{2}} \\ t \end{bmatrix}$$

Since there is one free Variable, we have found one eigenvector associated with this eigenvalue. The above can be written as

$$\begin{bmatrix} -\frac{t}{1+\sqrt{2}} \\ t \end{bmatrix} = t \begin{bmatrix} -\frac{1}{1+\sqrt{2}} \\ 1 \end{bmatrix}$$

Let $t = 1$ the eigenvector becomes

$$\begin{bmatrix} -\frac{t}{1+\sqrt{2}} \\ t \end{bmatrix} = \begin{bmatrix} -\frac{1}{1+\sqrt{2}} \\ 1 \end{bmatrix}$$

Which is normalized to

$$\begin{bmatrix} -\frac{t}{1+\sqrt{2}} \\ t \end{bmatrix} = \begin{bmatrix} -\frac{1}{1+\sqrt{2}} \\ 1 \end{bmatrix}$$

The following table gives a summary of this result. It shows for each eigenvalue the algebraic multiplicity m, and its geometric multiplicity k and the eigenvectors associated with the eigenvalue. If $m > k$ then the eigenvalue is defective which means the number of normal linearly independent eigenvectors associated with this eigenvalue (called the geometric multiplicity k) does not equal the algebraic multiplicity m, and we need to determine an additional $m - k$ generalized eigenvectors for this eigenvalue.

eigenvalue	multiplicity		defective?	eigenvectors
	algebraic m	geometric k		
$\sqrt{2}$	1	1	No	$\left[\begin{array}{c}\frac{1}{\sqrt{2}-1} \\ 1\end{array}\right]$
$-\sqrt{2}$	1	1	No	$\left[\begin{array}{c}\frac{1}{-1-\sqrt{2}} \\ 1\end{array}\right]$

Now that we found the eigenvalues and associated eigenvectors, we will go over each eigenvalue and generate the solution basis. The only problem we need to take care of is if the eigenvalue is defective. Since eigenvalue $\sqrt{2}$ is real and distinct then the corresponding eigenvector solution is

$$
\begin{aligned}
\vec{x}_1(t) &= \vec{v}_1 e^{\sqrt{2}\, t} \\
&= \left[\begin{array}{c}\frac{1}{\sqrt{2}-1} \\ 1\end{array}\right] e^{\sqrt{2}\, t}
\end{aligned}
$$

Since eigenvalue $-\sqrt{2}$ is real and distinct then the corresponding eigenvector solution is

$$
\begin{aligned}
\vec{x}_2(t) &= \vec{v}_2 e^{-\sqrt{2}\, t} \\
&= \left[\begin{array}{c}\frac{1}{-1-\sqrt{2}} \\ 1\end{array}\right] e^{-\sqrt{2}\, t}
\end{aligned}
$$

Therefore the homogeneous solution is

$$
\vec{x}_h(t) = c_1 \vec{x}_1(t) + c_2 \vec{x}_2(t)
$$

Which is written as

$$
\left[\begin{array}{c}x_1(t) \\ x_2(t)\end{array}\right] = c_1 \left[\begin{array}{c}\frac{e^{\sqrt{2}\, t}}{\sqrt{2}-1} \\ e^{\sqrt{2}\, t}\end{array}\right] + c_2 \left[\begin{array}{c}\frac{e^{-\sqrt{2}\, t}}{-1-\sqrt{2}} \\ e^{-\sqrt{2}\, t}\end{array}\right]
$$

Now that we found homogeneous solution above, we need to find a particular solution $\vec{x}_p(t)$. We will use Variation of parameters. The fundamental matrix is

$$
\Phi = \left[\begin{array}{ccc}\vec{x}_1 & \vec{x}_2 & \cdots\end{array}\right]
$$

Where $\vec{x}_i$ are the solution basis found above. Therefore the fundamental matrix is

$$\Phi(t) = \begin{bmatrix} \frac{e^{\sqrt{2}\, t}}{\sqrt{2}-1} & \frac{e^{-\sqrt{2}\, t}}{-1-\sqrt{2}} \\ e^{\sqrt{2}\, t} & e^{-\sqrt{2}\, t} \end{bmatrix}$$

The particular solution is then given by

$$\vec{x}_p(t) = \Phi \int \Phi^{-1} \vec{G}(t)\, dt$$

But

$$\Phi^{-1} = \begin{bmatrix} \frac{\sqrt{2}\, e^{-\sqrt{2}\, t}}{4} & \frac{\sqrt{2}\left(\sqrt{2}-1\right) e^{-\sqrt{2}\, t}}{4} \\ -\frac{\sqrt{2}\, e^{\sqrt{2}\, t}}{4} & \frac{\sqrt{2}\, e^{\sqrt{2}\, t}\left(1+\sqrt{2}\right)}{4} \end{bmatrix}$$

Hence

$$\begin{aligned} \vec{x}_p(t) &= \begin{bmatrix} \frac{e^{\sqrt{2}\, t}}{\sqrt{2}-1} & \frac{e^{-\sqrt{2}\, t}}{-1-\sqrt{2}} \\ e^{\sqrt{2}\, t} & e^{-\sqrt{2}\, t} \end{bmatrix} \int \begin{bmatrix} \frac{\sqrt{2}\, e^{-\sqrt{2}\, t}}{4} & \frac{\sqrt{2}\left(\sqrt{2}-1\right) e^{-\sqrt{2}\, t}}{4} \\ -\frac{\sqrt{2}\, e^{\sqrt{2}\, t}}{4} & \frac{\sqrt{2}\, e^{\sqrt{2}\, t}\left(1+\sqrt{2}\right)}{4} \end{bmatrix} \begin{bmatrix} -2 \\ 0 \end{bmatrix} dt \\ &= \begin{bmatrix} \frac{e^{\sqrt{2}\, t}}{\sqrt{2}-1} & \frac{e^{-\sqrt{2}\, t}}{-1-\sqrt{2}} \\ e^{\sqrt{2}\, t} & e^{-\sqrt{2}\, t} \end{bmatrix} \int \begin{bmatrix} -\frac{\sqrt{2}\, e^{-\sqrt{2}\, t}}{2} \\ \frac{\sqrt{2}\, e^{\sqrt{2}\, t}}{2} \end{bmatrix} dt \\ &= \begin{bmatrix} \frac{e^{\sqrt{2}\, t}}{\sqrt{2}-1} & \frac{e^{-\sqrt{2}\, t}}{-1-\sqrt{2}} \\ e^{\sqrt{2}\, t} & e^{-\sqrt{2}\, t} \end{bmatrix} \begin{bmatrix} \frac{e^{-\sqrt{2}\, t}}{2} \\ \frac{e^{\sqrt{2}\, t}}{2} \end{bmatrix} \\ &= \begin{bmatrix} 1 \\ 1 \end{bmatrix} \end{aligned}$$

Now that we found particular solution, the final solution is

$$\begin{aligned} \vec{x}(t) &= \vec{x}_h(t) + \vec{x}_p(t) \\ \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} &= \begin{bmatrix} \frac{c_1 e^{\sqrt{2}\, t}}{\sqrt{2}-1} \\ c_1 e^{\sqrt{2}\, t} \end{bmatrix} + \begin{bmatrix} \frac{c_2 e^{-\sqrt{2}\, t}}{-1-\sqrt{2}} \\ c_2 e^{-\sqrt{2}\, t} \end{bmatrix} + \begin{bmatrix} 1 \\ 1 \end{bmatrix} \end{aligned}$$

Which becomes

$$\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = \begin{bmatrix} -c_2\left(\sqrt{2}-1\right) e^{-\sqrt{2}\, t} + 1 + c_1\left(1+\sqrt{2}\right) e^{\sqrt{2}\, t} \\ c_1 e^{\sqrt{2}\, t} + c_2 e^{-\sqrt{2}\, t} + 1 \end{bmatrix}$$

The following is the phase plot of the system.

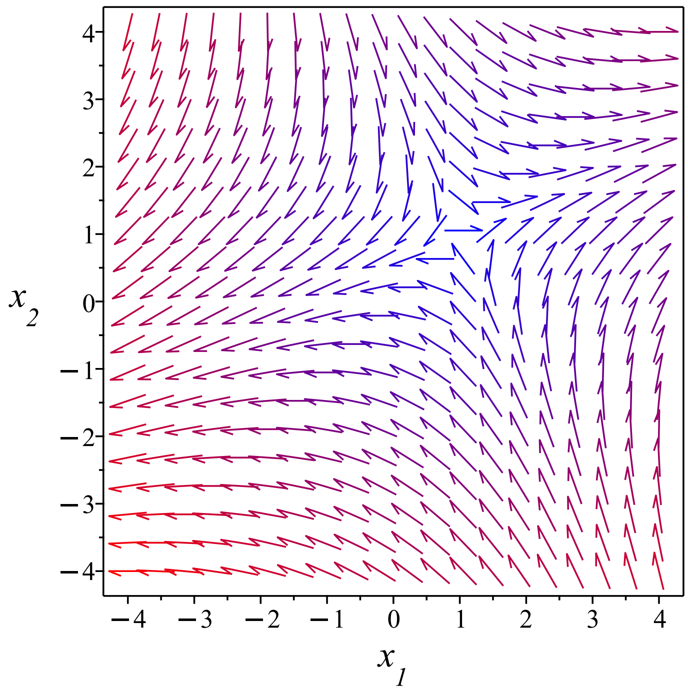


Figure 554: Phase plot

19.13.3 Maple step by step solution

Let's solve

$[x_1'(t) = x_1(t) + x_2(t) - 2, x_2'(t) = x_1(t) - x_2(t)]$

- Define vector

$$\underline{x}^{\rightarrow}(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}$$

- Convert system into a vector equation

$$\underline{x}^{\rightarrow}{}'(t) = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \cdot \underline{x}^{\rightarrow}(t) + \begin{bmatrix} -2 \\ 0 \end{bmatrix}$$

- System to solve

$$\underline{x}^{\rightarrow}{}'(t) = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \cdot \underline{x}^{\rightarrow}(t) + \begin{bmatrix} -2 \\ 0 \end{bmatrix}$$

- Define the forcing function

$$\vec{f}(t) = \begin{bmatrix} -2 \\ 0 \end{bmatrix}$$

- Define the coefficient matrix

$$A = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$$

- Rewrite the system as

$$\underline{\vec{x}}'(t) = A \cdot \underline{\vec{x}}(t) + \vec{f}$$

- To solve the system, find the eigenvalues and eigenvectors of A
- Eigenpairs of A

$$\left[\left[\sqrt{2}, \begin{bmatrix} \frac{1}{\sqrt{2}-1} \\ 1 \end{bmatrix}\right], \left[-\sqrt{2}, \begin{bmatrix} \frac{1}{-1-\sqrt{2}} \\ 1 \end{bmatrix}\right]\right]$$

- Consider eigenpair

$$\left[\sqrt{2}, \begin{bmatrix} \frac{1}{\sqrt{2}-1} \\ 1 \end{bmatrix}\right]$$

- Solution to homogeneous system from eigenpair

$$\underline{\vec{x}}_1 = e^{\sqrt{2}\,t} \cdot \begin{bmatrix} \frac{1}{\sqrt{2}-1} \\ 1 \end{bmatrix}$$

- Consider eigenpair

$$\left[-\sqrt{2}, \begin{bmatrix} \frac{1}{-1-\sqrt{2}} \\ 1 \end{bmatrix}\right]$$

- Solution to homogeneous system from eigenpair

$$\underline{\vec{x}}_2 = e^{-\sqrt{2}\,t} \cdot \begin{bmatrix} \frac{1}{-1-\sqrt{2}} \\ 1 \end{bmatrix}$$

- General solution of the system of ODEs can be written in terms of the particular solution $\underline{\vec{x}}$

$$\underline{\vec{x}}(t) = c_1\underline{\vec{x}}_1 + c_2\underline{\vec{x}}_2 + \underline{\vec{x}}_p(t)$$

- ☐ Fundamental matrix

- Let $\phi(t)$ be the matrix whose columns are the independent solutions of the homogeneous syst

$$\phi(t)=\begin{bmatrix}\frac{e^{\sqrt{2}\,t}}{\sqrt{2}-1} & \frac{e^{-\sqrt{2}\,t}}{-1-\sqrt{2}}\\ e^{\sqrt{2}\,t} & e^{-\sqrt{2}\,t}\end{bmatrix}$$

- The fundamental matrix, $\Phi(t)$ is a normalized version of $\phi(t)$ satisfying $\Phi(0)=I$ where I is th

$$\Phi(t)=\phi(t)\cdot\frac{1}{\phi(0)}$$

- Substitute the value of $\phi(t)$ and $\phi(0)$

$$\Phi(t)=\begin{bmatrix}\frac{e^{\sqrt{2}\,t}}{\sqrt{2}-1} & \frac{e^{-\sqrt{2}\,t}}{-1-\sqrt{2}}\\ e^{\sqrt{2}\,t} & e^{-\sqrt{2}\,t}\end{bmatrix}\cdot\frac{1}{\begin{bmatrix}\frac{1}{\sqrt{2}-1} & \frac{1}{-1-\sqrt{2}}\\ 1 & 1\end{bmatrix}}$$

- Evaluate and simplify to get the fundamental matrix

$$\Phi(t)=\begin{bmatrix}\frac{\left(\left(\sqrt{2}-1\right)e^{-\sqrt{2}\,t}+e^{\sqrt{2}\,t}\left(1+\sqrt{2}\right)\right)\sqrt{2}}{4} & \frac{\left(-e^{-\sqrt{2}\,t}+e^{\sqrt{2}\,t}\right)\sqrt{2}}{4}\\ \frac{\left(-e^{-\sqrt{2}\,t}+e^{\sqrt{2}\,t}\right)\sqrt{2}}{4} & \frac{\left(2+\sqrt{2}\right)e^{-\sqrt{2}\,t}}{4}-\frac{e^{\sqrt{2}\,t}\left(-2+\sqrt{2}\right)}{4}\end{bmatrix}$$

☐ Find a particular solution of the system of ODEs using variation of parameters

- Let the particular solution be the fundamental matrix multiplied by $\vec{v}(t)$ and solve for $\vec{v}(t)$

$$\vec{x}_p(t)=\Phi(t)\cdot\vec{v}(t)$$

- Take the derivative of the particular solution

$$\vec{x}_p'(t)=\Phi'(t)\cdot\vec{v}(t)+\Phi(t)\cdot\vec{v}'(t)$$

- Substitute particular solution and its derivative into the system of ODEs

$$\Phi'(t)\cdot\vec{v}(t)+\Phi(t)\cdot\vec{v}'(t)=A\cdot\Phi(t)\cdot\vec{v}(t)+\vec{f}(t)$$

- The fundamental matrix has columns that are solutions to the homogeneous system so its der

$$A\cdot\Phi(t)\cdot\vec{v}(t)+\Phi(t)\cdot\vec{v}'(t)=A\cdot\Phi(t)\cdot\vec{v}(t)+\vec{f}(t)$$

- Cancel like terms

$$\Phi(t)\cdot\vec{v}'(t)=\vec{f}(t)$$

- Multiply by the inverse of the fundamental matrix

$$\vec{v}'(t)=\frac{1}{\Phi(t)}\cdot\vec{f}(t)$$

- Integrate to solve for $\vec{v}(t)$

$$\vec{v}(t) = \int_0^t \frac{1}{\Phi(s)} \cdot \vec{f}(s)\, ds$$

- Plug $\vec{v}(t)$ into the equation for the particular solution

$$\underline{x}^{\rightarrow}_p(t) = \Phi(t) \cdot \left(\int_0^t \frac{1}{\Phi(s)} \cdot \vec{f}(s)\, ds\right)$$

- Plug in the fundamental matrix and the forcing function and compute

$$\underline{x}^{\rightarrow}_p(t) = \left[\begin{array}{c} \frac{\left(\sqrt{2}-1\right) e^{-\sqrt{2}\, t}}{2} + 1 + \frac{\left(-1-\sqrt{2}\right) e^{\sqrt{2}\, t}}{2} \\ 1 - \frac{e^{\sqrt{2}\, t}}{2} - \frac{e^{-\sqrt{2}\, t}}{2} \end{array}\right]$$

- Plug particular solution back into general solution

$$\underline{x}^{\rightarrow}(t) = c_1 \underline{x}^{\rightarrow}_1 + c_2 \underline{x}^{\rightarrow}_2 + \left[\begin{array}{c} \frac{\left(\sqrt{2}-1\right) e^{-\sqrt{2}\, t}}{2} + 1 + \frac{\left(-1-\sqrt{2}\right) e^{\sqrt{2}\, t}}{2} \\ 1 - \frac{e^{\sqrt{2}\, t}}{2} - \frac{e^{-\sqrt{2}\, t}}{2} \end{array}\right]$$

- Substitute in vector of dependent variables

$$\left[\begin{array}{c} x_1(t) \\ x_2(t) \end{array}\right] = \left[\begin{array}{c} \frac{\left((1-2c_2)\sqrt{2}-1+2c_2\right) e^{-\sqrt{2}\, t}}{2} + 1 + \frac{\left((-1+2c_1)\sqrt{2}+2c_1-1\right) e^{\sqrt{2}\, t}}{2} \\ c_1 e^{\sqrt{2}\, t} + c_2 e^{-\sqrt{2}\, t} + 1 - \frac{e^{\sqrt{2}\, t}}{2} - \frac{e^{-\sqrt{2}\, t}}{2} \end{array}\right]$$

- Solution to the system of ODEs

$$\left\{x_1(t) = \frac{\left((1-2c_2)\sqrt{2}-1+2c_2\right) e^{-\sqrt{2}\, t}}{2} + 1 + \frac{\left((-1+2c_1)\sqrt{2}+2c_1-1\right) e^{\sqrt{2}\, t}}{2}, x_2(t) = c_1 e^{\sqrt{2}\, t} + c_2 e^{-\sqrt{2}\, t} + 1 - \frac{e^{\sqrt{2}\, t}}{2}\right.$$

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 72

```
dsolve([diff(x__1(t),t)=1*x__1(t)+1*x__2(t)-2,diff(x__2(t),t)=1*x__1(t)-1*x__2(t)],singsol=a
```

$$x_1(t) = e^{\sqrt{2}\, t} c_2 + e^{-\sqrt{2}\, t} c_1 + 1$$
$$x_2(t) = \sqrt{2}\, e^{\sqrt{2}\, t} c_2 - \sqrt{2}\, e^{-\sqrt{2}\, t} c_1 - e^{\sqrt{2}\, t} c_2 - e^{-\sqrt{2}\, t} c_1 + 1$$

✓ Solution by Mathematica

Time used: 0.097 (sec). Leaf size: 160

```
DSolve[{x1'[t]==1*x1[t]+1*x2[t]-2,x2'[t]==1*x1[t]-1*x2[t]},{x1[t],x2[t]},t,IncludeSingularSo
```

$$\text{x1}(t) \to \frac{1}{4}e^{-\sqrt{2}t}\left(4e^{\sqrt{2}t} + \left(\left(2+\sqrt{2}\right)c_1 + \sqrt{2}c_2\right)e^{2\sqrt{2}t} - \left(\left(\sqrt{2}-2\right)c_1\right) - \sqrt{2}c_2\right)$$
$$\text{x2}(t) \to \frac{1}{4}e^{-\sqrt{2}t}\left(4e^{\sqrt{2}t} + \left(\sqrt{2}c_1 - \left(\sqrt{2}-2\right)c_2\right)e^{2\sqrt{2}t} - \sqrt{2}c_1 + \left(2+\sqrt{2}\right)c_2\right)$$

19.14 problem 14

Internal problem ID [805]
Internal file name [`OUTPUT/805_Sunday_June_05_2022_01_50_13_AM_61368531/index.tex`]

Book: Elementary differential equations and boundary value problems, 10th ed., Boyce and DiPrima
Section: Chapter 9.1, The Phase Plane: Linear Systems. page 505
Problem number: 14.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"system of linear ODEs"**

Solve

$$x_1'(t) = -2x_1(t) + x_2(t) - 2$$
$$x_2'(t) = x_1(t) - 2x_2(t) + 1$$

19.14.1 Solution using Matrix exponential method

In this method, we will assume we have found the matrix exponential e^{At} allready. There are different methods to determine this but will not be shown here. This is a system of linear ODE's given as

$$\vec{x}'(t) = A\,\vec{x}(t) + \vec{G}(t)$$

Or

$$\begin{bmatrix} x_1'(t) \\ x_2'(t) \end{bmatrix} = \begin{bmatrix} -2 & 1 \\ 1 & -2 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} + \begin{bmatrix} -2 \\ 1 \end{bmatrix}$$

Since the system is nonhomogeneous, then the solution is given by

$$\vec{x}(t) = \vec{x}_h(t) + \vec{x}_p(t)$$

Where $\vec{x}_h(t)$ is the homogeneous solution to $\vec{x}'(t) = A\,\vec{x}(t)$ and $\vec{x}_p(t)$ is a particular solution to $\vec{x}'(t) = A\,\vec{x}(t) + \vec{G}(t)$. The particular solution will be found using variation

of parameters method applied to the fundamental matrix. For the above matrix A, the matrix exponential can be found to be

$$e^{At} = \begin{bmatrix} \frac{e^{-3t}}{2} + \frac{e^{-t}}{2} & \frac{e^{-t}}{2} - \frac{e^{-3t}}{2} \\ \frac{e^{-t}}{2} - \frac{e^{-3t}}{2} & \frac{e^{-3t}}{2} + \frac{e^{-t}}{2} \end{bmatrix}$$

Therefore the homogeneous solution is

$$\begin{aligned} \vec{x}_h(t) &= e^{At}\vec{c} \\ &= \begin{bmatrix} \frac{e^{-3t}}{2} + \frac{e^{-t}}{2} & \frac{e^{-t}}{2} - \frac{e^{-3t}}{2} \\ \frac{e^{-t}}{2} - \frac{e^{-3t}}{2} & \frac{e^{-3t}}{2} + \frac{e^{-t}}{2} \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} \\ &= \begin{bmatrix} \left(\frac{e^{-3t}}{2} + \frac{e^{-t}}{2}\right) c_1 + \left(\frac{e^{-t}}{2} - \frac{e^{-3t}}{2}\right) c_2 \\ \left(\frac{e^{-t}}{2} - \frac{e^{-3t}}{2}\right) c_1 + \left(\frac{e^{-3t}}{2} + \frac{e^{-t}}{2}\right) c_2 \end{bmatrix} \\ &= \begin{bmatrix} \frac{(-c_2+c_1)e^{-3t}}{2} + \frac{e^{-t}(c_1+c_2)}{2} \\ \frac{(c_2-c_1)e^{-3t}}{2} + \frac{e^{-t}(c_1+c_2)}{2} \end{bmatrix} \end{aligned}$$

The particular solution given by

$$\vec{x}_p(t) = e^{At} \int e^{-At}\vec{G}(t)\,dt$$

But

$$\begin{aligned} e^{-At} &= (e^{At})^{-1} \\ &= \begin{bmatrix} \frac{(e^{2t}+1)e^{t}}{2} & -\frac{(e^{2t}-1)e^{t}}{2} \\ -\frac{(e^{2t}-1)e^{t}}{2} & \frac{(e^{2t}+1)e^{t}}{2} \end{bmatrix} \end{aligned}$$

Hence

$$\begin{aligned} \vec{x}_p(t) &= \begin{bmatrix} \frac{e^{-3t}}{2} + \frac{e^{-t}}{2} & \frac{e^{-t}}{2} - \frac{e^{-3t}}{2} \\ \frac{e^{-t}}{2} - \frac{e^{-3t}}{2} & \frac{e^{-3t}}{2} + \frac{e^{-t}}{2} \end{bmatrix} \int \begin{bmatrix} \frac{(e^{2t}+1)e^{t}}{2} & -\frac{(e^{2t}-1)e^{t}}{2} \\ -\frac{(e^{2t}-1)e^{t}}{2} & \frac{(e^{2t}+1)e^{t}}{2} \end{bmatrix} \begin{bmatrix} -2 \\ 1 \end{bmatrix} dt \\ &= \begin{bmatrix} \frac{e^{-3t}}{2} + \frac{e^{-t}}{2} & \frac{e^{-t}}{2} - \frac{e^{-3t}}{2} \\ \frac{e^{-t}}{2} - \frac{e^{-3t}}{2} & \frac{e^{-3t}}{2} + \frac{e^{-t}}{2} \end{bmatrix} \begin{bmatrix} -\frac{e^{3t}}{2} - \frac{e^{t}}{2} \\ -\frac{e^{t}}{2} + \frac{e^{3t}}{2} \end{bmatrix} \\ &= \begin{bmatrix} -1 \\ 0 \end{bmatrix} \end{aligned}$$

Hence the complete solution is

$$\vec{x}(t) = \vec{x}_h(t) + \vec{x}_p(t)$$

$$= \begin{bmatrix} \frac{(-c_2+c_1)e^{-3t}}{2} + \frac{e^{-t}(c_1+c_2)}{2} - 1 \\ \frac{(c_2-c_1)e^{-3t}}{2} + \frac{e^{-t}(c_1+c_2)}{2} \end{bmatrix}$$

19.14.2 Solution using explicit Eigenvalue and Eigenvector method

This is a system of linear ODE's given as

$$\vec{x}'(t) = A\,\vec{x}(t) + \vec{G}(t)$$

Or

$$\begin{bmatrix} x_1'(t) \\ x_2'(t) \end{bmatrix} = \begin{bmatrix} -2 & 1 \\ 1 & -2 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} + \begin{bmatrix} -2 \\ 1 \end{bmatrix}$$

Since the system is nonhomogeneous, then the solution is given by

$$\vec{x}(t) = \vec{x}_h(t) + \vec{x}_p(t)$$

Where $\vec{x}_h(t)$ is the homogeneous solution to $\vec{x}'(t) = A\,\vec{x}(t)$ and $\vec{x}_p(t)$ is a particular solution to $\vec{x}'(t) = A\,\vec{x}(t) + \vec{G}(t)$. The particular solution will be found using variation of parameters method applied to the fundamental matrix.

The first step is find the homogeneous solution. We start by finding the eigenvalues of A. This is done by solving the following equation for the eigenvalues λ

$$\det(A - \lambda I) = 0$$

Expanding gives

$$\det\left(\begin{bmatrix} -2 & 1 \\ 1 & -2 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right) = 0$$

Therefore

$$\det\left(\begin{bmatrix} -2-\lambda & 1 \\ 1 & -2-\lambda \end{bmatrix}\right) = 0$$

Which gives the characteristic equation

$$\lambda^2 + 4\lambda + 3 = 0$$

The roots of the above are the eigenvalues.

$$\lambda_1 = -3$$
$$\lambda_2 = -1$$

This table summarises the above result

eigenvalue	algebraic multiplicity	type of eigenvalue
-1	1	real eigenvalue
-3	1	real eigenvalue

Now the eigenvector for each eigenvalue are found.

Considering the eigenvalue $\lambda_1 = -3$

We need to solve $A\vec{v} = \lambda\vec{v}$ or $(A - \lambda I)\vec{v} = \vec{0}$ which becomes

$$\left(\begin{bmatrix} -2 & 1 \\ 1 & -2 \end{bmatrix} - (-3)\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right)\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$
$$\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Now forward elimination is applied to solve for the eigenvector $\vec{v}$. The augmented matrix is

$$\left[\begin{array}{cc|c} 1 & 1 & 0 \\ 1 & 1 & 0 \end{array}\right]$$

$$R_2 = R_2 - R_1 \Longrightarrow \left[\begin{array}{cc|c} 1 & 1 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

Therefore the system in Echelon form is

$$\begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

The free variables are $\{v_2\}$ and the leading variables are $\{v_1\}$. Let $v_2 = t$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\{v_1 = -t\}$

Hence the solution is

$$\begin{bmatrix} -t \\ t \end{bmatrix} = \begin{bmatrix} -t \\ t \end{bmatrix}$$

Since there is one free Variable, we have found one eigenvector associated with this eigenvalue. The above can be written as

$$\begin{bmatrix} -t \\ t \end{bmatrix} = t \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

Let $t = 1$ the eigenvector becomes

$$\begin{bmatrix} -t \\ t \end{bmatrix} = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

Considering the eigenvalue $\lambda_2 = -1$

We need to solve $A\vec{v} = \lambda\vec{v}$ or $(A - \lambda I)\vec{v} = \vec{0}$ which becomes

$$\left(\begin{bmatrix} -2 & 1 \\ 1 & -2 \end{bmatrix} - (-1) \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right) \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Now forward elimination is applied to solve for the eigenvector $\vec{v}$. The augmented matrix is

$$\left[\begin{array}{cc|c} -1 & 1 & 0 \\ 1 & -1 & 0 \end{array}\right]$$

$$R_2 = R_2 + R_1 \Longrightarrow \left[\begin{array}{cc|c} -1 & 1 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

Therefore the system in Echelon form is

$$\begin{bmatrix} -1 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

The free variables are $\{v_2\}$ and the leading variables are $\{v_1\}$. Let $v_2 = t$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\{v_1 = t\}$

Hence the solution is

$$\begin{bmatrix} t \\ t \end{bmatrix} = \begin{bmatrix} t \\ t \end{bmatrix}$$

Since there is one free Variable, we have found one eigenvector associated with this eigenvalue. The above can be written as

$$\begin{bmatrix} t \\ t \end{bmatrix} = t \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

Let $t = 1$ the eigenvector becomes

$$\begin{bmatrix} t \\ t \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

The following table gives a summary of this result. It shows for each eigenvalue the algebraic multiplicity m, and its geometric multiplicity k and the eigenvectors associated with the eigenvalue. If $m > k$ then the eigenvalue is defective which means the number of normal linearly independent eigenvectors associated with this eigenvalue (called the geometric multiplicity k) does not equal the algebraic multiplicity m, and we need to determine an additional $m - k$ generalized eigenvectors for this eigenvalue.

	multiplicity			
eigenvalue	algebraic m	geometric k	defective?	eigenvectors
-3	1	1	No	$\begin{bmatrix} -1 \\ 1 \end{bmatrix}$
-1	1	1	No	$\begin{bmatrix} 1 \\ 1 \end{bmatrix}$

Now that we found the eigenvalues and associated eigenvectors, we will go over each eigenvalue and generate the solution basis. The only problem we need to take care of is if the eigenvalue is defective. Since eigenvalue -3 is real and distinct then the corresponding eigenvector solution is

$$\begin{aligned} \vec{x}_1(t) &= \vec{v}_1 e^{-3t} \\ &= \begin{bmatrix} -1 \\ 1 \end{bmatrix} e^{-3t} \end{aligned}$$

Since eigenvalue -1 is real and distinct then the corresponding eigenvector solution is

$$\vec{x}_2(t) = \vec{v}_2 e^{-t}$$
$$= \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{-t}$$

Therefore the homogeneous solution is

$$\vec{x}_h(t) = c_1 \vec{x}_1(t) + c_2 \vec{x}_2(t)$$

Which is written as

$$\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = c_1 \begin{bmatrix} -\mathrm{e}^{-3t} \\ \mathrm{e}^{-3t} \end{bmatrix} + c_2 \begin{bmatrix} \mathrm{e}^{-t} \\ \mathrm{e}^{-t} \end{bmatrix}$$

Now that we found homogeneous solution above, we need to find a particular solution $\vec{x}_p(t)$. We will use Variation of parameters. The fundamental matrix is

$$\Phi = \begin{bmatrix} \vec{x}_1 & \vec{x}_2 & \cdots \end{bmatrix}$$

Where $\vec{x}_i$ are the solution basis found above. Therefore the fundamental matrix is

$$\Phi(t) = \begin{bmatrix} -\mathrm{e}^{-3t} & \mathrm{e}^{-t} \\ \mathrm{e}^{-3t} & \mathrm{e}^{-t} \end{bmatrix}$$

The particular solution is then given by

$$\vec{x}_p(t) = \Phi \int \Phi^{-1} \vec{G}(t) \, dt$$

But

$$\Phi^{-1} = \begin{bmatrix} -\frac{\mathrm{e}^{3t}}{2} & \frac{\mathrm{e}^{3t}}{2} \\ \frac{\mathrm{e}^{t}}{2} & \frac{\mathrm{e}^{t}}{2} \end{bmatrix}$$

Hence

$$
\begin{aligned}
\vec{x}_p(t) &= \begin{bmatrix} -\mathrm{e}^{-3t} & \mathrm{e}^{-t} \\ \mathrm{e}^{-3t} & \mathrm{e}^{-t} \end{bmatrix} \int \begin{bmatrix} -\frac{\mathrm{e}^{3t}}{2} & \frac{\mathrm{e}^{3t}}{2} \\ \frac{\mathrm{e}^{t}}{2} & \frac{\mathrm{e}^{t}}{2} \end{bmatrix} \begin{bmatrix} -2 \\ 1 \end{bmatrix} dt \\
&= \begin{bmatrix} -\mathrm{e}^{-3t} & \mathrm{e}^{-t} \\ \mathrm{e}^{-3t} & \mathrm{e}^{-t} \end{bmatrix} \int \begin{bmatrix} \frac{3\,\mathrm{e}^{3t}}{2} \\ -\frac{\mathrm{e}^{t}}{2} \end{bmatrix} dt \\
&= \begin{bmatrix} -\mathrm{e}^{-3t} & \mathrm{e}^{-t} \\ \mathrm{e}^{-3t} & \mathrm{e}^{-t} \end{bmatrix} \begin{bmatrix} \frac{\mathrm{e}^{3t}}{2} \\ -\frac{\mathrm{e}^{t}}{2} \end{bmatrix} \\
&= \begin{bmatrix} -1 \\ 0 \end{bmatrix}
\end{aligned}
$$

Now that we found particular solution, the final solution is

$$
\begin{aligned}
\vec{x}(t) &= \vec{x}_h(t) + \vec{x}_p(t) \\
\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} &= \begin{bmatrix} -c_1\mathrm{e}^{-3t} \\ c_1\mathrm{e}^{-3t} \end{bmatrix} + \begin{bmatrix} c_2\mathrm{e}^{-t} \\ c_2\mathrm{e}^{-t} \end{bmatrix} + \begin{bmatrix} -1 \\ 0 \end{bmatrix}
\end{aligned}
$$

Which becomes

$$
\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = \begin{bmatrix} -c_1\mathrm{e}^{-3t} + c_2\mathrm{e}^{-t} - 1 \\ c_1\mathrm{e}^{-3t} + c_2\mathrm{e}^{-t} \end{bmatrix}
$$

The following is the phase plot of the system.

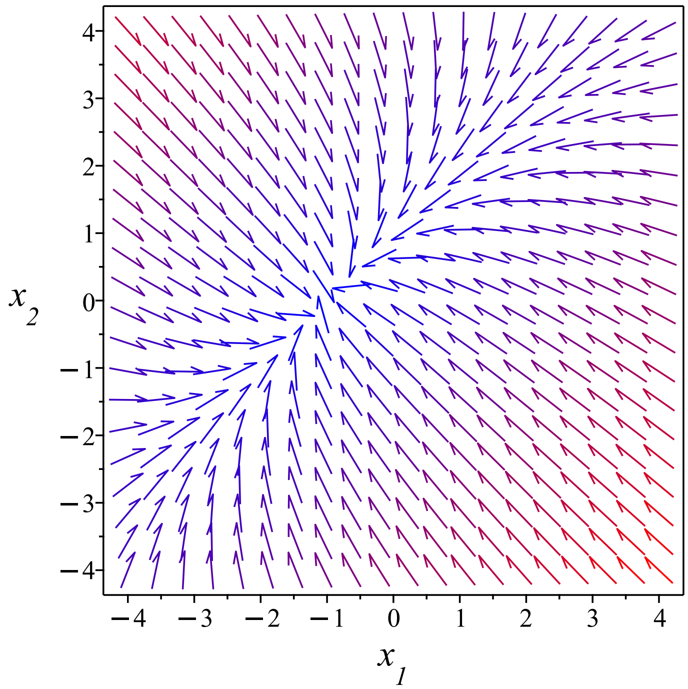


Figure 555: Phase plot

19.14.3 Maple step by step solution

Let's solve

$[x_1'(t) = -2x_1(t) + x_2(t) - 2, x_2'(t) = x_1(t) - 2x_2(t) + 1]$

- Define vector

$$\underline{x}^{\rightarrow}(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}$$

- Convert system into a vector equation

$$\underline{x}^{\rightarrow}{}'(t) = \begin{bmatrix} -2 & 1 \\ 1 & -2 \end{bmatrix} \cdot \underline{x}^{\rightarrow}(t) + \begin{bmatrix} -2 \\ 1 \end{bmatrix}$$

- System to solve

$$\underline{x}^{\rightarrow}{}'(t) = \begin{bmatrix} -2 & 1 \\ 1 & -2 \end{bmatrix} \cdot \underline{x}^{\rightarrow}(t) + \begin{bmatrix} -2 \\ 1 \end{bmatrix}$$

- Define the forcing function

$$\vec{f}(t) = \begin{bmatrix} -2 \\ 1 \end{bmatrix}$$

- Define the coefficient matrix

$$A = \begin{bmatrix} -2 & 1 \\ 1 & -2 \end{bmatrix}$$

- Rewrite the system as

$$\underline{x}^{\rightarrow\prime}(t) = A \cdot \underline{x}^{\rightarrow}(t) + \vec{f}$$

- To solve the system, find the eigenvalues and eigenvectors of A
- Eigenpairs of A

$$\left[\left[-3, \begin{bmatrix} -1 \\ 1 \end{bmatrix}\right], \left[-1, \begin{bmatrix} 1 \\ 1 \end{bmatrix}\right]\right]$$

- Consider eigenpair

$$\left[-3, \begin{bmatrix} -1 \\ 1 \end{bmatrix}\right]$$

- Solution to homogeneous system from eigenpair

$$\underline{x}^{\rightarrow}_{1} = e^{-3t} \cdot \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

- Consider eigenpair

$$\left[-1, \begin{bmatrix} 1 \\ 1 \end{bmatrix}\right]$$

- Solution to homogeneous system from eigenpair

$$\underline{x}^{\rightarrow}_{2} = e^{-t} \cdot \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

- General solution of the system of ODEs can be written in terms of the particular solution $\underline{x}^{\rightarrow}$

$$\underline{x}^{\rightarrow}(t) = c_1 \underline{x}^{\rightarrow}_{1} + c_2 \underline{x}^{\rightarrow}_{2} + \underline{x}^{\rightarrow}_{p}(t)$$

☐ Fundamental matrix

  - Let $\phi(t)$ be the matrix whose columns are the independent solutions of the homogeneous syst

$$\phi(t) = \begin{bmatrix} -e^{-3t} & e^{-t} \\ e^{-3t} & e^{-t} \end{bmatrix}$$

- The fundamental matrix, $\Phi(t)$ is a normalized version of $\phi(t)$ satisfying $\Phi(0) = I$ where I is th

  $\Phi(t) = \phi(t) \cdot \frac{1}{\phi(0)}$

- Substitute the value of $\phi(t)$ and $\phi(0)$

  $$\Phi(t) = \begin{bmatrix} -e^{-3t} & e^{-t} \\ e^{-3t} & e^{-t} \end{bmatrix} \cdot \frac{1}{\begin{bmatrix} -1 & 1 \\ 1 & 1 \end{bmatrix}}$$

- Evaluate and simplify to get the fundamental matrix

  $$\Phi(t) = \begin{bmatrix} \frac{e^{-3t}}{2} + \frac{e^{-t}}{2} & \frac{e^{-t}}{2} - \frac{e^{-3t}}{2} \\ \frac{e^{-t}}{2} - \frac{e^{-3t}}{2} & \frac{e^{-3t}}{2} + \frac{e^{-t}}{2} \end{bmatrix}$$

☐ Find a particular solution of the system of ODEs using variation of parameters

- Let the particular solution be the fundamental matrix multiplied by $\vec{v}(t)$ and solve for $\vec{v}(t)$

  $\vec{x}_p(t) = \Phi(t) \cdot \vec{v}(t)$

- Take the derivative of the particular solution

  $\vec{x}'_p(t) = \Phi'(t) \cdot \vec{v}(t) + \Phi(t) \cdot \vec{v}'(t)$

- Substitute particular solution and its derivative into the system of ODEs

  $\Phi'(t) \cdot \vec{v}(t) + \Phi(t) \cdot \vec{v}'(t) = A \cdot \Phi(t) \cdot \vec{v}(t) + \vec{f}(t)$

- The fundamental matrix has columns that are solutions to the homogeneous system so its der

  $A \cdot \Phi(t) \cdot \vec{v}(t) + \Phi(t) \cdot \vec{v}'(t) = A \cdot \Phi(t) \cdot \vec{v}(t) + \vec{f}(t)$

- Cancel like terms

  $\Phi(t) \cdot \vec{v}'(t) = \vec{f}(t)$

- Multiply by the inverse of the fundamental matrix

  $\vec{v}'(t) = \frac{1}{\Phi(t)} \cdot \vec{f}(t)$

- Integrate to solve for $\vec{v}(t)$

  $\vec{v}(t) = \int_0^t \frac{1}{\Phi(s)} \cdot \vec{f}(s)\, ds$

- Plug $\vec{v}(t)$ into the equation for the particular solution

$$\underline{x}^{\rightarrow}_p(t) = \Phi(t) \cdot \left(\int_0^t \frac{1}{\Phi(s)} \cdot \vec{f}(s)\, ds\right)$$

- Plug in the fundamental matrix and the forcing function and compute

$$\underline{x}^{\rightarrow}_p(t) = \left[\begin{array}{c} \frac{e^{-3t}}{2} - 1 + \frac{e^{-t}}{2} \\ \frac{e^{-t}}{2} - \frac{e^{-3t}}{2} \end{array}\right]$$

- Plug particular solution back into general solution

$$\underline{x}^{\rightarrow}(t) = c_1 \underline{x}^{\rightarrow}_1 + c_2 \underline{x}^{\rightarrow}_2 + \left[\begin{array}{c} \frac{e^{-3t}}{2} - 1 + \frac{e^{-t}}{2} \\ \frac{e^{-t}}{2} - \frac{e^{-3t}}{2} \end{array}\right]$$

- Substitute in vector of dependent variables

$$\left[\begin{array}{c} x_1(t) \\ x_2(t) \end{array}\right] = \left[\begin{array}{c} \frac{(-2c_1+1)e^{-3t}}{2} - 1 + \frac{(2c_2+1)e^{-t}}{2} \\ \frac{(-1+2c_1)e^{-3t}}{2} + \frac{(2c_2+1)e^{-t}}{2} \end{array}\right]$$

- Solution to the system of ODEs

$$\left\{x_1(t) = \frac{(-2c_1+1)e^{-3t}}{2} - 1 + \frac{(2c_2+1)e^{-t}}{2}, x_2(t) = \frac{(-1+2c_1)e^{-3t}}{2} + \frac{(2c_2+1)e^{-t}}{2}\right\}$$

✓ Solution by Maple

Time used: 0.015 (sec). Leaf size: 36

```
dsolve([diff(x__1(t),t)=-2*x__1(t)+1*x__2(t)-2,diff(x__2(t),t)=1*x__1(t)-2*x__2(t)+1],singso
```

$$x_1(t) = c_2 e^{-t} + c_1 e^{-3t} - 1$$
$$x_2(t) = c_2 e^{-t} - c_1 e^{-3t}$$

✓ Solution by Mathematica

Time used: 0.022 (sec). Leaf size: 72

```
DSolve[{x1'[t]==-2*x1[t]+1*x2[t]-2,x2'[t]==1*x1[t]-2*x2[t]+1},{x1[t],x2[t]},t,IncludeSingula
```

$$\text{x1}(t) \to \frac{1}{2} e^{-3t}\left(-2e^{3t} + (c_1 + c_2)e^{2t} + c_1 - c_2\right)$$
$$\text{x2}(t) \to \frac{1}{2} e^{-3t}\left(c_1\left(e^{2t} - 1\right) + c_2\left(e^{2t} + 1\right)\right)$$

19.15 problem 15

Internal problem ID [806]
Internal file name [`OUTPUT/806_Sunday_June_05_2022_01_50_15_AM_35575306/index.tex`]

Book: Elementary differential equations and boundary value problems, 10th ed., Boyce and DiPrima
Section: Chapter 9.1, The Phase Plane: Linear Systems. page 505
Problem number: 15.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"system of linear ODEs"**

Solve

$$x_1'(t) = -x_1(t) - x_2(t) - 1$$
$$x_2'(t) = 2x_1(t) - x_2(t) + 5$$

19.15.1 Solution using Matrix exponential method

In this method, we will assume we have found the matrix exponential e^{At} allready. There are different methods to determine this but will not be shown here. This is a system of linear ODE's given as

$$\vec{x}'(t) = A\,\vec{x}(t) + \vec{G}(t)$$

Or

$$\begin{bmatrix} x_1'(t) \\ x_2'(t) \end{bmatrix} = \begin{bmatrix} -1 & -1 \\ 2 & -1 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} + \begin{bmatrix} -1 \\ 5 \end{bmatrix}$$

Since the system is nonhomogeneous, then the solution is given by

$$\vec{x}(t) = \vec{x}_h(t) + \vec{x}_p(t)$$

Where $\vec{x}_h(t)$ is the homogeneous solution to $\vec{x}'(t) = A\,\vec{x}(t)$ and $\vec{x}_p(t)$ is a particular solution to $\vec{x}'(t) = A\,\vec{x}(t) + \vec{G}(t)$. The particular solution will be found using variation

of parameters method applied to the fundamental matrix. For the above matrix A, the matrix exponential can be found to be

$$e^{At} = \begin{bmatrix} e^{-t}\cos\left(\sqrt{2}\, t\right) & -\frac{\sqrt{2}\, e^{-t}\sin\left(\sqrt{2}\, t\right)}{2} \\ \sqrt{2}\, e^{-t}\sin\left(\sqrt{2}\, t\right) & e^{-t}\cos\left(\sqrt{2}\, t\right) \end{bmatrix}$$

Therefore the homogeneous solution is

$$\begin{aligned}
\vec{x}_h(t) &= e^{At}\vec{c} \\
&= \begin{bmatrix} e^{-t}\cos\left(\sqrt{2}\, t\right) & -\frac{\sqrt{2}\, e^{-t}\sin\left(\sqrt{2}\, t\right)}{2} \\ \sqrt{2}\, e^{-t}\sin\left(\sqrt{2}\, t\right) & e^{-t}\cos\left(\sqrt{2}\, t\right) \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} \\
&= \begin{bmatrix} e^{-t}\cos\left(\sqrt{2}\, t\right) c_1 - \frac{\sqrt{2}\, e^{-t}\sin\left(\sqrt{2}\, t\right) c_2}{2} \\ \sqrt{2}\, e^{-t}\sin\left(\sqrt{2}\, t\right) c_1 + e^{-t}\cos\left(\sqrt{2}\, t\right) c_2 \end{bmatrix} \\
&= \begin{bmatrix} e^{-t}\cos\left(\sqrt{2}\, t\right) c_1 - \frac{\sqrt{2}\, e^{-t}\sin\left(\sqrt{2}\, t\right) c_2}{2} \\ e^{-t}\left(\sqrt{2}\, \sin\left(\sqrt{2}\, t\right) c_1 + \cos\left(\sqrt{2}\, t\right) c_2\right) \end{bmatrix}
\end{aligned}$$

The particular solution given by

$$\vec{x}_p(t) = e^{At} \int e^{-At}\vec{G}(t)\, dt$$

But

$$\begin{aligned}
e^{-At} &= \left(e^{At}\right)^{-1} \\
&= \begin{bmatrix} \cos\left(\sqrt{2}\, t\right) e^{t} & \frac{\sin\left(\sqrt{2}\, t\right)\sqrt{2}\, e^{t}}{2} \\ -\sin\left(\sqrt{2}\, t\right)\sqrt{2}\, e^{t} & \cos\left(\sqrt{2}\, t\right) e^{t} \end{bmatrix}
\end{aligned}$$

Hence

$$\begin{aligned}
\vec{x}_p(t) &= \begin{bmatrix} e^{-t}\cos\left(\sqrt{2}\, t\right) & -\frac{\sqrt{2}\, e^{-t}\sin\left(\sqrt{2}\, t\right)}{2} \\ \sqrt{2}\, e^{-t}\sin\left(\sqrt{2}\, t\right) & e^{-t}\cos\left(\sqrt{2}\, t\right) \end{bmatrix} \int \begin{bmatrix} \cos\left(\sqrt{2}\, t\right) e^{t} & \frac{\sin\left(\sqrt{2}\, t\right)\sqrt{2}\, e^{t}}{2} \\ -\sin\left(\sqrt{2}\, t\right)\sqrt{2}\, e^{t} & \cos\left(\sqrt{2}\, t\right) e^{t} \end{bmatrix} \begin{bmatrix} -1 \\ 5 \end{bmatrix} dt \\
&= \begin{bmatrix} e^{-t}\cos\left(\sqrt{2}\, t\right) & -\frac{\sqrt{2}\, e^{-t}\sin\left(\sqrt{2}\, t\right)}{2} \\ \sqrt{2}\, e^{-t}\sin\left(\sqrt{2}\, t\right) & e^{-t}\cos\left(\sqrt{2}\, t\right) \end{bmatrix} \begin{bmatrix} \frac{e^{t}\left(\sqrt{2}\, \sin\left(\sqrt{2}\, t\right) - 4\cos\left(\sqrt{2}\, t\right)\right)}{2} \\ e^{t}\left(2\sqrt{2}\, \sin\left(\sqrt{2}\, t\right) + \cos\left(\sqrt{2}\, t\right)\right) \end{bmatrix} \\
&= \begin{bmatrix} -2 \\ 1 \end{bmatrix}
\end{aligned}$$

Hence the complete solution is

$$
\begin{aligned}
\vec{x}(t) &= \vec{x}_h(t) + \vec{x}_p(t) \\
&= \begin{bmatrix} \mathrm{e}^{-t}\cos\left(\sqrt{2}\,t\right)c_1 - \frac{\sqrt{2}\,\mathrm{e}^{-t}\sin\left(\sqrt{2}\,t\right)c_2}{2} - 2 \\ \sqrt{2}\,\mathrm{e}^{-t}\sin\left(\sqrt{2}\,t\right)c_1 + \mathrm{e}^{-t}\cos\left(\sqrt{2}\,t\right)c_2 + 1 \end{bmatrix}
\end{aligned}
$$

19.15.2 Solution using explicit Eigenvalue and Eigenvector method

This is a system of linear ODE's given as

$$\vec{x}'(t) = A\,\vec{x}(t) + \vec{G}(t)$$

Or

$$\begin{bmatrix} x_1'(t) \\ x_2'(t) \end{bmatrix} = \begin{bmatrix} -1 & -1 \\ 2 & -1 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} + \begin{bmatrix} -1 \\ 5 \end{bmatrix}$$

Since the system is nonhomogeneous, then the solution is given by

$$\vec{x}(t) = \vec{x}_h(t) + \vec{x}_p(t)$$

Where $\vec{x}_h(t)$ is the homogeneous solution to $\vec{x}'(t) = A\,\vec{x}(t)$ and $\vec{x}_p(t)$ is a particular solution to $\vec{x}'(t) = A\,\vec{x}(t) + \vec{G}(t)$. The particular solution will be found using variation of parameters method applied to the fundamental matrix.

The first step is find the homogeneous solution. We start by finding the eigenvalues of A. This is done by solving the following equation for the eigenvalues λ

$$\det\left(A - \lambda I\right) = 0$$

Expanding gives

$$\det\left(\begin{bmatrix} -1 & -1 \\ 2 & -1 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right) = 0$$

Therefore

$$\det\left(\begin{bmatrix} -1-\lambda & -1 \\ 2 & -1-\lambda \end{bmatrix}\right) = 0$$

Which gives the characteristic equation

$$\lambda^2 + 2\lambda + 3 = 0$$

The roots of the above are the eigenvalues.

$$\lambda_1 = -1 + i\sqrt{2}$$
$$\lambda_2 = -1 - i\sqrt{2}$$

This table summarises the above result

eigenvalue	algebraic multiplicity	type of eigenvalue
$-1 + i\sqrt{2}$	1	complex eigenvalue
$-1 - i\sqrt{2}$	1	complex eigenvalue

Now the eigenvector for each eigenvalue are found.

Considering the eigenvalue $\lambda_1 = -1 - i\sqrt{2}$

We need to solve $A\vec{v} = \lambda\vec{v}$ or $(A - \lambda I)\vec{v} = \vec{0}$ which becomes

$$\left(\begin{bmatrix} -1 & -1 \\ 2 & -1 \end{bmatrix} - \left(-1 - i\sqrt{2}\right)\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right)\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$
$$\begin{bmatrix} i\sqrt{2} & -1 \\ 2 & i\sqrt{2} \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Now forward elimination is applied to solve for the eigenvector $\vec{v}$. The augmented matrix is

$$\left[\begin{array}{cc|c} i\sqrt{2} & -1 & 0 \\ 2 & i\sqrt{2} & 0 \end{array}\right]$$

$$R_2 = R_2 + i\sqrt{2}\,R_1 \Longrightarrow \left[\begin{array}{cc|c} i\sqrt{2} & -1 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

Therefore the system in Echelon form is

$$\begin{bmatrix} i\sqrt{2} & -1 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

The free variables are $\{v_2\}$ and the leading variables are $\{v_1\}$. Let $v_2 = t$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\left\{v_1 = -\frac{it\sqrt{2}}{2}\right\}$

Hence the solution is

$$\begin{bmatrix} -\frac{\mathrm{I}}{2}t\sqrt{2} \\ t \end{bmatrix} = \begin{bmatrix} -\frac{it\sqrt{2}}{2} \\ t \end{bmatrix}$$

Since there is one free Variable, we have found one eigenvector associated with this eigenvalue. The above can be written as

$$\begin{bmatrix} -\frac{\mathrm{I}}{2}t\sqrt{2} \\ t \end{bmatrix} = t \begin{bmatrix} -\frac{i\sqrt{2}}{2} \\ 1 \end{bmatrix}$$

Let $t = 1$ the eigenvector becomes

$$\begin{bmatrix} -\frac{\mathrm{I}}{2}t\sqrt{2} \\ t \end{bmatrix} = \begin{bmatrix} -\frac{i\sqrt{2}}{2} \\ 1 \end{bmatrix}$$

Which is normalized to

$$\begin{bmatrix} -\frac{\mathrm{I}}{2}t\sqrt{2} \\ t \end{bmatrix} = \begin{bmatrix} -i\sqrt{2} \\ 2 \end{bmatrix}$$

Considering the eigenvalue $\lambda_2 = -1 + i\sqrt{2}$

We need to solve $A\vec{v} = \lambda\vec{v}$ or $(A - \lambda I)\vec{v} = \vec{0}$ which becomes

$$\left(\begin{bmatrix} -1 & -1 \\ 2 & -1 \end{bmatrix} - \left(-1 + i\sqrt{2}\right) \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right) \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} -i\sqrt{2} & -1 \\ 2 & -i\sqrt{2} \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Now forward elimination is applied to solve for the eigenvector $\vec{v}$. The augmented matrix is

$$\left[\begin{array}{cc|c} -i\sqrt{2} & -1 & 0 \\ 2 & -i\sqrt{2} & 0 \end{array}\right]$$

$$R_2 = R_2 - i\sqrt{2}\, R_1 \Longrightarrow \left[\begin{array}{cc|c} -i\sqrt{2} & -1 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

Therefore the system in Echelon form is

$$\begin{bmatrix} -i\sqrt{2} & -1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

The free variables are $\{v_2\}$ and the leading variables are $\{v_1\}$. Let $v_2 = t$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\left\{v_1 = \frac{it\sqrt{2}}{2}\right\}$

Hence the solution is

$$\begin{bmatrix} \frac{I}{2}t\sqrt{2} \\ t \end{bmatrix} = \begin{bmatrix} \frac{it\sqrt{2}}{2} \\ t \end{bmatrix}$$

Since there is one free Variable, we have found one eigenvector associated with this eigenvalue. The above can be written as

$$\begin{bmatrix} \frac{I}{2}t\sqrt{2} \\ t \end{bmatrix} = t \begin{bmatrix} \frac{i\sqrt{2}}{2} \\ 1 \end{bmatrix}$$

Let $t = 1$ the eigenvector becomes

$$\begin{bmatrix} \frac{I}{2}t\sqrt{2} \\ t \end{bmatrix} = \begin{bmatrix} \frac{i\sqrt{2}}{2} \\ 1 \end{bmatrix}$$

Which is normalized to

$$\begin{bmatrix} \frac{I}{2}t\sqrt{2} \\ t \end{bmatrix} = \begin{bmatrix} i\sqrt{2} \\ 2 \end{bmatrix}$$

The following table gives a summary of this result. It shows for each eigenvalue the algebraic multiplicity m, and its geometric multiplicity k and the eigenvectors associated with the eigenvalue. If $m > k$ then the eigenvalue is defective which means the number of normal linearly independent eigenvectors associated with this eigenvalue (called the geometric multiplicity k) does not equal the algebraic multiplicity m, and we need to determine an additional $m - k$ generalized eigenvectors for this eigenvalue.

eigenvalue	multiplicity: algebraic m	multiplicity: geometric k	defective?	eigenvectors
$-1+i\sqrt{2}$	1	1	No	$\left[\begin{array}{c} \frac{i\sqrt{2}}{2} \\ 1 \end{array}\right]$
$-1-i\sqrt{2}$	1	1	No	$\left[\begin{array}{c} -\frac{i\sqrt{2}}{2} \\ 1 \end{array}\right]$

Now that we found the eigenvalues and associated eigenvectors, we will go over each eigenvalue and generate the solution basis. The only problem we need to take care of is if the eigenvalue is defective. Therefore the homogeneous solution is

$$\vec{x}_h(t) = c_1\vec{x}_1(t) + c_2\vec{x}_2(t)$$

Which is written as

$$\left[\begin{array}{c} x_1(t) \\ x_2(t) \end{array}\right] = c_1 \left[\begin{array}{c} \frac{i\sqrt{2}\,\mathrm{e}^{\left(-1+i\sqrt{2}\right)t}}{2} \\ \mathrm{e}^{\left(-1+i\sqrt{2}\right)t} \end{array}\right] + c_2 \left[\begin{array}{c} -\frac{i\mathrm{e}^{\left(-1-i\sqrt{2}\right)t}\sqrt{2}}{2} \\ \mathrm{e}^{\left(-1-i\sqrt{2}\right)t} \end{array}\right]$$

Now that we found homogeneous solution above, we need to find a particular solution $\vec{x}_p(t)$. We will use Variation of parameters. The fundamental matrix is

$$\Phi = \left[\vec{x}_1 \quad \vec{x}_2 \quad \cdots\right]$$

Where $\vec{x}_i$ are the solution basis found above. Therefore the fundamental matrix is

$$\Phi(t) = \left[\begin{array}{cc} \frac{i\sqrt{2}\,\mathrm{e}^{\left(-1+i\sqrt{2}\right)t}}{2} & -\frac{i\mathrm{e}^{\left(-1-i\sqrt{2}\right)t}\sqrt{2}}{2} \\ \mathrm{e}^{\left(-1+i\sqrt{2}\right)t} & \mathrm{e}^{\left(-1-i\sqrt{2}\right)t} \end{array}\right]$$

The particular solution is then given by

$$\vec{x}_p(t) = \Phi \int \Phi^{-1}\vec{G}(t)\,dt$$

But

$$\Phi^{-1} = \left[\begin{array}{cc} -\frac{i\sqrt{2}\,\mathrm{e}^{-\left(-1+i\sqrt{2}\right)t}}{2} & \frac{\mathrm{e}^{-\left(-1+i\sqrt{2}\right)t}}{2} \\ \frac{i\sqrt{2}\,\mathrm{e}^{\left(1+i\sqrt{2}\right)t}}{2} & \frac{\mathrm{e}^{\left(1+i\sqrt{2}\right)t}}{2} \end{array}\right]$$

Hence

$$\vec{x}_p(t) = \begin{bmatrix} \frac{i\sqrt{2}\,\mathrm{e}^{\left(-1+i\sqrt{2}\right)t}}{2} & -\frac{i\mathrm{e}^{\left(-1-i\sqrt{2}\right)t}\sqrt{2}}{2} \\ \mathrm{e}^{\left(-1+i\sqrt{2}\right)t} & \mathrm{e}^{\left(-1-i\sqrt{2}\right)t} \end{bmatrix} \int \begin{bmatrix} -\frac{i\sqrt{2}\,\mathrm{e}^{-\left(-1+i\sqrt{2}\right)t}}{2} & \frac{\mathrm{e}^{-\left(-1+i\sqrt{2}\right)t}}{2} \\ \frac{i\sqrt{2}\,\mathrm{e}^{\left(1+i\sqrt{2}\right)t}}{2} & \frac{\mathrm{e}^{\left(1+i\sqrt{2}\right)t}}{2} \end{bmatrix} \begin{bmatrix} -1 \\ 5 \end{bmatrix} dt$$

$$= \begin{bmatrix} \frac{i\sqrt{2}\,\mathrm{e}^{\left(-1+i\sqrt{2}\right)t}}{2} & -\frac{i\mathrm{e}^{\left(-1-i\sqrt{2}\right)t}\sqrt{2}}{2} \\ \mathrm{e}^{\left(-1+i\sqrt{2}\right)t} & \mathrm{e}^{\left(-1-i\sqrt{2}\right)t} \end{bmatrix} \int \begin{bmatrix} \frac{\mathrm{e}^{-\left(-1+i\sqrt{2}\right)t}\left(i\sqrt{2}+5\right)}{2} \\ -\frac{\mathrm{e}^{\left(1+i\sqrt{2}\right)t}\left(i\sqrt{2}-5\right)}{2} \end{bmatrix} dt$$

$$= \begin{bmatrix} \frac{i\sqrt{2}\,\mathrm{e}^{\left(-1+i\sqrt{2}\right)t}}{2} & -\frac{i\mathrm{e}^{\left(-1-i\sqrt{2}\right)t}\sqrt{2}}{2} \\ \mathrm{e}^{\left(-1+i\sqrt{2}\right)t} & \mathrm{e}^{\left(-1-i\sqrt{2}\right)t} \end{bmatrix} \begin{bmatrix} \frac{\left(1+i\sqrt{2}\right)\mathrm{e}^{-\left(-1+i\sqrt{2}\right)t}\left(i\sqrt{2}+5\right)}{6} \\ -\frac{\mathrm{e}^{\left(1+i\sqrt{2}\right)t}\left(i+\sqrt{2}\right)\left(\sqrt{2}+5i\right)}{6} \end{bmatrix}$$

$$= \begin{bmatrix} -2 \\ 1 \end{bmatrix}$$

Now that we found particular solution, the final solution is

$$\vec{x}(t) = \vec{x}_h(t) + \vec{x}_p(t)$$

$$\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = \begin{bmatrix} \frac{ic_1\sqrt{2}\,\mathrm{e}^{\left(-1+i\sqrt{2}\right)t}}{2} \\ c_1\mathrm{e}^{\left(-1+i\sqrt{2}\right)t} \end{bmatrix} + \begin{bmatrix} -\frac{ic_2\mathrm{e}^{\left(-1-i\sqrt{2}\right)t}\sqrt{2}}{2} \\ c_2\mathrm{e}^{\left(-1-i\sqrt{2}\right)t} \end{bmatrix} + \begin{bmatrix} -2 \\ 1 \end{bmatrix}$$

Which becomes

$$\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = \begin{bmatrix} \frac{ic_1\sqrt{2}\,\mathrm{e}^{\left(-1+i\sqrt{2}\right)t}}{2} - \frac{ic_2\mathrm{e}^{-\left(1+i\sqrt{2}\right)t}\sqrt{2}}{2} - 2 \\ c_1\mathrm{e}^{\left(-1+i\sqrt{2}\right)t} + c_2\mathrm{e}^{-\left(1+i\sqrt{2}\right)t} + 1 \end{bmatrix}$$

The following is the phase plot of the system.

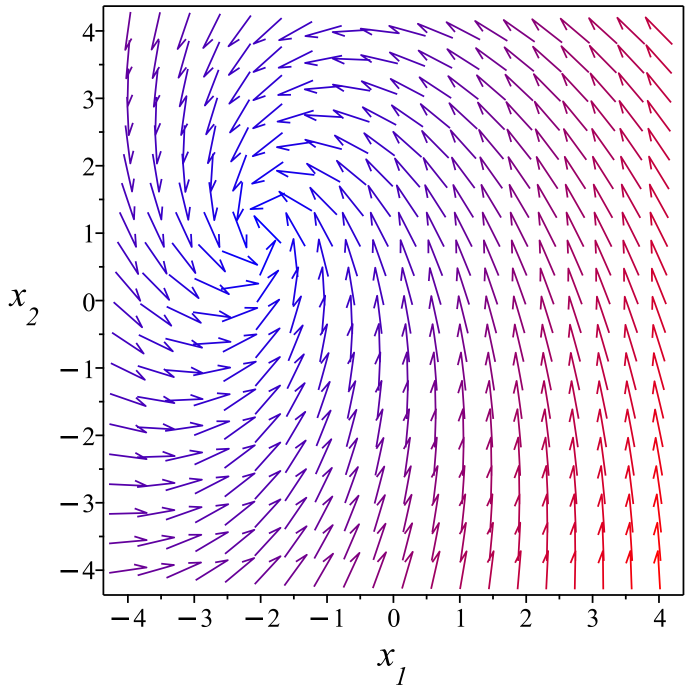


Figure 556: Phase plot

19.15.3 Maple step by step solution

Let's solve

$[x_1'(t) = -x_1(t) - x_2(t) - 1, x_2'(t) = 2x_1(t) - x_2(t) + 5]$

- Define vector

$$\underline{\vec{x}}(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}$$

- Convert system into a vector equation

$$\underline{\vec{x}}'(t) = \begin{bmatrix} -1 & -1 \\ 2 & -1 \end{bmatrix} \cdot \underline{\vec{x}}(t) + \begin{bmatrix} -1 \\ 5 \end{bmatrix}$$

- System to solve

$$\underline{\vec{x}}'(t) = \begin{bmatrix} -1 & -1 \\ 2 & -1 \end{bmatrix} \cdot \underline{\vec{x}}(t) + \begin{bmatrix} -1 \\ 5 \end{bmatrix}$$

- Define the forcing function

$$\vec{f}(t) = \begin{bmatrix} -1 \\ 5 \end{bmatrix}$$

- Define the coefficient matrix

$$A = \begin{bmatrix} -1 & -1 \\ 2 & -1 \end{bmatrix}$$

- Rewrite the system as

$$\underline{x}^{\rightarrow}{}'(t) = A \cdot \underline{x}^{\rightarrow}(t) + \vec{f}$$

- To solve the system, find the eigenvalues and eigenvectors of A
- Eigenpairs of A

$$\left[\left[-1 - \mathrm{I}\sqrt{2}, \begin{bmatrix} -\frac{\mathrm{I}}{2}\sqrt{2} \\ 1 \end{bmatrix}\right], \left[-1 + \mathrm{I}\sqrt{2}, \begin{bmatrix} \frac{\mathrm{I}}{2}\sqrt{2} \\ 1 \end{bmatrix}\right]\right]$$

- Consider complex eigenpair, complex conjugate eigenvalue can be ignored

$$\left[-1 - \mathrm{I}\sqrt{2}, \begin{bmatrix} -\frac{\mathrm{I}}{2}\sqrt{2} \\ 1 \end{bmatrix}\right]$$

- Solution from eigenpair

$$\mathrm{e}^{\left(-1-\mathrm{I}\sqrt{2}\right)t} \cdot \begin{bmatrix} -\frac{\mathrm{I}}{2}\sqrt{2} \\ 1 \end{bmatrix}$$

- Use Euler identity to write solution in terms of $\sin$ and $\cos$

$$\mathrm{e}^{-t} \cdot \left(\cos\left(\sqrt{2}\,t\right) - \mathrm{I}\sin\left(\sqrt{2}\,t\right)\right) \cdot \begin{bmatrix} -\frac{\mathrm{I}}{2}\sqrt{2} \\ 1 \end{bmatrix}$$

- Simplify expression

$$\mathrm{e}^{-t} \cdot \begin{bmatrix} -\frac{\mathrm{I}}{2}\left(\cos\left(\sqrt{2}\,t\right) - \mathrm{I}\sin\left(\sqrt{2}\,t\right)\right)\sqrt{2} \\ \cos\left(\sqrt{2}\,t\right) - \mathrm{I}\sin\left(\sqrt{2}\,t\right) \end{bmatrix}$$

- Both real and imaginary parts are solutions to the homogeneous system

$$\left[\underline{x}^{\rightarrow}{}_{1}(t) = \mathrm{e}^{-t} \cdot \begin{bmatrix} -\frac{\sqrt{2}\,\sin\left(\sqrt{2}\,t\right)}{2} \\ \cos\left(\sqrt{2}\,t\right) \end{bmatrix}, \underline{x}^{\rightarrow}{}_{2}(t) = \mathrm{e}^{-t} \cdot \begin{bmatrix} -\frac{\cos\left(\sqrt{2}\,t\right)\sqrt{2}}{2} \\ -\sin\left(\sqrt{2}\,t\right) \end{bmatrix}\right]$$

- General solution of the system of ODEs can be written in terms of the particular solution $\underline{\vec{x}}$
  $\underline{\vec{x}}(t) = c_1 \underline{\vec{x}}_1(t) + c_2 \underline{\vec{x}}_2(t) + \underline{\vec{x}}_p(t)$

☐ Fundamental matrix

- Let $\phi(t)$ be the matrix whose columns are the independent solutions of the homogeneous syst

$$\phi(t) = \begin{bmatrix} -\frac{\sqrt{2}\,\mathrm{e}^{-t}\sin(\sqrt{2}\,t)}{2} & -\frac{\sqrt{2}\,\mathrm{e}^{-t}\cos(\sqrt{2}\,t)}{2} \\ \mathrm{e}^{-t}\cos(\sqrt{2}\,t) & -\mathrm{e}^{-t}\sin(\sqrt{2}\,t) \end{bmatrix}$$

- The fundamental matrix, $\Phi(t)$ is a normalized version of $\phi(t)$ satisfying $\Phi(0) = I$ where I is th

$$\Phi(t) = \phi(t) \cdot \frac{1}{\phi(0)}$$

- Substitute the value of $\phi(t)$ and $\phi(0)$

$$\Phi(t) = \begin{bmatrix} -\frac{\sqrt{2}\,\mathrm{e}^{-t}\sin(\sqrt{2}\,t)}{2} & -\frac{\sqrt{2}\,\mathrm{e}^{-t}\cos(\sqrt{2}\,t)}{2} \\ \mathrm{e}^{-t}\cos(\sqrt{2}\,t) & -\mathrm{e}^{-t}\sin(\sqrt{2}\,t) \end{bmatrix} \cdot \frac{1}{\begin{bmatrix} 0 & -\frac{\sqrt{2}}{2} \\ 1 & 0 \end{bmatrix}}$$

- Evaluate and simplify to get the fundamental matrix

$$\Phi(t) = \begin{bmatrix} \mathrm{e}^{-t}\cos(\sqrt{2}\,t) & -\frac{\sqrt{2}\,\mathrm{e}^{-t}\sin(\sqrt{2}\,t)}{2} \\ \sqrt{2}\,\mathrm{e}^{-t}\sin(\sqrt{2}\,t) & \mathrm{e}^{-t}\cos(\sqrt{2}\,t) \end{bmatrix}$$

☐ Find a particular solution of the system of ODEs using variation of parameters

- Let the particular solution be the fundamental matrix multiplied by $\vec{v}(t)$ and solve for $\vec{v}(t)$

$$\underline{\vec{x}}_p(t) = \Phi(t) \cdot \vec{v}(t)$$

- Take the derivative of the particular solution

$$\underline{\vec{x}}_p'(t) = \Phi'(t) \cdot \vec{v}(t) + \Phi(t) \cdot \vec{v}'(t)$$

- Substitute particular solution and its derivative into the system of ODEs

$$\Phi'(t) \cdot \vec{v}(t) + \Phi(t) \cdot \vec{v}'(t) = A \cdot \Phi(t) \cdot \vec{v}(t) + \vec{f}(t)$$

- The fundamental matrix has columns that are solutions to the homogeneous system so its der

$$A \cdot \Phi(t) \cdot \vec{v}(t) + \Phi(t) \cdot \vec{v}'(t) = A \cdot \Phi(t) \cdot \vec{v}(t) + \vec{f}(t)$$

- Cancel like terms

$$\Phi(t) \cdot \vec{v}'(t) = \vec{f}(t)$$

- Multiply by the inverse of the fundamental matrix

$$\vec{v}'(t) = \frac{1}{\Phi(t)} \cdot \vec{f}(t)$$

- Integrate to solve for $\vec{v}(t)$

$$\vec{v}(t) = \int_0^t \frac{1}{\Phi(s)} \cdot \vec{f}(s)\, ds$$

- Plug $\vec{v}(t)$ into the equation for the particular solution

$$\vec{x}_p(t) = \Phi(t) \cdot \left(\int_0^t \frac{1}{\Phi(s)} \cdot \vec{f}(s)\, ds\right)$$

- Plug in the fundamental matrix and the forcing function and compute

$$\vec{x}_p(t) = \begin{bmatrix} \frac{\sqrt{2}\,e^{-t}\sin(\sqrt{2}\,t)}{2} - 2 + 2\,e^{-t}\cos(\sqrt{2}\,t) \\ 2\sqrt{2}\,e^{-t}\sin(\sqrt{2}\,t) + 1 - e^{-t}\cos(\sqrt{2}\,t) \end{bmatrix}$$

- Plug particular solution back into general solution

$$\vec{x}(t) = c_1\vec{x}_1(t) + c_2\vec{x}_2(t) + \begin{bmatrix} \frac{\sqrt{2}\,e^{-t}\sin(\sqrt{2}\,t)}{2} - 2 + 2\,e^{-t}\cos(\sqrt{2}\,t) \\ 2\sqrt{2}\,e^{-t}\sin(\sqrt{2}\,t) + 1 - e^{-t}\cos(\sqrt{2}\,t) \end{bmatrix}$$

- Substitute in vector of dependent variables

$$\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = \begin{bmatrix} \frac{(-\sqrt{2}\,c_2+4)e^{-t}\cos(\sqrt{2}\,t)}{2} - 2 - \frac{\sqrt{2}\,e^{-t}(c_1-1)\sin(\sqrt{2}\,t)}{2} \\ e^{-t}(c_1-1)\cos(\sqrt{2}\,t) + 1 - e^{-t}(c_2 - 2\sqrt{2})\sin(\sqrt{2}\,t) \end{bmatrix}$$

- Solution to the system of ODEs

$$\left\{x_1(t) = \frac{(-\sqrt{2}\,c_2+4)e^{-t}\cos(\sqrt{2}\,t)}{2} - 2 - \frac{\sqrt{2}\,e^{-t}(c_1-1)\sin(\sqrt{2}\,t)}{2}, x_2(t) = e^{-t}(c_1-1)\cos(\sqrt{2}\,t) + 1 - e^{-t}\right.$$

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 61

```
dsolve([diff(x__1(t),t)=-1*x__1(t)-1*x__2(t)-1,diff(x__2(t),t)=2*x__1(t)-1*x__2(t)+5],singso
```

$$x_1(t) = -2 + e^{-t}\left(\cos\left(\sqrt{2}\,t\right)c_1 + c_2\sin\left(\sqrt{2}\,t\right)\right)$$
$$x_2(t) = 1 - e^{-t}\sqrt{2}\left(c_2\cos\left(\sqrt{2}\,t\right) - c_1\sin\left(\sqrt{2}\,t\right)\right)$$

✓ Solution by Mathematica

Time used: 0.414 (sec). Leaf size: 85

```
DSolve[{x1'[t]==-1*x1[t]-1*x2[t]-1,x2'[t]==2*x1[t]-1*x2[t]+5},{x1[t],x2[t]},t,IncludeSingula
```

$$
\begin{aligned}
\mathrm{x1}(t) &\to c_1 e^{-t}\cos\left(\sqrt{2}t\right) - \frac{c_2 e^{-t}\sin\left(\sqrt{2}t\right)}{\sqrt{2}} - 2 \\
\mathrm{x2}(t) &\to e^{-t}\left(e^t + c_2\cos\left(\sqrt{2}t\right) + \sqrt{2}c_1\sin\left(\sqrt{2}t\right)\right)
\end{aligned}
$$

20 Chapter 9.2, Autonomous Systems and Stability. page 517

20.1 problem 1

Internal problem ID [807]
Internal file name [`OUTPUT/807_Sunday_June_05_2022_01_50_17_AM_1022039/index.tex`]

Book: Elementary differential equations and boundary value problems, 10th ed., Boyce and DiPrima
Section: Chapter 9.2, Autonomous Systems and Stability. page 517
Problem number: 1.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"system of linear ODEs"**

Solve

$$
\begin{aligned}
x'(t) &= -x(t) \\
y'(t) &= -2y(t)
\end{aligned}
$$

With initial conditions

$$
[x(0) = 4, y(0) = 2]
$$

20.1.1 Solution using Matrix exponential method

In this method, we will assume we have found the matrix exponential e^{At} allready. There are different methods to determine this but will not be shown here. This is a system of linear ODE's given as

$$
\vec{x}'(t) = A\,\vec{x}(t)
$$

Or

$$
\begin{bmatrix} x'(t) \\ y'(t) \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & -2 \end{bmatrix} \begin{bmatrix} x(t) \\ y(t) \end{bmatrix}
$$

For the above matrix A, the matrix exponential can be found to be

$$
e^{At} = \begin{bmatrix} \mathrm{e}^{-t} & 0 \\ 0 & \mathrm{e}^{-2t} \end{bmatrix}
$$

Therefore the homogeneous solution is

$$
\begin{aligned}
\vec{x}_h(t) &= e^{At}\vec{x}_0 \\
&= \begin{bmatrix} e^{-t} & 0 \\ 0 & e^{-2t} \end{bmatrix} \begin{bmatrix} 4 \\ 2 \end{bmatrix} \\
&= \begin{bmatrix} 4\,e^{-t} \\ 2\,e^{-2t} \end{bmatrix}
\end{aligned}
$$

Since no forcing function is given, then the final solution is $\vec{x}_h(t)$ above.

20.1.2 Solution using explicit Eigenvalue and Eigenvector method

This is a system of linear ODE's given as

$$
\vec{x}'(t) = A\,\vec{x}(t)
$$

Or

$$
\begin{bmatrix} x'(t) \\ y'(t) \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & -2 \end{bmatrix} \begin{bmatrix} x(t) \\ y(t) \end{bmatrix}
$$

The first step is find the homogeneous solution. We start by finding the eigenvalues of A. This is done by solving the following equation for the eigenvalues λ

$$
\det\left(A - \lambda I\right) = 0
$$

Expanding gives

$$
\det\left(\begin{bmatrix} -1 & 0 \\ 0 & -2 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right) = 0
$$

Therefore

$$
\det\left(\begin{bmatrix} -1-\lambda & 0 \\ 0 & -2-\lambda \end{bmatrix}\right) = 0
$$

Since the matrix A is triangular matrix, then the determinant is the product of the elements along the diagonal. Therefore the above becomes

$$
(-1-\lambda)(-2-\lambda) = 0
$$

The roots of the above are the eigenvalues.

$$
\begin{aligned}
\lambda_1 &= -2 \\
\lambda_2 &= -1
\end{aligned}
$$

This table summarises the above result

eigenvalue	algebraic multiplicity	type of eigenvalue
-1	1	real eigenvalue
-2	1	real eigenvalue

Now the eigenvector for each eigenvalue are found.

Considering the eigenvalue $\lambda_1 = -2$

We need to solve $A\vec{v} = \lambda\vec{v}$ or $(A - \lambda I)\vec{v} = \vec{0}$ which becomes

$$
\begin{aligned}
\left(\begin{bmatrix} -1 & 0 \\ 0 & -2 \end{bmatrix} - (-2)\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right)\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} &= \begin{bmatrix} 0 \\ 0 \end{bmatrix} \\
\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} &= \begin{bmatrix} 0 \\ 0 \end{bmatrix}
\end{aligned}
$$

Now forward elimination is applied to solve for the eigenvector $\vec{v}$. The augmented matrix is

$$
\left[\begin{array}{cc|c} 1 & 0 & 0 \\ 0 & 0 & 0 \end{array}\right]
$$

Therefore the system in Echelon form is

$$
\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}
$$

The free variables are $\{v_2\}$ and the leading variables are $\{v_1\}$. Let $v_2 = t$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\{v_1 = 0\}$

Hence the solution is

$$
\begin{bmatrix} 0 \\ t \end{bmatrix} = \begin{bmatrix} 0 \\ t \end{bmatrix}
$$

Since there is one free Variable, we have found one eigenvector associated with this eigenvalue. The above can be written as

$$\begin{bmatrix} 0 \\ t \end{bmatrix} = t \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

Let $t = 1$ the eigenvector becomes

$$\begin{bmatrix} 0 \\ t \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

Considering the eigenvalue $\lambda_2 = -1$

We need to solve $A\vec{v} = \lambda\vec{v}$ or $(A - \lambda I)\vec{v} = \vec{0}$ which becomes

$$\left(\begin{bmatrix} -1 & 0 \\ 0 & -2 \end{bmatrix} - (-1) \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right) \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$
$$\begin{bmatrix} 0 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Now forward elimination is applied to solve for the eigenvector $\vec{v}$. The augmented matrix is

$$\left[\begin{array}{cc|c} 0 & 0 & 0 \\ 0 & -1 & 0 \end{array}\right]$$

Since the current pivot $A(1,2)$ is zero, then the current pivot row is replaced with a row with a non-zero pivot. Swapping row 1 and row 2 gives

$$\left[\begin{array}{cc|c} 0 & -1 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

Therefore the system in Echelon form is

$$\begin{bmatrix} 0 & -1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

The free variables are $\{v_1\}$ and the leading variables are $\{v_2\}$. Let $v_1 = t$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\{v_2 = 0\}$

Hence the solution is

$$\begin{bmatrix} t \\ 0 \end{bmatrix} = \begin{bmatrix} t \\ 0 \end{bmatrix}$$

Since there is one free Variable, we have found one eigenvector associated with this eigenvalue. The above can be written as

$$\begin{bmatrix} t \\ 0 \end{bmatrix} = t \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

Let $t = 1$ the eigenvector becomes

$$\begin{bmatrix} t \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

The following table gives a summary of this result. It shows for each eigenvalue the algebraic multiplicity m, and its geometric multiplicity k and the eigenvectors associated with the eigenvalue. If $m > k$ then the eigenvalue is defective which means the number of normal linearly independent eigenvectors associated with this eigenvalue (called the geometric multiplicity k) does not equal the algebraic multiplicity m, and we need to determine an additional $m - k$ generalized eigenvectors for this eigenvalue.

	multiplicity			
eigenvalue	algebraic m	geometric k	defective?	eigenvectors
-2	1	1	No	$\begin{bmatrix} 0 \\ 1 \end{bmatrix}$
-1	1	1	No	$\begin{bmatrix} 1 \\ 0 \end{bmatrix}$

Now that we found the eigenvalues and associated eigenvectors, we will go over each eigenvalue and generate the solution basis. The only problem we need to take care of is if the eigenvalue is defective. Since eigenvalue -2 is real and distinct then the corresponding eigenvector solution is

$$\begin{aligned} \vec{x}_1(t) &= \vec{v}_1 e^{-2t} \\ &= \begin{bmatrix} 0 \\ 1 \end{bmatrix} e^{-2t} \end{aligned}$$

Since eigenvalue -1 is real and distinct then the corresponding eigenvector solution is

$$\vec{x}_2(t) = \vec{v}_2 e^{-t}$$
$$= \begin{bmatrix} 1 \\ 0 \end{bmatrix} e^{-t}$$

Therefore the final solution is

$$\vec{x}_h(t) = c_1 \vec{x}_1(t) + c_2 \vec{x}_2(t)$$

Which is written as

$$\begin{bmatrix} x(t) \\ y(t) \end{bmatrix} = c_1 \begin{bmatrix} 0 \\ \mathrm{e}^{-2t} \end{bmatrix} + c_2 \begin{bmatrix} \mathrm{e}^{-t} \\ 0 \end{bmatrix}$$

Which becomes

$$\begin{bmatrix} x(t) \\ y(t) \end{bmatrix} = \begin{bmatrix} c_2 \mathrm{e}^{-t} \\ c_1 \mathrm{e}^{-2t} \end{bmatrix}$$

Since initial conditions are given, the solution above needs to be updated by solving for the constants of integrations using the given initial conditions

$$\begin{bmatrix} x(0) = 4 \\ y(0) = 2 \end{bmatrix} \tag{1}$$

Substituting initial conditions into the above solution at $t = 0$ gives

$$\begin{bmatrix} 4 \\ 2 \end{bmatrix} = \begin{bmatrix} c_2 \\ c_1 \end{bmatrix}$$

Solving for the constants of integrations gives

$$\begin{bmatrix} c_1 = 2 \\ c_2 = 4 \end{bmatrix}$$

Substituting these constants back in original solution in Eq. (1) gives

$$\begin{bmatrix} x(t) \\ y(t) \end{bmatrix} = \begin{bmatrix} 4\,\mathrm{e}^{-t} \\ 2\,\mathrm{e}^{-2t} \end{bmatrix}$$

The following is the phase plot of the system.

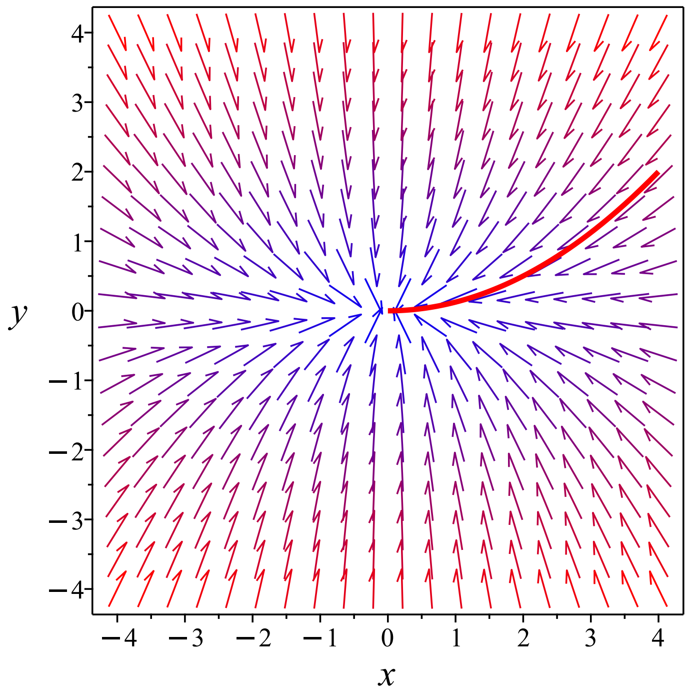
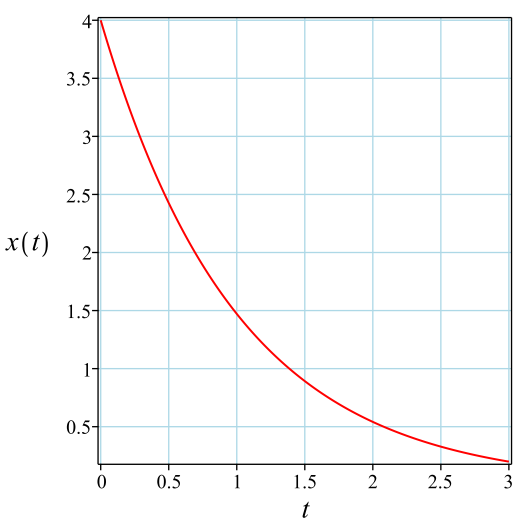
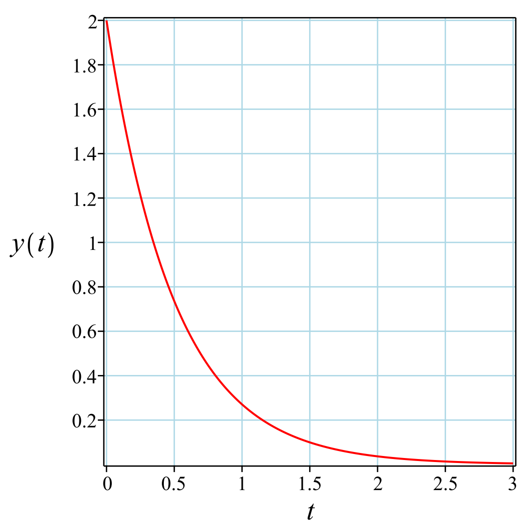


Figure 557: Phase plot

The following are plots of each solution.





✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 20

```
dsolve([diff(x(t),t) = -x(t), diff(y(t),t) = -2*y(t), x(0) = 4, y(0) = 2], singsol=all)
```

$$x(t) = 4\,\mathrm{e}^{-t}$$
$$y(t) = 2\,\mathrm{e}^{-2t}$$

✓ Solution by Mathematica

Time used: 0.038 (sec). Leaf size: 22

```
DSolve[{x'[t]==-1*x[t]+0*y[t],y'[t]==-2*y[t]},{x[0]==4,y[0]==2},{x[t],y[t]},t,IncludeSingula
```

$$x(t) \to 4e^{-t}$$
$$y(t) \to 2e^{-2t}$$

20.2 problem 2 part 1

Internal problem ID [808]
Internal file name [OUTPUT/808_Sunday_June_05_2022_01_50_18_AM_30327055/index.tex]

Book: Elementary differential equations and boundary value problems, 10th ed., Boyce and DiPrima
Section: Chapter 9.2, Autonomous Systems and Stability. page 517
Problem number: 2 part 1.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"system of linear ODEs"**

Solve

$$
\begin{aligned}
x'(t) &= -x(t) \\
y'(t) &= 2y(t)
\end{aligned}
$$

With initial conditions

$$[x(0) = 4, y(0) = 2]$$

20.2.1 Solution using Matrix exponential method

In this method, we will assume we have found the matrix exponential e^{At} allready. There are different methods to determine this but will not be shown here. This is a system of linear ODE's given as

$$\vec{x}'(t) = A\,\vec{x}(t)$$

Or

$$\begin{bmatrix} x'(t) \\ y'(t) \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} x(t) \\ y(t) \end{bmatrix}$$

For the above matrix A, the matrix exponential can be found to be

$$e^{At} = \begin{bmatrix} \mathrm{e}^{-t} & 0 \\ 0 & \mathrm{e}^{2t} \end{bmatrix}$$

Therefore the homogeneous solution is

$$
\begin{aligned}
\vec{x}_h(t) &= e^{At}\vec{x}_0 \\
&= \begin{bmatrix} e^{-t} & 0 \\ 0 & e^{2t} \end{bmatrix} \begin{bmatrix} 4 \\ 2 \end{bmatrix} \\
&= \begin{bmatrix} 4\,e^{-t} \\ 2\,e^{2t} \end{bmatrix}
\end{aligned}
$$

Since no forcing function is given, then the final solution is $\vec{x}_h(t)$ above.

20.2.2 Solution using explicit Eigenvalue and Eigenvector method

This is a system of linear ODE's given as

$$\vec{x}'(t) = A\,\vec{x}(t)$$

Or

$$\begin{bmatrix} x'(t) \\ y'(t) \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} x(t) \\ y(t) \end{bmatrix}$$

The first step is find the homogeneous solution. We start by finding the eigenvalues of A. This is done by solving the following equation for the eigenvalues λ

$$\det(A - \lambda I) = 0$$

Expanding gives

$$\det\left(\begin{bmatrix} -1 & 0 \\ 0 & 2 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right) = 0$$

Therefore

$$\det\left(\begin{bmatrix} -1-\lambda & 0 \\ 0 & 2-\lambda \end{bmatrix}\right) = 0$$

Since the matrix A is triangular matrix, then the determinant is the product of the elements along the diagonal. Therefore the above becomes

$$(-1-\lambda)(2-\lambda) = 0$$

The roots of the above are the eigenvalues.

$$\lambda_1 = -1$$
$$\lambda_2 = 2$$

This table summarises the above result

eigenvalue	algebraic multiplicity	type of eigenvalue
-1	1	real eigenvalue
2	1	real eigenvalue

Now the eigenvector for each eigenvalue are found.

Considering the eigenvalue $\lambda_1 = -1$

We need to solve $A\vec{v} = \lambda\vec{v}$ or $(A - \lambda I)\vec{v} = \vec{0}$ which becomes

$$\left(\begin{bmatrix} -1 & 0 \\ 0 & 2 \end{bmatrix} - (-1)\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right)\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$
$$\begin{bmatrix} 0 & 0 \\ 0 & 3 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Now forward elimination is applied to solve for the eigenvector $\vec{v}$. The augmented matrix is

$$\left[\begin{array}{cc|c} 0 & 0 & 0 \\ 0 & 3 & 0 \end{array}\right]$$

Since the current pivot $A(1,2)$ is zero, then the current pivot row is replaced with a row with a non-zero pivot. Swapping row 1 and row 2 gives

$$\left[\begin{array}{cc|c} 0 & 3 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

Therefore the system in Echelon form is

$$\begin{bmatrix} 0 & 3 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

The free variables are $\{v_1\}$ and the leading variables are $\{v_2\}$. Let $v_1 = t$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\{v_2 = 0\}$

Hence the solution is

$$\begin{bmatrix} t \\ 0 \end{bmatrix} = \begin{bmatrix} t \\ 0 \end{bmatrix}$$

Since there is one free Variable, we have found one eigenvector associated with this eigenvalue. The above can be written as

$$\begin{bmatrix} t \\ 0 \end{bmatrix} = t \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

Let $t = 1$ the eigenvector becomes

$$\begin{bmatrix} t \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

Considering the eigenvalue $\lambda_2 = 2$

We need to solve $A\vec{v} = \lambda\vec{v}$ or $(A - \lambda I)\vec{v} = \vec{0}$ which becomes

$$\left(\begin{bmatrix} -1 & 0 \\ 0 & 2 \end{bmatrix} - (2) \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right) \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} -3 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Now forward elimination is applied to solve for the eigenvector $\vec{v}$. The augmented matrix is

$$\left[\begin{array}{cc|c} -3 & 0 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

Therefore the system in Echelon form is

$$\begin{bmatrix} -3 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

The free variables are $\{v_2\}$ and the leading variables are $\{v_1\}$. Let $v_2 = t$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\{v_1 = 0\}$

Hence the solution is

$$\begin{bmatrix} 0 \\ t \end{bmatrix} = \begin{bmatrix} 0 \\ t \end{bmatrix}$$

Since there is one free Variable, we have found one eigenvector associated with this eigenvalue. The above can be written as

$$\begin{bmatrix} 0 \\ t \end{bmatrix} = t \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

Let $t = 1$ the eigenvector becomes

$$\begin{bmatrix} 0 \\ t \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

The following table gives a summary of this result. It shows for each eigenvalue the algebraic multiplicity m, and its geometric multiplicity k and the eigenvectors associated with the eigenvalue. If $m > k$ then the eigenvalue is defective which means the number of normal linearly independent eigenvectors associated with this eigenvalue (called the geometric multiplicity k) does not equal the algebraic multiplicity m, and we need to determine an additional $m - k$ generalized eigenvectors for this eigenvalue.

	multiplicity			
eigenvalue	algebraic m	geometric k	defective?	eigenvectors
-1	1	1	No	$\begin{bmatrix} 1 \\ 0 \end{bmatrix}$
2	1	1	No	$\begin{bmatrix} 0 \\ 1 \end{bmatrix}$

Now that we found the eigenvalues and associated eigenvectors, we will go over each eigenvalue and generate the solution basis. The only problem we need to take care of is if the eigenvalue is defective. Since eigenvalue -1 is real and distinct then the corresponding eigenvector solution is

$$\begin{aligned} \vec{x}_1(t) &= \vec{v}_1 e^{-t} \\ &= \begin{bmatrix} 1 \\ 0 \end{bmatrix} e^{-t} \end{aligned}$$

Since eigenvalue 2 is real and distinct then the corresponding eigenvector solution is

$$\begin{aligned}\vec{x}_2(t) &= \vec{v}_2 e^{2t} \\ &= \begin{bmatrix} 0 \\ 1 \end{bmatrix} e^{2t}\end{aligned}$$

Therefore the final solution is

$$\vec{x}_h(t) = c_1 \vec{x}_1(t) + c_2 \vec{x}_2(t)$$

Which is written as

$$\begin{bmatrix} x(t) \\ y(t) \end{bmatrix} = c_1 \begin{bmatrix} \mathrm{e}^{-t} \\ 0 \end{bmatrix} + c_2 \begin{bmatrix} 0 \\ \mathrm{e}^{2t} \end{bmatrix}$$

Which becomes

$$\begin{bmatrix} x(t) \\ y(t) \end{bmatrix} = \begin{bmatrix} c_1 \mathrm{e}^{-t} \\ c_2 \mathrm{e}^{2t} \end{bmatrix}$$

Since initial conditions are given, the solution above needs to be updated by solving for the constants of integrations using the given initial conditions

$$\begin{bmatrix} x(0) = 4 \\ y(0) = 2 \end{bmatrix} \tag{1}$$

Substituting initial conditions into the above solution at $t = 0$ gives

$$\begin{bmatrix} 4 \\ 2 \end{bmatrix} = \begin{bmatrix} c_1 \\ c_2 \end{bmatrix}$$

Solving for the constants of integrations gives

$$\begin{bmatrix} c_1 = 4 \\ c_2 = 2 \end{bmatrix}$$

Substituting these constants back in original solution in Eq. (1) gives

$$\begin{bmatrix} x(t) \\ y(t) \end{bmatrix} = \begin{bmatrix} 4\,\mathrm{e}^{-t} \\ 2\,\mathrm{e}^{2t} \end{bmatrix}$$

The following is the phase plot of the system.

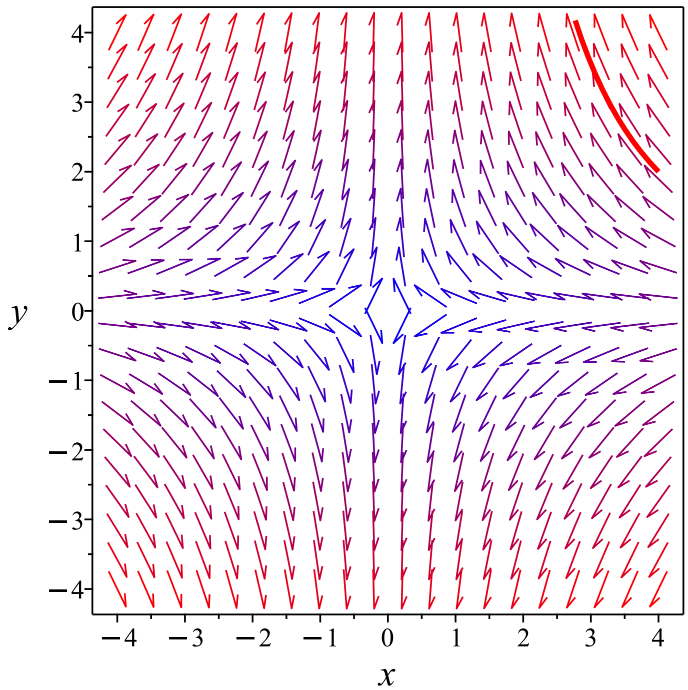
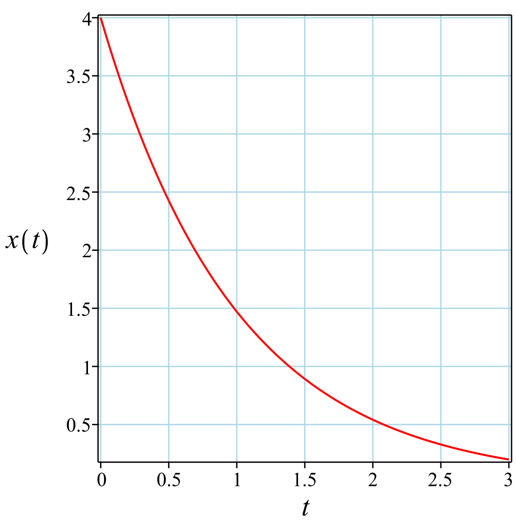
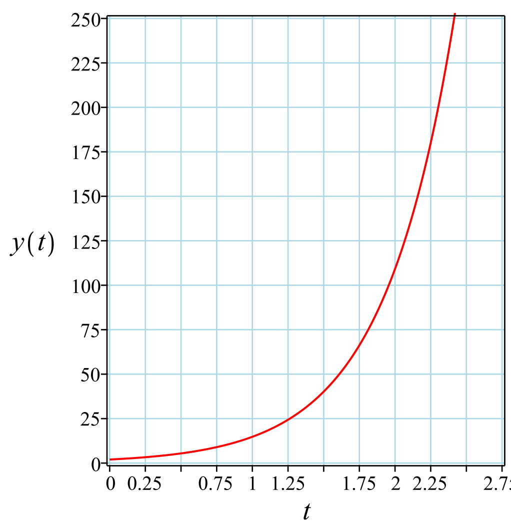


Figure 558: Phase plot

The following are plots of each solution.





✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 20

```
dsolve([diff(x(t),t) = -x(t), diff(y(t),t) = 2*y(t), x(0) = 4, y(0) = 2], singsol=all)
```

$$x(t) = 4\,\mathrm{e}^{-t}$$
$$y(t) = 2\,\mathrm{e}^{2t}$$

✓ Solution by Mathematica

Time used: 0.038 (sec). Leaf size: 22

```
DSolve[{x'[t]==-1*x[t]+0*y[t],y'[t]==0*x[t]+2*y[t]},{x[0]==4,y[0]==2},{x[t],y[t]},t,IncludeS
```

$$x(t) \to 4e^{-t}$$
$$y(t) \to 2e^{2t}$$

20.3 problem 2 part 2

Internal problem ID [809]
Internal file name [`OUTPUT/809_Sunday_June_05_2022_01_50_20_AM_21084214/index.tex`]

Book: Elementary differential equations and boundary value problems, 10th ed., Boyce and DiPrima
Section: Chapter 9.2, Autonomous Systems and Stability. page 517
Problem number: 2 part 2.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"system of linear ODEs"**

Solve

$$x'(t) = -x(t)$$
$$y'(t) = 2y(t)$$

With initial conditions

$$[x(0) = 4, y(0) = 0]$$

20.3.1 Solution using Matrix exponential method

In this method, we will assume we have found the matrix exponential e^{At} allready. There are different methods to determine this but will not be shown here. This is a system of linear ODE's given as

$$\vec{x}'(t) = A\,\vec{x}(t)$$

Or

$$\begin{bmatrix} x'(t) \\ y'(t) \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} x(t) \\ y(t) \end{bmatrix}$$

For the above matrix A, the matrix exponential can be found to be

$$e^{At} = \begin{bmatrix} e^{-t} & 0 \\ 0 & e^{2t} \end{bmatrix}$$

Therefore the homogeneous solution is

$$
\begin{aligned}
\vec{x}_h(t) &= e^{At}\vec{x}_0 \\
&= \begin{bmatrix} e^{-t} & 0 \\ 0 & e^{2t} \end{bmatrix} \begin{bmatrix} 4 \\ 0 \end{bmatrix} \\
&= \begin{bmatrix} 4\,e^{-t} \\ 0 \end{bmatrix}
\end{aligned}
$$

Since no forcing function is given, then the final solution is $\vec{x}_h(t)$ above.

20.3.2 Solution using explicit Eigenvalue and Eigenvector method

This is a system of linear ODE's given as

$$
\vec{x}'(t) = A\,\vec{x}(t)
$$

Or

$$
\begin{bmatrix} x'(t) \\ y'(t) \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} x(t) \\ y(t) \end{bmatrix}
$$

The first step is find the homogeneous solution. We start by finding the eigenvalues of A. This is done by solving the following equation for the eigenvalues λ

$$
\det(A - \lambda I) = 0
$$

Expanding gives

$$
\det\left(\begin{bmatrix} -1 & 0 \\ 0 & 2 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right) = 0
$$

Therefore

$$
\det\left(\begin{bmatrix} -1-\lambda & 0 \\ 0 & 2-\lambda \end{bmatrix}\right) = 0
$$

Since the matrix A is triangular matrix, then the determinant is the product of the elements along the diagonal. Therefore the above becomes

$$
(-1-\lambda)(2-\lambda) = 0
$$

The roots of the above are the eigenvalues.

$$\lambda_1 = -1$$
$$\lambda_2 = 2$$

This table summarises the above result

eigenvalue	algebraic multiplicity	type of eigenvalue
-1	1	real eigenvalue
2	1	real eigenvalue

Now the eigenvector for each eigenvalue are found.

Considering the eigenvalue $\lambda_1 = -1$

We need to solve $A\vec{v} = \lambda\vec{v}$ or $(A - \lambda I)\vec{v} = \vec{0}$ which becomes

$$\left(\begin{bmatrix} -1 & 0 \\ 0 & 2 \end{bmatrix} - (-1)\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right)\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$
$$\begin{bmatrix} 0 & 0 \\ 0 & 3 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Now forward elimination is applied to solve for the eigenvector $\vec{v}$. The augmented matrix is

$$\left[\begin{array}{cc|c} 0 & 0 & 0 \\ 0 & 3 & 0 \end{array}\right]$$

Since the current pivot $A(1,2)$ is zero, then the current pivot row is replaced with a row with a non-zero pivot. Swapping row 1 and row 2 gives

$$\left[\begin{array}{cc|c} 0 & 3 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

Therefore the system in Echelon form is

$$\begin{bmatrix} 0 & 3 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

The free variables are $\{v_1\}$ and the leading variables are $\{v_2\}$. Let $v_1 = t$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\{v_2 = 0\}$

Hence the solution is

$$\begin{bmatrix} t \\ 0 \end{bmatrix} = \begin{bmatrix} t \\ 0 \end{bmatrix}$$

Since there is one free Variable, we have found one eigenvector associated with this eigenvalue. The above can be written as

$$\begin{bmatrix} t \\ 0 \end{bmatrix} = t \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

Let $t = 1$ the eigenvector becomes

$$\begin{bmatrix} t \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

Considering the eigenvalue $\lambda_2 = 2$

We need to solve $A\vec{v} = \lambda\vec{v}$ or $(A - \lambda I)\vec{v} = \vec{0}$ which becomes

$$\left(\begin{bmatrix} -1 & 0 \\ 0 & 2 \end{bmatrix} - (2) \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right) \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} -3 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Now forward elimination is applied to solve for the eigenvector $\vec{v}$. The augmented matrix is

$$\left[\begin{array}{cc|c} -3 & 0 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

Therefore the system in Echelon form is

$$\begin{bmatrix} -3 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

The free variables are $\{v_2\}$ and the leading variables are $\{v_1\}$. Let $v_2 = t$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\{v_1 = 0\}$

Hence the solution is

$$\begin{bmatrix} 0 \\ t \end{bmatrix} = \begin{bmatrix} 0 \\ t \end{bmatrix}$$

Since there is one free Variable, we have found one eigenvector associated with this eigenvalue. The above can be written as

$$\begin{bmatrix} 0 \\ t \end{bmatrix} = t \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

Let $t = 1$ the eigenvector becomes

$$\begin{bmatrix} 0 \\ t \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

The following table gives a summary of this result. It shows for each eigenvalue the algebraic multiplicity m, and its geometric multiplicity k and the eigenvectors associated with the eigenvalue. If $m > k$ then the eigenvalue is defective which means the number of normal linearly independent eigenvectors associated with this eigenvalue (called the geometric multiplicity k) does not equal the algebraic multiplicity m, and we need to determine an additional $m - k$ generalized eigenvectors for this eigenvalue.

	multiplicity			
eigenvalue	algebraic m	geometric k	defective?	eigenvectors
-1	1	1	No	$\begin{bmatrix} 1 \\ 0 \end{bmatrix}$
2	1	1	No	$\begin{bmatrix} 0 \\ 1 \end{bmatrix}$

Now that we found the eigenvalues and associated eigenvectors, we will go over each eigenvalue and generate the solution basis. The only problem we need to take care of is if the eigenvalue is defective. Since eigenvalue -1 is real and distinct then the corresponding eigenvector solution is

$$\begin{aligned} \vec{x}_1(t) &= \vec{v}_1 e^{-t} \\ &= \begin{bmatrix} 1 \\ 0 \end{bmatrix} e^{-t} \end{aligned}$$

Since eigenvalue 2 is real and distinct then the corresponding eigenvector solution is

$$\vec{x}_2(t) = \vec{v}_2 e^{2t}$$
$$= \begin{bmatrix} 0 \\ 1 \end{bmatrix} e^{2t}$$

Therefore the final solution is

$$\vec{x}_h(t) = c_1 \vec{x}_1(t) + c_2 \vec{x}_2(t)$$

Which is written as

$$\begin{bmatrix} x(t) \\ y(t) \end{bmatrix} = c_1 \begin{bmatrix} \mathrm{e}^{-t} \\ 0 \end{bmatrix} + c_2 \begin{bmatrix} 0 \\ \mathrm{e}^{2t} \end{bmatrix}$$

Which becomes

$$\begin{bmatrix} x(t) \\ y(t) \end{bmatrix} = \begin{bmatrix} c_1 \mathrm{e}^{-t} \\ c_2 \mathrm{e}^{2t} \end{bmatrix}$$

Since initial conditions are given, the solution above needs to be updated by solving for the constants of integrations using the given initial conditions

$$\begin{bmatrix} x(0) = 4 \\ y(0) = 0 \end{bmatrix} \tag{1}$$

Substituting initial conditions into the above solution at $t = 0$ gives

$$\begin{bmatrix} 4 \\ 0 \end{bmatrix} = \begin{bmatrix} c_1 \\ c_2 \end{bmatrix}$$

Solving for the constants of integrations gives

$$\begin{bmatrix} c_1 = 4 \\ c_2 = 0 \end{bmatrix}$$

Substituting these constants back in original solution in Eq. (1) gives

$$\begin{bmatrix} x(t) \\ y(t) \end{bmatrix} = \begin{bmatrix} 4\,\mathrm{e}^{-t} \\ 0 \end{bmatrix}$$

The following is the phase plot of the system.

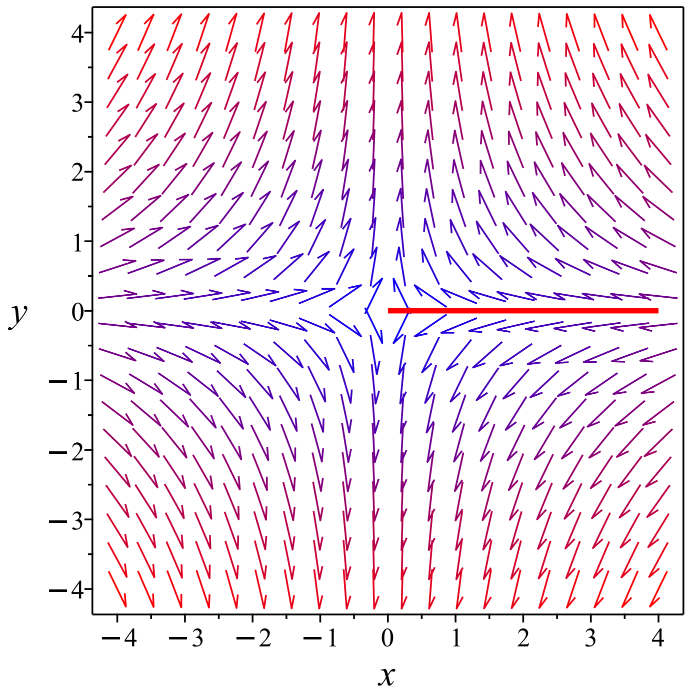
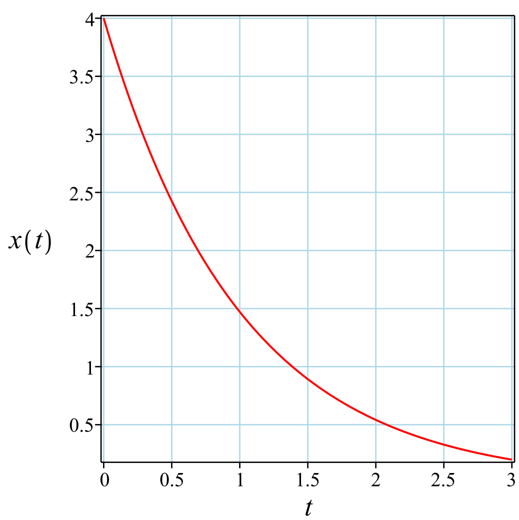
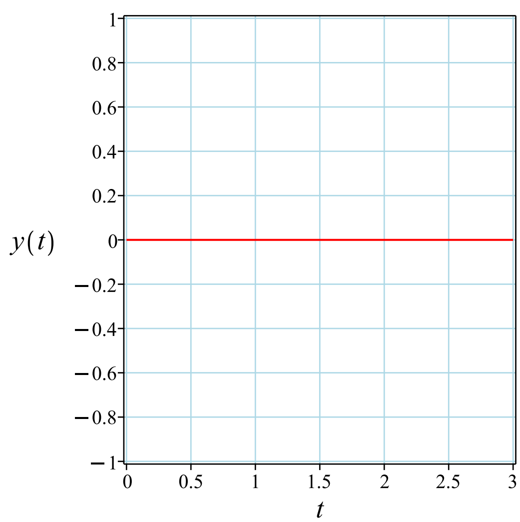


Figure 559: Phase plot

The following are plots of each solution.





✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 15

```
dsolve([diff(x(t),t) = -x(t), diff(y(t),t) = 2*y(t), x(0) = 4, y(0) = 0], singsol=all)
```

$$x(t) = 4\,\mathrm{e}^{-t}$$
$$y(t) = 0$$

✓ Solution by Mathematica

Time used: 0.042 (sec). Leaf size: 16

```
DSolve[{x'[t]==-1*x[t]+0*y[t],y'[t]==0*x[t]+2*y[t]},{x[0]==4,y[0]==0},{x[t],y[t]},t,IncludeS
```

$$x(t) \to 4e^{-t}$$
$$y(t) \to 0$$

20.4 problem 3 part 1

Internal problem ID [810]
Internal file name [OUTPUT/810_Sunday_June_05_2022_01_50_21_AM_58122514/index.tex]

Book: Elementary differential equations and boundary value problems, 10th ed., Boyce and DiPrima
Section: Chapter 9.2, Autonomous Systems and Stability. page 517
Problem number: 3 part 1.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"system of linear ODEs"**

Solve

$$
\begin{aligned}
x'(t) &= -y(t) \\
y'(t) &= x(t)
\end{aligned}
$$

With initial conditions

$$
[x(0) = 4, y(0) = 0]
$$

20.4.1 Solution using Matrix exponential method

In this method, we will assume we have found the matrix exponential e^{At} allready. There are different methods to determine this but will not be shown here. This is a system of linear ODE's given as

$$
\vec{x}'(t) = A\,\vec{x}(t)
$$

Or

$$
\begin{bmatrix} x'(t) \\ y'(t) \end{bmatrix} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} x(t) \\ y(t) \end{bmatrix}
$$

For the above matrix A, the matrix exponential can be found to be

$$
e^{At} = \begin{bmatrix} \cos(t) & -\sin(t) \\ \sin(t) & \cos(t) \end{bmatrix}
$$

Therefore the homogeneous solution is

$$
\begin{aligned}
\vec{x}_h(t) &= e^{At}\vec{x}_0 \\
&= \begin{bmatrix} \cos(t) & -\sin(t) \\ \sin(t) & \cos(t) \end{bmatrix} \begin{bmatrix} 4 \\ 0 \end{bmatrix} \\
&= \begin{bmatrix} 4\cos(t) \\ 4\sin(t) \end{bmatrix}
\end{aligned}
$$

Since no forcing function is given, then the final solution is $\vec{x}_h(t)$ above.

20.4.2 Solution using explicit Eigenvalue and Eigenvector method

This is a system of linear ODE's given as

$$\vec{x}'(t) = A\,\vec{x}(t)$$

Or

$$\begin{bmatrix} x'(t) \\ y'(t) \end{bmatrix} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} x(t) \\ y(t) \end{bmatrix}$$

The first step is find the homogeneous solution. We start by finding the eigenvalues of A. This is done by solving the following equation for the eigenvalues λ

$$\det(A - \lambda I) = 0$$

Expanding gives

$$\det\left(\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right) = 0$$

Therefore

$$\det\left(\begin{bmatrix} -\lambda & -1 \\ 1 & -\lambda \end{bmatrix}\right) = 0$$

Which gives the characteristic equation

$$\lambda^2 + 1 = 0$$

The roots of the above are the eigenvalues.

$$\lambda_1 = i$$
$$\lambda_2 = -i$$

This table summarises the above result

eigenvalue	algebraic multiplicity	type of eigenvalue
$-i$	1	complex eigenvalue
i	1	complex eigenvalue

Now the eigenvector for each eigenvalue are found.

Considering the eigenvalue $\lambda_1 = -i$

We need to solve $A\vec{v} = \lambda\vec{v}$ or $(A - \lambda I)\vec{v} = \vec{0}$ which becomes

$$\left(\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} - (-i)\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right)\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$
$$\begin{bmatrix} i & -1 \\ 1 & i \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Now forward elimination is applied to solve for the eigenvector $\vec{v}$. The augmented matrix is

$$\left[\begin{array}{cc|c} i & -1 & 0 \\ 1 & i & 0 \end{array}\right]$$

$$R_2 = iR_1 + R_2 \Longrightarrow \left[\begin{array}{cc|c} i & -1 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

Therefore the system in Echelon form is

$$\begin{bmatrix} i & -1 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

The free variables are $\{v_2\}$ and the leading variables are $\{v_1\}$. Let $v_2 = t$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\{v_1 = -it\}$

Hence the solution is

$$\begin{bmatrix} \text{-I}t \\ t \end{bmatrix} = \begin{bmatrix} -it \\ t \end{bmatrix}$$

Since there is one free Variable, we have found one eigenvector associated with this eigenvalue. The above can be written as

$$\begin{bmatrix} \text{-I}t \\ t \end{bmatrix} = t \begin{bmatrix} -i \\ 1 \end{bmatrix}$$

Let $t = 1$ the eigenvector becomes

$$\begin{bmatrix} \text{-I}t \\ t \end{bmatrix} = \begin{bmatrix} -i \\ 1 \end{bmatrix}$$

Considering the eigenvalue $\lambda_2 = i$

We need to solve $A\vec{v} = \lambda\vec{v}$ or $(A - \lambda I)\vec{v} = \vec{0}$ which becomes

$$\left(\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} - (i) \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right) \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} -i & -1 \\ 1 & -i \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Now forward elimination is applied to solve for the eigenvector $\vec{v}$. The augmented matrix is

$$\left[\begin{array}{cc|c} -i & -1 & 0 \\ 1 & -i & 0 \end{array}\right]$$

$$R_2 = -iR_1 + R_2 \Longrightarrow \left[\begin{array}{cc|c} -i & -1 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

Therefore the system in Echelon form is

$$\begin{bmatrix} -i & -1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

The free variables are $\{v_2\}$ and the leading variables are $\{v_1\}$. Let $v_2 = t$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\{v_1 = it\}$

Hence the solution is

$$\begin{bmatrix} \mathrm{I}t \\ t \end{bmatrix} = \begin{bmatrix} it \\ t \end{bmatrix}$$

Since there is one free Variable, we have found one eigenvector associated with this eigenvalue. The above can be written as

$$\begin{bmatrix} \mathrm{I}t \\ t \end{bmatrix} = t \begin{bmatrix} i \\ 1 \end{bmatrix}$$

Let $t = 1$ the eigenvector becomes

$$\begin{bmatrix} \mathrm{I}t \\ t \end{bmatrix} = \begin{bmatrix} i \\ 1 \end{bmatrix}$$

The following table gives a summary of this result. It shows for each eigenvalue the algebraic multiplicity m, and its geometric multiplicity k and the eigenvectors associated with the eigenvalue. If $m > k$ then the eigenvalue is defective which means the number of normal linearly independent eigenvectors associated with this eigenvalue (called the geometric multiplicity k) does not equal the algebraic multiplicity m, and we need to determine an additional $m - k$ generalized eigenvectors for this eigenvalue.

	multiplicity			
eigenvalue	algebraic m	geometric k	defective?	eigenvectors
i	1	1	No	$\begin{bmatrix} i \\ 1 \end{bmatrix}$
$-i$	1	1	No	$\begin{bmatrix} -i \\ 1 \end{bmatrix}$

Now that we found the eigenvalues and associated eigenvectors, we will go over each eigenvalue and generate the solution basis. The only problem we need to take care of is if the eigenvalue is defective. Therefore the final solution is

$$\vec{x}_h(t) = c_1 \vec{x}_1(t) + c_2 \vec{x}_2(t)$$

Which is written as

$$\begin{bmatrix} x(t) \\ y(t) \end{bmatrix} = c_1 \begin{bmatrix} i\mathrm{e}^{it} \\ \mathrm{e}^{it} \end{bmatrix} + c_2 \begin{bmatrix} -i\mathrm{e}^{-it} \\ \mathrm{e}^{-it} \end{bmatrix}$$

Which becomes

$$\begin{bmatrix} x(t) \\ y(t) \end{bmatrix} = \begin{bmatrix} -i(c_2 e^{-it} - c_1 e^{it}) \\ c_1 e^{it} + c_2 e^{-it} \end{bmatrix}$$

Since initial conditions are given, the solution above needs to be updated by solving for the constants of integrations using the given initial conditions

$$\begin{bmatrix} x(0) = 4 \\ y(0) = 0 \end{bmatrix} \tag{1}$$

Substituting initial conditions into the above solution at $t = 0$ gives

$$\begin{bmatrix} 4 \\ 0 \end{bmatrix} = \begin{bmatrix} i(c_1 - c_2) \\ c_1 + c_2 \end{bmatrix}$$

Solving for the constants of integrations gives

$$\begin{bmatrix} c_1 = -2i \\ c_2 = 2i \end{bmatrix}$$

Substituting these constants back in original solution in Eq. (1) gives

$$\begin{bmatrix} x(t) \\ y(t) \end{bmatrix} = \begin{bmatrix} -i(2i e^{-it} + 2i e^{it}) \\ -2i e^{it} + 2i e^{-it} \end{bmatrix}$$

The following is the phase plot of the system.

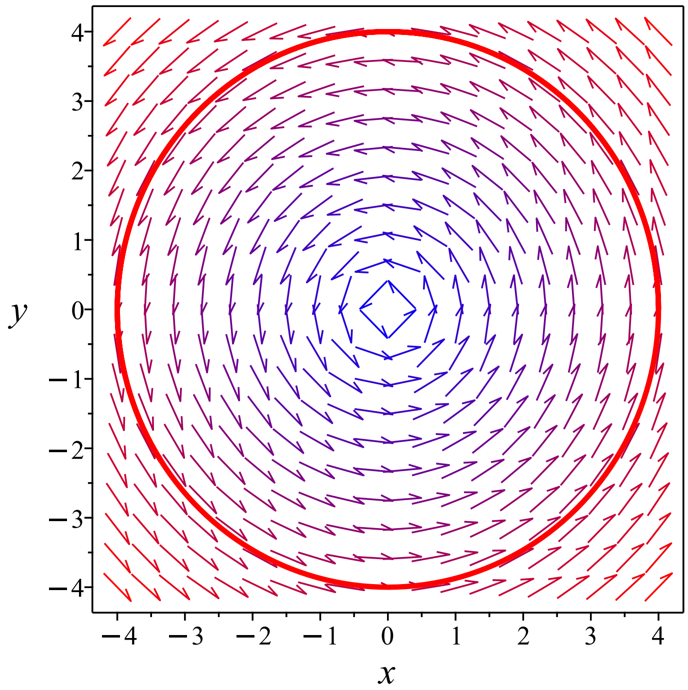


Figure 560: Phase plot

The following are plots of each solution.

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 16

```
dsolve([diff(x(t),t) = -y(t), diff(y(t),t) = x(t), x(0) = 4, y(0) = 0], singsol=all)
```

$$x(t) = 4\cos(t)$$
$$y(t) = 4\sin(t)$$

✓ Solution by Mathematica

Time used: 0.002 (sec). Leaf size: 16

```
DSolve[{x'[t]==-0*x[t]-1*y[t],y'[t]==1*x[t]+0*y[t]},{x[0]==4,y[0]==0},{x[t],y[t]},t,IncludeS
```

$$x(t) \to 4\cos(t)$$
$$y(t) \to 4\sin(t)$$

20.5 problem 3 part 2

Internal problem ID [811]
Internal file name [`OUTPUT/811_Sunday_June_05_2022_01_50_22_AM_62168446/index.tex`]

Book: Elementary differential equations and boundary value problems, 10th ed., Boyce and DiPrima
Section: Chapter 9.2, Autonomous Systems and Stability. page 517
Problem number: 3 part 2.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"system of linear ODEs"**

Solve

$$x'(t) = -y(t)$$
$$y'(t) = x(t)$$

With initial conditions

$$[x(0) = 0, y(0) = 4]$$

20.5.1 Solution using Matrix exponential method

In this method, we will assume we have found the matrix exponential e^{At} allready. There are different methods to determine this but will not be shown here. This is a system of linear ODE's given as

$$\vec{x}'(t) = A\,\vec{x}(t)$$

Or

$$\begin{bmatrix} x'(t) \\ y'(t) \end{bmatrix} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} x(t) \\ y(t) \end{bmatrix}$$

For the above matrix A, the matrix exponential can be found to be

$$e^{At} = \begin{bmatrix} \cos(t) & -\sin(t) \\ \sin(t) & \cos(t) \end{bmatrix}$$

Therefore the homogeneous solution is

$$\begin{aligned}
\vec{x}_h(t) &= e^{At}\vec{x}_0 \\
&= \begin{bmatrix} \cos(t) & -\sin(t) \\ \sin(t) & \cos(t) \end{bmatrix} \begin{bmatrix} 0 \\ 4 \end{bmatrix} \\
&= \begin{bmatrix} -4\sin(t) \\ 4\cos(t) \end{bmatrix}
\end{aligned}$$

Since no forcing function is given, then the final solution is $\vec{x}_h(t)$ above.

20.5.2 Solution using explicit Eigenvalue and Eigenvector method

This is a system of linear ODE's given as

$$\vec{x}'(t) = A\,\vec{x}(t)$$

Or

$$\begin{bmatrix} x'(t) \\ y'(t) \end{bmatrix} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} x(t) \\ y(t) \end{bmatrix}$$

The first step is find the homogeneous solution. We start by finding the eigenvalues of A. This is done by solving the following equation for the eigenvalues λ

$$\det(A - \lambda I) = 0$$

Expanding gives

$$\det\left(\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right) = 0$$

Therefore

$$\det\left(\begin{bmatrix} -\lambda & -1 \\ 1 & -\lambda \end{bmatrix}\right) = 0$$

Which gives the characteristic equation

$$\lambda^2 + 1 = 0$$

The roots of the above are the eigenvalues.

$$\lambda_1 = i$$
$$\lambda_2 = -i$$

This table summarises the above result

eigenvalue	algebraic multiplicity	type of eigenvalue
$-i$	1	complex eigenvalue
i	1	complex eigenvalue

Now the eigenvector for each eigenvalue are found.

Considering the eigenvalue $\lambda_1 = -i$

We need to solve $A\vec{v} = \lambda\vec{v}$ or $(A - \lambda I)\vec{v} = \vec{0}$ which becomes

$$\left(\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} - (-i)\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right)\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$
$$\begin{bmatrix} i & -1 \\ 1 & i \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Now forward elimination is applied to solve for the eigenvector $\vec{v}$. The augmented matrix is

$$\left[\begin{array}{cc|c} i & -1 & 0 \\ 1 & i & 0 \end{array}\right]$$

$$R_2 = iR_1 + R_2 \Longrightarrow \left[\begin{array}{cc|c} i & -1 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

Therefore the system in Echelon form is

$$\begin{bmatrix} i & -1 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

The free variables are $\{v_2\}$ and the leading variables are $\{v_1\}$. Let $v_2 = t$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\{v_1 = -it\}$

Hence the solution is

$$\begin{bmatrix} \text{-I}\,t \\ t \end{bmatrix} = \begin{bmatrix} -it \\ t \end{bmatrix}$$

Since there is one free Variable, we have found one eigenvector associated with this eigenvalue. The above can be written as

$$\begin{bmatrix} \text{-I}\,t \\ t \end{bmatrix} = t \begin{bmatrix} -i \\ 1 \end{bmatrix}$$

Let $t = 1$ the eigenvector becomes

$$\begin{bmatrix} \text{-I}\,t \\ t \end{bmatrix} = \begin{bmatrix} -i \\ 1 \end{bmatrix}$$

Considering the eigenvalue $\lambda_2 = i$

We need to solve $A\vec{v} = \lambda\vec{v}$ or $(A - \lambda I)\vec{v} = \vec{0}$ which becomes

$$\left(\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} - (i) \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right) \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} -i & -1 \\ 1 & -i \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Now forward elimination is applied to solve for the eigenvector $\vec{v}$. The augmented matrix is

$$\left[\begin{array}{cc|c} -i & -1 & 0 \\ 1 & -i & 0 \end{array}\right]$$

$$R_2 = -iR_1 + R_2 \Longrightarrow \left[\begin{array}{cc|c} -i & -1 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

Therefore the system in Echelon form is

$$\begin{bmatrix} -i & -1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

The free variables are $\{v_2\}$ and the leading variables are $\{v_1\}$. Let $v_2 = t$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\{v_1 = it\}$

Hence the solution is

$$\begin{bmatrix} \mathrm{I}t \\ t \end{bmatrix} = \begin{bmatrix} it \\ t \end{bmatrix}$$

Since there is one free Variable, we have found one eigenvector associated with this eigenvalue. The above can be written as

$$\begin{bmatrix} \mathrm{I}t \\ t \end{bmatrix} = t \begin{bmatrix} i \\ 1 \end{bmatrix}$$

Let $t = 1$ the eigenvector becomes

$$\begin{bmatrix} \mathrm{I}t \\ t \end{bmatrix} = \begin{bmatrix} i \\ 1 \end{bmatrix}$$

The following table gives a summary of this result. It shows for each eigenvalue the algebraic multiplicity m, and its geometric multiplicity k and the eigenvectors associated with the eigenvalue. If $m > k$ then the eigenvalue is defective which means the number of normal linearly independent eigenvectors associated with this eigenvalue (called the geometric multiplicity k) does not equal the algebraic multiplicity m, and we need to determine an additional $m - k$ generalized eigenvectors for this eigenvalue.

	multiplicity			
eigenvalue	algebraic m	geometric k	defective?	eigenvectors
i	1	1	No	$\begin{bmatrix} i \\ 1 \end{bmatrix}$
$-i$	1	1	No	$\begin{bmatrix} -i \\ 1 \end{bmatrix}$

Now that we found the eigenvalues and associated eigenvectors, we will go over each eigenvalue and generate the solution basis. The only problem we need to take care of is if the eigenvalue is defective. Therefore the final solution is

$$\vec{x}_h(t) = c_1 \vec{x}_1(t) + c_2 \vec{x}_2(t)$$

Which is written as

$$\begin{bmatrix} x(t) \\ y(t) \end{bmatrix} = c_1 \begin{bmatrix} i\mathrm{e}^{it} \\ \mathrm{e}^{it} \end{bmatrix} + c_2 \begin{bmatrix} -i\mathrm{e}^{-it} \\ \mathrm{e}^{-it} \end{bmatrix}$$

Which becomes

$$\begin{bmatrix} x(t) \\ y(t) \end{bmatrix} = \begin{bmatrix} i(c_1 e^{it} - c_2 e^{-it}) \\ c_1 e^{it} + c_2 e^{-it} \end{bmatrix}$$

Since initial conditions are given, the solution above needs to be updated by solving for the constants of integrations using the given initial conditions

$$\begin{bmatrix} x(0) = 0 \\ y(0) = 4 \end{bmatrix} \tag{1}$$

Substituting initial conditions into the above solution at $t = 0$ gives

$$\begin{bmatrix} 0 \\ 4 \end{bmatrix} = \begin{bmatrix} i(c_1 - c_2) \\ c_1 + c_2 \end{bmatrix}$$

Solving for the constants of integrations gives

$$\begin{bmatrix} c_1 = 2 \\ c_2 = 2 \end{bmatrix}$$

Substituting these constants back in original solution in Eq. (1) gives

$$\begin{bmatrix} x(t) \\ y(t) \end{bmatrix} = \begin{bmatrix} i(2\,e^{it} - 2\,e^{-it}) \\ 2\,e^{it} + 2\,e^{-it} \end{bmatrix}$$

The following is the phase plot of the system.

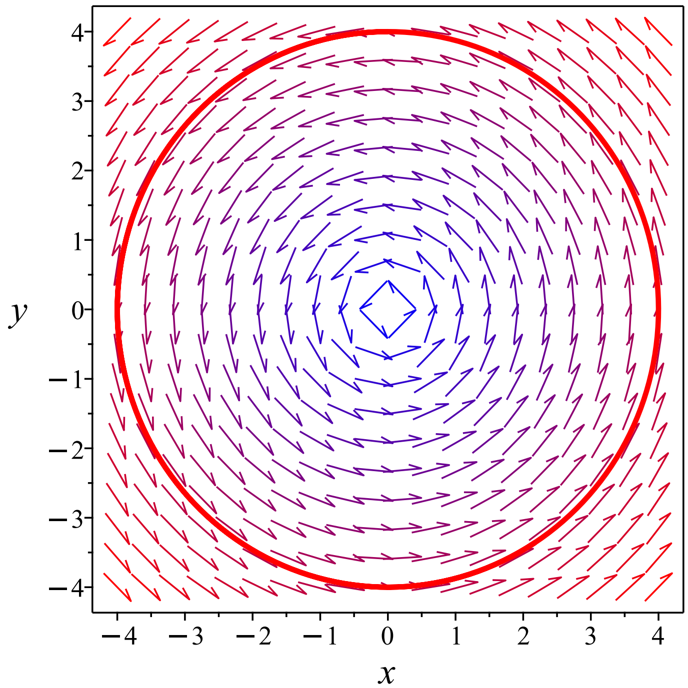


Figure 561: Phase plot

The following are plots of each solution.

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 16

```
dsolve([diff(x(t),t) = -y(t), diff(y(t),t) = x(t), x(0) = 0, y(0) = 4], singsol=all)
```

$$x(t) = -4 \sin(t)$$
$$y(t) = 4 \cos(t)$$

✓ Solution by Mathematica

Time used: 0.002 (sec). Leaf size: 16

```
DSolve[{x'[t]==-0*x[t]-1*y[t],y'[t]==1*x[t]+0*y[t]},{x[0]==0,y[0]==4},{x[t],y[t]},t,IncludeS
```

$$x(t) \to -4 \sin(t)$$
$$y(t) \to 4 \cos(t)$$