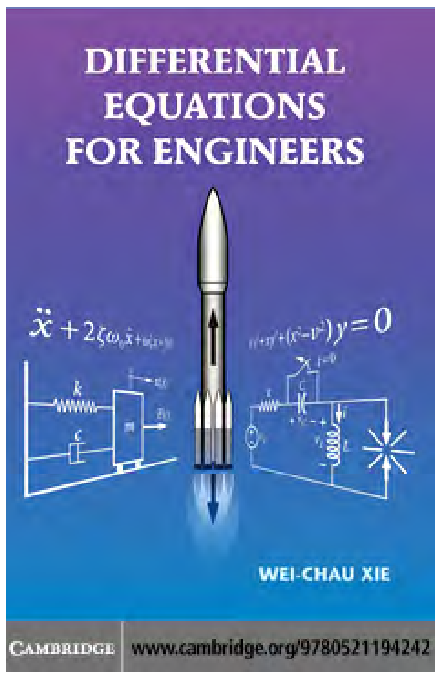
A Solution Manual For

Differential equations for engineers by Wei-Chau XIE, Cambridge Press 2010



Nasser M. Abbasi

May 15, 2024

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1 Chapter 2. First-Order and Simple Higher-Order Differential Equations. Page 78

1.1 problem 1

Internal problem ID [3146]
Internal file name [`OUTPUT/2638_Sunday_June_05_2022_08_37_51_AM_88841125/index.tex`]

Book: Differential equations for engineers by Wei-Chau XIE, Cambridge Press 2010
Section: Chapter 2. First-Order and Simple Higher-Order Differential Equations. Page 78
Problem number: 1.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program :

Maple gives the following as the ode type

`[_separable]`

$$\boxed{\cos(y)^2 + \left(1+\mathrm{e}^{-x}\right)\sin(y)\,y' = 0}$$

1.1.1 Solving as separable ode

In canonical form the ODE is

$$\begin{aligned} y' &= F(x,y) \\ &= f(x)g(y) \\ &= -\frac{\cos(y)\cot(y)}{1+\mathrm{e}^{-x}} \end{aligned}$$

Where $f(x) = -\frac{1}{1+\mathrm{e}^{-x}}$ and $g(y) = \cos(y)\cot(y)$. Integrating both sides gives

$$\begin{aligned} \frac{1}{\cos(y)\cot(y)}\,dy &= -\frac{1}{1+\mathrm{e}^{-x}}\,dx \\ \int \frac{1}{\cos(y)\cot(y)}\,dy &= \int -\frac{1}{1+\mathrm{e}^{-x}}\,dx \\ \frac{1}{\cos(y)} &= -\ln\left(1+\mathrm{e}^{-x}\right) + \ln\left(\mathrm{e}^{-x}\right) + c_1 \end{aligned}$$

Which results in

$$y = \pi - \arccos\left(\frac{1}{\ln\left(\left(e^{x}+1\right)e^{-x}\right)+x-c_1}\right)$$

Summary

The solution(s) found are the following

$$y = \pi - \arccos\left(\frac{1}{\ln\left(\left(e^{x}+1\right)e^{-x}\right)+x-c_1}\right) \tag{1}$$

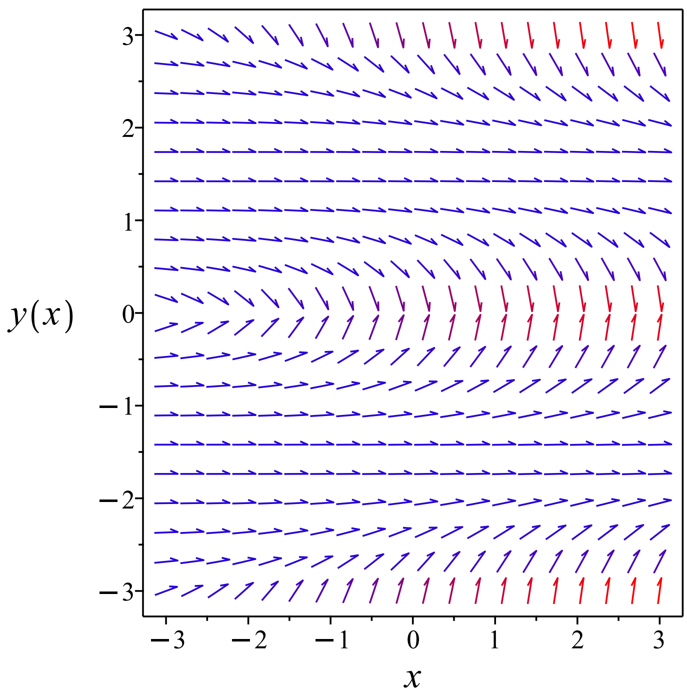


Figure 1: Slope field plot

Verification of solutions

$$y = \pi - \arccos\left(\frac{1}{\ln\left(\left(e^{x}+1\right)e^{-x}\right)+x-c_1}\right)$$

Verified OK.

1.1.2 Maple step by step solution

Let's solve

$\cos(y)^2 + (1+e^{-x})\sin(y)\,y' = 0$

- Highest derivative means the order of the ODE is 1

  y'

- Separate variables

  $\frac{y'\sin(y)}{\cos(y)^2} = -\frac{1}{1+e^{-x}}$

- Integrate both sides with respect to x

  $\int \frac{y'\sin(y)}{\cos(y)^2}dx = \int -\frac{1}{1+e^{-x}}dx + c_1$

- Evaluate integral

  $\frac{1}{\cos(y)} = -\ln(1+e^{-x}) + \ln(e^{-x}) + c_1$

- Solve for y

  $y = \pi - \arccos\left(\frac{1}{\ln\left(\frac{e^x+1}{e^x}\right)+x-c_1}\right)$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
<- separable successful`
```

✓ Solution by Maple

Time used: 0.015 (sec). Leaf size: 18

```
dsolve(cos(y(x))^2+(1+exp(-x))*sin(y(x))*diff(y(x),x)=0,y(x), singsol=all)
```

$$y(x) = \frac{\pi}{2} + \arcsin\left(\frac{1}{\ln(1+e^x)+c_1}\right)$$

✓ Solution by Mathematica

Time used: 0.95 (sec). Leaf size: 57

```
DSolve[Cos[y[x]]^2+(1+Exp[-x])*Sin[y[x]]*y'[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to -\sec^{-1}\left(-\log\left(e^x+1\right)+2c_1\right)$$
$$y(x) \to \sec^{-1}\left(-\log\left(e^x+1\right)+2c_1\right)$$
$$y(x) \to -\frac{\pi}{2}$$
$$y(x) \to \frac{\pi}{2}$$

1.2 problem 2

Internal problem ID [3147]
Internal file name [OUTPUT/2639_Sunday_June_05_2022_08_37_52_AM_99009684/index.tex]

Book: Differential equations for engineers by Wei-Chau XIE, Cambridge Press 2010
Section: Chapter 2. First-Order and Simple Higher-Order Differential Equations. Page 78
Problem number: 2.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program :

Maple gives the following as the ode type

`[_separable]`

$$\boxed{y' - \frac{x^3 e^{x^2}}{y \ln(y)} = 0}$$

1.2.1 Solving as separable ode

In canonical form the ODE is

$$\begin{aligned} y' &= F(x, y) \\ &= f(x)g(y) \\ &= \frac{x^3 e^{x^2}}{y \ln(y)} \end{aligned}$$

Where $f(x) = x^3 e^{x^2}$ and $g(y) = \frac{1}{y \ln(y)}$. Integrating both sides gives

$$\frac{1}{\frac{1}{y \ln(y)}}\, dy = x^3 e^{x^2}\, dx$$

$$\int \frac{1}{\frac{1}{y \ln(y)}}\, dy = \int x^3 e^{x^2}\, dx$$

$$\frac{y^2 \ln(y)}{2} - \frac{y^2}{4} = \frac{(x^2 - 1)\, e^{x^2}}{2} + c_1$$

Which results in

$$y = e^{\frac{\mathrm{LambertW}\left(2\left(x^2 e^{x^2} - e^{x^2} + 2c_1\right)e^{-1}\right)}{2} + \frac{1}{2}}$$

Summary

The solution(s) found are the following

$$y = e^{\frac{\mathrm{LambertW}\left(2\left(x^2 e^{x^2} - e^{x^2} + 2c_1\right)e^{-1}\right)}{2} + \frac{1}{2}} \tag{1}$$

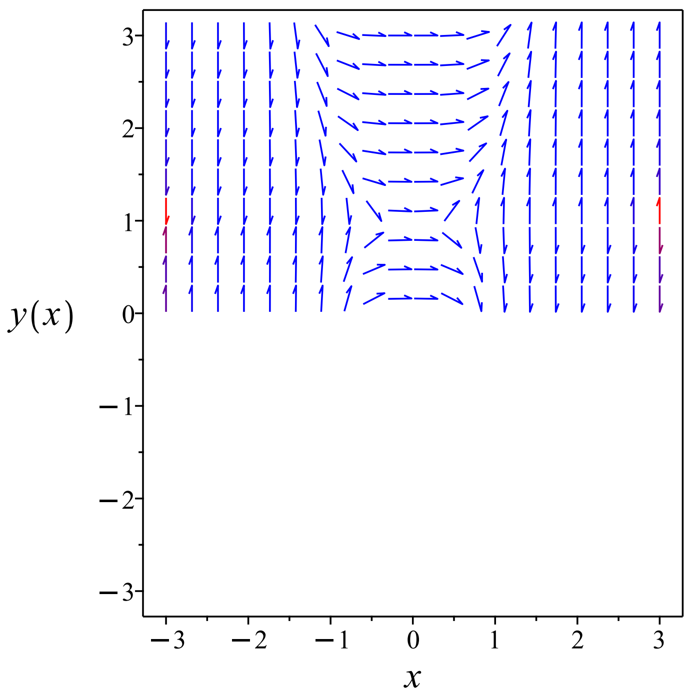


Figure 2: Slope field plot

Verification of solutions

$$y = e^{\frac{\mathrm{LambertW}\left(2\left(x^2 e^{x^2} - e^{x^2} + 2c_1\right)e^{-1}\right)}{2} + \frac{1}{2}}$$

Verified OK.

1.2.2 Maple step by step solution

Let's solve

$y' - \frac{x^3 e^{x^2}}{y \ln(y)} = 0$

- Highest derivative means the order of the ODE is 1

  y'

- Separate variables

  $y' y \ln(y) = x^3 e^{x^2}$

- Integrate both sides with respect to x

  $\int y' y \ln(y) dx = \int x^3 e^{x^2} dx + c_1$

- Evaluate integral

  $\frac{y^2 \ln(y)}{2} - \frac{y^2}{4} = \frac{(x^2-1) e^{x^2}}{2} + c_1$

- Solve for y

  $y = e^{\frac{LambertW\left(\frac{2\left(x^2 e^{x^2} - e^{x^2} + 2c_1\right)}{e}\right)}{2} + \frac{1}{2}}$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
<- separable successful`
```

✓ Solution by Maple

Time used: 0.031 (sec). Leaf size: 54

```
dsolve(diff(y(x),x)=(x^3*exp(x^2))/(y(x)*ln(y(x))),y(x), singsol=all)
```

$$y(x) = \sqrt{2}\sqrt{\frac{e^{x^2} x^2 - e^{x^2} + 2c_1}{\mathrm{LambertW}\left(2\left(e^{x^2} x^2 - e^{x^2} + 2c_1\right) e^{-1}\right)}}$$

✓ Solution by Mathematica

Time used: 60.191 (sec). Leaf size: 106

```
DSolve[y'[x]==(x^3*Exp[x^2])/(y[x]*Log[y[x]]),y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to -\frac{\sqrt{2e^{x^2}\left(x^2-1\right)+4c_1}}{\sqrt{W\left(\frac{2e^{x^2}(x^2-1)+4c_1}{e}\right)}}$$
$$y(x) \to \frac{\sqrt{2e^{x^2}\left(x^2-1\right)+4c_1}}{\sqrt{W\left(\frac{2e^{x^2}(x^2-1)+4c_1}{e}\right)}}$$

1.3 problem 3

Internal problem ID [3148]
Internal file name [`OUTPUT/2640_Sunday_June_05_2022_08_37_53_AM_87021584/index.tex`]

Book: Differential equations for engineers by Wei-Chau XIE, Cambridge Press 2010
Section: Chapter 2. First-Order and Simple Higher-Order Differential Equations. Page 78
Problem number: 3.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program :

Maple gives the following as the ode type

`[_separable]`

$$\boxed{x\cos(y)^2 + e^x \tan(y)\, y' = 0}$$

1.3.1 Solving as separable ode

In canonical form the ODE is

$$\begin{aligned} y' &= F(x,y) \\ &= f(x)g(y) \\ &= -x\,e^{-x}\cos(y)^2\cot(y) \end{aligned}$$

Where $f(x) = -x\,e^{-x}$ and $g(y) = \cos(y)^2\cot(y)$. Integrating both sides gives

$$\begin{aligned} \frac{1}{\cos(y)^2\cot(y)}\,dy &= -x\,e^{-x}\,dx \\ \int \frac{1}{\cos(y)^2\cot(y)}\,dy &= \int -x\,e^{-x}\,dx \\ \frac{1}{2\cot(y)^2} &= (x+1)\,e^{-x} + c_1 \end{aligned}$$

Which results in

$$y = \operatorname{arccot}\left(\frac{\sqrt{2}\,\sqrt{(c_1 \mathrm{e}^{x} + x + 1)\,\mathrm{e}^{x}}}{2c_1 \mathrm{e}^{x} + 2x + 2}\right)$$

$$y = \pi - \operatorname{arccot}\left(\frac{\sqrt{2}\,\sqrt{(c_1 \mathrm{e}^{x} + x + 1)\,\mathrm{e}^{x}}}{2c_1 \mathrm{e}^{x} + 2x + 2}\right)$$

Summary
The solution(s) found are the following

$$y = \operatorname{arccot}\left(\frac{\sqrt{2}\,\sqrt{(c_1 \mathrm{e}^{x} + x + 1)\,\mathrm{e}^{x}}}{2c_1 \mathrm{e}^{x} + 2x + 2}\right) \tag{1}$$

$$y = \pi - \operatorname{arccot}\left(\frac{\sqrt{2}\,\sqrt{(c_1 \mathrm{e}^{x} + x + 1)\,\mathrm{e}^{x}}}{2c_1 \mathrm{e}^{x} + 2x + 2}\right) \tag{2}$$

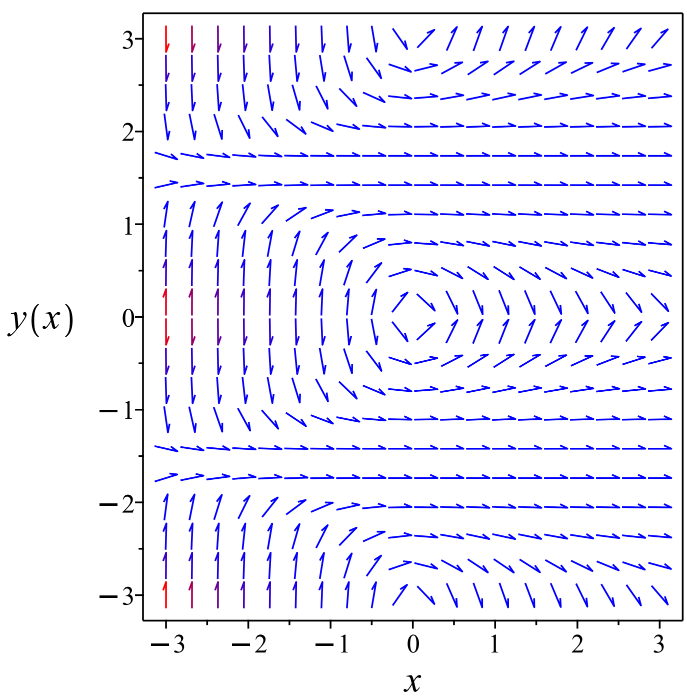


Figure 3: Slope field plot

Verification of solutions

$$y = \operatorname{arccot}\left(\frac{\sqrt{2}\,\sqrt{\left(c_1 \mathrm{e}^{x}+x+1\right) \mathrm{e}^{x}}}{2 c_1 \mathrm{e}^{x}+2 x+2}\right)$$

Verified OK.

$$y = \pi - \operatorname{arccot}\left(\frac{\sqrt{2}\,\sqrt{\left(c_1 \mathrm{e}^{x}+x+1\right) \mathrm{e}^{x}}}{2 c_1 \mathrm{e}^{x}+2 x+2}\right)$$

Verified OK.

1.3.2 Maple step by step solution

Let's solve

$x \cos\left(y\right)^{2}+\mathrm{e}^{x} \tan\left(y\right) y' = 0$

- Highest derivative means the order of the ODE is 1

  y'

- Separate variables

  $\frac{y' \tan(y)}{\cos(y)^{2}} = -\frac{x}{\mathrm{e}^{x}}$

- Integrate both sides with respect to x

  $\int \frac{y' \tan(y)}{\cos(y)^{2}} dx = \int -\frac{x}{\mathrm{e}^{x}} dx + c_1$

- Evaluate integral

  $\frac{\tan(y)^{2}}{2} = \frac{x+1}{\mathrm{e}^{x}} + c_1$

- Solve for y

  $\left\{y = -\arctan\left(\frac{\sqrt{2}\,\sqrt{(c_1 \mathrm{e}^{x}+x+1)\mathrm{e}^{x}}}{\mathrm{e}^{x}}\right), y = \arctan\left(\frac{\sqrt{2}\,\sqrt{(c_1 \mathrm{e}^{x}+x+1)\mathrm{e}^{x}}}{\mathrm{e}^{x}}\right)\right\}$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
<- separable successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 77

```
dsolve(x*cos(y(x))^2+exp(x)*tan(y(x))*diff(y(x),x)=0,y(x), singsol=all)
```

$$y(x) = \pi - \operatorname{arccot}\left(\frac{\sqrt{2}\,\sqrt{(-\mathrm{e}^{x}c_1 + x + 1)\,\mathrm{e}^{x}}}{-2\,\mathrm{e}^{x}c_1 + 2x + 2}\right)$$
$$y(x) = \frac{\pi}{2} - \arctan\left(\frac{\sqrt{2}\,\sqrt{(-\mathrm{e}^{x}c_1 + x + 1)\,\mathrm{e}^{x}}}{-2\,\mathrm{e}^{x}c_1 + 2x + 2}\right)$$

✓ Solution by Mathematica

Time used: 15.741 (sec). Leaf size: 149

```
DSolve[x*Cos[y[x]]^2+Exp[x]*Tan[y[x]]*y'[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to -\sec^{-1}\left(-\sqrt{2}\sqrt{e^{-x}\left(x + 4c_1e^{x} + 1\right)}\right)$$
$$y(x) \to \sec^{-1}\left(-\sqrt{2}\sqrt{e^{-x}\left(x + 4c_1e^{x} + 1\right)}\right)$$
$$y(x) \to -\sec^{-1}\left(\sqrt{2}\sqrt{e^{-x}\left(x + 4c_1e^{x} + 1\right)}\right)$$
$$y(x) \to \sec^{-1}\left(\sqrt{2}\sqrt{e^{-x}\left(x + 4c_1e^{x} + 1\right)}\right)$$
$$y(x) \to -\frac{\pi}{2}$$
$$y(x) \to \frac{\pi}{2}$$

1.4 problem 4

Internal problem ID [3149]
Internal file name [OUTPUT/2641_Sunday_June_05_2022_08_37_59_AM_8997525/index.tex]

Book: Differential equations for engineers by Wei-Chau XIE, Cambridge Press 2010
Section: Chapter 2. First-Order and Simple Higher-Order Differential Equations. Page 78
Problem number: 4.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program :

Maple gives the following as the ode type

`[_separable]`

$$\boxed{x(y^2+1)+(2y+1)\,\mathrm{e}^{-x}y'=0}$$

1.4.1 Solving as separable ode

In canonical form the ODE is

$$\begin{aligned} y' &= F(x,y) \\ &= f(x)g(y) \\ &= -\frac{x\,\mathrm{e}^{x}(y^2+1)}{2y+1} \end{aligned}$$

Where $f(x)=-x\,\mathrm{e}^{x}$ and $g(y)=\frac{y^2+1}{2y+1}$. Integrating both sides gives

$$\frac{1}{\frac{y^2+1}{2y+1}}\,dy = -x\,\mathrm{e}^{x}\,dx$$

$$\int \frac{1}{\frac{y^2+1}{2y+1}}\,dy = \int -x\,\mathrm{e}^{x}\,dx$$

$$\ln\left(y^2+1\right)+\arctan(y) = -(x-1)\,\mathrm{e}^{x}+c_1$$

Which results in

$$y = \tan\left(\mathrm{RootOf}\left(-x\,\mathrm{e}^{x} + \mathrm{e}^{x} - \ln\left(\frac{1}{\cos\left(\_Z\right)^{2}}\right) + c_1 - \_Z\right)\right)$$

Summary

The solution(s) found are the following

$$y = \tan\left(\mathrm{RootOf}\left(-x\,\mathrm{e}^{x} + \mathrm{e}^{x} - \ln\left(\frac{1}{\cos\left(\_Z\right)^{2}}\right) + c_1 - \_Z\right)\right) \tag{1}$$

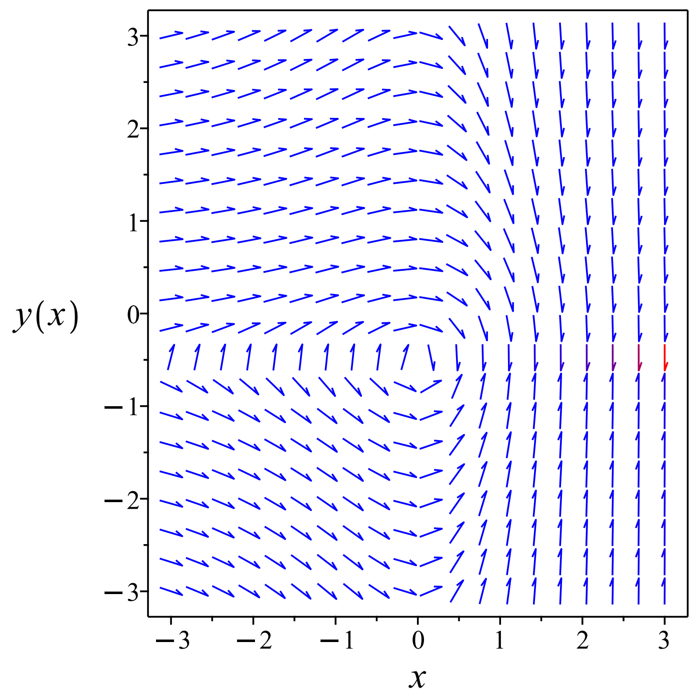


Figure 4: Slope field plot

Verification of solutions

$$y = \tan\left(\mathrm{RootOf}\left(-x\,\mathrm{e}^{x} + \mathrm{e}^{x} - \ln\left(\frac{1}{\cos\left(\_Z\right)^{2}}\right) + c_1 - \_Z\right)\right)$$

Verified OK.

1.4.2 Maple step by step solution

Let's solve

$x(y^2+1)+(2y+1)\,\mathrm{e}^{-x}y'=0$

- Highest derivative means the order of the ODE is 1

  y'

- Separate variables

  $\frac{y'(2y+1)}{y^2+1}=-\frac{x}{\mathrm{e}^{-x}}$

- Integrate both sides with respect to x

  $\int \frac{y'(2y+1)}{y^2+1}dx=\int -\frac{x}{\mathrm{e}^{-x}}dx+c_1$

- Evaluate integral

  $\ln(y^2+1)+\arctan(y)=-\frac{x-1}{\mathrm{e}^{-x}}+c_1$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
<- separable successful`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 28

```
dsolve(x*(y(x)^2+1)+(2*y(x)+1)*exp(-x)*diff(y(x),x)=0,y(x), singsol=all)
```

$$y(x)=\tan\left(\mathrm{RootOf}\left(x\,\mathrm{e}^{x}-\mathrm{e}^{x}+\ln(2)+\ln\left(\frac{1}{1+\cos(2\_Z)}\right)+\_Z+c_1\right)\right)$$

✓ Solution by Mathematica

Time used: 0.627 (sec). Leaf size: 43

```
DSolve[x*(y[x]^2+1)+(2*y[x]+1)*Exp[-x]*y'[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \text{InverseFunction}\left[\log\left(\#1^2+1\right)+\arctan(\#1)\&\right]\left[-e^x(x-1)+c_1\right]$$
$$y(x) \to -i$$
$$y(x) \to i$$

1.5 problem 5

Internal problem ID [3150]
Internal file name [OUTPUT/2642_Sunday_June_05_2022_08_37_59_AM_14399028/index.tex]

Book: Differential equations for engineers by Wei-Chau XIE, Cambridge Press 2010
Section: Chapter 2. First-Order and Simple Higher-Order Differential Equations. Page 78
Problem number: 5.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program :

Maple gives the following as the ode type

`[_separable]`

$$\boxed{x y^3 + \mathrm{e}^{x^2} y' = 0}$$

1.5.1 Solving as separable ode

In canonical form the ODE is

$$\begin{aligned} y' &= F(x, y) \\ &= f(x)g(y) \\ &= -x\, y^3 \mathrm{e}^{-x^2} \end{aligned}$$

Where $f(x) = -x\,\mathrm{e}^{-x^2}$ and $g(y) = y^3$. Integrating both sides gives

$$\frac{1}{y^3}\, dy = -x\,\mathrm{e}^{-x^2}\, dx$$

$$\int \frac{1}{y^3}\, dy = \int -x\,\mathrm{e}^{-x^2}\, dx$$

$$-\frac{1}{2y^2} = \frac{\mathrm{e}^{-x^2}}{2} + c_1$$

Which results in

$$y = \frac{\sqrt{-\left(2c_1 \mathrm{e}^{x^2}+1\right)\mathrm{e}^{x^2}}}{2c_1 \mathrm{e}^{x^2}+1}$$

$$y = -\frac{\sqrt{-\left(2c_1 \mathrm{e}^{x^2}+1\right)\mathrm{e}^{x^2}}}{2c_1 \mathrm{e}^{x^2}+1}$$

Summary

The solution(s) found are the following

$$y = \frac{\sqrt{-\left(2c_1 \mathrm{e}^{x^2}+1\right)\mathrm{e}^{x^2}}}{2c_1 \mathrm{e}^{x^2}+1} \tag{1}$$

$$y = -\frac{\sqrt{-\left(2c_1 \mathrm{e}^{x^2}+1\right)\mathrm{e}^{x^2}}}{2c_1 \mathrm{e}^{x^2}+1} \tag{2}$$

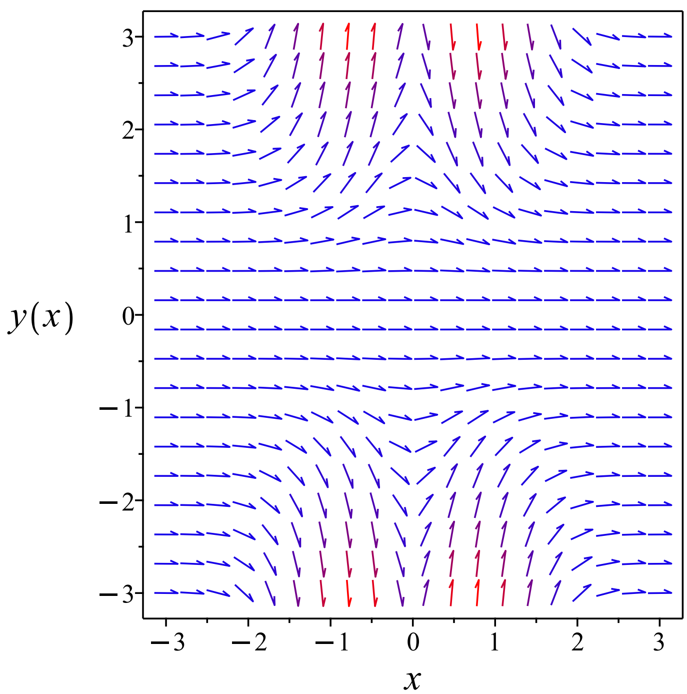


Figure 5: Slope field plot

Verification of solutions

$$y = \frac{\sqrt{-\left(2c_1 e^{x^2}+1\right) e^{x^2}}}{2c_1 e^{x^2}+1}$$

Verified OK.

$$y = -\frac{\sqrt{-\left(2c_1 e^{x^2}+1\right) e^{x^2}}}{2c_1 e^{x^2}+1}$$

Verified OK.

1.5.2 Maple step by step solution

Let's solve

$x y^3 + e^{x^2} y' = 0$

- Highest derivative means the order of the ODE is 1

  y'

- Separate variables

  $\frac{y'}{y^3} = -\frac{x}{e^{x^2}}$

- Integrate both sides with respect to x

  $\int \frac{y'}{y^3} dx = \int -\frac{x}{e^{x^2}} dx + c_1$

- Evaluate integral

  $-\frac{1}{2y^2} = \frac{1}{2 e^{x^2}} + c_1$

- Solve for y

  $\left\{y = \frac{\sqrt{-\left(2c_1 e^{x^2}+1\right) e^{x^2}}}{2c_1 e^{x^2}+1}, y = -\frac{\sqrt{-\left(2c_1 e^{x^2}+1\right) e^{x^2}}}{2c_1 e^{x^2}+1}\right\}$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
<- Bernoulli successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 33

```
dsolve(x*y(x)^3+exp(x^2)*diff(y(x),x)=0,y(x), singsol=all)
```

$$y(x) = \frac{1}{\sqrt{c_1 - e^{-x^2}}}$$
$$y(x) = -\frac{1}{\sqrt{c_1 - e^{-x^2}}}$$

✓ Solution by Mathematica

Time used: 7.124 (sec). Leaf size: 70

```
DSolve[x*y[x]^3+Exp[x^2]*y'[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to -\frac{ie^{\frac{x^2}{2}}}{\sqrt{1 + 2c_1 e^{x^2}}}$$
$$y(x) \to \frac{ie^{\frac{x^2}{2}}}{\sqrt{1 + 2c_1 e^{x^2}}}$$
$$y(x) \to 0$$

1.6 problem 6

Internal problem ID [3151]
Internal file name [`OUTPUT/2643_Sunday_June_05_2022_08_37_59_AM_34888727/index.tex`]

Book: Differential equations for engineers by Wei-Chau XIE, Cambridge Press 2010
Section: Chapter 2. First-Order and Simple Higher-Order Differential Equations. Page 78
Problem number: 6.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program :

Maple gives the following as the ode type

`[_separable]`

$$\boxed{x\cos(y)^2+\tan(y)\,y'=0}$$

1.6.1 Solving as separable ode

In canonical form the ODE is

$$\begin{aligned}
y' &= F(x,y)\\
&= f(x)g(y)\\
&= -x\cos(y)^2\cot(y)
\end{aligned}$$

Where $f(x)=-x$ and $g(y)=\cos(y)^2\cot(y)$. Integrating both sides gives

$$\begin{aligned}
\frac{1}{\cos(y)^2\cot(y)}\,dy &= -x\,dx\\
\int\frac{1}{\cos(y)^2\cot(y)}\,dy &= \int -x\,dx\\
\frac{1}{2\cot(y)^2} &= -\frac{x^2}{2}+c_1
\end{aligned}$$

Which results in

$$y = \operatorname{arccot}\left(\frac{1}{\sqrt{-x^2 + 2c_1}}\right)$$
$$y = \pi - \operatorname{arccot}\left(\frac{1}{\sqrt{-x^2 + 2c_1}}\right)$$

Summary

The solution(s) found are the following

$$y = \operatorname{arccot}\left(\frac{1}{\sqrt{-x^2 + 2c_1}}\right) \tag{1}$$
$$y = \pi - \operatorname{arccot}\left(\frac{1}{\sqrt{-x^2 + 2c_1}}\right) \tag{2}$$

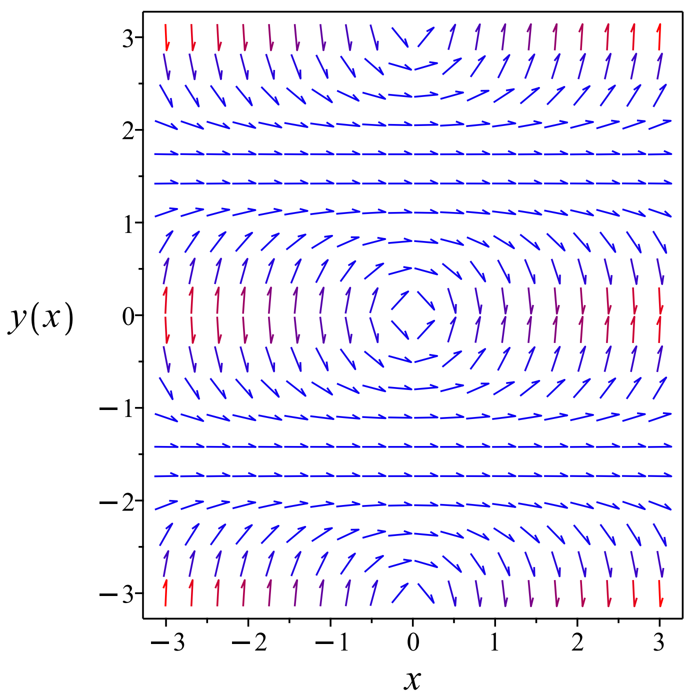


Figure 6: Slope field plot

Verification of solutions

$$y = \operatorname{arccot}\left(\frac{1}{\sqrt{-x^2+2c_1}}\right)$$

Verified OK.

$$y = \pi - \operatorname{arccot}\left(\frac{1}{\sqrt{-x^2+2c_1}}\right)$$

Verified OK.

1.6.2 Maple step by step solution

Let's solve

$x\cos(y)^2 + \tan(y)\,y' = 0$

- Highest derivative means the order of the ODE is 1

  y'

- Separate variables

  $\frac{y'\tan(y)}{\cos(y)^2} = -x$

- Integrate both sides with respect to x

  $\int \frac{y'\tan(y)}{\cos(y)^2}dx = \int -xdx + c_1$

- Evaluate integral

  $\frac{\tan(y)^2}{2} = -\frac{x^2}{2} + c_1$

- Solve for y

  $\left\{y = -\arctan\left(\sqrt{-x^2+2c_1}\right), y = \arctan\left(\sqrt{-x^2+2c_1}\right)\right\}$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
<- separable successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 35

```
dsolve(x*cos(y(x))^2+tan(y(x))*diff(y(x),x)=0,y(x), singsol=all)
```

$$y(x) = \operatorname{arccot}\left(\frac{1}{\sqrt{-x^2-2c_1}}\right)$$
$$y(x) = \frac{\pi}{2} + \arctan\left(\frac{1}{\sqrt{-x^2-2c_1}}\right)$$

✓ Solution by Mathematica

Time used: 1.202 (sec). Leaf size: 103

```
DSolve[x*Cos[y[x]]^2+Tan[y[x]]*y'[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to -\sec^{-1}\left(-\sqrt{-x^2+8c_1}\right)$$
$$y(x) \to \sec^{-1}\left(-\sqrt{-x^2+8c_1}\right)$$
$$y(x) \to -\sec^{-1}\left(\sqrt{-x^2+8c_1}\right)$$
$$y(x) \to \sec^{-1}\left(\sqrt{-x^2+8c_1}\right)$$
$$y(x) \to -\frac{\pi}{2}$$
$$y(x) \to \frac{\pi}{2}$$

1.7 problem 7

Internal problem ID [3152]
Internal file name [OUTPUT/2644_Sunday_June_05_2022_08_38_01_AM_44925495/index.tex]

Book: Differential equations for engineers by Wei-Chau XIE, Cambridge Press 2010
Section: Chapter 2. First-Order and Simple Higher-Order Differential Equations. Page 78
Problem number: 7.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program :

Maple gives the following as the ode type

`[_separable]`

$$\boxed{xy^3 + (y+1)\,\mathrm{e}^{-x} y' = 0}$$

1.7.1 Solving as separable ode

In canonical form the ODE is

$$\begin{aligned} y' &= F(x,y) \\ &= f(x)g(y) \\ &= -\frac{x\,y^3 \mathrm{e}^x}{y+1} \end{aligned}$$

Where $f(x) = -x\,\mathrm{e}^x$ and $g(y) = \frac{y^3}{y+1}$. Integrating both sides gives

$$\begin{aligned} \frac{1}{\frac{y^3}{y+1}}\,dy &= -x\,\mathrm{e}^x\,dx \\ \int \frac{1}{\frac{y^3}{y+1}}\,dy &= \int -x\,\mathrm{e}^x\,dx \\ -\frac{1}{y} - \frac{1}{2y^2} &= -(x-1)\,\mathrm{e}^x + c_1 \end{aligned}$$

Which results in

$$y = -\frac{e^{-x} - \sqrt{-2c_1 e^{-2x} + e^{-2x} + 2x\, e^{-x} - 2\, e^{-x}}}{2\,(c_1 e^{-x} - x + 1)}$$

$$y = -\frac{e^{-x} + \sqrt{-2c_1 e^{-2x} + e^{-2x} + 2x\, e^{-x} - 2\, e^{-x}}}{2\,(c_1 e^{-x} - x + 1)}$$

Summary

The solution(s) found are the following

$$y = -\frac{e^{-x} - \sqrt{-2c_1 e^{-2x} + e^{-2x} + 2x\, e^{-x} - 2\, e^{-x}}}{2\,(c_1 e^{-x} - x + 1)} \tag{1}$$

$$y = -\frac{e^{-x} + \sqrt{-2c_1 e^{-2x} + e^{-2x} + 2x\, e^{-x} - 2\, e^{-x}}}{2\,(c_1 e^{-x} - x + 1)} \tag{2}$$

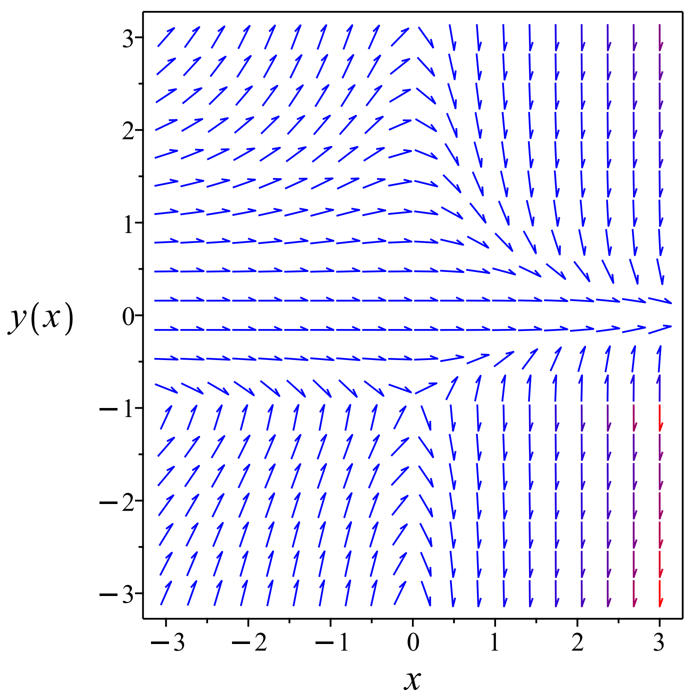


Figure 7: Slope field plot

Verification of solutions

$$y = -\frac{e^{-x} - \sqrt{-2c_1 e^{-2x} + e^{-2x} + 2x\, e^{-x} - 2\, e^{-x}}}{2\,(c_1 e^{-x} - x + 1)}$$

Verified OK.

$$y = -\frac{e^{-x} + \sqrt{-2c_1 e^{-2x} + e^{-2x} + 2x\, e^{-x} - 2\, e^{-x}}}{2\,(c_1 e^{-x} - x + 1)}$$

Verified OK.

1.7.2 Maple step by step solution

Let's solve

$xy^3 + (y+1)\, e^{-x} y' = 0$

- Highest derivative means the order of the ODE is 1

  y'

- Separate variables

  $\frac{y'(y+1)}{y^3} = -\frac{x}{e^{-x}}$

- Integrate both sides with respect to x

  $\int \frac{y'(y+1)}{y^3} dx = \int -\frac{x}{e^{-x}} dx + c_1$

- Evaluate integral

  $-\frac{1}{y} - \frac{1}{2y^2} = -\frac{x-1}{e^{-x}} + c_1$

- Solve for y

  $\left\{y = -\frac{e^{-x} - \sqrt{-2c_1 (e^{-x})^2 + (e^{-x})^2 + 2x\, e^{-x} - 2\, e^{-x}}}{2(c_1 e^{-x} - x + 1)}, y = -\frac{e^{-x} + \sqrt{-2c_1 (e^{-x})^2 + (e^{-x})^2 + 2x\, e^{-x} - 2\, e^{-x}}}{2(c_1 e^{-x} - x + 1)}\right\}$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
<- separable successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 73

```
dsolve(x*y(x)^3+(y(x)+1)*exp(-x)*diff(y(x),x)=0,y(x), singsol=all)
```

$$y(x) = \frac{1 - \sqrt{(2x-2)\,\mathrm{e}^{x} + 2c_1 + 1}}{(2x-2)\,\mathrm{e}^{x} + 2c_1}$$
$$y(x) = \frac{1 + \sqrt{(2x-2)\,\mathrm{e}^{x} + 2c_1 + 1}}{(2x-2)\,\mathrm{e}^{x} + 2c_1}$$

✓ Solution by Mathematica

Time used: 9.963 (sec). Leaf size: 88

```
DSolve[x*y[x]^3+(y[x]+1)*Exp[-x]*y'[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \frac{1 - \sqrt{2e^{x}(x-1) + 1 - 2c_1}}{2e^{x}(x-1) - 2c_1}$$
$$y(x) \to \frac{1 + \sqrt{2e^{x}(x-1) + 1 - 2c_1}}{2e^{x}(x-1) - 2c_1}$$
$$y(x) \to 0$$

1.8 problem 8

Internal problem ID [3153]
Internal file name [`OUTPUT/2645_Sunday_June_05_2022_08_38_02_AM_12623809/index.tex`]

Book: Differential equations for engineers by Wei-Chau XIE, Cambridge Press 2010
Section: Chapter 2. First-Order and Simple Higher-Order Differential Equations. Page 78
Problem number: 8.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"homogeneousTypeD2", "first_order_ode_lie_symmetry_calculated"**

Maple gives the following as the ode type

```
[[_homogeneous, `class A`], _rational, [_Abel, `2nd type`, `
   class A`]]
```

$$\boxed{y' + \frac{x}{y} = -2}$$

1.8.1 Solving as homogeneousTypeD2 ode

Using the change of variables $y = u(x)\,x$ on the above ode results in new ode in $u(x)$

$$u'(x)\,x + u(x) + \frac{1}{u\left(x\right)} = -2$$

In canonical form the ODE is

$$\begin{aligned} u' &= F(x,u) \\ &= f(x)g(u) \\ &= -\frac{(u+1)^2}{xu} \end{aligned}$$

Where $f(x) = -\frac{1}{x}$ and $g(u) = \frac{(u+1)^2}{u}$. Integrating both sides gives

$$
\begin{aligned}
\frac{1}{\frac{(u+1)^2}{u}}\,du &= -\frac{1}{x}\,dx \\
\int \frac{1}{\frac{(u+1)^2}{u}}\,du &= \int -\frac{1}{x}\,dx \\
\ln(u+1) + \frac{1}{u+1} &= -\ln(x) + c_2
\end{aligned}
$$

The solution is

$$
\ln(u(x)+1) + \frac{1}{u(x)+1} + \ln(x) - c_2 = 0
$$

Replacing $u(x)$ in the above solution by $\frac{y}{x}$ results in the solution for y in implicit form

$$
\begin{aligned}
\ln\left(\frac{y}{x}+1\right) + \frac{1}{\frac{y}{x}+1} + \ln(x) - c_2 &= 0 \\
\ln\left(\frac{y+x}{x}\right) + \frac{x}{y+x} + \ln(x) - c_2 &= 0
\end{aligned}
$$

Summary

The solution(s) found are the following

$$
\ln\left(\frac{y+x}{x}\right) + \frac{x}{y+x} + \ln(x) - c_2 = 0 \tag{1}
$$

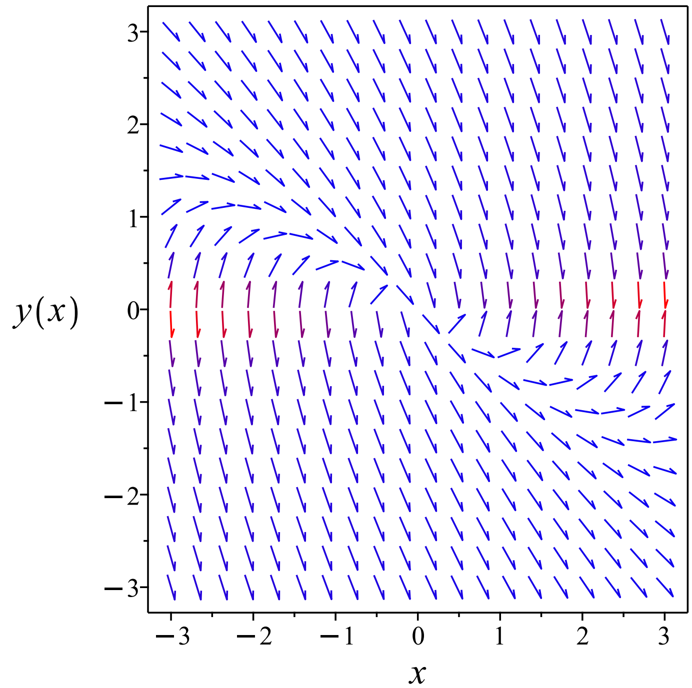


Figure 8: Slope field plot

Verification of solutions

$$\ln\left(\frac{y+x}{x}\right)+\frac{x}{y+x}+\ln(x)-c_2=0$$

Verified OK.

1.8.2 Solving as first order ode lie symmetry calculated ode

Writing the ode as

$$y' = -\frac{x+2y}{y}$$
$$y' = \omega(x,y)$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x+\omega(\eta_y-\xi_x)-\omega^2\xi_y-\omega_x\xi-\omega_y\eta=0 \tag{A}$$

The type of this ode is not in the lookup table. To determine ξ,η then (A) is solved using ansatz. Making bivariate polynomials of degree 1 to use as anstaz gives

$$\xi = xa_2+ya_3+a_1 \tag{1E}$$
$$\eta = xb_2+yb_3+b_1 \tag{2E}$$

Where the unknown coefficients are

$$\{a_1, a_2, a_3, b_1, b_2, b_3\}$$

Substituting equations (1E,2E) and ω into (A) gives

$$b_2 - \frac{(x+2y)(b_3-a_2)}{y} - \frac{(x+2y)^2 a_3}{y^2} + \frac{xa_2 + ya_3 + a_1}{y} - \left(-\frac{2}{y} + \frac{x+2y}{y^2}\right)(xb_2 + yb_3 + b_1) = 0 \tag{5E}$$

Putting the above in normal form gives

$$-\frac{x^2a_3 + x^2b_2 - 2xya_2 + 4xya_3 + 2xyb_3 - 2y^2a_2 + 3y^2a_3 - b_2y^2 + 2y^2b_3 + xb_1 - ya_1}{y^2} = 0$$

Setting the numerator to zero gives

$$-x^2a_3 - x^2b_2 + 2xya_2 - 4xya_3 - 2xyb_3 + 2y^2a_2 - 3y^2a_3 + b_2y^2 - 2y^2b_3 - xb_1 + ya_1 = 0 \tag{6E}$$

Looking at the above PDE shows the following are all the terms with $\{x, y\}$ in them.

$$\{x, y\}$$

The following substitution is now made to be able to collect on all terms with $\{x, y\}$ in them

$$\{x = v_1, y = v_2\}$$

The above PDE (6E) now becomes

$$2a_2v_1v_2 + 2a_2v_2^2 - a_3v_1^2 - 4a_3v_1v_2 - 3a_3v_2^2 - b_2v_1^2 + b_2v_2^2 - 2b_3v_1v_2 - 2b_3v_2^2 + a_1v_2 - b_1v_1 = 0 \tag{7E}$$

Collecting the above on the terms v_i introduced, and these are

$$\{v_1, v_2\}$$

Equation (7E) now becomes

$$(-a_3 - b_2)\, v_1^2 + (2a_2 - 4a_3 - 2b_3)\, v_1 v_2 - b_1 v_1 + (2a_2 - 3a_3 + b_2 - 2b_3)\, v_2^2 + a_1 v_2 = 0 \quad (8E)$$

Setting each coefficients in (8E) to zero gives the following equations to solve

$$\begin{aligned} a_1 &= 0 \\ -b_1 &= 0 \\ -a_3 - b_2 &= 0 \\ 2a_2 - 4a_3 - 2b_3 &= 0 \\ 2a_2 - 3a_3 + b_2 - 2b_3 &= 0 \end{aligned}$$

Solving the above equations for the unknowns gives

$$\begin{aligned} a_1 &= 0 \\ a_2 &= -2b_2 + b_3 \\ a_3 &= -b_2 \\ b_1 &= 0 \\ b_2 &= b_2 \\ b_3 &= b_3 \end{aligned}$$

Substituting the above solution in the anstaz (1E,2E) (using 1 as arbitrary value for any unknown in the RHS) gives

$$\begin{aligned} \xi &= x \\ \eta &= y \end{aligned}$$

Shifting is now applied to make $\xi = 0$ in order to simplify the rest of the computation

$$\begin{aligned} \eta &= \eta - \omega(x, y)\, \xi \\ &= y - \left(-\frac{x + 2y}{y}\right)(x) \\ &= \frac{x^2 + 2xy + y^2}{y} \\ \xi &= 0 \end{aligned}$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x, y) \to (R, S)$ where (R, S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}$$

The above comes from the requirements that $\left(\xi\frac{\partial}{\partial x} + \eta\frac{\partial}{\partial y}\right) S(x,y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$R = x$$

S is found from

$$\begin{aligned} S &= \int \frac{1}{\eta} dy \\ &= \int \frac{1}{\frac{x^2+2xy+y^2}{y}} dy \end{aligned}$$

Which results in

$$S = \ln(y+x) + \frac{x}{y+x}$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x,y)S_y}{R_x + \omega(x,y)R_y} \tag{2}$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x,y)$ is the right hand side of the original ode given by

$$\omega(x,y) = -\frac{x+2y}{y}$$

Evaluating all the partial derivatives gives

$$\begin{aligned} R_x &= 1 \\ R_y &= 0 \\ S_x &= \frac{x+2y}{(y+x)^2} \\ S_y &= \frac{y}{(y+x)^2} \end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = 0 \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = 0$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$S(R) = c_1 \tag{4}$$

To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$\frac{(y+x)\ln(y+x)+x}{y+x} = c_1$$

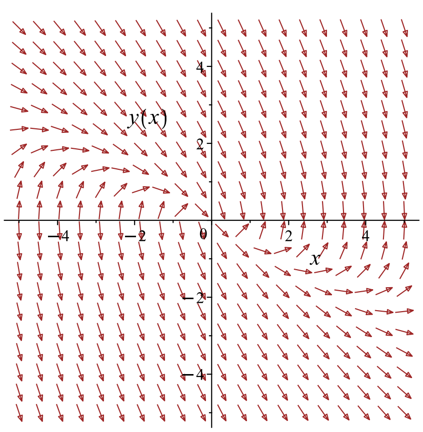
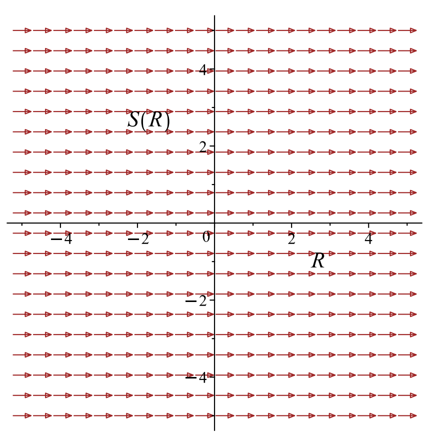
Which simplifies to

$$\frac{(y+x)\ln(y+x)+x}{y+x} = c_1$$

Which gives

$$y = e^{\operatorname{LambertW}\left(-x\,e^{-c_1}\right)+c_1} - x$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in x, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dx} = -\frac{x+2y}{y}$ 	$R = x$ $S = \frac{(y+x)\ln(y+x)+x}{y+x}$	$\frac{dS}{dR} = 0$ 

Summary

The solution(s) found are the following

$$y = \mathrm{e}^{\mathrm{LambertW}\left(-x\,\mathrm{e}^{-c_1}\right)+c_1} - x \tag{1}$$

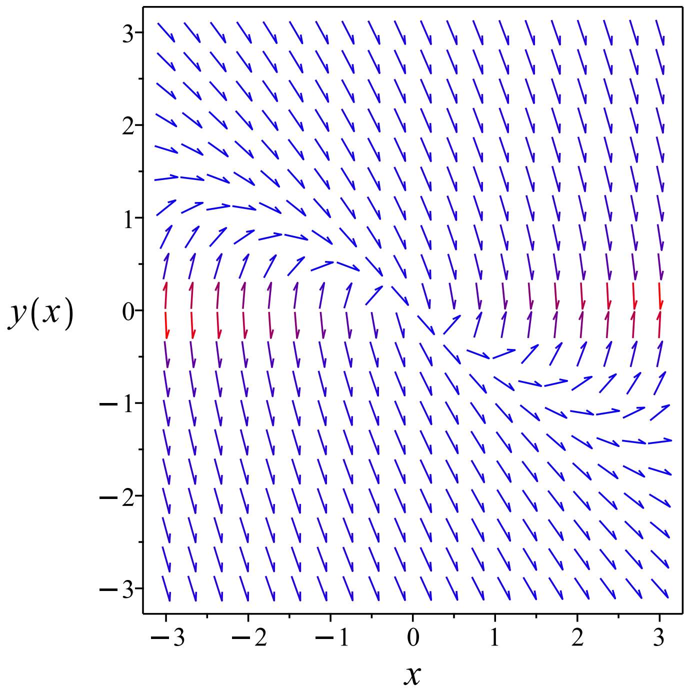


Figure 9: Slope field plot

Verification of solutions

$$y = \mathrm{e}^{\mathrm{LambertW}\left(-x\,\mathrm{e}^{-c_1}\right)+c_1} - x$$

Verified OK.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
trying homogeneous types:
trying homogeneous D
<- homogeneous successful`
```

✓ Solution by Maple

Time used: 0.031 (sec). Leaf size: 21

```
dsolve(diff(y(x),x)+x/y(x)+2=0,y(x), singsol=all)
```

$$y(x) = -\frac{x(\text{LambertW}(-c_1 x) + 1)}{\text{LambertW}(-c_1 x)}$$

✓ Solution by Mathematica

Time used: 0.125 (sec). Leaf size: 31

```
DSolve[y'[x]+x/y[x]+2==0,y[x],x,IncludeSingularSolutions -> True]
```

$$\text{Solve}\left[\frac{1}{\frac{y(x)}{x}+1} + \log\left(\frac{y(x)}{x}+1\right) = -\log(x) + c_1, y(x)\right]$$

1.9 problem 9

Internal problem ID [3154]
Internal file name [`OUTPUT/2646_Sunday_June_05_2022_08_38_03_AM_51971513/index.tex`]

Book: Differential equations for engineers by Wei-Chau XIE, Cambridge Press 2010
Section: Chapter 2. First-Order and Simple Higher-Order Differential Equations. Page 78
Problem number: 9.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"homogeneousTypeD", "homogeneousTypeD2", "first_order_ode_lie_symmetry_lookup"**

Maple gives the following as the ode type

```
[[_homogeneous, `class A`], _dAlembert]
```

$$\boxed{xy' - y - x\cot\left(\frac{y}{x}\right) = 0}$$

1.9.1 Solving as homogeneousTypeD ode

Writing the ode as

$$y' = \cot\left(\frac{y}{x}\right) + \frac{y}{x} \tag{A}$$

The given ode has the form

$$y' = \frac{y}{x} + g(x)\, f\left(b\frac{y}{x}\right)^{\frac{n}{m}} \tag{1}$$

Where b is scalar and $g(x)$ is function of x and n, m are integers. The solution is given in Kamke page 20. Using the substitution $y(x) = u(x)\, x$ then

$$\frac{dy}{dx} = \frac{du}{dx}x + u$$

Hence the given ode becomes

$$\frac{du}{dx}x + u = u + g(x)\, f(bu)^{\frac{n}{m}}$$

$$u' = \frac{1}{x}g(x)\, f(bu)^{\frac{n}{m}} \tag{2}$$

The above ode is always separable. This is easily solved for u assuming the integration can be resolved, and then the solution to the original ode becomes $y = ux$. Comapring the given ode (A) with the form (1) shows that

$$\begin{aligned} g(x) &= 1 \\ b &= 1 \\ f\left(\frac{bx}{y}\right) &= \cot\left(\frac{y}{x}\right) \end{aligned}$$

Substituting the above in (2) results in the $u(x)$ ode as

$$u'(x) = \frac{\cot(u(x))}{x}$$

Which is now solved as separable In canonical form the ODE is

$$\begin{aligned} u' &= F(x, u) \\ &= f(x)g(u) \\ &= \frac{\cot(u)}{x} \end{aligned}$$

Where $f(x) = \frac{1}{x}$ and $g(u) = \cot(u)$. Integrating both sides gives

$$\begin{aligned} \frac{1}{\cot(u)}\, du &= \frac{1}{x}\, dx \\ \int \frac{1}{\cot(u)}\, du &= \int \frac{1}{x}\, dx \\ -\ln(\cos(u)) &= \ln(x) + c_1 \end{aligned}$$

Raising both side to exponential gives

$$\frac{1}{\cos(u)} = e^{\ln(x)+c_1}$$

Which simplifies to

$$\sec(u) = c_2 x$$

Therefore the solution is

$$\begin{aligned} y &= ux \\ &= x \operatorname{arcsec}\left(c_2 e^{c_1} x\right) \end{aligned}$$

Summary

The solution(s) found are the following

$$y = x \operatorname{arcsec}\left(c_2 e^{c_1} x\right) \tag{1}$$

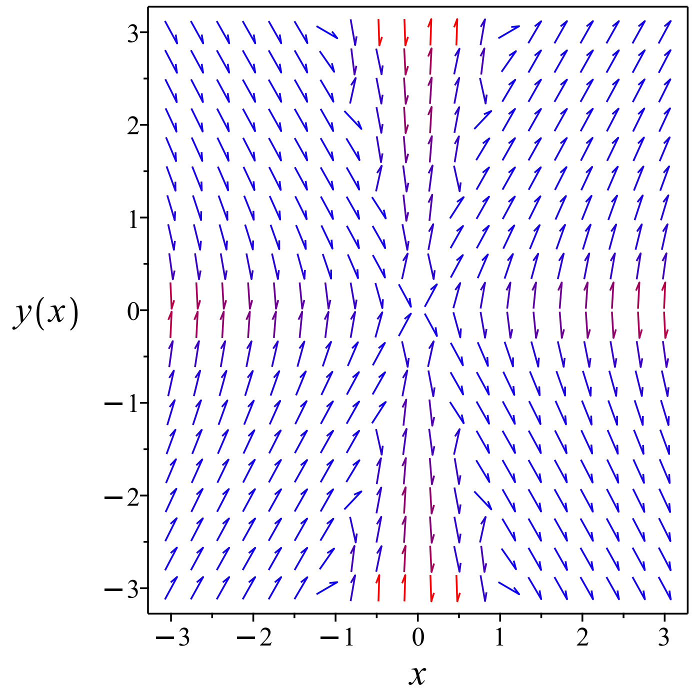


Figure 10: Slope field plot

Verification of solutions

$$y = x \operatorname{arcsec}\left(c_2 e^{c_1} x\right)$$

Verified OK.

1.9.2 Solving as homogeneousTypeD2 ode

Using the change of variables $y = u(x)\, x$ on the above ode results in new ode in $u(x)$

$$x(u'(x)\, x + u(x)) - u(x)\, x - x \cot\left(u(x)\right) = 0$$

In canonical form the ODE is

$$\begin{aligned} u' &= F(x,u) \\ &= f(x)g(u) \\ &= \frac{\cot\left(u\right)}{x} \end{aligned}$$

Where $f(x) = \frac{1}{x}$ and $g(u) = \cot(u)$. Integrating both sides gives

$$
\begin{aligned}
\frac{1}{\cot(u)}\,du &= \frac{1}{x}\,dx \\
\int \frac{1}{\cot(u)}\,du &= \int \frac{1}{x}\,dx \\
-\ln(\cos(u)) &= \ln(x) + c_2
\end{aligned}
$$

Raising both side to exponential gives

$$
\frac{1}{\cos(u)} = \mathrm{e}^{\ln(x)+c_2}
$$

Which simplifies to

$$
\sec(u) = c_3 x
$$

Therefore the solution y is

$$
\begin{aligned}
y &= xu \\
&= x \operatorname{arcsec}\left(c_3 \mathrm{e}^{c_2} x\right)
\end{aligned}
$$

Summary

The solution(s) found are the following

$$
y = x \operatorname{arcsec}\left(c_3 \mathrm{e}^{c_2} x\right) \tag{1}
$$

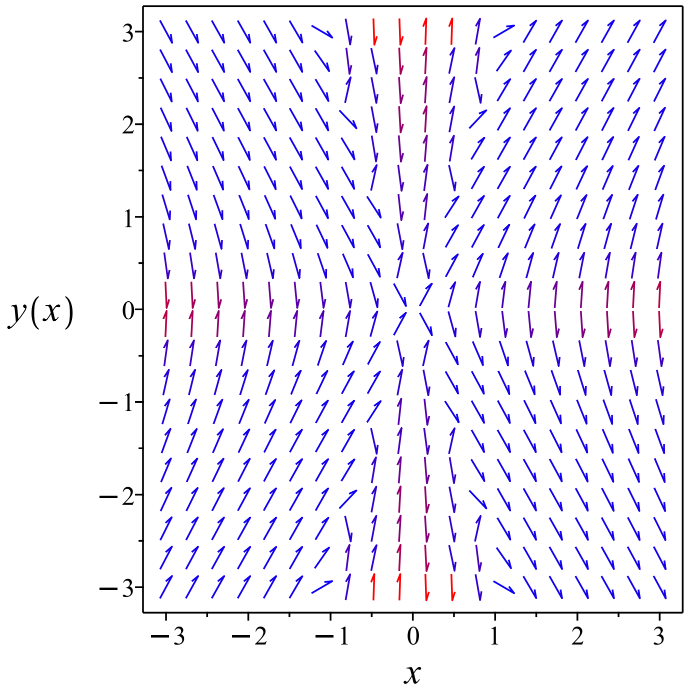


Figure 11: Slope field plot

Verification of solutions

$$y = x \operatorname{arcsec}\left(c_3 e^{c_2} x\right)$$

Verified OK.

1.9.3 Solving as first order ode lie symmetry lookup ode

Writing the ode as

$$y' = \frac{y + x \cot\left(\frac{y}{x}\right)}{x}$$
$$y' = \omega(x, y)$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x + \omega(\eta_y - \xi_x) - \omega^2 \xi_y - \omega_x \xi - \omega_y \eta = 0 \tag{A}$$

The type of this ode is known. It is of type `homogeneous Type D`. Therefore we do not need to solve the PDE (A), and can just use the lookup table shown below to find ξ, η

Table 8: Lie symmetry infinitesimal lookup table for known first order ODE's

ODE class	Form	ξ	η
linear ode	$y' = f(x)y(x) + g(x)$	0	$e^{\int f dx}$
separable ode	$y' = f(x)\, g(y)$	$\frac{1}{f}$	0
quadrature ode	$y' = f(x)$	0	1
quadrature ode	$y' = g(y)$	1	0
homogeneous ODEs of Class A	$y' = f\left(\frac{y}{x}\right)$	x	y
homogeneous ODEs of Class C	$y' = (a + bx + cy)^{\frac{n}{m}}$	1	$-\frac{b}{c}$
homogeneous class D	$y' = \frac{y}{x} + g(x)\, F\left(\frac{y}{x}\right)$	x^2	xy
First order special form ID 1	$y' = g(x)\, e^{h(x)+by} + f(x)$	$\frac{e^{-\int bf(x)dx - h(x)}}{g(x)}$	$\frac{f(x)e^{-\int bf(x)dx - h(x)}}{g(x)}$
polynomial type ode	$y' = \frac{a_1x+b_1y+c_1}{a_2x+b_2y+c_2}$	$\frac{a_1b_2x - a_2b_1x - b_1c_2 + b_2c_1}{a_1b_2 - a_2b_1}$	$\frac{a_1b_2y - a_2b_1y - a_1c_2 - a_2c_1}{a_1b_2 - a_2b_1}$
Bernoulli ode	$y' = f(x)\, y + g(x)\, y^n$	0	$e^{-\int (n-1)f(x)dx} y^n$
Reduced Riccati	$y' = f_1(x)\, y + f_2(x)\, y^2$	0	$e^{-\int f_1 dx}$

The above table shows that

$$\begin{aligned} \xi(x, y) &= x^2 \\ \eta(x, y) &= xy \end{aligned} \tag{A1}$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x, y) \to (R, S)$ where (R, S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y}\right) S(x, y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the

canonical coordinates, where $S(R)$. Therefore

$$
\begin{aligned}
\frac{dy}{dx} &= \frac{\eta}{\xi} \\
&= \frac{xy}{x^2} \\
&= \frac{y}{x}
\end{aligned}
$$

This is easily solved to give

$$
y = c_1 x
$$

Where now the coordinate R is taken as the constant of integration. Hence

$$
R = \frac{y}{x}
$$

And S is found from

$$
\begin{aligned}
dS &= \frac{dx}{\xi} \\
&= \frac{dx}{x^2}
\end{aligned}
$$

Integrating gives

$$
\begin{aligned}
S &= \int \frac{dx}{T} \\
&= -\frac{1}{x}
\end{aligned}
$$

Where the constant of integration is set to zero as we just need one solution. Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$
\frac{dS}{dR} = \frac{S_x + \omega(x,y) S_y}{R_x + \omega(x,y) R_y} \tag{2}
$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x,y)$ is the right hand side of the original ode given by

$$
\omega(x,y) = \frac{y + x \cot\left(\frac{y}{x}\right)}{x}
$$

Evaluating all the partial derivatives gives

$$
\begin{aligned}
R_x &= -\frac{y}{x^2} \\
R_y &= \frac{1}{x} \\
S_x &= \frac{1}{x^2} \\
S_y &= 0
\end{aligned}
$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = \frac{\tan\left(\frac{y}{x}\right)}{x} \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = -\tan(R)\, S(R)$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$S(R) = c_1 \cos(R) \tag{4}$$

To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$-\frac{1}{x} = c_1 \cos\left(\frac{y}{x}\right)$$

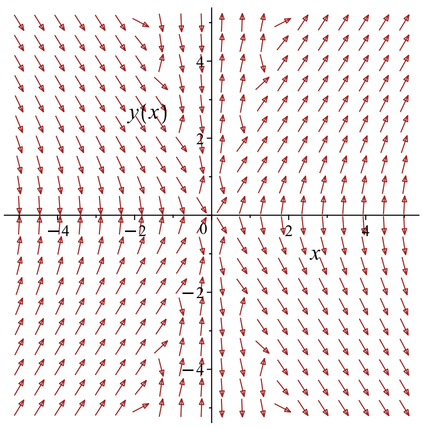
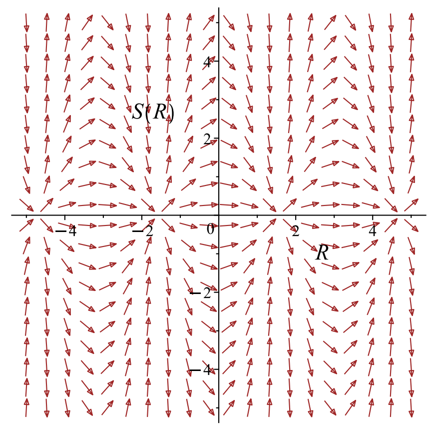
Which simplifies to

$$-\frac{1}{x} = c_1 \cos\left(\frac{y}{x}\right)$$

Which gives

$$y = \left(\pi - \arccos\left(\frac{1}{c_1 x}\right)\right) x$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in x, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dx} = \frac{y + x\cot\left(\frac{y}{x}\right)}{x}$ 	$R = \frac{y}{x}$ $S = -\frac{1}{x}$	$\frac{dS}{dR} = -\tan(R)\, S(R)$ 

Summary

The solution(s) found are the following

$$y = \left(\pi - \arccos\left(\frac{1}{c_1 x}\right)\right) x \tag{1}$$

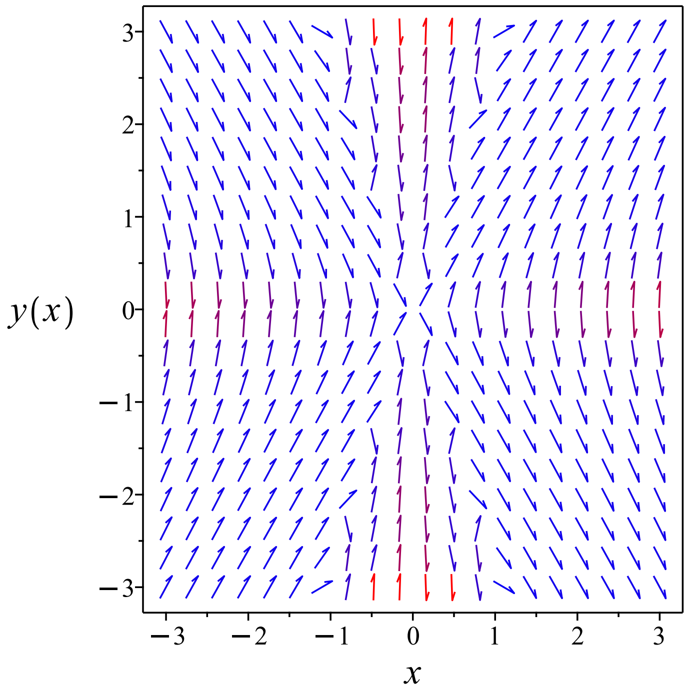


Figure 12: Slope field plot

Verification of solutions

$$y = \left(\pi - \arccos\left(\frac{1}{c_1 x}\right)\right) x$$

Verified OK.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
trying homogeneous types:
trying homogeneous D
<- homogeneous successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 14

```
dsolve(x*diff(y(x),x)-y(x)=x*cot(y(x)/x),y(x), singsol=all)
```

$$y(x) = x \arccos\left(\frac{1}{c_1 x}\right)$$

✓ Solution by Mathematica

Time used: 25.917 (sec). Leaf size: 56

```
DSolve[x*y'[x]-y[x]==x*Cot[y[x]/x],y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to -x \arccos\left(\frac{e^{-c_1}}{x}\right)$$
$$y(x) \to x \arccos\left(\frac{e^{-c_1}}{x}\right)$$
$$y(x) \to -\frac{\pi x}{2}$$
$$y(x) \to \frac{\pi x}{2}$$

1.10 problem 10

Internal problem ID [3155]
Internal file name [`OUTPUT/2647_Sunday_June_05_2022_08_38_04_AM_4756193/index.tex`]

Book: Differential equations for engineers by Wei-Chau XIE, Cambridge Press 2010
Section: Chapter 2. First-Order and Simple Higher-Order Differential Equations. Page 78
Problem number: 10.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"homogeneousTypeD", "homogeneousTypeD2", "first_order_ode_lie_symmetry_lookup"**

Maple gives the following as the ode type

```
[[_homogeneous, `class A`], _dAlembert]
```

$$\boxed{x\cos\left(\frac{y}{x}\right)^2 - y + xy' = 0}$$

1.10.1 Solving as homogeneousTypeD ode

Writing the ode as

$$y' = -\cos\left(\frac{y}{x}\right)^2 + \frac{y}{x} \tag{A}$$

The given ode has the form

$$y' = \frac{y}{x} + g(x)\, f\left(b\frac{y}{x}\right)^{\frac{n}{m}} \tag{1}$$

Where b is scalar and $g(x)$ is function of x and n, m are integers. The solution is given in Kamke page 20. Using the substitution $y(x) = u(x)\, x$ then

$$\frac{dy}{dx} = \frac{du}{dx}x + u$$

Hence the given ode becomes

$$\frac{du}{dx}x + u = u + g(x)\, f(bu)^{\frac{n}{m}}$$
$$u' = \frac{1}{x}g(x)\, f(bu)^{\frac{n}{m}} \tag{2}$$

The above ode is always separable. This is easily solved for u assuming the integration can be resolved, and then the solution to the original ode becomes $y = ux$. Comapring the given ode (A) with the form (1) shows that

$$g(x) = -1$$
$$b = 1$$
$$f\left(\frac{bx}{y}\right) = \cos\left(\frac{y}{x}\right)$$

Substituting the above in (2) results in the $u(x)$ ode as

$$u'(x) = -\frac{\cos\left(u(x)\right)^{2}}{x}$$

Which is now solved as separable In canonical form the ODE is

$$\begin{aligned} u' &= F(x,u) \\ &= f(x)g(u) \\ &= -\frac{\cos\left(u\right)^{2}}{x} \end{aligned}$$

Where $f(x) = -\frac{1}{x}$ and $g(u) = \cos\left(u\right)^{2}$. Integrating both sides gives

$$\frac{1}{\cos\left(u\right)^{2}}\, du = -\frac{1}{x}\, dx$$
$$\int \frac{1}{\cos\left(u\right)^{2}}\, du = \int -\frac{1}{x}\, dx$$
$$\tan\left(u\right) = -\ln\left(x\right) + c_1$$

The solution is

$$\tan\left(u(x)\right) + \ln\left(x\right) - c_1 = 0$$

Therefore the solution is found using $y = ux$. Hence

$$\tan\left(\frac{y}{x}\right) + \ln\left(x\right) - c_1 = 0$$

Summary

The solution(s) found are the following

$$\tan\left(\frac{y}{x}\right)+\ln\left(x\right)-c_1=0 \tag{1}$$

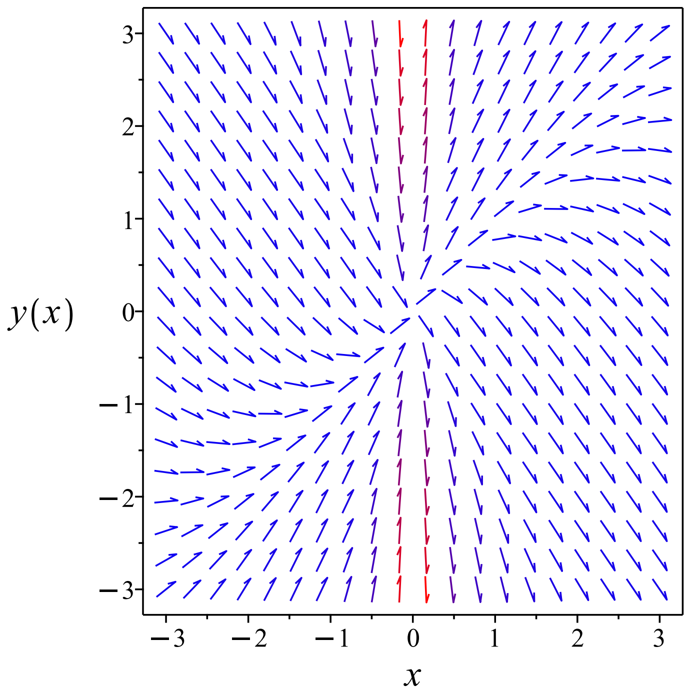


Figure 13: Slope field plot

Verification of solutions

$$\tan\left(\frac{y}{x}\right)+\ln\left(x\right)-c_1=0$$

Verified OK.

1.10.2 Solving as homogeneousTypeD2 ode

Using the change of variables $y = u(x)\,x$ on the above ode results in new ode in $u(x)$

$$x\cos\left(u(x)\right)^2-u(x)\,x+x(u'(x)\,x+u(x))=0$$

In canonical form the ODE is

$$\begin{aligned} u' &= F(x,u) \\ &= f(x)g(u) \\ &= -\frac{\cos\left(u\right)^2}{x} \end{aligned}$$

Where $f(x) = -\frac{1}{x}$ and $g(u) = \cos(u)^2$. Integrating both sides gives

$$\frac{1}{\cos(u)^2}\,du = -\frac{1}{x}\,dx$$
$$\int \frac{1}{\cos(u)^2}\,du = \int -\frac{1}{x}\,dx$$
$$\tan(u) = -\ln(x) + c_2$$

The solution is

$$\tan(u(x)) + \ln(x) - c_2 = 0$$

Replacing $u(x)$ in the above solution by $\frac{y}{x}$ results in the solution for y in implicit form

$$\tan\left(\frac{y}{x}\right) + \ln(x) - c_2 = 0$$
$$\tan\left(\frac{y}{x}\right) + \ln(x) - c_2 = 0$$

Summary

The solution(s) found are the following

$$\tan\left(\frac{y}{x}\right) + \ln(x) - c_2 = 0 \tag{1}$$

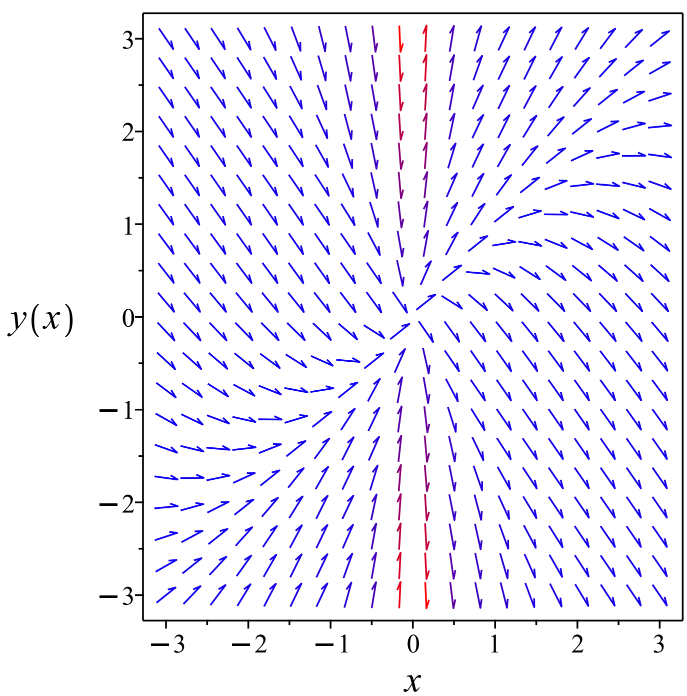


Figure 14: Slope field plot

Verification of solutions

$$\tan\left(\frac{y}{x}\right) + \ln(x) - c_2 = 0$$

Verified OK.

1.10.3 Solving as first order ode lie symmetry lookup ode

Writing the ode as

$$y' = -\frac{x\cos\left(\frac{y}{x}\right)^2 - y}{x}$$
$$y' = \omega(x, y)$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x + \omega(\eta_y - \xi_x) - \omega^2\xi_y - \omega_x\xi - \omega_y\eta = 0 \tag{A}$$

The type of this ode is known. It is of type `homogeneous Type D`. Therefore we do not need to solve the PDE (A), and can just use the lookup table shown below to find ξ, η

Table 10: Lie symmetry infinitesimal lookup table for known first order ODE's

ODE class	Form	ξ	η
linear ode	$y' = f(x)y(x) + g(x)$	0	$e^{\int f dx}$
separable ode	$y' = f(x)\, g(y)$	$\frac{1}{f}$	0
quadrature ode	$y' = f(x)$	0	1
quadrature ode	$y' = g(y)$	1	0
homogeneous ODEs of Class A	$y' = f\left(\frac{y}{x}\right)$	x	y
homogeneous ODEs of Class C	$y' = (a + bx + cy)^{\frac{n}{m}}$	1	$-\frac{b}{c}$
homogeneous class D	$y' = \frac{y}{x} + g(x)\, F\left(\frac{y}{x}\right)$	x^2	xy
First order special form ID 1	$y' = g(x)\, e^{h(x)+by} + f(x)$	$\frac{e^{-\int bf(x)dx-h(x)}}{g(x)}$	$\frac{f(x)e^{-\int bf(x)dx-h(x)}}{g(x)}$
polynomial type ode	$y' = \frac{a_1x+b_1y+c_1}{a_2x+b_2y+c_2}$	$\frac{a_1b_2x-a_2b_1x-b_1c_2+b_2c_1}{a_1b_2-a_2b_1}$	$\frac{a_1b_2y-a_2b_1y-a_1c_2-a_2c_1}{a_1b_2-a_2b_1}$
Bernoulli ode	$y' = f(x)\, y + g(x)\, y^n$	0	$e^{-\int (n-1)f(x)dx}y^n$
Reduced Riccati	$y' = f_1(x)\, y + f_2(x)\, y^2$	0	$e^{-\int f_1 dx}$

The above table shows that

$$
\begin{aligned}
\xi(x, y) &= x^2 \\
\eta(x, y) &= xy
\end{aligned} \tag{A1}
$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x, y) \to (R, S)$ where (R, S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$
\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}
$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y}\right) S(x, y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the

canonical coordinates, where $S(R)$. Therefore

$$\begin{aligned}\frac{dy}{dx} &= \frac{\eta}{\xi}\\ &= \frac{xy}{x^2}\\ &= \frac{y}{x}\end{aligned}$$

This is easily solved to give

$$y = c_1 x$$

Where now the coordinate R is taken as the constant of integration. Hence

$$R = \frac{y}{x}$$

And S is found from

$$\begin{aligned}dS &= \frac{dx}{\xi}\\ &= \frac{dx}{x^2}\end{aligned}$$

Integrating gives

$$\begin{aligned}S &= \int \frac{dx}{T}\\ &= -\frac{1}{x}\end{aligned}$$

Where the constant of integration is set to zero as we just need one solution. Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x,y) S_y}{R_x + \omega(x,y) R_y} \tag{2}$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x,y)$ is the right hand side of the original ode given by

$$\omega(x,y) = -\frac{x\cos\left(\frac{y}{x}\right)^2 - y}{x}$$

Evaluating all the partial derivatives gives

$$\begin{aligned}R_x &= -\frac{y}{x^2}\\ R_y &= \frac{1}{x}\\ S_x &= \frac{1}{x^2}\\ S_y &= 0\end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = -\frac{\sec\left(\frac{y}{x}\right)^2}{x} \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = \sec(R)^2 S(R)$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$S(R) = c_1 e^{\tan(R)} \tag{4}$$

To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$-\frac{1}{x} = c_1 e^{\tan\left(\frac{y}{x}\right)}$$

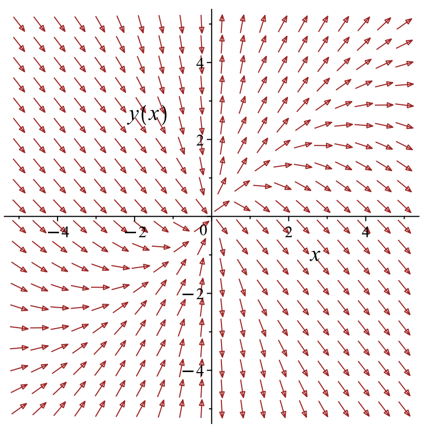
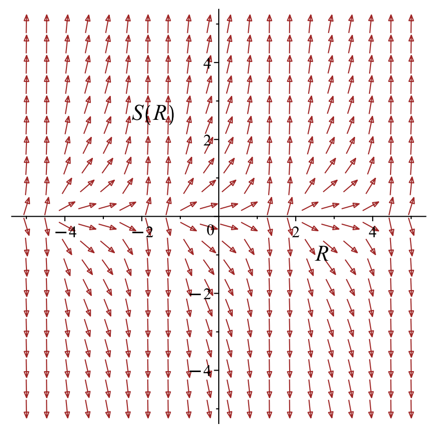
Which simplifies to

$$-\frac{1}{x} = c_1 e^{\tan\left(\frac{y}{x}\right)}$$

Which gives

$$y = \arctan\left(\ln\left(-\frac{1}{c_1 x}\right)\right) x$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

<table>
<tr><th>Original ode in x, y coordinates</th><th>Canonical coordinates transformation</th><th>ODE in canonical coordinates (R, S)</th></tr>
<tr><td>$\frac{dy}{dx} = -\frac{x \cos\left(\frac{y}{x}\right)^2 - y}{x}$

</td><td>$R = \frac{y}{x}$
$S = -\frac{1}{x}$</td><td>$\frac{dS}{dR} = \sec(R)^2 S(R)$

</td></tr>
</table>

Summary

The solution(s) found are the following

$$y = \arctan\left(\ln\left(-\frac{1}{c_1 x}\right)\right) x \tag{1}$$

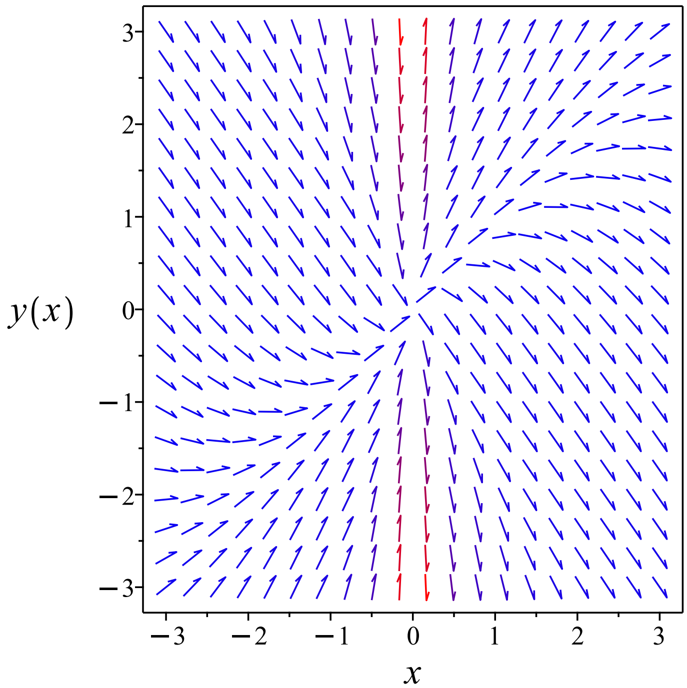


Figure 15: Slope field plot

Verification of solutions

$$y = \arctan\left(\ln\left(-\frac{1}{c_1 x}\right)\right) x$$

Verified OK.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
trying homogeneous types:
trying homogeneous D
<- homogeneous successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 12

```
dsolve((x*cos(y(x)/x)^2-y(x))+x*diff(y(x),x)=0,y(x), singsol=all)
```

$$y(x) = -\arctan(\ln(x) + c_1)\, x$$

✓ Solution by Mathematica

Time used: 0.5 (sec). Leaf size: 37

```
DSolve[(x*Cos[y[x]/x]^2-y[x])+x*y'[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to x \arctan(-\log(x) + 2c_1)$$
$$y(x) \to -\frac{\pi x}{2}$$
$$y(x) \to \frac{\pi x}{2}$$

1.11 problem 11

Internal problem ID [3156]
Internal file name [`OUTPUT/2648_Sunday_June_05_2022_08_38_05_AM_95895671/index.tex`]

Book: Differential equations for engineers by Wei-Chau XIE, Cambridge Press 2010
Section: Chapter 2. First-Order and Simple Higher-Order Differential Equations. Page 78
Problem number: 11.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"exactByInspection", "first_order_ode_lie_symmetry_calculated"**

Maple gives the following as the ode type

```
[[_homogeneous, `class A`], _dAlembert]
```

$$\boxed{xy' - y(1 + \ln(y) - \ln(x)) = 0}$$

1.11.1 Solving as first order ode lie symmetry calculated ode

Writing the ode as

$$y' = -\frac{y(-1 + \ln(x) - \ln(y))}{x}$$
$$y' = \omega(x, y)$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x + \omega(\eta_y - \xi_x) - \omega^2 \xi_y - \omega_x \xi - \omega_y \eta = 0 \tag{A}$$

The type of this ode is not in the lookup table. To determine ξ, η then (A) is solved using ansatz. Making bivariate polynomials of degree 1 to use as anstaz gives

$$\xi = xa_2 + ya_3 + a_1 \tag{1E}$$
$$\eta = xb_2 + yb_3 + b_1 \tag{2E}$$

Where the unknown coefficients are

$$\{a_1, a_2, a_3, b_1, b_2, b_3\}$$

Substituting equations (1E,2E) and ω into (A) gives

$$b_2 - \frac{y(-1+\ln(x)-\ln(y))(b_3-a_2)}{x} - \frac{y^2(-1+\ln(x)-\ln(y))^2 a_3}{x^2} - \left(-\frac{y}{x^2} + \frac{y(-1+\ln(x)-\ln(y))}{x^2}\right)(xa_2+ya_3+a_1) - \left(-\frac{-1+\ln(x)-\ln(y)}{x} + \frac{1}{x}\right)(xb_2+yb_3+b_1) = 0 \tag{5E}$$

Putting the above in normal form gives

$$-\frac{\ln(x)^2 y^2 a_3 - 2\ln(x)\ln(y) y^2 a_3 + \ln(y)^2 y^2 a_3 - \ln(x) x^2 b_2 - \ln(x) y^2 a_3 + \ln(y) x^2 b_2 + \ln(y) y^2 a_3 - \ln}{x^2} = 0$$

Setting the numerator to zero gives

$$-\ln(x)^2 y^2 a_3 + 2\ln(x)\ln(y) y^2 a_3 - \ln(y)^2 y^2 a_3 + \ln(x) x^2 b_2 + \ln(x) y^2 a_3 - \ln(y) x^2 b_2 - \ln(y) y^2 a_3 + \ln(x) x b_1 - \ln(x) y a_1 - \ln(y) x b_1 + \ln(y) y a_1 - b_2 x^2 + x y a_2 - x y b_3 + y^2 a_3 - 2 x b_1 + 2 y a_1 = 0 \tag{6E}$$

Looking at the above PDE shows the following are all the terms with $\{x, y\}$ in them.

$$\{x, y, \ln(x), \ln(y)\}$$

The following substitution is now made to be able to collect on all terms with $\{x, y\}$ in them

$$\{x = v_1, y = v_2, \ln(x) = v_3, \ln(y) = v_4\}$$

The above PDE (6E) now becomes

$$-v_3^2 v_2^2 a_3 + 2v_3 v_4 v_2^2 a_3 - v_4^2 v_2^2 a_3 + v_3 v_2^2 a_3 - v_4 v_2^2 a_3 + v_3 v_1^2 b_2 - v_4 v_1^2 b_2 - v_3 v_2 a_1 + v_4 v_2 a_1 + v_1 v_2 a_2 + v_2^2 a_3 + v_3 v_1 b_1 - v_4 v_1 b_1 - b_2 v_1^2 - v_1 v_2 b_3 + 2v_2 a_1 - 2v_1 b_1 = 0 \tag{7E}$$

Collecting the above on the terms v_i introduced, and these are

$$\{v_1, v_2, v_3, v_4\}$$

Equation (7E) now becomes

$$\begin{aligned} v_3 v_1^2 b_2 - v_4 v_1^2 b_2 - b_2 v_1^2 + (-b_3 + a_2) v_1 v_2 + v_3 v_1 b_1 - v_4 v_1 b_1 - 2 v_1 b_1 - v_3^2 v_2^2 a_3 \\ + 2 v_3 v_4 v_2^2 a_3 + v_3 v_2^2 a_3 - v_4^2 v_2^2 a_3 - v_4 v_2^2 a_3 + v_2^2 a_3 - v_3 v_2 a_1 + v_4 v_2 a_1 + 2 v_2 a_1 = 0 \end{aligned} \tag{8E}$$

Setting each coefficients in (8E) to zero gives the following equations to solve

$$\begin{aligned} a_1 &= 0 \\ a_3 &= 0 \\ b_1 &= 0 \\ b_2 &= 0 \\ -a_1 &= 0 \\ 2a_1 &= 0 \\ -a_3 &= 0 \\ 2a_3 &= 0 \\ -2b_1 &= 0 \\ -b_1 &= 0 \\ -b_2 &= 0 \\ -b_3 + a_2 &= 0 \end{aligned}$$

Solving the above equations for the unknowns gives

$$\begin{aligned} a_1 &= 0 \\ a_2 &= b_3 \\ a_3 &= 0 \\ b_1 &= 0 \\ b_2 &= 0 \\ b_3 &= b_3 \end{aligned}$$

Substituting the above solution in the anstaz (1E,2E) (using 1 as arbitrary value for any unknown in the RHS) gives

$$\begin{aligned} \xi &= x \\ \eta &= y \end{aligned}$$

Shifting is now applied to make $\xi = 0$ in order to simplify the rest of the computation

$$\begin{aligned}
\eta &= \eta - \omega(x, y)\,\xi \\
&= y - \left(-\frac{y(-1 + \ln(x) - \ln(y))}{x}\right)(x) \\
&= \ln(x)\,y - \ln(y)\,y \\
\xi &= 0
\end{aligned}$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x, y) \to (R, S)$ where (R, S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y}\right) S(x, y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$R = x$$

S is found from

$$\begin{aligned}
S &= \int \frac{1}{\eta} dy \\
&= \int \frac{1}{\ln(x)\,y - \ln(y)\,y} dy
\end{aligned}$$

Which results in

$$S = -\ln(\ln(x) - \ln(y))$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x, y) S_y}{R_x + \omega(x, y) R_y} \tag{2}$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x, y)$ is the right hand side of the original ode given by

$$\omega(x, y) = -\frac{y(-1 + \ln(x) - \ln(y))}{x}$$

Evaluating all the partial derivatives gives

$$R_x = 1$$
$$R_y = 0$$
$$S_x = -\frac{1}{x\left(\ln\left(x\right)-\ln\left(y\right)\right)}$$
$$S_y = \frac{1}{y\left(\ln\left(x\right)-\ln\left(y\right)\right)}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = -\frac{1}{x} \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = -\frac{1}{R}$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$S(R) = -\ln\left(R\right)+c_1 \tag{4}$$

To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$-\ln\left(\ln\left(x\right)-\ln\left(y\right)\right) = -\ln\left(x\right)+c_1$$

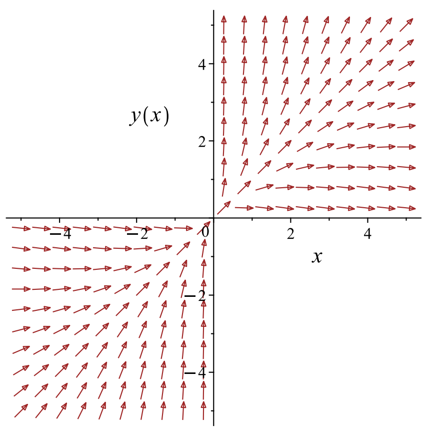
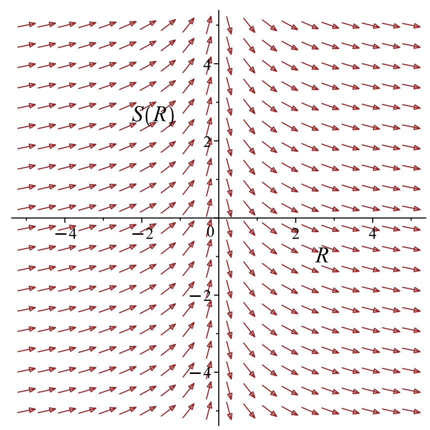
Which simplifies to

$$-\ln\left(\ln\left(x\right)-\ln\left(y\right)\right) = -\ln\left(x\right)+c_1$$

Which gives

$$y = \mathrm{e}^{\left(\mathrm{e}^{c_1}\ln(x)-x\right)\mathrm{e}^{-c_1}}$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in x, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dx} = -\frac{y(-1+\ln(x)-\ln(y))}{x}$ 	$R = x$ $S = -\ln(\ln(x) - \ln(y))$	$\frac{dS}{dR} = -\frac{1}{R}$ 

Summary

The solution(s) found are the following

$$y = e^{(e^{c_1} \ln(x) - x) e^{-c_1}} \tag{1}$$

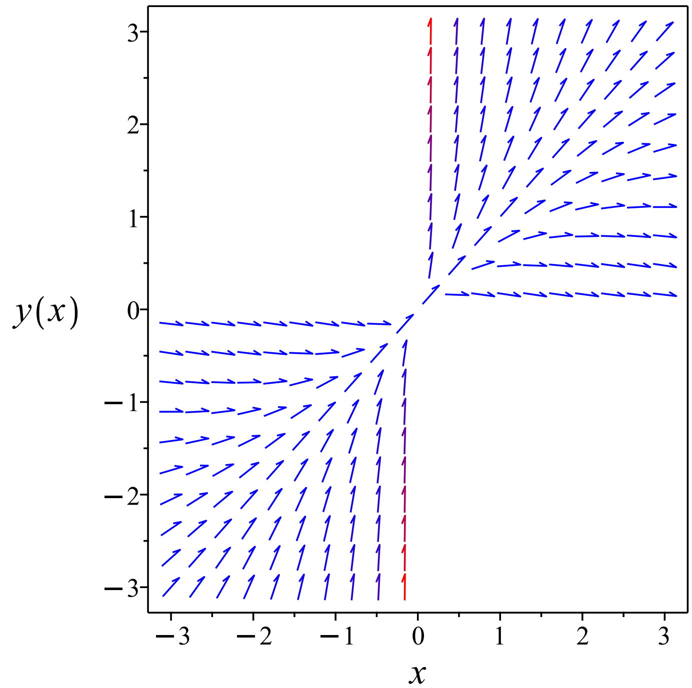


Figure 16: Slope field plot

Verification of solutions

$$y = e^{(e^{c_1} \ln(x) - x) e^{-c_1}}$$

Verified OK.

1.11.2 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x, y) + N(x, y) \frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x, y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx} \phi(x, y) = 0$$

Hence

$$\frac{\partial \phi}{\partial x} + \frac{\partial \phi}{\partial y} \frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$
\begin{aligned}
\frac{\partial\phi}{\partial x} &= M \\
\frac{\partial\phi}{\partial y} &= N
\end{aligned}
$$

But since $\frac{\partial^2\phi}{\partial x\partial y} = \frac{\partial^2\phi}{\partial y\partial x}$ then for the above to be valid, we require that

$$
\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}
$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x,y)$ but at least we know now that we can do that since the condition $\frac{\partial^2\phi}{\partial x\partial y} = \frac{\partial^2\phi}{\partial y\partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$
M(x,y)\,\mathrm{d}x + N(x,y)\,\mathrm{d}y = 0 \tag{1A}
$$

Therefore

$$
\begin{aligned}
(x)\,\mathrm{d}y &= (y(1+\ln(y)-\ln(x)))\,\mathrm{d}x \\
(-y(1+\ln(y)-\ln(x)))\,\mathrm{d}x + (x)\,\mathrm{d}y &= 0 \qquad (2A)
\end{aligned}
$$

Comparing (1A) and (2A) shows that

$$
\begin{aligned}
M(x,y) &= -y(1+\ln(y)-\ln(x)) \\
N(x,y) &= x
\end{aligned}
$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$
\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}
$$

Using result found above gives

$$
\begin{aligned}
\frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}(-y(1+\ln(y)-\ln(x))) \\
&= -2+\ln(x)-\ln(y)
\end{aligned}
$$

And

$$
\begin{aligned}
\frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}(x) \\
&= 1
\end{aligned}
$$

Since $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$, then the ODE is not exact. By inspection $\frac{1}{x^2 y}$ is an integrating factor. Therefore by multiplying $M = -y(1 + \ln(y) - \ln(x))$ and $N = x$ by this integrating factor the ode becomes exact. The new M, N are

$$M = -\frac{1 + \ln(y) - \ln(x)}{x^2}$$
$$N = \frac{1}{xy}$$

To solve an ode of the form

$$M(x,y) + N(x,y)\frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x,y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y) = 0$$

Hence

$$\frac{\partial \phi}{\partial x} + \frac{\partial \phi}{\partial y}\frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$\frac{\partial \phi}{\partial x} = M$$
$$\frac{\partial \phi}{\partial y} = N$$

But since $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x,y)$ but at least we know now that we can do that since the condition $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x,y)\,\mathrm{d}x + N(x,y)\,\mathrm{d}y = 0 \tag{1A}$$

Therefore

$$\left(\frac{1}{xy}\right) \mathrm{d}y = \left(\frac{1+\ln(y)-\ln(x)}{x^2}\right) \mathrm{d}x$$

$$\left(-\frac{1+\ln(y)-\ln(x)}{x^2}\right) \mathrm{d}x + \left(\frac{1}{xy}\right) \mathrm{d}y = 0 \tag{2A}$$

Comparing (1A) and (2A) shows that

$$M(x,y) = -\frac{1+\ln(y)-\ln(x)}{x^2}$$
$$N(x,y) = \frac{1}{xy}$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Using result found above gives

$$\frac{\partial M}{\partial y} = \frac{\partial}{\partial y}\left(-\frac{1+\ln(y)-\ln(x)}{x^2}\right)$$
$$= -\frac{1}{x^2 y}$$

And

$$\frac{\partial N}{\partial x} = \frac{\partial}{\partial x}\left(\frac{1}{xy}\right)$$
$$= -\frac{1}{x^2 y}$$

Since $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$, then the ODE is exact The following equations are now set up to solve for the function $\phi(x,y)$

$$\frac{\partial \phi}{\partial x} = M \tag{1}$$
$$\frac{\partial \phi}{\partial y} = N \tag{2}$$

Integrating (1) w.r.t. x gives

$$\int \frac{\partial \phi}{\partial x} \mathrm{d}x = \int M \mathrm{d}x$$
$$\int \frac{\partial \phi}{\partial x} \mathrm{d}x = \int -\frac{1+\ln(y)-\ln(x)}{x^2} \mathrm{d}x$$
$$\phi = \frac{\ln(y)-\ln(x)}{x} + f(y) \tag{3}$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$\frac{\partial \phi}{\partial y} = \frac{1}{xy} + f'(y) \tag{4}$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = \frac{1}{xy}$. Therefore equation (4) becomes

$$\frac{1}{xy} = \frac{1}{xy} + f'(y) \tag{5}$$

Solving equation (5) for $f'(y)$ gives

$$f'(y) = 0$$

Therefore

$$f(y) = c_1$$

Where c_1 is constant of integration. Substituting this result for $f(y)$ into equation (3) gives ϕ

$$\phi = \frac{\ln(y) - \ln(x)}{x} + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = \frac{\ln(y) - \ln(x)}{x}$$

The solution becomes

$$y = \mathrm{e}^{c_1 x} x$$

Summary

The solution(s) found are the following

$$y = \mathrm{e}^{c_1 x} x \tag{1}$$

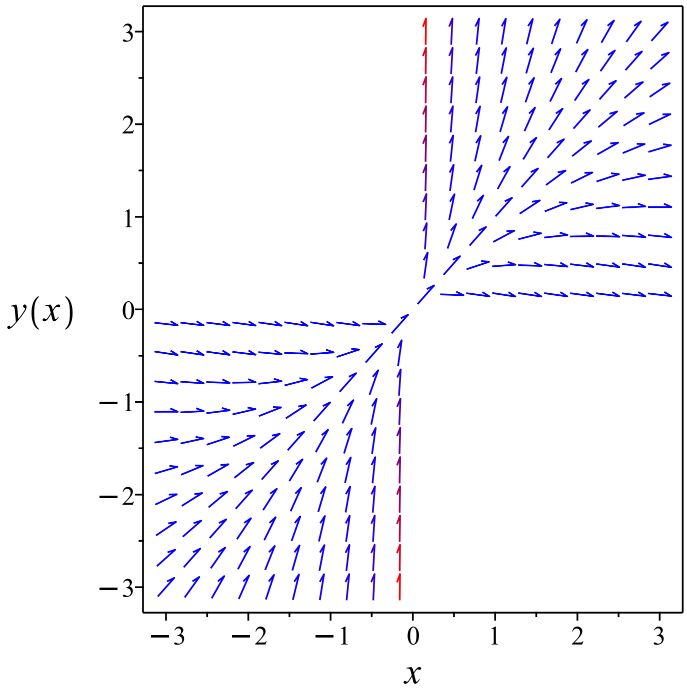


Figure 17: Slope field plot

Verification of solutions

$$y = e^{c_1 x} x$$

Verified OK.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
trying homogeneous types:
trying homogeneous G
1st order, trying the canonical coordinates of the invariance group
<- 1st order, canonical coordinates successful
<- homogeneous successful`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 11

```
dsolve(x*diff(y(x),x)=y(x)*(1+ln(y(x))-ln(x)),y(x), singsol=all)
```

$$y(x) = x\,\mathrm{e}^{-c_1 x}$$

✓ Solution by Mathematica

Time used: 0.228 (sec). Leaf size: 20

```
DSolve[x*y'[x]==y[x]*(1+Log[y[x]]-Log[x]),y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to x e^{e^{c_1 x}}$$
$$y(x) \to x$$

1.12 problem 12

Internal problem ID [3157]
Internal file name [`OUTPUT/2649_Sunday_June_05_2022_08_38_05_AM_83109919/index.tex`]

Book: Differential equations for engineers by Wei-Chau XIE, Cambridge Press 2010
Section: Chapter 2. First-Order and Simple Higher-Order Differential Equations. Page 78
Problem number: 12.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"homogeneousTypeD2", "exactWithIntegrationFactor", "first_order_ode_lie_symmetry_calculated"**

Maple gives the following as the ode type

```
[[_homogeneous, `class A`], _rational, _dAlembert]
```

$$\boxed{yx + (y^2 + x^2)\, y' = 0}$$

1.12.1 Solving as homogeneousTypeD2 ode

Using the change of variables $y = u(x)\, x$ on the above ode results in new ode in $u(x)$

$$u(x)\, x^2 + \left(u(x)^2\, x^2 + x^2\right)\left(u'(x)\, x + u(x)\right) = 0$$

In canonical form the ODE is

$$\begin{aligned} u' &= F(x,u) \\ &= f(x)g(u) \\ &= -\frac{u(u^2+2)}{x\,(u^2+1)} \end{aligned}$$

Where $f(x) = -\frac{1}{x}$ and $g(u) = \frac{u(u^2+2)}{u^2+1}$. Integrating both sides gives

$$\frac{1}{\frac{u(u^2+2)}{u^2+1}}\, du = -\frac{1}{x}\, dx$$

$$\int \frac{1}{\frac{u(u^2+2)}{u^2+1}} du = \int -\frac{1}{x} dx$$

$$\frac{\ln (u^2+2)}{4} + \frac{\ln (u)}{2} = -\ln (x) + c_2$$

Raising both side to exponential gives

$$e^{\frac{\ln\left(u^2+2\right)}{4}+\frac{\ln(u)}{2}} = e^{-\ln(x)+c_2}$$

Which simplifies to

$$\left(u^2+2\right)^{\frac{1}{4}} \sqrt{u} = \frac{c_3}{x}$$

The solution is

$$\left(u(x)^2+2\right)^{\frac{1}{4}} \sqrt{u(x)} = \frac{c_3}{x}$$

Replacing $u(x)$ in the above solution by $\frac{y}{x}$ results in the solution for y in implicit form

$$\left(\frac{y^2}{x^2}+2\right)^{\frac{1}{4}} \sqrt{\frac{y}{x}} = \frac{c_3}{x}$$

$$\left(\frac{y^2+2x^2}{x^2}\right)^{\frac{1}{4}} \sqrt{\frac{y}{x}} = \frac{c_3}{x}$$

Summary

The solution(s) found are the following

$$\left(\frac{y^2+2x^2}{x^2}\right)^{\frac{1}{4}} \sqrt{\frac{y}{x}} = \frac{c_3}{x} \tag{1}$$

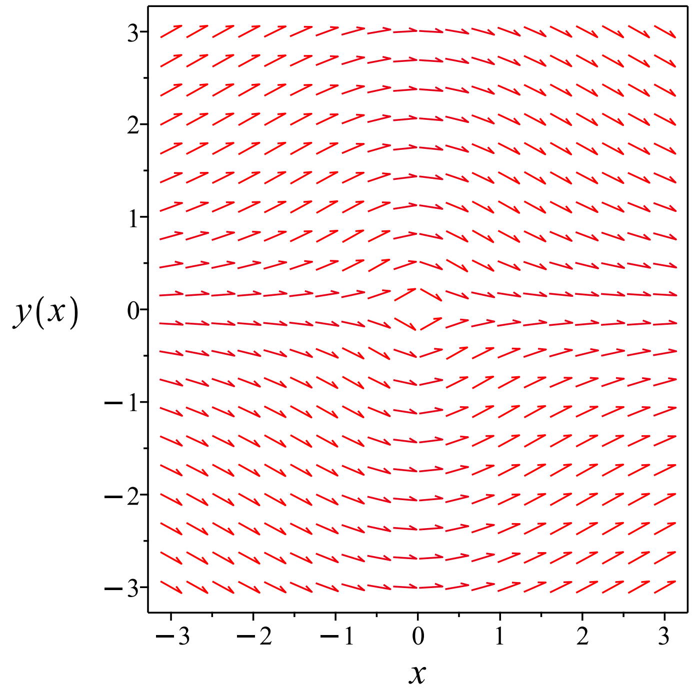


Figure 18: Slope field plot

Verification of solutions

$$\left(\frac{y^2+2x^2}{x^2}\right)^{\frac{1}{4}}\sqrt{\frac{y}{x}}=\frac{c_3}{x}$$

Verified OK.

1.12.2 Solving as first order ode lie symmetry calculated ode

Writing the ode as

$$y'=-\frac{yx}{x^2+y^2}$$
$$y'=\omega(x,y)$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x+\omega(\eta_y-\xi_x)-\omega^2\xi_y-\omega_x\xi-\omega_y\eta=0 \tag{A}$$

The type of this ode is not in the lookup table. To determine ξ,η then (A) is solved using ansatz. Making bivariate polynomials of degree 1 to use as anstaz gives

$$\xi=xa_2+ya_3+a_1 \tag{1E}$$

$$\eta=xb_2+yb_3+b_1 \tag{2E}$$

Where the unknown coefficients are

$$\{a_1, a_2, a_3, b_1, b_2, b_3\}$$

Substituting equations (1E,2E) and ω into (A) gives

$$b_2 - \frac{yx(b_3 - a_2)}{x^2 + y^2} - \frac{y^2 x^2 a_3}{(x^2 + y^2)^2} - \left(\frac{2x^2 y}{(x^2 + y^2)^2} - \frac{y}{x^2 + y^2}\right)(x a_2 + y a_3 + a_1) - \left(-\frac{x}{x^2 + y^2} + \frac{2y^2 x}{(x^2 + y^2)^2}\right)(x b_2 + y b_3 + b_1) = 0 \tag{5E}$$

Putting the above in normal form gives

$$\frac{2x^4 b_2 - 2y^2 x^2 a_3 + x^2 y^2 b_2 + 2x\,y^3 a_2 - 2x\,y^3 b_3 + y^4 a_3 + y^4 b_2 + x^3 b_1 - x^2 y a_1 - x\,y^2 b_1 + y^3 a_1}{(x^2 + y^2)^2} = 0$$

Setting the numerator to zero gives

$$2x^4 b_2 - 2y^2 x^2 a_3 + x^2 y^2 b_2 + 2x\,y^3 a_2 - 2x\,y^3 b_3 + y^4 a_3 + y^4 b_2 + x^3 b_1 - x^2 y a_1 - x\,y^2 b_1 + y^3 a_1 = 0 \tag{6E}$$

Looking at the above PDE shows the following are all the terms with $\{x, y\}$ in them.

$$\{x, y\}$$

The following substitution is now made to be able to collect on all terms with $\{x, y\}$ in them

$$\{x = v_1, y = v_2\}$$

The above PDE (6E) now becomes

$$2a_2 v_1 v_2^3 - 2a_3 v_1^2 v_2^2 + a_3 v_2^4 + 2b_2 v_1^4 + b_2 v_1^2 v_2^2 + b_2 v_2^4 - 2b_3 v_1 v_2^3 - a_1 v_1^2 v_2 + a_1 v_2^3 + b_1 v_1^3 - b_1 v_1 v_2^2 = 0 \tag{7E}$$

Collecting the above on the terms v_i introduced, and these are

$$\{v_1, v_2\}$$

Equation (7E) now becomes

$$
\begin{gathered}
2b_2v_1^4 + b_1v_1^3 + (-2a_3 + b_2)\, v_1^2v_2^2 - a_1v_1^2v_2 \\
+ (2a_2 - 2b_3)\, v_1v_2^3 - b_1v_1v_2^2 + (a_3 + b_2)\, v_2^4 + a_1v_2^3 = 0
\end{gathered} \tag{8E}
$$

Setting each coefficients in (8E) to zero gives the following equations to solve

$$
\begin{aligned}
a_1 &= 0 \\
b_1 &= 0 \\
-a_1 &= 0 \\
-b_1 &= 0 \\
2b_2 &= 0 \\
2a_2 - 2b_3 &= 0 \\
-2a_3 + b_2 &= 0 \\
a_3 + b_2 &= 0
\end{aligned}
$$

Solving the above equations for the unknowns gives

$$
\begin{aligned}
a_1 &= 0 \\
a_2 &= b_3 \\
a_3 &= 0 \\
b_1 &= 0 \\
b_2 &= 0 \\
b_3 &= b_3
\end{aligned}
$$

Substituting the above solution in the anstaz (1E,2E) (using 1 as arbitrary value for any unknown in the RHS) gives

$$
\begin{aligned}
\xi &= x \\
\eta &= y
\end{aligned}
$$

Shifting is now applied to make $\xi = 0$ in order to simplify the rest of the computation

$$
\begin{aligned}
\eta &= \eta - \omega(x, y)\, \xi \\
&= y - \left(-\frac{yx}{x^2 + y^2}\right)(x) \\
&= \frac{2x^2y + y^3}{x^2 + y^2} \\
\xi &= 0
\end{aligned}
$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x, y) \to (R, S)$ where (R, S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y}\right) S(x, y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$R = x$$

S is found from

$$\begin{aligned} S &= \int \frac{1}{\eta} dy \\ &= \int \frac{1}{\frac{2x^2y+y^3}{x^2+y^2}} dy \end{aligned}$$

Which results in

$$S = \frac{\ln(2x^2 + y^2)}{4} + \frac{\ln(y)}{2}$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x, y)S_y}{R_x + \omega(x, y)R_y} \tag{2}$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x, y)$ is the right hand side of the original ode given by

$$\omega(x, y) = -\frac{yx}{x^2 + y^2}$$

Evaluating all the partial derivatives gives

$$\begin{aligned} R_x &= 1 \\ R_y &= 0 \\ S_x &= \frac{x}{2x^2 + y^2} \\ S_y &= \frac{x^2 + y^2}{2x^2y + y^3} \end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = 0 \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = 0$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$S(R) = c_1 \tag{4}$$

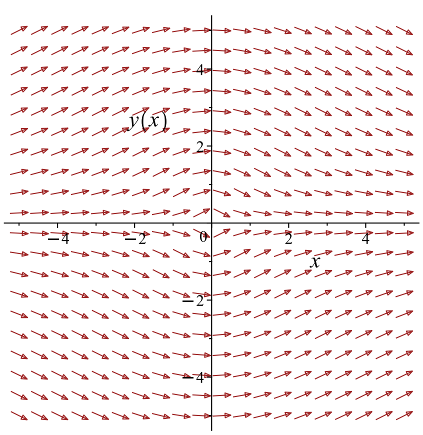
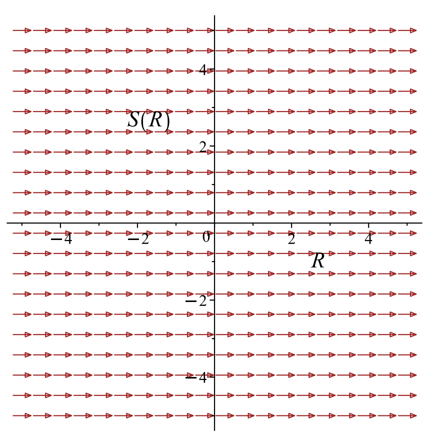
To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$\frac{\ln(y^2 + 2x^2)}{4} + \frac{\ln(y)}{2} = c_1$$

Which simplifies to

$$\frac{\ln(y^2 + 2x^2)}{4} + \frac{\ln(y)}{2} = c_1$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in x, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dx} = -\frac{yx}{x^2+y^2}$ 	$R = x$ $S = \frac{\ln(2x^2+y^2)}{4} + \frac{\ln(y}{2}$	$\frac{dS}{dR} = 0$ 

Summary

The solution(s) found are the following

$$\frac{\ln(y^2+2x^2)}{4}+\frac{\ln(y)}{2}=c_1 \tag{1}$$

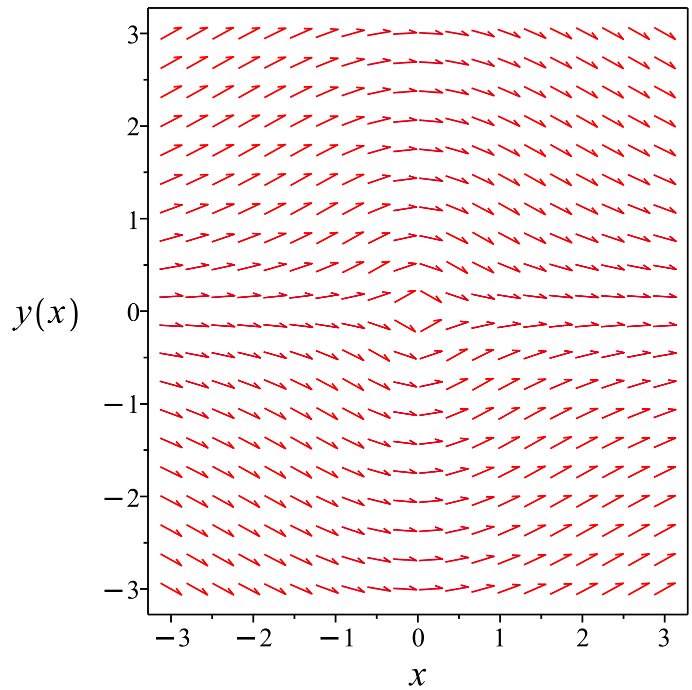


Figure 19: Slope field plot

Verification of solutions

$$\frac{\ln(y^2+2x^2)}{4}+\frac{\ln(y)}{2}=c_1$$

Verified OK.

1.12.3 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x,y)+N(x,y)\frac{dy}{dx}=0 \tag{A}$$

We assume there exists a function $\phi(x,y)=c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y)=0$$

Hence

$$\frac{\partial \phi}{\partial x}+\frac{\partial \phi}{\partial y}\frac{dy}{dx}=0 \tag{B}$$

Comparing (A,B) shows that

$$\begin{aligned}\frac{\partial \phi}{\partial x}&=M\\ \frac{\partial \phi}{\partial y}&=N\end{aligned}$$

But since $\frac{\partial^2\phi}{\partial x\partial y}=\frac{\partial^2\phi}{\partial y\partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y}=\frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x,y)$ but at least we know now that we can do that since the condition $\frac{\partial^2\phi}{\partial x\partial y}=\frac{\partial^2\phi}{\partial y\partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x,y)\,\mathrm{d}x+N(x,y)\,\mathrm{d}y=0 \tag{1A}$$

Therefore

$$\begin{aligned}\left(x^2+y^2\right)\mathrm{d}y&=(-xy)\,\mathrm{d}x\\ (xy)\,\mathrm{d}x+\left(x^2+y^2\right)\mathrm{d}y&=0 \qquad (2A)\end{aligned}$$

Comparing (1A) and (2A) shows that

$$\begin{aligned}M(x,y)&=xy\\ N(x,y)&=x^2+y^2\end{aligned}$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y}=\frac{\partial N}{\partial x}$$

Using result found above gives

$$\begin{aligned}\frac{\partial M}{\partial y}&=\frac{\partial}{\partial y}(xy)\\ &=x\end{aligned}$$

And

$$\begin{aligned}\frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}(x^2+y^2)\\ &= 2x\end{aligned}$$

Since $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$, then the ODE is <u>not exact</u>. Since the ODE is not exact, we will try to find an integrating factor to make it exact. Let

$$\begin{aligned}A &= \frac{1}{N}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right)\\ &= \frac{1}{x^2+y^2}((x)-(2x))\\ &= -\frac{x}{x^2+y^2}\end{aligned}$$

Since A depends on y, it can not be used to obtain an integrating factor. We will now try a second method to find an integrating factor. Let

$$\begin{aligned}B &= \frac{1}{M}\left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right)\\ &= \frac{1}{xy}((2x)-(x))\\ &= \frac{1}{y}\end{aligned}$$

Since B does not depend on x, it can be used to obtain an integrating factor. Let the integrating factor be μ. Then

$$\begin{aligned}\mu &= e^{\int B \, \mathrm{d}y}\\ &= e^{\int \frac{1}{y} \, \mathrm{d}y}\end{aligned}$$

The result of integrating gives

$$\begin{aligned}\mu &= e^{\ln(y)}\\ &= y\end{aligned}$$

M and N are now multiplied by this integrating factor, giving new M and new N which are called $\overline{M}$ and $\overline{N}$ so not to confuse them with the original M and N.

$$\begin{aligned}\overline{M} &= \mu M\\ &= y(xy)\\ &= x\,y^2\end{aligned}$$

And

$$
\begin{aligned}
\overline{N} &= \mu N \\
&= y(x^2+y^2) \\
&= y(x^2+y^2)
\end{aligned}
$$

So now a modified ODE is obtained from the original ODE which will be exact and can be solved using the standard method. The modified ODE is

$$
\begin{aligned}
\overline{M} + \overline{N}\frac{\mathrm{d}y}{\mathrm{d}x} &= 0 \\
(x\,y^2) + \left(y(x^2+y^2)\right)\frac{\mathrm{d}y}{\mathrm{d}x} &= 0
\end{aligned}
$$

The following equations are now set up to solve for the function $\phi(x,y)$

$$
\frac{\partial\phi}{\partial x} = \overline{M} \tag{1}
$$

$$
\frac{\partial\phi}{\partial y} = \overline{N} \tag{2}
$$

Integrating (1) w.r.t. x gives

$$
\begin{aligned}
\int \frac{\partial\phi}{\partial x}\,\mathrm{d}x &= \int \overline{M}\,\mathrm{d}x \\
\int \frac{\partial\phi}{\partial x}\,\mathrm{d}x &= \int x\,y^2\,\mathrm{d}x
\end{aligned}
$$

$$
\phi = \frac{y^2x^2}{2} + f(y) \tag{3}
$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$
\frac{\partial\phi}{\partial y} = x^2y + f'(y) \tag{4}
$$

But equation (2) says that $\frac{\partial\phi}{\partial y} = y(x^2+y^2)$. Therefore equation (4) becomes

$$
y(x^2+y^2) = x^2y + f'(y) \tag{5}
$$

Solving equation (5) for $f'(y)$ gives

$$
f'(y) = y^3
$$

Integrating the above w.r.t y gives

$$\int f'(y)\,\mathrm{d}y = \int \left(y^3\right)\mathrm{d}y$$

$$f(y) = \frac{y^4}{4} + c_1$$

Where c_1 is constant of integration. Substituting result found above for $f(y)$ into equation (3) gives ϕ

$$\phi = \frac{1}{2}y^2x^2 + \frac{1}{4}y^4 + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = \frac{1}{2}y^2x^2 + \frac{1}{4}y^4$$

Summary

The solution(s) found are the following

$$\frac{y^2x^2}{2} + \frac{y^4}{4} = c_1 \tag{1}$$

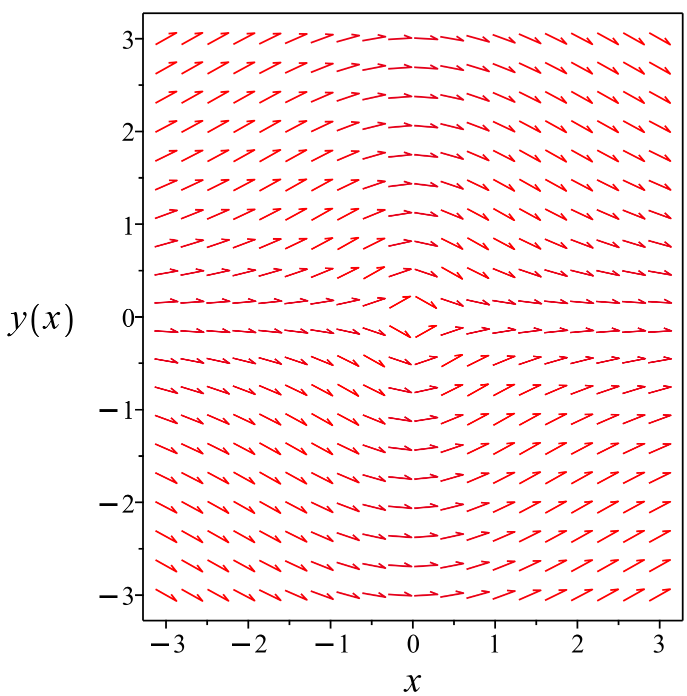


Figure 20: Slope field plot

Verification of solutions

$$\frac{y^2x^2}{2}+\frac{y^4}{4}=c_1$$

Verified OK.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
trying homogeneous types:
trying homogeneous D
<- homogeneous successful`
```

✓ Solution by Maple

Time used: 0.688 (sec). Leaf size: 221

```
dsolve(x*y(x)+(x^2+y(x)^2)*diff(y(x),x)=0,y(x), singsol=all)
```

$$y(x)=\frac{\sqrt{x^2c_1\left(c_1x^2-\sqrt{c_1^2x^4+1}\right)}}{x\left(c_1x^2-\sqrt{c_1^2x^4+1}\right)c_1}$$

$$y(x)=\frac{\sqrt{x^2c_1\left(c_1x^2+\sqrt{c_1^2x^4+1}\right)}}{x\left(c_1x^2+\sqrt{c_1^2x^4+1}\right)c_1}$$

$$y(x)=\frac{\sqrt{x^2c_1\left(c_1x^2-\sqrt{c_1^2x^4+1}\right)}}{x\left(-c_1x^2+\sqrt{c_1^2x^4+1}\right)c_1}$$

$$y(x)=-\frac{\sqrt{x^2c_1\left(c_1x^2+\sqrt{c_1^2x^4+1}\right)}}{x\left(c_1x^2+\sqrt{c_1^2x^4+1}\right)c_1}$$

✓ Solution by Mathematica

Time used: 9.087 (sec). Leaf size: 218

```
DSolve[x*y[x]+(x^2+y[x]^2)*y'[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to -\sqrt{-x^2 - \sqrt{x^4 + e^{4c_1}}}$$
$$y(x) \to \sqrt{-x^2 - \sqrt{x^4 + e^{4c_1}}}$$
$$y(x) \to -\sqrt{-x^2 + \sqrt{x^4 + e^{4c_1}}}$$
$$y(x) \to \sqrt{-x^2 + \sqrt{x^4 + e^{4c_1}}}$$
$$y(x) \to 0$$
$$y(x) \to -\sqrt{-\sqrt{x^4} - x^2}$$
$$y(x) \to \sqrt{-\sqrt{x^4} - x^2}$$
$$y(x) \to -\sqrt{\sqrt{x^4} - x^2}$$
$$y(x) \to \sqrt{\sqrt{x^4} - x^2}$$

1.13 problem 13

Internal problem ID [3158]
Internal file name [`OUTPUT/2650_Sunday_June_05_2022_08_38_06_AM_30676466/index.tex`]

Book: Differential equations for engineers by Wei-Chau XIE, Cambridge Press 2010
Section: Chapter 2. First-Order and Simple Higher-Order Differential Equations. Page 78
Problem number: 13.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"homogeneousTypeD2", "first_order_ode_lie_symmetry_calculated"**

Maple gives the following as the ode type

```
[[_homogeneous, `class A`], _dAlembert]
```

$$\boxed{\left(1-\mathrm{e}^{-\frac{y}{x}}\right) y'-\frac{y}{x}=-1}$$

1.13.1 Solving as homogeneousTypeD2 ode

Using the change of variables $y = u(x)\, x$ on the above ode results in new ode in $u(x)$

$$\left(1-\mathrm{e}^{-u(x)}\right) \left(u'(x)\, x+u(x)\right)-u(x)=-1$$

In canonical form the ODE is

$$\begin{aligned} u' &= F(x,u) \\ &= f(x)g(u) \\ &= -\frac{\mathrm{e}^{-u} u-1}{\left(\mathrm{e}^{-u}-1\right) x} \end{aligned}$$

Where $f(x)=-\frac{1}{x}$ and $g(u)=\frac{\mathrm{e}^{-u} u-1}{\mathrm{e}^{-u}-1}$. Integrating both sides gives

$$\frac{1}{\frac{\mathrm{e}^{-u} u-1}{\mathrm{e}^{-u}-1}}\, du=-\frac{1}{x}\, dx$$

$$\int \frac{1}{\frac{e^{-u}u-1}{e^{-u}-1}} du = \int -\frac{1}{x} dx$$

$$u + \ln\left(e^{-u}u - 1\right) = -\ln(x) + c_2$$

The solution is

$$u(x) + \ln\left(e^{-u(x)}u(x) - 1\right) + \ln(x) - c_2 = 0$$

Replacing $u(x)$ in the above solution by $\frac{y}{x}$ results in the solution for y in implicit form

$$\frac{y}{x} + \ln\left(\frac{e^{-\frac{y}{x}}y}{x} - 1\right) + \ln(x) - c_2 = 0$$

$$\frac{\ln\left(\frac{y\,e^{-\frac{y}{x}}-x}{x}\right)x + \ln(x)\,x - c_2x + y}{x} = 0$$

<u>Summary</u>

The solution(s) found are the following

$$\frac{\ln\left(\frac{y\,e^{-\frac{y}{x}}-x}{x}\right)x + \ln(x)\,x - c_2x + y}{x} = 0 \tag{1}$$

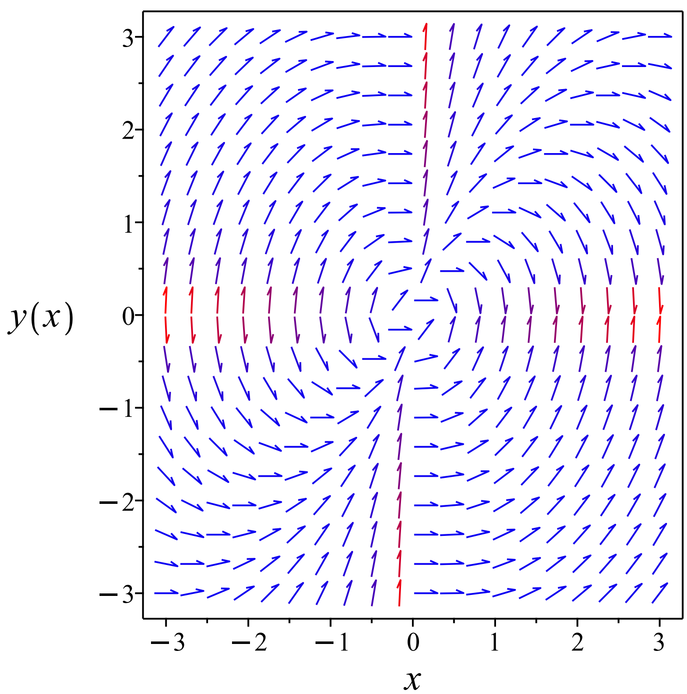


Figure 21: Slope field plot

Verification of solutions

$$\frac{\ln\left(\frac{y\,\mathrm{e}^{-\frac{y}{x}}-x}{x}\right) x+\ln\left(x\right) x-c_2 x+y}{x}=0$$

Verified OK.

1.13.2 Solving as first order ode lie symmetry calculated ode

Writing the ode as

$$\begin{aligned} y' &= -\frac{y-x}{\left(-1+\mathrm{e}^{-\frac{y}{x}}\right) x} \\ y' &= \omega(x,y) \end{aligned}$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x+\omega(\eta_y-\xi_x)-\omega^2\xi_y-\omega_x\xi-\omega_y\eta=0 \tag{A}$$

The type of this ode is not in the lookup table. To determine ξ,η then (A) is solved using ansatz. Making bivariate polynomials of degree 1 to use as anstaz gives

$$\xi = x a_2+y a_3+a_1 \tag{1E}$$

$$\eta = x b_2+y b_3+b_1 \tag{2E}$$

Where the unknown coefficients are

$$\{a_1, a_2, a_3, b_1, b_2, b_3\}$$

Substituting equations (1E,2E) and ω into (A) gives

$$\begin{aligned} &b_2-\frac{\left(y-x\right)\left(b_3-a_2\right)}{\left(-1+\mathrm{e}^{-\frac{y}{x}}\right) x}-\frac{\left(y-x\right)^2 a_3}{\left(-1+\mathrm{e}^{-\frac{y}{x}}\right)^2 x^2} \\ &-\left(\frac{1}{x\left(-1+\mathrm{e}^{-\frac{y}{x}}\right)}+\frac{\left(y-x\right) y\,\mathrm{e}^{-\frac{y}{x}}}{\left(-1+\mathrm{e}^{-\frac{y}{x}}\right)^2 x^3}+\frac{y-x}{\left(-1+\mathrm{e}^{-\frac{y}{x}}\right) x^2}\right)\left(x a_2+y a_3+a_1\right) \\ &-\left(-\frac{1}{x\left(-1+\mathrm{e}^{-\frac{y}{x}}\right)}-\frac{\left(y-x\right)\mathrm{e}^{-\frac{y}{x}}}{\left(-1+\mathrm{e}^{-\frac{y}{x}}\right)^2 x^2}\right)\left(x b_2+y b_3+b_1\right)=0 \end{aligned} \tag{5E}$$

Putting the above in normal form gives

$$\frac{\mathrm{e}^{-\frac{2y}{x}} x^3 b_2-\mathrm{e}^{-\frac{y}{x}} x^3 a_2-2\,\mathrm{e}^{-\frac{y}{x}} x^3 b_2+\mathrm{e}^{-\frac{y}{x}} x^3 b_3+\mathrm{e}^{-\frac{y}{x}} x^2 y a_2+\mathrm{e}^{-\frac{y}{x}} x^2 y b_2-\mathrm{e}^{-\frac{y}{x}} x^2 y b_3-\mathrm{e}^{-\frac{y}{x}} x\,y^2 a_2+\mathrm{e}^{-\frac{y}{x}} x\,y^2 b_3}{\left(-1+\mathrm{e}^{-\frac{y}{x}}\right)^2 x^3}$$

$$=0$$

Setting the numerator to zero gives

$$
\begin{aligned}
&\mathrm{e}^{-\frac{2y}{x}} x^3 b_2 - \mathrm{e}^{-\frac{y}{x}} x^3 a_2 - 2\,\mathrm{e}^{-\frac{y}{x}} x^3 b_2 + \mathrm{e}^{-\frac{y}{x}} x^3 b_3 + \mathrm{e}^{-\frac{y}{x}} x^2 y a_2 + \mathrm{e}^{-\frac{y}{x}} x^2 y b_2 \\
&\quad - \mathrm{e}^{-\frac{y}{x}} x^2 y b_3 - \mathrm{e}^{-\frac{y}{x}} x\, y^2 a_2 + \mathrm{e}^{-\frac{y}{x}} x\, y^2 b_3 - \mathrm{e}^{-\frac{y}{x}} y^3 a_3 + \mathrm{e}^{-\frac{y}{x}} x y b_1 \\
&\quad - \mathrm{e}^{-\frac{y}{x}} y^2 a_1 + x^3 a_2 - x^3 a_3 - x^3 b_3 + 2x^2 y a_3 - x^2 b_1 + x y a_1 = 0
\end{aligned}
\tag{6E}
$$

Simplifying the above gives

$$
\begin{aligned}
&\mathrm{e}^{-\frac{2y}{x}} x^3 b_2 - \mathrm{e}^{-\frac{y}{x}} x^3 a_2 - 2\,\mathrm{e}^{-\frac{y}{x}} x^3 b_2 + \mathrm{e}^{-\frac{y}{x}} x^3 b_3 + \mathrm{e}^{-\frac{y}{x}} x^2 y a_2 + \mathrm{e}^{-\frac{y}{x}} x^2 y b_2 \\
&\quad - \mathrm{e}^{-\frac{y}{x}} x^2 y b_3 - \mathrm{e}^{-\frac{y}{x}} x\, y^2 a_2 + \mathrm{e}^{-\frac{y}{x}} x\, y^2 b_3 - \mathrm{e}^{-\frac{y}{x}} y^3 a_3 + \mathrm{e}^{-\frac{y}{x}} x y b_1 \\
&\quad - \mathrm{e}^{-\frac{y}{x}} y^2 a_1 + x^3 a_2 - x^3 a_3 - x^3 b_3 + 2x^2 y a_3 - x^2 b_1 + x y a_1 = 0
\end{aligned}
\tag{6E}
$$

Looking at the above PDE shows the following are all the terms with $\{x, y\}$ in them.

$$
\left\{x, y, \mathrm{e}^{-\frac{2y}{x}}, \mathrm{e}^{-\frac{y}{x}}\right\}
$$

The following substitution is now made to be able to collect on all terms with $\{x, y\}$ in them

$$
\left\{x = v_1, y = v_2, \mathrm{e}^{-\frac{2y}{x}} = v_3, \mathrm{e}^{-\frac{y}{x}} = v_4\right\}
$$

The above PDE (6E) now becomes

$$
\begin{aligned}
&-v_4 v_1^3 a_2 + v_4 v_1^2 v_2 a_2 - v_4 v_1 v_2^2 a_2 - v_4 v_2^3 a_3 + v_3 v_1^3 b_2 - 2 v_4 v_1^3 b_2 \\
&\quad + v_4 v_1^2 v_2 b_2 + v_4 v_1^3 b_3 - v_4 v_1^2 v_2 b_3 + v_4 v_1 v_2^2 b_3 - v_4 v_2^2 a_1 + v_1^3 a_2 \\
&\quad - v_1^3 a_3 + 2 v_1^2 v_2 a_3 + v_4 v_1 v_2 b_1 - v_1^3 b_3 + v_1 v_2 a_1 - v_1^2 b_1 = 0
\end{aligned}
\tag{7E}
$$

Collecting the above on the terms v_i introduced, and these are

$$
\{v_1, v_2, v_3, v_4\}
$$

Equation (7E) now becomes

$$
\begin{aligned}
&v_3 v_1^3 b_2 + (-a_2 - 2b_2 + b_3)\, v_1^3 v_4 + (a_2 - a_3 - b_3)\, v_1^3 + (a_2 + b_2 - b_3)\, v_1^2 v_2 v_4 \\
&\quad + 2 v_1^2 v_2 a_3 - v_1^2 b_1 + (b_3 - a_2)\, v_1 v_2^2 v_4 + v_4 v_1 v_2 b_1 + v_1 v_2 a_1 - v_4 v_2^3 a_3 - v_4 v_2^2 a_1 = 0
\end{aligned}
\tag{8E}
$$

Setting each coefficients in (8E) to zero gives the following equations to solve

$$
\begin{aligned}
a_1 &= 0 \\
b_1 &= 0 \\
b_2 &= 0 \\
-a_1 &= 0 \\
-a_3 &= 0 \\
2a_3 &= 0 \\
-b_1 &= 0 \\
b_3 - a_2 &= 0 \\
-a_2 - 2b_2 + b_3 &= 0 \\
a_2 - a_3 - b_3 &= 0 \\
a_2 + b_2 - b_3 &= 0
\end{aligned}
$$

Solving the above equations for the unknowns gives

$$
\begin{aligned}
a_1 &= 0 \\
a_2 &= b_3 \\
a_3 &= 0 \\
b_1 &= 0 \\
b_2 &= 0 \\
b_3 &= b_3
\end{aligned}
$$

Substituting the above solution in the anstaz (1E,2E) (using 1 as arbitrary value for any unknown in the RHS) gives

$$
\begin{aligned}
\xi &= x \\
\eta &= y
\end{aligned}
$$

Shifting is now applied to make $\xi = 0$ in order to simplify the rest of the computation

$$
\begin{aligned}
\eta &= \eta - \omega(x, y)\, \xi \\
&= y - \left(-\frac{y - x}{\left(-1 + \mathrm{e}^{-\frac{y}{x}}\right) x}\right) (x) \\
&= \frac{y\, \mathrm{e}^{-\frac{y}{x}} - x}{-1 + \mathrm{e}^{-\frac{y}{x}}} \\
\xi &= 0
\end{aligned}
$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x, y) \to (R, S)$ where (R, S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y}\right) S(x, y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$R = x$$

S is found from

$$\begin{aligned} S &= \int \frac{1}{\eta} dy \\ &= \int \frac{1}{\frac{y\,\mathrm{e}^{-\frac{y}{x}} - x}{-1 + \mathrm{e}^{-\frac{y}{x}}}} dy \end{aligned}$$

Which results in

$$S = \ln\left(x\,\mathrm{e}^{\frac{y}{x}} - y\right)$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x, y) S_y}{R_x + \omega(x, y) R_y} \tag{2}$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x, y)$ is the right hand side of the original ode given by

$$\omega(x, y) = -\frac{y - x}{\left(-1 + \mathrm{e}^{-\frac{y}{x}}\right) x}$$

Evaluating all the partial derivatives gives

$$\begin{aligned} R_x &= 1 \\ R_y &= 0 \\ S_x &= \frac{(-y + x)\,\mathrm{e}^{\frac{y}{x}}}{x\left(x\,\mathrm{e}^{\frac{y}{x}} - y\right)} \\ S_y &= \frac{\mathrm{e}^{\frac{y}{x}} - 1}{x\,\mathrm{e}^{\frac{y}{x}} - y} \end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = 0 \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = 0$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$S(R) = c_1 \tag{4}$$

To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$\ln\left(x\,\mathrm{e}^{\frac{y}{x}} - y\right) = c_1$$

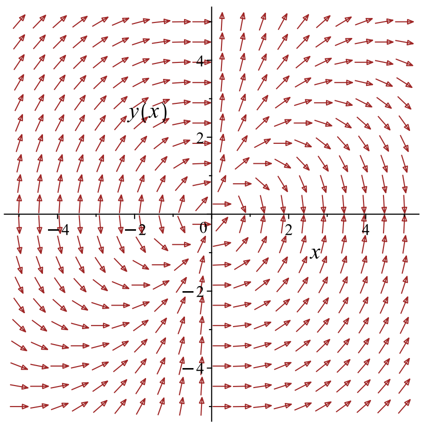
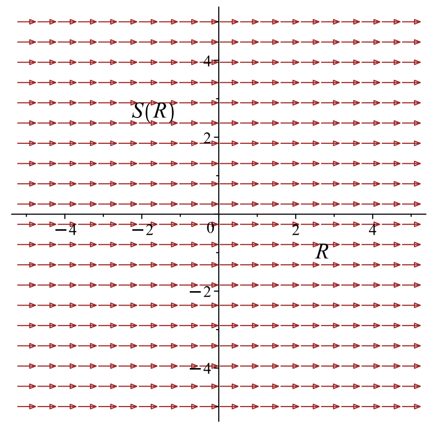
Which simplifies to

$$\ln\left(x\,\mathrm{e}^{\frac{y}{x}} - y\right) = c_1$$

Which gives

$$y = -x\,\mathrm{LambertW}\left(-\mathrm{e}^{-\frac{\mathrm{e}^{c_1}}{x}}\right) - \mathrm{e}^{c_1}$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in x, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dx} = -\frac{y-x}{\left(-1+\mathrm{e}^{-\frac{y}{x}}\right)x}$ 	$R = x$ $S = \ln\left(x\,\mathrm{e}^{\frac{y}{x}} - y\right)$	$\frac{dS}{dR} = 0$ 

Summary

The solution(s) found are the following

$$y = -x \operatorname{LambertW}\left(-\mathrm{e}^{-\frac{\mathrm{e}^{c_1}}{x}}\right) - \mathrm{e}^{c_1} \tag{1}$$

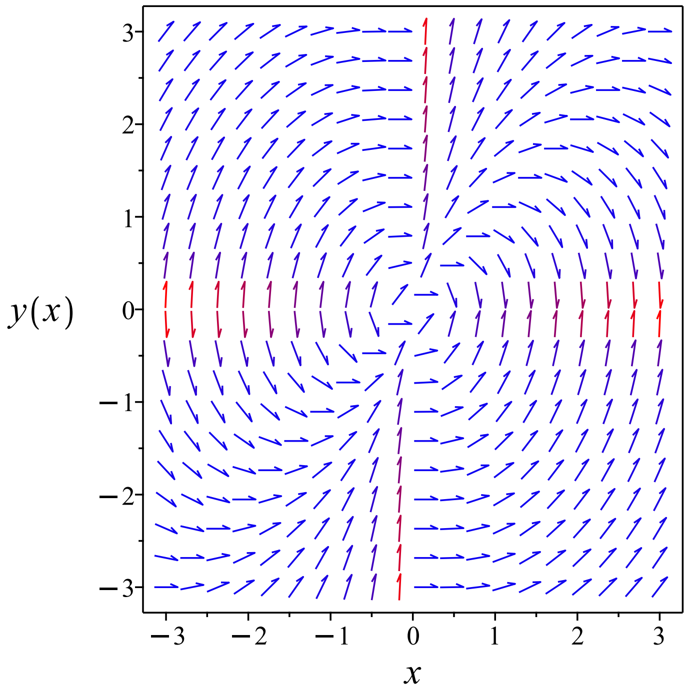


Figure 22: Slope field plot

Verification of solutions

$$y = -x \,\mathrm{LambertW}\left(-\mathrm{e}^{-\frac{\mathrm{e}^{c_1}}{x}}\right) - \mathrm{e}^{c_1}$$

Verified OK.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
trying homogeneous types:
trying homogeneous D
<- homogeneous successful`
```

✓ Solution by Maple

Time used: 0.015 (sec). Leaf size: 26

```
dsolve((1-exp(- y(x)/x))*diff(y(x),x)+(1- y(x)/x)=0,y(x), singsol=all)
```

$$y(x) = \frac{-c_1 \operatorname{LambertW}\left(-\mathrm{e}^{-\frac{1}{c_1 x}}\right) x - 1}{c_1}$$

✓ Solution by Mathematica

Time used: 60.202 (sec). Leaf size: 29

```
DSolve[(1-Exp[-y[x]/x])*y'[x]+(1-y[x]/x)==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to -xW\left(-e^{-\frac{e^{c_1}}{x}}\right) - e^{c_1}$$

1.14 problem 14

Internal problem ID [3159]
Internal file name [`OUTPUT/2651_Sunday_June_05_2022_08_38_07_AM_32855839/index.tex`]

Book: Differential equations for engineers by Wei-Chau XIE, Cambridge Press 2010
Section: Chapter 2. First-Order and Simple Higher-Order Differential Equations. Page 78
Problem number: 14.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"homogeneousTypeD2", "first_order_ode_lie_symmetry_calculated"**

Maple gives the following as the ode type

```
[[_homogeneous, `class A`], _rational, [_Abel, `2nd type`, `
   class B`]]
```

$$\boxed{-yx + y^2 - xyy' = -x^2}$$

1.14.1 Solving as homogeneousTypeD2 ode

Using the change of variables $y = u(x)\, x$ on the above ode results in new ode in $u(x)$

$$-u(x)\, x^2 + u(x)^2\, x^2 - x^2 u(x)\, (u'(x)\, x + u(x)) = -x^2$$

In canonical form the ODE is

$$\begin{aligned} u' &= F(x, u) \\ &= f(x) g(u) \\ &= -\frac{u-1}{ux} \end{aligned}$$

Where $f(x) = -\frac{1}{x}$ and $g(u) = \frac{u-1}{u}$. Integrating both sides gives

$$\frac{1}{\frac{u-1}{u}}\, du = -\frac{1}{x}\, dx$$

$$\int \frac{1}{\frac{u-1}{u}}\, du = \int -\frac{1}{x}\, dx$$

$$u + \ln(u-1) = -\ln(x) + c_2$$

The solution is

$$u(x) + \ln(u(x) - 1) + \ln(x) - c_2 = 0$$

Replacing $u(x)$ in the above solution by $\frac{y}{x}$ results in the solution for y in implicit form

$$\frac{y}{x} + \ln\left(\frac{y}{x} - 1\right) + \ln(x) - c_2 = 0$$
$$\frac{y}{x} + \ln\left(\frac{y - x}{x}\right) + \ln(x) - c_2 = 0$$

Summary

The solution(s) found are the following

$$\frac{y}{x} + \ln\left(\frac{y - x}{x}\right) + \ln(x) - c_2 = 0 \tag{1}$$

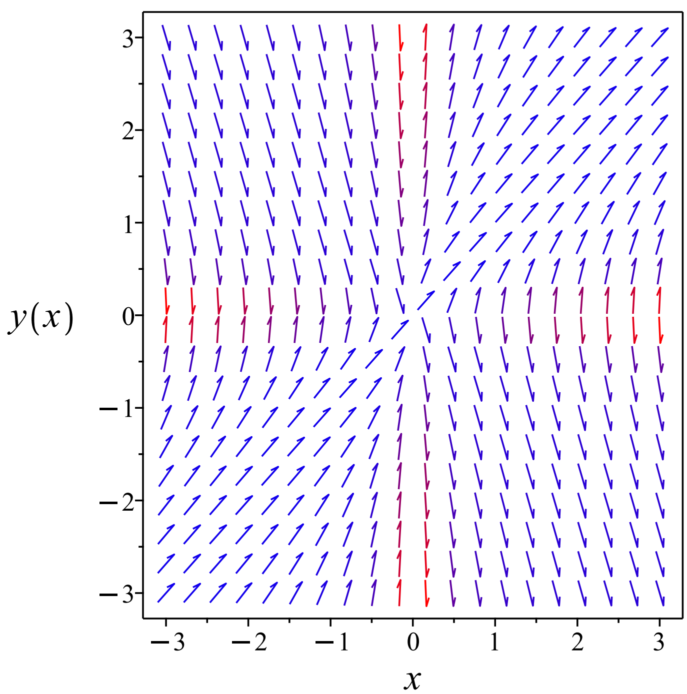


Figure 23: Slope field plot

Verification of solutions

$$\frac{y}{x} + \ln\left(\frac{y - x}{x}\right) + \ln(x) - c_2 = 0$$

Verified OK.

1.14.2 Solving as first order ode lie symmetry calculated ode

Writing the ode as

$$
\begin{aligned}
y' &= \frac{x^{2}-x y+y^{2}}{x y}\\
y' &= \omega(x,y)
\end{aligned}
$$

The condition of Lie symmetry is the linearized PDE given by

$$
\eta_{x}+\omega(\eta_{y}-\xi_{x})-\omega^{2}\xi_{y}-\omega_{x}\xi-\omega_{y}\eta=0 \tag{A}
$$

The type of this ode is not in the lookup table. To determine ξ, η then (A) is solved using ansatz. Making bivariate polynomials of degree 1 to use as anstaz gives

$$
\xi = x a_{2}+y a_{3}+a_{1} \tag{1E}
$$

$$
\eta = x b_{2}+y b_{3}+b_{1} \tag{2E}
$$

Where the unknown coefficients are

$$
\{a_{1}, a_{2}, a_{3}, b_{1}, b_{2}, b_{3}\}
$$

Substituting equations (1E,2E) and ω into (A) gives

$$
\begin{aligned}
&b_{2}+\frac{\left(x^{2}-x y+y^{2}\right)\left(b_{3}-a_{2}\right)}{x y}-\frac{\left(x^{2}-x y+y^{2}\right)^{2} a_{3}}{x^{2} y^{2}}\\
&-\left(\frac{2x-y}{x y}-\frac{x^{2}-x y+y^{2}}{x^{2} y}\right)\left(x a_{2}+y a_{3}+a_{1}\right)\\
&-\left(\frac{-x+2y}{x y}-\frac{x^{2}-x y+y^{2}}{x\,y^{2}}\right)\left(x b_{2}+y b_{3}+b_{1}\right)=0
\end{aligned} \tag{5E}
$$

Putting the above in normal form gives

$$
-\frac{x^{4} a_{3}-x^{4} b_{2}+2 x^{3} y a_{2}-2 x^{3} y a_{3}-2 x^{3} y b_{3}-x^{2} y^{2} a_{2}+4 x^{2} y^{2} a_{3}+x^{2} y^{2} b_{3}-2 x\,y^{3} a_{3}-x^{3} b_{1}+x^{2} y a_{1}+x\,y^{2}}{y^{2} x^{2}} = 0
$$

Setting the numerator to zero gives

$$
\begin{aligned}
&-x^{4} a_{3}+x^{4} b_{2}-2 x^{3} y a_{2}+2 x^{3} y a_{3}+2 x^{3} y b_{3}+x^{2} y^{2} a_{2}-4 x^{2} y^{2} a_{3}\\
&-x^{2} y^{2} b_{3}+2 x\,y^{3} a_{3}+x^{3} b_{1}-x^{2} y a_{1}-x\,y^{2} b_{1}+y^{3} a_{1}=0
\end{aligned} \tag{6E}
$$

Looking at the above PDE shows the following are all the terms with $\{x, y\}$ in them.

$$\{x, y\}$$

The following substitution is now made to be able to collect on all terms with $\{x, y\}$ in them

$$\{x = v_1, y = v_2\}$$

The above PDE (6E) now becomes

$$\begin{aligned} &-2a_2v_1^3v_2 + a_2v_1^2v_2^2 - a_3v_1^4 + 2a_3v_1^3v_2 - 4a_3v_1^2v_2^2 + 2a_3v_1v_2^3 \\ &+ b_2v_1^4 + 2b_3v_1^3v_2 - b_3v_1^2v_2^2 - a_1v_1^2v_2 + a_1v_2^3 + b_1v_1^3 - b_1v_1v_2^2 = 0 \end{aligned} \tag{7E}$$

Collecting the above on the terms v_i introduced, and these are

$$\{v_1, v_2\}$$

Equation (7E) now becomes

$$\begin{aligned} &(-a_3 + b_2)\, v_1^4 + (-2a_2 + 2a_3 + 2b_3)\, v_1^3v_2 + b_1v_1^3 \\ &+ (a_2 - 4a_3 - b_3)\, v_1^2v_2^2 - a_1v_1^2v_2 + 2a_3v_1v_2^3 - b_1v_1v_2^2 + a_1v_2^3 = 0 \end{aligned} \tag{8E}$$

Setting each coefficients in (8E) to zero gives the following equations to solve

$$\begin{aligned} a_1 &= 0 \\ b_1 &= 0 \\ -a_1 &= 0 \\ 2a_3 &= 0 \\ -b_1 &= 0 \\ -a_3 + b_2 &= 0 \\ -2a_2 + 2a_3 + 2b_3 &= 0 \\ a_2 - 4a_3 - b_3 &= 0 \end{aligned}$$

Solving the above equations for the unknowns gives

$$
\begin{aligned}
a_1 &= 0 \\
a_2 &= b_3 \\
a_3 &= 0 \\
b_1 &= 0 \\
b_2 &= 0 \\
b_3 &= b_3
\end{aligned}
$$

Substituting the above solution in the anstaz (1E,2E) (using 1 as arbitrary value for any unknown in the RHS) gives

$$
\begin{aligned}
\xi &= x \\
\eta &= y
\end{aligned}
$$

Shifting is now applied to make $\xi = 0$ in order to simplify the rest of the computation

$$
\begin{aligned}
\eta &= \eta - \omega(x, y)\,\xi \\
&= y - \left(\frac{x^2 - xy + y^2}{xy}\right)(x) \\
&= \frac{-x^2 + xy}{y} \\
\xi &= 0
\end{aligned}
$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x, y) \to (R, S)$ where (R, S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$
\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}
$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y}\right) S(x, y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$
R = x
$$

S is found from

$$
\begin{aligned}
S &= \int \frac{1}{\eta} dy \\
&= \int \frac{1}{\frac{-x^2 + xy}{y}} dy
\end{aligned}
$$

Which results in

$$S = \ln(y - x) + \frac{y}{x}$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x, y) S_y}{R_x + \omega(x, y) R_y} \tag{2}$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x, y)$ is the right hand side of the original ode given by

$$\omega(x, y) = \frac{x^2 - xy + y^2}{xy}$$

Evaluating all the partial derivatives gives

$$\begin{aligned} R_x &= 1 \\ R_y &= 0 \\ S_x &= \frac{x^2 - xy + y^2}{x^2 (-y + x)} \\ S_y &= -\frac{y}{x (-y + x)} \end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = 0 \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = 0$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$S(R) = c_1 \tag{4}$$

To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$\frac{\ln(y - x) x + y}{x} = c_1$$

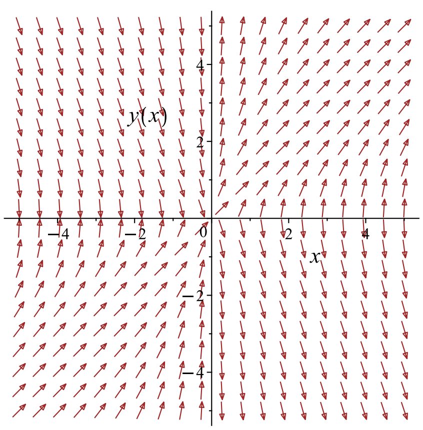
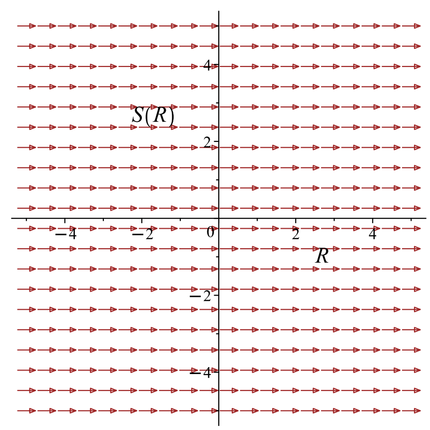
Which simplifies to

$$\frac{\ln(y-x)\,x+y}{x}=c_1$$

Which gives

$$y = x\,\mathrm{LambertW}\left(\frac{e^{c_1-1}}{x}\right)+x$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in x, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dx}=\frac{x^2-xy+y^2}{xy}$ 	$R = x$ $S=\frac{\ln(y-x)\,x+y}{x}$	$\frac{dS}{dR}=0$ 

Summary
The solution(s) found are the following

$$y = x\,\mathrm{LambertW}\left(\frac{e^{c_1-1}}{x}\right)+x \tag{1}$$

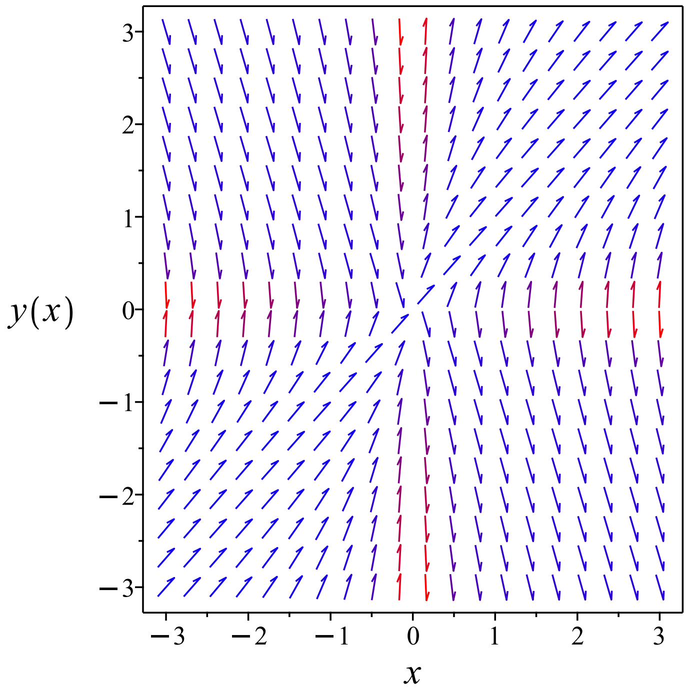


Figure 24: Slope field plot

Verification of solutions

$$y = x \,\mathrm{LambertW}\left(\frac{\mathrm{e}^{c_1-1}}{x}\right)+x$$

Verified OK.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
trying homogeneous types:
trying homogeneous D
<- homogeneous successful`
```

✓ Solution by Maple

Time used: 0.031 (sec). Leaf size: 19

```
dsolve((x^2-x*y(x)+y(x)^2)-x*y(x)*diff(y(x),x)=0,y(x), singsol=all)
```

$$y(x) = x\left(1 + \text{LambertW}\left(\frac{e^{-c_1-1}}{x}\right)\right)$$

✓ Solution by Mathematica

Time used: 3.69 (sec). Leaf size: 25

```
DSolve[(x^2-x*y[x]+y[x]^2)-x*y[x]*y'[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to x\left(1 + W\left(\frac{e^{-1+c_1}}{x}\right)\right)$$
$$y(x) \to x$$

1.15 problem 15

Internal problem ID [3160]
Internal file name [`OUTPUT/2652_Sunday_June_05_2022_08_38_08_AM_83933171/index.tex`]

Book: Differential equations for engineers by Wei-Chau XIE, Cambridge Press 2010
Section: Chapter 2. First-Order and Simple Higher-Order Differential Equations. Page 78
Problem number: 15.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"first_order_ode_lie_symmetry_calculated"**

Maple gives the following as the ode type

```
[[_homogeneous, `class C`], _rational, [_Abel, `2nd type`, `
   class A`]]
```

$$\boxed{(3+2x+4y)\,y'-2y=x+1}$$

1.15.1 Solving as first order ode lie symmetry calculated ode

Writing the ode as

$$y'=\frac{x+2y+1}{3+2x+4y}$$
$$y'=\omega(x,y)$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x+\omega(\eta_y-\xi_x)-\omega^2\xi_y-\omega_x\xi-\omega_y\eta=0 \tag{A}$$

The type of this ode is not in the lookup table. To determine ξ,η then (A) is solved using ansatz. Making bivariate polynomials of degree 1 to use as anstaz gives

$$\xi=xa_2+ya_3+a_1 \tag{1E}$$
$$\eta=xb_2+yb_3+b_1 \tag{2E}$$

Where the unknown coefficients are

$$\{a_1, a_2, a_3, b_1, b_2, b_3\}$$

Substituting equations (1E,2E) and ω into (A) gives

$$\begin{aligned} & b_2 + \frac{(x+2y+1)(b_3 - a_2)}{3+2x+4y} - \frac{(x+2y+1)^2 a_3}{(3+2x+4y)^2} \\ & - \left(\frac{1}{3+2x+4y} - \frac{2(x+2y+1)}{(3+2x+4y)^2}\right)(xa_2 + ya_3 + a_1) \\ & - \left(\frac{2}{3+2x+4y} - \frac{4(x+2y+1)}{(3+2x+4y)^2}\right)(xb_2 + yb_3 + b_1) = 0 \end{aligned} \tag{5E}$$

Putting the above in normal form gives

$$-\frac{2x^2a_2 + x^2a_3 - 4x^2b_2 - 2x^2b_3 + 8xya_2 + 4xya_3 - 16xyb_2 - 8xyb_3 + 8y^2a_2 + 4y^2a_3 - 16y^2b_2 - 8y^2b_3 + \cdots}{(3+2x+4\cdots}$$
$$= 0$$

Setting the numerator to zero gives

$$\begin{aligned} & -2x^2a_2 - x^2a_3 + 4x^2b_2 + 2x^2b_3 - 8xya_2 - 4xya_3 + 16xyb_2 + 8xyb_3 \\ & - 8y^2a_2 - 4y^2a_3 + 16y^2b_2 + 8y^2b_3 - 6xa_2 - 2xa_3 + 10xb_2 + 5xb_3 \\ & - 10ya_2 - 5ya_3 + 24yb_2 + 8yb_3 - a_1 - 3a_2 - a_3 - 2b_1 + 9b_2 + 3b_3 = 0 \end{aligned} \tag{6E}$$

Looking at the above PDE shows the following are all the terms with $\{x, y\}$ in them.

$$\{x, y\}$$

The following substitution is now made to be able to collect on all terms with $\{x, y\}$ in them

$$\{x = v_1, y = v_2\}$$

The above PDE (6E) now becomes

$$\begin{aligned} & -2a_2v_1^2 - 8a_2v_1v_2 - 8a_2v_2^2 - a_3v_1^2 - 4a_3v_1v_2 - 4a_3v_2^2 + 4b_2v_1^2 + 16b_2v_1v_2 \\ & + 16b_2v_2^2 + 2b_3v_1^2 + 8b_3v_1v_2 + 8b_3v_2^2 - 6a_2v_1 - 10a_2v_2 - 2a_3v_1 - 5a_3v_2 \\ & + 10b_2v_1 + 24b_2v_2 + 5b_3v_1 + 8b_3v_2 - a_1 - 3a_2 - a_3 - 2b_1 + 9b_2 + 3b_3 = 0 \end{aligned} \tag{7E}$$

Collecting the above on the terms v_i introduced, and these are

$$\{v_1, v_2\}$$

Equation (7E) now becomes

$$\begin{aligned} &(-2a_2 - a_3 + 4b_2 + 2b_3) v_1^2 + (-8a_2 - 4a_3 + 16b_2 + 8b_3) v_1 v_2 \\ &+ (-6a_2 - 2a_3 + 10b_2 + 5b_3) v_1 + (-8a_2 - 4a_3 + 16b_2 + 8b_3) v_2^2 \\ &+ (-10a_2 - 5a_3 + 24b_2 + 8b_3) v_2 - a_1 - 3a_2 - a_3 - 2b_1 + 9b_2 + 3b_3 = 0 \end{aligned} \tag{8E}$$

Setting each coefficients in (8E) to zero gives the following equations to solve

$$\begin{aligned} -10a_2 - 5a_3 + 24b_2 + 8b_3 &= 0 \\ -8a_2 - 4a_3 + 16b_2 + 8b_3 &= 0 \\ -6a_2 - 2a_3 + 10b_2 + 5b_3 &= 0 \\ -2a_2 - a_3 + 4b_2 + 2b_3 &= 0 \\ -a_1 - 3a_2 - a_3 - 2b_1 + 9b_2 + 3b_3 &= 0 \end{aligned}$$

Solving the above equations for the unknowns gives

$$\begin{aligned} a_1 &= 5b_2 - 2b_1 \\ a_2 &= 2b_2 \\ a_3 &= 4b_2 \\ b_1 &= b_1 \\ b_2 &= b_2 \\ b_3 &= 2b_2 \end{aligned}$$

Substituting the above solution in the anstaz (1E,2E) (using 1 as arbitrary value for any unknown in the RHS) gives

$$\begin{aligned} \xi &= -2 \\ \eta &= 1 \end{aligned}$$

Shifting is now applied to make $\xi = 0$ in order to simplify the rest of the computation

$$\begin{aligned} \eta &= \eta - \omega(x, y)\, \xi \\ &= 1 - \left(\frac{x + 2y + 1}{3 + 2x + 4y}\right)(-2) \\ &= \frac{4x + 8y + 5}{3 + 2x + 4y} \\ \xi &= 0 \end{aligned}$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x, y) \to (R, S)$ where (R, S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y}\right) S(x, y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$R = x$$

S is found from

$$\begin{aligned} S &= \int \frac{1}{\eta} dy \\ &= \int \frac{1}{\frac{4x+8y+5}{3+2x+4y}} dy \end{aligned}$$

Which results in

$$S = \frac{y}{2} + \frac{\ln(4x + 8y + 5)}{16}$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x, y) S_y}{R_x + \omega(x, y) R_y} \tag{2}$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x, y)$ is the right hand side of the original ode given by

$$\omega(x, y) = \frac{x + 2y + 1}{3 + 2x + 4y}$$

Evaluating all the partial derivatives gives

$$\begin{aligned} R_x &= 1 \\ R_y &= 0 \\ S_x &= \frac{1}{16x + 32y + 20} \\ S_y &= \frac{3 + 2x + 4y}{4x + 8y + 5} \end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = \frac{1}{4} \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = \frac{1}{4}$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$S(R) = \frac{R}{4} + c_1 \tag{4}$$

To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$\frac{y}{2} + \frac{\ln(4x + 8y + 5)}{16} = \frac{x}{4} + c_1$$

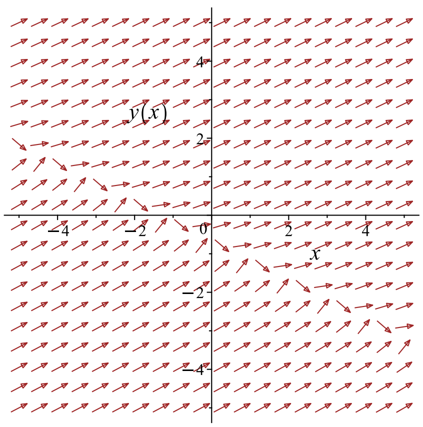
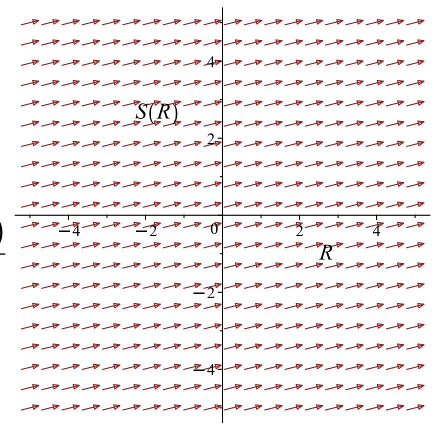
Which simplifies to

$$\frac{y}{2} + \frac{\ln(4x + 8y + 5)}{16} = \frac{x}{4} + c_1$$

Which gives

$$y = \frac{\operatorname{LambertW}\left(\mathrm{e}^{8x+5+16c_1}\right)}{8} - \frac{x}{2} - \frac{5}{8}$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in x, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dx} = \frac{x+2y+1}{3+2x+4y}$ 	$R = x$ $S = \frac{y}{2} + \frac{\ln(4x+8y+5)}{16}$	$\frac{dS}{dR} = \frac{1}{4}$ 

Summary

The solution(s) found are the following

$$y = \frac{\text{LambertW}\left(e^{8x+5+16c_1}\right)}{8} - \frac{x}{2} - \frac{5}{8} \tag{1}$$

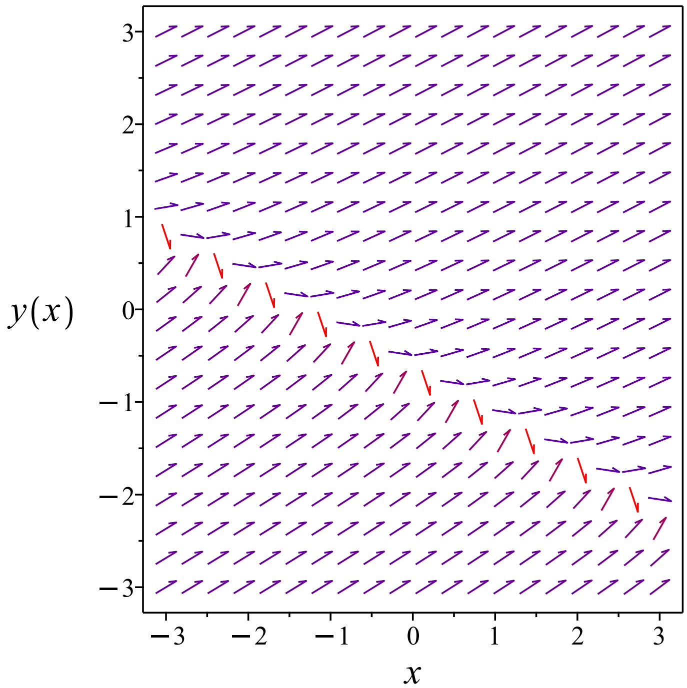


Figure 25: Slope field plot

Verification of solutions

$$y = \frac{\text{LambertW}\left(e^{8x+5+16c_1}\right)}{8} - \frac{x}{2} - \frac{5}{8}$$

Verified OK.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
trying homogeneous types:
trying homogeneous C
1st order, trying the canonical coordinates of the invariance group
<- 1st order, canonical coordinates successful
<- homogeneous successful`
```

✓ Solution by Maple

Time used: 0.032 (sec). Leaf size: 20

```
dsolve((3+2*x+4*y(x))*diff(y(x),x)=1+x+2*y(x),y(x), singsol=all)
```

$$y(x) = -\frac{x}{2} + \frac{\operatorname{LambertW}\left(c_1 e^{5+8x}\right)}{8} - \frac{5}{8}$$

✓ Solution by Mathematica

Time used: 4.849 (sec). Leaf size: 39

```
DSolve[(3+2*x+4*y[x])*y'[x]==1+x+2*y[x],y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \frac{1}{8}\left(W\left(-e^{8x-1+c_1}\right) - 4x - 5\right)$$
$$y(x) \to \frac{1}{8}(-4x - 5)$$

1.16 problem 16

Internal problem ID [3161]
Internal file name [`OUTPUT/2653_Sunday_June_05_2022_08_38_08_AM_66885656/index.tex`]

Book: Differential equations for engineers by Wei-Chau XIE, Cambridge Press 2010
Section: Chapter 2. First-Order and Simple Higher-Order Differential Equations. Page 78
Problem number: 16.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"homogeneousTypeMapleC", "first_order_ode_lie_symmetry_calculated"**

Maple gives the following as the ode type

```
[[_homogeneous, `class C`], _rational, [_Abel, `2nd type`, `
   class A`]]
```

$$\boxed{y' - \frac{2x + y - 1}{x - y - 2} = 0}$$

1.16.1 Solving as homogeneousTypeMapleC ode

Let $Y = y + y_0$ and $X = x + x_0$ then the above is transformed to new ode in $Y(X)$

$$\frac{d}{dX}Y(X) = -\frac{2X + 2x_0 + Y(X) + y_0 - 1}{-X - x_0 + Y(X) + y_0 + 2}$$

Solving for possible values of x_0 and y_0 which makes the above ode a homogeneous ode results in

$$x_0 = 1$$
$$y_0 = -1$$

Using these values now it is possible to easily solve for $Y(X)$. The above ode now becomes

$$\frac{d}{dX}Y(X) = -\frac{2X + Y(X)}{-X + Y(X)}$$

In canonical form, the ODE is

$$
\begin{aligned}
Y' &= F(X, Y) \\
&= -\frac{2X + Y}{-X + Y}
\end{aligned}
\tag{1}
$$

An ode of the form $Y' = \frac{M(X,Y)}{N(X,Y)}$ is called homogeneous if the functions $M(X, Y)$ and $N(X, Y)$ are both homogeneous functions and of the same order. Recall that a function $f(X, Y)$ is homogeneous of order n if

$$
f(t^n X, t^n Y) = t^n f(X, Y)
$$

In this case, it can be seen that both $M = 2X + Y$ and $N = X - Y$ are both homogeneous and of the same order $n = 1$. Therefore this is a homogeneous ode. Since this ode is homogeneous, it is converted to separable ODE using the substitution $u = \frac{Y}{X}$, or $Y = uX$. Hence

$$
\frac{\mathrm{d}Y}{\mathrm{d}X} = \frac{\mathrm{d}u}{\mathrm{d}X} X + u
$$

Applying the transformation $Y = uX$ to the above ODE in (1) gives

$$
\begin{aligned}
\frac{\mathrm{d}u}{\mathrm{d}X} X + u &= \frac{-u - 2}{u - 1} \\
\frac{\mathrm{d}u}{\mathrm{d}X} &= \frac{\frac{-u(X)-2}{u(X)-1} - u(X)}{X}
\end{aligned}
$$

Or

$$
\frac{d}{dX} u(X) - \frac{\frac{-u(X)-2}{u(X)-1} - u(X)}{X} = 0
$$

Or

$$
\left(\frac{d}{dX} u(X)\right) X u(X) - \left(\frac{d}{dX} u(X)\right) X + u(X)^2 + 2 = 0
$$

Or

$$
X(u(X) - 1)\left(\frac{d}{dX} u(X)\right) + u(X)^2 + 2 = 0
$$

Which is now solved as separable in $u(X)$. Which is now solved in $u(X)$. In canonical form the ODE is

$$
\begin{aligned}
u' &= F(X, u) \\
&= f(X) g(u) \\
&= -\frac{u^2 + 2}{X (u - 1)}
\end{aligned}
$$

Where $f(X) = -\frac{1}{X}$ and $g(u) = \frac{u^2+2}{u-1}$. Integrating both sides gives

$$\frac{1}{\frac{u^2+2}{u-1}}\,du = -\frac{1}{X}\,dX$$

$$\int \frac{1}{\frac{u^2+2}{u-1}}\,du = \int -\frac{1}{X}\,dX$$

$$\frac{\ln(u^2+2)}{2} - \frac{\sqrt{2}\,\arctan\left(\frac{\sqrt{2}\,u}{2}\right)}{2} = -\ln(X) + c_2$$

The solution is

$$\frac{\ln\left(u(X)^2+2\right)}{2} - \frac{\sqrt{2}\,\arctan\left(\frac{\sqrt{2}\,u(X)}{2}\right)}{2} + \ln(X) - c_2 = 0$$

Now u in the above solution is replaced back by Y using $u = \frac{Y}{X}$ which results in the solution

$$\frac{\ln\left(\frac{Y(X)^2}{X^2}+2\right)}{2} - \frac{\sqrt{2}\,\arctan\left(\frac{\sqrt{2}\,Y(X)}{2X}\right)}{2} + \ln(X) - c_2 = 0$$

Using the solution for $Y(X)$

$$\frac{\ln\left(\frac{Y(X)^2}{X^2}+2\right)}{2} - \frac{\sqrt{2}\,\arctan\left(\frac{\sqrt{2}\,Y(X)}{2X}\right)}{2} + \ln(X) - c_2 = 0$$

And replacing back terms in the above solution using

$$Y = y + y_0$$
$$X = x + x_0$$

Or

$$Y = y - 1$$
$$X = x + 1$$

Then the solution in y becomes

$$\frac{\ln\left(\frac{(y+1)^2}{(x-1)^2}+2\right)}{2} - \frac{\sqrt{2}\,\arctan\left(\frac{\sqrt{2}\,(y+1)}{2x-2}\right)}{2} + \ln(x-1) - c_2 = 0$$

Summary

The solution(s) found are the following

$$\frac{\ln\left(\frac{(y+1)^2}{(x-1)^2}+2\right)}{2}-\frac{\sqrt{2}\,\arctan\left(\frac{\sqrt{2}\,(y+1)}{2x-2}\right)}{2}+\ln(x-1)-c_2=0 \tag{1}$$

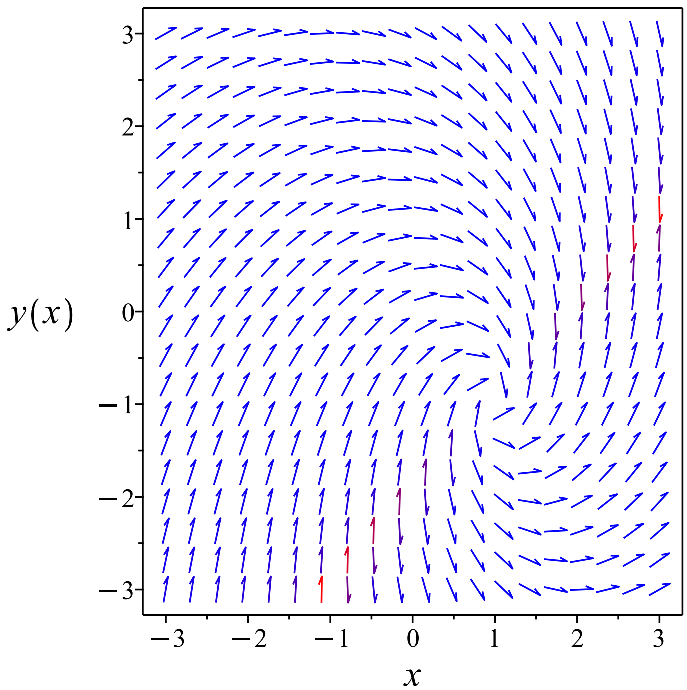


Figure 26: Slope field plot

Verification of solutions

$$\frac{\ln\left(\frac{(y+1)^2}{(x-1)^2}+2\right)}{2}-\frac{\sqrt{2}\,\arctan\left(\frac{\sqrt{2}\,(y+1)}{2x-2}\right)}{2}+\ln(x-1)-c_2=0$$

Verified OK.

1.16.2 Solving as first order ode lie symmetry calculated ode

Writing the ode as

$$y'=-\frac{2x+y-1}{-x+y+2}$$
$$y'=\omega(x,y)$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x+\omega(\eta_y-\xi_x)-\omega^2\xi_y-\omega_x\xi-\omega_y\eta=0 \tag{A}$$

The type of this ode is not in the lookup table. To determine ξ, η then (A) is solved using ansatz. Making bivariate polynomials of degree 1 to use as anstaz gives

$$\xi = xa_2 + ya_3 + a_1 \tag{1E}$$

$$\eta = xb_2 + yb_3 + b_1 \tag{2E}$$

Where the unknown coefficients are

$$\{a_1, a_2, a_3, b_1, b_2, b_3\}$$

Substituting equations (1E,2E) and ω into (A) gives

$$\begin{aligned} & b_2 - \frac{(2x+y-1)(b_3-a_2)}{-x+y+2} - \frac{(2x+y-1)^2 a_3}{(-x+y+2)^2} \\ & - \left(-\frac{2}{-x+y+2} - \frac{2x+y-1}{(-x+y+2)^2}\right)(xa_2+ya_3+a_1) \\ & - \left(-\frac{1}{-x+y+2} + \frac{2x+y-1}{(-x+y+2)^2}\right)(xb_2+yb_3+b_1) = 0 \end{aligned} \tag{5E}$$

Putting the above in normal form gives

$$-\frac{2x^2a_2 + 4x^2a_3 + 2x^2b_2 - 2x^2b_3 - 4xya_2 + 4xya_3 + 2xyb_2 + 4xyb_3 - y^2a_2 - 2y^2a_3 - y^2b_2 + y^2b_3 - 8xa_2}{(x - y -}$$
$$= 0$$

Setting the numerator to zero gives

$$\begin{aligned} & -2x^2a_2 - 4x^2a_3 - 2x^2b_2 + 2x^2b_3 + 4xya_2 - 4xya_3 - 2xyb_2 - 4xyb_3 \\ & + y^2a_2 + 2y^2a_3 + y^2b_2 - y^2b_3 + 8xa_2 + 4xa_3 - 3xb_1 - xb_2 - 5xb_3 + 3ya_1 \\ & + ya_2 + 5ya_3 + 4yb_2 + 2yb_3 + 3a_1 - 2a_2 - a_3 + 3b_1 + 4b_2 + 2b_3 = 0 \end{aligned} \tag{6E}$$

Looking at the above PDE shows the following are all the terms with $\{x, y\}$ in them.

$$\{x, y\}$$

The following substitution is now made to be able to collect on all terms with $\{x, y\}$ in them

$$\{x = v_1, y = v_2\}$$

The above PDE (6E) now becomes

$$
\begin{aligned}
&-2a_2v_1^2+4a_2v_1v_2+a_2v_2^2-4a_3v_1^2-4a_3v_1v_2+2a_3v_2^2-2b_2v_1^2-2b_2v_1v_2+b_2v_2^2 \\
&+2b_3v_1^2-4b_3v_1v_2-b_3v_2^2+3a_1v_2+8a_2v_1+a_2v_2+4a_3v_1+5a_3v_2-3b_1v_1 \\
&-b_2v_1+4b_2v_2-5b_3v_1+2b_3v_2+3a_1-2a_2-a_3+3b_1+4b_2+2b_3=0
\end{aligned} \tag{7E}
$$

Collecting the above on the terms v_i introduced, and these are

$$
\{v_1, v_2\}
$$

Equation (7E) now becomes

$$
\begin{aligned}
&(-2a_2-4a_3-2b_2+2b_3)\,v_1^2+(4a_2-4a_3-2b_2-4b_3)\,v_1v_2 \\
&+(8a_2+4a_3-3b_1-b_2-5b_3)\,v_1+(a_2+2a_3+b_2-b_3)\,v_2^2 \\
&+(3a_1+a_2+5a_3+4b_2+2b_3)\,v_2+3a_1-2a_2-a_3+3b_1+4b_2+2b_3=0
\end{aligned} \tag{8E}
$$

Setting each coefficients in (8E) to zero gives the following equations to solve

$$
\begin{aligned}
-2a_2-4a_3-2b_2+2b_3&=0 \\
a_2+2a_3+b_2-b_3&=0 \\
4a_2-4a_3-2b_2-4b_3&=0 \\
3a_1+a_2+5a_3+4b_2+2b_3&=0 \\
8a_2+4a_3-3b_1-b_2-5b_3&=0 \\
3a_1-2a_2-a_3+3b_1+4b_2+2b_3&=0
\end{aligned}
$$

Solving the above equations for the unknowns gives

$$
\begin{aligned}
a_1&=a_3-b_3 \\
a_2&=b_3 \\
a_3&=a_3 \\
b_1&=2a_3+b_3 \\
b_2&=-2a_3 \\
b_3&=b_3
\end{aligned}
$$

Substituting the above solution in the anstaz (1E,2E) (using 1 as arbitrary value for any unknown in the RHS) gives

$$
\begin{aligned}
\xi&=x-1 \\
\eta&=y+1
\end{aligned}
$$

Shifting is now applied to make $\xi = 0$ in order to simplify the rest of the computation

$$\begin{aligned}\eta &= \eta - \omega(x,y)\,\xi \\ &= y + 1 - \left(-\frac{2x + y - 1}{-x + y + 2}\right)(x - 1) \\ &= \frac{-2x^2 - y^2 + 4x - 2y - 3}{x - y - 2} \\ \xi &= 0\end{aligned}$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x, y) \to (R, S)$ where (R, S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y}\right) S(x,y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$R = x$$

S is found from

$$\begin{aligned}S &= \int \frac{1}{\eta} dy \\ &= \int \frac{1}{\frac{-2x^2 - y^2 + 4x - 2y - 3}{x - y - 2}} dy\end{aligned}$$

Which results in

$$S = \frac{\ln(2x^2 + y^2 - 4x + 2y + 3)}{2} + \frac{(1 - x)\sqrt{2}\,\arctan\left(\frac{(2+2y)\sqrt{2}}{4x-4}\right)}{2x - 2}$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x,y)S_y}{R_x + \omega(x,y)R_y} \tag{2}$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x,y)$ is the right hand side of the original ode given by

$$\omega(x,y) = -\frac{2x + y - 1}{-x + y + 2}$$

Evaluating all the partial derivatives gives

$$
\begin{aligned}
R_x &= 1 \\
R_y &= 0 \\
S_x &= \frac{2x + y - 1}{2x^2 + y^2 - 4x + 2y + 3} \\
S_y &= \frac{-x + y + 2}{2x^2 + y^2 - 4x + 2y + 3}
\end{aligned}
$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$
\frac{dS}{dR} = 0 \tag{2A}
$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$
\frac{dS}{dR} = 0
$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$
S(R) = c_1 \tag{4}
$$

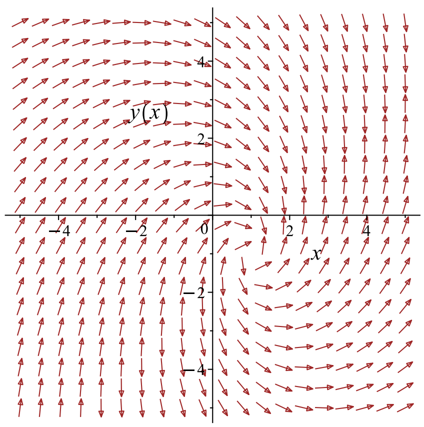
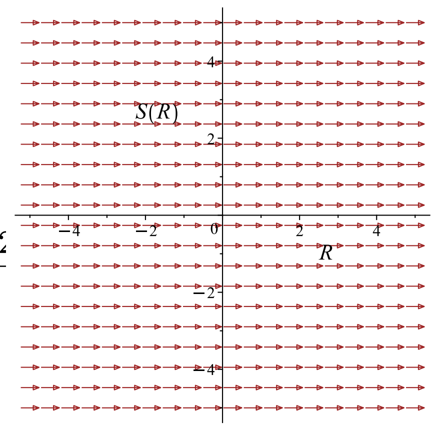
To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$
\frac{\ln \left(y^2 + 2x^2 + 2y - 4x + 3\right)}{2} - \frac{\sqrt{2}\, \arctan\left(\frac{\sqrt{2}\,(y+1)}{2x-2}\right)}{2} = c_1
$$

Which simplifies to

$$
\frac{\ln \left(y^2 + 2x^2 + 2y - 4x + 3\right)}{2} - \frac{\sqrt{2}\, \arctan\left(\frac{\sqrt{2}\,(y+1)}{2x-2}\right)}{2} = c_1
$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in x, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dx} = -\frac{2x+y-1}{-x+y+2}$ 	$R = x$ $S = \frac{\ln(2x^2 + y^2 - 4x + 2}{2}$	$\frac{dS}{dR} = 0$ 

Summary

The solution(s) found are the following

$$\frac{\ln(y^2 + 2x^2 + 2y - 4x + 3)}{2} - \frac{\sqrt{2}\arctan\left(\frac{\sqrt{2}(y+1)}{2x-2}\right)}{2} = c_1 \tag{1}$$

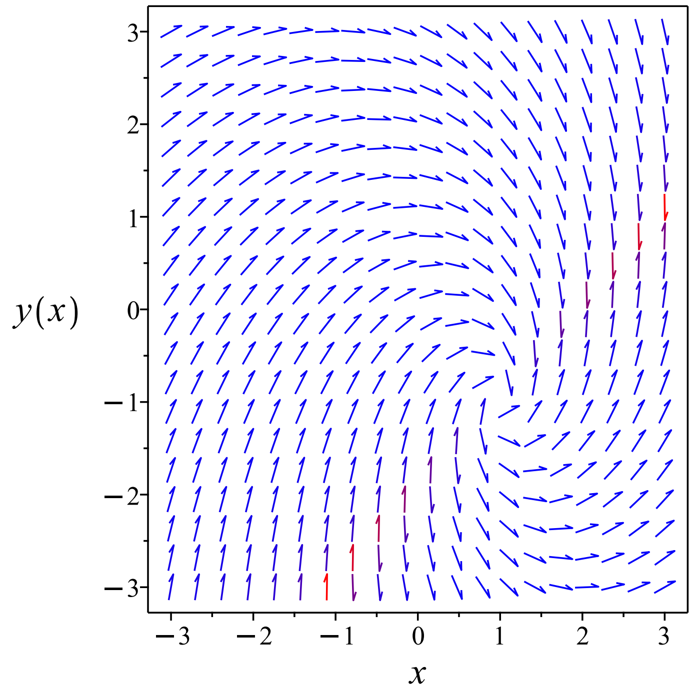


Figure 27: Slope field plot

Verification of solutions

$$\frac{\ln \left(y^2 + 2x^2 + 2y - 4x + 3\right)}{2} - \frac{\sqrt{2}\, \arctan\left(\frac{\sqrt{2}\,(y+1)}{2x-2}\right)}{2} = c_1$$

Verified OK.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
trying homogeneous types:
trying homogeneous C
trying homogeneous types:
trying homogeneous D
<- homogeneous successful
<- homogeneous successful`
```

✓ Solution by Maple

Time used: 0.297 (sec). Leaf size: 47

```
dsolve(diff(y(x),x)=(2*x+y(x)-1)/(x-y(x)-2),y(x), singsol=all)
```

$$y(x) = -1 - \tan\left(\mathrm{RootOf}\left(\sqrt{2}\,\ln\left(\sec\left(\_Z\right)^{2}\left(x-1\right)^{2}\right)+\sqrt{2}\,\ln\left(2\right)+2\sqrt{2}\,c_1+2\_Z\right)\right)\left(x-1\right)\sqrt{2}$$

✓ Solution by Mathematica

Time used: 0.125 (sec). Leaf size: 75

```
DSolve[y'[x]==(2*x+y[x]-1)/(x-y[x]-2),y[x],x,IncludeSingularSolutions -> True]
```

$$\mathrm{Solve}\left[2\sqrt{2}\arctan\left(\frac{y(x)+2x-1}{\sqrt{2}(-y(x)+x-2)}\right)+\log(9)=2\log\left(\frac{2x^2+y(x)^2+2y(x)-4x+3}{(x-1)^2}\right)+4\log(x-1)+3c_1,y(x)\right]$$

1.17 problem 17

Internal problem ID [3162]
Internal file name [`OUTPUT/2654_Sunday_June_05_2022_08_38_09_AM_50529257/index.tex`]

Book: Differential equations for engineers by Wei-Chau XIE, Cambridge Press 2010
Section: Chapter 2. First-Order and Simple Higher-Order Differential Equations. Page 78
Problem number: 17.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"homogeneousTypeMapleC", "exactWithIntegrationFactor", "first_order_ode_lie_symmetry_calculated"**

Maple gives the following as the ode type

```
[[_homogeneous, `class C`], _rational, [_Abel, `2nd type`, `
   class A`]]
```

$$\boxed{y-(2x+y-4)\,y'=-2}$$

1.17.1 Solving as homogeneousTypeMapleC ode

Let $Y = y + y_0$ and $X = x + x_0$ then the above is transformed to new ode in $Y(X)$

$$\frac{d}{dX}Y(X) = \frac{Y(X) + y_0 + 2}{2X + 2x_0 + Y(X) + y_0 - 4}$$

Solving for possible values of x_0 and y_0 which makes the above ode a homogeneous ode results in

$$x_0 = 3$$
$$y_0 = -2$$

Using these values now it is possible to easily solve for $Y(X)$. The above ode now becomes

$$\frac{d}{dX}Y(X) = \frac{Y(X)}{2X + Y(X)}$$

In canonical form, the ODE is

$$
\begin{aligned}
Y' &= F(X, Y) \\
&= \frac{Y}{2X + Y}
\end{aligned}
\tag{1}
$$

An ode of the form $Y' = \frac{M(X,Y)}{N(X,Y)}$ is called homogeneous if the functions $M(X, Y)$ and $N(X, Y)$ are both homogeneous functions and of the same order. Recall that a function $f(X, Y)$ is homogeneous of order n if

$$
f(t^n X, t^n Y) = t^n f(X, Y)
$$

In this case, it can be seen that both $M = Y$ and $N = 2X + Y$ are both homogeneous and of the same order $n = 1$. Therefore this is a homogeneous ode. Since this ode is homogeneous, it is converted to separable ODE using the substitution $u = \frac{Y}{X}$, or $Y = uX$. Hence

$$
\frac{\mathrm{d}Y}{\mathrm{d}X} = \frac{\mathrm{d}u}{\mathrm{d}X} X + u
$$

Applying the transformation $Y = uX$ to the above ODE in (1) gives

$$
\begin{aligned}
\frac{\mathrm{d}u}{\mathrm{d}X} X + u &= \frac{u}{u + 2} \\
\frac{\mathrm{d}u}{\mathrm{d}X} &= \frac{\frac{u(X)}{u(X)+2} - u(X)}{X}
\end{aligned}
$$

Or

$$
\frac{d}{dX} u(X) - \frac{\frac{u(X)}{u(X)+2} - u(X)}{X} = 0
$$

Or

$$
\left(\frac{d}{dX} u(X)\right) X u(X) + 2\left(\frac{d}{dX} u(X)\right) X + u(X)^2 + u(X) = 0
$$

Or

$$
X(u(X) + 2)\left(\frac{d}{dX} u(X)\right) + u(X)^2 + u(X) = 0
$$

Which is now solved as separable in $u(X)$. Which is now solved in $u(X)$. In canonical form the ODE is

$$
\begin{aligned}
u' &= F(X, u) \\
&= f(X) g(u) \\
&= -\frac{u(u + 1)}{X\,(u + 2)}
\end{aligned}
$$

Where $f(X) = -\frac{1}{X}$ and $g(u) = \frac{u(u+1)}{u+2}$. Integrating both sides gives

$$\frac{1}{\frac{u(u+1)}{u+2}} du = -\frac{1}{X} dX$$

$$\int \frac{1}{\frac{u(u+1)}{u+2}} du = \int -\frac{1}{X} dX$$

$$-\ln(u+1) + 2\ln(u) = -\ln(X) + c_2$$

Raising both side to exponential gives

$$e^{-\ln(u+1)+2\ln(u)} = e^{-\ln(X)+c_2}$$

Which simplifies to

$$\frac{u^2}{u+1} = \frac{c_3}{X}$$

The solution is

$$\frac{u(X)^2}{u(X)+1} = \frac{c_3}{X}$$

Now u in the above solution is replaced back by Y using $u = \frac{Y}{X}$ which results in the solution

$$\frac{Y(X)^2}{\left(\frac{Y(X)}{X}+1\right) X^2} = \frac{c_3}{X}$$

Which simplifies to

$$\frac{Y(X)^2}{Y(X)+X} = c_3$$

Using the solution for $Y(X)$

$$\frac{Y(X)^2}{Y(X)+X} = c_3$$

And replacing back terms in the above solution using

$$Y = y + y_0$$
$$X = x + x_0$$

Or

$$Y = y - 2$$
$$X = x + 3$$

Then the solution in y becomes

$$\frac{(y+2)^2}{y+x-1} = c_3$$

Summary
The solution(s) found are the following

$$\frac{(y+2)^2}{y+x-1} = c_3 \tag{1}$$

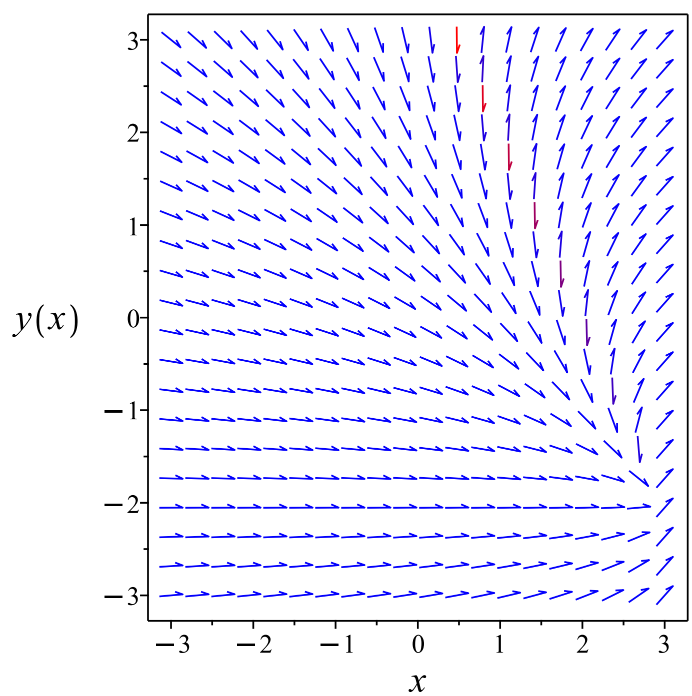


Figure 28: Slope field plot

Verification of solutions

$$\frac{(y+2)^2}{y+x-1} = c_3$$

Verified OK.

1.17.2 Solving as first order ode lie symmetry calculated ode

Writing the ode as

$$
\begin{aligned}
y' &= \frac{y+2}{2x+y-4} \\
y' &= \omega(x,y)
\end{aligned}
$$

The condition of Lie symmetry is the linearized PDE given by

$$
\eta_x + \omega(\eta_y - \xi_x) - \omega^2 \xi_y - \omega_x \xi - \omega_y \eta = 0 \tag{A}
$$

The type of this ode is not in the lookup table. To determine ξ, η then (A) is solved using ansatz. Making bivariate polynomials of degree 1 to use as anstaz gives

$$
\xi = x a_2 + y a_3 + a_1 \tag{1E}
$$

$$
\eta = x b_2 + y b_3 + b_1 \tag{2E}
$$

Where the unknown coefficients are

$$
\{a_1, a_2, a_3, b_1, b_2, b_3\}
$$

Substituting equations (1E,2E) and ω into (A) gives

$$
\begin{aligned}
b_2 &+ \frac{(y+2)(b_3 - a_2)}{2x+y-4} - \frac{(y+2)^2 a_3}{(2x+y-4)^2} + \frac{2(y+2)(x a_2 + y a_3 + a_1)}{(2x+y-4)^2} \\
&- \left(\frac{1}{2x+y-4} - \frac{y+2}{(2x+y-4)^2}\right)(x b_2 + y b_3 + b_1) = 0
\end{aligned} \tag{5E}
$$

Putting the above in normal form gives

$$
\frac{2x^2 b_2 + 4xy b_2 - y^2 a_2 + y^2 a_3 + y^2 b_2 + y^2 b_3 - 2x b_1 - 10x b_2 + 4x b_3 + 2y a_1 + 2y a_2 - 8y b_2 + 4y b_3 + 4a_1 + 8}{(2x+y-4)^2} = 0
$$

Setting the numerator to zero gives

$$
\begin{aligned}
&2x^2 b_2 + 4xy b_2 - y^2 a_2 + y^2 a_3 + y^2 b_2 + y^2 b_3 - 2x b_1 - 10x b_2 + 4x b_3 \\
&+ 2y a_1 + 2y a_2 - 8y b_2 + 4y b_3 + 4a_1 + 8a_2 - 4a_3 + 6b_1 + 16b_2 - 8b_3 = 0
\end{aligned} \tag{6E}
$$

Looking at the above PDE shows the following are all the terms with $\{x, y\}$ in them.

$$\{x, y\}$$

The following substitution is now made to be able to collect on all terms with $\{x, y\}$ in them

$$\{x = v_1, y = v_2\}$$

The above PDE (6E) now becomes

$$\begin{aligned} -a_2v_2^2 + a_3v_2^2 + 2b_2v_1^2 + 4b_2v_1v_2 + b_2v_2^2 + b_3v_2^2 + 2a_1v_2 + 2a_2v_2 - 2b_1v_1 \\ - 10b_2v_1 - 8b_2v_2 + 4b_3v_1 + 4b_3v_2 + 4a_1 + 8a_2 - 4a_3 + 6b_1 + 16b_2 - 8b_3 = 0 \end{aligned} \tag{7E}$$

Collecting the above on the terms v_i introduced, and these are

$$\{v_1, v_2\}$$

Equation (7E) now becomes

$$\begin{aligned} 2b_2v_1^2 + 4b_2v_1v_2 + (-2b_1 - 10b_2 + 4b_3)\, v_1 + (-a_2 + a_3 + b_2 + b_3)\, v_2^2 \\ + (2a_1 + 2a_2 - 8b_2 + 4b_3)\, v_2 + 4a_1 + 8a_2 - 4a_3 + 6b_1 + 16b_2 - 8b_3 = 0 \end{aligned} \tag{8E}$$

Setting each coefficients in (8E) to zero gives the following equations to solve

$$\begin{aligned} 2b_2 &= 0 \\ 4b_2 &= 0 \\ -2b_1 - 10b_2 + 4b_3 &= 0 \\ 2a_1 + 2a_2 - 8b_2 + 4b_3 &= 0 \\ -a_2 + a_3 + b_2 + b_3 &= 0 \\ 4a_1 + 8a_2 - 4a_3 + 6b_1 + 16b_2 - 8b_3 &= 0 \end{aligned}$$

Solving the above equations for the unknowns gives

$$\begin{aligned} a_1 &= -a_3 - 3b_3 \\ a_2 &= a_3 + b_3 \\ a_3 &= a_3 \\ b_1 &= 2b_3 \\ b_2 &= 0 \\ b_3 &= b_3 \end{aligned}$$

Substituting the above solution in the anstaz (1E,2E) (using 1 as arbitrary value for any unknown in the RHS) gives

$$\xi = x - 3$$
$$\eta = y + 2$$

Shifting is now applied to make $\xi = 0$ in order to simplify the rest of the computation

$$\begin{aligned}
\eta &= \eta - \omega(x, y)\,\xi \\
&= y + 2 - \left(\frac{y + 2}{2x + y - 4}\right)(x - 3) \\
&= \frac{xy + y^2 + 2x + y - 2}{2x + y - 4} \\
\xi &= 0
\end{aligned}$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x, y) \to (R, S)$ where (R, S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y}\right) S(x, y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$R = x$$

S is found from

$$\begin{aligned}
S &= \int \frac{1}{\eta} dy \\
&= \int \frac{1}{\frac{xy + y^2 + 2x + y - 2}{2x + y - 4}} dy
\end{aligned}$$

Which results in

$$S = 2\ln(y + 2) - \ln(x - 1 + y)$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x, y) S_y}{R_x + \omega(x, y) R_y} \tag{2}$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x, y)$ is the right hand side of the original ode given by

$$\omega(x, y) = \frac{y+2}{2x+y-4}$$

Evaluating all the partial derivatives gives

$$\begin{aligned}
R_x &= 1 \\
R_y &= 0 \\
S_x &= -\frac{1}{x-1+y} \\
S_y &= \frac{2x+y-4}{(y+2)(x-1+y)}
\end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = 0 \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = 0$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$S(R) = c_1 \tag{4}$$

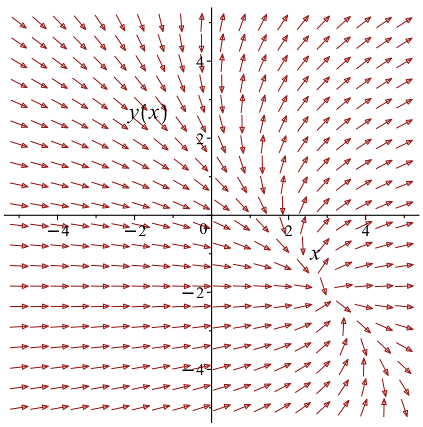
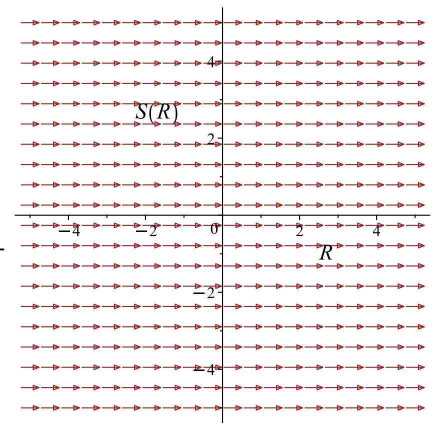
To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$2\ln(y+2) - \ln(y+x-1) = c_1$$

Which simplifies to

$$2\ln(y+2) - \ln(y+x-1) = c_1$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in x, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dx} = \frac{y+2}{2x+y-4}$ 	$R = x$ $S = 2\ln(y+2) - \ln(x -$	$\frac{dS}{dR} = 0$ 

Summary

The solution(s) found are the following

$$2\ln(y+2) - \ln(y+x-1) = c_1 \tag{1}$$

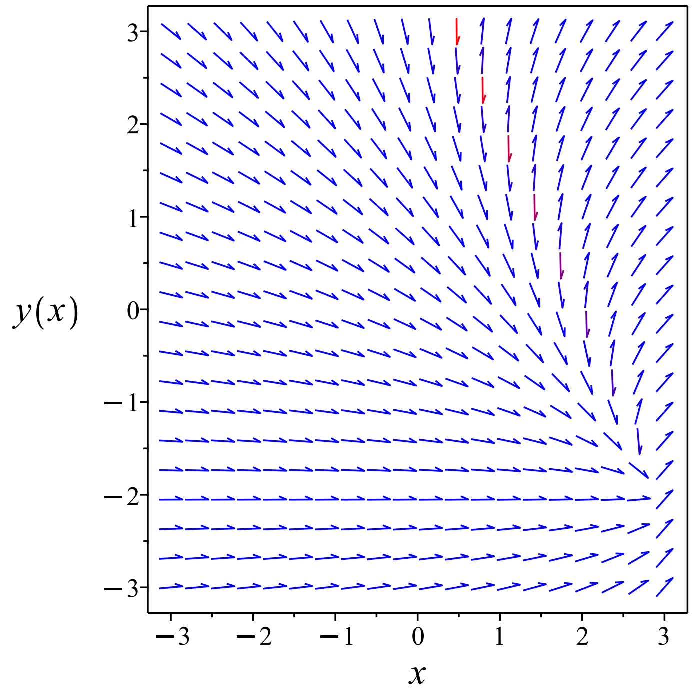


Figure 29: Slope field plot

Verification of solutions

$$2\ln(y+2)-\ln(y+x-1)=c_1$$

Verified OK.

1.17.3 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x,y)+N(x,y)\frac{dy}{dx}=0 \tag{A}$$

We assume there exists a function $\phi(x,y)=c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y)=0$$

Hence

$$\frac{\partial\phi}{\partial x}+\frac{\partial\phi}{\partial y}\frac{dy}{dx}=0 \tag{B}$$

Comparing (A,B) shows that

$$\begin{aligned}\frac{\partial \phi}{\partial x} &= M\\ \frac{\partial \phi}{\partial y} &= N\end{aligned}$$

But since $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x, y)$ but at least we know now that we can do that since the condition $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x, y)\,\mathrm{d}x + N(x, y)\,\mathrm{d}y = 0 \tag{1A}$$

Therefore

$$\begin{aligned}(-2x - y + 4)\,\mathrm{d}y &= (-y - 2)\,\mathrm{d}x\\ (y + 2)\,\mathrm{d}x + (-2x - y + 4)\,\mathrm{d}y &= 0 \qquad (2A)\end{aligned}$$

Comparing (1A) and (2A) shows that

$$\begin{aligned}M(x, y) &= y + 2\\ N(x, y) &= -2x - y + 4\end{aligned}$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Using result found above gives

$$\begin{aligned}\frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}(y + 2)\\ &= 1\end{aligned}$$

And

$$\begin{aligned}\frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}(-2x - y + 4)\\ &= -2\end{aligned}$$

Since $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$, then the ODE is not exact. Since the ODE is not exact, we will try to find an integrating factor to make it exact. Let

$$\begin{aligned}
A &= \frac{1}{N}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right) \\
&= \frac{1}{-2x - y + 4}((1) - (-2)) \\
&= -\frac{3}{2x + y - 4}
\end{aligned}$$

Since A depends on y, it can not be used to obtain an integrating factor. We will now try a second method to find an integrating factor. Let

$$\begin{aligned}
B &= \frac{1}{M}\left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) \\
&= \frac{1}{y + 2}((-2) - (1)) \\
&= -\frac{3}{y + 2}
\end{aligned}$$

Since B does not depend on x, it can be used to obtain an integrating factor. Let the integrating factor be μ. Then

$$\begin{aligned}
\mu &= e^{\int B \, \mathrm{d}y} \\
&= e^{\int -\frac{3}{y+2} \, \mathrm{d}y}
\end{aligned}$$

The result of integrating gives

$$\begin{aligned}
\mu &= e^{-3\ln(y+2)} \\
&= \frac{1}{(y + 2)^3}
\end{aligned}$$

M and N are now multiplied by this integrating factor, giving new M and new N which are called $\overline{M}$ and $\overline{N}$ so not to confuse them with the original M and N.

$$\begin{aligned}
\overline{M} &= \mu M \\
&= \frac{1}{(y + 2)^3}(y + 2) \\
&= \frac{1}{(y + 2)^2}
\end{aligned}$$

And

$$
\begin{aligned}
\overline{N} &= \mu N \\
&= \frac{1}{(y+2)^3}(-2x-y+4) \\
&= \frac{-2x-y+4}{(y+2)^3}
\end{aligned}
$$

So now a modified ODE is obtained from the original ODE which will be exact and can be solved using the standard method. The modified ODE is

$$
\begin{aligned}
\overline{M} + \overline{N}\frac{\mathrm{d}y}{\mathrm{d}x} &= 0 \\
\left(\frac{1}{(y+2)^2}\right) + \left(\frac{-2x-y+4}{(y+2)^3}\right)\frac{\mathrm{d}y}{\mathrm{d}x} &= 0
\end{aligned}
$$

The following equations are now set up to solve for the function $\phi(x, y)$

$$
\frac{\partial \phi}{\partial x} = \overline{M} \tag{1}
$$

$$
\frac{\partial \phi}{\partial y} = \overline{N} \tag{2}
$$

Integrating (1) w.r.t. x gives

$$
\begin{aligned}
\int \frac{\partial \phi}{\partial x}\,\mathrm{d}x &= \int \overline{M}\,\mathrm{d}x \\
\int \frac{\partial \phi}{\partial x}\,\mathrm{d}x &= \int \frac{1}{(y+2)^2}\,\mathrm{d}x
\end{aligned}
$$

$$
\phi = \frac{x}{(y+2)^2} + f(y) \tag{3}
$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$
\frac{\partial \phi}{\partial y} = -\frac{2x}{(y+2)^3} + f'(y) \tag{4}
$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = \frac{-2x-y+4}{(y+2)^3}$. Therefore equation (4) becomes

$$
\frac{-2x-y+4}{(y+2)^3} = -\frac{2x}{(y+2)^3} + f'(y) \tag{5}
$$

Solving equation (5) for $f'(y)$ gives

$$f'(y) = -\frac{-4+y}{(y+2)^3}$$

Integrating the above w.r.t y gives

$$\int f'(y)\,\mathrm{d}y = \int \left(\frac{4-y}{(y+2)^3}\right)\mathrm{d}y$$

$$f(y) = \frac{1}{y+2} - \frac{3}{(y+2)^2} + c_1$$

Where c_1 is constant of integration. Substituting result found above for $f(y)$ into equation (3) gives ϕ

$$\phi = \frac{x}{(y+2)^2} + \frac{1}{y+2} - \frac{3}{(y+2)^2} + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = \frac{x}{(y+2)^2} + \frac{1}{y+2} - \frac{3}{(y+2)^2}$$

Summary

The solution(s) found are the following

$$\frac{x}{(y+2)^2} + \frac{1}{y+2} - \frac{3}{(y+2)^2} = c_1 \tag{1}$$

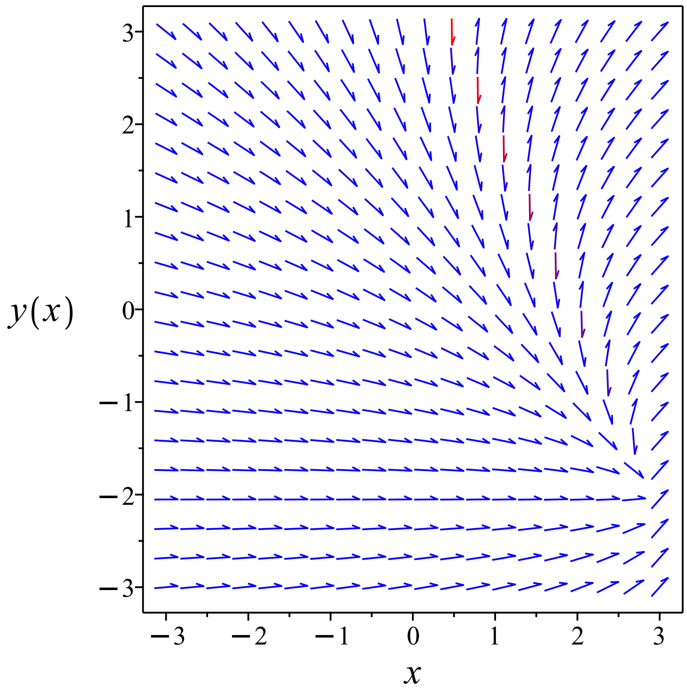


Figure 30: Slope field plot

Verification of solutions

$$\frac{x}{(y+2)^2} + \frac{1}{y+2} - \frac{3}{(y+2)^2} = c_1$$

Verified OK.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
<- 1st order linear successful
<- inverse linear successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 49

```
dsolve(y(x)+2=(2*x+y(x)-4)*diff(y(x),x),y(x), singsol=all)
```

$$y(x) = \frac{-4c_1 + 1 + \sqrt{1 + 4(x-3)\,c_1}}{2c_1}$$
$$y(x) = \frac{-4c_1 + 1 - \sqrt{1 + 4(x-3)\,c_1}}{2c_1}$$

✓ Solution by Mathematica

Time used: 0.28 (sec). Leaf size: 82

```
DSolve[y[x]+2==(2*x+y[x]-4)*y'[x],y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \rightarrow -\frac{\sqrt{1 + 4c_1(x-3)} - 1 + 4c_1}{2c_1}$$
$$y(x) \rightarrow \frac{\sqrt{1 + 4c_1(x-3)} + 1 - 4c_1}{2c_1}$$
$$y(x) \rightarrow -2$$
$$y(x) \rightarrow \text{Indeterminate}$$
$$y(x) \rightarrow 1 - x$$

1.18 problem 18

Internal problem ID [3163]
Internal file name [`OUTPUT/2655_Sunday_June_05_2022_08_38_10_AM_51036450/index.tex`]

Book: Differential equations for engineers by Wei-Chau XIE, Cambridge Press 2010
Section: Chapter 2. First-Order and Simple Higher-Order Differential Equations. Page 78
Problem number: 18.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"first_order_ode_lie_symmetry_calculated"**

Maple gives the following as the ode type

```
[[_homogeneous, `class C`], _dAlembert]
```

$$\boxed{y' - \sin(-y+x)^2 = 0}$$

1.18.1 Solving as first order ode lie symmetry calculated ode

Writing the ode as

$$\begin{aligned} y' &= \sin(-y+x)^2 \\ y' &= \omega(x,y) \end{aligned}$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x + \omega(\eta_y - \xi_x) - \omega^2 \xi_y - \omega_x \xi - \omega_y \eta = 0 \tag{A}$$

The type of this ode is not in the lookup table. To determine ξ, η then (A) is solved using ansatz. Making bivariate polynomials of degree 1 to use as anstaz gives

$$\xi = x a_2 + y a_3 + a_1 \tag{1E}$$

$$\eta = x b_2 + y b_3 + b_1 \tag{2E}$$

Where the unknown coefficients are

$$\{a_1, a_2, a_3, b_1, b_2, b_3\}$$

Substituting equations (1E,2E) and ω into (A) gives

$$\begin{aligned}&b_2 + \sin(-y+x)^2 (b_3 - a_2) - \sin(-y+x)^4 a_3 \\ &- 2\sin(-y+x)\cos(-y+x)(x a_2 + y a_3 + a_1) \\ &+ 2\sin(-y+x)\cos(-y+x)(x b_2 + y b_3 + b_1) = 0\end{aligned} \tag{5E}$$

Putting the above in normal form gives

$$\begin{aligned}&-\sin(-y+x)^4 a_3 - 2\sin(-y+x)\cos(-y+x) x a_2 \\ &+ 2\sin(-y+x)\cos(-y+x) x b_2 - 2\sin(-y+x)\cos(-y+x) y a_3 \\ &+ 2\sin(-y+x)\cos(-y+x) y b_3 - \sin(-y+x)^2 a_2 + \sin(-y+x)^2 b_3 \\ &- 2\sin(-y+x)\cos(-y+x) a_1 + 2\sin(-y+x)\cos(-y+x) b_1 + b_2 = 0\end{aligned}$$

Setting the numerator to zero gives

$$\begin{aligned}&-\sin(-y+x)^4 a_3 - 2\sin(-y+x)\cos(-y+x) x a_2 \\ &+ 2\sin(-y+x)\cos(-y+x) x b_2 - 2\sin(-y+x)\cos(-y+x) y a_3 \\ &+ 2\sin(-y+x)\cos(-y+x) y b_3 - \sin(-y+x)^2 a_2 + \sin(-y+x)^2 b_3 \\ &- 2\sin(-y+x)\cos(-y+x) a_1 + 2\sin(-y+x)\cos(-y+x) b_1 + b_2 = 0\end{aligned} \tag{6E}$$

Simplifying the above gives

$$\begin{aligned}&b_2 - \frac{3a_3}{8} - \frac{a_2}{2} + \frac{b_3}{2} + \frac{a_3 \cos(-2y+2x)}{2} - \frac{a_3 \cos(-4y+4x)}{8} \\ &- x a_2 \sin(-2y+2x) + x b_2 \sin(-2y+2x) - y a_3 \sin(-2y+2x) \\ &+ y b_3 \sin(-2y+2x) + \frac{a_2 \cos(-2y+2x)}{2} - \frac{b_3 \cos(-2y+2x)}{2} \\ &- a_1 \sin(-2y+2x) + b_1 \sin(-2y+2x) = 0\end{aligned} \tag{6E}$$

Looking at the above PDE shows the following are all the terms with $\{x, y\}$ in them.

$$\{x, y, \cos(-4y+4x), \cos(-2y+2x), \sin(-2y+2x)\}$$

The following substitution is now made to be able to collect on all terms with $\{x, y\}$ in them

$$\{x = v_1, y = v_2, \cos(-4y+4x) = v_3, \cos(-2y+2x) = v_4, \sin(-2y+2x) = v_5\}$$

The above PDE (6E) now becomes

$$
\begin{aligned}
& b_2 - \frac{3}{8}a_3 - \frac{1}{2}a_2 + \frac{1}{2}b_3 + \frac{1}{2}a_3v_4 - \frac{1}{8}a_3v_3 - v_1a_2v_5 + v_1b_2v_5 \\
& - v_2a_3v_5 + v_2b_3v_5 + \frac{1}{2}a_2v_4 - \frac{1}{2}b_3v_4 - a_1v_5 + b_1v_5 = 0
\end{aligned} \tag{7E}
$$

Collecting the above on the terms v_i introduced, and these are

$$
\{v_1, v_2, v_3, v_4, v_5\}
$$

Equation (7E) now becomes

$$
\begin{aligned}
& b_2 - \frac{3a_3}{8} - \frac{a_2}{2} + \frac{b_3}{2} + (-a_2 + b_2)\, v_5v_1 + (-a_3 + b_3)\, v_5v_2 \\
& - \frac{a_3v_3}{8} + \left(\frac{a_3}{2} + \frac{a_2}{2} - \frac{b_3}{2}\right) v_4 + (-a_1 + b_1)\, v_5 = 0
\end{aligned} \tag{8E}
$$

Setting each coefficients in (8E) to zero gives the following equations to solve

$$
\begin{aligned}
-\frac{a_3}{8} &= 0 \\
-a_1 + b_1 &= 0 \\
-a_2 + b_2 &= 0 \\
-a_3 + b_3 &= 0 \\
\frac{a_3}{2} + \frac{a_2}{2} - \frac{b_3}{2} &= 0 \\
b_2 - \frac{3a_3}{8} - \frac{a_2}{2} + \frac{b_3}{2} &= 0
\end{aligned}
$$

Solving the above equations for the unknowns gives

$$
\begin{aligned}
a_1 &= b_1 \\
a_2 &= 0 \\
a_3 &= 0 \\
b_1 &= b_1 \\
b_2 &= 0 \\
b_3 &= 0
\end{aligned}
$$

Substituting the above solution in the anstaz (1E,2E) (using 1 as arbitrary value for any unknown in the RHS) gives

$$\xi = 1$$
$$\eta = 1$$

Shifting is now applied to make $\xi = 0$ in order to simplify the rest of the computation

$$\begin{aligned}
\eta &= \eta - \omega(x, y)\,\xi \\
&= 1 - \left(\sin\left(-y + x\right)^2\right)(1) \\
&= 1 - \cos\left(x\right)^2 \sin\left(y\right)^2 + 2\cos\left(x\right)\sin\left(y\right)\sin\left(x\right)\cos\left(y\right) - \sin\left(x\right)^2 \cos\left(y\right)^2 \\
\xi &= 0
\end{aligned}$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x, y) \to (R, S)$ where (R, S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y}\right) S(x, y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$R = x$$

S is found from

$$\begin{aligned}
S &= \int \frac{1}{\eta} dy \\
&= \int \frac{1}{1 - \cos\left(x\right)^2 \sin\left(y\right)^2 + 2\cos\left(x\right)\sin\left(y\right)\sin\left(x\right)\cos\left(y\right) - \sin\left(x\right)^2 \cos\left(y\right)^2} dy
\end{aligned}$$

Which results in

$$S = -\tan\left(-y + x\right)$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x, y) S_y}{R_x + \omega(x, y) R_y} \tag{2}$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x, y)$ is the right hand side of the original ode given by

$$\omega(x, y) = \sin(-y + x)^2$$

Evaluating all the partial derivatives gives

$$\begin{aligned}
R_x &= 1 \\
R_y &= 0 \\
S_x &= -\sec(-y + x)^2 \\
S_y &= \sec(-y + x)^2
\end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = -1 \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = -1$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$S(R) = -R + c_1 \tag{4}$$

To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$-\tan(-y + x) = -x + c_1$$

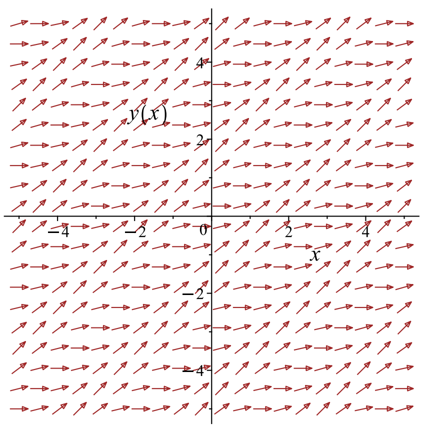
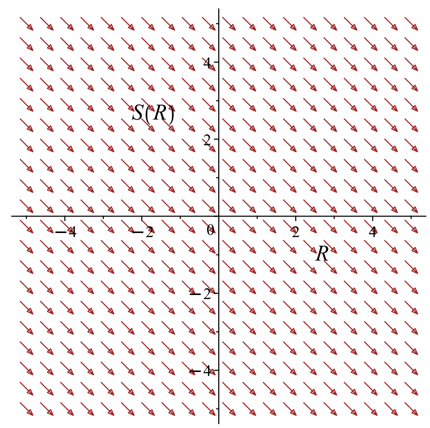
Which simplifies to

$$-\tan(-y + x) = -x + c_1$$

Which gives

$$y = x + \arctan(-x + c_1)$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in x, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dx} = \sin(-y+x)^2$ 	$R = x$ $S = -\tan(-y+x)$	$\frac{dS}{dR} = -1$ 

Summary

The solution(s) found are the following

$$y = x + \arctan(-x + c_1) \tag{1}$$

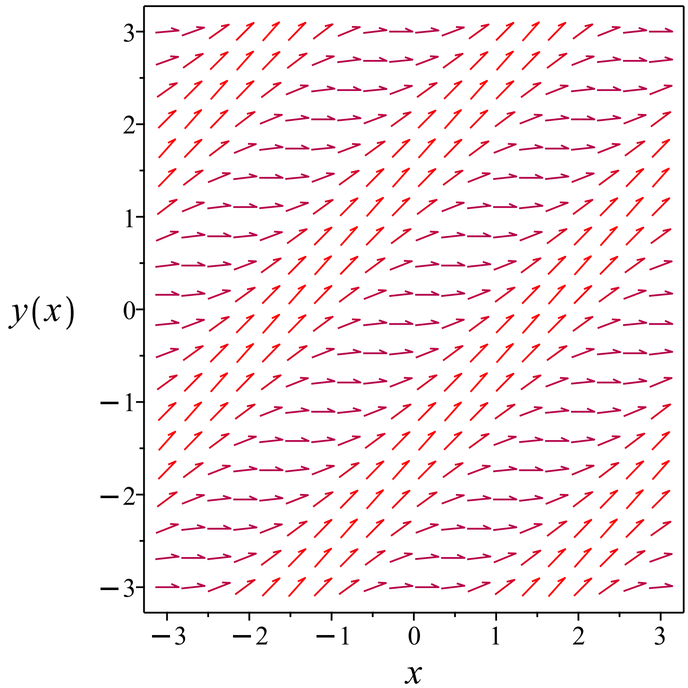


Figure 31: Slope field plot

Verification of solutions

$$y = x + \arctan(-x + c_1)$$

Verified OK.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
trying homogeneous types:
trying homogeneous C
1st order, trying the canonical coordinates of the invariance group
<- 1st order, canonical coordinates successful
<- homogeneous successful`
```

✓ Solution by Maple

Time used: 0.046 (sec). Leaf size: 12

```
dsolve(diff(y(x),x)=sin(x-y(x))^2,y(x), singsol=all)
```

$$y(x) = x + \arctan(c_1 - x)$$

✓ Solution by Mathematica

Time used: 0.199 (sec). Leaf size: 31

```
DSolve[y'[x]==Sin[x-y[x]]^2,y[x],x,IncludeSingularSolutions -> True]
```

$$\text{Solve}[2y(x) - 2(\tan(x - y(x)) - \arctan(\tan(x - y(x)))) = c_1, y(x)]$$

1.19 problem 19

Internal problem ID [3164]
Internal file name [`OUTPUT/2656_Sunday_June_05_2022_08_38_13_AM_76858072/index.tex`]

Book: Differential equations for engineers by Wei-Chau XIE, Cambridge Press 2010
Section: Chapter 2. First-Order and Simple Higher-Order Differential Equations. Page 78
Problem number: 19.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"riccati", "first_order_ode_lie_symmetry_calculated"**

Maple gives the following as the ode type

```
[[_homogeneous, `class C`], _Riccati]
```

$$\boxed{y' - (4y+1)^2 - 8yx = (x+1)^2 + 1}$$

1.19.1 Solving as first order ode lie symmetry calculated ode

Writing the ode as

$$y' = x^2 + 8xy + 16y^2 + 2x + 8y + 3$$
$$y' = \omega(x,y)$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x + \omega(\eta_y - \xi_x) - \omega^2\xi_y - \omega_x\xi - \omega_y\eta = 0 \tag{A}$$

The type of this ode is not in the lookup table. To determine ξ, η then (A) is solved using ansatz. Making bivariate polynomials of degree 1 to use as anstaz gives

$$\xi = xa_2 + ya_3 + a_1 \tag{1E}$$
$$\eta = xb_2 + yb_3 + b_1 \tag{2E}$$

Where the unknown coefficients are

$$\{a_1, a_2, a_3, b_1, b_2, b_3\}$$

Substituting equations (1E,2E) and ω into (A) gives

$$b_2 + \left(x^2 + 8xy + 16y^2 + 2x + 8y + 3\right)(b_3 - a_2) - \left(x^2 + 8xy + 16y^2 + 2x + 8y + 3\right)^2 a_3 - (2x + 8y + 2)(xa_2 + ya_3 + a_1) - (8 + 32y + 8x)(xb_2 + yb_3 + b_1) = 0 \tag{5E}$$

Putting the above in normal form gives

$$-x^4a_3 - 16x^3ya_3 - 96x^2y^2a_3 - 256x\,y^3a_3 - 256y^4a_3 - 4x^3a_3 - 48x^2ya_3 - 192x\,y^2a_3 - 256y^3a_3 - 3x^2a_2 - 10x^2a_3 - 8x^2b_2 + x^2b_3 - 16xya_2 - 82xya_3 - 32xyb_2 - 16y^2a_2 - 168y^2a_3 - 16y^2b_3 - 2xa_1 - 4xa_2 - 12xa_3 - 8xb_1 - 8xb_2 + 2xb_3 - 8ya_1 - 8ya_2 - 50ya_3 - 32yb_1 - 2a_1 - 3a_2 - 9a_3 - 8b_1 + b_2 + 3b_3 = 0$$

Setting the numerator to zero gives

$$-x^4a_3 - 16x^3ya_3 - 96x^2y^2a_3 - 256x\,y^3a_3 - 256y^4a_3 - 4x^3a_3 - 48x^2ya_3 - 192x\,y^2a_3 - 256y^3a_3 - 3x^2a_2 - 10x^2a_3 - 8x^2b_2 + x^2b_3 - 16xya_2 - 82xya_3 - 32xyb_2 - 16y^2a_2 - 168y^2a_3 - 16y^2b_3 - 2xa_1 - 4xa_2 - 12xa_3 - 8xb_1 - 8xb_2 + 2xb_3 - 8ya_1 - 8ya_2 - 50ya_3 - 32yb_1 - 2a_1 - 3a_2 - 9a_3 - 8b_1 + b_2 + 3b_3 = 0 \tag{6E}$$

Looking at the above PDE shows the following are all the terms with $\{x, y\}$ in them.

$$\{x, y\}$$

The following substitution is now made to be able to collect on all terms with $\{x, y\}$ in them

$$\{x = v_1, y = v_2\}$$

The above PDE (6E) now becomes

$$-a_3v_1^4 - 16a_3v_1^3v_2 - 96a_3v_1^2v_2^2 - 256a_3v_1v_2^3 - 256a_3v_2^4 - 4a_3v_1^3 - 48a_3v_1^2v_2 - 192a_3v_1v_2^2 - 256a_3v_2^3 - 3a_2v_1^2 - 16a_2v_1v_2 - 16a_2v_2^2 - 10a_3v_1^2 - 82a_3v_1v_2 - 168a_3v_2^2 - 8b_2v_1^2 - 32b_2v_1v_2 + b_3v_1^2 - 16b_3v_2^2 - 2a_1v_1 - 8a_1v_2 - 4a_2v_1 - 8a_2v_2 - 12a_3v_1 - 50a_3v_2 - 8b_1v_1 - 32b_1v_2 - 8b_2v_1 + 2b_3v_1 - 2a_1 - 3a_2 - 9a_3 - 8b_1 + b_2 + 3b_3 = 0 \tag{7E}$$

Collecting the above on the terms v_i introduced, and these are

$$\{v_1, v_2\}$$

Equation (7E) now becomes

$$
\begin{aligned}
&-a_3v_1^4 - 16a_3v_1^3v_2 - 4a_3v_1^3 - 96a_3v_1^2v_2^2 - 48a_3v_1^2v_2 \\
&+ (-3a_2 - 10a_3 - 8b_2 + b_3)\, v_1^2 - 256a_3v_1v_2^3 - 192a_3v_1v_2^2 \\
&+ (-16a_2 - 82a_3 - 32b_2)\, v_1v_2 + (-2a_1 - 4a_2 - 12a_3 - 8b_1 - 8b_2 + 2b_3)\, v_1 \\
&- 256a_3v_2^4 - 256a_3v_2^3 + (-16a_2 - 168a_3 - 16b_3)\, v_2^2 \\
&+ (-8a_1 - 8a_2 - 50a_3 - 32b_1)\, v_2 - 2a_1 - 3a_2 - 9a_3 - 8b_1 + b_2 + 3b_3 = 0
\end{aligned} \tag{8E}
$$

Setting each coefficients in (8E) to zero gives the following equations to solve

$$
\begin{aligned}
-256a_3 &= 0 \\
-192a_3 &= 0 \\
-96a_3 &= 0 \\
-48a_3 &= 0 \\
-16a_3 &= 0 \\
-4a_3 &= 0 \\
-a_3 &= 0 \\
-16a_2 - 168a_3 - 16b_3 &= 0 \\
-16a_2 - 82a_3 - 32b_2 &= 0 \\
-8a_1 - 8a_2 - 50a_3 - 32b_1 &= 0 \\
-3a_2 - 10a_3 - 8b_2 + b_3 &= 0 \\
-2a_1 - 4a_2 - 12a_3 - 8b_1 - 8b_2 + 2b_3 &= 0 \\
-2a_1 - 3a_2 - 9a_3 - 8b_1 + b_2 + 3b_3 &= 0
\end{aligned}
$$

Solving the above equations for the unknowns gives

$$
\begin{aligned}
a_1 &= -4b_1 \\
a_2 &= 0 \\
a_3 &= 0 \\
b_1 &= b_1 \\
b_2 &= 0 \\
b_3 &= 0
\end{aligned}
$$

Substituting the above solution in the anstaz (1E,2E) (using 1 as arbitrary value for any unknown in the RHS) gives

$$
\begin{aligned}
\xi &= -4 \\
\eta &= 1
\end{aligned}
$$

Shifting is now applied to make $\xi = 0$ in order to simplify the rest of the computation

$$\begin{aligned}\eta &= \eta - \omega(x,y)\,\xi \\ &= 1 - \left(x^2 + 8xy + 16y^2 + 2x + 8y + 3\right)(-4) \\ &= 4x^2 + 32xy + 64y^2 + 8x + 32y + 13 \\ \xi &= 0\end{aligned}$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x, y) \to (R, S)$ where (R, S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y}\right) S(x, y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$R = x$$

S is found from

$$\begin{aligned}S &= \int \frac{1}{\eta} dy \\ &= \int \frac{1}{4x^2 + 32xy + 64y^2 + 8x + 32y + 13} dy\end{aligned}$$

Which results in

$$S = \frac{\arctan\left(\frac{8y}{3} + \frac{2x}{3} + \frac{2}{3}\right)}{24}$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x, y) S_y}{R_x + \omega(x, y) R_y} \tag{2}$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x, y)$ is the right hand side of the original ode given by

$$\omega(x, y) = x^2 + 8xy + 16y^2 + 2x + 8y + 3$$

Evaluating all the partial derivatives gives

$$
\begin{aligned}
R_x &= 1 \\
R_y &= 0 \\
S_x &= \frac{1}{36\left(\frac{8y}{3}+\frac{2x}{3}+\frac{2}{3}\right)^2+36} \\
S_y &= \frac{1}{4x^2+(8+32y)\,x+64y^2+32y+13}
\end{aligned}
$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$
\frac{dS}{dR} = \frac{1}{4} \tag{2A}
$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$
\frac{dS}{dR} = \frac{1}{4}
$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$
S(R) = \frac{R}{4} + c_1 \tag{4}
$$

To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$
\frac{\arctan\left(\frac{8y}{3}+\frac{2x}{3}+\frac{2}{3}\right)}{24} = \frac{x}{4} + c_1
$$

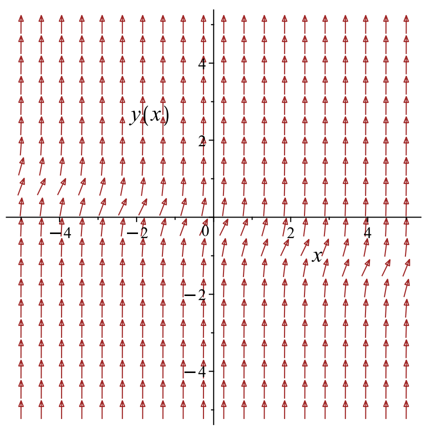
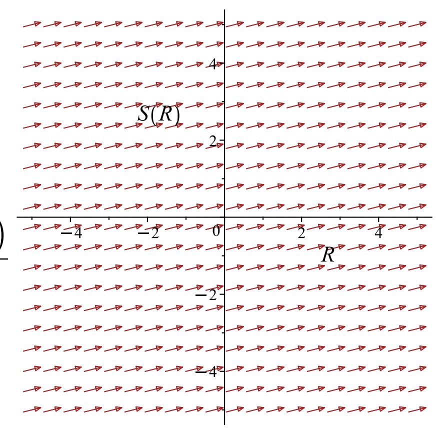
Which simplifies to

$$
\frac{\arctan\left(\frac{8y}{3}+\frac{2x}{3}+\frac{2}{3}\right)}{24} = \frac{x}{4} + c_1
$$

Which gives

$$
y = -\frac{x}{4} - \frac{1}{4} + \frac{3\tan\left(6x+24c_1\right)}{8}
$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in x, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dx} = x^2 + 8xy + 16y^2 + 2x + 8y + 3$ 	$R = x$ $S = \frac{\arctan\left(\frac{8y}{3} + \frac{2x}{3} + \frac{2}{3}\right)}{24}$	$\frac{dS}{dR} = \frac{1}{4}$ 

Summary

The solution(s) found are the following

$$y = -\frac{x}{4} - \frac{1}{4} + \frac{3\tan(6x + 24c_1)}{8} \tag{1}$$

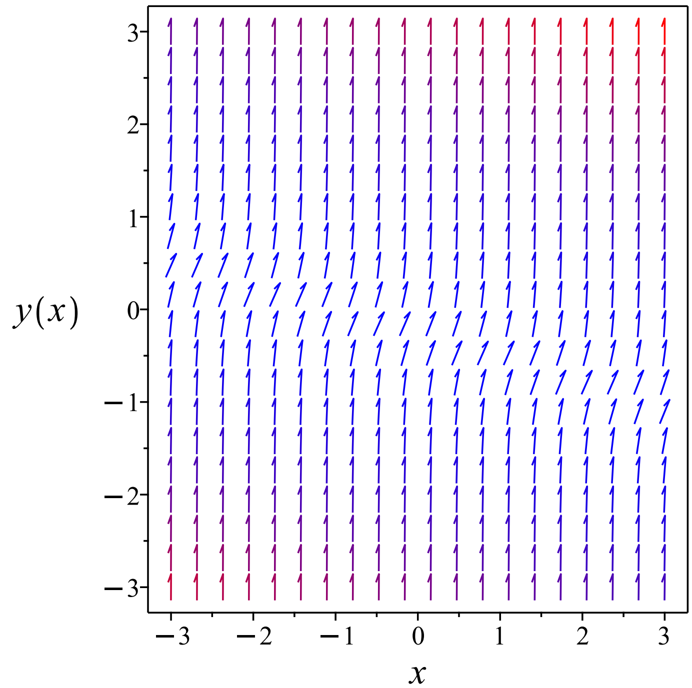


Figure 32: Slope field plot

Verification of solutions

$$y = -\frac{x}{4} - \frac{1}{4} + \frac{3\tan(6x + 24c_1)}{8}$$

Verified OK.

1.19.2 Solving as riccati ode

In canonical form the ODE is

$$\begin{aligned} y' &= F(x, y) \\ &= x^2 + 8xy + 16y^2 + 2x + 8y + 3 \end{aligned}$$

This is a Riccati ODE. Comparing the ODE to solve

$$y' = x^2 + 8xy + 16y^2 + 2x + 8y + 3$$

With Riccati ODE standard form

$$y' = f_0(x) + f_1(x)y + f_2(x)y^2$$

Shows that $f_0(x) = x^2 + 2x + 3,\ f_1(x) = 8x + 8$ and $f_2(x) = 16$. Let

$$\begin{aligned} y &= \frac{-u'}{f_2 u} \\ &= \frac{-u'}{16u} \end{aligned} \tag{1}$$

Using the above substitution in the given ODE results (after some simplification)in a second order ODE to solve for $u(x)$ which is

$$f_2 u''(x) - (f_2' + f_1 f_2)\, u'(x) + f_2^2 f_0 u(x) = 0 \tag{2}$$

But

$$\begin{aligned} f_2' &= 0 \\ f_1 f_2 &= 128x + 128 \\ f_2^2 f_0 &= 256x^2 + 512x + 768 \end{aligned}$$

Substituting the above terms back in equation (2) gives

$$16u''(x) - (128x + 128)\, u'(x) + \left(256x^2 + 512x + 768\right) u(x) = 0$$

Solving the above ODE (this ode solved using Maple, not this program), gives

$$u(x) = \mathrm{e}^{2x(x+2)} (c_1 \cos(6x) + c_2 \sin(6x))$$

The above shows that

$$u'(x) = 4\,\mathrm{e}^{2x(x+2)} \left(\left(c_1(x+1) + \frac{3c_2}{2} \right) \cos(6x) + \sin(6x) \left(-\frac{3c_1}{2} + (x+1)\, c_2 \right) \right)$$

Using the above in (1) gives the solution

$$y = -\frac{\left(c_1(x+1) + \frac{3c_2}{2}\right) \cos(6x) + \sin(6x) \left(-\frac{3c_1}{2} + (x+1)\, c_2\right)}{4\,(c_1 \cos(6x) + c_2 \sin(6x))}$$

Dividing both numerator and denominator by c_1 gives, after renaming the constant $\frac{c_2}{c_1} = c_3$ the following solution

$$y = \frac{(-3 + (-2 - 2x)\, c_3) \cos(6x) - 2 \sin(6x) \left(-\frac{3c_3}{2} + x + 1\right)}{8c_3 \cos(6x) + 8 \sin(6x)}$$

Summary
The solution(s) found are the following

$$y = \frac{(-3 + (-2 - 2x) c_3) \cos(6x) - 2 \sin(6x) \left(-\frac{3c_3}{2} + x + 1\right)}{8c_3 \cos(6x) + 8 \sin(6x)} \tag{1}$$

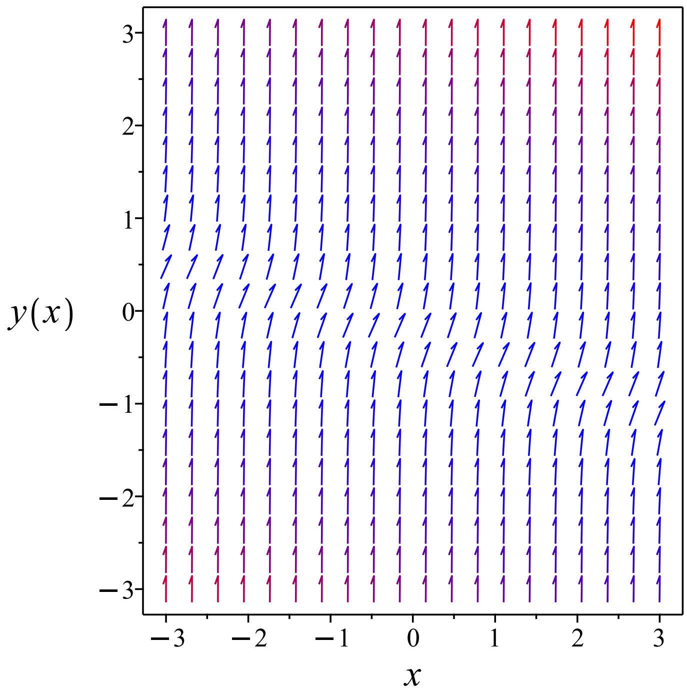


Figure 33: Slope field plot

Verification of solutions

$$y = \frac{(-3 + (-2 - 2x) c_3) \cos(6x) - 2 \sin(6x) \left(-\frac{3c_3}{2} + x + 1\right)}{8c_3 \cos(6x) + 8 \sin(6x)}$$

Verified OK.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
trying homogeneous types:
trying homogeneous C
1st order, trying the canonical coordinates of the invariance group
   -> Calling odsolve with the ODE`, diff(y(x), x) = -1/4, y(x)`      *** Sublevel 2 ***
      Methods for first order ODEs:
      --- Trying classification methods ---
      trying a quadrature
      trying 1st order linear
      <- 1st order linear successful
<- 1st order, canonical coordinates successful
<- homogeneous successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 19

```
dsolve(diff(y(x),x)=(x+1)^2+(4*y(x)+1)^2+8*x*y(x)+1,y(x), singsol=all)
```

$$y(x) = -\frac{x}{4} - \frac{1}{4} - \frac{3\tan(-6x + 6c_1)}{8}$$

✓ Solution by Mathematica

Time used: 0.18 (sec). Leaf size: 49

```
DSolve[y'[x]==(x+1)^2+(4*y[x]+1)^2+8*x*y[x]+1,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \frac{1}{16}\left(-4x + \frac{1}{c_1 e^{12ix} - \frac{i}{12}} - (4+6i)\right)$$
$$y(x) \to \frac{1}{8}(-2x - (2+3i))$$

1.20 problem 20

Internal problem ID [3165]
Internal file name [`OUTPUT/2657_Sunday_June_05_2022_08_38_14_AM_94601019/index.tex`]

Book: Differential equations for engineers by Wei-Chau XIE, Cambridge Press 2010
Section: Chapter 2. First-Order and Simple Higher-Order Differential Equations. Page 78
Problem number: 20.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"exact"**

Maple gives the following as the ode type

```
[_exact, _rational]
```

$$\boxed{6y^2x + (6x^2y + 4y^3)\, y' = -3x^2}$$

1.20.1 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x,y) + N(x,y)\frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x,y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y) = 0$$

Hence

$$\frac{\partial \phi}{\partial x} + \frac{\partial \phi}{\partial y}\frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$\begin{aligned}\frac{\partial \phi}{\partial x} &= M\\ \frac{\partial \phi}{\partial y} &= N\end{aligned}$$

But since $\frac{\partial^2\phi}{\partial x\partial y} = \frac{\partial^2\phi}{\partial y\partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x, y)$ but at least we know now that we can do that since the condition $\frac{\partial^2\phi}{\partial x\partial y} = \frac{\partial^2\phi}{\partial y\partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x, y)\,\mathrm{d}x + N(x, y)\,\mathrm{d}y = 0 \tag{1A}$$

Therefore

$$\begin{aligned}(6x^2y + 4y^3)\,\mathrm{d}y &= (-6x\,y^2 - 3x^2)\,\mathrm{d}x \\ (6x\,y^2 + 3x^2)\,\mathrm{d}x + (6x^2y + 4y^3)\,\mathrm{d}y &= 0 \end{aligned} \tag{2A}$$

Comparing (1A) and (2A) shows that

$$\begin{aligned} M(x, y) &= 6x\,y^2 + 3x^2 \\ N(x, y) &= 6x^2y + 4y^3 \end{aligned}$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Using result found above gives

$$\begin{aligned} \frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}(6x\,y^2 + 3x^2) \\ &= 12xy \end{aligned}$$

And

$$\begin{aligned} \frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}(6x^2y + 4y^3) \\ &= 12xy \end{aligned}$$

Since $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$, then the ODE is <u>exact</u> The following equations are now set up to solve for the function $\phi(x, y)$

$$\frac{\partial\phi}{\partial x} = M \tag{1}$$

$$\frac{\partial\phi}{\partial y} = N \tag{2}$$

Integrating (1) w.r.t. x gives

$$\int \frac{\partial \phi}{\partial x}\, \mathrm{d}x = \int M\, \mathrm{d}x$$
$$\int \frac{\partial \phi}{\partial x}\, \mathrm{d}x = \int 6x\, y^2 + 3x^2\, \mathrm{d}x$$
$$\phi = x^2(3y^2 + x) + f(y) \tag{3}$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$\frac{\partial \phi}{\partial y} = 6x^2y + f'(y) \tag{4}$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = 6x^2y + 4y^3$. Therefore equation (4) becomes

$$6x^2y + 4y^3 = 6x^2y + f'(y) \tag{5}$$

Solving equation (5) for $f'(y)$ gives

$$f'(y) = 4y^3$$

Integrating the above w.r.t y gives

$$\int f'(y)\, \mathrm{d}y = \int (4y^3)\, \mathrm{d}y$$
$$f(y) = y^4 + c_1$$

Where c_1 is constant of integration. Substituting result found above for $f(y)$ into equation (3) gives ϕ

$$\phi = x^2(3y^2 + x) + y^4 + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = x^2(3y^2 + x) + y^4$$

Summary

The solution(s) found are the following

$$x^2(3y^2 + x) + y^4 = c_1 \tag{1}$$

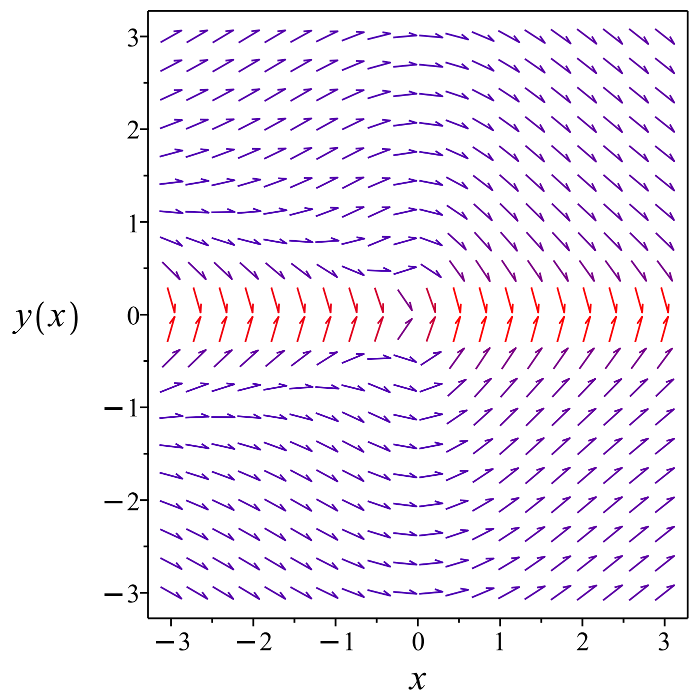


Figure 34: Slope field plot

Verification of solutions

$$x^2(3y^2 + x) + y^4 = c_1$$

Verified OK.

1.20.2 Maple step by step solution

Let's solve

$6y^2x + (6x^2y + 4y^3)\,y' = -3x^2$

- Highest derivative means the order of the ODE is 1

  y'

□ Check if ODE is exact

  ◦ ODE is exact if the lhs is the total derivative of a C^2 function

$F'(x,y)=0$

- Compute derivative of lhs

  $F'(x,y)+\left(\frac{\partial}{\partial y}F(x,y)\right)y'=0$

- Evaluate derivatives

  $12xy=12xy$

- Condition met, ODE is exact

- Exact ODE implies solution will be of this form

  $\left[F(x,y)=c_1, M(x,y)=F'(x,y), N(x,y)=\frac{\partial}{\partial y}F(x,y)\right]$

- Solve for $F(x,y)$ by integrating $M(x,y)$ with respect to x

  $F(x,y)=\int(6x\,y^2+3x^2)\,dx+f_1(y)$

- Evaluate integral

  $F(x,y)=3y^2x^2+x^3+f_1(y)$

- Take derivative of $F(x,y)$ with respect to y

  $N(x,y)=\frac{\partial}{\partial y}F(x,y)$

- Compute derivative

  $6x^2y+4y^3=6x^2y+\frac{d}{dy}f_1(y)$

- Isolate for $\frac{d}{dy}f_1(y)$

  $\frac{d}{dy}f_1(y)=4y^3$

- Solve for $f_1(y)$

  $f_1(y)=y^4$

- Substitute $f_1(y)$ into equation for $F(x,y)$

  $F(x,y)=3y^2x^2+y^4+x^3$

- Substitute $F(x,y)$ into the solution of the ODE

  $3y^2x^2+y^4+x^3=c_1$

- Solve for y

  $\left\{y=-\frac{\sqrt{-6x^2-2\sqrt{9x^4-4x^3+4c_1}}}{2}, y=\frac{\sqrt{-6x^2-2\sqrt{9x^4-4x^3+4c_1}}}{2}, y=-\frac{\sqrt{-6x^2+2\sqrt{9x^4-4x^3+4c_1}}}{2}, y=\frac{\sqrt{-6x^2+2\sqrt{}}}{}\right.$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
trying homogeneous types:
trying Chini
differential order: 1; looking for linear symmetries
trying exact
<- exact successful`
```

✓ Solution by Maple

Time used: 0.031 (sec). Leaf size: 125

```
dsolve((3*x^2+6*x*y(x)^2)+(6*x^2*y(x)+4*y(x)^3)*diff(y(x),x)=0,y(x), singsol=all)
```

$$y(x) = -\frac{\sqrt{-6x^2 - 2\sqrt{9x^4 - 4x^3 - 4c_1}}}{2}$$
$$y(x) = \frac{\sqrt{-6x^2 - 2\sqrt{9x^4 - 4x^3 - 4c_1}}}{2}$$
$$y(x) = -\frac{\sqrt{-6x^2 + 2\sqrt{9x^4 - 4x^3 - 4c_1}}}{2}$$
$$y(x) = \frac{\sqrt{-6x^2 + 2\sqrt{9x^4 - 4x^3 - 4c_1}}}{2}$$

✓ Solution by Mathematica

Time used: 6.017 (sec). Leaf size: 163

```
DSolve[(3*x^2+6*x*y[x]^2)+(6*x^2*y[x]+4*y[x]^3)*y'[x]==0,y[x],x,IncludeSingularSolutions ->
```

$$y(x) \to -\frac{\sqrt{-3x^2 - \sqrt{9x^4 - 4x^3 + 4c_1}}}{\sqrt{2}}$$
$$y(x) \to \frac{\sqrt{-3x^2 - \sqrt{9x^4 - 4x^3 + 4c_1}}}{\sqrt{2}}$$
$$y(x) \to -\frac{\sqrt{-3x^2 + \sqrt{9x^4 - 4x^3 + 4c_1}}}{\sqrt{2}}$$
$$y(x) \to \frac{\sqrt{-3x^2 + \sqrt{9x^4 - 4x^3 + 4c_1}}}{\sqrt{2}}$$

1.21 problem 21

Internal problem ID [3166]
Internal file name [`OUTPUT/2658_Sunday_June_05_2022_08_38_14_AM_48662186/index.tex`]

Book: Differential equations for engineers by Wei-Chau XIE, Cambridge Press 2010
Section: Chapter 2. First-Order and Simple Higher-Order Differential Equations. Page 78
Problem number: 21.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"exact"**

Maple gives the following as the ode type

```
[_exact, _rational, [_Abel, `2nd type`, `class B`]]
```

$$\boxed{-y^{2}x-2y-\left(x^{2}y+2x\right)y'=-2x^{2}-3}$$

1.21.1 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x,y)+N(x,y)\frac{dy}{dx}=0 \tag{A}$$

We assume there exists a function $\phi(x,y)=c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y)=0$$

Hence

$$\frac{\partial\phi}{\partial x}+\frac{\partial\phi}{\partial y}\frac{dy}{dx}=0 \tag{B}$$

Comparing (A,B) shows that

$$\frac{\partial\phi}{\partial x}=M$$
$$\frac{\partial\phi}{\partial y}=N$$

But since $\frac{\partial^2\phi}{\partial x\partial y}=\frac{\partial^2\phi}{\partial y\partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y}=\frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x,y)$ but at least we know now that we can do that since the condition $\frac{\partial^2\phi}{\partial x\partial y}=\frac{\partial^2\phi}{\partial y\partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x,y)\,\mathrm{d}x+N(x,y)\,\mathrm{d}y=0 \tag{1A}$$

Therefore

$$\begin{aligned}\left(-x^2y-2x\right)\mathrm{d}y&=\left(x\,y^2-2x^2+2y-3\right)\mathrm{d}x\\ \left(-x\,y^2+2x^2-2y+3\right)\mathrm{d}x+\left(-x^2y-2x\right)\mathrm{d}y&=0 \end{aligned}\tag{2A}$$

Comparing (1A) and (2A) shows that

$$\begin{aligned}M(x,y)&=-x\,y^2+2x^2-2y+3\\ N(x,y)&=-x^2y-2x\end{aligned}$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y}=\frac{\partial N}{\partial x}$$

Using result found above gives

$$\begin{aligned}\frac{\partial M}{\partial y}&=\frac{\partial}{\partial y}\left(-x\,y^2+2x^2-2y+3\right)\\ &=-2xy-2\end{aligned}$$

And

$$\begin{aligned}\frac{\partial N}{\partial x}&=\frac{\partial}{\partial x}\left(-x^2y-2x\right)\\ &=-2xy-2\end{aligned}$$

Since $\frac{\partial M}{\partial y}=\frac{\partial N}{\partial x}$, then the ODE is <u>exact</u> The following equations are now set up to solve for the function $\phi(x,y)$

$$\frac{\partial\phi}{\partial x}=M \tag{1}$$

$$\frac{\partial\phi}{\partial y}=N \tag{2}$$

Integrating (1) w.r.t. x gives

$$\int \frac{\partial \phi}{\partial x} \mathrm{d}x = \int M \,\mathrm{d}x$$

$$\int \frac{\partial \phi}{\partial x} \mathrm{d}x = \int -x\,y^2 + 2x^2 - 2y + 3 \,\mathrm{d}x$$

$$\phi = \frac{2x^3}{3} - \frac{y^2x^2}{2} + (-2y+3)\,x + f(y) \tag{3}$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$\frac{\partial \phi}{\partial y} = -x^2y - 2x + f'(y) \tag{4}$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = -x^2y - 2x$. Therefore equation (4) becomes

$$-x^2y - 2x = -x^2y - 2x + f'(y) \tag{5}$$

Solving equation (5) for $f'(y)$ gives

$$f'(y) = 0$$

Therefore

$$f(y) = c_1$$

Where c_1 is constant of integration. Substituting this result for $f(y)$ into equation (3) gives ϕ

$$\phi = \frac{2x^3}{3} - \frac{y^2x^2}{2} + (-2y+3)\,x + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = \frac{2x^3}{3} - \frac{y^2x^2}{2} + (-2y+3)\,x$$

Summary

The solution(s) found are the following

$$\frac{2x^3}{3} - \frac{y^2x^2}{2} + (-2y+3)\,x = c_1 \tag{1}$$

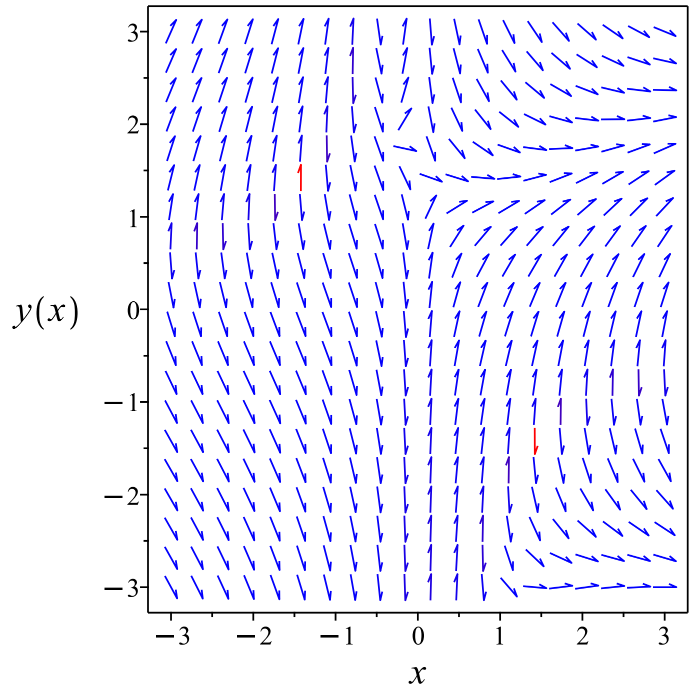


Figure 35: Slope field plot

Verification of solutions

$$\frac{2x^3}{3} - \frac{y^2x^2}{2} + (-2y + 3)\,x = c_1$$

Verified OK.

1.21.2 Maple step by step solution

Let's solve

$-y^2x - 2y - (x^2y + 2x)\,y' = -2x^2 - 3$

- Highest derivative means the order of the ODE is 1

  y'

- □ Check if ODE is exact

  - ◦ ODE is exact if the lhs is the total derivative of a C^2 function

    $F'(x, y) = 0$

  - ◦ Compute derivative of lhs

    $F'(x, y) + \left(\frac{\partial}{\partial y}F(x, y)\right) y' = 0$

- Evaluate derivatives

  $-2xy - 2 = -2xy - 2$

- Condition met, ODE is exact

- Exact ODE implies solution will be of this form

  $\left[F(x,y) = c_1, M(x,y) = F'(x,y), N(x,y) = \frac{\partial}{\partial y}F(x,y)\right]$

- Solve for $F(x,y)$ by integrating $M(x,y)$ with respect to x

  $F(x,y) = \int (-x\,y^2 + 2x^2 - 2y + 3)\,dx + f_1(y)$

- Evaluate integral

  $F(x,y) = -\frac{y^2x^2}{2} + \frac{2x^3}{3} - 2xy + 3x + f_1(y)$

- Take derivative of $F(x,y)$ with respect to y

  $N(x,y) = \frac{\partial}{\partial y}F(x,y)$

- Compute derivative

  $-x^2y - 2x = -x^2y - 2x + \frac{d}{dy}f_1(y)$

- Isolate for $\frac{d}{dy}f_1(y)$

  $\frac{d}{dy}f_1(y) = 0$

- Solve for $f_1(y)$

  $f_1(y) = 0$

- Substitute $f_1(y)$ into equation for $F(x,y)$

  $F(x,y) = -\frac{1}{2}y^2x^2 + \frac{2}{3}x^3 - 2xy + 3x$

- Substitute $F(x,y)$ into the solution of the ODE

  $-\frac{1}{2}y^2x^2 + \frac{2}{3}x^3 - 2xy + 3x = c_1$

- Solve for y

  $\left\{y = \frac{-2-\frac{\sqrt{12x^3-18c_1+54x+36}}{3}}{x}, y = \frac{-2+\frac{\sqrt{12x^3-18c_1+54x+36}}{3}}{x}\right\}$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
trying homogeneous types:
trying Chini
differential order: 1; looking for linear symmetries
trying exact
<- exact successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 53

```
dsolve((2*x^2-x*y(x)^2-2*y(x)+3)-(x^2*y(x)+2*x)*diff(y(x),x)=0,y(x), singsol=all)
```

$$y(x) = \frac{-6 - \sqrt{12x^3 + 18c_1 + 54x + 36}}{3x}$$

$$y(x) = \frac{-6 + \sqrt{12x^3 + 18c_1 + 54x + 36}}{3x}$$

✓ Solution by Mathematica

Time used: 0.646 (sec). Leaf size: 87

```
DSolve[(2*x^2-x*y[x]^2-2*y[x]+3)-(x^2*y[x]+2*x)*y'[x]==0,y[x],x,IncludeSingularSolutions ->
```

$$y(x) \to -\frac{6x + \sqrt{3}\sqrt{x^2 (4x^3 + 18x + 12 + 3c_1)}}{3x^2}$$

$$y(x) \to \frac{-6x + \sqrt{3}\sqrt{x^2 (4x^3 + 18x + 12 + 3c_1)}}{3x^2}$$

1.22 problem 22

Internal problem ID [3167]
Internal file name [`OUTPUT/2659_Sunday_June_05_2022_08_38_15_AM_76938598/index.tex`]

Book: Differential equations for engineers by Wei-Chau XIE, Cambridge Press 2010
Section: Chapter 2. First-Order and Simple Higher-Order Differential Equations. Page 78
Problem number: 22.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"exact"**

Maple gives the following as the ode type

```
[_exact, _rational, [_Abel, `2nd type`, `class B`]]
```

$$\boxed{y^2x - 2y + (x^2y - 2y - 2x)\, y' = -x - 3}$$

1.22.1 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x,y) + N(x,y)\frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x,y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y) = 0$$

Hence

$$\frac{\partial\phi}{\partial x} + \frac{\partial\phi}{\partial y}\frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$\frac{\partial\phi}{\partial x} = M$$
$$\frac{\partial\phi}{\partial y} = N$$

But since $\frac{\partial^2\phi}{\partial x\partial y} = \frac{\partial^2\phi}{\partial y\partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x, y)$ but at least we know now that we can do that since the condition $\frac{\partial^2\phi}{\partial x\partial y} = \frac{\partial^2\phi}{\partial y\partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x, y)\,\mathrm{d}x + N(x, y)\,\mathrm{d}y = 0 \tag{1A}$$

Therefore

$$\begin{aligned}
(x^2y - 2x - 2y)\,\mathrm{d}y &= (-x\,y^2 - x + 2y - 3)\,\mathrm{d}x \\
(x\,y^2 + x - 2y + 3)\,\mathrm{d}x + (x^2y - 2x - 2y)\,\mathrm{d}y &= 0 \qquad (2A)
\end{aligned}$$

Comparing (1A) and (2A) shows that

$$\begin{aligned}
M(x, y) &= x\,y^2 + x - 2y + 3 \\
N(x, y) &= x^2y - 2x - 2y
\end{aligned}$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Using result found above gives

$$\begin{aligned}
\frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}(x\,y^2 + x - 2y + 3) \\
&= 2xy - 2
\end{aligned}$$

And

$$\begin{aligned}
\frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}(x^2y - 2x - 2y) \\
&= 2xy - 2
\end{aligned}$$

Since $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$, then the ODE is exact The following equations are now set up to solve for the function $\phi(x, y)$

$$\frac{\partial \phi}{\partial x} = M \tag{1}$$

$$\frac{\partial \phi}{\partial y} = N \tag{2}$$

Integrating (1) w.r.t. x gives

$$\int \frac{\partial \phi}{\partial x}\,\mathrm{d}x = \int M\,\mathrm{d}x$$
$$\int \frac{\partial \phi}{\partial x}\,\mathrm{d}x = \int x\,y^2 + x - 2y + 3\,\mathrm{d}x$$
$$\phi = \frac{x(x\,y^2 + x - 4y + 6)}{2} + f(y) \tag{3}$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$\frac{\partial \phi}{\partial y} = \frac{x(2xy - 4)}{2} + f'(y) \tag{4}$$
$$= x(xy - 2) + f'(y)$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = x^2y - 2x - 2y$. Therefore equation (4) becomes

$$x^2y - 2x - 2y = x(xy - 2) + f'(y) \tag{5}$$

Solving equation (5) for $f'(y)$ gives

$$f'(y) = -2y$$

Integrating the above w.r.t y gives

$$\int f'(y)\,\mathrm{d}y = \int (-2y)\,\mathrm{d}y$$
$$f(y) = -y^2 + c_1$$

Where c_1 is constant of integration. Substituting result found above for $f(y)$ into equation (3) gives ϕ

$$\phi = \frac{x(x\,y^2 + x - 4y + 6)}{2} - y^2 + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = \frac{x(x\,y^2 + x - 4y + 6)}{2} - y^2$$

Summary

The solution(s) found are the following

$$\frac{x(y^2x + x - 4y + 6)}{2} - y^2 = c_1 \tag{1}$$

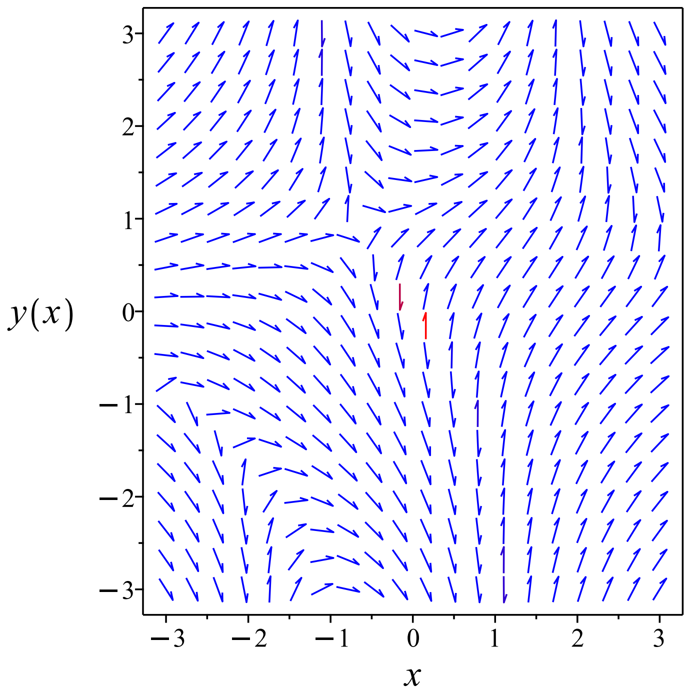


Figure 36: Slope field plot

Verification of solutions

$$\frac{x(y^2x + x - 4y + 6)}{2} - y^2 = c_1$$

Verified OK.

1.22.2 Maple step by step solution

Let's solve

$y^2x - 2y + (x^2y - 2y - 2x)\,y' = -x - 3$

- Highest derivative means the order of the ODE is 1

  y'

- □ Check if ODE is exact

- ODE is exact if the lhs is the total derivative of a C^2 function

  $F'(x,y) = 0$

- Compute derivative of lhs

  $F'(x,y) + \left(\frac{\partial}{\partial y}F(x,y)\right)y' = 0$

- Evaluate derivatives

  $2xy - 2 = 2xy - 2$

- Condition met, ODE is exact

- Exact ODE implies solution will be of this form

  $\left[F(x,y) = c_1, M(x,y) = F'(x,y), N(x,y) = \frac{\partial}{\partial y}F(x,y)\right]$

- Solve for $F(x,y)$ by integrating $M(x,y)$ with respect to x

  $F(x,y) = \int (x\,y^2 + x - 2y + 3)\,dx + f_1(y)$

- Evaluate integral

  $F(x,y) = \frac{y^2x^2}{2} + \frac{x^2}{2} - 2xy + 3x + f_1(y)$

- Take derivative of $F(x,y)$ with respect to y

  $N(x,y) = \frac{\partial}{\partial y}F(x,y)$

- Compute derivative

  $x^2y - 2x - 2y = x^2y - 2x + \frac{d}{dy}f_1(y)$

- Isolate for $\frac{d}{dy}f_1(y)$

  $\frac{d}{dy}f_1(y) = -2y$

- Solve for $f_1(y)$

  $f_1(y) = -y^2$

- Substitute $f_1(y)$ into equation for $F(x,y)$

  $F(x,y) = \frac{1}{2}y^2x^2 + \frac{1}{2}x^2 - 2xy + 3x - y^2$

- Substitute $F(x,y)$ into the solution of the ODE

  $\frac{1}{2}y^2x^2 + \frac{1}{2}x^2 - 2xy + 3x - y^2 = c_1$

- Solve for y

  $\left\{y = \frac{2x+\sqrt{-x^4+2c_1x^2-6x^3+6x^2-4c_1+12x}}{x^2-2}, y = -\frac{-2x+\sqrt{-x^4+2c_1x^2-6x^3+6x^2-4c_1+12x}}{x^2-2}\right\}$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
trying homogeneous types:
trying Chini
differential order: 1; looking for linear symmetries
trying exact
<- exact successful`
```

✓ Solution by Maple

Time used: 0.015 (sec). Leaf size: 89

```
dsolve((x*y(x)^2+x-2*y(x)+3)+(x^2*y(x)-2*(x+y(x)))*diff(y(x),x)=0,y(x), singsol=all)
```

$$y(x) = \frac{2x + \sqrt{-x^4 - 6x^3 + (-2c_1 + 6)\, x^2 + 12x + 4c_1}}{x^2 - 2}$$
$$y(x) = \frac{2x - \sqrt{-x^4 - 6x^3 + (-2c_1 + 6)\, x^2 + 12x + 4c_1}}{x^2 - 2}$$

✓ Solution by Mathematica

Time used: 0.549 (sec). Leaf size: 95

```
DSolve[(x*y[x]^2+x-2*y[x]+3)+(x^2*y[x]-2*(x+y[x]))*y'[x]==0,y[x],x,IncludeSingularSolutions
```

$$y(x) \to \frac{2x - \sqrt{-x^4 - 6x^3 + (6 + c_1)x^2 + 12x - 2c_1}}{x^2 - 2}$$
$$y(x) \to \frac{2x + \sqrt{-x^4 - 6x^3 + (6 + c_1)x^2 + 12x - 2c_1}}{x^2 - 2}$$

1.23 problem 23

Internal problem ID [3168]
Internal file name [`OUTPUT/2660_Sunday_June_05_2022_08_38_15_AM_29060004/index.tex`]

Book: Differential equations for engineers by Wei-Chau XIE, Cambridge Press 2010
Section: Chapter 2. First-Order and Simple Higher-Order Differential Equations. Page 78
Problem number: 23.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"exact", "differentialType"**

Maple gives the following as the ode type

```
[_exact, _rational, [_1st_order, `_with_symmetry_[F(x)*G(y)
   ,0]`], [_Abel, `2nd type`, `class A`]]
```

$$\boxed{3(x^2-1)\,y+(x^3+8y-3x)\,y'=0}$$

1.23.1 Solving as differentialType ode

Writing the ode as

$$y'=-\frac{3(x^2-1)\,y}{x^3+8y-3x} \tag{1}$$

Which becomes

$$(8y)\,dy=(-x^3+3x)\,dy+(-3y(x^2-1))\,dx \tag{2}$$

But the RHS is complete differential because

$$(-x^3+3x)\,dy+(-3y(x^2-1))\,dx=d\left(-3y\left(\frac{1}{3}x^3-x\right)\right)$$

Hence (2) becomes

$$(8y)\,dy=d\left(-3y\left(\frac{1}{3}x^3-x\right)\right)$$

Integrating both sides gives gives these solutions

$$y = -\frac{x^3}{8} + \frac{3x}{8} + \frac{\sqrt{x^6 - 6x^4 + 9x^2 + 16c_1}}{8} + c_1$$
$$y = -\frac{x^3}{8} + \frac{3x}{8} - \frac{\sqrt{x^6 - 6x^4 + 9x^2 + 16c_1}}{8} + c_1$$

Summary

The solution(s) found are the following

$$y = -\frac{x^3}{8} + \frac{3x}{8} + \frac{\sqrt{x^6 - 6x^4 + 9x^2 + 16c_1}}{8} + c_1 \tag{1}$$
$$y = -\frac{x^3}{8} + \frac{3x}{8} - \frac{\sqrt{x^6 - 6x^4 + 9x^2 + 16c_1}}{8} + c_1 \tag{2}$$

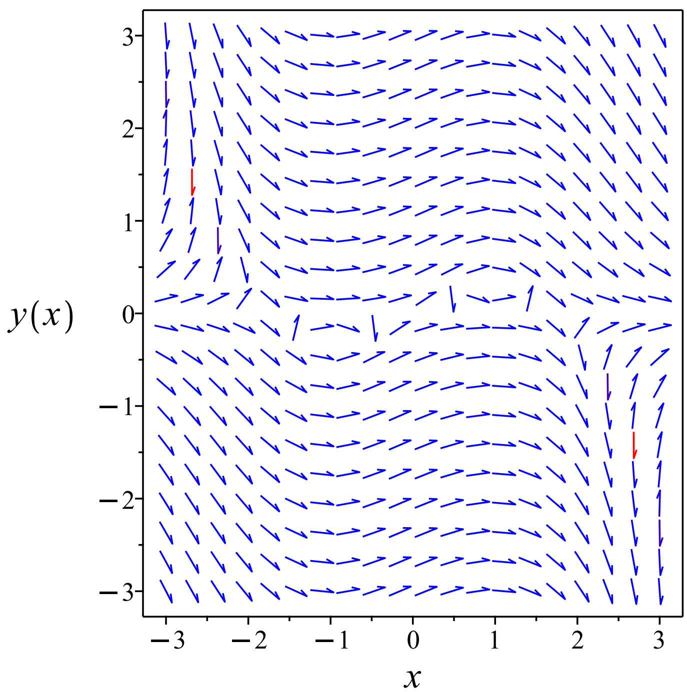


Figure 37: Slope field plot

Verification of solutions

$$y = -\frac{x^3}{8} + \frac{3x}{8} + \frac{\sqrt{x^6 - 6x^4 + 9x^2 + 16c_1}}{8} + c_1$$

Verified OK.

$$y = -\frac{x^3}{8} + \frac{3x}{8} - \frac{\sqrt{x^6 - 6x^4 + 9x^2 + 16c_1}}{8} + c_1$$

Verified OK.

1.23.2 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x, y) + N(x, y)\frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x, y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x, y) = 0$$

Hence

$$\frac{\partial \phi}{\partial x} + \frac{\partial \phi}{\partial y}\frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$\frac{\partial \phi}{\partial x} = M$$
$$\frac{\partial \phi}{\partial y} = N$$

But since $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x, y)$ but at least we know now that we can do that since the condition $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x, y)\,\mathrm{d}x + N(x, y)\,\mathrm{d}y = 0 \tag{1A}$$

Therefore

$$
\begin{aligned}
(x^3 - 3x + 8y)\,\mathrm{d}y &= \left(-3y(x^2 - 1)\right)\mathrm{d}x \\
\left(3y(x^2 - 1)\right)\mathrm{d}x + (x^3 - 3x + 8y)\,\mathrm{d}y &= 0 \qquad \text{(2A)}
\end{aligned}
$$

Comparing (1A) and (2A) shows that

$$
\begin{aligned}
M(x, y) &= 3y(x^2 - 1) \\
N(x, y) &= x^3 - 3x + 8y
\end{aligned}
$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$
\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}
$$

Using result found above gives

$$
\begin{aligned}
\frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}\left(3y(x^2 - 1)\right) \\
&= 3x^2 - 3
\end{aligned}
$$

And

$$
\begin{aligned}
\frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}(x^3 - 3x + 8y) \\
&= 3x^2 - 3
\end{aligned}
$$

Since $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$, then the ODE is <u>exact</u> The following equations are now set up to solve for the function $\phi(x, y)$

$$
\frac{\partial \phi}{\partial x} = M \qquad (1)
$$

$$
\frac{\partial \phi}{\partial y} = N \qquad (2)
$$

Integrating (1) w.r.t. x gives

$$
\begin{aligned}
\int \frac{\partial \phi}{\partial x}\,\mathrm{d}x &= \int M\,\mathrm{d}x \\
\int \frac{\partial \phi}{\partial x}\,\mathrm{d}x &= \int 3y(x^2 - 1)\,\mathrm{d}x \\
\phi &= yx(x^2 - 3) + f(y) \qquad (3)
\end{aligned}
$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$\frac{\partial \phi}{\partial y} = x(x^2 - 3) + f'(y) \tag{4}$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = x^3 - 3x + 8y$. Therefore equation (4) becomes

$$x^3 - 3x + 8y = x(x^2 - 3) + f'(y) \tag{5}$$

Solving equation (5) for $f'(y)$ gives

$$f'(y) = 8y$$

Integrating the above w.r.t y gives

$$\int f'(y)\,\mathrm{d}y = \int (8y)\,\mathrm{d}y$$
$$f(y) = 4y^2 + c_1$$

Where c_1 is constant of integration. Substituting result found above for $f(y)$ into equation (3) gives ϕ

$$\phi = yx(x^2 - 3) + 4y^2 + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = yx(x^2 - 3) + 4y^2$$

Summary

The solution(s) found are the following

$$yx(x^2 - 3) + 4y^2 = c_1 \tag{1}$$

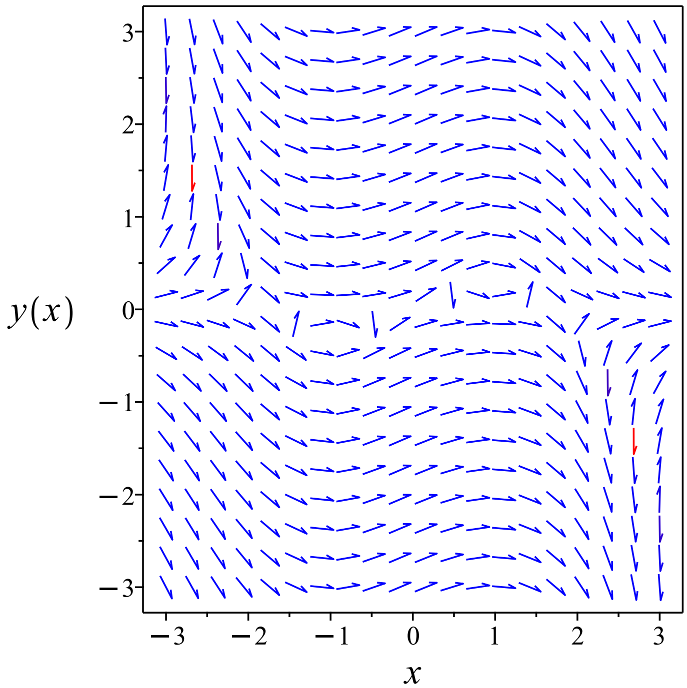


Figure 38: Slope field plot

Verification of solutions

$$yx(x^2-3)+4y^2=c_1$$

Verified OK.

1.23.3 Maple step by step solution

Let's solve

$3(x^2-1)\,y+(x^3+8y-3x)\,y'=0$

- Highest derivative means the order of the ODE is 1

  y'

- □ Check if ODE is exact

  - ◦ ODE is exact if the lhs is the total derivative of a C^2 function

    $F'(x,y)=0$

  - ◦ Compute derivative of lhs

    $F'(x,y)+\left(\frac{\partial}{\partial y}F(x,y)\right)y'=0$

- Evaluate derivatives

  $3x^2 - 3 = 3x^2 - 3$

- Condition met, ODE is exact

- Exact ODE implies solution will be of this form

  $\left[F(x,y) = c_1, M(x,y) = F'(x,y), N(x,y) = \frac{\partial}{\partial y}F(x,y)\right]$

- Solve for $F(x,y)$ by integrating $M(x,y)$ with respect to x

  $F(x,y) = \int 3y(x^2-1)\, dx + f_1(y)$

- Evaluate integral

  $F(x,y) = 3y\left(\frac{1}{3}x^3 - x\right) + f_1(y)$

- Take derivative of $F(x,y)$ with respect to y

  $N(x,y) = \frac{\partial}{\partial y}F(x,y)$

- Compute derivative

  $x^3 - 3x + 8y = x^3 - 3x + \frac{d}{dy}f_1(y)$

- Isolate for $\frac{d}{dy}f_1(y)$

  $\frac{d}{dy}f_1(y) = 8y$

- Solve for $f_1(y)$

  $f_1(y) = 4y^2$

- Substitute $f_1(y)$ into equation for $F(x,y)$

  $F(x,y) = 3y\left(\frac{1}{3}x^3 - x\right) + 4y^2$

- Substitute $F(x,y)$ into the solution of the ODE

  $3y\left(\frac{1}{3}x^3 - x\right) + 4y^2 = c_1$

- Solve for y

  $\left\{y = -\frac{x^3}{8} + \frac{3x}{8} - \frac{\sqrt{x^6-6x^4+9x^2+16c_1}}{8}, y = -\frac{x^3}{8} + \frac{3x}{8} + \frac{\sqrt{x^6-6x^4+9x^2+16c_1}}{8}\right\}$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
trying homogeneous types:
trying Chini
differential order: 1; looking for linear symmetries
trying exact
<- exact successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 67

```
dsolve((3*y(x)*(x^2-1))+(x^3+8*y(x)-3*x)*diff(y(x),x)=0,y(x), singsol=all)
```

$$y(x) = -\frac{x^3}{8} + \frac{3x}{8} - \frac{\sqrt{x^6 - 6x^4 + 9x^2 - 16c_1}}{8}$$
$$y(x) = -\frac{x^3}{8} + \frac{3x}{8} + \frac{\sqrt{x^6 - 6x^4 + 9x^2 - 16c_1}}{8}$$

✓ Solution by Mathematica

Time used: 0.17 (sec). Leaf size: 86

```
DSolve[(3*y[x]*(x^2-1))+(x^3+8*y[x]-3*x)*y'[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \frac{1}{8}\left(-x^3 - \sqrt{x^6 - 6x^4 + 9x^2 + 64c_1} + 3x\right)$$
$$y(x) \to \frac{1}{8}\left(-x^3 + \sqrt{x^6 - 6x^4 + 9x^2 + 64c_1} + 3x\right)$$
$$y(x) \to 0$$

1.24 problem 24

Internal problem ID [3169]
Internal file name [`OUTPUT/2661_Sunday_June_05_2022_08_38_16_AM_88832495/index.tex`]

Book: Differential equations for engineers by Wei-Chau XIE, Cambridge Press 2010
Section: Chapter 2. First-Order and Simple Higher-Order Differential Equations. Page 78
Problem number: 24.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"exact", "first_order_ode_lie_symmetry_calculated"**

Maple gives the following as the ode type

```
[_exact, [_1st_order, `_with_symmetry_[F(x),G(x)*y+H(x)]`]]
```

$$\boxed{\ln(y) = -x^2 - \frac{xy'}{y}}$$

1.24.1 Solving as first order ode lie symmetry calculated ode

Writing the ode as

$$y' = -\frac{(x^2 + \ln(y))\, y}{x}$$
$$y' = \omega(x, y)$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x + \omega(\eta_y - \xi_x) - \omega^2 \xi_y - \omega_x \xi - \omega_y \eta = 0 \tag{A}$$

The type of this ode is not in the lookup table. To determine ξ, η then (A) is solved using ansatz. Making bivariate polynomials of degree 3 to use as anstaz gives

$$\xi = x^3 a_7 + x^2 y a_8 + x\, y^2 a_9 + y^3 a_{10} + x^2 a_4 + x y a_5 + y^2 a_6 + x a_2 + y a_3 + a_1 \tag{1E}$$
$$\eta = x^3 b_7 + x^2 y b_8 + x\, y^2 b_9 + y^3 b_{10} + x^2 b_4 + x y b_5 + y^2 b_6 + x b_2 + y b_3 + b_1 \tag{2E}$$

Where the unknown coefficients are

$$\{a_1, a_2, a_3, a_4, a_5, a_6, a_7, a_8, a_9, a_{10}, b_1, b_2, b_3, b_4, b_5, b_6, b_7, b_8, b_9, b_{10}\}$$

Substituting equations (1E,2E) and ω into (A) gives

$$\begin{aligned}
&3x^2b_7 + 2xyb_8 + y^2b_9 + 2xb_4 + yb_5 + b_2 \\
&- \frac{(x^2 + \ln(y))\, y(-3x^2a_7 + x^2b_8 - 2xya_8 + 2xyb_9 - y^2a_9 + 3y^2b_{10} - 2xa_4 + xb_5 - ya_5 + 2yb_6 - a_2 + b_3)}{x} \\
&- \frac{(x^2 + \ln(y))^2\, y^2(x^2a_8 + 2xya_9 + 3y^2a_{10} + xa_5 + 2ya_6 + a_3)}{x^2} \\
&- \left(-2y + \frac{(x^2 + \ln(y))\, y}{x^2}\right)(x^3a_7 + x^2ya_8 + x\,y^2a_9 + y^3a_{10} + x^2a_4 \\
&+ xya_5 + y^2a_6 + xa_2 + ya_3 + a_1) - \left(-\frac{1}{x} - \frac{x^2 + \ln(y)}{x}\right)(x^3b_7 \\
&+ x^2yb_8 + x\,y^2b_9 + y^3b_{10} + x^2b_4 + xyb_5 + y^2b_6 + xb_2 + yb_3 + b_1) = 0
\end{aligned} \tag{5E}$$

Putting the above in normal form gives

$$-\frac{2\ln(y)\, x^2y^2a_3 + 2\ln(y)\, x^3y^2a_5 + 4\ln(y)\, x^2y^3a_6 + \ln(y)^2\, x\,y^2a_5 - \ln(y)\, x^2ya_4 + \ln(y)\, x\,y^2b_6 - 2\ln(y)}{} = 0$$

Setting the numerator to zero gives

$$\begin{aligned}
&-2\ln(y)\, x^2y^2a_3 - 2\ln(y)\, x^3y^2a_5 - 4\ln(y)\, x^2y^3a_6 - \ln(y)^2\, x\,y^2a_5 \\
&+ \ln(y)\, x^2ya_4 - \ln(y)\, x\,y^2b_6 + 2\ln(y)\, x^3ya_7 + \ln(y)\, x^2y^2a_8 \\
&- \ln(y)\, x^2y^2b_9 - 2\ln(y)\, x\,y^3b_{10} - 2\ln(y)\, x^4y^2a_8 - 4\ln(y)\, x^3y^3a_9 \\
&- 6\ln(y)\, x^2y^4a_{10} - \ln(y)^2\, x^2y^2a_8 - 2\ln(y)^2\, x\,y^3a_9 + 2b_2x^2 \\
&+ x^4b_2 + x^3b_1 + xb_1 + x^5b_4 + 3x^3b_4 + 4x^4b_7 + x^6b_7 + 3x^3yb_8 \\
&+ 2y^2b_9x^2 - x^3y^2b_6 + x^2y^3a_6 + x\,y^2b_6 - 2\ln(y)^2\, y^3a_6 + \ln(y)\, x^3b_4 \\
&- x^5y^2a_5 - 2x^4y^3a_6 + 3x^4ya_4 + 2x^3y^2a_5 + 2yb_5x^2 - \ln(y)\, y^2a_3 \\
&+ \ln(y)\, xb_1 - \ln(y)\, ya_1 - x^4y^2a_3 + 2x^3ya_2 + x^2y^2a_3 + x^2ya_1 \\
&+ xyb_3 - \ln(y)^2\, y^2a_3 + \ln(y)\, x^2b_2 - \ln(y)\, y^3a_6 + 4x^5ya_7 + 3x^4y^2a_8 \\
&- x^4y^2b_9 + 2x^3y^3a_9 - 2x^3y^3b_{10} - x^6y^2a_8 - 2x^5y^3a_9 - 3x^4y^4a_{10} \\
&+ x^2y^4a_{10} + x\,y^3b_{10} - 3\ln(y)^2\, y^4a_{10} - \ln(y)\, y^4a_{10} + \ln(y)\, x^4b_7 = 0
\end{aligned} \tag{6E}$$

Looking at the above PDE shows the following are all the terms with $\{x, y\}$ in them.

$$\{x, y, \ln(y)\}$$

The following substitution is now made to be able to collect on all terms with $\{x, y\}$ in them

$$\{x = v_1, y = v_2, \ln(y) = v_3\}$$

The above PDE (6E) now becomes

$$\begin{aligned}&-2v_3v_1^3v_2^2a_5-4v_3v_1^2v_2^3a_6-v_3^2v_1v_2^2a_5+v_3v_1^2v_2a_4-v_3v_1v_2^2b_6+2v_3v_1^3v_2a_7\\&+v_3v_1^2v_2^2a_8-v_3v_1^2v_2^2b_9-2v_3v_1v_2^3b_{10}-2v_3v_1^4v_2^2a_8-4v_3v_1^3v_2^3a_9\\&-6v_3v_1^2v_2^4a_{10}-v_3^2v_1^2v_2^2a_8-2v_3^2v_1v_2^3a_9-2v_3v_1^2v_2^2a_3+2b_2v_2v_1^2+v_1^4b_2\\&+v_1^3b_1+v_1b_1+v_1^5b_4+3v_1^3b_4+4v_1^4b_7+v_1^6b_7-2v_1^3v_2^3b_{10}-v_1^6v_2^2a_8\\&-2v_1^5v_2^3a_9-3v_1^4v_2^4a_{10}+v_1^2v_2^4a_{10}+v_1v_2^3b_{10}-3v_3^2v_2^4a_{10}-v_3v_2^4a_{10}\\&+v_3v_1^4b_7+3v_1^3v_2b_8+2v_2^2b_9v_1^2-v_1^3v_2^2b_6+v_1^2v_2^3a_6+v_1v_2^2b_6-2v_3^2v_2^3a_6\\&+v_3v_1^3b_4-v_1^5v_2^2a_5-2v_1^4v_2^3a_6+3v_1^4v_2a_4+2v_1^3v_2^2a_5+2v_2b_5v_1^2-v_3v_2^2a_3\\&+v_3v_1b_1-v_3v_2a_1-v_1^4v_2^2a_3+2v_1^3v_2a_2+v_1^2v_2^2a_3+v_1^2v_2a_1+v_1v_2b_3\\&-v_3^2v_2^2a_3+v_3v_1^2b_2-v_3v_2^3a_6+4v_1^5v_2a_7+3v_1^4v_2^2a_8-v_1^4v_2^2b_9+2v_1^3v_2^3a_9=0\end{aligned} \tag{7E}$$

Collecting the above on the terms v_i introduced, and these are

$$\{v_1, v_2, v_3\}$$

Equation (7E) now becomes

$$\begin{aligned}&(a_1+2b_5)\,v_1^2v_2+(-a_3+3a_8-b_9)\,v_2^2v_1^4+(-2b_{10}+2a_9)\,v_2^3v_1^3\\&+(3b_8+2a_2)\,v_2v_1^3+(2b_9+a_3)\,v_1^2v_2^2+(2a_5-b_6)\,v_2^2v_1^3-2v_3v_1^3v_2^2a_5\\&-4v_3v_1^2v_2^3a_6-v_3^2v_1v_2^2a_5+v_3v_1^2v_2a_4-v_3v_1v_2^2b_6+2v_3v_1^3v_2a_7\\&-2v_3v_1v_2^3b_{10}-2v_3v_1^4v_2^2a_8-4v_3v_1^3v_2^3a_9-6v_3v_1^2v_2^4a_{10}-v_3^2v_1^2v_2^2a_8\\&-2v_3^2v_1v_2^3a_9+(a_8-b_9-2a_3)\,v_2^2v_1^2v_3+2b_2v_2v_1^2+v_1b_1+v_1^5b_4+v_1^6b_7\\&-v_1^6v_2^2a_8-2v_1^5v_2^3a_9-3v_1^4v_2^4a_{10}+v_1^2v_2^4a_{10}+v_1v_2^3b_{10}-3v_3^2v_2^4a_{10}\\&-v_3v_2^4a_{10}+v_3v_1^4b_7+v_1^2v_2^3a_6+v_1v_2^2b_6-2v_3^2v_2^3a_6+v_3v_1^3b_4-v_1^5v_2^2a_5\\&-2v_1^4v_2^3a_6+3v_1^4v_2a_4-v_3v_2^2a_3+v_3v_1b_1-v_3v_2a_1+v_1v_2b_3-v_3^2v_2^2a_3\\&+v_3v_1^2b_2-v_3v_2^3a_6+4v_1^5v_2a_7+(b_2+4b_7)\,v_1^4+(b_1+3b_4)\,v_1^3=0\end{aligned} \tag{8E}$$

Setting each coefficients in (8E) to zero gives the following equations to solve

$$
\begin{aligned}
a_4 &= 0\\
a_6 &= 0\\
a_{10} &= 0\\
b_1 &= 0\\
b_2 &= 0\\
b_3 &= 0\\
b_4 &= 0\\
b_6 &= 0\\
b_7 &= 0\\
b_{10} &= 0\\
-a_1 &= 0\\
-a_3 &= 0\\
3a_4 &= 0\\
-2a_5 &= 0\\
-a_5 &= 0\\
-4a_6 &= 0\\
-2a_6 &= 0\\
-a_6 &= 0\\
2a_7 &= 0\\
4a_7 &= 0\\
-2a_8 &= 0\\
-a_8 &= 0\\
-4a_9 &= 0\\
-2a_9 &= 0\\
-6a_{10} &= 0\\
-3a_{10} &= 0\\
-a_{10} &= 0\\
2b_2 &= 0\\
-b_6 &= 0\\
-2b_{10} &= 0\\
a_1 + 2b_5 &= 0\\
2a_5 - b_6 &= 0\\
b_1 + 3b_4 &= 0\\
b_2 + 4b_7 &= 0\\
3b_8 + 2a_2 &= 0\\
2b_9 + a_3 &= 0
\end{aligned}
$$

Solving the above equations for the unknowns gives

$$
\begin{aligned}
a_1 &= 0 \\
a_2 &= -\frac{3b_8}{2} \\
a_3 &= 0 \\
a_4 &= 0 \\
a_5 &= 0 \\
a_6 &= 0 \\
a_7 &= 0 \\
a_8 &= 0 \\
a_9 &= 0 \\
a_{10} &= 0 \\
b_1 &= 0 \\
b_2 &= 0 \\
b_3 &= 0 \\
b_4 &= 0 \\
b_5 &= 0 \\
b_6 &= 0 \\
b_7 &= 0 \\
b_8 &= b_8 \\
b_9 &= 0 \\
b_{10} &= 0
\end{aligned}
$$

Substituting the above solution in the anstaz (1E,2E) (using 1 as arbitrary value for any unknown in the RHS) gives

$$
\begin{aligned}
\xi &= -\frac{3x}{2} \\
\eta &= x^2 y
\end{aligned}
$$

Shifting is now applied to make $\xi = 0$ in order to simplify the rest of the computation

$$\begin{aligned}\eta &= \eta - \omega(x,y)\,\xi \\ &= x^2y - \left(-\frac{(x^2+\ln(y))\,y}{x}\right)\left(-\frac{3x}{2}\right) \\ &= -\frac{x^2y}{2} - \frac{3\ln(y)\,y}{2} \\ \xi &= 0\end{aligned}$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x,y) \to (R,S)$ where (R,S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}$$

The above comes from the requirements that $\left(\xi\frac{\partial}{\partial x} + \eta\frac{\partial}{\partial y}\right)S(x,y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$R = x$$

S is found from

$$\begin{aligned}S &= \int \frac{1}{\eta} dy \\ &= \int \frac{1}{-\frac{x^2y}{2} - \frac{3\ln(y)y}{2}} dy\end{aligned}$$

Which results in

$$S = -\frac{2\ln(x^2+3\ln(y))}{3}$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x,y)S_y}{R_x + \omega(x,y)R_y} \tag{2}$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x,y)$ is the right hand side of the original ode given by

$$\omega(x,y) = -\frac{(x^2+\ln(y))\,y}{x}$$

Evaluating all the partial derivatives gives

$$
\begin{aligned}
R_x &= 1 \\
R_y &= 0 \\
S_x &= -\frac{4x}{3x^2 + 9\ln(y)} \\
S_y &= -\frac{2}{y\,(x^2 + 3\ln(y))}
\end{aligned}
$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$
\frac{dS}{dR} = \frac{2}{3x} \tag{2A}
$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$
\frac{dS}{dR} = \frac{2}{3R}
$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$
S(R) = \frac{2\ln(R)}{3} + c_1 \tag{4}
$$

To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$
-\frac{2\ln(x^2 + 3\ln(y))}{3} = \frac{2\ln(x)}{3} + c_1
$$

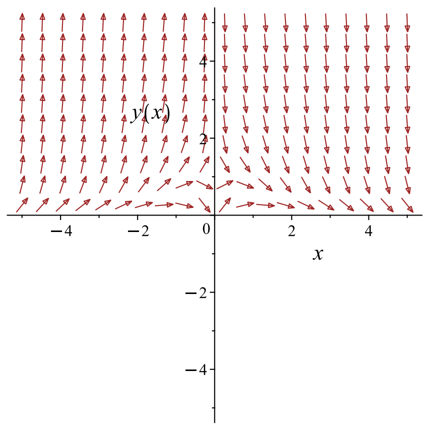
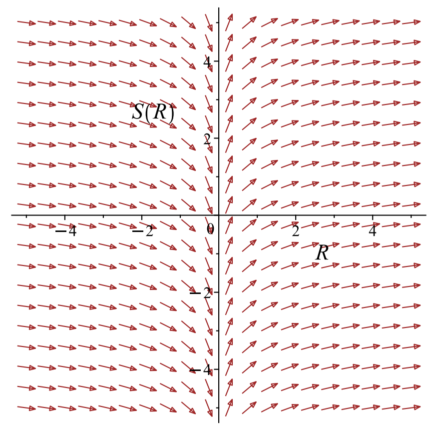
Which simplifies to

$$
-\frac{2\ln(x^2 + 3\ln(y))}{3} = \frac{2\ln(x)}{3} + c_1
$$

Which gives

$$
y = \mathrm{e}^{\frac{-x^3 + \mathrm{e}^{-\frac{3c_1}{2}}}{3x}}
$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in x, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dx} = -\frac{(x^2+\ln(y))y}{x}$ 	$R = x$ $S = -\frac{2\ln(x^2+3\ln(y))}{3}$	$\frac{dS}{dR} = \frac{2}{3R}$ 

Summary

The solution(s) found are the following

$$y = e^{\frac{-x^3+e^{-\frac{3c_1}{2}}}{3x}} \tag{1}$$

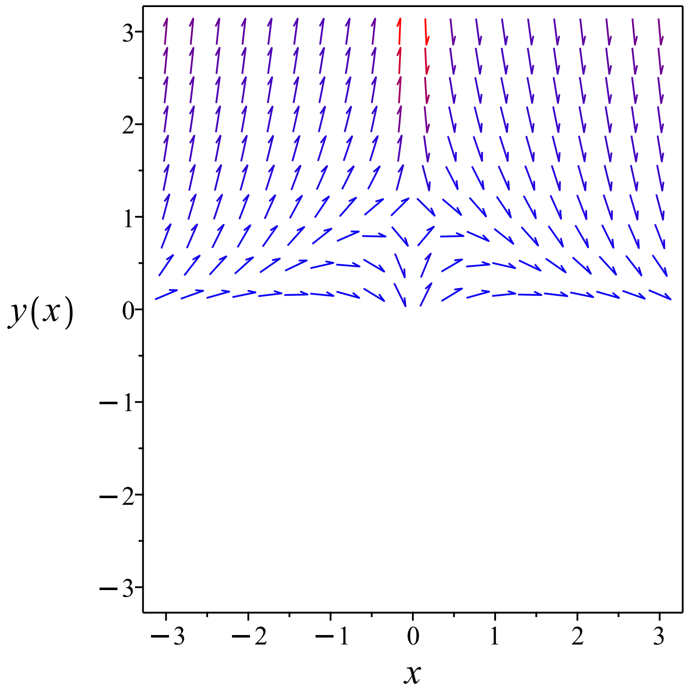


Figure 39: Slope field plot

Verification of solutions

$$y = e^{\frac{-x^3+e^{-\frac{3c_1}{2}}}{3x}}$$

Verified OK.

1.24.2 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x,y)+N(x,y)\frac{dy}{dx}=0 \tag{A}$$

We assume there exists a function $\phi(x,y)=c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y)=0$$

Hence

$$\frac{\partial \phi}{\partial x}+\frac{\partial \phi}{\partial y}\frac{dy}{dx}=0 \tag{B}$$

Comparing (A,B) shows that

$$\begin{aligned}\frac{\partial \phi}{\partial x} &= M \\ \frac{\partial \phi}{\partial y} &= N\end{aligned}$$

But since $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x, y)$ but at least we know now that we can do that since the condition $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x, y)\, \mathrm{d}x + N(x, y)\, \mathrm{d}y = 0 \tag{1A}$$

Therefore

$$\begin{aligned}\left(\frac{x}{y}\right) \mathrm{d}y &= \left(-x^2 - \ln(y)\right) \mathrm{d}x \\ \left(x^2 + \ln(y)\right) \mathrm{d}x + \left(\frac{x}{y}\right) \mathrm{d}y &= 0 \qquad (2A)\end{aligned}$$

Comparing (1A) and (2A) shows that

$$\begin{aligned}M(x, y) &= x^2 + \ln(y) \\ N(x, y) &= \frac{x}{y}\end{aligned}$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Using result found above gives

$$\begin{aligned}\frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}\left(x^2 + \ln(y)\right) \\ &= \frac{1}{y}\end{aligned}$$

And

$$\begin{aligned}\frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}\left(\frac{x}{y}\right)\\ &= \frac{1}{y}\end{aligned}$$

Since $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$, then the ODE is exact The following equations are now set up to solve for the function $\phi(x, y)$

$$\frac{\partial \phi}{\partial x} = M \tag{1}$$

$$\frac{\partial \phi}{\partial y} = N \tag{2}$$

Integrating (1) w.r.t. x gives

$$\begin{aligned}\int \frac{\partial \phi}{\partial x}\,\mathrm{d}x &= \int M\,\mathrm{d}x\\ \int \frac{\partial \phi}{\partial x}\,\mathrm{d}x &= \int x^2 + \ln(y)\,\mathrm{d}x\end{aligned}$$

$$\phi = \frac{x^3}{3} + \ln(y)\,x + f(y) \tag{3}$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$\frac{\partial \phi}{\partial y} = \frac{x}{y} + f'(y) \tag{4}$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = \frac{x}{y}$. Therefore equation (4) becomes

$$\frac{x}{y} = \frac{x}{y} + f'(y) \tag{5}$$

Solving equation (5) for $f'(y)$ gives

$$f'(y) = 0$$

Therefore

$$f(y) = c_1$$

Where c_1 is constant of integration. Substituting this result for $f(y)$ into equation (3) gives ϕ

$$\phi = \frac{x^3}{3} + \ln(y)\, x + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = \frac{x^3}{3} + \ln(y)\, x$$

The solution becomes

$$y = \mathrm{e}^{\frac{-x^3+3c_1}{3x}}$$

Summary

The solution(s) found are the following

$$y = \mathrm{e}^{\frac{-x^3+3c_1}{3x}} \tag{1}$$

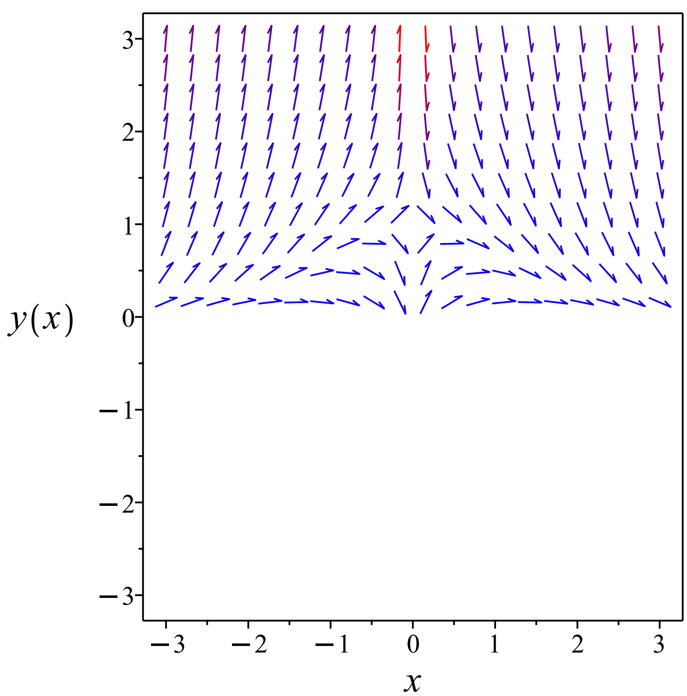


Figure 40: Slope field plot

Verification of solutions

$$y = e^{\frac{-x^3+3c_1}{3x}}$$

Verified OK.

1.24.3 Maple step by step solution

Let's solve

$\ln(y) = -x^2 - \frac{xy'}{y}$

- Highest derivative means the order of the ODE is 1

  y'

- □ Check if ODE is exact

  - ODE is exact if the lhs is the total derivative of a C^2 function

    $F'(x,y) = 0$

  - Compute derivative of lhs

    $F'(x,y) + \left(\frac{\partial}{\partial y}F(x,y)\right)y' = 0$

  - Evaluate derivatives

    $\frac{1}{y} = \frac{1}{y}$

  - Condition met, ODE is exact

- Exact ODE implies solution will be of this form

  $\left[F(x,y) = c_1, M(x,y) = F'(x,y), N(x,y) = \frac{\partial}{\partial y}F(x,y)\right]$

- Solve for $F(x,y)$ by integrating $M(x,y)$ with respect to x

  $F(x,y) = \int (x^2 + \ln(y))\, dx + f_1(y)$

- Evaluate integral

  $F(x,y) = \frac{x^3}{3} + \ln(y)\, x + f_1(y)$

- Take derivative of $F(x,y)$ with respect to y

  $N(x,y) = \frac{\partial}{\partial y}F(x,y)$

- Compute derivative

  $\frac{x}{y} = \frac{x}{y} + \frac{d}{dy}f_1(y)$

- Isolate for $\frac{d}{dy}f_1(y)$

$$\frac{d}{dy}f_1(y) = 0$$

- Solve for $f_1(y)$

$$f_1(y) = 0$$

- Substitute $f_1(y)$ into equation for $F(x, y)$

$$F(x, y) = \frac{x^3}{3} + \ln(y)\, x$$

- Substitute $F(x, y)$ into the solution of the ODE

$$\frac{x^3}{3} + \ln(y)\, x = c_1$$

- Solve for y

$$y = e^{\frac{-x^3+3c_1}{3x}}$$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
trying homogeneous types:
trying Chini
differential order: 1; looking for linear symmetries
trying exact
<- exact successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 17

```
dsolve((x^2+ln(y(x)))+(x/y(x))*diff(y(x),x)=0,y(x), singsol=all)
```

$$y(x) = e^{-\frac{x^3+3c_1}{3x}}$$

✓ Solution by Mathematica

Time used: 0.247 (sec). Leaf size: 21

```
DSolve[(x^2+Log[y[x]])+(x/y[x])*y'[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to e^{-\frac{x^2}{3}+\frac{c_1}{x}}$$

1.25 problem 25

Internal problem ID [3170]
Internal file name [`OUTPUT/2662_Sunday_June_05_2022_08_38_16_AM_88983900/index.tex`]

Book: Differential equations for engineers by Wei-Chau XIE, Cambridge Press 2010
Section: Chapter 2. First-Order and Simple Higher-Order Differential Equations. Page 78
Problem number: 25.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"exact"**

Maple gives the following as the ode type

`[_exact]`

$$\boxed{2x\left(3x+y-y\,\mathrm{e}^{-x^2}\right)+\left(x^2+3y^2+\mathrm{e}^{-x^2}\right)y'=0}$$

1.25.1 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x,y)+N(x,y)\,\frac{dy}{dx}=0 \tag{A}$$

We assume there exists a function $\phi(x,y)=c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y)=0$$

Hence

$$\frac{\partial\phi}{\partial x}+\frac{\partial\phi}{\partial y}\frac{dy}{dx}=0 \tag{B}$$

Comparing (A,B) shows that

$$\frac{\partial\phi}{\partial x}=M$$
$$\frac{\partial\phi}{\partial y}=N$$

But since $\frac{\partial^2\phi}{\partial x\partial y} = \frac{\partial^2\phi}{\partial y\partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x, y)$ but at least we know now that we can do that since the condition $\frac{\partial^2\phi}{\partial x\partial y} = \frac{\partial^2\phi}{\partial y\partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x, y)\,\mathrm{d}x + N(x, y)\,\mathrm{d}y = 0 \tag{1A}$$

Therefore

$$\left(x^2 + 3y^2 + \mathrm{e}^{-x^2}\right)\mathrm{d}y = \left(-2x\left(3x + y - y\,\mathrm{e}^{-x^2}\right)\right)\mathrm{d}x$$

$$\left(2x\left(3x + y - y\,\mathrm{e}^{-x^2}\right)\right)\mathrm{d}x + \left(x^2 + 3y^2 + \mathrm{e}^{-x^2}\right)\mathrm{d}y = 0 \tag{2A}$$

Comparing (1A) and (2A) shows that

$$\begin{aligned} M(x, y) &= 2x\left(3x + y - y\,\mathrm{e}^{-x^2}\right) \\ N(x, y) &= x^2 + 3y^2 + \mathrm{e}^{-x^2} \end{aligned}$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Using result found above gives

$$\begin{aligned} \frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}\left(2x\left(3x + y - y\,\mathrm{e}^{-x^2}\right)\right) \\ &= -2x\left(\mathrm{e}^{-x^2} - 1\right) \end{aligned}$$

And

$$\begin{aligned} \frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}\left(x^2 + 3y^2 + \mathrm{e}^{-x^2}\right) \\ &= -2x\left(\mathrm{e}^{-x^2} - 1\right) \end{aligned}$$

Since $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$, then the ODE is exact The following equations are now set up to solve for the function $\phi(x, y)$

$$\frac{\partial \phi}{\partial x} = M \tag{1}$$
$$\frac{\partial \phi}{\partial y} = N \tag{2}$$

Integrating (1) w.r.t. x gives

$$\int \frac{\partial \phi}{\partial x} \, \mathrm{d}x = \int M \, \mathrm{d}x$$
$$\int \frac{\partial \phi}{\partial x} \, \mathrm{d}x = \int 2x\left(3x + y - y\,\mathrm{e}^{-x^2}\right) \mathrm{d}x$$
$$\phi = x^2 y + 2x^3 + y\,\mathrm{e}^{-x^2} + f(y) \tag{3}$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$\frac{\partial \phi}{\partial y} = x^2 + \mathrm{e}^{-x^2} + f'(y) \tag{4}$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = x^2 + 3y^2 + \mathrm{e}^{-x^2}$. Therefore equation (4) becomes

$$x^2 + 3y^2 + \mathrm{e}^{-x^2} = x^2 + \mathrm{e}^{-x^2} + f'(y) \tag{5}$$

Solving equation (5) for $f'(y)$ gives

$$f'(y) = 3y^2$$

Integrating the above w.r.t y gives

$$\int f'(y) \, \mathrm{d}y = \int (3y^2) \, \mathrm{d}y$$
$$f(y) = y^3 + c_1$$

Where c_1 is constant of integration. Substituting result found above for $f(y)$ into equation (3) gives ϕ

$$\phi = x^2 y + 2x^3 + y\,\mathrm{e}^{-x^2} + y^3 + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = x^2y + 2x^3 + y\,\mathrm{e}^{-x^2} + y^3$$

Summary

The solution(s) found are the following

$$x^2y + 2x^3 + y\,\mathrm{e}^{-x^2} + y^3 = c_1 \tag{1}$$

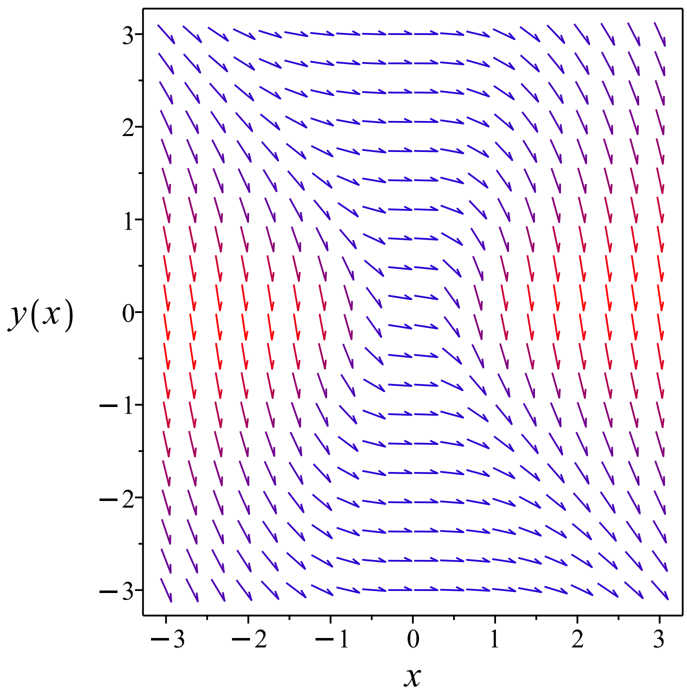


Figure 41: Slope field plot

Verification of solutions

$$x^2y + 2x^3 + y\,\mathrm{e}^{-x^2} + y^3 = c_1$$

Verified OK.

1.25.2 Maple step by step solution

Let's solve

$2x\left(3x+y-y\,\mathrm{e}^{-x^2}\right)+\left(x^2+3y^2+\mathrm{e}^{-x^2}\right)y'=0$

- Highest derivative means the order of the ODE is 1

  y'

□ Check if ODE is exact

  ◦ ODE is exact if the lhs is the total derivative of a C^2 function

    $F'(x,y)=0$

  ◦ Compute derivative of lhs

    $F'(x,y)+\left(\frac{\partial}{\partial y}F(x,y)\right)y'=0$

  ◦ Evaluate derivatives

    $2x\left(-\mathrm{e}^{-x^2}+1\right)=2x-2x\,\mathrm{e}^{-x^2}$

  ◦ Simplify

    $-2x\left(\mathrm{e}^{-x^2}-1\right)=-2x\left(\mathrm{e}^{-x^2}-1\right)$

  ◦ Condition met, ODE is exact

- Exact ODE implies solution will be of this form

  $\left[F(x,y)=c_1, M(x,y)=F'(x,y), N(x,y)=\frac{\partial}{\partial y}F(x,y)\right]$

- Solve for $F(x,y)$ by integrating $M(x,y)$ with respect to x

  $F(x,y)=\int 2x\left(3x+y-y\,\mathrm{e}^{-x^2}\right)dx+f_1(y)$

- Evaluate integral

  $F(x,y)=x^2y+2x^3+\frac{y}{\mathrm{e}^{x^2}}+f_1(y)$

- Take derivative of $F(x,y)$ with respect to y

  $N(x,y)=\frac{\partial}{\partial y}F(x,y)$

- Compute derivative

  $x^2+3y^2+\mathrm{e}^{-x^2}=x^2+\frac{1}{\mathrm{e}^{x^2}}+\frac{d}{dy}f_1(y)$

- Isolate for $\frac{d}{dy}f_1(y)$

  $\frac{d}{dy}f_1(y)=3y^2+\mathrm{e}^{-x^2}-\frac{1}{\mathrm{e}^{x^2}}$

- Solve for $f_1(y)$

  $f_1(y) = \frac{y^3 e^{x^2} + e^{-x^2} e^{x^2} y - y}{e^{x^2}}$

- Substitute $f_1(y)$ into equation for $F(x, y)$

  $F(x, y) = x^2 y + 2x^3 + \frac{y}{e^{x^2}} + \frac{y^3 e^{x^2} + e^{-x^2} e^{x^2} y - y}{e^{x^2}}$

- Substitute $F(x, y)$ into the solution of the ODE

  $x^2 y + 2x^3 + \frac{y}{e^{x^2}} + \frac{y^3 e^{x^2} + e^{-x^2} e^{x^2} y - y}{e^{x^2}} = c_1$

- Solve for y

$$\left\{ y = \frac{\left(-216x^3 + 108c_1 + 12\sqrt{336x^6 + 36x^4 e^{-x^2} + 36\left(e^{-x^2}\right)^2 x^2 + 12\left(e^{-x^2}\right)^3 - 324c_1 x^3 + 81c_1^2}\right)^{\frac{1}{3}}}{6} - \frac{}{\left(-216x^3 + 108c_1 + 12\sqrt{336x^6 \cdots}\right.} \right.$$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
trying homogeneous types:
trying Chini
differential order: 1; looking for linear symmetries
trying exact
<- exact successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 634

```
dsolve((2*x*(3*x+y(x)-y(x)*exp(-x^2)))+(x^2+3*y(x)^2+exp(-x^2))*diff(y(x),x)=0,y(x), singsol
```

$$y(x) = -\frac{12^{\frac{1}{3}}\left(-\left(\sqrt{3}\,\mathrm{e}^{2x^2}\sqrt{\left(4+\left(112x^6+108c_1x^3+27c_1^2\right)\mathrm{e}^{3x^2}+12\,\mathrm{e}^{2x^2}x^4+12\,\mathrm{e}^{x^2}x^2\right)\mathrm{e}^{-x^2}}-18\,\mathrm{e}^{3x^2}\left(x^3+\frac{c_1}{2}\right)\right.\right.}{6\left(\sqrt{3}\,\mathrm{e}^{2x^2}\sqrt{\left(4+\left(112x^6+108c_1x^3+27c_1^2\right)\mathrm{e}^{3x^2}+12\,\mathrm{e}^{2x^2}x^4+12\,\mathrm{e}^{x^2}x^2\right)\mathrm{e}^{-x^2}}-18\,\mathrm{e}^{3x^2}\right.}$$

$$y(x) = -\frac{3^{\frac{1}{3}}\left(\mathrm{e}^{-x^2}\left(1+i\sqrt{3}\right)\left(\sqrt{3}\,\mathrm{e}^{2x^2}\sqrt{\left(4+\left(112x^6+108c_1x^3+27c_1^2\right)\mathrm{e}^{3x^2}+12\,\mathrm{e}^{2x^2}x^4+12\,\mathrm{e}^{x^2}x^2\right)\mathrm{e}^{-x^2}}-18\,\mathrm{e}^{3\right.\right.}{12\left(\sqrt{3}\,\mathrm{e}^{2x^2}\sqrt{\left(4+\left(112x^6+108c_1x^3+27c_1^2\right)\mathrm{e}^{3x^2}+12\,\mathrm{e}^{2x^2}x^4+12\,\mathrm{e}^{x^2}x^2\right)\mathrm{e}^{-x}}\right.}$$

$$y(x) = \frac{3^{\frac{1}{3}}\left(\left(i\sqrt{3}-1\right)\mathrm{e}^{-x^2}\left(\sqrt{3}\,\mathrm{e}^{2x^2}\sqrt{\left(4+\left(112x^6+108c_1x^3+27c_1^2\right)\mathrm{e}^{3x^2}+12\,\mathrm{e}^{2x^2}x^4+12\,\mathrm{e}^{x^2}x^2\right)\mathrm{e}^{-x^2}}-18\,\mathrm{e}^{3x^2}\right.\right.}{12\left(\sqrt{3}\,\mathrm{e}^{2x^2}\sqrt{\left(4+\left(112x^6+108c_1x^3+27c_1^2\right)\mathrm{e}^{3x^2}+12\,\mathrm{e}^{2x^2}x^4+12\,\mathrm{e}^{x^2}x^2\right)\mathrm{e}^{-x^2}}\right.}$$

✓ Solution by Mathematica

Time used: 37.566 (sec). Leaf size: 416

```
DSolve[(2*x*(3*x+y[x]-y[x]*Exp[-x^2]))+(x^2+3*y[x]^2+Exp[-x^2])*y'[x]==0,y[x],x,IncludeSingu
```

$$y(x) \to \frac{-6\sqrt[3]{2}\left(x^2+e^{-x^2}\right)+2^{2/3}\left(-54x^3+\sqrt{108\left(x^2+e^{-x^2}\right)^3+729\left(-2x^3+c_1\right)^2}+27c_1\right)^{2/3}}{6\sqrt[3]{-54x^3+\sqrt{108\left(x^2+e^{-x^2}\right)^3+729\left(-2x^3+c_1\right)^2}+27c_1}}$$

$$y(x) \to \frac{\left(1+i\sqrt{3}\right)\left(x^2+e^{-x^2}\right)}{2^{2/3}\sqrt[3]{-54x^3+\sqrt{108\left(x^2+e^{-x^2}\right)^3+729\left(-2x^3+c_1\right)^2}+27c_1}} + \frac{\left(-1+i\sqrt{3}\right)\sqrt[3]{-54x^3+\sqrt{108\left(x^2+e^{-x^2}\right)^3+729\left(-2x^3+c_1\right)^2}+27c_1}}{6\sqrt[3]{2}}$$

$$y(x) \to \frac{\left(1-i\sqrt{3}\right)\left(x^2+e^{-x^2}\right)}{2^{2/3}\sqrt[3]{-54x^3+\sqrt{108\left(x^2+e^{-x^2}\right)^3+729\left(-2x^3+c_1\right)^2}+27c_1}} - \frac{\left(1+i\sqrt{3}\right)\sqrt[3]{-54x^3+\sqrt{108\left(x^2+e^{-x^2}\right)^3+729\left(-2x^3+c_1\right)^2}+27c_1}}{6\sqrt[3]{2}}$$

1.26 problem 26

Internal problem ID [3171]
Internal file name [`OUTPUT/2663_Sunday_June_05_2022_08_38_18_AM_55902060/index.tex`]

Book: Differential equations for engineers by Wei-Chau XIE, Cambridge Press 2010
Section: Chapter 2. First-Order and Simple Higher-Order Differential Equations. Page 78
Problem number: 26.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"exact"**

Maple gives the following as the ode type

```
[_exact, [_Abel, `2nd type`, `class B`]]
```

$$\boxed{y + 2y^2 \sin(x)^2 + (x + 2yx - y\sin(2x))\, y' = -3}$$

1.26.1 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x,y) + N(x,y)\frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x,y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y) = 0$$

Hence

$$\frac{\partial \phi}{\partial x} + \frac{\partial \phi}{\partial y}\frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$\frac{\partial \phi}{\partial x} = M$$
$$\frac{\partial \phi}{\partial y} = N$$

But since $\frac{\partial^2\phi}{\partial x\partial y} = \frac{\partial^2\phi}{\partial y\partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x,y)$ but at least we know now that we can do that since the condition $\frac{\partial^2\phi}{\partial x\partial y} = \frac{\partial^2\phi}{\partial y\partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x,y)\,\mathrm{d}x + N(x,y)\,\mathrm{d}y = 0 \tag{1A}$$

Therefore

$$\begin{aligned}(x + 2xy - y\sin(2x))\,\mathrm{d}y &= \left(-3 - y - 2y^2\sin(x)^2\right)\mathrm{d}x\\ \left(2y^2\sin(x)^2 + y + 3\right)\mathrm{d}x + (x + 2xy - y\sin(2x))\,\mathrm{d}y &= 0 \qquad (2A)\end{aligned}$$

Comparing (1A) and (2A) shows that

$$\begin{aligned}M(x,y) &= 2y^2\sin(x)^2 + y + 3\\ N(x,y) &= x + 2xy - y\sin(2x)\end{aligned}$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Using result found above gives

$$\begin{aligned}\frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}\left(2y^2\sin(x)^2 + y + 3\right)\\ &= 4\sin(x)^2 y + 1\end{aligned}$$

And

$$\begin{aligned}\frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}(x + 2xy - y\sin(2x))\\ &= 1 + 2y - 2y\cos(2x)\end{aligned}$$

Since $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$, then the ODE is <u>exact</u> The following equations are now set up to solve for the function $\phi(x,y)$

$$\frac{\partial\phi}{\partial x} = M \tag{1}$$

$$\frac{\partial\phi}{\partial y} = N \tag{2}$$

Integrating (1) w.r.t. x gives

$$\int \frac{\partial \phi}{\partial x} \mathrm{d}x = \int M \mathrm{d}x$$

$$\int \frac{\partial \phi}{\partial x} \mathrm{d}x = \int 2y^2 \sin(x)^2 + y + 3 \mathrm{d}x$$

$$\phi = -\frac{\sin(2x) y^2}{2} + x\left(y^2 + y + 3\right) + f(y) \tag{3}$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$\frac{\partial \phi}{\partial y} = -y\sin(2x) + x(2y+1) + f'(y) \tag{4}$$

$$= x + 2xy - y\sin(2x) + f'(y)$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = x + 2xy - y\sin(2x)$. Therefore equation (4) becomes

$$x + 2xy - y\sin(2x) = x + 2xy - y\sin(2x) + f'(y) \tag{5}$$

Solving equation (5) for $f'(y)$ gives

$$f'(y) = 0$$

Therefore

$$f(y) = c_1$$

Where c_1 is constant of integration. Substituting this result for $f(y)$ into equation (3) gives ϕ

$$\phi = -\frac{\sin(2x) y^2}{2} + x\left(y^2 + y + 3\right) + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = -\frac{\sin(2x) y^2}{2} + x\left(y^2 + y + 3\right)$$

Summary

The solution(s) found are the following

$$-\frac{\sin(2x)\,y^2}{2}+x\left(y^2+y+3\right)=c_1 \tag{1}$$

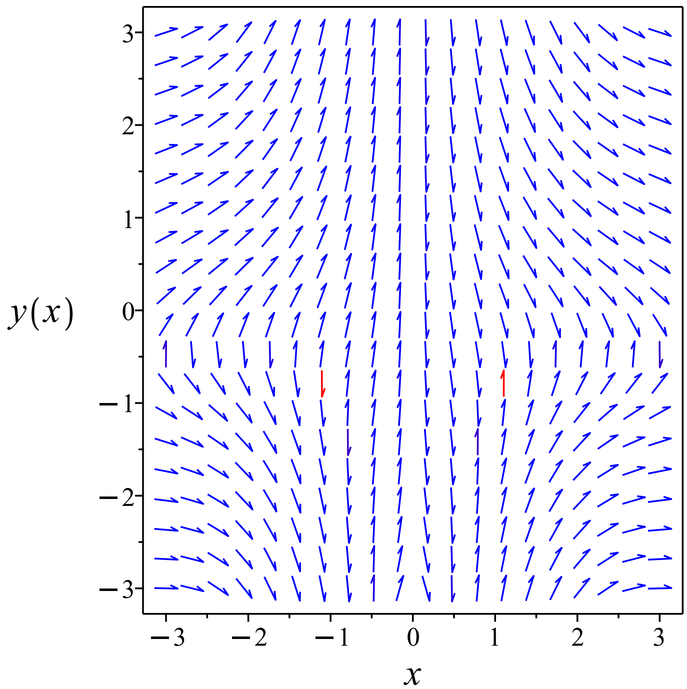


Figure 42: Slope field plot

Verification of solutions

$$-\frac{\sin(2x)\,y^2}{2}+x\left(y^2+y+3\right)=c_1$$

Verified OK.

1.26.2 Maple step by step solution

Let's solve

$y+2y^2\sin(x)^2+(x+2yx-y\sin(2x))\,y'=-3$

- Highest derivative means the order of the ODE is 1

  y'

- □ Check if ODE is exact

- ODE is exact if the lhs is the total derivative of a C^2 function

  $F'(x, y) = 0$
- Compute derivative of lhs

  $F'(x, y) + \left(\frac{\partial}{\partial y}F(x, y)\right) y' = 0$
- Evaluate derivatives

  $4 \sin(x)^2 y + 1 = 1 + 2y - 2y \cos(2x)$
- Simplify

  $4 \sin(x)^2 y + 1 = 1 + 2y - 2y \cos(2x)$
- Condition met, ODE is exact

- Exact ODE implies solution will be of this form

  $\left[F(x, y) = c_1, M(x, y) = F'(x, y), N(x, y) = \frac{\partial}{\partial y}F(x, y)\right]$
- Solve for $F(x, y)$ by integrating $M(x, y)$ with respect to x

  $F(x, y) = \int (2y^2 \sin(x)^2 + y + 3) \, dx + f_1(y)$
- Evaluate integral

  $F(x, y) = xy + 3x + 2y^2 \left(-\frac{\sin(x)\cos(x)}{2} + \frac{x}{2}\right) + f_1(y)$
- Take derivative of $F(x, y)$ with respect to y

  $N(x, y) = \frac{\partial}{\partial y}F(x, y)$
- Compute derivative

  $x + 2xy - y \sin(2x) = x + 4y\left(-\frac{\sin(x)\cos(x)}{2} + \frac{x}{2}\right) + \frac{d}{dy}f_1(y)$
- Isolate for $\frac{d}{dy}f_1(y)$

  $\frac{d}{dy}f_1(y) = 2xy - y \sin(2x) - 4y\left(-\frac{\sin(x)\cos(x)}{2} + \frac{x}{2}\right)$
- Solve for $f_1(y)$

  $f_1(y) = \frac{y^2(2\sin(x)\cos(x) - \sin(2x))}{2}$
- Substitute $f_1(y)$ into equation for $F(x, y)$

  $F(x, y) = xy + 3x + 2y^2\left(-\frac{\sin(x)\cos(x)}{2} + \frac{x}{2}\right) + \frac{y^2(2\sin(x)\cos(x) - \sin(2x))}{2}$
- Substitute $F(x, y)$ into the solution of the ODE

  $xy + 3x + 2y^2\left(-\frac{\sin(x)\cos(x)}{2} + \frac{x}{2}\right) + \frac{y^2(2\sin(x)\cos(x) - \sin(2x))}{2} = c_1$

- Solve for y

$$\left\{y=\frac{x+\sqrt{-2c_1\sin(2x)+6x\sin(2x)+4c_1x-11x^2}}{-2x+\sin(2x)}, y=-\frac{-x+\sqrt{-2c_1\sin(2x)+6x\sin(2x)+4c_1x-11x^2}}{-2x+\sin(2x)}\right\}$$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
trying homogeneous types:
trying Chini
differential order: 1; looking for linear symmetries
trying exact
<- exact successful`
```

✓ Solution by Maple
Time used: 0.016 (sec). Leaf size: 83

```
dsolve((3+y(x)+2*y(x)^2*sin(x)^2)+(x+2*x*y(x)-y(x)*sin(2*x))*diff(y(x),x)=0,y(x), singsol=al
```

$$y(x)=\frac{x+\sqrt{(2c_1+6x)\sin(2x)-11x^2-4c_1x}}{\sin(2x)-2x}$$
$$y(x)=\frac{x-\sqrt{(2c_1+6x)\sin(2x)-11x^2-4c_1x}}{\sin(2x)-2x}$$

✓ Solution by Mathematica
Time used: 1.378 (sec). Leaf size: 97

```
DSolve[(3+y[x]+2*y[x]^2*Sin[x]^2)+(x+2*x*y[x]-y[x]*Sin[2*x])*y'[x]==0,y[x],x,IncludeSingular
```

$$y(x)\to\frac{x-i\sqrt{x(11x+2c_1)-(6x+c_1)\sin(2x)}}{\sin(2x)-2x}$$
$$y(x)\to\frac{x+i\sqrt{x(11x+2c_1)-(6x+c_1)\sin(2x)}}{\sin(2x)-2x}$$

1.27 problem 27

Internal problem ID [3172]
Internal file name [`OUTPUT/2664_Sunday_June_05_2022_08_38_31_AM_86629003/index.tex`]

Book: Differential equations for engineers by Wei-Chau XIE, Cambridge Press 2010
Section: Chapter 2. First-Order and Simple Higher-Order Differential Equations. Page 78
Problem number: 27.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"homogeneousTypeD2", "first_order_ode_lie_symmetry_calculated"**

Maple gives the following as the ode type

```
[[_homogeneous, `class A`], _rational, _dAlembert]
```

$$\boxed{2yx + (x^2 + 2yx + y^2)\, y' = 0}$$

1.27.1 Solving as homogeneousTypeD2 ode

Using the change of variables $y = u(x)\, x$ on the above ode results in new ode in $u(x)$

$$2u(x)\, x^2 + \left(x^2 + 2u(x)\, x^2 + u(x)^2\, x^2\right) \left(u'(x)\, x + u(x)\right) = 0$$

In canonical form the ODE is

$$\begin{aligned} u' &= F(x,u) \\ &= f(x)g(u) \\ &= -\frac{u(u^2 + 2u + 3)}{x\,(u+1)^2} \end{aligned}$$

Where $f(x) = -\frac{1}{x}$ and $g(u) = \frac{u(u^2+2u+3)}{(u+1)^2}$. Integrating both sides gives

$$\frac{1}{\frac{u(u^2+2u+3)}{(u+1)^2}}\, du = -\frac{1}{x}\, dx$$

$$\int \frac{1}{\frac{u(u^2+2u+3)}{(u+1)^2}}\,du = \int -\frac{1}{x}\,dx$$

$$\frac{\ln(u)}{3} + \frac{\ln(u^2+2u+3)}{3} + \frac{\sqrt{2}\arctan\left(\frac{(2u+2)\sqrt{2}}{4}\right)}{3} = -\ln(x) + c_2$$

The solution is

$$\frac{\ln(u(x))}{3} + \frac{\ln\left(u(x)^2+2u(x)+3\right)}{3} + \frac{\sqrt{2}\arctan\left(\frac{(2u(x)+2)\sqrt{2}}{4}\right)}{3} + \ln(x) - c_2 = 0$$

Replacing $u(x)$ in the above solution by $\frac{y}{x}$ results in the solution for y in implicit form

$$\frac{\ln\left(\frac{y}{x}\right)}{3} + \frac{\ln\left(\frac{y^2}{x^2}+\frac{2y}{x}+3\right)}{3} + \frac{\sqrt{2}\arctan\left(\frac{\left(\frac{2y}{x}+2\right)\sqrt{2}}{4}\right)}{3} + \ln(x) - c_2 = 0$$

$$\frac{\ln\left(\frac{y}{x}\right)}{3} + \frac{\ln\left(\frac{y^2}{x^2}+\frac{2y}{x}+3\right)}{3} + \frac{\sqrt{2}\arctan\left(\frac{(y+x)\sqrt{2}}{2x}\right)}{3} + \ln(x) - c_2 = 0$$

Summary

The solution(s) found are the following

$$\frac{\ln\left(\frac{y}{x}\right)}{3} + \frac{\ln\left(\frac{y^2}{x^2}+\frac{2y}{x}+3\right)}{3} + \frac{\sqrt{2}\arctan\left(\frac{(y+x)\sqrt{2}}{2x}\right)}{3} + \ln(x) - c_2 = 0 \tag{1}$$

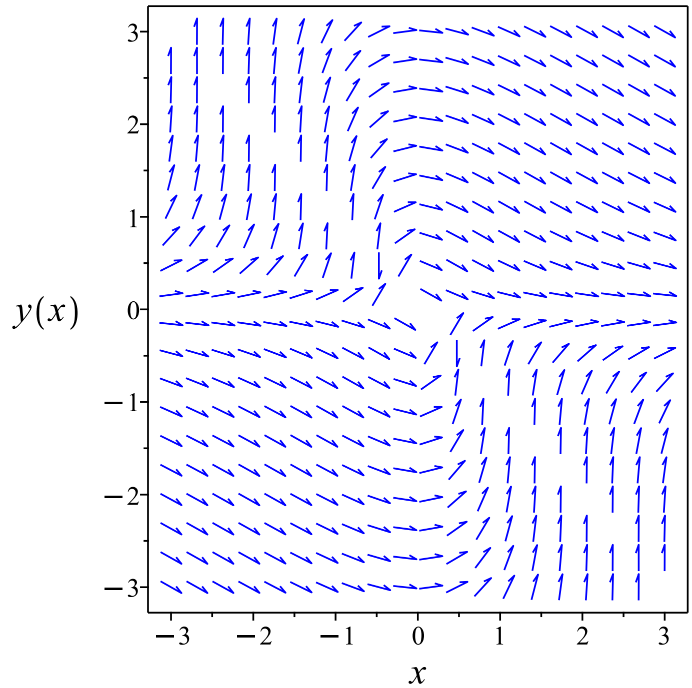


Figure 43: Slope field plot

Verification of solutions

$$\frac{\ln\left(\frac{y}{x}\right)}{3}+\frac{\ln\left(\frac{y^2}{x^2}+\frac{2y}{x}+3\right)}{3}+\frac{\sqrt{2}\,\arctan\left(\frac{(y+x)\sqrt{2}}{2x}\right)}{3}+\ln(x)-c_2=0$$

Verified OK.

1.27.2 Solving as first order ode lie symmetry calculated ode

Writing the ode as

$$y'=-\frac{2yx}{x^2+2xy+y^2}$$
$$y'=\omega(x,y)$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x+\omega(\eta_y-\xi_x)-\omega^2\xi_y-\omega_x\xi-\omega_y\eta=0 \tag{A}$$

The type of this ode is not in the lookup table. To determine ξ,η then (A) is solved using ansatz. Making bivariate polynomials of degree 1 to use as anstaz gives

$$\xi=xa_2+ya_3+a_1 \tag{1E}$$
$$\eta=xb_2+yb_3+b_1 \tag{2E}$$

Where the unknown coefficients are

$$\{a_1, a_2, a_3, b_1, b_2, b_3\}$$

Substituting equations (1E,2E) and ω into (A) gives

$$\begin{aligned} & b_2 - \frac{2yx(b_3 - a_2)}{x^2 + 2xy + y^2} - \frac{4y^2x^2a_3}{(x^2 + 2xy + y^2)^2} \\ & - \left(-\frac{2y}{x^2 + 2xy + y^2} + \frac{2yx(2x + 2y)}{(x^2 + 2xy + y^2)^2}\right)(xa_2 + ya_3 + a_1) \\ & - \left(-\frac{2x}{x^2 + 2xy + y^2} + \frac{2yx(2x + 2y)}{(x^2 + 2xy + y^2)^2}\right)(xb_2 + yb_3 + b_1) = 0 \end{aligned} \tag{5E}$$

Putting the above in normal form gives

$$\frac{3x^4b_2 + 4x^3yb_2 + 4x^2y^2a_2 - 6y^2x^2a_3 + 4x^2y^2b_2 - 4x^2y^2b_3 + 4x\,y^3a_2 + 4x\,y^3b_2 - 4x\,y^3b_3 + 2y^4a_3 + y^4b_2 +}{(x^2 + 2xy + y^2)^2}$$

$$= 0$$

Setting the numerator to zero gives

$$\begin{aligned} & 3x^4b_2 + 4x^3yb_2 + 4x^2y^2a_2 - 6y^2x^2a_3 + 4x^2y^2b_2 - 4x^2y^2b_3 + 4x\,y^3a_2 \\ & + 4x\,y^3b_2 - 4x\,y^3b_3 + 2y^4a_3 + y^4b_2 + 2x^3b_1 - 2x^2ya_1 - 2x\,y^2b_1 + 2y^3a_1 = 0 \end{aligned} \tag{6E}$$

Looking at the above PDE shows the following are all the terms with $\{x, y\}$ in them.

$$\{x, y\}$$

The following substitution is now made to be able to collect on all terms with $\{x, y\}$ in them

$$\{x = v_1, y = v_2\}$$

The above PDE (6E) now becomes

$$\begin{aligned} & 4a_2v_1^2v_2^2 + 4a_2v_1v_2^3 - 6a_3v_1^2v_2^2 + 2a_3v_2^4 + 3b_2v_1^4 + 4b_2v_1^3v_2 + 4b_2v_1^2v_2^2 + 4b_2v_1v_2^3 \\ & + b_2v_2^4 - 4b_3v_1^2v_2^2 - 4b_3v_1v_2^3 - 2a_1v_1^2v_2 + 2a_1v_2^3 + 2b_1v_1^3 - 2b_1v_1v_2^2 = 0 \end{aligned} \tag{7E}$$

Collecting the above on the terms v_i introduced, and these are

$$\{v_1, v_2\}$$

Equation (7E) now becomes

$$3b_2v_1^4 + 4b_2v_1^3v_2 + 2b_1v_1^3 + (4a_2 - 6a_3 + 4b_2 - 4b_3)\, v_1^2v_2^2 - 2a_1v_1^2v_2 \\ + (4a_2 + 4b_2 - 4b_3)\, v_1v_2^3 - 2b_1v_1v_2^2 + (2a_3 + b_2)\, v_2^4 + 2a_1v_2^3 = 0 \tag{8E}$$

Setting each coefficients in (8E) to zero gives the following equations to solve

$$\begin{aligned} -2a_1 &= 0 \\ 2a_1 &= 0 \\ -2b_1 &= 0 \\ 2b_1 &= 0 \\ 3b_2 &= 0 \\ 4b_2 &= 0 \\ 2a_3 + b_2 &= 0 \\ 4a_2 + 4b_2 - 4b_3 &= 0 \\ 4a_2 - 6a_3 + 4b_2 - 4b_3 &= 0 \end{aligned}$$

Solving the above equations for the unknowns gives

$$\begin{aligned} a_1 &= 0 \\ a_2 &= b_3 \\ a_3 &= 0 \\ b_1 &= 0 \\ b_2 &= 0 \\ b_3 &= b_3 \end{aligned}$$

Substituting the above solution in the anstaz (1E,2E) (using 1 as arbitrary value for any unknown in the RHS) gives

$$\begin{aligned} \xi &= x \\ \eta &= y \end{aligned}$$

Shifting is now applied to make $\xi = 0$ in order to simplify the rest of the computation

$$
\begin{aligned}
\eta &= \eta - \omega(x,y)\,\xi \\
&= y - \left(-\frac{2yx}{x^2+2xy+y^2}\right)(x) \\
&= \frac{3x^2y+2x\,y^2+y^3}{x^2+2xy+y^2} \\
\xi &= 0
\end{aligned}
$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x,y) \to (R,S)$ where (R,S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$
\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}
$$

The above comes from the requirements that $\left(\xi\frac{\partial}{\partial x} + \eta\frac{\partial}{\partial y}\right) S(x,y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$
R = x
$$

S is found from

$$
\begin{aligned}
S &= \int \frac{1}{\eta} dy \\
&= \int \frac{1}{\frac{3x^2y+2x\,y^2+y^3}{x^2+2xy+y^2}} dy
\end{aligned}
$$

Which results in

$$
S = \frac{\ln\left(3x^2+2xy+y^2\right)}{3} + \frac{\sqrt{2}\,\arctan\left(\frac{(2x+2y)\sqrt{2}}{4x}\right)}{3} + \frac{\ln(y)}{3}
$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$
\frac{dS}{dR} = \frac{S_x + \omega(x,y)S_y}{R_x + \omega(x,y)R_y} \tag{2}
$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x,y)$ is the right hand side of the original ode given by

$$
\omega(x,y) = -\frac{2yx}{x^2+2xy+y^2}
$$

Evaluating all the partial derivatives gives

$$
\begin{aligned}
R_x &= 1 \\
R_y &= 0 \\
S_x &= \frac{2x}{3x^2 + 2xy + y^2} \\
S_y &= \frac{(y+x)^2}{3x^2y + 2x\,y^2 + y^3}
\end{aligned}
$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$
\frac{dS}{dR} = 0 \tag{2A}
$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$
\frac{dS}{dR} = 0
$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$
S(R) = c_1 \tag{4}
$$

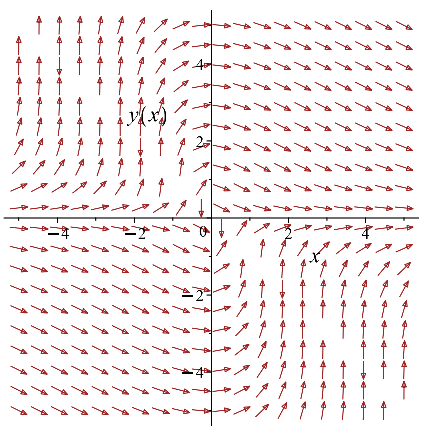
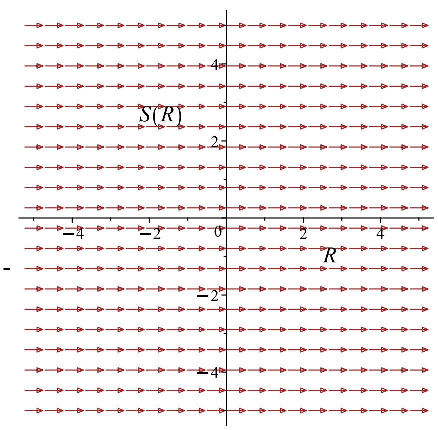
To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$
\frac{\ln(y^2 + 2yx + 3x^2)}{3} + \frac{\sqrt{2}\arctan\left(\frac{(y+x)\sqrt{2}}{2x}\right)}{3} + \frac{\ln(y)}{3} = c_1
$$

Which simplifies to

$$
\frac{\ln(y^2 + 2yx + 3x^2)}{3} + \frac{\sqrt{2}\arctan\left(\frac{(y+x)\sqrt{2}}{2x}\right)}{3} + \frac{\ln(y)}{3} = c_1
$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in x, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dx} = -\frac{2yx}{x^2+2xy+y^2}$ 	$R = x$ $S = \frac{\ln(3x^2+2xy+y^2)}{3}$ -	$\frac{dS}{dR} = 0$ 

Summary

The solution(s) found are the following

$$\frac{\ln(y^2+2yx+3x^2)}{3} + \frac{\sqrt{2}\arctan\left(\frac{(y+x)\sqrt{2}}{2x}\right)}{3} + \frac{\ln(y)}{3} = c_1 \tag{1}$$

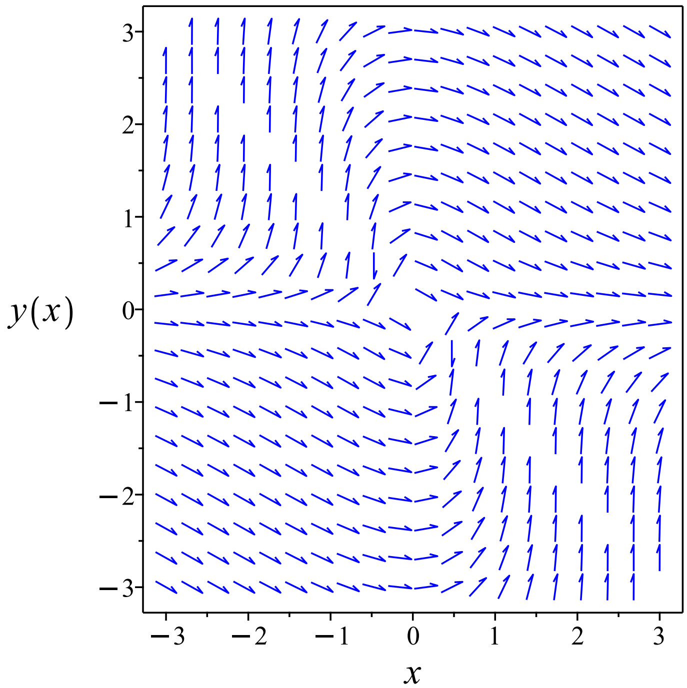


Figure 44: Slope field plot

Verification of solutions

$$\frac{\ln\left(y^2+2yx+3x^2\right)}{3}+\frac{\sqrt{2}\,\arctan\left(\frac{(y+x)\sqrt{2}}{2x}\right)}{3}+\frac{\ln\left(y\right)}{3}=c_1$$

Verified OK.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
trying homogeneous types:
trying homogeneous D
<- homogeneous successful`
```

✓ Solution by Maple

Time used: 0.328 (sec). Leaf size: 53

```
dsolve((2*x*y(x))+(x^2+2*x*y(x)+y(x)^2)*diff(y(x),x)=0,y(x), singsol=all)
```

$$y(x) = x\Big(-1 + \sqrt{2}\,\tan\left(\operatorname{RootOf}\left(2\sqrt{2}\,\ln\left(-\sec(\_Z)^2\left(\sqrt{2}-2\tan(\_Z)\right)x^3\right)+\sqrt{2}\,\ln(2)+6\sqrt{2}\,c_1+4\_Z\right)\right)\Big)$$

✓ Solution by Mathematica

Time used: 0.191 (sec). Leaf size: 62

```
DSolve[(2*x*y[x])+(x^2+2*x*y[x]+y[x]^2)*y'[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$\text{Solve}\left[\frac{1}{3}\left(\sqrt{2}\arctan\left(\frac{\frac{y(x)}{x}+1}{\sqrt{2}}\right)+\log\left(\frac{y(x)^2}{x^2}+\frac{2y(x)}{x}+3\right)+\log\left(\frac{y(x)}{x}\right)\right) = -\log(x)+c_1, y(x)\right]$$

1.28 problem 28

Internal problem ID [3173]
Internal file name [`OUTPUT/2665_Sunday_June_05_2022_08_38_31_AM_44988939/index.tex`]

Book: Differential equations for engineers by Wei-Chau XIE, Cambridge Press 2010
Section: Chapter 2. First-Order and Simple Higher-Order Differential Equations. Page 78
Problem number: 28.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"exactWithIntegrationFactor"**

Maple gives the following as the ode type

```
[`y=_G(x,y')`]
```

$$\boxed{-\sin(y)^2 + x\sin(2y)\, y' = -x^2}$$

1.28.1 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x,y) + N(x,y)\frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x,y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y) = 0$$

Hence

$$\frac{\partial\phi}{\partial x} + \frac{\partial\phi}{\partial y}\frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$\frac{\partial\phi}{\partial x} = M$$
$$\frac{\partial\phi}{\partial y} = N$$

But since $\frac{\partial^2\phi}{\partial x\partial y} = \frac{\partial^2\phi}{\partial y\partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x, y)$ but at least we know now that we can do that since the condition $\frac{\partial^2\phi}{\partial x\partial y} = \frac{\partial^2\phi}{\partial y\partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x, y)\,\mathrm{d}x + N(x, y)\,\mathrm{d}y = 0 \tag{1A}$$

Therefore

$$\begin{aligned}(\sin(2y)\,x)\,\mathrm{d}y &= \left(-x^2 + \sin(y)^2\right)\mathrm{d}x\\ \left(x^2 - \sin(y)^2\right)\mathrm{d}x + (\sin(2y)\,x)\,\mathrm{d}y &= 0 \end{aligned} \tag{2A}$$

Comparing (1A) and (2A) shows that

$$\begin{aligned}M(x, y) &= x^2 - \sin(y)^2\\ N(x, y) &= \sin(2y)\,x\end{aligned}$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Using result found above gives

$$\begin{aligned}\frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}\left(x^2 - \sin(y)^2\right)\\ &= -\sin(2y)\end{aligned}$$

And

$$\begin{aligned}\frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}(\sin(2y)\,x)\\ &= \sin(2y)\end{aligned}$$

Since $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$, then the ODE is <u>not exact</u>. Since the ODE is not exact, we will try to find an integrating factor to make it exact. Let

$$\begin{aligned}A &= \frac{1}{N}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right)\\ &= \frac{\csc(2y)}{x}\left((-2\cos(y)\sin(y)) - (\sin(2y))\right)\\ &= -\frac{2}{x}\end{aligned}$$

Since A does not depend on y, then it can be used to find an integrating factor. The integrating factor μ is

$$\begin{aligned}\mu &= e^{\int A \, dx} \\ &= e^{\int -\frac{2}{x} \, dx}\end{aligned}$$

The result of integrating gives

$$\begin{aligned}\mu &= e^{-2\ln(x)} \\ &= \frac{1}{x^2}\end{aligned}$$

M and N are multiplied by this integrating factor, giving new M and new N which are called $\overline{M}$ and $\overline{N}$ for now so not to confuse them with the original M and N.

$$\begin{aligned}\overline{M} &= \mu M \\ &= \frac{1}{x^2}\left(x^2 - \sin(y)^2\right) \\ &= \frac{x^2 - \sin(y)^2}{x^2}\end{aligned}$$

And

$$\begin{aligned}\overline{N} &= \mu N \\ &= \frac{1}{x^2}(\sin(2y)\, x) \\ &= \frac{\sin(2y)}{x}\end{aligned}$$

Now a modified ODE is ontained from the original ODE, which is exact and can be solved. The modified ODE is

$$\begin{aligned}\overline{M} + \overline{N}\frac{dy}{dx} &= 0 \\ \left(\frac{x^2 - \sin(y)^2}{x^2}\right) + \left(\frac{\sin(2y)}{x}\right)\frac{dy}{dx} &= 0\end{aligned}$$

The following equations are now set up to solve for the function $\phi(x, y)$

$$\frac{\partial \phi}{\partial x} = \overline{M} \tag{1}$$

$$\frac{\partial \phi}{\partial y} = \overline{N} \tag{2}$$

Integrating (1) w.r.t. x gives

$$\int \frac{\partial \phi}{\partial x} \mathrm{d}x = \int \overline{M} \mathrm{d}x$$
$$\int \frac{\partial \phi}{\partial x} \mathrm{d}x = \int \frac{x^2 - \sin(y)^2}{x^2} \mathrm{d}x$$
$$\phi = x + \frac{\sin(y)^2}{x} + f(y) \tag{3}$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$\frac{\partial \phi}{\partial y} = \frac{2\sin(y)\cos(y)}{x} + f'(y) \tag{4}$$
$$= \frac{\sin(2y)}{x} + f'(y)$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = \frac{\sin(2y)}{x}$. Therefore equation (4) becomes

$$\frac{\sin(2y)}{x} = \frac{\sin(2y)}{x} + f'(y) \tag{5}$$

Solving equation (5) for $f'(y)$ gives

$$f'(y) = 0$$

Therefore

$$f(y) = c_1$$

Where c_1 is constant of integration. Substituting this result for $f(y)$ into equation (3) gives ϕ

$$\phi = x + \frac{\sin(y)^2}{x} + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = x + \frac{\sin(y)^2}{x}$$

Summary

The solution(s) found are the following

$$x + \frac{\sin(y)^2}{x} = c_1 \tag{1}$$

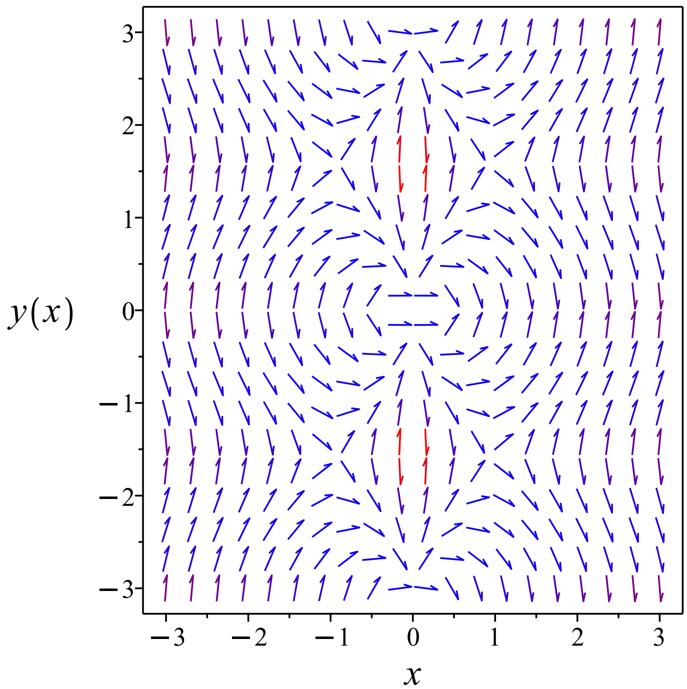


Figure 45: Slope field plot

Verification of solutions

$$x + \frac{\sin(y)^2}{x} = c_1$$

Verified OK.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
trying homogeneous types:
trying Chini
differential order: 1; looking for linear symmetries
trying exact
<- exact successful`
```

✓ Solution by Maple

Time used: 0.032 (sec). Leaf size: 27

```
dsolve((x^2-sin(y(x))^2)+(x*sin(2*y(x)))*diff(y(x),x)=0,y(x), singsol=all)
```

$$y(x) = \arcsin\left(\sqrt{-\left(c_1 + x\right) x}\right)$$
$$y(x) = -\arcsin\left(\sqrt{-\left(c_1 + x\right) x}\right)$$

✓ Solution by Mathematica

Time used: 6.502 (sec). Leaf size: 39

```
DSolve[(x^2-Sin[y[x]]^2)+(x*Sin[2*y[x]])*y'[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to -\arcsin\left(\sqrt{-x(x + 2c_1)}\right)$$
$$y(x) \to \arcsin\left(\sqrt{-x(x + 2c_1)}\right)$$

1.29 problem 29

Internal problem ID [3174]
Internal file name [`OUTPUT/2666_Sunday_June_05_2022_08_38_33_AM_32086448/index.tex`]

Book: Differential equations for engineers by Wei-Chau XIE, Cambridge Press 2010
Section: Chapter 2. First-Order and Simple Higher-Order Differential Equations. Page 78
Problem number: 29.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"homogeneousTypeD2", "exactWithIntegrationFactor"**

Maple gives the following as the ode type

```
[[_homogeneous, `class D`], _rational, [_Abel, `2nd type`, `
   class A`]]
```

$$\boxed{y(2x - y + 2) + 2(-y + x)\, y' = 0}$$

1.29.1 Solving as homogeneousTypeD2 ode

Using the change of variables $y = u(x)\, x$ on the above ode results in new ode in $u(x)$

$$u(x)\, x(2x - u(x)\, x + 2) + 2(-u(x)\, x + x)\, (u'(x)\, x + u(x)) = 0$$

In canonical form the ODE is

$$\begin{aligned} u' &= F(x,u) \\ &= f(x)g(u) \\ &= -\frac{(x+2)\, u(u-2)}{x\,(2u-2)} \end{aligned}$$

Where $f(x) = -\frac{x+2}{x}$ and $g(u) = \frac{u(u-2)}{2u-2}$. Integrating both sides gives

$$\frac{1}{\frac{u(u-2)}{2u-2}}\, du = -\frac{x+2}{x}\, dx$$

$$\int \frac{1}{\frac{u(u-2)}{2u-2}}\, du = \int -\frac{x+2}{x}\, dx$$
$$\ln\left(u(u-2)\right) = -x - 2\ln\left(x\right) + c_2$$

Raising both side to exponential gives

$$u(u-2) = e^{-x-2\ln(x)+c_2}$$

Which simplifies to

$$u(u-2) = c_3 e^{-x-2\ln(x)}$$

Which simplifies to

$$u(x)\left(u(x)-2\right) = \frac{c_3 e^{-x} e^{c_2}}{x^2}$$

The solution is

$$u(x)\left(u(x)-2\right) = \frac{c_3 e^{-x} e^{c_2}}{x^2}$$

Replacing $u(x)$ in the above solution by $\frac{y}{x}$ results in the solution for y in implicit form

$$\frac{y\left(-2+\frac{y}{x}\right)}{x} = \frac{c_3 e^{-x} e^{c_2}}{x^2}$$
$$\frac{-2yx+y^2}{x^2} = \frac{c_3 e^{-x+c_2}}{x^2}$$

Which simplifies to

$$-2yx + y^2 = c_3 e^{-x+c_2}$$

Summary

The solution(s) found are the following

$$-2yx + y^2 = c_3 e^{-x+c_2} \tag{1}$$

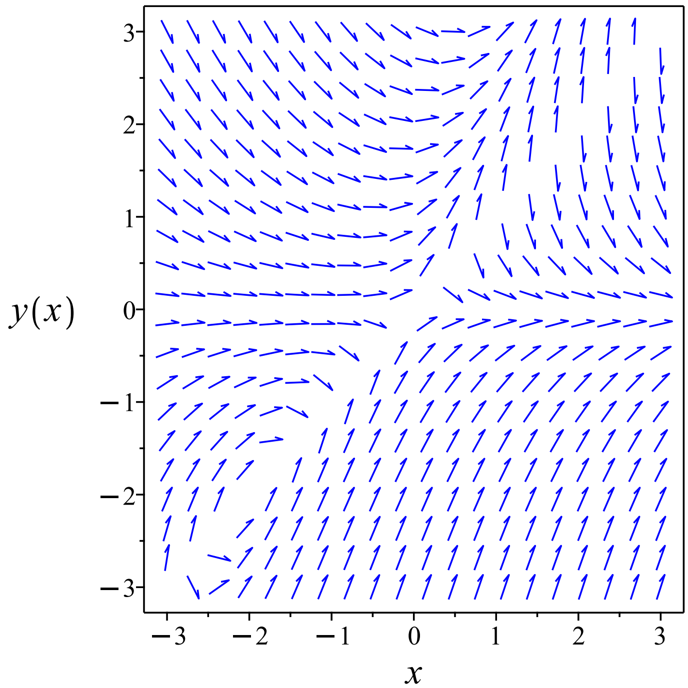


Figure 46: Slope field plot

Verification of solutions

$$-2yx + y^2 = c_3 e^{-x+c_2}$$

Verified OK.

1.29.2 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x,y) + N(x,y)\frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x,y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y) = 0$$

Hence

$$\frac{\partial \phi}{\partial x} + \frac{\partial \phi}{\partial y}\frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$\frac{\partial \phi}{\partial x} = M$$
$$\frac{\partial \phi}{\partial y} = N$$

But since $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x, y)$ but at least we know now that we can do that since the condition $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x, y)\, \mathrm{d}x + N(x, y)\, \mathrm{d}y = 0 \tag{1A}$$

Therefore

$$\begin{aligned} (-2y + 2x)\, \mathrm{d}y &= (-y(2x - y + 2))\, \mathrm{d}x \\ (y(2x - y + 2))\, \mathrm{d}x + (-2y + 2x)\, \mathrm{d}y &= 0 \end{aligned} \tag{2A}$$

Comparing (1A) and (2A) shows that

$$\begin{aligned} M(x, y) &= y(2x - y + 2) \\ N(x, y) &= -2y + 2x \end{aligned}$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Using result found above gives

$$\begin{aligned} \frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}(y(2x - y + 2)) \\ &= 2x - 2y + 2 \end{aligned}$$

And

$$\begin{aligned} \frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}(-2y + 2x) \\ &= 2 \end{aligned}$$

Since $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$, then the ODE is <u>not exact</u>. Since the ODE is not exact, we will try to find an integrating factor to make it exact. Let

$$\begin{aligned} A &= \frac{1}{N}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right) \\ &= \frac{1}{-2y+2x}((2x-2y+2)-(2)) \\ &= 1 \end{aligned}$$

Since A does not depend on y, then it can be used to find an integrating factor. The integrating factor μ is

$$\begin{aligned} \mu &= e^{\int A \, \mathrm{d}x} \\ &= e^{\int 1 \, \mathrm{d}x} \end{aligned}$$

The result of integrating gives

$$\begin{aligned} \mu &= e^{x} \\ &= \mathrm{e}^{x} \end{aligned}$$

M and N are multiplied by this integrating factor, giving new M and new N which are called $\overline{M}$ and $\overline{N}$ for now so not to confuse them with the original M and N.

$$\begin{aligned} \overline{M} &= \mu M \\ &= \mathrm{e}^{x}(y(2x-y+2)) \\ &= y(2x-y+2)\,\mathrm{e}^{x} \end{aligned}$$

And

$$\begin{aligned} \overline{N} &= \mu N \\ &= \mathrm{e}^{x}(-2y+2x) \\ &= 2(-y+x)\,\mathrm{e}^{x} \end{aligned}$$

Now a modified ODE is ontained from the original ODE, which is exact and can be solved. The modified ODE is

$$\begin{aligned} \overline{M} + \overline{N}\frac{\mathrm{d}y}{\mathrm{d}x} &= 0 \\ \left(y(2x-y+2)\,\mathrm{e}^{x}\right) + \left(2(-y+x)\,\mathrm{e}^{x}\right)\frac{\mathrm{d}y}{\mathrm{d}x} &= 0 \end{aligned}$$

The following equations are now set up to solve for the function $\phi(x,y)$

$$\frac{\partial \phi}{\partial x} = \overline{M} \tag{1}$$

$$\frac{\partial \phi}{\partial y} = \overline{N} \tag{2}$$

Integrating (1) w.r.t. x gives

$$\int \frac{\partial \phi}{\partial x} \,\mathrm{d}x = \int \overline{M} \,\mathrm{d}x$$

$$\int \frac{\partial \phi}{\partial x} \,\mathrm{d}x = \int y(2x - y + 2)\,\mathrm{e}^{x} \,\mathrm{d}x$$

$$\phi = (2x - y)\,y\,\mathrm{e}^{x} + f(y) \tag{3}$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$\frac{\partial \phi}{\partial y} = -\mathrm{e}^{x} y + (2x - y)\,\mathrm{e}^{x} + f'(y) \tag{4}$$

$$= 2(-y + x)\,\mathrm{e}^{x} + f'(y)$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = 2(-y + x)\,\mathrm{e}^{x}$. Therefore equation (4) becomes

$$2(-y + x)\,\mathrm{e}^{x} = 2(-y + x)\,\mathrm{e}^{x} + f'(y) \tag{5}$$

Solving equation (5) for $f'(y)$ gives

$$f'(y) = 0$$

Therefore

$$f(y) = c_1$$

Where c_1 is constant of integration. Substituting this result for $f(y)$ into equation (3) gives ϕ

$$\phi = (2x - y)\,y\,\mathrm{e}^{x} + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = (2x - y)\,y\,\mathrm{e}^{x}$$

Summary

The solution(s) found are the following

$$(2x - y)\, y\, \mathrm{e}^{x} = c_1 \tag{1}$$

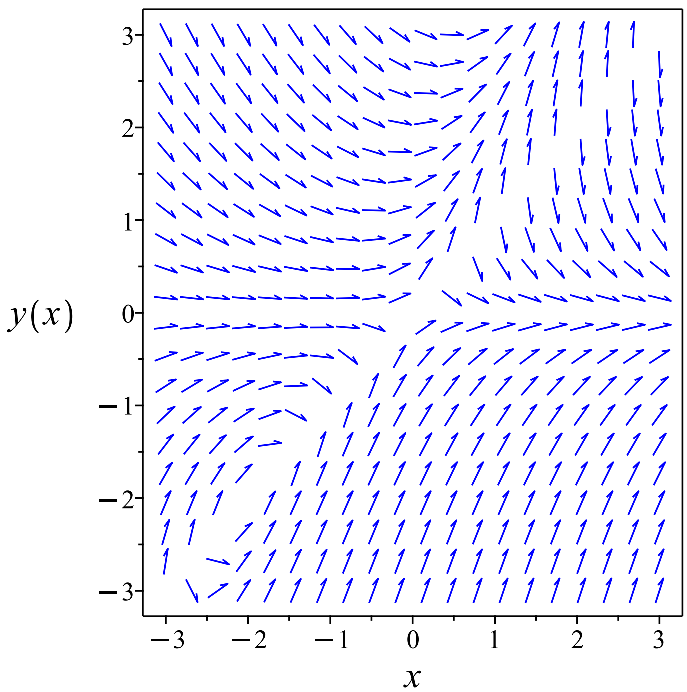


Figure 47: Slope field plot

Verification of solutions

$$(2x - y)\, y\, \mathrm{e}^{x} = c_1$$

Verified OK.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
trying homogeneous types:
trying homogeneous D
<- homogeneous successful`
```

✓ Solution by Maple

Time used: 0.063 (sec). Leaf size: 64

```
dsolve(y(x)*(2*x-y(x)+2)+2*(x-y(x))*diff(y(x),x)=0,y(x), singsol=all)
```

$$y(x) = \frac{c_1 x - \sqrt{e^x c_1 (e^x c_1 x^2 + 1)}\, e^{-x}}{c_1}$$
$$y(x) = \frac{c_1 x + \sqrt{e^x c_1 (e^x c_1 x^2 + 1)}\, e^{-x}}{c_1}$$

✓ Solution by Mathematica

Time used: 43.224 (sec). Leaf size: 125

```
DSolve[y[x]*(2*x-y[x]+2)+2*(x-y[x])*y'[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to x - e^{-x}\sqrt{e^x (e^x x^2 - e^{2c_1})}$$
$$y(x) \to x + e^{-x}\sqrt{e^x (e^x x^2 - e^{2c_1})}$$
$$y(x) \to x - e^{-x}\sqrt{e^{2x} x^2}$$
$$y(x) \to e^{-x}\sqrt{e^{2x} x^2} + x$$

1.30 problem 30

Internal problem ID [3175]
Internal file name [`OUTPUT/2667_Sunday_June_05_2022_08_38_34_AM_86752118/index.tex`]

Book: Differential equations for engineers by Wei-Chau XIE, Cambridge Press 2010
Section: Chapter 2. First-Order and Simple Higher-Order Differential Equations. Page 78
Problem number: 30.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"exactWithIntegrationFactor"**

Maple gives the following as the ode type

```
[_rational, [_Abel, `2nd type`, `class B`]]
```

$$\boxed{4yx + 3y^2 + x(x + 2y)\, y' = x}$$

1.30.1 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x, y) + N(x, y) \frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x, y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x, y) = 0$$

Hence

$$\frac{\partial \phi}{\partial x} + \frac{\partial \phi}{\partial y}\frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$\begin{aligned}
\frac{\partial \phi}{\partial x} &= M \\
\frac{\partial \phi}{\partial y} &= N
\end{aligned}$$

But since $\frac{\partial^2\phi}{\partial x\partial y} = \frac{\partial^2\phi}{\partial y\partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x,y)$ but at least we know now that we can do that since the condition $\frac{\partial^2\phi}{\partial x\partial y} = \frac{\partial^2\phi}{\partial y\partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x,y)\,\mathrm{d}x + N(x,y)\,\mathrm{d}y = 0 \tag{1A}$$

Therefore

$$\begin{aligned}(x(x+2y))\,\mathrm{d}y &= \left(-4xy - 3y^2 + x\right)\mathrm{d}x\\ \left(4xy + 3y^2 - x\right)\mathrm{d}x + (x(x+2y))\,\mathrm{d}y &= 0 \qquad (2A)\end{aligned}$$

Comparing (1A) and (2A) shows that

$$\begin{aligned}M(x,y) &= 4xy + 3y^2 - x\\ N(x,y) &= x(x+2y)\end{aligned}$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Using result found above gives

$$\begin{aligned}\frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}\left(4xy + 3y^2 - x\right)\\ &= 4x + 6y\end{aligned}$$

And

$$\begin{aligned}\frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}(x(x+2y))\\ &= 2x + 2y\end{aligned}$$

Since $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$, then the ODE is not exact. Since the ODE is not exact, we will try to find an integrating factor to make it exact. Let

$$\begin{aligned}A &= \frac{1}{N}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right)\\ &= \frac{1}{x\,(x+2y)}((4x+6y) - (2x+2y))\\ &= \frac{2}{x}\end{aligned}$$

Since A does not depend on y, then it can be used to find an integrating factor. The integrating factor μ is

$$\begin{aligned}\mu &= e^{\int A \, \mathrm{d}x} \\ &= e^{\int \frac{2}{x} \, \mathrm{d}x}\end{aligned}$$

The result of integrating gives

$$\begin{aligned}\mu &= e^{2\ln(x)} \\ &= x^2\end{aligned}$$

M and N are multiplied by this integrating factor, giving new M and new N which are called $\overline{M}$ and $\overline{N}$ for now so not to confuse them with the original M and N.

$$\begin{aligned}\overline{M} &= \mu M \\ &= x^2\left(4xy + 3y^2 - x\right) \\ &= (4y - 1)\, x^3 + 3y^2x^2\end{aligned}$$

And

$$\begin{aligned}\overline{N} &= \mu N \\ &= x^2(x(x + 2y)) \\ &= x^3(x + 2y)\end{aligned}$$

Now a modified ODE is ontained from the original ODE, which is exact and can be solved. The modified ODE is

$$\begin{aligned}\overline{M} + \overline{N}\frac{\mathrm{d}y}{\mathrm{d}x} &= 0 \\ \left((4y - 1)\, x^3 + 3y^2x^2\right) + \left(x^3(x + 2y)\right)\frac{\mathrm{d}y}{\mathrm{d}x} &= 0\end{aligned}$$

The following equations are now set up to solve for the function $\phi(x, y)$

$$\frac{\partial \phi}{\partial x} = \overline{M} \tag{1}$$

$$\frac{\partial \phi}{\partial y} = \overline{N} \tag{2}$$

Integrating (1) w.r.t. x gives

$$\begin{aligned}\int \frac{\partial \phi}{\partial x} \, \mathrm{d}x &= \int \overline{M} \, \mathrm{d}x \\ \int \frac{\partial \phi}{\partial x} \, \mathrm{d}x &= \int (4y - 1)\, x^3 + 3y^2x^2 \, \mathrm{d}x\end{aligned}$$

$$\phi = \frac{(4y - 1)\, x^4}{4} + x^3y^2 + f(y) \tag{3}$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$\begin{aligned} \frac{\partial \phi}{\partial y} &= x^4 + 2x^3 y + f'(y) \\ &= x^3(x + 2y) + f'(y) \end{aligned} \tag{4}$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = x^3(x + 2y)$. Therefore equation (4) becomes

$$x^3(x + 2y) = x^3(x + 2y) + f'(y) \tag{5}$$

Solving equation (5) for $f'(y)$ gives

$$f'(y) = 0$$

Therefore

$$f(y) = c_1$$

Where c_1 is constant of integration. Substituting this result for $f(y)$ into equation (3) gives ϕ

$$\phi = \frac{(4y - 1)\, x^4}{4} + x^3 y^2 + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = \frac{(4y - 1)\, x^4}{4} + x^3 y^2$$

Summary

The solution(s) found are the following

$$\frac{(4y - 1)\, x^4}{4} + y^2 x^3 = c_1 \tag{1}$$

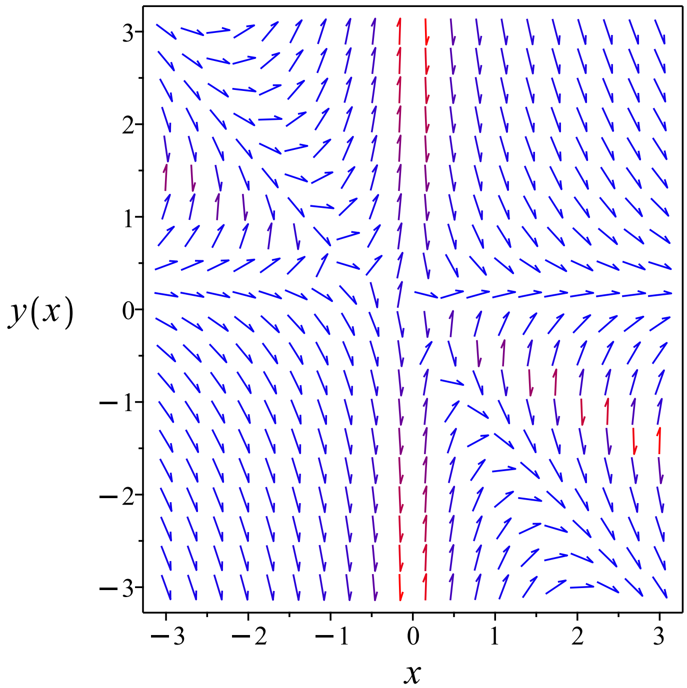


Figure 48: Slope field plot

Verification of solutions

$$\frac{(4y-1)\,x^4}{4}+y^2x^3=c_1$$

Verified OK.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
trying homogeneous types:
trying Chini
differential order: 1; looking for linear symmetries
trying exact
<- exact successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 59

```
dsolve((4*x*y(x)+3*y(x)^2-x)+x*(x+2*y(x))*diff(y(x),x)=0,y(x), singsol=all)
```

$$y(x) = \frac{-x^3 + \sqrt{x\,(x^5 + x^4 - 4c_1)}}{2x^2}$$
$$y(x) = \frac{-x^3 - \sqrt{x\,(x^5 + x^4 - 4c_1)}}{2x^2}$$

✓ Solution by Mathematica

Time used: 0.621 (sec). Leaf size: 80

```
DSolve[(4*x*y[x]+3*y[x]^2-x)+x*(x+2*y[x])*y'[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to -\frac{x^4 + \sqrt{x^2}\sqrt{x^6 + x^5 + 4c_1x}}{2x^3}$$
$$y(x) \to -\frac{x}{2} + \frac{\sqrt{x^2}\sqrt{x^6 + x^5 + 4c_1x}}{2x^3}$$

1.31 problem 31

Internal problem ID [3176]
Internal file name [OUTPUT/2668_Sunday_June_05_2022_08_38_34_AM_20579265/index.tex]

Book: Differential equations for engineers by Wei-Chau XIE, Cambridge Press 2010
Section: Chapter 2. First-Order and Simple Higher-Order Differential Equations. Page 78
Problem number: 31.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"exactWithIntegrationFactor", "first_order_ode_lie_symmetry_calculated"**

Maple gives the following as the ode type

```
[[_1st_order, `_with_symmetry_[F(x),G(y)]`]]
```

$$\boxed{y + x(y^2 + \ln(x))\, y' = 0}$$

1.31.1 Solving as first order ode lie symmetry calculated ode

Writing the ode as

$$\begin{aligned} y' &= -\frac{y}{x\,(y^2 + \ln(x))} \\ y' &= \omega(x, y) \end{aligned}$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x + \omega(\eta_y - \xi_x) - \omega^2 \xi_y - \omega_x \xi - \omega_y \eta = 0 \tag{A}$$

The type of this ode is not in the lookup table. To determine ξ, η then (A) is solved using ansatz. Making bivariate polynomials of degree 3 to use as anstaz gives

$$\xi = x^3 a_7 + x^2 y a_8 + x\, y^2 a_9 + y^3 a_{10} + x^2 a_4 + x y a_5 + y^2 a_6 + x a_2 + y a_3 + a_1 \tag{1E}$$

$$\eta = x^3 b_7 + x^2 y b_8 + x\, y^2 b_9 + y^3 b_{10} + x^2 b_4 + x y b_5 + y^2 b_6 + x b_2 + y b_3 + b_1 \tag{2E}$$

Where the unknown coefficients are

$$\{a_1, a_2, a_3, a_4, a_5, a_6, a_7, a_8, a_9, a_{10}, b_1, b_2, b_3, b_4, b_5, b_6, b_7, b_8, b_9, b_{10}\}$$

Substituting equations (1E,2E) and ω into (A) gives

$$3x^2b_7 + 2xyb_8 + y^2b_9 + 2xb_4 + yb_5 + b_2 - \frac{y(-3x^2a_7 + x^2b_8 - 2xya_8 + 2xyb_9 - y^2a_9 + 3y^2b_{10} - 2xa_4 + xb_5 - ya_5 + 2yb_6 - a_2 + b_3)}{x\left(y^2 + \ln(x)\right)} - \frac{y^2(x^2a_8 + 2xya_9 + 3y^2a_{10} + xa_5 + 2ya_6 + a_3)}{x^2\left(y^2 + \ln(x)\right)^2} - \left(\frac{y}{x^2\left(y^2 + \ln(x)\right)} + \frac{y}{x^2\left(y^2 + \ln(x)\right)^2}\right)\left(x^3a_7 + x^2ya_8 + x\,y^2a_9 + y^3a_{10} + x^2a_4 + xya_5 + y^2a_6 + xa_2 + ya_3 + a_1\right) - \left(-\frac{1}{x\left(y^2 + \ln(x)\right)} + \frac{2y^2}{x\left(y^2 + \ln(x)\right)^2}\right)\left(x^3b_7 + x^2yb_8 + x\,y^2b_9 + y^3b_{10} + x^2b_4 + xyb_5 + y^2b_6 + xb_2 + yb_3 + b_1\right) = 0 \tag{5E}$$

Putting the above in normal form gives

$$\frac{3x^4y^4b_7 + 2x^3y^5b_8 + x^2y^6b_9 + 2x^3y^3a_7 - 2x^3y^3b_8 + x^2y^4a_8 - 3x^2y^4b_9 - 4x\,y^5b_{10} - 2x^2y^2a_8 - 3x\,y^3a_9 - x^3}{} = 0$$

Setting the numerator to zero gives

$$3x^4y^4b_7 + 2x^3y^5b_8 + x^2y^6b_9 + 2x^3y^3a_7 - 2x^3y^3b_8 + x^2y^4a_8 - 3x^2y^4b_9 - 4x\,y^5b_{10} - 2x^2y^2a_8 - 3x\,y^3a_9 - x^3ya_7 - x^4y^2b_7 - x^2ya_4 - 2x\,y^2a_5 - x^3y^2b_4 + x^2y^3a_4 - 2x^2y^3b_5 - 3x\,y^4b_6 + 2\ln(x)^2 x^3b_4 + \ln(x)\,x^3b_4 - \ln(x)\,y^3a_6 - 2y^2a_3 + 2\ln(x)\,x^2y^2b_2 + 4\ln(x)\,x^3y^2b_4 + 2\ln(x)\,x^2y^3b_5 + \ln(x)^2 x^2yb_5 + \ln(x)\,x^2ya_4 - \ln(x)\,x\,y^2b_6 + 6\ln(x)\,x^4y^2b_7 + 4\ln(x)\,x^3y^3b_8 + 2\ln(x)^2 x^3yb_8 + 2\ln(x)\,x^2y^4b_9 + \ln(x)^2 x^2y^2b_9 + 2\ln(x)\,x^3ya_7 + \ln(x)\,x^2y^2a_8 - \ln(x)\,x^2y^2b_9 - 2\ln(x)\,x\,y^3b_{10} + 2x^3y^4b_4 + x^2y^5b_5 - 4y^4a_{10} - y^6a_{10} + 3\ln(x)^2 x^4b_7 - \ln(x)\,y^4a_{10} + \ln(x)\,x^4b_7 - y^4a_3 - y^3a_1 - ya_1 + x^2y^4b_2 - x^2y^2b_2 - 2x\,y^3b_3 - x\,y^2b_1 - xya_2 + \ln(x)^2 x^2b_2 + \ln(x)\,x^2b_2 - \ln(x)\,y^2a_3 + \ln(x)\,xb_1 - \ln(x)\,ya_1 - y^5a_6 - 3y^3a_6 = 0 \tag{6E}$$

Looking at the above PDE shows the following are all the terms with $\{x, y\}$ in them.

$$\{x, y, \ln(x)\}$$

The following substitution is now made to be able to collect on all terms with $\{x, y\}$ in them

$$\{x = v_1, y = v_2, \ln(x) = v_3\}$$

The above PDE (6E) now becomes

$$\begin{aligned}
&-v_3v_2a_1 + v_1^2v_2^4b_2 - v_1^2v_2^2b_2 - 2v_1v_2^3b_3 - v_1v_2^2b_1 - v_1v_2a_2 + v_3^2v_1^2b_2 \\
&+ v_3v_1^2b_2 - v_3v_2^2a_3 + v_3v_1b_1 + 3v_1^4v_2^4b_7 + 2v_1^3v_2^5b_8 + v_1^2v_2^6b_9 + 2v_1^3v_2^3a_7 \\
&- 2v_1^3v_2^3b_8 + v_1^2v_2^4a_8 - 3v_1^2v_2^4b_9 - 4v_1v_2^5b_{10} - 2v_1^2v_2^2a_8 - 3v_1v_2^3a_9 \\
&- v_1^3v_2a_7 - v_1^4v_2^2b_7 - v_1^2v_2a_4 - 2v_1v_2^2a_5 - v_1^3v_2^2b_4 + v_1^2v_2^3a_4 - 2v_1^2v_2^3b_5 \\
&+ v_3^2v_1^2v_2^2b_9 + 2v_3v_1^3v_2a_7 + v_3v_1^2v_2^2a_8 - v_3v_1^2v_2^2b_9 - 2v_3v_1v_2^3b_{10} \\
&+ v_3^2v_1^2v_2b_5 + v_3v_1^2v_2a_4 - v_3v_1v_2^2b_6 + 6v_3v_1^4v_2^2b_7 + 4v_3v_1^3v_2^3b_8 + 2v_3^2v_1^3v_2b_8 \\
&+ 2v_3v_1^2v_2^4b_9 + 4v_3v_1^3v_2^2b_4 + 2v_3v_1^2v_2^3b_5 + 2v_3v_1^2v_2^2b_2 - 3v_1v_2^4b_6 + 2v_3^2v_1^3b_4 \\
&+ v_3v_1^3b_4 - v_3v_2^3a_6 + 2v_1^3v_2^4b_4 + v_1^2v_2^5b_5 + 3v_3^2v_1^4b_7 - v_3v_2^4a_{10} + v_3v_1^4b_7 \\
&- 2v_2^2a_3 - 4v_2^4a_{10} - v_2^6a_{10} - v_2^4a_3 - v_2^3a_1 - v_2a_1 - v_2^5a_6 - 3v_2^3a_6 = 0
\end{aligned} \tag{7E}$$

Collecting the above on the terms v_i introduced, and these are

$$\{v_1, v_2, v_3\}$$

Equation (7E) now becomes

$$\begin{aligned}
&-v_3v_2a_1 - v_1v_2a_2 + v_3^2v_1^2b_2 + v_3v_1^2b_2 - v_3v_2^2a_3 + v_3v_1b_1 + 3v_1^4v_2^4b_7 \\
&+ 2v_1^3v_2^5b_8 + v_1^2v_2^6b_9 - 4v_1v_2^5b_{10} - v_1^3v_2a_7 - v_1^4v_2^2b_7 - v_1^2v_2a_4 - v_1^3v_2^2b_4 \\
&+ (a_8 - b_9 + 2b_2)\, v_2^2v_1^2v_3 + (2a_7 - 2b_8)\, v_2^3v_1^3 + (a_4 - 2b_5)\, v_2^3v_1^2 \\
&+ (b_2 + a_8 - 3b_9)\, v_2^4v_1^2 + (-b_2 - 2a_8)\, v_2^2v_1^2 + (-2b_3 - 3a_9)\, v_2^3v_1 \\
&+ (-2a_5 - b_1)\, v_2^2v_1 + v_3^2v_1^2v_2^2b_9 + 2v_3v_1^3v_2a_7 - 2v_3v_1v_2^3b_{10} + v_3^2v_1^2v_2b_5 \\
&+ v_3v_1^2v_2a_4 - v_3v_1v_2^2b_6 + 6v_3v_1^4v_2^2b_7 + 4v_3v_1^3v_2^3b_8 + 2v_3^2v_1^3v_2b_8 \\
&+ 2v_3v_1^2v_2^4b_9 + 4v_3v_1^3v_2^2b_4 + 2v_3v_1^2v_2^3b_5 - 3v_1v_2^4b_6 + 2v_3^2v_1^3b_4 + v_3v_1^3b_4 \\
&- v_3v_2^3a_6 + 2v_1^3v_2^4b_4 + v_1^2v_2^5b_5 + 3v_3^2v_1^4b_7 - v_3v_2^4a_{10} + v_3v_1^4b_7 \\
&+ (-4a_{10} - a_3)\, v_2^4 + (-a_1 - 3a_6)\, v_2^3 - 2v_2^2a_3 - v_2^6a_{10} - v_2a_1 - v_2^5a_6 = 0
\end{aligned} \tag{8E}$$

Setting each coefficients in (8E) to zero gives the following equations to solve

$$
\begin{aligned}
a_4 &= 0 \\
b_1 &= 0 \\
b_2 &= 0 \\
b_4 &= 0 \\
b_5 &= 0 \\
b_7 &= 0 \\
b_9 &= 0 \\
-a_1 &= 0 \\
-a_2 &= 0 \\
-2a_3 &= 0 \\
-a_3 &= 0 \\
-a_4 &= 0 \\
-a_6 &= 0 \\
-a_7 &= 0 \\
2a_7 &= 0 \\
-a_{10} &= 0 \\
-b_4 &= 0 \\
2b_4 &= 0 \\
4b_4 &= 0 \\
2b_5 &= 0 \\
-3b_6 &= 0 \\
-b_6 &= 0 \\
-b_7 &= 0 \\
3b_7 &= 0 \\
6b_7 &= 0 \\
2b_8 &= 0 \\
4b_8 &= 0 \\
2b_9 &= 0 \\
-4b_{10} &= 0 \\
-2b_{10} &= 0 \\
-a_1 - 3a_6 &= 0 \\
a_4 - 2b_5 &= 0 \\
-2a_5 - b_1 &= 0 \\
2a_7 - 2b_8 &= 0 \\
-4a_{10} - a_3 &= 0 \\
-b_2 - 2a_8 &= 0
\end{aligned}
$$

Solving the above equations for the unknowns gives

$$
\begin{aligned}
a_1 &= 0 \\
a_2 &= 0 \\
a_3 &= 0 \\
a_4 &= 0 \\
a_5 &= 0 \\
a_6 &= 0 \\
a_7 &= 0 \\
a_8 &= 0 \\
a_9 &= -\frac{2b_3}{3} \\
a_{10} &= 0 \\
b_1 &= 0 \\
b_2 &= 0 \\
b_3 &= b_3 \\
b_4 &= 0 \\
b_5 &= 0 \\
b_6 &= 0 \\
b_7 &= 0 \\
b_8 &= 0 \\
b_9 &= 0 \\
b_{10} &= 0
\end{aligned}
$$

Substituting the above solution in the anstaz (1E,2E) (using 1 as arbitrary value for any unknown in the RHS) gives

$$
\begin{aligned}
\xi &= -\frac{2x\,y^2}{3} \\
\eta &= y
\end{aligned}
$$

Shifting is now applied to make $\xi = 0$ in order to simplify the rest of the computation

$$\begin{aligned}
\eta &= \eta - \omega(x,y)\,\xi \\
&= y - \left(-\frac{y}{x\,(y^2+\ln(x))}\right)\left(-\frac{2x\,y^2}{3}\right) \\
&= \frac{y^3+3\ln(x)\,y}{3y^2+3\ln(x)} \\
\xi &= 0
\end{aligned}$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x,y) \to (R,S)$ where (R,S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}$$

The above comes from the requirements that $\left(\xi\frac{\partial}{\partial x} + \eta\frac{\partial}{\partial y}\right) S(x,y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$R = x$$

S is found from

$$\begin{aligned}
S &= \int \frac{1}{\eta} dy \\
&= \int \frac{1}{\frac{y^3+3\ln(x)y}{3y^2+3\ln(x)}} dy
\end{aligned}$$

Which results in

$$S = \ln\left(y\left(y^2+3\ln(x)\right)\right)$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x,y)S_y}{R_x + \omega(x,y)R_y} \tag{2}$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x,y)$ is the right hand side of the original ode given by

$$\omega(x,y) = -\frac{y}{x\,(y^2+\ln(x))}$$

Evaluating all the partial derivatives gives

$$\begin{aligned}
R_x &= 1 \\
R_y &= 0 \\
S_x &= \frac{3}{x\left(y^2+3\ln\left(x\right)\right)} \\
S_y &= \frac{1}{y}+\frac{2y}{y^2+3\ln\left(x\right)}
\end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = 0 \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = 0$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$S(R) = c_1 \tag{4}$$

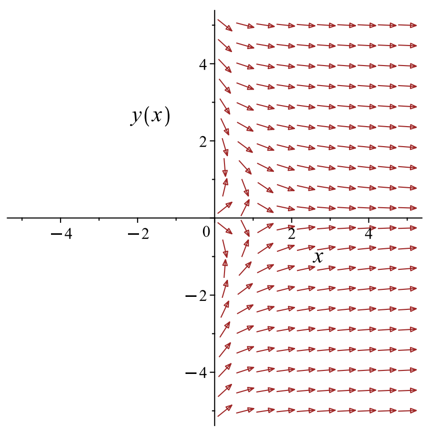
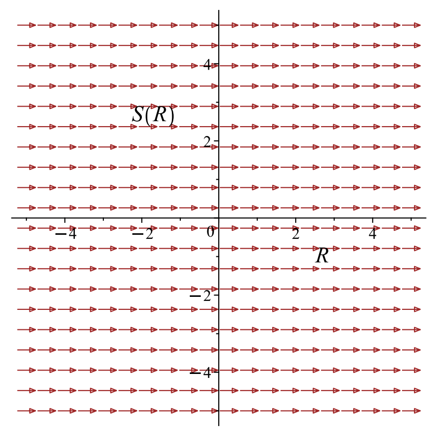
To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$\ln\left(y\right)+\ln\left(y^2+3\ln\left(x\right)\right) = c_1$$

Which simplifies to

$$\ln\left(y\right)+\ln\left(y^2+3\ln\left(x\right)\right) = c_1$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in x, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dx} = -\frac{y}{x(y^2+\ln(x))}$ 	$R = x$ $S = \ln(y) + \ln\left(y^2 + 3\ln\right.$	$\frac{dS}{dR} = 0$ 

Summary

The solution(s) found are the following

$$\ln(y) + \ln\left(y^2 + 3\ln(x)\right) = c_1 \tag{1}$$

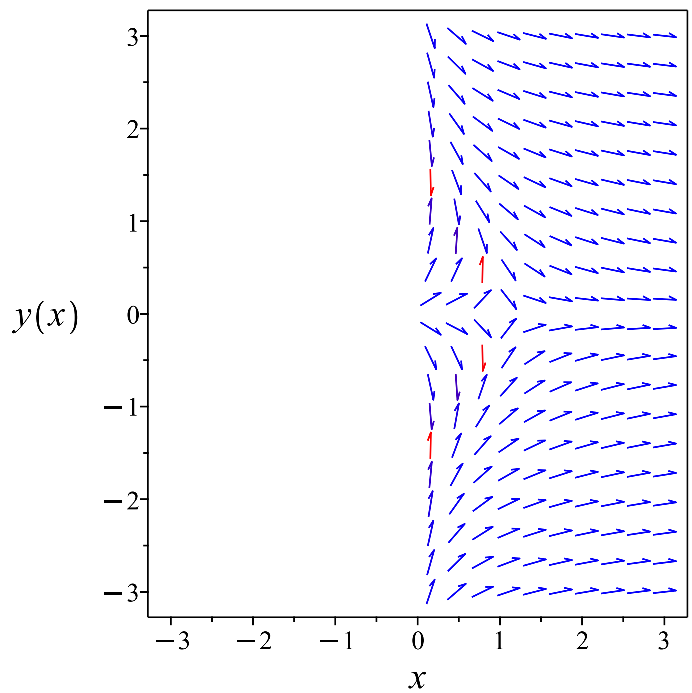


Figure 49: Slope field plot

Verification of solutions

$$\ln(y) + \ln\left(y^2 + 3\ln(x)\right) = c_1$$

Verified OK.

1.31.2 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x,y) + N(x,y)\frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x,y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y) = 0$$

Hence

$$\frac{\partial\phi}{\partial x} + \frac{\partial\phi}{\partial y}\frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$
\begin{aligned}
\frac{\partial \phi}{\partial x} &= M \\
\frac{\partial \phi}{\partial y} &= N
\end{aligned}
$$

But since $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ then for the above to be valid, we require that

$$
\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}
$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x, y)$ but at least we know now that we can do that since the condition $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$
M(x, y)\,\mathrm{d}x + N(x, y)\,\mathrm{d}y = 0 \tag{1A}
$$

Therefore

$$
\begin{aligned}
\left(x\left(y^2 + \ln(x)\right)\right)\mathrm{d}y &= (-y)\,\mathrm{d}x \\
(y)\,\mathrm{d}x + \left(x\left(y^2 + \ln(x)\right)\right)\mathrm{d}y &= 0 \qquad \text{(2A)}
\end{aligned}
$$

Comparing (1A) and (2A) shows that

$$
\begin{aligned}
M(x, y) &= y \\
N(x, y) &= x\left(y^2 + \ln(x)\right)
\end{aligned}
$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$
\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}
$$

Using result found above gives

$$
\begin{aligned}
\frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}(y) \\
&= 1
\end{aligned}
$$

And

$$
\begin{aligned}
\frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}\left(x\left(y^2 + \ln(x)\right)\right) \\
&= y^2 + \ln(x) + 1
\end{aligned}
$$

Since $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$, then the ODE is not exact. Since the ODE is not exact, we will try to find an integrating factor to make it exact. Let

$$
\begin{aligned}
A &= \frac{1}{N}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right) \\
&= \frac{1}{x\left(y^2 + \ln\left(x\right)\right)}\left(\left(1\right) - \left(y^2 + \ln\left(x\right) + 1\right)\right) \\
&= -\frac{1}{x}
\end{aligned}
$$

Since A does not depend on y, then it can be used to find an integrating factor. The integrating factor μ is

$$
\begin{aligned}
\mu &= e^{\int A \, \mathrm{d}x} \\
&= e^{\int -\frac{1}{x} \, \mathrm{d}x}
\end{aligned}
$$

The result of integrating gives

$$
\begin{aligned}
\mu &= e^{-\ln(x)} \\
&= \frac{1}{x}
\end{aligned}
$$

M and N are multiplied by this integrating factor, giving new M and new N which are called $\overline{M}$ and $\overline{N}$ for now so not to confuse them with the original M and N.

$$
\begin{aligned}
\overline{M} &= \mu M \\
&= \frac{1}{x}(y) \\
&= \frac{y}{x}
\end{aligned}
$$

And

$$
\begin{aligned}
\overline{N} &= \mu N \\
&= \frac{1}{x}\left(x\left(y^2 + \ln\left(x\right)\right)\right) \\
&= y^2 + \ln\left(x\right)
\end{aligned}
$$

Now a modified ODE is ontained from the original ODE, which is exact and can be solved. The modified ODE is

$$
\begin{aligned}
\overline{M} + \overline{N}\frac{\mathrm{d}y}{\mathrm{d}x} &= 0 \\
\left(\frac{y}{x}\right) + \left(y^2 + \ln\left(x\right)\right)\frac{\mathrm{d}y}{\mathrm{d}x} &= 0
\end{aligned}
$$

The following equations are now set up to solve for the function $\phi(x, y)$

$$\frac{\partial \phi}{\partial x} = \overline{M} \tag{1}$$
$$\frac{\partial \phi}{\partial y} = \overline{N} \tag{2}$$

Integrating (1) w.r.t. x gives

$$\int \frac{\partial \phi}{\partial x} \, \mathrm{d}x = \int \overline{M} \, \mathrm{d}x$$
$$\int \frac{\partial \phi}{\partial x} \, \mathrm{d}x = \int \frac{y}{x} \, \mathrm{d}x$$
$$\phi = \ln(x) \, y + f(y) \tag{3}$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$\frac{\partial \phi}{\partial y} = \ln(x) + f'(y) \tag{4}$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = y^2 + \ln(x)$. Therefore equation (4) becomes

$$y^2 + \ln(x) = \ln(x) + f'(y) \tag{5}$$

Solving equation (5) for $f'(y)$ gives

$$f'(y) = y^2$$

Integrating the above w.r.t y gives

$$\int f'(y) \, \mathrm{d}y = \int (y^2) \, \mathrm{d}y$$
$$f(y) = \frac{y^3}{3} + c_1$$

Where c_1 is constant of integration. Substituting result found above for $f(y)$ into equation (3) gives ϕ

$$\phi = \ln(x) \, y + \frac{y^3}{3} + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = \ln(x)\, y + \frac{y^3}{3}$$

Summary
The solution(s) found are the following

$$y \ln(x) + \frac{y^3}{3} = c_1 \tag{1}$$

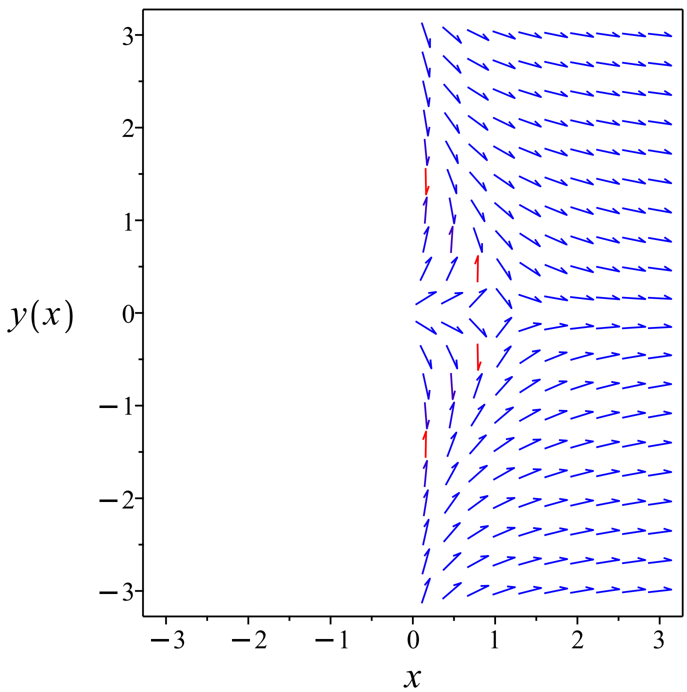


Figure 50: Slope field plot

Verification of solutions

$$y \ln(x) + \frac{y^3}{3} = c_1$$

Verified OK.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
trying homogeneous types:
trying Chini
differential order: 1; looking for linear symmetries
trying exact
<- exact successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 237

```
dsolve((y(x))+x*(y(x)^2+ln(x))*diff(y(x),x)=0,y(x), singsol=all)
```

$$y(x) = \frac{\left(-12c_1 + 4\sqrt{4\ln(x)^3 + 9c_1^2}\right)^{\frac{2}{3}} - 4\ln(x)}{2\left(-12c_1 + 4\sqrt{4\ln(x)^3 + 9c_1^2}\right)^{\frac{1}{3}}}$$

$$y(x) = -\frac{i\left(-12c_1 + 4\sqrt{4\ln(x)^3 + 9c_1^2}\right)^{\frac{2}{3}}\sqrt{3} + 4i\ln(x)\sqrt{3} + \left(-12c_1 + 4\sqrt{4\ln(x)^3 + 9c_1^2}\right)^{\frac{2}{3}} - 4\ln(x)}{4\left(-12c_1 + 4\sqrt{4\ln(x)^3 + 9c_1^2}\right)^{\frac{1}{3}}}$$

$$y(x) = \frac{i\left(-12c_1 + 4\sqrt{4\ln(x)^3 + 9c_1^2}\right)^{\frac{2}{3}}\sqrt{3} + 4i\ln(x)\sqrt{3} - \left(-12c_1 + 4\sqrt{4\ln(x)^3 + 9c_1^2}\right)^{\frac{2}{3}} + 4\ln(x)}{4\left(-12c_1 + 4\sqrt{4\ln(x)^3 + 9c_1^2}\right)^{\frac{1}{3}}}$$

✓ Solution by Mathematica

Time used: 1.211 (sec). Leaf size: 272

```
DSolve[(y[x])+x*(y[x]^2+Log[x])*y'[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \frac{\sqrt[3]{\sqrt{4\log^3(x)+9c_1{}^2}+3c_1}}{\sqrt[3]{2}} - \frac{\sqrt[3]{2}\log(x)}{\sqrt[3]{\sqrt{4\log^3(x)+9c_1{}^2}+3c_1}}$$

$$y(x) \to \frac{\sqrt[3]{2}\left(2+2i\sqrt{3}\right)\log(x)+i2^{2/3}\left(\sqrt{3}+i\right)\left(\sqrt{4\log^3(x)+9c_1{}^2}+3c_1\right)^{2/3}}{4\sqrt[3]{\sqrt{4\log^3(x)+9c_1{}^2}+3c_1}}$$

$$y(x) \to \frac{\left(1-i\sqrt{3}\right)\log(x)}{2^{2/3}\sqrt[3]{\sqrt{4\log^3(x)+9c_1{}^2}+3c_1}} - \frac{\left(1+i\sqrt{3}\right)\sqrt[3]{\sqrt{4\log^3(x)+9c_1{}^2}+3c_1}}{2\sqrt[3]{2}}$$

$$y(x) \to 0$$

1.32 problem 32

Internal problem ID [3177]
Internal file name [OUTPUT/2669_Sunday_June_05_2022_08_38_35_AM_57792872/index.tex]

Book: Differential equations for engineers by Wei-Chau XIE, Cambridge Press 2010
Section: Chapter 2. First-Order and Simple Higher-Order Differential Equations. Page 78
Problem number: 32.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"exactWithIntegrationFactor"**

Maple gives the following as the ode type

```
[_rational, [_1st_order, `_with_symmetry_[F(x),G(x)]`], [_Abel,
    `2nd type`, `class B`]]
```

$$\boxed{y + (3x^2y - x)\, y' = -x^2 - 2x}$$

1.32.1 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x,y) + N(x,y)\frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x,y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y) = 0$$

Hence

$$\frac{\partial\phi}{\partial x} + \frac{\partial\phi}{\partial y}\frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$\frac{\partial\phi}{\partial x} = M$$
$$\frac{\partial\phi}{\partial y} = N$$

But since $\frac{\partial^2\phi}{\partial x\partial y} = \frac{\partial^2\phi}{\partial y\partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x, y)$ but at least we know now that we can do that since the condition $\frac{\partial^2\phi}{\partial x\partial y} = \frac{\partial^2\phi}{\partial y\partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x, y)\,\mathrm{d}x + N(x, y)\,\mathrm{d}y = 0 \tag{1A}$$

Therefore

$$\begin{aligned}
\left(3x^2y - x\right)\mathrm{d}y &= \left(-x^2 - 2x - y\right)\mathrm{d}x \\
\left(x^2 + 2x + y\right)\mathrm{d}x + \left(3x^2y - x\right)\mathrm{d}y &= 0 \qquad (2A)
\end{aligned}$$

Comparing (1A) and (2A) shows that

$$\begin{aligned}
M(x, y) &= x^2 + 2x + y \\
N(x, y) &= 3x^2y - x
\end{aligned}$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Using result found above gives

$$\begin{aligned}
\frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}\left(x^2 + 2x + y\right) \\
&= 1
\end{aligned}$$

And

$$\begin{aligned}
\frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}\left(3x^2y - x\right) \\
&= 6xy - 1
\end{aligned}$$

Since $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$, then the ODE is <u>not exact</u>. Since the ODE is not exact, we will try to find an integrating factor to make it exact. Let

$$\begin{aligned}
A &= \frac{1}{N}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right) \\
&= \frac{1}{3x^2y - x}((1) - (6xy - 1)) \\
&= -\frac{2}{x}
\end{aligned}$$

Since A does not depend on y, then it can be used to find an integrating factor. The integrating factor μ is

$$\begin{aligned}\mu &= e^{\int A \,\mathrm{d}x} \\ &= e^{\int -\frac{2}{x} \,\mathrm{d}x}\end{aligned}$$

The result of integrating gives

$$\begin{aligned}\mu &= e^{-2\ln(x)} \\ &= \frac{1}{x^2}\end{aligned}$$

M and N are multiplied by this integrating factor, giving new M and new N which are called $\overline{M}$ and $\overline{N}$ for now so not to confuse them with the original M and N.

$$\begin{aligned}\overline{M} &= \mu M \\ &= \frac{1}{x^2}\left(x^2 + 2x + y\right) \\ &= \frac{x^2 + 2x + y}{x^2}\end{aligned}$$

And

$$\begin{aligned}\overline{N} &= \mu N \\ &= \frac{1}{x^2}\left(3x^2 y - x\right) \\ &= \frac{3xy - 1}{x}\end{aligned}$$

Now a modified ODE is ontained from the original ODE, which is exact and can be solved. The modified ODE is

$$\begin{aligned}\overline{M} + \overline{N}\frac{\mathrm{d}y}{\mathrm{d}x} &= 0 \\ \left(\frac{x^2 + 2x + y}{x^2}\right) + \left(\frac{3xy - 1}{x}\right)\frac{\mathrm{d}y}{\mathrm{d}x} &= 0\end{aligned}$$

The following equations are now set up to solve for the function $\phi(x, y)$

$$\frac{\partial \phi}{\partial x} = \overline{M} \tag{1}$$

$$\frac{\partial \phi}{\partial y} = \overline{N} \tag{2}$$

Integrating (1) w.r.t. x gives

$$\int \frac{\partial \phi}{\partial x} \,\mathrm{d}x = \int \overline{M} \,\mathrm{d}x$$
$$\int \frac{\partial \phi}{\partial x} \,\mathrm{d}x = \int \frac{x^2 + 2x + y}{x^2} \,\mathrm{d}x$$
$$\phi = x - \frac{y}{x} + 2\ln(x) + f(y) \tag{3}$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$\frac{\partial \phi}{\partial y} = -\frac{1}{x} + f'(y) \tag{4}$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = \frac{3xy-1}{x}$. Therefore equation (4) becomes

$$\frac{3xy - 1}{x} = -\frac{1}{x} + f'(y) \tag{5}$$

Solving equation (5) for $f'(y)$ gives

$$f'(y) = 3y$$

Integrating the above w.r.t y gives

$$\int f'(y) \,\mathrm{d}y = \int (3y) \,\mathrm{d}y$$
$$f(y) = \frac{3y^2}{2} + c_1$$

Where c_1 is constant of integration. Substituting result found above for $f(y)$ into equation (3) gives ϕ

$$\phi = x - \frac{y}{x} + 2\ln(x) + \frac{3y^2}{2} + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = x - \frac{y}{x} + 2\ln(x) + \frac{3y^2}{2}$$

Summary
The solution(s) found are the following

$$x - \frac{y}{x} + 2\ln(x) + \frac{3y^2}{2} = c_1 \tag{1}$$

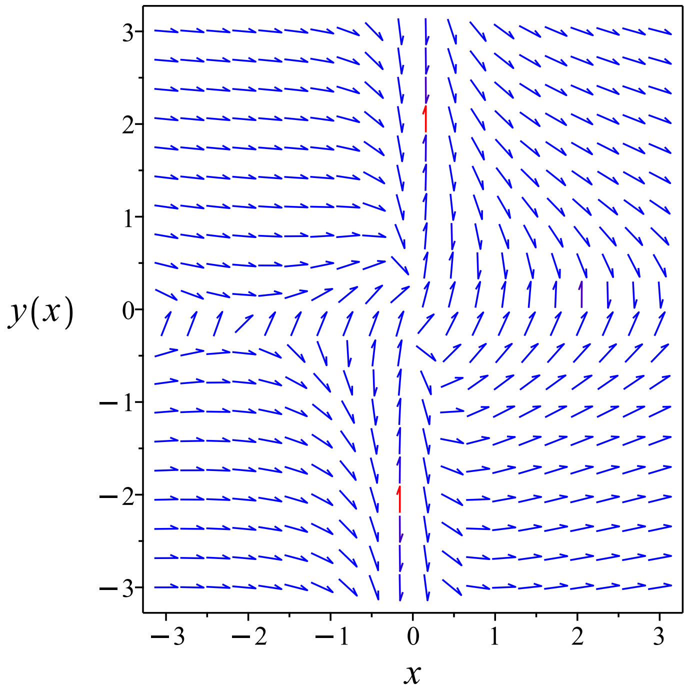


Figure 51: Slope field plot

Verification of solutions

$$x - \frac{y}{x} + 2\ln(x) + \frac{3y^2}{2} = c_1$$

Verified OK.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
trying homogeneous types:
trying Chini
differential order: 1; looking for linear symmetries
trying exact
<- exact successful`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 67

```
dsolve((x^2+2*x+y(x))+(3*x^2*y(x)-x)*diff(y(x),x)=0,y(x), singsol=all)
```

$$y(x) = \frac{1 - \sqrt{-12 \ln(x) x^2 - 6c_1 x^2 - 6x^3 + 1}}{3x}$$
$$y(x) = \frac{1 + \sqrt{-12 \ln(x) x^2 - 6c_1 x^2 - 6x^3 + 1}}{3x}$$

✓ Solution by Mathematica

Time used: 0.543 (sec). Leaf size: 96

```
DSolve[(x^2+2*x+y[x])+(3*x^2*y[x]-x)*y'[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \frac{1 - \sqrt{\frac{1}{x^2}} x \sqrt{-6x^3 - 12x^2 \log(x) + 9c_1 x^2 + 1}}{3x}$$
$$y(x) \to \frac{1 + \sqrt{\frac{1}{x^2}} x \sqrt{-6x^3 - 12x^2 \log(x) + 9c_1 x^2 + 1}}{3x}$$

1.33 problem 33

Internal problem ID [3178]
Internal file name [`OUTPUT/2670_Sunday_June_05_2022_08_38_35_AM_89941550/index.tex`]

Book: Differential equations for engineers by Wei-Chau XIE, Cambridge Press 2010
Section: Chapter 2. First-Order and Simple Higher-Order Differential Equations. Page 78
Problem number: 33.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"exactWithIntegrationFactor", "first_order_ode_lie_symmetry_calculated"**

Maple gives the following as the ode type

```
[[_1st_order, _with_linear_symmetries], _rational]
```

$$\boxed{y^2 + (yx + y^2 - 1) y' = 0}$$

1.33.1 Solving as first order ode lie symmetry calculated ode

Writing the ode as

$$y' = -\frac{y^2}{xy + y^2 - 1}$$
$$y' = \omega(x, y)$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x + \omega(\eta_y - \xi_x) - \omega^2 \xi_y - \omega_x \xi - \omega_y \eta = 0 \tag{A}$$

The type of this ode is not in the lookup table. To determine ξ, η then (A) is solved using ansatz. Making bivariate polynomials of degree 1 to use as anstaz gives

$$\xi = xa_2 + ya_3 + a_1 \tag{1E}$$
$$\eta = xb_2 + yb_3 + b_1 \tag{2E}$$

Where the unknown coefficients are

$$\{a_1, a_2, a_3, b_1, b_2, b_3\}$$

Substituting equations (1E,2E) and ω into (A) gives

$$\begin{aligned} &b_2 - \frac{y^2(b_3 - a_2)}{xy + y^2 - 1} - \frac{y^4 a_3}{(xy + y^2 - 1)^2} - \frac{y^3(x a_2 + y a_3 + a_1)}{(xy + y^2 - 1)^2} \\ &- \left(-\frac{2y}{xy + y^2 - 1} + \frac{y^2(x + 2y)}{(xy + y^2 - 1)^2}\right)(x b_2 + y b_3 + b_1) = 0 \end{aligned} \tag{5E}$$

Putting the above in normal form gives

$$\frac{2x^2y^2b_2 + 2x\,y^3b_2 + y^4a_2 - 2y^4a_3 + y^4b_2 - y^4b_3 + x\,y^2b_1 - y^3a_1 - 4xyb_2 - y^2a_2 - 2y^2b_2 - y^2b_3 - 2yb_1 + b_2}{(xy + y^2 - 1)^2} = 0$$

Setting the numerator to zero gives

$$\begin{aligned} &2x^2y^2b_2 + 2x\,y^3b_2 + y^4a_2 - 2y^4a_3 + y^4b_2 - y^4b_3 + x\,y^2b_1 \\ &- y^3a_1 - 4xyb_2 - y^2a_2 - 2y^2b_2 - y^2b_3 - 2yb_1 + b_2 = 0 \end{aligned} \tag{6E}$$

Looking at the above PDE shows the following are all the terms with $\{x, y\}$ in them.

$$\{x, y\}$$

The following substitution is now made to be able to collect on all terms with $\{x, y\}$ in them

$$\{x = v_1, y = v_2\}$$

The above PDE (6E) now becomes

$$\begin{aligned} &a_2v_2^4 - 2a_3v_2^4 + 2b_2v_1^2v_2^2 + 2b_2v_1v_2^3 + b_2v_2^4 - b_3v_2^4 - a_1v_2^3 \\ &+ b_1v_1v_2^2 - a_2v_2^2 - 4b_2v_1v_2 - 2b_2v_2^2 - b_3v_2^2 - 2b_1v_2 + b_2 = 0 \end{aligned} \tag{7E}$$

Collecting the above on the terms v_i introduced, and these are

$$\{v_1, v_2\}$$

Equation (7E) now becomes

$$
\begin{aligned}
&2b_2 v_1^2 v_2^2 + 2b_2 v_1 v_2^3 + b_1 v_1 v_2^2 - 4b_2 v_1 v_2 + (a_2 - 2a_3 + b_2 - b_3)\, v_2^4 \\
&- a_1 v_2^3 + (-a_2 - 2b_2 - b_3)\, v_2^2 - 2b_1 v_2 + b_2 = 0
\end{aligned} \tag{8E}
$$

Setting each coefficients in (8E) to zero gives the following equations to solve

$$
\begin{aligned}
b_1 &= 0 \\
b_2 &= 0 \\
-a_1 &= 0 \\
-2b_1 &= 0 \\
-4b_2 &= 0 \\
2b_2 &= 0 \\
-a_2 - 2b_2 - b_3 &= 0 \\
a_2 - 2a_3 + b_2 - b_3 &= 0
\end{aligned}
$$

Solving the above equations for the unknowns gives

$$
\begin{aligned}
a_1 &= 0 \\
a_2 &= -b_3 \\
a_3 &= -b_3 \\
b_1 &= 0 \\
b_2 &= 0 \\
b_3 &= b_3
\end{aligned}
$$

Substituting the above solution in the anstaz (1E,2E) (using 1 as arbitrary value for any unknown in the RHS) gives

$$
\begin{aligned}
\xi &= -y - x \\
\eta &= y
\end{aligned}
$$

Shifting is now applied to make $\xi = 0$ in order to simplify the rest of the computation

$$
\begin{aligned}
\eta &= \eta - \omega(x, y)\, \xi \\
&= y - \left(-\frac{y^2}{xy + y^2 - 1}\right)(-y - x) \\
&= -\frac{y}{xy + y^2 - 1} \\
\xi &= 0
\end{aligned}
$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x, y) \to (R, S)$ where (R, S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y}\right) S(x, y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$R = x$$

S is found from

$$\begin{aligned} S &= \int \frac{1}{\eta} dy \\ &= \int \frac{1}{-\frac{y}{xy+y^2-1}} dy \end{aligned}$$

Which results in

$$S = -\frac{y^2}{2} - xy + \ln(y)$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x, y) S_y}{R_x + \omega(x, y) R_y} \tag{2}$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x, y)$ is the right hand side of the original ode given by

$$\omega(x, y) = -\frac{y^2}{xy + y^2 - 1}$$

Evaluating all the partial derivatives gives

$$\begin{aligned} R_x &= 1 \\ R_y &= 0 \\ S_x &= -y \\ S_y &= -y - x + \frac{1}{y} \end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = 0 \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = 0$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$S(R) = c_1 \tag{4}$$

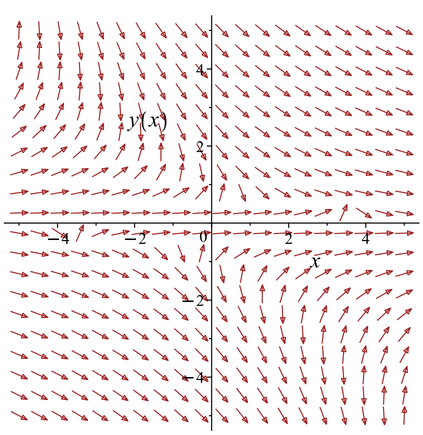
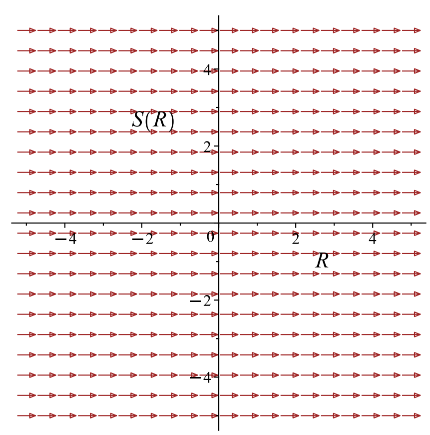
To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$-\frac{y^2}{2} - yx + \ln(y) = c_1$$

Which simplifies to

$$-\frac{y^2}{2} - yx + \ln(y) = c_1$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in x, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dx} = -\frac{y^2}{xy+y^2-1}$ 	$R = x$ $S = -\frac{y^2}{2} - xy + \ln(y)$	$\frac{dS}{dR} = 0$ 

Summary
The solution(s) found are the following

$$-\frac{y^2}{2} - yx + \ln(y) = c_1 \tag{1}$$

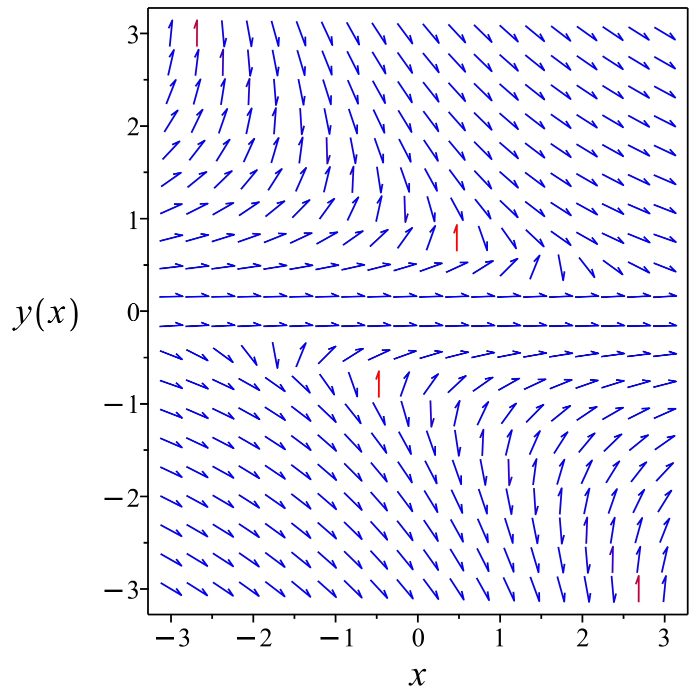


Figure 52: Slope field plot

Verification of solutions

$$-\frac{y^2}{2} - yx + \ln(y) = c_1$$

Verified OK.

1.33.2 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x,y) + N(x,y)\frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x,y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y) = 0$$

Hence

$$\frac{\partial \phi}{\partial x} + \frac{\partial \phi}{\partial y}\frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$\begin{aligned}\frac{\partial \phi}{\partial x} &= M \\ \frac{\partial \phi}{\partial y} &= N\end{aligned}$$

But since $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x, y)$ but at least we know now that we can do that since the condition $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x, y) \, \mathrm{d}x + N(x, y) \, \mathrm{d}y = 0 \tag{1A}$$

Therefore

$$\begin{aligned}(xy + y^2 - 1) \, \mathrm{d}y &= (-y^2) \, \mathrm{d}x \\ (y^2) \, \mathrm{d}x + (xy + y^2 - 1) \, \mathrm{d}y &= 0 \end{aligned} \tag{2A}$$

Comparing (1A) and (2A) shows that

$$\begin{aligned}M(x, y) &= y^2 \\ N(x, y) &= xy + y^2 - 1\end{aligned}$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Using result found above gives

$$\begin{aligned}\frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}(y^2) \\ &= 2y\end{aligned}$$

And

$$
\begin{aligned}
\frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}\left(xy + y^2 - 1\right) \\
&= y
\end{aligned}
$$

Since $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$, then the ODE is <u>not exact</u>. Since the ODE is not exact, we will try to find an integrating factor to make it exact. Let

$$
\begin{aligned}
A &= \frac{1}{N}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right) \\
&= \frac{1}{xy + y^2 - 1}((2y) - (y)) \\
&= \frac{y}{xy + y^2 - 1}
\end{aligned}
$$

Since A depends on y, it can not be used to obtain an integrating factor. We will now try a second method to find an integrating factor. Let

$$
\begin{aligned}
B &= \frac{1}{M}\left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) \\
&= \frac{1}{y^2}((y) - (2y)) \\
&= -\frac{1}{y}
\end{aligned}
$$

Since B does not depend on x, it can be used to obtain an integrating factor. Let the integrating factor be μ. Then

$$
\begin{aligned}
\mu &= e^{\int B \, \mathrm{d}y} \\
&= e^{\int -\frac{1}{y} \, \mathrm{d}y}
\end{aligned}
$$

The result of integrating gives

$$
\begin{aligned}
\mu &= e^{-\ln(y)} \\
&= \frac{1}{y}
\end{aligned}
$$

M and N are now multiplied by this integrating factor, giving new M and new N which are called $\overline{M}$ and $\overline{N}$ so not to confuse them with the original M and N.

$$
\begin{aligned}
\overline{M} &= \mu M \\
&= \frac{1}{y}\left(y^2\right) \\
&= y
\end{aligned}
$$

And

$$
\begin{aligned}
\overline{N} &= \mu N \\
&= \frac{1}{y}(xy + y^2 - 1) \\
&= \frac{xy + y^2 - 1}{y}
\end{aligned}
$$

So now a modified ODE is obtained from the original ODE which will be exact and can be solved using the standard method. The modified ODE is

$$
\begin{aligned}
\overline{M} + \overline{N}\frac{\mathrm{d}y}{\mathrm{d}x} &= 0 \\
(y) + \left(\frac{xy + y^2 - 1}{y}\right)\frac{\mathrm{d}y}{\mathrm{d}x} &= 0
\end{aligned}
$$

The following equations are now set up to solve for the function $\phi(x, y)$

$$
\frac{\partial \phi}{\partial x} = \overline{M} \tag{1}
$$

$$
\frac{\partial \phi}{\partial y} = \overline{N} \tag{2}
$$

Integrating (1) w.r.t. x gives

$$
\begin{aligned}
\int \frac{\partial \phi}{\partial x}\,\mathrm{d}x &= \int \overline{M}\,\mathrm{d}x \\
\int \frac{\partial \phi}{\partial x}\,\mathrm{d}x &= \int y\,\mathrm{d}x \\
\phi &= xy + f(y) \qquad (3)
\end{aligned}
$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$
\frac{\partial \phi}{\partial y} = x + f'(y) \tag{4}
$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = \frac{xy+y^2-1}{y}$. Therefore equation (4) becomes

$$
\frac{xy + y^2 - 1}{y} = x + f'(y) \tag{5}
$$

Solving equation (5) for $f'(y)$ gives

$$f'(y) = \frac{y^2 - 1}{y}$$

Integrating the above w.r.t y gives

$$\int f'(y)\,\mathrm{d}y = \int \left(\frac{y^2 - 1}{y}\right) \mathrm{d}y$$

$$f(y) = \frac{y^2}{2} - \ln(y) + c_1$$

Where c_1 is constant of integration. Substituting result found above for $f(y)$ into equation (3) gives ϕ

$$\phi = xy + \frac{y^2}{2} - \ln(y) + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = xy + \frac{y^2}{2} - \ln(y)$$

Summary

The solution(s) found are the following

$$\frac{y^2}{2} + yx - \ln(y) = c_1 \tag{1}$$

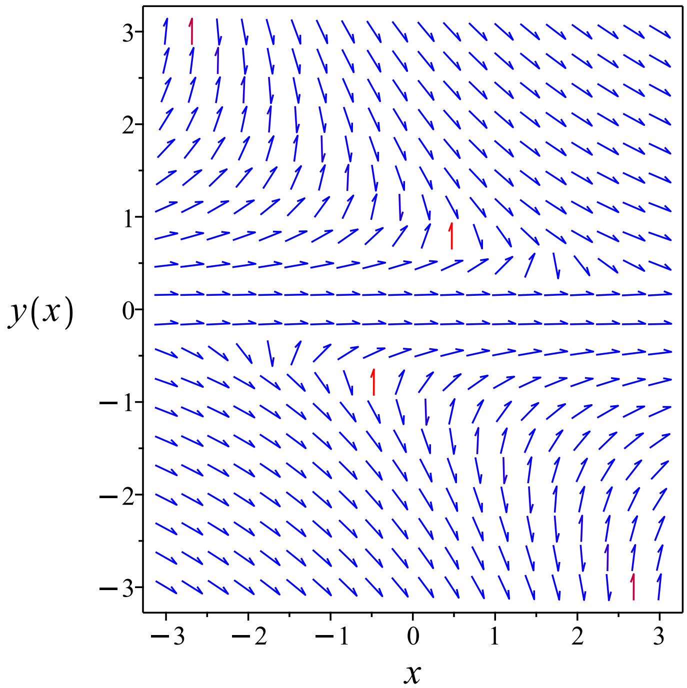


Figure 53: Slope field plot

Verification of solutions

$$\frac{y^2}{2} + yx - \ln(y) = c_1$$

Verified OK.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
<- 1st order linear successful
<- inverse linear successful`
```

✓ Solution by Maple

Time used: 0.015 (sec). Leaf size: 24

```
dsolve((y(x)^2)+(x*y(x)+y(x)^2-1)*diff(y(x),x)=0,y(x), singsol=all)
```

$$y(x) = e^{\text{RootOf}\left(-e^{2\_Z}-2\,e^{\_Z}x+2c_1+2\_Z\right)}$$

✓ Solution by Mathematica

Time used: 0.15 (sec). Leaf size: 30

```
DSolve[(y[x]^2)+(x*y[x]+y[x]^2-1)*y'[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$\text{Solve}\left[x = \frac{\log(y(x)) - \frac{y(x)^2}{2}}{y(x)} + \frac{c_1}{y(x)}, y(x)\right]$$

1.34 problem 34

Internal problem ID [3179]
Internal file name [`OUTPUT/2671_Sunday_June_05_2022_08_38_36_AM_81695143/index.tex`]

Book: Differential equations for engineers by Wei-Chau XIE, Cambridge Press 2010
Section: Chapter 2. First-Order and Simple Higher-Order Differential Equations. Page 78
Problem number: 34.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"exactWithIntegrationFactor"**

Maple gives the following as the ode type

`[_rational]`

$$\boxed{3y^2 + x\left(x^2 + 3y^2 + 6y\right) y' = -3x^2}$$

1.34.1 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x,y) + N(x,y)\,\frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x,y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y) = 0$$

Hence

$$\frac{\partial \phi}{\partial x} + \frac{\partial \phi}{\partial y}\frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$\frac{\partial \phi}{\partial x} = M$$
$$\frac{\partial \phi}{\partial y} = N$$

But since $\frac{\partial^2\phi}{\partial x\partial y} = \frac{\partial^2\phi}{\partial y\partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x, y)$ but at least we know now that we can do that since the condition $\frac{\partial^2\phi}{\partial x\partial y} = \frac{\partial^2\phi}{\partial y\partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x, y)\,\mathrm{d}x + N(x, y)\,\mathrm{d}y = 0 \tag{1A}$$

Therefore

$$\begin{aligned}\left(x\left(x^2+3y^2+6y\right)\right)\mathrm{d}y &= \left(-3x^2-3y^2\right)\mathrm{d}x\\ \left(3x^2+3y^2\right)\mathrm{d}x + \left(x\left(x^2+3y^2+6y\right)\right)\mathrm{d}y &= 0 \qquad (2A)\end{aligned}$$

Comparing (1A) and (2A) shows that

$$\begin{aligned}M(x, y) &= 3x^2+3y^2\\ N(x, y) &= x\left(x^2+3y^2+6y\right)\end{aligned}$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Using result found above gives

$$\begin{aligned}\frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}\left(3x^2+3y^2\right)\\ &= 6y\end{aligned}$$

And

$$\begin{aligned}\frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}\left(x\left(x^2+3y^2+6y\right)\right)\\ &= 3x^2+3y^2+6y\end{aligned}$$

Since $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$, then the ODE is <u>not exact</u>. Since the ODE is not exact, we will try to find an integrating factor to make it exact. Let

$$
\begin{aligned}
A &= \frac{1}{N}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right) \\
&= \frac{1}{x\left(x^2+3y^2+6y\right)}\left((6y) - \left(3x^2+3y^2+6y\right)\right) \\
&= \frac{-3x^2-3y^2}{x\left(x^2+3y^2+6y\right)}
\end{aligned}
$$

Since A depends on y, it can not be used to obtain an integrating factor. We will now try a second method to find an integrating factor. Let

$$
\begin{aligned}
B &= \frac{1}{M}\left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) \\
&= \frac{1}{3x^2+3y^2}\left(\left(3x^2+3y^2+6y\right) - (6y)\right) \\
&= 1
\end{aligned}
$$

Since B does not depend on x, it can be used to obtain an integrating factor. Let the integrating factor be μ. Then

$$
\begin{aligned}
\mu &= e^{\int B \, \mathrm{d}y} \\
&= e^{\int 1 \, \mathrm{d}y}
\end{aligned}
$$

The result of integrating gives

$$
\begin{aligned}
\mu &= e^{y} \\
&= \mathrm{e}^{y}
\end{aligned}
$$

M and N are now multiplied by this integrating factor, giving new M and new N which are called $\overline{M}$ and $\overline{N}$ so not to confuse them with the original M and N.

$$
\begin{aligned}
\overline{M} &= \mu M \\
&= \mathrm{e}^{y}\left(3x^2+3y^2\right) \\
&= 3\left(x^2+y^2\right)\mathrm{e}^{y}
\end{aligned}
$$

And

$$
\begin{aligned}
\overline{N} &= \mu N \\
&= \mathrm{e}^{y}\left(x\left(x^2+3y^2+6y\right)\right) \\
&= x\left(x^2+3y^2+6y\right)\mathrm{e}^{y}
\end{aligned}
$$

So now a modified ODE is obtained from the original ODE which will be exact and can be solved using the standard method. The modified ODE is

$$\overline{M} + \overline{N}\frac{\mathrm{d}y}{\mathrm{d}x} = 0$$
$$\left(3\left(x^2+y^2\right)\mathrm{e}^y\right) + \left(x\left(x^2+3y^2+6y\right)\mathrm{e}^y\right)\frac{\mathrm{d}y}{\mathrm{d}x} = 0$$

The following equations are now set up to solve for the function $\phi(x,y)$

$$\frac{\partial \phi}{\partial x} = \overline{M} \tag{1}$$
$$\frac{\partial \phi}{\partial y} = \overline{N} \tag{2}$$

Integrating (1) w.r.t. x gives

$$\int \frac{\partial \phi}{\partial x}\,\mathrm{d}x = \int \overline{M}\,\mathrm{d}x$$
$$\int \frac{\partial \phi}{\partial x}\,\mathrm{d}x = \int 3\left(x^2+y^2\right)\mathrm{e}^y\,\mathrm{d}x$$
$$\phi = \mathrm{e}^y x\left(x^2+3y^2\right) + f(y) \tag{3}$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$\frac{\partial \phi}{\partial y} = \mathrm{e}^y x\left(x^2+3y^2\right) + 6\,\mathrm{e}^y xy + f'(y) \tag{4}$$
$$= x\left(x^2+3y^2+6y\right)\mathrm{e}^y + f'(y)$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = x(x^2+3y^2+6y)\,\mathrm{e}^y$. Therefore equation (4) becomes

$$x\left(x^2+3y^2+6y\right)\mathrm{e}^y = x\left(x^2+3y^2+6y\right)\mathrm{e}^y + f'(y) \tag{5}$$

Solving equation (5) for $f'(y)$ gives

$$f'(y) = 0$$

Therefore

$$f(y) = c_1$$

Where c_1 is constant of integration. Substituting this result for $f(y)$ into equation (3) gives ϕ

$$\phi = e^y x(x^2 + 3y^2) + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = e^y x(x^2 + 3y^2)$$

Summary

The solution(s) found are the following

$$e^y x(x^2 + 3y^2) = c_1 \tag{1}$$

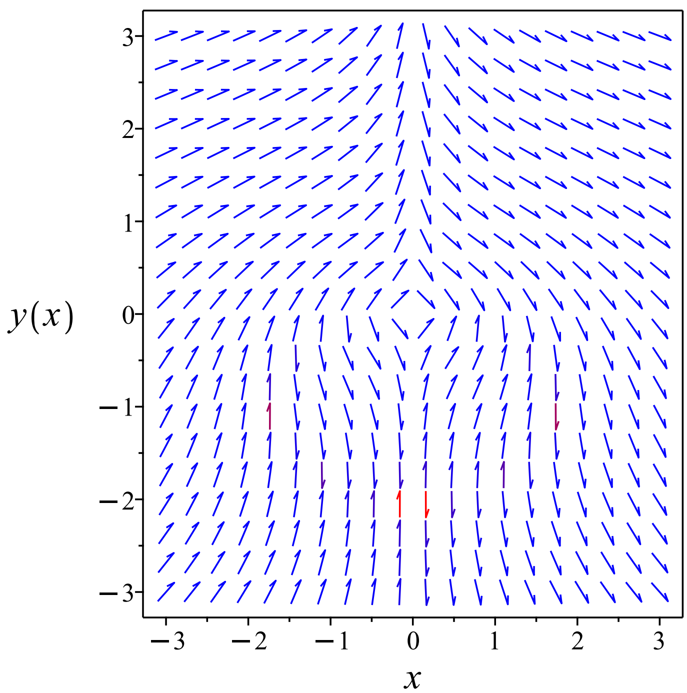


Figure 54: Slope field plot

Verification of solutions

$$e^y x(x^2 + 3y^2) = c_1$$

Verified OK.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
trying homogeneous types:
trying Chini
differential order: 1; looking for linear symmetries
trying exact
<- exact successful`
```

✓ Solution by Maple

Time used: 0.047 (sec). Leaf size: 22

```
dsolve(3*(x^2+y(x)^2)+x*(x^2+3*y(x)^2+6*y(x))*diff(y(x),x)=0,y(x), singsol=all)
```

$$c_1 + \frac{e^{y(x)}x^3}{3} + e^{y(x)}xy(x)^2 = 0$$

✓ Solution by Mathematica

Time used: 0.149 (sec). Leaf size: 26

```
DSolve[3*(x^2+y[x]^2)+x*(x^2+3*y[x]^2+6*y[x])*y'[x]==0,y[x],x,IncludeSingularSolutions -> Tr
```

$$\text{Solve}\left[x^3e^{y(x)} + 3xe^{y(x)}y(x)^2 = c_1, y(x)\right]$$

1.35 problem 35

Internal problem ID [3180]
Internal file name [OUTPUT/2672_Sunday_June_05_2022_08_38_37_AM_1189778/index.tex]

Book: Differential equations for engineers by Wei-Chau XIE, Cambridge Press 2010
Section: Chapter 2. First-Order and Simple Higher-Order Differential Equations. Page 78
Problem number: 35.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"exactWithIntegrationFactor", "first_order_ode_lie_symmetry_calculated"**

Maple gives the following as the ode type

```
[[_1st_order, _with_linear_symmetries], _rational]
```

$$\boxed{2y(x+y+2)+\left(y^2-x^2-4x-1\right)y'=0}$$

1.35.1 Solving as first order ode lie symmetry calculated ode

Writing the ode as

$$\begin{aligned} y' &= -\frac{2y(x+y+2)}{-x^2+y^2-4x-1} \\ y' &= \omega(x,y) \end{aligned}$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x+\omega(\eta_y-\xi_x)-\omega^2\xi_y-\omega_x\xi-\omega_y\eta=0 \tag{A}$$

The type of this ode is not in the lookup table. To determine ξ,η then (A) is solved using ansatz. Making bivariate polynomials of degree 1 to use as anstaz gives

$$\xi = xa_2+ya_3+a_1 \tag{1E}$$

$$\eta = xb_2+yb_3+b_1 \tag{2E}$$

Where the unknown coefficients are

$$\{a_1, a_2, a_3, b_1, b_2, b_3\}$$

Substituting equations (1E,2E) and ω into (A) gives

$$\begin{aligned} b_2 &- \frac{2y(x+y+2)(b_3-a_2)}{-x^2+y^2-4x-1} - \frac{4y^2(x+y+2)^2 a_3}{(-x^2+y^2-4x-1)^2} \\ &- \left(-\frac{2y}{-x^2+y^2-4x-1} + \frac{2y(x+y+2)(-2x-4)}{(-x^2+y^2-4x-1)^2}\right)(xa_2+ya_3+a_1) \\ &- \left(-\frac{2(x+y+2)}{-x^2+y^2-4x-1} - \frac{2y}{-x^2+y^2-4x-1} \right. \\ &\left. + \frac{4y^2(x+y+2)}{(-x^2+y^2-4x-1)^2}\right)(xb_2+yb_3+b_1) = 0 \end{aligned} \tag{5E}$$

Putting the above in normal form gives

$$-\frac{x^4b_2 + 4x^3yb_2 - 2x^2y^2a_2 + 2x^2y^2a_3 + 4x^2y^2b_2 + 2x^2y^2b_3 - 4x\,y^3a_2 + 4x\,y^3a_3 + 4x\,y^3b_3 - 2y^4a_2 + 2y^4a_3}{}$$
$$= 0$$

Setting the numerator to zero gives

$$\begin{aligned} &-x^4b_2 - 4x^3yb_2 + 2x^2y^2a_2 - 2x^2y^2a_3 - 4x^2y^2b_2 - 2x^2y^2b_3 + 4x\,y^3a_2 \\ &- 4x\,y^3a_3 - 4x\,y^3b_3 + 2y^4a_2 - 2y^4a_3 + y^4b_2 - 2y^4b_3 - 2x^3b_1 - 4x^3b_2 \\ &+ 2x^2ya_1 - 4x^2ya_2 - 4x^2yb_1 - 16x^2yb_2 + 4x\,y^2a_1 - 8x\,y^2a_3 - 2x\,y^2b_1 \\ &- 12x\,y^2b_2 - 8x\,y^2b_3 + 2y^3a_1 + 4y^3a_2 - 8y^3a_3 - 8y^3b_3 - 12x^2b_1 \\ &+ 8xya_1 - 4xya_2 - 16xyb_1 - 4xyb_2 + 8y^2a_1 - 2y^2a_2 - 2y^2a_3 - 4y^2b_1 \\ &- 2y^2b_2 - 2y^2b_3 - 18xb_1 + 4xb_2 + 14ya_1 - 4ya_2 - 4yb_1 - 4b_1 + b_2 = 0 \end{aligned} \tag{6E}$$

Looking at the above PDE shows the following are all the terms with $\{x, y\}$ in them.

$$\{x, y\}$$

The following substitution is now made to be able to collect on all terms with $\{x, y\}$ in them

$$\{x = v_1, y = v_2\}$$

The above PDE (6E) now becomes

$$
\begin{aligned}
&2a_2v_1^2v_2^2+4a_2v_1v_2^3+2a_2v_2^4-2a_3v_1^2v_2^2-4a_3v_1v_2^3-2a_3v_2^4-b_2v_1^4-4b_2v_1^3v_2\\
&-4b_2v_1^2v_2^2+b_2v_2^4-2b_3v_1^2v_2^2-4b_3v_1v_2^3-2b_3v_2^4+2a_1v_1^2v_2+4a_1v_1v_2^2\\
&+2a_1v_2^3-4a_2v_1^2v_2+4a_2v_2^3-8a_3v_1v_2^2-8a_3v_2^3-2b_1v_1^3-4b_1v_1^2v_2-2b_1v_1v_2^2\\
&-4b_2v_1^3-16b_2v_1^2v_2-12b_2v_1v_2^2-8b_3v_1v_2^2-8b_3v_2^3+8a_1v_1v_2+8a_1v_2^2\\
&-4a_2v_1v_2-2a_2v_2^2-2a_3v_2^2-12b_1v_1^2-16b_1v_1v_2-4b_1v_2^2-4b_2v_1v_2-2b_2v_2^2\\
&-2b_3v_2^2+14a_1v_2-4a_2v_2-18b_1v_1-4b_1v_2+4b_2v_1-4b_1+b_2=0
\end{aligned}
\tag{7E}
$$

Collecting the above on the terms v_i introduced, and these are

$$
\{v_1, v_2\}
$$

Equation (7E) now becomes

$$
\begin{aligned}
&-b_2v_1^4-4b_2v_1^3v_2+(-2b_1-4b_2)\,v_1^3+(2a_2-2a_3-4b_2-2b_3)\,v_1^2v_2^2\\
&+(2a_1-4a_2-4b_1-16b_2)\,v_1^2v_2-12b_1v_1^2+(4a_2-4a_3-4b_3)\,v_1v_2^3\\
&+(4a_1-8a_3-2b_1-12b_2-8b_3)\,v_1v_2^2+(8a_1-4a_2-16b_1-4b_2)\,v_1v_2\\
&+(-18b_1+4b_2)\,v_1+(2a_2-2a_3+b_2-2b_3)\,v_2^4+(2a_1+4a_2-8a_3-8b_3)\,v_2^3\\
&+(8a_1-2a_2-2a_3-4b_1-2b_2-2b_3)\,v_2^2+(14a_1-4a_2-4b_1)\,v_2-4b_1+b_2=0
\end{aligned}
\tag{8E}
$$

Setting each coefficients in (8E) to zero gives the following equations to solve

$$
\begin{aligned}
-12b_1&=0\\
-4b_2&=0\\
-b_2&=0\\
-18b_1+4b_2&=0\\
-4b_1+b_2&=0\\
-2b_1-4b_2&=0\\
14a_1-4a_2-4b_1&=0\\
4a_2-4a_3-4b_3&=0\\
2a_1-4a_2-4b_1-16b_2&=0\\
2a_1+4a_2-8a_3-8b_3&=0\\
8a_1-4a_2-16b_1-4b_2&=0\\
2a_2-2a_3-4b_2-2b_3&=0\\
2a_2-2a_3+b_2-2b_3&=0\\
4a_1-8a_3-2b_1-12b_2-8b_3&=0\\
8a_1-2a_2-2a_3-4b_1-2b_2-2b_3&=0
\end{aligned}
$$

Solving the above equations for the unknowns gives

$$\begin{aligned}a_1 &= 0\\ a_2 &= 0\\ a_3 &= -b_3\\ b_1 &= 0\\ b_2 &= 0\\ b_3 &= b_3\end{aligned}$$

Substituting the above solution in the anstaz (1E,2E) (using 1 as arbitrary value for any unknown in the RHS) gives

$$\begin{aligned}\xi &= -y\\ \eta &= y\end{aligned}$$

Shifting is now applied to make $\xi = 0$ in order to simplify the rest of the computation

$$\begin{aligned}\eta &= \eta - \omega(x,y)\,\xi\\ &= y - \left(-\frac{2y(x+y+2)}{-x^2+y^2-4x-1}\right)(-y)\\ &= \frac{x^2y + 2x\,y^2 + y^3 + 4xy + 4y^2 + y}{x^2 - y^2 + 4x + 1}\\ \xi &= 0\end{aligned}$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x,y) \to (R,S)$ where (R,S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}$$

The above comes from the requirements that $\left(\xi\frac{\partial}{\partial x} + \eta\frac{\partial}{\partial y}\right)S(x,y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$R = x$$

S is found from

$$\begin{aligned}S &= \int \frac{1}{\eta} dy\\ &= \int \frac{1}{\frac{x^2y+2x\,y^2+y^3+4xy+4y^2+y}{x^2-y^2+4x+1}} dy\end{aligned}$$

Which results in

$$S = -\ln\left(x^2 + 2xy + y^2 + 4x + 4y + 1\right) + \ln(y)$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x,y)S_y}{R_x + \omega(x,y)R_y} \tag{2}$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x,y)$ is the right hand side of the original ode given by

$$\omega(x,y) = -\frac{2y(x+y+2)}{-x^2 + y^2 - 4x - 1}$$

Evaluating all the partial derivatives gives

$$\begin{aligned} R_x &= 1 \\ R_y &= 0 \\ S_x &= \frac{-2x - 2y - 4}{x^2 + (2y+4)x + y^2 + 4y + 1} \\ S_y &= \frac{x^2 - y^2 + 4x + 1}{y\left(x^2 + 2xy + y^2 + 4x + 4y + 1\right)} \end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = 0 \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = 0$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$S(R) = c_1 \tag{4}$$

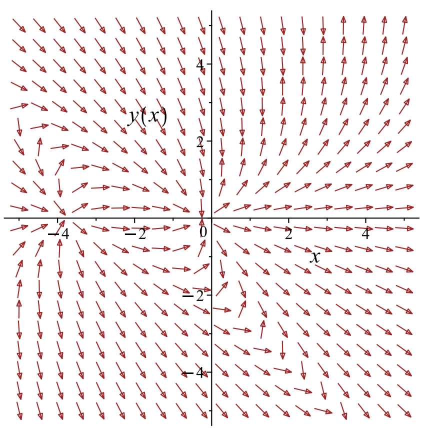
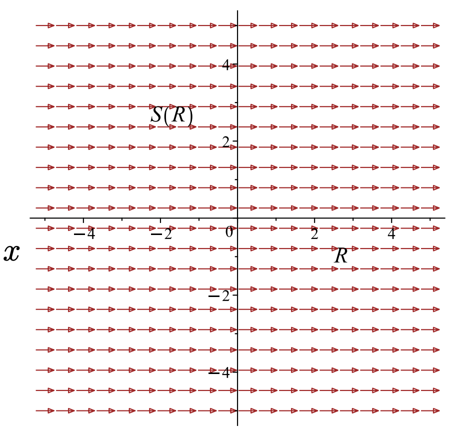
To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$-\ln\left(x^2 + (2y+4)x + y^2 + 4y + 1\right) + \ln(y) = c_1$$

Which simplifies to

$$-\ln\left(x^2+(2y+4)\,x+y^2+4y+1\right)+\ln\left(y\right)=c_1$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in x, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dx}=-\frac{2y(x+y+2)}{-x^2+y^2-4x-1}$ 	$R = x$ $S = -\ln\left(x^2+(2y+4)\,x\right.$	$\frac{dS}{dR}=0$ 

Summary

The solution(s) found are the following

$$-\ln\left(x^2+(2y+4)\,x+y^2+4y+1\right)+\ln\left(y\right)=c_1 \tag{1}$$

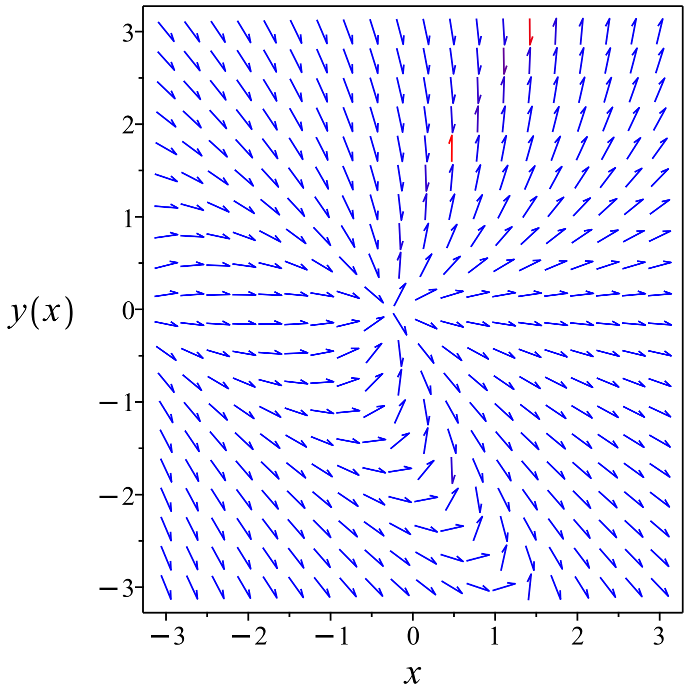


Figure 55: Slope field plot

Verification of solutions

$$-\ln\left(x^2+(2y+4)\,x+y^2+4y+1\right)+\ln(y)=c_1$$

Verified OK.

1.35.2 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x,y)+N(x,y)\,\frac{dy}{dx}=0 \tag{A}$$

We assume there exists a function $\phi(x,y)=c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y)=0$$

Hence

$$\frac{\partial\phi}{\partial x}+\frac{\partial\phi}{\partial y}\frac{dy}{dx}=0 \tag{B}$$

Comparing (A,B) shows that

$$\frac{\partial \phi}{\partial x} = M$$
$$\frac{\partial \phi}{\partial y} = N$$

But since $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x, y)$ but at least we know now that we can do that since the condition $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x, y)\,\mathrm{d}x + N(x, y)\,\mathrm{d}y = 0 \tag{1A}$$

Therefore

$$\begin{aligned} \left(-x^2 + y^2 - 4x - 1\right) \mathrm{d}y &= \left(-2y(x + y + 2)\right) \mathrm{d}x \\ \left(2y(x + y + 2)\right) \mathrm{d}x + \left(-x^2 + y^2 - 4x - 1\right) \mathrm{d}y &= 0 \end{aligned} \tag{2A}$$

Comparing (1A) and (2A) shows that

$$\begin{aligned} M(x, y) &= 2y(x + y + 2) \\ N(x, y) &= -x^2 + y^2 - 4x - 1 \end{aligned}$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Using result found above gives

$$\begin{aligned} \frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}\left(2y(x + y + 2)\right) \\ &= 2x + 4y + 4 \end{aligned}$$

And

$$\begin{aligned} \frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}\left(-x^2 + y^2 - 4x - 1\right) \\ &= -2x - 4 \end{aligned}$$

Since $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$, then the ODE is not exact. Since the ODE is not exact, we will try to find an integrating factor to make it exact. Let

$$\begin{aligned}
A &= \frac{1}{N}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right) \\
&= \frac{1}{-x^2 + y^2 - 4x - 1}((2x + 4y + 4) - (-2x - 4)) \\
&= \frac{-4x - 4y - 8}{x^2 - y^2 + 4x + 1}
\end{aligned}$$

Since A depends on y, it can not be used to obtain an integrating factor. We will now try a second method to find an integrating factor. Let

$$\begin{aligned}
B &= \frac{1}{M}\left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) \\
&= \frac{1}{2y\,(x + y + 2)}((-2x - 4) - (2x + 4y + 4)) \\
&= -\frac{2}{y}
\end{aligned}$$

Since B does not depend on x, it can be used to obtain an integrating factor. Let the integrating factor be μ. Then

$$\begin{aligned}
\mu &= e^{\int B \,\mathrm{d}y} \\
&= e^{\int -\frac{2}{y} \,\mathrm{d}y}
\end{aligned}$$

The result of integrating gives

$$\begin{aligned}
\mu &= e^{-2\ln(y)} \\
&= \frac{1}{y^2}
\end{aligned}$$

M and N are now multiplied by this integrating factor, giving new M and new N which are called $\overline{M}$ and $\overline{N}$ so not to confuse them with the original M and N.

$$\begin{aligned}
\overline{M} &= \mu M \\
&= \frac{1}{y^2}(2y(x + y + 2)) \\
&= \frac{2x + 2y + 4}{y}
\end{aligned}$$

And

$$\begin{aligned}\overline{N} &= \mu N \\ &= \frac{1}{y^2}(-x^2 + y^2 - 4x - 1) \\ &= \frac{-x^2 + y^2 - 4x - 1}{y^2}\end{aligned}$$

So now a modified ODE is obtained from the original ODE which will be exact and can be solved using the standard method. The modified ODE is

$$\begin{aligned}\overline{M} + \overline{N}\frac{\mathrm{d}y}{\mathrm{d}x} &= 0 \\ \left(\frac{2x + 2y + 4}{y}\right) + \left(\frac{-x^2 + y^2 - 4x - 1}{y^2}\right)\frac{\mathrm{d}y}{\mathrm{d}x} &= 0\end{aligned}$$

The following equations are now set up to solve for the function $\phi(x, y)$

$$\frac{\partial \phi}{\partial x} = \overline{M} \tag{1}$$

$$\frac{\partial \phi}{\partial y} = \overline{N} \tag{2}$$

Integrating (1) w.r.t. x gives

$$\begin{aligned}\int \frac{\partial \phi}{\partial x}\,\mathrm{d}x &= \int \overline{M}\,\mathrm{d}x \\ \int \frac{\partial \phi}{\partial x}\,\mathrm{d}x &= \int \frac{2x + 2y + 4}{y}\,\mathrm{d}x\end{aligned}$$

$$\phi = \frac{x(x + 2y + 4)}{y} + f(y) \tag{3}$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$\frac{\partial \phi}{\partial y} = \frac{2x}{y} - \frac{x(x + 2y + 4)}{y^2} + f'(y) \tag{4}$$

$$= -\frac{x(x + 4)}{y^2} + f'(y)$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = \frac{-x^2+y^2-4x-1}{y^2}$. Therefore equation (4) becomes

$$\frac{-x^2 + y^2 - 4x - 1}{y^2} = -\frac{x(x + 4)}{y^2} + f'(y) \tag{5}$$

Solving equation (5) for $f'(y)$ gives

$$f'(y) = \frac{y^2 - 1}{y^2}$$

Integrating the above w.r.t y gives

$$\int f'(y)\, \mathrm{d}y = \int \left(\frac{y^2 - 1}{y^2}\right) \mathrm{d}y$$

$$f(y) = y + \frac{1}{y} + c_1$$

Where c_1 is constant of integration. Substituting result found above for $f(y)$ into equation (3) gives ϕ

$$\phi = \frac{x(x + 2y + 4)}{y} + y + \frac{1}{y} + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = \frac{x(x + 2y + 4)}{y} + y + \frac{1}{y}$$

Summary

The solution(s) found are the following

$$\frac{x(x + 2y + 4)}{y} + y + \frac{1}{y} = c_1 \tag{1}$$

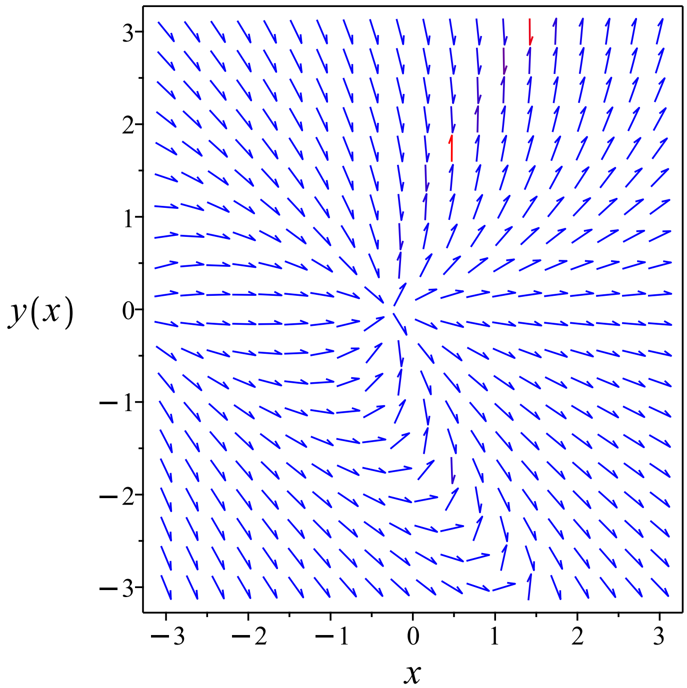


Figure 56: Slope field plot

Verification of solutions

$$\frac{x(x+2y+4)}{y}+y+\frac{1}{y}=c_1$$

Verified OK.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
trying homogeneous types:
trying Chini
differential order: 1; looking for linear symmetries
differential order: 1; found: 1 linear symmetries. Trying reduction of order
1st order, trying the canonical coordinates of the invariance group
<- 1st order, canonical coordinates successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 55

```
dsolve(2*y(x)*(x+y(x)+2)+(y(x)^2-x^2-4*x-1)*diff(y(x),x)=0,y(x), singsol=all)
```

$$y(x) = -x - 2 + \frac{c_1}{2} - \frac{\sqrt{12 + c_1^2 + (-4x - 8)\, c_1}}{2}$$
$$y(x) = -x - 2 + \frac{c_1}{2} + \frac{\sqrt{12 + c_1^2 + (-4x - 8)\, c_1}}{2}$$

✓ Solution by Mathematica

Time used: 0.462 (sec). Leaf size: 74

```
DSolve[2*y[x]*(x+y[x]+2)+(y[x]^2-x^2-4*x-1)*y'[x]==0,y[x],x,IncludeSingularSolutions -> True
```

$$y(x) \to \frac{1}{2}\left(-2x - \sqrt{4(-4 + c_1)x - 4 + c_1{}^2} - c_1\right)$$
$$y(x) \to \frac{1}{2}\left(-2x + \sqrt{4(-4 + c_1)x - 4 + c_1{}^2} - c_1\right)$$
$$y(x) \to 0$$

1.36 problem 36

Internal problem ID [3181]
Internal file name [`OUTPUT/2673_Sunday_June_05_2022_08_38_37_AM_46697388/index.tex`]

Book: Differential equations for engineers by Wei-Chau XIE, Cambridge Press 2010
Section: Chapter 2. First-Order and Simple Higher-Order Differential Equations. Page 78
Problem number: 36.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"bernoulli", "exactWithIntegrationFactor", "first_order_ode_lie_symmetry_lookup"**

Maple gives the following as the ode type

```
[_rational, _Bernoulli]
```

$$\boxed{y^2 + 2yy' = -2 - 2x}$$

1.36.1 Solving as first order ode lie symmetry lookup ode

Writing the ode as

$$y' = -\frac{y^2 + 2x + 2}{2y}$$
$$y' = \omega(x, y)$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x + \omega(\eta_y - \xi_x) - \omega^2\xi_y - \omega_x\xi - \omega_y\eta = 0 \tag{A}$$

The type of this ode is known. It is of type `Bernoulli`. Therefore we do not need to solve the PDE (A), and can just use the lookup table shown below to find ξ, η

Table 19: Lie symmetry infinitesimal lookup table for known first order ODE's

ODE class	Form	ξ	η
linear ode	$y' = f(x)y(x) + g(x)$	0	$e^{\int f dx}$
separable ode	$y' = f(x)\,g(y)$	$\frac{1}{f}$	0
quadrature ode	$y' = f(x)$	0	1
quadrature ode	$y' = g(y)$	1	0
homogeneous ODEs of Class A	$y' = f\left(\frac{y}{x}\right)$	x	y
homogeneous ODEs of Class C	$y' = (a + bx + cy)^{\frac{n}{m}}$	1	$-\frac{b}{c}$
homogeneous class D	$y' = \frac{y}{x} + g(x)\,F\left(\frac{y}{x}\right)$	x^2	xy
First order special form ID 1	$y' = g(x)\,e^{h(x)+by} + f(x)$	$\frac{e^{-\int bf(x)dx - h(x)}}{g(x)}$	$\frac{f(x)e^{-\int bf(x)dx - h(x)}}{g(x)}$
polynomial type ode	$y' = \frac{a_1x+b_1y+c_1}{a_2x+b_2y+c_2}$	$\frac{a_1b_2x - a_2b_1x - b_1c_2 + b_2c_1}{a_1b_2 - a_2b_1}$	$\frac{a_1b_2y - a_2b_1y - a_1c_2 - a_2c_1}{a_1b_2 - a_2b_1}$
Bernoulli ode	$y' = f(x)\,y + g(x)\,y^n$	0	$e^{-\int (n-1)f(x)dx}y^n$
Reduced Riccati	$y' = f_1(x)\,y + f_2(x)\,y^2$	0	$e^{-\int f_1 dx}$

The above table shows that

$$
\begin{aligned}
\xi(x, y) &= 0 \\
\eta(x, y) &= \frac{e^{-x}}{y} \qquad \text{(A1)}
\end{aligned}
$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x, y) \to (R, S)$ where (R, S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$
\frac{dx}{\xi} = \frac{dy}{\eta} = dS \qquad (1)
$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y}\right) S(x, y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the

canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$R = x$$

S is found from

$$\begin{aligned} S &= \int \frac{1}{\eta} dy \\ &= \int \frac{1}{\frac{e^{-x}}{y}} dy \end{aligned}$$

Which results in

$$S = \frac{e^{x} y^{2}}{2}$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x,y) S_y}{R_x + \omega(x,y) R_y} \tag{2}$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x, y)$ is the right hand side of the original ode given by

$$\omega(x,y) = -\frac{y^2 + 2x + 2}{2y}$$

Evaluating all the partial derivatives gives

$$\begin{aligned} R_x &= 1 \\ R_y &= 0 \\ S_x &= \frac{e^{x} y^{2}}{2} \\ S_y &= e^{x} y \end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = e^{x}(-x - 1) \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = e^{R}(-R - 1)$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordinates R, S. Integrating the above gives

$$S(R) = -\mathrm{e}^{R} R + c_1 \tag{4}$$

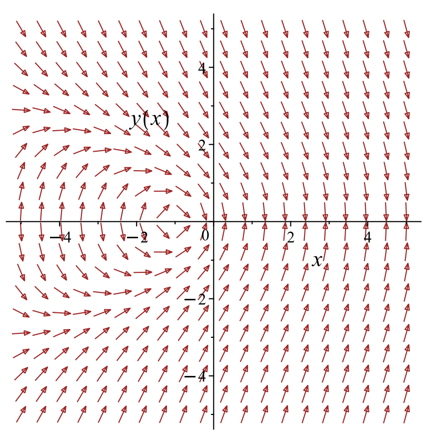
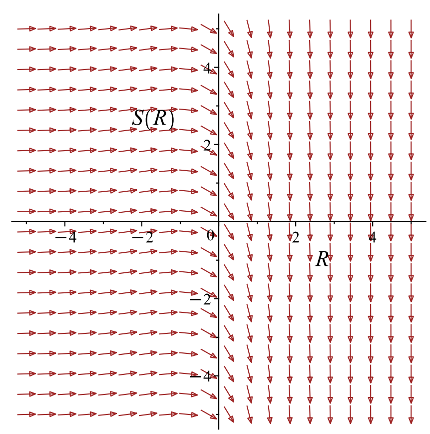
To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$\frac{\mathrm{e}^{x} y^{2}}{2} = -x\,\mathrm{e}^{x} + c_1$$

Which simplifies to

$$\frac{\mathrm{e}^{x} y^{2}}{2} = -x\,\mathrm{e}^{x} + c_1$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in x, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dx} = -\frac{y^2+2x+2}{2y}$ 	$R = x$ $S = \frac{\mathrm{e}^{x} y^{2}}{2}$	$\frac{dS}{dR} = \mathrm{e}^{R}(-R-1)$ 

Summary

The solution(s) found are the following

$$\frac{\mathrm{e}^{x} y^{2}}{2} = -x\,\mathrm{e}^{x} + c_1 \tag{1}$$

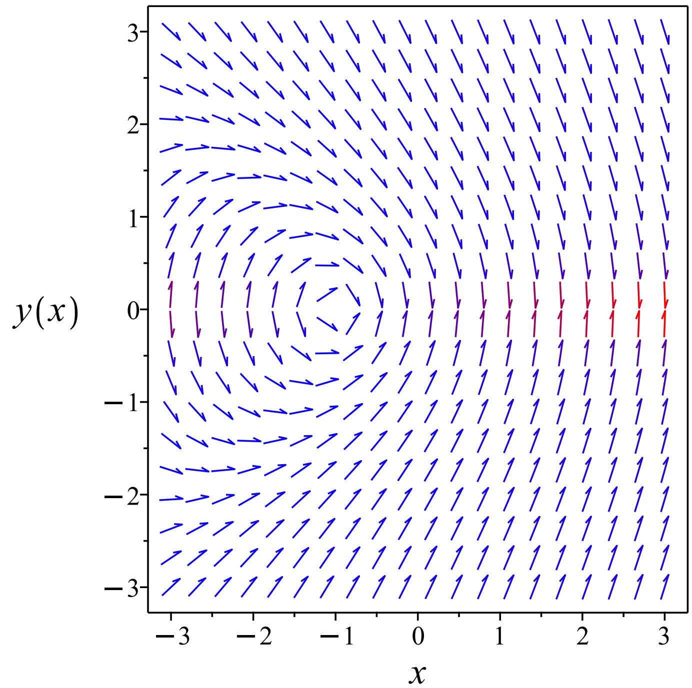


Figure 57: Slope field plot

Verification of solutions

$$\frac{e^{x} y^{2}}{2} = -x\,e^{x} + c_1$$

Verified OK.

1.36.2 Solving as bernoulli ode

In canonical form, the ODE is

$$\begin{aligned} y' &= F(x, y) \\ &= -\frac{y^{2} + 2x + 2}{2y} \end{aligned}$$

This is a Bernoulli ODE.

$$y' = -\frac{1}{2}y - x - 1\frac{1}{y} \tag{1}$$

The standard Bernoulli ODE has the form

$$y' = f_0(x)y + f_1(x)y^{n} \tag{2}$$

The first step is to divide the above equation by y^n which gives

$$\frac{y'}{y^{n}} = f_0(x)y^{1-n} + f_1(x) \tag{3}$$

The next step is use the substitution $w = y^{1-n}$ in equation (3) which generates a new ODE in $w(x)$ which will be linear and can be easily solved using an integrating factor. Backsubstitution then gives the solution $y(x)$ which is what we want.

This method is now applied to the ODE at hand. Comparing the ODE (1) With (2) Shows that

$$
\begin{aligned}
f_0(x) &= -\frac{1}{2} \\
f_1(x) &= -x - 1 \\
n &= -1
\end{aligned}
$$

Dividing both sides of ODE (1) by $y^n = \frac{1}{y}$ gives

$$
y'y = -\frac{y^2}{2} - x - 1 \tag{4}
$$

Let

$$
\begin{aligned}
w &= y^{1-n} \\
&= y^2 \qquad (5)
\end{aligned}
$$

Taking derivative of equation (5) w.r.t x gives

$$
w' = 2yy' \tag{6}
$$

Substituting equations (5) and (6) into equation (4) gives

$$
\begin{aligned}
\frac{w'(x)}{2} &= -\frac{w(x)}{2} - x - 1 \\
w' &= -w - 2 - 2x \qquad (7)
\end{aligned}
$$

The above now is a linear ODE in $w(x)$ which is now solved.

Entering Linear first order ODE solver. In canonical form a linear first order is

$$
w'(x) + p(x)w(x) = q(x)
$$

Where here

$$
\begin{aligned}
p(x) &= 1 \\
q(x) &= -2 - 2x
\end{aligned}
$$

Hence the ode is

$$
w'(x) + w(x) = -2 - 2x
$$

The integrating factor μ is

$$\mu = e^{\int 1dx}$$
$$= e^x$$

The ode becomes

$$\frac{d}{dx}(\mu w) = (\mu)(-2 - 2x)$$
$$\frac{d}{dx}(e^x w) = (e^x)(-2 - 2x)$$
$$d(e^x w) = (-2\,e^x(x+1))\,dx$$

Integrating gives

$$e^x w = \int -2\,e^x(x+1)\,dx$$
$$e^x w = -2x\,e^x + c_1$$

Dividing both sides by the integrating factor $\mu = e^x$ results in

$$w(x) = -2\,e^{-x} x\,e^x + c_1 e^{-x}$$

which simplifies to

$$w(x) = -2x + c_1 e^{-x}$$

Replacing w in the above by y^2 using equation (5) gives the final solution.

$$y^2 = -2x + c_1 e^{-x}$$

Solving for y gives

$$y(x) = \sqrt{-2x + c_1 e^{-x}}$$
$$y(x) = -\sqrt{-2x + c_1 e^{-x}}$$

<u>Summary</u>

The solution(s) found are the following

$$y = \sqrt{-2x + c_1 e^{-x}} \tag{1}$$
$$y = -\sqrt{-2x + c_1 e^{-x}} \tag{2}$$

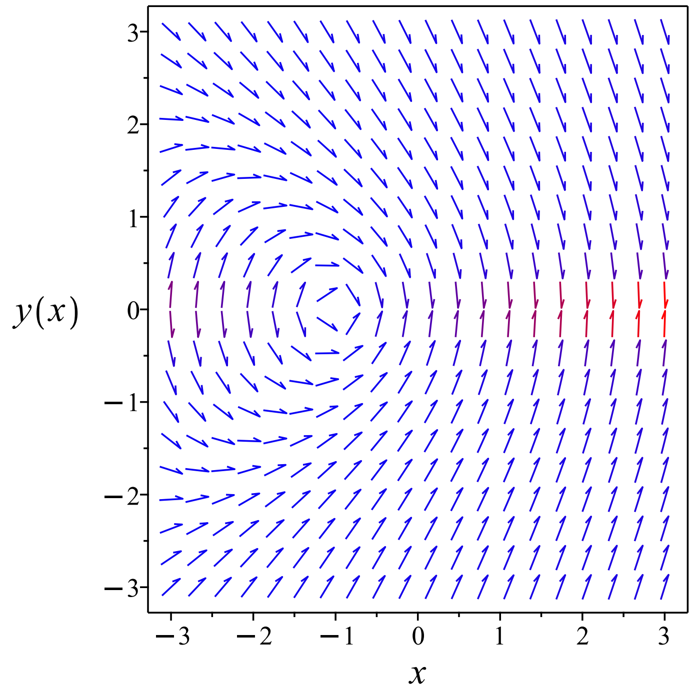


Figure 58: Slope field plot

Verification of solutions

$$y = \sqrt{-2x + c_1 e^{-x}}$$

Verified OK.

$$y = -\sqrt{-2x + c_1 e^{-x}}$$

Verified OK.

1.36.3 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x, y) + N(x, y) \frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x, y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x, y) = 0$$

Hence

$$\frac{\partial \phi}{\partial x} + \frac{\partial \phi}{\partial y}\frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$\begin{aligned}\frac{\partial \phi}{\partial x} &= M \\ \frac{\partial \phi}{\partial y} &= N\end{aligned}$$

But since $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x,y)$ but at least we know now that we can do that since the condition $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x,y)\,\mathrm{d}x + N(x,y)\,\mathrm{d}y = 0 \tag{1A}$$

Therefore

$$\begin{aligned}(2y)\,\mathrm{d}y &= \left(-y^2 - 2x - 2\right)\mathrm{d}x \\ \left(y^2 + 2x + 2\right)\mathrm{d}x + (2y)\,\mathrm{d}y &= 0 \qquad (2A)\end{aligned}$$

Comparing (1A) and (2A) shows that

$$\begin{aligned}M(x,y) &= y^2 + 2x + 2 \\ N(x,y) &= 2y\end{aligned}$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Using result found above gives

$$\begin{aligned}\frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}\left(y^2 + 2x + 2\right) \\ &= 2y\end{aligned}$$

And

$$\begin{aligned}\frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}(2y)\\ &= 0\end{aligned}$$

Since $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$, then the ODE is not exact. Since the ODE is not exact, we will try to find an integrating factor to make it exact. Let

$$\begin{aligned}A &= \frac{1}{N}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right)\\ &= \frac{1}{2y}((2y) - (0))\\ &= 1\end{aligned}$$

Since A does not depend on y, then it can be used to find an integrating factor. The integrating factor μ is

$$\begin{aligned}\mu &= e^{\int A \, \mathrm{d}x}\\ &= e^{\int 1 \, \mathrm{d}x}\end{aligned}$$

The result of integrating gives

$$\begin{aligned}\mu &= e^{x}\\ &= \mathrm{e}^{x}\end{aligned}$$

M and N are multiplied by this integrating factor, giving new M and new N which are called $\overline{M}$ and $\overline{N}$ for now so not to confuse them with the original M and N.

$$\begin{aligned}\overline{M} &= \mu M\\ &= \mathrm{e}^{x}\left(y^{2}+2x+2\right)\\ &= \mathrm{e}^{x}\left(y^{2}+2x+2\right)\end{aligned}$$

And

$$\begin{aligned}\overline{N} &= \mu N\\ &= \mathrm{e}^{x}(2y)\\ &= 2\,\mathrm{e}^{x} y\end{aligned}$$

Now a modified ODE is ontained from the original ODE, which is exact and can be solved. The modified ODE is

$$\begin{aligned}\overline{M} + \overline{N}\frac{\mathrm{d}y}{\mathrm{d}x} &= 0\\ \left(\mathrm{e}^{x}\left(y^{2}+2x+2\right)\right) + \left(2\,\mathrm{e}^{x} y\right)\frac{\mathrm{d}y}{\mathrm{d}x} &= 0\end{aligned}$$

The following equations are now set up to solve for the function $\phi(x, y)$

$$\frac{\partial \phi}{\partial x} = \overline{M} \tag{1}$$
$$\frac{\partial \phi}{\partial y} = \overline{N} \tag{2}$$

Integrating (1) w.r.t. x gives

$$\int \frac{\partial \phi}{\partial x} \,\mathrm{d}x = \int \overline{M} \,\mathrm{d}x$$
$$\int \frac{\partial \phi}{\partial x} \,\mathrm{d}x = \int \mathrm{e}^{x}\left(y^{2} + 2x + 2\right) \mathrm{d}x$$
$$\phi = \left(y^{2} + 2x\right) \mathrm{e}^{x} + f(y) \tag{3}$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$\frac{\partial \phi}{\partial y} = 2\,\mathrm{e}^{x} y + f'(y) \tag{4}$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = 2\,\mathrm{e}^{x} y$. Therefore equation (4) becomes

$$2\,\mathrm{e}^{x} y = 2\,\mathrm{e}^{x} y + f'(y) \tag{5}$$

Solving equation (5) for $f'(y)$ gives

$$f'(y) = 0$$

Therefore

$$f(y) = c_1$$

Where c_1 is constant of integration. Substituting this result for $f(y)$ into equation (3) gives ϕ

$$\phi = \left(y^{2} + 2x\right) \mathrm{e}^{x} + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = \left(y^{2} + 2x\right) \mathrm{e}^{x}$$

Summary

The solution(s) found are the following

$$\left(y^2 + 2x\right) e^x = c_1 \tag{1}$$

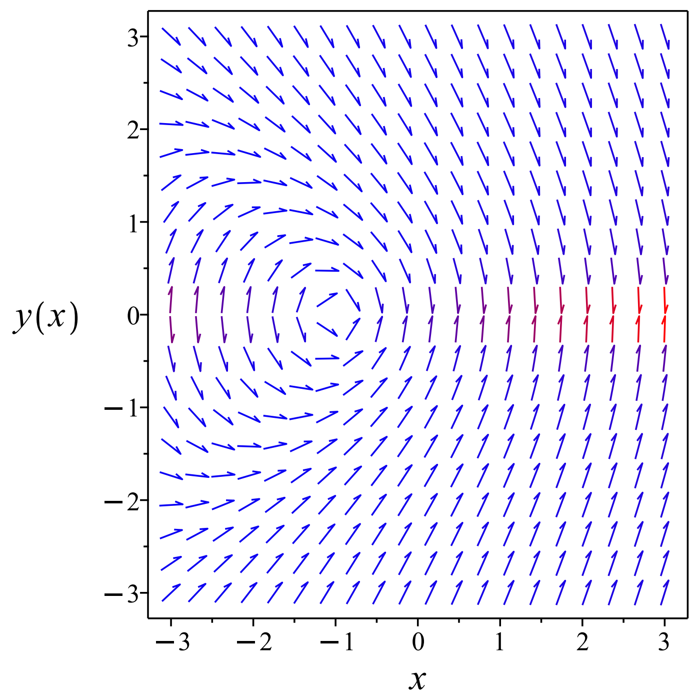


Figure 59: Slope field plot

Verification of solutions

$$\left(y^2 + 2x\right) e^x = c_1$$

Verified OK.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
<- Bernoulli successful`
```

✓ Solution by Maple

Time used: 0.015 (sec). Leaf size: 33

```
dsolve((2+y(x)^2+2*x)+(2*y(x))*diff(y(x),x)=0,y(x), singsol=all)
```

$$y(x) = \sqrt{e^{-x}c_1 - 2x}$$
$$y(x) = -\sqrt{e^{-x}c_1 - 2x}$$

✓ Solution by Mathematica

Time used: 3.531 (sec). Leaf size: 43

```
DSolve[(2+y[x]^2+2*x)+(2*y[x])*y'[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \rightarrow -\sqrt{-2x + c_1 e^{-x}}$$
$$y(x) \rightarrow \sqrt{-2x + c_1 e^{-x}}$$

1.37 problem 37

Internal problem ID [3182]
Internal file name [`OUTPUT/2674_Sunday_June_05_2022_08_38_38_AM_98414307/index.tex`]

Book: Differential equations for engineers by Wei-Chau XIE, Cambridge Press 2010
Section: Chapter 2. First-Order and Simple Higher-Order Differential Equations. Page 78
Problem number: 37.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"exactWithIntegrationFactor"**

Maple gives the following as the ode type

`[_rational]`

$$\boxed{2y^2x - y + (y^2 + x + y)\, y' = 0}$$

1.37.1 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x,y) + N(x,y)\,\frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x,y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y) = 0$$

Hence

$$\frac{\partial \phi}{\partial x} + \frac{\partial \phi}{\partial y}\frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$\frac{\partial \phi}{\partial x} = M$$
$$\frac{\partial \phi}{\partial y} = N$$

But since $\frac{\partial^2\phi}{\partial x\partial y} = \frac{\partial^2\phi}{\partial y\partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x,y)$ but at least we know now that we can do that since the condition $\frac{\partial^2\phi}{\partial x\partial y} = \frac{\partial^2\phi}{\partial y\partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x,y)\,\mathrm{d}x + N(x,y)\,\mathrm{d}y = 0 \tag{1A}$$

Therefore

$$\begin{aligned}\left(y^2+x+y\right)\mathrm{d}y &= \left(-2x\,y^2+y\right)\mathrm{d}x\\ \left(2x\,y^2-y\right)\mathrm{d}x + \left(y^2+x+y\right)\mathrm{d}y &= 0 \end{aligned}\tag{2A}$$

Comparing (1A) and (2A) shows that

$$\begin{aligned}M(x,y) &= 2x\,y^2 - y\\ N(x,y) &= y^2 + x + y\end{aligned}$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Using result found above gives

$$\begin{aligned}\frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}\left(2x\,y^2 - y\right)\\ &= 4xy - 1\end{aligned}$$

And

$$\begin{aligned}\frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}\left(y^2 + x + y\right)\\ &= 1\end{aligned}$$

Since $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$, then the ODE is <u>not exact</u>. Since the ODE is not exact, we will try to find an integrating factor to make it exact. Let

$$\begin{aligned}A &= \frac{1}{N}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right)\\ &= \frac{1}{y^2+x+y}((4xy-1)-(1))\\ &= \frac{4xy-2}{y^2+x+y}\end{aligned}$$

Since A depends on y, it can not be used to obtain an integrating factor. We will now try a second method to find an integrating factor. Let

$$\begin{aligned} B &= \frac{1}{M}\left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) \\ &= \frac{1}{2x\,y^2 - y}((1) - (4xy - 1)) \\ &= -\frac{2}{y} \end{aligned}$$

Since B does not depend on x, it can be used to obtain an integrating factor. Let the integrating factor be μ. Then

$$\begin{aligned} \mu &= e^{\int B \,\mathrm{d}y} \\ &= e^{\int -\frac{2}{y} \,\mathrm{d}y} \end{aligned}$$

The result of integrating gives

$$\begin{aligned} \mu &= e^{-2\ln(y)} \\ &= \frac{1}{y^2} \end{aligned}$$

M and N are now multiplied by this integrating factor, giving new M and new N which are called $\overline{M}$ and $\overline{N}$ so not to confuse them with the original M and N.

$$\begin{aligned} \overline{M} &= \mu M \\ &= \frac{1}{y^2}(2x\,y^2 - y) \\ &= \frac{2xy - 1}{y} \end{aligned}$$

And

$$\begin{aligned} \overline{N} &= \mu N \\ &= \frac{1}{y^2}(y^2 + x + y) \\ &= \frac{y^2 + x + y}{y^2} \end{aligned}$$

So now a modified ODE is obtained from the original ODE which will be exact and can be solved using the standard method. The modified ODE is

$$\begin{aligned} \overline{M} + \overline{N}\frac{\mathrm{d}y}{\mathrm{d}x} &= 0 \\ \left(\frac{2xy - 1}{y}\right) + \left(\frac{y^2 + x + y}{y^2}\right)\frac{\mathrm{d}y}{\mathrm{d}x} &= 0 \end{aligned}$$

The following equations are now set up to solve for the function $\phi(x,y)$

$$\frac{\partial \phi}{\partial x} = \overline{M} \tag{1}$$

$$\frac{\partial \phi}{\partial y} = \overline{N} \tag{2}$$

Integrating (1) w.r.t. x gives

$$\int \frac{\partial \phi}{\partial x} \, \mathrm{d}x = \int \overline{M} \, \mathrm{d}x$$

$$\int \frac{\partial \phi}{\partial x} \, \mathrm{d}x = \int \frac{2xy-1}{y} \, \mathrm{d}x$$

$$\phi = \frac{x(xy-1)}{y} + f(y) \tag{3}$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$\frac{\partial \phi}{\partial y} = \frac{x^2}{y} - \frac{x(xy-1)}{y^2} + f'(y) \tag{4}$$

$$= \frac{x}{y^2} + f'(y)$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = \frac{y^2+x+y}{y^2}$. Therefore equation (4) becomes

$$\frac{y^2+x+y}{y^2} = \frac{x}{y^2} + f'(y) \tag{5}$$

Solving equation (5) for $f'(y)$ gives

$$f'(y) = \frac{y+1}{y}$$

Integrating the above w.r.t y gives

$$\int f'(y) \, \mathrm{d}y = \int \left(\frac{y+1}{y}\right) \mathrm{d}y$$

$$f(y) = y + \ln(y) + c_1$$

Where c_1 is constant of integration. Substituting result found above for $f(y)$ into equation (3) gives ϕ

$$\phi = \frac{x(xy-1)}{y} + y + \ln(y) + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = \frac{x(xy-1)}{y} + y + \ln(y)$$

Summary

The solution(s) found are the following

$$\frac{x(yx-1)}{y} + y + \ln(y) = c_1 \tag{1}$$

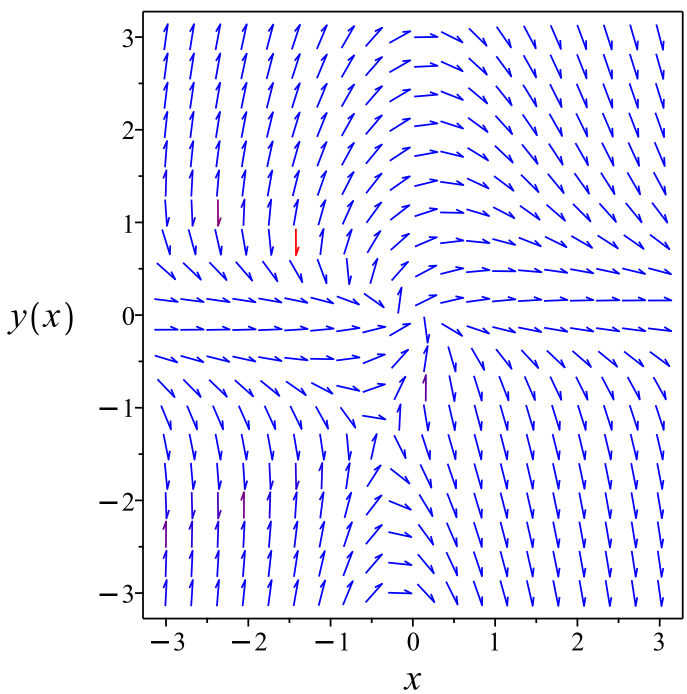


Figure 60: Slope field plot

Verification of solutions

$$\frac{x(yx-1)}{y} + y + \ln(y) = c_1$$

Verified OK.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
trying homogeneous types:
trying Chini
differential order: 1; looking for linear symmetries
trying exact
<- exact successful`
```

✓ Solution by Maple

Time used: 0.031 (sec). Leaf size: 28

```
dsolve((2*x*y(x)^2-y(x))+(y(x)^2+x+y(x))*diff(y(x),x)=0,y(x), singsol=all)
```

$$y(x) = \mathrm{e}^{\mathrm{RootOf}\left(x^2 \mathrm{e}^{\_Z} + \mathrm{e}^{2\_Z} + c_1 \mathrm{e}^{\_Z} + \_Z \mathrm{e}^{\_Z} - x\right)}$$

✓ Solution by Mathematica

Time used: 0.18 (sec). Leaf size: 22

```
DSolve[(2*x*y[x]^2-y[x])+(y[x]^2+x+y[x])*y'[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$\text{Solve}\left[x^2 - \frac{x}{y(x)} + y(x) + \log(y(x)) = c_1, y(x)\right]$$

1.38 problem 38

Internal problem ID [3183]
Internal file name [`OUTPUT/2675_Sunday_June_05_2022_08_38_39_AM_82102619/index.tex`]

Book: Differential equations for engineers by Wei-Chau XIE, Cambridge Press 2010
Section: Chapter 2. First-Order and Simple Higher-Order Differential Equations. Page 78
Problem number: 38.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"exactWithIntegrationFactor"**

Maple gives the following as the ode type

```
[_rational, [_Abel, `2nd type`, `class A`]]
```

$$\boxed{y(y+x)+(x+2y-1)\,y'=0}$$

1.38.1 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x,y)+N(x,y)\,\frac{dy}{dx}=0 \tag{A}$$

We assume there exists a function $\phi(x,y)=c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y)=0$$

Hence

$$\frac{\partial\phi}{\partial x}+\frac{\partial\phi}{\partial y}\frac{dy}{dx}=0 \tag{B}$$

Comparing (A,B) shows that

$$\frac{\partial\phi}{\partial x}=M$$
$$\frac{\partial\phi}{\partial y}=N$$

But since $\frac{\partial^2\phi}{\partial x\partial y} = \frac{\partial^2\phi}{\partial y\partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x,y)$ but at least we know now that we can do that since the condition $\frac{\partial^2\phi}{\partial x\partial y} = \frac{\partial^2\phi}{\partial y\partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x,y)\,\mathrm{d}x + N(x,y)\,\mathrm{d}y = 0 \tag{1A}$$

Therefore

$$\begin{aligned}(x+2y-1)\,\mathrm{d}y &= (-y(y+x))\,\mathrm{d}x\\ (y(y+x))\,\mathrm{d}x + (x+2y-1)\,\mathrm{d}y &= 0 \end{aligned} \tag{2A}$$

Comparing (1A) and (2A) shows that

$$\begin{aligned}M(x,y) &= y(y+x)\\ N(x,y) &= x+2y-1\end{aligned}$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Using result found above gives

$$\begin{aligned}\frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}(y(y+x))\\ &= x+2y\end{aligned}$$

And

$$\begin{aligned}\frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}(x+2y-1)\\ &= 1\end{aligned}$$

Since $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$, then the ODE is <u>not exact</u>. Since the ODE is not exact, we will try to find an integrating factor to make it exact. Let

$$\begin{aligned}A &= \frac{1}{N}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right)\\ &= \frac{1}{x+2y-1}((x+2y)-(1))\\ &= 1\end{aligned}$$

Since A does not depend on y, then it can be used to find an integrating factor. The integrating factor μ is

$$\begin{aligned}\mu &= e^{\int A\,dx}\\ &= e^{\int 1\,dx}\end{aligned}$$

The result of integrating gives

$$\begin{aligned}\mu &= e^{x}\\ &= \mathrm{e}^{x}\end{aligned}$$

M and N are multiplied by this integrating factor, giving new M and new N which are called $\overline{M}$ and $\overline{N}$ for now so not to confuse them with the original M and N.

$$\begin{aligned}\overline{M} &= \mu M\\ &= \mathrm{e}^{x}(y(y+x))\\ &= y(y+x)\,\mathrm{e}^{x}\end{aligned}$$

And

$$\begin{aligned}\overline{N} &= \mu N\\ &= \mathrm{e}^{x}(x+2y-1)\\ &= (x+2y-1)\,\mathrm{e}^{x}\end{aligned}$$

Now a modified ODE is ontained from the original ODE, which is exact and can be solved. The modified ODE is

$$\begin{aligned}\overline{M}+\overline{N}\frac{\mathrm{d}y}{\mathrm{d}x} &= 0\\ \left(y(y+x)\,\mathrm{e}^{x}\right)+\left((x+2y-1)\,\mathrm{e}^{x}\right)\frac{\mathrm{d}y}{\mathrm{d}x} &= 0\end{aligned}$$

The following equations are now set up to solve for the function $\phi(x,y)$

$$\frac{\partial\phi}{\partial x} = \overline{M} \tag{1}$$

$$\frac{\partial\phi}{\partial y} = \overline{N} \tag{2}$$

Integrating (1) w.r.t. x gives

$$\begin{aligned}\int\frac{\partial\phi}{\partial x}\,\mathrm{d}x &= \int\overline{M}\,\mathrm{d}x\\ \int\frac{\partial\phi}{\partial x}\,\mathrm{d}x &= \int y(y+x)\,\mathrm{e}^{x}\,\mathrm{d}x\end{aligned}$$

$$\phi = (x-1+y)\,y\,\mathrm{e}^{x}+f(y) \tag{3}$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$\begin{aligned} \frac{\partial \phi}{\partial y} &= \mathrm{e}^x y + (x - 1 + y)\,\mathrm{e}^x + f'(y) \\ &= (x + 2y - 1)\,\mathrm{e}^x + f'(y) \end{aligned} \tag{4}$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = (x + 2y - 1)\,\mathrm{e}^x$. Therefore equation (4) becomes

$$(x + 2y - 1)\,\mathrm{e}^x = (x + 2y - 1)\,\mathrm{e}^x + f'(y) \tag{5}$$

Solving equation (5) for $f'(y)$ gives

$$f'(y) = 0$$

Therefore

$$f(y) = c_1$$

Where c_1 is constant of integration. Substituting this result for $f(y)$ into equation (3) gives ϕ

$$\phi = (x - 1 + y)\,y\,\mathrm{e}^x + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = (x - 1 + y)\,y\,\mathrm{e}^x$$

Summary

The solution(s) found are the following

$$(y + x - 1)\,y\,\mathrm{e}^x = c_1 \tag{1}$$

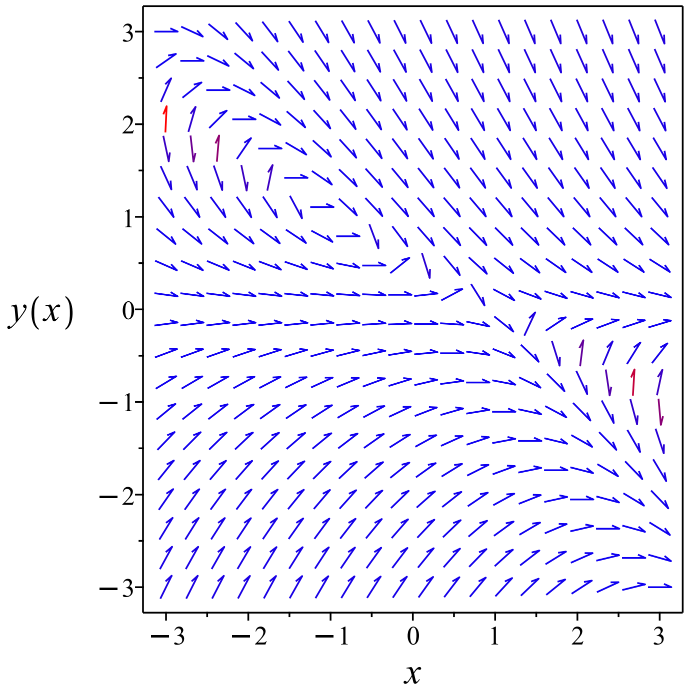


Figure 61: Slope field plot

Verification of solutions

$$(y + x - 1)\, y\, e^{x} = c_1$$

Verified OK.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
trying homogeneous types:
trying Chini
differential order: 1; looking for linear symmetries
trying exact
<- exact successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 63

```
dsolve(y(x)*(x+y(x))+(x+2*y(x)-1)*diff(y(x),x)=0,y(x), singsol=all)
```

$$y(x) = -\frac{x}{2}+\frac{1}{2}-\frac{\sqrt{\mathrm{e}^{x}\left(\left(x-1\right)^{2}\mathrm{e}^{x}-4c_{1}\right)}\,\mathrm{e}^{-x}}{2}$$
$$y(x) = -\frac{x}{2}+\frac{1}{2}+\frac{\sqrt{\mathrm{e}^{x}\left(\left(x-1\right)^{2}\mathrm{e}^{x}-4c_{1}\right)}\,\mathrm{e}^{-x}}{2}$$

✓ Solution by Mathematica

Time used: 11.91 (sec). Leaf size: 80

```
DSolve[y[x]*(x+y[x])+(x+2*y[x]-1)*y'[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \frac{1}{2}\left(-x-\frac{\sqrt{e^{x}(x-1)^{2}+4c_{1}}}{\sqrt{e^{x}}}+1\right)$$
$$y(x) \to \frac{1}{2}\left(-x+\frac{\sqrt{e^{x}(x-1)^{2}+4c_{1}}}{\sqrt{e^{x}}}+1\right)$$

1.39 problem 39

Internal problem ID [3184]
Internal file name [OUTPUT/2676_Sunday_June_05_2022_08_38_39_AM_35960000/index.tex]

Book: Differential equations for engineers by Wei-Chau XIE, Cambridge Press 2010
Section: Chapter 2. First-Order and Simple Higher-Order Differential Equations. Page 78
Problem number: 39.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"exactWithIntegrationFactor"**

Maple gives the following as the ode type

```
[`y=_G(x,y')`]
```

$$\boxed{2x\left(x^{2}-\sin\left(y\right)+1\right)+\left(x^{2}+1\right)\cos\left(y\right)y'=0}$$

1.39.1 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x,y)+N(x,y)\frac{dy}{dx}=0 \tag{A}$$

We assume there exists a function $\phi(x,y)=c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y)=0$$

Hence

$$\frac{\partial\phi}{\partial x}+\frac{\partial\phi}{\partial y}\frac{dy}{dx}=0 \tag{B}$$

Comparing (A,B) shows that

$$\frac{\partial\phi}{\partial x}=M$$
$$\frac{\partial\phi}{\partial y}=N$$

But since $\frac{\partial^2\phi}{\partial x\partial y} = \frac{\partial^2\phi}{\partial y\partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x,y)$ but at least we know now that we can do that since the condition $\frac{\partial^2\phi}{\partial x\partial y} = \frac{\partial^2\phi}{\partial y\partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x,y)\,\mathrm{d}x + N(x,y)\,\mathrm{d}y = 0 \tag{1A}$$

Therefore

$$\begin{aligned}
\left((x^2+1)\cos(y)\right)\mathrm{d}y &= \left(-2x\left(x^2-\sin(y)+1\right)\right)\mathrm{d}x \\
\left(2x\left(x^2-\sin(y)+1\right)\right)\mathrm{d}x + \left((x^2+1)\cos(y)\right)\mathrm{d}y &= 0 \qquad (2A)
\end{aligned}$$

Comparing (1A) and (2A) shows that

$$\begin{aligned}
M(x,y) &= 2x\left(x^2-\sin(y)+1\right) \\
N(x,y) &= \left(x^2+1\right)\cos(y)
\end{aligned}$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Using result found above gives

$$\begin{aligned}
\frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}\left(2x\left(x^2-\sin(y)+1\right)\right) \\
&= -2x\cos(y)
\end{aligned}$$

And

$$\begin{aligned}
\frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}\left(\left(x^2+1\right)\cos(y)\right) \\
&= 2x\cos(y)
\end{aligned}$$

Since $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$, then the ODE is <u>not exact</u>. Since the ODE is not exact, we will try to find an integrating factor to make it exact. Let

$$\begin{aligned}
A &= \frac{1}{N}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right) \\
&= \frac{\sec(y)}{x^2+1}\left(\left(-2x\cos(y)\right) - \left(2x\cos(y)\right)\right) \\
&= -\frac{4x}{x^2+1}
\end{aligned}$$

Since A does not depend on y, then it can be used to find an integrating factor. The integrating factor μ is

$$\mu = e^{\int A \, \mathrm{d}x}$$
$$= e^{\int -\frac{4x}{x^2+1} \, \mathrm{d}x}$$

The result of integrating gives

$$\mu = e^{-2\ln(x^2+1)}$$
$$= \frac{1}{(x^2+1)^2}$$

M and N are multiplied by this integrating factor, giving new M and new N which are called $\overline{M}$ and $\overline{N}$ for now so not to confuse them with the original M and N.

$$\overline{M} = \mu M$$
$$= \frac{1}{(x^2+1)^2} \left(2x\left(x^2 - \sin(y) + 1\right)\right)$$
$$= -\frac{2x(-x^2 + \sin(y) - 1)}{(x^2+1)^2}$$

And

$$\overline{N} = \mu N$$
$$= \frac{1}{(x^2+1)^2} \left(\left(x^2+1\right)\cos(y)\right)$$
$$= \frac{\cos(y)}{x^2+1}$$

Now a modified ODE is ontained from the original ODE, which is exact and can be solved. The modified ODE is

$$\overline{M} + \overline{N}\frac{\mathrm{d}y}{\mathrm{d}x} = 0$$
$$\left(-\frac{2x(-x^2 + \sin(y) - 1)}{(x^2+1)^2}\right) + \left(\frac{\cos(y)}{x^2+1}\right)\frac{\mathrm{d}y}{\mathrm{d}x} = 0$$

The following equations are now set up to solve for the function $\phi(x, y)$

$$\frac{\partial \phi}{\partial x} = \overline{M} \tag{1}$$
$$\frac{\partial \phi}{\partial y} = \overline{N} \tag{2}$$

Integrating (1) w.r.t. x gives

$$\int \frac{\partial \phi}{\partial x} \, dx = \int \overline{M} \, dx$$

$$\int \frac{\partial \phi}{\partial x} \, dx = \int -\frac{2x(-x^2 + \sin(y) - 1)}{(x^2+1)^2} \, dx$$

$$\phi = \ln(x^2+1) + \frac{\sin(y)}{x^2+1} + f(y) \tag{3}$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$\frac{\partial \phi}{\partial y} = \frac{\cos(y)}{x^2+1} + f'(y) \tag{4}$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = \frac{\cos(y)}{x^2+1}$. Therefore equation (4) becomes

$$\frac{\cos(y)}{x^2+1} = \frac{\cos(y)}{x^2+1} + f'(y) \tag{5}$$

Solving equation (5) for $f'(y)$ gives

$$f'(y) = 0$$

Therefore

$$f(y) = c_1$$

Where c_1 is constant of integration. Substituting this result for $f(y)$ into equation (3) gives ϕ

$$\phi = \ln(x^2+1) + \frac{\sin(y)}{x^2+1} + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = \ln(x^2+1) + \frac{\sin(y)}{x^2+1}$$

Summary

The solution(s) found are the following

$$\ln(x^2+1) + \frac{\sin(y)}{x^2+1} = c_1 \tag{1}$$

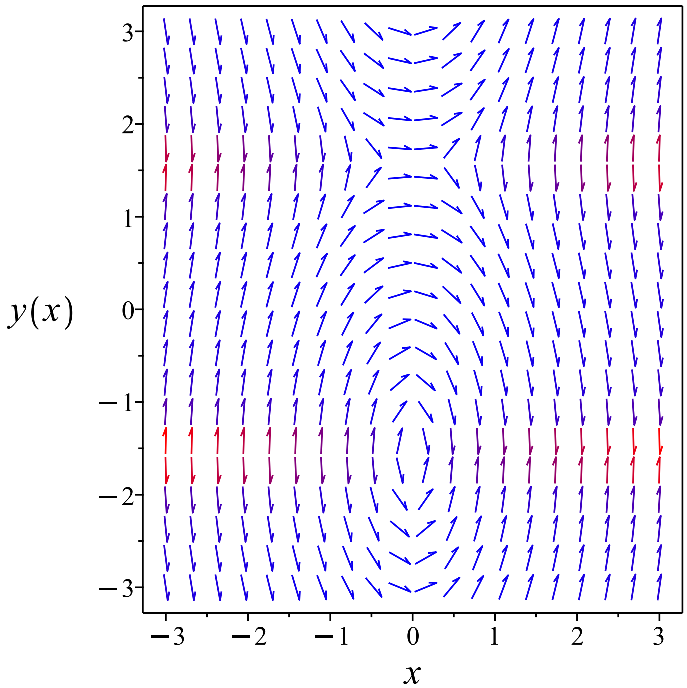


Figure 62: Slope field plot

Verification of solutions

$$\ln\left(x^2+1\right)+\frac{\sin\left(y\right)}{x^2+1}=c_1$$

Verified OK.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
trying homogeneous types:
trying Chini
differential order: 1; looking for linear symmetries
trying exact
<- exact successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 21

```
dsolve(2*x*(x^2-sin(y(x))+1)+(x^2+1)*cos(y(x))*diff(y(x),x)=0,y(x), singsol=all)
```

$$y(x) = -\arcsin\left(\left(x^2+1\right)\left(c_1+\ln\left(x^2+1\right)\right)\right)$$

✓ Solution by Mathematica

Time used: 7.478 (sec). Leaf size: 25

```
DSolve[2*x*(x^2-Sin[y[x]]+1)+(x^2+1)*Cos[y[x]]*y'[x]==0,y[x],x,IncludeSingularSolutions -> T
```

$$y(x) \to -\arcsin\left(\left(x^2+1\right)\left(\log\left(x^2+1\right)+8c_1\right)\right)$$

1.40 problem 41

Internal problem ID [3185]
Internal file name [`OUTPUT/2677_Sunday_June_05_2022_08_38_40_AM_78153908/index.tex`]

Book: Differential equations for engineers by Wei-Chau XIE, Cambridge Press 2010
Section: Chapter 2. First-Order and Simple Higher-Order Differential Equations. Page 78
Problem number: 41.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"riccati", "exactByInspection", "homogeneousTypeD2"**

Maple gives the following as the ode type

```
[[_homogeneous, `class D`], _rational, _Riccati]
```

$$\boxed{y + y^2 - xy' = -x^2}$$

1.40.1 Solving as homogeneousTypeD2 ode

Using the change of variables $y = u(x)\,x$ on the above ode results in new ode in $u(x)$

$$u(x)\,x + u(x)^2\,x^2 - x(u'(x)\,x + u(x)) = -x^2$$

Integrating both sides gives

$$\int \frac{1}{u^2+1}du = x + c_2$$
$$\arctan(u) = x + c_2$$

Solving for u gives these solutions

$$u_1 = \tan(x + c_2)$$

Therefore the solution y is

$$\begin{aligned} y &= xu \\ &= x\tan(x + c_2) \end{aligned}$$

Summary
The solution(s) found are the following

$$y = x \tan (x + c_2) \tag{1}$$

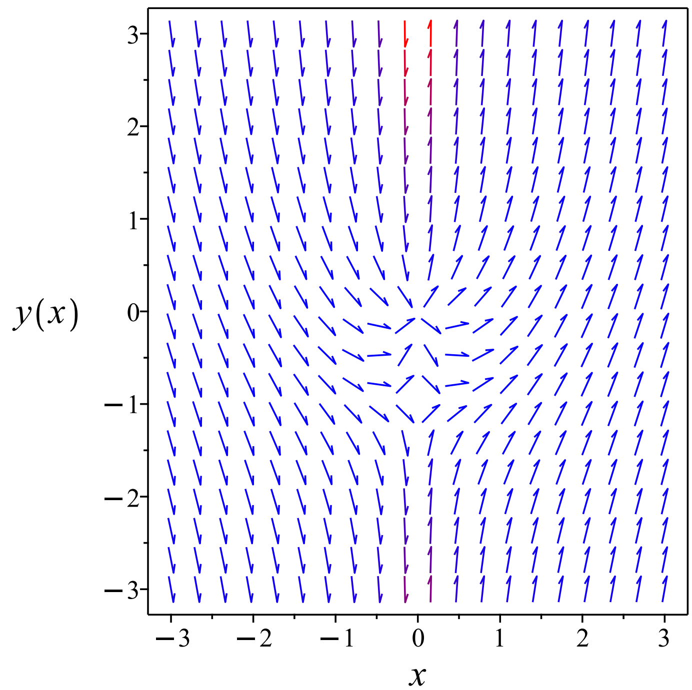


Figure 63: Slope field plot

Verification of solutions

$$y = x \tan (x + c_2)$$

Verified OK.

1.40.2 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x,y) + N(x,y)\frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x,y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y) = 0$$

Hence

$$\frac{\partial \phi}{\partial x} + \frac{\partial \phi}{\partial y}\frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$\begin{aligned}\frac{\partial \phi}{\partial x} &= M \\ \frac{\partial \phi}{\partial y} &= N\end{aligned}$$

But since $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x,y)$ but at least we know now that we can do that since the condition $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x,y)\,\mathrm{d}x + N(x,y)\,\mathrm{d}y = 0 \tag{1A}$$

Therefore

$$\begin{aligned}(-x)\,\mathrm{d}y &= \left(-x^2 - y^2 - y\right)\mathrm{d}x \\ \left(x^2 + y^2 + y\right)\mathrm{d}x + (-x)\,\mathrm{d}y &= 0 \qquad \text{(2A)}\end{aligned}$$

Comparing (1A) and (2A) shows that

$$\begin{aligned}M(x,y) &= x^2 + y^2 + y \\ N(x,y) &= -x\end{aligned}$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Using result found above gives

$$\begin{aligned}\frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}\left(x^2 + y^2 + y\right) \\ &= 2y + 1\end{aligned}$$

And

$$\begin{aligned}\frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}(-x)\\ &= -1\end{aligned}$$

Since $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$, then the ODE is not exact. By inspection $\frac{1}{y^2+x^2}$ is an integrating factor. Therefore by multiplying $M = x^2 + y + y^2$ and $N = -x$ by this integrating factor the ode becomes exact. The new M, N are

$$\begin{aligned}M &= \frac{x^2 + y + y^2}{y^2 + x^2}\\ N &= -\frac{x}{y^2 + x^2}\end{aligned}$$

To solve an ode of the form

$$M(x,y) + N(x,y)\frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x,y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y) = 0$$

Hence

$$\frac{\partial \phi}{\partial x} + \frac{\partial \phi}{\partial y}\frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$\begin{aligned}\frac{\partial \phi}{\partial x} &= M\\ \frac{\partial \phi}{\partial y} &= N\end{aligned}$$

But since $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x,y)$ but at least we know now that we can do that since the condition $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x,y)\,\mathrm{d}x + N(x,y)\,\mathrm{d}y = 0 \tag{1A}$$

Therefore

$$\left(-\frac{x}{x^2+y^2}\right)\mathrm{d}y = \left(-\frac{x^2+y^2+y}{x^2+y^2}\right)\mathrm{d}x$$

$$\left(\frac{x^2+y^2+y}{x^2+y^2}\right)\mathrm{d}x + \left(-\frac{x}{x^2+y^2}\right)\mathrm{d}y = 0 \tag{2A}$$

Comparing (1A) and (2A) shows that

$$M(x,y) = \frac{x^2+y^2+y}{x^2+y^2}$$
$$N(x,y) = -\frac{x}{x^2+y^2}$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Using result found above gives

$$\begin{aligned}\frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}\left(\frac{x^2+y^2+y}{x^2+y^2}\right)\\ &= \frac{x^2-y^2}{(x^2+y^2)^2}\end{aligned}$$

And

$$\begin{aligned}\frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}\left(-\frac{x}{x^2+y^2}\right)\\ &= \frac{x^2-y^2}{(x^2+y^2)^2}\end{aligned}$$

Since $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$, then the ODE is exact The following equations are now set up to solve for the function $\phi(x,y)$

$$\frac{\partial \phi}{\partial x} = M \tag{1}$$

$$\frac{\partial \phi}{\partial y} = N \tag{2}$$

Integrating (1) w.r.t. x gives

$$\int \frac{\partial \phi}{\partial x} \, \mathrm{d}x = \int M \, \mathrm{d}x$$

$$\int \frac{\partial \phi}{\partial x} \, \mathrm{d}x = \int \frac{x^2 + y^2 + y}{x^2 + y^2} \, \mathrm{d}x$$

$$\phi = x + \arctan\left(\frac{x}{y}\right) + f(y) \tag{3}$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$\begin{aligned} \frac{\partial \phi}{\partial y} &= -\frac{x}{y^2 \left(\frac{x^2}{y^2} + 1\right)} + f'(y) \qquad (4) \\ &= -\frac{x}{x^2 + y^2} + f'(y) \end{aligned}$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = -\frac{x}{x^2+y^2}$. Therefore equation (4) becomes

$$-\frac{x}{x^2 + y^2} = -\frac{x}{x^2 + y^2} + f'(y) \tag{5}$$

Solving equation (5) for $f'(y)$ gives

$$f'(y) = 0$$

Therefore

$$f(y) = c_1$$

Where c_1 is constant of integration. Substituting this result for $f(y)$ into equation (3) gives ϕ

$$\phi = x + \arctan\left(\frac{x}{y}\right) + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = x + \arctan\left(\frac{x}{y}\right)$$

The solution becomes

$$y = \frac{x}{\tan(-x + c_1)}$$

Summary

The solution(s) found are the following

$$y = \frac{x}{\tan(-x + c_1)} \tag{1}$$

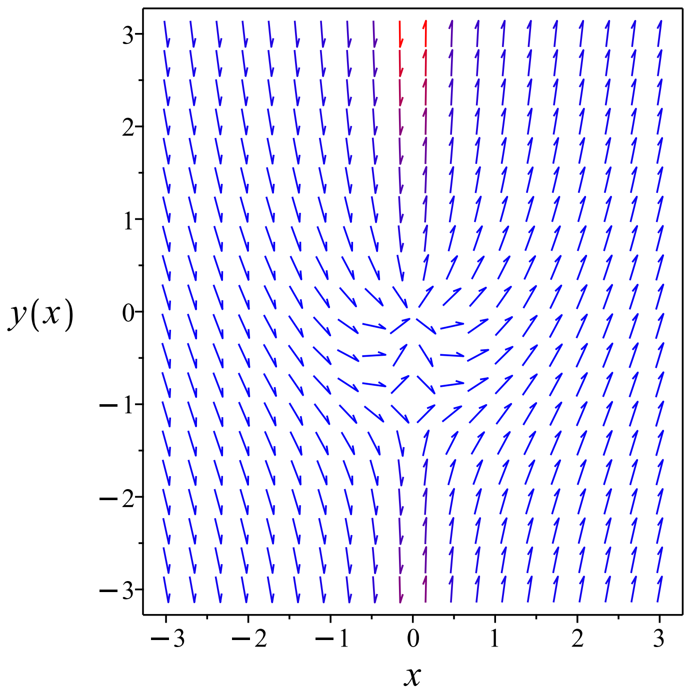


Figure 64: Slope field plot

Verification of solutions

$$y = \frac{x}{\tan(-x + c_1)}$$

Verified OK.

1.40.3 Solving as riccati ode

In canonical form the ODE is

$$\begin{aligned} y' &= F(x,y) \\ &= \frac{x^2+y^2+y}{x} \end{aligned}$$

This is a Riccati ODE. Comparing the ODE to solve

$$y' = x + \frac{y^2}{x} + \frac{y}{x}$$

With Riccati ODE standard form

$$y' = f_0(x) + f_1(x)y + f_2(x)y^2$$

Shows that $f_0(x) = x$, $f_1(x) = \frac{1}{x}$ and $f_2(x) = \frac{1}{x}$. Let

$$\begin{aligned} y &= \frac{-u'}{f_2 u} \\ &= \frac{-u'}{\frac{u}{x}} \end{aligned} \tag{1}$$

Using the above substitution in the given ODE results (after some simplification)in a second order ODE to solve for $u(x)$ which is

$$f_2 u''(x) - (f_2' + f_1 f_2)\, u'(x) + f_2^2 f_0 u(x) = 0 \tag{2}$$

But

$$\begin{aligned} f_2' &= -\frac{1}{x^2} \\ f_1 f_2 &= \frac{1}{x^2} \\ f_2^2 f_0 &= \frac{1}{x} \end{aligned}$$

Substituting the above terms back in equation (2) gives

$$\frac{u''(x)}{x} + \frac{u(x)}{x} = 0$$

Solving the above ODE (this ode solved using Maple, not this program), gives

$$u(x) = c_1 \sin(x) + c_2 \cos(x)$$

The above shows that

$$u'(x) = c_1 \cos(x) - c_2 \sin(x)$$

Using the above in (1) gives the solution

$$y = -\frac{(c_1 \cos(x) - c_2 \sin(x))\, x}{c_1 \sin(x) + c_2 \cos(x)}$$

Dividing both numerator and denominator by c_1 gives, after renaming the constant $\frac{c_2}{c_1} = c_3$ the following solution

$$y = \frac{(-c_3 \cos(x) + \sin(x))\, x}{c_3 \sin(x) + \cos(x)}$$

Summary

The solution(s) found are the following

$$y = \frac{(-c_3 \cos(x) + \sin(x))\, x}{c_3 \sin(x) + \cos(x)} \tag{1}$$

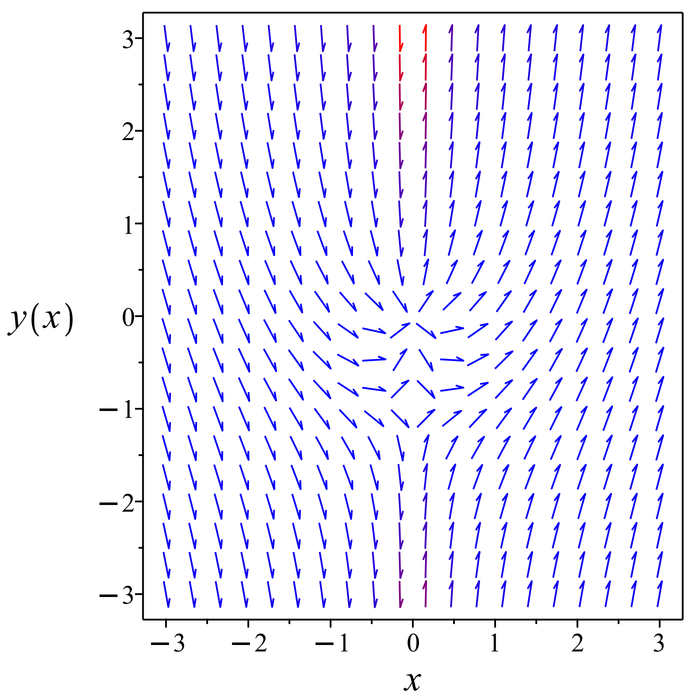


Figure 65: Slope field plot

Verification of solutions

$$y = \frac{(-c_3 \cos(x) + \sin(x))\, x}{c_3 \sin(x) + \cos(x)}$$

Verified OK.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
trying homogeneous types:
trying homogeneous D
<- homogeneous successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 10

```
dsolve((x^2+y(x)+y(x)^2)-x*diff(y(x),x)=0,y(x), singsol=all)
```

$$y(x) = \tan(c_1 + x)\, x$$

✓ Solution by Mathematica

Time used: 0.207 (sec). Leaf size: 12

```
DSolve[(x^2+y[x]+y[x]^2)-x*y'[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to x \tan(x + c_1)$$

1.41 problem 42

Internal problem ID [3186]
Internal file name [`OUTPUT/2678_Sunday_June_05_2022_08_38_41_AM_40029954/index.tex`]

Book: Differential equations for engineers by Wei-Chau XIE, Cambridge Press 2010
Section: Chapter 2. First-Order and Simple Higher-Order Differential Equations. Page 78
Problem number: 42.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"first_order_ode_lie_symmetry_calculated"**

Maple gives the following as the ode type

```
[[_homogeneous, `class G`], _dAlembert]
```

$$
-\sqrt{y^2+x^2}+\left(y-\sqrt{y^2+x^2}\right)y' = -x
$$

1.41.1 Solving as first order ode lie symmetry calculated ode

Writing the ode as

$$
\begin{aligned}
y' &= \frac{\sqrt{x^2+y^2}-x}{y-\sqrt{x^2+y^2}} \\
y' &= \omega(x,y)
\end{aligned}
$$

The condition of Lie symmetry is the linearized PDE given by

$$
\eta_x+\omega(\eta_y-\xi_x)-\omega^2\xi_y-\omega_x\xi-\omega_y\eta=0 \tag{A}
$$

The type of this ode is not in the lookup table. To determine ξ,η then (A) is solved using ansatz. Making bivariate polynomials of degree 1 to use as anstaz gives

$$
\xi = xa_2+ya_3+a_1 \tag{1E}
$$

$$
\eta = xb_2+yb_3+b_1 \tag{2E}
$$

Where the unknown coefficients are

$$
\{a_1,a_2,a_3,b_1,b_2,b_3\}
$$

Substituting equations (1E,2E) and ω into (A) gives

$$
\begin{aligned}
&b_2 + \frac{\left(\sqrt{x^2+y^2}-x\right)(b_3-a_2)}{y-\sqrt{x^2+y^2}} - \frac{\left(\sqrt{x^2+y^2}-x\right)^2 a_3}{\left(y-\sqrt{x^2+y^2}\right)^2} \\
&- \left(\frac{\frac{x}{\sqrt{x^2+y^2}}-1}{y-\sqrt{x^2+y^2}} + \frac{\left(\sqrt{x^2+y^2}-x\right)x}{\left(y-\sqrt{x^2+y^2}\right)^2\sqrt{x^2+y^2}}\right)(xa_2+ya_3+a_1) \\
&- \left(\frac{y}{\sqrt{x^2+y^2}\,\left(y-\sqrt{x^2+y^2}\right)} \right. \\
&\left. - \frac{\left(\sqrt{x^2+y^2}-x\right)\left(1-\frac{y}{\sqrt{x^2+y^2}}\right)}{\left(y-\sqrt{x^2+y^2}\right)^2}\right)(xb_2+yb_3+b_1) = 0
\end{aligned}
\tag{5E}
$$

Putting the above in normal form gives

$$
\frac{2\sqrt{x^2+y^2}\,xya_2 - 2\sqrt{x^2+y^2}\,xyb_3 + 2x^2yb_3 - 2x\,y^2a_2 - 2x^2ya_2 - x^2yb_2 + x\,y^2a_3 + 2x\,y^2b_3 - xya_1 + xyb}{}
$$

$$= 0$$

Setting the numerator to zero gives

$$
\begin{aligned}
&2\sqrt{x^2+y^2}\,xya_2 - 2\sqrt{x^2+y^2}\,xyb_3 + 2x^2yb_3 - 2x\,y^2a_2 - 2x^2ya_2 - x^2yb_2 \\
&+ x\,y^2a_3 + 2x\,y^2b_3 - xya_1 + xyb_1 + 2x^3a_3 + x^3b_2 + x^3b_3 - y^3a_2 - y^3a_3 \\
&- 2y^3b_2 + x^2b_1 - y^2a_1 - x^3a_2 + y^3b_3 + \left(x^2+y^2\right)^{\frac{3}{2}} a_2 - \left(x^2+y^2\right)^{\frac{3}{2}} a_3 \\
&+ \left(x^2+y^2\right)^{\frac{3}{2}} b_2 - \left(x^2+y^2\right)^{\frac{3}{2}} b_3 - \sqrt{x^2+y^2}\,x^2a_3 - \sqrt{x^2+y^2}\,x^2b_2 \\
&+ \sqrt{x^2+y^2}\,y^2a_3 + \sqrt{x^2+y^2}\,y^2b_2 - \sqrt{x^2+y^2}\,xb_1 + \sqrt{x^2+y^2}\,ya_1 = 0
\end{aligned}
\tag{6E}
$$

Simplifying the above gives

$$
\begin{aligned}
&2\sqrt{x^2+y^2}\,xya_2 - 2\sqrt{x^2+y^2}\,xyb_3 - x^2ya_2 + x^2ya_3 + x^2yb_2 - x\,y^2a_3 \\
&- x\,y^2b_2 + x\,y^2b_3 - xya_1 + xyb_1 - 2\left(x^2+y^2\right)xa_2 + x^3a_2 - y^3b_3 \\
&+ x^2a_1 - y^2b_1 + \left(x^2+y^2\right)^{\frac{3}{2}} a_2 - \left(x^2+y^2\right)^{\frac{3}{2}} a_3 + \left(x^2+y^2\right)^{\frac{3}{2}} b_2 \\
&- \left(x^2+y^2\right)^{\frac{3}{2}} b_3 - \left(x^2+y^2\right)a_1 + \left(x^2+y^2\right)b_1 + 2\left(x^2+y^2\right)xa_3 \\
&+ \left(x^2+y^2\right)xb_2 + \left(x^2+y^2\right)xb_3 - \left(x^2+y^2\right)ya_2 - \left(x^2+y^2\right)ya_3 \\
&- 2\left(x^2+y^2\right)yb_2 + 2\left(x^2+y^2\right)yb_3 - \sqrt{x^2+y^2}\,x^2a_3 - \sqrt{x^2+y^2}\,x^2b_2 \\
&+ \sqrt{x^2+y^2}\,y^2a_3 + \sqrt{x^2+y^2}\,y^2b_2 - \sqrt{x^2+y^2}\,xb_1 + \sqrt{x^2+y^2}\,ya_1 = 0
\end{aligned}
\tag{6E}
$$

Since the PDE has radicals, simplifying gives

$$
\begin{aligned}
&2\sqrt{x^2+y^2}\,xya_2 - 2\sqrt{x^2+y^2}\,xyb_3 + x^2\sqrt{x^2+y^2}\,a_2 - x^2\sqrt{x^2+y^2}\,b_3 + 2x^2yb_3 \\
&- 2x\,y^2a_2 - 2x^2ya_2 - x^2yb_2 + x\,y^2a_3 + 2x\,y^2b_3 - xya_1 + xyb_1 + \sqrt{x^2+y^2}\,y^2a_2 \\
&- \sqrt{x^2+y^2}\,y^2b_3 + 2x^3a_3 + x^3b_2 + x^3b_3 - y^3a_2 - y^3a_3 - 2y^3b_2 + x^2b_1 - y^2a_1 - x^3a_2 \\
&+ y^3b_3 - 2\sqrt{x^2+y^2}\,x^2a_3 + 2\sqrt{x^2+y^2}\,y^2b_2 - \sqrt{x^2+y^2}\,xb_1 + \sqrt{x^2+y^2}\,ya_1 = 0
\end{aligned}
$$

Looking at the above PDE shows the following are all the terms with $\{x, y\}$ in them.

$$
\left\{x, y, \sqrt{x^2+y^2}\right\}
$$

The following substitution is now made to be able to collect on all terms with $\{x, y\}$ in them

$$
\left\{x = v_1, y = v_2, \sqrt{x^2+y^2} = v_3\right\}
$$

The above PDE (6E) now becomes

$$
\begin{aligned}
&-v_1^3a_2 - 2v_1^2v_2a_2 + v_1^2v_3a_2 - 2v_1v_2^2a_2 + 2v_3v_1v_2a_2 - v_2^3a_2 + v_3v_2^2a_2 \\
&+ 2v_1^3a_3 - 2v_3v_1^2a_3 + v_1v_2^2a_3 - v_2^3a_3 + v_1^3b_2 - v_1^2v_2b_2 - 2v_2^3b_2 \\
&+ 2v_3v_2^2b_2 + v_1^3b_3 + 2v_1^2v_2b_3 - v_1^2v_3b_3 + 2v_1v_2^2b_3 - 2v_3v_1v_2b_3 + v_2^3b_3 \\
&- v_3v_2^2b_3 - v_1v_2a_1 - v_2^2a_1 + v_3v_2a_1 + v_1^2b_1 + v_1v_2b_1 - v_3v_1b_1 = 0
\end{aligned}
\tag{7E}
$$

Collecting the above on the terms v_i introduced, and these are

$$
\{v_1, v_2, v_3\}
$$

Equation (7E) now becomes

$$
\begin{aligned}
&(-a_2 + 2a_3 + b_2 + b_3)\,v_1^3 + (-2a_2 - b_2 + 2b_3)\,v_1^2v_2 + (a_2 - 2a_3 - b_3)\,v_1^2v_3 \\
&+ v_1^2b_1 + (-2a_2 + a_3 + 2b_3)\,v_1v_2^2 + (2a_2 - 2b_3)\,v_1v_2v_3 + (-a_1 + b_1)\,v_1v_2 \\
&- v_3v_1b_1 + (-a_2 - a_3 - 2b_2 + b_3)\,v_2^3 + (a_2 + 2b_2 - b_3)\,v_2^2v_3 - v_2^2a_1 + v_3v_2a_1 = 0
\end{aligned}
\tag{8E}
$$

Setting each coefficients in (8E) to zero gives the following equations to solve

$$
\begin{aligned}
a_1 &= 0\\
b_1 &= 0\\
-a_1 &= 0\\
-b_1 &= 0\\
-a_1 + b_1 &= 0\\
2a_2 - 2b_3 &= 0\\
-2a_2 + a_3 + 2b_3 &= 0\\
-2a_2 - b_2 + 2b_3 &= 0\\
a_2 - 2a_3 - b_3 &= 0\\
a_2 + 2b_2 - b_3 &= 0\\
-a_2 - a_3 - 2b_2 + b_3 &= 0\\
-a_2 + 2a_3 + b_2 + b_3 &= 0
\end{aligned}
$$

Solving the above equations for the unknowns gives

$$
\begin{aligned}
a_1 &= 0\\
a_2 &= b_3\\
a_3 &= 0\\
b_1 &= 0\\
b_2 &= 0\\
b_3 &= b_3
\end{aligned}
$$

Substituting the above solution in the anstaz (1E,2E) (using 1 as arbitrary value for any unknown in the RHS) gives

$$
\begin{aligned}
\xi &= x\\
\eta &= y
\end{aligned}
$$

Shifting is now applied to make $\xi = 0$ in order to simplify the rest of the computation

$$
\begin{aligned}
\eta &= \eta - \omega(x, y)\,\xi\\
&= y - \left(\frac{\sqrt{x^2+y^2} - x}{y - \sqrt{x^2+y^2}}\right)(x)\\
&= \frac{x\sqrt{x^2+y^2} + y\sqrt{x^2+y^2} - x^2 - y^2}{-y + \sqrt{x^2+y^2}}\\
\xi &= 0
\end{aligned}
$$

The next step is to determine the canonical coordinates R,S. The canonical coordinates map $(x,y)\to(R,S)$ where (R,S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$\frac{dx}{\xi}=\frac{dy}{\eta}=dS \tag{1}$$

The above comes from the requirements that $\left(\xi\frac{\partial}{\partial x}+\eta\frac{\partial}{\partial y}\right)S(x,y)=1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the canonical coordinates, where $S(R)$. Since $\xi=0$ then in this special case

$$R=x$$

S is found from

$$\begin{aligned}S&=\int\frac{1}{\eta}dy\\&=\int\frac{1}{\frac{x\sqrt{x^2+y^2}+y\sqrt{x^2+y^2}-x^2-y^2}{-y+\sqrt{x^2+y^2}}}dy\end{aligned}$$

Which results in

$$S=-\frac{\ln\left(y+\sqrt{x^2+y^2}\right)}{2}-\frac{x\ln\left(\frac{2x^2+2\sqrt{x^2}\,\sqrt{x^2+y^2}}{y}\right)}{2\sqrt{x^2}}+\frac{\ln(y)}{2}$$

Now that R,S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR}=\frac{S_x+\omega(x,y)S_y}{R_x+\omega(x,y)R_y} \tag{2}$$

Where in the above R_x,R_y,S_x,S_y are all partial derivatives and $\omega(x,y)$ is the right hand side of the original ode given by

$$\omega(x,y)=\frac{\sqrt{x^2+y^2}-x}{y-\sqrt{x^2+y^2}}$$

Evaluating all the partial derivatives gives

$$\begin{aligned}R_x&=1\\R_y&=0\\S_x&=\frac{(-y-x)\sqrt{x^2+y^2}-2x^2-xy-y^2}{2\sqrt{x^2+y^2}\,x\left(y+\sqrt{x^2+y^2}\right)}\\S_y&=\frac{(2x-y)\sqrt{x^2+y^2}+2x^2-xy+y^2}{2\sqrt{x^2+y^2}\,y\left(\sqrt{x^2+y^2}+x\right)}\end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = -\frac{3\left(x\sqrt{x^2+y^2}+x^2+y^2\right)}{2x\sqrt{x^2+y^2}\,\left(\sqrt{x^2+y^2}+x\right)} \tag{2A}$$

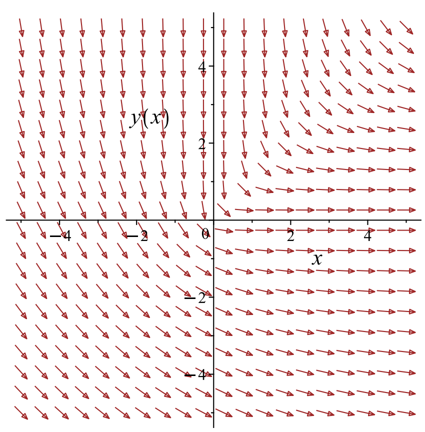
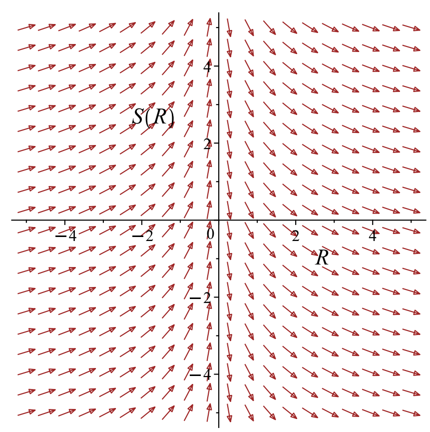
We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = -\frac{3}{2R}$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordinates R, S. Integrating the above gives

$$S(R) = -\frac{3\ln(R)}{2} + c_1 \tag{4}$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in x, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dx} = \frac{\sqrt{x^2+y^2}-x}{y-\sqrt{x^2+y^2}}$ 	$R = x$ $S = -\frac{\ln\left(y+\sqrt{x^2+y^2}\right)}{2}$	$\frac{dS}{dR} = -\frac{3}{2R}$ 

Summary
The solution(s) found are the following

$$-\frac{\ln\left(y+\sqrt{y^2+x^2}\right)}{2} - \frac{\ln(2)}{2} - \frac{\ln(x)}{2} - \frac{\ln\left(x+\sqrt{y^2+x^2}\right)}{2} + \ln(y) = -\frac{3\ln(x)}{2} + c_1 \tag{1}$$

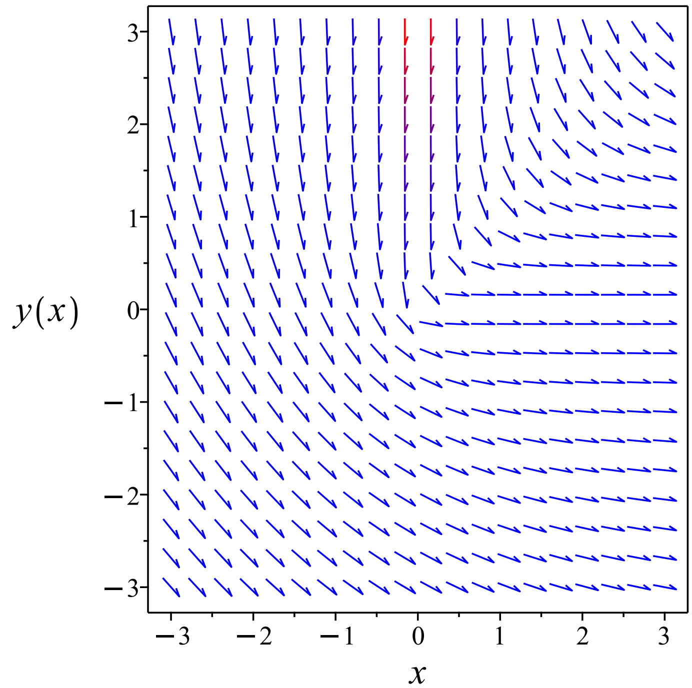


Figure 66: Slope field plot

Verification of solutions

$$-\frac{\ln\left(y+\sqrt{y^{2}+x^{2}}\right)}{2}-\frac{\ln\left(2\right)}{2}-\frac{\ln\left(x\right)}{2}-\frac{\ln\left(x+\sqrt{y^{2}+x^{2}}\right)}{2}+\ln\left(y\right)=-\frac{3\ln\left(x\right)}{2}+c_{1}$$

Verified OK.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying homogeneous types:
trying homogeneous G
1st order, trying the canonical coordinates of the invariance group
<- 1st order, canonical coordinates successful
<- homogeneous successful`
```

✓ Solution by Maple

Time used: 0.063 (sec). Leaf size: 47

```
dsolve((x-sqrt(x^2+y(x)^2))+(y(x)-sqrt(x^2+y(x)^2))*diff(y(x),x)=0,y(x), singsol=all)
```

$$\frac{(x+y(x))\sqrt{x^2+y(x)^2}+(-c_1x^2+1)\,y(x)^2+xy(x)+x^2}{y(x)^2\,x^2}=0$$

✓ Solution by Mathematica

Time used: 0.834 (sec). Leaf size: 34

```
DSolve[(x-Sqrt[x^2+y[x]^2])+(y[x]-Sqrt[x^2+y[x]^2])*y'[x]==0,y[x],x,IncludeSingularSolutions
```

$$\begin{aligned} y(x) &\to -\frac{e^{c_1}(2x+e^{c_1})}{2(x+e^{c_1})} \\ y(x) &\to 0 \end{aligned}$$

1.42 problem 43

Internal problem ID [3187]
Internal file name [OUTPUT/2679_Sunday_June_05_2022_08_38_41_AM_85031265/index.tex]

Book: Differential equations for engineers by Wei-Chau XIE, Cambridge Press 2010
Section: Chapter 2. First-Order and Simple Higher-Order Differential Equations. Page 78
Problem number: 43.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"exactWithIntegrationFactor"**

Maple gives the following as the ode type

```
[[_1st_order, `_with_symmetry_[F(x)*G(y),0]`]]
```

$$\boxed{y\sqrt{y^2+1}+\left(x\sqrt{y^2+1}-y\right)y'=0}$$

1.42.1 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x,y)+N(x,y)\frac{dy}{dx}=0 \tag{A}$$

We assume there exists a function $\phi(x,y)=c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y)=0$$

Hence

$$\frac{\partial\phi}{\partial x}+\frac{\partial\phi}{\partial y}\frac{dy}{dx}=0 \tag{B}$$

Comparing (A,B) shows that

$$\frac{\partial\phi}{\partial x}=M$$
$$\frac{\partial\phi}{\partial y}=N$$

But since $\frac{\partial^2\phi}{\partial x\partial y} = \frac{\partial^2\phi}{\partial y\partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x,y)$ but at least we know now that we can do that since the condition $\frac{\partial^2\phi}{\partial x\partial y} = \frac{\partial^2\phi}{\partial y\partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x,y)\,\mathrm{d}x + N(x,y)\,\mathrm{d}y = 0 \tag{1A}$$

Therefore

$$\begin{aligned}\left(x\sqrt{y^2+1}-y\right)\mathrm{d}y &= \left(-y\sqrt{y^2+1}\right)\mathrm{d}x\\ \left(y\sqrt{y^2+1}\right)\mathrm{d}x + \left(x\sqrt{y^2+1}-y\right)\mathrm{d}y &= 0 \qquad (2A)\end{aligned}$$

Comparing (1A) and (2A) shows that

$$\begin{aligned}M(x,y) &= y\sqrt{y^2+1}\\ N(x,y) &= x\sqrt{y^2+1}-y\end{aligned}$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Using result found above gives

$$\begin{aligned}\frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}\left(y\sqrt{y^2+1}\right)\\ &= \frac{2y^2+1}{\sqrt{y^2+1}}\end{aligned}$$

And

$$\begin{aligned}\frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}\left(x\sqrt{y^2+1}-y\right)\\ &= \sqrt{y^2+1}\end{aligned}$$

Since $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$, then the ODE is not exact. Since the ODE is not exact, we will try to find an integrating factor to make it exact. Let

$$
\begin{aligned}
A &= \frac{1}{N}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right) \\
&= \frac{1}{x\sqrt{y^2+1} - y}\left(\left(\sqrt{y^2+1} + \frac{y^2}{\sqrt{y^2+1}}\right) - \left(\sqrt{y^2+1}\right)\right) \\
&= \frac{y^2}{\left(x\sqrt{y^2+1} - y\right)\sqrt{y^2+1}}
\end{aligned}
$$

Since A depends on y, it can not be used to obtain an integrating factor. We will now try a second method to find an integrating factor. Let

$$
\begin{aligned}
B &= \frac{1}{M}\left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) \\
&= \frac{1}{y\sqrt{y^2+1}}\left(\left(\sqrt{y^2+1}\right) - \left(\sqrt{y^2+1} + \frac{y^2}{\sqrt{y^2+1}}\right)\right) \\
&= -\frac{y}{y^2+1}
\end{aligned}
$$

Since B does not depend on x, it can be used to obtain an integrating factor. Let the integrating factor be μ. Then

$$
\begin{aligned}
\mu &= e^{\int B \, \mathrm{d}y} \\
&= e^{\int -\frac{y}{y^2+1} \, \mathrm{d}y}
\end{aligned}
$$

The result of integrating gives

$$
\begin{aligned}
\mu &= e^{-\frac{\ln\left(y^2+1\right)}{2}} \\
&= \frac{1}{\sqrt{y^2+1}}
\end{aligned}
$$

M and N are now multiplied by this integrating factor, giving new M and new N which are called $\overline{M}$ and $\overline{N}$ so not to confuse them with the original M and N.

$$
\begin{aligned}
\overline{M} &= \mu M \\
&= \frac{1}{\sqrt{y^2+1}}\left(y\sqrt{y^2+1}\right) \\
&= y
\end{aligned}
$$

And

$$
\begin{aligned}
\overline{N} &= \mu N \\
&= \frac{1}{\sqrt{y^2+1}}\left(x\sqrt{y^2+1}-y\right) \\
&= \frac{x\sqrt{y^2+1}-y}{\sqrt{y^2+1}}
\end{aligned}
$$

So now a modified ODE is obtained from the original ODE which will be exact and can be solved using the standard method. The modified ODE is

$$
\begin{aligned}
\overline{M} + \overline{N}\frac{\mathrm{d}y}{\mathrm{d}x} &= 0 \\
(y) + \left(\frac{x\sqrt{y^2+1}-y}{\sqrt{y^2+1}}\right)\frac{\mathrm{d}y}{\mathrm{d}x} &= 0
\end{aligned}
$$

The following equations are now set up to solve for the function $\phi(x, y)$

$$
\frac{\partial \phi}{\partial x} = \overline{M} \tag{1}
$$

$$
\frac{\partial \phi}{\partial y} = \overline{N} \tag{2}
$$

Integrating (1) w.r.t. x gives

$$
\begin{aligned}
\int \frac{\partial \phi}{\partial x}\,\mathrm{d}x &= \int \overline{M}\,\mathrm{d}x \\
\int \frac{\partial \phi}{\partial x}\,\mathrm{d}x &= \int y\,\mathrm{d}x \\
\phi &= xy + f(y) \qquad (3)
\end{aligned}
$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$
\frac{\partial \phi}{\partial y} = x + f'(y) \tag{4}
$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = \frac{x\sqrt{y^2+1}-y}{\sqrt{y^2+1}}$. Therefore equation (4) becomes

$$
\frac{x\sqrt{y^2+1}-y}{\sqrt{y^2+1}} = x + f'(y) \tag{5}
$$

Solving equation (5) for $f'(y)$ gives

$$f'(y) = -\frac{y}{\sqrt{y^2+1}}$$

Integrating the above w.r.t y gives

$$\int f'(y)\,\mathrm{d}y = \int \left(-\frac{y}{\sqrt{y^2+1}}\right)\mathrm{d}y$$
$$f(y) = -\sqrt{y^2+1} + c_1$$

Where c_1 is constant of integration. Substituting result found above for $f(y)$ into equation (3) gives ϕ

$$\phi = xy - \sqrt{y^2+1} + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = xy - \sqrt{y^2+1}$$

Summary

The solution(s) found are the following

$$yx - \sqrt{y^2+1} = c_1 \tag{1}$$

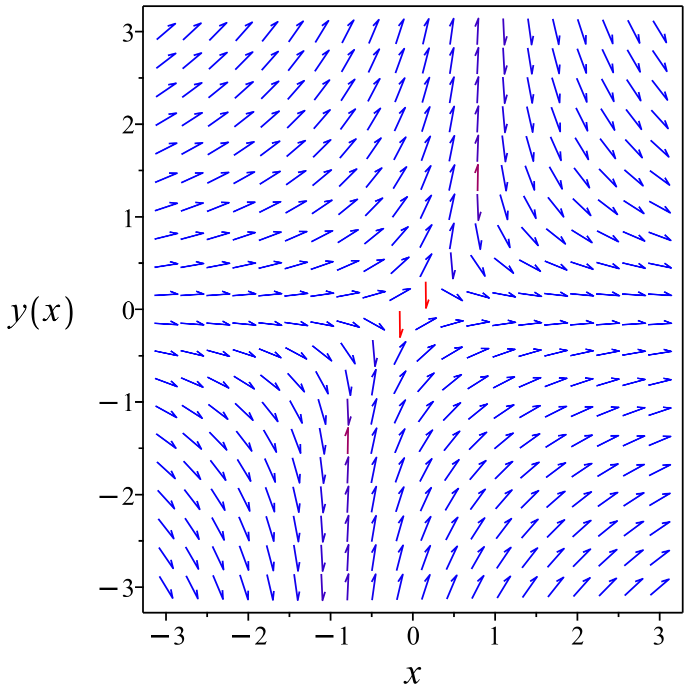


Figure 67: Slope field plot

Verification of solutions

$$yx - \sqrt{y^2 + 1} = c_1$$

Verified OK.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
<- 1st order linear successful
<- inverse linear successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 26

```
dsolve((y(x)*sqrt(1+y(x)^2))+(x*sqrt(1+y(x)^2)-y(x))*diff(y(x),x)=0,y(x), singsol=all)
```

$$\frac{xy(x)-\sqrt{y(x)^2+1}-c_1}{y(x)}=0$$

✓ Solution by Mathematica

Time used: 0.479 (sec). Leaf size: 82

```
DSolve[(y[x]*Sqrt[1+y[x]^2])+(x*Sqrt[1+y[x]^2]-y[x])*y'[x]==0,y[x],x,IncludeSingularSolution
```

$$y(x)\to\frac{c_1x-\sqrt{x^2-1+c_1{}^2}}{x^2-1}$$
$$y(x)\to\frac{\sqrt{x^2-1+c_1{}^2}+c_1x}{x^2-1}$$
$$y(x)\to 0$$
$$y(x)\to -i$$
$$y(x)\to i$$

1.43 problem 44

Internal problem ID [3188]
Internal file name [`OUTPUT/2680_Sunday_June_05_2022_08_38_42_AM_5433064/index.tex`]

Book: Differential equations for engineers by Wei-Chau XIE, Cambridge Press 2010
Section: Chapter 2. First-Order and Simple Higher-Order Differential Equations. Page 78
Problem number: 44.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"exactWithIntegrationFactor", "first_order_ode_lie_symmetry_calculated"**

Maple gives the following as the ode type

```
[[_homogeneous, `class G`], _rational, [_Abel, `2nd type`, `
   class B`]]
```

$$\boxed{y^2 - (yx + x^3)\, y' = 0}$$

1.43.1 Solving as first order ode lie symmetry calculated ode

Writing the ode as

$$y' = \frac{y^2}{x\,(x^2 + y)}$$
$$y' = \omega(x, y)$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x + \omega(\eta_y - \xi_x) - \omega^2 \xi_y - \omega_x \xi - \omega_y \eta = 0 \tag{A}$$

The type of this ode is not in the lookup table. To determine ξ, η then (A) is solved using ansatz. Making bivariate polynomials of degree 1 to use as anstaz gives

$$\xi = xa_2 + ya_3 + a_1 \tag{1E}$$
$$\eta = xb_2 + yb_3 + b_1 \tag{2E}$$

Where the unknown coefficients are

$$\{a_1, a_2, a_3, b_1, b_2, b_3\}$$

Substituting equations (1E,2E) and ω into (A) gives

$$b_2 + \frac{y^2(b_3 - a_2)}{x(x^2 + y)} - \frac{y^4 a_3}{x^2 (x^2 + y)^2} - \left(-\frac{y^2}{x^2 (x^2 + y)} - \frac{2y^2}{(x^2 + y)^2}\right)(x a_2 + y a_3 + a_1) \\ - \left(\frac{2y}{x(x^2 + y)} - \frac{y^2}{x(x^2 + y)^2}\right)(x b_2 + y b_3 + b_1) = 0 \tag{5E}$$

Putting the above in normal form gives

$$\frac{x^6 b_2 + 2x^3 y^2 a_2 - x^3 y^2 b_3 + 3x^2 y^3 a_3 - 2x^3 y b_1 + 3x^2 y^2 a_1 - x\,y^2 b_1 + y^3 a_1}{x^2 (x^2 + y)^2} = 0$$

Setting the numerator to zero gives

$$x^6 b_2 + 2x^3 y^2 a_2 - x^3 y^2 b_3 + 3x^2 y^3 a_3 - 2x^3 y b_1 + 3x^2 y^2 a_1 - x\,y^2 b_1 + y^3 a_1 = 0 \tag{6E}$$

Looking at the above PDE shows the following are all the terms with $\{x, y\}$ in them.

$$\{x, y\}$$

The following substitution is now made to be able to collect on all terms with $\{x, y\}$ in them

$$\{x = v_1, y = v_2\}$$

The above PDE (6E) now becomes

$$b_2 v_1^6 + 2a_2 v_1^3 v_2^2 + 3a_3 v_1^2 v_2^3 - b_3 v_1^3 v_2^2 + 3a_1 v_1^2 v_2^2 - 2b_1 v_1^3 v_2 + a_1 v_2^3 - b_1 v_1 v_2^2 = 0 \tag{7E}$$

Collecting the above on the terms v_i introduced, and these are

$$\{v_1, v_2\}$$

Equation (7E) now becomes

$$b_2 v_1^6 + (2a_2 - b_3) v_1^3 v_2^2 - 2b_1 v_1^3 v_2 + 3a_3 v_1^2 v_2^3 + 3a_1 v_1^2 v_2^2 - b_1 v_1 v_2^2 + a_1 v_2^3 = 0 \tag{8E}$$

Setting each coefficients in (8E) to zero gives the following equations to solve

$$
\begin{aligned}
a_1 &= 0 \\
b_2 &= 0 \\
3a_1 &= 0 \\
3a_3 &= 0 \\
-2b_1 &= 0 \\
-b_1 &= 0 \\
2a_2 - b_3 &= 0
\end{aligned}
$$

Solving the above equations for the unknowns gives

$$
\begin{aligned}
a_1 &= 0 \\
a_2 &= a_2 \\
a_3 &= 0 \\
b_1 &= 0 \\
b_2 &= 0 \\
b_3 &= 2a_2
\end{aligned}
$$

Substituting the above solution in the anstaz (1E,2E) (using 1 as arbitrary value for any unknown in the RHS) gives

$$
\begin{aligned}
\xi &= x \\
\eta &= 2y
\end{aligned}
$$

Shifting is now applied to make $\xi = 0$ in order to simplify the rest of the computation

$$
\begin{aligned}
\eta &= \eta - \omega(x, y)\,\xi \\
&= 2y - \left(\frac{y^2}{x\,(x^2 + y)}\right)(x) \\
&= \frac{2x^2 y + y^2}{x^2 + y} \\
\xi &= 0
\end{aligned}
$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x, y) \to (R, S)$ where (R, S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$
\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}
$$

The above comes from the requirements that $\left(\xi\frac{\partial}{\partial x}+\eta\frac{\partial}{\partial y}\right)S(x,y)=1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the canonical coordinates, where $S(R)$. Since $\xi=0$ then in this special case

$$R = x$$

S is found from

$$\begin{aligned}S &= \int \frac{1}{\eta} dy\\ &= \int \frac{1}{\frac{2x^2y+y^2}{x^2+y}} dy\end{aligned}$$

Which results in

$$S = \frac{\ln\left(y(2x^2+y)\right)}{2}$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x+\omega(x,y)S_y}{R_x+\omega(x,y)R_y} \tag{2}$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x,y)$ is the right hand side of the original ode given by

$$\omega(x,y) = \frac{y^2}{x\left(x^2+y\right)}$$

Evaluating all the partial derivatives gives

$$\begin{aligned}R_x &= 1\\ R_y &= 0\\ S_x &= \frac{2x}{2x^2+y}\\ S_y &= \frac{x^2+y}{2x^2y+y^2}\end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = \frac{1}{x} \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = \frac{1}{R}$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$S(R) = \ln(R) + c_1 \tag{4}$$

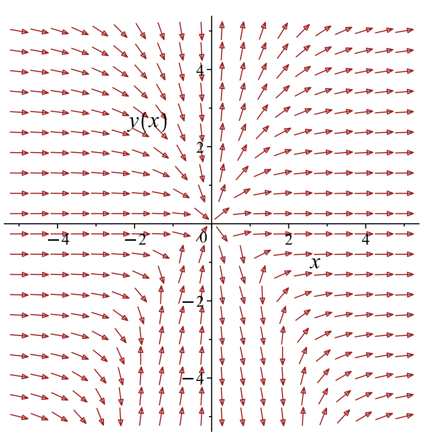
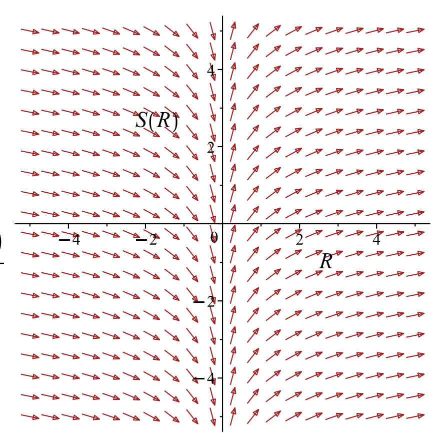
To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$\frac{\ln(y)}{2} + \frac{\ln(2x^2 + y)}{2} = \ln(x) + c_1$$

Which simplifies to

$$\frac{\ln(y)}{2} + \frac{\ln(2x^2 + y)}{2} = \ln(x) + c_1$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in x, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dx} = \frac{y^2}{x(x^2+y)}$ 	$R = x$ $S = \frac{\ln(y)}{2} + \frac{\ln(2x^2 + y)}{2}$	$\frac{dS}{dR} = \frac{1}{R}$ 

Summary

The solution(s) found are the following

$$\frac{\ln(y)}{2}+\frac{\ln(2x^2+y)}{2}=\ln(x)+c_1 \tag{1}$$

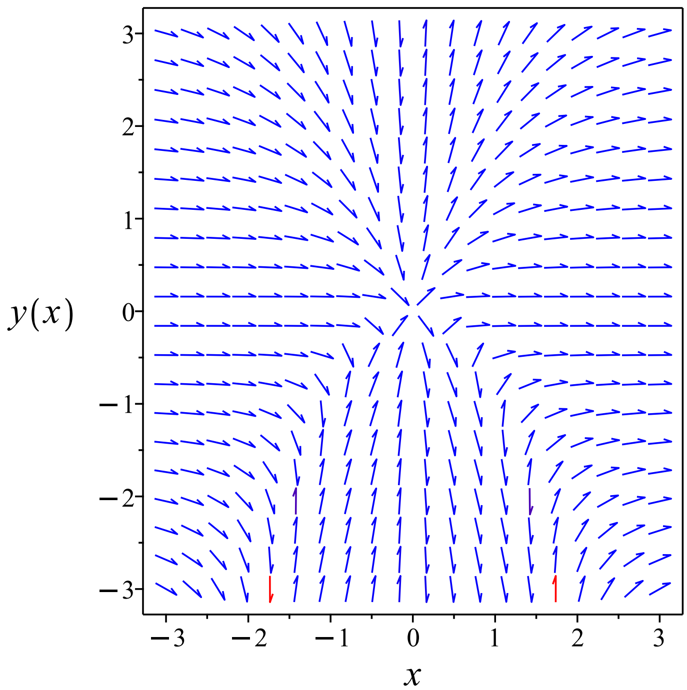


Figure 68: Slope field plot

Verification of solutions

$$\frac{\ln(y)}{2}+\frac{\ln(2x^2+y)}{2}=\ln(x)+c_1$$

Verified OK.

1.43.2 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x,y)+N(x,y)\frac{dy}{dx}=0 \tag{A}$$

We assume there exists a function $\phi(x,y)=c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y)=0$$

Hence

$$\frac{\partial\phi}{\partial x}+\frac{\partial\phi}{\partial y}\frac{dy}{dx}=0 \tag{B}$$

Comparing (A,B) shows that

$$\begin{aligned}\frac{\partial\phi}{\partial x}&=M\\ \frac{\partial\phi}{\partial y}&=N\end{aligned}$$

But since $\frac{\partial^2\phi}{\partial x\partial y}=\frac{\partial^2\phi}{\partial y\partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y}=\frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x,y)$ but at least we know now that we can do that since the condition $\frac{\partial^2\phi}{\partial x\partial y}=\frac{\partial^2\phi}{\partial y\partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x,y)\,\mathrm{d}x+N(x,y)\,\mathrm{d}y=0 \tag{1A}$$

Therefore

$$\begin{aligned}(-x^3-xy)\,\mathrm{d}y&=(-y^2)\,\mathrm{d}x\\ (y^2)\,\mathrm{d}x+(-x^3-xy)\,\mathrm{d}y&=0\end{aligned} \tag{2A}$$

Comparing (1A) and (2A) shows that

$$\begin{aligned}M(x,y)&=y^2\\ N(x,y)&=-x^3-xy\end{aligned}$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y}=\frac{\partial N}{\partial x}$$

Using result found above gives

$$\begin{aligned}\frac{\partial M}{\partial y}&=\frac{\partial}{\partial y}(y^2)\\ &=2y\end{aligned}$$

And

$$
\begin{aligned}
\frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}\left(-x^{3}-xy\right) \\
&= -3x^{2}-y
\end{aligned}
$$

Since $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$, then the ODE is <u>not exact</u>. Since the ODE is not exact, we will try to find an integrating factor to make it exact. Let

$$
\begin{aligned}
A &= \frac{1}{N}\left(\frac{\partial M}{\partial y}-\frac{\partial N}{\partial x}\right) \\
&= -\frac{1}{x\left(x^{2}+y\right)}\left(\left(2y\right)-\left(-3x^{2}-y\right)\right) \\
&= -\frac{3}{x}
\end{aligned}
$$

Since A does not depend on y, then it can be used to find an integrating factor. The integrating factor μ is

$$
\begin{aligned}
\mu &= e^{\int A \,\mathrm{d}x} \\
&= e^{\int -\frac{3}{x} \,\mathrm{d}x}
\end{aligned}
$$

The result of integrating gives

$$
\begin{aligned}
\mu &= e^{-3\ln(x)} \\
&= \frac{1}{x^{3}}
\end{aligned}
$$

M and N are multiplied by this integrating factor, giving new M and new N which are called $\overline{M}$ and $\overline{N}$ for now so not to confuse them with the original M and N.

$$
\begin{aligned}
\overline{M} &= \mu M \\
&= \frac{1}{x^{3}}\left(y^{2}\right) \\
&= \frac{y^{2}}{x^{3}}
\end{aligned}
$$

And

$$
\begin{aligned}
\overline{N} &= \mu N \\
&= \frac{1}{x^{3}}\left(-x^{3}-xy\right) \\
&= \frac{-x^{2}-y}{x^{2}}
\end{aligned}
$$

Now a modified ODE is ontained from the original ODE, which is exact and can be solved. The modified ODE is

$$\overline{M} + \overline{N}\frac{\mathrm{d}y}{\mathrm{d}x} = 0$$
$$\left(\frac{y^2}{x^3}\right) + \left(\frac{-x^2 - y}{x^2}\right)\frac{\mathrm{d}y}{\mathrm{d}x} = 0$$

The following equations are now set up to solve for the function $\phi(x, y)$

$$\frac{\partial\phi}{\partial x} = \overline{M} \tag{1}$$
$$\frac{\partial\phi}{\partial y} = \overline{N} \tag{2}$$

Integrating (1) w.r.t. x gives

$$\int \frac{\partial\phi}{\partial x}\,\mathrm{d}x = \int \overline{M}\,\mathrm{d}x$$
$$\int \frac{\partial\phi}{\partial x}\,\mathrm{d}x = \int \frac{y^2}{x^3}\,\mathrm{d}x$$
$$\phi = -\frac{y^2}{2x^2} + f(y) \tag{3}$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$\frac{\partial\phi}{\partial y} = -\frac{y}{x^2} + f'(y) \tag{4}$$

But equation (2) says that $\frac{\partial\phi}{\partial y} = \frac{-x^2-y}{x^2}$. Therefore equation (4) becomes

$$\frac{-x^2 - y}{x^2} = -\frac{y}{x^2} + f'(y) \tag{5}$$

Solving equation (5) for $f'(y)$ gives

$$f'(y) = -1$$

Integrating the above w.r.t y gives

$$\int f'(y)\,\mathrm{d}y = \int (-1)\,\mathrm{d}y$$
$$f(y) = -y + c_1$$

Where c_1 is constant of integration. Substituting result found above for $f(y)$ into equation (3) gives ϕ

$$\phi = -\frac{y^2}{2x^2} - y + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = -\frac{y^2}{2x^2} - y$$

<u>Summary</u>

The solution(s) found are the following

$$-\frac{y^2}{2x^2} - y = c_1 \tag{1}$$

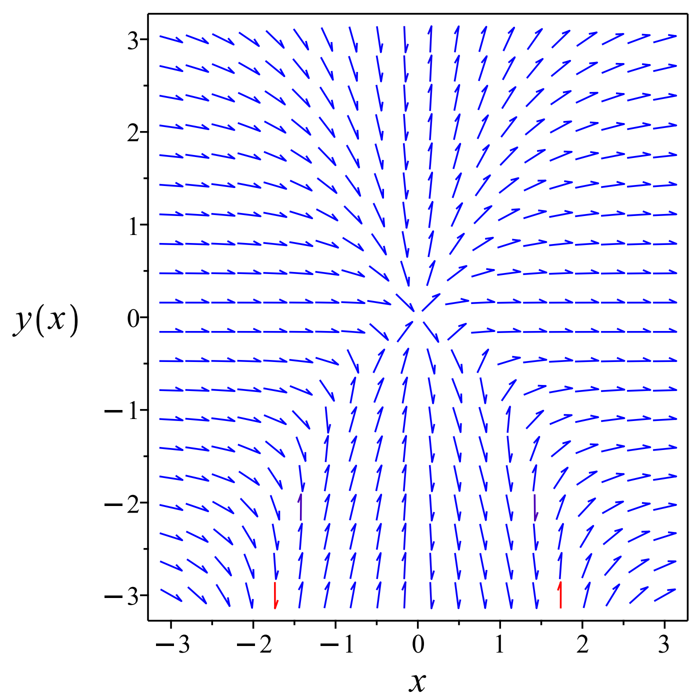


Figure 69: Slope field plot

<u>Verification of solutions</u>

$$-\frac{y^2}{2x^2} - y = c_1$$

Verified OK.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
trying homogeneous types:
trying homogeneous G
<- homogeneous successful`
```

✓ Solution by Maple

Time used: 0.062 (sec). Leaf size: 35

```
dsolve((y(x)^2)-(x*y(x)+x^3)*diff(y(x),x)=0,y(x), singsol=all)
```

$$y(x) = \left(-x - \sqrt{x^2 + c_1}\right) x$$
$$y(x) = \left(-x + \sqrt{x^2 + c_1}\right) x$$

✓ Solution by Mathematica

Time used: 0.551 (sec). Leaf size: 67

```
DSolve[(y[x]^2)-(x*y[x]+x^3)*y'[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to -x^2 \left(1 + \sqrt{\frac{1}{x^3}} \sqrt{x (x^2 + c_1)}\right)$$
$$y(x) \to x^2 \left(-1 + \sqrt{\frac{1}{x^3}} \sqrt{x (x^2 + c_1)}\right)$$
$$y(x) \to 0$$

1.44 problem 45

Internal problem ID [3189]
Internal file name [`OUTPUT/2681_Sunday_June_05_2022_08_38_43_AM_6111692/index.tex`]

Book: Differential equations for engineers by Wei-Chau XIE, Cambridge Press 2010
Section: Chapter 2. First-Order and Simple Higher-Order Differential Equations. Page 78
Problem number: 45.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"homogeneousTypeD", "homogeneousTypeD2", "first_order_ode_lie_symmetry_lookup"**

Maple gives the following as the ode type

```
[[_homogeneous, `class D`]]
```

$$\boxed{y - 2x^3 \tan\left(\frac{y}{x}\right) - xy' = 0}$$

1.44.1 Solving as homogeneousTypeD ode

Writing the ode as

$$y' = -2\tan\left(\frac{y}{x}\right)x^2 + \frac{y}{x} \tag{A}$$

The given ode has the form

$$y' = \frac{y}{x} + g(x)\, f\left(b\frac{y}{x}\right)^{\frac{n}{m}} \tag{1}$$

Where b is scalar and $g(x)$ is function of x and n, m are integers. The solution is given in Kamke page 20. Using the substitution $y(x) = u(x)\, x$ then

$$\frac{dy}{dx} = \frac{du}{dx}x + u$$

Hence the given ode becomes

$$\begin{aligned} \frac{du}{dx}x + u &= u + g(x)\, f(bu)^{\frac{n}{m}} \\ u' &= \frac{1}{x}g(x)\, f(bu)^{\frac{n}{m}} \qquad (2) \end{aligned}$$

The above ode is always separable. This is easily solved for u assuming the integration can be resolved, and then the solution to the original ode becomes $y = ux$. Comapring the given ode (A) with the form (1) shows that

$$\begin{aligned} g(x) &= -2x^2 \\ b &= 1 \\ f\left(\frac{bx}{y}\right) &= \tan\left(\frac{y}{x}\right) \end{aligned}$$

Substituting the above in (2) results in the $u(x)$ ode as

$$u'(x) = -2x\tan(u(x))$$

Which is now solved as separable In canonical form the ODE is

$$\begin{aligned} u' &= F(x, u) \\ &= f(x)g(u) \\ &= -2x\tan(u) \end{aligned}$$

Where $f(x) = -2x$ and $g(u) = \tan(u)$. Integrating both sides gives

$$\begin{aligned} \frac{1}{\tan(u)}\,du &= -2x\,dx \\ \int \frac{1}{\tan(u)}\,du &= \int -2x\,dx \\ \ln(\sin(u)) &= -x^2 + c_1 \end{aligned}$$

Raising both side to exponential gives

$$\sin(u) = \mathrm{e}^{-x^2+c_1}$$

Which simplifies to

$$\sin(u) = c_2\mathrm{e}^{-x^2}$$

Therefore the solution is

$$\begin{aligned} y &= ux \\ &= x\arcsin\left(c_2\mathrm{e}^{-x^2+c_1}\right) \end{aligned}$$

Summary
The solution(s) found are the following

$$y = x \arcsin\left(c_2 e^{-x^2+c_1}\right) \tag{1}$$

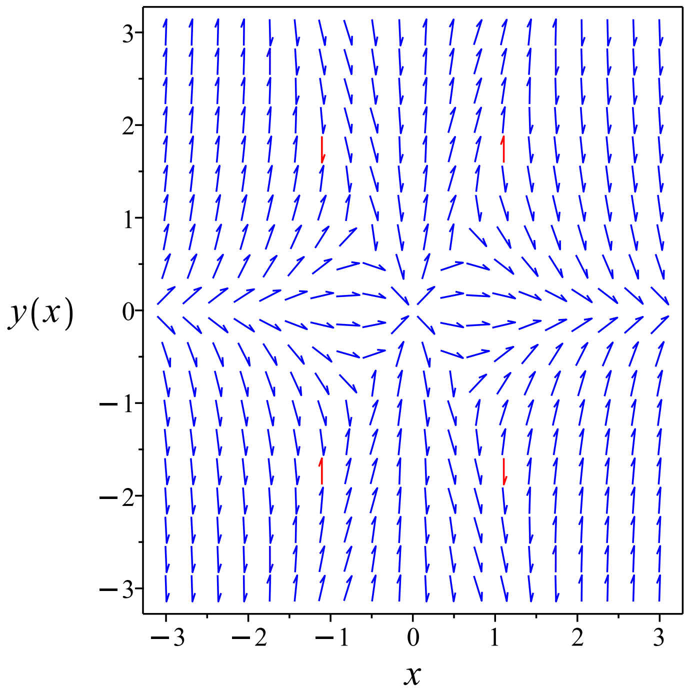


Figure 70: Slope field plot

Verification of solutions

$$y = x \arcsin\left(c_2 e^{-x^2+c_1}\right)$$

Verified OK.

1.44.2 Solving as homogeneousTypeD2 ode

Using the change of variables $y = u(x)\,x$ on the above ode results in new ode in $u(x)$

$$u(x)\,x - 2x^3 \tan(u(x)) - x(u'(x)\,x + u(x)) = 0$$

In canonical form the ODE is

$$\begin{aligned} u' &= F(x,u) \\ &= f(x)g(u) \\ &= -2\tan(u)\,x \end{aligned}$$

Where $f(x) = -2x$ and $g(u) = \tan(u)$. Integrating both sides gives

$$\frac{1}{\tan(u)}\, du = -2x\, dx$$
$$\int \frac{1}{\tan(u)}\, du = \int -2x\, dx$$
$$\ln(\sin(u)) = -x^2 + c_2$$

Raising both side to exponential gives

$$\sin(u) = \mathrm{e}^{-x^2 + c_2}$$

Which simplifies to

$$\sin(u) = c_3 \mathrm{e}^{-x^2}$$

Therefore the solution y is

$$\begin{aligned} y &= xu \\ &= x \arcsin\left(c_3 \mathrm{e}^{-x^2 + c_2}\right) \end{aligned}$$

Summary
The solution(s) found are the following

$$y = x \arcsin\left(c_3 \mathrm{e}^{-x^2 + c_2}\right) \tag{1}$$

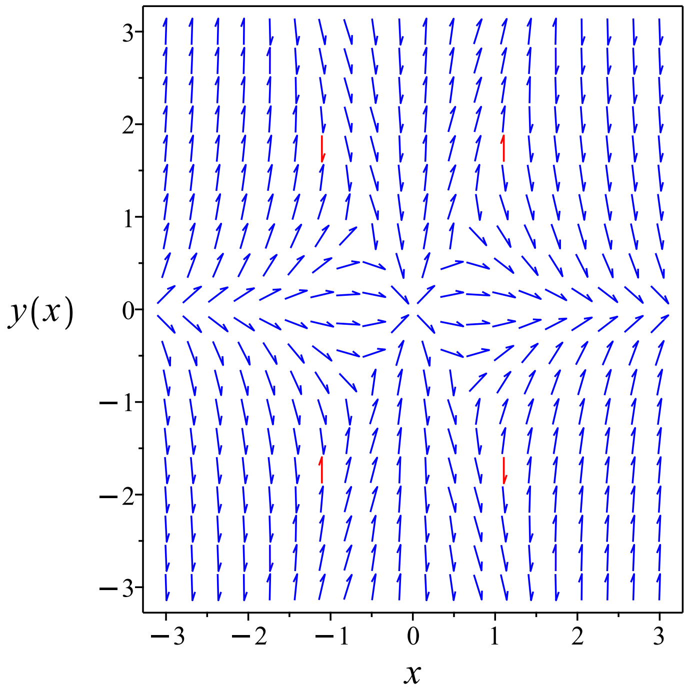


Figure 71: Slope field plot

Verification of solutions

$$y = x \arcsin\left(c_3 e^{-x^2+c_2}\right)$$

Verified OK.

1.44.3 Solving as first order ode lie symmetry lookup ode

Writing the ode as

$$y' = -\frac{-y + 2x^3 \tan\left(\frac{y}{x}\right)}{x}$$
$$y' = \omega(x, y)$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x + \omega(\eta_y - \xi_x) - \omega^2 \xi_y - \omega_x \xi - \omega_y \eta = 0 \tag{A}$$

The type of this ode is known. It is of type `homogeneous Type D`. Therefore we do not need to solve the PDE (A), and can just use the lookup table shown below to find ξ, η

Table 21: Lie symmetry infinitesimal lookup table for known first order ODE's

ODE class	Form	ξ	η
linear ode	$y' = f(x)y(x) + g(x)$	0	$e^{\int f dx}$
separable ode	$y' = f(x)\, g(y)$	$\frac{1}{f}$	0
quadrature ode	$y' = f(x)$	0	1
quadrature ode	$y' = g(y)$	1	0
homogeneous ODEs of Class A	$y' = f\left(\frac{y}{x}\right)$	x	y
homogeneous ODEs of Class C	$y' = (a + bx + cy)^{\frac{n}{m}}$	1	$-\frac{b}{c}$
homogeneous class D	$y' = \frac{y}{x} + g(x)\, F\left(\frac{y}{x}\right)$	x^2	xy
First order special form ID 1	$y' = g(x)\, e^{h(x)+by} + f(x)$	$\frac{e^{-\int bf(x)dx-h(x)}}{g(x)}$	$\frac{f(x)e^{-\int bf(x)dx-h(x)}}{g(x)}$
polynomial type ode	$y' = \frac{a_1x+b_1y+c_1}{a_2x+b_2y+c_2}$	$\frac{a_1b_2x-a_2b_1x-b_1c_2+b_2c_1}{a_1b_2-a_2b_1}$	$\frac{a_1b_2y-a_2b_1y-a_1c_2-a_2c_1}{a_1b_2-a_2b_1}$
Bernoulli ode	$y' = f(x)\, y + g(x)\, y^n$	0	$e^{-\int(n-1)f(x)dx}y^n$
Reduced Riccati	$y' = f_1(x)\, y + f_2(x)\, y^2$	0	$e^{-\int f_1 dx}$

The above table shows that

$$\begin{aligned} \xi(x, y) &= x^2 \\ \eta(x, y) &= xy \end{aligned} \tag{A1}$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x, y) \to (R, S)$ where (R, S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y}\right) S(x, y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the

canonical coordinates, where $S(R)$. Therefore

$$
\begin{aligned}
\frac{dy}{dx} &= \frac{\eta}{\xi} \\
&= \frac{xy}{x^2} \\
&= \frac{y}{x}
\end{aligned}
$$

This is easily solved to give

$$
y = c_1 x
$$

Where now the coordinate R is taken as the constant of integration. Hence

$$
R = \frac{y}{x}
$$

And S is found from

$$
\begin{aligned}
dS &= \frac{dx}{\xi} \\
&= \frac{dx}{x^2}
\end{aligned}
$$

Integrating gives

$$
\begin{aligned}
S &= \int \frac{dx}{T} \\
&= -\frac{1}{x}
\end{aligned}
$$

Where the constant of integration is set to zero as we just need one solution. Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$
\frac{dS}{dR} = \frac{S_x + \omega(x,y) S_y}{R_x + \omega(x,y) R_y} \tag{2}
$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x,y)$ is the right hand side of the original ode given by

$$
\omega(x,y) = -\frac{-y + 2x^3 \tan\left(\frac{y}{x}\right)}{x}
$$

Evaluating all the partial derivatives gives

$$
\begin{aligned}
R_x &= -\frac{y}{x^2} \\
R_y &= \frac{1}{x} \\
S_x &= \frac{1}{x^2} \\
S_y &= 0
\end{aligned}
$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = -\frac{\cot\left(\frac{y}{x}\right)}{2x^3} \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = \frac{\cot(R)\, S(R)^3}{2}$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordiates R, S. Integrating the above gives these solutions

$$\begin{aligned} S(R) &= \frac{1}{\sqrt{c_1 - \ln(\sin(R))}} \\ S(R) &= -\frac{1}{\sqrt{c_1 - \ln(\sin(R))}} \end{aligned} \tag{4}$$

Each will now be processed. To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$-\frac{1}{x} = \frac{1}{\sqrt{c_1 - \ln\left(\sin\left(\frac{y}{x}\right)\right)}}$$

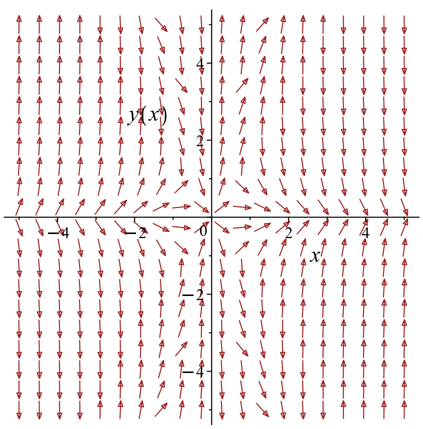
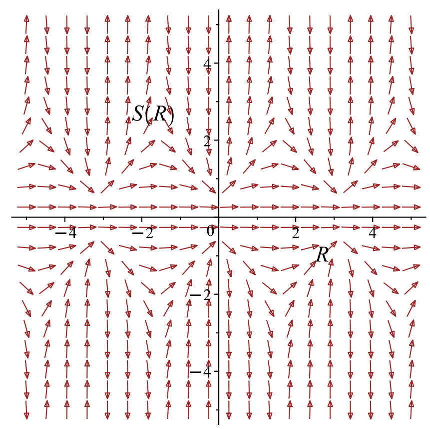
Which simplifies to

$$-\frac{1}{x} = \frac{1}{\sqrt{c_1 - \ln\left(\sin\left(\frac{y}{x}\right)\right)}}$$

Which gives

$$y = \arcsin\left(\mathrm{e}^{-x^2+c_1}\right) x$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in x, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dx} = -\frac{-y+2x^3\tan\left(\frac{y}{x}\right)}{x}$ 	$R = \frac{y}{x}$ $S = -\frac{1}{x}$	$\frac{dS}{dR} = \frac{\cot(R)S(R)^3}{2}$ 

To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$-\frac{1}{x} = -\frac{1}{\sqrt{c_1 - \ln\left(\sin\left(\frac{y}{x}\right)\right)}}$$

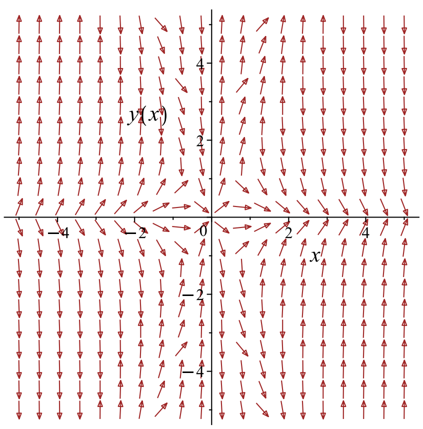
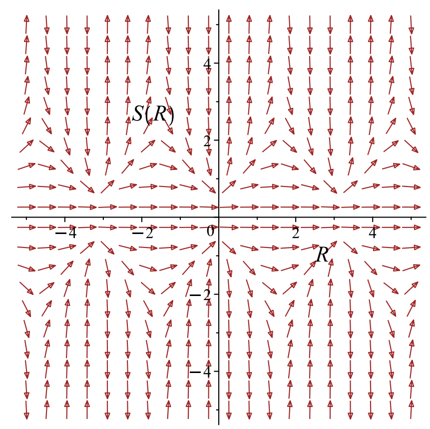
Which simplifies to

$$-\frac{1}{x} = -\frac{1}{\sqrt{c_1 - \ln\left(\sin\left(\frac{y}{x}\right)\right)}}$$

Which gives

$$y = \arcsin\left(e^{-x^2+c_1}\right)x$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in x, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dx} = -\frac{-y+2x^3\tan\left(\frac{y}{x}\right)}{x}$ 	$R = \frac{y}{x}$ $S = -\frac{1}{x}$	$\frac{dS}{dR} = \frac{\cot(R)S(R)^3}{2}$ 

Summary

The solution(s) found are the following

$$y = \arcsin\left(\mathrm{e}^{-x^2+c_1}\right) x \tag{1}$$

$$y = \arcsin\left(\mathrm{e}^{-x^2+c_1}\right) x \tag{2}$$

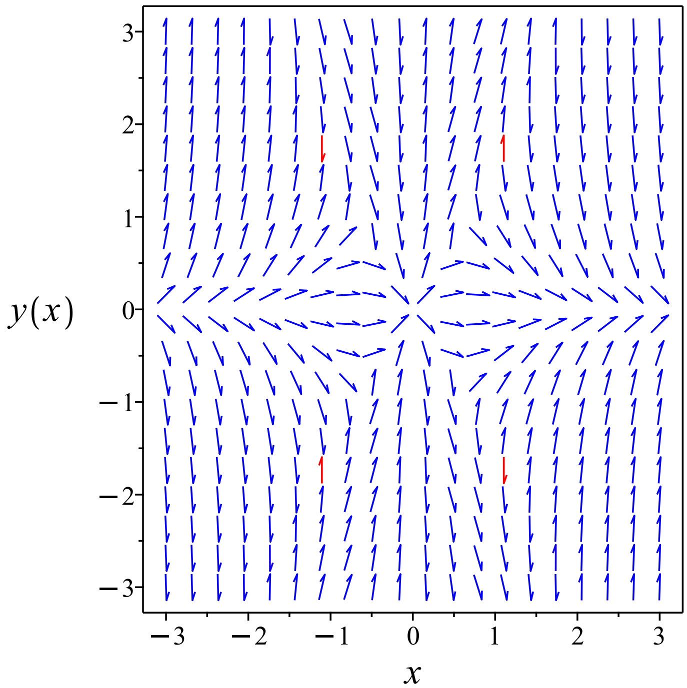


Figure 72: Slope field plot

Verification of solutions

$$y = \arcsin\left(\mathrm{e}^{-x^2+c_1}\right) x$$

Verified OK.

$$y = \arcsin\left(\mathrm{e}^{-x^2+c_1}\right) x$$

Verified OK.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
trying homogeneous types:
trying homogeneous D
<- homogeneous successful`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 15

```
dsolve(y(x)-2*x^3*tan(y(x)/x)-x*diff(y(x),x)=0,y(x), singsol=all)
```

$$y(x) = \arcsin\left(c_1 e^{-x^2}\right) x$$

✓ Solution by Mathematica

Time used: 59.679 (sec). Leaf size: 23

```
DSolve[y[x]-2*x^3*Tan[y[x]/x]-x*y'[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to x \arcsin\left(e^{-x^2+c_1}\right)$$
$$y(x) \to 0$$

1.45 problem 46

Internal problem ID [3190]
Internal file name [`OUTPUT/2682_Sunday_June_05_2022_08_38_44_AM_51242300/index.tex`]

Book: Differential equations for engineers by Wei-Chau XIE, Cambridge Press 2010
Section: Chapter 2. First-Order and Simple Higher-Order Differential Equations. Page 78
Problem number: 46.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"exactWithIntegrationFactor", "first_order_ode_lie_symmetry_calculated"**

Maple gives the following as the ode type

```
[[_homogeneous, `class G`], _rational, [_Abel, `2nd type`, `
   class B`]]
```

$$\boxed{2y^2x^2 + y + (yx^3 - x)\, y' = 0}$$

1.45.1 Solving as first order ode lie symmetry calculated ode

Writing the ode as

$$\begin{aligned} y' &= -\frac{y(2x^2y + 1)}{x\,(x^2y - 1)} \\ y' &= \omega(x, y) \end{aligned}$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x + \omega(\eta_y - \xi_x) - \omega^2\xi_y - \omega_x\xi - \omega_y\eta = 0 \tag{A}$$

The type of this ode is not in the lookup table. To determine ξ, η then (A) is solved using ansatz. Making bivariate polynomials of degree 1 to use as anstaz gives

$$\xi = xa_2 + ya_3 + a_1 \tag{1E}$$

$$\eta = xb_2 + yb_3 + b_1 \tag{2E}$$

Where the unknown coefficients are

$$\{a_1, a_2, a_3, b_1, b_2, b_3\}$$

Substituting equations (1E,2E) and ω into (A) gives

$$\begin{aligned}
&b_2 - \frac{y(2x^2y+1)(b_3-a_2)}{x(x^2y-1)} - \frac{y^2(2x^2y+1)^2 a_3}{x^2(x^2y-1)^2} \\
&- \left(-\frac{4y^2}{x^2y-1} + \frac{y(2x^2y+1)}{x^2(x^2y-1)} + \frac{2y^2(2x^2y+1)}{(x^2y-1)^2}\right)(xa_2 + ya_3 + a_1) \\
&- \left(-\frac{2x^2y+1}{x(x^2y-1)} - \frac{2yx}{x^2y-1} + \frac{y(2x^2y+1)x}{(x^2y-1)^2}\right)(xb_2 + yb_3 + b_1) = 0
\end{aligned} \tag{5E}$$

Putting the above in normal form gives

$$\frac{3x^6y^2b_2 - 6x^4y^4a_3 + 2x^5y^2b_1 - 2x^4y^3a_1 - 6x^4yb_2 - 6x^3y^2a_2 - 3x^3y^2b_3 - 9x^2y^3a_3 - 4x^3yb_1 - 5x^2y^2a_1 - x}{(x^2y-1)^2 x^2}$$
$$= 0$$

Setting the numerator to zero gives

$$\begin{aligned}
&3x^6y^2b_2 - 6x^4y^4a_3 + 2x^5y^2b_1 - 2x^4y^3a_1 - 6x^4yb_2 - 6x^3y^2a_2 \\
&- 3x^3y^2b_3 - 9x^2y^3a_3 - 4x^3yb_1 - 5x^2y^2a_1 - xb_1 + ya_1 = 0
\end{aligned} \tag{6E}$$

Looking at the above PDE shows the following are all the terms with $\{x, y\}$ in them.

$$\{x, y\}$$

The following substitution is now made to be able to collect on all terms with $\{x, y\}$ in them

$$\{x = v_1, y = v_2\}$$

The above PDE (6E) now becomes

$$\begin{aligned}
&-6a_3v_1^4v_2^4 + 3b_2v_1^6v_2^2 - 2a_1v_1^4v_2^3 + 2b_1v_1^5v_2^2 - 6a_2v_1^3v_2^2 - 9a_3v_1^2v_2^3 \\
&- 6b_2v_1^4v_2 - 3b_3v_1^3v_2^2 - 5a_1v_1^2v_2^2 - 4b_1v_1^3v_2 + a_1v_2 - b_1v_1 = 0
\end{aligned} \tag{7E}$$

Collecting the above on the terms v_i introduced, and these are

$$\{v_1, v_2\}$$

Equation (7E) now becomes

$$\begin{aligned}
&3b_2v_1^6v_2^2 + 2b_1v_1^5v_2^2 - 6a_3v_1^4v_2^4 - 2a_1v_1^4v_2^3 - 6b_2v_1^4v_2 + (-6a_2 - 3b_3)\,v_1^3v_2^2 \\
&- 4b_1v_1^3v_2 - 9a_3v_1^2v_2^3 - 5a_1v_1^2v_2^2 - b_1v_1 + a_1v_2 = 0
\end{aligned} \tag{8E}$$

Setting each coefficients in (8E) to zero gives the following equations to solve

$$\begin{aligned}
a_1 &= 0 \\
-5a_1 &= 0 \\
-2a_1 &= 0 \\
-9a_3 &= 0 \\
-6a_3 &= 0 \\
-4b_1 &= 0 \\
-b_1 &= 0 \\
2b_1 &= 0 \\
-6b_2 &= 0 \\
3b_2 &= 0 \\
-6a_2 - 3b_3 &= 0
\end{aligned}$$

Solving the above equations for the unknowns gives

$$\begin{aligned}
a_1 &= 0 \\
a_2 &= a_2 \\
a_3 &= 0 \\
b_1 &= 0 \\
b_2 &= 0 \\
b_3 &= -2a_2
\end{aligned}$$

Substituting the above solution in the anstaz (1E,2E) (using 1 as arbitrary value for any unknown in the RHS) gives

$$\begin{aligned}
\xi &= x \\
\eta &= -2y
\end{aligned}$$

Shifting is now applied to make $\xi = 0$ in order to simplify the rest of the computation

$$
\begin{aligned}
\eta &= \eta - \omega(x,y)\,\xi \\
&= -2y - \left(-\frac{y(2x^2y+1)}{x\,(x^2y-1)}\right)(x) \\
&= \frac{3y}{x^2y-1} \\
\xi &= 0
\end{aligned}
$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x,y) \to (R,S)$ where (R,S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$
\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}
$$

The above comes from the requirements that $\left(\xi\frac{\partial}{\partial x} + \eta\frac{\partial}{\partial y}\right) S(x,y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$
R = x
$$

S is found from

$$
\begin{aligned}
S &= \int \frac{1}{\eta} dy \\
&= \int \frac{1}{\frac{3y}{x^2y-1}} dy
\end{aligned}
$$

Which results in

$$
S = \frac{x^2y}{3} - \frac{\ln(y)}{3}
$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$
\frac{dS}{dR} = \frac{S_x + \omega(x,y)S_y}{R_x + \omega(x,y)R_y} \tag{2}
$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x,y)$ is the right hand side of the original ode given by

$$
\omega(x,y) = -\frac{y(2x^2y+1)}{x\,(x^2y-1)}
$$

Evaluating all the partial derivatives gives

$$
\begin{aligned}
R_x &= 1 \\
R_y &= 0 \\
S_x &= \frac{2xy}{3} \\
S_y &= \frac{x^2 y - 1}{3y}
\end{aligned}
$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = -\frac{1}{3x} \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = -\frac{1}{3R}$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$S(R) = -\frac{\ln(R)}{3} + c_1 \tag{4}$$

To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$\frac{x^2 y}{3} - \frac{\ln(y)}{3} = -\frac{\ln(x)}{3} + c_1$$

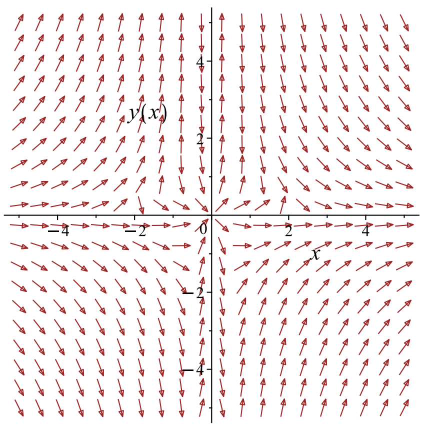
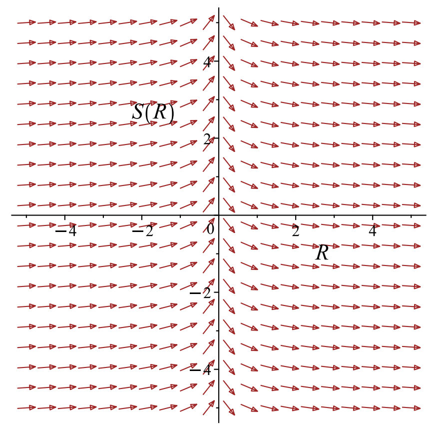
Which simplifies to

$$\frac{x^2 y}{3} - \frac{\ln(y)}{3} = -\frac{\ln(x)}{3} + c_1$$

Which gives

$$y = -\frac{\mathrm{LambertW}\left(-\mathrm{e}^{-3c_1} x^3\right)}{x^2}$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in x, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dx} = -\frac{y(2x^2y+1)}{x(x^2y-1)}$ 	$R = x$ $S = \frac{x^2y}{3} - \frac{\ln(y)}{3}$	$\frac{dS}{dR} = -\frac{1}{3R}$ 

Summary
The solution(s) found are the following

$$y = -\frac{\mathrm{LambertW}\left(-\mathrm{e}^{-3c_1}x^3\right)}{x^2} \tag{1}$$

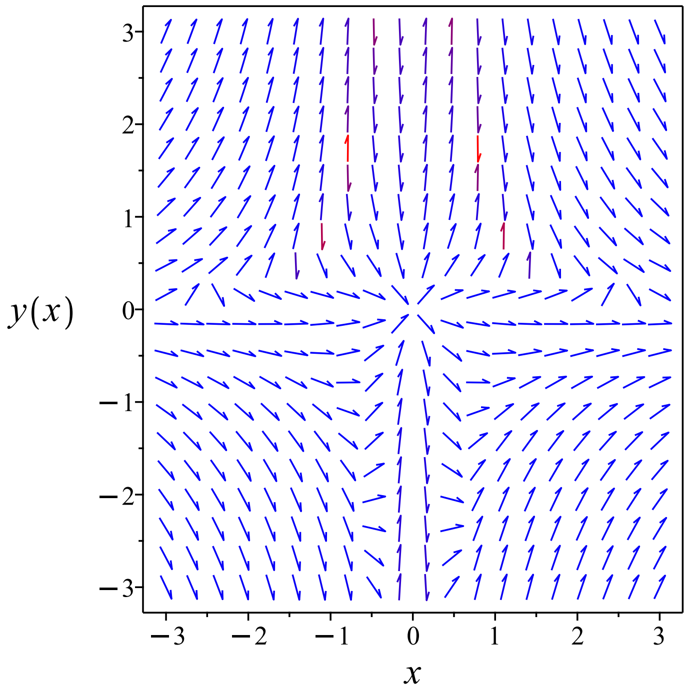


Figure 73: Slope field plot

Verification of solutions

$$y = -\frac{\text{LambertW}\left(-\mathrm{e}^{-3c_1}x^3\right)}{x^2}$$

Verified OK.

1.45.2 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x,y) + N(x,y)\frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x,y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y) = 0$$

Hence

$$\frac{\partial \phi}{\partial x} + \frac{\partial \phi}{\partial y}\frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$\frac{\partial \phi}{\partial x} = M$$
$$\frac{\partial \phi}{\partial y} = N$$

But since $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x,y)$ but at least we know now that we can do that since the condition $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x,y)\,\mathrm{d}x + N(x,y)\,\mathrm{d}y = 0 \tag{1A}$$

Therefore

$$\begin{aligned}\left(x^3 y - x\right)\mathrm{d}y &= \left(-2y^2x^2 - y\right)\mathrm{d}x\\ \left(2y^2x^2 + y\right)\mathrm{d}x + \left(x^3 y - x\right)\mathrm{d}y &= 0 \qquad (2A)\end{aligned}$$

Comparing (1A) and (2A) shows that

$$M(x,y) = 2y^2x^2 + y$$
$$N(x,y) = x^3 y - x$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Using result found above gives

$$\begin{aligned}\frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}\left(2y^2x^2 + y\right)\\ &= 4x^2 y + 1\end{aligned}$$

And

$$\begin{aligned}\frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}\left(x^3 y - x\right)\\ &= 3x^2 y - 1\end{aligned}$$

Since $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$, then the ODE is <u>not exact</u>. Since the ODE is not exact, we will try to find an integrating factor to make it exact. Let

$$
\begin{aligned}
A &= \frac{1}{N}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right) \\
&= \frac{1}{x^3y - x}\left(\left(4x^2y + 1\right) - \left(3x^2y - 1\right)\right) \\
&= \frac{x^2y + 2}{x^3y - x}
\end{aligned}
$$

Since A depends on y, it can not be used to obtain an integrating factor. We will now try a second method to find an integrating factor. Let

$$
\begin{aligned}
B &= \frac{1}{M}\left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) \\
&= \frac{1}{2y^2x^2 + y}\left(\left(3x^2y - 1\right) - \left(4x^2y + 1\right)\right) \\
&= \frac{-x^2y - 2}{2y^2x^2 + y}
\end{aligned}
$$

Since B depends on x, it can not be used to obtain an integrating factor.We will now try a third method to find an integrating factor. Let

$$
R = \frac{\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}}{xM - yN}
$$

R is now checked to see if it is a function of only $t = xy$. Therefore

$$
\begin{aligned}
R &= \frac{\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}}{xM - yN} \\
&= \frac{(3x^2y - 1) - (4x^2y + 1)}{x\,(2y^2x^2 + y) - y\,(x^3y - x)} \\
&= -\frac{1}{xy}
\end{aligned}
$$

Replacing all powers of terms xy by t gives

$$
R = -\frac{1}{t}
$$

Since R depends on t only, then it can be used to find an integrating factor. Let the integrating factor be μ then

$$
\begin{aligned}
\mu &= e^{\int R \,\mathrm{d}t} \\
&= e^{\int \left(-\frac{1}{t}\right) \mathrm{d}t}
\end{aligned}
$$

The result of integrating gives

$$\begin{aligned}\mu &= e^{-\ln(t)} \\ &= \frac{1}{t}\end{aligned}$$

Now t is replaced back with xy giving

$$\mu = \frac{1}{xy}$$

Multiplying M and N by this integrating factor gives new M and new N which are called $\overline{M}$ and $\overline{N}$ so not to confuse them with the original M and N

$$\begin{aligned}\overline{M} &= \mu M \\ &= \frac{1}{xy}(2y^2x^2 + y) \\ &= \frac{2x^2y + 1}{x}\end{aligned}$$

And

$$\begin{aligned}\overline{N} &= \mu N \\ &= \frac{1}{xy}(x^3y - x) \\ &= \frac{x^2y - 1}{y}\end{aligned}$$

A modified ODE is now obtained from the original ODE, which is exact and can solved. The modified ODE is

$$\begin{aligned}\overline{M} + \overline{N}\frac{\mathrm{d}y}{\mathrm{d}x} &= 0 \\ \left(\frac{2x^2y + 1}{x}\right) + \left(\frac{x^2y - 1}{y}\right)\frac{\mathrm{d}y}{\mathrm{d}x} &= 0\end{aligned}$$

The following equations are now set up to solve for the function $\phi(x, y)$

$$\frac{\partial \phi}{\partial x} = \overline{M} \tag{1}$$

$$\frac{\partial \phi}{\partial y} = \overline{N} \tag{2}$$

Integrating (1) w.r.t. x gives

$$
\begin{aligned}
\int \frac{\partial \phi}{\partial x} \mathrm{d}x &= \int \overline{M} \mathrm{d}x \\
\int \frac{\partial \phi}{\partial x} \mathrm{d}x &= \int \frac{2x^2 y + 1}{x} \mathrm{d}x \\
\phi &= x^2 y + \ln(x) + f(y) \qquad (3)
\end{aligned}
$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$
\frac{\partial \phi}{\partial y} = x^2 + f'(y) \tag{4}
$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = \frac{x^2 y - 1}{y}$. Therefore equation (4) becomes

$$
\frac{x^2 y - 1}{y} = x^2 + f'(y) \tag{5}
$$

Solving equation (5) for $f'(y)$ gives

$$
f'(y) = -\frac{1}{y}
$$

Integrating the above w.r.t y gives

$$
\begin{aligned}
\int f'(y) \mathrm{d}y &= \int \left(-\frac{1}{y}\right) \mathrm{d}y \\
f(y) &= -\ln(y) + c_1
\end{aligned}
$$

Where c_1 is constant of integration. Substituting result found above for $f(y)$ into equation (3) gives ϕ

$$
\phi = x^2 y + \ln(x) - \ln(y) + c_1
$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$
c_1 = x^2 y + \ln(x) - \ln(y)
$$

The solution becomes

$$y = -\frac{\text{LambertW}\left(-x^3 \mathrm{e}^{-c_1}\right)}{x^2}$$

Summary

The solution(s) found are the following

$$y = -\frac{\text{LambertW}\left(-x^3 \mathrm{e}^{-c_1}\right)}{x^2} \tag{1}$$

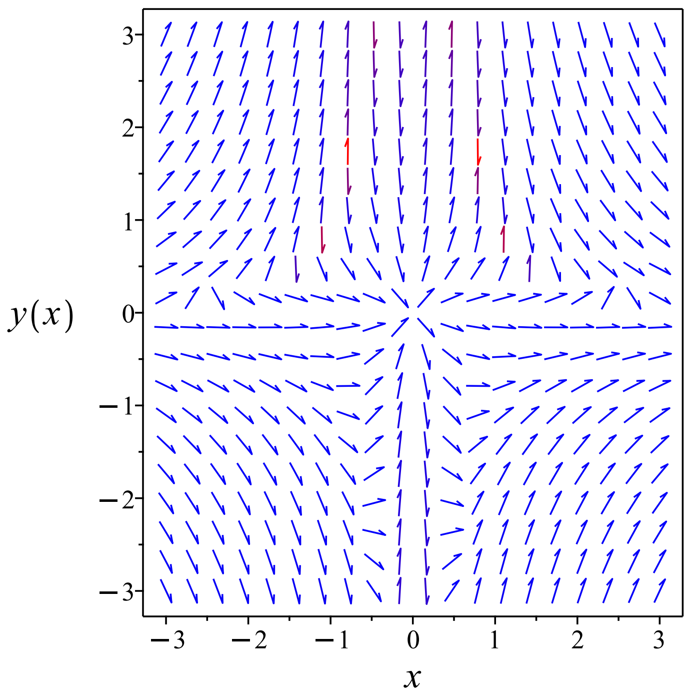


Figure 74: Slope field plot

Verification of solutions

$$y = -\frac{\text{LambertW}\left(-x^3 \mathrm{e}^{-c_1}\right)}{x^2}$$

Verified OK.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
trying homogeneous types:
trying homogeneous G
<- homogeneous successful`
```

✓ Solution by Maple

Time used: 0.062 (sec). Leaf size: 19

```
dsolve((2*x^2*y(x)^2+y(x))+(x^3*y(x)-x)*diff(y(x),x)=0,y(x), singsol=all)
```

$$y(x) = -\frac{\text{LambertW}\left(-x^3 e^{-3c_1}\right)}{x^2}$$

✓ Solution by Mathematica

Time used: 2.365 (sec). Leaf size: 33

```
DSolve[(2*x^2*y[x]^2+y[x])+(x^3*y[x]-x)*y'[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to -\frac{W\left(e^{-1+\frac{9c_1}{2^{2/3}}}x^3\right)}{x^2}$$
$$y(x) \to 0$$

1.46 problem 47

Internal problem ID [3191]
Internal file name [`OUTPUT/2683_Sunday_June_05_2022_08_38_44_AM_51986505/index.tex`]

Book: Differential equations for engineers by Wei-Chau XIE, Cambridge Press 2010
Section: Chapter 2. First-Order and Simple Higher-Order Differential Equations. Page 78
Problem number: 47.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"exactWithIntegrationFactor", "first_order_ode_lie_symmetry_calculated"**

Maple gives the following as the ode type

```
[[_homogeneous, `class G`]]
```

$$\boxed{y^2 + (yx + \tan(yx))\, y' = 0}$$

1.46.1 Solving as first order ode lie symmetry calculated ode

Writing the ode as

$$\begin{aligned} y' &= -\frac{y^2}{xy + \tan(xy)} \\ y' &= \omega(x, y) \end{aligned}$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x + \omega(\eta_y - \xi_x) - \omega^2 \xi_y - \omega_x \xi - \omega_y \eta = 0 \tag{A}$$

The type of this ode is not in the lookup table. To determine ξ, η then (A) is solved using ansatz. Making bivariate polynomials of degree 1 to use as anstaz gives

$$\xi = x a_2 + y a_3 + a_1 \tag{1E}$$

$$\eta = x b_2 + y b_3 + b_1 \tag{2E}$$

Where the unknown coefficients are

$$\{a_1, a_2, a_3, b_1, b_2, b_3\}$$

Substituting equations (1E,2E) and ω into (A) gives

$$\begin{aligned} &b_2 - \frac{y^2(b_3 - a_2)}{xy + \tan(xy)} - \frac{y^4 a_3}{(xy + \tan(xy))^2} \\ &- \frac{y^2(y + y(1 + \tan(xy)^2))(xa_2 + ya_3 + a_1)}{(xy + \tan(xy))^2} \\ &- \left(-\frac{2y}{xy + \tan(xy)} + \frac{y^2(x + x(1 + \tan(xy)^2))}{(xy + \tan(xy))^2}\right)(xb_2 + yb_3 + b_1) = 0 \end{aligned} \tag{5E}$$

Putting the above in normal form gives

$$-\frac{\tan(xy)^2 x^2y^2b_2 + \tan(xy)^2 x\,y^3a_2 + \tan(xy)^2 x\,y^3b_3 + \tan(xy)^2 y^4a_3 + \tan(xy)^2 x\,y^2b_1 + \tan(xy)^2 y^3a}{}$$

$$= 0$$

Setting the numerator to zero gives

$$\begin{aligned} &-\tan(xy)^2 x^2y^2b_2 - \tan(xy)^2 x\,y^3a_2 - \tan(xy)^2 x\,y^3b_3 \\ &- \tan(xy)^2 y^4a_3 - \tan(xy)^2 x\,y^2b_1 - \tan(xy)^2 y^3a_1 + x^2y^2b_2 \\ &- x\,y^3a_2 - x\,y^3b_3 - 3y^4a_3 + 4\tan(xy)\,xyb_2 + \tan(xy)\,y^2a_2 \\ &+ \tan(xy)\,y^2b_3 - 2y^3a_1 + \tan(xy)^2 b_2 + 2\tan(xy)\,yb_1 = 0 \end{aligned} \tag{6E}$$

Looking at the above PDE shows the following are all the terms with $\{x, y\}$ in them.

$$\{x, y, \tan(xy)\}$$

The following substitution is now made to be able to collect on all terms with $\{x, y\}$ in them

$$\{x = v_1, y = v_2, \tan(xy) = v_3\}$$

The above PDE (6E) now becomes

$$\begin{aligned} &-v_3^2v_1v_2^3a_2 - v_3^2v_2^4a_3 - v_3^2v_1^2v_2^2b_2 - v_3^2v_1v_2^3b_3 - v_3^2v_2^3a_1 - v_3^2v_1v_2^2b_1 - v_1v_2^3a_2 - 3v_2^4a_3 \\ &+ v_1^2v_2^2b_2 - v_1v_2^3b_3 - 2v_2^3a_1 + v_3v_2^2a_2 + 4v_3v_1v_2b_2 + v_3v_2^2b_3 + 2v_3v_2b_1 + v_3^2b_2 = 0 \end{aligned} \tag{7E}$$

Collecting the above on the terms v_i introduced, and these are

$$\{v_1, v_2, v_3\}$$

Equation (7E) now becomes

$$-v_3^2v_1^2v_2^2b_2+v_1^2v_2^2b_2+(-a_2-b_3)\,v_1v_2^3v_3^2+(-a_2-b_3)\,v_1v_2^3-v_3^2v_1v_2^2b_1+4v_3v_1v_2b_2 \\ -v_3^2v_2^4a_3-3v_2^4a_3-v_3^2v_2^3a_1-2v_2^3a_1+(a_2+b_3)\,v_2^2v_3+2v_3v_2b_1+v_3^2b_2=0 \tag{8E}$$

Setting each coefficients in (8E) to zero gives the following equations to solve

$$\begin{aligned}
b_2 &= 0\\
-2a_1 &= 0\\
-a_1 &= 0\\
-3a_3 &= 0\\
-a_3 &= 0\\
-b_1 &= 0\\
2b_1 &= 0\\
-b_2 &= 0\\
4b_2 &= 0\\
-a_2-b_3 &= 0\\
a_2+b_3 &= 0
\end{aligned}$$

Solving the above equations for the unknowns gives

$$\begin{aligned}
a_1 &= 0\\
a_2 &= -b_3\\
a_3 &= 0\\
b_1 &= 0\\
b_2 &= 0\\
b_3 &= b_3
\end{aligned}$$

Substituting the above solution in the anstaz (1E,2E) (using 1 as arbitrary value for any unknown in the RHS) gives

$$\begin{aligned}
\xi &= -x\\
\eta &= y
\end{aligned}$$

Shifting is now applied to make $\xi = 0$ in order to simplify the rest of the computation

$$\begin{aligned}\eta &= \eta - \omega(x,y)\,\xi \\ &= y - \left(-\frac{y^2}{xy + \tan(xy)}\right)(-x) \\ &= \frac{\tan(xy)\,y}{xy + \tan(xy)} \\ \xi &= 0\end{aligned}$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x,y) \to (R,S)$ where (R,S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}$$

The above comes from the requirements that $\left(\xi\frac{\partial}{\partial x} + \eta\frac{\partial}{\partial y}\right) S(x,y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$R = x$$

S is found from

$$\begin{aligned}S &= \int \frac{1}{\eta} dy \\ &= \int \frac{1}{\frac{\tan(xy)y}{xy+\tan(xy)}} dy\end{aligned}$$

Which results in

$$S = \ln(\sin(xy)) + \ln(xy)$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x,y)S_y}{R_x + \omega(x,y)R_y} \tag{2}$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x,y)$ is the right hand side of the original ode given by

$$\omega(x,y) = -\frac{y^2}{xy + \tan(xy)}$$

Evaluating all the partial derivatives gives

$$R_x = 1$$
$$R_y = 0$$
$$S_x = \cot(xy)\, y + \frac{1}{x}$$
$$S_y = \cot(xy)\, x + \frac{1}{y}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = \frac{1}{x} \tag{2A}$$

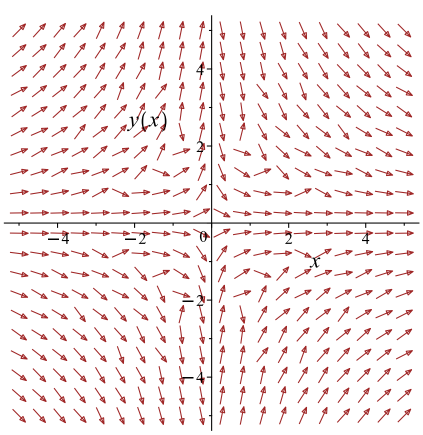
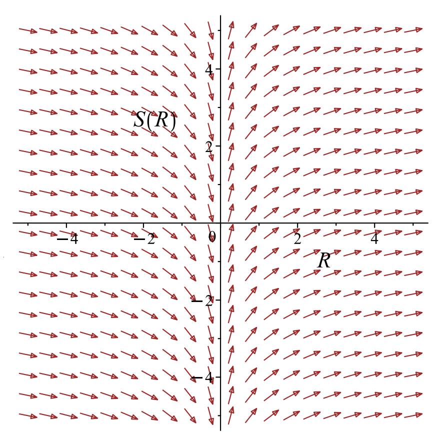
We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = \frac{1}{R}$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$S(R) = \ln(R) + c_1 \tag{4}$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in x, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dx} = -\frac{y^2}{xy+\tan(xy)}$ 	$R = x$ $S = \ln(\sin(xy)) + \ln(x)$	$\frac{dS}{dR} = \frac{1}{R}$ 

Summary
The solution(s) found are the following

$$\ln(\sin(yx)) + \ln(x) + \ln(y) = \ln(x) + c_1 \tag{1}$$

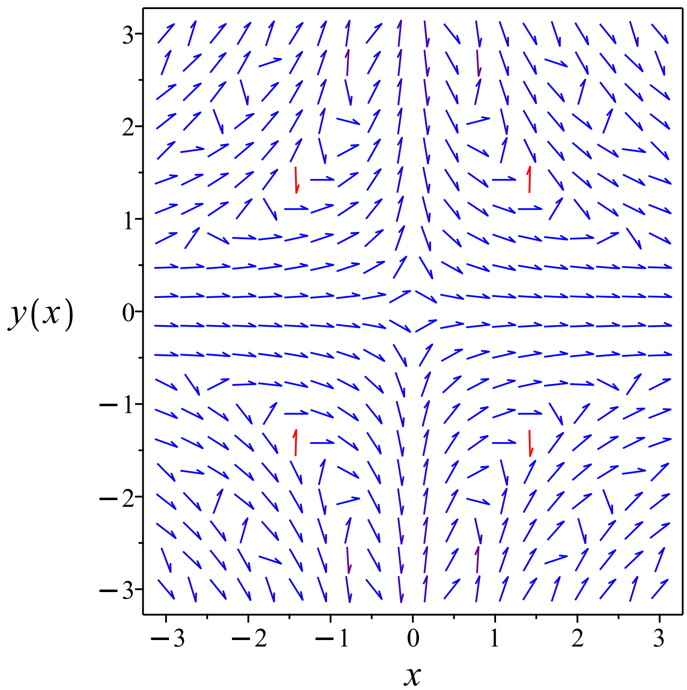


Figure 75: Slope field plot

Verification of solutions

$$\ln(\sin(yx)) + \ln(x) + \ln(y) = \ln(x) + c_1$$

Verified OK.

1.46.2 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x,y) + N(x,y)\frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x,y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y) = 0$$

Hence

$$\frac{\partial \phi}{\partial x} + \frac{\partial \phi}{\partial y}\frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$\begin{aligned}\frac{\partial \phi}{\partial x} &= M \\ \frac{\partial \phi}{\partial y} &= N\end{aligned}$$

But since $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x, y)$ but at least we know now that we can do that since the condition $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x, y)\,\mathrm{d}x + N(x, y)\,\mathrm{d}y = 0 \tag{1A}$$

Therefore

$$\begin{aligned}(xy + \tan(xy))\,\mathrm{d}y &= \left(-y^2\right)\mathrm{d}x \\ \left(y^2\right)\mathrm{d}x + (xy + \tan(xy))\,\mathrm{d}y &= 0 \qquad \text{(2A)}\end{aligned}$$

Comparing (1A) and (2A) shows that

$$\begin{aligned}M(x, y) &= y^2 \\ N(x, y) &= xy + \tan(xy)\end{aligned}$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Using result found above gives

$$\begin{aligned}\frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}\left(y^2\right) \\ &= 2y\end{aligned}$$

And

$$\begin{aligned}\frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}(xy + \tan(xy)) \\ &= y\left(1 + \sec(xy)^2\right)\end{aligned}$$

Since $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$, then the ODE is <u>not exact</u>. Since the ODE is not exact, we will try to find an integrating factor to make it exact. Let

$$\begin{aligned}A &= \frac{1}{N}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right) \\ &= \frac{1}{xy + \tan(xy)}\left((2y) - \left(y + y\left(1 + \tan(xy)^2\right)\right)\right) \\ &= -\frac{y\tan(xy)^2}{xy + \tan(xy)}\end{aligned}$$

Since A depends on y, it can not be used to obtain an integrating factor. We will now try a second method to find an integrating factor. Let

$$\begin{aligned}B &= \frac{1}{M}\left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) \\ &= \frac{1}{y^2}\left(\left(y + y\left(1 + \tan(xy)^2\right)\right) - (2y)\right) \\ &= \frac{\tan(xy)^2}{y}\end{aligned}$$

Since B depends on x, it can not be used to obtain an integrating factor.We will now try a third method to find an integrating factor. Let

$$R = \frac{\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}}{xM - yN}$$

R is now checked to see if it is a function of only $t = xy$. Therefore

$$\begin{aligned}R &= \frac{\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}}{xM - yN} \\ &= \frac{\left(y + y\left(1 + \tan(xy)^2\right)\right) - (2y)}{x(y^2) - y(xy + \tan(xy))} \\ &= -\tan(xy)\end{aligned}$$

Replacing all powers of terms xy by t gives

$$R = -\tan(t)$$

Since R depends on t only, then it can be used to find an integrating factor. Let the integrating factor be μ then

$$\begin{aligned}\mu &= e^{\int R \,\mathrm{d}t} \\ &= e^{\int (-\tan(t)) \,\mathrm{d}t}\end{aligned}$$

The result of integrating gives

$$\begin{aligned}\mu &= e^{\ln(\cos(t))} \\ &= \cos(t)\end{aligned}$$

Now t is replaced back with xy giving

$$\mu = \cos(xy)$$

Multiplying M and N by this integrating factor gives new M and new N which are called $\overline{M}$ and $\overline{N}$ so not to confuse them with the original M and N

$$\begin{aligned}\overline{M} &= \mu M \\ &= \cos(xy)\left(y^2\right) \\ &= y^2\cos(xy)\end{aligned}$$

And

$$\begin{aligned}\overline{N} &= \mu N \\ &= \cos(xy)\left(xy + \tan(xy)\right) \\ &= y\cos(xy)\,x + \sin(xy)\end{aligned}$$

A modified ODE is now obtained from the original ODE, which is exact and can solved. The modified ODE is

$$\begin{aligned}\overline{M} + \overline{N}\frac{\mathrm{d}y}{\mathrm{d}x} &= 0 \\ \left(y^2\cos(xy)\right) + \left(y\cos(xy)\,x + \sin(xy)\right)\frac{\mathrm{d}y}{\mathrm{d}x} &= 0\end{aligned}$$

The following equations are now set up to solve for the function $\phi(x,y)$

$$\frac{\partial \phi}{\partial x} = \overline{M} \tag{1}$$

$$\frac{\partial \phi}{\partial y} = \overline{N} \tag{2}$$

Integrating (1) w.r.t. x gives

$$\int \frac{\partial \phi}{\partial x} \mathrm{d}x = \int \overline{M} \mathrm{d}x$$
$$\int \frac{\partial \phi}{\partial x} \mathrm{d}x = \int y^2 \cos(xy) \mathrm{d}x$$
$$\phi = y \sin(xy) + f(y) \tag{3}$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$\frac{\partial \phi}{\partial y} = y \cos(xy) x + \sin(xy) + f'(y) \tag{4}$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = y \cos(xy) x + \sin(xy)$. Therefore equation (4) becomes

$$y \cos(xy) x + \sin(xy) = y \cos(xy) x + \sin(xy) + f'(y) \tag{5}$$

Solving equation (5) for $f'(y)$ gives

$$f'(y) = 0$$

Therefore

$$f(y) = c_1$$

Where c_1 is constant of integration. Substituting this result for $f(y)$ into equation (3) gives ϕ

$$\phi = y \sin(xy) + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = y \sin(xy)$$

Summary

The solution(s) found are the following

$$y \sin(yx) = c_1 \tag{1}$$

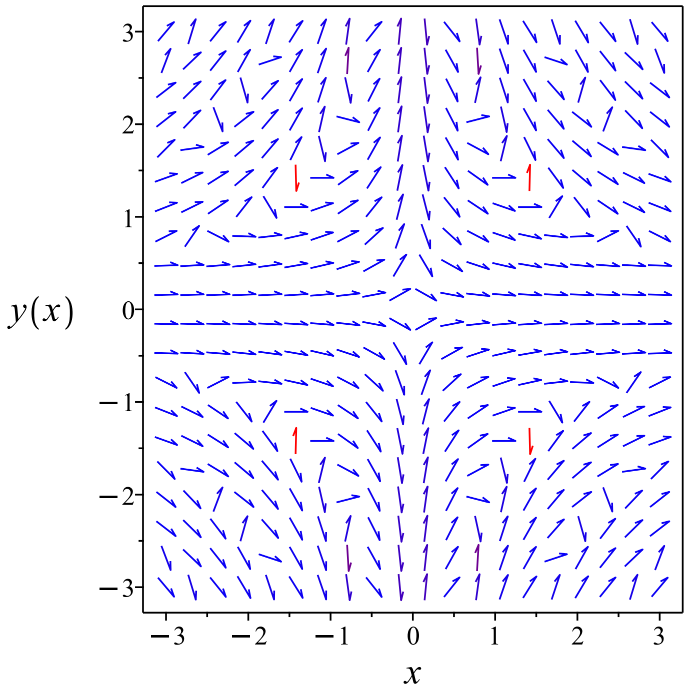


Figure 76: Slope field plot

Verification of solutions

$$y\sin(yx) = c_1$$

Verified OK.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
trying homogeneous types:
trying homogeneous G
<- homogeneous successful`
```

✓ Solution by Maple

Time used: 0.062 (sec). Leaf size: 18

```
dsolve((y(x)^2)+(x*y(x)+tan(x*y(x)))*diff(y(x),x)=0,y(x), singsol=all)
```

$$y(x) = \frac{\text{RootOf}\left(\_Zc_1 \sin\left(\_Z\right) - x\right)}{x}$$

✓ Solution by Mathematica

Time used: 0.271 (sec). Leaf size: 14

```
DSolve[(y[x]^2)+(x*y[x]+Tan[x*y[x]])*y'[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$\text{Solve}[y(x)\sin(xy(x)) = c_1, y(x)]$$

1.47 problem 48

Internal problem ID [3192]
Internal file name [`OUTPUT/2684_Sunday_June_05_2022_08_38_45_AM_41111291/index.tex`]

Book: Differential equations for engineers by Wei-Chau XIE, Cambridge Press 2010
Section: Chapter 2. First-Order and Simple Higher-Order Differential Equations. Page 78
Problem number: 48.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"unknown"**

Maple gives the following as the ode type

`[_rational]`

Unable to solve or complete the solution.

$$\boxed{2y^4x - y + \left(4y^3x^3 - x\right)y' = 0}$$

Unable to determine ODE type.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
trying homogeneous types:
trying Chini
differential order: 1; looking for linear symmetries
trying exact
Looking for potential symmetries
trying inverse_Riccati
trying an equivalence to an Abel ODE
equivalence obtained to this Abel ODE: diff(y(x),x) = -3*y(x)/x+(-16*x+6)*y(x)^2+(-48*x^3+24
trying to solve the Abel ODE ...
Looking for potential symmetries
Looking for potential symmetries
differential order: 1; trying a linearization to 2nd order
--- trying a change of variables {x -> y(x), y(x) -> x}
differential order: 1; trying a linearization to 2nd order
trying 1st order ODE linearizable_by_differentiation
--- Trying Lie symmetry methods, 1st order ---
`, `-> Computing symmetries using: way = 2
`, `-> Computing symmetries using: way = 3
`, `-> Computing symmetries using: way = 4
trying symmetry patterns for 1st order ODEs
-> trying a symmetry pattern of the form [F(x)*G(y), 0]
-> trying a symmetry pattern of the form [0, F(x)*G(y)]
-> trying symmetry patterns of the forms [F(x),G(y)] and [G(y),F(x)]
-> trying a symmetry pattern of the form [F(x),G(x)]
-> trying a symmetry pattern of the form [F(y),G(y)]
-> trying a symmetry pattern of the form [F(x)+G(y), 0]
-> trying a symmetry pattern of the form [0, F(x)+G(y)]
-> trying a symmetry pattern of the form [F(x),G(x)*y+H(x)]
-> trying a symmetry pattern of conformal type`
```

✗ Solution by Maple

```
dsolve((2*x*y(x)^4-y(x))+(4*x^3*y(x)^3-x)*diff(y(x),x)=0,y(x), singsol=all)
```

No solution found

✗ Solution by Mathematica

Time used: 0.0 (sec). Leaf size: 0

```
DSolve[(2*x*y[x]^4-y[x])+(4*x^3*y[x]^3-x)*y'[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

Not solved

1.48 problem 49

Internal problem ID [3193]
Internal file name [`OUTPUT/2685_Sunday_June_05_2022_08_38_47_AM_81204172/index.tex`]

Book: Differential equations for engineers by Wei-Chau XIE, Cambridge Press 2010
Section: Chapter 2. First-Order and Simple Higher-Order Differential Equations. Page 78
Problem number: 49.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"unknown"**

Maple gives the following as the ode type

`[_rational]`

Unable to solve or complete the solution.

$$\boxed{y^3 + y + (x^3 + y^2 - x)\, y' = -x^2}$$

Unable to determine ODE type.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
trying homogeneous types:
trying Chini
differential order: 1; looking for linear symmetries
trying exact
Looking for potential symmetries
trying inverse_Riccati
trying an equivalence to an Abel ODE
differential order: 1; trying a linearization to 2nd order
--- trying a change of variables {x -> y(x), y(x) -> x}
differential order: 1; trying a linearization to 2nd order
trying 1st order ODE linearizable_by_differentiation
--- Trying Lie symmetry methods, 1st order ---
`, `-> Computing symmetries using: way = 2
`, `-> Computing symmetries using: way = 3
`, `-> Computing symmetries using: way = 4
trying symmetry patterns for 1st order ODEs
-> trying a symmetry pattern of the form [F(x)*G(y), 0]
-> trying a symmetry pattern of the form [0, F(x)*G(y)]
-> trying symmetry patterns of the forms [F(x),G(y)] and [G(y),F(x)]
-> trying a symmetry pattern of the form [F(x),G(x)]
-> trying a symmetry pattern of the form [F(y),G(y)]
-> trying a symmetry pattern of the form [F(x)+G(y), 0]
-> trying a symmetry pattern of the form [0, F(x)+G(y)]
-> trying a symmetry pattern of the form [F(x),G(x)*y+H(x)]
-> trying a symmetry pattern of conformal type`
```

✗ Solution by Maple

```
dsolve((x^2+y(x)^3+y(x))+( x^3+y(x)^2-x )*diff(y(x),x)=0,y(x), singsol=all)
```

No solution found

✗ Solution by Mathematica

Time used: 0.0 (sec). Leaf size: 0

```
DSolve[(x^2+y[x]^3+y[x])+( x^3+y[x]^2-x )*y'[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

Not solved

1.49 problem 50

Internal problem ID [3194]
Internal file name [`OUTPUT/2686_Sunday_June_05_2022_08_38_47_AM_79769673/index.tex`]

Book: Differential equations for engineers by Wei-Chau XIE, Cambridge Press 2010
Section: Chapter 2. First-Order and Simple Higher-Order Differential Equations. Page 78
Problem number: 50.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"first_order_ode_lie_symmetry_calculated"**

Maple gives the following as the ode type

```
[_rational, [_1st_order, `_with_symmetry_[F(x)*G(y),0]`]]
```

$$\boxed{y(y^2+1)+x(y^2-x+1)\,y'=0}$$

1.49.1 Solving as first order ode lie symmetry calculated ode

Writing the ode as

$$\begin{aligned} y' &= -\frac{y(y^2+1)}{x\,(y^2-x+1)} \\ y' &= \omega(x,y) \end{aligned}$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x+\omega(\eta_y-\xi_x)-\omega^2\xi_y-\omega_x\xi-\omega_y\eta=0 \tag{A}$$

The type of this ode is not in the lookup table. To determine ξ,η then (A) is solved using ansatz. Making bivariate polynomials of degree 3 to use as anstaz gives

$$\xi = x^3a_7+x^2ya_8+x\,y^2a_9+y^3a_{10}+x^2a_4+xya_5+y^2a_6+xa_2+ya_3+a_1 \tag{1E}$$

$$\eta = x^3b_7+x^2yb_8+x\,y^2b_9+y^3b_{10}+x^2b_4+xyb_5+y^2b_6+xb_2+yb_3+b_1 \tag{2E}$$

Where the unknown coefficients are

$$\{a_1,a_2,a_3,a_4,a_5,a_6,a_7,a_8,a_9,a_{10},b_1,b_2,b_3,b_4,b_5,b_6,b_7,b_8,b_9,b_{10}\}$$

Substituting equations (1E,2E) and ω into (A) gives

$$3x^2b_7+2xyb_8+y^2b_9+2xb_4+yb_5+b_2 -\frac{y(y^2+1)\left(-3x^2a_7+x^2b_8-2xya_8+2xyb_9-y^2a_9+3y^2b_{10}-2xa_4+xb_5-ya_5+2yb_6-a_2+b_3\right)}{x\left(y^2-x+1\right)} -\frac{y^2(y^2+1)^2\left(x^2a_8+2xya_9+3y^2a_{10}+xa_5+2ya_6+a_3\right)}{x^2\left(y^2-x+1\right)^2} -\left(\frac{y(y^2+1)}{x^2\left(y^2-x+1\right)}-\frac{y(y^2+1)}{x\left(y^2-x+1\right)^2}\right)\left(x^3a_7+x^2ya_8 +x\,y^2a_9+y^3a_{10}+x^2a_4+xya_5+y^2a_6+xa_2+ya_3+a_1\right) -\left(-\frac{y^2+1}{x\left(y^2-x+1\right)}-\frac{2y^2}{x\left(y^2-x+1\right)}+\frac{2y^2(y^2+1)}{x\left(y^2-x+1\right)^2}\right)\left(x^3b_7 +x^2yb_8+x\,y^2b_9+y^3b_{10}+x^2b_4+xyb_5+y^2b_6+xb_2+yb_3+b_1\right)=0 \tag{5E}$$

Putting the above in normal form gives

$$\frac{x^2y^5a_9+4x^3y^3a_7-4x\,y^5a_9-4x\,y^5b_{10}-x^4ya_7+x^2y^3a_9+2x^2y^3b_{10}+2x^3ya_7-2x\,y^3a_9-2x\,y^3b_{10}+2x\,y}{} =0$$

Setting the numerator to zero gives

$$x^2y^5a_9+4x^3y^3a_7-4x\,y^5a_9-4x\,y^5b_{10}-x^4ya_7+x^2y^3a_9+2x^2y^3b_{10} +2x^3ya_7-2x\,y^3a_9-2x\,y^3b_{10}+2x\,y^6a_{10}+2x\,y^4a_{10}+3x^3y^4b_4 +x^2y^5a_4+x^2y^5b_5+2b_2x^2-2y^2a_3+2x^2y^4b_2-5x^3y^2b_2+x^2y^3a_2 -2x^2y^3b_3+2x\,y^4a_3+x\,y^4b_1-3x^2y^2b_1+4x^2y^2b_2+2x\,y^3a_1 +x^2ya_2+2x\,y^2a_3+2x\,y^2b_1+2xya_1+3x^6b_7-7x^5b_7-4y^8a_{10} -8y^6a_{10}-4y^4a_{10}+2x^3yb_8+yb_5x^2+4x^4b_7+4x^4y^4b_7-9x^5y^2b_7 +8x^4y^2b_7+2x^3y^5b_8-6x^4y^3b_8+2x^5yb_8+4x^3y^3b_8-4x^4yb_8 -3x^3y^4b_9+x^4y^2b_9-x^3y^2b_9+2x^3y^5a_7-2x\,y^7a_9-2x\,y^7b_{10} -x^4y^3a_7-2y^6a_3-y^5a_1+x^4b_2-4y^4a_3-3x^3b_2-2y^3a_1-x^2b_1 +xb_1-x\,y^2b_6-6y^5a_6-5x^4b_4-3y^3a_6-3y^7a_6+2x^5b_4-4x^3y^3b_5 +x^2y^4a_5-x^2y^4b_6+2x\,y^5a_6+x^2y^2b_6+2x\,y^3a_6+x^2ya_4-x\,y^2a_5 +x^4yb_5+6x^3y^2b_4+2x^2y^3a_4+2x^2y^3b_5-x\,y^6a_5-x\,y^6b_6-7x^4y^2b_4 +3x^3b_4-2x\,y^4a_5-2x\,y^4b_6-2x^3yb_5+x^2y^2a_5-ya_1=0 \tag{6E}$$

Looking at the above PDE shows the following are all the terms with $\{x,y\}$ in them.

$$\{x,y\}$$

The following substitution is now made to be able to collect on all terms with $\{x, y\}$ in them

$$\{x = v_1, y = v_2\}$$

The above PDE (6E) now becomes

$$\begin{aligned}
&2b_2v_1^2 - 2v_2^2a_3 + 3v_1^6b_7 - 7v_1^5b_7 - 4v_2^8a_{10} - 8v_2^6a_{10} - 4v_2^4a_{10} + 4v_1^4b_7 \\
&- 2v_2^6a_3 - v_2^5a_1 + v_1^4b_2 - 4v_2^4a_3 - 3v_1^3b_2 - 2v_2^3a_1 - v_1^2b_1 + v_1b_1 - 6v_2^5a_6 \\
&- 5v_1^4b_4 - 3v_2^3a_6 - 3v_2^7a_6 + 2v_1^5b_4 + 3v_1^3b_4 - v_2a_1 + v_1^2v_2^5a_9 + 4v_1^3v_2^3a_7 \\
&- 4v_1v_2^5a_9 - 4v_1v_2^5b_{10} - v_1^4v_2a_7 + v_1^2v_2^3a_9 + 2v_1^2v_2^3b_{10} + 2v_1^3v_2a_7 \\
&- 2v_1v_2^3a_9 - 2v_1v_2^3b_{10} + 2v_1v_2^6a_{10} + 2v_1v_2^4a_{10} + 3v_1^3v_2^4b_4 + v_1^2v_2^5a_4 \\
&+ v_1^2v_2^5b_5 + 2v_1^2v_2^4b_2 - 5v_1^3v_2^2b_2 + v_1^2v_2^3a_2 - 2v_1^2v_2^3b_3 + 2v_1v_2^4a_3 + v_1v_2^4b_1 \\
&- 3v_1^2v_2^2b_1 + 4v_1^2v_2^2b_2 + 2v_1v_2^3a_1 - 2v_1v_2^7a_9 - 2v_1v_2^7b_{10} - v_1^4v_2^3a_7 \\
&- v_1v_2^2b_6 - 4v_1^3v_2^3b_5 + v_1^2v_2^4a_5 - v_1^2v_2^4b_6 + 2v_1v_2^5a_6 + v_1^2v_2^2b_6 + 2v_1v_2^3a_6 \\
&+ v_1^2v_2a_4 - v_1v_2^2a_5 + v_1^4v_2b_5 + 6v_1^3v_2^2b_4 + 2v_1^2v_2^3a_4 + 2v_1^2v_2^3b_5 - v_1v_2^6a_5 \\
&- v_1v_2^6b_6 - 7v_1^4v_2^2b_4 - 2v_1v_2^4a_5 - 2v_1v_2^4b_6 - 2v_1^3v_2b_5 + v_1^2v_2^2a_5 \\
&+ v_1^2v_2a_2 + 2v_1v_2^2a_3 + 2v_1v_2^2b_1 + 2v_1v_2a_1 + 2v_1^3v_2b_8 + v_2b_5v_1^2 \\
&+ 4v_1^4v_2^4b_7 - 9v_1^5v_2^2b_7 + 8v_1^4v_2^2b_7 + 2v_1^3v_2^5b_8 - 6v_1^4v_2^3b_8 + 2v_1^5v_2b_8 \\
&+ 4v_1^3v_2^3b_8 - 4v_1^4v_2b_8 - 3v_1^3v_2^4b_9 + v_1^4v_2^2b_9 - v_1^3v_2^2b_9 + 2v_1^3v_2^5a_7 = 0
\end{aligned} \tag{7E}$$

Collecting the above on the terms v_i introduced, and these are

$$\{v_1, v_2\}$$

Equation (7E) now becomes

$$\begin{aligned}
&-2v_2^2a_3 + 3v_1^6b_7 - 4v_2^8a_{10} + v_1b_1 - 3v_2^7a_6 - v_2a_1 + (a_9 + 2b_{10} + a_2 - 2b_3 + 2a_4 + 2b_5)\, v_2^3v_1^2 \\
&+ (2a_7 - 2b_5 + 2b_8)\, v_2v_1^3 + (-2a_9 - 2b_{10} + 2a_1 + 2a_6)\, v_2^3v_1 + (2a_{10} - a_5 - b_6)\, v_2^6v_1 \\
&+ (2a_{10} + 2a_3 + b_1 - 2a_5 - 2b_6)\, v_2^4v_1 + (3b_4 - 3b_9)\, v_2^4v_1^3 + (a_5 + 2b_2 - b_6)\, v_2^4v_1^2 \\
&+ (-5b_2 + 6b_4 - b_9)\, v_2^2v_1^3 + (a_5 - 3b_1 + 4b_2 + b_6)\, v_2^2v_1^2 + (-2a_9 - 2b_{10})\, v_2^7v_1 \\
&+ (-a_7 - 6b_8)\, v_2^3v_1^4 + (a_9 + a_4 + b_5)\, v_2^5v_1^2 + (4a_7 - 4b_5 + 4b_8)\, v_2^3v_1^3 + (-4a_9 - 4b_{10} + 2a_6)\, v_2^5v_1 \\
&+ (-a_7 + b_5 - 4b_8)\, v_2v_1^4 + (2a_3 - a_5 + 2b_1 - b_6)\, v_2^2v_1 + (a_2 + a_4 + b_5)\, v_2v_1^2 \\
&+ (-7b_4 + 8b_7 + b_9)\, v_2^2v_1^4 + (2b_8 + 2a_7)\, v_2^5v_1^3 + (-3b_2 + 3b_4)\, v_1^3 + (-2a_1 - 3a_6)\, v_2^3 \\
&+ (-b_1 + 2b_2)\, v_1^2 + (-7b_7 + 2b_4)\, v_1^5 + (-8a_{10} - 2a_3)\, v_2^6 + (-4a_{10} - 4a_3)\, v_2^4 \\
&+ (4b_7 + b_2 - 5b_4)\, v_1^4 + (-a_1 - 6a_6)\, v_2^5 + 2v_1v_2a_1 + 4v_1^4v_2^4b_7 - 9v_1^5v_2^2b_7 + 2v_1^5v_2b_8 = 0
\end{aligned} \tag{8E}$$

Setting each coefficients in (8E) to zero gives the following equations to solve

$$
\begin{aligned}
b_1 &= 0 \\
-a_1 &= 0 \\
2a_1 &= 0 \\
-2a_3 &= 0 \\
-3a_6 &= 0 \\
-4a_{10} &= 0 \\
-9b_7 &= 0 \\
3b_7 &= 0 \\
4b_7 &= 0 \\
2b_8 &= 0 \\
-2a_1 - 3a_6 &= 0 \\
-a_1 - 6a_6 &= 0 \\
-a_7 - 6b_8 &= 0 \\
-2a_9 - 2b_{10} &= 0 \\
-8a_{10} - 2a_3 &= 0 \\
-4a_{10} - 4a_3 &= 0 \\
-b_1 + 2b_2 &= 0 \\
-3b_2 + 3b_4 &= 0 \\
3b_4 - 3b_9 &= 0 \\
-7b_7 + 2b_4 &= 0 \\
2b_8 + 2a_7 &= 0 \\
a_2 + a_4 + b_5 &= 0 \\
a_5 + 2b_2 - b_6 &= 0 \\
-a_7 + b_5 - 4b_8 &= 0 \\
2a_7 - 2b_5 + 2b_8 &= 0 \\
4a_7 - 4b_5 + 4b_8 &= 0 \\
-4a_9 - 4b_{10} + 2a_6 &= 0 \\
a_9 + a_4 + b_5 &= 0 \\
2a_{10} - a_5 - b_6 &= 0 \\
-5b_2 + 6b_4 - b_9 &= 0 \\
-7b_4 + 8b_7 + b_9 &= 0 \\
4b_7 + b_2 - 5b_4 &= 0 \\
2a_3 - a_5 + 2b_1 - b_6 &= 0 \\
a_5 - 3b_1 + 4b_2 + b_6 &= 0 \\
-2a_9 - 2b_{10} + 2a_1 + 2a_6 &= 0 \\
2a_{10} + 2a_3 + b_1 - 2a_5 - 2b_6 &= 0
\end{aligned}
$$

Solving the above equations for the unknowns gives

$$
\begin{aligned}
a_1 &= 0 \\
a_2 &= -b_{10} \\
a_3 &= 0 \\
a_4 &= b_{10} \\
a_5 &= 0 \\
a_6 &= 0 \\
a_7 &= 0 \\
a_8 &= a_8 \\
a_9 &= -b_{10} \\
a_{10} &= 0 \\
b_1 &= 0 \\
b_2 &= 0 \\
b_3 &= b_{10} \\
b_4 &= 0 \\
b_5 &= 0 \\
b_6 &= 0 \\
b_7 &= 0 \\
b_8 &= 0 \\
b_9 &= 0 \\
b_{10} &= b_{10}
\end{aligned}
$$

Substituting the above solution in the anstaz (1E,2E) (using 1 as arbitrary value for any unknown in the RHS) gives

$$
\begin{aligned}
\xi &= x^2 y \\
\eta &= 0
\end{aligned}
$$

Shifting is now applied to make $\xi = 0$ in order to simplify the rest of the computation

$$
\begin{aligned}
\eta &= \eta - \omega(x, y)\,\xi \\
&= 0 - \left(-\frac{y(y^2+1)}{x\,(y^2-x+1)}\right)(x^2 y) \\
&= \frac{-y^4 x - x\,y^2}{-y^2+x-1} \\
\xi &= 0
\end{aligned}
$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x, y) \to (R, S)$ where (R, S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y}\right) S(x, y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$R = x$$

S is found from

$$\begin{aligned} S &= \int \frac{1}{\eta} dy \\ &= \int \frac{1}{\frac{-y^4 x - x\, y^2}{-y^2 + x - 1}} dy \end{aligned}$$

Which results in

$$S = \arctan(y) + \frac{1}{y} - \frac{1}{xy}$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x, y) S_y}{R_x + \omega(x, y) R_y} \tag{2}$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x, y)$ is the right hand side of the original ode given by

$$\omega(x, y) = -\frac{y(y^2 + 1)}{x\,(y^2 - x + 1)}$$

Evaluating all the partial derivatives gives

$$\begin{aligned} R_x &= 1 \\ R_y &= 0 \\ S_x &= \frac{1}{x^2 y} \\ S_y &= \frac{y^2 - x + 1}{y^2\,(y^2 + 1)\, x} \end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = 0 \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = 0$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$S(R) = c_1 \tag{4}$$

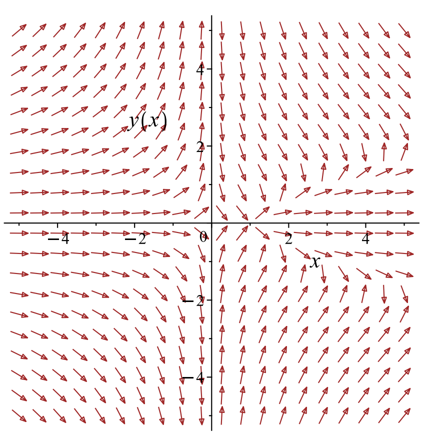
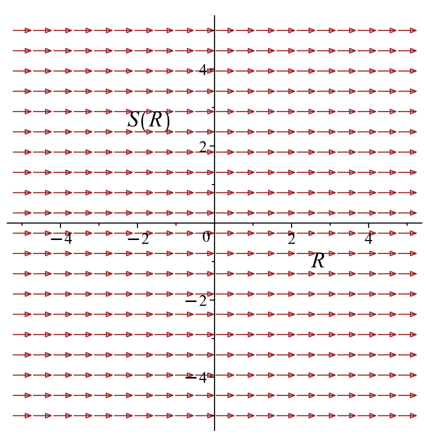
To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$\frac{\arctan(y)\, xy + x - 1}{xy} = c_1$$

Which simplifies to

$$\frac{\arctan(y)\, xy + x - 1}{xy} = c_1$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in x, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dx} = -\frac{y(y^2+1)}{x(y^2-x+1)}$ 	$R = x$ $S = \frac{\arctan(y)\, xy + x - 1}{xy}$	$\frac{dS}{dR} = 0$ 

Summary

The solution(s) found are the following

$$\frac{\arctan(y)\, xy + x - 1}{xy} = c_1 \tag{1}$$

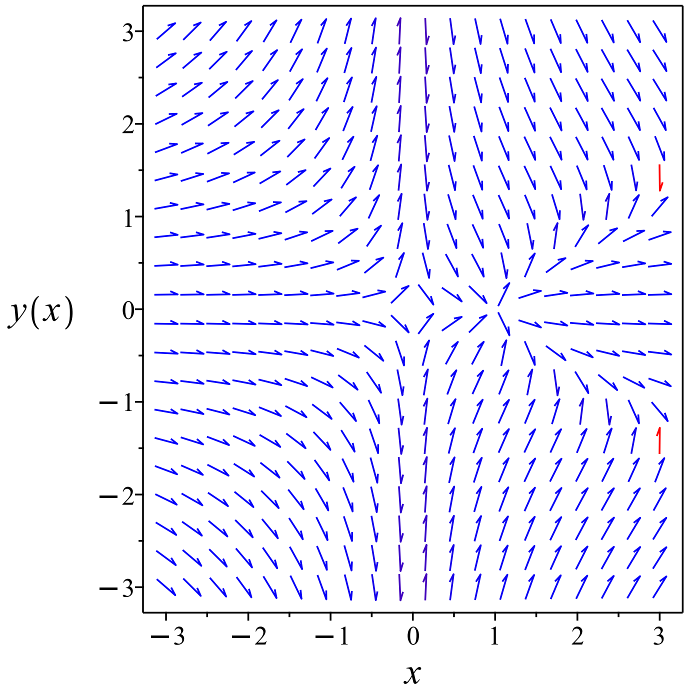


Figure 77: Slope field plot

Verification of solutions

$$\frac{\arctan(y)\, xy + x - 1}{xy} = c_1$$

Verified OK.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
trying homogeneous types:
trying Chini
differential order: 1; looking for linear symmetries
trying exact
Looking for potential symmetries
trying inverse_Riccati
trying an equivalence to an Abel ODE
equivalence obtained to this Abel ODE: diff(y(x),x) = 2/x/(x-1)*y(x)+(2+x)/(x-1)^2/x*y(x)^2+
trying to solve the Abel ODE ...
<- Abel successful
equivalence to an Abel ODE successful, Abel ODE has been solved`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 138

```
dsolve((y(x)*(y(x)^2+1))+( x*(y(x)^2-x+1))*diff(y(x),x)=0,y(x), singsol=all)
```

$$-\frac{\left(\operatorname{arctanh}\left(\frac{\sqrt{\frac{x^2 y(x)^2}{(x-1)\left(y(x)^2-x+1\right)}}\,(x-1)}{\sqrt{\frac{x-1}{x-1-y(x)^2}}\,x}\right)-c_1\right)\sqrt{\frac{x^2 y(x)^2}{(x-1)\left(y(x)^2-x+1\right)}}-\frac{\sqrt{\frac{2x-2}{x-1-y(x)^2}}\,\sqrt{2}}{2}}{\sqrt{\frac{x^2 y(x)^2}{(x-1)\left(y(x)^2-x+1\right)}}}=0$$

✓ Solution by Mathematica

Time used: 0.077 (sec). Leaf size: 34

```
DSolve[(y[x]*(y[x]^2+1))+( x*(y[x]^2-x+1))*y'[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$\text{Solve}\left[\frac{1}{2}\left(-\arctan(y(x))-\frac{1}{y(x)}\right)+\frac{1}{2xy(x)}=c_1,y(x)\right]$$

1.50 problem 51

Internal problem ID [3195]
Internal file name [OUTPUT/2687_Sunday_June_05_2022_08_38_48_AM_23160464/index.tex]

Book: Differential equations for engineers by Wei-Chau XIE, Cambridge Press 2010
Section: Chapter 2. First-Order and Simple Higher-Order Differential Equations. Page 78
Problem number: 51.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"first_order_ode_lie_symmetry_calculated"**

Maple gives the following as the ode type

```
[[_1st_order, _with_linear_symmetries], [_Abel, `2nd type`, `
   class A`]]
```

$$\boxed{y^2 + (-y + e^x)\, y' = 0}$$

1.50.1 Solving as first order ode lie symmetry calculated ode

Writing the ode as

$$\begin{aligned} y' &= -\frac{y^2}{-y + e^x} \\ y' &= \omega(x, y) \end{aligned}$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x + \omega(\eta_y - \xi_x) - \omega^2 \xi_y - \omega_x \xi - \omega_y \eta = 0 \tag{A}$$

The type of this ode is not in the lookup table. To determine ξ, η then (A) is solved using ansatz. Making bivariate polynomials of degree 1 to use as anstaz gives

$$\xi = x a_2 + y a_3 + a_1 \tag{1E}$$

$$\eta = x b_2 + y b_3 + b_1 \tag{2E}$$

Where the unknown coefficients are

$$\{a_1, a_2, a_3, b_1, b_2, b_3\}$$

Substituting equations (1E,2E) and ω into (A) gives

$$\begin{aligned} b_2 - \frac{y^2(b_3 - a_2)}{-y + \mathrm{e}^x} - \frac{y^4 a_3}{(-y + \mathrm{e}^x)^2} - \frac{y^2 \mathrm{e}^x (x a_2 + y a_3 + a_1)}{(-y + \mathrm{e}^x)^2} \\ - \left(-\frac{2y}{-y + \mathrm{e}^x} - \frac{y^2}{(-y + \mathrm{e}^x)^2}\right)(x b_2 + y b_3 + b_1) = 0 \end{aligned} \tag{5E}$$

Putting the above in normal form gives

$$\frac{-\mathrm{e}^x x\, y^2 a_2 - \mathrm{e}^x y^3 a_3 - y^4 a_3 + 2\,\mathrm{e}^x x y b_2 - \mathrm{e}^x y^2 a_1 + \mathrm{e}^x y^2 a_2 + \mathrm{e}^x y^2 b_3 - x\, y^2 b_2 - y^3 a_2 + \mathrm{e}^{2x} b_2 + 2\,\mathrm{e}^x y b_1 - 2\,\mathrm{e}}{(-y + \mathrm{e}^x)^2}$$
$$= 0$$

Setting the numerator to zero gives

$$\begin{aligned} -\mathrm{e}^x x\, y^2 a_2 - \mathrm{e}^x y^3 a_3 - y^4 a_3 + 2\,\mathrm{e}^x x y b_2 - \mathrm{e}^x y^2 a_1 + \mathrm{e}^x y^2 a_2 + \mathrm{e}^x y^2 b_3 \\ - x\, y^2 b_2 - y^3 a_2 + \mathrm{e}^{2x} b_2 + 2\,\mathrm{e}^x y b_1 - 2\,\mathrm{e}^x y b_2 - y^2 b_1 + y^2 b_2 = 0 \end{aligned} \tag{6E}$$

Simplifying the above gives

$$\begin{aligned} -\mathrm{e}^x x\, y^2 a_2 - \mathrm{e}^x y^3 a_3 - y^4 a_3 + 2\,\mathrm{e}^x x y b_2 - \mathrm{e}^x y^2 a_1 + \mathrm{e}^x y^2 a_2 + \mathrm{e}^x y^2 b_3 \\ - x\, y^2 b_2 - y^3 a_2 + \mathrm{e}^{2x} b_2 + 2\,\mathrm{e}^x y b_1 - 2\,\mathrm{e}^x y b_2 - y^2 b_1 + y^2 b_2 = 0 \end{aligned} \tag{6E}$$

Looking at the above PDE shows the following are all the terms with $\{x, y\}$ in them.

$$\{x, y, \mathrm{e}^x, \mathrm{e}^{2x}\}$$

The following substitution is now made to be able to collect on all terms with $\{x, y\}$ in them

$$\{x = v_1, y = v_2, \mathrm{e}^x = v_3, \mathrm{e}^{2x} = v_4\}$$

The above PDE (6E) now becomes

$$\begin{aligned} -v_3 v_1 v_2^2 a_2 - v_2^4 a_3 - v_3 v_2^3 a_3 - v_3 v_2^2 a_1 - v_2^3 a_2 + v_3 v_2^2 a_2 - v_1 v_2^2 b_2 \\ + 2 v_3 v_1 v_2 b_2 + v_3 v_2^2 b_3 - v_2^2 b_1 + 2 v_3 v_2 b_1 + v_2^2 b_2 - 2 v_3 v_2 b_2 + v_4 b_2 = 0 \end{aligned} \tag{7E}$$

Collecting the above on the terms v_i introduced, and these are

$$\{v_1, v_2, v_3, v_4\}$$

Equation (7E) now becomes

$$\begin{aligned} & -v_3 v_1 v_2^2 a_2 - v_1 v_2^2 b_2 + 2 v_3 v_1 v_2 b_2 - v_2^4 a_3 - v_3 v_2^3 a_3 - v_2^3 a_2 \\ & + (-a_1 + a_2 + b_3) v_2^2 v_3 + (-b_1 + b_2) v_2^2 + (2b_1 - 2b_2) v_2 v_3 + v_4 b_2 = 0 \end{aligned} \tag{8E}$$

Setting each coefficients in (8E) to zero gives the following equations to solve

$$\begin{aligned} b_2 &= 0 \\ -a_2 &= 0 \\ -a_3 &= 0 \\ -b_2 &= 0 \\ 2b_2 &= 0 \\ -b_1 + b_2 &= 0 \\ 2b_1 - 2b_2 &= 0 \\ -a_1 + a_2 + b_3 &= 0 \end{aligned}$$

Solving the above equations for the unknowns gives

$$\begin{aligned} a_1 &= b_3 \\ a_2 &= 0 \\ a_3 &= 0 \\ b_1 &= 0 \\ b_2 &= 0 \\ b_3 &= b_3 \end{aligned}$$

Substituting the above solution in the anstaz (1E,2E) (using 1 as arbitrary value for any unknown in the RHS) gives

$$\begin{aligned} \xi &= 1 \\ \eta &= y \end{aligned}$$

Shifting is now applied to make $\xi = 0$ in order to simplify the rest of the computation

$$\begin{aligned}\eta &= \eta - \omega(x,y)\,\xi \\ &= y - \left(-\frac{y^2}{-y+\mathrm{e}^x}\right)(1) \\ &= \frac{y\,\mathrm{e}^x}{-y+\mathrm{e}^x} \\ \xi &= 0\end{aligned}$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x,y) \to (R,S)$ where (R,S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}$$

The above comes from the requirements that $\left(\xi\frac{\partial}{\partial x} + \eta\frac{\partial}{\partial y}\right)S(x,y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$R = x$$

S is found from

$$\begin{aligned}S &= \int \frac{1}{\eta} dy \\ &= \int \frac{1}{\frac{y\,\mathrm{e}^x}{-y+\mathrm{e}^x}} dy\end{aligned}$$

Which results in

$$S = \ln(y) - \mathrm{e}^{-x} y$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x,y)S_y}{R_x + \omega(x,y)R_y} \tag{2}$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x,y)$ is the right hand side of the original ode given by

$$\omega(x,y) = -\frac{y^2}{-y+\mathrm{e}^x}$$

Evaluating all the partial derivatives gives

$$
\begin{aligned}
R_x &= 1 \\
R_y &= 0 \\
S_x &= \mathrm{e}^{-x} y \\
S_y &= \frac{-\mathrm{e}^{-x} y + 1}{y}
\end{aligned}
$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$
\frac{dS}{dR} = 0 \tag{2A}
$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$
\frac{dS}{dR} = 0
$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$
S(R) = c_1 \tag{4}
$$

To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$
\ln(y) - \mathrm{e}^{-x} y = c_1
$$

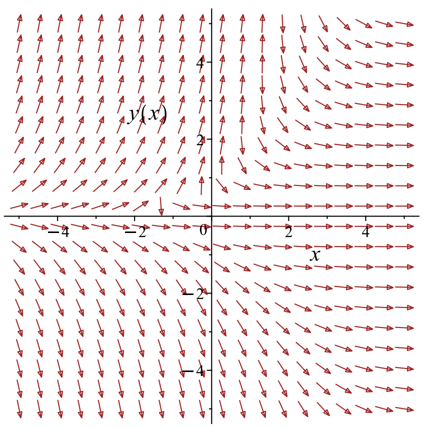
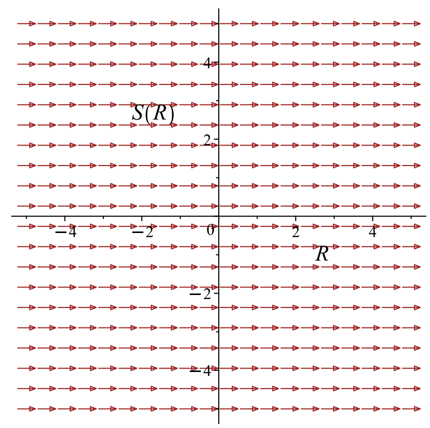
Which simplifies to

$$
\ln(y) - \mathrm{e}^{-x} y = c_1
$$

Which gives

$$
y = \mathrm{e}^{-\operatorname{LambertW}\left(-\mathrm{e}^{-x+c_1}\right)+c_1}
$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in x, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dx} = -\frac{y^2}{-y+e^x}$ 	$R = x$ $S = \ln(y) - e^{-x}y$	$\frac{dS}{dR} = 0$ 

Summary

The solution(s) found are the following

$$y = e^{-\operatorname{LambertW}\left(-e^{-x+c_1}\right)+c_1} \tag{1}$$

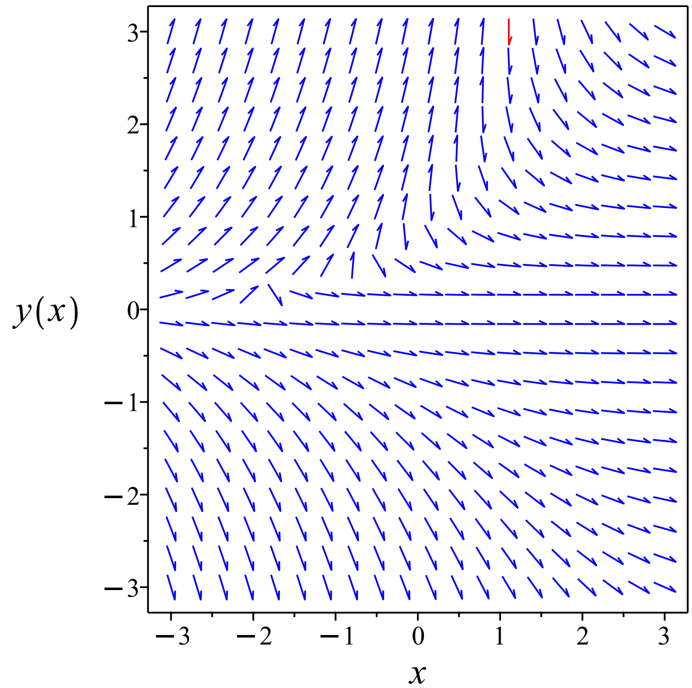


Figure 78: Slope field plot

Verification of solutions

$$y = e^{-\mathrm{LambertW}\left(-e^{-x+c_1}\right)+c_1}$$

Verified OK.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
trying homogeneous types:
trying Chini
differential order: 1; looking for linear symmetries
differential order: 1; found: 1 linear symmetries. Trying reduction of order
1st order, trying the canonical coordinates of the invariance group
<- 1st order, canonical coordinates successful`
```

✓ Solution by Maple

Time used: 0.078 (sec). Leaf size: 16

```
dsolve((y(x)^2)+( exp(x)-y(x))*diff(y(x),x)=0,y(x), singsol=all)
```

$$y(x) = -\mathrm{e}^{x} \operatorname{LambertW}\left(-\mathrm{e}^{-x} c_1\right)$$

✓ Solution by Mathematica

Time used: 6.706 (sec). Leaf size: 306

```
DSolve[(y[x]^2)+( Exp[x]-y[x])*y'[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$\text{Solve}\left[\frac{1}{9}2^{2/3}\left(\frac{\left(\frac{e^x-\frac{3e^{2x}}{e^x-y(x)}}{\sqrt[3]{e^{3x}}}+2\right)\left(\frac{e^x(y(x)+2e^x)}{\sqrt[3]{e^{3x}}(e^x-y(x))}+1\right)\left(\left(\frac{e^x-\frac{3e^{2x}}{e^x-y(x)}}{\sqrt[3]{e^{3x}}}-1\right)\log\left(2^{2/3}\left(\frac{e^x-\frac{3e^{2x}}{e^x-y(x)}}{\sqrt[3]{e^{3x}}}+2\right)\right)+\left(\right.\right.}{\frac{(y(x)+2e^x)^3}{(e^x-y(x))^3}-\frac{3e^x(y(x)+2e^x)}{\sqrt[3]{e^{3x}}(e^x-y(x))}-2}\right.\right.$$

1.51 problem 52

Internal problem ID [3196]
Internal file name [`OUTPUT/2688_Sunday_June_05_2022_08_38_48_AM_12918249/index.tex`]

Book: Differential equations for engineers by Wei-Chau XIE, Cambridge Press 2010
Section: Chapter 2. First-Order and Simple Higher-Order Differential Equations. Page 78
Problem number: 52.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"first_order_ode_lie_symmetry_calculated"**

Maple gives the following as the ode type

```
[[_homogeneous, `class G`], _rational, [_Abel, `2nd type`, `
   class B`]]
```

$$\boxed{y^2x^2 - 2y + (yx^3 - x)\, y' = 0}$$

1.51.1 Solving as first order ode lie symmetry calculated ode

Writing the ode as

$$\begin{aligned} y' &= -\frac{y(x^2y - 2)}{x\,(x^2y - 1)} \\ y' &= \omega(x, y) \end{aligned}$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x + \omega(\eta_y - \xi_x) - \omega^2\xi_y - \omega_x\xi - \omega_y\eta = 0 \tag{A}$$

The type of this ode is not in the lookup table. To determine ξ, η then (A) is solved using ansatz. Making bivariate polynomials of degree 1 to use as anstaz gives

$$\xi = xa_2 + ya_3 + a_1 \tag{1E}$$

$$\eta = xb_2 + yb_3 + b_1 \tag{2E}$$

Where the unknown coefficients are

$$\{a_1, a_2, a_3, b_1, b_2, b_3\}$$

Substituting equations (1E,2E) and ω into (A) gives

$$b_2 - \frac{y(x^2y-2)(b_3-a_2)}{x(x^2y-1)} - \frac{y^2(x^2y-2)^2 a_3}{x^2(x^2y-1)^2} - \left(-\frac{2y^2}{x^2y-1} + \frac{y(x^2y-2)}{x^2(x^2y-1)} + \frac{2y^2(x^2y-2)}{(x^2y-1)^2}\right)(xa_2+ya_3+a_1) - \left(-\frac{x^2y-2}{x(x^2y-1)} - \frac{yx}{x^2y-1} + \frac{y(x^2y-2)x}{(x^2y-1)^2}\right)(xb_2+yb_3+b_1) = 0 \tag{5E}$$

Putting the above in normal form gives

$$\frac{2x^6y^2b_2 - 2x^4y^4a_3 + x^5y^2b_1 - x^4y^3a_1 - 4x^4yb_2 + 2x^3y^2a_2 + x^3y^2b_3 + 9x^2y^3a_3 - 2x^3yb_1 + 5x^2y^2a_1 + 3b_2x}{(x^2y-1)^2 x^2} = 0$$

Setting the numerator to zero gives

$$2x^6y^2b_2 - 2x^4y^4a_3 + x^5y^2b_1 - x^4y^3a_1 - 4x^4yb_2 + 2x^3y^2a_2 + x^3y^2b_3 + 9x^2y^3a_3 - 2x^3yb_1 + 5x^2y^2a_1 + 3b_2x^2 - 6y^2a_3 + 2xb_1 - 2ya_1 = 0 \tag{6E}$$

Looking at the above PDE shows the following are all the terms with $\{x, y\}$ in them.

$$\{x, y\}$$

The following substitution is now made to be able to collect on all terms with $\{x, y\}$ in them

$$\{x = v_1, y = v_2\}$$

The above PDE (6E) now becomes

$$-2a_3v_1^4v_2^4 + 2b_2v_1^6v_2^2 - a_1v_1^4v_2^3 + b_1v_1^5v_2^2 + 2a_2v_1^3v_2^2 + 9a_3v_1^2v_2^3 - 4b_2v_1^4v_2 + b_3v_1^3v_2^2 + 5a_1v_1^2v_2^2 - 2b_1v_1^3v_2 - 6a_3v_2^2 + 3b_2v_1^2 - 2a_1v_2 + 2b_1v_1 = 0 \tag{7E}$$

Collecting the above on the terms v_i introduced, and these are

$$\{v_1, v_2\}$$

Equation (7E) now becomes

$$\begin{aligned} &2b_2v_1^6v_2^2 + b_1v_1^5v_2^2 - 2a_3v_1^4v_2^4 - a_1v_1^4v_2^3 - 4b_2v_1^4v_2 + (2a_2 + b_3)\, v_1^3v_2^2 \\ &- 2b_1v_1^3v_2 + 9a_3v_1^2v_2^3 + 5a_1v_1^2v_2^2 + 3b_2v_1^2 + 2b_1v_1 - 6a_3v_2^2 - 2a_1v_2 = 0 \end{aligned} \tag{8E}$$

Setting each coefficients in (8E) to zero gives the following equations to solve

$$\begin{aligned} b_1 &= 0 \\ -2a_1 &= 0 \\ -a_1 &= 0 \\ 5a_1 &= 0 \\ -6a_3 &= 0 \\ -2a_3 &= 0 \\ 9a_3 &= 0 \\ -2b_1 &= 0 \\ 2b_1 &= 0 \\ -4b_2 &= 0 \\ 2b_2 &= 0 \\ 3b_2 &= 0 \\ 2a_2 + b_3 &= 0 \end{aligned}$$

Solving the above equations for the unknowns gives

$$\begin{aligned} a_1 &= 0 \\ a_2 &= a_2 \\ a_3 &= 0 \\ b_1 &= 0 \\ b_2 &= 0 \\ b_3 &= -2a_2 \end{aligned}$$

Substituting the above solution in the anstaz (1E,2E) (using 1 as arbitrary value for any unknown in the RHS) gives

$$\begin{aligned} \xi &= x \\ \eta &= -2y \end{aligned}$$

Shifting is now applied to make $\xi = 0$ in order to simplify the rest of the computation

$$
\begin{aligned}
\eta &= \eta - \omega(x,y)\,\xi \\
&= -2y - \left(-\frac{y(x^2y-2)}{x\,(x^2y-1)}\right)(x) \\
&= -\frac{y^2x^2}{x^2y-1} \\
\xi &= 0
\end{aligned}
$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x,y) \to (R,S)$ where (R,S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$
\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}
$$

The above comes from the requirements that $\left(\xi\frac{\partial}{\partial x} + \eta\frac{\partial}{\partial y}\right)S(x,y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$
R = x
$$

S is found from

$$
\begin{aligned}
S &= \int \frac{1}{\eta} dy \\
&= \int \frac{1}{-\frac{y^2x^2}{x^2y-1}} dy
\end{aligned}
$$

Which results in

$$
S = -\frac{1}{x^2y} - \ln(y)
$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$
\frac{dS}{dR} = \frac{S_x + \omega(x,y)S_y}{R_x + \omega(x,y)R_y} \tag{2}
$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x,y)$ is the right hand side of the original ode given by

$$
\omega(x,y) = -\frac{y(x^2y-2)}{x\,(x^2y-1)}
$$

Evaluating all the partial derivatives gives

$$\begin{aligned}
R_x &= 1\\
R_y &= 0\\
S_x &= \frac{2}{x^3 y}\\
S_y &= \frac{-x^2 y + 1}{y^2 x^2}
\end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = \frac{1}{x} \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = \frac{1}{R}$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$S(R) = \ln(R) + c_1 \tag{4}$$

To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$\frac{-\ln(y)\, x^2 y - 1}{x^2 y} = \ln(x) + c_1$$

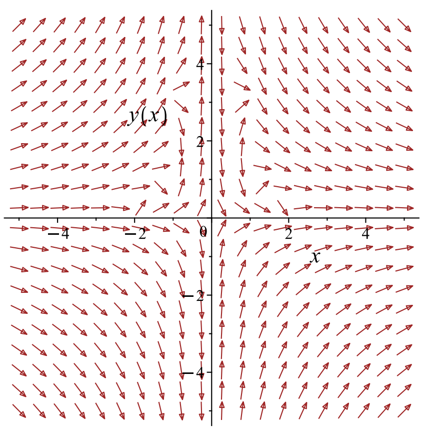
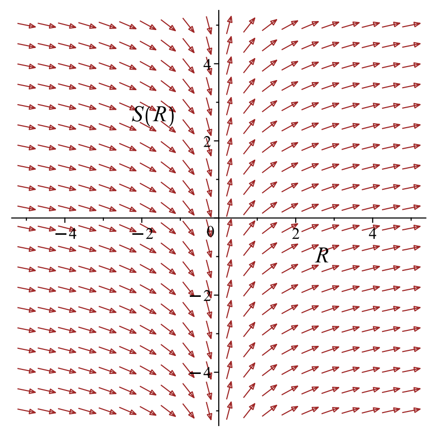
Which simplifies to

$$\frac{-\ln(y)\, x^2 y - 1}{x^2 y} = \ln(x) + c_1$$

Which gives

$$y = -\frac{1}{x^2 \operatorname{LambertW}\left(-\frac{e^{c_1}}{x}\right)}$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in x, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dx} = -\frac{y(x^2y-2)}{x(x^2y-1)}$ 	$R = x$ $S = \frac{-\ln(y)\, x^2 y - 1}{x^2 y}$	$\frac{dS}{dR} = \frac{1}{R}$ 

Summary
The solution(s) found are the following

$$y = -\frac{1}{x^2 \operatorname{LambertW}\left(-\frac{e^{c_1}}{x}\right)} \tag{1}$$

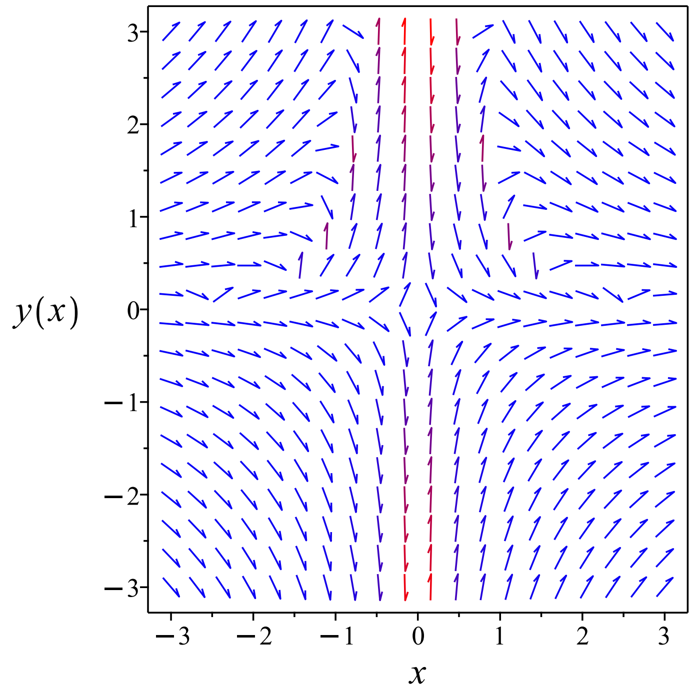


Figure 79: Slope field plot

Verification of solutions

$$y = -\frac{1}{x^2 \operatorname{LambertW}\left(-\frac{e^{c_1}}{x}\right)}$$

Verified OK.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
trying homogeneous types:
trying homogeneous G
<- homogeneous successful`
```

✓ Solution by Maple

Time used: 0.031 (sec). Leaf size: 18

```
dsolve((x^2*y(x)^2-2*y(x))+( x^3*y(x)-x)*diff(y(x),x)=0,y(x), singsol=all)
```

$$y(x) = -\frac{1}{\text{LambertW}\left(-\frac{c_1}{x}\right) x^2}$$

✓ Solution by Mathematica

Time used: 6.74 (sec). Leaf size: 35

```
DSolve[(x^2*y[x]^2-2*y[x])+( x^3*y[x]-x)*y'[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to -\frac{1}{x^2 W\left(\frac{e^{-1+\frac{9c_1}{2^{2/3}}}}{x}\right)}$$
$$y(x) \to 0$$

1.52 problem 53

Internal problem ID [3197]
Internal file name [`OUTPUT/2689_Sunday_June_05_2022_08_38_48_AM_50370268/index.tex`]

Book: Differential equations for engineers by Wei-Chau XIE, Cambridge Press 2010
Section: Chapter 2. First-Order and Simple Higher-Order Differential Equations. Page 78
Problem number: 53.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"first_order_ode_lie_symmetry_calculated"**

Maple gives the following as the ode type

```
[[_homogeneous, `class G`], _rational]
```

$$\boxed{2yx^3 + y^3 - (x^4 + 2y^2x)\, y' = 0}$$

1.52.1 Solving as first order ode lie symmetry calculated ode

Writing the ode as

$$\begin{aligned} y' &= \frac{y(2x^3 + y^2)}{x\,(x^3 + 2y^2)} \\ y' &= \omega(x, y) \end{aligned}$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x + \omega(\eta_y - \xi_x) - \omega^2 \xi_y - \omega_x \xi - \omega_y \eta = 0 \tag{A}$$

The type of this ode is not in the lookup table. To determine ξ, η then (A) is solved using ansatz. Making bivariate polynomials of degree 1 to use as anstaz gives

$$\xi = xa_2 + ya_3 + a_1 \tag{1E}$$

$$\eta = xb_2 + yb_3 + b_1 \tag{2E}$$

Where the unknown coefficients are

$$\{a_1, a_2, a_3, b_1, b_2, b_3\}$$

Substituting equations (1E,2E) and ω into (A) gives

$$
\begin{aligned}
& b_2+\frac{y(2x^3+y^2)\,(b_3-a_2)}{x\,(x^3+2y^2)}-\frac{y^2(2x^3+y^2)^2\,a_3}{x^2\,(x^3+2y^2)^2} \\
& -\left(\frac{6yx}{x^3+2y^2}-\frac{y(2x^3+y^2)}{x^2\,(x^3+2y^2)}-\frac{3y(2x^3+y^2)\,x}{(x^3+2y^2)^2}\right)(xa_2+ya_3+a_1) \\
& -\left(\frac{2x^3+y^2}{x\,(x^3+2y^2)}+\frac{2y^2}{x\,(x^3+2y^2)}-\frac{4y^2(2x^3+y^2)}{x\,(x^3+2y^2)^2}\right)(xb_2+yb_3+b_1)=0
\end{aligned} \tag{5E}
$$

Putting the above in normal form gives

$$
-\frac{x^8b_2+2x^6y^2a_3+2x^7b_1-2x^6ya_1-5x^5y^2b_2+9x^4y^3a_2-6x^4y^3b_3+8x^3y^4a_3-x^4y^2b_1+4x^3y^3a_1-2x^2y}{(x^3+2y^2)^2\,x^2}
$$

$=0$

Setting the numerator to zero gives

$$
\begin{aligned}
& -x^8b_2-2x^6y^2a_3-2x^7b_1+2x^6ya_1+5x^5y^2b_2-9x^4y^3a_2+6x^4y^3b_3 \\
& -8x^3y^4a_3+x^4y^2b_1-4x^3y^3a_1+2x^2y^4b_2+y^6a_3-2x\,y^4b_1+2y^5a_1=0
\end{aligned} \tag{6E}
$$

Looking at the above PDE shows the following are all the terms with $\{x,y\}$ in them.

$$
\{x,y\}
$$

The following substitution is now made to be able to collect on all terms with $\{x,y\}$ in them

$$
\{x=v_1,y=v_2\}
$$

The above PDE (6E) now becomes

$$
\begin{aligned}
& -2a_3v_1^6v_2^2-b_2v_1^8+2a_1v_1^6v_2-9a_2v_1^4v_2^3-8a_3v_1^3v_2^4-2b_1v_1^7+5b_2v_1^5v_2^2 \\
& +6b_3v_1^4v_2^3-4a_1v_1^3v_2^3+a_3v_2^6+b_1v_1^4v_2^2+2b_2v_1^2v_2^4+2a_1v_2^5-2b_1v_1v_2^4=0
\end{aligned} \tag{7E}
$$

Collecting the above on the terms v_i introduced, and these are

$$
\{v_1,v_2\}
$$

Equation (7E) now becomes

$$
\begin{aligned}
&-b_2 v_1^8 - 2b_1 v_1^7 - 2a_3 v_1^6 v_2^2 + 2a_1 v_1^6 v_2 + 5b_2 v_1^5 v_2^2 + (-9a_2 + 6b_3) v_1^4 v_2^3 \\
&+ b_1 v_1^4 v_2^2 - 8a_3 v_1^3 v_2^4 - 4a_1 v_1^3 v_2^3 + 2b_2 v_1^2 v_2^4 - 2b_1 v_1 v_2^4 + a_3 v_2^6 + 2a_1 v_2^5 = 0
\end{aligned} \tag{8E}
$$

Setting each coefficients in (8E) to zero gives the following equations to solve

$$
\begin{aligned}
a_3 &= 0 \\
b_1 &= 0 \\
-4a_1 &= 0 \\
2a_1 &= 0 \\
-8a_3 &= 0 \\
-2a_3 &= 0 \\
-2b_1 &= 0 \\
-b_2 &= 0 \\
2b_2 &= 0 \\
5b_2 &= 0 \\
-9a_2 + 6b_3 &= 0
\end{aligned}
$$

Solving the above equations for the unknowns gives

$$
\begin{aligned}
a_1 &= 0 \\
a_2 &= a_2 \\
a_3 &= 0 \\
b_1 &= 0 \\
b_2 &= 0 \\
b_3 &= \frac{3a_2}{2}
\end{aligned}
$$

Substituting the above solution in the anstaz (1E,2E) (using 1 as arbitrary value for any unknown in the RHS) gives

$$
\begin{aligned}
\xi &= x \\
\eta &= \frac{3y}{2}
\end{aligned}
$$

Shifting is now applied to make $\xi = 0$ in order to simplify the rest of the computation

$$\begin{aligned}\eta &= \eta - \omega(x,y)\,\xi \\ &= \frac{3y}{2} - \left(\frac{y(2x^3+y^2)}{x\,(x^3+2y^2)}\right)(x) \\ &= \frac{-x^3y+4y^3}{2x^3+4y^2} \\ \xi &= 0\end{aligned}$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x,y) \to (R,S)$ where (R,S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y}\right) S(x,y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$R = x$$

S is found from

$$\begin{aligned}S &= \int \frac{1}{\eta} dy \\ &= \int \frac{1}{\frac{-x^3y+4y^3}{2x^3+4y^2}} dy\end{aligned}$$

Which results in

$$S = \frac{3\ln(-x^3+4y^2)}{2} - 2\ln(y)$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x,y)S_y}{R_x + \omega(x,y)R_y} \tag{2}$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x,y)$ is the right hand side of the original ode given by

$$\omega(x,y) = \frac{y(2x^3+y^2)}{x\,(x^3+2y^2)}$$

Evaluating all the partial derivatives gives

$$
\begin{aligned}
R_x &= 1 \\
R_y &= 0 \\
S_x &= \frac{9x^2}{2x^3 - 8y^2} \\
S_y &= -\frac{12y}{x^3 - 4y^2} - \frac{2}{y}
\end{aligned}
$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$
\frac{dS}{dR} = \frac{1}{2x} \tag{2A}
$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$
\frac{dS}{dR} = \frac{1}{2R}
$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$
S(R) = \frac{\ln(R)}{2} + c_1 \tag{4}
$$

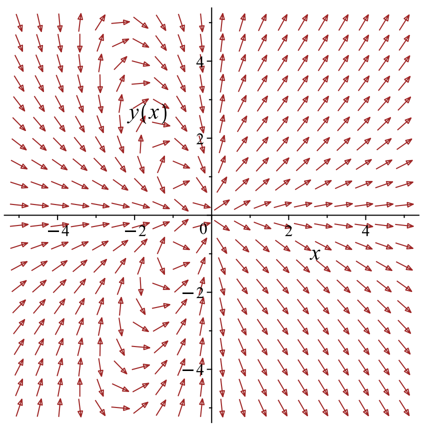
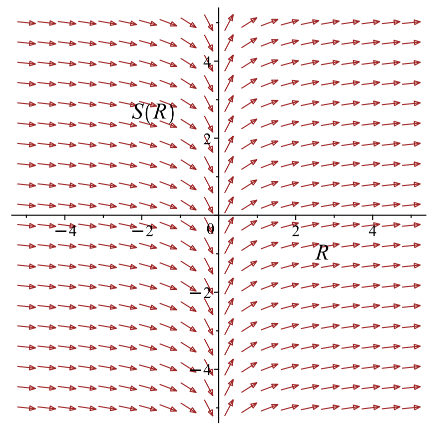
To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$
\frac{3\ln(-x^3 + 4y^2)}{2} - 2\ln(y) = \frac{\ln(x)}{2} + c_1
$$

Which simplifies to

$$
\frac{3\ln(-x^3 + 4y^2)}{2} - 2\ln(y) = \frac{\ln(x)}{2} + c_1
$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in x, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dx} = \frac{y(2x^3+y^2)}{x(x^3+2y^2)}$ 	$R = x$ $S = \frac{3\ln(-x^3+4y^2)}{2} - 2$	$\frac{dS}{dR} = \frac{1}{2R}$ 

Summary

The solution(s) found are the following

$$\frac{3\ln(-x^3+4y^2)}{2} - 2\ln(y) = \frac{\ln(x)}{2} + c_1 \tag{1}$$

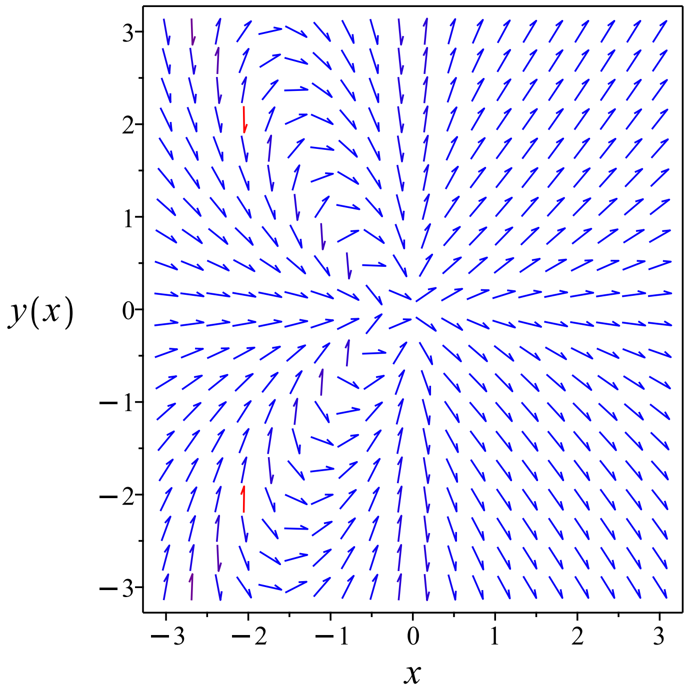


Figure 80: Slope field plot

Verification of solutions

$$\frac{3\ln\left(-x^3+4y^2\right)}{2}-2\ln\left(y\right)=\frac{\ln\left(x\right)}{2}+c_1$$

Verified OK.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
trying homogeneous types:
trying homogeneous G
<- homogeneous successful`
```

✓ Solution by Maple

Time used: 1.281 (sec). Leaf size: 149

```
dsolve((2*x^3*y(x)+y(x)^3)-( x^4+2*x*y(x)^2)*diff(y(x),x)=0,y(x), singsol=all)
```

$$y(x) = \frac{-x^{\frac{3}{2}} \operatorname{RootOf}\left(-16+x^{7} c_{1}\_Z^{12}-4 c_{1} x^{\frac{11}{2}}\_Z^{10}+6 c_{1} x^{4}\_Z^{8}+\left(128 x^{\frac{9}{2}}-4 x^{\frac{5}{2}} c_{1}\right)\_Z^{6}+\left(-192 x^{3}+c_{1} x\right)\_}{2 \operatorname{RootOf}\left(-16+x^{7} c_{1}\_Z^{12}-4 c_{1} x^{\frac{11}{2}}\_Z^{10}+6 c_{1} x^{4}\_Z^{8}+\left(128 x^{\frac{9}{2}}-4 x^{\frac{5}{2}} c_{1}\right)\_Z^{6}+\left(-192 x^{3}+c_{1} x\right)\_}$$

✓ Solution by Mathematica

Time used: 60.151 (sec). Leaf size: 2023

```
DSolve[(2*x^3*y[x]+y[x]^3)-( x^4+2*x*y[x]^2)*y'[x]==0,y[x],x,IncludeSingularSolutions -> Tru
```

$y(x) \rightarrow$

$$-\frac{\sqrt{48x^3+\frac{e^{4c_1}x^2}{\sqrt[3]{-3456e^{2c_1}x^7+144e^{4c_1}x^5-e^{6c_1}x^3+192\sqrt{3}\sqrt{-e^{4c_1}x^{12}\left(-108x^2+e^{2c_1}\right)}}}+\sqrt[3]{-3456e^{2c_1}x}}}{}$$

$y(x)$

$$\rightarrow \frac{\sqrt{48x^3+\frac{e^{4c_1}x^2}{\sqrt[3]{-3456e^{2c_1}x^7+144e^{4c_1}x^5-e^{6c_1}x^3+192\sqrt{3}\sqrt{-e^{4c_1}x^{12}\left(-108x^2+e^{2c_1}\right)}}}+\sqrt[3]{-3456e^{2c_1}x^7}}}{}$$

$y(x) \rightarrow$

$$-\frac{\sqrt{\frac{i\left(\sqrt{3}+i\right)e^{4c_1}x^2+96x^3\sqrt[3]{-3456e^{2c_1}x^7+144e^{4c_1}x^5-e^{6c_1}x^3+192\sqrt{3}\sqrt{-e^{4c_1}x^{12}\left(-108x^2+e^{2c_1}\right)}}-2e^{2c_1}x\left(\right.}{\sqrt[3]{-3456e}}}}{}$$

$y(x)$

$$\rightarrow \frac{\sqrt{\frac{i\left(\sqrt{3}+i\right)e^{4c_1}x^2+96x^3\sqrt[3]{-3456e^{2c_1}x^7+144e^{4c_1}x^5-e^{6c_1}x^3+192\sqrt{3}\sqrt{-e^{4c_1}x^{12}\left(-108x^2+e^{2c_1}\right)}}-2e^{2c_1}x\left(48i\right.}{\sqrt[3]{-3456e^{2c}}}}}{}$$

$y(x) \rightarrow$

$$-\frac{\sqrt{\frac{-i\left(\sqrt{3}-i\right)e^{4c_1}x^2+96x^3\sqrt[3]{-3456e^{2c_1}x^7+144e^{4c_1}x^5-e^{6c_1}x^3+192\sqrt{3}\sqrt{-e^{4c_1}x^{12}\left(-108x^2+e^{2c_1}\right)}}+i\left(\sqrt{3}+\right.}{\sqrt[3]{-3456e}}}}{}$$

$y(x)$

$$\rightarrow \frac{\sqrt{\frac{-i\left(\sqrt{3}-i\right)e^{4c_1}x^2+96x^3\sqrt[3]{-3456e^{2c_1}x^7+144e^{4c_1}x^5-e^{6c_1}x^3+192\sqrt{3}\sqrt{-e^{4c_1}x^{12}\left(-108x^2+e^{2c_1}\right)}}+i\left(\sqrt{3}+i\right)\left(\right.}{\sqrt[3]{-3456e^{2c}}}}}{}$$

1.53 problem 54

Internal problem ID [3198]
Internal file name [`OUTPUT/2690_Sunday_June_05_2022_08_38_51_AM_15825738/index.tex`]

Book: Differential equations for engineers by Wei-Chau XIE, Cambridge Press 2010
Section: Chapter 2. First-Order and Simple Higher-Order Differential Equations. Page 78
Problem number: 54.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program :

Maple gives the following as the ode type

`[_linear]`

$$\boxed{\cos(x) y - y' \sin(x) = -1}$$

1.53.1 Solving as linear ode

Entering Linear first order ODE solver. In canonical form a linear first order is

$$y' + p(x)y = q(x)$$

Where here

$$\begin{aligned} p(x) &= -\cot(x) \\ q(x) &= \csc(x) \end{aligned}$$

Hence the ode is

$$y' - y\cot(x) = \csc(x)$$

The integrating factor μ is

$$\begin{aligned} \mu &= e^{\int -\cot(x)dx} \\ &= \frac{1}{\sin(x)} \end{aligned}$$

Which simplifies to

$$\mu = \csc(x)$$

The ode becomes

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}x}(\mu y) &= (\mu)(\csc(x)) \\
\frac{\mathrm{d}}{\mathrm{d}x}(\csc(x)\, y) &= (\csc(x))(\csc(x)) \\
\mathrm{d}(\csc(x)\, y) &= \csc(x)^2 \,\mathrm{d}x
\end{aligned}$$

Integrating gives

$$\begin{aligned}
\csc(x)\, y &= \int \csc(x)^2 \,\mathrm{d}x \\
\csc(x)\, y &= -\cot(x) + c_1
\end{aligned}$$

Dividing both sides by the integrating factor $\mu = \csc(x)$ results in

$$y = -\cot(x)\sin(x) + c_1 \sin(x)$$

which simplifies to

$$y = c_1 \sin(x) - \cos(x)$$

Summary
The solution(s) found are the following

$$y = c_1 \sin(x) - \cos(x) \tag{1}$$

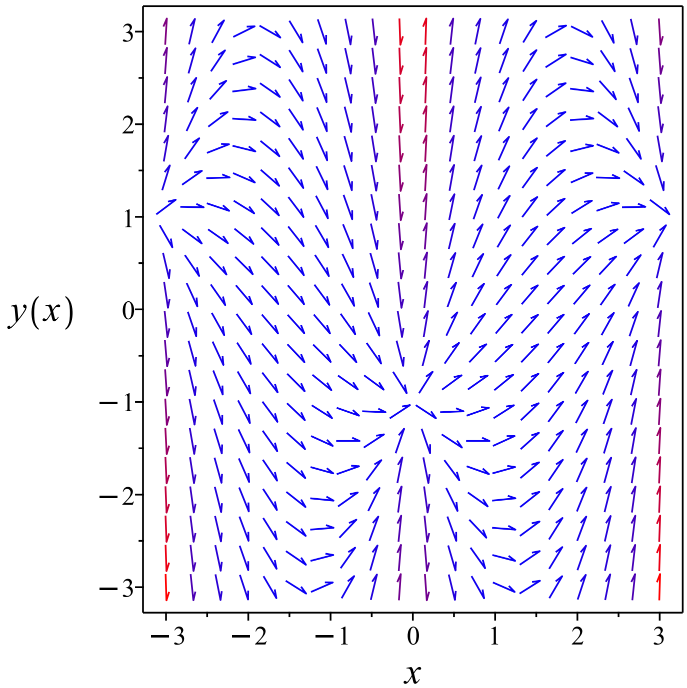


Figure 81: Slope field plot

Verification of solutions

$$y = c_1 \sin(x) - \cos(x)$$

Verified OK.

1.53.2 Maple step by step solution

Let's solve

$\cos(x) y - y' \sin(x) = -1$

- Highest derivative means the order of the ODE is 1

  y'

- Isolate the derivative

  $y' = \frac{\cos(x)y}{\sin(x)} + \frac{1}{\sin(x)}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE

  $y' - \frac{\cos(x)y}{\sin(x)} = \frac{1}{\sin(x)}$

- The ODE is linear; multiply by an integrating factor $\mu(x)$

$$\mu(x)\left(y' - \frac{\cos(x)y}{\sin(x)}\right) = \frac{\mu(x)}{\sin(x)}$$

- Assume the lhs of the ODE is the total derivative $\frac{d}{dx}(\mu(x)\,y)$

$$\mu(x)\left(y' - \frac{\cos(x)y}{\sin(x)}\right) = \mu'(x)\,y + \mu(x)\,y'$$

- Isolate $\mu'(x)$

$$\mu'(x) = -\frac{\mu(x)\cos(x)}{\sin(x)}$$

- Solve to find the integrating factor

$$\mu(x) = \frac{1}{\sin(x)}$$

- Integrate both sides with respect to x

$$\int \left(\frac{d}{dx}(\mu(x)\,y)\right) dx = \int \frac{\mu(x)}{\sin(x)} dx + c_1$$

- Evaluate the integral on the lhs

$$\mu(x)\,y = \int \frac{\mu(x)}{\sin(x)} dx + c_1$$

- Solve for y

$$y = \frac{\int \frac{\mu(x)}{\sin(x)} dx + c_1}{\mu(x)}$$

- Substitute $\mu(x) = \frac{1}{\sin(x)}$

$$y = \sin(x)\left(\int \frac{1}{\sin(x)^2} dx + c_1\right)$$

- Evaluate the integrals on the rhs

$$y = \sin(x)\,(-\cot(x) + c_1)$$

- Simplify

$$y = c_1 \sin(x) - \cos(x)$$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
<- 1st order linear successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 13

```
dsolve((1+y(x)*cos(x))-( sin(x))*diff(y(x),x)=0,y(x), singsol=all)
```

$$y(x) = c_1 \sin(x) - \cos(x)$$

✓ Solution by Mathematica

Time used: 0.042 (sec). Leaf size: 15

```
DSolve[(1+y[x]*Cos[x])-( Sin[x])*y'[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to -\cos(x) + c_1 \sin(x)$$

1.54 problem 55

Internal problem ID [3199]
Internal file name [`OUTPUT/2691_Sunday_June_05_2022_08_38_51_AM_59993157/index.tex`]

Book: Differential equations for engineers by Wei-Chau XIE, Cambridge Press 2010
Section: Chapter 2. First-Order and Simple Higher-Order Differential Equations. Page 78
Problem number: 55.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"quadrature"**

Maple gives the following as the ode type

`[_quadrature]`

$$\boxed{\left(\sin \left(y\right)^{2}+x \cot \left(y\right)\right) y^{\prime}=0}$$

1.54.1 Solving as quadrature ode

Integrating both sides gives

$$\begin{aligned} y &= \int 0 \, \mathrm{d}x \\ &= c_1 \end{aligned}$$

Summary
The solution(s) found are the following

$$y = c_1 \tag{1}$$

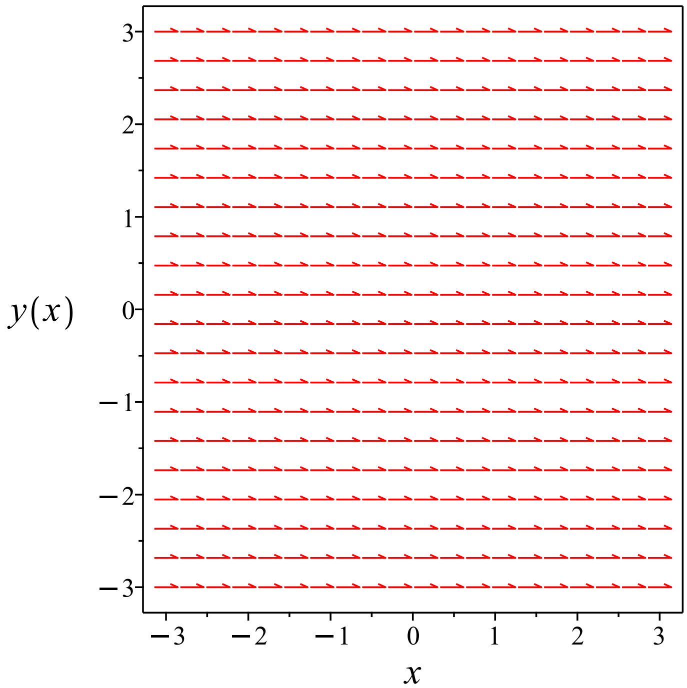


Figure 82: Slope field plot

Verification of solutions

$$y = c_1$$

Verified OK.

1.54.2 Maple step by step solution

Let's solve

$\left(\sin(y)^2 + x\cot(y)\right) y' = 0$

- Highest derivative means the order of the ODE is 1

  y'

- Integrate both sides with respect to x

  $\int \left(\sin(y)^2 + x\cot(y)\right) y' dx = \int 0 dx + c_1$

- Cannot compute integral

  $\int \left(\sin(y)^2 + x\cot(y)\right) y' dx = c_1$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
<- 1st order linear successful`
```

✓ Solution by Maple

Time used: 0.047 (sec). Leaf size: 1635

```
dsolve((sin(y(x))^2+x*cot(y(x)))*diff(y(x),x)=0,y(x), singsol=all)
```

$$y(x) = \arctan\left(-\frac{\sqrt{\frac{\left(108x^2+12\sqrt{3}\,\sqrt{4x^6+27x^4}\right)^{\frac{2}{3}}-12x^2}{\left(108x^2+12\sqrt{3}\,\sqrt{4x^6+27x^4}\right)^{\frac{1}{3}}}}}{6}, \frac{\sqrt{\frac{\left(108x^2+12\sqrt{3}\,\sqrt{4x^6+27x^4}\right)^{\frac{2}{3}}-12x^2}{\left(108x^2+12\sqrt{3}\,\sqrt{4x^6+27x^4}\right)^{\frac{1}{3}}}}\left(\left(108x^2+12\sqrt{3}\,\sqrt{4x^6+27x^4}\right.\right.}{36x\left(108x^2+12\sqrt{3}\,\sqrt{4x^6+27x^4}\right)^{\frac{1}{3}}}\right.$$

$$y(x) = \arctan\left(\frac{\sqrt{\frac{\left(108x^2+12\sqrt{3}\,\sqrt{4x^6+27x^4}\right)^{\frac{2}{3}}-12x^2}{\left(108x^2+12\sqrt{3}\,\sqrt{4x^6+27x^4}\right)^{\frac{1}{3}}}}}{6}, -\frac{\sqrt{\frac{\left(108x^2+12\sqrt{3}\,\sqrt{4x^6+27x^4}\right)^{\frac{2}{3}}-12x^2}{\left(108x^2+12\sqrt{3}\,\sqrt{4x^6+27x^4}\right)^{\frac{1}{3}}}}\left(\left(108x^2+12\sqrt{3}\,\sqrt{4x^6+27x^4}\right)^{\frac{2}{3}}-12x^2\right)}{36x\left(108x^2+12\sqrt{3}\,\sqrt{4x^6+27x^4}\right)^{\frac{1}{3}}}\right)$$

$$y(x) = \arctan\left(-\frac{\sqrt{\frac{i\left(-\left(108x^2+12\sqrt{3}\,\sqrt{4x^6+27x^4}\right)^{\frac{2}{3}}-12x^2\right)\sqrt{3}-\left(108x^2+12\sqrt{3}\,\sqrt{4x^6+27x^4}\right)^{\frac{2}{3}}+12x^2}{\left(108x^2+12\sqrt{3}\,\sqrt{4x^6+27x^4}\right)^{\frac{1}{3}}}}}{6}, \frac{\left(-i\left(\left(108x^2+12\sqrt{3}\,\sqrt{4x\right.\right.\right.}{}\right.$$

$$y(x) = \arctan\left(\frac{\sqrt{\frac{i\left(-\left(108x^2+12\sqrt{3}\,\sqrt{4x^6+27x^4}\right)^{\frac{2}{3}}-12x^2\right)\sqrt{3}-\left(108x^2+12\sqrt{3}\,\sqrt{4x^6+27x^4}\right)^{\frac{2}{3}}+12x^2}{\left(108x^2+12\sqrt{3}\,\sqrt{4x^6+27x^4}\right)^{\frac{1}{3}}}}}{6}, \frac{\sqrt{\frac{i\left(-\left(108x^2+12\sqrt{3}\,\sqrt{4x^6+27x^4}\right)^{\frac{2}{3}}\right.}{\left(108x^2\right.}}}{}\right.$$

$$y(x) = \arctan\left(-\frac{\sqrt{\frac{i\left(\left(108x^2+12\sqrt{3}\,\sqrt{4x^6+27x^4}\right)^{\frac{2}{3}}+12x^2\right)\sqrt{3}-\left(108x^2+12\sqrt{3}\,\sqrt{4x^6+27x^4}\right)^{\frac{2}{3}}+12x^2}{\left(108x^2+12\sqrt{3}\,\sqrt{4x^6+27x^4}\right)^{\frac{1}{3}}}}}{6}, \frac{\sqrt{\frac{i\left(\left(108x^2+12\sqrt{3}\,\sqrt{4x^6+27x^4}\right)^{\frac{2}{3}}\right.}{\left(108x^2\right.}}}{}\right.$$

1.55 problem 56

Internal problem ID [3200]
Internal file name [OUTPUT/2692_Sunday_June_05_2022_08_38_53_AM_37364604/index.tex]

Book: Differential equations for engineers by Wei-Chau XIE, Cambridge Press 2010
Section: Chapter 2. First-Order and Simple Higher-Order Differential Equations. Page 78
Problem number: 56.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program :

Maple gives the following as the ode type

`[_separable]`

$$\boxed{-(y - 2yx)\, y' = -1}$$

1.55.1 Solving as separable ode

In canonical form the ODE is

$$\begin{aligned} y' &= F(x, y) \\ &= f(x)g(y) \\ &= -\frac{1}{y\,(2x - 1)} \end{aligned}$$

Where $f(x) = -\frac{1}{2x-1}$ and $g(y) = \frac{1}{y}$. Integrating both sides gives

$$\begin{aligned} \frac{1}{\frac{1}{y}}\, dy &= -\frac{1}{2x - 1}\, dx \\ \int \frac{1}{\frac{1}{y}}\, dy &= \int -\frac{1}{2x - 1}\, dx \\ \frac{y^2}{2} &= -\frac{\ln(2x - 1)}{2} + c_1 \end{aligned}$$

Which results in

$$y = \sqrt{-\ln(2x-1) + 2c_1}$$
$$y = -\sqrt{-\ln(2x-1) + 2c_1}$$

Summary

The solution(s) found are the following

$$y = \sqrt{-\ln(2x-1) + 2c_1} \tag{1}$$
$$y = -\sqrt{-\ln(2x-1) + 2c_1} \tag{2}$$

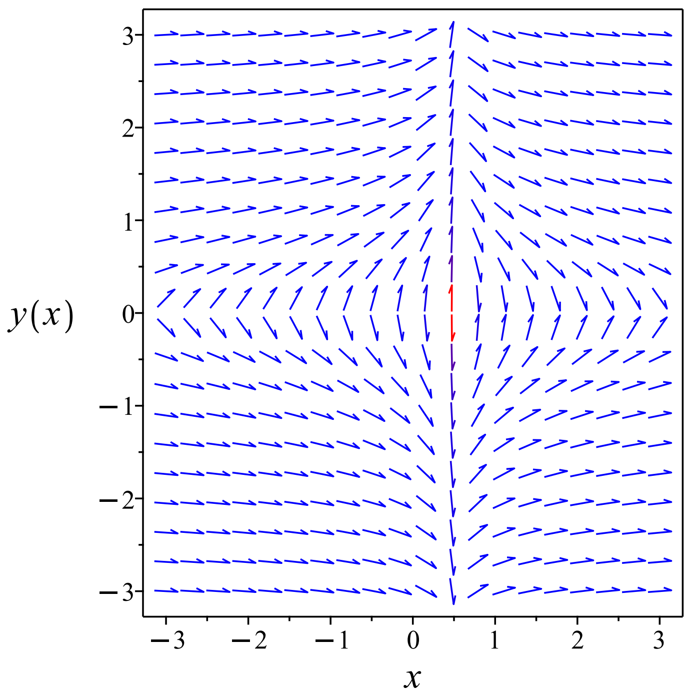


Figure 83: Slope field plot

Verification of solutions

$$y = \sqrt{-\ln(2x-1) + 2c_1}$$

Verified OK.

$$y = -\sqrt{-\ln(2x-1) + 2c_1}$$

Verified OK.

1.55.2 Maple step by step solution

Let's solve

$-(y-2yx)\,y'=-1$

- Highest derivative means the order of the ODE is 1

  y'

- Separate variables

  $yy'=-\frac{1}{2x-1}$

- Integrate both sides with respect to x

  $\int yy'dx=\int -\frac{1}{2x-1}dx+c_1$

- Evaluate integral

  $\frac{y^2}{2}=-\frac{\ln(2x-1)}{2}+c_1$

- Solve for y

  $\left\{y=\sqrt{-\ln(2x-1)+2c_1}, y=-\sqrt{-\ln(2x-1)+2c_1}\right\}$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
<- Bernoulli successful`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 33

```
dsolve(1-(y(x)-2*x*y(x))*diff(y(x),x)=0,y(x), singsol=all)
```

$$y(x)=\sqrt{-\ln(2x-1)+c_1}$$
$$y(x)=-\sqrt{-\ln(2x-1)+c_1}$$

✓ Solution by Mathematica

Time used: 0.013 (sec). Leaf size: 45

```
DSolve[1-(y[x]-2*x*y[x])*y'[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to -\sqrt{-\log(1-2x)+2c_1}$$
$$y(x) \to \sqrt{-\log(1-2x)+2c_1}$$

1.56 problem 57

Internal problem ID [3201]
Internal file name [`OUTPUT/2693_Sunday_June_05_2022_08_38_54_AM_62708748/index.tex`]

Book: Differential equations for engineers by Wei-Chau XIE, Cambridge Press 2010
Section: Chapter 2. First-Order and Simple Higher-Order Differential Equations. Page 78
Problem number: 57.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"exactWithIntegrationFactor"**

Maple gives the following as the ode type

```
[[_1st_order, `_with_symmetry_[F(x)*G(y),0]`]]
```

$$\boxed{-(1+2x\tan(y))\,y'=-1}$$

1.56.1 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x,y)+N(x,y)\frac{dy}{dx}=0 \tag{A}$$

We assume there exists a function $\phi(x,y)=c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y)=0$$

Hence

$$\frac{\partial\phi}{\partial x}+\frac{\partial\phi}{\partial y}\frac{dy}{dx}=0 \tag{B}$$

Comparing (A,B) shows that

$$\begin{aligned}\frac{\partial\phi}{\partial x}&=M\\ \frac{\partial\phi}{\partial y}&=N\end{aligned}$$

But since $\frac{\partial^2\phi}{\partial x\partial y} = \frac{\partial^2\phi}{\partial y\partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x,y)$ but at least we know now that we can do that since the condition $\frac{\partial^2\phi}{\partial x\partial y} = \frac{\partial^2\phi}{\partial y\partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x,y)\,\mathrm{d}x + N(x,y)\,\mathrm{d}y = 0 \tag{1A}$$

Therefore

$$\begin{aligned}(-1-2x\tan(y))\,\mathrm{d}y &= (-1)\,\mathrm{d}x\\ (1)\,\mathrm{d}x + (-1-2x\tan(y))\,\mathrm{d}y &= 0 \qquad (2A)\end{aligned}$$

Comparing (1A) and (2A) shows that

$$\begin{aligned}M(x,y) &= 1\\ N(x,y) &= -1-2x\tan(y)\end{aligned}$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Using result found above gives

$$\begin{aligned}\frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}(1)\\ &= 0\end{aligned}$$

And

$$\begin{aligned}\frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}(-1-2x\tan(y))\\ &= -2\tan(y)\end{aligned}$$

Since $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$, then the ODE is not exact. Since the ODE is not exact, we will try to find an integrating factor to make it exact. Let

$$\begin{aligned}
A &= \frac{1}{N}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right) \\
&= \frac{1}{-1 - 2x\tan(y)}((0) - (-2\tan(y))) \\
&= -\frac{2\tan(y)}{1 + 2x\tan(y)}
\end{aligned}$$

Since A depends on y, it can not be used to obtain an integrating factor. We will now try a second method to find an integrating factor. Let

$$\begin{aligned}
B &= \frac{1}{M}\left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) \\
&= 1((-2\tan(y)) - (0)) \\
&= -2\tan(y)
\end{aligned}$$

Since B does not depend on x, it can be used to obtain an integrating factor. Let the integrating factor be μ. Then

$$\begin{aligned}
\mu &= e^{\int B \, \mathrm{d}y} \\
&= e^{\int -2\tan(y) \, \mathrm{d}y}
\end{aligned}$$

The result of integrating gives

$$\begin{aligned}
\mu &= e^{2\ln(\cos(y))} \\
&= \cos(y)^2
\end{aligned}$$

M and N are now multiplied by this integrating factor, giving new M and new N which are called $\overline{M}$ and $\overline{N}$ so not to confuse them with the original M and N.

$$\begin{aligned}
\overline{M} &= \mu M \\
&= \cos(y)^2 (1) \\
&= \cos(y)^2
\end{aligned}$$

And

$$\begin{aligned}
\overline{N} &= \mu N \\
&= \cos(y)^2 (-1 - 2x\tan(y)) \\
&= (-1 - 2x\tan(y))\cos(y)^2
\end{aligned}$$

So now a modified ODE is obtained from the original ODE which will be exact and can be solved using the standard method. The modified ODE is

$$
\begin{aligned}
\overline{M} + \overline{N}\frac{\mathrm{d}y}{\mathrm{d}x} &= 0 \\
\left(\cos\left(y\right)^{2}\right) + \left(\left(-1-2x\tan\left(y\right)\right)\cos\left(y\right)^{2}\right)\frac{\mathrm{d}y}{\mathrm{d}x} &= 0
\end{aligned}
$$

The following equations are now set up to solve for the function $\phi(x,y)$

$$
\frac{\partial\phi}{\partial x} = \overline{M} \tag{1}
$$

$$
\frac{\partial\phi}{\partial y} = \overline{N} \tag{2}
$$

Integrating (1) w.r.t. x gives

$$
\begin{aligned}
\int \frac{\partial\phi}{\partial x}\,\mathrm{d}x &= \int \overline{M}\,\mathrm{d}x \\
\int \frac{\partial\phi}{\partial x}\,\mathrm{d}x &= \int \cos\left(y\right)^{2}\mathrm{d}x
\end{aligned}
$$

$$
\phi = x\cos\left(y\right)^{2} + f(y) \tag{3}
$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$
\frac{\partial\phi}{\partial y} = -2\sin\left(y\right)\cos\left(y\right)x + f'(y) \tag{4}
$$

But equation (2) says that $\frac{\partial\phi}{\partial y} = \left(-1-2x\tan\left(y\right)\right)\cos\left(y\right)^{2}$. Therefore equation (4) becomes

$$
\left(-1-2x\tan\left(y\right)\right)\cos\left(y\right)^{2} = -x\sin\left(2y\right) + f'(y) \tag{5}
$$

Solving equation (5) for $f'(y)$ gives

$$
\begin{aligned}
f'(y) &= -2\cos\left(y\right)^{2}x\tan\left(y\right) - \cos\left(y\right)^{2} + x\sin\left(2y\right) \\
&= -\cos\left(y\right)^{2}
\end{aligned}
$$

Integrating the above w.r.t y results in

$$\int f'(y)\, \mathrm{d}y = \int \left(-\cos(y)^2\right) \mathrm{d}y$$
$$f(y) = -\frac{\cos(y)\sin(y)}{2} - \frac{y}{2} + c_1$$

Where c_1 is constant of integration. Substituting result found above for $f(y)$ into equation (3) gives ϕ

$$\phi = x\cos(y)^2 - \frac{\cos(y)\sin(y)}{2} - \frac{y}{2} + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = x\cos(y)^2 - \frac{\cos(y)\sin(y)}{2} - \frac{y}{2}$$

Summary

The solution(s) found are the following

$$x\cos(y)^2 - \frac{\cos(y)\sin(y)}{2} - \frac{y}{2} = c_1 \tag{1}$$

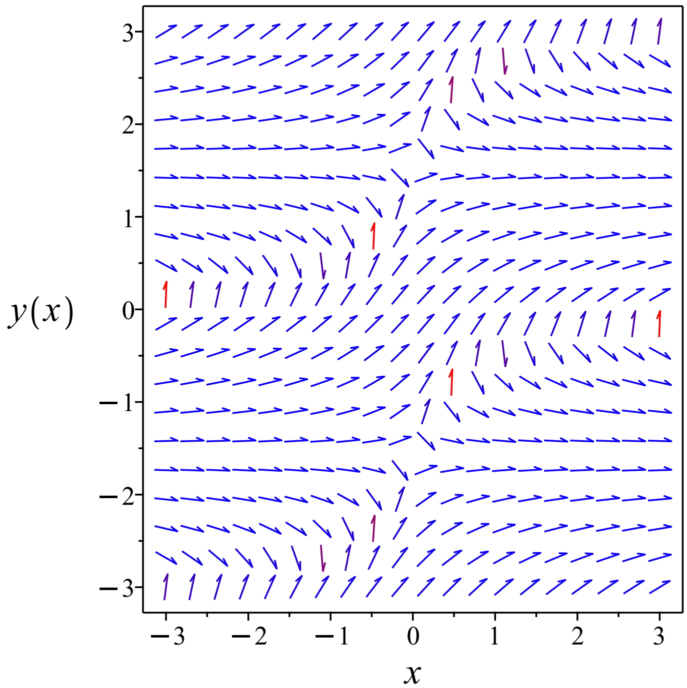


Figure 84: Slope field plot

Verification of solutions

$$x\cos(y)^2 - \frac{\cos(y)\sin(y)}{2} - \frac{y}{2} = c_1$$

Verified OK.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
<- 1st order linear successful
<- inverse linear successful`
```

✓ Solution by Maple

Time used: 0.032 (sec). Leaf size: 39

```
dsolve(1-(1+2*x*tan(y(x)))*diff(y(x),x)=0,y(x), singsol=all)
```

$$\frac{2x\cos(2y(x)) - 2y(x) - \sin(2y(x)) + c_1 + 2x}{2\cos(2y(x)) + 2} = 0$$

✓ Solution by Mathematica

Time used: 0.145 (sec). Leaf size: 36

```
DSolve[1-(1+2*x*Tan[y[x]])*y'[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$\text{Solve}\left[x = \left(\frac{y(x)}{2} + \frac{1}{4}\sin(2y(x))\right)\sec^2(y(x)) + c_1\sec^2(y(x)), y(x)\right]$$

1.57 problem 58

Internal problem ID [3202]
Internal file name [`OUTPUT/2694_Sunday_June_05_2022_08_38_54_AM_98552836/index.tex`]

Book: Differential equations for engineers by Wei-Chau XIE, Cambridge Press 2010
Section: Chapter 2. First-Order and Simple Higher-Order Differential Equations. Page 78
Problem number: 58.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"exactWithIntegrationFactor", "first_order_ode_lie_symmetry_calculated"**

Maple gives the following as the ode type

```
[[_homogeneous, `class G`], _rational]
```

$$\boxed{\left(y^3+\frac{x}{y}\right)y'=1}$$

1.57.1 Solving as first order ode lie symmetry calculated ode

Writing the ode as

$$\begin{aligned} y' &= \frac{y}{y^4+x} \\ y' &= \omega(x,y) \end{aligned}$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x+\omega(\eta_y-\xi_x)-\omega^2\xi_y-\omega_x\xi-\omega_y\eta=0 \tag{A}$$

The type of this ode is not in the lookup table. To determine ξ,η then (A) is solved using ansatz. Making bivariate polynomials of degree 1 to use as anstaz gives

$$\xi = xa_2+ya_3+a_1 \tag{1E}$$

$$\eta = xb_2+yb_3+b_1 \tag{2E}$$

Where the unknown coefficients are

$$\{a_1, a_2, a_3, b_1, b_2, b_3\}$$

Substituting equations (1E,2E) and ω into (A) gives

$$b_2 + \frac{y(b_3 - a_2)}{y^4 + x} - \frac{y^2 a_3}{(y^4 + x)^2} + \frac{y(x a_2 + y a_3 + a_1)}{(y^4 + x)^2} - \left(\frac{1}{y^4 + x} - \frac{4y^4}{(y^4 + x)^2}\right)(x b_2 + y b_3 + b_1) = 0 \tag{5E}$$

Putting the above in normal form gives

$$\frac{y^8 b_2 + 5x\, y^4 b_2 - y^5 a_2 + 4y^5 b_3 + 3y^4 b_1 - x b_1 + y a_1}{(y^4 + x)^2} = 0$$

Setting the numerator to zero gives

$$y^8 b_2 + 5x\, y^4 b_2 - y^5 a_2 + 4y^5 b_3 + 3y^4 b_1 - x b_1 + y a_1 = 0 \tag{6E}$$

Looking at the above PDE shows the following are all the terms with $\{x, y\}$ in them.

$$\{x, y\}$$

The following substitution is now made to be able to collect on all terms with $\{x, y\}$ in them

$$\{x = v_1, y = v_2\}$$

The above PDE (6E) now becomes

$$b_2 v_2^8 - a_2 v_2^5 + 5b_2 v_1 v_2^4 + 4b_3 v_2^5 + 3b_1 v_2^4 + a_1 v_2 - b_1 v_1 = 0 \tag{7E}$$

Collecting the above on the terms v_i introduced, and these are

$$\{v_1, v_2\}$$

Equation (7E) now becomes

$$5b_2 v_1 v_2^4 - b_1 v_1 + b_2 v_2^8 + (-a_2 + 4b_3)\, v_2^5 + 3b_1 v_2^4 + a_1 v_2 = 0 \tag{8E}$$

Setting each coefficients in (8E) to zero gives the following equations to solve

$$
\begin{aligned}
a_1 &= 0 \\
b_2 &= 0 \\
-b_1 &= 0 \\
3b_1 &= 0 \\
5b_2 &= 0 \\
-a_2 + 4b_3 &= 0
\end{aligned}
$$

Solving the above equations for the unknowns gives

$$
\begin{aligned}
a_1 &= 0 \\
a_2 &= 4b_3 \\
a_3 &= a_3 \\
b_1 &= 0 \\
b_2 &= 0 \\
b_3 &= b_3
\end{aligned}
$$

Substituting the above solution in the anstaz (1E,2E) (using 1 as arbitrary value for any unknown in the RHS) gives

$$
\begin{aligned}
\xi &= y \\
\eta &= 0
\end{aligned}
$$

Shifting is now applied to make $\xi = 0$ in order to simplify the rest of the computation

$$
\begin{aligned}
\eta &= \eta - \omega(x, y)\,\xi \\
&= 0 - \left(\frac{y}{y^4 + x}\right)(y) \\
&= -\frac{y^2}{y^4 + x} \\
\xi &= 0
\end{aligned}
$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x, y) \to (R, S)$ where (R, S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$
\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}
$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y}\right) S(x,y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$R = x$$

S is found from

$$\begin{aligned} S &= \int \frac{1}{\eta} dy \\ &= \int \frac{1}{-\frac{y^2}{y^4+x}} dy \end{aligned}$$

Which results in

$$S = -\frac{y^3}{3} + \frac{x}{y}$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x,y) S_y}{R_x + \omega(x,y) R_y} \tag{2}$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x,y)$ is the right hand side of the original ode given by

$$\omega(x,y) = \frac{y}{y^4 + x}$$

Evaluating all the partial derivatives gives

$$\begin{aligned} R_x &= 1 \\ R_y &= 0 \\ S_x &= \frac{1}{y} \\ S_y &= \frac{-y^4 - x}{y^2} \end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = 0 \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = 0$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$S(R) = c_1 \tag{4}$$

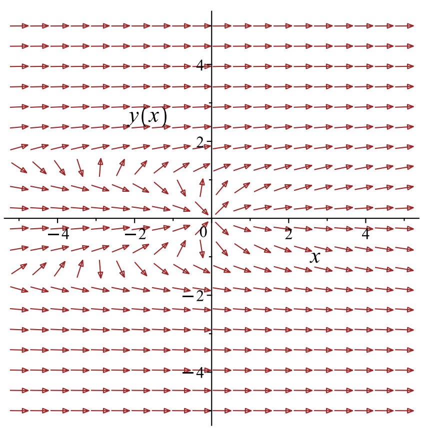
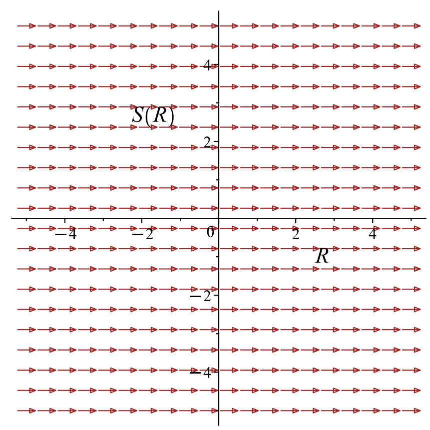
To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$-\frac{y^3}{3} + \frac{x}{y} = c_1$$

Which simplifies to

$$-\frac{y^3}{3} + \frac{x}{y} = c_1$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in x, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dx} = \frac{y}{y^4+x}$ 	$R = x$ $S = -\frac{y^3}{3} + \frac{x}{y}$	$\frac{dS}{dR} = 0$ 

Summary

The solution(s) found are the following

$$-\frac{y^3}{3} + \frac{x}{y} = c_1 \tag{1}$$

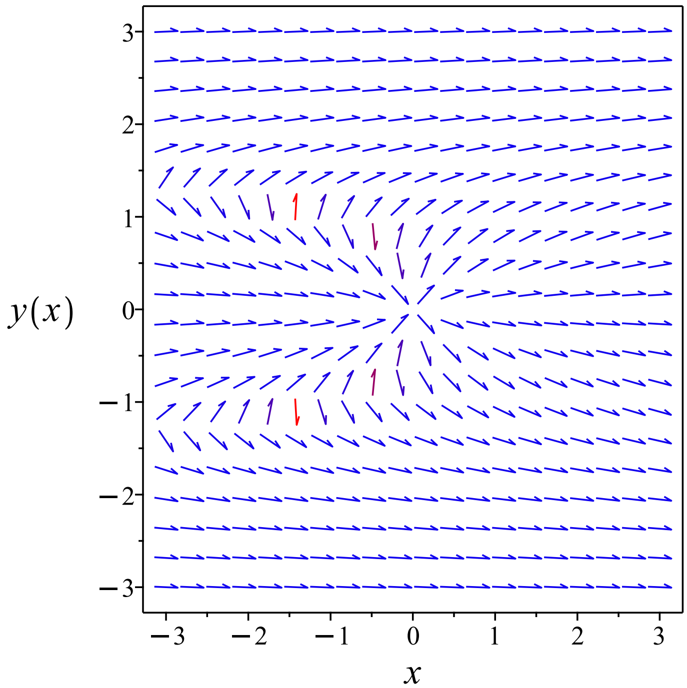


Figure 85: Slope field plot

Verification of solutions

$$-\frac{y^3}{3}+\frac{x}{y}=c_1$$

Verified OK.

1.57.2 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x,y)+N(x,y)\frac{dy}{dx}=0 \tag{A}$$

We assume there exists a function $\phi(x,y)=c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y)=0$$

Hence

$$\frac{\partial\phi}{\partial x}+\frac{\partial\phi}{\partial y}\frac{dy}{dx}=0 \tag{B}$$

Comparing (A,B) shows that

$$
\begin{aligned}
\frac{\partial \phi}{\partial x} &= M \\
\frac{\partial \phi}{\partial y} &= N
\end{aligned}
$$

But since $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ then for the above to be valid, we require that

$$
\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}
$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x, y)$ but at least we know now that we can do that since the condition $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$
M(x, y)\, \mathrm{d}x + N(x, y)\, \mathrm{d}y = 0 \tag{1A}
$$

Therefore

$$
\begin{aligned}
\left(y^3 + \frac{x}{y}\right) \mathrm{d}y &= \mathrm{d}x \\
-\mathrm{d}x + \left(y^3 + \frac{x}{y}\right) \mathrm{d}y &= 0 \qquad (2\mathrm{A})
\end{aligned}
$$

Comparing (1A) and (2A) shows that

$$
\begin{aligned}
M(x, y) &= -1 \\
N(x, y) &= y^3 + \frac{x}{y}
\end{aligned}
$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$
\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}
$$

Using result found above gives

$$
\begin{aligned}
\frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}(-1) \\
&= 0
\end{aligned}
$$

And

$$\begin{aligned}\frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}\left(y^3 + \frac{x}{y}\right)\\ &= \frac{1}{y}\end{aligned}$$

Since $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$, then the ODE is <u>not exact</u>. Since the ODE is not exact, we will try to find an integrating factor to make it exact. Let

$$\begin{aligned}A &= \frac{1}{N}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right)\\ &= \frac{y}{y^4 + x}\left((0) - \left(\frac{1}{y}\right)\right)\\ &= -\frac{1}{y^4 + x}\end{aligned}$$

Since A depends on y, it can not be used to obtain an integrating factor. We will now try a second method to find an integrating factor. Let

$$\begin{aligned}B &= \frac{1}{M}\left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right)\\ &= -1\left(\left(\frac{1}{y}\right) - (0)\right)\\ &= -\frac{1}{y}\end{aligned}$$

Since B does not depend on x, it can be used to obtain an integrating factor. Let the integrating factor be μ. Then

$$\begin{aligned}\mu &= e^{\int B \, \mathrm{d}y}\\ &= e^{\int -\frac{1}{y} \, \mathrm{d}y}\end{aligned}$$

The result of integrating gives

$$\begin{aligned}\mu &= e^{-\ln(y)}\\ &= \frac{1}{y}\end{aligned}$$

M and N are now multiplied by this integrating factor, giving new M and new N which are called $\overline{M}$ and $\overline{N}$ so not to confuse them with the original M and N.

$$\begin{aligned}\overline{M} &= \mu M\\ &= \frac{1}{y}(-1)\\ &= -\frac{1}{y}\end{aligned}$$

And

$$
\begin{aligned}
\overline{N} &= \mu N \\
&= \frac{1}{y}\left(y^3 + \frac{x}{y}\right) \\
&= \frac{y^4 + x}{y^2}
\end{aligned}
$$

So now a modified ODE is obtained from the original ODE which will be exact and can be solved using the standard method. The modified ODE is

$$
\begin{aligned}
\overline{M} + \overline{N}\frac{\mathrm{d}y}{\mathrm{d}x} &= 0 \\
\left(-\frac{1}{y}\right) + \left(\frac{y^4 + x}{y^2}\right)\frac{\mathrm{d}y}{\mathrm{d}x} &= 0
\end{aligned}
$$

The following equations are now set up to solve for the function $\phi(x, y)$

$$
\frac{\partial \phi}{\partial x} = \overline{M} \tag{1}
$$

$$
\frac{\partial \phi}{\partial y} = \overline{N} \tag{2}
$$

Integrating (1) w.r.t. x gives

$$
\begin{aligned}
\int \frac{\partial \phi}{\partial x}\,\mathrm{d}x &= \int \overline{M}\,\mathrm{d}x \\
\int \frac{\partial \phi}{\partial x}\,\mathrm{d}x &= \int -\frac{1}{y}\,\mathrm{d}x
\end{aligned}
$$

$$
\phi = -\frac{x}{y} + f(y) \tag{3}
$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$
\frac{\partial \phi}{\partial y} = \frac{x}{y^2} + f'(y) \tag{4}
$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = \frac{y^4+x}{y^2}$. Therefore equation (4) becomes

$$
\frac{y^4 + x}{y^2} = \frac{x}{y^2} + f'(y) \tag{5}
$$

Solving equation (5) for $f'(y)$ gives

$$f'(y) = y^2$$

Integrating the above w.r.t y gives

$$\int f'(y)\,\mathrm{d}y = \int (y^2)\,\mathrm{d}y$$
$$f(y) = \frac{y^3}{3} + c_1$$

Where c_1 is constant of integration. Substituting result found above for $f(y)$ into equation (3) gives ϕ

$$\phi = -\frac{x}{y} + \frac{y^3}{3} + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = -\frac{x}{y} + \frac{y^3}{3}$$

Summary
The solution(s) found are the following

$$\frac{y^3}{3} - \frac{x}{y} = c_1 \tag{1}$$

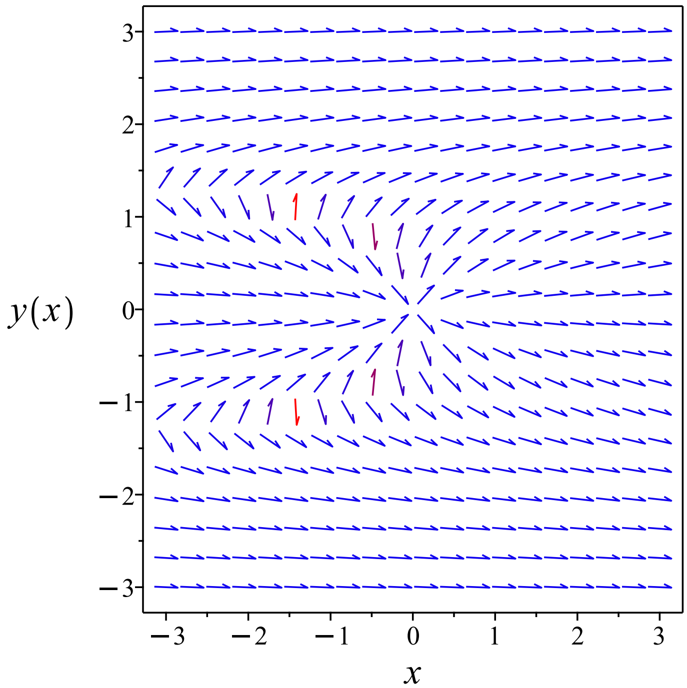


Figure 86: Slope field plot

Verification of solutions

$$\frac{y^3}{3} - \frac{x}{y} = c_1$$

Verified OK.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
<- 1st order linear successful
<- inverse linear successful`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 16

```
dsolve((y(x)^3+x/y(x))*diff(y(x),x)=1,y(x), singsol=all)
```

$$-c_1 y(x) + x - \frac{y(x)^4}{3} = 0$$

✓ Solution by Mathematica

Time used: 0.107 (sec). Leaf size: 997

```
DSolve[(y[x]^3+x/y[x])*y'[x]==1,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \frac{1}{2}\sqrt{\frac{\sqrt[3]{9c_1^2 - \sqrt{256x^3 + 81c_1^4}}}{\sqrt[3]{2}} - \frac{4\sqrt[3]{2}x}{\sqrt[3]{9c_1^2 - \sqrt{256x^3 + 81c_1^4}}}}$$
$$-\frac{1}{2}\sqrt{\frac{4\sqrt[3]{2}x}{\sqrt[3]{9c_1^2 - \sqrt{256x^3 + 81c_1^4}}} - \frac{\sqrt[3]{9c_1^2 - \sqrt{256x^3 + 81c_1^4}}}{\sqrt[3]{2}} - \frac{6c_1}{\sqrt{\frac{\sqrt[3]{9c_1^2 - \sqrt{256x^3 + 81c_1^4}}}{\sqrt[3]{2}} - \frac{}{\sqrt[3]{9c_1}}}}}$$

$$y(x) \to \frac{1}{2}\sqrt{\frac{\sqrt[3]{9c_1^2 - \sqrt{256x^3 + 81c_1^4}}}{\sqrt[3]{2}} - \frac{4\sqrt[3]{2}x}{\sqrt[3]{9c_1^2 - \sqrt{256x^3 + 81c_1^4}}}}$$
$$+\frac{1}{2}\sqrt{\frac{4\sqrt[3]{2}x}{\sqrt[3]{9c_1^2 - \sqrt{256x^3 + 81c_1^4}}} - \frac{\sqrt[3]{9c_1^2 - \sqrt{256x^3 + 81c_1^4}}}{\sqrt[3]{2}} - \frac{6c_1}{\sqrt{\frac{\sqrt[3]{9c_1^2 - \sqrt{256x^3 + 81c_1^4}}}{\sqrt[3]{2}} - \frac{}{\sqrt[3]{9c_1}}}}}$$

$$y(x) \to -\frac{1}{2}\sqrt{\frac{\sqrt[3]{9c_1^2 - \sqrt{256x^3 + 81c_1^4}}}{\sqrt[3]{2}} - \frac{4\sqrt[3]{2}x}{\sqrt[3]{9c_1^2 - \sqrt{256x^3 + 81c_1^4}}}}$$
$$-\frac{1}{2}\sqrt{\frac{4\sqrt[3]{2}x}{\sqrt[3]{9c_1^2 - \sqrt{256x^3 + 81c_1^4}}} - \frac{\sqrt[3]{9c_1^2 - \sqrt{256x^3 + 81c_1^4}}}{\sqrt[3]{2}} + \frac{6c_1}{\sqrt{\frac{\sqrt[3]{9c_1^2 - \sqrt{256x^3 + 81c_1^4}}}{\sqrt[3]{2}} - \frac{}{\sqrt[3]{9c_1}}}}}$$

$$y(x)$$
$$\to \frac{1}{2}\sqrt{\frac{4\sqrt[3]{2}x}{\sqrt[3]{9c_1^2 - \sqrt{256x^3 + 81c_1^4}}} - \frac{\sqrt[3]{9c_1^2 - \sqrt{256x^3 + 81c_1^4}}}{\sqrt[3]{2}} + \frac{6c_1}{\sqrt{\frac{\sqrt[3]{9c_1^2 - \sqrt{256x^3 + 81c_1^4}}}{\sqrt[3]{2}} - \frac{}{\sqrt[3]{9c_1^2}}}}}$$
$$-\frac{1}{2}\sqrt{\frac{\sqrt[3]{9c_1^2 - \sqrt{256x^3 + 81c_1^4}}}{\sqrt[3]{2}} - \frac{4\sqrt[3]{2}x}{\sqrt[3]{9c_1^2 - \sqrt{256x^3 + 81c_1^4}}}}$$

1.58 problem 59

Internal problem ID [3203]
Internal file name [`OUTPUT/2695_Sunday_June_05_2022_08_38_55_AM_97845022/index.tex`]

Book: Differential equations for engineers by Wei-Chau XIE, Cambridge Press 2010
Section: Chapter 2. First-Order and Simple Higher-Order Differential Equations. Page 78
Problem number: 59.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"exactWithIntegrationFactor", "first_order_ode_lie_symmetry_calculated"**

Maple gives the following as the ode type

```
[[_1st_order, _with_exponential_symmetries]]
```

$$\boxed{(x - y^2)\, y' = -1}$$

1.58.1 Solving as first order ode lie symmetry calculated ode

Writing the ode as

$$y' = \frac{1}{y^2 - x}$$
$$y' = \omega(x, y)$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x + \omega(\eta_y - \xi_x) - \omega^2 \xi_y - \omega_x \xi - \omega_y \eta = 0 \tag{A}$$

The type of this ode is not in the lookup table. To determine ξ, η then (A) is solved using ansatz. Making bivariate polynomials of degree 1 to use as anstaz gives

$$\xi = xa_2 + ya_3 + a_1 \tag{1E}$$
$$\eta = xb_2 + yb_3 + b_1 \tag{2E}$$

Where the unknown coefficients are

$$\{a_1, a_2, a_3, b_1, b_2, b_3\}$$

Substituting equations (1E,2E) and ω into (A) gives

$$b_2 + \frac{b_3 - a_2}{y^2 - x} - \frac{a_3}{(y^2 - x)^2} - \frac{xa_2 + ya_3 + a_1}{(y^2 - x)^2} + \frac{2y(xb_2 + yb_3 + b_1)}{(y^2 - x)^2} = 0 \tag{5E}$$

Putting the above in normal form gives

$$\frac{y^4 b_2 - 2x\, y^2 b_2 + x^2 b_2 + 2xyb_2 - y^2 a_2 + 3y^2 b_3 - xb_3 - ya_3 + 2yb_1 - a_1 - a_3}{(-y^2 + x)^2} = 0$$

Setting the numerator to zero gives

$$y^4 b_2 - 2x\, y^2 b_2 + x^2 b_2 + 2xyb_2 - y^2 a_2 + 3y^2 b_3 - xb_3 - ya_3 + 2yb_1 - a_1 - a_3 = 0 \tag{6E}$$

Looking at the above PDE shows the following are all the terms with $\{x, y\}$ in them.

$$\{x, y\}$$

The following substitution is now made to be able to collect on all terms with $\{x, y\}$ in them

$$\{x = v_1, y = v_2\}$$

The above PDE (6E) now becomes

$$b_2 v_2^4 - 2b_2 v_1 v_2^2 - a_2 v_2^2 + b_2 v_1^2 + 2b_2 v_1 v_2 + 3b_3 v_2^2 - a_3 v_2 + 2b_1 v_2 - b_3 v_1 - a_1 - a_3 = 0 \tag{7E}$$

Collecting the above on the terms v_i introduced, and these are

$$\{v_1, v_2\}$$

Equation (7E) now becomes

$$b_2 v_1^2 - 2b_2 v_1 v_2^2 + 2b_2 v_1 v_2 - b_3 v_1 + b_2 v_2^4 + (-a_2 + 3b_3)\, v_2^2 + (-a_3 + 2b_1)\, v_2 - a_1 - a_3 = 0 \tag{8E}$$

Setting each coefficients in (8E) to zero gives the following equations to solve

$$\begin{aligned}
b_2 &= 0\\
-2b_2 &= 0\\
2b_2 &= 0\\
-b_3 &= 0\\
-a_1 - a_3 &= 0\\
-a_2 + 3b_3 &= 0\\
-a_3 + 2b_1 &= 0
\end{aligned}$$

Solving the above equations for the unknowns gives

$$\begin{aligned}
a_1 &= -2b_1\\
a_2 &= 0\\
a_3 &= 2b_1\\
b_1 &= b_1\\
b_2 &= 0\\
b_3 &= 0
\end{aligned}$$

Substituting the above solution in the anstaz (1E,2E) (using 1 as arbitrary value for any unknown in the RHS) gives

$$\begin{aligned}
\xi &= 2y - 2\\
\eta &= 1
\end{aligned}$$

Shifting is now applied to make $\xi = 0$ in order to simplify the rest of the computation

$$\begin{aligned}
\eta &= \eta - \omega(x, y)\,\xi\\
&= 1 - \left(\frac{1}{y^2 - x}\right)(2y - 2)\\
&= \frac{-y^2 + x + 2y - 2}{-y^2 + x}\\
\xi &= 0
\end{aligned}$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x, y) \to (R, S)$ where (R, S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial x}+\eta \frac{\partial}{\partial y}\right) S(x, y)=1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the canonical coordinates, where $S(R)$. Since $\xi=0$ then in this special case

$$R=x$$

S is found from

$$\begin{aligned} S &= \int \frac{1}{\eta} d y \\ &= \int \frac{1}{\frac{-y^{2}+x+2 y-2}{-y^{2}+x}} d y \end{aligned}$$

Which results in

$$S=y+\ln \left(y^{2}-x-2 y+2\right)$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{d S}{d R}=\frac{S_{x}+\omega(x, y) S_{y}}{R_{x}+\omega(x, y) R_{y}} \tag{2}$$

Where in the above $R_{x}, R_{y}, S_{x}, S_{y}$ are all partial derivatives and $\omega(x, y)$ is the right hand side of the original ode given by

$$\omega(x, y)=\frac{1}{y^{2}-x}$$

Evaluating all the partial derivatives gives

$$\begin{aligned} R_{x} &= 1 \\ R_{y} &= 0 \\ S_{x} &= \frac{1}{-y^{2}+x+2 y-2} \\ S_{y} &= \frac{-y^{2}+x}{-y^{2}+x+2 y-2} \end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{d S}{d R}=0 \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{d S}{d R}=0$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordinates R, S. Integrating the above gives

$$S(R) = c_1 \tag{4}$$

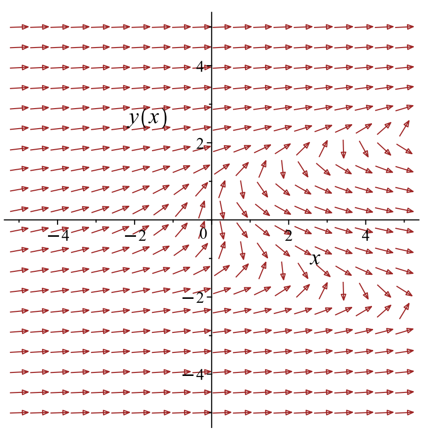
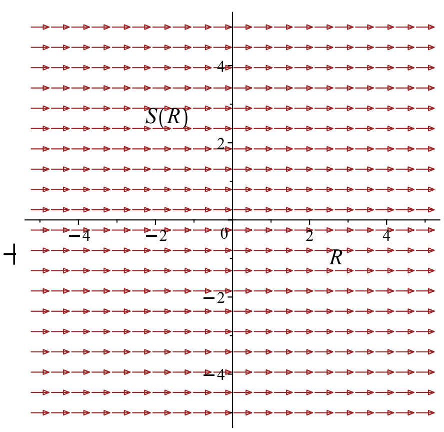
To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$y + \ln\left(y^2 - x - 2y + 2\right) = c_1$$

Which simplifies to

$$y + \ln\left(y^2 - x - 2y + 2\right) = c_1$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in x, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dx} = \frac{1}{y^2 - x}$ 	$R = x$ $S = y + \ln\left(y^2 - x - 2y +\right.$	$\frac{dS}{dR} = 0$ 

Summary
The solution(s) found are the following

$$y + \ln\left(y^2 - x - 2y + 2\right) = c_1 \tag{1}$$

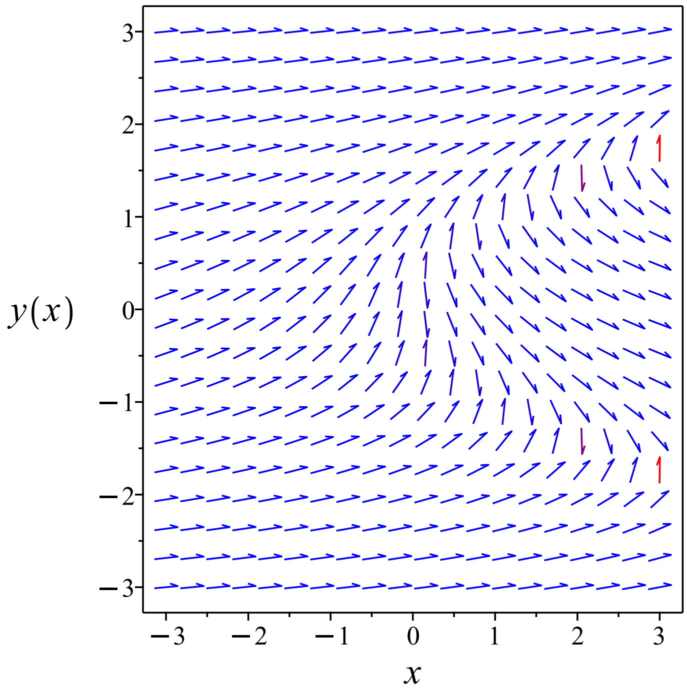


Figure 87: Slope field plot

Verification of solutions

$$y+\ln\left(y^2-x-2y+2\right)=c_1$$

Verified OK.

1.58.2 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x,y)+N(x,y)\frac{dy}{dx}=0 \tag{A}$$

We assume there exists a function $\phi(x,y)=c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y)=0$$

Hence

$$\frac{\partial\phi}{\partial x}+\frac{\partial\phi}{\partial y}\frac{dy}{dx}=0 \tag{B}$$

Comparing (A,B) shows that

$$\frac{\partial \phi}{\partial x} = M$$
$$\frac{\partial \phi}{\partial y} = N$$

But since $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x, y)$ but at least we know now that we can do that since the condition $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x, y)\,\mathrm{d}x + N(x, y)\,\mathrm{d}y = 0 \tag{1A}$$

Therefore

$$\begin{aligned}(-y^2 + x)\,\mathrm{d}y &= (-1)\,\mathrm{d}x\\ (1)\,\mathrm{d}x + (-y^2 + x)\,\mathrm{d}y &= 0 \qquad (2A)\end{aligned}$$

Comparing (1A) and (2A) shows that

$$\begin{aligned}M(x, y) &= 1\\ N(x, y) &= -y^2 + x\end{aligned}$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Using result found above gives

$$\begin{aligned}\frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}(1)\\ &= 0\end{aligned}$$

And

$$\begin{aligned}\frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}(-y^2 + x)\\ &= 1\end{aligned}$$

Since $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$, then the ODE is <u>not exact</u>. Since the ODE is not exact, we will try to find an integrating factor to make it exact. Let

$$
\begin{aligned}
A &= \frac{1}{N}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right) \\
&= \frac{1}{-y^2 + x}((0) - (1)) \\
&= -\frac{1}{-y^2 + x}
\end{aligned}
$$

Since A depends on y, it can not be used to obtain an integrating factor. We will now try a second method to find an integrating factor. Let

$$
\begin{aligned}
B &= \frac{1}{M}\left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) \\
&= 1((1) - (0)) \\
&= 1
\end{aligned}
$$

Since B does not depend on x, it can be used to obtain an integrating factor. Let the integrating factor be μ. Then

$$
\begin{aligned}
\mu &= e^{\int B \, \mathrm{d}y} \\
&= e^{\int 1 \, \mathrm{d}y}
\end{aligned}
$$

The result of integrating gives

$$
\begin{aligned}
\mu &= e^{y} \\
&= \mathrm{e}^{y}
\end{aligned}
$$

M and N are now multiplied by this integrating factor, giving new M and new N which are called $\overline{M}$ and $\overline{N}$ so not to confuse them with the original M and N.

$$
\begin{aligned}
\overline{M} &= \mu M \\
&= \mathrm{e}^{y}(1) \\
&= \mathrm{e}^{y}
\end{aligned}
$$

And

$$
\begin{aligned}
\overline{N} &= \mu N \\
&= \mathrm{e}^{y}\left(-y^2 + x\right) \\
&= \left(-y^2 + x\right)\mathrm{e}^{y}
\end{aligned}
$$

So now a modified ODE is obtained from the original ODE which will be exact and can be solved using the standard method. The modified ODE is

$$
\begin{aligned}
\overline{M} + \overline{N}\frac{\mathrm{d}y}{\mathrm{d}x} &= 0 \\
(\mathrm{e}^{y}) + \left(\left(-y^{2}+x\right)\mathrm{e}^{y}\right)\frac{\mathrm{d}y}{\mathrm{d}x} &= 0
\end{aligned}
$$

The following equations are now set up to solve for the function $\phi(x,y)$

$$
\frac{\partial \phi}{\partial x} = \overline{M} \tag{1}
$$

$$
\frac{\partial \phi}{\partial y} = \overline{N} \tag{2}
$$

Integrating (1) w.r.t. x gives

$$
\begin{aligned}
\int \frac{\partial \phi}{\partial x}\,\mathrm{d}x &= \int \overline{M}\,\mathrm{d}x \\
\int \frac{\partial \phi}{\partial x}\,\mathrm{d}x &= \int \mathrm{e}^{y}\,\mathrm{d}x
\end{aligned}
$$

$$
\phi = x\,\mathrm{e}^{y} + f(y) \tag{3}
$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$
\frac{\partial \phi}{\partial y} = x\,\mathrm{e}^{y} + f'(y) \tag{4}
$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = \left(-y^{2}+x\right)\mathrm{e}^{y}$. Therefore equation (4) becomes

$$
\left(-y^{2}+x\right)\mathrm{e}^{y} = x\,\mathrm{e}^{y} + f'(y) \tag{5}
$$

Solving equation (5) for $f'(y)$ gives

$$
f'(y) = -\mathrm{e}^{y} y^{2}
$$

Integrating the above w.r.t y gives

$$
\begin{aligned}
\int f'(y)\,\mathrm{d}y &= \int \left(-\mathrm{e}^{y} y^{2}\right)\mathrm{d}y \\
f(y) &= -\left(y^{2}-2y+2\right)\mathrm{e}^{y} + c_{1}
\end{aligned}
$$

Where c_1 is constant of integration. Substituting result found above for $f(y)$ into equation (3) gives ϕ

$$\phi = x\,e^{y} - \left(y^{2}-2y+2\right)e^{y} + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = x\,e^{y} - \left(y^{2}-2y+2\right)e^{y}$$

Summary

The solution(s) found are the following

$$x\,e^{y} - \left(y^{2}-2y+2\right)e^{y} = c_1 \tag{1}$$

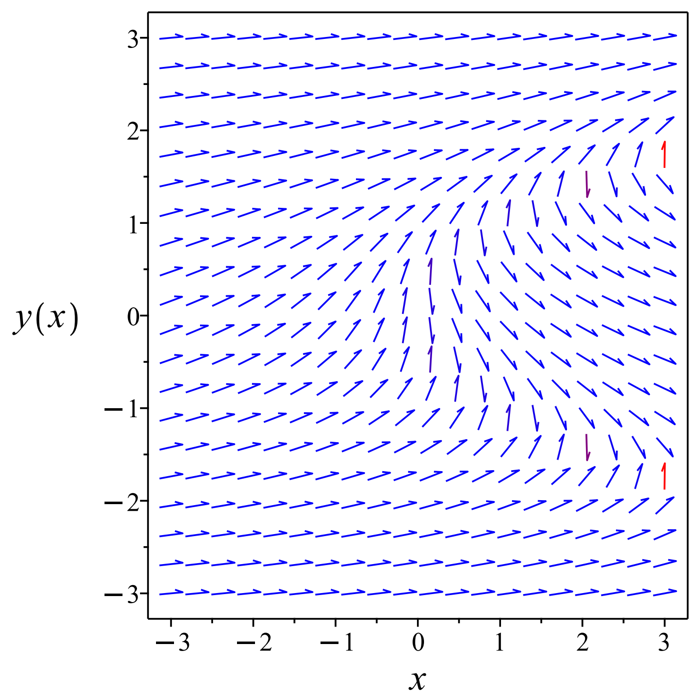


Figure 88: Slope field plot

Verification of solutions

$$x\,e^{y} - \left(y^{2}-2y+2\right)e^{y} = c_1$$

Verified OK.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
<- 1st order linear successful
<- inverse linear successful`
```

✓ Solution by Maple
Time used: 0.0 (sec). Leaf size: 24

```
dsolve(1+(x-y(x)^2)*diff(y(x),x)=0,y(x), singsol=all)
```

$$x - y(x)^2 + 2y(x) - 2 - e^{-y(x)}c_1 = 0$$

✓ Solution by Mathematica
Time used: 0.116 (sec). Leaf size: 24

```
DSolve[1+(x-y[x]^2)*y'[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$\text{Solve}\left[x = y(x)^2 - 2y(x) + c_1 e^{-y(x)} + 2, y(x)\right]$$

1.59 problem 60

Internal problem ID [3204]
Internal file name [`OUTPUT/2696_Sunday_June_05_2022_08_38_56_AM_86314611/index.tex`]

Book: Differential equations for engineers by Wei-Chau XIE, Cambridge Press 2010
Section: Chapter 2. First-Order and Simple Higher-Order Differential Equations. Page 78
Problem number: 60.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"exactWithIntegrationFactor", "first_order_ode_lie_symmetry_calculated"**

Maple gives the following as the ode type

```
[[_1st_order, _with_linear_symmetries], _rational]
```

$$\boxed{y^2 + (yx + y^2 - 1)\, y' = 0}$$

1.59.1 Solving as first order ode lie symmetry calculated ode

Writing the ode as

$$\begin{aligned} y' &= -\frac{y^2}{xy + y^2 - 1} \\ y' &= \omega(x, y) \end{aligned}$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x + \omega(\eta_y - \xi_x) - \omega^2 \xi_y - \omega_x \xi - \omega_y \eta = 0 \tag{A}$$

The type of this ode is not in the lookup table. To determine ξ, η then (A) is solved using ansatz. Making bivariate polynomials of degree 1 to use as anstaz gives

$$\xi = x a_2 + y a_3 + a_1 \tag{1E}$$

$$\eta = x b_2 + y b_3 + b_1 \tag{2E}$$

Where the unknown coefficients are

$$
\{a_1, a_2, a_3, b_1, b_2, b_3\}
$$

Substituting equations (1E,2E) and ω into (A) gives

$$
\begin{aligned}
b_2 - \frac{y^2(b_3 - a_2)}{xy + y^2 - 1} - \frac{y^4 a_3}{(xy + y^2 - 1)^2} - \frac{y^3(x a_2 + y a_3 + a_1)}{(xy + y^2 - 1)^2} \\
- \left(-\frac{2y}{xy + y^2 - 1} + \frac{y^2(x + 2y)}{(xy + y^2 - 1)^2}\right)(x b_2 + y b_3 + b_1) = 0
\end{aligned} \tag{5E}
$$

Putting the above in normal form gives

$$
\frac{2x^2y^2b_2 + 2x\,y^3b_2 + y^4a_2 - 2y^4a_3 + y^4b_2 - y^4b_3 + x\,y^2b_1 - y^3a_1 - 4xyb_2 - y^2a_2 - 2y^2b_2 - y^2b_3 - 2yb_1 + b}{(xy + y^2 - 1)^2}
$$
$$
= 0
$$

Setting the numerator to zero gives

$$
\begin{aligned}
2x^2y^2b_2 + 2x\,y^3b_2 + y^4a_2 - 2y^4a_3 + y^4b_2 - y^4b_3 + x\,y^2b_1 \\
- y^3a_1 - 4xyb_2 - y^2a_2 - 2y^2b_2 - y^2b_3 - 2yb_1 + b_2 = 0
\end{aligned} \tag{6E}
$$

Looking at the above PDE shows the following are all the terms with $\{x, y\}$ in them.

$$
\{x, y\}
$$

The following substitution is now made to be able to collect on all terms with $\{x, y\}$ in them

$$
\{x = v_1, y = v_2\}
$$

The above PDE (6E) now becomes

$$
\begin{aligned}
a_2v_2^4 - 2a_3v_2^4 + 2b_2v_1^2v_2^2 + 2b_2v_1v_2^3 + b_2v_2^4 - b_3v_2^4 - a_1v_2^3 \\
+ b_1v_1v_2^2 - a_2v_2^2 - 4b_2v_1v_2 - 2b_2v_2^2 - b_3v_2^2 - 2b_1v_2 + b_2 = 0
\end{aligned} \tag{7E}
$$

Collecting the above on the terms v_i introduced, and these are

$$
\{v_1, v_2\}
$$

Equation (7E) now becomes

$$\begin{aligned}&2b_2v_1^2v_2^2 + 2b_2v_1v_2^3 + b_1v_1v_2^2 - 4b_2v_1v_2 + (a_2 - 2a_3 + b_2 - b_3)\, v_2^4 \\ &- a_1v_2^3 + (-a_2 - 2b_2 - b_3)\, v_2^2 - 2b_1v_2 + b_2 = 0\end{aligned} \tag{8E}$$

Setting each coefficients in (8E) to zero gives the following equations to solve

$$\begin{aligned}b_1 &= 0\\ b_2 &= 0\\ -a_1 &= 0\\ -2b_1 &= 0\\ -4b_2 &= 0\\ 2b_2 &= 0\\ -a_2 - 2b_2 - b_3 &= 0\\ a_2 - 2a_3 + b_2 - b_3 &= 0\end{aligned}$$

Solving the above equations for the unknowns gives

$$\begin{aligned}a_1 &= 0\\ a_2 &= -b_3\\ a_3 &= -b_3\\ b_1 &= 0\\ b_2 &= 0\\ b_3 &= b_3\end{aligned}$$

Substituting the above solution in the anstaz (1E,2E) (using 1 as arbitrary value for any unknown in the RHS) gives

$$\begin{aligned}\xi &= -y - x\\ \eta &= y\end{aligned}$$

Shifting is now applied to make $\xi = 0$ in order to simplify the rest of the computation

$$\begin{aligned}\eta &= \eta - \omega(x, y)\, \xi\\ &= y - \left(-\frac{y^2}{xy + y^2 - 1}\right)(-y - x)\\ &= -\frac{y}{xy + y^2 - 1}\\ \xi &= 0\end{aligned}$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x, y) \to (R, S)$ where (R, S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y}\right) S(x, y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$R = x$$

S is found from

$$\begin{aligned} S &= \int \frac{1}{\eta} dy \\ &= \int \frac{1}{-\frac{y}{xy+y^2-1}} dy \end{aligned}$$

Which results in

$$S = -\frac{y^2}{2} - xy + \ln(y)$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x, y) S_y}{R_x + \omega(x, y) R_y} \tag{2}$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x, y)$ is the right hand side of the original ode given by

$$\omega(x, y) = -\frac{y^2}{xy + y^2 - 1}$$

Evaluating all the partial derivatives gives

$$\begin{aligned} R_x &= 1 \\ R_y &= 0 \\ S_x &= -y \\ S_y &= -y - x + \frac{1}{y} \end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = 0 \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = 0$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$S(R) = c_1 \tag{4}$$

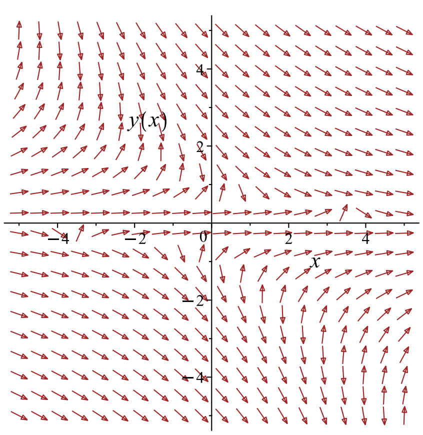
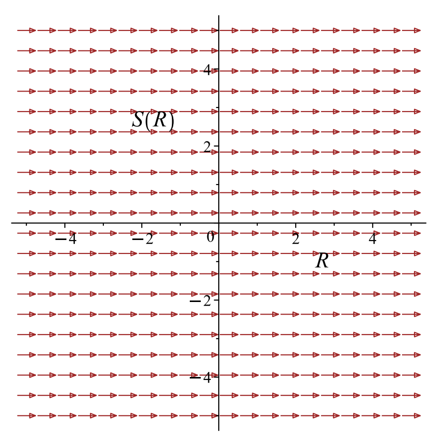
To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$-\frac{y^2}{2} - yx + \ln(y) = c_1$$

Which simplifies to

$$-\frac{y^2}{2} - yx + \ln(y) = c_1$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in x, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dx} = -\frac{y^2}{xy+y^2-1}$ 	$R = x$ $S = -\frac{y^2}{2} - xy + \ln(y)$	$\frac{dS}{dR} = 0$ 

Summary

The solution(s) found are the following

$$-\frac{y^2}{2} - yx + \ln(y) = c_1 \tag{1}$$

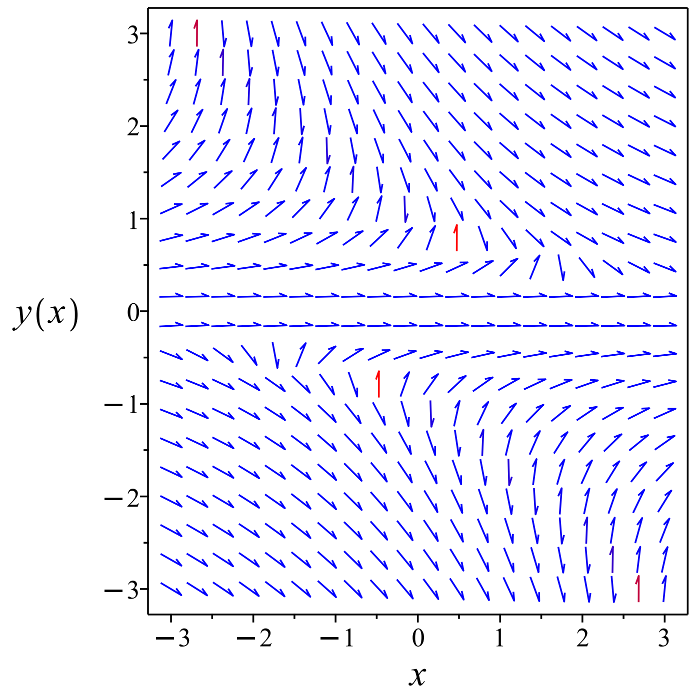


Figure 89: Slope field plot

Verification of solutions

$$-\frac{y^2}{2} - yx + \ln(y) = c_1$$

Verified OK.

1.59.2 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x,y) + N(x,y)\frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x,y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y) = 0$$

Hence

$$\frac{\partial \phi}{\partial x} + \frac{\partial \phi}{\partial y}\frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$\begin{aligned}\frac{\partial \phi}{\partial x} &= M \\ \frac{\partial \phi}{\partial y} &= N\end{aligned}$$

But since $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x, y)$ but at least we know now that we can do that since the condition $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x, y)\,\mathrm{d}x + N(x, y)\,\mathrm{d}y = 0 \tag{1A}$$

Therefore

$$\begin{aligned}(xy + y^2 - 1)\,\mathrm{d}y &= (-y^2)\,\mathrm{d}x \\ (y^2)\,\mathrm{d}x + (xy + y^2 - 1)\,\mathrm{d}y &= 0 \qquad \text{(2A)}\end{aligned}$$

Comparing (1A) and (2A) shows that

$$\begin{aligned}M(x, y) &= y^2 \\ N(x, y) &= xy + y^2 - 1\end{aligned}$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Using result found above gives

$$\begin{aligned}\frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}(y^2) \\ &= 2y\end{aligned}$$

And

$$
\begin{aligned}
\frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}\left(xy + y^2 - 1\right) \\
&= y
\end{aligned}
$$

Since $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$, then the ODE is <u>not exact</u>. Since the ODE is not exact, we will try to find an integrating factor to make it exact. Let

$$
\begin{aligned}
A &= \frac{1}{N}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right) \\
&= \frac{1}{xy + y^2 - 1}((2y) - (y)) \\
&= \frac{y}{xy + y^2 - 1}
\end{aligned}
$$

Since A depends on y, it can not be used to obtain an integrating factor. We will now try a second method to find an integrating factor. Let

$$
\begin{aligned}
B &= \frac{1}{M}\left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) \\
&= \frac{1}{y^2}((y) - (2y)) \\
&= -\frac{1}{y}
\end{aligned}
$$

Since B does not depend on x, it can be used to obtain an integrating factor. Let the integrating factor be μ. Then

$$
\begin{aligned}
\mu &= e^{\int B \, \mathrm{d}y} \\
&= e^{\int -\frac{1}{y} \, \mathrm{d}y}
\end{aligned}
$$

The result of integrating gives

$$
\begin{aligned}
\mu &= e^{-\ln(y)} \\
&= \frac{1}{y}
\end{aligned}
$$

M and N are now multiplied by this integrating factor, giving new M and new N which are called $\overline{M}$ and $\overline{N}$ so not to confuse them with the original M and N.

$$
\begin{aligned}
\overline{M} &= \mu M \\
&= \frac{1}{y}\left(y^2\right) \\
&= y
\end{aligned}
$$

And

$$
\begin{aligned}
\overline{N} &= \mu N \\
&= \frac{1}{y}(xy + y^2 - 1) \\
&= \frac{xy + y^2 - 1}{y}
\end{aligned}
$$

So now a modified ODE is obtained from the original ODE which will be exact and can be solved using the standard method. The modified ODE is

$$
\begin{aligned}
\overline{M} + \overline{N}\frac{\mathrm{d}y}{\mathrm{d}x} &= 0 \\
(y) + \left(\frac{xy + y^2 - 1}{y}\right)\frac{\mathrm{d}y}{\mathrm{d}x} &= 0
\end{aligned}
$$

The following equations are now set up to solve for the function $\phi(x, y)$

$$
\frac{\partial \phi}{\partial x} = \overline{M} \tag{1}
$$

$$
\frac{\partial \phi}{\partial y} = \overline{N} \tag{2}
$$

Integrating (1) w.r.t. x gives

$$
\begin{aligned}
\int \frac{\partial \phi}{\partial x}\,\mathrm{d}x &= \int \overline{M}\,\mathrm{d}x \\
\int \frac{\partial \phi}{\partial x}\,\mathrm{d}x &= \int y\,\mathrm{d}x \\
\phi &= xy + f(y) \qquad (3)
\end{aligned}
$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$
\frac{\partial \phi}{\partial y} = x + f'(y) \tag{4}
$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = \frac{xy+y^2-1}{y}$. Therefore equation (4) becomes

$$
\frac{xy + y^2 - 1}{y} = x + f'(y) \tag{5}
$$

Solving equation (5) for $f'(y)$ gives

$$f'(y) = \frac{y^2 - 1}{y}$$

Integrating the above w.r.t y gives

$$\int f'(y)\,\mathrm{d}y = \int \left(\frac{y^2-1}{y}\right)\mathrm{d}y$$
$$f(y) = \frac{y^2}{2} - \ln(y) + c_1$$

Where c_1 is constant of integration. Substituting result found above for $f(y)$ into equation (3) gives ϕ

$$\phi = xy + \frac{y^2}{2} - \ln(y) + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = xy + \frac{y^2}{2} - \ln(y)$$

Summary
The solution(s) found are the following

$$\frac{y^2}{2} + yx - \ln(y) = c_1 \tag{1}$$

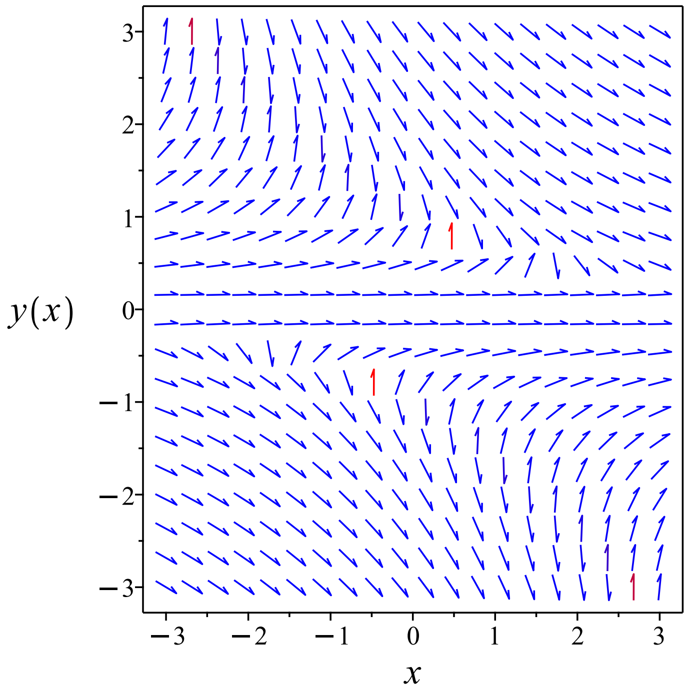


Figure 90: Slope field plot

Verification of solutions

$$\frac{y^2}{2} + yx - \ln(y) = c_1$$

Verified OK.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
<- 1st order linear successful
<- inverse linear successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 24

```
dsolve(y(x)^2+(x*y(x)+y(x)^2-1)*diff(y(x),x)=0,y(x), singsol=all)
```

$$y(x) = e^{\text{RootOf}\left(-e^{2\_Z}-2\,e^{\_Z}x+2c_1+2\_Z\right)}$$

✓ Solution by Mathematica

Time used: 0.135 (sec). Leaf size: 30

```
DSolve[y[x]^2+(x*y[x]+y[x]^2-1)*y'[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$\text{Solve}\left[x = \frac{\log(y(x)) - \frac{y(x)^2}{2}}{y(x)} + \frac{c_1}{y(x)}, y(x)\right]$$

1.60 problem 61

Internal problem ID [3205]
Internal file name [OUTPUT/2697_Sunday_June_05_2022_08_38_56_AM_58537924/index.tex]

Book: Differential equations for engineers by Wei-Chau XIE, Cambridge Press 2010
Section: Chapter 2. First-Order and Simple Higher-Order Differential Equations. Page 78
Problem number: 61.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"exactWithIntegrationFactor"**

Maple gives the following as the ode type

```
[[_1st_order, `_with_symmetry_[F(x)*G(y),0]`]]
```

$$\boxed{y - (e^y + 2yx - 2x)\, y' = 0}$$

1.60.1 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x, y) + N(x, y)\frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x, y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x, y) = 0$$

Hence

$$\frac{\partial \phi}{\partial x} + \frac{\partial \phi}{\partial y}\frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$\begin{aligned}\frac{\partial \phi}{\partial x} &= M \\ \frac{\partial \phi}{\partial y} &= N\end{aligned}$$

But since $\frac{\partial^2\phi}{\partial x\partial y} = \frac{\partial^2\phi}{\partial y\partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x,y)$ but at least we know now that we can do that since the condition $\frac{\partial^2\phi}{\partial x\partial y} = \frac{\partial^2\phi}{\partial y\partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x,y)\,\mathrm{d}x + N(x,y)\,\mathrm{d}y = 0 \tag{1A}$$

Therefore

$$\begin{aligned}(-\mathrm{e}^y - 2xy + 2x)\,\mathrm{d}y &= (-y)\,\mathrm{d}x \\ (y)\,\mathrm{d}x + (-\mathrm{e}^y - 2xy + 2x)\,\mathrm{d}y &= 0 \end{aligned} \tag{2A}$$

Comparing (1A) and (2A) shows that

$$\begin{aligned}M(x,y) &= y \\ N(x,y) &= -\mathrm{e}^y - 2xy + 2x\end{aligned}$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Using result found above gives

$$\begin{aligned}\frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}(y) \\ &= 1\end{aligned}$$

And

$$\begin{aligned}\frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}(-\mathrm{e}^y - 2xy + 2x) \\ &= -2y + 2\end{aligned}$$

Since $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$, then the ODE is <u>not exact</u>. Since the ODE is not exact, we will try to find an integrating factor to make it exact. Let

$$\begin{aligned}A &= \frac{1}{N}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right) \\ &= -\frac{1}{\mathrm{e}^y + (2y-2)\,x}((1) - (-2y+2)) \\ &= \frac{1-2y}{\mathrm{e}^y + (2y-2)\,x}\end{aligned}$$

Since A depends on y, it can not be used to obtain an integrating factor. We will now try a second method to find an integrating factor. Let

$$\begin{aligned}
B &= \frac{1}{M}\left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) \\
&= \frac{1}{y}((-2y+2) - (1)) \\
&= \frac{1-2y}{y}
\end{aligned}$$

Since B does not depend on x, it can be used to obtain an integrating factor. Let the integrating factor be μ. Then

$$\begin{aligned}
\mu &= e^{\int B \, \mathrm{d}y} \\
&= e^{\int \frac{1-2y}{y} \, \mathrm{d}y}
\end{aligned}$$

The result of integrating gives

$$\begin{aligned}
\mu &= e^{-2y+\ln(y)} \\
&= y \, \mathrm{e}^{-2y}
\end{aligned}$$

M and N are now multiplied by this integrating factor, giving new M and new N which are called $\overline{M}$ and $\overline{N}$ so not to confuse them with the original M and N.

$$\begin{aligned}
\overline{M} &= \mu M \\
&= y \, \mathrm{e}^{-2y}(y) \\
&= y^2 \mathrm{e}^{-2y}
\end{aligned}$$

And

$$\begin{aligned}
\overline{N} &= \mu N \\
&= y \, \mathrm{e}^{-2y}(-\mathrm{e}^{y} - 2xy + 2x) \\
&= -(\mathrm{e}^{y} + 2xy - 2x) \, y \, \mathrm{e}^{-2y}
\end{aligned}$$

So now a modified ODE is obtained from the original ODE which will be exact and can be solved using the standard method. The modified ODE is

$$\begin{aligned}
\overline{M} + \overline{N}\frac{\mathrm{d}y}{\mathrm{d}x} &= 0 \\
\left(y^2 \mathrm{e}^{-2y}\right) + \left(-(\mathrm{e}^{y} + 2xy - 2x) \, y \, \mathrm{e}^{-2y}\right) \frac{\mathrm{d}y}{\mathrm{d}x} &= 0
\end{aligned}$$

The following equations are now set up to solve for the function $\phi(x,y)$

$$\frac{\partial \phi}{\partial x} = \overline{M} \tag{1}$$
$$\frac{\partial \phi}{\partial y} = \overline{N} \tag{2}$$

Integrating (1) w.r.t. x gives

$$\begin{aligned}\int \frac{\partial \phi}{\partial x} \, \mathrm{d}x &= \int \overline{M} \, \mathrm{d}x \\ \int \frac{\partial \phi}{\partial x} \, \mathrm{d}x &= \int y^2 \mathrm{e}^{-2y} \, \mathrm{d}x \\ \phi &= y^2 \mathrm{e}^{-2y} x + f(y) \end{aligned} \tag{3}$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$\begin{aligned}\frac{\partial \phi}{\partial y} &= 2y \, \mathrm{e}^{-2y} x - 2y^2 \mathrm{e}^{-2y} x + f'(y) \quad (4) \\ &= -2x \, \mathrm{e}^{-2y} y(y-1) + f'(y)\end{aligned}$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = -(\mathrm{e}^y + 2xy - 2x) \, y \, \mathrm{e}^{-2y}$. Therefore equation (4) becomes

$$-(\mathrm{e}^y + 2xy - 2x) \, y \, \mathrm{e}^{-2y} = -2x \, \mathrm{e}^{-2y} y(y-1) + f'(y) \tag{5}$$

Solving equation (5) for $f'(y)$ gives

$$\begin{aligned}f'(y) &= -\mathrm{e}^y \mathrm{e}^{-2y} y \\ &= -\mathrm{e}^{-y} y\end{aligned}$$

Integrating the above w.r.t y results in

$$\begin{aligned}\int f'(y) \, \mathrm{d}y &= \int \left(-\mathrm{e}^{-y} y\right) \mathrm{d}y \\ f(y) &= (y+1) \, \mathrm{e}^{-y} + c_1\end{aligned}$$

Where c_1 is constant of integration. Substituting result found above for $f(y)$ into equation (3) gives ϕ

$$\phi = y^2 e^{-2y} x + (y+1)\,e^{-y} + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = y^2 e^{-2y} x + (y+1)\,e^{-y}$$

Summary

The solution(s) found are the following

$$y^2 e^{-2y} x + (y+1)\,e^{-y} = c_1 \tag{1}$$

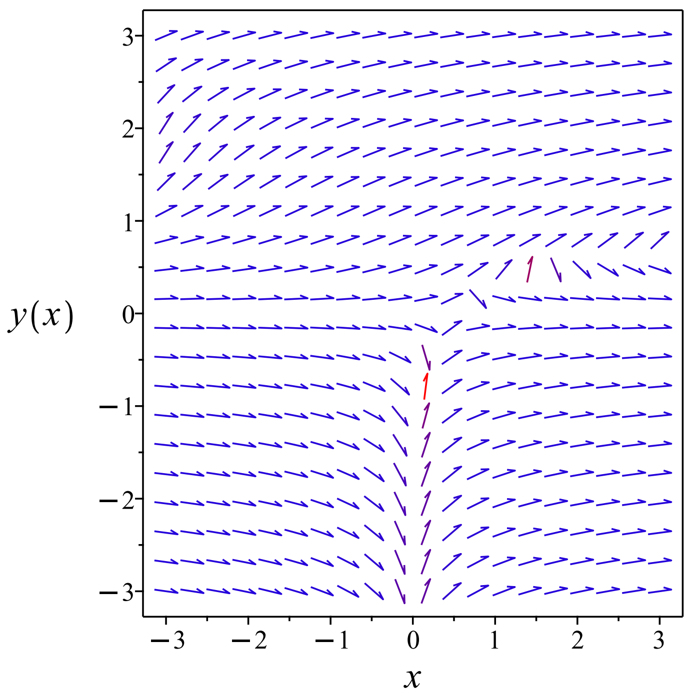


Figure 91: Slope field plot

Verification of solutions

$$y^2 e^{-2y} x + (y+1)\,e^{-y} = c_1$$

Verified OK.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
<- 1st order linear successful
<- inverse linear successful`
```

✓ Solution by Maple

Time used: 0.079 (sec). Leaf size: 62

```
dsolve(y(x)=(exp(y(x))+2*x*y(x)-2*x)*diff(y(x),x),y(x), singsol=all)
```

$$y(x) = \mathrm{RootOf}\left(x\_Z^2 - c_1 + \_Z + \mathrm{e}^{\mathrm{RootOf}\left(-x\,\mathrm{e}^{2\_Z}\_Z^2 + \_Z\,\mathrm{e}^{\_Z} + c_1 - \mathrm{e}^{\_Z}\right)}\right) \mathrm{e}^{-\mathrm{RootOf}\left(-x\,\mathrm{e}^{2\_Z}\_Z^2 + \_Z\,\mathrm{e}^{\_Z} + c_1 - \mathrm{e}^{\_Z}\right)}$$

✓ Solution by Mathematica

Time used: 0.294 (sec). Leaf size: 34

```
DSolve[y[x]==(Exp[y[x]]+2*x*y[x]-2*x)*y'[x],y[x],x,IncludeSingularSolutions -> True]
```

$$\text{Solve}\left[x = \frac{e^{y(x)}(-y(x) - 1)}{y(x)^2} + \frac{c_1 e^{2y(x)}}{y(x)^2}, y(x)\right]$$

1.61 problem 62

Internal problem ID [3206]
Internal file name [`OUTPUT/2698_Sunday_June_05_2022_08_38_57_AM_38294634/index.tex`]

Book: Differential equations for engineers by Wei-Chau XIE, Cambridge Press 2010
Section: Chapter 2. First-Order and Simple Higher-Order Differential Equations. Page 78
Problem number: 62.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program :

Maple gives the following as the ode type

`[_linear]`

$$\boxed{(2x+3)\,y'-y=\sqrt{2x+3}}$$

1.61.1 Solving as linear ode

Entering Linear first order ODE solver. In canonical form a linear first order is

$$y'+p(x)y=q(x)$$

Where here

$$\begin{aligned} p(x) &= -\frac{1}{2x+3} \\ q(x) &= \frac{1}{\sqrt{2x+3}} \end{aligned}$$

Hence the ode is

$$y'-\frac{y}{2x+3}=\frac{1}{\sqrt{2x+3}}$$

The integrating factor μ is

$$\begin{aligned} \mu &= \mathrm{e}^{\int -\frac{1}{2x+3}dx} \\ &= \frac{1}{\sqrt{2x+3}} \end{aligned}$$

The ode becomes

$$\frac{d}{dx}(\mu y) = (\mu)\left(\frac{1}{\sqrt{2x+3}}\right)$$

$$\frac{d}{dx}\left(\frac{y}{\sqrt{2x+3}}\right) = \left(\frac{1}{\sqrt{2x+3}}\right)\left(\frac{1}{\sqrt{2x+3}}\right)$$

$$d\left(\frac{y}{\sqrt{2x+3}}\right) = \frac{1}{2x+3}\,dx$$

Integrating gives

$$\frac{y}{\sqrt{2x+3}} = \int \frac{1}{2x+3}\,dx$$

$$\frac{y}{\sqrt{2x+3}} = \frac{\ln(2x+3)}{2} + c_1$$

Dividing both sides by the integrating factor $\mu = \frac{1}{\sqrt{2x+3}}$ results in

$$y = \frac{\sqrt{2x+3}\,\ln(2x+3)}{2} + c_1\sqrt{2x+3}$$

which simplifies to

$$y = \left(\frac{\ln(2x+3)}{2} + c_1\right)\sqrt{2x+3}$$

Summary

The solution(s) found are the following

$$y = \left(\frac{\ln(2x+3)}{2} + c_1\right)\sqrt{2x+3} \tag{1}$$

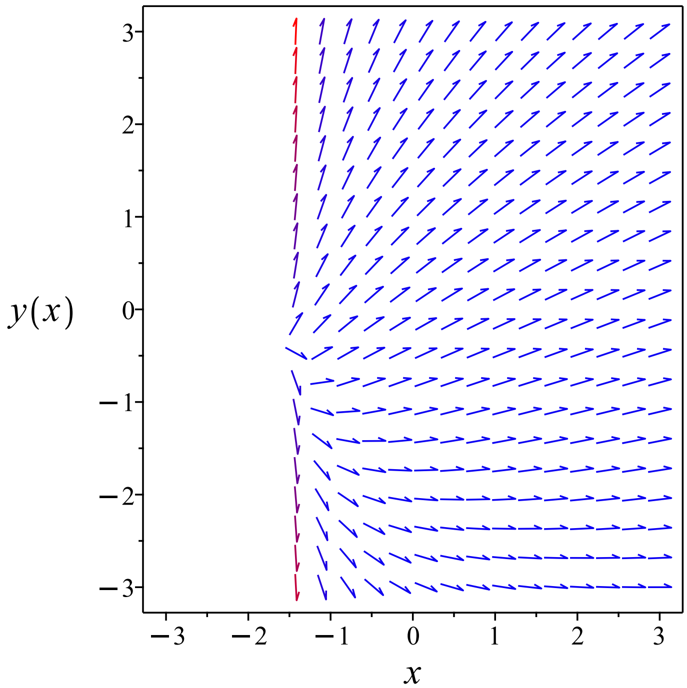


Figure 92: Slope field plot

Verification of solutions

$$y = \left(\frac{\ln(2x+3)}{2} + c_1\right)\sqrt{2x+3}$$

Verified OK.

1.61.2 Maple step by step solution

Let's solve

$(2x+3)\,y' - y = \sqrt{2x+3}$

- Highest derivative means the order of the ODE is 1

  y'

- Isolate the derivative

  $y' = \frac{y}{2x+3} + \frac{1}{\sqrt{2x+3}}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE

  $y' - \frac{y}{2x+3} = \frac{1}{\sqrt{2x+3}}$

- The ODE is linear; multiply by an integrating factor $\mu(x)$

$$\mu(x)\left(y' - \frac{y}{2x+3}\right) = \frac{\mu(x)}{\sqrt{2x+3}}$$

- Assume the lhs of the ODE is the total derivative $\frac{d}{dx}(\mu(x)\,y)$

$$\mu(x)\left(y' - \frac{y}{2x+3}\right) = \mu'(x)\,y + \mu(x)\,y'$$

- Isolate $\mu'(x)$

$$\mu'(x) = -\frac{\mu(x)}{2x+3}$$

- Solve to find the integrating factor

$$\mu(x) = \frac{1}{\sqrt{2x+3}}$$

- Integrate both sides with respect to x

$$\int \left(\frac{d}{dx}(\mu(x)\,y)\right) dx = \int \frac{\mu(x)}{\sqrt{2x+3}} dx + c_1$$

- Evaluate the integral on the lhs

$$\mu(x)\,y = \int \frac{\mu(x)}{\sqrt{2x+3}} dx + c_1$$

- Solve for y

$$y = \frac{\int \frac{\mu(x)}{\sqrt{2x+3}} dx + c_1}{\mu(x)}$$

- Substitute $\mu(x) = \frac{1}{\sqrt{2x+3}}$

$$y = \sqrt{2x+3}\left(\int \frac{1}{2x+3} dx + c_1\right)$$

- Evaluate the integrals on the rhs

$$y = \left(\frac{\ln(2x+3)}{2} + c_1\right)\sqrt{2x+3}$$

- Simplify

$$y = \frac{(\ln(2x+3)+2c_1)\sqrt{2x+3}}{2}$$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
<- 1st order linear successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 23

```
dsolve((2*x+3)*diff(y(x),x)=y(x)+sqrt(2*x+3),y(x), singsol=all)
```

$$y(x) = \frac{(\ln(3+2x) + 2c_1)\sqrt{3+2x}}{2}$$

✓ Solution by Mathematica

Time used: 0.049 (sec). Leaf size: 29

```
DSolve[(2*x+3)*y'[x]==y[x]+Sqrt[2*x+3],y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \frac{1}{2}\sqrt{2x+3}(\log(2x+3) + 2c_1)$$

1.62 problem 63

Internal problem ID [3207]
Internal file name [`OUTPUT/2699_Sunday_June_05_2022_08_38_57_AM_41906111/index.tex`]

Book: Differential equations for engineers by Wei-Chau XIE, Cambridge Press 2010
Section: Chapter 2. First-Order and Simple Higher-Order Differential Equations. Page 78
Problem number: 63.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"exactWithIntegrationFactor", "first_order_ode_lie_symmetry_calculated"**

Maple gives the following as the ode type

```
[[_1st_order, _with_linear_symmetries]]
```

$$\boxed{y + \left(y^2 \mathrm{e}^y - x\right) y' = 0}$$

1.62.1 Solving as first order ode lie symmetry calculated ode

Writing the ode as

$$y' = -\frac{y}{\mathrm{e}^y y^2 - x}$$
$$y' = \omega(x, y)$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x + \omega(\eta_y - \xi_x) - \omega^2 \xi_y - \omega_x \xi - \omega_y \eta = 0 \tag{A}$$

The type of this ode is not in the lookup table. To determine ξ, η then (A) is solved using ansatz. Making bivariate polynomials of degree 1 to use as anstaz gives

$$\xi = x a_2 + y a_3 + a_1 \tag{1E}$$
$$\eta = x b_2 + y b_3 + b_1 \tag{2E}$$

Where the unknown coefficients are

$$\{a_1, a_2, a_3, b_1, b_2, b_3\}$$

Substituting equations (1E,2E) and ω into (A) gives

$$b_2 - \frac{y(b_3 - a_2)}{\mathrm{e}^y y^2 - x} - \frac{y^2 a_3}{\left(\mathrm{e}^y y^2 - x\right)^2} + \frac{y(x a_2 + y a_3 + a_1)}{\left(\mathrm{e}^y y^2 - x\right)^2} - \left(-\frac{1}{\mathrm{e}^y y^2 - x} + \frac{y(\mathrm{e}^y y^2 + 2y\,\mathrm{e}^y)}{\left(\mathrm{e}^y y^2 - x\right)^2}\right)(x b_2 + y b_3 + b_1) = 0 \tag{5E}$$

Putting the above in normal form gives

$$\frac{\mathrm{e}^{2y} y^4 b_2 - \mathrm{e}^y x\, y^3 b_2 - \mathrm{e}^y y^4 b_3 - 3\,\mathrm{e}^y x\, y^2 b_2 + \mathrm{e}^y y^3 a_2 - \mathrm{e}^y y^3 b_1 - 2\,\mathrm{e}^y y^3 b_3 - \mathrm{e}^y y^2 b_1 - x b_1 + y a_1}{\left(\mathrm{e}^y y^2 - x\right)^2} = 0$$

Setting the numerator to zero gives

$$\mathrm{e}^{2y} y^4 b_2 - \mathrm{e}^y x\, y^3 b_2 - \mathrm{e}^y y^4 b_3 - 3\,\mathrm{e}^y x\, y^2 b_2 + \mathrm{e}^y y^3 a_2 - \mathrm{e}^y y^3 b_1 - 2\,\mathrm{e}^y y^3 b_3 - \mathrm{e}^y y^2 b_1 - x b_1 + y a_1 = 0 \tag{6E}$$

Simplifying the above gives

$$\mathrm{e}^{2y} y^4 b_2 - \mathrm{e}^y x\, y^3 b_2 - \mathrm{e}^y y^4 b_3 - 3\,\mathrm{e}^y x\, y^2 b_2 + \mathrm{e}^y y^3 a_2 - \mathrm{e}^y y^3 b_1 - 2\,\mathrm{e}^y y^3 b_3 - \mathrm{e}^y y^2 b_1 - x b_1 + y a_1 = 0 \tag{6E}$$

Looking at the above PDE shows the following are all the terms with $\{x, y\}$ in them.

$$\{x, y, \mathrm{e}^y, \mathrm{e}^{2y}\}$$

The following substitution is now made to be able to collect on all terms with $\{x, y\}$ in them

$$\{x = v_1, y = v_2, \mathrm{e}^y = v_3, \mathrm{e}^{2y} = v_4\}$$

The above PDE (6E) now becomes

$$-v_3 v_1 v_2^3 b_2 + v_4 v_2^4 b_2 - v_3 v_2^4 b_3 + v_3 v_2^3 a_2 - v_3 v_2^3 b_1 - 3 v_3 v_1 v_2^2 b_2 - 2 v_3 v_2^3 b_3 - v_3 v_2^2 b_1 + v_2 a_1 - v_1 b_1 = 0 \tag{7E}$$

Collecting the above on the terms v_i introduced, and these are

$$\{v_1, v_2, v_3, v_4\}$$

Equation (7E) now becomes

$$-v_3 v_1 v_2^3 b_2 - 3v_3 v_1 v_2^2 b_2 - v_1 b_1 - v_3 v_2^4 b_3 + v_4 v_2^4 b_2 + (a_2 - b_1 - 2b_3) v_2^3 v_3 - v_3 v_2^2 b_1 + v_2 a_1 = 0 \tag{8E}$$

Setting each coefficients in (8E) to zero gives the following equations to solve

$$\begin{aligned} a_1 &= 0 \\ b_2 &= 0 \\ -b_1 &= 0 \\ -3b_2 &= 0 \\ -b_2 &= 0 \\ -b_3 &= 0 \\ a_2 - b_1 - 2b_3 &= 0 \end{aligned}$$

Solving the above equations for the unknowns gives

$$\begin{aligned} a_1 &= 0 \\ a_2 &= 0 \\ a_3 &= a_3 \\ b_1 &= 0 \\ b_2 &= 0 \\ b_3 &= 0 \end{aligned}$$

Substituting the above solution in the anstaz (1E,2E) (using 1 as arbitrary value for any unknown in the RHS) gives

$$\begin{aligned} \xi &= y \\ \eta &= 0 \end{aligned}$$

Shifting is now applied to make $\xi = 0$ in order to simplify the rest of the computation

$$\begin{aligned} \eta &= \eta - \omega(x,y)\,\xi \\ &= 0 - \left(-\frac{y}{e^y y^2 - x}\right)(y) \\ &= \frac{y^2}{e^y y^2 - x} \\ \xi &= 0 \end{aligned}$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x, y) \to (R, S)$ where (R, S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y}\right) S(x, y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$R = x$$

S is found from

$$\begin{aligned} S &= \int \frac{1}{\eta} dy \\ &= \int \frac{1}{\frac{y^2}{e^y y^2 - x}} dy \end{aligned}$$

Which results in

$$S = \frac{x}{y} + e^y$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x, y) S_y}{R_x + \omega(x, y) R_y} \tag{2}$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x, y)$ is the right hand side of the original ode given by

$$\omega(x, y) = -\frac{y}{e^y y^2 - x}$$

Evaluating all the partial derivatives gives

$$\begin{aligned} R_x &= 1 \\ R_y &= 0 \\ S_x &= \frac{1}{y} \\ S_y &= \frac{e^y y^2 - x}{y^2} \end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = 0 \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = 0$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$S(R) = c_1 \tag{4}$$

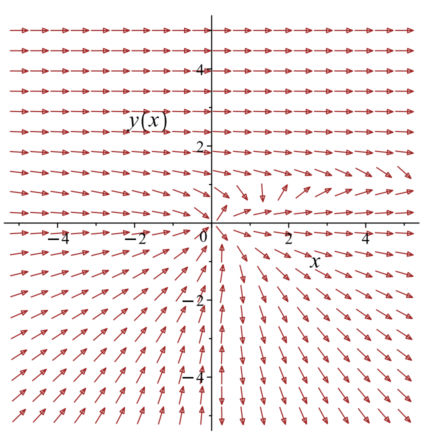
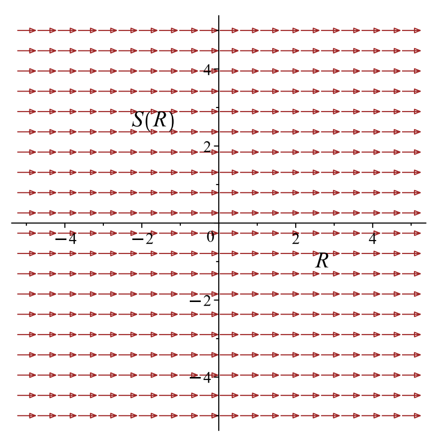
To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$\frac{e^y y + x}{y} = c_1$$

Which simplifies to

$$\frac{e^y y + x}{y} = c_1$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in x, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dx} = -\frac{y}{e^y y^2 - x}$ 	$R = x$ $S = \frac{y\,e^y + x}{y}$	$\frac{dS}{dR} = 0$ 

Summary
The solution(s) found are the following

$$\frac{e^{y}y+x}{y}=c_1 \tag{1}$$

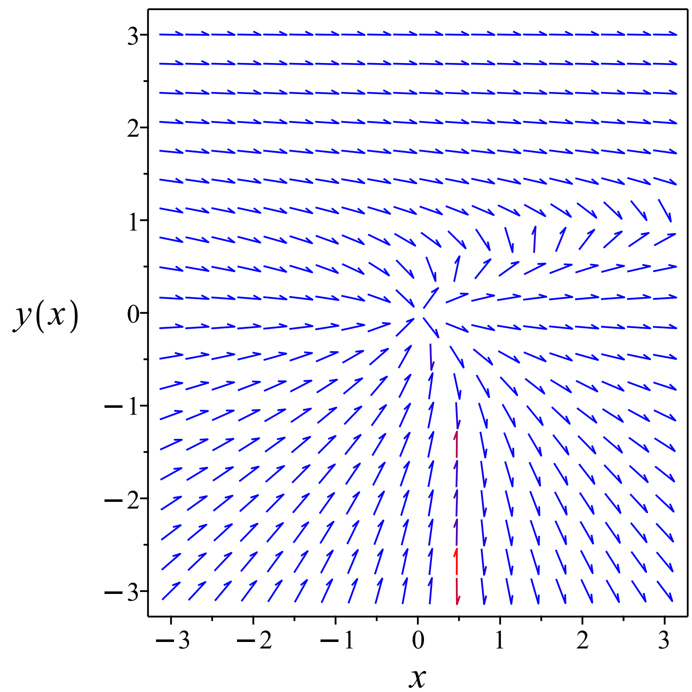


Figure 93: Slope field plot

Verification of solutions

$$\frac{e^{y}y+x}{y}=c_1$$

Verified OK.

1.62.2 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x,y)+N(x,y)\frac{dy}{dx}=0 \tag{A}$$

We assume there exists a function $\phi(x,y)=c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y)=0$$

Hence

$$\frac{\partial \phi}{\partial x} + \frac{\partial \phi}{\partial y}\frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$\begin{aligned}\frac{\partial \phi}{\partial x} &= M \\ \frac{\partial \phi}{\partial y} &= N\end{aligned}$$

But since $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x,y)$ but at least we know now that we can do that since the condition $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x,y)\,\mathrm{d}x + N(x,y)\,\mathrm{d}y = 0 \tag{1A}$$

Therefore

$$\begin{aligned}\left(\mathrm{e}^{y} y^2 - x\right)\mathrm{d}y &= (-y)\,\mathrm{d}x \\ (y)\,\mathrm{d}x + \left(\mathrm{e}^{y} y^2 - x\right)\mathrm{d}y &= 0 \end{aligned}\tag{2A}$$

Comparing (1A) and (2A) shows that

$$\begin{aligned}M(x,y) &= y \\ N(x,y) &= \mathrm{e}^{y} y^2 - x\end{aligned}$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Using result found above gives

$$\begin{aligned}\frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}(y) \\ &= 1\end{aligned}$$

And

$$\frac{\partial N}{\partial x} = \frac{\partial}{\partial x}(\mathrm{e}^y y^2 - x)$$
$$= -1$$

Since $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$, then the ODE is not exact. Since the ODE is not exact, we will try to find an integrating factor to make it exact. Let

$$A = \frac{1}{N}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right)$$
$$= \frac{1}{\mathrm{e}^y y^2 - x}((1) - (-1))$$
$$= -\frac{2}{-\mathrm{e}^y y^2 + x}$$

Since A depends on y, it can not be used to obtain an integrating factor. We will now try a second method to find an integrating factor. Let

$$B = \frac{1}{M}\left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right)$$
$$= \frac{1}{y}((-1) - (1))$$
$$= -\frac{2}{y}$$

Since B does not depend on x, it can be used to obtain an integrating factor. Let the integrating factor be μ. Then

$$\mu = e^{\int B \, \mathrm{d}y}$$
$$= e^{\int -\frac{2}{y} \, \mathrm{d}y}$$

The result of integrating gives

$$\mu = e^{-2\ln(y)}$$
$$= \frac{1}{y^2}$$

M and N are now multiplied by this integrating factor, giving new M and new N which are called $\overline{M}$ and $\overline{N}$ so not to confuse them with the original M and N.

$$\overline{M} = \mu M$$
$$= \frac{1}{y^2}(y)$$
$$= \frac{1}{y}$$

And

$$
\begin{aligned}
\overline{N} &= \mu N \\
&= \frac{1}{y^2}\left(e^y y^2 - x\right) \\
&= \frac{e^y y^2 - x}{y^2}
\end{aligned}
$$

So now a modified ODE is obtained from the original ODE which will be exact and can be solved using the standard method. The modified ODE is

$$
\begin{aligned}
\overline{M} + \overline{N}\frac{dy}{dx} &= 0 \\
\left(\frac{1}{y}\right) + \left(\frac{e^y y^2 - x}{y^2}\right)\frac{dy}{dx} &= 0
\end{aligned}
$$

The following equations are now set up to solve for the function $\phi(x, y)$

$$
\frac{\partial \phi}{\partial x} = \overline{M} \tag{1}
$$

$$
\frac{\partial \phi}{\partial y} = \overline{N} \tag{2}
$$

Integrating (1) w.r.t. x gives

$$
\begin{aligned}
\int \frac{\partial \phi}{\partial x}\, dx &= \int \overline{M}\, dx \\
\int \frac{\partial \phi}{\partial x}\, dx &= \int \frac{1}{y}\, dx
\end{aligned}
$$

$$
\phi = \frac{x}{y} + f(y) \tag{3}
$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$
\frac{\partial \phi}{\partial y} = -\frac{x}{y^2} + f'(y) \tag{4}
$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = \frac{e^y y^2 - x}{y^2}$. Therefore equation (4) becomes

$$
\frac{e^y y^2 - x}{y^2} = -\frac{x}{y^2} + f'(y) \tag{5}
$$

Solving equation (5) for $f'(y)$ gives

$$f'(y) = e^y$$

Integrating the above w.r.t y gives

$$\int f'(y)\, dy = \int (e^y)\, dy$$
$$f(y) = e^y + c_1$$

Where c_1 is constant of integration. Substituting result found above for $f(y)$ into equation (3) gives ϕ

$$\phi = \frac{x}{y} + e^y + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = \frac{x}{y} + e^y$$

Summary
The solution(s) found are the following

$$\frac{x}{y} + e^y = c_1 \tag{1}$$

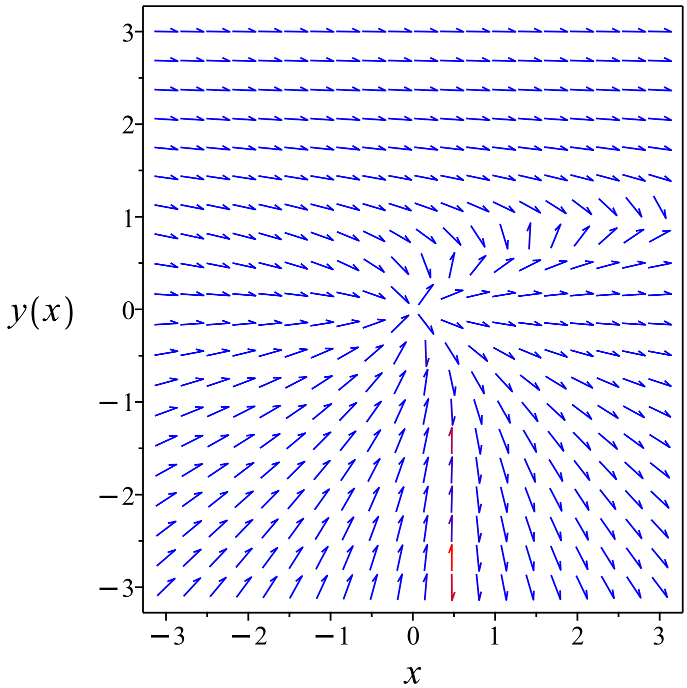


Figure 94: Slope field plot

Verification of solutions

$$\frac{x}{y} + e^{y} = c_1$$

Verified OK.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
<- 1st order linear successful
<- inverse linear successful`
```

✓ Solution by Maple

Time used: 0.031 (sec). Leaf size: 16

```
dsolve(y(x)+(y(x)^2*exp(y(x))-x)*diff(y(x),x)=0,y(x), singsol=all)
```

$$e^{y(x)}y(x) - c_1 y(x) + x = 0$$

✓ Solution by Mathematica

Time used: 0.195 (sec). Leaf size: 19

```
DSolve[y[x]+(y[x]^2*Exp[y[x]]-x)*y'[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$\text{Solve}\left[x = -e^{y(x)}y(x) + c_1 y(x), y(x)\right]$$

1.63 problem 64

Internal problem ID [3208]
Internal file name [`OUTPUT/2700_Sunday_June_05_2022_08_38_58_AM_75200012/index.tex`]

Book: Differential equations for engineers by Wei-Chau XIE, Cambridge Press 2010
Section: Chapter 2. First-Order and Simple Higher-Order Differential Equations. Page 78
Problem number: 64.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program :

Maple gives the following as the ode type

`[_linear]`

$$\boxed{y' - 3y\tan(x) = 1}$$

1.63.1 Solving as linear ode

Entering Linear first order ODE solver. In canonical form a linear first order is

$$y' + p(x)y = q(x)$$

Where here

$$\begin{aligned} p(x) &= -3\tan(x) \\ q(x) &= 1 \end{aligned}$$

Hence the ode is

$$y' - 3y\tan(x) = 1$$

The integrating factor μ is

$$\begin{aligned} \mu &= e^{\int -3\tan(x)dx} \\ &= \cos(x)^3 \end{aligned}$$

The ode becomes

$$
\begin{aligned}
\frac{d}{dx}(\mu y) &= \mu \\
\frac{d}{dx}\left(\cos(x)^3 y\right) &= \cos(x)^3 \\
d\left(\cos(x)^3 y\right) &= \cos(x)^3\, dx
\end{aligned}
$$

Integrating gives

$$
\begin{aligned}
\cos(x)^3 y &= \int \cos(x)^3\, dx \\
\cos(x)^3 y &= \frac{\left(2+\cos(x)^2\right)\sin(x)}{3} + c_1
\end{aligned}
$$

Dividing both sides by the integrating factor $\mu = \cos(x)^3$ results in

$$
y = \frac{\sec(x)^3 \left(2+\cos(x)^2\right)\sin(x)}{3} + c_1 \sec(x)^3
$$

which simplifies to

$$
y = \frac{\tan(x)}{3} + \frac{2\tan(x)\sec(x)^2}{3} + c_1 \sec(x)^3
$$

Summary

The solution(s) found are the following

$$
y = \frac{\tan(x)}{3} + \frac{2\tan(x)\sec(x)^2}{3} + c_1 \sec(x)^3 \tag{1}
$$

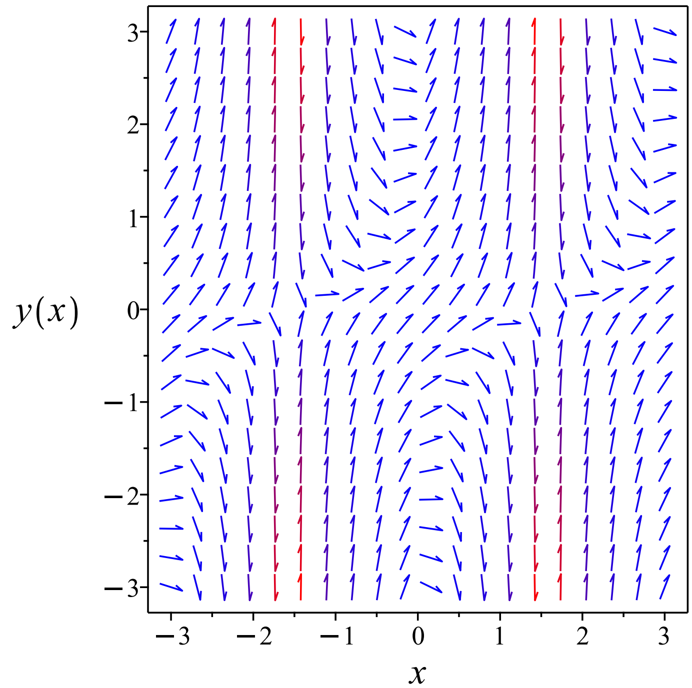


Figure 95: Slope field plot

Verification of solutions

$$y = \frac{\tan(x)}{3} + \frac{2\tan(x)\sec(x)^2}{3} + c_1 \sec(x)^3$$

Verified OK.

1.63.2 Maple step by step solution

Let's solve

$y' - 3y\tan(x) = 1$

- Highest derivative means the order of the ODE is 1

  y'

- Isolate the derivative

  $y' = 1 + 3y\tan(x)$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE

  $y' - 3y\tan(x) = 1$

- The ODE is linear; multiply by an integrating factor $\mu(x)$

$$\mu(x)\left(y' - 3y\tan(x)\right) = \mu(x)$$

- Assume the lhs of the ODE is the total derivative $\frac{d}{dx}(\mu(x)\, y)$

$$\mu(x)\left(y' - 3y\tan(x)\right) = \mu'(x)\, y + \mu(x)\, y'$$

- Isolate $\mu'(x)$

$$\mu'(x) = -3\mu(x)\tan(x)$$

- Solve to find the integrating factor

$$\mu(x) = \cos(x)^3$$

- Integrate both sides with respect to x

$$\int \left(\frac{d}{dx}(\mu(x)\, y)\right) dx = \int \mu(x)\, dx + c_1$$

- Evaluate the integral on the lhs

$$\mu(x)\, y = \int \mu(x)\, dx + c_1$$

- Solve for y

$$y = \frac{\int \mu(x)dx + c_1}{\mu(x)}$$

- Substitute $\mu(x) = \cos(x)^3$

$$y = \frac{\int \cos(x)^3 dx + c_1}{\cos(x)^3}$$

- Evaluate the integrals on the rhs

$$y = \frac{\frac{\left(2+\cos(x)^2\right)\sin(x)}{3} + c_1}{\cos(x)^3}$$

- Simplify

$$y = \frac{\tan(x)}{3} + \frac{2\tan(x)\sec(x)^2}{3} + c_1 \sec(x)^3$$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
<- 1st order linear successful`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 23

```
dsolve(diff(y(x),x)=1+3*y(x)*tan(x),y(x), singsol=all)
```

$$y(x) = \frac{\tan(x)}{3} + \sec(x)^3 c_1 + \frac{2\sec(x)^2\tan(x)}{3}$$

✓ Solution by Mathematica

Time used: 0.049 (sec). Leaf size: 26

```
DSolve[y'[x]==1+3*y[x]*Tan[x],y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \frac{1}{12}\sec^3(x)(9\sin(x) + \sin(3x) + 12c_1)$$

1.64 problem 65

Internal problem ID [3209]
Internal file name [`OUTPUT/2701_Sunday_June_05_2022_08_38_59_AM_98974110/index.tex`]

Book: Differential equations for engineers by Wei-Chau XIE, Cambridge Press 2010
Section: Chapter 2. First-Order and Simple Higher-Order Differential Equations. Page 78
Problem number: 65.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program :

Maple gives the following as the ode type

`[_linear]`

$$\boxed{(\cos(x)+1)\,y' - \sin(x)\,(\sin(x)+\sin(x)\cos(x)-y) = 0}$$

1.64.1 Solving as linear ode

Entering Linear first order ODE solver. In canonical form a linear first order is

$$y' + p(x)y = q(x)$$

Where here

$$p(x) = \frac{\sin(x)}{\cos(x)+1}$$
$$q(x) = \sin(x)^2$$

Hence the ode is

$$y' + \frac{\sin(x)\,y}{\cos(x)+1} = \sin(x)^2$$

The integrating factor μ is

$$\mu = \mathrm{e}^{\int \frac{\sin(x)}{\cos(x)+1}dx}$$
$$= \frac{1}{\cos(x)+1}$$

The ode becomes

$$
\begin{aligned}
\frac{d}{dx}(\mu y) &= (\mu)\left(\sin(x)^2\right) \\
\frac{d}{dx}\left(\frac{y}{\cos(x)+1}\right) &= \left(\frac{1}{\cos(x)+1}\right)\left(\sin(x)^2\right) \\
d\left(\frac{y}{\cos(x)+1}\right) &= \left(\frac{\sin(x)^2}{\cos(x)+1}\right) dx
\end{aligned}
$$

Integrating gives

$$
\begin{aligned}
\frac{y}{\cos(x)+1} &= \int \frac{\sin(x)^2}{\cos(x)+1}\, dx \\
\frac{y}{\cos(x)+1} &= -\frac{2\tan\left(\frac{x}{2}\right)}{1+\tan\left(\frac{x}{2}\right)^2} + x + c_1
\end{aligned}
$$

Dividing both sides by the integrating factor $\mu = \frac{1}{\cos(x)+1}$ results in

$$
y = (\cos(x)+1)\left(-\frac{2\tan\left(\frac{x}{2}\right)}{1+\tan\left(\frac{x}{2}\right)^2} + x\right) + c_1(\cos(x)+1)
$$

which simplifies to

$$
y = (\cos(x)+1)\,(c_1 + x - \sin(x))
$$

Summary

The solution(s) found are the following

$$
y = (\cos(x)+1)\,(c_1 + x - \sin(x)) \tag{1}
$$

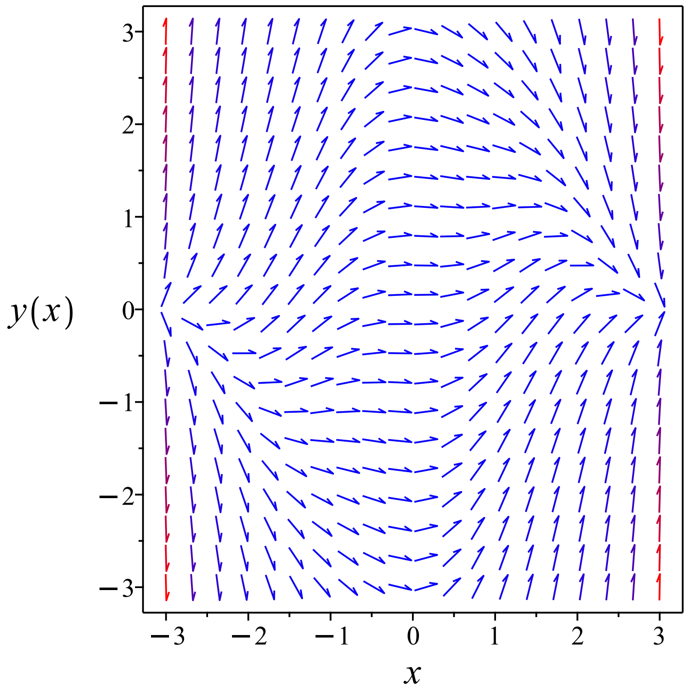


Figure 96: Slope field plot

Verification of solutions

$$y = (\cos(x) + 1)(c_1 + x - \sin(x))$$

Verified OK.

1.64.2 Maple step by step solution

Let's solve

$(\cos(x) + 1) y' - \sin(x)(\sin(x) + \sin(x)\cos(x) - y) = 0$

- Highest derivative means the order of the ODE is 1

  y'

- Isolate the derivative

  $y' = -\frac{\sin(x)y}{\cos(x)+1} + \sin(x)^2$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE

  $y' + \frac{\sin(x)y}{\cos(x)+1} = \sin(x)^2$

- The ODE is linear; multiply by an integrating factor $\mu(x)$

$$\mu(x)\left(y'+\frac{\sin(x)y}{\cos(x)+1}\right)=\mu(x)\sin(x)^2$$

- Assume the lhs of the ODE is the total derivative $\frac{d}{dx}(\mu(x)\,y)$

$$\mu(x)\left(y'+\frac{\sin(x)y}{\cos(x)+1}\right)=\mu'(x)\,y+\mu(x)\,y'$$

- Isolate $\mu'(x)$

$$\mu'(x)=\frac{\mu(x)\sin(x)}{\cos(x)+1}$$

- Solve to find the integrating factor

$$\mu(x)=\frac{1}{\cos(x)+1}$$

- Integrate both sides with respect to x

$$\int\left(\frac{d}{dx}(\mu(x)\,y)\right)dx=\int\mu(x)\sin(x)^2\,dx+c_1$$

- Evaluate the integral on the lhs

$$\mu(x)\,y=\int\mu(x)\sin(x)^2\,dx+c_1$$

- Solve for y

$$y=\frac{\int\mu(x)\sin(x)^2dx+c_1}{\mu(x)}$$

- Substitute $\mu(x)=\frac{1}{\cos(x)+1}$

$$y=(\cos(x)+1)\left(\int\frac{\sin(x)^2}{\cos(x)+1}dx+c_1\right)$$

- Evaluate the integrals on the rhs

$$y=(\cos(x)+1)\left(-\frac{2\tan\left(\frac{x}{2}\right)}{1+\tan\left(\frac{x}{2}\right)^2}+x+c_1\right)$$

- Simplify

$$y=(\cos(x)+1)\,(c_1+x-\sin(x))$$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
<- 1st order linear successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 16

```
dsolve((1+cos(x))*diff(y(x),x)=sin(x)*( sin(x)+sin(x)*cos(x)-y(x) ),y(x), singsol=all)
```

$$y(x) = (-\sin(x) + x + c_1)(\cos(x) + 1)$$

✓ Solution by Mathematica

Time used: 0.096 (sec). Leaf size: 24

```
DSolve[(1+Cos[x])*y'[x]==Sin[x]*( Sin[x]+Sin[x]*Cos[x]-y[x] ),y[x],x,IncludeSingularSolution
```

$$y(x) \to \cos^2\left(\frac{x}{2}\right)(2x - 2\sin(x) + c_1)$$

1.65 problem 66

Internal problem ID [3210]
Internal file name [`OUTPUT/2702_Sunday_June_05_2022_08_38_59_AM_39178359/index.tex`]

Book: Differential equations for engineers by Wei-Chau XIE, Cambridge Press 2010
Section: Chapter 2. First-Order and Simple Higher-Order Differential Equations. Page 78
Problem number: 66.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program :

Maple gives the following as the ode type

`[_linear]`

$$\boxed{y' - \left(\sin \left(x\right)^{2} - y\right) \cos \left(x\right) = 0}$$

1.65.1 Solving as linear ode

Entering Linear first order ODE solver. In canonical form a linear first order is

$$y' + p(x)y = q(x)$$

Where here

$$\begin{aligned} p(x) &= \cos \left(x\right) \\ q(x) &= \sin \left(x\right)^{2} \cos \left(x\right) \end{aligned}$$

Hence the ode is

$$y' + \cos \left(x\right) y = \sin \left(x\right)^{2} \cos \left(x\right)$$

The integrating factor μ is

$$\begin{aligned} \mu &= \mathrm{e}^{\int \cos(x)dx} \\ &= \mathrm{e}^{\sin(x)} \end{aligned}$$

The ode becomes

$$\frac{d}{dx}(\mu y) = (\mu)\left(\sin(x)^2\cos(x)\right)$$
$$\frac{d}{dx}\left(e^{\sin(x)}y\right) = \left(e^{\sin(x)}\right)\left(\sin(x)^2\cos(x)\right)$$
$$d\left(e^{\sin(x)}y\right) = \left(\sin(x)^2\cos(x)\,e^{\sin(x)}\right)\,dx$$

Integrating gives

$$e^{\sin(x)}y = \int \sin(x)^2\cos(x)\,e^{\sin(x)}\,dx$$
$$e^{\sin(x)}y = \sin(x)^2 e^{\sin(x)} - 2\sin(x)\,e^{\sin(x)} + 2\,e^{\sin(x)} + c_1$$

Dividing both sides by the integrating factor $\mu = e^{\sin(x)}$ results in

$$y = e^{-\sin(x)}\left(\sin(x)^2 e^{\sin(x)} - 2\sin(x)\,e^{\sin(x)} + 2\,e^{\sin(x)}\right) + c_1 e^{-\sin(x)}$$

which simplifies to

$$y = \sin(x)^2 - 2\sin(x) + 2 + c_1 e^{-\sin(x)}$$

<u>Summary</u>

The solution(s) found are the following

$$y = \sin(x)^2 - 2\sin(x) + 2 + c_1 e^{-\sin(x)} \tag{1}$$

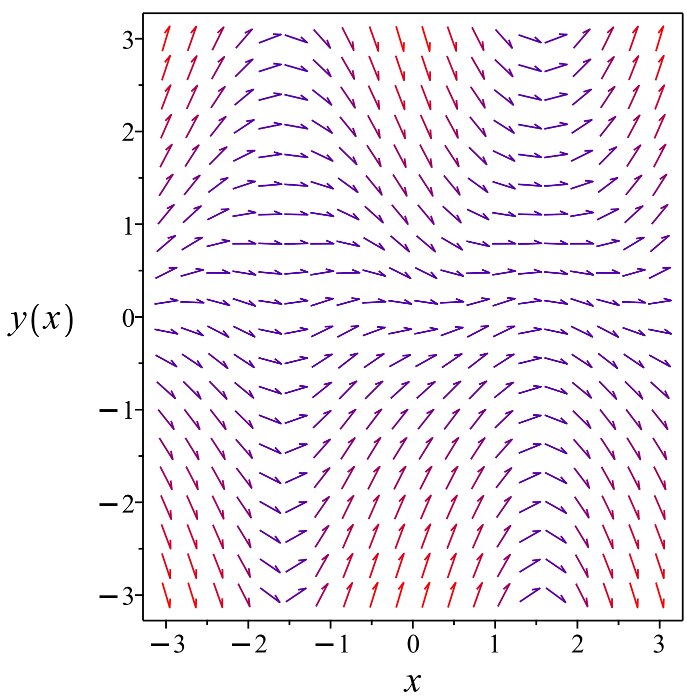


Figure 97: Slope field plot

Verification of solutions

$$y = \sin(x)^2 - 2\sin(x) + 2 + c_1 \mathrm{e}^{-\sin(x)}$$

Verified OK.

1.65.2 Maple step by step solution

Let's solve

$y' - (\sin(x)^2 - y)\cos(x) = 0$

- Highest derivative means the order of the ODE is 1

  y'

- Isolate the derivative

  $y' = -\cos(x)\, y + \sin(x)^2 \cos(x)$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE

  $y' + \cos(x)\, y = \sin(x)^2 \cos(x)$

- The ODE is linear; multiply by an integrating factor $\mu(x)$

  $\mu(x)\,(y' + \cos(x)\, y) = \mu(x)\sin(x)^2 \cos(x)$

- Assume the lhs of the ODE is the total derivative $\frac{d}{dx}(\mu(x)\, y)$

  $\mu(x)\,(y' + \cos(x)\, y) = \mu'(x)\, y + \mu(x)\, y'$

- Isolate $\mu'(x)$

  $\mu'(x) = \mu(x)\cos(x)$

- Solve to find the integrating factor

  $\mu(x) = \mathrm{e}^{\sin(x)}$

- Integrate both sides with respect to x

  $\int \left(\frac{d}{dx}(\mu(x)\, y)\right) dx = \int \mu(x)\sin(x)^2 \cos(x)\, dx + c_1$

- Evaluate the integral on the lhs

  $\mu(x)\, y = \int \mu(x)\sin(x)^2 \cos(x)\, dx + c_1$

- Solve for y

  $y = \frac{\int \mu(x)\sin(x)^2 \cos(x) dx + c_1}{\mu(x)}$

- Substitute $\mu(x) = \mathrm{e}^{\sin(x)}$

  $y = \frac{\int \sin(x)^2 \cos(x) \mathrm{e}^{\sin(x)} dx + c_1}{\mathrm{e}^{\sin(x)}}$

- Evaluate the integrals on the rhs

  $y = \frac{\sin(x)^2 e^{\sin(x)} - 2\sin(x)e^{\sin(x)} + 2\,e^{\sin(x)} + c_1}{e^{\sin(x)}}$

- Simplify

  $y = -2\sin(x) - \cos(x)^2 + 3 + c_1 e^{-\sin(x)}$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
<- 1st order linear successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 23

```
dsolve(diff(y(x),x)=( sin(x)^2-y(x))*cos(x),y(x), singsol=all)
```

$$y(x) = 3 + e^{-\sin(x)}c_1 - \cos(x)^2 - 2\sin(x)$$

✓ Solution by Mathematica

Time used: 0.147 (sec). Leaf size: 30

```
DSolve[y'[x]==( Sin[x]^2-y[x])*Cos[x],y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to -2\sin(x) - \frac{1}{2}\cos(2x) + c_1 e^{-\sin(x)} + \frac{5}{2}$$

1.66 problem 68

Internal problem ID [3211]
Internal file name [`OUTPUT/2703_Sunday_June_05_2022_08_39_00_AM_74786523/index.tex`]

Book: Differential equations for engineers by Wei-Chau XIE, Cambridge Press 2010
Section: Chapter 2. First-Order and Simple Higher-Order Differential Equations. Page 78
Problem number: 68.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program :

Maple gives the following as the ode type

`[_linear]`

$$\boxed{(x+1)\,y'-y=x(x+1)^2}$$

1.66.1 Solving as linear ode

Entering Linear first order ODE solver. In canonical form a linear first order is

$$y'+p(x)y=q(x)$$

Where here

$$\begin{aligned}p(x)&=-\frac{1}{x+1}\\ q(x)&=x(x+1)\end{aligned}$$

Hence the ode is

$$y'-\frac{y}{x+1}=x(x+1)$$

The integrating factor μ is

$$\begin{aligned}\mu&=e^{\int -\frac{1}{x+1}dx}\\ &=\frac{1}{x+1}\end{aligned}$$

The ode becomes

$$
\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}x}(\mu y) &= (\mu)\,(x(x+1)) \\
\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{y}{x+1}\right) &= \left(\frac{1}{x+1}\right)(x(x+1)) \\
\mathrm{d}\left(\frac{y}{x+1}\right) &= x\,\mathrm{d}x
\end{aligned}
$$

Integrating gives

$$
\begin{aligned}
\frac{y}{x+1} &= \int x\,\mathrm{d}x \\
\frac{y}{x+1} &= \frac{x^2}{2} + c_1
\end{aligned}
$$

Dividing both sides by the integrating factor $\mu = \frac{1}{x+1}$ results in

$$
y = \frac{x^2(x+1)}{2} + c_1(x+1)
$$

which simplifies to

$$
y = \frac{(x+1)\,(x^2+2c_1)}{2}
$$

Summary
The solution(s) found are the following

$$
y = \frac{(x+1)\,(x^2+2c_1)}{2} \tag{1}
$$

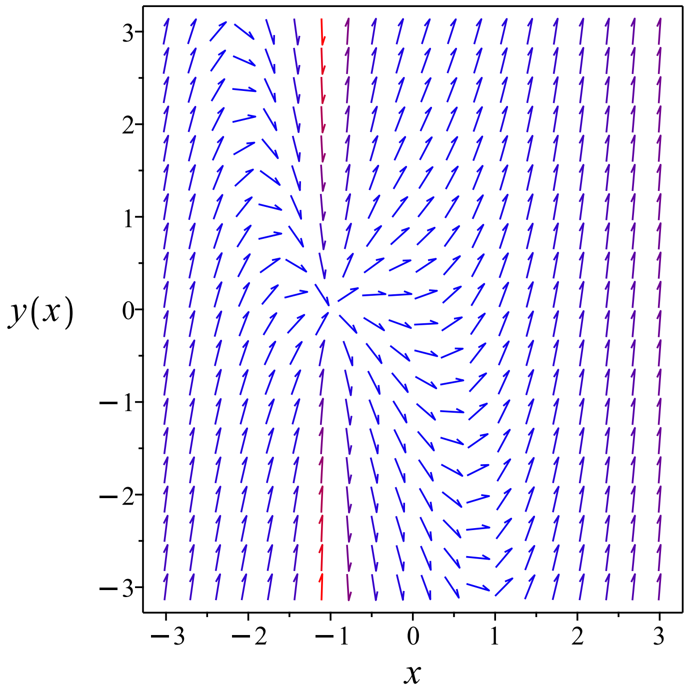


Figure 98: Slope field plot

Verification of solutions

$$y = \frac{(x+1)(x^2+2c_1)}{2}$$

Verified OK.

1.66.2 Maple step by step solution

Let's solve

$(x+1)\,y' - y = x(x+1)^2$

- Highest derivative means the order of the ODE is 1

  y'

- Isolate the derivative

  $y' = \frac{y}{x+1} + x(x+1)$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE

  $y' - \frac{y}{x+1} = x(x+1)$

- The ODE is linear; multiply by an integrating factor $\mu(x)$

$$\mu(x)\left(y'-\frac{y}{x+1}\right)=\mu(x)\,x(x+1)$$

- Assume the lhs of the ODE is the total derivative $\frac{d}{dx}(\mu(x)\,y)$

$$\mu(x)\left(y'-\frac{y}{x+1}\right)=\mu'(x)\,y+\mu(x)\,y'$$

- Isolate $\mu'(x)$

$$\mu'(x)=-\frac{\mu(x)}{x+1}$$

- Solve to find the integrating factor

$$\mu(x)=\frac{1}{x+1}$$

- Integrate both sides with respect to x

$$\int\left(\frac{d}{dx}(\mu(x)\,y)\right)dx=\int\mu(x)\,x(x+1)\,dx+c_1$$

- Evaluate the integral on the lhs

$$\mu(x)\,y=\int\mu(x)\,x(x+1)\,dx+c_1$$

- Solve for y

$$y=\frac{\int\mu(x)x(x+1)dx+c_1}{\mu(x)}$$

- Substitute $\mu(x)=\frac{1}{x+1}$

$$y=(x+1)\left(\int xdx+c_1\right)$$

- Evaluate the integrals on the rhs

$$y=(x+1)\left(\frac{x^2}{2}+c_1\right)$$

- Simplify

$$y=\frac{(x+1)(x^2+2c_1)}{2}$$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
<- 1st order linear successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 16

```
dsolve((1+x)*diff(y(x),x)-y(x)=x*(1+x)^2,y(x), singsol=all)
```

$$y(x) = \frac{(x^2 + 2c_1)(x + 1)}{2}$$

✓ Solution by Mathematica

Time used: 0.029 (sec). Leaf size: 20

```
DSolve[(1+x)*y'[x]-y[x]==x*(1+x)^2,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \frac{1}{2}(x + 1)(x^2 + 2c_1)$$

1.67 problem 69

Internal problem ID [3212]
Internal file name [`OUTPUT/2704_Sunday_June_05_2022_08_39_00_AM_6394563/index.tex`]

Book: Differential equations for engineers by Wei-Chau XIE, Cambridge Press 2010
Section: Chapter 2. First-Order and Simple Higher-Order Differential Equations. Page 78
Problem number: 69.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"exact", "differentialType", "first_order_ode_lie_symmetry_calculated"**

Maple gives the following as the ode type

```
[_exact, _rational, [_1st_order, `_with_symmetry_[F(x)*G(y)
   ,0]`]]
```

$$\boxed{y + \left(x - y(y+1)^2\right) y' = -1}$$

1.67.1 Solving as differentialType ode

Writing the ode as

$$y' = \frac{-1-y}{x - y\,(y+1)^2} \tag{1}$$

Which becomes

$$\left(-y^3 - 2y^2 - y\right) dy = (-x)\, dy + (-y-1)\, dx \tag{2}$$

But the RHS is complete differential because

$$(-x)\, dy + (-y-1)\, dx = d(-(y+1)\, x)$$

Hence (2) becomes

$$\left(-y^3 - 2y^2 - y\right) dy = d(-(y+1)\, x)$$

Integrating both sides gives gives the solution as

$$-\frac{y^4}{4} - \frac{2y^3}{3} - \frac{y^2}{2} = -(y+1)\,x + c_1$$

Summary
The solution(s) found are the following

$$-\frac{y^4}{4} - \frac{2y^3}{3} - \frac{y^2}{2} = -(y+1)\,x + c_1 \tag{1}$$

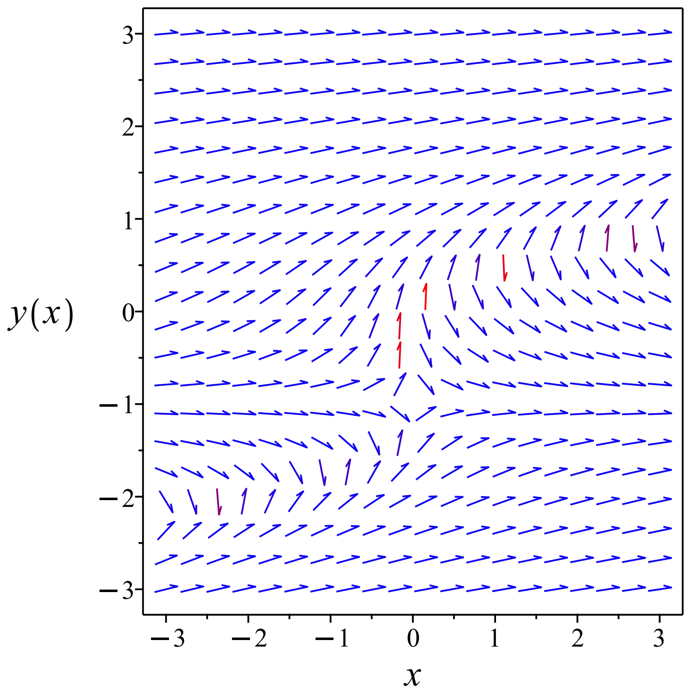


Figure 99: Slope field plot

Verification of solutions

$$-\frac{y^4}{4} - \frac{2y^3}{3} - \frac{y^2}{2} = -(y+1)\,x + c_1$$

Verified OK.

1.67.2 Solving as first order ode lie symmetry calculated ode

Writing the ode as

$$
\begin{aligned}
y' &= \frac{y+1}{y^3+2y^2-x+y} \\
y' &= \omega(x,y)
\end{aligned}
$$

The condition of Lie symmetry is the linearized PDE given by

$$
\eta_x + \omega(\eta_y - \xi_x) - \omega^2 \xi_y - \omega_x \xi - \omega_y \eta = 0 \tag{A}
$$

The type of this ode is not in the lookup table. To determine ξ, η then (A) is solved using ansatz. Making bivariate polynomials of degree 2 to use as anstaz gives

$$
\xi = x^2 a_4 + xy a_5 + y^2 a_6 + x a_2 + y a_3 + a_1 \tag{1E}
$$

$$
\eta = x^2 b_4 + xy b_5 + y^2 b_6 + x b_2 + y b_3 + b_1 \tag{2E}
$$

Where the unknown coefficients are

$$
\{a_1, a_2, a_3, a_4, a_5, a_6, b_1, b_2, b_3, b_4, b_5, b_6\}
$$

Substituting equations (1E,2E) and ω into (A) gives

$$
\begin{aligned}
&2x b_4 + y b_5 + b_2 + \frac{(y+1)(-2x a_4 + x b_5 - y a_5 + 2y b_6 - a_2 + b_3)}{y^3+2y^2-x+y} \\
&- \frac{(y+1)^2 (x a_5 + 2y a_6 + a_3)}{(y^3+2y^2-x+y)^2} \\
&- \frac{(y+1)(x^2 a_4 + xy a_5 + y^2 a_6 + x a_2 + y a_3 + a_1)}{(y^3+2y^2-x+y)^2} - \left(\frac{1}{y^3+2y^2-x+y} \right. \\
&\left. - \frac{(y+1)(3y^2+4y+1)}{(y^3+2y^2-x+y)^2}\right)(x^2 b_4 + xy b_5 + y^2 b_6 + x b_2 + y b_3 + b_1) = 0
\end{aligned} \tag{5E}
$$

Putting the above in normal form gives

$$
\frac{2x\,y^6 b_4 + y^7 b_5 + 8x\,y^5 b_4 + y^6 b_2 + 4y^6 b_5 - 2x^2 y^3 b_4 - 2x\,y^4 a_4 + 12x\,y^4 b_4 + x\,y^4 b_5 - y^5 a_5 + 4y^5 b_2 + 6y^5 b_5 +}{}
$$

$$
= 0
$$

Setting the numerator to zero gives

$$
\begin{aligned}
&2x\,y^6 b_4+y^7 b_5+8x\,y^5 b_4+y^6 b_2+4y^6 b_5-2x^2y^3 b_4-2x\,y^4 a_4+12x\,y^4 b_4\\
&+x\,y^4 b_5-y^5 a_5+4y^5 b_2+6y^5 b_5+4y^5 b_6-3x^2y^2 b_4-6x\,y^3 a_4+8x\,y^3 b_4\\
&+4x\,y^3 b_5-y^4 a_2-3y^4 a_5+6y^4 b_2+3y^4 b_3+4y^4 b_5+11y^4 b_6+3x^3 b_4\\
&+x^2y a_4+x^2y b_5-6x\,y^2 a_4-x\,y^2 a_5+x\,y^2 b_2+2x\,y^2 b_4+5x\,y^2 b_5-x\,y^2 b_6\\
&-3y^3 a_2-3y^3 a_5-3y^3 a_6+2y^3 b_1+4y^3 b_2+8y^3 b_3+y^3 b_5+10y^3 b_6+x^2 a_4\\
&+2x^2 b_2+x^2 b_4-x^2 b_5-2xya_4-2xya_5+2xyb_2+2xyb_5-2xyb_6-3y^2 a_2\\
&-2y^2 a_3-y^2 a_5-5y^2 a_6+5y^2 b_1+y^2 b_2+7y^2 b_3+3y^2 b_6-xa_5+xb_1\\
&+xb_2-xb_3-ya_1-ya_2-3ya_3-2ya_6+4yb_1+2yb_3-a_1-a_3+b_1=0
\end{aligned}
\tag{6E}
$$

Looking at the above PDE shows the following are all the terms with $\{x, y\}$ in them.

$$\{x, y\}$$

The following substitution is now made to be able to collect on all terms with $\{x, y\}$ in them

$$\{x = v_1, y = v_2\}$$

The above PDE (6E) now becomes

$$
\begin{aligned}
&2b_4v_1v_2^6+b_5v_2^7+b_2v_2^6+8b_4v_1v_2^5+4b_5v_2^6-2a_4v_1v_2^4-a_5v_2^5+4b_2v_2^5\\
&-2b_4v_1^2v_2^3+12b_4v_1v_2^4+b_5v_1v_2^4+6b_5v_2^5+4b_6v_2^5-a_2v_2^4-6a_4v_1v_2^3\\
&-3a_5v_2^4+6b_2v_2^4+3b_3v_2^4-3b_4v_1^2v_2^2+8b_4v_1v_2^3+4b_5v_1v_2^3+4b_5v_2^4\\
&+11b_6v_2^4-3a_2v_2^3+a_4v_1^2v_2-6a_4v_1v_2^2-a_5v_1v_2^2-3a_5v_2^3-3a_6v_2^3\\
&+2b_1v_2^3+b_2v_1v_2^2+4b_2v_2^3+8b_3v_2^3+3b_4v_1^3+2b_4v_1v_2^2+b_5v_1^2v_2+5b_5v_1v_2^2\\
&+b_5v_2^3-b_6v_1v_2^2+10b_6v_2^3-3a_2v_2^2-2a_3v_2^2+a_4v_1^2-2a_4v_1v_2-2a_5v_1v_2\\
&-a_5v_2^2-5a_6v_2^2+5b_1v_2^2+2b_2v_1^2+2b_2v_1v_2+b_2v_2^2+7b_3v_2^2+b_4v_1^2\\
&-b_5v_1^2+2b_5v_1v_2-2b_6v_1v_2+3b_6v_2^2-a_1v_2-a_2v_2-3a_3v_2-a_5v_1\\
&-2a_6v_2+b_1v_1+4b_1v_2+b_2v_1-b_3v_1+2b_3v_2-a_1-a_3+b_1=0
\end{aligned}
\tag{7E}
$$

Collecting the above on the terms v_i introduced, and these are

$$\{v_1, v_2\}$$

Equation (7E) now becomes

$$
\begin{aligned}
&3b_4v_1^3 - 2b_4v_1^2v_2^3 - 3b_4v_1^2v_2^2 + (a_4 + b_5)\,v_1^2v_2 + (a_4 + 2b_2 + b_4 - b_5)\,v_1^2 \\
&+ 2b_4v_1v_2^6 + 8b_4v_1v_2^5 + (-2a_4 + 12b_4 + b_5)\,v_1v_2^4 \\
&+ (-6a_4 + 8b_4 + 4b_5)\,v_1v_2^3 + (-6a_4 - a_5 + b_2 + 2b_4 + 5b_5 - b_6)\,v_1v_2^2 \\
&+ (-2a_4 - 2a_5 + 2b_2 + 2b_5 - 2b_6)\,v_1v_2 + (-a_5 + b_1 + b_2 - b_3)\,v_1 \\
&+ b_5v_2^7 + (b_2 + 4b_5)\,v_2^6 + (-a_5 + 4b_2 + 6b_5 + 4b_6)\,v_2^5 \\
&+ (-a_2 - 3a_5 + 6b_2 + 3b_3 + 4b_5 + 11b_6)\,v_2^4 \\
&+ (-3a_2 - 3a_5 - 3a_6 + 2b_1 + 4b_2 + 8b_3 + b_5 + 10b_6)\,v_2^3 \\
&+ (-3a_2 - 2a_3 - a_5 - 5a_6 + 5b_1 + b_2 + 7b_3 + 3b_6)\,v_2^2 \\
&+ (-a_1 - a_2 - 3a_3 - 2a_6 + 4b_1 + 2b_3)\,v_2 - a_1 - a_3 + b_1 = 0
\end{aligned} \tag{8E}
$$

Setting each coefficients in (8E) to zero gives the following equations to solve

$$
\begin{aligned}
b_5 &= 0 \\
-3b_4 &= 0 \\
-2b_4 &= 0 \\
2b_4 &= 0 \\
3b_4 &= 0 \\
8b_4 &= 0 \\
a_4 + b_5 &= 0 \\
b_2 + 4b_5 &= 0 \\
-a_1 - a_3 + b_1 &= 0 \\
-6a_4 + 8b_4 + 4b_5 &= 0 \\
-2a_4 + 12b_4 + b_5 &= 0 \\
a_4 + 2b_2 + b_4 - b_5 &= 0 \\
-a_5 + b_1 + b_2 - b_3 &= 0 \\
-a_5 + 4b_2 + 6b_5 + 4b_6 &= 0 \\
-2a_4 - 2a_5 + 2b_2 + 2b_5 - 2b_6 &= 0 \\
-a_1 - a_2 - 3a_3 - 2a_6 + 4b_1 + 2b_3 &= 0 \\
-a_2 - 3a_5 + 6b_2 + 3b_3 + 4b_5 + 11b_6 &= 0 \\
-6a_4 - a_5 + b_2 + 2b_4 + 5b_5 - b_6 &= 0 \\
-3a_2 - 2a_3 - a_5 - 5a_6 + 5b_1 + b_2 + 7b_3 + 3b_6 &= 0 \\
-3a_2 - 3a_5 - 3a_6 + 2b_1 + 4b_2 + 8b_3 + b_5 + 10b_6 &= 0
\end{aligned}
$$

Solving the above equations for the unknowns gives

$$
\begin{aligned}
a_1 &= \frac{b_3}{3} \\
a_2 &= 3b_3 \\
a_3 &= \frac{2b_3}{3} \\
a_4 &= 0 \\
a_5 &= 0 \\
a_6 &= \frac{b_3}{3} \\
b_1 &= b_3 \\
b_2 &= 0 \\
b_3 &= b_3 \\
b_4 &= 0 \\
b_5 &= 0 \\
b_6 &= 0
\end{aligned}
$$

Substituting the above solution in the anstaz (1E,2E) (using 1 as arbitrary value for any unknown in the RHS) gives

$$
\begin{aligned}
\xi &= \frac{1}{3}y^2 + 3x + \frac{2}{3}y + \frac{1}{3} \\
\eta &= y + 1
\end{aligned}
$$

Shifting is now applied to make $\xi = 0$ in order to simplify the rest of the computation

$$
\begin{aligned}
\eta &= \eta - \omega(x,y)\,\xi \\
&= y + 1 - \left(\frac{y+1}{y^3 + 2y^2 - x + y}\right)\left(\frac{1}{3}y^2 + 3x + \frac{2}{3}y + \frac{1}{3}\right) \\
&= \frac{-3y^4 - 8y^3 + 12xy - 6y^2 + 12x + 1}{-3y^3 - 6y^2 + 3x - 3y} \\
\xi &= 0
\end{aligned}
$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x, y) \to (R, S)$ where (R, S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$
\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}
$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y}\right) S(x,y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$R = x$$

S is found from

$$\begin{aligned} S &= \int \frac{1}{\eta} dy \\ &= \int \frac{1}{\frac{-3y^4-8y^3+12xy-6y^2+12x+1}{-3y^3-6y^2+3x-3y}} dy \end{aligned}$$

Which results in

$$S = \frac{\ln\left(3y^4+8y^3-12xy+6y^2-12x-1\right)}{4}$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x,y) S_y}{R_x + \omega(x,y) R_y} \tag{2}$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x,y)$ is the right hand side of the original ode given by

$$\omega(x,y) = \frac{y+1}{y^3+2y^2-x+y}$$

Evaluating all the partial derivatives gives

$$\begin{aligned} R_x &= 1 \\ R_y &= 0 \\ S_x &= \frac{3}{-3y^3-5y^2+12x-y+1} \\ S_y &= \frac{-3y^3-6y^2+3x-3y}{(y+1)\left(-3y^3-5y^2+12x-y+1\right)} \end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = 0 \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = 0$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$S(R) = c_1 \tag{4}$$

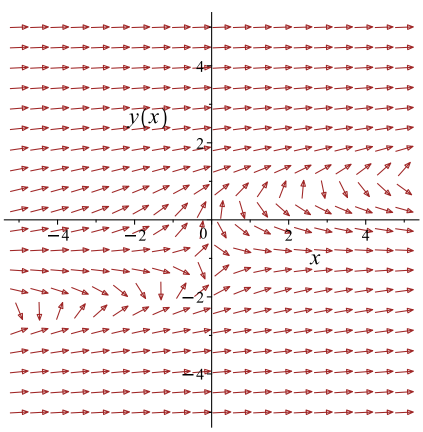
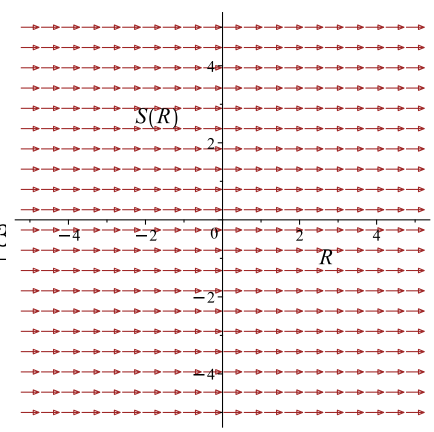
To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$\frac{\ln\left(-1-y\right)}{4}+\frac{\ln\left(-3y^{3}-5y^{2}+12x-y+1\right)}{4} = c_1$$

Which simplifies to

$$\frac{\ln\left(-1-y\right)}{4}+\frac{\ln\left(-3y^{3}-5y^{2}+12x-y+1\right)}{4} = c_1$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in x, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dx} = \frac{y+1}{y^{3}+2y^{2}-x+y}$ 	$R = x$ $S = \frac{\ln\left(-y-1\right)}{4} + \ln(-3$	$\frac{dS}{dR} = 0$ 

Summary
The solution(s) found are the following

$$\frac{\ln\left(-1-y\right)}{4}+\frac{\ln\left(-3y^{3}-5y^{2}+12x-y+1\right)}{4} = c_1 \tag{1}$$

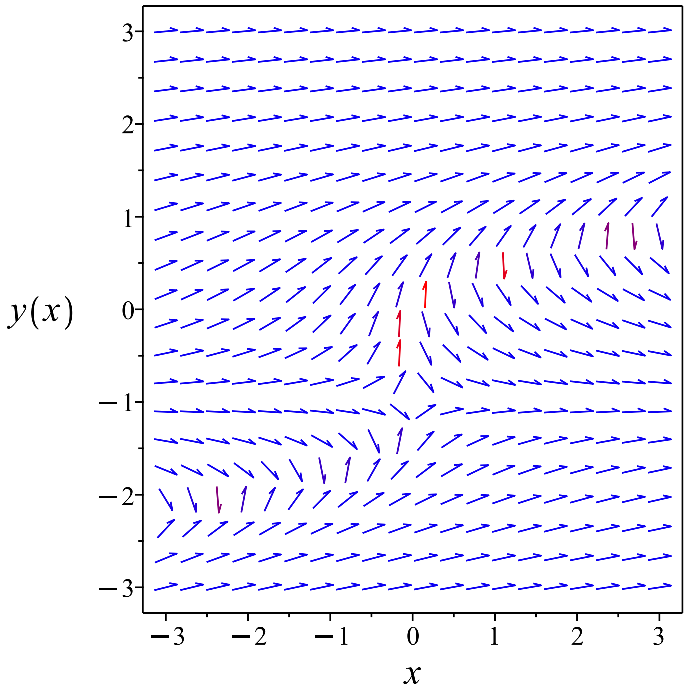


Figure 100: Slope field plot

Verification of solutions

$$\frac{\ln(-1-y)}{4} + \frac{\ln(-3y^3 - 5y^2 + 12x - y + 1)}{4} = c_1$$

Verified OK.

1.67.3 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x,y) + N(x,y)\frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x,y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y) = 0$$

Hence

$$\frac{\partial \phi}{\partial x} + \frac{\partial \phi}{\partial y}\frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$\begin{aligned}\frac{\partial \phi}{\partial x} &= M \\ \frac{\partial \phi}{\partial y} &= N\end{aligned}$$

But since $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x, y)$ but at least we know now that we can do that since the condition $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x, y)\,\mathrm{d}x + N(x, y)\,\mathrm{d}y = 0 \tag{1A}$$

Therefore

$$\begin{aligned}\left(x - y(y+1)^2\right)\mathrm{d}y &= (-y-1)\,\mathrm{d}x \\ (y+1)\,\mathrm{d}x + \left(x - y(y+1)^2\right)\mathrm{d}y &= 0 \qquad (2A)\end{aligned}$$

Comparing (1A) and (2A) shows that

$$\begin{aligned}M(x, y) &= y + 1 \\ N(x, y) &= x - y(y+1)^2\end{aligned}$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Using result found above gives

$$\begin{aligned}\frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}(y+1) \\ &= 1\end{aligned}$$

And

$$\begin{aligned}\frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}\left(x - y(y+1)^2\right) \\ &= 1\end{aligned}$$

Since $\frac{\partial M}{\partial y}=\frac{\partial N}{\partial x}$, then the ODE is <u>exact</u> The following equations are now set up to solve for the function $\phi(x,y)$

$$\frac{\partial \phi}{\partial x}=M \tag{1}$$
$$\frac{\partial \phi}{\partial y}=N \tag{2}$$

Integrating (1) w.r.t. x gives

$$\int \frac{\partial \phi}{\partial x}\,\mathrm{d}x=\int M\,\mathrm{d}x$$
$$\int \frac{\partial \phi}{\partial x}\,\mathrm{d}x=\int y+1\,\mathrm{d}x$$
$$\phi=(y+1)\,x+f(y) \tag{3}$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$\frac{\partial \phi}{\partial y}=x+f'(y) \tag{4}$$

But equation (2) says that $\frac{\partial \phi}{\partial y}=x-y(y+1)^2$. Therefore equation (4) becomes

$$x-y(y+1)^2=x+f'(y) \tag{5}$$

Solving equation (5) for $f'(y)$ gives

$$f'(y)=-y(y+1)^2$$

Integrating the above w.r.t y gives

$$\int f'(y)\,\mathrm{d}y=\int \left(-y(y+1)^2\right)\mathrm{d}y$$
$$f(y)=-\frac{1}{4}y^4-\frac{2}{3}y^3-\frac{1}{2}y^2+c_1$$

Where c_1 is constant of integration. Substituting result found above for $f(y)$ into equation (3) gives ϕ

$$\phi=(y+1)\,x-\frac{y^4}{4}-\frac{2y^3}{3}-\frac{y^2}{2}+c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = (y+1)\,x - \frac{y^4}{4} - \frac{2y^3}{3} - \frac{y^2}{2}$$

Summary
The solution(s) found are the following

$$-\frac{y^4}{4} - \frac{2y^3}{3} - \frac{y^2}{2} + (y+1)\,x = c_1 \tag{1}$$

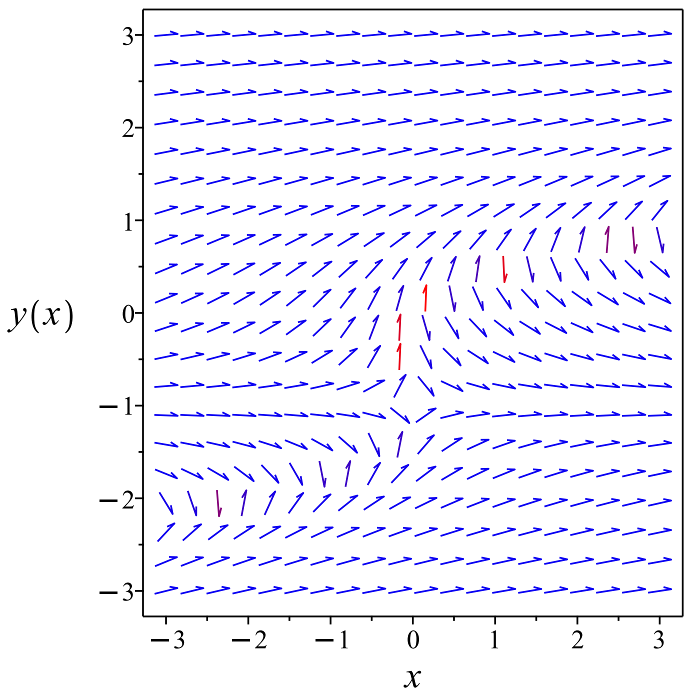


Figure 101: Slope field plot

Verification of solutions

$$-\frac{y^4}{4} - \frac{2y^3}{3} - \frac{y^2}{2} + (y+1)\,x = c_1$$

Verified OK.

1.67.4 Maple step by step solution

Let's solve

$y + \left(x - y(y+1)^2\right) y' = -1$

- Highest derivative means the order of the ODE is 1

  y'

□ Check if ODE is exact

  - ODE is exact if the lhs is the total derivative of a C^2 function

    $F'(x, y) = 0$

  - Compute derivative of lhs

    $F'(x, y) + \left(\frac{\partial}{\partial y} F(x, y)\right) y' = 0$

  - Evaluate derivatives

    $1 = 1$

  - Condition met, ODE is exact

- Exact ODE implies solution will be of this form

  $\left[F(x, y) = c_1, M(x, y) = F'(x, y), N(x, y) = \frac{\partial}{\partial y} F(x, y)\right]$

- Solve for $F(x, y)$ by integrating $M(x, y)$ with respect to x

  $F(x, y) = \int (y+1)\, dx + f_1(y)$

- Evaluate integral

  $F(x, y) = (y+1)\, x + f_1(y)$

- Take derivative of $F(x, y)$ with respect to y

  $N(x, y) = \frac{\partial}{\partial y} F(x, y)$

- Compute derivative

  $x - y(y+1)^2 = x + \frac{d}{dy} f_1(y)$

- Isolate for $\frac{d}{dy} f_1(y)$

  $\frac{d}{dy} f_1(y) = -y(y+1)^2$

- Solve for $f_1(y)$

  $f_1(y) = -\frac{1}{4} y^4 - \frac{2}{3} y^3 - \frac{1}{2} y^2$

- Substitute $f_1(y)$ into equation for $F(x, y)$

$$F(x,y) = (y+1)\,x - \frac{y^4}{4} - \frac{2y^3}{3} - \frac{y^2}{2}$$

- Substitute $F(x,y)$ into the solution of the ODE

$$(y+1)\,x - \frac{y^4}{4} - \frac{2y^3}{3} - \frac{y^2}{2} = c_1$$

- Solve for y

$$y = \mathit{RootOf}\left(3\_Z^4 + 8\_Z^3 + 6\_Z^2 - 12\_Zx + 12c_1 - 12x\right)$$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
<- 1st order linear successful
<- inverse linear successful`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 36

```
dsolve((1+y(x))+(x-y(x)*(1+y(x))^2)* diff(y(x),x)=0,y(x), singsol=all)
```

$$x + \frac{-3y(x)^4 - 8y(x)^3 - 6y(x)^2 - 12c_1}{12y(x) + 12} = 0$$

✓ Solution by Mathematica

Time used: 33.714 (sec). Leaf size: 1594

```
DSolve[(1+y[x])+(x-y[x]*(1+y[x])^2)* y'[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \rightarrow \frac{1}{6}\left(-\sqrt{\frac{-24x+6+72c_1}{\sqrt[3]{27x^2-\frac{1}{432}\sqrt{186624\left(27x^2+1+12c_1\right)^2-4(-144x+36+432c_1)^3}+1+12c_1}}+6\sqrt[3]{27x^2 -} \right.$$
$$-3\sqrt{-\frac{3\left(32x+\frac{64}{27}\right)}{4\sqrt{\frac{-24x+6+72c_1}{\sqrt[3]{27x^2-\frac{1}{432}\sqrt{186624\left(27x^2+1+12c_1\right)^2-4(-144x+36+432c_1)^3}+1+12c_1}}+6\sqrt[3]{27}}}}$$
$$\left. -4 \right)$$

$$y(x) \rightarrow \frac{1}{6}\left(-\sqrt{\frac{-24x+6+72c_1}{\sqrt[3]{27x^2-\frac{1}{432}\sqrt{186624\left(27x^2+1+12c_1\right)^2-4(-144x+36+432c_1)^3}+1+12c_1}}+6\sqrt[3]{27x^2 -} \right.$$
$$+3\sqrt{-\frac{3\left(32x+\frac{64}{27}\right)}{4\sqrt{\frac{-24x+6+72c_1}{\sqrt[3]{27x^2-\frac{1}{432}\sqrt{186624\left(27x^2+1+12c_1\right)^2-4(-144x+36+432c_1)^3}+1+12c_1}}+6\sqrt[3]{27}}}}$$
$$\left. -4 \right)$$

$y(x)$

1.68 problem 71.1

Internal problem ID [3213]
Internal file name [OUTPUT/2705_Sunday_June_05_2022_08_39_01_AM_70722247/index.tex]

Book: Differential equations for engineers by Wei-Chau XIE, Cambridge Press 2010
Section: Chapter 2. First-Order and Simple Higher-Order Differential Equations. Page 78
Problem number: 71.1.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program :

Maple gives the following as the ode type

```
[_Riccati]
```

$$\boxed{y' + y^2 = x^2 + 1}$$

1.68.1 Solving as riccati ode

In canonical form the ODE is

$$\begin{aligned} y' &= F(x,y) \\ &= x^2 - y^2 + 1 \end{aligned}$$

This is a Riccati ODE. Comparing the ODE to solve

$$y' = x^2 - y^2 + 1$$

With Riccati ODE standard form

$$y' = f_0(x) + f_1(x)y + f_2(x)y^2$$

Shows that $f_0(x) = x^2 + 1$, $f_1(x) = 0$ and $f_2(x) = -1$. Let

$$\begin{aligned} y &= \frac{-u'}{f_2 u} \\ &= \frac{-u'}{-u} \end{aligned} \tag{1}$$

Using the above substitution in the given ODE results (after some simplification)in a second order ODE to solve for $u(x)$ which is

$$f_2 u''(x) - (f_2' + f_1 f_2) u'(x) + f_2^2 f_0 u(x) = 0 \tag{2}$$

But

$$\begin{aligned} f_2' &= 0 \\ f_1 f_2 &= 0 \\ f_2^2 f_0 &= x^2 + 1 \end{aligned}$$

Substituting the above terms back in equation (2) gives

$$-u''(x) + (x^2 + 1) u(x) = 0$$

Solving the above ODE (this ode solved using Maple, not this program), gives

$$u(x) = \mathrm{e}^{\frac{x^2}{2}} (c_1 + \mathrm{erf}(x) c_2)$$

The above shows that

$$u'(x) = \frac{\left(x\sqrt{\pi}\,(c_1 + \mathrm{erf}(x) c_2)\,\mathrm{e}^{x^2} + 2c_2\right)\mathrm{e}^{-\frac{x^2}{2}}}{\sqrt{\pi}}$$

Using the above in (1) gives the solution

$$y = \frac{\left(x\sqrt{\pi}\,(c_1 + \mathrm{erf}(x) c_2)\,\mathrm{e}^{x^2} + 2c_2\right)\left(\mathrm{e}^{-\frac{x^2}{2}}\right)^2}{\sqrt{\pi}\,(c_1 + \mathrm{erf}(x) c_2)}$$

Dividing both numerator and denominator by c_1 gives, after renaming the constant $\frac{c_2}{c_1} = c_3$ the following solution

$$y = \frac{2\,\mathrm{e}^{-x^2} + x\sqrt{\pi}\,(c_3 + \mathrm{erf}(x))}{\sqrt{\pi}\,(c_3 + \mathrm{erf}(x))}$$

Summary

The solution(s) found are the following

$$y = \frac{2\,\mathrm{e}^{-x^2} + x\sqrt{\pi}\,(c_3 + \mathrm{erf}(x))}{\sqrt{\pi}\,(c_3 + \mathrm{erf}(x))} \tag{1}$$

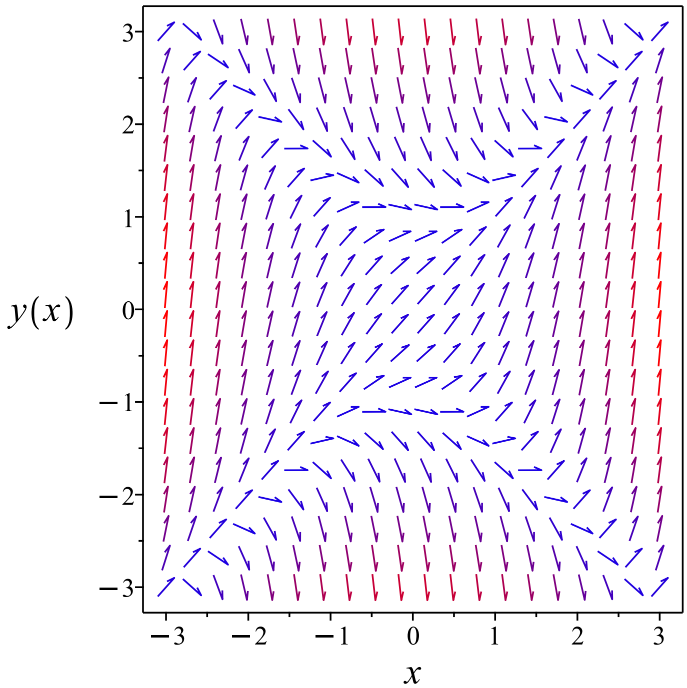


Figure 102: Slope field plot

Verification of solutions

$$y = \frac{2\,e^{-x^2} + x\sqrt{\pi}\,(c_3 + \operatorname{erf}(x))}{\sqrt{\pi}\,(c_3 + \operatorname{erf}(x))}$$

Verified OK.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
trying homogeneous types:
trying Chini
differential order: 1; looking for linear symmetries
trying exact
Looking for potential symmetries
trying Riccati
trying Riccati Special
trying Riccati sub-methods:
   <- Riccati particular polynomial solution successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 37

```
dsolve(diff(y(x),x)+y(x)^2=1+x^2,y(x), singsol=all)
```

$$y(x) = \frac{\sqrt{\pi}\, \operatorname{erf}(x)\, x - 2c_1 x + 2\,\mathrm{e}^{-x^2}}{\sqrt{\pi}\, \operatorname{erf}(x) - 2c_1}$$

✓ Solution by Mathematica

Time used: 0.136 (sec). Leaf size: 36

```
DSolve[y'[x]+y[x]^2==1+x^2,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to x + \frac{2e^{-x^2}}{\sqrt{\pi}\operatorname{erf}(x) + 2c_1}$$
$$y(x) \to x$$

1.69 problem 72

Internal problem ID [3214]
Internal file name [OUTPUT/2706_Sunday_June_05_2022_08_39_01_AM_942369/index.tex]

Book: Differential equations for engineers by Wei-Chau XIE, Cambridge Press 2010
Section: Chapter 2. First-Order and Simple Higher-Order Differential Equations. Page 78
Problem number: 72.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program :

Maple gives the following as the ode type

```
[_Bernoulli]
```

$$\boxed{3xy' - 3xy^4 \ln(x) - y = 0}$$

1.69.1 Solving as bernoulli ode

In canonical form, the ODE is

$$\begin{aligned} y' &= F(x,y) \\ &= \frac{y(3x\,y^3 \ln(x) + 1)}{3x} \end{aligned}$$

This is a Bernoulli ODE.

$$y' = \frac{1}{3x}y + \ln(x)\,y^4 \tag{1}$$

The standard Bernoulli ODE has the form

$$y' = f_0(x)y + f_1(x)y^n \tag{2}$$

The first step is to divide the above equation by y^n which gives

$$\frac{y'}{y^n} = f_0(x)y^{1-n} + f_1(x) \tag{3}$$

The next step is use the substitution $w = y^{1-n}$ in equation (3) which generates a new ODE in $w(x)$ which will be linear and can be easily solved using an integrating factor. Backsubstitution then gives the solution $y(x)$ which is what we want.

This method is now applied to the ODE at hand. Comparing the ODE (1) With (2) Shows that

$$\begin{aligned}
f_0(x) &= \frac{1}{3x} \\
f_1(x) &= \ln(x) \\
n &= 4
\end{aligned}$$

Dividing both sides of ODE (1) by $y^n = y^4$ gives

$$y' \frac{1}{y^4} = \frac{1}{3x\,y^3} + \ln(x) \tag{4}$$

Let

$$\begin{aligned}
w &= y^{1-n} \\
&= \frac{1}{y^3}
\end{aligned} \tag{5}$$

Taking derivative of equation (5) w.r.t x gives

$$w' = -\frac{3}{y^4} y' \tag{6}$$

Substituting equations (5) and (6) into equation (4) gives

$$\begin{aligned}
-\frac{w'(x)}{3} &= \frac{w(x)}{3x} + \ln(x) \\
w' &= -\frac{w}{x} - 3\ln(x)
\end{aligned} \tag{7}$$

The above now is a linear ODE in $w(x)$ which is now solved.

Entering Linear first order ODE solver. In canonical form a linear first order is

$$w'(x) + p(x)w(x) = q(x)$$

Where here

$$\begin{aligned}
p(x) &= \frac{1}{x} \\
q(x) &= -3\ln(x)
\end{aligned}$$

Hence the ode is

$$w'(x) + \frac{w(x)}{x} = -3\ln(x)$$

The integrating factor μ is

$$\mu = e^{\int \frac{1}{x}dx} \\ = x$$

The ode becomes

$$\frac{d}{dx}(\mu w) = (\mu)(-3\ln(x))$$
$$\frac{d}{dx}(xw) = (x)(-3\ln(x))$$
$$d(xw) = (-3\ln(x)\,x)\,dx$$

Integrating gives

$$xw = \int -3\ln(x)\,x\,dx$$
$$xw = -\frac{3\ln(x)\,x^2}{2} + \frac{3x^2}{4} + c_1$$

Dividing both sides by the integrating factor $\mu = x$ results in

$$w(x) = \frac{-\frac{3\ln(x)x^2}{2} + \frac{3x^2}{4}}{x} + \frac{c_1}{x}$$

which simplifies to

$$w(x) = \frac{-6\ln(x)\,x^2 + 3x^2 + 4c_1}{4x}$$

Replacing w in the above by $\frac{1}{y^3}$ using equation (5) gives the final solution.

$$\frac{1}{y^3} = \frac{-6\ln(x)\,x^2 + 3x^2 + 4c_1}{4x}$$

Solving for y gives

$$y(x) = \frac{2^{\frac{2}{3}}\left(-x(6\ln(x)\,x^2 - 3x^2 - 4c_1)^2\right)^{\frac{1}{3}}}{6\ln(x)\,x^2 - 3x^2 - 4c_1}$$
$$y(x) = -\frac{2^{\frac{2}{3}}\left(-x(6\ln(x)\,x^2 - 3x^2 - 4c_1)^2\right)^{\frac{1}{3}}(1 + i\sqrt{3})}{12\ln(x)\,x^2 - 6x^2 - 8c_1}$$
$$y(x) = \frac{2^{\frac{2}{3}}\left(-x(6\ln(x)\,x^2 - 3x^2 - 4c_1)^2\right)^{\frac{1}{3}}(-1 + i\sqrt{3})}{12\ln(x)\,x^2 - 6x^2 - 8c_1}$$

Summary

The solution(s) found are the following

$$y = \frac{2^{\frac{2}{3}} \left(-x(6 \ln(x) x^2 - 3x^2 - 4c_1)^2\right)^{\frac{1}{3}}}{6 \ln(x) x^2 - 3x^2 - 4c_1} \tag{1}$$

$$y = -\frac{2^{\frac{2}{3}} \left(-x(6 \ln(x) x^2 - 3x^2 - 4c_1)^2\right)^{\frac{1}{3}} (1 + i\sqrt{3})}{12 \ln(x) x^2 - 6x^2 - 8c_1} \tag{2}$$

$$y = \frac{2^{\frac{2}{3}} \left(-x(6 \ln(x) x^2 - 3x^2 - 4c_1)^2\right)^{\frac{1}{3}} (-1 + i\sqrt{3})}{12 \ln(x) x^2 - 6x^2 - 8c_1} \tag{3}$$

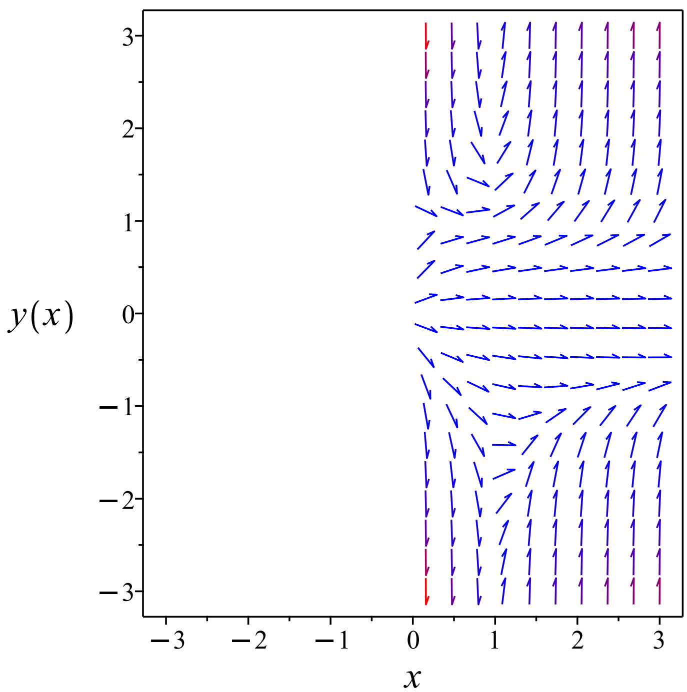


Figure 103: Slope field plot

Verification of solutions

$$y = \frac{2^{\frac{2}{3}} \left(-x(6 \ln (x) x^2 - 3x^2 - 4c_1)^2\right)^{\frac{1}{3}}}{6 \ln (x) x^2 - 3x^2 - 4c_1}$$

Verified OK.

$$y = -\frac{2^{\frac{2}{3}} \left(-x(6 \ln (x) x^2 - 3x^2 - 4c_1)^2\right)^{\frac{1}{3}} (1 + i\sqrt{3})}{12 \ln (x) x^2 - 6x^2 - 8c_1}$$

Verified OK.

$$y = \frac{2^{\frac{2}{3}} \left(-x(6 \ln (x) x^2 - 3x^2 - 4c_1)^2\right)^{\frac{1}{3}} (-1 + i\sqrt{3})}{12 \ln (x) x^2 - 6x^2 - 8c_1}$$

Verified OK.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
<- Bernoulli successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 162

```
dsolve(3*x*diff(y(x),x)-3*x*y(x)^4*ln(x)-y(x)=0,y(x), singsol=all)
```

$$y(x) = \frac{2^{\frac{2}{3}} \left(-x(6 \ln (x) x^2 - 3x^2 - 4c_1)^2\right)^{\frac{1}{3}}}{6 \ln (x) x^2 - 3x^2 - 4c_1}$$

$$y(x) = -\frac{2^{\frac{2}{3}} \left(-x(6 \ln (x) x^2 - 3x^2 - 4c_1)^2\right)^{\frac{1}{3}} (1 + i\sqrt{3})}{12 \ln (x) x^2 - 6x^2 - 8c_1}$$

$$y(x) = \frac{2^{\frac{2}{3}} \left(-x(6 \ln (x) x^2 - 3x^2 - 4c_1)^2\right)^{\frac{1}{3}} (i\sqrt{3} - 1)}{12 \ln (x) x^2 - 6x^2 - 8c_1}$$

✓ Solution by Mathematica

Time used: 0.25 (sec). Leaf size: 120

```
DSolve[3*x*y'[x]-3*x*y[x]^4*Log[x]-y[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \frac{(-2)^{2/3}\sqrt[3]{x}}{\sqrt[3]{3x^2 - 6x^2\log(x) + 4c_1}}$$
$$y(x) \to \frac{2^{2/3}\sqrt[3]{x}}{\sqrt[3]{3x^2 - 6x^2\log(x) + 4c_1}}$$
$$y(x) \to -\frac{\sqrt[3]{-1}2^{2/3}\sqrt[3]{x}}{\sqrt[3]{3x^2 - 6x^2\log(x) + 4c_1}}$$
$$y(x) \to 0$$

1.70 problem 73

Internal problem ID [3215]
Internal file name [`OUTPUT/2707_Sunday_June_05_2022_08_39_02_AM_17707126/index.tex`]

Book: Differential equations for engineers by Wei-Chau XIE, Cambridge Press 2010
Section: Chapter 2. First-Order and Simple Higher-Order Differential Equations. Page 78
Problem number: 73.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"exactWithIntegrationFactor", "first_order_ode_lie_symmetry_calculated"**

Maple gives the following as the ode type

```
[[_homogeneous, `class G`], _rational, [_Abel, `2nd type`, `
   class B`]]
```

$$\boxed{y' - \frac{4x^3y^2}{yx^4 + 2} = 0}$$

1.70.1 Solving as first order ode lie symmetry calculated ode

Writing the ode as

$$y' = \frac{4x^3y^2}{x^4y + 2}$$
$$y' = \omega(x, y)$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x + \omega(\eta_y - \xi_x) - \omega^2\xi_y - \omega_x\xi - \omega_y\eta = 0 \tag{A}$$

The type of this ode is not in the lookup table. To determine ξ, η then (A) is solved using ansatz. Making bivariate polynomials of degree 1 to use as anstaz gives

$$\xi = xa_2 + ya_3 + a_1 \tag{1E}$$
$$\eta = xb_2 + yb_3 + b_1 \tag{2E}$$

Where the unknown coefficients are

$$\{a_1, a_2, a_3, b_1, b_2, b_3\}$$

Substituting equations (1E,2E) and ω into (A) gives

$$b_2 + \frac{4x^3y^2(b_3 - a_2)}{x^4y + 2} - \frac{16x^6y^4a_3}{(x^4y+2)^2} - \left(\frac{12x^2y^2}{x^4y+2} - \frac{16x^6y^3}{(x^4y+2)^2}\right)(xa_2 + ya_3 + a_1) - \left(\frac{8x^3y}{x^4y+2} - \frac{4x^7y^2}{(x^4y+2)^2}\right)(xb_2 + yb_3 + b_1) = 0 \tag{5E}$$

Putting the above in normal form gives

$$-\frac{3x^8y^2b_2 + 12x^6y^4a_3 + 4x^7y^2b_1 - 4x^6y^3a_1 + 12x^4yb_2 + 32x^3y^2a_2 + 8x^3y^2b_3 + 24x^2y^3a_3 + 16x^3yb_1 + 24x}{(x^4y+2)^2} = 0$$

Setting the numerator to zero gives

$$-3x^8y^2b_2 - 12x^6y^4a_3 - 4x^7y^2b_1 + 4x^6y^3a_1 - 12x^4yb_2 - 32x^3y^2a_2 - 8x^3y^2b_3 - 24x^2y^3a_3 - 16x^3yb_1 - 24x^2y^2a_1 + 4b_2 = 0 \tag{6E}$$

Looking at the above PDE shows the following are all the terms with $\{x, y\}$ in them.

$$\{x, y\}$$

The following substitution is now made to be able to collect on all terms with $\{x, y\}$ in them

$$\{x = v_1, y = v_2\}$$

The above PDE (6E) now becomes

$$-12a_3v_1^6v_2^4 - 3b_2v_1^8v_2^2 + 4a_1v_1^6v_2^3 - 4b_1v_1^7v_2^2 - 32a_2v_1^3v_2^2 - 24a_3v_1^2v_2^3 - 12b_2v_1^4v_2 - 8b_3v_1^3v_2^2 - 24a_1v_1^2v_2^2 - 16b_1v_1^3v_2 + 4b_2 = 0 \tag{7E}$$

Collecting the above on the terms v_i introduced, and these are

$$\{v_1, v_2\}$$

Equation (7E) now becomes

$$
\begin{aligned}
&-3b_2 v_1^8 v_2^2 - 4b_1 v_1^7 v_2^2 - 12a_3 v_1^6 v_2^4 + 4a_1 v_1^6 v_2^3 - 12b_2 v_1^4 v_2 \\
&+ (-32a_2 - 8b_3) v_1^3 v_2^2 - 16b_1 v_1^3 v_2 - 24a_3 v_1^2 v_2^3 - 24a_1 v_1^2 v_2^2 + 4b_2 = 0
\end{aligned} \tag{8E}
$$

Setting each coefficients in (8E) to zero gives the following equations to solve

$$
\begin{aligned}
-24a_1 &= 0 \\
4a_1 &= 0 \\
-24a_3 &= 0 \\
-12a_3 &= 0 \\
-16b_1 &= 0 \\
-4b_1 &= 0 \\
-12b_2 &= 0 \\
-3b_2 &= 0 \\
4b_2 &= 0 \\
-32a_2 - 8b_3 &= 0
\end{aligned}
$$

Solving the above equations for the unknowns gives

$$
\begin{aligned}
a_1 &= 0 \\
a_2 &= a_2 \\
a_3 &= 0 \\
b_1 &= 0 \\
b_2 &= 0 \\
b_3 &= -4a_2
\end{aligned}
$$

Substituting the above solution in the anstaz (1E,2E) (using 1 as arbitrary value for any unknown in the RHS) gives

$$
\begin{aligned}
\xi &= x \\
\eta &= -4y
\end{aligned}
$$

Shifting is now applied to make $\xi = 0$ in order to simplify the rest of the computation

$$\begin{aligned}
\eta &= \eta - \omega(x, y)\,\xi \\
&= -4y - \left(\frac{4x^3y^2}{x^4y+2}\right)(x) \\
&= \frac{-8x^4y^2 - 8y}{x^4y+2} \\
\xi &= 0
\end{aligned}$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x, y) \to (R, S)$ where (R, S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}$$

The above comes from the requirements that $\left(\xi\frac{\partial}{\partial x} + \eta\frac{\partial}{\partial y}\right)S(x, y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$R = x$$

S is found from

$$\begin{aligned}
S &= \int \frac{1}{\eta} dy \\
&= \int \frac{1}{\frac{-8x^4y^2-8y}{x^4y+2}} dy
\end{aligned}$$

Which results in

$$S = \frac{\ln(x^4y+1)}{8} - \frac{\ln(y)}{4}$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x, y)S_y}{R_x + \omega(x, y)R_y} \tag{2}$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x, y)$ is the right hand side of the original ode given by

$$\omega(x, y) = \frac{4x^3y^2}{x^4y+2}$$

Evaluating all the partial derivatives gives

$$
\begin{aligned}
R_x &= 1 \\
R_y &= 0 \\
S_x &= \frac{x^3 y}{2x^4 y + 2} \\
S_y &= \frac{-x^4 y - 2}{8y\left(x^4 y + 1\right)}
\end{aligned}
$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$
\frac{dS}{dR} = 0 \tag{2A}
$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$
\frac{dS}{dR} = 0
$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$
S(R) = c_1 \tag{4}
$$

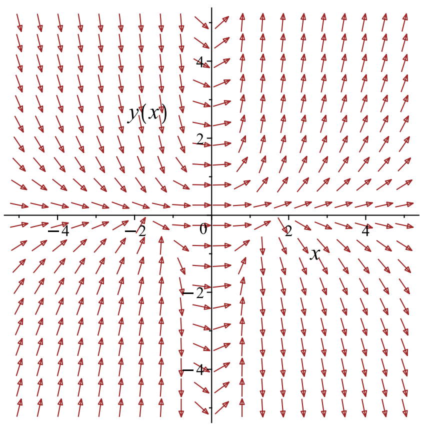
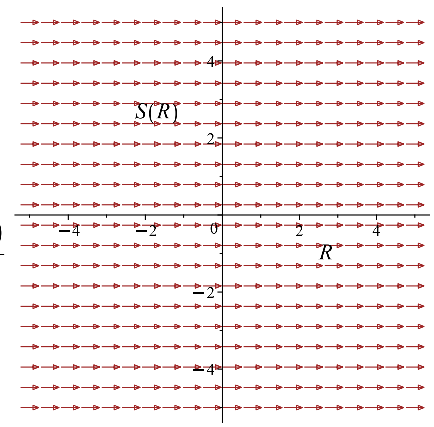
To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$
\frac{\ln\left(y x^4 + 1\right)}{8} - \frac{\ln\left(y\right)}{4} = c_1
$$

Which simplifies to

$$
\frac{\ln\left(y x^4 + 1\right)}{8} - \frac{\ln\left(y\right)}{4} = c_1
$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in x, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dx} = \frac{4x^3y^2}{x^4y+2}$ 	$R = x$ $S = \frac{\ln(x^4y+1)}{8} - \frac{\ln(y)}{4}$	$\frac{dS}{dR} = 0$ 

Summary

The solution(s) found are the following

$$\frac{\ln(yx^4+1)}{8} - \frac{\ln(y)}{4} = c_1 \tag{1}$$

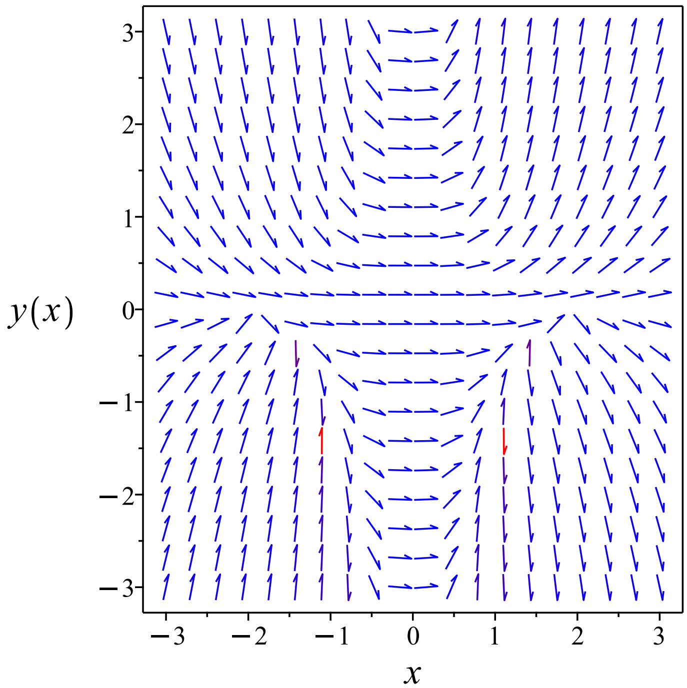


Figure 104: Slope field plot

Verification of solutions

$$\frac{\ln\left(y x^{4}+1\right)}{8}-\frac{\ln\left(y\right)}{4}=c_{1}$$

Verified OK.

1.70.2 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x,y)+N(x,y)\frac{dy}{dx}=0 \tag{A}$$

We assume there exists a function $\phi(x,y)=c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y)=0$$

Hence

$$\frac{\partial\phi}{\partial x}+\frac{\partial\phi}{\partial y}\frac{dy}{dx}=0 \tag{B}$$

Comparing (A,B) shows that

$$\frac{\partial \phi}{\partial x} = M$$
$$\frac{\partial \phi}{\partial y} = N$$

But since $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x, y)$ but at least we know now that we can do that since the condition $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x, y)\,\mathrm{d}x + N(x, y)\,\mathrm{d}y = 0 \tag{1A}$$

Therefore

$$\begin{aligned}
(x^4 y + 2)\,\mathrm{d}y &= (4x^3 y^2)\,\mathrm{d}x \\
(-4x^3 y^2)\,\mathrm{d}x + (x^4 y + 2)\,\mathrm{d}y &= 0 \qquad (2A)
\end{aligned}$$

Comparing (1A) and (2A) shows that

$$\begin{aligned}
M(x, y) &= -4x^3 y^2 \\
N(x, y) &= x^4 y + 2
\end{aligned}$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Using result found above gives

$$\begin{aligned}
\frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}(-4x^3 y^2) \\
&= -8x^3 y
\end{aligned}$$

And

$$\begin{aligned}
\frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}(x^4 y + 2) \\
&= 4x^3 y
\end{aligned}$$

Since $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$, then the ODE is not exact. Since the ODE is not exact, we will try to find an integrating factor to make it exact. Let

$$\begin{aligned} A &= \frac{1}{N}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right) \\ &= \frac{1}{x^4 y + 2}\left((-8x^3 y) - (4x^3 y)\right) \\ &= -\frac{12x^3 y}{x^4 y + 2} \end{aligned}$$

Since A depends on y, it can not be used to obtain an integrating factor. We will now try a second method to find an integrating factor. Let

$$\begin{aligned} B &= \frac{1}{M}\left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) \\ &= -\frac{1}{4x^3 y^2}\left((4x^3 y) - (-8x^3 y)\right) \\ &= -\frac{3}{y} \end{aligned}$$

Since B does not depend on x, it can be used to obtain an integrating factor. Let the integrating factor be μ. Then

$$\begin{aligned} \mu &= e^{\int B \, \mathrm{d}y} \\ &= e^{\int -\frac{3}{y} \, \mathrm{d}y} \end{aligned}$$

The result of integrating gives

$$\begin{aligned} \mu &= e^{-3\ln(y)} \\ &= \frac{1}{y^3} \end{aligned}$$

M and N are now multiplied by this integrating factor, giving new M and new N which are called $\overline{M}$ and $\overline{N}$ so not to confuse them with the original M and N.

$$\begin{aligned} \overline{M} &= \mu M \\ &= \frac{1}{y^3}\left(-4x^3 y^2\right) \\ &= -\frac{4x^3}{y} \end{aligned}$$

And

$$
\begin{aligned}
\overline{N} &= \mu N \\
&= \frac{1}{y^3}\left(x^4 y + 2\right) \\
&= \frac{x^4 y + 2}{y^3}
\end{aligned}
$$

So now a modified ODE is obtained from the original ODE which will be exact and can be solved using the standard method. The modified ODE is

$$
\begin{aligned}
\overline{M} + \overline{N}\frac{\mathrm{d}y}{\mathrm{d}x} &= 0 \\
\left(-\frac{4x^3}{y}\right) + \left(\frac{x^4 y + 2}{y^3}\right)\frac{\mathrm{d}y}{\mathrm{d}x} &= 0
\end{aligned}
$$

The following equations are now set up to solve for the function $\phi(x, y)$

$$
\frac{\partial \phi}{\partial x} = \overline{M} \tag{1}
$$

$$
\frac{\partial \phi}{\partial y} = \overline{N} \tag{2}
$$

Integrating (1) w.r.t. x gives

$$
\begin{aligned}
\int \frac{\partial \phi}{\partial x}\,\mathrm{d}x &= \int \overline{M}\,\mathrm{d}x \\
\int \frac{\partial \phi}{\partial x}\,\mathrm{d}x &= \int -\frac{4x^3}{y}\,\mathrm{d}x
\end{aligned}
$$

$$
\phi = -\frac{x^4}{y} + f(y) \tag{3}
$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$
\frac{\partial \phi}{\partial y} = \frac{x^4}{y^2} + f'(y) \tag{4}
$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = \frac{x^4 y+2}{y^3}$. Therefore equation (4) becomes

$$
\frac{x^4 y + 2}{y^3} = \frac{x^4}{y^2} + f'(y) \tag{5}
$$

Solving equation (5) for $f'(y)$ gives

$$f'(y) = \frac{2}{y^3}$$

Integrating the above w.r.t y gives

$$\int f'(y)\,\mathrm{d}y = \int \left(\frac{2}{y^3}\right) \mathrm{d}y$$
$$f(y) = -\frac{1}{y^2} + c_1$$

Where c_1 is constant of integration. Substituting result found above for $f(y)$ into equation (3) gives ϕ

$$\phi = -\frac{x^4}{y} - \frac{1}{y^2} + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = -\frac{x^4}{y} - \frac{1}{y^2}$$

<u>Summary</u>
The solution(s) found are the following

$$-\frac{x^4}{y} - \frac{1}{y^2} = c_1 \tag{1}$$

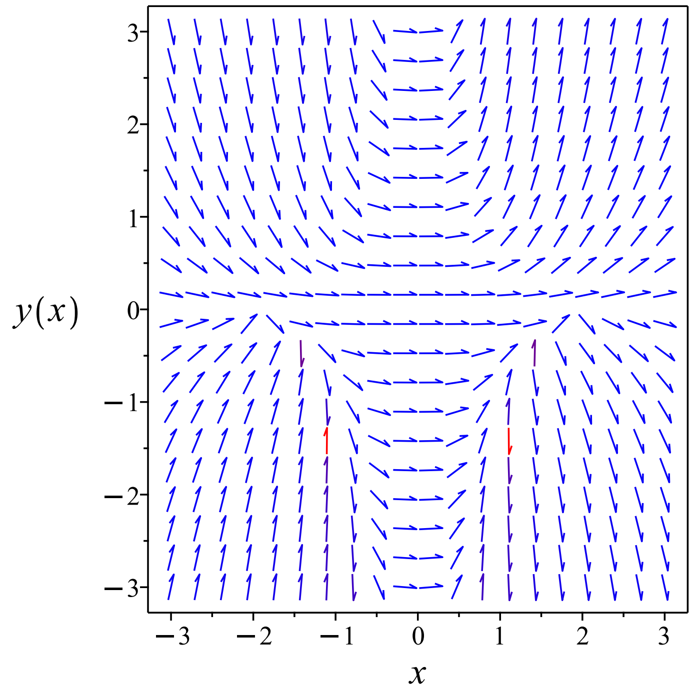


Figure 105: Slope field plot

Verification of solutions

$$-\frac{x^4}{y} - \frac{1}{y^2} = c_1$$

Verified OK.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
trying homogeneous types:
trying homogeneous G
<- homogeneous successful`
```

✓ Solution by Maple

Time used: 0.594 (sec). Leaf size: 45

```
dsolve(diff(y(x),x)=(4*x^3*y(x)^2)/(x^4*y(x)+2),y(x), singsol=all)
```

$$y(x) = \frac{x^4 - \sqrt{x^8 + 4c_1}}{2c_1}$$
$$y(x) = \frac{x^4 + \sqrt{x^8 + 4c_1}}{2c_1}$$

✓ Solution by Mathematica

Time used: 0.409 (sec). Leaf size: 56

```
DSolve[y'[x]==(4*x^3*y[x]^2)/(x^4*y[x]+2),y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \frac{2}{-x^4 + \sqrt{x^8 + 4c_1}}$$
$$y(x) \to -\frac{2}{x^4 + \sqrt{x^8 + 4c_1}}$$
$$y(x) \to 0$$

1.71 problem 74

Internal problem ID [3216]
Internal file name [OUTPUT/2708_Sunday_June_05_2022_08_39_03_AM_61626433/index.tex]

Book: Differential equations for engineers by Wei-Chau XIE, Cambridge Press 2010
Section: Chapter 2. First-Order and Simple Higher-Order Differential Equations. Page 78
Problem number: 74.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program :

Maple gives the following as the ode type

```
[_rational, _Bernoulli]
```

$$\boxed{y(6y^2 - x - 1) + 2xy' = 0}$$

1.71.1 Solving as bernoulli ode

In canonical form, the ODE is

$$\begin{aligned} y' &= F(x, y) \\ &= -\frac{y(6y^2 - x - 1)}{2x} \end{aligned}$$

This is a Bernoulli ODE.

$$y' = -\frac{-x-1}{2x}y - \frac{3}{x}y^3 \tag{1}$$

The standard Bernoulli ODE has the form

$$y' = f_0(x)y + f_1(x)y^n \tag{2}$$

The first step is to divide the above equation by y^n which gives

$$\frac{y'}{y^n} = f_0(x)y^{1-n} + f_1(x) \tag{3}$$

The next step is use the substitution $w = y^{1-n}$ in equation (3) which generates a new ODE in $w(x)$ which will be linear and can be easily solved using an integrating factor. Backsubstitution then gives the solution $y(x)$ which is what we want.

This method is now applied to the ODE at hand. Comparing the ODE (1) With (2) Shows that

$$
\begin{aligned}
f_0(x) &= -\frac{-x-1}{2x} \\
f_1(x) &= -\frac{3}{x} \\
n &= 3
\end{aligned}
$$

Dividing both sides of ODE (1) by $y^n = y^3$ gives

$$
y'\frac{1}{y^3} = -\frac{-x-1}{2x\,y^2} - \frac{3}{x} \tag{4}
$$

Let

$$
\begin{aligned}
w &= y^{1-n} \\
&= \frac{1}{y^2}
\end{aligned} \tag{5}
$$

Taking derivative of equation (5) w.r.t x gives

$$
w' = -\frac{2}{y^3}y' \tag{6}
$$

Substituting equations (5) and (6) into equation (4) gives

$$
\begin{aligned}
-\frac{w'(x)}{2} &= -\frac{(-x-1)\,w(x)}{2x} - \frac{3}{x} \\
w' &= \frac{(-x-1)\,w}{x} + \frac{6}{x}
\end{aligned} \tag{7}
$$

The above now is a linear ODE in $w(x)$ which is now solved.

Entering Linear first order ODE solver. In canonical form a linear first order is

$$
w'(x) + p(x)w(x) = q(x)
$$

Where here

$$
\begin{aligned}
p(x) &= -\frac{-x-1}{x} \\
q(x) &= \frac{6}{x}
\end{aligned}
$$

Hence the ode is

$$
w'(x) - \frac{(-x-1)\,w(x)}{x} = \frac{6}{x}
$$

The integrating factor μ is

$$\mu = \mathrm{e}^{\int -\frac{-x-1}{x}dx}$$
$$= \mathrm{e}^{x+\ln(x)}$$

Which simplifies to

$$\mu = x\,\mathrm{e}^{x}$$

The ode becomes

$$\frac{\mathrm{d}}{\mathrm{d}x}(\mu w) = (\mu)\left(\frac{6}{x}\right)$$
$$\frac{\mathrm{d}}{\mathrm{d}x}(x\,\mathrm{e}^{x} w) = (x\,\mathrm{e}^{x})\left(\frac{6}{x}\right)$$
$$\mathrm{d}(x\,\mathrm{e}^{x} w) = (6\,\mathrm{e}^{x})\ \mathrm{d}x$$

Integrating gives

$$x\,\mathrm{e}^{x} w = \int 6\,\mathrm{e}^{x}\,\mathrm{d}x$$
$$x\,\mathrm{e}^{x} w = 6\,\mathrm{e}^{x} + c_1$$

Dividing both sides by the integrating factor $\mu = x\,\mathrm{e}^{x}$ results in

$$w(x) = \frac{6\,\mathrm{e}^{-x}\mathrm{e}^{x}}{x} + \frac{c_1\mathrm{e}^{-x}}{x}$$

which simplifies to

$$w(x) = \frac{c_1\mathrm{e}^{-x} + 6}{x}$$

Replacing w in the above by $\frac{1}{y^2}$ using equation (5) gives the final solution.

$$\frac{1}{y^2} = \frac{c_1\mathrm{e}^{-x} + 6}{x}$$

Solving for y gives

$$y(x) = \frac{\sqrt{(c_1\mathrm{e}^{-x} + 6)\,x}}{c_1\mathrm{e}^{-x} + 6}$$
$$y(x) = -\frac{\sqrt{(c_1\mathrm{e}^{-x} + 6)\,x}}{c_1\mathrm{e}^{-x} + 6}$$

Summary
The solution(s) found are the following

$$y = \frac{\sqrt{(c_1 e^{-x} + 6) x}}{c_1 e^{-x} + 6} \tag{1}$$

$$y = -\frac{\sqrt{(c_1 e^{-x} + 6) x}}{c_1 e^{-x} + 6} \tag{2}$$

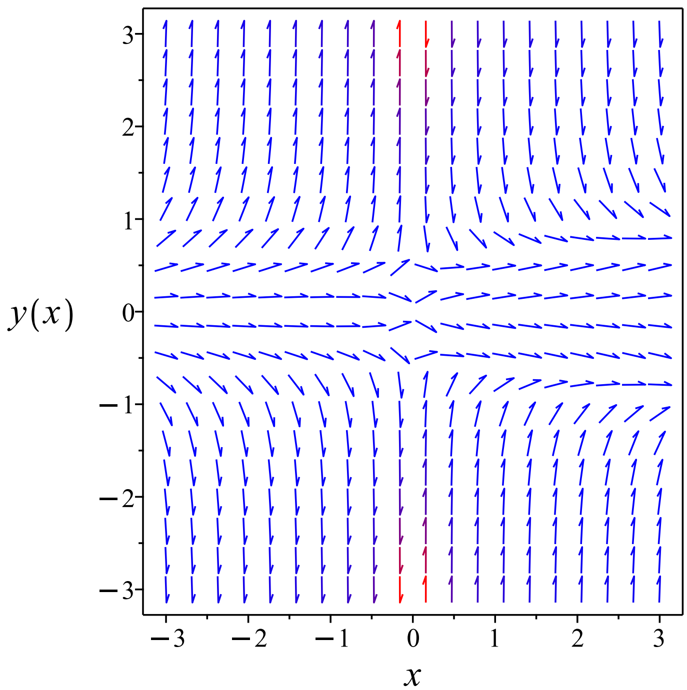


Figure 106: Slope field plot

Verification of solutions

$$y = \frac{\sqrt{(c_1 e^{-x} + 6) x}}{c_1 e^{-x} + 6}$$

Verified OK.

$$y = -\frac{\sqrt{(c_1 e^{-x} + 6) x}}{c_1 e^{-x} + 6}$$

Verified OK.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
<- Bernoulli successful`
```

✓ Solution by Maple

Time used: 0.031 (sec). Leaf size: 54

```
dsolve(y(x)*(6*y(x)^2-x-1)+2*x*diff(y(x),x)=0,y(x), singsol=all)
```

$$y(x) = \frac{\sqrt{(\mathrm{e}^{-x} c_1 + 6)\, x}}{\mathrm{e}^{-x} c_1 + 6}$$
$$y(x) = -\frac{\sqrt{(\mathrm{e}^{-x} c_1 + 6)\, x}}{\mathrm{e}^{-x} c_1 + 6}$$

✓ Solution by Mathematica

Time used: 0.709 (sec). Leaf size: 65

```
DSolve[y[x]*(6*y[x]^2-x-1)+2*x*y'[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to -\frac{e^{x/2}\sqrt{x}}{\sqrt{6e^x + c_1}}$$
$$y(x) \to \frac{e^{x/2}\sqrt{x}}{\sqrt{6e^x + c_1}}$$
$$y(x) \to 0$$

1.72 problem 75

Internal problem ID [3217]
Internal file name [OUTPUT/2709_Sunday_June_05_2022_08_39_04_AM_11853043/index.tex]

Book: Differential equations for engineers by Wei-Chau XIE, Cambridge Press 2010
Section: Chapter 2. First-Order and Simple Higher-Order Differential Equations. Page 78
Problem number: 75.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program :

Maple gives the following as the ode type

```
[[_1st_order, _with_linear_symmetries], _rational, _Bernoulli]
```

$$\boxed{(x+1)\left(y'+y^2\right)-y=0}$$

1.72.1 Solving as bernoulli ode

In canonical form, the ODE is

$$\begin{aligned} y' &= F(x,y) \\ &= -\frac{y(xy+y-1)}{x+1} \end{aligned}$$

This is a Bernoulli ODE.

$$y' = \frac{1}{x+1}y - y^2 \tag{1}$$

The standard Bernoulli ODE has the form

$$y' = f_0(x)y + f_1(x)y^n \tag{2}$$

The first step is to divide the above equation by y^n which gives

$$\frac{y'}{y^n} = f_0(x)y^{1-n} + f_1(x) \tag{3}$$

The next step is use the substitution $w = y^{1-n}$ in equation (3) which generates a new ODE in $w(x)$ which will be linear and can be easily solved using an integrating factor. Backsubstitution then gives the solution $y(x)$ which is what we want.

This method is now applied to the ODE at hand. Comparing the ODE (1) With (2) Shows that

$$\begin{aligned}
f_0(x) &= \frac{1}{x+1} \\
f_1(x) &= -1 \\
n &= 2
\end{aligned}$$

Dividing both sides of ODE (1) by $y^n = y^2$ gives

$$y' \frac{1}{y^2} = \frac{1}{(x+1)\, y} - 1 \tag{4}$$

Let

$$\begin{aligned}
w &= y^{1-n} \\
&= \frac{1}{y}
\end{aligned} \tag{5}$$

Taking derivative of equation (5) w.r.t x gives

$$w' = -\frac{1}{y^2} y' \tag{6}$$

Substituting equations (5) and (6) into equation (4) gives

$$\begin{aligned}
-w'(x) &= \frac{w(x)}{x+1} - 1 \\
w' &= -\frac{w}{x+1} + 1
\end{aligned} \tag{7}$$

The above now is a linear ODE in $w(x)$ which is now solved.

Entering Linear first order ODE solver. In canonical form a linear first order is

$$w'(x) + p(x)w(x) = q(x)$$

Where here

$$\begin{aligned}
p(x) &= \frac{1}{x+1} \\
q(x) &= 1
\end{aligned}$$

Hence the ode is

$$w'(x) + \frac{w(x)}{x+1} = 1$$

The integrating factor μ is

$$\mu = e^{\int \frac{1}{x+1}dx}$$
$$= x + 1$$

The ode becomes

$$\frac{d}{dx}(\mu w) = \mu$$
$$\frac{d}{dx}((x+1)\,w) = x + 1$$
$$d((x+1)\,w) = x + 1 dx$$

Integrating gives

$$(x+1)\,w = \int x + 1\,dx$$
$$(x+1)\,w = x + \frac{1}{2}x^2 + c_1$$

Dividing both sides by the integrating factor $\mu = x + 1$ results in

$$w(x) = \frac{x + \frac{1}{2}x^2}{x+1} + \frac{c_1}{x+1}$$

which simplifies to

$$w(x) = \frac{x^2 + 2c_1 + 2x}{2 + 2x}$$

Replacing w in the above by $\frac{1}{y}$ using equation (5) gives the final solution.

$$\frac{1}{y} = \frac{x^2 + 2c_1 + 2x}{2 + 2x}$$

Or

$$y = \frac{2 + 2x}{x^2 + 2c_1 + 2x}$$

Summary
The solution(s) found are the following

$$y = \frac{2 + 2x}{x^2 + 2c_1 + 2x} \tag{1}$$

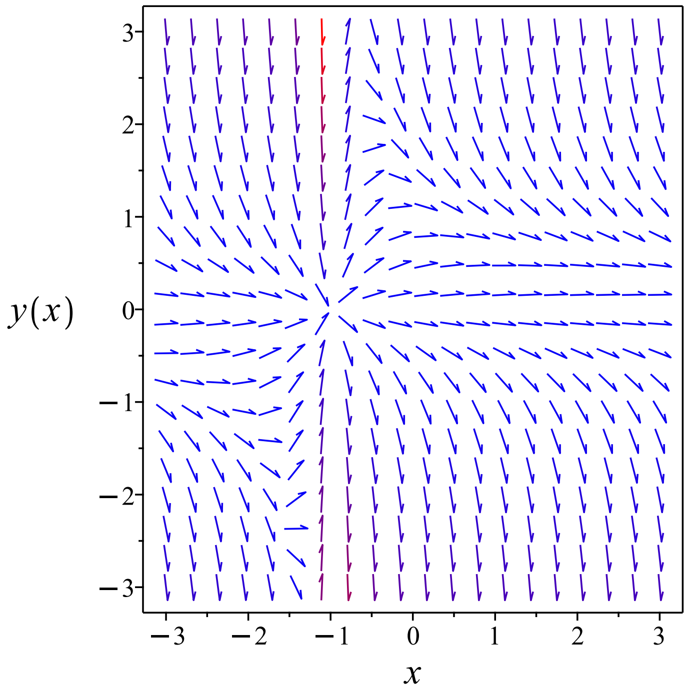


Figure 107: Slope field plot

Verification of solutions

$$y = \frac{2 + 2x}{x^2 + 2c_1 + 2x}$$

Verified OK.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
<- Bernoulli successful`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 22

```
dsolve((1+x)*(diff(y(x),x)+y(x)^2)-y(x)=0,y(x), singsol=all)
```

$$y(x) = \frac{2x+2}{x^2+2c_1+2x}$$

✓ Solution by Mathematica

Time used: 0.201 (sec). Leaf size: 28

```
DSolve[(1+x)*(y'[x]+y[x]^2)-y[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \frac{2(x+1)}{x^2+2x+2c_1}$$
$$y(x) \to 0$$

1.73 problem 76

Internal problem ID [3218]
Internal file name [OUTPUT/2710_Sunday_June_05_2022_08_39_05_AM_50437549/index.tex]

Book: Differential equations for engineers by Wei-Chau XIE, Cambridge Press 2010
Section: Chapter 2. First-Order and Simple Higher-Order Differential Equations. Page 78
Problem number: 76.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program :

Maple gives the following as the ode type

```
[_Bernoulli]
```

$$\boxed{xyy' + y^2 = \sin(x)}$$

1.73.1 Solving as bernoulli ode

In canonical form, the ODE is

$$\begin{aligned} y' &= F(x,y) \\ &= \frac{-y^2 + \sin(x)}{xy} \end{aligned}$$

This is a Bernoulli ODE.

$$y' = -\frac{1}{x}y + \frac{\sin(x)}{x}\frac{1}{y} \tag{1}$$

The standard Bernoulli ODE has the form

$$y' = f_0(x)y + f_1(x)y^n \tag{2}$$

The first step is to divide the above equation by y^n which gives

$$\frac{y'}{y^n} = f_0(x)y^{1-n} + f_1(x) \tag{3}$$

The next step is use the substitution $w = y^{1-n}$ in equation (3) which generates a new ODE in $w(x)$ which will be linear and can be easily solved using an integrating factor. Backsubstitution then gives the solution $y(x)$ which is what we want.

This method is now applied to the ODE at hand. Comparing the ODE (1) With (2) Shows that

$$
\begin{aligned}
f_0(x) &= -\frac{1}{x} \\
f_1(x) &= \frac{\sin(x)}{x} \\
n &= -1
\end{aligned}
$$

Dividing both sides of ODE (1) by $y^n = \frac{1}{y}$ gives

$$
y'y = -\frac{y^2}{x} + \frac{\sin(x)}{x} \tag{4}
$$

Let

$$
\begin{aligned}
w &= y^{1-n} \\
&= y^2
\end{aligned} \tag{5}
$$

Taking derivative of equation (5) w.r.t x gives

$$
w' = 2yy' \tag{6}
$$

Substituting equations (5) and (6) into equation (4) gives

$$
\begin{aligned}
\frac{w'(x)}{2} &= -\frac{w(x)}{x} + \frac{\sin(x)}{x} \\
w' &= -\frac{2w}{x} + \frac{2\sin(x)}{x}
\end{aligned} \tag{7}
$$

The above now is a linear ODE in $w(x)$ which is now solved.

Entering Linear first order ODE solver. In canonical form a linear first order is

$$
w'(x) + p(x)w(x) = q(x)
$$

Where here

$$
\begin{aligned}
p(x) &= \frac{2}{x} \\
q(x) &= \frac{2\sin(x)}{x}
\end{aligned}
$$

Hence the ode is

$$
w'(x) + \frac{2w(x)}{x} = \frac{2\sin(x)}{x}
$$

The integrating factor μ is

$$\mu = e^{\int \frac{2}{x}dx} = x^2$$

The ode becomes

$$\frac{d}{dx}(\mu w) = (\mu)\left(\frac{2\sin(x)}{x}\right)$$
$$\frac{d}{dx}(x^2 w) = (x^2)\left(\frac{2\sin(x)}{x}\right)$$
$$d(x^2 w) = (2x\sin(x))\,dx$$

Integrating gives

$$x^2 w = \int 2x\sin(x)\,dx$$
$$x^2 w = 2\sin(x) - 2\cos(x)\,x + c_1$$

Dividing both sides by the integrating factor $\mu = x^2$ results in

$$w(x) = \frac{2\sin(x) - 2\cos(x)\,x}{x^2} + \frac{c_1}{x^2}$$

which simplifies to

$$w(x) = \frac{2\sin(x) - 2\cos(x)\,x + c_1}{x^2}$$

Replacing w in the above by y^2 using equation (5) gives the final solution.

$$y^2 = \frac{2\sin(x) - 2\cos(x)\,x + c_1}{x^2}$$

Solving for y gives

$$y(x) = \frac{\sqrt{2\sin(x) - 2\cos(x)\,x + c_1}}{x}$$
$$y(x) = -\frac{\sqrt{2\sin(x) - 2\cos(x)\,x + c_1}}{x}$$

Summary

The solution(s) found are the following

$$y = \frac{\sqrt{2\sin(x) - 2\cos(x)\,x + c_1}}{x} \tag{1}$$
$$y = -\frac{\sqrt{2\sin(x) - 2\cos(x)\,x + c_1}}{x} \tag{2}$$

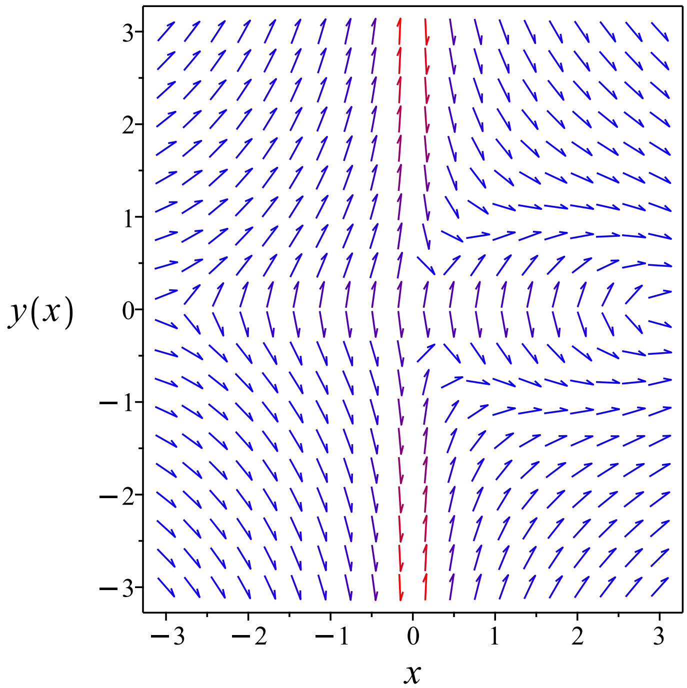


Figure 108: Slope field plot

Verification of solutions

$$y = \frac{\sqrt{2\sin(x) - 2\cos(x)\,x + c_1}}{x}$$

Verified OK.

$$y = -\frac{\sqrt{2\sin(x) - 2\cos(x)\,x + c_1}}{x}$$

Verified OK.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
<- Bernoulli successful`
```

✓ Solution by Maple

Time used: 0.047 (sec). Leaf size: 42

```
dsolve(x*y(x)*diff(y(x),x)+y(x)^2-sin(x)=0,y(x), singsol=all)
```

$$y(x) = \frac{\sqrt{2\sin(x) - 2x\cos(x) + c_1}}{x}$$
$$y(x) = -\frac{\sqrt{2\sin(x) - 2x\cos(x) + c_1}}{x}$$

✓ Solution by Mathematica

Time used: 0.367 (sec). Leaf size: 50

```
DSolve[x*y[x]*y'[x]+y[x]^2-Sin[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to -\frac{\sqrt{2\sin(x) - 2x\cos(x) + c_1}}{x}$$
$$y(x) \to \frac{\sqrt{2\sin(x) - 2x\cos(x) + c_1}}{x}$$

1.74 problem 77

Internal problem ID [3219]
Internal file name [OUTPUT/2711_Sunday_June_05_2022_08_39_06_AM_73609574/index.tex]

Book: Differential equations for engineers by Wei-Chau XIE, Cambridge Press 2010
Section: Chapter 2. First-Order and Simple Higher-Order Differential Equations. Page 78
Problem number: 77.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program :

Maple gives the following as the ode type

```
[[_homogeneous, `class D`], _rational, _Bernoulli]
```

$$\boxed{-y^4 + xy^3y' = -2x^3}$$

1.74.1 Solving as bernoulli ode

In canonical form, the ODE is

$$\begin{aligned} y' &= F(x,y) \\ &= \frac{y^4 - 2x^3}{x\,y^3} \end{aligned}$$

This is a Bernoulli ODE.

$$y' = \frac{1}{x}y - 2x^2\frac{1}{y^3} \tag{1}$$

The standard Bernoulli ODE has the form

$$y' = f_0(x)y + f_1(x)y^n \tag{2}$$

The first step is to divide the above equation by y^n which gives

$$\frac{y'}{y^n} = f_0(x)y^{1-n} + f_1(x) \tag{3}$$

The next step is use the substitution $w = y^{1-n}$ in equation (3) which generates a new ODE in $w(x)$ which will be linear and can be easily solved using an integrating factor. Backsubstitution then gives the solution $y(x)$ which is what we want.

This method is now applied to the ODE at hand. Comparing the ODE (1) With (2) Shows that

$$
\begin{aligned}
f_0(x) &= \frac{1}{x} \\
f_1(x) &= -2x^2 \\
n &= -3
\end{aligned}
$$

Dividing both sides of ODE (1) by $y^n = \frac{1}{y^3}$ gives

$$
y'y^3 = \frac{y^4}{x} - 2x^2 \tag{4}
$$

Let

$$
\begin{aligned}
w &= y^{1-n} \\
&= y^4
\end{aligned} \tag{5}
$$

Taking derivative of equation (5) w.r.t x gives

$$
w' = 4y^3y' \tag{6}
$$

Substituting equations (5) and (6) into equation (4) gives

$$
\begin{aligned}
\frac{w'(x)}{4} &= \frac{w(x)}{x} - 2x^2 \\
w' &= \frac{4w}{x} - 8x^2
\end{aligned} \tag{7}
$$

The above now is a linear ODE in $w(x)$ which is now solved.

Entering Linear first order ODE solver. In canonical form a linear first order is

$$
w'(x) + p(x)w(x) = q(x)
$$

Where here

$$
\begin{aligned}
p(x) &= -\frac{4}{x} \\
q(x) &= -8x^2
\end{aligned}
$$

Hence the ode is

$$
w'(x) - \frac{4w(x)}{x} = -8x^2
$$

The integrating factor μ is

$$
\begin{aligned}
\mu &= \mathrm{e}^{\int -\frac{4}{x}dx} \\
&= \frac{1}{x^4}
\end{aligned}
$$

The ode becomes

$$
\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}x}(\mu w) &= (\mu)\left(-8x^2\right) \\
\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{w}{x^4}\right) &= \left(\frac{1}{x^4}\right)\left(-8x^2\right) \\
\mathrm{d}\left(\frac{w}{x^4}\right) &= \left(-\frac{8}{x^2}\right)\mathrm{d}x
\end{aligned}
$$

Integrating gives

$$
\begin{aligned}
\frac{w}{x^4} &= \int -\frac{8}{x^2}\,\mathrm{d}x \\
\frac{w}{x^4} &= \frac{8}{x} + c_1
\end{aligned}
$$

Dividing both sides by the integrating factor $\mu = \frac{1}{x^4}$ results in

$$
w(x) = c_1x^4 + 8x^3
$$

Replacing w in the above by y^4 using equation (5) gives the final solution.

$$
y^4 = c_1x^4 + 8x^3
$$

Solving for y gives

$$
\begin{aligned}
y(x) &= \left(x^3(c_1x+8)\right)^{\frac{1}{4}} \\
y(x) &= i\left(x^3(c_1x+8)\right)^{\frac{1}{4}} \\
y(x) &= -\left(x^3(c_1x+8)\right)^{\frac{1}{4}} \\
y(x) &= -i\left(x^3(c_1x+8)\right)^{\frac{1}{4}}
\end{aligned}
$$

Summary

The solution(s) found are the following

$$
y = \left(x^3(c_1x+8)\right)^{\frac{1}{4}} \tag{1}
$$

$$
y = i\left(x^3(c_1x+8)\right)^{\frac{1}{4}} \tag{2}
$$

$$
y = -\left(x^3(c_1x+8)\right)^{\frac{1}{4}} \tag{3}
$$

$$
y = -i\left(x^3(c_1x+8)\right)^{\frac{1}{4}} \tag{4}
$$

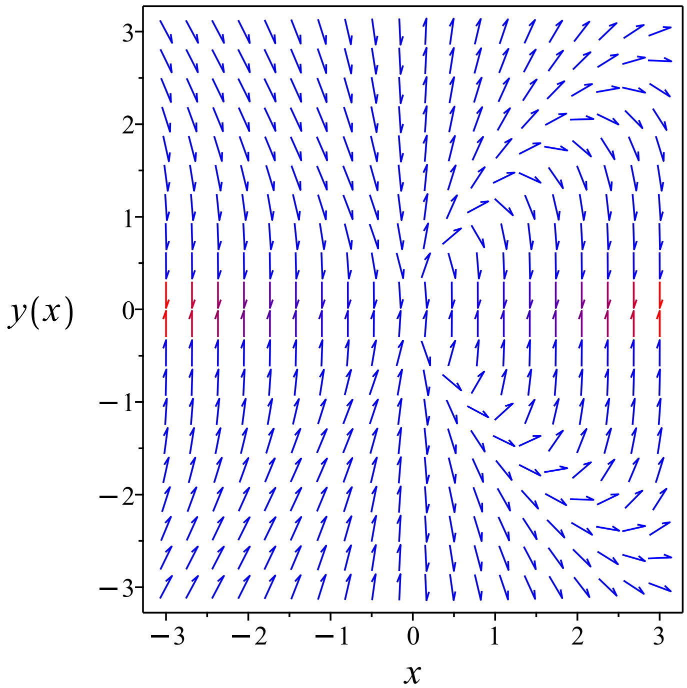


Figure 109: Slope field plot

Verification of solutions

$$y = \left(x^3(c_1x + 8)\right)^{\frac{1}{4}}$$

Verified OK.

$$y = i\left(x^3(c_1x + 8)\right)^{\frac{1}{4}}$$

Verified OK.

$$y = -\left(x^3(c_1x + 8)\right)^{\frac{1}{4}}$$

Verified OK.

$$y = -i\left(x^3(c_1x + 8)\right)^{\frac{1}{4}}$$

Verified OK.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
<- Bernoulli successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 65

```
dsolve((2*x^3-y(x)^4)+(x*y(x)^3)*diff(y(x),x)=0,y(x), singsol=all)
```

$$y(x) = \left(x^3(c_1x+8)\right)^{\frac{1}{4}}$$
$$y(x) = -\left(x^3(c_1x+8)\right)^{\frac{1}{4}}$$
$$y(x) = -i\left(x^3(c_1x+8)\right)^{\frac{1}{4}}$$
$$y(x) = i\left(x^3(c_1x+8)\right)^{\frac{1}{4}}$$

✓ Solution by Mathematica

Time used: 0.243 (sec). Leaf size: 88

```
DSolve[(2*x^3-y[x]^4)+(x*y[x]^3)*y'[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to -x^{3/4}\sqrt[4]{8+c_1x}$$
$$y(x) \to -ix^{3/4}\sqrt[4]{8+c_1x}$$
$$y(x) \to ix^{3/4}\sqrt[4]{8+c_1x}$$
$$y(x) \to x^{3/4}\sqrt[4]{8+c_1x}$$

1.75 problem 78

Internal problem ID [3220]
Internal file name [OUTPUT/2712_Sunday_June_05_2022_08_39_06_AM_24935528/index.tex]

Book: Differential equations for engineers by Wei-Chau XIE, Cambridge Press 2010
Section: Chapter 2. First-Order and Simple Higher-Order Differential Equations. Page 78
Problem number: 78.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program :

Maple gives the following as the ode type

`[_Bernoulli]`

$$\boxed{y' - y\tan(x) + y^2\cos(x) = 0}$$

1.75.1 Solving as bernoulli ode

In canonical form, the ODE is

$$\begin{aligned} y' &= F(x, y) \\ &= \tan(x)\, y - \cos(x)\, y^2 \end{aligned}$$

This is a Bernoulli ODE.

$$y' = \tan(x)\, y - \cos(x)\, y^2 \tag{1}$$

The standard Bernoulli ODE has the form

$$y' = f_0(x)y + f_1(x)y^n \tag{2}$$

The first step is to divide the above equation by y^n which gives

$$\frac{y'}{y^n} = f_0(x)y^{1-n} + f_1(x) \tag{3}$$

The next step is use the substitution $w = y^{1-n}$ in equation (3) which generates a new ODE in $w(x)$ which will be linear and can be easily solved using an integrating factor. Backsubstitution then gives the solution $y(x)$ which is what we want.

This method is now applied to the ODE at hand. Comparing the ODE (1) With (2) Shows that

$$
\begin{aligned}
f_0(x) &= \tan(x) \\
f_1(x) &= -\cos(x) \\
n &= 2
\end{aligned}
$$

Dividing both sides of ODE (1) by $y^n = y^2$ gives

$$
y' \frac{1}{y^2} = \frac{\tan(x)}{y} - \cos(x) \tag{4}
$$

Let

$$
\begin{aligned}
w &= y^{1-n} \\
&= \frac{1}{y}
\end{aligned} \tag{5}
$$

Taking derivative of equation (5) w.r.t x gives

$$
w' = -\frac{1}{y^2} y' \tag{6}
$$

Substituting equations (5) and (6) into equation (4) gives

$$
\begin{aligned}
-w'(x) &= \tan(x)\, w(x) - \cos(x) \\
w' &= -\tan(x)\, w + \cos(x)
\end{aligned} \tag{7}
$$

The above now is a linear ODE in $w(x)$ which is now solved.

Entering Linear first order ODE solver. In canonical form a linear first order is

$$
w'(x) + p(x)w(x) = q(x)
$$

Where here

$$
\begin{aligned}
p(x) &= \tan(x) \\
q(x) &= \cos(x)
\end{aligned}
$$

Hence the ode is

$$
w'(x) + \tan(x)\, w(x) = \cos(x)
$$

The integrating factor μ is

$$
\begin{aligned}
\mu &= e^{\int \tan(x) dx} \\
&= \frac{1}{\cos(x)}
\end{aligned}
$$

Which simplifies to

$$\mu = \sec(x)$$

The ode becomes

$$\begin{aligned}\frac{d}{dx}(\mu w) &= (\mu)(\cos(x)) \\ \frac{d}{dx}(\sec(x) w) &= (\sec(x))(\cos(x)) \\ d(\sec(x) w) &= dx\end{aligned}$$

Integrating gives

$$\begin{aligned}\sec(x) w &= \int dx \\ \sec(x) w &= x + c_1\end{aligned}$$

Dividing both sides by the integrating factor $\mu = \sec(x)$ results in

$$w(x) = \cos(x) x + c_1 \cos(x)$$

which simplifies to

$$w(x) = \cos(x)(x + c_1)$$

Replacing w in the above by $\frac{1}{y}$ using equation (5) gives the final solution.

$$\frac{1}{y} = \cos(x)(x + c_1)$$

Or

$$y = \frac{1}{\cos(x)(x + c_1)}$$

Summary
The solution(s) found are the following

$$y = \frac{1}{\cos(x)(x + c_1)} \tag{1}$$

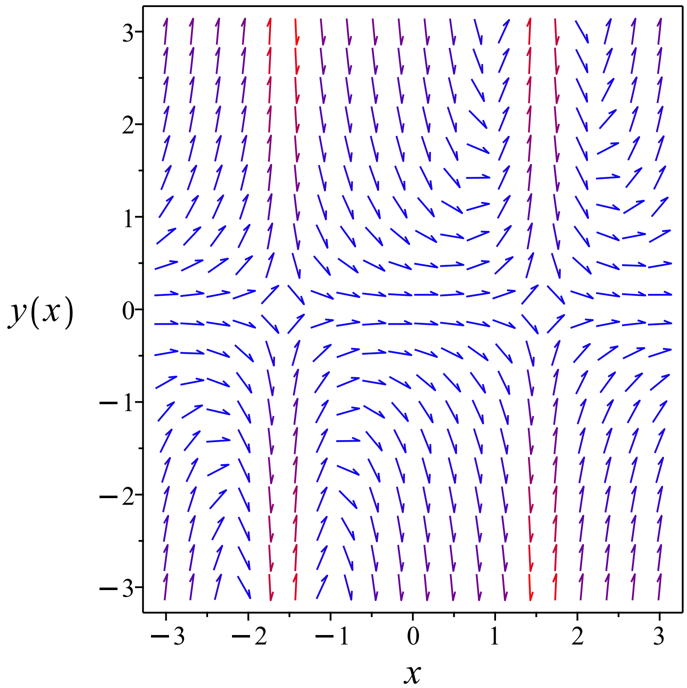


Figure 110: Slope field plot

Verification of solutions

$$y = \frac{1}{\cos(x)(x + c_1)}$$

Verified OK.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
<- Bernoulli successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 12

```
dsolve(diff(y(x),x)-y(x)*tan(x)+y(x)^2*cos(x)=0,y(x), singsol=all)
```

$$y(x) = \frac{\sec(x)}{c_1 + x}$$

✓ Solution by Mathematica

Time used: 0.22 (sec). Leaf size: 19

```
DSolve[y'[x]-y[x]*Tan[x]+y[x]^2*Cos[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \frac{\sec(x)}{x + c_1}$$
$$y(x) \to 0$$

1.76 problem 79

Internal problem ID [3221]
Internal file name [`OUTPUT/2713_Sunday_June_05_2022_08_39_07_AM_44701922/index.tex`]

Book: Differential equations for engineers by Wei-Chau XIE, Cambridge Press 2010
Section: Chapter 2. First-Order and Simple Higher-Order Differential Equations. Page 78
Problem number: 79.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"first_order_ode_lie_symmetry_calculated"**

Maple gives the following as the ode type

```
[[_homogeneous, `class G`], _rational, [_Abel, `2nd type`, `
   class B`]]
```

$$\boxed{6y^2 - x\left(2x^3 + y\right) y' = 0}$$

1.76.1 Solving as first order ode lie symmetry calculated ode

Writing the ode as

$$\begin{aligned} y' &= \frac{6y^2}{x\left(2x^3 + y\right)} \\ y' &= \omega(x, y) \end{aligned}$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x + \omega(\eta_y - \xi_x) - \omega^2 \xi_y - \omega_x \xi - \omega_y \eta = 0 \tag{A}$$

The type of this ode is not in the lookup table. To determine ξ, η then (A) is solved using ansatz. Making bivariate polynomials of degree 1 to use as anstaz gives

$$\xi = x a_2 + y a_3 + a_1 \tag{1E}$$

$$\eta = x b_2 + y b_3 + b_1 \tag{2E}$$

Where the unknown coefficients are

$$\{a_1, a_2, a_3, b_1, b_2, b_3\}$$

Substituting equations (1E,2E) and ω into (A) gives

$$\begin{aligned} & b_2 + \frac{6y^2(b_3 - a_2)}{x\,(2x^3+y)} - \frac{36y^4 a_3}{x^2\,(2x^3+y)^2} \\ & - \left(-\frac{6y^2}{x^2\,(2x^3+y)} - \frac{36y^2 x}{(2x^3+y)^2}\right)(xa_2 + ya_3 + a_1) \\ & - \left(\frac{12y}{x\,(2x^3+y)} - \frac{6y^2}{x\,(2x^3+y)^2}\right)(xb_2 + yb_3 + b_1) = 0 \end{aligned} \tag{5E}$$

Putting the above in normal form gives

$$\frac{4x^8b_2 - 20x^5yb_2 + 36x^4y^2a_2 - 12x^4y^2b_3 + 48x^3y^3a_3 - 24x^4yb_1 + 48x^3y^2a_1 - 5x^2y^2b_2 - 30y^4a_3 - 6x\,y^2b_1}{x^2\,(2x^3+y)^2} = 0$$

Setting the numerator to zero gives

$$\begin{aligned} & 4x^8b_2 - 20x^5yb_2 + 36x^4y^2a_2 - 12x^4y^2b_3 + 48x^3y^3a_3 - 24x^4yb_1 \\ & + 48x^3y^2a_1 - 5x^2y^2b_2 - 30y^4a_3 - 6x\,y^2b_1 + 6y^3a_1 = 0 \end{aligned} \tag{6E}$$

Looking at the above PDE shows the following are all the terms with $\{x, y\}$ in them.

$$\{x, y\}$$

The following substitution is now made to be able to collect on all terms with $\{x, y\}$ in them

$$\{x = v_1, y = v_2\}$$

The above PDE (6E) now becomes

$$\begin{aligned} & 4b_2v_1^8 + 36a_2v_1^4v_2^2 + 48a_3v_1^3v_2^3 - 20b_2v_1^5v_2 - 12b_3v_1^4v_2^2 + 48a_1v_1^3v_2^2 \\ & - 24b_1v_1^4v_2 - 30a_3v_2^4 - 5b_2v_1^2v_2^2 + 6a_1v_2^3 - 6b_1v_1v_2^2 = 0 \end{aligned} \tag{7E}$$

Collecting the above on the terms v_i introduced, and these are

$$\{v_1, v_2\}$$

Equation (7E) now becomes

$$\begin{aligned} 4b_2v_1^8 - 20b_2v_1^5v_2 + (36a_2 - 12b_3)\, v_1^4v_2^2 - 24b_1v_1^4v_2 + 48a_3v_1^3v_2^3 \\ + 48a_1v_1^3v_2^2 - 5b_2v_1^2v_2^2 - 6b_1v_1v_2^2 - 30a_3v_2^4 + 6a_1v_2^3 = 0 \end{aligned} \tag{8E}$$

Setting each coefficients in (8E) to zero gives the following equations to solve

$$\begin{aligned} 6a_1 &= 0 \\ 48a_1 &= 0 \\ -30a_3 &= 0 \\ 48a_3 &= 0 \\ -24b_1 &= 0 \\ -6b_1 &= 0 \\ -20b_2 &= 0 \\ -5b_2 &= 0 \\ 4b_2 &= 0 \\ 36a_2 - 12b_3 &= 0 \end{aligned}$$

Solving the above equations for the unknowns gives

$$\begin{aligned} a_1 &= 0 \\ a_2 &= a_2 \\ a_3 &= 0 \\ b_1 &= 0 \\ b_2 &= 0 \\ b_3 &= 3a_2 \end{aligned}$$

Substituting the above solution in the anstaz (1E,2E) (using 1 as arbitrary value for any unknown in the RHS) gives

$$\begin{aligned} \xi &= x \\ \eta &= 3y \end{aligned}$$

Shifting is now applied to make $\xi = 0$ in order to simplify the rest of the computation

$$\begin{aligned}
\eta &= \eta - \omega(x,y)\,\xi \\
&= 3y - \left(\frac{6y^2}{x\,(2x^3+y)}\right)(x) \\
&= \frac{6x^3y - 3y^2}{2x^3+y} \\
\xi &= 0
\end{aligned}$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x,y) \to (R,S)$ where (R,S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}$$

The above comes from the requirements that $\left(\xi\frac{\partial}{\partial x} + \eta\frac{\partial}{\partial y}\right) S(x,y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$R = x$$

S is found from

$$\begin{aligned}
S &= \int \frac{1}{\eta} dy \\
&= \int \frac{1}{\frac{6x^3y-3y^2}{2x^3+y}} dy
\end{aligned}$$

Which results in

$$S = -\frac{2\ln(-2x^3+y)}{3} + \frac{\ln(y)}{3}$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x,y)S_y}{R_x + \omega(x,y)R_y} \tag{2}$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x,y)$ is the right hand side of the original ode given by

$$\omega(x,y) = \frac{6y^2}{x\,(2x^3+y)}$$

Evaluating all the partial derivatives gives

$$
\begin{aligned}
R_x &= 1 \\
R_y &= 0 \\
S_x &= -\frac{4x^2}{2x^3 - y} \\
S_y &= \frac{2}{6x^3 - 3y} + \frac{1}{3y}
\end{aligned}
$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$
\frac{dS}{dR} = -\frac{2}{x} \tag{2A}
$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$
\frac{dS}{dR} = -\frac{2}{R}
$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$
S(R) = -2\ln(R) + c_1 \tag{4}
$$

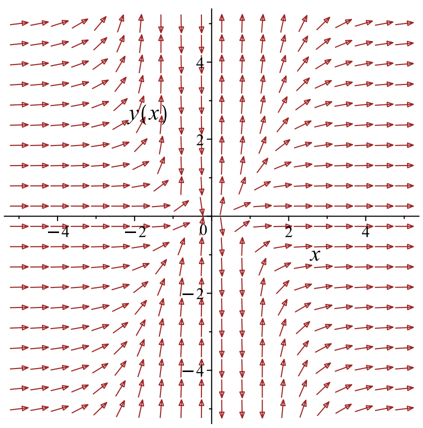
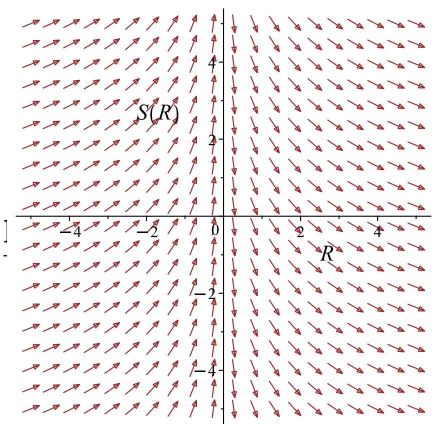
To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$
-\frac{2\ln(-2x^3 + y)}{3} + \frac{\ln(y)}{3} = -2\ln(x) + c_1
$$

Which simplifies to

$$
-\frac{2\ln(-2x^3 + y)}{3} + \frac{\ln(y)}{3} = -2\ln(x) + c_1
$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in x, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dx} = \frac{6y^2}{x(2x^3+y)}$ 	$R = x$ $S = -\frac{2\ln(-2x^3+y)}{3} +$	$\frac{dS}{dR} = -\frac{2}{R}$ 

Summary

The solution(s) found are the following

$$-\frac{2\ln(-2x^3+y)}{3} + \frac{\ln(y)}{3} = -2\ln(x) + c_1 \tag{1}$$

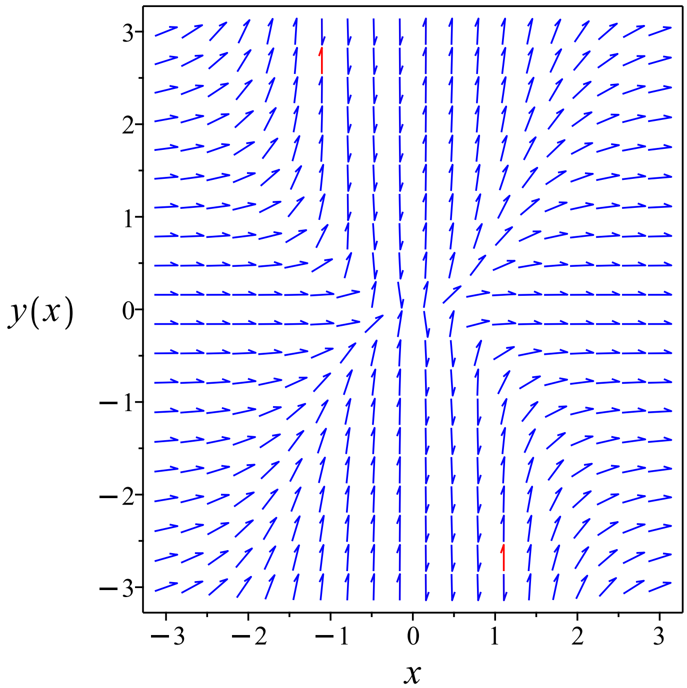


Figure 111: Slope field plot

Verification of solutions

$$-\frac{2\ln\left(-2x^3+y\right)}{3}+\frac{\ln\left(y\right)}{3}=-2\ln\left(x\right)+c_1$$

Verified OK.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
trying homogeneous types:
trying homogeneous G
<- homogeneous successful`
```

✓ Solution by Maple

Time used: 0.578 (sec). Leaf size: 193

```
dsolve(6*y(x)^2-(x*(2*x^3+y(x)))*diff(y(x),x)=0,y(x), singsol=all)
```

$$y(x) = -\frac{x^3\left(-x^3 + \sqrt{x^3\left(x^3 + 8c_1\right)} - 4c_1\right)}{2c_1}$$
$$y(x) = \frac{x^3\left(x^3 + \sqrt{x^3\left(x^3 + 8c_1\right)} + 4c_1\right)}{2c_1}$$
$$y(x) = -\frac{x^3\left(-x^3 + \sqrt{x^3\left(x^3 + 8c_1\right)} - 4c_1\right)}{2c_1}$$
$$y(x) = \frac{x^3\left(x^3 + \sqrt{x^3\left(x^3 + 8c_1\right)} + 4c_1\right)}{2c_1}$$
$$y(x) = -\frac{x^3\left(-x^3 + \sqrt{x^3\left(x^3 + 8c_1\right)} - 4c_1\right)}{2c_1}$$
$$y(x) = \frac{x^3\left(x^3 + \sqrt{x^3\left(x^3 + 8c_1\right)} + 4c_1\right)}{2c_1}$$

✓ Solution by Mathematica

Time used: 1.396 (sec). Leaf size: 123

```
DSolve[6*y[x]^2-(x*(2*x^3+y[x]))*y'[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to 2x^3\left(-1 + \frac{2}{1 - \frac{4x^{3/2}}{\sqrt{16x^3+c_1}}}\right)$$
$$y(x) \to 2x^3\left(-1 + \frac{2}{1 + \frac{4x^{3/2}}{\sqrt{16x^3+c_1}}}\right)$$
$$y(x) \to 0$$
$$y(x) \to 2x^3$$
$$y(x) \to \frac{2\left(\left(x^3\right)^{3/2} - x^{9/2}\right)}{x^{3/2} + \sqrt{x^3}}$$

1.77 problem 80

Internal problem ID [3222]
Internal file name [OUTPUT/2714_Sunday_June_05_2022_08_39_08_AM_40232232/index.tex]

Book: Differential equations for engineers by Wei-Chau XIE, Cambridge Press 2010
Section: Chapter 2. First-Order and Simple Higher-Order Differential Equations. Page 78
Problem number: 80.
ODE order: 1.
ODE degree: 3.

The type(s) of ODE detected by this program : **"clairaut"**

Maple gives the following as the ode type

```
[[_1st_order, _with_linear_symmetries], _Clairaut]
```

$$\boxed{xy'^3 - yy'^2 = -1}$$

1.77.1 Solving as clairaut ode

This is Clairaut ODE. It has the form

$$y = xy' + g(y')$$

Where g is function of $y'(x)$. Let $p = y'$ the ode becomes

$$x\,p^3 - y\,p^2 = -1$$

Solving for y from the above results in

$$y = \frac{x\,p^3 + 1}{p^2} \tag{1A}$$

The above ode is a Clairaut ode which is now solved. We start by replacing y' by p which gives

$$\begin{aligned} y &= px + \frac{1}{p^2} \\ &= px + \frac{1}{p^2} \end{aligned}$$

Writing the ode as

$$y = px + g(p)$$

We now write $g \equiv g(p)$ to make notation simpler but we should always remember that g is function of p which in turn is function of x. Hence the above becomes

$$y = px + g \tag{1}$$

Then we see that

$$g = \frac{1}{p^2}$$

Taking derivative of (1) w.r.t. x gives

$$\begin{aligned}
p &= \frac{d}{dx}(xp + g) \\
p &= \left(p + x\frac{dp}{dx}\right) + \left(g'\frac{dp}{dx}\right) \\
p &= p + (x + g')\frac{dp}{dx} \\
0 &= (x + g')\frac{dp}{dx}
\end{aligned}$$

Where g' is derivative of $g(p)$ w.r.t. p. The general solution is given by

$$\begin{aligned}
\frac{dp}{dx} &= 0 \\
p &= c_1
\end{aligned}$$

Substituting this in (1) gives the general solution as

$$y = c_1 x + \frac{1}{c_1^2}$$

The singular solution is found from solving for p from

$$x + g'(p) = 0$$

And substituting the result back in (1). Since we found above that $g = \frac{1}{p^2}$, then the above equation becomes

$$\begin{aligned}
x + g'(p) &= x - \frac{2}{p^3} \\
&= 0
\end{aligned}$$

Solving the above for p results in

$$
\begin{aligned}
p_1 &= \frac{2^{\frac{1}{3}} \left(x^2\right)^{\frac{1}{3}}}{x} \\
p_2 &= -\frac{2^{\frac{1}{3}} \left(x^2\right)^{\frac{1}{3}}}{2x} + \frac{i\sqrt{3}\, 2^{\frac{1}{3}} \left(x^2\right)^{\frac{1}{3}}}{2x} \\
p_3 &= -\frac{2^{\frac{1}{3}} \left(x^2\right)^{\frac{1}{3}}}{2x} - \frac{i\sqrt{3}\, 2^{\frac{1}{3}} \left(x^2\right)^{\frac{1}{3}}}{2x}
\end{aligned}
$$

Substituting the above back in (1) results in

$$
\begin{aligned}
y_1 &= \frac{3x^2 2^{\frac{1}{3}}}{2 \left(x^2\right)^{\frac{2}{3}}} \\
y_2 &= -\frac{3x^2 2^{\frac{1}{3}}}{\left(x^2\right)^{\frac{2}{3}} \left(1 + i\sqrt{3}\right)} \\
y_3 &= \frac{3x^2 2^{\frac{1}{3}}}{\left(x^2\right)^{\frac{2}{3}} \left(-1 + i\sqrt{3}\right)}
\end{aligned}
$$

<u>Summary</u>

The solution(s) found are the following

$$
y = c_1 x + \frac{1}{c_1^2} \tag{1}
$$

$$
y = \frac{3x^2 2^{\frac{1}{3}}}{2 \left(x^2\right)^{\frac{2}{3}}} \tag{2}
$$

$$
y = -\frac{3x^2 2^{\frac{1}{3}}}{\left(x^2\right)^{\frac{2}{3}} \left(1 + i\sqrt{3}\right)} \tag{3}
$$

$$
y = \frac{3x^2 2^{\frac{1}{3}}}{\left(x^2\right)^{\frac{2}{3}} \left(-1 + i\sqrt{3}\right)} \tag{4}
$$

Verification of solutions

$$y = c_1 x + \frac{1}{c_1^2}$$

Verified OK.

$$y = \frac{3x^2 2^{\frac{1}{3}}}{2 \left(x^2\right)^{\frac{2}{3}}}$$

Verified OK.

$$y = -\frac{3x^2 2^{\frac{1}{3}}}{\left(x^2\right)^{\frac{2}{3}} \left(1 + i\sqrt{3}\right)}$$

Verified OK.

$$y = \frac{3x^2 2^{\frac{1}{3}}}{\left(x^2\right)^{\frac{2}{3}} \left(-1 + i\sqrt{3}\right)}$$

Verified OK.

Maple trace

```
`Methods for first order ODEs:
   *** Sublevel 2 ***
   Methods for first order ODEs:
   -> Solving 1st order ODE of high degree, 1st attempt
   trying 1st order WeierstrassP solution for high degree ODE
   trying 1st order WeierstrassPPrime solution for high degree ODE
   trying 1st order JacobiSN solution for high degree ODE
   trying 1st order ODE linearizable_by_differentiation
   trying differential order: 1; missing variables
   trying dAlembert
   <- dAlembert successful`
```

✓ Solution by Maple

Time used: 0.125 (sec). Leaf size: 66

```
dsolve(x*(diff(y(x),x))^3-y(x)*(diff(y(x),x))^2+1=0,y(x), singsol=all)
```

$$y(x) = \frac{3\,2^{\frac{1}{3}}(x^2)^{\frac{1}{3}}}{2}$$
$$y(x) = -\frac{3\,2^{\frac{1}{3}}(x^2)^{\frac{1}{3}}\left(1+i\sqrt{3}\right)}{4}$$
$$y(x) = \frac{3\,2^{\frac{1}{3}}(x^2)^{\frac{1}{3}}\left(i\sqrt{3}-1\right)}{4}$$
$$y(x) = c_1x + \frac{1}{c_1^2}$$

✓ Solution by Mathematica

Time used: 0.011 (sec). Leaf size: 69

```
DSolve[x*(y'[x])^3-y[x]*(y'[x])^2+1==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to c_1x + \frac{1}{c_1{}^2}$$
$$y(x) \to 3\left(-\frac{1}{2}\right)^{2/3} x^{2/3}$$
$$y(x) \to \frac{3x^{2/3}}{2^{2/3}}$$
$$y(x) \to -\frac{3\sqrt[3]{-1}x^{2/3}}{2^{2/3}}$$

1.78 problem 81

Internal problem ID [3223]
Internal file name [`OUTPUT/2715_Sunday_June_05_2022_08_39_10_AM_81734703/index.tex`]

Book: Differential equations for engineers by Wei-Chau XIE, Cambridge Press 2010
Section: Chapter 2. First-Order and Simple Higher-Order Differential Equations. Page 78
Problem number: 81.
ODE order: 1.
ODE degree: 3.

The type(s) of ODE detected by this program : **"clairaut"**

Maple gives the following as the ode type

```
[[_1st_order, _with_linear_symmetries], _Clairaut]
```

$$\boxed{y - xy' - y'^3 = 0}$$

1.78.1 Solving as clairaut ode

This is Clairaut ODE. It has the form

$$y = xy' + g(y')$$

Where g is function of $y'(x)$. Let $p = y'$ the ode becomes

$$-p^3 - xp + y = 0$$

Solving for y from the above results in

$$y = p^3 + xp \tag{1A}$$

The above ode is a Clairaut ode which is now solved. We start by replacing y' by p which gives

$$\begin{aligned} y &= p^3 + xp \\ &= p^3 + xp \end{aligned}$$

Writing the ode as

$$y = xp + g(p)$$

We now write $g \equiv g(p)$ to make notation simpler but we should always remember that g is function of p which in turn is function of x. Hence the above becomes

$$y = xp + g \tag{1}$$

Then we see that

$$g = p^3$$

Taking derivative of (1) w.r.t. x gives

$$\begin{aligned}
p &= \frac{d}{dx}(xp + g) \\
p &= \left(p + x\frac{dp}{dx}\right) + \left(g'\frac{dp}{dx}\right) \\
p &= p + (x + g')\frac{dp}{dx} \\
0 &= (x + g')\frac{dp}{dx}
\end{aligned}$$

Where g' is derivative of $g(p)$ w.r.t. p. The general solution is given by

$$\begin{aligned}
\frac{dp}{dx} &= 0 \\
p &= c_1
\end{aligned}$$

Substituting this in (1) gives the general solution as

$$y = c_1^3 + c_1 x$$

The singular solution is found from solving for p from

$$x + g'(p) = 0$$

And substituting the result back in (1). Since we found above that $g = p^3$, then the above equation becomes

$$\begin{aligned}
x + g'(p) &= 3p^2 + x \\
&= 0
\end{aligned}$$

Solving the above for p results in

$$p_1 = \frac{\sqrt{-3x}}{3}$$
$$p_2 = -\frac{\sqrt{-3x}}{3}$$

Substituting the above back in (1) results in

$$y_1 = -\frac{2\sqrt{3}\,(-x)^{\frac{3}{2}}}{9}$$
$$y_2 = \frac{2\sqrt{3}\,(-x)^{\frac{3}{2}}}{9}$$

Summary

The solution(s) found are the following

$$y = c_1^3 + c_1 x \tag{1}$$
$$y = -\frac{2\sqrt{3}\,(-x)^{\frac{3}{2}}}{9} \tag{2}$$
$$y = \frac{2\sqrt{3}\,(-x)^{\frac{3}{2}}}{9} \tag{3}$$

Verification of solutions

$$y = c_1^3 + c_1 x$$

Verified OK.

$$y = -\frac{2\sqrt{3}\,(-x)^{\frac{3}{2}}}{9}$$

Verified OK.

$$y = \frac{2\sqrt{3}\,(-x)^{\frac{3}{2}}}{9}$$

Verified OK.

Maple trace

```
`Methods for first order ODEs:
   *** Sublevel 2 ***
   Methods for first order ODEs:
   -> Solving 1st order ODE of high degree, 1st attempt
   trying 1st order WeierstrassP solution for high degree ODE
   trying 1st order WeierstrassPPrime solution for high degree ODE
   trying 1st order JacobiSN solution for high degree ODE
   trying 1st order ODE linearizable_by_differentiation
   trying differential order: 1; missing variables
   trying dAlembert
   <- dAlembert successful`
```

✓ Solution by Maple

Time used: 0.094 (sec). Leaf size: 37

```
dsolve(y(x)=x*diff(y(x),x)+(diff(y(x),x))^3,y(x), singsol=all)
```

$$y(x) = \frac{2\sqrt{3}\,(-x)^{\frac{3}{2}}}{9}$$
$$y(x) = -\frac{2\sqrt{3}\,(-x)^{\frac{3}{2}}}{9}$$
$$y(x) = c_1\left(c_1^2 + x\right)$$

✓ Solution by Mathematica

Time used: 0.008 (sec). Leaf size: 54

```
DSolve[y[x]==x*y'[x]+(y'[x])^3,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to c_1\left(x + c_1{}^2\right)$$
$$y(x) \to -\frac{2ix^{3/2}}{3\sqrt{3}}$$
$$y(x) \to \frac{2ix^{3/2}}{3\sqrt{3}}$$

1.79 problem 82

Internal problem ID [3224]
Internal file name [`OUTPUT/2716_Sunday_June_05_2022_08_39_11_AM_40883080/index.tex`]

Book: Differential equations for engineers by Wei-Chau XIE, Cambridge Press 2010
Section: Chapter 2. First-Order and Simple Higher-Order Differential Equations. Page 78
Problem number: 82.
ODE order: 1.
ODE degree: 2.

The type(s) of ODE detected by this program : **"quadrature"**

Maple gives the following as the ode type

`[_quadrature]`

$$\boxed{x\left(-1+{y'}^{2}\right)-2y'=0}$$

Solving the given ode for y' results in 2 differential equations to solve. Each one of these will generate a solution. The equations generated are

$$y' = \frac{1+\sqrt{x^2+1}}{x} \tag{1}$$

$$y' = -\frac{-1+\sqrt{x^2+1}}{x} \tag{2}$$

Now each one of the above ODE is solved.

Solving equation (1)

Integrating both sides gives

$$\begin{aligned} y &= \int \frac{1+\sqrt{x^2+1}}{x}\,\mathrm{d}x \\ &= \int \frac{1+\sqrt{x^2+1}}{x}dx + c_1 \end{aligned}$$

Summary
The solution(s) found are the following

$$y = \int \frac{1+\sqrt{x^2+1}}{x}dx + c_1 \tag{1}$$

Verification of solutions

$$y = \int \frac{1+\sqrt{x^2+1}}{x}dx + c_1$$

Verified OK.

Solving equation (2)

Integrating both sides gives

$$\begin{aligned} y &= \int -\frac{-1+\sqrt{x^2+1}}{x}\,\mathrm{d}x \\ &= \int -\frac{-1+\sqrt{x^2+1}}{x}dx + c_2 \end{aligned}$$

Summary

The solution(s) found are the following

$$y = \int -\frac{-1+\sqrt{x^2+1}}{x}dx + c_2 \tag{1}$$

Verification of solutions

$$y = \int -\frac{-1+\sqrt{x^2+1}}{x}dx + c_2$$

Verified OK.

1.79.1 Maple step by step solution

Let's solve

$x(-1+y'^2) - 2y' = 0$

- Highest derivative means the order of the ODE is 1

  y'

- Integrate both sides with respect to x

  $\int \left(x(-1+y'^2) - 2y'\right) dx = \int 0dx + c_1$

- Cannot compute integral

  $\int \left(x(-1+y'^2) - 2y'\right) dx = c_1$

Maple trace

```
`Methods for first order ODEs:
-> Solving 1st order ODE of high degree, 1st attempt
trying 1st order WeierstrassP solution for high degree ODE
trying 1st order WeierstrassPPrime solution for high degree ODE
trying 1st order JacobiSN solution for high degree ODE
trying 1st order ODE linearizable_by_differentiation
trying differential order: 1; missing variables
<- differential order: 1; missing  y(x)  successful`
```

✓ Solution by Maple

Time used: 0.062 (sec). Leaf size: 49

```
dsolve(x*( (diff(y(x),x))^2-1)=2*diff(y(x),x) ,y(x), singsol=all)
```

$$y(x) = \sqrt{x^2+1} - \operatorname{arctanh}\left(\frac{1}{\sqrt{x^2+1}}\right) + \ln(x) + c_1$$
$$y(x) = -\sqrt{x^2+1} + \operatorname{arctanh}\left(\frac{1}{\sqrt{x^2+1}}\right) + \ln(x) + c_1$$

✓ Solution by Mathematica

Time used: 0.031 (sec). Leaf size: 59

```
DSolve[x*( (y'[x])^2-1)==2*y'[x],y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \sqrt{x^2+1} + \log\left(\sqrt{x^2+1} - 1\right) + c_1$$
$$y(x) \to -\sqrt{x^2+1} + \log\left(\sqrt{x^2+1} + 1\right) + c_1$$

1.80 problem 83

Internal problem ID [3225]
Internal file name [OUTPUT/2717_Sunday_June_05_2022_08_39_11_AM_45878433/index.tex]

Book: Differential equations for engineers by Wei-Chau XIE, Cambridge Press 2010
Section: Chapter 2. First-Order and Simple Higher-Order Differential Equations. Page 78
Problem number: 83.
ODE order: 1.
ODE degree: 2.

The type(s) of ODE detected by this program : **"dAlembert"**

Maple gives the following as the ode type

```
[[_homogeneous, `class A`], _rational, _dAlembert]
```

$$\boxed{xy'(y'+2) - y = 0}$$

1.80.1 Solving as dAlembert ode

Let $p = y'$ the ode becomes

$$xp(p+2) - y = 0$$

Solving for y from the above results in

$$y = xp(p+2) \tag{1A}$$

This has the form

$$y = xf(p) + g(p) \tag{*}$$

Where f, g are functions of $p = y'(x)$. The above ode is dAlembert ode which is now solved. Taking derivative of (*) w.r.t. x gives

$$p = f + (xf' + g')\frac{dp}{dx}$$
$$p - f = (xf' + g')\frac{dp}{dx} \tag{2}$$

Comparing the form $y = xf + g$ to (1A) shows that

$$
\begin{aligned}
f &= p(p+2) \\
g &= 0
\end{aligned}
$$

Hence (2) becomes

$$
p - p(p+2) = x(2p+2)\,p'(x) \tag{2A}
$$

The singular solution is found by setting $\frac{dp}{dx} = 0$ in the above which gives

$$
p - p(p+2) = 0
$$

Solving for p from the above gives

$$
\begin{aligned}
p &= -1 \\
p &= 0
\end{aligned}
$$

Substituting these in (1A) gives

$$
\begin{aligned}
y &= -x \\
y &= 0
\end{aligned}
$$

The general solution is found when $\frac{\mathrm{d}p}{\mathrm{d}x} \neq 0$. From eq. (2A). This results in

$$
p'(x) = \frac{p(x) - p(x)\,(p(x)+2)}{x\,(2p\,(x)+2)} \tag{3}
$$

This ODE is now solved for $p(x)$.

Entering Linear first order ODE solver. In canonical form a linear first order is

$$
p'(x) + p(x)p(x) = q(x)
$$

Where here

$$
\begin{aligned}
p(x) &= \frac{1}{2x} \\
q(x) &= 0
\end{aligned}
$$

Hence the ode is

$$
p'(x) + \frac{p(x)}{2x} = 0
$$

The integrating factor μ is

$$\mu = e^{\int \frac{1}{2x}dx}$$
$$= \sqrt{x}$$

The ode becomes

$$\frac{d}{dx}\mu p = 0$$
$$\frac{d}{dx}(\sqrt{x}\, p) = 0$$

Integrating gives

$$\sqrt{x}\, p = c_1$$

Dividing both sides by the integrating factor $\mu = \sqrt{x}$ results in

$$p(x) = \frac{c_1}{\sqrt{x}}$$

Substituing the above solution for p in (2A) gives

$$y = \sqrt{x}\, c_1 \left(\frac{c_1}{\sqrt{x}} + 2\right)$$

Summary

The solution(s) found are the following

$$y = -x \tag{1}$$
$$y = 0 \tag{2}$$
$$y = \sqrt{x}\, c_1 \left(\frac{c_1}{\sqrt{x}} + 2\right) \tag{3}$$

Verification of solutions

$$y = -x$$

Verified OK.

$$y = 0$$

Verified OK.

$$y = \sqrt{x}\, c_1 \left(\frac{c_1}{\sqrt{x}} + 2\right)$$

Verified OK.

Maple trace

```
`Methods for first order ODEs:
   *** Sublevel 2 ***
   Methods for first order ODEs:
   -> Solving 1st order ODE of high degree, 1st attempt
   trying 1st order WeierstrassP solution for high degree ODE
   trying 1st order WeierstrassPPrime solution for high degree ODE
   trying 1st order JacobiSN solution for high degree ODE
   trying 1st order ODE linearizable_by_differentiation
   trying differential order: 1; missing variables
   trying dAlembert
   <- dAlembert successful`
```

✓ Solution by Maple

Time used: 0.063 (sec). Leaf size: 40

```
dsolve(x*diff(y(x),x)*(diff(y(x),x)+2)=y(x),y(x), singsol=all)
```

$$y(x) = -x$$
$$y(x) = \frac{\sqrt{c_1 x}\left(\sqrt{c_1 x}+2x\right)}{x}$$
$$y(x) = -2\sqrt{c_1 x}+c_1$$

✓ Solution by Mathematica

Time used: 0.186 (sec). Leaf size: 63

```
DSolve[x*y'[x]*(y'[x]+2)==y[x],y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to e^{c_1}-2e^{\frac{c_1}{2}}\sqrt{x}$$
$$y(x) \to 2e^{-\frac{c_1}{2}}\sqrt{x}+e^{-c_1}$$
$$y(x) \to 0$$
$$y(x) \to -x$$

1.81 problem 84

Internal problem ID [3226]
Internal file name [`OUTPUT/2718_Sunday_June_05_2022_08_39_12_AM_61153006/index.tex`]

Book: Differential equations for engineers by Wei-Chau XIE, Cambridge Press 2010
Section: Chapter 2. First-Order and Simple Higher-Order Differential Equations. Page 78
Problem number: 84.
ODE order: 1.
ODE degree: 4.

The type(s) of ODE detected by this program : **"quadrature"**

Maple gives the following as the ode type

`[_quadrature]`

$$\boxed{-y'\sqrt{1+y'^2}=-x}$$

Solving the given ode for y' results in 4 differential equations to solve. Each one of these will generate a solution. The equations generated are

$$y' = \frac{\sqrt{2\sqrt{4x^2+1}-2}}{2} \tag{1}$$

$$y' = -\frac{\sqrt{2\sqrt{4x^2+1}-2}}{2} \tag{2}$$

$$y' = \frac{\sqrt{-2-2\sqrt{4x^2+1}}}{2} \tag{3}$$

$$y' = -\frac{\sqrt{-2-2\sqrt{4x^2+1}}}{2} \tag{4}$$

Now each one of the above ODE is solved.

<u>Solving equation (1)</u>

Integrating both sides gives

$$\begin{aligned} y &= \int \frac{\sqrt{2\sqrt{4x^2+1}-2}}{2}\,\mathrm{d}x \\ &= \int \frac{\sqrt{2\sqrt{4x^2+1}-2}}{2} dx + c_1 \end{aligned}$$

Summary

The solution(s) found are the following

$$y = \int \frac{\sqrt{2\sqrt{4x^2+1}-2}}{2}dx + c_1 \tag{1}$$

Verification of solutions

$$y = \int \frac{\sqrt{2\sqrt{4x^2+1}-2}}{2}dx + c_1$$

Verified OK.

Solving equation (2)

Integrating both sides gives

$$\begin{aligned} y &= \int -\frac{\sqrt{2\sqrt{4x^2+1}-2}}{2}\,\mathrm{d}x \\ &= \int -\frac{\sqrt{2\sqrt{4x^2+1}-2}}{2}dx + c_2 \end{aligned}$$

Summary

The solution(s) found are the following

$$y = \int -\frac{\sqrt{2\sqrt{4x^2+1}-2}}{2}dx + c_2 \tag{1}$$

Verification of solutions

$$y = \int -\frac{\sqrt{2\sqrt{4x^2+1}-2}}{2}dx + c_2$$

Verified OK.

Solving equation (3)

Integrating both sides gives

$$\begin{aligned} y &= \int \frac{\sqrt{-2-2\sqrt{4x^2+1}}}{2}\,\mathrm{d}x \\ &= -\frac{i\sqrt{2}\left(-\frac{256\sqrt{2}\,\sqrt{\pi}\,x^3\cosh\left(\frac{3\ \mathrm{arcsinh}(2x)}{2}\right)}{3}-\frac{8\sqrt{2}\,\sqrt{\pi}\,\left(-\frac{64}{3}x^4-\frac{8}{3}x^2+\frac{2}{3}\right)\sinh\left(\frac{3\ \mathrm{arcsinh}(2x)}{2}\right)}{\sqrt{4x^2+1}}\right)}{32\sqrt{\pi}} + c_3 \end{aligned}$$

Summary
The solution(s) found are the following

$$y = -\frac{i\sqrt{2}\left(-\frac{256\sqrt{2}\,\sqrt{\pi}\,x^3\cosh\left(\frac{3\,\mathrm{arcsinh}(2x)}{2}\right)}{3}-\frac{8\sqrt{2}\,\sqrt{\pi}\,\left(-\frac{64}{3}x^4-\frac{8}{3}x^2+\frac{2}{3}\right)\sinh\left(\frac{3\,\mathrm{arcsinh}(2x)}{2}\right)}{\sqrt{4x^2+1}}\right)}{32\sqrt{\pi}}+c_3 \quad (1)$$

Verification of solutions

$$y = -\frac{i\sqrt{2}\left(-\frac{256\sqrt{2}\,\sqrt{\pi}\,x^3\cosh\left(\frac{3\,\mathrm{arcsinh}(2x)}{2}\right)}{3}-\frac{8\sqrt{2}\,\sqrt{\pi}\,\left(-\frac{64}{3}x^4-\frac{8}{3}x^2+\frac{2}{3}\right)\sinh\left(\frac{3\,\mathrm{arcsinh}(2x)}{2}\right)}{\sqrt{4x^2+1}}\right)}{32\sqrt{\pi}}+c_3$$

Verified OK.

Solving equation (4)

Integrating both sides gives

$$\begin{aligned} y &= \int -\frac{\sqrt{-2-2\sqrt{4x^2+1}}}{2}\,\mathrm{d}x \\ &= \frac{i\sqrt{2}\left(-\frac{256\sqrt{2}\,\sqrt{\pi}\,x^3\cosh\left(\frac{3\,\mathrm{arcsinh}(2x)}{2}\right)}{3}-\frac{8\sqrt{2}\,\sqrt{\pi}\,\left(-\frac{64}{3}x^4-\frac{8}{3}x^2+\frac{2}{3}\right)\sinh\left(\frac{3\,\mathrm{arcsinh}(2x)}{2}\right)}{\sqrt{4x^2+1}}\right)}{32\sqrt{\pi}}+c_4 \end{aligned}$$

Summary
The solution(s) found are the following

$$y = \frac{i\sqrt{2}\left(-\frac{256\sqrt{2}\,\sqrt{\pi}\,x^3\cosh\left(\frac{3\,\mathrm{arcsinh}(2x)}{2}\right)}{3}-\frac{8\sqrt{2}\,\sqrt{\pi}\,\left(-\frac{64}{3}x^4-\frac{8}{3}x^2+\frac{2}{3}\right)\sinh\left(\frac{3\,\mathrm{arcsinh}(2x)}{2}\right)}{\sqrt{4x^2+1}}\right)}{32\sqrt{\pi}}+c_4 \quad (1)$$

Verification of solutions

$$y = \frac{i\sqrt{2}\left(-\frac{256\sqrt{2}\,\sqrt{\pi}\,x^3\cosh\left(\frac{3\,\mathrm{arcsinh}(2x)}{2}\right)}{3}-\frac{8\sqrt{2}\,\sqrt{\pi}\,\left(-\frac{64}{3}x^4-\frac{8}{3}x^2+\frac{2}{3}\right)\sinh\left(\frac{3\,\mathrm{arcsinh}(2x)}{2}\right)}{\sqrt{4x^2+1}}\right)}{32\sqrt{\pi}}+c_4$$

Verified OK.

1.81.1 Maple step by step solution

Let's solve

$-y'\sqrt{1+y'^2} = -x$

- Highest derivative means the order of the ODE is 1

  y'

- Integrate both sides with respect to x

  $\int -y'\sqrt{1+y'^2}dx = \int -xdx + c_1$

- Cannot compute integral

  $\int -y'\sqrt{1+y'^2}dx = -\frac{x^2}{2} + c_1$

Maple trace

```
`Methods for first order ODEs:
-> Solving 1st order ODE of high degree, 1st attempt
trying 1st order WeierstrassP solution for high degree ODE
trying 1st order WeierstrassPPrime solution for high degree ODE
trying 1st order JacobiSN solution for high degree ODE
trying 1st order ODE linearizable_by_differentiation
trying differential order: 1; missing variables
<- differential order: 1; missing  y(x)  successful`
```

✓ Solution by Maple

Time used: 0.125 (sec). Leaf size: 147

```
dsolve(x=diff(y(x),x)*sqrt( (diff(y(x),x))^2+1),y(x), singsol=all)
```

$$y(x) = -\frac{i(-32x^4 - 4x^2 + 1)\sinh\left(\frac{3\,\mathrm{arcsinh}(2x)}{2}\right)}{3\sqrt{4x^2+1}} - \frac{16ix^3\cosh\left(\frac{3\,\mathrm{arcsinh}(2x)}{2}\right)}{3} + c_1$$

$$y(x) = \frac{i(-32x^4 - 4x^2 + 1)\sinh\left(\frac{3\,\mathrm{arcsinh}(2x)}{2}\right)}{3\sqrt{4x^2+1}} + \frac{16ix^3\cosh\left(\frac{3\,\mathrm{arcsinh}(2x)}{2}\right)}{3} + c_1$$

$$y(x) = -\frac{\left(\int \sqrt{2\sqrt{4x^2+1} - 2}dx\right)}{2} + c_1$$

$$y(x) = \frac{\left(\int \sqrt{2\sqrt{4x^2+1} - 2}dx\right)}{2} + c_1$$

✓ Solution by Mathematica

Time used: 0.161 (sec). Leaf size: 207

```
DSolve[x==y'[x]*Sqrt[ (y'[x])^2+1],y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to -\frac{\sqrt{2}x\left(\sqrt{4x^2+1}-2\right)}{3\sqrt{\sqrt{4x^2+1}-1}}+c_1$$

$$y(x) \to \frac{\sqrt{2}x\left(\sqrt{4x^2+1}-2\right)}{3\sqrt{\sqrt{4x^2+1}-1}}+c_1$$

$$y(x) \to -\frac{\sqrt{2}x\left(4x^2+3\sqrt{4x^2+1}+3\right)}{3\left(-\sqrt{4x^2+1}-1\right)^{3/2}}+c_1$$

$$y(x) \to \frac{\sqrt{2}x\left(4x^2+3\sqrt{4x^2+1}+3\right)}{3\left(-\sqrt{4x^2+1}-1\right)^{3/2}}+c_1$$

1.82 problem 85

Internal problem ID [3227]
Internal file name [OUTPUT/2719_Sunday_June_05_2022_08_39_14_AM_16286413/index.tex]

Book: Differential equations for engineers by Wei-Chau XIE, Cambridge Press 2010
Section: Chapter 2. First-Order and Simple Higher-Order Differential Equations. Page 78
Problem number: 85.
ODE order: 1.
ODE degree: 3.

The type(s) of ODE detected by this program : **"clairaut"**

Maple gives the following as the ode type

```
[[_1st_order, _with_linear_symmetries], _Clairaut]
```

$$\boxed{2y'^2(-xy'+y)=1}$$

1.82.1 Solving as clairaut ode

This is Clairaut ODE. It has the form

$$y=xy'+g(y')$$

Where g is function of $y'(x)$. Let $p=y'$ the ode becomes

$$2p^2(-xp+y)=1$$

Solving for y from the above results in

$$y=\frac{2p^3x+1}{2p^2} \tag{1A}$$

The above ode is a Clairaut ode which is now solved. We start by replacing y' by p which gives

$$\begin{aligned} y &= xp+\frac{1}{2p^2} \\ &= xp+\frac{1}{2p^2} \end{aligned}$$

Writing the ode as

$$y = xp + g(p)$$

We now write $g \equiv g(p)$ to make notation simpler but we should always remember that g is function of p which in turn is function of x. Hence the above becomes

$$y = xp + g \tag{1}$$

Then we see that

$$g = \frac{1}{2p^2}$$

Taking derivative of (1) w.r.t. x gives

$$\begin{aligned}
p &= \frac{d}{dx}(xp + g) \\
p &= \left(p + x\frac{dp}{dx}\right) + \left(g'\frac{dp}{dx}\right) \\
p &= p + (x + g')\frac{dp}{dx} \\
0 &= (x + g')\frac{dp}{dx}
\end{aligned}$$

Where g' is derivative of $g(p)$ w.r.t. p. The general solution is given by

$$\begin{aligned}
\frac{dp}{dx} &= 0 \\
p &= c_1
\end{aligned}$$

Substituting this in (1) gives the general solution as

$$y = c_1 x + \frac{1}{2c_1^2}$$

The singular solution is found from solving for p from

$$x + g'(p) = 0$$

And substituting the result back in (1). Since we found above that $g = \frac{1}{2p^2}$, then the above equation becomes

$$\begin{aligned}
x + g'(p) &= x - \frac{1}{p^3} \\
&= 0
\end{aligned}$$

Solving the above for p results in

$$
\begin{aligned}
p_1 &= \frac{1}{x^{\frac{1}{3}}} \\
p_2 &= -\frac{1}{2x^{\frac{1}{3}}} + \frac{i\sqrt{3}}{2x^{\frac{1}{3}}} \\
p_3 &= -\frac{1}{2x^{\frac{1}{3}}} - \frac{i\sqrt{3}}{2x^{\frac{1}{3}}}
\end{aligned}
$$

Substituting the above back in (1) results in

$$
\begin{aligned}
y_1 &= \frac{3x^{\frac{2}{3}}}{2} \\
y_2 &= -\frac{3x^{\frac{2}{3}}}{1 + i\sqrt{3}} \\
y_3 &= \frac{3x^{\frac{2}{3}}}{-1 + i\sqrt{3}}
\end{aligned}
$$

Summary

The solution(s) found are the following

$$
y = c_1 x + \frac{1}{2c_1^2} \tag{1}
$$

$$
y = \frac{3x^{\frac{2}{3}}}{2} \tag{2}
$$

$$
y = -\frac{3x^{\frac{2}{3}}}{1 + i\sqrt{3}} \tag{3}
$$

$$
y = \frac{3x^{\frac{2}{3}}}{-1 + i\sqrt{3}} \tag{4}
$$

Verification of solutions

$$y = c_1 x + \frac{1}{2c_1^2}$$

Verified OK.

$$y = \frac{3x^{\frac{2}{3}}}{2}$$

Verified OK.

$$y = -\frac{3x^{\frac{2}{3}}}{1 + i\sqrt{3}}$$

Verified OK.

$$y = \frac{3x^{\frac{2}{3}}}{-1 + i\sqrt{3}}$$

Verified OK.

Maple trace

```
`Methods for first order ODEs:
   *** Sublevel 2 ***
   Methods for first order ODEs:
   -> Solving 1st order ODE of high degree, 1st attempt
   trying 1st order WeierstrassP solution for high degree ODE
   trying 1st order WeierstrassPPrime solution for high degree ODE
   trying 1st order JacobiSN solution for high degree ODE
   trying 1st order ODE linearizable_by_differentiation
   trying differential order: 1; missing variables
   trying dAlembert
   <- dAlembert successful`
```

✓ Solution by Maple

Time used: 0.125 (sec). Leaf size: 53

```
dsolve(2*(diff(y(x),x))^2*(y(x)-x*diff(y(x),x))=1,y(x), singsol=all)
```

$$y(x) = \frac{3x^{\frac{2}{3}}}{2}$$
$$y(x) = -\frac{3x^{\frac{2}{3}}\left(1 + i\sqrt{3}\right)}{4}$$
$$y(x) = \frac{3x^{\frac{2}{3}}\left(i\sqrt{3} - 1\right)}{4}$$
$$y(x) = c_1 x + \frac{1}{2c_1^2}$$

✓ Solution by Mathematica

Time used: 0.012 (sec). Leaf size: 67

```
DSolve[2*(y'[x])^2*(y[x]-x*y'[x])==1,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to c_1 x + \frac{1}{2{c_1}^2}$$
$$y(x) \to \frac{3x^{2/3}}{2}$$
$$y(x) \to -\frac{3}{2}\sqrt[3]{-1}x^{2/3}$$
$$y(x) \to \frac{3}{2}(-1)^{2/3}x^{2/3}$$

1.83 problem 86

Internal problem ID [3228]
Internal file name [`OUTPUT/2720_Sunday_June_05_2022_08_39_16_AM_49114467/index.tex`]

Book: Differential equations for engineers by Wei-Chau XIE, Cambridge Press 2010
Section: Chapter 2. First-Order and Simple Higher-Order Differential Equations. Page 78
Problem number: 86.
ODE order: 1.
ODE degree: 3.

The type(s) of ODE detected by this program : **"first_order_ode_lie_symmetry_calculated"**

Maple gives the following as the ode type

`[[_1st_order, _with_linear_symmetries]]`

$$\boxed{y - 2xy' - y^2 {y'}^3 = 0}$$

Solving the given ode for y' results in 3 differential equations to solve. Each one of these will generate a solution. The equations generated are

$$y' = \frac{\left(108y^2 + 12\sqrt{3}\,\sqrt{27y^4 + 32x^3}\right)^{\frac{1}{3}}}{6y} - \frac{4x}{y\left(108y^2 + 12\sqrt{3}\,\sqrt{27y^4 + 32x^3}\right)^{\frac{1}{3}}} \tag{1}$$

$$y' = -\frac{\left(108y^2 + 12\sqrt{3}\,\sqrt{27y^4 + 32x^3}\right)^{\frac{1}{3}}}{12y} + \frac{2x}{y\left(108y^2 + 12\sqrt{3}\,\sqrt{27y^4 + 32x^3}\right)^{\frac{1}{3}}} + \frac{i\sqrt{3}\left(\frac{\left(108y^2+12\sqrt{3}\,\sqrt{27y^4+32x}\right.}{6y}\right.}{} \tag{2}$$

$$y' = -\frac{\left(108y^2 + 12\sqrt{3}\,\sqrt{27y^4 + 32x^3}\right)^{\frac{1}{3}}}{12y} + \frac{2x}{y\left(108y^2 + 12\sqrt{3}\,\sqrt{27y^4 + 32x^3}\right)^{\frac{1}{3}}} - \frac{i\sqrt{3}\left(\frac{\left(108y^2+12\sqrt{3}\,\sqrt{27y^4+32x}\right.}{6y}\right.}{} \tag{3}$$

Now each one of the above ODE is solved.

Solving equation (1)

Writing the ode as

$$
\begin{aligned}
y' &= \frac{\left(108y^2 + 12\sqrt{3}\,\sqrt{27y^4 + 32x^3}\right)^{\frac{2}{3}} - 24x}{6y\left(108y^2 + 12\sqrt{3}\,\sqrt{27y^4 + 32x^3}\right)^{\frac{1}{3}}} \\
y' &= \omega(x, y)
\end{aligned}
$$

The condition of Lie symmetry is the linearized PDE given by

$$
\eta_x + \omega(\eta_y - \xi_x) - \omega^2 \xi_y - \omega_x \xi - \omega_y \eta = 0 \tag{A}
$$

The type of this ode is not in the lookup table. To determine ξ, η then (A) is solved using ansatz. Making bivariate polynomials of degree 1 to use as anstaz gives

$$
\xi = x a_2 + y a_3 + a_1 \tag{1E}
$$

$$
\eta = x b_2 + y b_3 + b_1 \tag{2E}
$$

Where the unknown coefficients are

$$
\{a_1, a_2, a_3, b_1, b_2, b_3\}
$$

Substituting equations (1E,2E) and ω into (A) gives

$$b_2+\frac{\left(\left(108y^2+12\sqrt{3}\,\sqrt{27y^4+32x^3}\right)^{\frac{2}{3}}-24x\right)(b_3-a_2)}{6y\left(108y^2+12\sqrt{3}\,\sqrt{27y^4+32x^3}\right)^{\frac{1}{3}}}-\frac{\left(\left(108y^2+12\sqrt{3}\,\sqrt{27y^4+32x^3}\right)^{\frac{2}{3}}-24x\right)^2 a_3}{36y^2\left(108y^2+12\sqrt{3}\,\sqrt{27y^4+32x^3}\right)^{\frac{2}{3}}}-\left(\frac{\frac{384\sqrt{3}\,x^2}{\left(108y^2+12\sqrt{3}\,\sqrt{27y^4+32x^3}\right)^{\frac{1}{3}}\sqrt{27y^4+32x^3}}-24}{6y\left(108y^2+12\sqrt{3}\,\sqrt{27y^4+32x^3}\right)^{\frac{1}{3}}}-\frac{32\left(\left(108y^2+12\sqrt{3}\,\sqrt{27y^4+32x^3}\right)^{\frac{2}{3}}-24x\right)\sqrt{3}\,x^2}{y\left(108y^2+12\sqrt{3}\,\sqrt{27y^4+32x^3}\right)^{\frac{4}{3}}\sqrt{27y^4+32x^3}}\right)(xa_2+ya_3+a_1)-\left(\frac{216y+\frac{648\sqrt{3}\,y^3}{\sqrt{27y^4+32x^3}}}{9y\left(108y^2+12\sqrt{3}\,\sqrt{27y^4+32x^3}\right)^{\frac{2}{3}}}-\frac{\left(108y^2+12\sqrt{3}\,\sqrt{27y^4+32x^3}\right)^{\frac{2}{3}}-24x}{6y^2\left(108y^2+12\sqrt{3}\,\sqrt{27y^4+32x^3}\right)^{\frac{1}{3}}}-\frac{\left(\left(108y^2+12\sqrt{3}\,\sqrt{27y^4+32x^3}\right)^{\frac{2}{3}}-24x\right)\left(216y+\frac{648\sqrt{3}\,y^3}{\sqrt{27y^4+32x^3}}\right)}{18y\left(108y^2+12\sqrt{3}\,\sqrt{27y^4+32x^3}\right)^{\frac{4}{3}}}\right)(xb_2+yb_3+b_1)=0 \tag{5E}$$

Putting the above in normal form gives

$$-\frac{216\left(108y^2+12\sqrt{3}\,\sqrt{27y^4+32x^3}\right)^{\frac{2}{3}}\sqrt{3}\,y^5b_3-\sqrt{27y^4+32x^3}\left(108y^2+12\sqrt{3}\,\sqrt{27y^4+32x^3}\right)^{\frac{5}{3}}xb_2+\sqrt{27}}{}$$

$$=0$$

Setting the numerator to zero gives

$$
\begin{aligned}
&-216\left(108y^2+12\sqrt{3}\,\sqrt{27y^4+32x^3}\right)^{\frac{2}{3}}\sqrt{3}\,y^5b_3\\
&+\sqrt{27y^4+32x^3}\left(108y^2+12\sqrt{3}\,\sqrt{27y^4+32x^3}\right)^{\frac{5}{3}}xb_2\\
&-\sqrt{27y^4+32x^3}\left(108y^2+12\sqrt{3}\,\sqrt{27y^4+32x^3}\right)^{\frac{5}{3}}ya_2\\
&+2\sqrt{27y^4+32x^3}\left(108y^2+12\sqrt{3}\,\sqrt{27y^4+32x^3}\right)^{\frac{5}{3}}yb_3\\
&+6b_2y^2\left(108y^2+12\sqrt{3}\,\sqrt{27y^4+32x^3}\right)^{\frac{4}{3}}\sqrt{27y^4+32x^3}\\
&-216\left(108y^2+12\sqrt{3}\,\sqrt{27y^4+32x^3}\right)^{\frac{2}{3}}\sqrt{3}\,y^4b_1\\
&-12960\sqrt{3}\,x^2y^4b_2-20736\sqrt{3}\,x\,y^5b_3\\
&+8\sqrt{27y^4+32x^3}\left(108y^2+12\sqrt{3}\,\sqrt{27y^4+32x^3}\right)^{\frac{4}{3}}xa_3\\
&-72\sqrt{27y^4+32x^3}\left(108y^2\right.\\
&\left.+12\sqrt{3}\,\sqrt{27y^4+32x^3}\right)^{\frac{2}{3}}y^3b_3\\
&+13824\sqrt{3}\,x^4ya_2-9216\sqrt{3}\,x^3y^2a_3\\
&-12960\sqrt{3}\,x\,y^4b_1-96\sqrt{27y^4+32x^3}\left(108y^2\right.\\
&\left.+12\sqrt{3}\,\sqrt{27y^4+32x^3}\right)^{\frac{2}{3}}x^2a_3\\
&-72\sqrt{27y^4+32x^3}\left(108y^2+12\sqrt{3}\,\sqrt{27y^4+32x^3}\right)^{\frac{2}{3}}y^2b_1\\
&-4320\sqrt{27y^4+32x^3}\,x^2y^2b_2\\
&-6912\sqrt{27y^4+32x^3}\,x\,y^3b_3+4608\sqrt{3}\,x^3ya_1\\
&-4320\sqrt{27y^4+32x^3}\,x\,y^2b_1+15552\sqrt{3}\,x\,y^5a_2\\
&-18432\sqrt{3}\,x^4yb_3+5184\sqrt{27y^4+32x^3}\,x\,y^3a_2\\
&-216\left(108y^2+12\sqrt{3}\,\sqrt{27y^4+32x^3}\right)^{\frac{2}{3}}\sqrt{3}\,x\,y^4b_2\\
&-192\left(108y^2+12\sqrt{3}\,\sqrt{27y^4+32x^3}\right)^{\frac{2}{3}}\sqrt{3}\,x^3ya_2\\
&-192\left(108y^2+12\sqrt{3}\,\sqrt{27y^4+32x^3}\right)^{\frac{2}{3}}\sqrt{3}\,x^2y^2a_3\\
&-72\sqrt{27y^4+32x^3}\left(108y^2\right.\\
&\left.+12\sqrt{3}\,\sqrt{27y^4+32x^3}\right)^{\frac{2}{3}}x\,y^2b_2\\
&-192\left(108y^2+12\sqrt{3}\,\sqrt{27y^4+32x^3}\right)^{\frac{2}{3}}\sqrt{3}\,x^2ya_1\\
&-3888\sqrt{3}\,y^6a_3-9216\sqrt{3}\,x^5b_2\\
&+7776\sqrt{3}\,y^5a_1-9216\sqrt{3}\,x^4b_1\\
&+\sqrt{27y^4+32x^3}\left(108y^2+12\sqrt{3}\,\sqrt{27y^4+32x^3}\right)^{\frac{5}{3}}b_1
\end{aligned}
\qquad (6E)
$$

Simplifying the above gives

$$\text{Expression too large to display} \tag{6E}$$

Since the PDE has radicals, simplifying gives

$$\text{Expression too large to display}$$

Looking at the above PDE shows the following are all the terms with $\{x, y\}$ in them.

$$\left\{x, y, \left(108y^2+12\sqrt{3}\,\sqrt{27y^4+32x^3}\right)^{\frac{1}{3}}, \left(108y^2+12\sqrt{3}\,\sqrt{27y^4+32x^3}\right)^{\frac{2}{3}}, \sqrt{27y^4+32x^3}\right\}$$

The following substitution is now made to be able to collect on all terms with $\{x, y\}$ in them

$$\left\{x = v_1, y = v_2, \left(108y^2 + 12\sqrt{3}\,\sqrt{27y^4+32x^3}\right)^{\frac{1}{3}} = v_3, \left(108y^2 + 12\sqrt{3}\,\sqrt{27y^4+32x^3}\right)^{\frac{2}{3}} = v_4, \sqrt{27y^4+32x^3} = v_5\right\}$$

The above PDE (6E) now becomes

$$\begin{gathered}
648v_4\sqrt{3}\,v_2^4b_1 - 77760\sqrt{3}\,v_1^2v_2^4b_2 - 124416\sqrt{3}\,v_1v_2^5b_3 + 864v_5v_4v_2^3b_3 \\
+ 82944\sqrt{3}\,v_1^4v_2a_2 - 55296\sqrt{3}\,v_1^3v_2^2a_3 - 77760\sqrt{3}\,v_1v_2^4b_1 \\
- 576v_5v_4v_1^2a_3 + 216v_5v_4v_2^2b_1 - 25920v_5v_1^2v_2^2b_2 - 41472v_5v_1v_2^3b_3 \\
+ 27648\sqrt{3}\,v_1^3v_2a_1 - 25920v_5v_1v_2^2b_1 + 93312\sqrt{3}\,v_1v_2^5a_2 \\
- 110592\sqrt{3}\,v_1^4v_2b_3 - 648v_4v_5v_2^3a_2 + 3888v_3v_5v_2^4b_2 \\
+ 2304\sqrt{3}\,v_1^3v_4b_1 - 1944\sqrt{3}\,v_4v_2^5a_2 + 11664\sqrt{3}\,v_3v_2^6b_2 \\
+ 2304\sqrt{3}\,v_1^4v_4b_2 + 18432\sqrt{3}\,v_1^4v_3a_3 + 31104v_5v_1v_2^3a_2 \\
+ 2592v_4\sqrt{3}\,v_2^5b_3 + 5184v_1v_3v_5v_2^2a_3 + 4608\sqrt{3}\,v_1^3v_4v_2b_3 \\
+ 13824\sqrt{3}\,v_1^3v_3v_2^2b_2 + 15552\sqrt{3}\,v_1v_3v_2^4a_3 + 648v_4\sqrt{3}\,v_1v_2^4b_2 \\
- 3456v_4\sqrt{3}\,v_1^3v_2a_2 - 1152v_4\sqrt{3}\,v_1^2v_2^2a_3 + 216v_5v_4v_1v_2^2b_2 \\
- 1152v_4\sqrt{3}\,v_1^2v_2a_1 - 13824v_1^3v_5a_3 - 23328\sqrt{3}\,v_2^6a_3 - 55296\sqrt{3}\,v_1^5b_2 \\
+ 46656\sqrt{3}\,v_2^5a_1 - 55296\sqrt{3}\,v_1^4b_1 - 7776v_5v_2^4a_3 + 15552v_5v_2^3a_1 = 0
\end{gathered} \tag{7E}$$

Collecting the above on the terms v_i introduced, and these are

$$\{v_1, v_2, v_3, v_4, v_5\}$$

Equation (7E) now becomes

$$
\begin{aligned}
&648v_4\sqrt{3}\,v_2^4b_1 - 77760\sqrt{3}\,v_1^2v_2^4b_2 - 55296\sqrt{3}\,v_1^3v_2^2a_3 \\
&- 77760\sqrt{3}\,v_1v_2^4b_1 - 576v_5v_4v_1^2a_3 + 216v_5v_4v_2^2b_1 - 25920v_5v_1^2v_2^2b_2 \\
&+ 27648\sqrt{3}\,v_1^3v_2a_1 - 25920v_5v_1v_2^2b_1 + 3888v_3v_5v_2^4b_2 \\
&+ 2304\sqrt{3}\,v_1^3v_4b_1 + 11664\sqrt{3}\,v_3v_2^6b_2 + 2304\sqrt{3}\,v_1^4v_4b_2 \\
&+ 18432\sqrt{3}\,v_1^4v_3a_3 + \left(82944\sqrt{3}\,a_2 - 110592\sqrt{3}\,b_3\right)v_1^4v_2 \\
&+ \left(93312\sqrt{3}\,a_2 - 124416\sqrt{3}\,b_3\right)v_1v_2^5 \\
&+ \left(-1944\sqrt{3}\,a_2 + 2592\sqrt{3}\,b_3\right)v_2^5v_4 + 5184v_1v_3v_5v_2^2a_3 \\
&+ 13824\sqrt{3}\,v_1^3v_3v_2^2b_2 + 15552\sqrt{3}\,v_1v_3v_2^4a_3 + 648v_4\sqrt{3}\,v_1v_2^4b_2 \\
&- 1152v_4\sqrt{3}\,v_1^2v_2^2a_3 + 216v_5v_4v_1v_2^2b_2 - 1152v_4\sqrt{3}\,v_1^2v_2a_1 \\
&+ (-648a_2 + 864b_3)\,v_2^3v_4v_5 + \left(-3456\sqrt{3}\,a_2 + 4608\sqrt{3}\,b_3\right)v_1^3v_2v_4 \\
&+ (31104a_2 - 41472b_3)\,v_1v_2^3v_5 - 13824v_1^3v_5a_3 \\
&- 23328\sqrt{3}\,v_2^6a_3 - 55296\sqrt{3}\,v_1^5b_2 + 46656\sqrt{3}\,v_2^5a_1 \\
&- 55296\sqrt{3}\,v_1^4b_1 - 7776v_5v_2^4a_3 + 15552v_5v_2^3a_1 = 0
\end{aligned}
\tag{8E}
$$

Setting each coefficients in (8E) to zero gives the following equations to solve

$$
\begin{aligned}
15552a_1 &= 0\\
-13824a_3 &= 0\\
-7776a_3 &= 0\\
-576a_3 &= 0\\
5184a_3 &= 0\\
-25920b_1 &= 0\\
216b_1 &= 0\\
-25920b_2 &= 0\\
216b_2 &= 0\\
3888b_2 &= 0\\
-1152\sqrt{3}\,a_1 &= 0\\
27648\sqrt{3}\,a_1 &= 0\\
46656\sqrt{3}\,a_1 &= 0\\
-55296\sqrt{3}\,a_3 &= 0\\
-23328\sqrt{3}\,a_3 &= 0\\
-1152\sqrt{3}\,a_3 &= 0\\
15552\sqrt{3}\,a_3 &= 0\\
18432\sqrt{3}\,a_3 &= 0\\
-77760\sqrt{3}\,b_1 &= 0\\
-55296\sqrt{3}\,b_1 &= 0\\
648\sqrt{3}\,b_1 &= 0\\
2304\sqrt{3}\,b_1 &= 0\\
-77760\sqrt{3}\,b_2 &= 0\\
-55296\sqrt{3}\,b_2 &= 0\\
648\sqrt{3}\,b_2 &= 0\\
2304\sqrt{3}\,b_2 &= 0\\
11664\sqrt{3}\,b_2 &= 0\\
13824\sqrt{3}\,b_2 &= 0\\
-648a_2 + 864b_3 &= 0\\
31104a_2 - 41472b_3 &= 0\\
-3456\sqrt{3}\,a_2 + 4608\sqrt{3}\,b_3 &= 0\\
-1944\sqrt{3}\,a_2 + 2592\sqrt{3}\,b_3 &= 0\\
82944\sqrt{3}\,a_2 - 110592\sqrt{3}\,b_3 &= 0\\
93312\sqrt{3}\,a_2 - 124416\sqrt{3}\,b_3 &= 0
\end{aligned}
$$

Solving the above equations for the unknowns gives

$$
\begin{aligned}
a_1 &= 0\\
a_2 &= \frac{4b_3}{3}\\
a_3 &= 0\\
b_1 &= 0\\
b_2 &= 0\\
b_3 &= b_3
\end{aligned}
$$

Substituting the above solution in the anstaz (1E,2E) (using 1 as arbitrary value for any unknown in the RHS) gives

$$
\begin{aligned}
\xi &= \frac{4x}{3}\\
\eta &= y
\end{aligned}
$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x, y) \to (R, S)$ where (R, S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$
\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}
$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y}\right) S(x, y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the canonical coordinates, where $S(R)$. Therefore

$$
\begin{aligned}
\frac{dy}{dx} &= \frac{\eta}{\xi}\\
&= \frac{y}{\frac{4x}{3}}\\
&= \frac{3y}{4x}
\end{aligned}
$$

This is easily solved to give

$$
y = c_1 x^{\frac{3}{4}}
$$

Where now the coordinate R is taken as the constant of integration. Hence

$$
R = \frac{y}{x^{\frac{3}{4}}}
$$

And S is found from

$$\begin{aligned} dS &= \frac{dx}{\xi} \\ &= \frac{dx}{\frac{4x}{3}} \end{aligned}$$

Integrating gives

$$\begin{aligned} S &= \int \frac{dx}{T} \\ &= \frac{3\ln(x)}{4} \end{aligned}$$

Where the constant of integration is set to zero as we just need one solution. Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x,y)S_y}{R_x + \omega(x,y)R_y} \tag{2}$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x,y)$ is the right hand side of the original ode given by

$$\omega(x,y) = \frac{\left(108y^2 + 12\sqrt{3}\,\sqrt{27y^4 + 32x^3}\right)^{\frac{2}{3}} - 24x}{6y\left(108y^2 + 12\sqrt{3}\,\sqrt{27y^4 + 32x^3}\right)^{\frac{1}{3}}}$$

Evaluating all the partial derivatives gives

$$\begin{aligned} R_x &= -\frac{3y}{4x^{\frac{7}{4}}} \\ R_y &= \frac{1}{x^{\frac{3}{4}}} \\ S_x &= \frac{3}{4x} \\ S_y &= 0 \end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = \frac{9x^{\frac{3}{4}}y\left(108y^2 + 12\sqrt{3}\,\sqrt{27y^4 + 32x^3}\right)^{\frac{1}{3}}}{2\left(108y^2 + 12\sqrt{3}\,\sqrt{27y^4 + 32x^3}\right)^{\frac{2}{3}} x - 9y^2\left(108y^2 + 12\sqrt{3}\,\sqrt{27y^4 + 32x^3}\right)^{\frac{1}{3}} - 48x^2} \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = \frac{9R12^{\frac{1}{3}}\left(\sqrt{3}\,\sqrt{27R^4+32}+9R^2\right)^{\frac{1}{3}}}{2\,12^{\frac{2}{3}}\left(\sqrt{3}\,\sqrt{27R^4+32}+9R^2\right)^{\frac{2}{3}} - 9\,12^{\frac{1}{3}}\left(\sqrt{3}\,\sqrt{27R^4+32}+9R^2\right)^{\frac{1}{3}}R^2 - 48}$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$S(R) = \int \frac{9R\left(12\sqrt{81R^4+96}+108R^2\right)^{\frac{1}{3}}}{4\,18^{\frac{1}{3}}\left(\left(\sqrt{81R^4+96}+9R^2\right)^2\right)^{\frac{1}{3}} - 9R^2\left(12\sqrt{81R^4+96}+108R^2\right)^{\frac{1}{3}} - 48}dR + c_1 \tag{4}$$

To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$\frac{3\ln(x)}{4} = \int^{\frac{y}{x^{\frac{3}{4}}}} \frac{9\_a\left(12\sqrt{81\_a^4+96}+108\_a^2\right)^{\frac{1}{3}}}{4\,18^{\frac{1}{3}}\left(\left(\sqrt{81\_a^4+96}+9\_a^2\right)^2\right)^{\frac{1}{3}} - 9\_a^2\left(12\sqrt{81\_a^4+96}+108\_a^2\right)^{\frac{1}{3}} - 48}d\_a + c_1$$

Which simplifies to

$$\frac{3\ln(x)}{4} = \int^{\frac{y}{x^{\frac{3}{4}}}} \frac{9\_a\left(12\sqrt{81\_a^4+96}+108\_a^2\right)^{\frac{1}{3}}}{4\,18^{\frac{1}{3}}\left(\left(\sqrt{81\_a^4+96}+9\_a^2\right)^2\right)^{\frac{1}{3}} - 9\_a^2\left(12\sqrt{81\_a^4+96}+108\_a^2\right)^{\frac{1}{3}} - 48}d\_a + c_1$$

Summary

The solution(s) found are the following

$$\frac{3\ln(x)}{4} = \int^{\frac{y}{x^{\frac{3}{4}}}} \frac{9\_a\left(12\sqrt{81\_a^4+96}+108\_a^2\right)^{\frac{1}{3}}}{4\,18^{\frac{1}{3}}\left(\left(\sqrt{81\_a^4+96}+9\_a^2\right)^2\right)^{\frac{1}{3}} - 9\_a^2\left(12\sqrt{81\_a^4+96}+108\_a^2\right)^{\frac{1}{3}} - 48}d\_a + c_1 \tag{1}$$

Verification of solutions

$$\frac{3\ln(x)}{4} = \int^{\frac{y}{x^{\frac{3}{4}}}} \frac{9\_a\left(12\sqrt{81\_a^4+96}+108\_a^2\right)^{\frac{1}{3}}}{4\,18^{\frac{1}{3}}\left(\left(\sqrt{81\_a^4+96}+9\_a^2\right)^2\right)^{\frac{1}{3}} - 9\_a^2\left(12\sqrt{81\_a^4+96}+108\_a^2\right)^{\frac{1}{3}} - 48} d\_a + c_1$$

Verified OK.

Solving equation (2)

Writing the ode as

$$y' = \frac{i\sqrt{3}\left(108y^2+12\sqrt{3}\sqrt{27y^4+32x^3}\right)^{\frac{2}{3}} + 24i\sqrt{3}\,x - \left(108y^2+12\sqrt{3}\sqrt{27y^4+32x^3}\right)^{\frac{2}{3}} + 24x}{12y\left(108y^2+12\sqrt{3}\sqrt{27y^4+32x^3}\right)^{\frac{1}{3}}}$$

$$y' = \omega(x,y)$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x + \omega(\eta_y - \xi_x) - \omega^2\xi_y - \omega_x\xi - \omega_y\eta = 0 \tag{A}$$

The type of this ode is not in the lookup table. To determine ξ, η then (A) is solved using ansatz. Making bivariate polynomials of degree 1 to use as anstaz gives

$$\xi = xa_2 + ya_3 + a_1 \tag{1E}$$

$$\eta = xb_2 + yb_3 + b_1 \tag{2E}$$

Where the unknown coefficients are

$$\{a_1, a_2, a_3, b_1, b_2, b_3\}$$

Substituting equations (1E,2E) and ω into (A) gives

$$b_2 \tag{5E}$$
$$+\frac{\left(i\sqrt{3}\left(108y^2+12\sqrt{3}\sqrt{27y^4+32x^3}\right)^{\frac{2}{3}}+24i\sqrt{3}\,x-\left(108y^2+12\sqrt{3}\sqrt{27y^4+32x^3}\right)^{\frac{2}{3}}+24x\right)(b_3-a_2)}{12y\left(108y^2+12\sqrt{3}\sqrt{27y^4+32x^3}\right)^{\frac{1}{3}}}$$
$$-\frac{\left(i\sqrt{3}\left(108y^2+12\sqrt{3}\sqrt{27y^4+32x^3}\right)^{\frac{2}{3}}+24i\sqrt{3}\,x-\left(108y^2+12\sqrt{3}\sqrt{27y^4+32x^3}\right)^{\frac{2}{3}}+24x\right)^2 a_3}{144y^2\left(108y^2+12\sqrt{3}\sqrt{27y^4+32x^3}\right)^{\frac{2}{3}}}$$
$$-\left(\frac{\frac{1152ix^2}{\left(108y^2+12\sqrt{3}\sqrt{27y^4+32x^3}\right)^{\frac{1}{3}}\sqrt{27y^4+32x^3}}+24i\sqrt{3}-\frac{384\sqrt{3}\,x^2}{\left(108y^2+12\sqrt{3}\sqrt{27y^4+32x^3}\right)^{\frac{1}{3}}\sqrt{27y^4+32x^3}}+24}{12y\left(108y^2+12\sqrt{3}\sqrt{27y^4+32x^3}\right)^{\frac{1}{3}}}\right.$$
$$\left.-\frac{16\left(i\sqrt{3}\left(108y^2+12\sqrt{3}\sqrt{27y^4+32x^3}\right)^{\frac{2}{3}}+24i\sqrt{3}\,x-\left(108y^2+12\sqrt{3}\sqrt{27y^4+32x^3}\right)^{\frac{2}{3}}+24x\right)\sqrt{3}\,x^2}{y\left(108y^2+12\sqrt{3}\sqrt{27y^4+32x^3}\right)^{\frac{4}{3}}\sqrt{27y^4+32x^3}}\right)$$
$$+ya_3+a_1)-\left(\frac{\frac{2i\sqrt{3}\left(216y+\frac{648\sqrt{3}\,y^3}{\sqrt{27y^4+32x^3}}\right)}{3\left(108y^2+12\sqrt{3}\sqrt{27y^4+32x^3}\right)^{\frac{1}{3}}}-\frac{2\left(216y+\frac{648\sqrt{3}\,y^3}{\sqrt{27y^4+32x^3}}\right)}{3\left(108y^2+12\sqrt{3}\sqrt{27y^4+32x^3}\right)^{\frac{1}{3}}}}{12y\left(108y^2+12\sqrt{3}\sqrt{27y^4+32x^3}\right)^{\frac{1}{3}}}\right.$$
$$-\frac{i\sqrt{3}\left(108y^2+12\sqrt{3}\sqrt{27y^4+32x^3}\right)^{\frac{2}{3}}+24i\sqrt{3}\,x-\left(108y^2+12\sqrt{3}\sqrt{27y^4+32x^3}\right)^{\frac{2}{3}}+24x}{12y^2\left(108y^2+12\sqrt{3}\sqrt{27y^4+32x^3}\right)^{\frac{1}{3}}}$$
$$-\frac{\left(i\sqrt{3}\left(108y^2+12\sqrt{3}\sqrt{27y^4+32x^3}\right)^{\frac{2}{3}}+24i\sqrt{3}\,x-\left(108y^2+12\sqrt{3}\sqrt{27y^4+32x^3}\right)^{\frac{2}{3}}+24x\right)\left(216y+\frac{}{\sqrt{}}\right.}{36y\left(108y^2+12\sqrt{3}\sqrt{27y^4+32x^3}\right)^{\frac{4}{3}}}$$
$$+yb_3+b_1)=0$$

Putting the above in normal form gives

Expression too large to display

Setting the numerator to zero gives

$$\text{Expression too large to display} \tag{6E}$$

Simplifying the above gives

$$\text{Expression too large to display} \tag{6E}$$

Since the PDE has radicals, simplifying gives

$$\text{Expression too large to display}$$

Looking at the above PDE shows the following are all the terms with $\{x, y\}$ in them.

$$\left\{x, y, \left(108y^2+12\sqrt{3}\,\sqrt{27y^4+32x^3}\right)^{\frac{1}{3}}, \left(108y^2+12\sqrt{3}\,\sqrt{27y^4+32x^3}\right)^{\frac{2}{3}}, \sqrt{27y^4+32x^3}\right\}$$

The following substitution is now made to be able to collect on all terms with $\{x, y\}$ in them

$$\left\{x = v_1, y = v_2, \left(108y^2+12\sqrt{3}\,\sqrt{27y^4+32x^3}\right)^{\frac{1}{3}} = v_3, \left(108y^2 + 12\sqrt{3}\,\sqrt{27y^4+32x^3}\right)^{\frac{2}{3}} = v_4, \sqrt{27y^4+32x^3} = v_5\right\}$$

The above PDE (6E) now becomes

$$
\begin{aligned}
&27648v_1^3v_5a_3 + 15552v_5v_2^4a_3 - 31104v_5v_2^3a_1\\
&+ 46656\sqrt{3}\,v_2^6a_3 + 110592\sqrt{3}\,v_1^5b_2 - 93312\sqrt{3}\,v_2^5a_1\\
&+ 110592\sqrt{3}\,v_1^4b_1 + 139968iv_2^6a_3 + 331776iv_1^5b_2\\
&- 279936iv_2^5a_1 + 331776iv_1^4b_1 - 1296i\sqrt{3}\,v_5v_4v_2^3a_2\\
&+ 1728iv_5v_4\sqrt{3}\,v_2^3b_3 + 432iv_5v_4\sqrt{3}\,v_2^2b_1\\
&+ 51840iv_5\sqrt{3}\,v_1^2v_2^2b_2 - 62208iv_5\sqrt{3}\,v_1v_2^3a_2\\
&+ 82944iv_5\sqrt{3}\,v_1v_2^3b_3 + 51840iv_5\sqrt{3}\,v_1v_2^2b_1\\
&- 1152iv_5v_4\sqrt{3}\,v_1^2a_3 + 432iv_5v_4\sqrt{3}\,v_1v_2^2b_2\\
&- 6912iv_4v_1^2v_2^2a_3 - 6912iv_4v_1^2v_2a_1 - 31104iv_5\sqrt{3}\,v_2^3a_1\\
&+ 3888iv_4v_1v_2^4b_2 - 20736iv_4v_1^3v_2a_2 + 27648iv_1^3v_4v_2b_3\\
&+ 15552iv_5\sqrt{3}\,v_2^4a_3 + 27648i\sqrt{3}\,v_1^3v_5a_3\\
&- 1296v_4\sqrt{3}\,v_1v_2^4b_2 + 6912v_4\sqrt{3}\,v_1^3v_2a_2\\
&+ 2304v_4\sqrt{3}\,v_1^2v_2^2a_3 - 432v_5v_4v_1v_2^2b_2 + 2304v_4\sqrt{3}\,v_1^2v_2a_1\\
&- 9216\sqrt{3}\,v_1^3v_4v_2b_3 + 55296\sqrt{3}\,v_1^3v_3v_2^2b_2\\
&+ 62208\sqrt{3}\,v_1v_3v_2^4a_3 + 20736v_1v_5v_3v_2^2a_3\\
&+ 663552iv_1^4v_2b_3 + 331776iv_1^3v_2^2a_3 + 466560iv_1v_2^4b_1\\
&- 11664iv_4v_2^5a_2 + 15552iv_4v_2^5b_3 + 13824iv_1^4v_4b_2\\
&+ 3888iv_4v_2^4b_1 + 13824iv_1^3v_4b_1 + 155520\sqrt{3}\,v_1^2v_2^4b_2\\
&+ 248832\sqrt{3}\,v_1v_2^5b_3 - 1728v_5v_4v_2^3b_3 - 165888\sqrt{3}\,v_1^4v_2a_2\\
&+ 110592\sqrt{3}\,v_1^3v_2^2a_3 + 155520\sqrt{3}\,v_1v_2^4b_1\\
&- 432v_5v_4v_2^2b_1 + 51840v_5v_1^2v_2^2b_2 + 82944v_5v_1v_2^3b_3\\
&- 55296\sqrt{3}\,v_1^3v_2a_1 + 51840v_5v_1v_2^2b_1 - 62208v_5v_1v_2^3a_2\\
&- 186624\sqrt{3}\,v_1v_2^5a_2 + 221184\sqrt{3}\,v_1^4v_2b_3\\
&+ 73728\sqrt{3}\,v_1^4v_3a_3 - 4608\sqrt{3}\,v_1^3v_4b_1 + 3888\sqrt{3}\,v_4v_2^5a_2\\
&+ 46656\sqrt{3}\,v_3v_2^6b_2 - 4608\sqrt{3}\,v_1^4v_4b_2 - 165888iv_1^3v_2a_1\\
&+ 466560iv_1^2v_2^4b_2 - 559872iv_1v_2^5a_2 + 746496iv_1v_2^5b_3\\
&- 497664iv_1^4v_2a_2 + 15552v_5v_3v_2^4b_2 + 1152v_5v_4v_1^2a_3\\
&- 5184v_4\sqrt{3}\,v_2^5b_3 - 1296v_4\sqrt{3}\,v_2^4b_1 + 1296v_5v_4v_2^3a_2 = 0
\end{aligned}
\tag{7E}
$$

Collecting the above on the terms v_i introduced, and these are

$$\{v_1, v_2, v_3, v_4, v_5\}$$

Equation (7E) now becomes

$$
\begin{aligned}
&\Big(-497664ia_2 + 663552ib_3 - 165888\sqrt{3}\,a_2 \\
&+ 221184\sqrt{3}\,b_3\Big)\, v_1^4 v_2 + \Big(13824ib_2 - 4608\sqrt{3}\,b_2\Big)\, v_1^4 v_4 \\
&+ \Big(331776ia_3 + 110592\sqrt{3}\,a_3\Big)\, v_1^3 v_2^2 \\
&+ \Big(-165888ia_1 - 55296\sqrt{3}\,a_1\Big)\, v_1^3 v_2 \\
&+ \Big(13824ib_1 - 4608\sqrt{3}\,b_1\Big)\, v_1^3 v_4 \\
&+ \Big(27648i\sqrt{3}\,a_3 + 27648a_3\Big)\, v_1^3 v_5 \\
&+ \Big(466560ib_2 + 155520\sqrt{3}\,b_2\Big)\, v_1^2 v_2^4 \\
&+ \Big(-559872ia_2 + 746496ib_3 - 186624\sqrt{3}\,a_2 \\
&+ 248832\sqrt{3}\,b_3\Big)\, v_1 v_2^5 + \Big(466560ib_1 + 155520\sqrt{3}\,b_1\Big)\, v_1 v_2^4 \\
&+ \Big(-11664ia_2 + 15552ib_3 + 3888\sqrt{3}\,a_2 - 5184\sqrt{3}\,b_3\Big)\, v_2^5 v_4 \\
&+ \Big(3888ib_1 - 1296\sqrt{3}\,b_1\Big)\, v_2^4 v_4 \\
&+ \Big(15552i\sqrt{3}\,a_3 + 15552a_3\Big)\, v_2^4 v_5 \\
&+ \Big(-31104i\sqrt{3}\,a_1 - 31104a_1\Big)\, v_2^3 v_5 \\
&+ \Big(432i\sqrt{3}\,b_2 - 432b_2\Big)\, v_1 v_2^2 v_4 v_5 \\
&+ \Big(331776ib_2 + 110592\sqrt{3}\,b_2\Big)\, v_1^5 \\
&+ \Big(331776ib_1 + 110592\sqrt{3}\,b_1\Big)\, v_1^4 \\
&+ \Big(139968ia_3 + 46656\sqrt{3}\,a_3\Big)\, v_2^6 \\
&+ \Big(-279936ia_1 - 93312\sqrt{3}\,a_1\Big)\, v_2^5 \\
&+ \Big(-1296i\sqrt{3}\,a_2 + 1728i\sqrt{3}\,b_3 + 1296a_2 - 1728b_3\Big)\, v_2^3 v_4 v_5 \\
&+ \Big(432i\sqrt{3}\,b_1 - 432b_1\Big)\, v_2^2 v_4 v_5 + 55296\sqrt{3}\, v_1^3 v_3 v_2^2 b_2 \\
&+ 62208\sqrt{3}\, v_1 v_3 v_2^4 a_3 + 20736 v_1 v_5 v_3 v_2^2 a_3 \\
&+ 73728\sqrt{3}\, v_1^4 v_3 a_3 + 46656\sqrt{3}\, v_3 v_2^6 b_2 + 15552 v_5 v_3 v_2^4 b_2 \\
&+ \Big(-20736ia_2 + 27648ib_3 + 6912\sqrt{3}\,a_2 \\
&- 9216\sqrt{3}\,b_3\Big)\, v_1^3 v_2 v_4 + \Big(-6912ia_3 + 2304\sqrt{3}\,a_3\Big)\, v_1^2 v_2^2 v_4 \\
&+ \Big(51840i\sqrt{3}\,b_2 + 51840b_2\Big)\, v_1^2 v_2^2 v_5 \\
&+ \Big(-6912ia_1 + 2304\sqrt{3}\,a_1\Big)\, v_1^2 v_2 v_4 \\
&+ \Big(-1152i\sqrt{3}\,a_3 + 1152a_3\Big)\, v_1^2 v_4 v_5 \\
&+ \Big(3888ib_2 - 1296\sqrt{3}\,b_2\Big)\, v_1 v_2^4 v_4 + \Big(-62208i\sqrt{3}\,a_2
\end{aligned}
\tag{8E}
$$

Setting each coefficients in (8E) to zero gives the following equations to solve

$$20736a_3 = 0$$
$$15552b_2 = 0$$
$$62208\sqrt{3}\,a_3 = 0$$
$$73728\sqrt{3}\,a_3 = 0$$
$$46656\sqrt{3}\,b_2 = 0$$
$$55296\sqrt{3}\,b_2 = 0$$
$$-279936ia_1 - 93312\sqrt{3}\,a_1 = 0$$
$$-165888ia_1 - 55296\sqrt{3}\,a_1 = 0$$
$$-6912ia_1 + 2304\sqrt{3}\,a_1 = 0$$
$$-6912ia_3 + 2304\sqrt{3}\,a_3 = 0$$
$$3888ib_1 - 1296\sqrt{3}\,b_1 = 0$$
$$3888ib_2 - 1296\sqrt{3}\,b_2 = 0$$
$$13824ib_1 - 4608\sqrt{3}\,b_1 = 0$$
$$13824ib_2 - 4608\sqrt{3}\,b_2 = 0$$
$$139968ia_3 + 46656\sqrt{3}\,a_3 = 0$$
$$331776ia_3 + 110592\sqrt{3}\,a_3 = 0$$
$$331776ib_1 + 110592\sqrt{3}\,b_1 = 0$$
$$331776ib_2 + 110592\sqrt{3}\,b_2 = 0$$
$$466560ib_1 + 155520\sqrt{3}\,b_1 = 0$$
$$466560ib_2 + 155520\sqrt{3}\,b_2 = 0$$
$$-31104i\sqrt{3}\,a_1 - 31104a_1 = 0$$
$$-1152i\sqrt{3}\,a_3 + 1152a_3 = 0$$
$$432i\sqrt{3}\,b_1 - 432b_1 = 0$$
$$432i\sqrt{3}\,b_2 - 432b_2 = 0$$
$$15552i\sqrt{3}\,a_3 + 15552a_3 = 0$$
$$27648i\sqrt{3}\,a_3 + 27648a_3 = 0$$
$$51840i\sqrt{3}\,b_1 + 51840b_1 = 0$$
$$51840i\sqrt{3}\,b_2 + 51840b_2 = 0$$
$$-559872ia_2 + 746496ib_3 - 186624\sqrt{3}\,a_2 + 248832\sqrt{3}\,b_3 = 0$$
$$-497664ia_2 + 663552ib_3 - 165888\sqrt{3}\,a_2 + 221184\sqrt{3}\,b_3 = 0$$
$$-20736ia_2 + 27648ib_3 + 6912\sqrt{3}\,a_2 - 9216\sqrt{3}\,b_3 = 0$$
$$-11664ia_2 + 15552ib_3 + 3888\sqrt{3}\,a_2 - 5184\sqrt{3}\,b_3 = 0$$
$$-62208i\sqrt{3}\,a_2 + 82944i\sqrt{3}\,b_3 - 62208a_2 + 82944b_3 = 0$$
$$-1296i\sqrt{3}\,a_2 + 1728i\sqrt{3}\,b_3 + 1296a_2 - 1728b_3 = 0$$

Solving the above equations for the unknowns gives

$$
\begin{aligned}
a_1 &= 0 \\
a_2 &= \frac{4b_3}{3} \\
a_3 &= 0 \\
b_1 &= 0 \\
b_2 &= 0 \\
b_3 &= b_3
\end{aligned}
$$

Substituting the above solution in the anstaz (1E,2E) (using 1 as arbitrary value for any unknown in the RHS) gives

$$
\begin{aligned}
\xi &= \frac{4x}{3} \\
\eta &= y
\end{aligned}
$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x, y) \to (R, S)$ where (R, S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$
\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}
$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y}\right) S(x, y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the canonical coordinates, where $S(R)$. Unable to determine R. Terminating

Unable to determine ODE type.

Solving equation (3)

Writing the ode as

$$
y' = -\frac{i\sqrt{3}\left(108y^2 + 12\sqrt{3}\,\sqrt{27y^4 + 32x^3}\right)^{\frac{2}{3}} + 24i\sqrt{3}\,x + \left(108y^2 + 12\sqrt{3}\,\sqrt{27y^4 + 32x^3}\right)^{\frac{2}{3}} - 24x}{12y\left(108y^2 + 12\sqrt{3}\,\sqrt{27y^4 + 32x^3}\right)^{\frac{1}{3}}}
$$

$$
y' = \omega(x, y)
$$

The condition of Lie symmetry is the linearized PDE given by

$$
\eta_x + \omega(\eta_y - \xi_x) - \omega^2 \xi_y - \omega_x \xi - \omega_y \eta = 0 \tag{A}
$$

The type of this ode is not in the lookup table. To determine ξ, η then (A) is solved using ansatz. Making bivariate polynomials of degree 1 to use as anstaz gives

$$\xi = xa_2 + ya_3 + a_1 \tag{1E}$$

$$\eta = xb_2 + yb_3 + b_1 \tag{2E}$$

Where the unknown coefficients are

$$\{a_1, a_2, a_3, b_1, b_2, b_3\}$$

Substituting equations (1E,2E) and ω into (A) gives

$$b_2 - \frac{\left(i\sqrt{3}\left(108y^2+12\sqrt{3}\sqrt{27y^4+32x^3}\right)^{\frac{2}{3}}+24i\sqrt{3}\,x+\left(108y^2+12\sqrt{3}\sqrt{27y^4+32x^3}\right)^{\frac{2}{3}}-24x\right)(b_3-a_2)}{12y\left(108y^2+12\sqrt{3}\sqrt{27y^4+32x^3}\right)^{\frac{1}{3}}} \tag{5E}$$

$$- \frac{\left(i\sqrt{3}\left(108y^2+12\sqrt{3}\sqrt{27y^4+32x^3}\right)^{\frac{2}{3}}+24i\sqrt{3}\,x+\left(108y^2+12\sqrt{3}\sqrt{27y^4+32x^3}\right)^{\frac{2}{3}}-24x\right)^2 a_3}{144y^2\left(108y^2+12\sqrt{3}\sqrt{27y^4+32x^3}\right)^{\frac{2}{3}}}$$

$$-\left(-\frac{-\frac{1152ix^2}{\left(108y^2+12\sqrt{3}\sqrt{27y^4+32x^3}\right)^{\frac{1}{3}}\sqrt{27y^4+32x^3}}+24i\sqrt{3}+\frac{384\sqrt{3}\,x^2}{\left(108y^2+12\sqrt{3}\sqrt{27y^4+32x^3}\right)^{\frac{1}{3}}\sqrt{27y^4+32x^3}}-24}{12y\left(108y^2+12\sqrt{3}\sqrt{27y^4+32x^3}\right)^{\frac{1}{3}}}\right.$$

$$\left.+\frac{16\left(i\sqrt{3}\left(108y^2+12\sqrt{3}\sqrt{27y^4+32x^3}\right)^{\frac{2}{3}}+24i\sqrt{3}\,x+\left(108y^2+12\sqrt{3}\sqrt{27y^4+32x^3}\right)^{\frac{2}{3}}-24x\right)\sqrt{3}\,x^2}{y\left(108y^2+12\sqrt{3}\sqrt{27y^4+32x^3}\right)^{\frac{4}{3}}\sqrt{27y^4+32x^3}}\right)(\cdots$$

$$+ya_3+a_1)-\left(-\frac{\frac{2i\sqrt{3}\left(216y+\frac{648\sqrt{3}\,y^3}{\sqrt{27y^4+32x^3}}\right)}{3\left(108y^2+12\sqrt{3}\sqrt{27y^4+32x^3}\right)^{\frac{1}{3}}}+\frac{144y+\frac{432\sqrt{3}\,y^3}{\sqrt{27y^4+32x^3}}}{\left(108y^2+12\sqrt{3}\sqrt{27y^4+32x^3}\right)^{\frac{1}{3}}}}{12y\left(108y^2+12\sqrt{3}\sqrt{27y^4+32x^3}\right)^{\frac{1}{3}}}\right.$$

$$+\frac{i\sqrt{3}\left(108y^2+12\sqrt{3}\sqrt{27y^4+32x^3}\right)^{\frac{2}{3}}+24i\sqrt{3}\,x+\left(108y^2+12\sqrt{3}\sqrt{27y^4+32x^3}\right)^{\frac{2}{3}}-24x}{12y^2\left(108y^2+12\sqrt{3}\sqrt{27y^4+32x^3}\right)^{\frac{1}{3}}}$$

$$+\frac{\left(i\sqrt{3}\left(108y^2+12\sqrt{3}\sqrt{27y^4+32x^3}\right)^{\frac{2}{3}}+24i\sqrt{3}\,x+\left(108y^2+12\sqrt{3}\sqrt{27y^4+32x^3}\right)^{\frac{2}{3}}-24x\right)\left(216y+\frac{6\cdots}{\sqrt{2\cdots}}\right.}{36y\left(108y^2+12\sqrt{3}\sqrt{27y^4+32x^3}\right)^{\frac{4}{3}}}$$

$$+yb_3+b_1)=0$$

Putting the above in normal form gives

Expression too large to display

Setting the numerator to zero gives

$$\text{Expression too large to display} \tag{6E}$$

Simplifying the above gives

$$\text{Expression too large to display} \tag{6E}$$

Since the PDE has radicals, simplifying gives

$$\text{Expression too large to display}$$

Looking at the above PDE shows the following are all the terms with $\{x, y\}$ in them.

$$\left\{x, y, \left(108y^2+12\sqrt{3}\,\sqrt{27y^4+32x^3}\right)^{\frac{1}{3}}, \left(108y^2+12\sqrt{3}\,\sqrt{27y^4+32x^3}\right)^{\frac{2}{3}}, \sqrt{27y^4+32x^3}\right\}$$

The following substitution is now made to be able to collect on all terms with $\{x, y\}$ in them

$$\left\{x = v_1, y = v_2, \left(108y^2 + 12\sqrt{3}\,\sqrt{27y^4+32x^3}\right)^{\frac{1}{3}} = v_3, \left(108y^2 + 12\sqrt{3}\,\sqrt{27y^4+32x^3}\right)^{\frac{2}{3}} = v_4, \sqrt{27y^4+32x^3} = v_5\right\}$$

The above PDE (6E) now becomes

$$
\begin{aligned}
&62208iv_5\sqrt{3}\,v_1v_2^3a_2 - 82944iv_5\sqrt{3}\,v_1v_2^3b_3\\
&- 51840iv_5\sqrt{3}\,v_1v_2^2b_1 + 1296i\sqrt{3}\,v_5v_4v_2^3a_2\\
&- 1728iv_5v_4\sqrt{3}\,v_2^3b_3 - 432iv_5v_4\sqrt{3}\,v_2^2b_1\\
&+ 1152iv_5v_4\sqrt{3}\,v_1^2a_3 - 51840iv_5\sqrt{3}\,v_1^2v_2^2b_2\\
&- 27648i\sqrt{3}\,v_1^3v_5a_3 - 3888iv_4v_1v_2^4b_2 + 20736iv_4v_1^3v_2a_2\\
&- 27648iv_1^3v_4v_2b_3 + 6912iv_4v_1^2v_2^2a_3 + 6912iv_4v_1^2v_2a_1\\
&+ 20736v_1v_5v_3v_2^2a_3 - 9216\sqrt{3}\,v_1^3v_4v_2b_3\\
&+ 55296\sqrt{3}\,v_1^3v_3v_2^2b_2 + 62208\sqrt{3}\,v_1v_3v_2^4a_3\\
&- 1296v_4\sqrt{3}\,v_1v_2^4b_2 + 6912v_4\sqrt{3}\,v_1^3v_2a_2\\
&+ 2304v_4\sqrt{3}\,v_1^2v_2^2a_3 - 432v_5v_4v_1v_2^2b_2 + 2304v_4\sqrt{3}\,v_1^2v_2a_1\\
&+ 31104iv_5\sqrt{3}\,v_2^3a_1 - 15552iv_5\sqrt{3}\,v_2^4a_3\\
&- 4608\sqrt{3}\,v_1^3v_4b_1 + 3888\sqrt{3}\,v_4v_2^5a_2 + 46656\sqrt{3}\,v_3v_2^6b_2\\
&+ 1296v_5v_4v_2^3a_2 + 15552v_5v_3v_2^4b_2 - 186624\sqrt{3}\,v_1v_2^5a_2\\
&+ 221184\sqrt{3}\,v_1^4v_2b_3 - 62208v_5v_1v_2^3a_2 + 1152v_5v_4v_1^2a_3\\
&- 5184v_4\sqrt{3}\,v_2^5b_3 - 1296v_4\sqrt{3}\,v_2^4b_1 + 155520\sqrt{3}\,v_1^2v_2^4b_2\\
&+ 248832\sqrt{3}\,v_1v_2^5b_3 - 1728v_5v_4v_2^3b_3 - 165888\sqrt{3}\,v_1^4v_2a_2\\
&+ 110592\sqrt{3}\,v_1^3v_2^2a_3 + 155520\sqrt{3}\,v_1v_2^4b_1\\
&- 432v_5v_4v_2^2b_1 + 51840v_5v_1^2v_2^2b_2 + 82944v_5v_1v_2^3b_3\\
&- 55296\sqrt{3}\,v_1^3v_2a_1 + 51840v_5v_1v_2^2b_1 + 11664iv_4v_2^5a_2\\
&- 15552iv_4v_2^5b_3 - 13824iv_1^4v_4b_2 - 3888iv_4v_2^4b_1\\
&- 13824iv_1^3v_4b_1 - 466560iv_1^2v_2^4b_2 + 73728\sqrt{3}\,v_1^4v_3a_3\\
&- 4608\sqrt{3}\,v_1^4v_4b_2 - 432iv_5v_4\sqrt{3}\,v_1v_2^2b_2\\
&+ 559872iv_1v_2^5a_2 - 746496iv_1v_2^5b_3 + 497664iv_1^4v_2a_2\\
&- 663552iv_1^4v_2b_3 - 331776iv_1^3v_2^2a_3 - 466560iv_1v_2^4b_1\\
&+ 165888iv_1^3v_2a_1 - 139968iv_2^6a_3 - 331776iv_1^5b_2\\
&+ 279936iv_2^5a_1 - 331776iv_1^4b_1 + 27648v_1^3v_5a_3\\
&+ 110592\sqrt{3}\,v_1^5b_2 - 93312\sqrt{3}\,v_2^5a_1 + 110592\sqrt{3}\,v_1^4b_1\\
&+ 15552v_5v_2^4a_3 - 31104v_5v_2^3a_1 + 46656\sqrt{3}\,v_2^6a_3 = 0
\end{aligned} \tag{7E}
$$

Collecting the above on the terms v_i introduced, and these are

$$\{v_1, v_2, v_3, v_4, v_5\}$$

Equation (7E) now becomes

$$
\begin{aligned}
&\left(-331776ib_2+110592\sqrt{3}\,b_2\right)v_1^5\\
&+\left(-331776ib_1+110592\sqrt{3}\,b_1\right)v_1^4\\
&+\left(-139968ia_3+46656\sqrt{3}\,a_3\right)v_2^6\\
&+\left(279936ia_1-93312\sqrt{3}\,a_1\right)v_2^5\\
&+\left(-15552i\sqrt{3}\,a_3+15552a_3\right)v_2^4v_5\\
&+\left(31104i\sqrt{3}\,a_1-31104a_1\right)v_2^3v_5+\Big(497664ia_2\\
&-663552ib_3-165888\sqrt{3}\,a_2+221184\sqrt{3}\,b_3\Big)v_1^4v_2\\
&+\left(-13824ib_2-4608\sqrt{3}\,b_2\right)v_1^4v_4\\
&+\left(-331776ia_3+110592\sqrt{3}\,a_3\right)v_1^3v_2^2\\
&+\left(165888ia_1-55296\sqrt{3}\,a_1\right)v_1^3v_2\\
&+\left(-13824ib_1-4608\sqrt{3}\,b_1\right)v_1^3v_4\\
&+\left(-27648i\sqrt{3}\,a_3+27648a_3\right)v_1^3v_5\\
&+\left(-466560ib_2+155520\sqrt{3}\,b_2\right)v_1^2v_2^4+\Big(559872ia_2\\
&-746496ib_3-186624\sqrt{3}\,a_2+248832\sqrt{3}\,b_3\Big)v_1v_2^5\\
&+\left(-466560ib_1+155520\sqrt{3}\,b_1\right)v_1v_2^4 \qquad\qquad (8E)\\
&+\left(11664ia_2-15552ib_3+3888\sqrt{3}\,a_2-5184\sqrt{3}\,b_3\right)v_2^5v_4\\
&+\left(-3888ib_1-1296\sqrt{3}\,b_1\right)v_2^4v_4+20736v_1v_5v_3v_2^2a_3\\
&+55296\sqrt{3}\,v_1^3v_3v_2^2b_2+62208\sqrt{3}\,v_1v_3v_2^4a_3\\
&+46656\sqrt{3}\,v_3v_2^6b_2+15552v_5v_3v_2^4b_2+73728\sqrt{3}\,v_1^4v_3a_3\\
&+\left(20736ia_2-27648ib_3+6912\sqrt{3}\,a_2-9216\sqrt{3}\,b_3\right)v_1^3v_2v_4\\
&+\left(6912ia_3+2304\sqrt{3}\,a_3\right)v_1^2v_2^2v_4\\
&+\left(-51840i\sqrt{3}\,b_2+51840b_2\right)v_1^2v_2^2v_5\\
&+\left(6912ia_1+2304\sqrt{3}\,a_1\right)v_1^2v_2v_4\\
&+\left(1152i\sqrt{3}\,a_3+1152a_3\right)v_1^2v_4v_5\\
&+\left(-3888ib_2-1296\sqrt{3}\,b_2\right)v_1v_2^4v_4+\Big(62208i\sqrt{3}\,a_2\\
&-82944i\sqrt{3}\,b_3-62208a_2+82944b_3\Big)v_1v_2^3v_5\\
&+\left(-51840i\sqrt{3}\,b_1+51840b_1\right)v_1v_2^2v_5\\
&+\left(1296i\sqrt{3}\,a_2-1728i\sqrt{3}\,b_3+1296a_2-1728b_3\right)v_2^3v_4v_5
\end{aligned}
$$

Setting each coefficients in (8E) to zero gives the following equations to solve

$$20736a_3 = 0$$
$$15552b_2 = 0$$
$$62208\sqrt{3}\, a_3 = 0$$
$$73728\sqrt{3}\, a_3 = 0$$
$$46656\sqrt{3}\, b_2 = 0$$
$$55296\sqrt{3}\, b_2 = 0$$
$$-466560ib_1 + 155520\sqrt{3}\, b_1 = 0$$
$$-466560ib_2 + 155520\sqrt{3}\, b_2 = 0$$
$$-331776ia_3 + 110592\sqrt{3}\, a_3 = 0$$
$$-331776ib_1 + 110592\sqrt{3}\, b_1 = 0$$
$$-331776ib_2 + 110592\sqrt{3}\, b_2 = 0$$
$$-139968ia_3 + 46656\sqrt{3}\, a_3 = 0$$
$$-13824ib_1 - 4608\sqrt{3}\, b_1 = 0$$
$$-13824ib_2 - 4608\sqrt{3}\, b_2 = 0$$
$$-3888ib_1 - 1296\sqrt{3}\, b_1 = 0$$
$$-3888ib_2 - 1296\sqrt{3}\, b_2 = 0$$
$$6912ia_1 + 2304\sqrt{3}\, a_1 = 0$$
$$6912ia_3 + 2304\sqrt{3}\, a_3 = 0$$
$$165888ia_1 - 55296\sqrt{3}\, a_1 = 0$$
$$279936ia_1 - 93312\sqrt{3}\, a_1 = 0$$
$$-51840i\sqrt{3}\, b_1 + 51840b_1 = 0$$
$$-51840i\sqrt{3}\, b_2 + 51840b_2 = 0$$
$$-27648i\sqrt{3}\, a_3 + 27648a_3 = 0$$
$$-15552i\sqrt{3}\, a_3 + 15552a_3 = 0$$
$$-432i\sqrt{3}\, b_1 - 432b_1 = 0$$
$$-432i\sqrt{3}\, b_2 - 432b_2 = 0$$
$$1152i\sqrt{3}\, a_3 + 1152a_3 = 0$$
$$31104i\sqrt{3}\, a_1 - 31104a_1 = 0$$
$$11664ia_2 - 15552ib_3 + 3888\sqrt{3}\, a_2 - 5184\sqrt{3}\, b_3 = 0$$
$$20736ia_2 - 27648ib_3 + 6912\sqrt{3}\, a_2 - 9216\sqrt{3}\, b_3 = 0$$
$$497664ia_2 - 663552ib_3 - 165888\sqrt{3}\, a_2 + 221184\sqrt{3}\, b_3 = 0$$
$$559872ia_2 - 746496ib_3 - 186624\sqrt{3}\, a_2 + 248832\sqrt{3}\, b_3 = 0$$
$$1296i\sqrt{3}\, a_2 - 1728i\sqrt{3}\, b_3 + 1296a_2 - 1728b_3 = 0$$
$$62208i\sqrt{3}\, a_2 - 82944i\sqrt{3}\, b_3 - 62208a_2 + 82944b_3 = 0$$

Solving the above equations for the unknowns gives

$$
\begin{aligned}
a_1 &= 0 \\
a_2 &= \frac{4b_3}{3} \\
a_3 &= 0 \\
b_1 &= 0 \\
b_2 &= 0 \\
b_3 &= b_3
\end{aligned}
$$

Substituting the above solution in the anstaz (1E,2E) (using 1 as arbitrary value for any unknown in the RHS) gives

$$
\begin{aligned}
\xi &= \frac{4x}{3} \\
\eta &= y
\end{aligned}
$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x, y) \to (R, S)$ where (R, S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$
\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}
$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y}\right) S(x, y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the canonical coordinates, where $S(R)$. Unable to determine R. Terminating

Unable to determine ODE type.

Maple trace

```
`Methods for first order ODEs:
   *** Sublevel 2 ***
   Methods for first order ODEs:
   -> Solving 1st order ODE of high degree, 1st attempt
   trying 1st order WeierstrassP solution for high degree ODE
   trying 1st order WeierstrassPPrime solution for high degree ODE
   trying 1st order JacobiSN solution for high degree ODE
   trying 1st order ODE linearizable_by_differentiation
   trying differential order: 1; missing variables
   trying simple symmetries for implicit equations
   Successful isolation of dy/dx: 3 solutions were found. Trying to solve each resulting ODE
      *** Sublevel 3 ***
      Methods for first order ODEs:
      --- Trying classification methods ---
      trying homogeneous types:
      trying exact
      Looking for potential symmetries
      trying an equivalence to an Abel ODE
      trying 1st order ODE linearizable_by_differentiation
   -> Solving 1st order ODE of high degree, Lie methods, 1st trial
   `, `-> Computing symmetries using: way = 2
   `, `-> Computing symmetries using: way = 2
   -> Solving 1st order ODE of high degree, 2nd attempt. Trying parametric methods
   trying dAlembert
   -> Calling odsolve with the ODE`, diff(y(x), x) = (-2*y(x)^2*x^3-y(x))/(2*x^4*y(x)+x), y(
      Methods for first order ODEs:
      --- Trying classification methods ---
      trying a quadrature
      trying 1st order linear
      <- 1st order linear successful
   <- 1st order, parametric methods successful`
```

✓ Solution by Maple

Time used: 0.172 (sec). Leaf size: 97

```
dsolve(y(x)=2*x*diff(y(x),x)+y(x)^2*(diff(y(x),x))^3,y(x), singsol=all)
```

$$y(x) = -\frac{2(-x^3)^{\frac{1}{4}} 6^{\frac{1}{4}}}{3}$$
$$y(x) = \frac{2(-x^3)^{\frac{1}{4}} 6^{\frac{1}{4}}}{3}$$
$$y(x) = -\frac{2i(-x^3)^{\frac{1}{4}} 6^{\frac{1}{4}}}{3}$$
$$y(x) = \frac{2i(-x^3)^{\frac{1}{4}} 6^{\frac{1}{4}}}{3}$$
$$y(x) = 0$$
$$y(x) = \sqrt{c_1 (c_1^2 + 2x)}$$
$$y(x) = -\sqrt{c_1 (c_1^2 + 2x)}$$

✓ Solution by Mathematica

Time used: 0.147 (sec). Leaf size: 119

```
DSolve[y[x]==2*x*y'[x]+y[x]^2*(y'[x])^3,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to -\sqrt{2c_1x + c_1{}^3}$$
$$y(x) \to \sqrt{2c_1x + c_1{}^3}$$
$$y(x) \to (-1 - i) \left(\frac{2}{3}\right)^{3/4} x^{3/4}$$
$$y(x) \to (1 - i) \left(\frac{2}{3}\right)^{3/4} x^{3/4}$$
$$y(x) \to (-1 + i) \left(\frac{2}{3}\right)^{3/4} x^{3/4}$$
$$y(x) \to (1 + i) \left(\frac{2}{3}\right)^{3/4} x^{3/4}$$

1.84 problem 87

Internal problem ID [3229]
Internal file name [`OUTPUT/2721_Sunday_June_05_2022_08_39_18_AM_77622382/index.tex`]

Book: Differential equations for engineers by Wei-Chau XIE, Cambridge Press 2010
Section: Chapter 2. First-Order and Simple Higher-Order Differential Equations. Page 78
Problem number: 87.
ODE order: 1.
ODE degree: 3.

The type(s) of ODE detected by this program : **"first_order_ode_lie_symmetry_calculated"**

Maple gives the following as the ode type

```
[[_1st_order, _with_linear_symmetries]]
```

$$\boxed{y'^3 + y^2 - xyy' = 0}$$

Solving the given ode for y' results in 3 differential equations to solve. Each one of these will generate a solution. The equations generated are

$$y' = \frac{\left(-108y^2 + 12\sqrt{-12y^3x^3 + 81y^4}\right)^{\frac{1}{3}}}{6} + \frac{2yx}{\left(-108y^2 + 12\sqrt{-12y^3x^3 + 81y^4}\right)^{\frac{1}{3}}} \tag{1}$$

$$y' = -\frac{\left(-108y^2 + 12\sqrt{-12y^3x^3 + 81y^4}\right)^{\frac{1}{3}}}{12} - \frac{yx}{\left(-108y^2 + 12\sqrt{-12y^3x^3 + 81y^4}\right)^{\frac{1}{3}}} + \frac{i\sqrt{3}\left(\frac{\left(-108y^2+12\sqrt{-12y^3}\right.}{6}\right.}{} \tag{2}$$

$$y' = -\frac{\left(-108y^2 + 12\sqrt{-12y^3x^3 + 81y^4}\right)^{\frac{1}{3}}}{12} - \frac{yx}{\left(-108y^2 + 12\sqrt{-12y^3x^3 + 81y^4}\right)^{\frac{1}{3}}} - \frac{i\sqrt{3}\left(\frac{\left(-108y^2+12\sqrt{-12y^3}\right.}{6}\right.}{} \tag{3}$$

Now each one of the above ODE is solved.

Solving equation (1)

Writing the ode as

$$
\begin{aligned}
y' &= \frac{\left(-108y^2+12\sqrt{-12x^3y^3+81y^4}\right)^{\frac{2}{3}}+12xy}{6\left(-108y^2+12\sqrt{-12x^3y^3+81y^4}\right)^{\frac{1}{3}}} \\
y' &= \omega(x,y)
\end{aligned}
$$

The condition of Lie symmetry is the linearized PDE given by

$$
\eta_x+\omega(\eta_y-\xi_x)-\omega^2\xi_y-\omega_x\xi-\omega_y\eta=0 \tag{A}
$$

The type of this ode is not in the lookup table. To determine ξ,η then (A) is solved using ansatz. Making bivariate polynomials of degree 1 to use as anstaz gives

$$
\xi = xa_2+ya_3+a_1 \tag{1E}
$$

$$
\eta = xb_2+yb_3+b_1 \tag{2E}
$$

Where the unknown coefficients are

$$
\{a_1,a_2,a_3,b_1,b_2,b_3\}
$$

Substituting equations (1E,2E) and ω into (A) gives

$$b_2 + \frac{\left(\left(-108y^2 + 12\sqrt{-12x^3y^3 + 81y^4}\right)^{\frac{2}{3}} + 12xy\right)(b_3 - a_2)}{6\left(-108y^2 + 12\sqrt{-12x^3y^3 + 81y^4}\right)^{\frac{1}{3}}} - \frac{\left(\left(-108y^2 + 12\sqrt{-12x^3y^3 + 81y^4}\right)^{\frac{2}{3}} + 12xy\right)^2 a_3}{36\left(-108y^2 + 12\sqrt{-12x^3y^3 + 81y^4}\right)^{\frac{2}{3}}} - \left(\frac{-\frac{144y^3x^2}{\left(-108y^2+12\sqrt{-12x^3y^3+81y^4}\right)^{\frac{1}{3}}\sqrt{-12x^3y^3+81y^4}} + 12y}{6\left(-108y^2 + 12\sqrt{-12x^3y^3 + 81y^4}\right)^{\frac{1}{3}}} + \frac{12\left(\left(-108y^2 + 12\sqrt{-12x^3y^3 + 81y^4}\right)^{\frac{2}{3}} + 12xy\right)y^3x^2}{\left(-108y^2 + 12\sqrt{-12x^3y^3 + 81y^4}\right)^{\frac{4}{3}}\sqrt{-12x^3y^3 + 81y^4}}\right)(xa_2 + ya_3 + a_1) - \left(\frac{\frac{-144y+\frac{2\left(-216x^3y^2+1944y^3\right)}{3\sqrt{-12x^3y^3+81y^4}}}{\left(-108y^2+12\sqrt{-12x^3y^3+81y^4}\right)^{\frac{1}{3}}} + 12x}{6\left(-108y^2 + 12\sqrt{-12x^3y^3 + 81y^4}\right)^{\frac{1}{3}}} - \frac{\left(\left(-108y^2 + 12\sqrt{-12x^3y^3 + 81y^4}\right)^{\frac{2}{3}} + 12xy\right)\left(-216y + \frac{-216x^3y^2+1944y^3}{\sqrt{-12x^3y^3+81y^4}}\right)}{18\left(-108y^2 + 12\sqrt{-12x^3y^3 + 81y^4}\right)^{\frac{4}{3}}}\right)(xb_2 + yb_3 + b_1) = 0 \tag{5E}$$

Putting the above in normal form gives

$$-\frac{4\left(-108y^2 + 12\sqrt{-12x^3y^3 + 81y^4}\right)^{\frac{4}{3}}\sqrt{-12x^3y^3 + 81y^4}\,xya_3 + 24\left(-108y^2 + 12\sqrt{-12x^3y^3 + 81y^4}\right)^{\frac{2}{3}}\sqrt{-}}{} = 0$$

Setting the numerator to zero gives

$$
\begin{aligned}
&-4\left(-108y^2 \right.\\
&\left.+12\sqrt{-12x^3y^3+81y^4}\right)^{\frac{4}{3}}\sqrt{-12x^3y^3+81y^4}\,xya_3\\
&-24\left(-108y^2 \right.\\
&\left.+12\sqrt{-12x^3y^3+81y^4}\right)^{\frac{2}{3}}\sqrt{-12x^3y^3+81y^4}\,x^2y^2a_3\\
&+72\left(-108y^2 \right.\\
&\left.+12\sqrt{-12x^3y^3+81y^4}\right)^{\frac{2}{3}}\sqrt{-12x^3y^3+81y^4}\,xyb_2\\
&+72\left(-108y^2+12\sqrt{-12x^3y^3+81y^4}\right)^{\frac{2}{3}}x^4y^2b_2\\
&+72\left(-108y^2+12\sqrt{-12x^3y^3+81y^4}\right)^{\frac{2}{3}}x^3y^3a_2\\
&+72\left(-108y^2+12\sqrt{-12x^3y^3+81y^4}\right)^{\frac{2}{3}}x^3y^3b_3\\
&+72\left(-108y^2+12\sqrt{-12x^3y^3+81y^4}\right)^{\frac{2}{3}}x^2y^4a_3\\
&+72\left(-108y^2+12\sqrt{-12x^3y^3+81y^4}\right)^{\frac{2}{3}}x^3y^2b_1\\
&+72\left(-108y^2+12\sqrt{-12x^3y^3+81y^4}\right)^{\frac{2}{3}}x^2y^3a_1\\
&-648\left(-108y^2+12\sqrt{-12x^3y^3+81y^4}\right)^{\frac{2}{3}}x\,y^3b_2\\
&+72\left(-108y^2 \right.\\
&\left.+12\sqrt{-12x^3y^3+81y^4}\right)^{\frac{2}{3}}\sqrt{-12x^3y^3+81y^4}\,y^2b_3\\
&+432\sqrt{-12x^3y^3+81y^4}\,x^2y^2b_2\\
&+2592\sqrt{-12x^3y^3+81y^4}\,x\,y^3a_2\\
&-864\sqrt{-12x^3y^3+81y^4}\,x\,y^3b_3+72\left(-108y^2 \right.\\
&\left.+12\sqrt{-12x^3y^3+81y^4}\right)^{\frac{2}{3}}\sqrt{-12x^3y^3+81y^4}\,yb_1\\
&+432\sqrt{-12x^3y^3+81y^4}\,x\,y^2b_1\\
&-24\left(-12x^3y^3+81y^4\right)^{\frac{3}{2}}a_3+23328y^6a_3-11664y^5a_1\\
&-23328x\,y^5a_2+2592x^4y^4a_2-4320x^3y^5a_3\\
&+864x^3y^4a_1+864x^5y^3b_2-864x^4y^4b_3+864x^4y^3b_1\\
&-3888x^2y^4b_2+7776x\,y^5b_3-3888x\,y^4b_1-\left(-108y^2 \right.\\
&\left.+12\sqrt{-12x^3y^3+81y^4}\right)^{\frac{5}{3}}\sqrt{-12x^3y^3+81y^4}\,a_2\\
&+\left(-108y^2 \right.\\
&\left.+12\sqrt{-12x^3y^3+81y^4}\right)^{\frac{5}{3}}\sqrt{-12x^3y^3+81y^4}\,b_3
\end{aligned}
\tag{6E}
$$

Simplifying the above gives

$$\text{Expression too large to display} \tag{6E}$$

Since the PDE has radicals, simplifying gives

$$\text{Expression too large to display}$$

Looking at the above PDE shows the following are all the terms with $\{x, y\}$ in them.

$$\left\{x, y, \sqrt{-y^3 (4x^3 - 27y)}, \left(-108y^2 + 12\sqrt{3}\, \sqrt{-y^3 (4x^3 - 27y)}\right)^{\frac{1}{3}}, \left(-108y^2 + 12\sqrt{3}\, \sqrt{-y^3 (4x^3 - 27y)}\right)^{\frac{2}{3}}\right\}$$

The following substitution is now made to be able to collect on all terms with $\{x, y\}$ in them

$$\left\{x = v_1, y = v_2, \sqrt{-y^3 (4x^3 - 27y)} = v_3, \left(-108y^2 + 12\sqrt{3}\, \sqrt{-y^3 (4x^3 - 27y)}\right)^{\frac{1}{3}} = v_4, \left(-108y^2 + 12\sqrt{3}\, \sqrt{-y^3 (4x^3 - 27y)}\right)^{\frac{2}{3}} = v_5\right\}$$

The above PDE (6E) now becomes

$$\begin{aligned}
-72v_2\Big(&-24v_3\sqrt{3}\, v_1^3v_2^2a_3 - 6v_5v_3\sqrt{3}\, v_1b_2 - 9v_5v_3\sqrt{3}\, v_2a_2 \\
&+ 3v_5v_3\sqrt{3}\, v_2b_3 - 36v_3\sqrt{3}\, v_1^2v_2b_2 - 216v_3\sqrt{3}\, v_1v_2^2a_2 + 72v_3\sqrt{3}\, v_1v_2^2b_3 \\
&+ 54v_4v_3\sqrt{3}\, v_2b_2 - 36v_3\sqrt{3}\, v_1v_2b_1 - 108v_3\sqrt{3}\, v_2^2a_1 - 48v_4v_1^4v_2^3a_3 \\
&- 6v_5v_1^4v_2b_2 - 18v_5v_1^3v_2^2a_2 + 6v_5v_1^3v_2^2b_3 - 6v_5v_1^2v_2^3a_3 - 6v_5v_1^3v_2b_1 \\
&- 6v_5v_1^2v_2^2a_1 + 72v_4v_1^3v_2^2b_2 + 324v_4v_1v_2^4a_3 + 54v_5v_1v_2^2b_2 + 216v_3\sqrt{3}\, v_2^3a_3 \\
&- 6v_5v_3\sqrt{3}\, b_1 - 1944v_2^5a_3 + 972v_2^4a_1 + 324v_2^3v_1^2b_2 + 1944v_2^4v_1a_2 \\
&+ 324v_2^3v_1b_1 - 72v_1^5v_2^2b_2 - 216v_1^4v_2^3a_2 + 72v_1^4v_2^3b_3 + 360v_1^3v_2^4a_3 \\
&- 72v_1^4v_2^2b_1 - 72v_1^3v_2^3a_1 - 648v_1v_2^4b_3 + 81v_5v_2^3a_2 - 27v_5v_2^3b_3 \\
&+ 54v_5v_2^2b_1 - 486v_4v_2^3b_2 + 2v_5v_3\sqrt{3}\, v_1^2v_2a_3 - 36v_4v_3\sqrt{3}\, v_1v_2^2a_3\Big) = 0
\end{aligned} \tag{7E}$$

Collecting the above on the terms v_i introduced, and these are

$$\{v_1, v_2, v_3, v_4, v_5\}$$

Equation (7E) now becomes

$$
\begin{aligned}
&-5184b_2v_4v_1^3v_2^3 + (1296a_2 - 432b_3)\, v_5v_1^3v_2^3 \\
&+ 432b_1v_5v_1^3v_2^2 + 432a_3v_5v_1^2v_2^4 + 432a_1v_5v_1^2v_2^3 \\
&+ 3456a_3v_4v_1^4v_2^4 + (15552a_2 - 5184b_3)\, v_1^4v_2^4 + 432b_2v_5v_1^4v_2^2 \\
&+ \left(648\sqrt{3}\, a_2 - 216\sqrt{3}\, b_3\right) v_3v_5v_2^2 + 2592\sqrt{3}\, b_2v_3v_1^2v_2^2 \\
&+ 2592\sqrt{3}\, b_1v_3v_1v_2^2 - 3888\sqrt{3}\, b_2v_3v_4v_2^2 + 432v_5v_3\sqrt{3}\, b_1v_2 \\
&- 23328a_3v_4v_1v_2^5 + (-139968a_2 + 46656b_3)\, v_1v_2^5 \\
&+ \left(15552\sqrt{3}\, a_2 - 5184\sqrt{3}\, b_3\right) v_3v_1v_2^3 - 3888b_2v_5v_1v_2^3 \\
&- 15552\sqrt{3}\, a_3v_3v_2^4 + 7776\sqrt{3}\, a_1v_3v_2^3 + 1728\sqrt{3}\, a_3v_3v_1^3v_2^3 \\
&+ 139968a_3v_2^6 - 69984a_1v_2^5 - 144\sqrt{3}\, a_3v_3v_5v_1^2v_2^2 \\
&+ 2592\sqrt{3}\, a_3v_3v_4v_1v_2^3 + 432\sqrt{3}\, b_2v_3v_5v_1v_2 - 23328b_1v_1v_2^4 \\
&+ 34992b_2v_4v_2^4 + (-5832a_2 + 1944b_3)\, v_5v_2^4 - 3888b_1v_5v_2^3 + 5184v_2^3b_2v_1^5 \\
&+ 5184b_1v_1^4v_2^3 - 25920a_3v_1^3v_2^5 + 5184a_1v_1^3v_2^4 - 23328b_2v_1^2v_2^4 = 0
\end{aligned}
\tag{8E}
$$

Setting each coefficients in (8E) to zero gives the following equations to solve

$$
\begin{aligned}
-69984a_1 &= 0\\
432a_1 &= 0\\
5184a_1 &= 0\\
-25920a_3 &= 0\\
-23328a_3 &= 0\\
432a_3 &= 0\\
3456a_3 &= 0\\
139968a_3 &= 0\\
-23328b_1 &= 0\\
-3888b_1 &= 0\\
432b_1 &= 0\\
5184b_1 &= 0\\
-23328b_2 &= 0\\
-5184b_2 &= 0\\
-3888b_2 &= 0\\
432b_2 &= 0\\
5184b_2 &= 0\\
34992b_2 &= 0\\
7776\sqrt{3}\, a_1 &= 0\\
-15552\sqrt{3}\, a_3 &= 0\\
-144\sqrt{3}\, a_3 &= 0\\
1728\sqrt{3}\, a_3 &= 0\\
2592\sqrt{3}\, a_3 &= 0\\
432\sqrt{3}\, b_1 &= 0\\
2592\sqrt{3}\, b_1 &= 0\\
-3888\sqrt{3}\, b_2 &= 0\\
432\sqrt{3}\, b_2 &= 0\\
2592\sqrt{3}\, b_2 &= 0\\
-139968a_2 + 46656b_3 &= 0\\
-5832a_2 + 1944b_3 &= 0\\
1296a_2 - 432b_3 &= 0\\
15552a_2 - 5184b_3 &= 0\\
648\sqrt{3}\, a_2 - 216\sqrt{3}\, b_3 &= 0\\
15552\sqrt{3}\, a_2 - 5184\sqrt{3}\, b_3 &= 0
\end{aligned}
$$

Solving the above equations for the unknowns gives

$$
\begin{aligned}
a_1 &= 0 \\
a_2 &= a_2 \\
a_3 &= 0 \\
b_1 &= 0 \\
b_2 &= 0 \\
b_3 &= 3a_2
\end{aligned}
$$

Substituting the above solution in the anstaz (1E,2E) (using 1 as arbitrary value for any unknown in the RHS) gives

$$
\begin{aligned}
\xi &= x \\
\eta &= 3y
\end{aligned}
$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x, y) \to (R, S)$ where (R, S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$
\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}
$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y}\right) S(x, y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the canonical coordinates, where $S(R)$. Therefore

$$
\begin{aligned}
\frac{dy}{dx} &= \frac{\eta}{\xi} \\
&= \frac{3y}{x} \\
&= \frac{3y}{x}
\end{aligned}
$$

This is easily solved to give

$$
y = c_1 x^3
$$

Where now the coordinate R is taken as the constant of integration. Hence

$$
R = \frac{y}{x^3}
$$

And S is found from

$$
\begin{aligned}
dS &= \frac{dx}{\xi} \\
&= \frac{dx}{x}
\end{aligned}
$$

Integrating gives

$$
\begin{aligned}
S &= \int \frac{dx}{T} \\
&= \ln(x)
\end{aligned}
$$

Where the constant of integration is set to zero as we just need one solution. Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$
\frac{dS}{dR} = \frac{S_x + \omega(x,y) S_y}{R_x + \omega(x,y) R_y} \tag{2}
$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x,y)$ is the right hand side of the original ode given by

$$
\omega(x,y) = \frac{\left(-108y^2 + 12\sqrt{-12x^3y^3 + 81y^4}\right)^{\frac{2}{3}} + 12xy}{6\left(-108y^2 + 12\sqrt{-12x^3y^3 + 81y^4}\right)^{\frac{1}{3}}}
$$

Evaluating all the partial derivatives gives

$$
\begin{aligned}
R_x &= -\frac{3y}{x^4} \\
R_y &= \frac{1}{x^3} \\
S_x &= \frac{1}{x} \\
S_y &= 0
\end{aligned}
$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$
\frac{dS}{dR} = \frac{6x^3\left(-108y^2 + 12\sqrt{3}\,\sqrt{-4x^3y^3 + 27y^4}\right)^{\frac{1}{3}}}{\left(-108y^2 + 12\sqrt{3}\,\sqrt{-4x^3y^3 + 27y^4}\right)^{\frac{2}{3}} x + 12x^2y - 18y\left(-108y^2 + 12\sqrt{3}\,\sqrt{-4x^3y^3 + 27y^4}\right)^{\frac{1}{3}}} \tag{2A}
$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$
\frac{dS}{dR} = \frac{6\,12^{\frac{1}{3}}\left(\sqrt{3}\,\sqrt{27R-4} - 9\sqrt{R}\right)^{\frac{1}{3}}}{\sqrt{R}\left(12^{\frac{2}{3}}\left(\sqrt{3}\,\sqrt{27R-4} - 9\sqrt{R}\right)^{\frac{2}{3}} - 18\,12^{\frac{1}{3}}\sqrt{R}\left(\sqrt{3}\,\sqrt{27R-4} - 9\sqrt{R}\right)^{\frac{1}{3}} + 12\right)}
$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$S(R) = \int \frac{6\left(12\sqrt{81R-12}-108\sqrt{R}\right)^{\frac{1}{3}}}{\left(2\,18^{\frac{1}{3}}\left(\left(\sqrt{81R-12}-9\sqrt{R}\right)^{2}\right)^{\frac{1}{3}}-18\sqrt{R}\,\left(12\sqrt{81R-12}-108\sqrt{R}\right)^{\frac{1}{3}}+12\right)\sqrt{R}}dR + c_1 \tag{4}$$

To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$\ln(x) = \int^{\frac{y}{x^3}} \frac{6\left(12\sqrt{81\_a-12}-108\sqrt{\_a}\right)^{\frac{1}{3}}}{\left(2\,18^{\frac{1}{3}}\left(\left(\sqrt{81\_a-12}-9\sqrt{\_a}\right)^{2}\right)^{\frac{1}{3}}-18\sqrt{\_a}\,\left(12\sqrt{81\_a-12}-108\sqrt{\_a}\right)^{\frac{1}{3}}+12\right)\sqrt{\_a}}d\_$$

Which simplifies to

$$\ln(x) = \int^{\frac{y}{x^3}} \frac{6\left(12\sqrt{81\_a-12}-108\sqrt{\_a}\right)^{\frac{1}{3}}}{\left(2\,18^{\frac{1}{3}}\left(\left(\sqrt{81\_a-12}-9\sqrt{\_a}\right)^{2}\right)^{\frac{1}{3}}-18\sqrt{\_a}\,\left(12\sqrt{81\_a-12}-108\sqrt{\_a}\right)^{\frac{1}{3}}+12\right)\sqrt{\_a}}d\_$$

Summary

The solution(s) found are the following

$$\ln(x) = \int^{\frac{y}{x^3}} \frac{6\left(12\sqrt{81\_a-12}-108\sqrt{\_a}\right)^{\frac{1}{3}}}{\left(2\,18^{\frac{1}{3}}\left(\left(\sqrt{81\_a-12}-9\sqrt{\_a}\right)^{2}\right)^{\frac{1}{3}}-18\sqrt{\_a}\,\left(12\sqrt{81\_a-12}-108\sqrt{\_a}\right)^{\frac{1}{3}}+12\right)\sqrt{\_a}}d\_a + c_1 \tag{1}$$

Verification of solutions

$$\ln(x) = \int^{\frac{y}{x^3}} \frac{6\left(12\sqrt{81\_a-12}-108\sqrt{\_a}\right)^{\frac{1}{3}}}{\left(2\,18^{\frac{1}{3}}\left(\left(\sqrt{81\_a-12}-9\sqrt{\_a}\right)^{2}\right)^{\frac{1}{3}}-18\sqrt{\_a}\,\left(12\sqrt{81\_a-12}-108\sqrt{\_a}\right)^{\frac{1}{3}}+12\right)\sqrt{\_a}}d\_a + c_1$$

Verified OK.

Solving equation (2)

Writing the ode as

$$y' = \frac{i\sqrt{3}\left(-108y^2+12\sqrt{-12x^3y^3+81y^4}\right)^{\frac{2}{3}}-12i\sqrt{3}\,yx-\left(-108y^2+12\sqrt{-12x^3y^3+81y^4}\right)^{\frac{2}{3}}-12xy}{12\left(-108y^2+12\sqrt{-12x^3y^3+81y^4}\right)^{\frac{1}{3}}}$$

$$y' = \omega(x,y)$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x + \omega(\eta_y - \xi_x) - \omega^2\xi_y - \omega_x\xi - \omega_y\eta = 0 \tag{A}$$

The type of this ode is not in the lookup table. To determine ξ, η then (A) is solved using ansatz. Making bivariate polynomials of degree 1 to use as anstaz gives

$$\xi = xa_2 + ya_3 + a_1 \tag{1E}$$

$$\eta = xb_2 + yb_3 + b_1 \tag{2E}$$

Where the unknown coefficients are

$$\{a_1, a_2, a_3, b_1, b_2, b_3\}$$

Substituting equations (1E,2E) and ω into (A) gives

$$b_2 + \frac{\left(i\sqrt{3}\left(-108y^2+12\sqrt{-12x^3y^3+81y^4}\right)^{\frac{2}{3}} - 12i\sqrt{3}\,yx - \left(-108y^2+12\sqrt{-12x^3y^3+81y^4}\right)^{\frac{2}{3}} - 12xy\right)(b_3 -}{12\left(-108y^2+12\sqrt{-12x^3y^3+81y^4}\right)^{\frac{1}{3}}} \tag{5E}$$

$$- \frac{\left(i\sqrt{3}\left(-108y^2+12\sqrt{-12x^3y^3+81y^4}\right)^{\frac{2}{3}} - 12i\sqrt{3}\,yx - \left(-108y^2+12\sqrt{-12x^3y^3+81y^4}\right)^{\frac{2}{3}} - 12xy\right)^2 a_3}{144\left(-108y^2+12\sqrt{-12x^3y^3+81y^4}\right)^{\frac{2}{3}}}$$

$$- \left(\frac{-\frac{144i\sqrt{3}\,y^3x^2}{\left(-108y^2+12\sqrt{-12x^3y^3+81y^4}\right)^{\frac{1}{3}}\sqrt{-12x^3y^3+81y^4}} - 12i\sqrt{3}\,y + \frac{144y^3x^2}{\left(-108y^2+12\sqrt{-12x^3y^3+81y^4}\right)^{\frac{1}{3}}\sqrt{-12x^3y^3+81y^4}} - 12y}{12\left(-108y^2+12\sqrt{-12x^3y^3+81y^4}\right)^{\frac{1}{3}}} \right.$$

$$+ \frac{6\left(i\sqrt{3}\left(-108y^2+12\sqrt{-12x^3y^3+81y^4}\right)^{\frac{2}{3}} - 12i\sqrt{3}\,yx - \left(-108y^2+12\sqrt{-12x^3y^3+81y^4}\right)^{\frac{2}{3}} - 12xy\right)y^3x}{\left(-108y^2+12\sqrt{-12x^3y^3+81y^4}\right)^{\frac{4}{3}}\sqrt{-12x^3y^3+81y^4}}$$

$$+ ya_3 + a_1)$$

$$- \left(\frac{\frac{2i\sqrt{3}\left(-216y+\frac{-216x^3y^2+1944y^3}{\sqrt{-12x^3y^3+81y^4}}\right)}{3\left(-108y^2+12\sqrt{-12x^3y^3+81y^4}\right)^{\frac{1}{3}}} - 12i\sqrt{3}\,x - \frac{2\left(-216y+\frac{-216x^3y^2+1944y^3}{\sqrt{-12x^3y^3+81y^4}}\right)}{3\left(-108y^2+12\sqrt{-12x^3y^3+81y^4}\right)^{\frac{1}{3}}} - 12x}{12\left(-108y^2+12\sqrt{-12x^3y^3+81y^4}\right)^{\frac{1}{3}}} \right.$$

$$- \frac{\left(i\sqrt{3}\left(-108y^2+12\sqrt{-12x^3y^3+81y^4}\right)^{\frac{2}{3}} - 12i\sqrt{3}\,yx - \left(-108y^2+12\sqrt{-12x^3y^3+81y^4}\right)^{\frac{2}{3}} - 12xy\right)\left(-\right.}{36\left(-108y^2+12\sqrt{-12x^3y^3+81y^4}\right)^{\frac{4}{3}}}$$

$$+ yb_3 + b_1) = 0$$

Putting the above in normal form gives

Expression too large to display

Setting the numerator to zero gives

$$\text{Expression too large to display} \tag{6E}$$

Simplifying the above gives

$$\text{Expression too large to display} \tag{6E}$$

Since the PDE has radicals, simplifying gives

$$\text{Expression too large to display}$$

Looking at the above PDE shows the following are all the terms with $\{x, y\}$ in them.

$$\left\{x, y, \sqrt{-y^3 (4x^3 - 27y)}, \left(-108y^2 + 12\sqrt{3}\,\sqrt{-y^3 (4x^3 - 27y)}\right)^{\frac{1}{3}}, \left(-108y^2 + 12\sqrt{3}\,\sqrt{-y^3 (4x^3 - 27y)}\right)^{\frac{2}{3}}\right\}$$

The following substitution is now made to be able to collect on all terms with $\{x, y\}$ in them

$$\left\{x = v_1, y = v_2, \sqrt{-y^3 (4x^3 - 27y)} = v_3, \left(-108y^2 + 12\sqrt{3}\,\sqrt{-y^3 (4x^3 - 27y)}\right)^{\frac{1}{3}} = v_4, \left(-108y^2 + 12\sqrt{3}\,\sqrt{-y^3 (4x^3 - 27y)}\right)^{\frac{2}{3}} = v_5\right\}$$

The above PDE (6E) now becomes

$$
\begin{aligned}
144v_2\Big(&-144v_4v_1^3v_2^2b_2 - 648v_4v_1v_2^4a_3 + 216\sqrt{3}\,v_3v_2^3a_3 \\
&+ 54v_5v_1v_2^2b_2 - 6\sqrt{3}\,v_5v_3b_1 - 108\sqrt{3}\,v_3v_2^2a_1 - 1944i\sqrt{3}\,v_2^5a_3 \\
&+ 972i\sqrt{3}\,v_2^4a_1 + 648iv_3v_2^3a_3 + 18iv_5v_3b_1 - 324iv_3v_2^2a_1 \\
&+ 6i\sqrt{3}\,v_5v_1^2v_2^2a_1 - 54i\sqrt{3}\,v_5v_1v_2^2b_2 - 6iv_5v_3v_1^2v_2a_3 \\
&+ 324v_2^3v_1^2b_2 + 1944v_2^4v_1a_2 + 324v_2^3v_1b_1 - 216v_1^4v_2^3a_2 \\
&+ 72v_1^4v_2^3b_3 + 360v_1^3v_2^4a_3 - 72v_1^4v_2^2b_1 - 72v_1^3v_2^3a_1 - 648v_1v_2^4b_3 \\
&+ 81v_5v_2^3a_2 - 27v_5v_2^3b_3 + 54v_5v_2^2b_1 + 972v_4v_2^3b_2 - 72v_1^5v_2^2b_2 \\
&- 6v_5v_1^4v_2b_2 - 18v_5v_1^3v_2^2a_2 + 6v_5v_1^3v_2^2b_3 - 6v_5v_1^2v_2^3a_3 \\
&- 6v_5v_1^3v_2b_1 - 6v_5v_1^2v_2^2a_1 + 96v_4v_1^4v_2^3a_3 - 1944v_2^5a_3 \\
&+ 972v_2^4a_1 - 6\sqrt{3}\,v_5v_3v_1b_2 - 9\sqrt{3}\,v_5v_3v_2a_2 + 3\sqrt{3}\,v_5v_3v_2b_3 \\
&- 36\sqrt{3}\,v_3v_1^2v_2b_2 - 216\sqrt{3}\,v_3v_1v_2^2a_2 + 72\sqrt{3}\,v_3v_1v_2^2b_3 \\
&- 108\sqrt{3}\,v_4v_3v_2b_2 - 36\sqrt{3}\,v_3v_1v_2b_1 - 24\sqrt{3}\,v_3v_1^3v_2^2a_3 \\
&- 72i\sqrt{3}\,v_1^5v_2^2b_2 - 216i\sqrt{3}\,v_1^4v_2^3a_2 + 72i\sqrt{3}\,v_1^4v_2^3b_3 \\
&+ 360i\sqrt{3}\,v_1^3v_2^4a_3 - 72i\sqrt{3}\,v_1^4v_2^2b_1 - 72i\sqrt{3}\,v_1^3v_2^3a_1 \\
&- 81i\sqrt{3}\,v_5v_2^3a_2 + 27i\sqrt{3}\,v_5v_2^3b_3 + 324i\sqrt{3}\,v_2^3v_1^2b_2 \\
&+ 1944i\sqrt{3}\,v_2^4v_1a_2 - 648i\sqrt{3}\,v_1v_2^4b_3 - 72iv_3v_1^3v_2^2a_3 \\
&- 54i\sqrt{3}\,v_5v_2^2b_1 + 324i\sqrt{3}\,v_2^3v_1b_1 + 18iv_5v_3v_1b_2 \\
&+ 27iv_5v_3v_2a_2 - 9iv_5v_3v_2b_3 - 108iv_3v_1^2v_2b_2 - 648iv_3v_1v_2^2a_2 \\
&+ 216iv_3v_1v_2^2b_3 - 108iv_3v_1v_2b_1 + 2\sqrt{3}\,v_5v_3v_1^2v_2a_3 \\
&+ 72\sqrt{3}\,v_4v_3v_1v_2^2a_3 + 6i\sqrt{3}\,v_5v_1^4v_2b_2 + 18i\sqrt{3}\,v_5v_1^3v_2^2a_2 \\
&- 6i\sqrt{3}\,v_5v_1^3v_2^2b_3 + 6i\sqrt{3}\,v_5v_1^2v_2^3a_3 + 6i\sqrt{3}\,v_5v_1^3v_2b_1\Big) = 0
\end{aligned}
\tag{7E}
$$

Collecting the above on the terms v_i introduced, and these are

$$\{v_1, v_2, v_3, v_4, v_5\}$$

Equation (7E) now becomes

$$
\begin{aligned}
&139968 b_2 v_4 v_2^4 + \left(279936 i\sqrt{3}\, a_2 - 93312 i\sqrt{3}\, b_3 + 279936 a_2\right.\\
&\left. - 93312 b_3\right) v_1 v_2^5 + \left(46656 i\sqrt{3}\, b_1 + 46656 b_1\right) v_1 v_2^4\\
&+ \left(46656 i\sqrt{3}\, b_2 + 46656 b_2\right) v_1^2 v_2^4\\
&+ \left(93312 i a_3 + 31104\sqrt{3}\, a_3\right) v_3 v_2^4\\
&+ \left(-11664 i\sqrt{3}\, a_2 + 3888 i\sqrt{3}\, b_3 + 11664 a_2 - 3888 b_3\right) v_5 v_2^4\\
&+ \left(-46656 i a_1 - 15552\sqrt{3}\, a_1\right) v_3 v_2^3\\
&+ \left(-7776 i\sqrt{3}\, b_1 + 7776 b_1\right) v_5 v_2^3\\
&+ \left(-10368 i\sqrt{3}\, b_2 - 10368 b_2\right) v_1^5 v_2^3\\
&+ \left(-31104 i\sqrt{3}\, a_2 + 10368 i\sqrt{3}\, b_3 - 31104 a_2\right.\\
&\left. + 10368 b_3\right) v_1^4 v_2^4 + \left(-10368 i\sqrt{3}\, b_1 - 10368 b_1\right) v_1^4 v_2^3\\
&+ \left(51840 i\sqrt{3}\, a_3 + 51840 a_3\right) v_1^3 v_2^5\\
&+ \left(-10368 i\sqrt{3}\, a_1 - 10368 a_1\right) v_1^3 v_2^4\\
&+ \left(-7776 i\sqrt{3}\, b_2 + 7776 b_2\right) v_5 v_1 v_2^3\\
&+ \left(-15552 i b_1 - 5184\sqrt{3}\, b_1\right) v_3 v_1 v_2^2 + 13824 a_3 v_4 v_1^4 v_2^4 \qquad (8E)\\
&- 20736 b_2 v_4 v_1^3 v_2^3 + \left(-279936 i\sqrt{3}\, a_3 - 279936 a_3\right) v_2^6\\
&+ \left(139968 i\sqrt{3}\, a_1 + 139968 a_1\right) v_2^5\\
&+ 10368\sqrt{3}\, a_3 v_3 v_4 v_1 v_2^3 - 15552\sqrt{3}\, b_2 v_3 v_4 v_2^2\\
&- 93312 a_3 v_4 v_1 v_2^5 + \left(-864 i a_3 + 288\sqrt{3}\, a_3\right) v_3 v_5 v_1^2 v_2^2\\
&+ \left(2592 i b_2 - 864\sqrt{3}\, b_2\right) v_3 v_5 v_1 v_2\\
&+ \left(864 i\sqrt{3}\, b_2 - 864 b_2\right) v_5 v_1^4 v_2^2\\
&+ \left(-10368 i a_3 - 3456\sqrt{3}\, a_3\right) v_3 v_1^3 v_2^3\\
&+ \left(2592 i\sqrt{3}\, a_2 - 864 i\sqrt{3}\, b_3 - 2592 a_2 + 864 b_3\right) v_5 v_1^3 v_2^3\\
&+ \left(864 i\sqrt{3}\, b_1 - 864 b_1\right) v_5 v_1^3 v_2^2\\
&+ \left(864 i\sqrt{3}\, a_3 - 864 a_3\right) v_5 v_1^2 v_2^4\\
&+ \left(864 i\sqrt{3}\, a_1 - 864 a_1\right) v_5 v_1^2 v_2^3\\
&+ \left(-15552 i b_2 - 5184\sqrt{3}\, b_2\right) v_3 v_1^2 v_2^2\\
&+ \left(3888 i a_2 - 1296 i b_3 - 1296\sqrt{3}\, a_2 + 432\sqrt{3}\, b_3\right) v_3 v_5 v_2^2\\
&+ \left(2592 i b_1 - 864\sqrt{3}\, b_1\right) v_3 v_5 v_2 + \left(-93312 i a_2\right.
\end{aligned}
$$

Setting each coefficients in (8E) to zero gives the following equations to solve

$$
\begin{aligned}
-93312a_3 &= 0\\
13824a_3 &= 0\\
-20736b_2 &= 0\\
139968b_2 &= 0\\
10368\sqrt{3}\,a_3 &= 0\\
-15552\sqrt{3}\,b_2 &= 0\\
-46656ia_1 - 15552\sqrt{3}\,a_1 &= 0\\
-15552ib_1 - 5184\sqrt{3}\,b_1 &= 0\\
-15552ib_2 - 5184\sqrt{3}\,b_2 &= 0\\
-10368ia_3 - 3456\sqrt{3}\,a_3 &= 0\\
-864ia_3 + 288\sqrt{3}\,a_3 &= 0\\
2592ib_1 - 864\sqrt{3}\,b_1 &= 0\\
2592ib_2 - 864\sqrt{3}\,b_2 &= 0\\
93312ia_3 + 31104\sqrt{3}\,a_3 &= 0\\
-279936i\sqrt{3}\,a_3 - 279936a_3 &= 0\\
-10368i\sqrt{3}\,a_1 - 10368a_1 &= 0\\
-10368i\sqrt{3}\,b_1 - 10368b_1 &= 0\\
-10368i\sqrt{3}\,b_2 - 10368b_2 &= 0\\
-7776i\sqrt{3}\,b_1 + 7776b_1 &= 0\\
-7776i\sqrt{3}\,b_2 + 7776b_2 &= 0\\
864i\sqrt{3}\,a_1 - 864a_1 &= 0\\
864i\sqrt{3}\,a_3 - 864a_3 &= 0\\
864i\sqrt{3}\,b_1 - 864b_1 &= 0\\
864i\sqrt{3}\,b_2 - 864b_2 &= 0\\
46656i\sqrt{3}\,b_1 + 46656b_1 &= 0\\
46656i\sqrt{3}\,b_2 + 46656b_2 &= 0\\
51840i\sqrt{3}\,a_3 + 51840a_3 &= 0\\
139968i\sqrt{3}\,a_1 + 139968a_1 &= 0\\
-93312ia_2 + 31104ib_3 - 31104\sqrt{3}\,a_2 + 10368\sqrt{3}\,b_3 &= 0\\
3888ia_2 - 1296ib_3 - 1296\sqrt{3}\,a_2 + 432\sqrt{3}\,b_3 &= 0\\
-31104i\sqrt{3}\,a_2 + 10368i\sqrt{3}\,b_3 - 31104a_2 + 10368b_3 &= 0\\
-11664i\sqrt{3}\,a_2 + 3888i\sqrt{3}\,b_3 + 11664a_2 - 3888b_3 &= 0\\
2592i\sqrt{3}\,a_2 - 864i\sqrt{3}\,b_3 - 2592a_2 + 864b_3 &= 0\\
279936i\sqrt{3}\,a_2 - 93312i\sqrt{3}\,b_3 + 279936a_2 - 93312b_3 &= 0
\end{aligned}
$$

Solving the above equations for the unknowns gives

$$\begin{aligned} a_1 &= 0 \\ a_2 &= a_2 \\ a_3 &= 0 \\ b_1 &= 0 \\ b_2 &= 0 \\ b_3 &= 3a_2 \end{aligned}$$

Substituting the above solution in the anstaz (1E,2E) (using 1 as arbitrary value for any unknown in the RHS) gives

$$\begin{aligned} \xi &= x \\ \eta &= 3y \end{aligned}$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x, y) \to (R, S)$ where (R, S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y}\right) S(x, y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the canonical coordinates, where $S(R)$. Unable to determine R. Terminating

Unable to determine ODE type.

Solving equation (3)

Writing the ode as

$$y' = -\frac{i\sqrt{3}\left(-108y^2 + 12\sqrt{-12x^3y^3 + 81y^4}\right)^{\frac{2}{3}} - 12i\sqrt{3}\, yx + \left(-108y^2 + 12\sqrt{-12x^3y^3 + 81y^4}\right)^{\frac{2}{3}} + 12xy}{12\left(-108y^2 + 12\sqrt{-12x^3y^3 + 81y^4}\right)^{\frac{1}{3}}}$$

$$y' = \omega(x, y)$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x + \omega(\eta_y - \xi_x) - \omega^2 \xi_y - \omega_x \xi - \omega_y \eta = 0 \tag{A}$$

The type of this ode is not in the lookup table. To determine ξ, η then (A) is solved using ansatz. Making bivariate polynomials of degree 1 to use as anstaz gives

$$\xi = xa_2 + ya_3 + a_1 \tag{1E}$$

$$\eta = xb_2 + yb_3 + b_1 \tag{2E}$$

Where the unknown coefficients are

$$\{a_1, a_2, a_3, b_1, b_2, b_3\}$$

Substituting equations (1E,2E) and ω into (A) gives

$$b_2 \tag{5E}$$

$$-\frac{\left(i\sqrt{3}\left(-108y^2+12\sqrt{-12x^3y^3+81y^4}\right)^{\frac{2}{3}}-12i\sqrt{3}\,yx+\left(-108y^2+12\sqrt{-12x^3y^3+81y^4}\right)^{\frac{2}{3}}+12xy\right)(b_3}{12\left(-108y^2+12\sqrt{-12x^3y^3+81y^4}\right)^{\frac{1}{3}}}$$

$$-\frac{\left(i\sqrt{3}\left(-108y^2+12\sqrt{-12x^3y^3+81y^4}\right)^{\frac{2}{3}}-12i\sqrt{3}\,yx+\left(-108y^2+12\sqrt{-12x^3y^3+81y^4}\right)^{\frac{2}{3}}+12xy\right)^2 a_3}{144\left(-108y^2+12\sqrt{-12x^3y^3+81y^4}\right)^{\frac{2}{3}}}$$

$$-\left(-\frac{-\frac{144i\sqrt{3}\,y^3x^2}{\left(-108y^2+12\sqrt{-12x^3y^3+81y^4}\right)^{\frac{1}{3}}\sqrt{-12x^3y^3+81y^4}}-12i\sqrt{3}\,y-\frac{144y^3x^2}{\left(-108y^2+12\sqrt{-12x^3y^3+81y^4}\right)^{\frac{1}{3}}\sqrt{-12x^3y^3+81y^4}}+12y}{12\left(-108y^2+12\sqrt{-12x^3y^3+81y^4}\right)^{\frac{1}{3}}}\right.$$

$$-\frac{6\left(i\sqrt{3}\left(-108y^2+12\sqrt{-12x^3y^3+81y^4}\right)^{\frac{2}{3}}-12i\sqrt{3}\,yx+\left(-108y^2+12\sqrt{-12x^3y^3+81y^4}\right)^{\frac{2}{3}}+12xy\right)y^3}{\left(-108y^2+12\sqrt{-12x^3y^3+81y^4}\right)^{\frac{4}{3}}\sqrt{-12x^3y^3+81y^4}}$$

$$+ya_3+a_1)$$

$$-\left(-\frac{\frac{2i\sqrt{3}\left(-216y+\frac{-216x^3y^2+1944y^3}{\sqrt{-12x^3y^3+81y^4}}\right)}{3\left(-108y^2+12\sqrt{-12x^3y^3+81y^4}\right)^{\frac{1}{3}}}-12i\sqrt{3}\,x+\frac{-144y+\frac{2\left(-216x^3y^2+1944y^3\right)}{3\sqrt{-12x^3y^3+81y^4}}}{\left(-108y^2+12\sqrt{-12x^3y^3+81y^4}\right)^{\frac{1}{3}}}+12x}{12\left(-108y^2+12\sqrt{-12x^3y^3+81y^4}\right)^{\frac{1}{3}}}\right.$$

$$+\frac{\left(i\sqrt{3}\left(-108y^2+12\sqrt{-12x^3y^3+81y^4}\right)^{\frac{2}{3}}-12i\sqrt{3}\,yx+\left(-108y^2+12\sqrt{-12x^3y^3+81y^4}\right)^{\frac{2}{3}}+12xy\right)\left(-2\right.}{36\left(-108y^2+12\sqrt{-12x^3y^3+81y^4}\right)^{\frac{4}{3}}}$$

$$+yb_3+b_1)=0$$

Putting the above in normal form gives

Expression too large to display

Setting the numerator to zero gives

Expression too large to display (6E)

Simplifying the above gives

Expression too large to display (6E)

Since the PDE has radicals, simplifying gives

Expression too large to display

Looking at the above PDE shows the following are all the terms with $\{x, y\}$ in them.

$$\left\{x, y, \sqrt{-y^3 (4x^3 - 27y)}, \left(-108y^2 + 12\sqrt{3}\, \sqrt{-y^3 (4x^3 - 27y)}\right)^{\frac{1}{3}}, \left(-108y^2 + 12\sqrt{3}\, \sqrt{-y^3 (4x^3 - 27y)}\right)^{\frac{2}{3}}\right\}$$

The following substitution is now made to be able to collect on all terms with $\{x, y\}$ in them

$$\left\{x = v_1, y = v_2, \sqrt{-y^3 (4x^3 - 27y)} = v_3, \left(-108y^2 + 12\sqrt{3}\, \sqrt{-y^3 (4x^3 - 27y)}\right)^{\frac{1}{3}} = v_4, \left(-108y^2 + 12\sqrt{3}\, \sqrt{-y^3 (4x^3 - 27y)}\right)^{\frac{2}{3}} = v_5\right\}$$

The above PDE (6E) now becomes

$$
\begin{aligned}
&-144v_2\Big(-1944v_2^4v_1a_2 + 72v_1^4v_2^2b_1 + 648v_1v_2^4b_3\\
&- 360v_1^3v_2^4a_3 - 324v_2^3v_1b_1 - 324v_2^3v_1^2b_2 + 72v_1^3v_2^3a_1\\
&- 972v_4v_2^3b_2 + 27v_5v_2^3b_3 - 54v_5v_2^2b_1 - 81v_5v_2^3a_2\\
&- 1944iv_2^5\sqrt{3}\,a_3 + 972iv_2^4\sqrt{3}\,a_1 + 648iv_3v_2^3a_3 + 18iv_5v_3b_1\\
&- 324iv_3v_2^2a_1 + 6v_5v_1^4v_2b_2 + 18v_5v_1^3v_2^2a_2 - 6v_5v_1^3v_2^2b_3\\
&+ 6v_5v_1^2v_2^3a_3 + 6v_5v_1^3v_2b_1 + 6v_5v_1^2v_2^2a_1 + 144v_4v_1^3v_2^2b_2\\
&+ 648v_4v_1v_2^4a_3 - 54v_5v_1v_2^2b_2 - 216\sqrt{3}\,v_3v_2^3a_3 + 6v_5\sqrt{3}\,v_3b_1\\
&+ 108\sqrt{3}\,v_3v_2^2a_1 + 24\sqrt{3}\,v_3v_1^3v_2^2a_3 + 6v_5\sqrt{3}\,v_3v_1b_2\\
&+ 9v_5\sqrt{3}\,v_3v_2a_2 - 3v_5\sqrt{3}\,v_3v_2b_3 + 36\sqrt{3}\,v_3v_1^2v_2b_2\\
&+ 216\sqrt{3}\,v_3v_1v_2^2a_2 - 72\sqrt{3}\,v_3v_1v_2^2b_3 + 108v_4\sqrt{3}\,v_3v_2b_2\\
&+ 36\sqrt{3}\,v_3v_1v_2b_1 + 324iv_2^3\sqrt{3}\,v_1^2b_2 + 1944iv_2^4\sqrt{3}\,v_1a_2\\
&- 648i\sqrt{3}\,v_1v_2^4b_3 - 72iv_3v_1^3v_2^2a_3 - 54iv_5\sqrt{3}\,v_2^2b_1\\
&+ 324iv_2^3\sqrt{3}\,v_1b_1 + 18iv_5v_3v_1b_2 + 27iv_5v_3v_2a_2\\
&- 9iv_5v_3v_2b_3 - 96v_4v_1^4v_2^3a_3 - 108iv_3v_1^2v_2b_2 - 648iv_3v_1v_2^2a_2\\
&+ 216iv_3v_1v_2^2b_3 - 108iv_3v_1v_2b_1 - 72i\sqrt{3}\,v_1^5v_2^2b_2\\
&- 216i\sqrt{3}\,v_1^4v_2^3a_2 + 72i\sqrt{3}\,v_1^4v_2^3b_3 + 360i\sqrt{3}\,v_1^3v_2^4a_3\\
&- 72i\sqrt{3}\,v_1^4v_2^2b_1 - 72i\sqrt{3}\,v_1^3v_2^3a_1 - 81iv_5\sqrt{3}\,v_2^3a_2\\
&+ 27iv_5\sqrt{3}\,v_2^3b_3 - 6iv_5\sqrt{3}\,v_1^3v_2^2b_3 + 6iv_5\sqrt{3}\,v_1^2v_2^3a_3\\
&+ 6iv_5\sqrt{3}\,v_1^3v_2b_1 + 6iv_5\sqrt{3}\,v_1^2v_2^2a_1 - 54iv_5\sqrt{3}\,v_1v_2^2b_2\\
&- 6iv_5v_3v_1^2v_2a_3 + 6iv_5\sqrt{3}\,v_1^4v_2b_2 + 18iv_5\sqrt{3}\,v_1^3v_2^2a_2\\
&- 2v_5\sqrt{3}\,v_3v_1^2v_2a_3 - 72v_4\sqrt{3}\,v_3v_1v_2^2a_3 + 216v_1^4v_2^3a_2\\
&+ 72v_1^5v_2^2b_2 - 72v_1^4v_2^3b_3 + 1944v_2^5a_3 - 972v_2^4a_1\Big) = 0
\end{aligned}
\tag{7E}
$$

Collecting the above on the terms v_i introduced, and these are

$$\{v_1, v_2, v_3, v_4, v_5\}$$

Equation (7E) now becomes

$$
\begin{aligned}
&\left(-93312ia_3+31104\sqrt{3}\,a_3\right)v_3v_2^4\\
&+\left(7776i\sqrt{3}\,b_1+7776b_1\right)v_5v_2^3\\
&+\left(-51840i\sqrt{3}\,a_3+51840a_3\right)v_1^3v_2^5\\
&+13824a_3v_4v_1^4v_2^4+\left(-2592ib_2-864\sqrt{3}\,b_2\right)v_3v_5v_1v_2\\
&+\left(864ia_3+288\sqrt{3}\,a_3\right)v_3v_5v_1^2v_2^2\\
&+\left(-3888ia_2+1296ib_3-1296\sqrt{3}\,a_2+432\sqrt{3}\,b_3\right)v_3v_5v_2^2\\
&+\left(-2592ib_1-864\sqrt{3}\,b_1\right)v_3v_5v_2+\Big(93312ia_2\\
&-31104ib_3-31104\sqrt{3}\,a_2+10368\sqrt{3}\,b_3\Big)v_3v_1v_2^3\\
&+\left(7776i\sqrt{3}\,b_2+7776b_2\right)v_5v_1v_2^3\\
&+\left(15552ib_1-5184\sqrt{3}\,b_1\right)v_3v_1v_2^2\\
&+\left(10368ia_3-3456\sqrt{3}\,a_3\right)v_3v_1^3v_2^3\\
&+\left(-2592i\sqrt{3}\,a_2+864i\sqrt{3}\,b_3-2592a_2+864b_3\right)v_5v_1^3v_2^3\\
&+\left(-864i\sqrt{3}\,b_1-864b_1\right)v_5v_1^3v_2^2\\
&+\left(-864i\sqrt{3}\,b_2-864b_2\right)v_5v_1^4v_2^2\\
&+\left(-864i\sqrt{3}\,a_3-864a_3\right)v_5v_1^2v_2^4\\
&+\left(-864i\sqrt{3}\,a_1-864a_1\right)v_5v_1^2v_2^3\\
&+\left(15552ib_2-5184\sqrt{3}\,b_2\right)v_3v_1^2v_2^2-93312a_3v_4v_1v_2^5\\
&-20736b_2v_4v_1^3v_2^3-15552\sqrt{3}\,b_2v_3v_4v_2^2\\
&+10368\sqrt{3}\,a_3v_3v_4v_1v_2^3+\left(10368i\sqrt{3}\,b_1-10368b_1\right)v_1^4v_2^3\\
&+\left(10368i\sqrt{3}\,b_2-10368b_2\right)v_1^5v_2^3\\
&+\left(46656ia_1-15552\sqrt{3}\,a_1\right)v_3v_2^3\\
&+\left(10368i\sqrt{3}\,a_1-10368a_1\right)v_1^3v_2^4\\
&+\left(11664i\sqrt{3}\,a_2-3888i\sqrt{3}\,b_3+11664a_2-3888b_3\right)v_5v_2^4\\
&+\left(279936i\sqrt{3}\,a_3-279936a_3\right)v_2^6\\
&+\left(-139968i\sqrt{3}\,a_1+139968a_1\right)v_2^5\\
&+\left(-46656i\sqrt{3}\,b_1+46656b_1\right)v_1v_2^4+\Big(-279936i\sqrt{3}\,a_2\\
&+93312i\sqrt{3}\,b_3+279936a_2-93312b_3\Big)v_1v_2^5\\
&+\left(31104i\sqrt{3}\,a_2-10368i\sqrt{3}\,b_3-31104a_2+10368b_3\right)v_1^4v_2^4
\end{aligned}
\tag{8E}
$$

Setting each coefficients in (8E) to zero gives the following equations to solve

$$-93312a_3 = 0$$
$$13824a_3 = 0$$
$$-20736b_2 = 0$$
$$139968b_2 = 0$$
$$10368\sqrt{3}\, a_3 = 0$$
$$-15552\sqrt{3}\, b_2 = 0$$
$$-93312ia_3 + 31104\sqrt{3}\, a_3 = 0$$
$$-2592ib_1 - 864\sqrt{3}\, b_1 = 0$$
$$-2592ib_2 - 864\sqrt{3}\, b_2 = 0$$
$$864ia_3 + 288\sqrt{3}\, a_3 = 0$$
$$10368ia_3 - 3456\sqrt{3}\, a_3 = 0$$
$$15552ib_1 - 5184\sqrt{3}\, b_1 = 0$$
$$15552ib_2 - 5184\sqrt{3}\, b_2 = 0$$
$$46656ia_1 - 15552\sqrt{3}\, a_1 = 0$$
$$-139968i\sqrt{3}\, a_1 + 139968a_1 = 0$$
$$-51840i\sqrt{3}\, a_3 + 51840a_3 = 0$$
$$-46656i\sqrt{3}\, b_1 + 46656b_1 = 0$$
$$-46656i\sqrt{3}\, b_2 + 46656b_2 = 0$$
$$-864i\sqrt{3}\, a_1 - 864a_1 = 0$$
$$-864i\sqrt{3}\, a_3 - 864a_3 = 0$$
$$-864i\sqrt{3}\, b_1 - 864b_1 = 0$$
$$-864i\sqrt{3}\, b_2 - 864b_2 = 0$$
$$7776i\sqrt{3}\, b_1 + 7776b_1 = 0$$
$$7776i\sqrt{3}\, b_2 + 7776b_2 = 0$$
$$10368i\sqrt{3}\, a_1 - 10368a_1 = 0$$
$$10368i\sqrt{3}\, b_1 - 10368b_1 = 0$$
$$10368i\sqrt{3}\, b_2 - 10368b_2 = 0$$
$$279936i\sqrt{3}\, a_3 - 279936a_3 = 0$$
$$-3888ia_2 + 1296ib_3 - 1296\sqrt{3}\, a_2 + 432\sqrt{3}\, b_3 = 0$$
$$93312ia_2 - 31104ib_3 - 31104\sqrt{3}\, a_2 + 10368\sqrt{3}\, b_3 = 0$$
$$-279936i\sqrt{3}\, a_2 + 93312i\sqrt{3}\, b_3 + 279936a_2 - 93312b_3 = 0$$
$$-2592i\sqrt{3}\, a_2 + 864i\sqrt{3}\, b_3 - 2592a_2 + 864b_3 = 0$$
$$11664i\sqrt{3}\, a_2 - 3888i\sqrt{3}\, b_3 + 11664a_2 - 3888b_3 = 0$$
$$31104i\sqrt{3}\, a_2 - 10368i\sqrt{3}\, b_3 - 31104a_2 + 10368b_3 = 0$$

Solving the above equations for the unknowns gives

$$
\begin{aligned}
a_1 &= 0 \\
a_2 &= a_2 \\
a_3 &= 0 \\
b_1 &= 0 \\
b_2 &= 0 \\
b_3 &= 3a_2
\end{aligned}
$$

Substituting the above solution in the anstaz (1E,2E) (using 1 as arbitrary value for any unknown in the RHS) gives

$$
\begin{aligned}
\xi &= x \\
\eta &= 3y
\end{aligned}
$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x, y) \to (R, S)$ where (R, S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$
\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}
$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y}\right) S(x, y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the canonical coordinates, where $S(R)$. Unable to determine R. Terminating

Unable to determine ODE type.

Maple trace

```
`Methods for first order ODEs:
-> Solving 1st order ODE of high degree, 1st attempt
trying 1st order WeierstrassP solution for high degree ODE
trying 1st order WeierstrassPPrime solution for high degree ODE
trying 1st order JacobiSN solution for high degree ODE
trying 1st order ODE linearizable_by_differentiation
trying differential order: 1; missing variables
trying simple symmetries for implicit equations
Successful isolation of dy/dx: 3 solutions were found. Trying to solve each resulting ODE.
   *** Sublevel 2 ***
   Methods for first order ODEs:
   --- Trying classification methods ---
   trying homogeneous types:
   trying exact
   Looking for potential symmetries
   trying an equivalence to an Abel ODE
   trying 1st order ODE linearizable_by_differentiation
-> Solving 1st order ODE of high degree, Lie methods, 1st trial
`, `-> Computing symmetries using: way = 2
`, `-> Computing symmetries using: way = 2
-> Solving 1st order ODE of high degree, 2nd attempt. Trying parametric methods
trying dAlembert
-> Calling odsolve with the ODE`, diff(y(x), x) = (2*y(x)*x^3-y(x)^3)/x^4, y(x)`   *** Suble
   Methods for first order ODEs:
   --- Trying classification methods ---
   trying a quadrature
   trying 1st order linear
   trying Bernoulli
   <- Bernoulli successful
<- 1st order, parametric methods successful`
```

✓ Solution by Maple

Time used: 0.141 (sec). Leaf size: 135

```
dsolve((diff(y(x),x))^3+y(x)^2=x*y(x)*diff(y(x),x),y(x), singsol=all)
```

$$y(x) = 0$$
$$y(x) = \frac{2x^3\sqrt{x^2+3c_1} - 2x^4 - 6x\sqrt{x^2+3c_1}\,c_1 + 3c_1x^2 - 9c_1^2}{-27x + 27\sqrt{x^2+3c_1}}$$
$$y(x) = \frac{2x^3\sqrt{x^2+3c_1} + 2x^4 - 6x\sqrt{x^2+3c_1}\,c_1 - 3c_1x^2 + 9c_1^2}{27x + 27\sqrt{x^2+3c_1}}$$

✗ Solution by Mathematica

Time used: 0.0 (sec). Leaf size: 0

```
DSolve[(y'[x])^3+y[x]^2==x*y[x]*y'[x],y[x],x,IncludeSingularSolutions -> True]
```

Timed out

1.85 problem 88

Internal problem ID [3230]
Internal file name [`OUTPUT/2722_Sunday_June_05_2022_08_39_20_AM_61568442/index.tex`]

Book: Differential equations for engineers by Wei-Chau XIE, Cambridge Press 2010
Section: Chapter 2. First-Order and Simple Higher-Order Differential Equations. Page 78
Problem number: 88.
ODE order: 1.
ODE degree: 0.

The type(s) of ODE detected by this program : **"first_order_ode_lie_symmetry_calculated"**

Maple gives the following as the ode type

`[[_1st_order, _with_linear_symmetries]]`

$$\boxed{2xy' - y - y' \ln (yy') = 0}$$

Solving the given ode for y' results in 1 differential equations to solve. Each one of these will generate a solution. The equations generated are

$$y' = -\frac{y}{\text{LambertW}\left(-y^2 \mathrm{e}^{-2x}\right)} \tag{1}$$

Now each one of the above ODE is solved.

Solving equation (1)

Writing the ode as

$$\begin{aligned} y' &= -\frac{y}{\text{LambertW}\left(-y^2 \mathrm{e}^{-2x}\right)} \\ y' &= \omega(x, y) \end{aligned}$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x + \omega(\eta_y - \xi_x) - \omega^2 \xi_y - \omega_x \xi - \omega_y \eta = 0 \tag{A}$$

The type of this ode is not in the lookup table. To determine ξ, η then (A) is solved using ansatz. Making bivariate polynomials of degree 1 to use as anstaz gives

$$\xi = xa_2 + ya_3 + a_1 \tag{1E}$$

$$\eta = xb_2 + yb_3 + b_1 \tag{2E}$$

Where the unknown coefficients are

$$\{a_1, a_2, a_3, b_1, b_2, b_3\}$$

Substituting equations (1E,2E) and ω into (A) gives

$$\begin{aligned} &b_2 - \frac{y(b_3 - a_2)}{\mathrm{LambertW}\left(-y^2 \mathrm{e}^{-2x}\right)} - \frac{y^2 a_3}{\mathrm{LambertW}\left(-y^2 \mathrm{e}^{-2x}\right)^2} \\ &+ \frac{2y(x a_2 + y a_3 + a_1)}{\mathrm{LambertW}\left(-y^2 \mathrm{e}^{-2x}\right)\left(1 + \mathrm{LambertW}\left(-y^2 \mathrm{e}^{-2x}\right)\right)} \\ &- \left(-\frac{1}{\mathrm{LambertW}\left(-y^2 \mathrm{e}^{-2x}\right)} \right. \\ &\left. + \frac{2}{\mathrm{LambertW}\left(-y^2 \mathrm{e}^{-2x}\right)\left(1 + \mathrm{LambertW}\left(-y^2 \mathrm{e}^{-2x}\right)\right)}\right)(x b_2 + y b_3 + b_1) = 0 \end{aligned} \tag{5E}$$

Putting the above in normal form gives

$$\frac{b_2 \mathrm{LambertW}\left(-y^2 \mathrm{e}^{-2x}\right)^3 + \mathrm{LambertW}\left(-y^2 \mathrm{e}^{-2x}\right)^2 x b_2 + \mathrm{LambertW}\left(-y^2 \mathrm{e}^{-2x}\right)^2 y a_2 + 2 \mathrm{LambertW}\left(-y^2\mathrm{e}\right.}{}$$

$$= 0$$

Setting the numerator to zero gives

$$\begin{aligned} &b_2 \mathrm{LambertW}\left(-y^2 \mathrm{e}^{-2x}\right)^3 + \mathrm{LambertW}\left(-y^2 \mathrm{e}^{-2x}\right)^2 x b_2 \\ &+ \mathrm{LambertW}\left(-y^2 \mathrm{e}^{-2x}\right)^2 y a_2 + 2 \mathrm{LambertW}\left(-y^2 \mathrm{e}^{-2x}\right) x y a_2 \\ &+ y^2 a_3 \mathrm{LambertW}\left(-y^2 \mathrm{e}^{-2x}\right) + \mathrm{LambertW}\left(-y^2 \mathrm{e}^{-2x}\right)^2 b_1 \\ &+ b_2 \mathrm{LambertW}\left(-y^2 \mathrm{e}^{-2x}\right)^2 - \mathrm{LambertW}\left(-y^2 \mathrm{e}^{-2x}\right) x b_2 \\ &+ 2 \mathrm{LambertW}\left(-y^2 \mathrm{e}^{-2x}\right) y a_1 + \mathrm{LambertW}\left(-y^2 \mathrm{e}^{-2x}\right) y a_2 \\ &- 2 \mathrm{LambertW}\left(-y^2 \mathrm{e}^{-2x}\right) y b_3 - y^2 a_3 - \mathrm{LambertW}\left(-y^2 \mathrm{e}^{-2x}\right) b_1 = 0 \end{aligned} \tag{6E}$$

Looking at the above PDE shows the following are all the terms with $\{x, y\}$ in them.

$$\left\{x, y, \mathrm{e}^{-2x}, \mathrm{LambertW}\left(-y^2 \mathrm{e}^{-2x}\right)\right\}$$

The following substitution is now made to be able to collect on all terms with $\{x, y\}$ in them

$$\left\{x = v_1, y = v_2, \mathrm{e}^{-2x} = v_3, \mathrm{LambertW}\left(-y^2 \mathrm{e}^{-2x}\right) = v_4\right\}$$

The above PDE (6E) now becomes

$$2v_4v_1v_2a_2 + v_4^2v_2a_2 + v_2^2a_3v_4 + v_4^2v_1b_2 + b_2v_4^3 + 2v_4v_2a_1 + v_4v_2a_2 - v_2^2a_3 + v_4^2b_1 - v_4v_1b_2 + b_2v_4^2 - 2v_4v_2b_3 - v_4b_1 = 0 \tag{7E}$$

Collecting the above on the terms v_i introduced, and these are

$$\{v_1, v_2, v_3, v_4\}$$

Equation (7E) now becomes

$$2v_4v_1v_2a_2 + v_4^2v_1b_2 - v_4v_1b_2 + v_2^2a_3v_4 - v_2^2a_3 + v_4^2v_2a_2 + (2a_1 + a_2 - 2b_3)v_2v_4 + b_2v_4^3 + (b_1 + b_2)v_4^2 - v_4b_1 = 0 \tag{8E}$$

Setting each coefficients in (8E) to zero gives the following equations to solve

$$\begin{aligned} a_2 &= 0 \\ a_3 &= 0 \\ b_2 &= 0 \\ 2a_2 &= 0 \\ -a_3 &= 0 \\ -b_1 &= 0 \\ -b_2 &= 0 \\ b_1 + b_2 &= 0 \\ 2a_1 + a_2 - 2b_3 &= 0 \end{aligned}$$

Solving the above equations for the unknowns gives

$$\begin{aligned} a_1 &= b_3 \\ a_2 &= 0 \\ a_3 &= 0 \\ b_1 &= 0 \\ b_2 &= 0 \\ b_3 &= b_3 \end{aligned}$$

Substituting the above solution in the anstaz (1E,2E) (using 1 as arbitrary value for any unknown in the RHS) gives

$$\xi = 1$$
$$\eta = y$$

Shifting is now applied to make $\xi = 0$ in order to simplify the rest of the computation

$$\begin{aligned}
\eta &= \eta - \omega(x,y)\,\xi \\
&= y - \left(-\frac{y}{\text{LambertW}\left(-y^2 \mathrm{e}^{-2x}\right)}\right) (1) \\
&= \frac{y + y\,\text{LambertW}\left(-y^2 \mathrm{e}^{-2x}\right)}{\text{LambertW}\left(-y^2 \mathrm{e}^{-2x}\right)} \\
\xi &= 0
\end{aligned}$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x,y) \to (R,S)$ where (R,S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y}\right) S(x,y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$R = x$$

S is found from

$$\begin{aligned}
S &= \int \frac{1}{\eta} dy \\
&= \int \frac{1}{\frac{y + y\,\text{LambertW}\left(-y^2 \mathrm{e}^{-2x}\right)}{\text{LambertW}\left(-y^2 \mathrm{e}^{-2x}\right)}} dy
\end{aligned}$$

Which results in

$$S = \frac{\text{LambertW}\left(-y^2 \mathrm{e}^{-2x}\right)}{2}$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x,y) S_y}{R_x + \omega(x,y) R_y} \tag{2}$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x, y)$ is the right hand side of the original ode given by

$$\omega(x, y) = -\frac{y}{\text{LambertW}\left(-y^2 \mathrm{e}^{-2x}\right)}$$

Evaluating all the partial derivatives gives

$$\begin{aligned}
R_x &= 1 \\
R_y &= 0 \\
S_x &= -\frac{\text{LambertW}\left(-y^2 \mathrm{e}^{-2x}\right)}{1 + \text{LambertW}\left(-y^2 \mathrm{e}^{-2x}\right)} \\
S_y &= \frac{\text{LambertW}\left(-y^2 \mathrm{e}^{-2x}\right)}{y \left(1 + \text{LambertW}\left(-y^2 \mathrm{e}^{-2x}\right)\right)}
\end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = -1 \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = -1$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$S(R) = -R + c_1 \tag{4}$$

To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$\frac{\text{LambertW}\left(-y^2 \mathrm{e}^{-2x}\right)}{2} = -x + c_1$$

Which simplifies to

$$\frac{\text{LambertW}\left(-y^2 \mathrm{e}^{-2x}\right)}{2} = -x + c_1$$

Summary

The solution(s) found are the following

$$\frac{\text{LambertW}\left(-y^2 \mathrm{e}^{-2x}\right)}{2} = -x + c_1 \tag{1}$$

Verification of solutions

$$\frac{\operatorname{LambertW}\left(-y^{2} \mathrm{e}^{-2x}\right)}{2} = -x + c_{1}$$

Verified OK.

Maple trace

```
`Methods for first order ODEs:
-> Solving 1st order ODE of high degree, 1st attempt
trying 1st order WeierstrassP solution for high degree ODE
trying 1st order WeierstrassPPrime solution for high degree ODE
trying 1st order JacobiSN solution for high degree ODE
trying 1st order ODE linearizable_by_differentiation
trying differential order: 1; missing variables
trying simple symmetries for implicit equations
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
trying homogeneous types:
trying exact
Looking for potential symmetries
trying an equivalence to an Abel ODE
trying 1st order ODE linearizable_by_differentiation
-> Solving 1st order ODE of high degree, Lie methods, 1st trial
`, `-> Computing symmetries using: way = 3`[1, y]
```

✓ Solution by Maple

Time used: 0.235 (sec). Leaf size: 68

```
dsolve(2*x*diff(y(x),x)-y(x)=diff(y(x),x)*ln(y(x)*diff(y(x),x)),y(x), singsol=all)
```

$$y(x) = \mathrm{e}^{x-\frac{1}{2}}$$
$$y(x) = -\mathrm{e}^{x-\frac{1}{2}}$$
$$y(x) = \sqrt{2}\,\sqrt{\mathrm{e}^{-2x+2c_{1}}\left(-c_{1}+x\right)}\,\mathrm{e}^{x}$$
$$y(x) = -\sqrt{2}\,\sqrt{\mathrm{e}^{-2x+2c_{1}}\left(-c_{1}+x\right)}\,\mathrm{e}^{x}$$

✓ Solution by Mathematica

Time used: 0.35 (sec). Leaf size: 59

```
DSolve[2*x*y'[x]-y[x]==y'[x]*Log[y[x]*y'[x]],y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to -e^{c_1}\sqrt{-2x + i\pi + 2c_1}$$
$$y(x) \to e^{c_1}\sqrt{-2x + i\pi + 2c_1}$$
$$y(x) \to 0$$

1.86 problem 89

Internal problem ID [3231]
Internal file name [`OUTPUT/2723_Sunday_June_05_2022_08_39_21_AM_24446730/index.tex`]

Book: Differential equations for engineers by Wei-Chau XIE, Cambridge Press 2010
Section: Chapter 2. First-Order and Simple Higher-Order Differential Equations. Page 78
Problem number: 89.
ODE order: 1.
ODE degree: 3.

The type(s) of ODE detected by this program : **"first_order_ode_lie_symmetry_calculated"**

Maple gives the following as the ode type

`[[_1st_order, _with_linear_symmetries]]`

$$\boxed{y - xy' + x^2{y'}^3 = 0}$$

Solving the given ode for y' results in 3 differential equations to solve. Each one of these will generate a solution. The equations generated are

$$y' = \frac{\left(\left(12\sqrt{3}\,\sqrt{27y^2-4x}-108y\right)x\right)^{\frac{1}{3}}}{6x} + \frac{2}{\left(\left(12\sqrt{3}\,\sqrt{27y^2-4x}-108y\right)x\right)^{\frac{1}{3}}} \tag{1}$$

$$y' = -\frac{\left(\left(12\sqrt{3}\,\sqrt{27y^2-4x}-108y\right)x\right)^{\frac{1}{3}}}{12x} - \frac{1}{\left(\left(12\sqrt{3}\,\sqrt{27y^2-4x}-108y\right)x\right)^{\frac{1}{3}}} + \frac{i\sqrt{3}\left(\frac{\left(\left(12\sqrt{3}\,\sqrt{27y^2-4x}-108y\right.\right.}{6x}\right.}{} \tag{2}$$

$$y' = -\frac{\left(\left(12\sqrt{3}\,\sqrt{27y^2-4x}-108y\right)x\right)^{\frac{1}{3}}}{12x} - \frac{1}{\left(\left(12\sqrt{3}\,\sqrt{27y^2-4x}-108y\right)x\right)^{\frac{1}{3}}} - \frac{i\sqrt{3}\left(\frac{\left(\left(12\sqrt{3}\,\sqrt{27y^2-4x}-108y\right.\right.}{6x}\right.}{} \tag{3}$$

Now each one of the above ODE is solved.

Solving equation (1)

Writing the ode as

$$
\begin{aligned}
y' &= \frac{\left(\left(12\sqrt{3}\,\sqrt{27y^2-4x}-108y\right)x\right)^{\frac{2}{3}}+12x}{6x\left(\left(12\sqrt{3}\,\sqrt{27y^2-4x}-108y\right)x\right)^{\frac{1}{3}}} \\
y' &= \omega(x,y)
\end{aligned}
$$

The condition of Lie symmetry is the linearized PDE given by

$$
\eta_x+\omega(\eta_y-\xi_x)-\omega^2\xi_y-\omega_x\xi-\omega_y\eta=0 \tag{A}
$$

The type of this ode is not in the lookup table. To determine ξ,η then (A) is solved using ansatz. Making bivariate polynomials of degree 1 to use as anstaz gives

$$
\xi = xa_2+ya_3+a_1 \tag{1E}
$$

$$
\eta = xb_2+yb_3+b_1 \tag{2E}
$$

Where the unknown coefficients are

$$
\{a_1,a_2,a_3,b_1,b_2,b_3\}
$$

Substituting equations (1E,2E) and ω into (A) gives

$$b_2+\frac{\left(\left(\left(12\sqrt{3}\,\sqrt{27y^2-4x}-108y\right)x\right)^{\frac{2}{3}}+12x\right)(b_3-a_2)}{6x\left(\left(12\sqrt{3}\,\sqrt{27y^2-4x}-108y\right)x\right)^{\frac{1}{3}}} \tag{5E}$$
$$-\frac{\left(\left(\left(12\sqrt{3}\,\sqrt{27y^2-4x}-108y\right)x\right)^{\frac{2}{3}}+12x\right)^2 a_3}{36x^2\left(\left(12\sqrt{3}\,\sqrt{27y^2-4x}-108y\right)x\right)^{\frac{2}{3}}}$$
$$-\left(\frac{\frac{-\frac{16\sqrt{3}\,x}{\sqrt{27y^2-4x}}+8\sqrt{3}\,\sqrt{27y^2-4x}-72y}{\left(\left(12\sqrt{3}\,\sqrt{27y^2-4x}-108y\right)x\right)^{\frac{1}{3}}}+12}{6x\left(\left(12\sqrt{3}\,\sqrt{27y^2-4x}-108y\right)x\right)^{\frac{1}{3}}}\right.$$
$$-\frac{\left(\left(12\sqrt{3}\,\sqrt{27y^2-4x}-108y\right)x\right)^{\frac{2}{3}}+12x}{6x^2\left(\left(12\sqrt{3}\,\sqrt{27y^2-4x}-108y\right)x\right)^{\frac{1}{3}}}$$
$$\left.-\frac{\left(\left(\left(12\sqrt{3}\,\sqrt{27y^2-4x}-108y\right)x\right)^{\frac{2}{3}}+12x\right)\left(-\frac{24\sqrt{3}\,x}{\sqrt{27y^2-4x}}+12\sqrt{3}\,\sqrt{27y^2-4x}-108y\right)}{18x\left(\left(12\sqrt{3}\,\sqrt{27y^2-4x}-108y\right)x\right)^{\frac{4}{3}}}\right)(xa_2$$
$$+ya_3+a_1)-\left(\frac{\frac{324\sqrt{3}\,y}{\sqrt{27y^2-4x}}-108}{9\left(\left(12\sqrt{3}\,\sqrt{27y^2-4x}-108y\right)x\right)^{\frac{2}{3}}}\right.$$
$$\left.-\frac{\left(\left(\left(12\sqrt{3}\,\sqrt{27y^2-4x}-108y\right)x\right)^{\frac{2}{3}}+12x\right)\left(\frac{324\sqrt{3}\,y}{\sqrt{27y^2-4x}}-108\right)}{18\left(\left(12\sqrt{3}\,\sqrt{27y^2-4x}-108y\right)x\right)^{\frac{4}{3}}}\right)(xb_2+yb_3+b_1)$$
$$=0$$

Putting the above in normal form gives

Expression too large to display

Setting the numerator to zero gives

Expression too large to display (6E)

Simplifying the above gives

Expression too large to display (6E)

Since the PDE has radicals, simplifying gives

Expression too large to display

Looking at the above PDE shows the following are all the terms with $\{x, y\}$ in them.

$$\left\{x, y, \left(\left(\sqrt{3}\,\sqrt{27y^2-4x}-9y\right)x\right)^{\frac{1}{3}}, \left(\left(\sqrt{3}\,\sqrt{27y^2-4x}-9y\right)x\right)^{\frac{2}{3}}, \sqrt{27y^2-4x}\right\}$$

The following substitution is now made to be able to collect on all terms with $\{x, y\}$ in them

$$\left\{x=v_1, y=v_2, \left(\left(\sqrt{3}\,\sqrt{27y^2-4x}-9y\right)x\right)^{\frac{1}{3}}=v_3, \left(\left(\sqrt{3}\,\sqrt{27y^2-4x}-9y\right)x\right)^{\frac{2}{3}}=v_4, \sqrt{27y^2-4x}=v_5\right\}$$

The above PDE (6E) now becomes

$$\begin{aligned}
&72v_1\Big(18\,12^{\frac{2}{3}}v_4\sqrt{3}\,v_2^3a_3+3v_5 12^{\frac{2}{3}}v_4v_1^2b_2-6v_5 12^{\frac{2}{3}}v_4v_2^2a_3\\
&+2\,12^{\frac{2}{3}}v_4\sqrt{3}\,v_1^2a_2-4\,12^{\frac{2}{3}}v_4\sqrt{3}\,v_1^2b_3+18\,12^{\frac{2}{3}}v_4\sqrt{3}\,v_2^2a_1\\
&-2v_5 12^{\frac{2}{3}}v_4v_1a_3+3v_5 12^{\frac{2}{3}}v_4v_1b_1-6v_5 12^{\frac{2}{3}}v_4v_2a_1\\
&-2\,12^{\frac{2}{3}}v_4\sqrt{3}\,v_1a_1-24\,12^{\frac{1}{3}}v_3\sqrt{3}\,v_1^3b_2+16\,12^{\frac{1}{3}}v_3\sqrt{3}\,v_1^2a_3\\
&-9\,12^{\frac{2}{3}}v_4\sqrt{3}\,v_1v_2^2a_2-9\,12^{\frac{2}{3}}v_4\sqrt{3}\,v_1^2v_2b_2+18\,12^{\frac{2}{3}}v_4\sqrt{3}\,v_1v_2^2b_3\\
&+3v_5 12^{\frac{2}{3}}v_4v_1v_2a_2-6v_5 12^{\frac{2}{3}}v_4v_1v_2b_3-2\,12^{\frac{2}{3}}v_4\sqrt{3}\,v_1v_2a_3\\
&-9\,12^{\frac{2}{3}}v_4\sqrt{3}\,v_1v_2b_1+162\,12^{\frac{1}{3}}v_3\sqrt{3}\,v_1^2v_2^2b_2-54v_5 12^{\frac{1}{3}}v_3v_1^2v_2b_2\\
&-108\,12^{\frac{1}{3}}v_3\sqrt{3}\,v_1v_2^2a_3+36v_5 12^{\frac{1}{3}}v_3v_1v_2a_3-216\sqrt{3}\,v_1^2v_2^2a_2\\
&+432\sqrt{3}\,v_1^2v_2^2b_3+1080\sqrt{3}\,v_1v_2^3a_3+72v_5v_1^2v_2a_2-144v_5v_1^2v_2b_3\\
&-360v_5v_1v_2^2a_3-168\sqrt{3}\,v_1^2v_2a_3+108\sqrt{3}\,v_1^2v_2b_1+108\sqrt{3}\,v_1v_2^2a_1\\
&-36v_5v_1v_2a_1+108\sqrt{3}\,v_1^3v_2b_2-48\sqrt{3}\,v_1^3b_3+24v_5v_1^2a_3\\
&-36v_5v_1^2b_1-24\sqrt{3}\,v_1^2a_1-36v_5v_1^3b_2+24\sqrt{3}\,v_1^3a_2\Big)=0
\end{aligned} \tag{7E}$$

Collecting the above on the terms v_i introduced, and these are

$$\{v_1, v_2, v_3, v_4, v_5\}$$

Equation (7E) now becomes

$$
\begin{aligned}
&-7776\,12^{\frac{1}{3}}\sqrt{3}\,a_3v_2^2v_3v_1^2 + 2592\,12^{\frac{1}{3}}a_3v_2v_3v_5v_1^2\\
&+1296\,12^{\frac{2}{3}}\sqrt{3}\,a_3v_2^3v_4v_1 - 432\,12^{\frac{2}{3}}a_3v_2^2v_4v_5v_1\\
&+1296\,12^{\frac{2}{3}}\sqrt{3}\,a_1v_2^2v_4v_1 - 432\,12^{\frac{2}{3}}a_1v_2v_4v_5v_1\\
&+11664\,12^{\frac{1}{3}}\sqrt{3}\,b_2v_2^2v_3v_1^3 - 3888\,12^{\frac{1}{3}}b_2v_2v_3v_5v_1^3\\
&-648\,12^{\frac{2}{3}}\sqrt{3}\,b_2v_2v_4v_1^3 + 7776\sqrt{3}\,b_2v_2v_1^4\\
&+(5184a_2 - 10368b_3)\,v_2v_5v_1^3 + 77760\sqrt{3}\,a_3v_2^3v_1^2\\
&-1728\,12^{\frac{1}{3}}\sqrt{3}\,b_2v_3v_1^4 + 1152\,12^{\frac{1}{3}}\sqrt{3}\,a_3v_3v_1^3 + 216\,12^{\frac{2}{3}}b_2v_4v_5v_1^3\\
&+\left(216\,12^{\frac{2}{3}}a_2 - 432\,12^{\frac{2}{3}}b_3\right)v_2v_4v_5v_1^2 - 144\,12^{\frac{2}{3}}\sqrt{3}\,a_1v_4v_1^2\\
&+\left(-648\,12^{\frac{2}{3}}\sqrt{3}\,a_2 + 1296\,12^{\frac{2}{3}}\sqrt{3}\,b_3\right)v_2^2v_4v_1^2 - 25920a_3v_2^2v_5v_1^2\\
&+7776\sqrt{3}\,a_1v_2^2v_1^2 + \left(-144\,12^{\frac{2}{3}}\sqrt{3}\,a_3 - 648\,12^{\frac{2}{3}}\sqrt{3}\,b_1\right)v_2v_4v_1^2\\
&-2592a_1v_2v_5v_1^2 + \left(-144\,12^{\frac{2}{3}}a_3 + 216\,12^{\frac{2}{3}}b_1\right)v_4v_5v_1^2\\
&+\left(-12096\sqrt{3}\,a_3 + 7776\sqrt{3}\,b_1\right)v_2v_1^3\\
&+\left(144\,12^{\frac{2}{3}}\sqrt{3}\,a_2 - 288\,12^{\frac{2}{3}}\sqrt{3}\,b_3\right)v_4v_1^3 + (1728a_3 - 2592b_1)\,v_5v_1^3\\
&-1728\sqrt{3}\,a_1v_1^3 - 2592b_2v_5v_1^4 + \left(1728\sqrt{3}\,a_2 - 3456\sqrt{3}\,b_3\right)v_1^4\\
&+\left(-15552\sqrt{3}\,a_2 + 31104\sqrt{3}\,b_3\right)v_2^2v_1^3 = 0
\end{aligned}
\tag{8E}
$$

Setting each coefficients in (8E) to zero gives the following equations to solve

$$
\begin{aligned}
-2592a_1 &= 0 \\
-25920a_3 &= 0 \\
-2592b_2 &= 0 \\
-1728\sqrt{3}\, a_1 &= 0 \\
7776\sqrt{3}\, a_1 &= 0 \\
77760\sqrt{3}\, a_3 &= 0 \\
7776\sqrt{3}\, b_2 &= 0 \\
2592\, 12^{\frac{1}{3}} a_3 &= 0 \\
-3888\, 12^{\frac{1}{3}} b_2 &= 0 \\
-432\, 12^{\frac{2}{3}} a_1 &= 0 \\
-432\, 12^{\frac{2}{3}} a_3 &= 0 \\
216\, 12^{\frac{2}{3}} b_2 &= 0 \\
-7776\, 12^{\frac{1}{3}} \sqrt{3}\, a_3 &= 0 \\
1152\, 12^{\frac{1}{3}} \sqrt{3}\, a_3 &= 0 \\
-1728\, 12^{\frac{1}{3}} \sqrt{3}\, b_2 &= 0 \\
11664\, 12^{\frac{1}{3}} \sqrt{3}\, b_2 &= 0 \\
-144\, 12^{\frac{2}{3}} \sqrt{3}\, a_1 &= 0 \\
1296\, 12^{\frac{2}{3}} \sqrt{3}\, a_1 &= 0 \\
1296\, 12^{\frac{2}{3}} \sqrt{3}\, a_3 &= 0 \\
-648\, 12^{\frac{2}{3}} \sqrt{3}\, b_2 &= 0 \\
5184a_2 - 10368b_3 &= 0 \\
1728a_3 - 2592b_1 &= 0 \\
-15552\sqrt{3}\, a_2 + 31104\sqrt{3}\, b_3 &= 0 \\
1728\sqrt{3}\, a_2 - 3456\sqrt{3}\, b_3 &= 0 \\
-12096\sqrt{3}\, a_3 + 7776\sqrt{3}\, b_1 &= 0 \\
216\, 12^{\frac{2}{3}} a_2 - 432\, 12^{\frac{2}{3}} b_3 &= 0 \\
-144\, 12^{\frac{2}{3}} a_3 + 216\, 12^{\frac{2}{3}} b_1 &= 0 \\
-648\, 12^{\frac{2}{3}} \sqrt{3}\, a_2 + 1296\, 12^{\frac{2}{3}} \sqrt{3}\, b_3 &= 0 \\
144\, 12^{\frac{2}{3}} \sqrt{3}\, a_2 - 288\, 12^{\frac{2}{3}} \sqrt{3}\, b_3 &= 0 \\
-144\, 12^{\frac{2}{3}} \sqrt{3}\, a_3 - 648\, 12^{\frac{2}{3}} \sqrt{3}\, b_1 &= 0
\end{aligned}
$$

Solving the above equations for the unknowns gives

$$
\begin{aligned}
a_1 &= 0 \\
a_2 &= 2b_3 \\
a_3 &= 0 \\
b_1 &= 0 \\
b_2 &= 0 \\
b_3 &= b_3
\end{aligned}
$$

Substituting the above solution in the anstaz (1E,2E) (using 1 as arbitrary value for any unknown in the RHS) gives

$$
\begin{aligned}
\xi &= 2x \\
\eta &= y
\end{aligned}
$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x, y) \to (R, S)$ where (R, S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$
\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}
$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y}\right) S(x, y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the canonical coordinates, where $S(R)$. Therefore

$$
\begin{aligned}
\frac{dy}{dx} &= \frac{\eta}{\xi} \\
&= \frac{y}{2x} \\
&= \frac{y}{2x}
\end{aligned}
$$

This is easily solved to give

$$
y = c_1 \sqrt{x}
$$

Where now the coordinate R is taken as the constant of integration. Hence

$$
R = \frac{y}{\sqrt{x}}
$$

And S is found from

$$\begin{aligned} dS &= \frac{dx}{\xi} \\ &= \frac{dx}{2x} \end{aligned}$$

Integrating gives

$$\begin{aligned} S &= \int \frac{dx}{T} \\ &= \frac{\ln(x)}{2} \end{aligned}$$

Where the constant of integration is set to zero as we just need one solution. Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x,y)S_y}{R_x + \omega(x,y)R_y} \tag{2}$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x,y)$ is the right hand side of the original ode given by

$$\omega(x,y) = \frac{\left(\left(12\sqrt{3}\,\sqrt{27y^2-4x}-108y\right)x\right)^{\frac{2}{3}}+12x}{6x\left(\left(12\sqrt{3}\,\sqrt{27y^2-4x}-108y\right)x\right)^{\frac{1}{3}}}$$

Evaluating all the partial derivatives gives

$$\begin{aligned} R_x &= -\frac{y}{2x^{\frac{3}{2}}} \\ R_y &= \frac{1}{\sqrt{x}} \\ S_x &= \frac{1}{2x} \\ S_y &= 0 \end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = \frac{3\sqrt{x}\left(\left(\sqrt{3}\,\sqrt{27y^2-4x}-9y\right)x\right)^{\frac{1}{3}}}{12^{\frac{2}{3}}x+12^{\frac{1}{3}}\left(\left(\sqrt{3}\,\sqrt{27y^2-4x}-9y\right)x\right)^{\frac{2}{3}}-3\left(\left(\sqrt{3}\,\sqrt{27y^2-4x}-9y\right)x\right)^{\frac{1}{3}}y} \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = \frac{3\left(\sqrt{3}\,\sqrt{27R^2-4}-9R\right)^{\frac{1}{3}}}{12^{\frac{1}{3}}\left(\sqrt{3}\,\sqrt{27R^2-4}-9R\right)^{\frac{2}{3}}+12^{\frac{2}{3}}-3\left(\sqrt{3}\,\sqrt{27R^2-4}-9R\right)^{\frac{1}{3}}R}$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$S(R) = \int \frac{3\left(\sqrt{81R^2-12}-9R\right)^{\frac{1}{3}}}{12^{\frac{1}{3}}\left(\left(\sqrt{81R^2-12}-9R\right)^2\right)^{\frac{1}{3}}+12^{\frac{2}{3}}-3\left(\sqrt{81R^2-12}-9R\right)^{\frac{1}{3}}R}dR + c_1 \tag{4}$$

To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$\frac{\ln(x)}{2} = \int^{\frac{y}{\sqrt{x}}} \frac{3\left(\sqrt{81\_a^2-12}-9\_a\right)^{\frac{1}{3}}}{12^{\frac{1}{3}}\left(\left(\sqrt{81\_a^2-12}-9\_a\right)^2\right)^{\frac{1}{3}}+12^{\frac{2}{3}}-3\left(\sqrt{81\_a^2-12}-9\_a\right)^{\frac{1}{3}}\_a}d\_a + c_1$$

Which simplifies to

$$\frac{\ln(x)}{2} = \int^{\frac{y}{\sqrt{x}}} \frac{3\left(\sqrt{81\_a^2-12}-9\_a\right)^{\frac{1}{3}}}{12^{\frac{1}{3}}\left(\left(\sqrt{81\_a^2-12}-9\_a\right)^2\right)^{\frac{1}{3}}+12^{\frac{2}{3}}-3\left(\sqrt{81\_a^2-12}-9\_a\right)^{\frac{1}{3}}\_a}d\_a + c_1$$

Summary

The solution(s) found are the following

$$\frac{\ln(x)}{2} = \int^{\frac{y}{\sqrt{x}}} \frac{3\left(\sqrt{81\_a^2-12}-9\_a\right)^{\frac{1}{3}}}{12^{\frac{1}{3}}\left(\left(\sqrt{81\_a^2-12}-9\_a\right)^2\right)^{\frac{1}{3}}+12^{\frac{2}{3}}-3\left(\sqrt{81\_a^2-12}-9\_a\right)^{\frac{1}{3}}\_a}d\_a + c_1 \tag{1}$$

Verification of solutions

$$\frac{\ln(x)}{2} = \int^{\frac{y}{\sqrt{x}}} \frac{3\left(\sqrt{81\_a^2-12}-9\_a\right)^{\frac{1}{3}}}{12^{\frac{1}{3}}\left(\left(\sqrt{81\_a^2-12}-9\_a\right)^2\right)^{\frac{1}{3}}+12^{\frac{2}{3}}-3\left(\sqrt{81\_a^2-12}-9\_a\right)^{\frac{1}{3}}\_a}d\_a + c_1$$

Verified OK.

Solving equation (2)

Writing the ode as

$$y' = \frac{i\sqrt{3}\left(\left(12\sqrt{3}\,\sqrt{27y^2-4x}-108y\right)x\right)^{\frac{2}{3}}-12i\sqrt{3}\,x-\left(\left(12\sqrt{3}\,\sqrt{27y^2-4x}-108y\right)x\right)^{\frac{2}{3}}-12x}{12x\left(\left(12\sqrt{3}\,\sqrt{27y^2-4x}-108y\right)x\right)^{\frac{1}{3}}}$$

$$y' = \omega(x,y)$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x + \omega(\eta_y - \xi_x) - \omega^2\xi_y - \omega_x\xi - \omega_y\eta = 0 \tag{A}$$

The type of this ode is not in the lookup table. To determine ξ, η then (A) is solved using ansatz. Making bivariate polynomials of degree 1 to use as anstaz gives

$$\xi = xa_2 + ya_3 + a_1 \tag{1E}$$

$$\eta = xb_2 + yb_3 + b_1 \tag{2E}$$

Where the unknown coefficients are

$$\{a_1, a_2, a_3, b_1, b_2, b_3\}$$

Substituting equations (1E,2E) and ω into (A) gives

$$b_2 \tag{5E}$$

$$+\frac{\left(i\sqrt{3}\left(\left(12\sqrt{3}\sqrt{27y^2-4x}-108y\right)x\right)^{\frac{2}{3}}-12i\sqrt{3}\,x-\left(\left(12\sqrt{3}\sqrt{27y^2-4x}-108y\right)x\right)^{\frac{2}{3}}-12x\right)(b_3-a_2)}{12x\left(\left(12\sqrt{3}\sqrt{27y^2-4x}-108y\right)x\right)^{\frac{1}{3}}}$$

$$-\frac{\left(i\sqrt{3}\left(\left(12\sqrt{3}\sqrt{27y^2-4x}-108y\right)x\right)^{\frac{2}{3}}-12i\sqrt{3}\,x-\left(\left(12\sqrt{3}\sqrt{27y^2-4x}-108y\right)x\right)^{\frac{2}{3}}-12x\right)^2 a_3}{144x^2\left(\left(12\sqrt{3}\sqrt{27y^2-4x}-108y\right)x\right)^{\frac{2}{3}}}$$

$$-\left(\frac{\frac{2i\sqrt{3}\left(-\frac{24\sqrt{3}\,x}{\sqrt{27y^2-4x}}+12\sqrt{3}\sqrt{27y^2-4x}-108y\right)}{3\left(\left(12\sqrt{3}\sqrt{27y^2-4x}-108y\right)x\right)^{\frac{1}{3}}}-12i\sqrt{3}-\frac{2\left(-\frac{24\sqrt{3}\,x}{\sqrt{27y^2-4x}}+12\sqrt{3}\sqrt{27y^2-4x}-108y\right)}{3\left(\left(12\sqrt{3}\sqrt{27y^2-4x}-108y\right)x\right)^{\frac{1}{3}}}-12}{12x\left(\left(12\sqrt{3}\sqrt{27y^2-4x}-108y\right)x\right)^{\frac{1}{3}}}\right.$$

$$-\frac{i\sqrt{3}\left(\left(12\sqrt{3}\sqrt{27y^2-4x}-108y\right)x\right)^{\frac{2}{3}}-12i\sqrt{3}\,x-\left(\left(12\sqrt{3}\sqrt{27y^2-4x}-108y\right)x\right)^{\frac{2}{3}}-12x}{12x^2\left(\left(12\sqrt{3}\sqrt{27y^2-4x}-108y\right)x\right)^{\frac{1}{3}}}$$

$$-\frac{\left(i\sqrt{3}\left(\left(12\sqrt{3}\sqrt{27y^2-4x}-108y\right)x\right)^{\frac{2}{3}}-12i\sqrt{3}\,x-\left(\left(12\sqrt{3}\sqrt{27y^2-4x}-108y\right)x\right)^{\frac{2}{3}}-12x\right)\left(-\frac{24\sqrt{3}}{\sqrt{27y^2-}}\right.}{36x\left(\left(12\sqrt{3}\sqrt{27y^2-4x}-108y\right)x\right)^{\frac{4}{3}}}$$

$$+ya_3+a_1\Bigg)-\left(\frac{\frac{2i\sqrt{3}\left(\frac{324\sqrt{3}\,y}{\sqrt{27y^2-4x}}-108\right)x}{3\left(\left(12\sqrt{3}\sqrt{27y^2-4x}-108y\right)x\right)^{\frac{1}{3}}}-\frac{2\left(\frac{324\sqrt{3}\,y}{\sqrt{27y^2-4x}}-108\right)x}{3\left(\left(12\sqrt{3}\sqrt{27y^2-4x}-108y\right)x\right)^{\frac{1}{3}}}}{12x\left(\left(12\sqrt{3}\sqrt{27y^2-4x}-108y\right)x\right)^{\frac{1}{3}}}\right.$$

$$-\frac{\left(i\sqrt{3}\left(\left(12\sqrt{3}\sqrt{27y^2-4x}-108y\right)x\right)^{\frac{2}{3}}-12i\sqrt{3}\,x-\left(\left(12\sqrt{3}\sqrt{27y^2-4x}-108y\right)x\right)^{\frac{2}{3}}-12x\right)\left(\frac{324\sqrt{3}\,y}{\sqrt{27y^2-4}}\right.}{36\left(\left(12\sqrt{3}\sqrt{27y^2-4x}-108y\right)x\right)^{\frac{4}{3}}}$$

$$+yb_3+b_1\Bigg)=0$$

Putting the above in normal form gives

Expression too large to display

Setting the numerator to zero gives

$$\text{Expression too large to display} \tag{6E}$$

Simplifying the above gives

$$\text{Expression too large to display} \tag{6E}$$

Since the PDE has radicals, simplifying gives

$$\text{Expression too large to display}$$

Looking at the above PDE shows the following are all the terms with $\{x, y\}$ in them.

$$\left\{x, y, \left(\left(\sqrt{3}\,\sqrt{27y^2-4x}-9y\right)x\right)^{\frac{1}{3}}, \left(\left(\sqrt{3}\,\sqrt{27y^2-4x}-9y\right)x\right)^{\frac{2}{3}}, \sqrt{27y^2-4x}\right\}$$

The following substitution is now made to be able to collect on all terms with $\{x, y\}$ in them

$$\left\{x=v_1, y=v_2, \left(\left(\sqrt{3}\,\sqrt{27y^2-4x}-9y\right)x\right)^{\frac{1}{3}}=v_3, \left(\left(\sqrt{3}\,\sqrt{27y^2-4x}-9y\right)x\right)^{\frac{2}{3}}=v_4, \sqrt{27y^2-4x}=v_5\right\}$$

The above PDE (6E) now becomes

$$\text{Expression too large to display} \tag{7E}$$

Collecting the above on the terms v_i introduced, and these are

$$\{v_1, v_2, v_3, v_4, v_5\}$$

Equation (7E) now becomes

$$
\begin{aligned}
&\left(-864i12^{\frac{2}{3}}\sqrt{3}\,a_3+864\,12^{\frac{2}{3}}a_3\right)v_2^2v_4v_5v_1\\
&+\left(-864i12^{\frac{2}{3}}\sqrt{3}\,a_1+864\,12^{\frac{2}{3}}a_1\right)v_2v_4v_5v_1\\
&+\left(432i12^{\frac{2}{3}}\sqrt{3}\,a_2-864i12^{\frac{2}{3}}\sqrt{3}\,b_3\right.\\
&\left.-432\,12^{\frac{2}{3}}a_2+864\,12^{\frac{2}{3}}b_3\right)v_2v_4v_5v_1^2\\
&+4608\,12^{\frac{1}{3}}\sqrt{3}\,a_3v_3v_1^3-6912\,12^{\frac{1}{3}}\sqrt{3}\,b_2v_3v_1^4\\
&+\left(-466560ia_3-155520\sqrt{3}\,a_3\right)v_2^3v_1^2\\
&+\left(-46656ia_1-15552\sqrt{3}\,a_1\right)v_2^2v_1^2\\
&+46656\,12^{\frac{1}{3}}\sqrt{3}\,b_2v_2^2v_3v_1^3-15552\,12^{\frac{1}{3}}b_2v_2v_3v_5v_1^3\\
&-31104\,12^{\frac{1}{3}}\sqrt{3}\,a_3v_2^2v_3v_1^2+10368\,12^{\frac{1}{3}}a_3v_2v_3v_5v_1^2\\
&+\left(-10368ia_2+20736ib_3-3456\sqrt{3}\,a_2+6912\sqrt{3}\,b_3\right)v_1^4\\
&+\left(10368ia_1+3456\sqrt{3}\,a_1\right)v_1^3\\
&+\left(-3888i12^{\frac{2}{3}}b_2+1296\,12^{\frac{2}{3}}\sqrt{3}\,b_2\right)v_2v_4v_1^3\\
&+\left(-10368i\sqrt{3}\,a_2+20736i\sqrt{3}\,b_3\right.\\
&\left.-10368a_2+20736b_3\right)v_2v_5v_1^3\\
&+\left(432i12^{\frac{2}{3}}\sqrt{3}\,b_2-432\,12^{\frac{2}{3}}b_2\right)v_4v_5v_1^3+\left(-3888i12^{\frac{2}{3}}a_2\right.\\
&\left.+7776i12^{\frac{2}{3}}b_3+1296\,12^{\frac{2}{3}}\sqrt{3}\,a_2-2592\,12^{\frac{2}{3}}\sqrt{3}\,b_3\right)v_2^2v_4v_1^2\\
&+\left(51840i\sqrt{3}\,a_3+51840a_3\right)v_2^2v_5v_1^2+\left(-864i12^{\frac{2}{3}}a_3\right.\\
&\left.-3888i12^{\frac{2}{3}}b_1+288\,12^{\frac{2}{3}}\sqrt{3}\,a_3+1296\,12^{\frac{2}{3}}\sqrt{3}\,b_1\right)v_2v_4v_1^2\\
&+\left(5184i\sqrt{3}\,a_1+5184a_1\right)v_2v_5v_1^2+\left(-288i12^{\frac{2}{3}}\sqrt{3}\,a_3\right.\\
&\left.+432i12^{\frac{2}{3}}\sqrt{3}\,b_1+288\,12^{\frac{2}{3}}a_3-432\,12^{\frac{2}{3}}b_1\right)v_4v_5v_1^2\\
&+\left(7776i12^{\frac{2}{3}}a_3-2592\,12^{\frac{2}{3}}\sqrt{3}\,a_3\right)v_2^3v_4v_1\\
&+\left(7776i12^{\frac{2}{3}}a_1-2592\,12^{\frac{2}{3}}\sqrt{3}\,a_1\right)v_2^2v_4v_1\\
&+\left(-864i12^{\frac{2}{3}}a_1+288\,12^{\frac{2}{3}}\sqrt{3}\,a_1\right)v_4v_1^2\\
&+\left(-46656ib_2-15552\sqrt{3}\,b_2\right)v_2v_1^4\\
&+\left(5184i\sqrt{3}\,b_2+5184b_2\right)v_5v_1^4+\left(93312ia_2\right.\\
&\left.-186624ib_3+31104\sqrt{3}\,a_2-62208\sqrt{3}\,b_3\right)v_2^2v_1^3\\
&+\left(72576ia_3-46656ib_1+24192\sqrt{3}\,a_3\right.\\
&\left.-15552\sqrt{3}\,b_1\right)v_2v_1^3+\left(864i12^{\frac{2}{3}}a_2-1728i12^{\frac{2}{3}}b_3\right.
\end{aligned}
\qquad (8E)
$$

Setting each coefficients in (8E) to zero gives the following equations to solve

$$10368\,12^{\frac{1}{3}} a_3 = 0$$
$$-15552\,12^{\frac{1}{3}} b_2 = 0$$
$$-31104\,12^{\frac{1}{3}} \sqrt{3}\, a_3 = 0$$
$$4608\,12^{\frac{1}{3}} \sqrt{3}\, a_3 = 0$$
$$-6912\,12^{\frac{1}{3}} \sqrt{3}\, b_2 = 0$$
$$46656\,12^{\frac{1}{3}} \sqrt{3}\, b_2 = 0$$
$$-466560 i a_3 - 155520\sqrt{3}\, a_3 = 0$$
$$-46656 i a_1 - 15552\sqrt{3}\, a_1 = 0$$
$$-46656 i b_2 - 15552\sqrt{3}\, b_2 = 0$$
$$10368 i a_1 + 3456\sqrt{3}\, a_1 = 0$$
$$-3888 i 12^{\frac{2}{3}} b_2 + 1296\,12^{\frac{2}{3}} \sqrt{3}\, b_2 = 0$$
$$-864 i 12^{\frac{2}{3}} a_1 + 288\,12^{\frac{2}{3}} \sqrt{3}\, a_1 = 0$$
$$5184 i \sqrt{3}\, a_1 + 5184 a_1 = 0$$
$$5184 i \sqrt{3}\, b_2 + 5184 b_2 = 0$$
$$7776 i 12^{\frac{2}{3}} a_1 - 2592\,12^{\frac{2}{3}} \sqrt{3}\, a_1 = 0$$
$$7776 i 12^{\frac{2}{3}} a_3 - 2592\,12^{\frac{2}{3}} \sqrt{3}\, a_3 = 0$$
$$51840 i \sqrt{3}\, a_3 + 51840 a_3 = 0$$
$$-864 i 12^{\frac{2}{3}} \sqrt{3}\, a_1 + 864\,12^{\frac{2}{3}} a_1 = 0$$
$$-864 i 12^{\frac{2}{3}} \sqrt{3}\, a_3 + 864\,12^{\frac{2}{3}} a_3 = 0$$
$$432 i 12^{\frac{2}{3}} \sqrt{3}\, b_2 - 432\,12^{\frac{2}{3}} b_2 = 0$$
$$-10368 i a_2 + 20736 i b_3 - 3456\sqrt{3}\, a_2 + 6912\sqrt{3}\, b_3 = 0$$
$$72576 i a_3 - 46656 i b_1 + 24192\sqrt{3}\, a_3 - 15552\sqrt{3}\, b_1 = 0$$
$$93312 i a_2 - 186624 i b_3 + 31104\sqrt{3}\, a_2 - 62208\sqrt{3}\, b_3 = 0$$
$$-10368 i \sqrt{3}\, a_2 + 20736 i \sqrt{3}\, b_3 - 10368 a_2 + 20736 b_3 = 0$$
$$-3888 i 12^{\frac{2}{3}} a_2 + 7776 i 12^{\frac{2}{3}} b_3 + 1296\,12^{\frac{2}{3}} \sqrt{3}\, a_2 - 2592\,12^{\frac{2}{3}} \sqrt{3}\, b_3 = 0$$
$$-3456 i \sqrt{3}\, a_3 + 5184 i \sqrt{3}\, b_1 - 3456 a_3 + 5184 b_1 = 0$$
$$-864 i 12^{\frac{2}{3}} a_3 - 3888 i 12^{\frac{2}{3}} b_1 + 288\,12^{\frac{2}{3}} \sqrt{3}\, a_3 + 1296\,12^{\frac{2}{3}} \sqrt{3}\, b_1 = 0$$
$$864 i 12^{\frac{2}{3}} a_2 - 1728 i 12^{\frac{2}{3}} b_3 - 288\,12^{\frac{2}{3}} \sqrt{3}\, a_2 + 576\,12^{\frac{2}{3}} \sqrt{3}\, b_3 = 0$$
$$-288 i 12^{\frac{2}{3}} \sqrt{3}\, a_3 + 432 i 12^{\frac{2}{3}} \sqrt{3}\, b_1 + 288\,12^{\frac{2}{3}} a_3 - 432\,12^{\frac{2}{3}} b_1 = 0$$
$$432 i 12^{\frac{2}{3}} \sqrt{3}\, a_2 - 864 i 12^{\frac{2}{3}} \sqrt{3}\, b_3 - 432\,12^{\frac{2}{3}} a_2 + 864\,12^{\frac{2}{3}} b_3 = 0$$

Solving the above equations for the unknowns gives

$$
\begin{aligned}
a_1 &= 0 \\
a_2 &= 2b_3 \\
a_3 &= 0 \\
b_1 &= 0 \\
b_2 &= 0 \\
b_3 &= b_3
\end{aligned}
$$

Substituting the above solution in the anstaz (1E,2E) (using 1 as arbitrary value for any unknown in the RHS) gives

$$
\begin{aligned}
\xi &= 2x \\
\eta &= y
\end{aligned}
$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x, y) \to (R, S)$ where (R, S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$
\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}
$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y}\right) S(x, y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the canonical coordinates, where $S(R)$. Unable to determine R. Terminating

Unable to determine ODE type.

Solving equation (3)

Writing the ode as

$$
\begin{aligned}
y' &= -\frac{i\sqrt{3}\left(\left(12\sqrt{3}\,\sqrt{27y^2-4x}-108y\right)x\right)^{\frac{2}{3}} - 12i\sqrt{3}\,x + \left(\left(12\sqrt{3}\,\sqrt{27y^2-4x}-108y\right)x\right)^{\frac{2}{3}} + 12x}{12x\left(\left(12\sqrt{3}\,\sqrt{27y^2-4x}-108y\right)x\right)^{\frac{1}{3}}} \\
y' &= \omega(x, y)
\end{aligned}
$$

The condition of Lie symmetry is the linearized PDE given by

$$
\eta_x + \omega(\eta_y - \xi_x) - \omega^2 \xi_y - \omega_x \xi - \omega_y \eta = 0 \tag{A}
$$

The type of this ode is not in the lookup table. To determine ξ, η then (A) is solved using ansatz. Making bivariate polynomials of degree 1 to use as anstaz gives

$$\xi = xa_2 + ya_3 + a_1 \tag{1E}$$

$$\eta = xb_2 + yb_3 + b_1 \tag{2E}$$

Where the unknown coefficients are

$$\{a_1, a_2, a_3, b_1, b_2, b_3\}$$

Substituting equations (1E,2E) and ω into (A) gives

$$
\begin{aligned}
&b_2 \qquad \text{(5E)}\\
&-\frac{\left(i\sqrt{3}\left(\left(12\sqrt{3}\,\sqrt{27y^2-4x}-108y\right)x\right)^{\frac{2}{3}}-12i\sqrt{3}\,x+\left(\left(12\sqrt{3}\,\sqrt{27y^2-4x}-108y\right)x\right)^{\frac{2}{3}}+12x\right)(b_3-a_2)}{12x\left(\left(12\sqrt{3}\,\sqrt{27y^2-4x}-108y\right)x\right)^{\frac{1}{3}}}\\
&-\frac{\left(i\sqrt{3}\left(\left(12\sqrt{3}\,\sqrt{27y^2-4x}-108y\right)x\right)^{\frac{2}{3}}-12i\sqrt{3}\,x+\left(\left(12\sqrt{3}\,\sqrt{27y^2-4x}-108y\right)x\right)^{\frac{2}{3}}+12x\right)^2 a_3}{144x^2\left(\left(12\sqrt{3}\,\sqrt{27y^2-4x}-108y\right)x\right)^{\frac{2}{3}}}\\
&-\left(-\frac{\frac{2i\sqrt{3}\left(-\frac{24\sqrt{3}\,x}{\sqrt{27y^2-4x}}+12\sqrt{3}\,\sqrt{27y^2-4x}-108y\right)}{3\left(\left(12\sqrt{3}\,\sqrt{27y^2-4x}-108y\right)x\right)^{\frac{1}{3}}}-12i\sqrt{3}+\frac{-\frac{16\sqrt{3}\,x}{\sqrt{27y^2-4x}}+8\sqrt{3}\,\sqrt{27y^2-4x}-72y}{\left(\left(12\sqrt{3}\,\sqrt{27y^2-4x}-108y\right)x\right)^{\frac{1}{3}}}+12}{12x\left(\left(12\sqrt{3}\,\sqrt{27y^2-4x}-108y\right)x\right)^{\frac{1}{3}}}\right.\\
&+\frac{i\sqrt{3}\left(\left(12\sqrt{3}\,\sqrt{27y^2-4x}-108y\right)x\right)^{\frac{2}{3}}-12i\sqrt{3}\,x+\left(\left(12\sqrt{3}\,\sqrt{27y^2-4x}-108y\right)x\right)^{\frac{2}{3}}+12x}{12x^2\left(\left(12\sqrt{3}\,\sqrt{27y^2-4x}-108y\right)x\right)^{\frac{1}{3}}}\\
&+\frac{\left(i\sqrt{3}\left(\left(12\sqrt{3}\,\sqrt{27y^2-4x}-108y\right)x\right)^{\frac{2}{3}}-12i\sqrt{3}\,x+\left(\left(12\sqrt{3}\,\sqrt{27y^2-4x}-108y\right)x\right)^{\frac{2}{3}}+12x\right)\left(-\frac{24\sqrt{3}\,x}{\sqrt{27y^2-}}\right.}{36x\left(\left(12\sqrt{3}\,\sqrt{27y^2-4x}-108y\right)x\right)^{\frac{4}{3}}}\\
&\left.+ya_3+a_1\right)-\left(-\frac{\frac{2i\sqrt{3}\left(\frac{324\sqrt{3}\,y}{\sqrt{27y^2-4x}}-108\right)x}{3\left(\left(12\sqrt{3}\,\sqrt{27y^2-4x}-108y\right)x\right)^{\frac{1}{3}}}+\frac{2\left(\frac{324\sqrt{3}\,y}{\sqrt{27y^2-4x}}-108\right)x}{3\left(\left(12\sqrt{3}\,\sqrt{27y^2-4x}-108y\right)x\right)^{\frac{1}{3}}}}{12x\left(\left(12\sqrt{3}\,\sqrt{27y^2-4x}-108y\right)x\right)^{\frac{1}{3}}}\right.\\
&+\frac{\left(i\sqrt{3}\left(\left(12\sqrt{3}\,\sqrt{27y^2-4x}-108y\right)x\right)^{\frac{2}{3}}-12i\sqrt{3}\,x+\left(\left(12\sqrt{3}\,\sqrt{27y^2-4x}-108y\right)x\right)^{\frac{2}{3}}+12x\right)\left(\frac{324\sqrt{3}\,y}{\sqrt{27y^2-4x}}\right.}{36\left(\left(12\sqrt{3}\,\sqrt{27y^2-4x}-108y\right)x\right)^{\frac{4}{3}}}\\
&\left.+yb_3+b_1\right)=0
\end{aligned}
$$

Putting the above in normal form gives

Expression too large to display

Setting the numerator to zero gives

$$\text{Expression too large to display} \tag{6E}$$

Simplifying the above gives

$$\text{Expression too large to display} \tag{6E}$$

Since the PDE has radicals, simplifying gives

$$\text{Expression too large to display}$$

Looking at the above PDE shows the following are all the terms with $\{x, y\}$ in them.

$$\left\{x, y, \left(\left(\sqrt{3}\,\sqrt{27y^2-4x}-9y\right)x\right)^{\frac{1}{3}}, \left(\left(\sqrt{3}\,\sqrt{27y^2-4x}-9y\right)x\right)^{\frac{2}{3}}, \sqrt{27y^2-4x}\right\}$$

The following substitution is now made to be able to collect on all terms with $\{x, y\}$ in them

$$\left\{x=v_1, y=v_2, \left(\left(\sqrt{3}\,\sqrt{27y^2-4x}-9y\right)x\right)^{\frac{1}{3}}=v_3, \left(\left(\sqrt{3}\,\sqrt{27y^2-4x}-9y\right)x\right)^{\frac{2}{3}}=v_4, \sqrt{27y^2-4x}=v_5\right\}$$

The above PDE (6E) now becomes

$$\text{Expression too large to display} \tag{7E}$$

Collecting the above on the terms v_i introduced, and these are

$$\{v_1, v_2, v_3, v_4, v_5\}$$

Equation (7E) now becomes

$$
\begin{aligned}
&\left(10368ia_2 - 20736ib_3 - 3456\sqrt{3}\,a_2 + 6912\sqrt{3}\,b_3\right) v_1^4 \\
&+ \left(-10368ia_1 + 3456\sqrt{3}\,a_1\right) v_1^3 - 6912\,12^{\frac{1}{3}}\sqrt{3}\,b_2 v_3 v_1^4 \\
&+ 4608\,12^{\frac{1}{3}}\sqrt{3}\,a_3 v_3 v_1^3 + \left(-432i12^{\frac{2}{3}}\sqrt{3}\,a_2\right. \\
&\left.+ 864i12^{\frac{2}{3}}\sqrt{3}\,b_3 - 432\,12^{\frac{2}{3}}a_2 + 864\,12^{\frac{2}{3}}b_3\right) v_2 v_4 v_5 v_1^2 \\
&+ \left(864i12^{\frac{2}{3}}\sqrt{3}\,a_3 + 864\,12^{\frac{2}{3}}a_3\right) v_2^2 v_4 v_5 v_1 \\
&+ \left(864i12^{\frac{2}{3}}\sqrt{3}\,a_1 + 864\,12^{\frac{2}{3}}a_1\right) v_2 v_4 v_5 v_1 \\
&+ \left(3888i12^{\frac{2}{3}}b_2 + 1296\,12^{\frac{2}{3}}\sqrt{3}\,b_2\right) v_2 v_4 v_1^3 \\
&+ \left(10368i\sqrt{3}\,a_2 - 20736i\sqrt{3}\,b_3\right. \\
&\left.- 10368a_2 + 20736b_3\right) v_2 v_5 v_1^3 \\
&+ \left(-432i12^{\frac{2}{3}}\sqrt{3}\,b_2 - 432\,12^{\frac{2}{3}}b_2\right) v_4 v_5 v_1^3 \\
&+ \left(3888i12^{\frac{2}{3}}a_2 - 7776i12^{\frac{2}{3}}b_3\right. \\
&\left.+ 1296\,12^{\frac{2}{3}}\sqrt{3}\,a_2 - 2592\,12^{\frac{2}{3}}\sqrt{3}\,b_3\right) v_2^2 v_4 v_1^2 \\
&+ \left(-51840i\sqrt{3}\,a_3 + 51840a_3\right) v_2^2 v_5 v_1^2 + \left(864i12^{\frac{2}{3}}a_3\right. \\
&\left.+ 3888i12^{\frac{2}{3}}b_1 + 288\,12^{\frac{2}{3}}\sqrt{3}\,a_3 + 1296\,12^{\frac{2}{3}}\sqrt{3}\,b_1\right) v_2 v_4 v_1^2 \qquad \text{(8E)} \\
&+ \left(-5184i\sqrt{3}\,a_1 + 5184a_1\right) v_2 v_5 v_1^2 + \left(288i12^{\frac{2}{3}}\sqrt{3}\,a_3\right. \\
&\left.- 432i12^{\frac{2}{3}}\sqrt{3}\,b_1 + 288\,12^{\frac{2}{3}}a_3 - 432\,12^{\frac{2}{3}}b_1\right) v_4 v_5 v_1^2 \\
&+ \left(-7776i12^{\frac{2}{3}}a_3 - 2592\,12^{\frac{2}{3}}\sqrt{3}\,a_3\right) v_2^3 v_4 v_1 \\
&+ \left(-7776i12^{\frac{2}{3}}a_1 - 2592\,12^{\frac{2}{3}}\sqrt{3}\,a_1\right) v_2^2 v_4 v_1 \\
&+ 10368\,12^{\frac{1}{3}}a_3 v_2 v_3 v_5 v_1^2 + 46656\,12^{\frac{1}{3}}\sqrt{3}\,b_2 v_2^2 v_3 v_1^3 \\
&- 15552\,12^{\frac{1}{3}}b_2 v_2 v_3 v_5 v_1^3 - 31104\,12^{\frac{1}{3}}\sqrt{3}\,a_3 v_2^2 v_3 v_1^2 \\
&+ \left(46656ib_2 - 15552\sqrt{3}\,b_2\right) v_2 v_1^4 \\
&+ \left(-5184i\sqrt{3}\,b_2 + 5184b_2\right) v_5 v_1^4 + \left(-93312ia_2\right. \\
&\left.+ 186624ib_3 + 31104\sqrt{3}\,a_2 - 62208\sqrt{3}\,b_3\right) v_2^2 v_1^3 \\
&+ \left(-72576ia_3 + 46656ib_1 + 24192\sqrt{3}\,a_3\right. \\
&\left.- 15552\sqrt{3}\,b_1\right) v_2 v_1^3 + \left(-864i12^{\frac{2}{3}}a_2 + 1728i12^{\frac{2}{3}}b_3\right. \\
&\left.- 288\,12^{\frac{2}{3}}\sqrt{3}\,a_2 + 576\,12^{\frac{2}{3}}\sqrt{3}\,b_3\right) v_4 v_1^3 \\
&+ \left(3456i\sqrt{3}\,a_3 - 5184i\sqrt{3}\,b_1 - 3456a_3 + 5184b_1\right) v_5 v_1^3 \\
&+ \left(466560ia_3 - 155520\sqrt{3}\,a_3\right) v_2^3 v_1^2
\end{aligned}
$$

Setting each coefficients in (8E) to zero gives the following equations to solve

$$10368\, 12^{\frac{1}{3}} a_3 = 0$$
$$-15552\, 12^{\frac{1}{3}} b_2 = 0$$
$$-31104\, 12^{\frac{1}{3}} \sqrt{3}\, a_3 = 0$$
$$4608\, 12^{\frac{1}{3}} \sqrt{3}\, a_3 = 0$$
$$-6912\, 12^{\frac{1}{3}} \sqrt{3}\, b_2 = 0$$
$$46656\, 12^{\frac{1}{3}} \sqrt{3}\, b_2 = 0$$
$$-10368 i a_1 + 3456\sqrt{3}\, a_1 = 0$$
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$$46656 i b_2 - 15552\sqrt{3}\, b_2 = 0$$
$$466560 i a_3 - 155520\sqrt{3}\, a_3 = 0$$
$$-51840 i\sqrt{3}\, a_3 + 51840 a_3 = 0$$
$$-7776 i 12^{\frac{2}{3}} a_1 - 2592\, 12^{\frac{2}{3}} \sqrt{3}\, a_1 = 0$$
$$-7776 i 12^{\frac{2}{3}} a_3 - 2592\, 12^{\frac{2}{3}} \sqrt{3}\, a_3 = 0$$
$$-5184 i\sqrt{3}\, a_1 + 5184 a_1 = 0$$
$$-5184 i\sqrt{3}\, b_2 + 5184 b_2 = 0$$
$$864 i 12^{\frac{2}{3}} a_1 + 288\, 12^{\frac{2}{3}} \sqrt{3}\, a_1 = 0$$
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$$-432 i 12^{\frac{2}{3}} \sqrt{3}\, b_2 - 432\, 12^{\frac{2}{3}} b_2 = 0$$
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$$-72576 i a_3 + 46656 i b_1 + 24192\sqrt{3}\, a_3 - 15552\sqrt{3}\, b_1 = 0$$
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$$3456 i\sqrt{3}\, a_3 - 5184 i\sqrt{3}\, b_1 - 3456 a_3 + 5184 b_1 = 0$$
$$3888 i 12^{\frac{2}{3}} a_2 - 7776 i 12^{\frac{2}{3}} b_3 + 1296\, 12^{\frac{2}{3}} \sqrt{3}\, a_2 - 2592\, 12^{\frac{2}{3}} \sqrt{3}\, b_3 = 0$$
$$10368 i\sqrt{3}\, a_2 - 20736 i\sqrt{3}\, b_3 - 10368 a_2 + 20736 b_3 = 0$$
$$-432 i 12^{\frac{2}{3}} \sqrt{3}\, a_2 + 864 i 12^{\frac{2}{3}} \sqrt{3}\, b_3 - 432\, 12^{\frac{2}{3}} a_2 + 864\, 12^{\frac{2}{3}} b_3 = 0$$
$$288 i 12^{\frac{2}{3}} \sqrt{3}\, a_3 - 432 i 12^{\frac{2}{3}} \sqrt{3}\, b_1 + 288\, 12^{\frac{2}{3}} a_3 - 432\, 12^{\frac{2}{3}} b_1 = 0$$

Solving the above equations for the unknowns gives

$$
\begin{aligned}
a_1 &= 0 \\
a_2 &= 2b_3 \\
a_3 &= 0 \\
b_1 &= 0 \\
b_2 &= 0 \\
b_3 &= b_3
\end{aligned}
$$

Substituting the above solution in the anstaz (1E,2E) (using 1 as arbitrary value for any unknown in the RHS) gives

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\begin{aligned}
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$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x, y) \to (R, S)$ where (R, S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$
\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}
$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y}\right) S(x, y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the canonical coordinates, where $S(R)$. Unable to determine R. Terminating

Unable to determine ODE type.

Maple trace

```
`Methods for first order ODEs:
-> Solving 1st order ODE of high degree, 1st attempt
trying 1st order WeierstrassP solution for high degree ODE
trying 1st order WeierstrassPPrime solution for high degree ODE
trying 1st order JacobiSN solution for high degree ODE
trying 1st order ODE linearizable_by_differentiation
trying differential order: 1; missing variables
trying dAlembert
trying simple symmetries for implicit equations
Successful isolation of dy/dx: 3 solutions were found. Trying to solve each resulting ODE.
   *** Sublevel 2 ***
   Methods for first order ODEs:
   --- Trying classification methods ---
   trying homogeneous types:
   trying exact
   Looking for potential symmetries
   trying an equivalence to an Abel ODE
   trying 1st order ODE linearizable_by_differentiation
-> Solving 1st order ODE of high degree, Lie methods, 1st trial
`, `-> Computing symmetries using: way = 2
`, `-> Computing symmetries using: way = 2
-> Solving 1st order ODE of high degree, 2nd attempt. Trying parametric methods
-> Calling odsolve with the ODE`, diff(y(x), x) = (3*y(x)*x-(-4*y(x)*x+1)^(1/2)-1)/(x^2*(1+(
   Methods for first order ODEs:
   --- Trying classification methods ---
   trying homogeneous types:
   trying homogeneous G
   <- homogeneous successful
-> Calling odsolve with the ODE`, diff(y(x), x) = (-3*y(x)*x-(-4*y(x)*x+1)^(1/2)+1)/(x^2*(-1
   Methods for first order ODEs:
   --- Trying classification methods ---
   trying homogeneous types:
   trying homogeneous G
   <- homogeneous successful
<- 1st order, parametric methods successful`
```

✓ Solution by Maple

Time used: 0.14 (sec). Leaf size: 123

```
dsolve(y(x)=x*diff(y(x),x)-x^2* (diff(y(x),x))^3,y(x), singsol=all)
```

$$
\begin{aligned}
y(x) = &-x^2 \operatorname{RootOf}\left(4\_Z^4 c_1 x^2+8\_Z^2 c_1 x-\_Z+4 c_1\right)^3 \\
&+x \operatorname{RootOf}\left(4\_Z^4 c_1 x^2+8\_Z^2 c_1 x-\_Z+4 c_1\right) \\
y(x) = &-x^2 \operatorname{RootOf}\left(4\_Z^4 c_1 x^2-16\_Z^2 c_1 x-\_Z+16 c_1\right)^3 \\
&+x \operatorname{RootOf}\left(4\_Z^4 c_1 x^2-16\_Z^2 c_1 x-\_Z+16 c_1\right)
\end{aligned}
$$

✗ Solution by Mathematica

Time used: 0.0 (sec). Leaf size: 0

```
DSolve[y[x]==x*y'[x]-x^2*(y'[x])^3,y[x],x,IncludeSingularSolutions -> True]
```

Timed out

1.87 problem 90

Internal problem ID [3232]
Internal file name [`OUTPUT/2724_Sunday_June_05_2022_08_39_25_AM_11199866/index.tex`]

Book: Differential equations for engineers by Wei-Chau XIE, Cambridge Press 2010
Section: Chapter 2. First-Order and Simple Higher-Order Differential Equations. Page 78
Problem number: 90.
ODE order: 1.
ODE degree: 3.

The type(s) of ODE detected by this program : **"exact", "first_order_ode_lie_symmetry_calculated"**

Maple gives the following as the ode type

```
[[_homogeneous, `class G`]]
```

$$\boxed{y(y-2xy')^3 - y'^2 = 0}$$

Solving the given ode for y' results in 3 differential equations to solve. Each one of these will generate a solution. The equations generated are

$$y' = \frac{\left(-216y^4x^4 + 24\sqrt{3}\,\sqrt{27y^2x^2-1}\,y^3x^3 + 36y^2x^2 - 1\right)^{\frac{1}{3}}}{24yx^3} - \frac{24y^2x^2-1}{24yx^3\left(-216y^4x^4 + 24\sqrt{3}\,\sqrt{27y^2x^2-1}\,y^3x^{\cdots}} \tag{1}$$

$$y' = -\frac{\left(-216y^4x^4 + 24\sqrt{3}\,\sqrt{27y^2x^2-1}\,y^3x^3 + 36y^2x^2 - 1\right)^{\frac{1}{3}}}{48yx^3} + \frac{24y^2x^2-1}{48yx^3\left(-216y^4x^4 + 24\sqrt{3}\,\sqrt{27y^2x^2-1}\,y^{\cdots}} \tag{2}$$

$$y' = -\frac{\left(-216y^4x^4 + 24\sqrt{3}\,\sqrt{27y^2x^2-1}\,y^3x^3 + 36y^2x^2 - 1\right)^{\frac{1}{3}}}{48yx^3} + \frac{24y^2x^2-1}{48yx^3\left(-216y^4x^4 + 24\sqrt{3}\,\sqrt{27y^2x^2-1}\,y^{\cdots}} \tag{3}$$

Now each one of the above ODE is solved.

Solving equation (1)

Writing the ode as

$$y' = \frac{12y^2x^2\left(-216x^4y^4+24\sqrt{3}\,\sqrt{27y^2x^2-1}\,y^3x^3+36y^2x^2-1\right)^{\frac{1}{3}}-24y^2x^2+\left(-216x^4y^4+24\sqrt{3}\,\sqrt{27y^2x}\right.}{24y\,x^3\left(-216x^4y^4+24\sqrt{3}\,\sqrt{27y^2x^2-\right.}}$$

$$y' = \omega(x,y)$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x+\omega(\eta_y-\xi_x)-\omega^2\xi_y-\omega_x\xi-\omega_y\eta=0 \tag{A}$$

The type of this ode is not in the lookup table. To determine ξ,η then (A) is solved using ansatz. Making bivariate polynomials of degree 1 to use as anstaz gives

$$\xi = xa_2+ya_3+a_1 \tag{1E}$$

$$\eta = xb_2+yb_3+b_1 \tag{2E}$$

Where the unknown coefficients are

$$\{a_1,a_2,a_3,b_1,b_2,b_3\}$$

Substituting equations (1E,2E) and ω into (A) gives

$$\text{Expression too large to display} \tag{5E}$$

Putting the above in normal form gives

$$\text{Expression too large to display}$$

Setting the numerator to zero gives

$$\text{Expression too large to display} \tag{6E}$$

Simplifying the above gives

$$\text{Expression too large to display} \tag{6E}$$

Since the PDE has radicals, simplifying gives

$$\text{Expression too large to display}$$

Looking at the above PDE shows the following are all the terms with $\{x, y\}$ in them.

$$\left\{x, y, \left(-216x^4y^4 + 24\sqrt{3}\,\sqrt{27y^2x^2 - 1}\,y^3x^3 + 36y^2x^2 - 1\right)^{\frac{1}{3}}, \left(-216x^4y^4 + 24\sqrt{3}\,\sqrt{27y^2x^2 - 1}\,y^3x^3 + 36y^2x^2 - 1\right)^{\frac{2}{3}}, \sqrt{27y^2x^2 - 1}\right\}$$

The following substitution is now made to be able to collect on all terms with $\{x, y\}$ in them

$$\left\{x = v_1, y = v_2, \left(-216x^4y^4 + 24\sqrt{3}\,\sqrt{27y^2x^2 - 1}\,y^3x^3 + 36y^2x^2 - 1\right)^{\frac{1}{3}} = v_3, \left(-216x^4y^4 + 24\sqrt{3}\,\sqrt{27y^2x^2 - 1}\,y^3x^3 + 36y^2x^2 - 1\right)^{\frac{2}{3}} = v_4, \sqrt{27y^2x^2 - 1} = v_5\right\}$$

The above PDE (6E) now becomes

$$\text{Expression too large to display} \tag{7E}$$

Collecting the above on the terms v_i introduced, and these are

$$\{v_1, v_2, v_3, v_4, v_5\}$$

Equation (7E) now becomes

$$\text{Expression too large to display} \tag{8E}$$

Setting each coefficients in (8E) to zero gives the following equations to solve

$$-62208a_1 = 0$$
$$-57024a_1 = 0$$
$$-8640a_1 = 0$$
$$-2880a_1 = 0$$
$$-72a_1 = 0$$
$$72a_1 = 0$$
$$2016a_1 = 0$$
$$3744a_1 = 0$$
$$25920a_1 = 0$$
$$290304a_1 = 0$$
$$-31104a_3 = 0$$
$$-3456a_3 = 0$$
$$-1584a_3 = 0$$
$$-108a_3 = 0$$
$$-3a_3 = 0$$
$$3a_3 = 0$$
$$72a_3 = 0$$
$$144a_3 = 0$$
$$504a_3 = 0$$
$$5184a_3 = 0$$
$$72576a_3 = 0$$
$$-5184b_1 = 0$$
$$-576b_1 = 0$$
$$-24b_1 = 0$$
$$24b_1 = 0$$
$$288b_1 = 0$$
$$864b_1 = 0$$
$$1728b_1 = 0$$
$$41472b_1 = 0$$
$$62208b_1 = 0$$
$$-62208b_2 = 0$$
$$-5184b_2 = 0$$
$$-1152b_2 = 0$$
$$-24b_2 = 0$$
$$24b_2 = 0$$
$$288b_2 = 0$$

Solving the above equations for the unknowns gives

$$
\begin{aligned}
a_1 &= 0 \\
a_2 &= -b_3 \\
a_3 &= 0 \\
b_1 &= 0 \\
b_2 &= 0 \\
b_3 &= b_3
\end{aligned}
$$

Substituting the above solution in the anstaz (1E,2E) (using 1 as arbitrary value for any unknown in the RHS) gives

$$
\begin{aligned}
\xi &= -x \\
\eta &= y
\end{aligned}
$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x, y) \to (R, S)$ where (R, S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$
\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}
$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y}\right) S(x, y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the canonical coordinates, where $S(R)$. Therefore

$$
\begin{aligned}
\frac{dy}{dx} &= \frac{\eta}{\xi} \\
&= \frac{y}{-x} \\
&= -\frac{y}{x}
\end{aligned}
$$

This is easily solved to give

$$
y = \frac{c_1}{x}
$$

Where now the coordinate R is taken as the constant of integration. Hence

$$
R = xy
$$

And S is found from

$$
\begin{aligned}
dS &= \frac{dx}{\xi} \\
&= \frac{dx}{-x}
\end{aligned}
$$

Integrating gives

$$
\begin{aligned}
S &= \int \frac{dx}{T} \\
&= -\ln(x)
\end{aligned}
$$

Where the constant of integration is set to zero as we just need one solution. Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$
\frac{dS}{dR} = \frac{S_x + \omega(x,y) S_y}{R_x + \omega(x,y) R_y} \tag{2}
$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x,y)$ is the right hand side of the original ode given by

$$
\omega(x,y) = \frac{12y^2x^2\left(-216x^4y^4 + 24\sqrt{3}\,\sqrt{27y^2x^2-1}\,y^3x^3 + 36y^2x^2 - 1\right)^{\frac{1}{3}} - 24y^2x^2 + \left(-216x^4y^4 + 24\sqrt{3}\,\sqrt{\cdots}\right.}{24y\,x^3\left(-216x^4y^4 + 24\sqrt{3}\,\sqrt{27y\cdots}\right.}
$$

Evaluating all the partial derivatives gives

$$
\begin{aligned}
R_x &= y \\
R_y &= x \\
S_x &= -\frac{1}{x} \\
S_y &= 0
\end{aligned}
$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$
\frac{dS}{dR} = -\frac{24yx\left(-216x^4y^4 + 24\sqrt{3}\,\sqrt{27y^2x\cdots}\right.}{36y^2x^2\left(-216x^4y^4 + 24\sqrt{3}\,\sqrt{27y^2x^2-1}\,y^3x^3 + 36y^2x^2 - 1\right)^{\frac{1}{3}} - 24y^2x^2 + \left(-216x^4y^4 + 24\sqrt{3}\,\sqrt{2\cdots}\right.} \tag{2A}
$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$
\frac{dS}{dR} = -\frac{24R\left(-216R^4 + 24\sqrt{3}\,\sqrt{27R^2-1}\,R^3 + \cdots\right.}{36R^2\left(-216R^4 + 24\sqrt{3}\,\sqrt{27R^2-1}\,R^3 + 36R^2 - 1\right)^{\frac{1}{3}} - 24R^2 + \left(-216R^4 + 24\sqrt{3}\,\sqrt{27R^2-1}\,R^3 \cdots\right.}
$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$S(R) = \int -\frac{24R\left(-216R^4 + 24R^3\sqrt{81R^2-3} + 36R \cdots\right.}{36R^2\left(-216R^4 + 24R^3\sqrt{81R^2-3} + 36R^2 - 1\right)^{\frac{1}{3}} - 24R^2 + \left(-216R^4 + 24R^3\sqrt{81R^2-3} + 3 \cdots\right.} \tag{4}$$

To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$-\ln(x) = \int^{yx} -\frac{24\_a\left(-216\_a^4 + 24\_a^3\sqrt{81\_ \cdots}\right.}{36\_a^2\left(-216\_a^4 + 24\_a^3\sqrt{81\_a^2-3} + 36\_a^2 - 1\right)^{\frac{1}{3}} - 24\_a^2 + \left(-216\_a^4 + 24\_a^3\sqrt{\cdots}\right.}$$

Which simplifies to

$$-\ln(x) = \int^{yx} -\frac{24\_a\left(-216\_a^4 + 24\_a^3\sqrt{81\_ \cdots}\right.}{36\_a^2\left(-216\_a^4 + 24\_a^3\sqrt{81\_a^2-3} + 36\_a^2 - 1\right)^{\frac{1}{3}} - 24\_a^2 + \left(-216\_a^4 + 24\_a^3\sqrt{\cdots}\right.}$$

Summary

The solution(s) found are the following

$$-\ln(x) = \int^{yx} \tag{1}$$

$$-\frac{24\_a\left(-216\_a^4 + 24\_a^3\sqrt{81\_a^2-3} + 36\_ \cdots\right.}{36\_a^2\left(-216\_a^4 + 24\_a^3\sqrt{81\_a^2-3} + 36\_a^2 - 1\right)^{\frac{1}{3}} - 24\_a^2 + \left(-216\_a^4 + 24\_a^3\sqrt{81\_a^2-3} + \cdots\right.}$$

$$+ c_1$$

Verification of solutions

$$-\ln(x) = \int^{yx}$$

$$-\frac{24\_a\left(-216\_a^4 + 24\_a^3\sqrt{81\_a^2-3} + 36\_ \cdots\right.}{36\_a^2\left(-216\_a^4 + 24\_a^3\sqrt{81\_a^2-3} + 36\_a^2 - 1\right)^{\frac{1}{3}} - 24\_a^2 + \left(-216\_a^4 + 24\_a^3\sqrt{81\_a^2-3} + \cdots\right.}$$

$$+ c_1$$

Verified OK.

Solving equation (2)

Writing the ode as

$$y' = \frac{24i\sqrt{3}\, y^2x^2 + 24y^2x^2\left(-216x^4y^4 + 24\sqrt{3}\,\sqrt{27y^2x^2-1}\, y^3x^3 + 36y^2x^2 - 1\right)^{\frac{1}{3}} + i\sqrt{3}\left(-216x^4y^4 + 24\sqrt{\right.}}{}$$

$$y' = \omega(x, y)$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x + \omega(\eta_y - \xi_x) - \omega^2\xi_y - \omega_x\xi - \omega_y\eta = 0 \tag{A}$$

The type of this ode is not in the lookup table. To determine ξ, η then (A) is solved using ansatz. Making bivariate polynomials of degree 1 to use as anstaz gives

$$\xi = xa_2 + ya_3 + a_1 \tag{1E}$$

$$\eta = xb_2 + yb_3 + b_1 \tag{2E}$$

Where the unknown coefficients are

$$\{a_1, a_2, a_3, b_1, b_2, b_3\}$$

Substituting equations (1E,2E) and ω into (A) gives

$$\text{Expression too large to display} \tag{5E}$$

Putting the above in normal form gives

$$\text{Expression too large to display}$$

Setting the numerator to zero gives

$$\text{Expression too large to display} \tag{6E}$$

Simplifying the above gives

$$\text{Expression too large to display} \tag{6E}$$

Since the PDE has radicals, simplifying gives

$$\text{Expression too large to display}$$

Looking at the above PDE shows the following are all the terms with $\{x, y\}$ in them.

$$\left\{x, y, \left(-216x^4y^4 + 24\sqrt{3}\,\sqrt{27y^2x^2 - 1}\,y^3x^3 + 36y^2x^2 - 1\right)^{\frac{1}{3}}, \left(-216x^4y^4 + 24\sqrt{3}\,\sqrt{27y^2x^2 - 1}\,y^3x^3 + 36y^2x^2 - 1\right)^{\frac{2}{3}}, \sqrt{27y^2x^2 - 1}\right\}$$

The following substitution is now made to be able to collect on all terms with $\{x, y\}$ in them

$$\left\{x = v_1, y = v_2, \left(-216x^4y^4 + 24\sqrt{3}\,\sqrt{27y^2x^2 - 1}\,y^3x^3 + 36y^2x^2 - 1\right)^{\frac{1}{3}} = v_3, \left(-216x^4y^4 + 24\sqrt{3}\,\sqrt{27y^2x^2 - 1}\,y^3x^3 + 36y^2x^2 - 1\right)^{\frac{2}{3}} = v_4, \sqrt{27y^2x^2 - 1} = v_5\right\}$$

The above PDE (6E) now becomes

$$\text{Expression too large to display} \tag{7E}$$

Collecting the above on the terms v_i introduced, and these are

$$\{v_1, v_2, v_3, v_4, v_5\}$$

Equation (7E) now becomes

$$\text{Expression too large to display} \tag{8E}$$

Setting each coefficients in (8E) to zero gives the following equations to solve

$$
\begin{aligned}
-248832a_1 &= 0 \\
-11520a_1 &= 0 \\
288a_1 &= 0 \\
103680a_1 &= 0 \\
-124416a_3 &= 0 \\
-432a_3 &= 0 \\
12a_3 &= 0 \\
2016a_3 &= 0 \\
20736a_3 &= 0 \\
-20736b_1 &= 0 \\
-2304b_1 &= 0 \\
96b_1 &= 0 \\
248832b_1 &= 0 \\
-248832b_2 &= 0 \\
-4608b_2 &= 0 \\
96b_2 &= 0 \\
62208b_2 &= 0 \\
-214272\sqrt{3}\,a_1 &= 0 \\
6912\sqrt{3}\,a_1 &= 0 \\
746496\sqrt{3}\,a_1 &= 0 \\
-13824\sqrt{3}\,a_3 &= 0 \\
-7776\sqrt{3}\,a_3 &= 0 \\
288\sqrt{3}\,a_3 &= 0 \\
373248\sqrt{3}\,a_3 &= 0 \\
-746496\sqrt{3}\,b_1 &= 0 \\
-34560\sqrt{3}\,b_1 &= 0 \\
2304\sqrt{3}\,b_1 &= 0 \\
-89856\sqrt{3}\,b_2 &= 0 \\
2304\sqrt{3}\,b_2 &= 0 \\
746496\sqrt{3}\,b_2 &= 0 \\
-6912a_2 - 6912b_3 &= 0 \\
192a_2 + 192b_3 &= 0 \\
41472a_2 + 41472b_3 &= 0 \\
-190080\sqrt{3}\,a_1 - 570240ia_1 &= 0 \\
-51840\sqrt{3}\,a_1 + 155520ia_1 &= 0
\end{aligned}
$$

Solving the above equations for the unknowns gives

$$
\begin{aligned}
a_1 &= 0 \\
a_2 &= -b_3 \\
a_3 &= 0 \\
b_1 &= 0 \\
b_2 &= 0 \\
b_3 &= b_3
\end{aligned}
$$

Substituting the above solution in the anstaz (1E,2E) (using 1 as arbitrary value for any unknown in the RHS) gives

$$
\begin{aligned}
\xi &= -x \\
\eta &= y
\end{aligned}
$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x, y) \to (R, S)$ where (R, S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$
\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}
$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y}\right) S(x, y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the canonical coordinates, where $S(R)$. Unable to determine R. Terminating

Unable to determine ODE type.

Solving equation (3)

Writing the ode as

$$
y' = -\frac{24i\sqrt{3}\, y^2x^2 - 24y^2x^2\left(-216x^4y^4 + 24\sqrt{3}\,\sqrt{27y^2x^2 - 1}\, y^3x^3 + 36y^2x^2 - 1\right)^{\frac{1}{3}} + i\sqrt{3}\left(-216x^4y^4 + 24\right.}{}
$$

$$
y' = \omega(x, y)
$$

The condition of Lie symmetry is the linearized PDE given by

$$
\eta_x + \omega(\eta_y - \xi_x) - \omega^2\xi_y - \omega_x\xi - \omega_y\eta = 0 \tag{A}
$$

The type of this ode is not in the lookup table. To determine ξ, η then (A) is solved using ansatz. Making bivariate polynomials of degree 1 to use as anstaz gives

$$\xi = xa_2 + ya_3 + a_1 \tag{1E}$$

$$\eta = xb_2 + yb_3 + b_1 \tag{2E}$$

Where the unknown coefficients are

$$\{a_1, a_2, a_3, b_1, b_2, b_3\}$$

Substituting equations (1E,2E) and ω into (A) gives

$$\text{Expression too large to display} \tag{5E}$$

Putting the above in normal form gives

$$\text{Expression too large to display}$$

Setting the numerator to zero gives

$$\text{Expression too large to display} \tag{6E}$$

Simplifying the above gives

$$\text{Expression too large to display} \tag{6E}$$

Since the PDE has radicals, simplifying gives

$$\text{Expression too large to display}$$

Looking at the above PDE shows the following are all the terms with $\{x, y\}$ in them.

$$\left\{x, y, \left(-216x^4y^4 + 24\sqrt{3}\,\sqrt{27y^2x^2-1}\,y^3x^3 + 36y^2x^2 - 1\right)^{\frac{1}{3}}, \left(-216x^4y^4 + 24\sqrt{3}\,\sqrt{27y^2x^2-1}\,y^3x^3 + 36y^2x^2 - 1\right)^{\frac{2}{3}}, \sqrt{27y^2x^2-1}\right\}$$

The following substitution is now made to be able to collect on all terms with $\{x, y\}$ in them

$$\left\{x = v_1, y = v_2, \left(-216x^4y^4 + 24\sqrt{3}\,\sqrt{27y^2x^2-1}\,y^3x^3 + 36y^2x^2 - 1\right)^{\frac{1}{3}} = v_3, \left(-216x^4y^4 + 24\sqrt{3}\,\sqrt{27y^2x^2-1}\,y^3x^3 + 36y^2x^2 - 1\right)^{\frac{2}{3}} = v_4, \sqrt{27y^2x^2-1} = v_5\right\}$$

The above PDE (6E) now becomes

$$\text{Expression too large to display} \tag{7E}$$

Collecting the above on the terms v_i introduced, and these are

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Equation (7E) now becomes

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Setting each coefficients in (8E) to zero gives the following equations to solve

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$$2016a_3 = 0$$
$$20736a_3 = 0$$
$$-20736b_1 = 0$$
$$-2304b_1 = 0$$
$$96b_1 = 0$$
$$248832b_1 = 0$$
$$-248832b_2 = 0$$
$$-4608b_2 = 0$$
$$96b_2 = 0$$
$$62208b_2 = 0$$
$$-214272\sqrt{3}\, a_1 = 0$$
$$6912\sqrt{3}\, a_1 = 0$$
$$746496\sqrt{3}\, a_1 = 0$$
$$-13824\sqrt{3}\, a_3 = 0$$
$$-7776\sqrt{3}\, a_3 = 0$$
$$288\sqrt{3}\, a_3 = 0$$
$$373248\sqrt{3}\, a_3 = 0$$
$$-746496\sqrt{3}\, b_1 = 0$$
$$-34560\sqrt{3}\, b_1 = 0$$
$$2304\sqrt{3}\, b_1 = 0$$
$$-89856\sqrt{3}\, b_2 = 0$$
$$2304\sqrt{3}\, b_2 = 0$$
$$746496\sqrt{3}\, b_2 = 0$$
$$-6912a_2 - 6912b_3 = 0$$
$$192a_2 + 192b_3 = 0$$
$$41472a_2 + 41472b_3 = 0$$
$$-190080\sqrt{3}\, a_1 + 570240ia_1 = 0$$
$$-51840\sqrt{3}\, a_1 - 155520ia_1 = 0$$

Solving the above equations for the unknowns gives

$$
\begin{aligned}
a_1 &= 0 \\
a_2 &= -b_3 \\
a_3 &= 0 \\
b_1 &= 0 \\
b_2 &= 0 \\
b_3 &= b_3
\end{aligned}
$$

Substituting the above solution in the anstaz (1E,2E) (using 1 as arbitrary value for any unknown in the RHS) gives

$$
\begin{aligned}
\xi &= -x \\
\eta &= y
\end{aligned}
$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x, y) \to (R, S)$ where (R, S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$
\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}
$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y}\right) S(x, y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the canonical coordinates, where $S(R)$. Unable to determine R. Terminating

Unable to determine ODE type.

Maple trace

```
`Methods for first order ODEs:
   *** Sublevel 2 ***
   Methods for first order ODEs:
   -> Solving 1st order ODE of high degree, 1st attempt
   trying 1st order WeierstrassP solution for high degree ODE
   trying 1st order WeierstrassPPrime solution for high degree ODE
   trying 1st order JacobiSN solution for high degree ODE
   trying 1st order ODE linearizable_by_differentiation
   trying differential order: 1; missing variables
   trying simple symmetries for implicit equations
   Successful isolation of dy/dx: 3 solutions were found. Trying to solve each resulting ODE
      *** Sublevel 3 ***
      Methods for first order ODEs:
      --- Trying classification methods ---
      trying homogeneous types:
      trying homogeneous G
      <- homogeneous successful
   -------------------
   * Tackling next ODE.
      *** Sublevel 3 ***
      Methods for first order ODEs:
      --- Trying classification methods ---
      trying homogeneous types:
      trying homogeneous G
      <- homogeneous successful
   -------------------
   * Tackling next ODE.
      *** Sublevel 3 ***
      Methods for first order ODEs:
      --- Trying classification methods ---
      trying homogeneous types:
      trying homogeneous G
      <- homogeneous successful`
```

✓ Solution by Maple

Time used: 0.156 (sec). Leaf size: 577

```
dsolve(y(x)* (y(x)-2*x*diff(y(x),x))^3= (diff(y(x),x))^2 ,y(x), singsol=all)
```

$$y(x) = -\frac{\sqrt{3}}{9x}$$
$$y(x) = \frac{\sqrt{3}}{9x}$$
$$y(x) = 0$$
$$y(x) = \frac{\mathrm{RootOf}\left(-\ln(x) + c_1 + 24\left(\int^{\_Z} \frac{\left(-216\_a^4+24\_a^3\sqrt{81\_a^2-3}-\right.}{36\left(-216\_a^4+24\_a^3\sqrt{81\_a^2-3}+36\_a^2-1\right)^{\frac{1}{3}}\_a^2+\left(-216\_a^4+24\_a^3\sqrt{81\_a^2-3}+\right.}\right.\right.}{x}$$
$$y(x) = \frac{\mathrm{RootOf}\left(-\ln(x) + c_1 - 48\left(\int^{\_Z} \frac{}{i\left(-216\_a^4+24\_a^3\sqrt{81\_a^2-3}+36\_a^2-1\right)^{\frac{2}{3}}\sqrt{3}+24i\sqrt{3}\_a^2-72\left(-216\_a^4+24\_a^3\sqrt{\right.}\right.\right.}{}$$
$$y(x) = \frac{\mathrm{RootOf}\left(-\ln(x) + c_1 + 48\left(\int^{\_Z} \frac{}{i\left(-216\_a^4+24\_a^3\sqrt{81\_a^2-3}+36\_a^2-1\right)^{\frac{2}{3}}\sqrt{3}+24i\sqrt{3}\_a^2+72\left(-216\_a^4+24\_a^3\sqrt{\right.}\right.\right.}{}$$

✗ Solution by Mathematica

Time used: 0.0 (sec). Leaf size: 0

```
DSolve[y[x]*(y[x]-2*x*y'[x])^3== (y'[x])^2,y[x],x,IncludeSingularSolutions -> True]
```

Timed out

1.88 problem 91

Internal problem ID [3233]
Internal file name [OUTPUT/2725_Sunday_June_05_2022_08_39_28_AM_12257417/index.tex]

Book: Differential equations for engineers by Wei-Chau XIE, Cambridge Press 2010
Section: Chapter 2. First-Order and Simple Higher-Order Differential Equations. Page 78
Problem number: 91.
ODE order: 1.
ODE degree: 2.

The type(s) of ODE detected by this program : **"dAlembert"**

Maple gives the following as the ode type

```
[[_homogeneous, `class G`], _dAlembert]
```

$$\boxed{xy' + y - 4\sqrt{y'} = 0}$$

1.88.1 Solving as dAlembert ode

Let $p = y'$ the ode becomes

$$xp + y - 4\sqrt{p} = 0$$

Solving for y from the above results in

$$y = -xp + 4\sqrt{p} \tag{1A}$$

This has the form

$$y = xf(p) + g(p) \tag{*}$$

Where f, g are functions of $p = y'(x)$. The above ode is dAlembert ode which is now solved. Taking derivative of (*) w.r.t. x gives

$$\begin{aligned} p &= f + (xf' + g')\frac{dp}{dx} \\ p - f &= (xf' + g')\frac{dp}{dx} \end{aligned} \tag{2}$$

Comparing the form $y = xf + g$ to (1A) shows that

$$
\begin{aligned}
f &= -p \\
g &= 4\sqrt{p}
\end{aligned}
$$

Hence (2) becomes

$$
2p = \left(-x + \frac{2}{\sqrt{p}}\right) p'(x) \tag{2A}
$$

The singular solution is found by setting $\frac{dp}{dx} = 0$ in the above which gives

$$
2p = 0
$$

Solving for p from the above gives

$$
p = 0
$$

Substituting these in (1A) gives

$$
y = 0
$$

The general solution is found when $\frac{\mathrm{d}p}{\mathrm{d}x} \neq 0$. From eq. (2A). This results in

$$
p'(x) = \frac{2p(x)}{-x + \frac{2}{\sqrt{p(x)}}} \tag{3}
$$

This ODE is now solved for $p(x)$.

Inverting the above ode gives

$$
\frac{d}{dp}x(p) = \frac{-x(p) + \frac{2}{\sqrt{p}}}{2p} \tag{4}
$$

This ODE is now solved for $x(p)$.

Entering Linear first order ODE solver. In canonical form a linear first order is

$$
\frac{d}{dp}x(p) + p(p)x(p) = q(p)
$$

Where here

$$
\begin{aligned}
p(p) &= \frac{1}{2p} \\
q(p) &= \frac{1}{p^{\frac{3}{2}}}
\end{aligned}
$$

Hence the ode is

$$\frac{d}{dp}x(p) + \frac{x(p)}{2p} = \frac{1}{p^{\frac{3}{2}}}$$

The integrating factor μ is

$$\begin{aligned}\mu &= e^{\int \frac{1}{2p}dp} \\ &= \sqrt{p}\end{aligned}$$

The ode becomes

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}p}(\mu x) &= (\mu)\left(\frac{1}{p^{\frac{3}{2}}}\right) \\ \frac{\mathrm{d}}{\mathrm{d}p}(x\sqrt{p}) &= (\sqrt{p})\left(\frac{1}{p^{\frac{3}{2}}}\right) \\ \mathrm{d}(x\sqrt{p}) &= \frac{1}{p}\,\mathrm{d}p\end{aligned}$$

Integrating gives

$$\begin{aligned}x\sqrt{p} &= \int \frac{1}{p}\,\mathrm{d}p \\ x\sqrt{p} &= \ln(p) + c_1\end{aligned}$$

Dividing both sides by the integrating factor $\mu = \sqrt{p}$ results in

$$x(p) = \frac{\ln(p)}{\sqrt{p}} + \frac{c_1}{\sqrt{p}}$$

which simplifies to

$$x(p) = \frac{\ln(p) + c_1}{\sqrt{p}}$$

Now we need to eliminate p between the above and (1A). One way to do this is by solving (1) for p. This results in

$$\begin{aligned}p &= \frac{\frac{8+4\sqrt{4-yx}}{x} - y}{x} \\ p &= \frac{-\frac{4(-2+\sqrt{4-yx})}{x} - y}{x}\end{aligned}$$

Substituting the above in the solution for x found above gives

$$x = \frac{\ln\left(\frac{-yx+4\sqrt{4-yx}+8}{x^2}\right)+c_1}{\sqrt{\frac{-yx+4\sqrt{4-yx}+8}{x^2}}}$$

$$x = \frac{\ln\left(\frac{-yx-4\sqrt{4-yx}+8}{x^2}\right)+c_1}{\sqrt{\frac{-yx-4\sqrt{4-yx}+8}{x^2}}}$$

Summary

The solution(s) found are the following

$$y = 0 \tag{1}$$

$$x = \frac{\ln\left(\frac{-yx+4\sqrt{4-yx}+8}{x^2}\right)+c_1}{\sqrt{\frac{-yx+4\sqrt{4-yx}+8}{x^2}}} \tag{2}$$

$$x = \frac{\ln\left(\frac{-yx-4\sqrt{4-yx}+8}{x^2}\right)+c_1}{\sqrt{\frac{-yx-4\sqrt{4-yx}+8}{x^2}}} \tag{3}$$

Verification of solutions

$$y = 0$$

Verified OK.

$$x = \frac{\ln\left(\frac{-yx+4\sqrt{4-yx}+8}{x^2}\right)+c_1}{\sqrt{\frac{-yx+4\sqrt{4-yx}+8}{x^2}}}$$

Verified OK.

$$x = \frac{\ln\left(\frac{-yx-4\sqrt{4-yx}+8}{x^2}\right)+c_1}{\sqrt{\frac{-yx-4\sqrt{4-yx}+8}{x^2}}}$$

Verified OK.

Maple trace

```
`Methods for first order ODEs:
-> Solving 1st order ODE of high degree, 1st attempt
trying 1st order WeierstrassP solution for high degree ODE
trying 1st order WeierstrassPPrime solution for high degree ODE
trying 1st order JacobiSN solution for high degree ODE
trying 1st order ODE linearizable_by_differentiation
trying differential order: 1; missing variables
trying dAlembert
<- dAlembert successful`
```

✓ Solution by Maple

Time used: 0.063 (sec). Leaf size: 67

```
dsolve(y(x)+x*diff(y(x),x) = 4*sqrt(diff(y(x),x)),y(x), singsol=all)
```

$$y(x) = \frac{8\sqrt{\frac{\text{LambertW}\left(-\frac{c_1 x}{2}\right)^2}{x^2}}\, x - 4\,\text{LambertW}\left(-\frac{c_1 x}{2}\right)^2}{x}$$

$$y(x) = \frac{-4\,\text{LambertW}\left(\frac{c_1 x}{2}\right)^2 + 8\sqrt{\frac{\text{LambertW}\left(\frac{c_1 x}{2}\right)^2}{x^2}}\, x}{x}$$

✓ Solution by Mathematica

Time used: 1.157 (sec). Leaf size: 94

```
DSolve[y[x]+x*y'[x]==4*Sqrt[y'[x]],y[x],x,IncludeSingularSolutions -> True]
```

$$\text{Solve}\left[\frac{2e^{-\frac{1}{2}\sqrt{4-xy(x)}}\left(-2\sqrt{4-xy(x)}-4\right)}{y(x)} = c_1, y(x)\right]$$

$$\text{Solve}\left[\frac{2e^{\frac{1}{2}\sqrt{4-xy(x)}}\left(2\sqrt{4-xy(x)}-4\right)}{y(x)} = c_1, y(x)\right]$$

$$y(x) \to 0$$

1.89 problem 92

Internal problem ID [3234]
Internal file name [OUTPUT/2726_Sunday_June_05_2022_08_39_43_AM_74450280/index.tex]

Book: Differential equations for engineers by Wei-Chau XIE, Cambridge Press 2010
Section: Chapter 2. First-Order and Simple Higher-Order Differential Equations. Page 78
Problem number: 92.
ODE order: 1.
ODE degree: 0.

The type(s) of ODE detected by this program : **"dAlembert"**

Maple gives the following as the ode type

```
[[_1st_order, _with_linear_symmetries], _dAlembert]
```

$$\boxed{2xy' - y - \ln(y') = 0}$$

1.89.1 Solving as dAlembert ode

Let $p = y'$ the ode becomes

$$2xp - y - \ln(p) = 0$$

Solving for y from the above results in

$$y = 2xp - \ln(p) \tag{1A}$$

This has the form

$$y = xf(p) + g(p) \tag{*}$$

Where f, g are functions of $p = y'(x)$. The above ode is dAlembert ode which is now solved. Taking derivative of (*) w.r.t. x gives

$$p = f + (xf' + g')\frac{dp}{dx}$$
$$p - f = (xf' + g')\frac{dp}{dx} \tag{2}$$

Comparing the form $y = xf + g$ to (1A) shows that

$$
\begin{aligned}
f &= 2p \\
g &= -\ln(p)
\end{aligned}
$$

Hence (2) becomes

$$
-p = \left(2x - \frac{1}{p}\right) p'(x) \tag{2A}
$$

The singular solution is found by setting $\frac{dp}{dx} = 0$ in the above which gives

$$
-p = 0
$$

Solving for p from the above gives

$$
p = 0
$$

Substituting these in (1A) gives

$$
y = \infty
$$

The general solution is found when $\frac{\mathrm{d}p}{\mathrm{d}x} \neq 0$. From eq. (2A). This results in

$$
p'(x) = -\frac{p(x)}{2x - \frac{1}{p(x)}} \tag{3}
$$

This ODE is now solved for $p(x)$.

Inverting the above ode gives

$$
\frac{d}{dp}x(p) = -\frac{2x(p) - \frac{1}{p}}{p} \tag{4}
$$

This ODE is now solved for $x(p)$.

Entering Linear first order ODE solver. In canonical form a linear first order is

$$
\frac{d}{dp}x(p) + p(p)x(p) = q(p)
$$

Where here

$$
\begin{aligned}
p(p) &= \frac{2}{p} \\
q(p) &= \frac{1}{p^2}
\end{aligned}
$$

Hence the ode is

$$\frac{d}{dp}x(p) + \frac{2x(p)}{p} = \frac{1}{p^2}$$

The integrating factor μ is

$$\begin{aligned}\mu &= e^{\int \frac{2}{p}dp} \\ &= p^2\end{aligned}$$

The ode becomes

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}p}(\mu x) &= (\mu)\left(\frac{1}{p^2}\right) \\ \frac{\mathrm{d}}{\mathrm{d}p}(p^2 x) &= (p^2)\left(\frac{1}{p^2}\right) \\ \mathrm{d}(p^2 x) &= \mathrm{d}p\end{aligned}$$

Integrating gives

$$\begin{aligned}p^2 x &= \int \mathrm{d}p \\ p^2 x &= p + c_1\end{aligned}$$

Dividing both sides by the integrating factor $\mu = p^2$ results in

$$x(p) = \frac{1}{p} + \frac{c_1}{p^2}$$

which simplifies to

$$x(p) = \frac{p + c_1}{p^2}$$

Now we need to eliminate p between the above and (1A). One way to do this is by solving (1) for p. This results in

$$p = e^{-\operatorname{LambertW}(-2x\,e^{-y})-y}$$

Substituting the above in the solution for x found above gives

$$x = -\frac{2(-2c_1 x + \operatorname{LambertW}(-2x\,e^{-y}))\,x}{\operatorname{LambertW}(-2x\,e^{-y})^2}$$

Summary
The solution(s) found are the following

$$y = \infty \tag{1}$$

$$x = -\frac{2(-2c_1x + \operatorname{LambertW}(-2x\,e^{-y}))\,x}{\operatorname{LambertW}(-2x\,e^{-y})^2} \tag{2}$$

Verification of solutions

$$y = \infty$$

Warning, solution could not be verified

$$x = -\frac{2(-2c_1x + \operatorname{LambertW}(-2x\,e^{-y}))\,x}{\operatorname{LambertW}(-2x\,e^{-y})^2}$$

Verified OK.

Maple trace

```
`Methods for first order ODEs:
-> Solving 1st order ODE of high degree, 1st attempt
trying 1st order WeierstrassP solution for high degree ODE
trying 1st order WeierstrassPPrime solution for high degree ODE
trying 1st order JacobiSN solution for high degree ODE
trying 1st order ODE linearizable_by_differentiation
trying differential order: 1; missing variables
trying dAlembert
<- dAlembert successful`
```

✓ Solution by Maple
Time used: 0.015 (sec). Leaf size: 69

```
dsolve(2*x*diff(y(x),x) -y(x) = ln(diff(y(x),x)),y(x), singsol=all)
```

$$y(x) = 1 + \sqrt{4c_1x + 1} + \ln(2) - \ln\left(\frac{1 + \sqrt{4c_1x + 1}}{x}\right)$$

$$y(x) = 1 - \sqrt{4c_1x + 1} + \ln(2) - \ln\left(\frac{1 - \sqrt{4c_1x + 1}}{x}\right)$$

✓ Solution by Mathematica

Time used: 0.106 (sec). Leaf size: 34

```
DSolve[2*x*y'[x] -y[x] == Log[y'[x]],y[x],x,IncludeSingularSolutions -> True]
```

$$\text{Solve}\left[W\left(-2xe^{-y(x)}\right) - \log\left(W\left(-2xe^{-y(x)}\right) + 2\right) + y(x) = c_1, y(x)\right]$$

1.90 problem 111

Internal problem ID [3235]
Internal file name [`OUTPUT/2727_Sunday_June_05_2022_08_39_46_AM_78403092/index.tex`]

Book: Differential equations for engineers by Wei-Chau XIE, Cambridge Press 2010
Section: Chapter 2. First-Order and Simple Higher-Order Differential Equations. Page 78
Problem number: 111.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"bernoulli", "exactWithIntegrationFactor", "first_order_ode_lie_symmetry_lookup"**

Maple gives the following as the ode type

```
[[_homogeneous, `class G`], _rational, _Bernoulli]
```

$$\boxed{xy^2(xy' + y) = 1}$$

1.90.1 Solving as first order ode lie symmetry lookup ode

Writing the ode as

$$y' = -\frac{x\,y^3 - 1}{y^2x^2}$$
$$y' = \omega(x, y)$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x + \omega(\eta_y - \xi_x) - \omega^2\xi_y - \omega_x\xi - \omega_y\eta = 0 \tag{A}$$

The type of this ode is known. It is of type `Bernoulli`. Therefore we do not need to solve the PDE (A), and can just use the lookup table shown below to find ξ, η

Table 34: Lie symmetry infinitesimal lookup table for known first order ODE's

ODE class	Form	ξ	η
linear ode	$y' = f(x)y(x) + g(x)$	0	$e^{\int f dx}$
separable ode	$y' = f(x)\, g(y)$	$\frac{1}{f}$	0
quadrature ode	$y' = f(x)$	0	1
quadrature ode	$y' = g(y)$	1	0
homogeneous ODEs of Class A	$y' = f\left(\frac{y}{x}\right)$	x	y
homogeneous ODEs of Class C	$y' = (a + bx + cy)^{\frac{n}{m}}$	1	$-\frac{b}{c}$
homogeneous class D	$y' = \frac{y}{x} + g(x)\, F\left(\frac{y}{x}\right)$	x^2	xy
First order special form ID 1	$y' = g(x)\, e^{h(x)+by} + f(x)$	$\frac{e^{-\int bf(x)dx - h(x)}}{g(x)}$	$\frac{f(x)e^{-\int bf(x)dx - h(x)}}{g(x)}$
polynomial type ode	$y' = \frac{a_1x+b_1y+c_1}{a_2x+b_2y+c_2}$	$\frac{a_1b_2x - a_2b_1x - b_1c_2 + b_2c_1}{a_1b_2 - a_2b_1}$	$\frac{a_1b_2y - a_2b_1y - a_1c_2 - a_2c_1}{a_1b_2 - a_2b_1}$
Bernoulli ode	$y' = f(x)\, y + g(x)\, y^n$	0	$e^{-\int (n-1)f(x)dx} y^n$
Reduced Riccati	$y' = f_1(x)\, y + f_2(x)\, y^2$	0	$e^{-\int f_1 dx}$

The above table shows that

$$\begin{aligned} \xi(x, y) &= 0 \\ \eta(x, y) &= \frac{1}{y^2 x^3} \end{aligned} \tag{A1}$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x, y) \to (R, S)$ where (R, S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y}\right) S(x, y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the

canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$R = x$$

S is found from

$$\begin{aligned} S &= \int \frac{1}{\eta} dy \\ &= \int \frac{1}{\frac{1}{y^2 x^3}} dy \end{aligned}$$

Which results in

$$S = \frac{x^3 y^3}{3}$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x,y) S_y}{R_x + \omega(x,y) R_y} \tag{2}$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x,y)$ is the right hand side of the original ode given by

$$\omega(x,y) = -\frac{x\,y^3 - 1}{y^2 x^2}$$

Evaluating all the partial derivatives gives

$$\begin{aligned} R_x &= 1 \\ R_y &= 0 \\ S_x &= y^3 x^2 \\ S_y &= x^3 y^2 \end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = x \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = R$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordinates R, S. Integrating the above gives

$$S(R) = \frac{R^2}{2} + c_1 \tag{4}$$

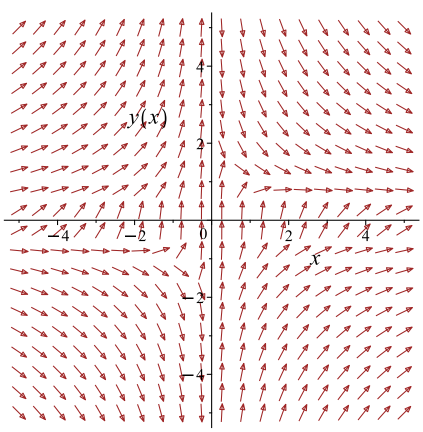
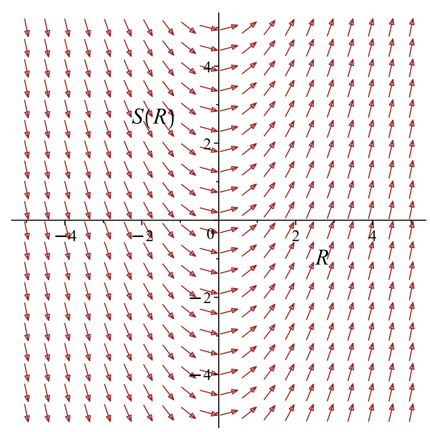
To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$\frac{y^3 x^3}{3} = \frac{x^2}{2} + c_1$$

Which simplifies to

$$\frac{y^3 x^3}{3} = \frac{x^2}{2} + c_1$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in x, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dx} = -\frac{x\,y^3-1}{y^2x^2}$ 	$R = x$ $S = \frac{x^3y^3}{3}$	$\frac{dS}{dR} = R$ 

Summary
The solution(s) found are the following

$$\frac{y^3 x^3}{3} = \frac{x^2}{2} + c_1 \tag{1}$$

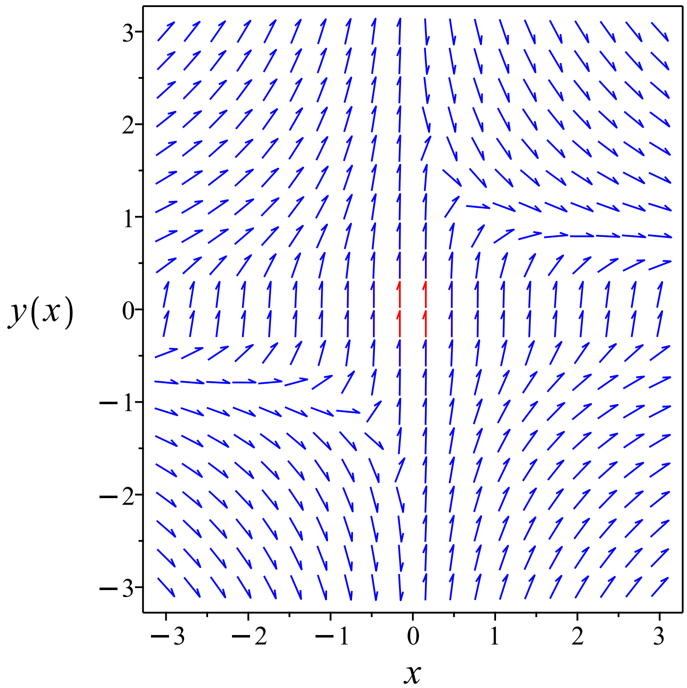


Figure 112: Slope field plot

Verification of solutions

$$\frac{y^3x^3}{3} = \frac{x^2}{2} + c_1$$

Verified OK.

1.90.2 Solving as bernoulli ode

In canonical form, the ODE is

$$\begin{aligned} y' &= F(x,y) \\ &= -\frac{x\,y^3 - 1}{y^2x^2} \end{aligned}$$

This is a Bernoulli ODE.

$$y' = -\frac{1}{x}y + \frac{1}{x^2}\frac{1}{y^2} \tag{1}$$

The standard Bernoulli ODE has the form

$$y' = f_0(x)y + f_1(x)y^n \tag{2}$$

The first step is to divide the above equation by y^n which gives

$$\frac{y'}{y^n} = f_0(x)y^{1-n} + f_1(x) \tag{3}$$

The next step is use the substitution $w = y^{1-n}$ in equation (3) which generates a new ODE in $w(x)$ which will be linear and can be easily solved using an integrating factor. Backsubstitution then gives the solution $y(x)$ which is what we want.

This method is now applied to the ODE at hand. Comparing the ODE (1) With (2) Shows that

$$\begin{aligned} f_0(x) &= -\frac{1}{x} \\ f_1(x) &= \frac{1}{x^2} \\ n &= -2 \end{aligned}$$

Dividing both sides of ODE (1) by $y^n = \frac{1}{y^2}$ gives

$$y'y^2 = -\frac{y^3}{x} + \frac{1}{x^2} \tag{4}$$

Let

$$\begin{aligned} w &= y^{1-n} \\ &= y^3 \end{aligned} \tag{5}$$

Taking derivative of equation (5) w.r.t x gives

$$w' = 3y^2y' \tag{6}$$

Substituting equations (5) and (6) into equation (4) gives

$$\begin{aligned} \frac{w'(x)}{3} &= -\frac{w(x)}{x} + \frac{1}{x^2} \\ w' &= -\frac{3w}{x} + \frac{3}{x^2} \end{aligned} \tag{7}$$

The above now is a linear ODE in $w(x)$ which is now solved.

Entering Linear first order ODE solver. In canonical form a linear first order is

$$w'(x) + p(x)w(x) = q(x)$$

Where here

$$\begin{aligned} p(x) &= \frac{3}{x} \\ q(x) &= \frac{3}{x^2} \end{aligned}$$

Hence the ode is

$$w'(x) + \frac{3w(x)}{x} = \frac{3}{x^2}$$

The integrating factor μ is

$$\begin{aligned}\mu &= e^{\int \frac{3}{x} dx}\\ &= x^3\end{aligned}$$

The ode becomes

$$\begin{aligned}\frac{d}{dx}(\mu w) &= (\mu)\left(\frac{3}{x^2}\right)\\ \frac{d}{dx}(x^3 w) &= (x^3)\left(\frac{3}{x^2}\right)\\ d(x^3 w) &= (3x)\, dx\end{aligned}$$

Integrating gives

$$\begin{aligned}x^3 w &= \int 3x\, dx\\ x^3 w &= \frac{3x^2}{2} + c_1\end{aligned}$$

Dividing both sides by the integrating factor $\mu = x^3$ results in

$$w(x) = \frac{3}{2x} + \frac{c_1}{x^3}$$

Replacing w in the above by y^3 using equation (5) gives the final solution.

$$y^3 = \frac{3}{2x} + \frac{c_1}{x^3}$$

Solving for y gives

$$\begin{aligned}y(x) &= \frac{(12x^2 + 8c_1)^{\frac{1}{3}}}{2x}\\ y(x) &= \frac{(12x^2 + 8c_1)^{\frac{1}{3}}\left(-1 + i\sqrt{3}\right)}{4x}\\ y(x) &= -\frac{(12x^2 + 8c_1)^{\frac{1}{3}}\left(1 + i\sqrt{3}\right)}{4x}\end{aligned}$$

Summary

The solution(s) found are the following

$$y = \frac{(12x^2 + 8c_1)^{\frac{1}{3}}}{2x} \tag{1}$$

$$y = \frac{(12x^2 + 8c_1)^{\frac{1}{3}} \left(-1 + i\sqrt{3}\right)}{4x} \tag{2}$$

$$y = -\frac{(12x^2 + 8c_1)^{\frac{1}{3}} \left(1 + i\sqrt{3}\right)}{4x} \tag{3}$$

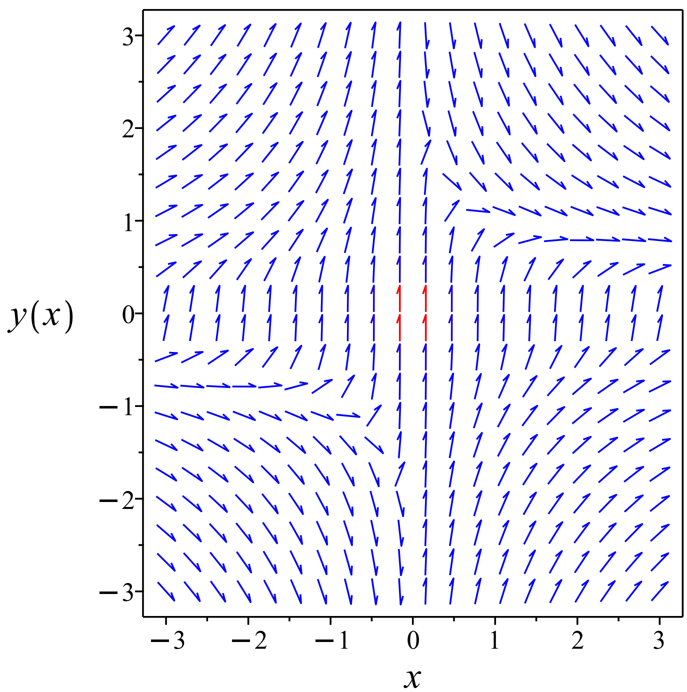


Figure 113: Slope field plot

Verification of solutions

$$y = \frac{(12x^2 + 8c_1)^{\frac{1}{3}}}{2x}$$

Verified OK.

$$y = \frac{(12x^2 + 8c_1)^{\frac{1}{3}} \left(-1 + i\sqrt{3}\right)}{4x}$$

Verified OK.

$$y = -\frac{(12x^2 + 8c_1)^{\frac{1}{3}} \left(1 + i\sqrt{3}\right)}{4x}$$

Verified OK.

1.90.3 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x,y) + N(x,y)\frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x,y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y) = 0$$

Hence

$$\frac{\partial \phi}{\partial x} + \frac{\partial \phi}{\partial y}\frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$\frac{\partial \phi}{\partial x} = M$$
$$\frac{\partial \phi}{\partial y} = N$$

But since $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x,y)$ but at least we know now that we can do that since the condition

$\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x, y)\,\mathrm{d}x + N(x, y)\,\mathrm{d}y = 0 \tag{1A}$$

Therefore

$$\begin{aligned}\left(y^2 x^2\right) \mathrm{d}y &= \left(-x\,y^3 + 1\right) \mathrm{d}x \\ \left(x\,y^3 - 1\right) \mathrm{d}x + \left(y^2 x^2\right) \mathrm{d}y &= 0 \end{aligned} \tag{2A}$$

Comparing (1A) and (2A) shows that

$$\begin{aligned} M(x, y) &= x\,y^3 - 1 \\ N(x, y) &= y^2 x^2 \end{aligned}$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Using result found above gives

$$\begin{aligned} \frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}\left(x\,y^3 - 1\right) \\ &= 3x\,y^2 \end{aligned}$$

And

$$\begin{aligned} \frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}\left(y^2 x^2\right) \\ &= 2x\,y^2 \end{aligned}$$

Since $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$, then the ODE is not exact. Since the ODE is not exact, we will try to find an integrating factor to make it exact. Let

$$\begin{aligned} A &= \frac{1}{N}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right) \\ &= \frac{1}{y^2 x^2}\left(\left(3x\,y^2\right) - \left(2x\,y^2\right)\right) \\ &= \frac{1}{x} \end{aligned}$$

Since A does not depend on y, then it can be used to find an integrating factor. The integrating factor μ is

$$\begin{aligned}\mu &= e^{\int A\,\mathrm{d}x} \\ &= e^{\int \frac{1}{x}\,\mathrm{d}x}\end{aligned}$$

The result of integrating gives

$$\begin{aligned}\mu &= e^{\ln(x)} \\ &= x\end{aligned}$$

M and N are multiplied by this integrating factor, giving new M and new N which are called $\overline{M}$ and $\overline{N}$ for now so not to confuse them with the original M and N.

$$\begin{aligned}\overline{M} &= \mu M \\ &= x\left(x\,y^3-1\right) \\ &= x\left(x\,y^3-1\right)\end{aligned}$$

And

$$\begin{aligned}\overline{N} &= \mu N \\ &= x\left(y^2x^2\right) \\ &= x^3y^2\end{aligned}$$

Now a modified ODE is ontained from the original ODE, which is exact and can be solved. The modified ODE is

$$\begin{aligned}\overline{M}+\overline{N}\frac{\mathrm{d}y}{\mathrm{d}x} &= 0 \\ \left(x\left(x\,y^3-1\right)\right)+\left(x^3y^2\right)\frac{\mathrm{d}y}{\mathrm{d}x} &= 0\end{aligned}$$

The following equations are now set up to solve for the function $\phi(x,y)$

$$\frac{\partial\phi}{\partial x} = \overline{M} \tag{1}$$

$$\frac{\partial\phi}{\partial y} = \overline{N} \tag{2}$$

Integrating (1) w.r.t. x gives

$$\begin{aligned}\int\frac{\partial\phi}{\partial x}\,\mathrm{d}x &= \int\overline{M}\,\mathrm{d}x \\ \int\frac{\partial\phi}{\partial x}\,\mathrm{d}x &= \int x\left(x\,y^3-1\right)\mathrm{d}x\end{aligned}$$

$$\phi = \frac{1}{3}x^3y^3-\frac{1}{2}x^2+f(y) \tag{3}$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$\frac{\partial \phi}{\partial y} = x^3 y^2 + f'(y) \tag{4}$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = x^3 y^2$. Therefore equation (4) becomes

$$x^3 y^2 = x^3 y^2 + f'(y) \tag{5}$$

Solving equation (5) for $f'(y)$ gives

$$f'(y) = 0$$

Therefore

$$f(y) = c_1$$

Where c_1 is constant of integration. Substituting this result for $f(y)$ into equation (3) gives ϕ

$$\phi = \frac{1}{3}x^3 y^3 - \frac{1}{2}x^2 + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = \frac{1}{3}x^3 y^3 - \frac{1}{2}x^2$$

Summary

The solution(s) found are the following

$$\frac{y^3 x^3}{3} - \frac{x^2}{2} = c_1 \tag{1}$$

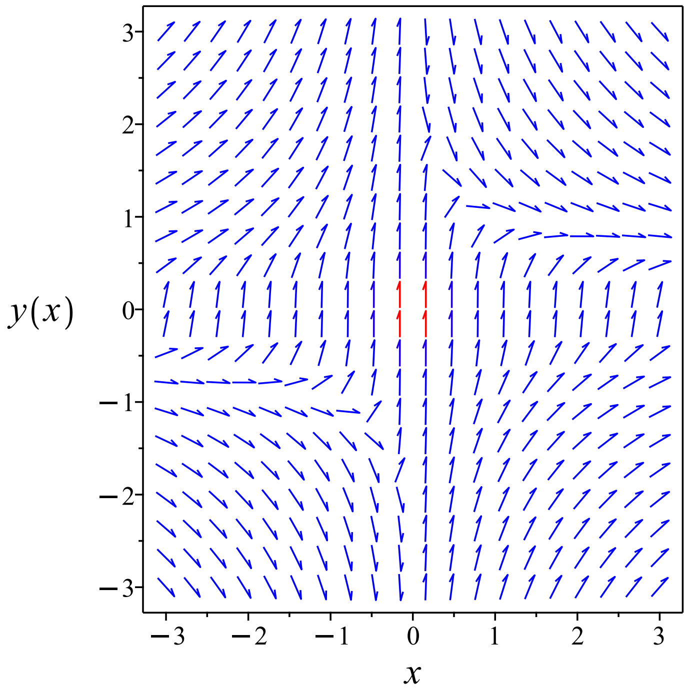


Figure 114: Slope field plot

Verification of solutions

$$\frac{y^3x^3}{3} - \frac{x^2}{2} = c_1$$

Verified OK.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
<- Bernoulli successful`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 74

```
dsolve(x*y(x)^2*(x*diff(y(x),x)+y(x) )=1,y(x), singsol=all)
```

$$y(x) = \frac{(12x^2 + 8c_1)^{\frac{1}{3}}}{2x}$$
$$y(x) = -\frac{(12x^2 + 8c_1)^{\frac{1}{3}} \left(1 + i\sqrt{3}\right)}{4x}$$
$$y(x) = \frac{(12x^2 + 8c_1)^{\frac{1}{3}} \left(i\sqrt{3} - 1\right)}{4x}$$

✓ Solution by Mathematica

Time used: 0.233 (sec). Leaf size: 80

```
DSolve[x*y[x]^2*(x*y'[x]+y[x])==1,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to -\frac{\sqrt[3]{-\frac{1}{2}}\sqrt[3]{3x^2 + 2c_1}}{x}$$
$$y(x) \to \frac{\sqrt[3]{\frac{3x^2}{2} + c_1}}{x}$$
$$y(x) \to \frac{(-1)^{2/3}\sqrt[3]{\frac{3x^2}{2} + c_1}}{x}$$

1.91 problem 112

Internal problem ID [3236]
Internal file name [`OUTPUT/2728_Sunday_June_05_2022_08_39_47_AM_99213743/index.tex`]

Book: Differential equations for engineers by Wei-Chau XIE, Cambridge Press 2010
Section: Chapter 2. First-Order and Simple Higher-Order Differential Equations. Page 78
Problem number: 112.
ODE order: 1.
ODE degree: 2.

The type(s) of ODE detected by this program : **"first_order_ode_lie_symmetry_calculated"**

Maple gives the following as the ode type

```
[[_homogeneous, `class G`]]
```

$$\boxed{5y + y'^2 - x(x + y') = 0}$$

Solving the given ode for y' results in 2 differential equations to solve. Each one of these will generate a solution. The equations generated are

$$y' = \frac{x}{2} + \frac{\sqrt{5x^2 - 20y}}{2} \tag{1}$$

$$y' = \frac{x}{2} - \frac{\sqrt{5x^2 - 20y}}{2} \tag{2}$$

Now each one of the above ODE is solved.

Solving equation (1)

Writing the ode as

$$y' = \frac{x}{2} + \frac{\sqrt{5x^2 - 20y}}{2}$$
$$y' = \omega(x, y)$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x + \omega(\eta_y - \xi_x) - \omega^2 \xi_y - \omega_x \xi - \omega_y \eta = 0 \tag{A}$$

The type of this ode is not in the lookup table. To determine ξ, η then (A) is solved using ansatz. Making bivariate polynomials of degree 1 to use as anstaz gives

$$\xi = x a_2 + y a_3 + a_1 \tag{1E}$$

$$\eta = x b_2 + y b_3 + b_1 \tag{2E}$$

Where the unknown coefficients are

$$\{a_1, a_2, a_3, b_1, b_2, b_3\}$$

Substituting equations (1E,2E) and ω into (A) gives

$$\begin{aligned} b_2 &+ \left(\frac{x}{2} + \frac{\sqrt{5x^2 - 20y}}{2}\right)(b_3 - a_2) - \left(\frac{x}{2} + \frac{\sqrt{5x^2 - 20y}}{2}\right)^2 a_3 \\ &- \left(\frac{1}{2} + \frac{5x}{2\sqrt{5x^2 - 20y}}\right)(x a_2 + y a_3 + a_1) + \frac{5x b_2 + 5y b_3 + 5b_1}{\sqrt{5x^2 - 20y}} = 0 \end{aligned} \tag{5E}$$

Putting the above in normal form gives

$$-\frac{(5x^2 - 20y)^{\frac{3}{2}} a_3 + \sqrt{5x^2 - 20y}\, x^2 a_3 + 10x^3 a_3 + 4\sqrt{5x^2 - 20y}\, x a_2 - 2\sqrt{5x^2 - 20y}\, x b_3 + 2\sqrt{5x^2 - 20y}\, y a}{4\sqrt{5}}$$

$$= 0$$

Setting the numerator to zero gives

$$\begin{aligned} &-(5x^2 - 20y)^{\frac{3}{2}} a_3 - \sqrt{5x^2 - 20y}\, x^2 a_3 - 10x^3 a_3 \\ &- 4\sqrt{5x^2 - 20y}\, x a_2 + 2\sqrt{5x^2 - 20y}\, x b_3 - 2\sqrt{5x^2 - 20y}\, y a_3 \\ &- 20x^2 a_2 + 10x^2 b_3 + 30xy a_3 - 2\sqrt{5x^2 - 20y}\, a_1 \\ &+ 4b_2\sqrt{5x^2 - 20y} - 10x a_1 + 20x b_2 + 40y a_2 - 20y b_3 + 20b_1 = 0 \end{aligned} \tag{6E}$$

Simplifying the above gives

$$\begin{aligned} &-(5x^2 - 20y)^{\frac{3}{2}} a_3 - 2(5x^2 - 20y)\, x a_3 - \sqrt{5x^2 - 20y}\, x^2 a_3 \\ &- 2(5x^2 - 20y)\, a_2 + 2(5x^2 - 20y)\, b_3 - 4\sqrt{5x^2 - 20y}\, x a_2 \\ &+ 2\sqrt{5x^2 - 20y}\, x b_3 - 2\sqrt{5x^2 - 20y}\, y a_3 - 10x^2 a_2 - 10xy a_3 \\ &- 2\sqrt{5x^2 - 20y}\, a_1 + 4b_2\sqrt{5x^2 - 20y} - 10x a_1 + 20x b_2 + 20y b_3 + 20b_1 = 0 \end{aligned} \tag{6E}$$

Since the PDE has radicals, simplifying gives

$$
\begin{aligned}
&-10x^3a_3-6\sqrt{5x^2-20y}\,x^2a_3-20x^2a_2+10x^2b_3-4\sqrt{5x^2-20y}\,xa_2\\
&+2\sqrt{5x^2-20y}\,xb_3+30xya_3+18\sqrt{5x^2-20y}\,ya_3-10xa_1+20xb_2\\
&-2\sqrt{5x^2-20y}\,a_1+4b_2\sqrt{5x^2-20y}+40ya_2-20yb_3+20b_1=0
\end{aligned}
$$

Looking at the above PDE shows the following are all the terms with $\{x,y\}$ in them.

$$
\left\{x,y,\sqrt{5x^2-20y}\right\}
$$

The following substitution is now made to be able to collect on all terms with $\{x,y\}$ in them

$$
\left\{x=v_1,y=v_2,\sqrt{5x^2-20y}=v_3\right\}
$$

The above PDE (6E) now becomes

$$
\begin{aligned}
&-10v_1^3a_3-6v_3v_1^2a_3-20v_1^2a_2-4v_3v_1a_2+30v_1v_2a_3+18v_3v_2a_3+10v_1^2b_3\\
&+2v_3v_1b_3-10v_1a_1-2v_3a_1+40v_2a_2+20v_1b_2+4b_2v_3-20v_2b_3+20b_1=0
\end{aligned}
\tag{7E}
$$

Collecting the above on the terms v_i introduced, and these are

$$
\{v_1,v_2,v_3\}
$$

Equation (7E) now becomes

$$
\begin{aligned}
&-10v_1^3a_3-6v_3v_1^2a_3+(-20a_2+10b_3)\,v_1^2+30v_1v_2a_3+(-4a_2+2b_3)\,v_1v_3\\
&+(-10a_1+20b_2)\,v_1+18v_3v_2a_3+(40a_2-20b_3)\,v_2+(-2a_1+4b_2)\,v_3+20b_1=0
\end{aligned}
\tag{8E}
$$

Setting each coefficients in (8E) to zero gives the following equations to solve

$$
\begin{aligned}
-10a_3 &= 0 \\
-6a_3 &= 0 \\
18a_3 &= 0 \\
30a_3 &= 0 \\
20b_1 &= 0 \\
-10a_1 + 20b_2 &= 0 \\
-2a_1 + 4b_2 &= 0 \\
-20a_2 + 10b_3 &= 0 \\
-4a_2 + 2b_3 &= 0 \\
40a_2 - 20b_3 &= 0
\end{aligned}
$$

Solving the above equations for the unknowns gives

$$
\begin{aligned}
a_1 &= 2b_2 \\
a_2 &= a_2 \\
a_3 &= 0 \\
b_1 &= 0 \\
b_2 &= b_2 \\
b_3 &= 2a_2
\end{aligned}
$$

Substituting the above solution in the anstaz (1E,2E) (using 1 as arbitrary value for any unknown in the RHS) gives

$$
\begin{aligned}
\xi &= 2 \\
\eta &= x
\end{aligned}
$$

Shifting is now applied to make $\xi = 0$ in order to simplify the rest of the computation

$$
\begin{aligned}
\eta &= \eta - \omega(x, y)\, \xi \\
&= x - \left(\frac{x}{2} + \frac{\sqrt{5x^2 - 20y}}{2}\right) (2) \\
&= -\sqrt{5x^2 - 20y} \\
\xi &= 0
\end{aligned}
$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x, y) \to (R, S)$ where (R, S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y}\right) S(x, y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$R = x$$

S is found from

$$\begin{aligned} S &= \int \frac{1}{\eta} dy \\ &= \int \frac{1}{-\sqrt{5x^2 - 20y}} dy \end{aligned}$$

Which results in

$$S = \frac{x^2 - 4y}{2\sqrt{5x^2 - 20y}}$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x, y) S_y}{R_x + \omega(x, y) R_y} \tag{2}$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x, y)$ is the right hand side of the original ode given by

$$\omega(x, y) = \frac{x}{2} + \frac{\sqrt{5x^2 - 20y}}{2}$$

Evaluating all the partial derivatives gives

$$\begin{aligned} R_x &= 1 \\ R_y &= 0 \\ S_x &= \frac{x}{2\sqrt{5x^2 - 20y}} \\ S_y &= -\frac{1}{\sqrt{5x^2 - 20y}} \end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = -\frac{1}{2} \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = -\frac{1}{2}$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$S(R) = -\frac{R}{2} + c_1 \tag{4}$$

To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$\frac{x^2 - 4y}{2\sqrt{5x^2 - 20y}} = -\frac{x}{2} + c_1$$

Which simplifies to

$$\frac{x^2 - 4y}{2\sqrt{5x^2 - 20y}} = -\frac{x}{2} + c_1$$

Which gives

$$y = -5c_1^2 + 5c_1 x - x^2$$

Summary

The solution(s) found are the following

$$y = -5c_1^2 + 5c_1 x - x^2 \tag{1}$$

Verification of solutions

$$y = -5c_1^2 + 5c_1 x - x^2$$

Verified OK.

Solving equation (2)

Writing the ode as

$$y' = \frac{x}{2} - \frac{\sqrt{5x^2 - 20y}}{2}$$
$$y' = \omega(x, y)$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x + \omega(\eta_y - \xi_x) - \omega^2 \xi_y - \omega_x \xi - \omega_y \eta = 0 \tag{A}$$

The type of this ode is not in the lookup table. To determine ξ, η then (A) is solved using ansatz. Making bivariate polynomials of degree 1 to use as anstaz gives

$$\xi = x a_2 + y a_3 + a_1 \tag{1E}$$

$$\eta = x b_2 + y b_3 + b_1 \tag{2E}$$

Where the unknown coefficients are

$$\{a_1, a_2, a_3, b_1, b_2, b_3\}$$

Substituting equations (1E,2E) and ω into (A) gives

$$\begin{aligned} &b_2 + \left(\frac{x}{2} - \frac{\sqrt{5x^2 - 20y}}{2}\right)(b_3 - a_2) - \left(\frac{x}{2} - \frac{\sqrt{5x^2 - 20y}}{2}\right)^2 a_3 \\ &- \left(\frac{1}{2} - \frac{5x}{2\sqrt{5x^2 - 20y}}\right)(x a_2 + y a_3 + a_1) - \frac{5(x b_2 + y b_3 + b_1)}{\sqrt{5x^2 - 20y}} = 0 \end{aligned} \tag{5E}$$

Putting the above in normal form gives

$$-\frac{(5x^2 - 20y)^{\frac{3}{2}} a_3 + \sqrt{5x^2 - 20y}\, x^2 a_3 - 10x^3 a_3 + 4\sqrt{5x^2 - 20y}\, x a_2 - 2\sqrt{5x^2 - 20y}\, x b_3 + 2\sqrt{5x^2 - 20y}\, y a}{4\sqrt{5}}$$

$$= 0$$

Setting the numerator to zero gives

$$\begin{aligned} &-(5x^2 - 20y)^{\frac{3}{2}} a_3 - \sqrt{5x^2 - 20y}\, x^2 a_3 + 10x^3 a_3 \\ &- 4\sqrt{5x^2 - 20y}\, x a_2 + 2\sqrt{5x^2 - 20y}\, x b_3 - 2\sqrt{5x^2 - 20y}\, y a_3 \\ &+ 20x^2 a_2 - 10x^2 b_3 - 30xy a_3 - 2\sqrt{5x^2 - 20y}\, a_1 \\ &+ 4b_2\sqrt{5x^2 - 20y} + 10x a_1 - 20x b_2 - 40y a_2 + 20y b_3 - 20b_1 = 0 \end{aligned} \tag{6E}$$

Simplifying the above gives

$$\begin{aligned} &-(5x^2 - 20y)^{\frac{3}{2}} a_3 + 2(5x^2 - 20y)\, x a_3 - \sqrt{5x^2 - 20y}\, x^2 a_3 \\ &+ 2(5x^2 - 20y)\, a_2 - 2(5x^2 - 20y)\, b_3 - 4\sqrt{5x^2 - 20y}\, x a_2 \\ &+ 2\sqrt{5x^2 - 20y}\, x b_3 - 2\sqrt{5x^2 - 20y}\, y a_3 + 10x^2 a_2 + 10xy a_3 \\ &- 2\sqrt{5x^2 - 20y}\, a_1 + 4b_2\sqrt{5x^2 - 20y} + 10x a_1 - 20x b_2 - 20y b_3 - 20b_1 = 0 \end{aligned} \tag{6E}$$

Since the PDE has radicals, simplifying gives

$$\begin{aligned}
&10x^3a_3-6\sqrt{5x^2-20y}\,x^2a_3+20x^2a_2-10x^2b_3-4\sqrt{5x^2-20y}\,xa_2\\
&+2\sqrt{5x^2-20y}\,xb_3-30xya_3+18\sqrt{5x^2-20y}\,ya_3+10xa_1-20xb_2\\
&-2\sqrt{5x^2-20y}\,a_1+4b_2\sqrt{5x^2-20y}-40ya_2+20yb_3-20b_1=0
\end{aligned}$$

Looking at the above PDE shows the following are all the terms with $\{x,y\}$ in them.

$$\left\{x,y,\sqrt{5x^2-20y}\right\}$$

The following substitution is now made to be able to collect on all terms with $\{x,y\}$ in them

$$\left\{x=v_1,y=v_2,\sqrt{5x^2-20y}=v_3\right\}$$

The above PDE (6E) now becomes

$$\begin{aligned}
&10v_1^3a_3-6v_3v_1^2a_3+20v_1^2a_2-4v_3v_1a_2-30v_1v_2a_3+18v_3v_2a_3-10v_1^2b_3\\
&+2v_3v_1b_3+10v_1a_1-2v_3a_1-40v_2a_2-20v_1b_2+4b_2v_3+20v_2b_3-20b_1=0
\end{aligned}\tag{7E}$$

Collecting the above on the terms v_i introduced, and these are

$$\{v_1,v_2,v_3\}$$

Equation (7E) now becomes

$$\begin{aligned}
&10v_1^3a_3-6v_3v_1^2a_3+(20a_2-10b_3)\,v_1^2-30v_1v_2a_3+(-4a_2+2b_3)\,v_1v_3\\
&+(10a_1-20b_2)\,v_1+18v_3v_2a_3+(-40a_2+20b_3)\,v_2+(-2a_1+4b_2)\,v_3-20b_1=0
\end{aligned}\tag{8E}$$

Setting each coefficients in (8E) to zero gives the following equations to solve

$$\begin{aligned}
-30a_3 &= 0 \\
-6a_3 &= 0 \\
10a_3 &= 0 \\
18a_3 &= 0 \\
-20b_1 &= 0 \\
-2a_1 + 4b_2 &= 0 \\
10a_1 - 20b_2 &= 0 \\
-40a_2 + 20b_3 &= 0 \\
-4a_2 + 2b_3 &= 0 \\
20a_2 - 10b_3 &= 0
\end{aligned}$$

Solving the above equations for the unknowns gives

$$\begin{aligned}
a_1 &= 2b_2 \\
a_2 &= a_2 \\
a_3 &= 0 \\
b_1 &= 0 \\
b_2 &= b_2 \\
b_3 &= 2a_2
\end{aligned}$$

Substituting the above solution in the anstaz (1E,2E) (using 1 as arbitrary value for any unknown in the RHS) gives

$$\begin{aligned}
\xi &= 2 \\
\eta &= x
\end{aligned}$$

Shifting is now applied to make $\xi = 0$ in order to simplify the rest of the computation

$$\begin{aligned}
\eta &= \eta - \omega(x, y)\,\xi \\
&= x - \left(\frac{x}{2} - \frac{\sqrt{5x^2 - 20y}}{2}\right)(2) \\
&= \sqrt{5x^2 - 20y} \\
\xi &= 0
\end{aligned}$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x, y) \to (R, S)$ where (R, S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}$$

The above comes from the requirements that $\left(\xi\frac{\partial}{\partial x} + \eta\frac{\partial}{\partial y}\right) S(x,y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$R = x$$

S is found from

$$\begin{aligned} S &= \int \frac{1}{\eta} dy \\ &= \int \frac{1}{\sqrt{5x^2 - 20y}} dy \end{aligned}$$

Which results in

$$S = -\frac{x^2 - 4y}{2\sqrt{5x^2 - 20y}}$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x,y)S_y}{R_x + \omega(x,y)R_y} \tag{2}$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x,y)$ is the right hand side of the original ode given by

$$\omega(x,y) = \frac{x}{2} - \frac{\sqrt{5x^2 - 20y}}{2}$$

Evaluating all the partial derivatives gives

$$\begin{aligned} R_x &= 1 \\ R_y &= 0 \\ S_x &= -\frac{x}{2\sqrt{5x^2 - 20y}} \\ S_y &= \frac{1}{\sqrt{5x^2 - 20y}} \end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = -\frac{1}{2} \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = -\frac{1}{2}$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$S(R) = -\frac{R}{2} + c_1 \tag{4}$$

To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$-\frac{x^2 - 4y}{2\sqrt{5x^2 - 20y}} = -\frac{x}{2} + c_1$$

Which simplifies to

$$-\frac{x^2 - 4y}{2\sqrt{5x^2 - 20y}} = -\frac{x}{2} + c_1$$

Which gives

$$y = -5c_1^2 + 5c_1 x - x^2$$

Summary

The solution(s) found are the following

$$y = -5c_1^2 + 5c_1 x - x^2 \tag{1}$$

Verification of solutions

$$y = -5c_1^2 + 5c_1 x - x^2$$

Verified OK.

Maple trace

```
`Methods for first order ODEs:
   *** Sublevel 2 ***
   Methods for first order ODEs:
   -> Solving 1st order ODE of high degree, 1st attempt
   trying 1st order WeierstrassP solution for high degree ODE
   trying 1st order WeierstrassPPrime solution for high degree ODE
   trying 1st order JacobiSN solution for high degree ODE
   trying 1st order ODE linearizable_by_differentiation
   trying differential order: 1; missing variables
   trying dAlembert
   trying simple symmetries for implicit equations
   Successful isolation of dy/dx: 2 solutions were found. Trying to solve each resulting ODE
      *** Sublevel 3 ***
      Methods for first order ODEs:
      --- Trying classification methods ---
      trying homogeneous types:
      trying homogeneous G
      1st order, trying the canonical coordinates of the invariance group
      <- 1st order, canonical coordinates successful
      <- homogeneous successful
   -------------------
   * Tackling next ODE.
      *** Sublevel 3 ***
      Methods for first order ODEs:
      --- Trying classification methods ---
      trying homogeneous types:
      trying homogeneous G
      1st order, trying the canonical coordinates of the invariance group
      <- 1st order, canonical coordinates successful
      <- homogeneous successful`
```

✓ Solution by Maple

Time used: 0.047 (sec). Leaf size: 91

```
dsolve(5*y(x)+(diff(y(x),x))^2=x*(x+diff(y(x),x)),y(x), singsol=all)
```

$$y(x) = \frac{x^2}{4}$$
$$y(x) = x\sqrt{5}\,\sqrt{-c_1} - x^2 + c_1$$
$$y(x) = -x\sqrt{5}\,\sqrt{-c_1} - x^2 + c_1$$
$$y(x) = -x\sqrt{5}\,\sqrt{-c_1} - x^2 + c_1$$
$$y(x) = x\sqrt{5}\,\sqrt{-c_1} - x^2 + c_1$$

✗ Solution by Mathematica

Time used: 0.0 (sec). Leaf size: 0

```
DSolve[5*y[x]+(y'[x])^2==x*(x+y'[x]),y[x],x,IncludeSingularSolutions -> True]
```

Timed out

1.92 problem 113

Internal problem ID [3237]
Internal file name [`OUTPUT/2729_Sunday_June_05_2022_08_39_48_AM_99015751/index.tex`]

Book: Differential equations for engineers by Wei-Chau XIE, Cambridge Press 2010
Section: Chapter 2. First-Order and Simple Higher-Order Differential Equations. Page 78
Problem number: 113.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"exact", "linear", "separable", "homogeneousTypeD2", "homogeneousTypeMapleC", "first_order_ode_lie_symmetry_lookup"**

Maple gives the following as the ode type

```
[_separable]
```

$$\boxed{y' - \frac{y+2}{x+1} = 0}$$

1.92.1 Solving as separable ode

In canonical form the ODE is

$$\begin{aligned}
y' &= F(x,y) \\
&= f(x)g(y) \\
&= \frac{y+2}{x+1}
\end{aligned}$$

Where $f(x) = \frac{1}{x+1}$ and $g(y) = y + 2$. Integrating both sides gives

$$\frac{1}{y+2}\,dy = \frac{1}{x+1}\,dx$$
$$\int \frac{1}{y+2}\,dy = \int \frac{1}{x+1}\,dx$$
$$\ln(y+2) = \ln(x+1) + c_1$$

Raising both side to exponential gives

$$y + 2 = e^{\ln(x+1)+c_1}$$

Which simplifies to

$$y + 2 = c_2(x+1)$$

Summary

The solution(s) found are the following

$$y = c_2 e^{\ln(x+1)+c_1} - 2 \tag{1}$$

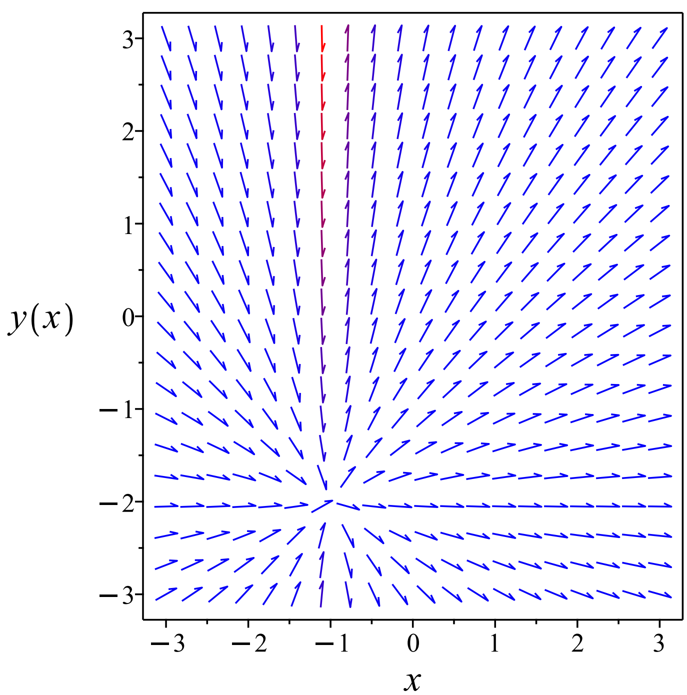


Figure 115: Slope field plot

Verification of solutions

$$y = c_2 e^{\ln(x+1)+c_1} - 2$$

Verified OK.

1.92.2 Solving as linear ode

Entering Linear first order ODE solver. In canonical form a linear first order is

$$y' + p(x)y = q(x)$$

Where here

$$p(x) = -\frac{1}{x+1}$$
$$q(x) = \frac{2}{x+1}$$

Hence the ode is

$$y' - \frac{y}{x+1} = \frac{2}{x+1}$$

The integrating factor μ is

$$\mu = e^{\int -\frac{1}{x+1}dx}$$
$$= \frac{1}{x+1}$$

The ode becomes

$$\frac{d}{dx}(\mu y) = (\mu)\left(\frac{2}{x+1}\right)$$
$$\frac{d}{dx}\left(\frac{y}{x+1}\right) = \left(\frac{1}{x+1}\right)\left(\frac{2}{x+1}\right)$$
$$d\left(\frac{y}{x+1}\right) = \left(\frac{2}{(x+1)^2}\right) dx$$

Integrating gives

$$\frac{y}{x+1} = \int \frac{2}{(x+1)^2} dx$$
$$\frac{y}{x+1} = -\frac{2}{x+1} + c_1$$

Dividing both sides by the integrating factor $\mu = \frac{1}{x+1}$ results in

$$y = -2 + c_1(x + 1)$$

which simplifies to

$$y = c_1 x + c_1 - 2$$

Summary

The solution(s) found are the following

$$y = c_1 x + c_1 - 2 \tag{1}$$

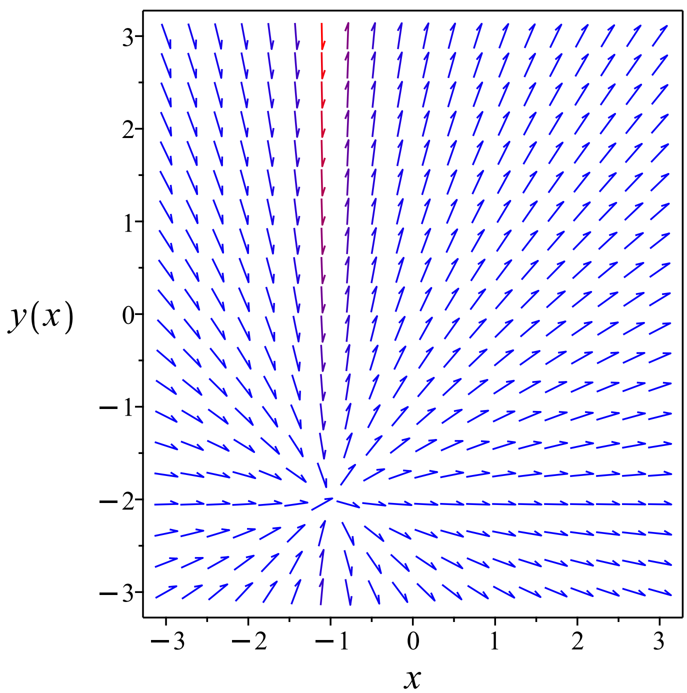


Figure 116: Slope field plot

Verification of solutions

$$y = c_1 x + c_1 - 2$$

Verified OK.

1.92.3 Solving as homogeneousTypeD2 ode

Using the change of variables $y = u(x)\,x$ on the above ode results in new ode in $u(x)$

$$u'(x)\,x + u(x) - \frac{u(x)\,x + 2}{x + 1} = 0$$

In canonical form the ODE is

$$\begin{aligned} u' &= F(x,u) \\ &= f(x)g(u) \\ &= \frac{-u + 2}{x\,(x + 1)} \end{aligned}$$

Where $f(x) = \frac{1}{x(x+1)}$ and $g(u) = -u + 2$. Integrating both sides gives

$$\begin{aligned} \frac{1}{-u + 2}\,du &= \frac{1}{x\,(x + 1)}\,dx \\ \int \frac{1}{-u + 2}\,du &= \int \frac{1}{x\,(x + 1)}\,dx \\ -\ln\left(u - 2\right) &= -\ln\left(x + 1\right) + \ln\left(x\right) + c_2 \end{aligned}$$

Raising both side to exponential gives

$$\frac{1}{u - 2} = \mathrm{e}^{-\ln(x+1)+\ln(x)+c_2}$$

Which simplifies to

$$\frac{1}{u - 2} = c_3\mathrm{e}^{-\ln(x+1)+\ln(x)}$$

Which simplifies to

$$u(x) = \frac{\left(\frac{2c_3\mathrm{e}^{c_2}x}{x+1} + 1\right)(x + 1)\,\mathrm{e}^{-c_2}}{c_3x}$$

Therefore the solution y is

$$\begin{aligned} y &= ux \\ &= \frac{\left(\frac{2c_3\mathrm{e}^{c_2}x}{x+1} + 1\right)(x + 1)\,\mathrm{e}^{-c_2}}{c_3} \end{aligned}$$

Summary
The solution(s) found are the following

$$y = \frac{\left(\frac{2c_3 e^{c_2} x}{x+1} + 1\right)(x+1)\,e^{-c_2}}{c_3} \tag{1}$$

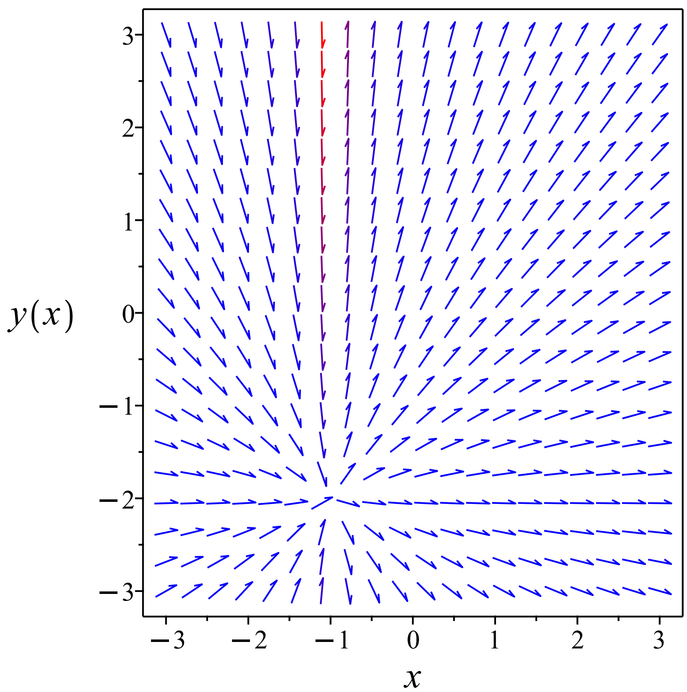


Figure 117: Slope field plot

Verification of solutions

$$y = \frac{\left(\frac{2c_3 e^{c_2} x}{x+1} + 1\right)(x+1)\,e^{-c_2}}{c_3}$$

Verified OK.

1.92.4 Solving as homogeneousTypeMapleC ode

Let $Y = y + y_0$ and $X = x + x_0$ then the above is transformed to new ode in $Y(X)$

$$\frac{d}{dX}Y(X) = \frac{Y(X) + y_0 + 2}{X + x_0 + 1}$$

Solving for possible values of x_0 and y_0 which makes the above ode a homogeneous ode results in

$$\begin{aligned}x_0 &= -1\\ y_0 &= -2\end{aligned}$$

Using these values now it is possible to easily solve for $Y(X)$. The above ode now becomes

$$\frac{d}{dX}Y(X) = \frac{Y(X)}{X}$$

In canonical form, the ODE is

$$\begin{aligned}Y' &= F(X,Y)\\ &= \frac{Y}{X} \end{aligned}\tag{1}$$

An ode of the form $Y' = \frac{M(X,Y)}{N(X,Y)}$ is called homogeneous if the functions $M(X,Y)$ and $N(X,Y)$ are both homogeneous functions and of the same order. Recall that a function $f(X,Y)$ is homogeneous of order n if

$$f(t^nX, t^nY) = t^n f(X,Y)$$

In this case, it can be seen that both $M = Y$ and $N = X$ are both homogeneous and of the same order $n = 1$. Therefore this is a homogeneous ode. Since this ode is homogeneous, it is converted to separable ODE using the substitution $u = \frac{Y}{X}$, or $Y = uX$. Hence

$$\frac{dY}{dX} = \frac{du}{dX}X + u$$

Applying the transformation $Y = uX$ to the above ODE in (1) gives

$$\begin{aligned}\frac{du}{dX}X + u &= u\\ \frac{du}{dX} &= 0\end{aligned}$$

Or

$$\frac{d}{dX}u(X) = 0$$

Which is now solved as separable in $u(X)$. Which is now solved in $u(X)$. Integrating both sides gives

$$\begin{aligned}u(X) &= \int 0 \, dX\\ &= c_2\end{aligned}$$

Now u in the above solution is replaced back by Y using $u = \frac{Y}{X}$ which results in the solution

$$Y(X) = Xc_2$$

Using the solution for $Y(X)$

$$Y(X) = Xc_2$$

And replacing back terms in the above solution using

$$Y = y + y_0$$
$$X = x + x_0$$

Or

$$Y = y - 2$$
$$X = x - 1$$

Then the solution in y becomes

$$y + 2 = c_2(x + 1)$$

<u>Summary</u>
The solution(s) found are the following

$$y + 2 = c_2(x + 1) \tag{1}$$

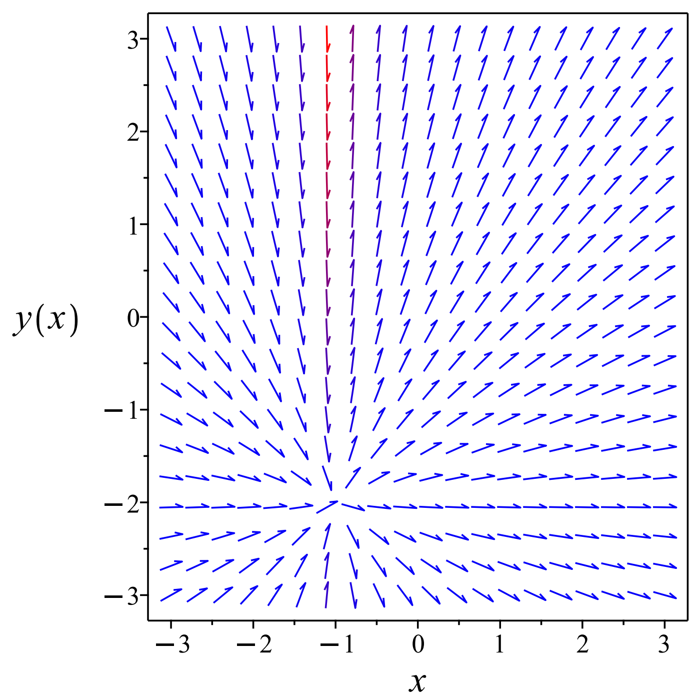


Figure 118: Slope field plot

Verification of solutions

$$y + 2 = c_2(x + 1)$$

Verified OK.

1.92.5 Solving as first order ode lie symmetry lookup ode

Writing the ode as

$$y' = \frac{y + 2}{x + 1}$$
$$y' = \omega(x, y)$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x + \omega(\eta_y - \xi_x) - \omega^2\xi_y - \omega_x\xi - \omega_y\eta = 0 \tag{A}$$

The type of this ode is known. It is of type `linear`. Therefore we do not need to solve the PDE (A), and can just use the lookup table shown below to find ξ, η

Table 36: Lie symmetry infinitesimal lookup table for known first order ODE's

ODE class	Form	ξ	η
linear ode	$y' = f(x)y(x) + g(x)$	0	$e^{\int f dx}$
separable ode	$y' = f(x)\, g(y)$	$\frac{1}{f}$	0
quadrature ode	$y' = f(x)$	0	1
quadrature ode	$y' = g(y)$	1	0
homogeneous ODEs of Class A	$y' = f\left(\frac{y}{x}\right)$	x	y
homogeneous ODEs of Class C	$y' = (a + bx + cy)^{\frac{n}{m}}$	1	$-\frac{b}{c}$
homogeneous class D	$y' = \frac{y}{x} + g(x)\, F\left(\frac{y}{x}\right)$	x^2	xy
First order special form ID 1	$y' = g(x)\, e^{h(x)+by} + f(x)$	$\frac{e^{-\int bf(x)dx-h(x)}}{g(x)}$	$\frac{f(x)e^{-\int bf(x)dx-h(x)}}{g(x)}$
polynomial type ode	$y' = \frac{a_1x+b_1y+c_1}{a_2x+b_2y+c_2}$	$\frac{a_1b_2x-a_2b_1x-b_1c_2+b_2c_1}{a_1b_2-a_2b_1}$	$\frac{a_1b_2y-a_2b_1y-a_1c_2-a_2c_1}{a_1b_2-a_2b_1}$
Bernoulli ode	$y' = f(x)\, y + g(x)\, y^n$	0	$e^{-\int (n-1)f(x)dx} y^n$
Reduced Riccati	$y' = f_1(x)\, y + f_2(x)\, y^2$	0	$e^{-\int f_1 dx}$

The above table shows that

$$\begin{aligned} \xi(x,y) &= 0 \\ \eta(x,y) &= x+1 \end{aligned} \tag{A1}$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x,y) \to (R,S)$ where (R,S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y}\right) S(x,y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the

canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$R = x$$

S is found from

$$\begin{aligned} S &= \int \frac{1}{\eta} dy \\ &= \int \frac{1}{x+1} dy \end{aligned}$$

Which results in

$$S = \frac{y}{x+1}$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x,y) S_y}{R_x + \omega(x,y) R_y} \tag{2}$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x,y)$ is the right hand side of the original ode given by

$$\omega(x,y) = \frac{y+2}{x+1}$$

Evaluating all the partial derivatives gives

$$\begin{aligned} R_x &= 1 \\ R_y &= 0 \\ S_x &= -\frac{y}{(x+1)^2} \\ S_y &= \frac{1}{x+1} \end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = \frac{2}{(x+1)^2} \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = \frac{2}{(R+1)^2}$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordinates R, S. Integrating the above gives

$$S(R) = -\frac{2}{R+1} + c_1 \tag{4}$$

To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$\frac{y}{x+1} = -\frac{2}{x+1} + c_1$$

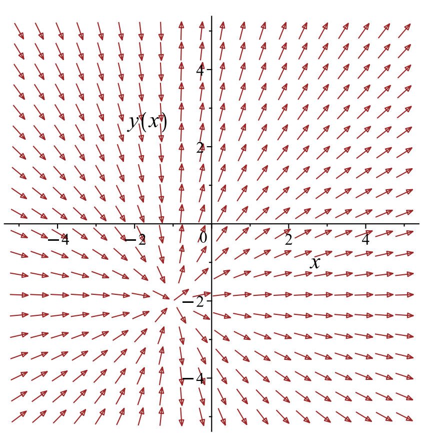
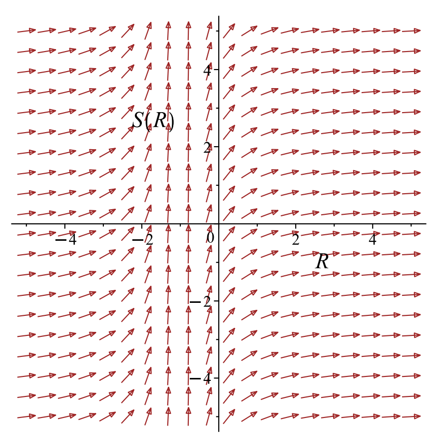
Which simplifies to

$$\frac{y}{x+1} = -\frac{2}{x+1} + c_1$$

Which gives

$$y = c_1 x + c_1 - 2$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in x, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dx} = \frac{y+2}{x+1}$ 	$R = x$ $S = \frac{y}{x+1}$	$\frac{dS}{dR} = \frac{2}{(R+1)^2}$ 

Summary

The solution(s) found are the following

$$y = c_1 x + c_1 - 2 \tag{1}$$

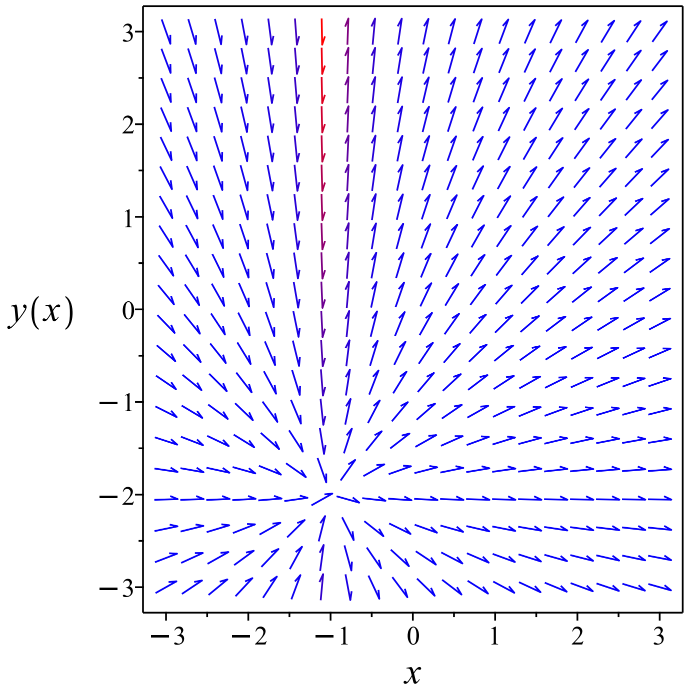


Figure 119: Slope field plot

Verification of solutions

$$y = c_1 x + c_1 - 2$$

Verified OK.

1.92.6 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x,y) + N(x,y)\frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x,y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y) = 0$$

Hence

$$\frac{\partial\phi}{\partial x} + \frac{\partial\phi}{\partial y}\frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$\begin{aligned}\frac{\partial \phi}{\partial x} &= M \\ \frac{\partial \phi}{\partial y} &= N\end{aligned}$$

But since $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x, y)$ but at least we know now that we can do that since the condition $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x, y)\,\mathrm{d}x + N(x, y)\,\mathrm{d}y = 0 \tag{1A}$$

Therefore

$$\begin{aligned}\left(\frac{1}{y+2}\right)\mathrm{d}y &= \left(\frac{1}{x+1}\right)\mathrm{d}x \\ \left(-\frac{1}{x+1}\right)\mathrm{d}x + \left(\frac{1}{y+2}\right)\mathrm{d}y &= 0 \qquad (2A)\end{aligned}$$

Comparing (1A) and (2A) shows that

$$\begin{aligned}M(x, y) &= -\frac{1}{x+1} \\ N(x, y) &= \frac{1}{y+2}\end{aligned}$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Using result found above gives

$$\begin{aligned}\frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}\left(-\frac{1}{x+1}\right) \\ &= 0\end{aligned}$$

And

$$
\begin{aligned}
\frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}\left(\frac{1}{y+2}\right) \\
&= 0
\end{aligned}
$$

Since $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$, then the ODE is exact The following equations are now set up to solve for the function $\phi(x, y)$

$$
\frac{\partial \phi}{\partial x} = M \tag{1}
$$

$$
\frac{\partial \phi}{\partial y} = N \tag{2}
$$

Integrating (1) w.r.t. x gives

$$
\begin{aligned}
\int \frac{\partial \phi}{\partial x} \mathrm{d}x &= \int M \mathrm{d}x \\
\int \frac{\partial \phi}{\partial x} \mathrm{d}x &= \int -\frac{1}{x+1} \mathrm{d}x
\end{aligned}
$$

$$
\phi = -\ln(x+1) + f(y) \tag{3}
$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$
\frac{\partial \phi}{\partial y} = 0 + f'(y) \tag{4}
$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = \frac{1}{y+2}$. Therefore equation (4) becomes

$$
\frac{1}{y+2} = 0 + f'(y) \tag{5}
$$

Solving equation (5) for $f'(y)$ gives

$$
f'(y) = \frac{1}{y+2}
$$

Integrating the above w.r.t y gives

$$
\begin{aligned}
\int f'(y) \mathrm{d}y &= \int \left(\frac{1}{y+2}\right) \mathrm{d}y \\
f(y) &= \ln(y+2) + c_1
\end{aligned}
$$

Where c_1 is constant of integration. Substituting result found above for $f(y)$ into equation (3) gives ϕ

$$\phi = -\ln(x+1) + \ln(y+2) + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = -\ln(x+1) + \ln(y+2)$$

The solution becomes

$$y = \mathrm{e}^{c_1}x + \mathrm{e}^{c_1} - 2$$

Summary

The solution(s) found are the following

$$y = \mathrm{e}^{c_1}x + \mathrm{e}^{c_1} - 2 \tag{1}$$

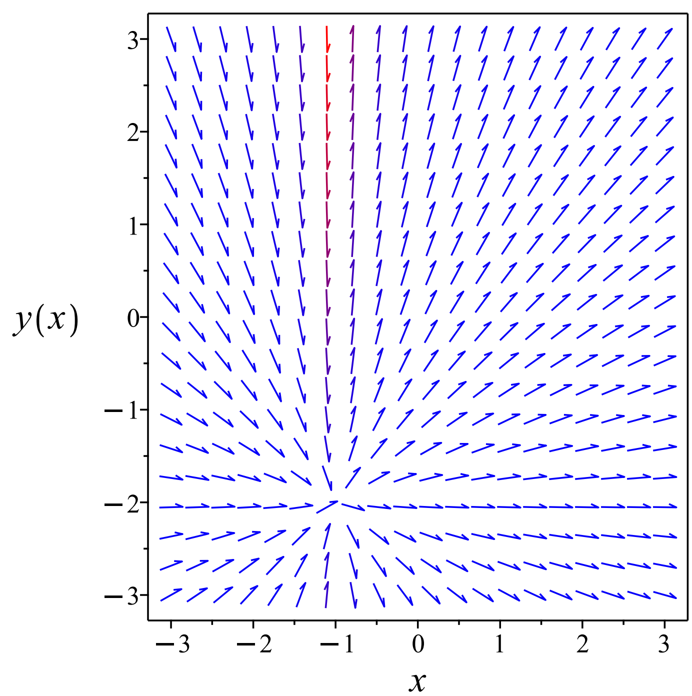


Figure 120: Slope field plot

Verification of solutions

$$y = \mathrm{e}^{c_1}x + \mathrm{e}^{c_1} - 2$$

Verified OK.

1.92.7 Maple step by step solution

Let's solve

$y' - \frac{y+2}{x+1} = 0$

- Highest derivative means the order of the ODE is 1

  y'

- Separate variables

  $\frac{y'}{y+2} = \frac{1}{x+1}$

- Integrate both sides with respect to x

  $\int \frac{y'}{y+2}dx = \int \frac{1}{x+1}dx + c_1$

- Evaluate integral

  $\ln(y+2) = \ln(x+1) + c_1$

- Solve for y

  $y = e^{c_1}x + e^{c_1} - 2$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
<- 1st order linear successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 10

```
dsolve(diff(y(x),x)= (y(x)+2)/(x+1),y(x), singsol=all)
```

$$y(x) = c_1x + c_1 - 2$$

✓ Solution by Mathematica

Time used: 0.044 (sec). Leaf size: 18

```
DSolve[y'[x]== (y[x]+2)/(x+1),y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to -2 + c_1(x+1)$$
$$y(x) \to -2$$

1.93 problem 115

Internal problem ID [3238]
Internal file name [`OUTPUT/2730_Sunday_June_05_2022_08_39_49_AM_26650830/index.tex`]

Book: Differential equations for engineers by Wei-Chau XIE, Cambridge Press 2010
Section: Chapter 2. First-Order and Simple Higher-Order Differential Equations. Page 78
Problem number: 115.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"homogeneousTypeD", "homogeneousTypeD2", "first_order_ode_lie_symmetry_lookup"**

Maple gives the following as the ode type

```
[[_homogeneous, `class A`], _dAlembert]
```

$$\boxed{xy' - y + x\,\mathrm{e}^{\frac{y}{x}} = 0}$$

1.93.1 Solving as homogeneousTypeD ode

Writing the ode as

$$y' = -\mathrm{e}^{\frac{y}{x}} + \frac{y}{x} \tag{A}$$

The given ode has the form

$$y' = \frac{y}{x} + g(x)\,f\left(b\frac{y}{x}\right)^{\frac{n}{m}} \tag{1}$$

Where b is scalar and $g(x)$ is function of x and n, m are integers. The solution is given in Kamke page 20. Using the substitution $y(x) = u(x)\,x$ then

$$\frac{dy}{dx} = \frac{du}{dx}x + u$$

Hence the given ode becomes

$$\frac{du}{dx}x + u = u + g(x)\,f(bu)^{\frac{n}{m}}$$

$$u' = \frac{1}{x}g(x)\,f(bu)^{\frac{n}{m}} \tag{2}$$

The above ode is always separable. This is easily solved for u assuming the integration can be resolved, and then the solution to the original ode becomes $y = ux$. Comapring the given ode (A) with the form (1) shows that

$$\begin{aligned} g(x) &= -1 \\ b &= 1 \\ f\left(\frac{bx}{y}\right) &= e^{\frac{y}{x}} \end{aligned}$$

Substituting the above in (2) results in the $u(x)$ ode as

$$u'(x) = -\frac{e^{u(x)}}{x}$$

Which is now solved as separable In canonical form the ODE is

$$\begin{aligned} u' &= F(x, u) \\ &= f(x)g(u) \\ &= -\frac{e^{u}}{x} \end{aligned}$$

Where $f(x) = -\frac{1}{x}$ and $g(u) = e^{u}$. Integrating both sides gives

$$\begin{aligned} \frac{1}{e^{u}}\,du &= -\frac{1}{x}\,dx \\ \int \frac{1}{e^{u}}\,du &= \int -\frac{1}{x}\,dx \\ -e^{-u} &= -\ln(x) + c_1 \end{aligned}$$

The solution is

$$-e^{-u(x)} + \ln(x) - c_1 = 0$$

Therefore the solution is found using $y = ux$. Hence

$$-e^{-\frac{y}{x}} + \ln(x) - c_1 = 0$$

Summary
The solution(s) found are the following

$$-e^{-\frac{y}{x}} + \ln(x) - c_1 = 0 \tag{1}$$

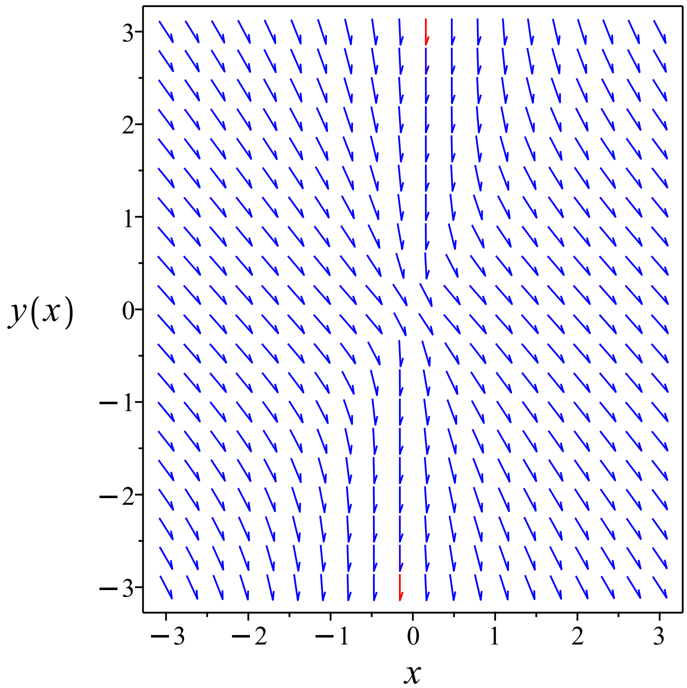


Figure 121: Slope field plot

Verification of solutions

$$-e^{-\frac{y}{x}} + \ln(x) - c_1 = 0$$

Verified OK.

1.93.2 Solving as homogeneousTypeD2 ode

Using the change of variables $y = u(x)\, x$ on the above ode results in new ode in $u(x)$

$$x(u'(x)\, x + u(x)) - u(x)\, x + x\, e^{u(x)} = 0$$

In canonical form the ODE is

$$\begin{aligned} u' &= F(x, u) \\ &= f(x)g(u) \\ &= -\frac{e^u}{x} \end{aligned}$$

Where $f(x) = -\frac{1}{x}$ and $g(u) = e^u$. Integrating both sides gives

$$\frac{1}{e^u}\, du = -\frac{1}{x}\, dx$$

$$\int \frac{1}{e^u}\,du = \int -\frac{1}{x}\,dx$$
$$-e^{-u} = -\ln(x) + c_2$$

The solution is

$$-e^{-u(x)} + \ln(x) - c_2 = 0$$

Replacing $u(x)$ in the above solution by $\frac{y}{x}$ results in the solution for y in implicit form

$$-e^{-\frac{y}{x}} + \ln(x) - c_2 = 0$$
$$-e^{-\frac{y}{x}} + \ln(x) - c_2 = 0$$

Summary

The solution(s) found are the following

$$-e^{-\frac{y}{x}} + \ln(x) - c_2 = 0 \tag{1}$$

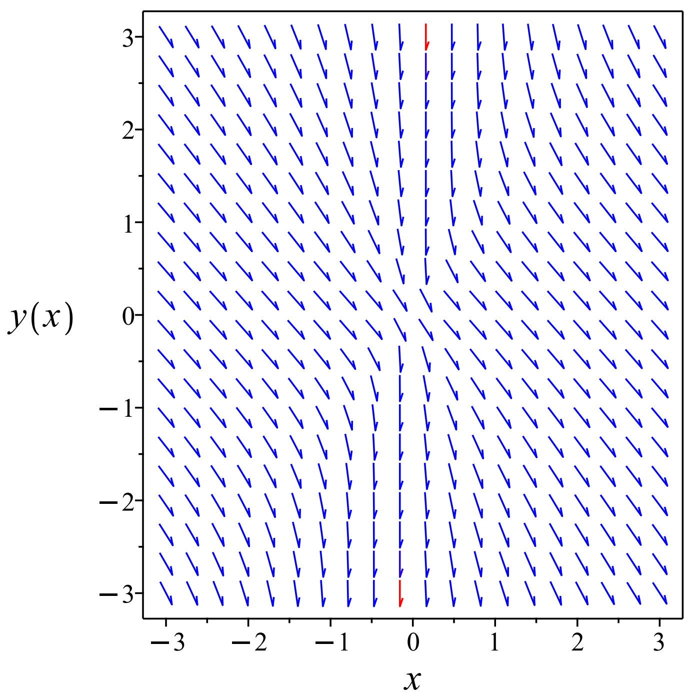


Figure 122: Slope field plot

Verification of solutions

$$-e^{-\frac{y}{x}} + \ln(x) - c_2 = 0$$

Verified OK.

1.93.3 Solving as first order ode lie symmetry lookup ode

Writing the ode as

$$
\begin{aligned}
y' &= -\frac{x\,\mathrm{e}^{\frac{y}{x}}-y}{x} \\
y' &= \omega(x,y)
\end{aligned}
$$

The condition of Lie symmetry is the linearized PDE given by

$$
\eta_x + \omega(\eta_y - \xi_x) - \omega^2 \xi_y - \omega_x \xi - \omega_y \eta = 0 \tag{A}
$$

The type of this ode is known. It is of type `homogeneous Type D`. Therefore we do not need to solve the PDE (A), and can just use the lookup table shown below to find ξ, η

Table 39: Lie symmetry infinitesimal lookup table for known first order ODE's

ODE class	Form	ξ	η
linear ode	$y' = f(x)y(x) + g(x)$	0	$e^{\int f dx}$
separable ode	$y' = f(x)\, g(y)$	$\frac{1}{f}$	0
quadrature ode	$y' = f(x)$	0	1
quadrature ode	$y' = g(y)$	1	0
homogeneous ODEs of Class A	$y' = f\left(\frac{y}{x}\right)$	x	y
homogeneous ODEs of Class C	$y' = (a + bx + cy)^{\frac{n}{m}}$	1	$-\frac{b}{c}$
homogeneous class D	$y' = \frac{y}{x} + g(x)\, F\left(\frac{y}{x}\right)$	x^2	xy
First order special form ID 1	$y' = g(x)\, e^{h(x)+by} + f(x)$	$\frac{e^{-\int bf(x)dx - h(x)}}{g(x)}$	$\frac{f(x)e^{-\int bf(x)dx - h(x)}}{g(x)}$
polynomial type ode	$y' = \frac{a_1x+b_1y+c_1}{a_2x+b_2y+c_2}$	$\frac{a_1b_2x-a_2b_1x-b_1c_2+b_2c_1}{a_1b_2-a_2b_1}$	$\frac{a_1b_2y-a_2b_1y-a_1c_2-a_2c_1}{a_1b_2-a_2b_1}$
Bernoulli ode	$y' = f(x)\, y + g(x)\, y^n$	0	$e^{-\int (n-1)f(x)dx} y^n$
Reduced Riccati	$y' = f_1(x)\, y + f_2(x)\, y^2$	0	$e^{-\int f_1 dx}$

The above table shows that

$$\begin{aligned} \xi(x,y) &= x^2 \\ \eta(x,y) &= xy \end{aligned} \tag{A1}$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x,y) \to (R,S)$ where (R,S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}$$

The above comes from the requirements that $\left(\xi\frac{\partial}{\partial x} + \eta\frac{\partial}{\partial y}\right) S(x,y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the canonical coordinates, where $S(R)$. Therefore

$$\begin{aligned} \frac{dy}{dx} &= \frac{\eta}{\xi} \\ &= \frac{xy}{x^2} \\ &= \frac{y}{x} \end{aligned}$$

This is easily solved to give

$$y = c_1 x$$

Where now the coordinate R is taken as the constant of integration. Hence

$$R = \frac{y}{x}$$

And S is found from

$$\begin{aligned} dS &= \frac{dx}{\xi} \\ &= \frac{dx}{x^2} \end{aligned}$$

Integrating gives

$$\begin{aligned} S &= \int \frac{dx}{T} \\ &= -\frac{1}{x} \end{aligned}$$

Where the constant of integration is set to zero as we just need one solution. Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x,y)S_y}{R_x + \omega(x,y)R_y} \tag{2}$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x,y)$ is the right hand side of the original ode given by

$$\omega(x,y) = -\frac{x\,\mathrm{e}^{\frac{y}{x}} - y}{x}$$

Evaluating all the partial derivatives gives

$$\begin{aligned} R_x &= -\frac{y}{x^2} \\ R_y &= \frac{1}{x} \\ S_x &= \frac{1}{x^2} \\ S_y &= 0 \end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = -\frac{\mathrm{e}^{-\frac{y}{x}}}{x} \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = S(R)\,\mathrm{e}^{-R}$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$S(R) = c_1 \mathrm{e}^{-\mathrm{e}^{-R}} \tag{4}$$

To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$-\frac{1}{x} = c_1 \mathrm{e}^{-\mathrm{e}^{-\frac{y}{x}}}$$

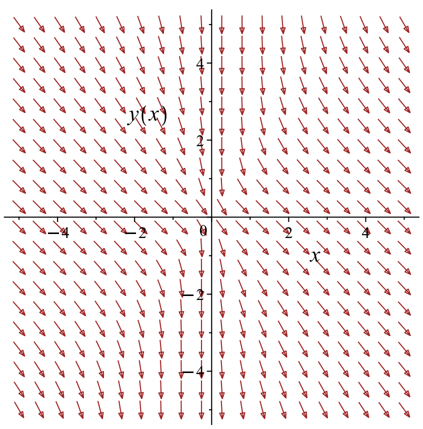
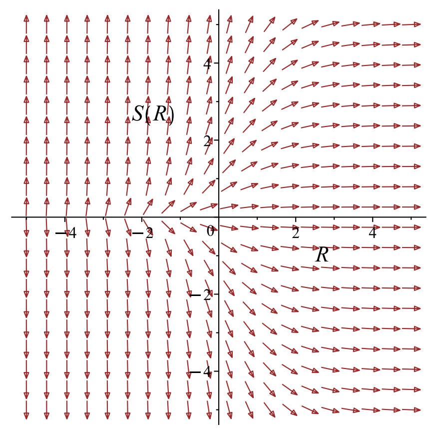
Which simplifies to

$$-\frac{1}{x} = c_1 \mathrm{e}^{-\mathrm{e}^{-\frac{y}{x}}}$$

Which gives

$$y = -\ln\left(-\ln\left(-\frac{1}{c_1 x}\right)\right) x$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

<table>
<tr><th>Original ode in x, y coordinates</th><th>Canonical coordinates transformation</th><th>ODE in canonical coordinates (R, S)</th></tr>
<tr><td>$\frac{dy}{dx} = -\frac{x\,\mathrm{e}^{\frac{y}{x}} - y}{x}$

</td><td>$R = \frac{y}{x}$
$S = -\frac{1}{x}$</td><td>$\frac{dS}{dR} = S(R)\,\mathrm{e}^{-R}$

</td></tr>
</table>

Summary
The solution(s) found are the following

$$y = -\ln\left(-\ln\left(-\frac{1}{c_1 x}\right)\right) x \tag{1}$$

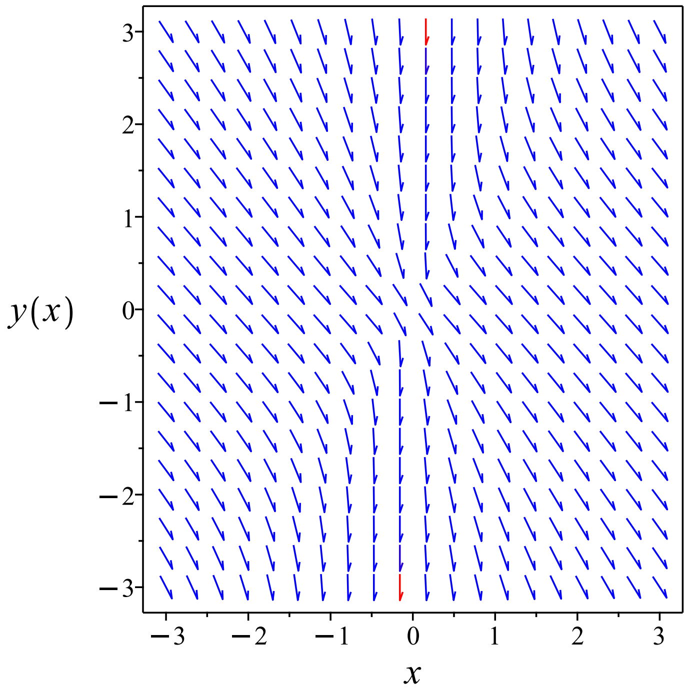


Figure 123: Slope field plot

Verification of solutions

$$y = -\ln\left(-\ln\left(-\frac{1}{c_1 x}\right)\right) x$$

Verified OK.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
trying homogeneous types:
trying homogeneous D
<- homogeneous successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 12

```
dsolve(x*diff(y(x),x)= y(x)-x*exp(y(x)/x),y(x), singsol=all)
```

$$y(x) = -\ln(\ln(x) + c_1)\, x$$

✓ Solution by Mathematica

Time used: 0.426 (sec). Leaf size: 16

```
DSolve[x*y'[x]== y[x]-x*Exp[y[x]/x],y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to -x \log(\log(x) - c_1)$$

1.94 problem 116

Internal problem ID [3239]
Internal file name [`OUTPUT/2731_Sunday_June_05_2022_08_39_50_AM_72756352/index.tex`]

Book: Differential equations for engineers by Wei-Chau XIE, Cambridge Press 2010
Section: Chapter 2. First-Order and Simple Higher-Order Differential Equations. Page 78
Problem number: 116.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"exact", "bernoulli", "first_order_ode_lie_symmetry_lookup"**

Maple gives the following as the ode type

```
[_exact, _Bernoulli]
```

$$\boxed{\sin (2x)\, y^2 - 2y\cos (x)^2 y' = -1}$$

1.94.1 Solving as first order ode lie symmetry lookup ode

Writing the ode as

$$y' = \frac{\sin (2x)\, y^2 + 1}{2y\cos (x)^2}$$
$$y' = \omega(x, y)$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x + \omega(\eta_y - \xi_x) - \omega^2 \xi_y - \omega_x \xi - \omega_y \eta = 0 \tag{A}$$

The type of this ode is known. It is of type `Bernoulli`. Therefore we do not need to solve the PDE (A), and can just use the lookup table shown below to find ξ, η

Table 41: Lie symmetry infinitesimal lookup table for known first order ODE's

ODE class	Form	ξ	η
linear ode	$y' = f(x)y(x) + g(x)$	0	$e^{\int f dx}$
separable ode	$y' = f(x)\, g(y)$	$\frac{1}{f}$	0
quadrature ode	$y' = f(x)$	0	1
quadrature ode	$y' = g(y)$	1	0
homogeneous ODEs of Class A	$y' = f\left(\frac{y}{x}\right)$	x	y
homogeneous ODEs of Class C	$y' = (a + bx + cy)^{\frac{n}{m}}$	1	$-\frac{b}{c}$
homogeneous class D	$y' = \frac{y}{x} + g(x)\, F\left(\frac{y}{x}\right)$	x^2	xy
First order special form ID 1	$y' = g(x)\, e^{h(x)+by} + f(x)$	$\frac{e^{-\int b f(x) dx - h(x)}}{g(x)}$	$\frac{f(x) e^{-\int b f(x) dx - h(x)}}{g(x)}$
polynomial type ode	$y' = \frac{a_1 x + b_1 y + c_1}{a_2 x + b_2 y + c_2}$	$\frac{a_1 b_2 x - a_2 b_1 x - b_1 c_2 + b_2 c_1}{a_1 b_2 - a_2 b_1}$	$\frac{a_1 b_2 y - a_2 b_1 y - a_1 c_2 - a_2 c_1}{a_1 b_2 - a_2 b_1}$
Bernoulli ode	$y' = f(x)\, y + g(x)\, y^n$	0	$e^{-\int (n-1) f(x) dx} y^n$
Reduced Riccati	$y' = f_1(x)\, y + f_2(x)\, y^2$	0	$e^{-\int f_1 dx}$

The above table shows that

$$
\begin{aligned}
\xi(x, y) &= 0 \\
\eta(x, y) &= \frac{1}{y \cos(x)^2}
\end{aligned} \tag{A1}
$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x, y) \to (R, S)$ where (R, S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$
\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}
$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y}\right) S(x, y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the

canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$R = x$$

S is found from

$$\begin{aligned} S &= \int \frac{1}{\eta} dy \\ &= \int \frac{1}{\frac{1}{y \cos(x)^2}} dy \end{aligned}$$

Which results in

$$S = \frac{y^2 \cos(x)^2}{2}$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x,y) S_y}{R_x + \omega(x,y) R_y} \tag{2}$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x,y)$ is the right hand side of the original ode given by

$$\omega(x,y) = \frac{\sin(2x) y^2 + 1}{2y \cos(x)^2}$$

Evaluating all the partial derivatives gives

$$\begin{aligned} R_x &= 1 \\ R_y &= 0 \\ S_x &= -\frac{\sin(2x) y^2}{2} \\ S_y &= y \cos(x)^2 \end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = \frac{1}{2} \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = \frac{1}{2}$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordinates R, S. Integrating the above gives

$$S(R) = \frac{R}{2} + c_1 \tag{4}$$

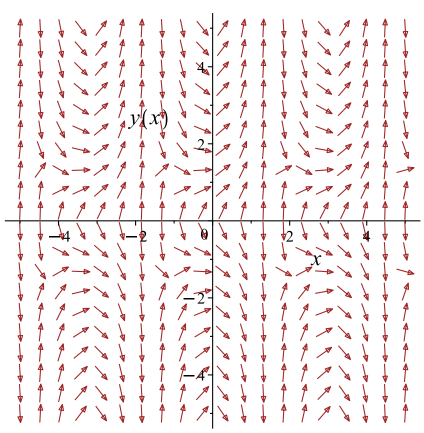
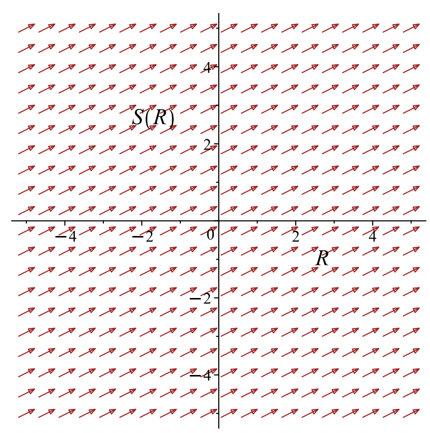
To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$\frac{y^2 \cos(x)^2}{2} = \frac{x}{2} + c_1$$

Which simplifies to

$$\frac{y^2 \cos(x)^2}{2} = \frac{x}{2} + c_1$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in x, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dx} = \frac{\sin(2x)y^2+1}{2y\cos(x)^2}$ 	$R = x$ $S = \frac{y^2 \cos(x)^2}{2}$	$\frac{dS}{dR} = \frac{1}{2}$ 

Summary
The solution(s) found are the following

$$\frac{y^2 \cos(x)^2}{2} = \frac{x}{2} + c_1 \tag{1}$$

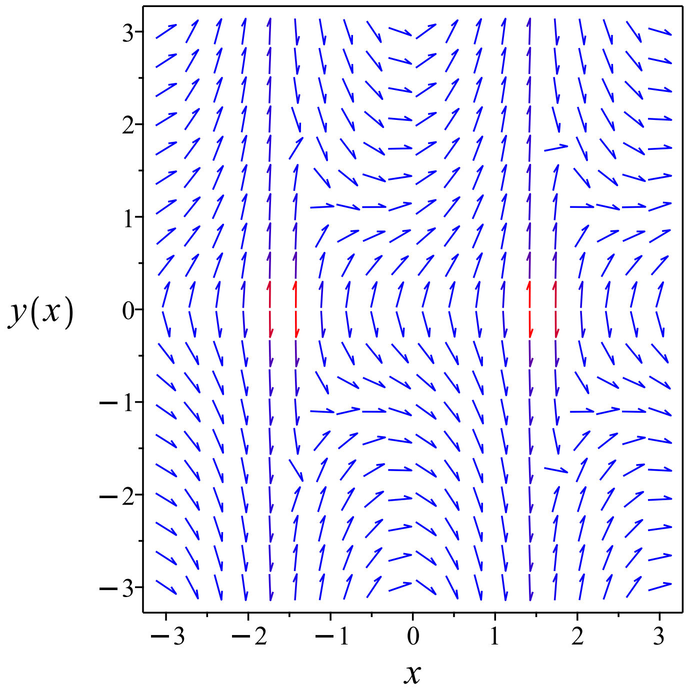


Figure 124: Slope field plot

Verification of solutions

$$\frac{y^2 \cos(x)^2}{2} = \frac{x}{2} + c_1$$

Verified OK.

1.94.2 Solving as bernoulli ode

In canonical form, the ODE is

$$\begin{aligned} y' &= F(x, y) \\ &= \frac{\sin(2x)\, y^2 + 1}{2y \cos(x)^2} \end{aligned}$$

This is a Bernoulli ODE.

$$y' = \frac{\sin(x)}{\cos(x)} y + \frac{1}{2\cos(x)^2} \frac{1}{y} \tag{1}$$

The standard Bernoulli ODE has the form

$$y' = f_0(x) y + f_1(x) y^n \tag{2}$$

The first step is to divide the above equation by y^n which gives

$$\frac{y'}{y^n} = f_0(x)y^{1-n} + f_1(x) \tag{3}$$

The next step is use the substitution $w = y^{1-n}$ in equation (3) which generates a new ODE in $w(x)$ which will be linear and can be easily solved using an integrating factor. Backsubstitution then gives the solution $y(x)$ which is what we want.

This method is now applied to the ODE at hand. Comparing the ODE (1) With (2) Shows that

$$\begin{aligned} f_0(x) &= \frac{\sin(x)}{\cos(x)} \\ f_1(x) &= \frac{1}{2\cos(x)^2} \\ n &= -1 \end{aligned}$$

Dividing both sides of ODE (1) by $y^n = \frac{1}{y}$ gives

$$y'y = \frac{\sin(x)\,y^2}{\cos(x)} + \frac{1}{2\cos(x)^2} \tag{4}$$

Let

$$\begin{aligned} w &= y^{1-n} \\ &= y^2 \end{aligned} \tag{5}$$

Taking derivative of equation (5) w.r.t x gives

$$w' = 2yy' \tag{6}$$

Substituting equations (5) and (6) into equation (4) gives

$$\begin{aligned} \frac{w'(x)}{2} &= \frac{\sin(x)\,w(x)}{\cos(x)} + \frac{1}{2\cos(x)^2} \\ w' &= \frac{2\sin(x)\,w}{\cos(x)} + \frac{1}{\cos(x)^2} \end{aligned} \tag{7}$$

The above now is a linear ODE in $w(x)$ which is now solved.

Entering Linear first order ODE solver. In canonical form a linear first order is

$$w'(x) + p(x)w(x) = q(x)$$

Where here

$$
\begin{aligned}
p(x) &= -2\tan(x) \\
q(x) &= \sec(x)^2
\end{aligned}
$$

Hence the ode is

$$
w'(x) - 2\tan(x)\, w(x) = \sec(x)^2
$$

The integrating factor μ is

$$
\begin{aligned}
\mu &= e^{\int -2\tan(x)dx} \\
&= \cos(x)^2
\end{aligned}
$$

The ode becomes

$$
\begin{aligned}
\frac{d}{dx}(\mu w) &= (\mu)\left(\sec(x)^2\right) \\
\frac{d}{dx}\left(\cos(x)^2 w\right) &= \left(\cos(x)^2\right)\left(\sec(x)^2\right) \\
d\left(\cos(x)^2 w\right) &= dx
\end{aligned}
$$

Integrating gives

$$
\begin{aligned}
\cos(x)^2 w &= \int dx \\
\cos(x)^2 w &= x + c_1
\end{aligned}
$$

Dividing both sides by the integrating factor $\mu = \cos(x)^2$ results in

$$
w(x) = \sec(x)^2 x + c_1 \sec(x)^2
$$

which simplifies to

$$
w(x) = \sec(x)^2 (x + c_1)
$$

Replacing w in the above by y^2 using equation (5) gives the final solution.

$$
y^2 = \sec(x)^2 (x + c_1)
$$

Solving for y gives

$$
\begin{aligned}
y(x) &= \sec(x)\sqrt{x + c_1} \\
y(x) &= -\sec(x)\sqrt{x + c_1}
\end{aligned}
$$

Summary

The solution(s) found are the following

$$y = \sec(x)\sqrt{x + c_1} \tag{1}$$
$$y = -\sec(x)\sqrt{x + c_1} \tag{2}$$

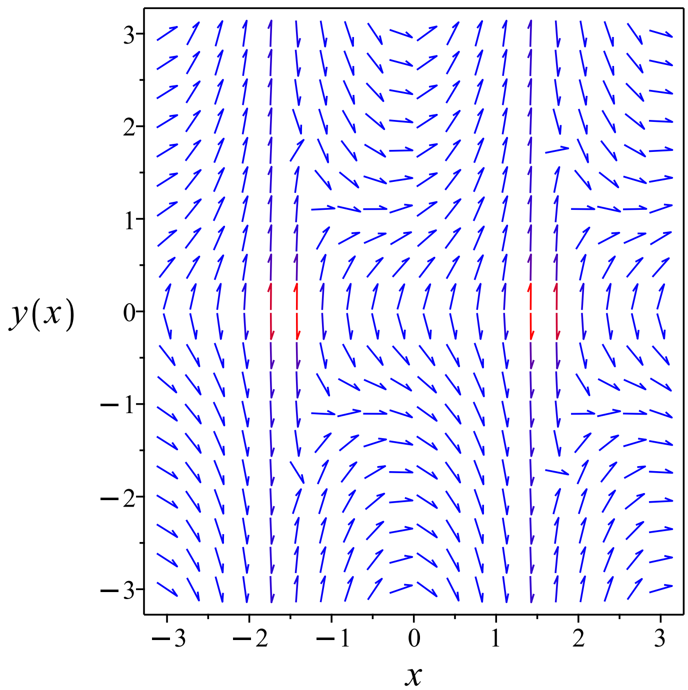


Figure 125: Slope field plot

Verification of solutions

$$y = \sec(x)\sqrt{x + c_1}$$

Verified OK.

$$y = -\sec(x)\sqrt{x + c_1}$$

Verified OK.

1.94.3 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x, y) + N(x, y)\frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x, y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x, y) = 0$$

Hence

$$\frac{\partial \phi}{\partial x} + \frac{\partial \phi}{\partial y}\frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$\begin{aligned}\frac{\partial \phi}{\partial x} &= M \\ \frac{\partial \phi}{\partial y} &= N\end{aligned}$$

But since $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x, y)$ but at least we know now that we can do that since the condition $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x, y)\,\mathrm{d}x + N(x, y)\,\mathrm{d}y = 0 \tag{1A}$$

Therefore

$$\begin{aligned}\left(-2y\cos(x)^2\right)\mathrm{d}y &= \left(-1 - \sin(2x)\,y^2\right)\mathrm{d}x \\ \left(\sin(2x)\,y^2 + 1\right)\mathrm{d}x + \left(-2y\cos(x)^2\right)\mathrm{d}y &= 0 \end{aligned} \tag{2A}$$

Comparing (1A) and (2A) shows that

$$\begin{aligned}M(x, y) &= \sin(2x)\,y^2 + 1 \\ N(x, y) &= -2y\cos(x)^2\end{aligned}$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Using result found above gives

$$\begin{aligned}\frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}\left(\sin(2x)\, y^2 + 1\right)\\ &= 2y\sin(2x)\end{aligned}$$

And

$$\begin{aligned}\frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}\left(-2y\cos(x)^2\right)\\ &= 2y\sin(2x)\end{aligned}$$

Since $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$, then the ODE is exact The following equations are now set up to solve for the function $\phi(x,y)$

$$\frac{\partial \phi}{\partial x} = M \tag{1}$$
$$\frac{\partial \phi}{\partial y} = N \tag{2}$$

Integrating (1) w.r.t. x gives

$$\int \frac{\partial \phi}{\partial x}\, \mathrm{d}x = \int M\, \mathrm{d}x$$

$$\int \frac{\partial \phi}{\partial x}\, \mathrm{d}x = \int \sin(2x)\, y^2 + 1\, \mathrm{d}x$$

$$\phi = x - \frac{\cos(2x)\, y^2}{2} + f(y) \tag{3}$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$\frac{\partial \phi}{\partial y} = -y\cos(2x) + f'(y) \tag{4}$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = -2y\cos(x)^2$. Therefore equation (4) becomes

$$-2y\cos(x)^2 = -y\cos(2x) + f'(y) \tag{5}$$

Solving equation (5) for $f'(y)$ gives

$$
\begin{aligned}
f'(y) &= -2y\cos(x)^2 + y\cos(2x) \\
&= -y
\end{aligned}
$$

Integrating the above w.r.t y results in

$$
\begin{aligned}
\int f'(y)\,\mathrm{d}y &= \int (-y)\,\mathrm{d}y \\
f(y) &= -\frac{y^2}{2} + c_1
\end{aligned}
$$

Where c_1 is constant of integration. Substituting result found above for $f(y)$ into equation (3) gives ϕ

$$
\phi = x - \frac{\cos(2x)\,y^2}{2} - \frac{y^2}{2} + c_1
$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$
c_1 = x - \frac{\cos(2x)\,y^2}{2} - \frac{y^2}{2}
$$

Summary
The solution(s) found are the following

$$
x - \frac{\cos(2x)\,y^2}{2} - \frac{y^2}{2} = c_1 \tag{1}
$$

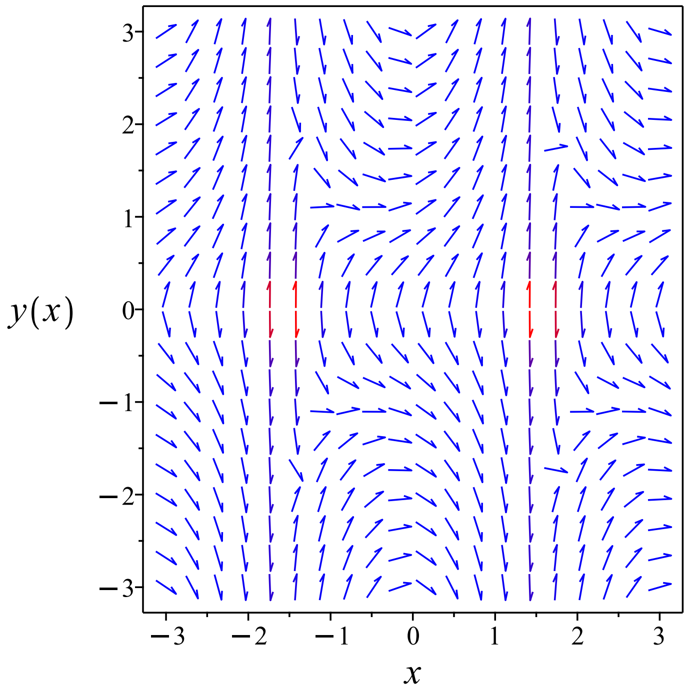


Figure 126: Slope field plot

Verification of solutions

$$x - \frac{\cos(2x)\, y^2}{2} - \frac{y^2}{2} = c_1$$

Verified OK.

1.94.4 Maple step by step solution

Let's solve

$\sin(2x)\, y^2 - 2y\cos(x)^2\, y' = -1$

- Highest derivative means the order of the ODE is 1

  y'

- □ Check if ODE is exact

  - ∘ ODE is exact if the lhs is the total derivative of a C^2 function

    $F'(x,y) = 0$

  - ∘ Compute derivative of lhs

    $F'(x,y) + \left(\frac{\partial}{\partial y} F(x,y)\right) y' = 0$

- Evaluate derivatives

  $2y \sin(2x) = 4y \sin(x) \cos(x)$

- Simplify

  $2y \sin(2x) = 2y \sin(2x)$

- Condition met, ODE is exact

- Exact ODE implies solution will be of this form

  $\left[F(x,y) = c_1, M(x,y) = F'(x,y), N(x,y) = \frac{\partial}{\partial y}F(x,y)\right]$

- Solve for $F(x,y)$ by integrating $M(x,y)$ with respect to x

  $F(x,y) = \int (\sin(2x)\, y^2 + 1)\, dx + f_1(y)$

- Evaluate integral

  $F(x,y) = x - \frac{\cos(2x)y^2}{2} + f_1(y)$

- Take derivative of $F(x,y)$ with respect to y

  $N(x,y) = \frac{\partial}{\partial y}F(x,y)$

- Compute derivative

  $-2y \cos(x)^2 = -y \cos(2x) + \frac{d}{dy}f_1(y)$

- Isolate for $\frac{d}{dy}f_1(y)$

  $\frac{d}{dy}f_1(y) = -2y \cos(x)^2 + y \cos(2x)$

- Solve for $f_1(y)$

  $f_1(y) = -\frac{y^2\left(2\cos(x)^2 - \cos(2x)\right)}{2}$

- Substitute $f_1(y)$ into equation for $F(x,y)$

  $F(x,y) = x - \frac{\cos(2x)y^2}{2} - \frac{y^2\left(2\cos(x)^2 - \cos(2x)\right)}{2}$

- Substitute $F(x,y)$ into the solution of the ODE

  $x - \frac{\cos(2x)y^2}{2} - \frac{y^2\left(2\cos(x)^2 - \cos(2x)\right)}{2} = c_1$

- Solve for y

  $\left\{y = \frac{\sqrt{x-c_1}}{\cos(x)}, y = -\frac{\sqrt{x-c_1}}{\cos(x)}\right\}$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
<- Bernoulli successful`
```

✓ Solution by Maple

Time used: 0.031 (sec). Leaf size: 24

```
dsolve((1+y(x)^2*sin(2*x))-(2*y(x)*cos(x)^2)*diff(y(x),x)=0,y(x), singsol=all)
```

$$y(x) = \sec(x)\sqrt{c_1 + x}$$
$$y(x) = -\sec(x)\sqrt{c_1 + x}$$

✓ Solution by Mathematica

Time used: 0.321 (sec). Leaf size: 32

```
DSolve[(1+y[x]^2*Sin[2*x])-(2*y[x]*Cos[x]^2)*y'[x]==0,y[x],x,IncludeSingularSolutions -> Tru
```

$$y(x) \to -\sqrt{x + c_1}\sec(x)$$
$$y(x) \to \sqrt{x + c_1}\sec(x)$$

1.95 problem 117

Internal problem ID [3240]
Internal file name [`OUTPUT/2732_Sunday_June_05_2022_08_39_51_AM_92742251/index.tex`]

Book: Differential equations for engineers by Wei-Chau XIE, Cambridge Press 2010
Section: Chapter 2. First-Order and Simple Higher-Order Differential Equations. Page 78
Problem number: 117.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"exactWithIntegrationFactor", "first_order_ode_lie_symmetry_calculated"**

Maple gives the following as the ode type

```
[[_homogeneous, `class A`], _dAlembert]
```

$$\boxed{2\sqrt{yx} - y - xy' = 0}$$

1.95.1 Solving as first order ode lie symmetry calculated ode

Writing the ode as

$$\begin{aligned} y' &= -\frac{-2\sqrt{xy} + y}{x} \\ y' &= \omega(x, y) \end{aligned}$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x + \omega(\eta_y - \xi_x) - \omega^2 \xi_y - \omega_x \xi - \omega_y \eta = 0 \tag{A}$$

The type of this ode is not in the lookup table. To determine ξ, η then (A) is solved using ansatz. Making bivariate polynomials of degree 1 to use as anstaz gives

$$\xi = xa_2 + ya_3 + a_1 \tag{1E}$$

$$\eta = xb_2 + yb_3 + b_1 \tag{2E}$$

Where the unknown coefficients are

$$\{a_1, a_2, a_3, b_1, b_2, b_3\}$$

Substituting equations (1E,2E) and ω into (A) gives

$$\begin{aligned} & b_2 - \frac{\left(-2\sqrt{xy}+y\right)\left(b_3-a_2\right)}{x} - \frac{\left(-2\sqrt{xy}+y\right)^2 a_3}{x^2} \\ & - \left(\frac{y}{\sqrt{xy}\,x} + \frac{-2\sqrt{xy}+y}{x^2}\right)\left(xa_2+ya_3+a_1\right) \\ & + \frac{\left(-\frac{x}{\sqrt{xy}}+1\right)\left(xb_2+yb_3+b_1\right)}{x} = 0 \end{aligned} \tag{5E}$$

Putting the above in normal form gives

$$-\frac{4(xy)^{\frac{3}{2}} a_3 - x^2 y b_3 - 5x\,y^2 a_3 - 2b_2\sqrt{xy}\,x^2 + 2\sqrt{xy}\,y^2 a_3 + x^3 b_2 + x^2 y a_2 - xya_1 - \sqrt{xy}\,xb_1 + \sqrt{xy}\,ya_1 +}{\sqrt{xy}\,x^2}$$
$$= 0$$

Setting the numerator to zero gives

$$\begin{aligned} & -4(xy)^{\frac{3}{2}} a_3 + 2b_2\sqrt{xy}\,x^2 - 2\sqrt{xy}\,y^2 a_3 - x^3 b_2 - x^2 y a_2 \\ & + x^2 y b_3 + 5x\,y^2 a_3 + \sqrt{xy}\,xb_1 - \sqrt{xy}\,ya_1 - x^2 b_1 + xya_1 = 0 \end{aligned} \tag{6E}$$

Since the PDE has radicals, simplifying gives

$$\begin{aligned} & -x^3 b_2 + 2b_2\sqrt{xy}\,x^2 - x^2 y a_2 + x^2 y b_3 - 4xy\sqrt{xy}\,a_3 + 5x\,y^2 a_3 \\ & - 2\sqrt{xy}\,y^2 a_3 - x^2 b_1 + \sqrt{xy}\,xb_1 + xya_1 - \sqrt{xy}\,ya_1 = 0 \end{aligned}$$

Looking at the above PDE shows the following are all the terms with $\{x, y\}$ in them.

$$\{x, y, \sqrt{xy}\}$$

The following substitution is now made to be able to collect on all terms with $\{x, y\}$ in them

$$\{x = v_1, y = v_2, \sqrt{xy} = v_3\}$$

The above PDE (6E) now becomes

$$
\begin{aligned}
&-v_1^2 v_2 a_2 + 5 v_1 v_2^2 a_3 - 4 v_1 v_2 v_3 a_3 - 2 v_3 v_2^2 a_3 - v_1^3 b_2 \\
&+ 2 b_2 v_3 v_1^2 + v_1^2 v_2 b_3 + v_1 v_2 a_1 - v_3 v_2 a_1 - v_1^2 b_1 + v_3 v_1 b_1 = 0
\end{aligned} \tag{7E}
$$

Collecting the above on the terms v_i introduced, and these are

$$
\{v_1, v_2, v_3\}
$$

Equation (7E) now becomes

$$
\begin{aligned}
&-v_1^3 b_2 + (b_3 - a_2) v_1^2 v_2 + 2 b_2 v_3 v_1^2 - v_1^2 b_1 + 5 v_1 v_2^2 a_3 \\
&- 4 v_1 v_2 v_3 a_3 + v_1 v_2 a_1 + v_3 v_1 b_1 - 2 v_3 v_2^2 a_3 - v_3 v_2 a_1 = 0
\end{aligned} \tag{8E}
$$

Setting each coefficients in (8E) to zero gives the following equations to solve

$$
\begin{aligned}
a_1 &= 0 \\
b_1 &= 0 \\
-a_1 &= 0 \\
-4a_3 &= 0 \\
-2a_3 &= 0 \\
5a_3 &= 0 \\
-b_1 &= 0 \\
-b_2 &= 0 \\
2b_2 &= 0 \\
b_3 - a_2 &= 0
\end{aligned}
$$

Solving the above equations for the unknowns gives

$$
\begin{aligned}
a_1 &= 0 \\
a_2 &= b_3 \\
a_3 &= 0 \\
b_1 &= 0 \\
b_2 &= 0 \\
b_3 &= b_3
\end{aligned}
$$

Substituting the above solution in the anstaz (1E,2E) (using 1 as arbitrary value for any unknown in the RHS) gives

$$
\begin{aligned}
\xi &= x \\
\eta &= y
\end{aligned}
$$

Shifting is now applied to make $\xi = 0$ in order to simplify the rest of the computation

$$
\begin{aligned}
\eta &= \eta - \omega(x,y)\,\xi \\
&= y - \left(-\frac{-2\sqrt{xy}+y}{x}\right)(x) \\
&= 2y - 2\sqrt{xy} \\
\xi &= 0
\end{aligned}
$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x,y) \to (R,S)$ where (R,S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$
\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}
$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y}\right) S(x,y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$
R = x
$$

S is found from

$$
\begin{aligned}
S &= \int \frac{1}{\eta} dy \\
&= \int \frac{1}{2y - 2\sqrt{xy}} dy
\end{aligned}
$$

Which results in

$$
S = \frac{\ln(y-x)}{2} - \frac{\ln\left(\sqrt{xy}+x\right)}{2} + \frac{\ln\left(\sqrt{xy}-x\right)}{2}
$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$
\frac{dS}{dR} = \frac{S_x + \omega(x,y)S_y}{R_x + \omega(x,y)R_y} \tag{2}
$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x, y)$ is the right hand side of the original ode given by

$$\omega(x, y) = -\frac{-2\sqrt{xy} + y}{x}$$

Evaluating all the partial derivatives gives

$$\begin{aligned}
R_x &= 1 \\
R_y &= 0 \\
S_x &= \frac{\sqrt{x} + \sqrt{y}}{\sqrt{x}\,(-2y + 2x)} \\
S_y &= -\frac{\sqrt{x} + \sqrt{y}}{\sqrt{y}\,(-2y + 2x)}
\end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = \frac{\left(\sqrt{x} + \sqrt{y}\right)\left(\sqrt{y}\,x - 2\sqrt{x}\,\sqrt{xy} + \sqrt{x}\,y\right)}{x^{\frac{3}{2}}\sqrt{y}\,(-2y + 2x)} \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = -\frac{1}{2R}$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$S(R) = -\frac{\ln(R)}{2} + c_1 \tag{4}$$

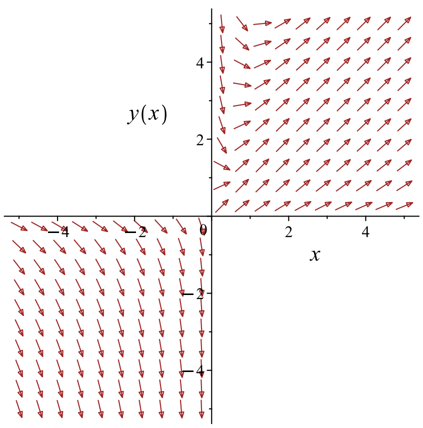
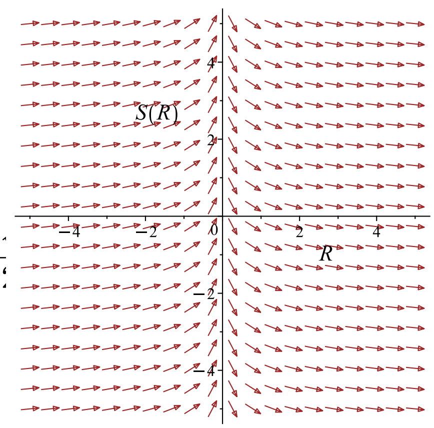
To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$\frac{\ln(y - x)}{2} - \frac{\ln\left(\sqrt{x}\,\sqrt{y} + x\right)}{2} + \frac{\ln\left(\sqrt{x}\,\sqrt{y} - x\right)}{2} = -\frac{\ln(x)}{2} + c_1$$

Which simplifies to

$$\frac{\ln(y - x)}{2} - \frac{\ln\left(\sqrt{x}\,\sqrt{y} + x\right)}{2} + \frac{\ln\left(\sqrt{x}\,\sqrt{y} - x\right)}{2} = -\frac{\ln(x)}{2} + c_1$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in x, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dx} = -\frac{-2\sqrt{xy}+y}{x}$ 	$R = x$ $S = \frac{\ln (y - x)}{2} - \frac{\ln \left(\sqrt{x} \right.}{2}$	$\frac{dS}{dR} = -\frac{1}{2R}$ 

Summary
The solution(s) found are the following

$$\frac{\ln (y - x)}{2} - \frac{\ln \left(\sqrt{x}\,\sqrt{y} + x\right)}{2} + \frac{\ln \left(\sqrt{x}\,\sqrt{y} - x\right)}{2} = -\frac{\ln (x)}{2} + c_1 \tag{1}$$

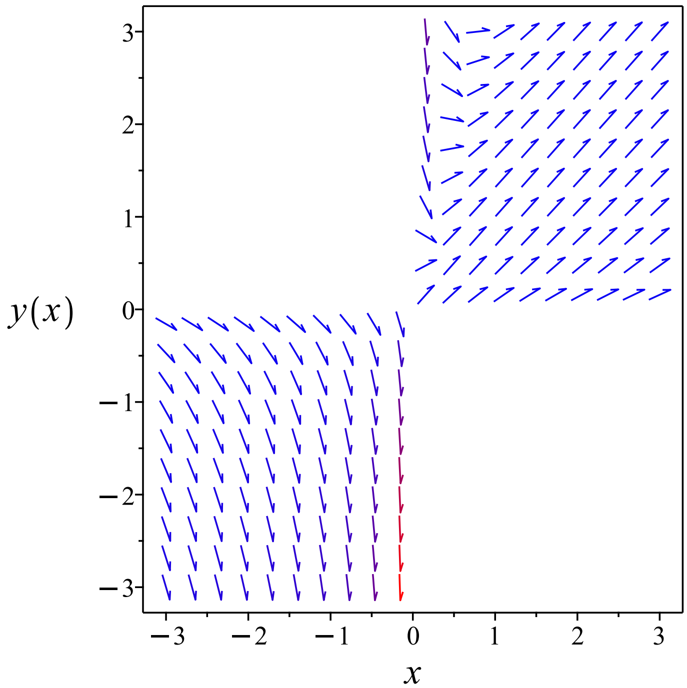


Figure 127: Slope field plot

Verification of solutions

$$\frac{\ln \left(y-x\right)}{2}-\frac{\ln \left(\sqrt{x}\, \sqrt{y}+x\right)}{2}+\frac{\ln \left(\sqrt{x}\, \sqrt{y}-x\right)}{2} = -\frac{\ln \left(x\right)}{2}+c_{1}$$

Verified OK.

1.95.2 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x,y)+N(x,y)\frac{dy}{dx}=0 \tag{A}$$

We assume there exists a function $\phi(x,y)=c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y)=0$$

Hence

$$\frac{\partial \phi}{\partial x}+\frac{\partial \phi}{\partial y}\frac{dy}{dx}=0 \tag{B}$$

Comparing (A,B) shows that

$$\frac{\partial \phi}{\partial x} = M$$
$$\frac{\partial \phi}{\partial y} = N$$

But since $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x, y)$ but at least we know now that we can do that since the condition $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x, y)\, \mathrm{d}x + N(x, y)\, \mathrm{d}y = 0 \tag{1A}$$

Therefore

$$(-x)\, \mathrm{d}y = (-2\sqrt{xy} + y)\, \mathrm{d}x$$
$$(2\sqrt{xy} - y)\, \mathrm{d}x + (-x)\, \mathrm{d}y = 0 \tag{2A}$$

Comparing (1A) and (2A) shows that

$$M(x, y) = 2\sqrt{xy} - y$$
$$N(x, y) = -x$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Using result found above gives

$$\begin{aligned}\frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}(2\sqrt{xy} - y)\\ &= \frac{x}{\sqrt{xy}} - 1\end{aligned}$$

And

$$\begin{aligned}\frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}(-x)\\ &= -1\end{aligned}$$

Since $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$, then the ODE is not exact. Since the ODE is not exact, we will try to find an integrating factor to make it exact. Let

$$\begin{aligned}
A &= \frac{1}{N}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right) \\
&= -\frac{1}{x}\left(\left(\frac{x}{\sqrt{xy}} - 1\right) - (-1)\right) \\
&= -\frac{1}{\sqrt{xy}}
\end{aligned}$$

Since A depends on y, it can not be used to obtain an integrating factor. We will now try a second method to find an integrating factor. Let

$$\begin{aligned}
B &= \frac{1}{M}\left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) \\
&= \frac{1}{2\sqrt{xy} - y}\left((-1) - \left(\frac{x}{\sqrt{xy}} - 1\right)\right) \\
&= \frac{x}{\left(-2\sqrt{xy} + y\right)\sqrt{xy}}
\end{aligned}$$

Since B depends on x, it can not be used to obtain an integrating factor.We will now try a third method to find an integrating factor. Let

$$R = \frac{\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}}{xM - yN}$$

R is now checked to see if it is a function of only $t = xy$. Therefore

$$\begin{aligned}
R &= \frac{\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}}{xM - yN} \\
&= \frac{(-1) - \left(\frac{x}{\sqrt{xy}} - 1\right)}{x\left(2\sqrt{xy} - y\right) - y\left(-x\right)} \\
&= -\frac{1}{2xy}
\end{aligned}$$

Replacing all powers of terms xy by t gives

$$R = -\frac{1}{2t}$$

Since R depends on t only, then it can be used to find an integrating factor. Let the integrating factor be μ then

$$\begin{aligned}
\mu &= e^{\int R \,\mathrm{d}t} \\
&= e^{\int \left(-\frac{1}{2t}\right) \mathrm{d}t}
\end{aligned}$$

The result of integrating gives

$$\begin{aligned}\mu &= e^{-\frac{\ln(t)}{2}} \\ &= \frac{1}{\sqrt{t}}\end{aligned}$$

Now t is replaced back with xy giving

$$\mu = \frac{1}{\sqrt{xy}}$$

Multiplying M and N by this integrating factor gives new M and new N which are called $\overline{M}$ and $\overline{N}$ so not to confuse them with the original M and N

$$\begin{aligned}\overline{M} &= \mu M \\ &= \frac{1}{\sqrt{xy}}(2\sqrt{xy} - y) \\ &= -\frac{-2\sqrt{xy} + y}{\sqrt{xy}}\end{aligned}$$

And

$$\begin{aligned}\overline{N} &= \mu N \\ &= \frac{1}{\sqrt{xy}}(-x) \\ &= -\frac{x}{\sqrt{xy}}\end{aligned}$$

A modified ODE is now obtained from the original ODE, which is exact and can solved. The modified ODE is

$$\begin{aligned}\overline{M} + \overline{N}\frac{\mathrm{d}y}{\mathrm{d}x} &= 0 \\ \left(-\frac{-2\sqrt{xy} + y}{\sqrt{xy}}\right) + \left(-\frac{x}{\sqrt{xy}}\right)\frac{\mathrm{d}y}{\mathrm{d}x} &= 0\end{aligned}$$

The following equations are now set up to solve for the function $\phi(x, y)$

$$\frac{\partial\phi}{\partial x} = \overline{M} \tag{1}$$

$$\frac{\partial\phi}{\partial y} = \overline{N} \tag{2}$$

Integrating (1) w.r.t. x gives

$$\int \frac{\partial \phi}{\partial x} \mathrm{d}x = \int \overline{M} \mathrm{d}x$$

$$\int \frac{\partial \phi}{\partial x} \mathrm{d}x = \int -\frac{-2\sqrt{xy} + y}{\sqrt{xy}} \mathrm{d}x$$

$$\phi = 2x - 2\sqrt{xy} + f(y) \tag{3}$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$\frac{\partial \phi}{\partial y} = -\frac{x}{\sqrt{xy}} + f'(y) \tag{4}$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = -\frac{x}{\sqrt{xy}}$. Therefore equation (4) becomes

$$-\frac{x}{\sqrt{xy}} = -\frac{x}{\sqrt{xy}} + f'(y) \tag{5}$$

Solving equation (5) for $f'(y)$ gives

$$f'(y) = 0$$

Therefore

$$f(y) = c_1$$

Where c_1 is constant of integration. Substituting this result for $f(y)$ into equation (3) gives ϕ

$$\phi = 2x - 2\sqrt{xy} + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = 2x - 2\sqrt{xy}$$

The solution becomes

$$y = \frac{c_1^2 - 4c_1 x + 4x^2}{4x}$$

Summary

The solution(s) found are the following

$$y = \frac{c_1^2 - 4c_1 x + 4x^2}{4x} \tag{1}$$

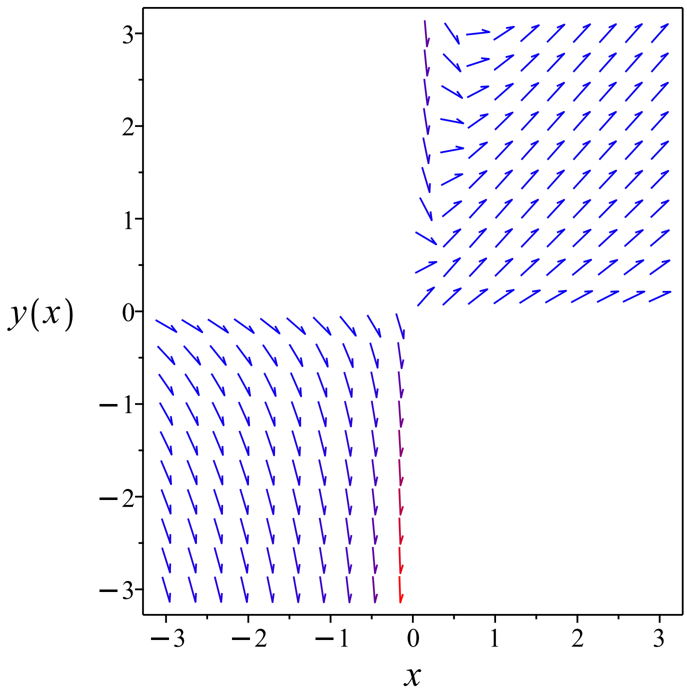


Figure 128: Slope field plot

Verification of solutions

$$y = \frac{c_1^2 - 4c_1 x + 4x^2}{4x}$$

Verified OK.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying homogeneous types:
trying homogeneous G
1st order, trying the canonical coordinates of the invariance group
<- 1st order, canonical coordinates successful
<- homogeneous successful`
```

✓ Solution by Maple

Time used: 0.015 (sec). Leaf size: 71

```
dsolve((2*sqrt(x*y(x))-y(x))-x*diff(y(x),x)=0,y(x), singsol=all)
```

$$\frac{x^{2}c_{1}y(x)-y(x)\sqrt{xy\left(x\right)}\,c_{1}x-c_{1}x^{3}+\sqrt{xy\left(x\right)}\,c_{1}x^{2}+x+\sqrt{xy\left(x\right)}}{\left(-x+y\left(x\right)\right)\left(\sqrt{xy\left(x\right)}-x\right)x}=0$$

✓ Solution by Mathematica

Time used: 0.209 (sec). Leaf size: 26

```
DSolve[(2*Sqrt[x*y[x]]-y[x])-x*y'[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x)\to\frac{\left(x+e^{\frac{c_1}{2}}\right)^2}{x}$$
$$y(x)\to x$$

1.96 problem 119

Internal problem ID [3241]
Internal file name [OUTPUT/2733_Sunday_June_05_2022_08_39_52_AM_49446695/index.tex]

Book: Differential equations for engineers by Wei-Chau XIE, Cambridge Press 2010
Section: Chapter 2. First-Order and Simple Higher-Order Differential Equations. Page 78
Problem number: 119.
ODE order: 1.
ODE degree: 0.

The type(s) of ODE detected by this program : **"dAlembert", "homogeneousTypeD2"**

Maple gives the following as the ode type

```
[[_homogeneous, `class A`], _dAlembert]
```

$$\boxed{y' = \mathrm{e}^{\frac{xy'}{y}}}$$

1.96.1 Solving as homogeneousTypeD2 ode

Using the change of variables $y = u(x)\,x$ on the above ode results in new ode in $u(x)$

$$u'(x)\,x + u(x) = \mathrm{e}^{\frac{\frac{d}{dx}(u(x)x)}{u(x)}}$$

In canonical form the ODE is

$$\begin{aligned}
u' &= F(x,u)\\
&= f(x)g(u)\\
&= \frac{-\operatorname{LambertW}\left(-\frac{1}{u}\right)u - u}{x}
\end{aligned}$$

Where $f(x) = \frac{1}{x}$ and $g(u) = -\operatorname{LambertW}\left(-\frac{1}{u}\right)u - u$. Integrating both sides gives

$$\begin{aligned}
\frac{1}{-\operatorname{LambertW}\left(-\frac{1}{u}\right)u - u}\,du &= \frac{1}{x}\,dx\\
\int \frac{1}{-\operatorname{LambertW}\left(-\frac{1}{u}\right)u - u}\,du &= \int \frac{1}{x}\,dx\\
\ln\left(\operatorname{LambertW}\left(-\frac{1}{u}\right)\right) &= \ln(x) + c_2
\end{aligned}$$

Raising both side to exponential gives

$$\text{LambertW}\left(-\frac{1}{u}\right) = \mathrm{e}^{\ln(x)+c_2}$$

Which simplifies to

$$\text{LambertW}\left(-\frac{1}{u}\right) = c_3 x$$

Therefore the solution y is

$$\begin{aligned} y &= xu \\ &= -\frac{\mathrm{e}^{-c_2}\mathrm{e}^{-c_3 \mathrm{e}^{c_2} x}}{c_3} \end{aligned}$$

<u>Summary</u>
The solution(s) found are the following

$$y = -\frac{\mathrm{e}^{-c_2}\mathrm{e}^{-c_3 \mathrm{e}^{c_2} x}}{c_3} \tag{1}$$

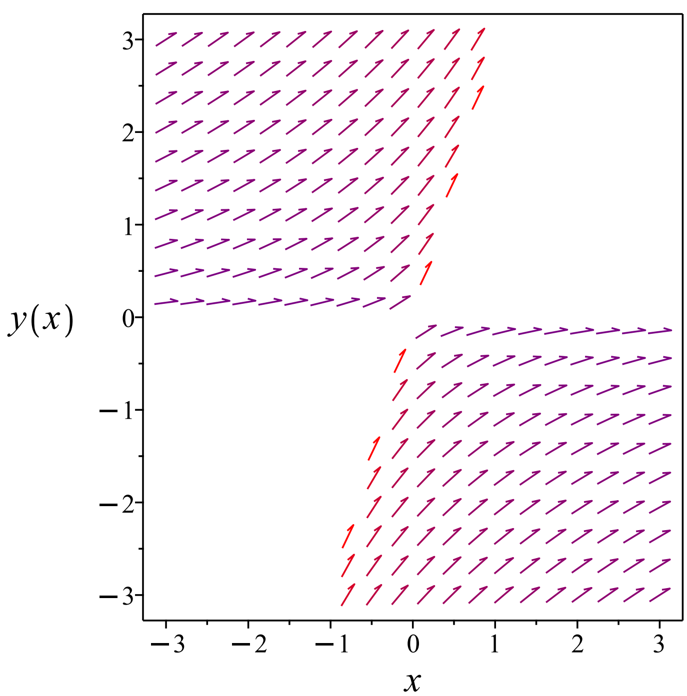


Figure 129: Slope field plot

Verification of solutions

$$y = -\frac{e^{-c_2} e^{-c_3 e^{c_2} x}}{c_3}$$

Verified OK.

1.96.2 Solving as dAlembert ode

Let $p = y'$ the ode becomes

$$p = e^{\frac{xp}{y}}$$

Solving for y from the above results in

$$y = \frac{xp}{\ln(p)} \tag{1A}$$

This has the form

$$y = xf(p) + g(p) \tag{*}$$

Where f, g are functions of $p = y'(x)$. The above ode is dAlembert ode which is now solved. Taking derivative of (*) w.r.t. x gives

$$\begin{aligned} p &= f + (xf' + g')\frac{dp}{dx} \\ p - f &= (xf' + g')\frac{dp}{dx} \end{aligned} \tag{2}$$

Comparing the form $y = xf + g$ to (1A) shows that

$$\begin{aligned} f &= \frac{p}{\ln(p)} \\ g &= 0 \end{aligned}$$

Hence (2) becomes

$$p - \frac{p}{\ln(p)} = x\left(\frac{1}{\ln(p)} - \frac{1}{\ln(p)^2}\right) p'(x) \tag{2A}$$

The singular solution is found by setting $\frac{dp}{dx} = 0$ in the above which gives

$$p - \frac{p}{\ln(p)} = 0$$

Solving for p from the above gives

$$p = e$$

Substituting these in (1A) gives

$$y = x\,\mathrm{e}$$

The general solution is found when $\frac{\mathrm{d}p}{\mathrm{d}x} \neq 0$. From eq. (2A). This results in

$$p'(x) = \frac{p(x) - \frac{p(x)}{\ln(p(x))}}{x\left(\frac{1}{\ln(p(x))} - \frac{1}{\ln(p(x))^2}\right)} \tag{3}$$

This ODE is now solved for $p(x)$. In canonical form the ODE is

$$\begin{aligned} p' &= F(x,p) \\ &= f(x)g(p) \\ &= \frac{\ln(p)\,p}{x} \end{aligned}$$

Where $f(x) = \frac{1}{x}$ and $g(p) = \ln(p)\,p$. Integrating both sides gives

$$\begin{aligned} \frac{1}{\ln(p)\,p}\,dp &= \frac{1}{x}\,dx \\ \int \frac{1}{\ln(p)\,p}\,dp &= \int \frac{1}{x}\,dx \\ \ln(\ln(p)) &= \ln(x) + c_1 \end{aligned}$$

Raising both side to exponential gives

$$\ln(p) = \mathrm{e}^{\ln(x)+c_1}$$

Which simplifies to

$$\ln(p) = c_2 x$$

Substituing the above solution for p in (2A) gives

$$y = \frac{x\,\mathrm{e}^{c_2 \mathrm{e}^{c_1} x}}{\ln\left(\mathrm{e}^{c_2 \mathrm{e}^{c_1} x}\right)}$$

<u>Summary</u>

The solution(s) found are the following

$$y = x\,\mathrm{e} \tag{1}$$

$$y = \frac{x\,\mathrm{e}^{c_2 \mathrm{e}^{c_1} x}}{\ln\left(\mathrm{e}^{c_2 \mathrm{e}^{c_1} x}\right)} \tag{2}$$

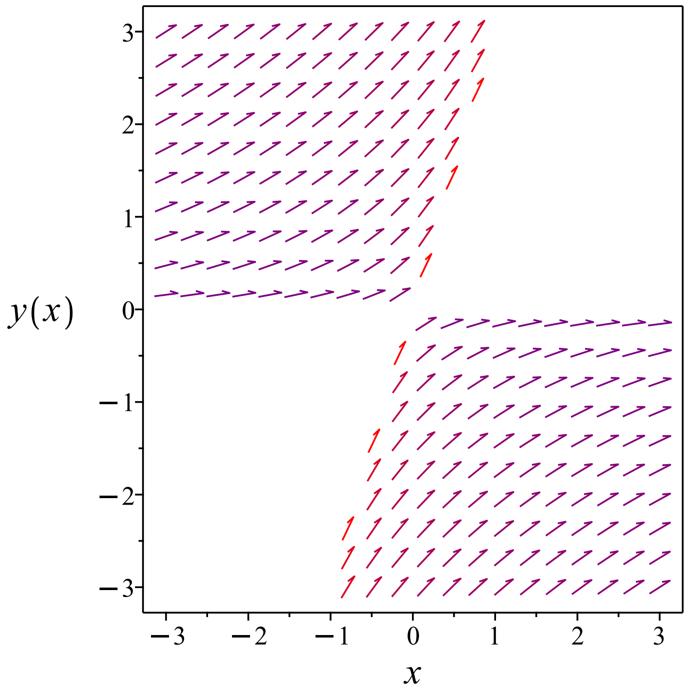


Figure 130: Slope field plot

Verification of solutions

$$y = x\,\mathrm{e}$$

Verified OK.

$$y = \frac{x\,\mathrm{e}^{c_2 \mathrm{e}^{c_1} x}}{\ln\left(\mathrm{e}^{c_2 \mathrm{e}^{c_1} x}\right)}$$

Verified OK.

Maple trace

```
`Methods for first order ODEs:
-> Solving 1st order ODE of high degree, 1st attempt
trying 1st order WeierstrassP solution for high degree ODE
trying 1st order WeierstrassPPrime solution for high degree ODE
trying 1st order JacobiSN solution for high degree ODE
trying 1st order ODE linearizable_by_differentiation
trying differential order: 1; missing variables
trying homogeneous B
trying homogeneous types:
trying homogeneous D
<- homogeneous successful`
```

✓ Solution by Maple
Time used: 0.047 (sec). Leaf size: 14

```
dsolve(diff(y(x),x)=exp(x*diff(y(x),x)/y(x)),y(x), singsol=all)
```

$$y(x) = -\frac{e^{-c_1 x}}{c_1}$$

✓ Solution by Mathematica
Time used: 0.078 (sec). Leaf size: 21

```
DSolve[y'[x]==Exp[x*y'[x]/y[x]],y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to -e^{c_1 - e^{-c_1} x}$$

2 Chapter 4. Linear Differential Equations. Page 183

2.1 problem 1

Internal problem ID [3242]
Internal file name [`OUTPUT/2734_Sunday_June_05_2022_08_39_53_AM_96870353/index.tex`]

Book: Differential equations for engineers by Wei-Chau XIE, Cambridge Press 2010
Section: Chapter 4. Linear Differential Equations. Page 183
Problem number: 1.
ODE order: 3.
ODE degree: 1.

The type(s) of ODE detected by this program : **"higher_order_linear_constant_coefficients_ODE"**

Maple gives the following as the ode type

```
[[_3rd_order, _missing_x]]
```

$$\boxed{y''' - 2y'' + y' - 2y = 0}$$

The characteristic equation is

$$\lambda^3 - 2\lambda^2 + \lambda - 2 = 0$$

The roots of the above equation are

$$\lambda_1 = 2$$
$$\lambda_2 = i$$
$$\lambda_3 = -i$$

Therefore the homogeneous solution is

$$y_h(x) = \mathrm{e}^{2x}c_1 + \mathrm{e}^{ix}c_2 + \mathrm{e}^{-ix}c_3$$

The fundamental set of solutions for the homogeneous solution are the following

$$y_1 = \mathrm{e}^{2x}$$
$$y_2 = \mathrm{e}^{ix}$$
$$y_3 = \mathrm{e}^{-ix}$$

Summary
The solution(s) found are the following

$$y = e^{2x}c_1 + e^{ix}c_2 + e^{-ix}c_3 \tag{1}$$

Verification of solutions

$$y = e^{2x}c_1 + e^{ix}c_2 + e^{-ix}c_3$$

Verified OK.

2.1.1 Maple step by step solution

Let's solve

$y''' - 2y'' + y' - 2y = 0$

- Highest derivative means the order of the ODE is 3

  y'''

□ Convert linear ODE into a system of first order ODEs

  ◦ Define new variable $y_1(x)$

    $y_1(x) = y$

  ◦ Define new variable $y_2(x)$

    $y_2(x) = y'$

  ◦ Define new variable $y_3(x)$

    $y_3(x) = y''$

  ◦ Isolate for $y_3'(x)$ using original ODE

    $y_3'(x) = 2y_3(x) - y_2(x) + 2y_1(x)$

  Convert linear ODE into a system of first order ODEs

  $[y_2(x) = y_1'(x), y_3(x) = y_2'(x), y_3'(x) = 2y_3(x) - y_2(x) + 2y_1(x)]$

- Define vector

  $\vec{y}(x) = \begin{bmatrix} y_1(x) \\ y_2(x) \\ y_3(x) \end{bmatrix}$

- System to solve

$$\overrightarrow{y}'(x) = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 2 & -1 & 2 \end{bmatrix} \cdot \overrightarrow{y}(x)$$

- Define the coefficient matrix

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 2 & -1 & 2 \end{bmatrix}$$

- Rewrite the system as

$$\overrightarrow{y}'(x) = A \cdot \overrightarrow{y}(x)$$

- To solve the system, find the eigenvalues and eigenvectors of A
- Eigenpairs of A

$$\left[\left[2, \begin{bmatrix} \frac{1}{4} \\ \frac{1}{2} \\ 1 \end{bmatrix}\right], \left[-\mathrm{I}, \begin{bmatrix} -1 \\ \mathrm{I} \\ 1 \end{bmatrix}\right], \left[\mathrm{I}, \begin{bmatrix} -1 \\ -\mathrm{I} \\ 1 \end{bmatrix}\right]\right]$$

- Consider eigenpair

$$\left[2, \begin{bmatrix} \frac{1}{4} \\ \frac{1}{2} \\ 1 \end{bmatrix}\right]$$

- Solution to homogeneous system from eigenpair

$$\overrightarrow{y}_1 = \mathrm{e}^{2x} \cdot \begin{bmatrix} \frac{1}{4} \\ \frac{1}{2} \\ 1 \end{bmatrix}$$

- Consider complex eigenpair, complex conjugate eigenvalue can be ignored

$$\left[-\mathrm{I}, \begin{bmatrix} -1 \\ \mathrm{I} \\ 1 \end{bmatrix}\right]$$

- Solution from eigenpair

$$e^{-Ix} \cdot \begin{bmatrix} -1 \\ I \\ 1 \end{bmatrix}$$

- Use Euler identity to write solution in terms of sin and cos

$$(\cos(x) - I\sin(x)) \cdot \begin{bmatrix} -1 \\ I \\ 1 \end{bmatrix}$$

- Simplify expression

$$\begin{bmatrix} -\cos(x) + I\sin(x) \\ I(\cos(x) - I\sin(x)) \\ \cos(x) - I\sin(x) \end{bmatrix}$$

- Both real and imaginary parts are solutions to the homogeneous system

$$\left[\vec{y}_2(x) = \begin{bmatrix} -\cos(x) \\ \sin(x) \\ \cos(x) \end{bmatrix}, \vec{y}_3(x) = \begin{bmatrix} \sin(x) \\ \cos(x) \\ -\sin(x) \end{bmatrix}\right]$$

- General solution to the system of ODEs

$\vec{y} = c_1\vec{y}_1 + c_2\vec{y}_2(x) + c_3\vec{y}_3(x)$

- Substitute solutions into the general solution

$$\vec{y} = e^{2x}c_1 \cdot \begin{bmatrix} \frac{1}{4} \\ \frac{1}{2} \\ 1 \end{bmatrix} + \begin{bmatrix} -c_2\cos(x) + c_3\sin(x) \\ c_2\sin(x) + c_3\cos(x) \\ c_2\cos(x) - c_3\sin(x) \end{bmatrix}$$

- First component of the vector is the solution to the ODE

$y = \frac{e^{2x}c_1}{4} + c_3\sin(x) - c_2\cos(x)$

Maple trace

```
`Methods for third order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
<- constant coefficients successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 19

```
dsolve(diff(y(x),x$3)-2*diff(y(x),x$2)+diff(y(x),x)-2*y(x)=0,y(x), singsol=all)
```

$$y(x) = c_1 e^{2x} + \sin(x) c_2 + \cos(x) c_3$$

✓ Solution by Mathematica

Time used: 0.006 (sec). Leaf size: 24

```
DSolve[y'''[x]-2*y''[x]+y'[x]-2*y[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to c_3 e^{2x} + c_1 \cos(x) + c_2 \sin(x)$$

2.2 problem 2

Internal problem ID [3243]
Internal file name [OUTPUT/2735_Sunday_June_05_2022_08_39_53_AM_88550952/index.tex]

Book: Differential equations for engineers by Wei-Chau XIE, Cambridge Press 2010
Section: Chapter 4. Linear Differential Equations. Page 183
Problem number: 2.
ODE order: 3.
ODE degree: 1.

The type(s) of ODE detected by this program : **"higher_order_linear_constant_coefficients_ODE"**

Maple gives the following as the ode type

```
[[_3rd_order, _missing_x]]
```

$$\boxed{y''' + y'' + 9y' + 9y = 0}$$

The characteristic equation is

$$\lambda^3 + \lambda^2 + 9\lambda + 9 = 0$$

The roots of the above equation are

$$\lambda_1 = -1$$
$$\lambda_2 = 3i$$
$$\lambda_3 = -3i$$

Therefore the homogeneous solution is

$$y_h(x) = c_1 \mathrm{e}^{-x} + \mathrm{e}^{-3ix} c_2 + \mathrm{e}^{3ix} c_3$$

The fundamental set of solutions for the homogeneous solution are the following

$$y_1 = \mathrm{e}^{-x}$$
$$y_2 = \mathrm{e}^{-3ix}$$
$$y_3 = \mathrm{e}^{3ix}$$

Summary
The solution(s) found are the following

$$y = c_1 e^{-x} + e^{-3ix} c_2 + e^{3ix} c_3 \tag{1}$$

Verification of solutions

$$y = c_1 e^{-x} + e^{-3ix} c_2 + e^{3ix} c_3$$

Verified OK.

2.2.1 Maple step by step solution

Let's solve

$y''' + y'' + 9y' + 9y = 0$

- Highest derivative means the order of the ODE is 3

  y'''

□ Convert linear ODE into a system of first order ODEs

  ◦ Define new variable $y_1(x)$

    $y_1(x) = y$

  ◦ Define new variable $y_2(x)$

    $y_2(x) = y'$

  ◦ Define new variable $y_3(x)$

    $y_3(x) = y''$

  ◦ Isolate for $y_3'(x)$ using original ODE

    $y_3'(x) = -y_3(x) - 9y_2(x) - 9y_1(x)$

  Convert linear ODE into a system of first order ODEs

  $[y_2(x) = y_1'(x), y_3(x) = y_2'(x), y_3'(x) = -y_3(x) - 9y_2(x) - 9y_1(x)]$

- Define vector

  $$\vec{y}(x) = \begin{bmatrix} y_1(x) \\ y_2(x) \\ y_3(x) \end{bmatrix}$$

- System to solve

$$\vec{y}'(x) = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -9 & -9 & -1 \end{bmatrix} \cdot \vec{y}(x)$$

- Define the coefficient matrix

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -9 & -9 & -1 \end{bmatrix}$$

- Rewrite the system as

$$\vec{y}'(x) = A \cdot \vec{y}(x)$$

- To solve the system, find the eigenvalues and eigenvectors of A
- Eigenpairs of A

$$\left[\left[-1, \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}\right], \left[-3\,\mathrm{I}, \begin{bmatrix} -\frac{1}{9} \\ \frac{\mathrm{I}}{3} \\ 1 \end{bmatrix}\right], \left[3\,\mathrm{I}, \begin{bmatrix} -\frac{1}{9} \\ -\frac{\mathrm{I}}{3} \\ 1 \end{bmatrix}\right]\right]$$

- Consider eigenpair

$$\left[-1, \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}\right]$$

- Solution to homogeneous system from eigenpair

$$\vec{y}_1 = \mathrm{e}^{-x} \cdot \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$$

- Consider complex eigenpair, complex conjugate eigenvalue can be ignored

$$\left[-3\,\mathrm{I}, \begin{bmatrix} -\frac{1}{9} \\ \frac{\mathrm{I}}{3} \\ 1 \end{bmatrix}\right]$$

- Solution from eigenpair

$$e^{-3\,\mathrm{I}x}\cdot\begin{bmatrix}-\frac{1}{9}\\ \frac{\mathrm{I}}{3}\\ 1\end{bmatrix}$$

- Use Euler identity to write solution in terms of sin and cos

$$(\cos(3x)-\mathrm{I}\sin(3x))\cdot\begin{bmatrix}-\frac{1}{9}\\ \frac{\mathrm{I}}{3}\\ 1\end{bmatrix}$$

- Simplify expression

$$\begin{bmatrix}-\frac{\cos(3x)}{9}+\frac{\mathrm{I}\sin(3x)}{9}\\ \frac{\mathrm{I}}{3}(\cos(3x)-\mathrm{I}\sin(3x))\\ \cos(3x)-\mathrm{I}\sin(3x)\end{bmatrix}$$

- Both real and imaginary parts are solutions to the homogeneous system

$$\left[\vec{y}_2(x)=\begin{bmatrix}-\frac{\cos(3x)}{9}\\ \frac{\sin(3x)}{3}\\ \cos(3x)\end{bmatrix},\vec{y}_3(x)=\begin{bmatrix}\frac{\sin(3x)}{9}\\ \frac{\cos(3x)}{3}\\ -\sin(3x)\end{bmatrix}\right]$$

- General solution to the system of ODEs

$$\vec{y}=c_1\vec{y}_1+c_2\vec{y}_2(x)+c_3\vec{y}_3(x)$$

- Substitute solutions into the general solution

$$\vec{y}=c_1e^{-x}\cdot\begin{bmatrix}1\\ -1\\ 1\end{bmatrix}+\begin{bmatrix}-\frac{c_2\cos(3x)}{9}+\frac{c_3\sin(3x)}{9}\\ \frac{c_2\sin(3x)}{3}+\frac{c_3\cos(3x)}{3}\\ c_2\cos(3x)-c_3\sin(3x)\end{bmatrix}$$

- First component of the vector is the solution to the ODE

$$y=c_1e^{-x}+\frac{c_3\sin(3x)}{9}-\frac{c_2\cos(3x)}{9}$$

Maple trace

```
`Methods for third order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
<- constant coefficients successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 23

```
dsolve(diff(y(x),x$3)+diff(y(x),x$2)+9*diff(y(x),x)+9*y(x)=0,y(x), singsol=all)
```

$$y(x) = e^{-x}c_1 + \sin(3x)\, c_2 + c_3 \cos(3x)$$

✓ Solution by Mathematica

Time used: 0.003 (sec). Leaf size: 28

```
DSolve[y'''[x]+y''[x]+9*y'[x]+9*y[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to c_3 e^{-x} + c_1 \cos(3x) + c_2 \sin(3x)$$

2.3 problem 3

Internal problem ID [3244]
Internal file name [`OUTPUT/2736_Sunday_June_05_2022_08_39_53_AM_94318382/index.tex`]

Book: Differential equations for engineers by Wei-Chau XIE, Cambridge Press 2010
Section: Chapter 4. Linear Differential Equations. Page 183
Problem number: 3.
ODE order: 3.
ODE degree: 1.

The type(s) of ODE detected by this program : **"higher_order_linear_constant_coefficients_ODE"**

Maple gives the following as the ode type

```
[[_3rd_order, _missing_x]]
```

$$\boxed{y''' + y'' - y' - y = 0}$$

The characteristic equation is

$$\lambda^3 + \lambda^2 - \lambda - 1 = 0$$

The roots of the above equation are

$$\lambda_1 = 1$$
$$\lambda_2 = -1$$
$$\lambda_3 = -1$$

Therefore the homogeneous solution is

$$y_h(x) = c_1 \mathrm{e}^{-x} + c_2 x \,\mathrm{e}^{-x} + c_3 \mathrm{e}^{x}$$

The fundamental set of solutions for the homogeneous solution are the following

$$y_1 = \mathrm{e}^{-x}$$
$$y_2 = x \,\mathrm{e}^{-x}$$
$$y_3 = \mathrm{e}^{x}$$

Summary
The solution(s) found are the following

$$y = c_1 e^{-x} + c_2 x\, e^{-x} + c_3 e^{x} \tag{1}$$

Verification of solutions

$$y = c_1 e^{-x} + c_2 x\, e^{-x} + c_3 e^{x}$$

Verified OK.

2.3.1 Maple step by step solution

Let's solve

$y''' + y'' - y' - y = 0$

- Highest derivative means the order of the ODE is 3

  y'''

□ Convert linear ODE into a system of first order ODEs

  ◦ Define new variable $y_1(x)$

    $y_1(x) = y$

  ◦ Define new variable $y_2(x)$

    $y_2(x) = y'$

  ◦ Define new variable $y_3(x)$

    $y_3(x) = y''$

  ◦ Isolate for $y_3'(x)$ using original ODE

    $y_3'(x) = -y_3(x) + y_2(x) + y_1(x)$

  Convert linear ODE into a system of first order ODEs

  $[y_2(x) = y_1'(x), y_3(x) = y_2'(x), y_3'(x) = -y_3(x) + y_2(x) + y_1(x)]$

- Define vector

  $\vec{y}(x) = \begin{bmatrix} y_1(x) \\ y_2(x) \\ y_3(x) \end{bmatrix}$

- System to solve

$$\vec{y}'(x) = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 1 & -1 \end{bmatrix} \cdot \vec{y}(x)$$

- Define the coefficient matrix

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 1 & -1 \end{bmatrix}$$

- Rewrite the system as

$$\vec{y}'(x) = A \cdot \vec{y}(x)$$

- To solve the system, find the eigenvalues and eigenvectors of A
- Eigenpairs of A

$$\left[\left[-1, \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}\right], \left[-1, \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}\right], \left[1, \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}\right]\right]$$

- Consider eigenpair, with eigenvalue of algebraic multiplicity 2

$$\left[-1, \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}\right]$$

- First solution from eigenvalue -1

$$\vec{y}_1(x) = \mathrm{e}^{-x} \cdot \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$$

- Form of the 2nd homogeneous solution where $\vec{p}$ is to be solved for, $\lambda = -1$ is the eigenvalue, a

$$\vec{y}_2(x) = \mathrm{e}^{\lambda x}\left(x\vec{v} + \vec{p}\right)$$

- Note that the x multiplying $\vec{v}$ makes this solution linearly independent to the 1st solution obt
- Substitute $\vec{y}_2(x)$ into the homogeneous system

$$\lambda\,\mathrm{e}^{\lambda x}\left(x\vec{v} + \vec{p}\right) + \mathrm{e}^{\lambda x}\vec{v} = \left(\mathrm{e}^{\lambda x} A\right) \cdot \left(x\vec{v} + \vec{p}\right)$$

- Use the fact that $\vec{v}$ is an eigenvector of A

$$\lambda\, e^{\lambda x}\left(x\vec{v}+\vec{p}\right)+e^{\lambda x}\vec{v}=e^{\lambda x}\left(\lambda x\vec{v}+A\cdot\vec{p}\right)$$

- Simplify equation

$$\lambda\vec{p}+\vec{v}=A\cdot\vec{p}$$

- Make use of the identity matrix I

$$(\lambda\cdot I)\cdot\vec{p}+\vec{v}=A\cdot\vec{p}$$

- Condition $\vec{p}$ must meet for $\vec{y}_2(x)$ to be a solution to the homogeneous system

$$(A-\lambda\cdot I)\cdot\vec{p}=\vec{v}$$

- Choose $\vec{p}$ to use in the second solution to the homogeneous system from eigenvalue -1

$$\left(\begin{bmatrix}0 & 1 & 0\\ 0 & 0 & 1\\ 1 & 1 & -1\end{bmatrix}-(-1)\cdot\begin{bmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{bmatrix}\right)\cdot\vec{p}=\begin{bmatrix}1\\ -1\\ 1\end{bmatrix}$$

- Choice of $\vec{p}$

$$\vec{p}=\begin{bmatrix}1\\ 0\\ 0\end{bmatrix}$$

- Second solution from eigenvalue -1

$$\vec{y}_2(x)=e^{-x}\cdot\left(x\cdot\begin{bmatrix}1\\ -1\\ 1\end{bmatrix}+\begin{bmatrix}1\\ 0\\ 0\end{bmatrix}\right)$$

- Consider eigenpair

$$\left[1,\begin{bmatrix}1\\ 1\\ 1\end{bmatrix}\right]$$

- Solution to homogeneous system from eigenpair

$$\vec{y}_3=e^{x}\cdot\begin{bmatrix}1\\ 1\\ 1\end{bmatrix}$$

- General solution to the system of ODEs

$$\vec{y} = c_1 \vec{y}_1(x) + c_2 \vec{y}_2(x) + c_3 \vec{y}_3$$

- Substitute solutions into the general solution

$$\vec{y} = c_1 e^{-x} \cdot \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} + c_2 e^{-x} \cdot \left(x \cdot \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \right) + c_3 e^{x} \cdot \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

- First component of the vector is the solution to the ODE

$$y = (c_2(x+1) + c_1)\, e^{-x} + c_3 e^{x}$$

Maple trace

```
`Methods for third order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
<- constant coefficients successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 19

```
dsolve(diff(y(x),x$3)+diff(y(x),x$2)-diff(y(x),x)-y(x)=0,y(x), singsol=all)
```

$$y(x) = (c_3 x + c_2)\, e^{-x} + e^{x} c_1$$

✓ Solution by Mathematica

Time used: 0.003 (sec). Leaf size: 26

```
DSolve[y'''[x]+y''[x]-y'[x]-y[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to e^{-x} \left(c_2 x + c_3 e^{2x} + c_1 \right)$$

2.4 problem 4

Internal problem ID [3245]
Internal file name [`OUTPUT/2737_Sunday_June_05_2022_08_39_53_AM_98960955/index.tex`]

Book: Differential equations for engineers by Wei-Chau XIE, Cambridge Press 2010
Section: Chapter 4. Linear Differential Equations. Page 183
Problem number: 4.
ODE order: 3.
ODE degree: 1.

The type(s) of ODE detected by this program : **"higher_order_linear_constant_coefficients_ODE"**

Maple gives the following as the ode type

```
[[_3rd_order, _missing_x]]
```

$$\boxed{y''' + 8y = 0}$$

The characteristic equation is

$$\lambda^3 + 8 = 0$$

The roots of the above equation are

$$\begin{aligned}\lambda_1 &= -2\\ \lambda_2 &= 1 - i\sqrt{3}\\ \lambda_3 &= 1 + i\sqrt{3}\end{aligned}$$

Therefore the homogeneous solution is

$$y_h(x) = c_1 \mathrm{e}^{-2x} + \mathrm{e}^{\left(1-i\sqrt{3}\right)x} c_2 + \mathrm{e}^{\left(1+i\sqrt{3}\right)x} c_3$$

The fundamental set of solutions for the homogeneous solution are the following

$$\begin{aligned}y_1 &= \mathrm{e}^{-2x}\\ y_2 &= \mathrm{e}^{\left(1-i\sqrt{3}\right)x}\\ y_3 &= \mathrm{e}^{\left(1+i\sqrt{3}\right)x}\end{aligned}$$

Summary
The solution(s) found are the following

$$y = c_1 \mathrm{e}^{-2x} + \mathrm{e}^{\left(1-i\sqrt{3}\right)x} c_2 + \mathrm{e}^{\left(1+i\sqrt{3}\right)x} c_3 \tag{1}$$

Verification of solutions

$$y = c_1 \mathrm{e}^{-2x} + \mathrm{e}^{\left(1-i\sqrt{3}\right)x} c_2 + \mathrm{e}^{\left(1+i\sqrt{3}\right)x} c_3$$

Verified OK.

2.4.1 Maple step by step solution

Let's solve

$y''' + 8y = 0$

- Highest derivative means the order of the ODE is 3

  y'''

□ Convert linear ODE into a system of first order ODEs

  ◦ Define new variable $y_1(x)$

    $y_1(x) = y$

  ◦ Define new variable $y_2(x)$

    $y_2(x) = y'$

  ◦ Define new variable $y_3(x)$

    $y_3(x) = y''$

  ◦ Isolate for $y_3'(x)$ using original ODE

    $y_3'(x) = -8y_1(x)$

  Convert linear ODE into a system of first order ODEs

  $[y_2(x) = y_1'(x), y_3(x) = y_2'(x), y_3'(x) = -8y_1(x)]$

- Define vector

  $$\vec{y}(x) = \begin{bmatrix} y_1(x) \\ y_2(x) \\ y_3(x) \end{bmatrix}$$

- System to solve

$$\vec{y}'(x) = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -8 & 0 & 0 \end{bmatrix} \cdot \vec{y}(x)$$

- Define the coefficient matrix

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -8 & 0 & 0 \end{bmatrix}$$

- Rewrite the system as

$$\vec{y}'(x) = A \cdot \vec{y}(x)$$

- To solve the system, find the eigenvalues and eigenvectors of A
- Eigenpairs of A

$$\left[\left[-2, \begin{bmatrix} \frac{1}{4} \\ -\frac{1}{2} \\ 1 \end{bmatrix}\right], \left[1 - \mathrm{I}\sqrt{3}, \begin{bmatrix} \frac{1}{\left(1-\mathrm{I}\sqrt{3}\right)^2} \\ \frac{1}{1-\mathrm{I}\sqrt{3}} \\ 1 \end{bmatrix}\right], \left[1 + \mathrm{I}\sqrt{3}, \begin{bmatrix} \frac{1}{\left(1+\mathrm{I}\sqrt{3}\right)^2} \\ \frac{1}{1+\mathrm{I}\sqrt{3}} \\ 1 \end{bmatrix}\right]\right]$$

- Consider eigenpair

$$\left[-2, \begin{bmatrix} \frac{1}{4} \\ -\frac{1}{2} \\ 1 \end{bmatrix}\right]$$

- Solution to homogeneous system from eigenpair

$$\vec{y}_1 = \mathrm{e}^{-2x} \cdot \begin{bmatrix} \frac{1}{4} \\ -\frac{1}{2} \\ 1 \end{bmatrix}$$

- Consider complex eigenpair, complex conjugate eigenvalue can be ignored

$$\left[1 - \mathrm{I}\sqrt{3}, \begin{bmatrix} \frac{1}{\left(1-\mathrm{I}\sqrt{3}\right)^2} \\ \frac{1}{1-\mathrm{I}\sqrt{3}} \\ 1 \end{bmatrix}\right]$$

- Solution from eigenpair

$$e^{\left(1-I\sqrt{3}\right)x} \cdot \left[\begin{array}{c} \frac{1}{\left(1-I\sqrt{3}\right)^{2}} \\ \frac{1}{1-I\sqrt{3}} \\ 1 \end{array}\right]$$

- Use Euler identity to write solution in terms of sin and cos

$$e^{x} \cdot \left(\cos\left(\sqrt{3}\,x\right) - I\sin\left(\sqrt{3}\,x\right)\right) \cdot \left[\begin{array}{c} \frac{1}{\left(1-I\sqrt{3}\right)^{2}} \\ \frac{1}{1-I\sqrt{3}} \\ 1 \end{array}\right]$$

- Simplify expression

$$e^{x} \cdot \left[\begin{array}{c} \frac{\cos\left(\sqrt{3}\,x\right)-I\sin\left(\sqrt{3}\,x\right)}{\left(1-I\sqrt{3}\right)^{2}} \\ \frac{\cos\left(\sqrt{3}\,x\right)-I\sin\left(\sqrt{3}\,x\right)}{1-I\sqrt{3}} \\ \cos\left(\sqrt{3}\,x\right) - I\sin\left(\sqrt{3}\,x\right) \end{array}\right]$$

- Both real and imaginary parts are solutions to the homogeneous system

$$\left[\vec{y}_{2}(x) = e^{x} \cdot \left[\begin{array}{c} -\frac{\cos\left(\sqrt{3}\,x\right)}{8} + \frac{\sin\left(\sqrt{3}\,x\right)\sqrt{3}}{8} \\ \frac{\cos\left(\sqrt{3}\,x\right)}{4} + \frac{\sin\left(\sqrt{3}\,x\right)\sqrt{3}}{4} \\ \cos\left(\sqrt{3}\,x\right) \end{array}\right], \vec{y}_{3}(x) = e^{x} \cdot \left[\begin{array}{c} \frac{\cos\left(\sqrt{3}\,x\right)\sqrt{3}}{8} + \frac{\sin\left(\sqrt{3}\,x\right)}{8} \\ \frac{\cos\left(\sqrt{3}\,x\right)\sqrt{3}}{4} - \frac{\sin\left(\sqrt{3}\,x\right)}{4} \\ -\sin\left(\sqrt{3}\,x\right) \end{array}\right]\right]$$

- General solution to the system of ODEs

$$\vec{y} = c_{1}\vec{y}_{1} + c_{2}\vec{y}_{2}(x) + c_{3}\vec{y}_{3}(x)$$

- Substitute solutions into the general solution

$$\vec{y} = c_{1}e^{-2x} \cdot \left[\begin{array}{c} \frac{1}{4} \\ -\frac{1}{2} \\ 1 \end{array}\right] + c_{2}e^{x} \cdot \left[\begin{array}{c} -\frac{\cos\left(\sqrt{3}\,x\right)}{8} + \frac{\sin\left(\sqrt{3}\,x\right)\sqrt{3}}{8} \\ \frac{\cos\left(\sqrt{3}\,x\right)}{4} + \frac{\sin\left(\sqrt{3}\,x\right)\sqrt{3}}{4} \\ \cos\left(\sqrt{3}\,x\right) \end{array}\right] + c_{3}e^{x} \cdot \left[\begin{array}{c} \frac{\cos\left(\sqrt{3}\,x\right)\sqrt{3}}{8} + \frac{\sin\left(\sqrt{3}\,x\right)}{8} \\ \frac{\cos\left(\sqrt{3}\,x\right)\sqrt{3}}{4} - \frac{\sin\left(\sqrt{3}\,x\right)}{4} \\ -\sin\left(\sqrt{3}\,x\right) \end{array}\right]$$

- First component of the vector is the solution to the ODE

$$y = \frac{\left(-\frac{e^{3x}\left(-\sqrt{3}\, c_3+c_2\right)\cos\left(\sqrt{3}\, x\right)}{2}+\frac{e^{3x}\left(c_2\sqrt{3}+c_3\right)\sin\left(\sqrt{3}\, x\right)}{2}+c_1\right)e^{-2x}}{4}$$

Maple trace

```
`Methods for third order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
<- constant coefficients successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 35

```
dsolve(diff(y(x),x$3)+8*y(x)=0,y(x), singsol=all)
```

$$y(x) = \left(c_2 e^{3x}\sin\left(\sqrt{3}\, x\right)+c_3 e^{3x}\cos\left(\sqrt{3}\, x\right)+c_1\right)e^{-2x}$$

✓ Solution by Mathematica

Time used: 0.003 (sec). Leaf size: 42

```
DSolve[y'''[x]+8*y[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to c_1 e^{-2x}+c_3 e^x\cos\left(\sqrt{3}x\right)+c_2 e^x\sin\left(\sqrt{3}x\right)$$

2.5 problem 5

Internal problem ID [3246]
Internal file name [`OUTPUT/2738_Sunday_June_05_2022_08_39_54_AM_43057242/index.tex`]

Book: Differential equations for engineers by Wei-Chau XIE, Cambridge Press 2010
Section: Chapter 4. Linear Differential Equations. Page 183
Problem number: 5.
ODE order: 3.
ODE degree: 1.

The type(s) of ODE detected by this program : **"higher_order_linear_constant_coefficients_ODE"**

Maple gives the following as the ode type

```
[[_3rd_order, _missing_x]]
```

$$\boxed{y''' - 8y = 0}$$

The characteristic equation is

$$\lambda^3 - 8 = 0$$

The roots of the above equation are

$$\lambda_1 = 2$$
$$\lambda_2 = -1 - i\sqrt{3}$$
$$\lambda_3 = -1 + i\sqrt{3}$$

Therefore the homogeneous solution is

$$y_h(x) = e^{2x}c_1 + e^{\left(-1+i\sqrt{3}\right)x}c_2 + e^{\left(-1-i\sqrt{3}\right)x}c_3$$

The fundamental set of solutions for the homogeneous solution are the following

$$y_1 = e^{2x}$$
$$y_2 = e^{\left(-1+i\sqrt{3}\right)x}$$
$$y_3 = e^{\left(-1-i\sqrt{3}\right)x}$$

Summary
The solution(s) found are the following

$$y = e^{2x} c_1 + e^{\left(-1+i\sqrt{3}\right)x} c_2 + e^{\left(-1-i\sqrt{3}\right)x} c_3 \tag{1}$$

Verification of solutions

$$y = e^{2x} c_1 + e^{\left(-1+i\sqrt{3}\right)x} c_2 + e^{\left(-1-i\sqrt{3}\right)x} c_3$$

Verified OK.

2.5.1 Maple step by step solution

Let's solve

$y''' - 8y = 0$

- Highest derivative means the order of the ODE is 3

  y'''

- □ Convert linear ODE into a system of first order ODEs
  - Define new variable $y_1(x)$

    $y_1(x) = y$
  - Define new variable $y_2(x)$

    $y_2(x) = y'$
  - Define new variable $y_3(x)$

    $y_3(x) = y''$
  - Isolate for $y_3'(x)$ using original ODE

    $y_3'(x) = 8y_1(x)$

  Convert linear ODE into a system of first order ODEs

  $[y_2(x) = y_1'(x), y_3(x) = y_2'(x), y_3'(x) = 8y_1(x)]$

- Define vector

  $$\vec{y}(x) = \begin{bmatrix} y_1(x) \\ y_2(x) \\ y_3(x) \end{bmatrix}$$

- System to solve

$$\vec{y}'(x) = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 8 & 0 & 0 \end{bmatrix} \cdot \vec{y}(x)$$

- Define the coefficient matrix

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 8 & 0 & 0 \end{bmatrix}$$

- Rewrite the system as

$$\vec{y}'(x) = A \cdot \vec{y}(x)$$

- To solve the system, find the eigenvalues and eigenvectors of A
- Eigenpairs of A

$$\left[\left[2, \begin{bmatrix} \frac{1}{4} \\ \frac{1}{2} \\ 1 \end{bmatrix}\right], \left[-1 - \mathrm{I}\sqrt{3}, \begin{bmatrix} \frac{1}{\left(-1-\mathrm{I}\sqrt{3}\right)^2} \\ \frac{1}{-1-\mathrm{I}\sqrt{3}} \\ 1 \end{bmatrix}\right], \left[-1 + \mathrm{I}\sqrt{3}, \begin{bmatrix} \frac{1}{\left(-1+\mathrm{I}\sqrt{3}\right)^2} \\ \frac{1}{-1+\mathrm{I}\sqrt{3}} \\ 1 \end{bmatrix}\right]\right]$$

- Consider eigenpair

$$\left[2, \begin{bmatrix} \frac{1}{4} \\ \frac{1}{2} \\ 1 \end{bmatrix}\right]$$

- Solution to homogeneous system from eigenpair

$$\vec{y}_1 = \mathrm{e}^{2x} \cdot \begin{bmatrix} \frac{1}{4} \\ \frac{1}{2} \\ 1 \end{bmatrix}$$

- Consider complex eigenpair, complex conjugate eigenvalue can be ignored

$$\left[-1 - \mathrm{I}\sqrt{3}, \begin{bmatrix} \frac{1}{\left(-1-\mathrm{I}\sqrt{3}\right)^2} \\ \frac{1}{-1-\mathrm{I}\sqrt{3}} \\ 1 \end{bmatrix}\right]$$

- Solution from eigenpair

$$e^{\left(-1-\mathrm{I}\sqrt{3}\right)x}\cdot\left[\begin{array}{c}\frac{1}{\left(-1-\mathrm{I}\sqrt{3}\right)^{2}}\\ \frac{1}{-1-\mathrm{I}\sqrt{3}}\\ 1\end{array}\right]$$

- Use Euler identity to write solution in terms of sin and cos

$$e^{-x}\cdot\left(\cos\left(\sqrt{3}\,x\right)-\mathrm{I}\sin\left(\sqrt{3}\,x\right)\right)\cdot\left[\begin{array}{c}\frac{1}{\left(-1-\mathrm{I}\sqrt{3}\right)^{2}}\\ \frac{1}{-1-\mathrm{I}\sqrt{3}}\\ 1\end{array}\right]$$

- Simplify expression

$$e^{-x}\cdot\left[\begin{array}{c}\frac{\cos\left(\sqrt{3}\,x\right)-\mathrm{I}\sin\left(\sqrt{3}\,x\right)}{\left(-1-\mathrm{I}\sqrt{3}\right)^{2}}\\ \frac{\cos\left(\sqrt{3}\,x\right)-\mathrm{I}\sin\left(\sqrt{3}\,x\right)}{-1-\mathrm{I}\sqrt{3}}\\ \cos\left(\sqrt{3}\,x\right)-\mathrm{I}\sin\left(\sqrt{3}\,x\right)\end{array}\right]$$

- Both real and imaginary parts are solutions to the homogeneous system

$$\left[\vec{y}_{2}(x)=e^{-x}\cdot\left[\begin{array}{c}-\frac{\cos\left(\sqrt{3}\,x\right)}{8}-\frac{\sin\left(\sqrt{3}\,x\right)\sqrt{3}}{8}\\ -\frac{\cos\left(\sqrt{3}\,x\right)}{4}+\frac{\sin\left(\sqrt{3}\,x\right)\sqrt{3}}{4}\\ \cos\left(\sqrt{3}\,x\right)\end{array}\right],\vec{y}_{3}(x)=e^{-x}\cdot\left[\begin{array}{c}-\frac{\cos\left(\sqrt{3}\,x\right)\sqrt{3}}{8}+\frac{\sin\left(\sqrt{3}\,x\right)}{8}\\ \frac{\cos\left(\sqrt{3}\,x\right)\sqrt{3}}{4}+\frac{\sin\left(\sqrt{3}\,x\right)}{4}\\ -\sin\left(\sqrt{3}\,x\right)\end{array}\right]\right]$$

- General solution to the system of ODEs

$$\vec{y}=c_{1}\vec{y}_{1}+c_{2}\vec{y}_{2}(x)+c_{3}\vec{y}_{3}(x)$$

- Substitute solutions into the general solution

$$\vec{y}=e^{2x}c_{1}\cdot\left[\begin{array}{c}\frac{1}{4}\\ \frac{1}{2}\\ 1\end{array}\right]+c_{2}e^{-x}\cdot\left[\begin{array}{c}-\frac{\cos\left(\sqrt{3}\,x\right)}{8}-\frac{\sin\left(\sqrt{3}\,x\right)\sqrt{3}}{8}\\ -\frac{\cos\left(\sqrt{3}\,x\right)}{4}+\frac{\sin\left(\sqrt{3}\,x\right)\sqrt{3}}{4}\\ \cos\left(\sqrt{3}\,x\right)\end{array}\right]+c_{3}e^{-x}\cdot\left[\begin{array}{c}-\frac{\cos\left(\sqrt{3}\,x\right)\sqrt{3}}{8}+\frac{\sin\left(\sqrt{3}\,x\right)}{8}\\ \frac{\cos\left(\sqrt{3}\,x\right)\sqrt{3}}{4}+\frac{\sin\left(\sqrt{3}\,x\right)}{4}\\ -\sin\left(\sqrt{3}\,x\right)\end{array}\right]$$

- First component of the vector is the solution to the ODE

$$y=-\frac{e^{-x}\left(\sqrt{3}\,c_{3}+c_{2}\right)\cos\left(\sqrt{3}\,x\right)}{8}-\frac{e^{-x}\left(c_{2}\sqrt{3}-c_{3}\right)\sin\left(\sqrt{3}\,x\right)}{8}+\frac{e^{2x}c_{1}}{4}$$

Maple trace

```
`Methods for third order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
<- constant coefficients successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 35

```
dsolve(diff(y(x),x$3)-8*y(x)=0,y(x), singsol=all)
```

$$y(x) = c_1 e^{2x} + c_2 e^{-x} \sin\left(\sqrt{3}\, x\right) + c_3 e^{-x} \cos\left(\sqrt{3}\, x\right)$$

✓ Solution by Mathematica

Time used: 0.003 (sec). Leaf size: 42

```
DSolve[y'''[x]-8*y[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to e^{-x}\left(c_1 e^{3x} + c_2 \cos\left(\sqrt{3}x\right) + c_3 \sin\left(\sqrt{3}x\right)\right)$$

2.6 problem 6

Internal problem ID [3247]
Internal file name [OUTPUT/2739_Sunday_June_05_2022_08_39_54_AM_97575065/index.tex]

Book: Differential equations for engineers by Wei-Chau XIE, Cambridge Press 2010
Section: Chapter 4. Linear Differential Equations. Page 183
Problem number: 6.
ODE order: 4.
ODE degree: 1.

The type(s) of ODE detected by this program : **"higher_order_linear_constant_coefficients_ODE"**

Maple gives the following as the ode type

```
[[_high_order, _missing_x]]
```

$$\boxed{y'''' + 4y = 0}$$

The characteristic equation is

$$\lambda^4 + 4 = 0$$

The roots of the above equation are

$$\begin{aligned}\lambda_1 &= 1 - i \\ \lambda_2 &= 1 + i \\ \lambda_3 &= -1 - i \\ \lambda_4 &= -1 + i\end{aligned}$$

Therefore the homogeneous solution is

$$y_h(x) = \mathrm{e}^{(-1+i)x} c_1 + \mathrm{e}^{(-1-i)x} c_2 + \mathrm{e}^{(1+i)x} c_3 + \mathrm{e}^{(1-i)x} c_4$$

The fundamental set of solutions for the homogeneous solution are the following

$$\begin{aligned}y_1 &= \mathrm{e}^{(-1+i)x} \\ y_2 &= \mathrm{e}^{(-1-i)x} \\ y_3 &= \mathrm{e}^{(1+i)x} \\ y_4 &= \mathrm{e}^{(1-i)x}\end{aligned}$$

Summary
The solution(s) found are the following

$$y = e^{(-1+i)x}c_1 + e^{(-1-i)x}c_2 + e^{(1+i)x}c_3 + e^{(1-i)x}c_4 \tag{1}$$

Verification of solutions

$$y = e^{(-1+i)x}c_1 + e^{(-1-i)x}c_2 + e^{(1+i)x}c_3 + e^{(1-i)x}c_4$$

Verified OK.

2.6.1 Maple step by step solution

Let's solve

$y'''' + 4y = 0$

- Highest derivative means the order of the ODE is 4

  y''''

- □ Convert linear ODE into a system of first order ODEs
  - ○ Define new variable $y_1(x)$

    $y_1(x) = y$
  - ○ Define new variable $y_2(x)$

    $y_2(x) = y'$
  - ○ Define new variable $y_3(x)$

    $y_3(x) = y''$
  - ○ Define new variable $y_4(x)$

    $y_4(x) = y'''$
  - ○ Isolate for $y_4'(x)$ using original ODE

    $y_4'(x) = -4y_1(x)$

  Convert linear ODE into a system of first order ODEs

  $[y_2(x) = y_1'(x), y_3(x) = y_2'(x), y_4(x) = y_3'(x), y_4'(x) = -4y_1(x)]$

- Define vector

$$\vec{y}(x) = \begin{bmatrix} y_1(x) \\ y_2(x) \\ y_3(x) \\ y_4(x) \end{bmatrix}$$

- System to solve

$$\vec{y}'(x) = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -4 & 0 & 0 & 0 \end{bmatrix} \cdot \vec{y}(x)$$

- Define the coefficient matrix

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -4 & 0 & 0 & 0 \end{bmatrix}$$

- Rewrite the system as

$$\vec{y}'(x) = A \cdot \vec{y}(x)$$

- To solve the system, find the eigenvalues and eigenvectors of A
- Eigenpairs of A

$$\left[\left[-1-\mathrm{I}, \begin{bmatrix} \frac{1}{4}+\frac{\mathrm{I}}{4} \\ -\frac{\mathrm{I}}{2} \\ -\frac{1}{2}+\frac{\mathrm{I}}{2} \\ 1 \end{bmatrix}\right], \left[-1+\mathrm{I}, \begin{bmatrix} \frac{1}{4}-\frac{\mathrm{I}}{4} \\ \frac{\mathrm{I}}{2} \\ -\frac{1}{2}-\frac{\mathrm{I}}{2} \\ 1 \end{bmatrix}\right], \left[1-\mathrm{I}, \begin{bmatrix} -\frac{1}{4}+\frac{\mathrm{I}}{4} \\ \frac{\mathrm{I}}{2} \\ \frac{1}{2}+\frac{\mathrm{I}}{2} \\ 1 \end{bmatrix}\right], \left[1+\mathrm{I}, \begin{bmatrix} -\frac{1}{4}-\frac{\mathrm{I}}{4} \\ -\frac{\mathrm{I}}{2} \\ \frac{1}{2}-\frac{\mathrm{I}}{2} \\ 1 \end{bmatrix}\right]\right]$$

- Consider complex eigenpair, complex conjugate eigenvalue can be ignored

$$\left[-1-\mathrm{I}, \begin{bmatrix} \frac{1}{4}+\frac{\mathrm{I}}{4} \\ -\frac{\mathrm{I}}{2} \\ -\frac{1}{2}+\frac{\mathrm{I}}{2} \\ 1 \end{bmatrix}\right]$$

- Solution from eigenpair

$$e^{(-1-I)x} \cdot \begin{bmatrix} \frac{1}{4}+\frac{I}{4} \\ -\frac{I}{2} \\ -\frac{1}{2}+\frac{I}{2} \\ 1 \end{bmatrix}$$

- Use Euler identity to write solution in terms of sin and cos

$$e^{-x} \cdot (\cos(x) - I\sin(x)) \cdot \begin{bmatrix} \frac{1}{4}+\frac{I}{4} \\ -\frac{I}{2} \\ -\frac{1}{2}+\frac{I}{2} \\ 1 \end{bmatrix}$$

- Simplify expression

$$e^{-x} \cdot \begin{bmatrix} \left(\frac{1}{4}+\frac{I}{4}\right)(\cos(x) - I\sin(x)) \\ -\frac{I}{2}(\cos(x) - I\sin(x)) \\ \left(-\frac{1}{2}+\frac{I}{2}\right)(\cos(x) - I\sin(x)) \\ \cos(x) - I\sin(x) \end{bmatrix}$$

- Both real and imaginary parts are solutions to the homogeneous system

$$\left[\vec{y}_1(x) = e^{-x} \cdot \begin{bmatrix} \frac{\cos(x)}{4}+\frac{\sin(x)}{4} \\ -\frac{\sin(x)}{2} \\ -\frac{\cos(x)}{2}+\frac{\sin(x)}{2} \\ \cos(x) \end{bmatrix}, \vec{y}_2(x) = e^{-x} \cdot \begin{bmatrix} \frac{\cos(x)}{4}-\frac{\sin(x)}{4} \\ -\frac{\cos(x)}{2} \\ \frac{\sin(x)}{2}+\frac{\cos(x)}{2} \\ -\sin(x) \end{bmatrix}\right]$$

- Consider complex eigenpair, complex conjugate eigenvalue can be ignored

$$\left[1-I, \begin{bmatrix} -\frac{1}{4}+\frac{I}{4} \\ \frac{I}{2} \\ \frac{1}{2}+\frac{I}{2} \\ 1 \end{bmatrix}\right]$$

- Solution from eigenpair

$$e^{(1-I)x}\cdot\begin{bmatrix}-\frac{1}{4}+\frac{I}{4}\\ \frac{I}{2}\\ \frac{1}{2}+\frac{I}{2}\\ 1\end{bmatrix}$$

- Use Euler identity to write solution in terms of sin and cos

$$e^{x}\cdot(\cos(x)-I\sin(x))\cdot\begin{bmatrix}-\frac{1}{4}+\frac{I}{4}\\ \frac{I}{2}\\ \frac{1}{2}+\frac{I}{2}\\ 1\end{bmatrix}$$

- Simplify expression

$$e^{x}\cdot\begin{bmatrix}\left(-\frac{1}{4}+\frac{I}{4}\right)(\cos(x)-I\sin(x))\\ \frac{I}{2}(\cos(x)-I\sin(x))\\ \left(\frac{1}{2}+\frac{I}{2}\right)(\cos(x)-I\sin(x))\\ \cos(x)-I\sin(x)\end{bmatrix}$$

- Both real and imaginary parts are solutions to the homogeneous system

$$\left[\vec{y}_3(x)=e^{x}\cdot\begin{bmatrix}-\frac{\cos(x)}{4}+\frac{\sin(x)}{4}\\ \frac{\sin(x)}{2}\\ \frac{\sin(x)}{2}+\frac{\cos(x)}{2}\\ \cos(x)\end{bmatrix},\vec{y}_4(x)=e^{x}\cdot\begin{bmatrix}\frac{\cos(x)}{4}+\frac{\sin(x)}{4}\\ \frac{\cos(x)}{2}\\ \frac{\cos(x)}{2}-\frac{\sin(x)}{2}\\ -\sin(x)\end{bmatrix}\right]$$

- General solution to the system of ODEs

$$\vec{y}=c_1\vec{y}_1(x)+c_2\vec{y}_2(x)+c_3\vec{y}_3(x)+c_4\vec{y}_4(x)$$

- Substitute solutions into the general solution

$$\vec{y}=c_1e^{-x}\cdot\begin{bmatrix}\frac{\cos(x)}{4}+\frac{\sin(x)}{4}\\ -\frac{\sin(x)}{2}\\ -\frac{\cos(x)}{2}+\frac{\sin(x)}{2}\\ \cos(x)\end{bmatrix}+c_2e^{-x}\cdot\begin{bmatrix}\frac{\cos(x)}{4}-\frac{\sin(x)}{4}\\ -\frac{\cos(x)}{2}\\ \frac{\sin(x)}{2}+\frac{\cos(x)}{2}\\ -\sin(x)\end{bmatrix}+c_3e^{x}\cdot\begin{bmatrix}-\frac{\cos(x)}{4}+\frac{\sin(x)}{4}\\ \frac{\sin(x)}{2}\\ \frac{\sin(x)}{2}+\frac{\cos(x)}{2}\\ \cos(x)\end{bmatrix}+c_4e^{x}\cdot$$

- First component of the vector is the solution to the ODE

$$y = \frac{((c_1+c_2)\cos(x)+\sin(x)(c_1-c_2))\,\mathrm{e}^{-x}}{4} - \frac{\mathrm{e}^{x}((c_3-c_4)\cos(x)-\sin(x)(c_3+c_4))}{4}$$

Maple trace

```
`Methods for high order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
<- constant coefficients successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 33

```
dsolve(diff(y(x),x$4)+4*y(x)=0,y(x), singsol=all)
```

$$y(x) = c_1 \sin(x)\,\mathrm{e}^{x} + c_2 \mathrm{e}^{x} \cos(x) + c_3 \mathrm{e}^{-x} \sin(x) + c_4 \mathrm{e}^{-x} \cos(x)$$

✓ Solution by Mathematica

Time used: 0.004 (sec). Leaf size: 40

```
DSolve[y''''[x]+4*y[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to e^{-x}\left(\left(c_4 e^{2x} + c_1\right)\cos(x) + \left(c_3 e^{2x} + c_2\right)\sin(x)\right)$$

2.7 problem 7

Internal problem ID [3248]
Internal file name [`OUTPUT/2740_Sunday_June_05_2022_08_39_55_AM_49799157/index.tex`]

Book: Differential equations for engineers by Wei-Chau XIE, Cambridge Press 2010
Section: Chapter 4. Linear Differential Equations. Page 183
Problem number: 7.
ODE order: 4.
ODE degree: 1.

The type(s) of ODE detected by this program : **"higher_order_linear_constant_coefficients_ODE"**

Maple gives the following as the ode type

```
[[_high_order, _missing_x]]
```

$$\boxed{y'''' + 18y'' + 81y = 0}$$

The characteristic equation is

$$\lambda^4 + 18\lambda^2 + 81 = 0$$

The roots of the above equation are

$$\begin{aligned} \lambda_1 &= 3i \\ \lambda_2 &= -3i \\ \lambda_3 &= 3i \\ \lambda_4 &= -3i \end{aligned}$$

Therefore the homogeneous solution is

$$y_h(x) = \mathrm{e}^{-3ix} c_1 + x\,\mathrm{e}^{-3ix} c_2 + \mathrm{e}^{3ix} c_3 + x\,\mathrm{e}^{3ix} c_4$$

The fundamental set of solutions for the homogeneous solution are the following

$$\begin{aligned} y_1 &= \mathrm{e}^{-3ix} \\ y_2 &= x\,\mathrm{e}^{-3ix} \\ y_3 &= \mathrm{e}^{3ix} \\ y_4 &= x\,\mathrm{e}^{3ix} \end{aligned}$$

Summary

The solution(s) found are the following

$$y = e^{-3ix} c_1 + x e^{-3ix} c_2 + e^{3ix} c_3 + x e^{3ix} c_4 \tag{1}$$

Verification of solutions

$$y = e^{-3ix} c_1 + x e^{-3ix} c_2 + e^{3ix} c_3 + x e^{3ix} c_4$$

Verified OK.

Maple trace

```
`Methods for high order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
<- constant coefficients successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 25

```
dsolve(diff(y(x),x$4)+18*diff(y(x),x$2)+81*y(x)=0,y(x), singsol=all)
```

$$y(x) = (c_4 x + c_2) \cos(3x) + \sin(3x)(c_3 x + c_1)$$

✓ Solution by Mathematica

Time used: 0.003 (sec). Leaf size: 30

```
DSolve[y''''[x]+18*y''[x]+81*y[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to (c_2 x + c_1) \cos(3x) + (c_4 x + c_3) \sin(3x)$$

2.8 problem 8

Internal problem ID [3249]
Internal file name [`OUTPUT/2741_Sunday_June_05_2022_08_39_55_AM_42962800/index.tex`]

Book: Differential equations for engineers by Wei-Chau XIE, Cambridge Press 2010
Section: Chapter 4. Linear Differential Equations. Page 183
Problem number: 8.
ODE order: 4.
ODE degree: 1.

The type(s) of ODE detected by this program : **"higher_order_linear_constant_coefficients_ODE"**

Maple gives the following as the ode type

```
[[_high_order, _missing_x]]
```

$$\boxed{y'''' - 4y'' + 16y = 0}$$

The characteristic equation is

$$\lambda^4 - 4\lambda^2 + 16 = 0$$

The roots of the above equation are

$$\lambda_1 = -i + \sqrt{3}$$
$$\lambda_2 = i - \sqrt{3}$$
$$\lambda_3 = \sqrt{3} + i$$
$$\lambda_4 = -i - \sqrt{3}$$

Therefore the homogeneous solution is

$$y_h(x) = \mathrm{e}^{\left(i-\sqrt{3}\right)x} c_1 + \mathrm{e}^{\left(-i+\sqrt{3}\right)x} c_2 + \mathrm{e}^{\left(-i-\sqrt{3}\right)x} c_3 + \mathrm{e}^{\left(\sqrt{3}+i\right)x} c_4$$

The fundamental set of solutions for the homogeneous solution are the following

$$y_1 = \mathrm{e}^{\left(i-\sqrt{3}\right)x}$$
$$y_2 = \mathrm{e}^{\left(-i+\sqrt{3}\right)x}$$
$$y_3 = \mathrm{e}^{\left(-i-\sqrt{3}\right)x}$$
$$y_4 = \mathrm{e}^{\left(\sqrt{3}+i\right)x}$$

Summary
The solution(s) found are the following

$$y = e^{\left(i-\sqrt{3}\right)x} c_1 + e^{\left(-i+\sqrt{3}\right)x} c_2 + e^{\left(-i-\sqrt{3}\right)x} c_3 + e^{\left(\sqrt{3}+i\right)x} c_4 \tag{1}$$

Verification of solutions

$$y = e^{\left(i-\sqrt{3}\right)x} c_1 + e^{\left(-i+\sqrt{3}\right)x} c_2 + e^{\left(-i-\sqrt{3}\right)x} c_3 + e^{\left(\sqrt{3}+i\right)x} c_4$$

Verified OK.

2.8.1 Maple step by step solution

Let's solve

$y'''' - 4y'' + 16y = 0$

- Highest derivative means the order of the ODE is 4

  y''''

- □ Convert linear ODE into a system of first order ODEs
  - ◦ Define new variable $y_1(x)$

    $y_1(x) = y$
  - ◦ Define new variable $y_2(x)$

    $y_2(x) = y'$
  - ◦ Define new variable $y_3(x)$

    $y_3(x) = y''$
  - ◦ Define new variable $y_4(x)$

    $y_4(x) = y'''$
  - ◦ Isolate for $y_4'(x)$ using original ODE

    $y_4'(x) = 4y_3(x) - 16y_1(x)$

  Convert linear ODE into a system of first order ODEs

  $[y_2(x) = y_1'(x), y_3(x) = y_2'(x), y_4(x) = y_3'(x), y_4'(x) = 4y_3(x) - 16y_1(x)]$

- Define vector

$$\vec{y}(x) = \begin{bmatrix} y_1(x) \\ y_2(x) \\ y_3(x) \\ y_4(x) \end{bmatrix}$$

- System to solve

$$\vec{y}'(x) = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -16 & 0 & 4 & 0 \end{bmatrix} \cdot \vec{y}(x)$$

- Define the coefficient matrix

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -16 & 0 & 4 & 0 \end{bmatrix}$$

- Rewrite the system as

$\vec{y}'(x) = A \cdot \vec{y}(x)$

- To solve the system, find the eigenvalues and eigenvectors of A
- Eigenpairs of A

$$\left[\left[-\mathrm{I}-\sqrt{3}, \begin{bmatrix} \frac{1}{\left(-\mathrm{I}-\sqrt{3}\right)^3} \\ \frac{1}{\left(-\mathrm{I}-\sqrt{3}\right)^2} \\ \frac{1}{-\mathrm{I}-\sqrt{3}} \\ 1 \end{bmatrix}\right], \left[-\mathrm{I}+\sqrt{3}, \begin{bmatrix} \frac{1}{\left(-\mathrm{I}+\sqrt{3}\right)^3} \\ \frac{1}{\left(-\mathrm{I}+\sqrt{3}\right)^2} \\ \frac{1}{-\mathrm{I}+\sqrt{3}} \\ 1 \end{bmatrix}\right], \left[\mathrm{I}-\sqrt{3}, \begin{bmatrix} \frac{1}{\left(\mathrm{I}-\sqrt{3}\right)^3} \\ \frac{1}{\left(\mathrm{I}-\sqrt{3}\right)^2} \\ \frac{1}{\mathrm{I}-\sqrt{3}} \\ 1 \end{bmatrix}\right], \left[\sqrt{3}+\mathrm{I}, \right.\right.$$

- Consider complex eigenpair, complex conjugate eigenvalue can be ignored

$$\left[-\mathrm{I}-\sqrt{3}, \left[\begin{array}{c} \frac{1}{\left(-\mathrm{I}-\sqrt{3}\right)^{3}} \\ \frac{1}{\left(-\mathrm{I}-\sqrt{3}\right)^{2}} \\ \frac{1}{-\mathrm{I}-\sqrt{3}} \\ 1 \end{array}\right]\right]$$

- Solution from eigenpair

$$e^{\left(-\mathrm{I}-\sqrt{3}\right)x} \cdot \left[\begin{array}{c} \frac{1}{\left(-\mathrm{I}-\sqrt{3}\right)^{3}} \\ \frac{1}{\left(-\mathrm{I}-\sqrt{3}\right)^{2}} \\ \frac{1}{-\mathrm{I}-\sqrt{3}} \\ 1 \end{array}\right]$$

- Use Euler identity to write solution in terms of sin and cos

$$e^{-\sqrt{3}\,x} \cdot \left(\cos\left(x\right)-\mathrm{I}\sin\left(x\right)\right) \cdot \left[\begin{array}{c} \frac{1}{\left(-\mathrm{I}-\sqrt{3}\right)^{3}} \\ \frac{1}{\left(-\mathrm{I}-\sqrt{3}\right)^{2}} \\ \frac{1}{-\mathrm{I}-\sqrt{3}} \\ 1 \end{array}\right]$$

- Simplify expression

$$e^{-\sqrt{3}\,x} \cdot \left[\begin{array}{c} \frac{\cos(x)-\mathrm{I}\sin(x)}{\left(-\mathrm{I}-\sqrt{3}\right)^{3}} \\ \frac{\cos(x)-\mathrm{I}\sin(x)}{\left(-\mathrm{I}-\sqrt{3}\right)^{2}} \\ \frac{\cos(x)-\mathrm{I}\sin(x)}{-\mathrm{I}-\sqrt{3}} \\ \cos\left(x\right)-\mathrm{I}\sin\left(x\right) \end{array}\right]$$

- Both real and imaginary parts are solutions to the homogeneous system

$$\left[\vec{y}_1(x) = e^{-\sqrt{3}\,x} \cdot \left[\begin{array}{c} \frac{\sin(x)}{8} \\ \frac{\cos(x)}{8} - \frac{\sqrt{3}\,\sin(x)}{8} \\ -\frac{\cos(x)\sqrt{3}}{4} + \frac{\sin(x)}{4} \\ \cos(x) \end{array}\right], \vec{y}_2(x) = e^{-\sqrt{3}\,x} \cdot \left[\begin{array}{c} \frac{\cos(x)}{8} \\ -\frac{\cos(x)\sqrt{3}}{8} - \frac{\sin(x)}{8} \\ \frac{\cos(x)}{4} + \frac{\sqrt{3}\,\sin(x)}{4} \\ -\sin(x) \end{array}\right]\right]$$

- Consider complex eigenpair, complex conjugate eigenvalue can be ignored

$$\left[-I + \sqrt{3}, \left[\begin{array}{c} \frac{1}{\left(-I+\sqrt{3}\right)^3} \\ \frac{1}{\left(-I+\sqrt{3}\right)^2} \\ \frac{1}{-I+\sqrt{3}} \\ 1 \end{array}\right]\right]$$

- Solution from eigenpair

$$e^{\left(-I+\sqrt{3}\right)x} \cdot \left[\begin{array}{c} \frac{1}{\left(-I+\sqrt{3}\right)^3} \\ \frac{1}{\left(-I+\sqrt{3}\right)^2} \\ \frac{1}{-I+\sqrt{3}} \\ 1 \end{array}\right]$$

- Use Euler identity to write solution in terms of $\sin$ and $\cos$

$$e^{\sqrt{3}\,x} \cdot (\cos(x) - I\sin(x)) \cdot \left[\begin{array}{c} \frac{1}{\left(-I+\sqrt{3}\right)^3} \\ \frac{1}{\left(-I+\sqrt{3}\right)^2} \\ \frac{1}{-I+\sqrt{3}} \\ 1 \end{array}\right]$$

- Simplify expression

$$e^{\sqrt{3}\,x} \cdot \begin{bmatrix} \frac{\cos(x)-I\sin(x)}{\left(-I+\sqrt{3}\right)^3} \\ \frac{\cos(x)-I\sin(x)}{\left(-I+\sqrt{3}\right)^2} \\ \frac{\cos(x)-I\sin(x)}{-I+\sqrt{3}} \\ \cos(x)-I\sin(x) \end{bmatrix}$$

- Both real and imaginary parts are solutions to the homogeneous system

$$\left[\vec{y}_3(x) = e^{\sqrt{3}\,x} \cdot \begin{bmatrix} \frac{\sin(x)}{8} \\ \frac{\cos(x)}{8}+\frac{\sqrt{3}\,\sin(x)}{8} \\ \frac{\cos(x)\sqrt{3}}{4}+\frac{\sin(x)}{4} \\ \cos(x) \end{bmatrix}, \vec{y}_4(x) = e^{\sqrt{3}\,x} \cdot \begin{bmatrix} \frac{\cos(x)}{8} \\ \frac{\cos(x)\sqrt{3}}{8}-\frac{\sin(x)}{8} \\ \frac{\cos(x)}{4}-\frac{\sqrt{3}\,\sin(x)}{4} \\ -\sin(x) \end{bmatrix}\right]$$

- General solution to the system of ODEs

$$\vec{y} = c_1\vec{y}_1(x) + c_2\vec{y}_2(x) + c_3\vec{y}_3(x) + c_4\vec{y}_4(x)$$

- Substitute solutions into the general solution

$$\vec{y} = c_1 e^{-\sqrt{3}\,x} \cdot \begin{bmatrix} \frac{\sin(x)}{8} \\ \frac{\cos(x)}{8}-\frac{\sqrt{3}\,\sin(x)}{8} \\ -\frac{\cos(x)\sqrt{3}}{4}+\frac{\sin(x)}{4} \\ \cos(x) \end{bmatrix} + c_2 e^{-\sqrt{3}\,x} \cdot \begin{bmatrix} \frac{\cos(x)}{8} \\ -\frac{\cos(x)\sqrt{3}}{8}-\frac{\sin(x)}{8} \\ \frac{\cos(x)}{4}+\frac{\sqrt{3}\,\sin(x)}{4} \\ -\sin(x) \end{bmatrix} + c_3 e^{\sqrt{3}\,x} \cdot \begin{bmatrix} \frac{\sin(x)}{8} \\ \frac{\cos(x)}{8}+\frac{\sqrt{3}\,s}{} \\ \frac{\cos(x)\sqrt{3}}{4}+\frac{s}{} \\ \cos(x) \end{bmatrix}$$

- First component of the vector is the solution to the ODE

$$y = \frac{(c_1\sin(x)+c_2\cos(x))\,e^{-\sqrt{3}\,x}}{8} + \frac{e^{\sqrt{3}\,x}(c_3\sin(x)+\cos(x)c_4)}{8}$$

Maple trace

```
`Methods for high order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
<- constant coefficients successful`
```

✓ Solution by Maple

Time used: 0.015 (sec). Leaf size: 48

```
dsolve(diff(y(x),x$4)-4*diff(y(x),x$2)+16*y(x)=0,y(x), singsol=all)
```

$$y(x) = -c_1 e^{\sqrt{3}\,x} \sin(x) + c_2 e^{-\sqrt{3}\,x} \sin(x) + c_3 e^{\sqrt{3}\,x} \cos(x) + c_4 e^{-\sqrt{3}\,x} \cos(x)$$

✓ Solution by Mathematica

Time used: 0.006 (sec). Leaf size: 55

```
DSolve[y''''[x]-4*y''[x]+16*y[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to e^{-\sqrt{3}x}\left(\left(c_3 e^{2\sqrt{3}x} + c_2\right)\cos(x) + \left(c_1 e^{2\sqrt{3}x} + c_4\right)\sin(x)\right)$$

2.9 problem 9

Internal problem ID [3250]
Internal file name [OUTPUT/2742_Sunday_June_05_2022_08_39_55_AM_83864776/index.tex]

Book: Differential equations for engineers by Wei-Chau XIE, Cambridge Press 2010
Section: Chapter 4. Linear Differential Equations. Page 183
Problem number: 9.
ODE order: 4.
ODE degree: 1.

The type(s) of ODE detected by this program : **"higher_order_linear_constant_coefficients_ODE"**

Maple gives the following as the ode type

```
[[_high_order, _missing_x]]
```

$$\boxed{y'''' - 2y''' + 2y'' - 2y' + y = 0}$$

The characteristic equation is

$$\lambda^4 - 2\lambda^3 + 2\lambda^2 - 2\lambda + 1 = 0$$

The roots of the above equation are

$$\lambda_1 = i$$
$$\lambda_2 = -i$$
$$\lambda_3 = 1$$
$$\lambda_4 = 1$$

Therefore the homogeneous solution is

$$y_h(x) = c_1 e^x + c_2 x\, e^x + e^{ix} c_3 + e^{-ix} c_4$$

The fundamental set of solutions for the homogeneous solution are the following

$$y_1 = e^x$$
$$y_2 = x\, e^x$$
$$y_3 = e^{ix}$$
$$y_4 = e^{-ix}$$

Summary

The solution(s) found are the following

$$y = c_1 e^x + c_2 x e^x + e^{ix} c_3 + e^{-ix} c_4 \tag{1}$$

Verification of solutions

$$y = c_1 e^x + c_2 x e^x + e^{ix} c_3 + e^{-ix} c_4$$

Verified OK.

2.9.1 Maple step by step solution

Let's solve

$y'''' - 2y''' + 2y'' - 2y' + y = 0$

- Highest derivative means the order of the ODE is 4

  y''''

□ Convert linear ODE into a system of first order ODEs

  ∘ Define new variable $y_1(x)$

    $y_1(x) = y$

  ∘ Define new variable $y_2(x)$

    $y_2(x) = y'$

  ∘ Define new variable $y_3(x)$

    $y_3(x) = y''$

  ∘ Define new variable $y_4(x)$

    $y_4(x) = y'''$

  ∘ Isolate for $y_4'(x)$ using original ODE

    $y_4'(x) = 2y_4(x) - 2y_3(x) + 2y_2(x) - y_1(x)$

  Convert linear ODE into a system of first order ODEs

  $[y_2(x) = y_1'(x), y_3(x) = y_2'(x), y_4(x) = y_3'(x), y_4'(x) = 2y_4(x) - 2y_3(x) + 2y_2(x) - y_1(x)]$

- Define vector

$$\vec{y}(x) = \begin{bmatrix} y_1(x) \\ y_2(x) \\ y_3(x) \\ y_4(x) \end{bmatrix}$$

- System to solve

$$\vec{y}'(x) = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & 2 & -2 & 2 \end{bmatrix} \cdot \vec{y}(x)$$

- Define the coefficient matrix

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & 2 & -2 & 2 \end{bmatrix}$$

- Rewrite the system as

$$\vec{y}'(x) = A \cdot \vec{y}(x)$$

- To solve the system, find the eigenvalues and eigenvectors of A
- Eigenpairs of A

$$\left[\left[1, \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}\right], \left[1, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}\right], \left[-\mathrm{I}, \begin{bmatrix} -\mathrm{I} \\ -1 \\ \mathrm{I} \\ 1 \end{bmatrix}\right], \left[\mathrm{I}, \begin{bmatrix} \mathrm{I} \\ -1 \\ -\mathrm{I} \\ 1 \end{bmatrix}\right]\right]$$

- Consider eigenpair, with eigenvalue of algebraic multiplicity 2

$$\left[1, \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}\right]$$

- First solution from eigenvalue 1

$$\overrightarrow{y}_1(x) = e^x \cdot \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$$

- Form of the 2nd homogeneous solution where $\overrightarrow{p}$ is to be solved for, $\lambda = 1$ is the eigenvalue, an

$$\overrightarrow{y}_2(x) = e^{\lambda x}\left(x\overrightarrow{v} + \overrightarrow{p}\right)$$

- Note that the x multiplying $\overrightarrow{v}$ makes this solution linearly independent to the 1st solution obt
- Substitute $\overrightarrow{y}_2(x)$ into the homogeneous system

$$\lambda\, e^{\lambda x}\left(x\overrightarrow{v} + \overrightarrow{p}\right) + e^{\lambda x}\overrightarrow{v} = \left(e^{\lambda x} A\right) \cdot \left(x\overrightarrow{v} + \overrightarrow{p}\right)$$

- Use the fact that $\overrightarrow{v}$ is an eigenvector of A

$$\lambda\, e^{\lambda x}\left(x\overrightarrow{v} + \overrightarrow{p}\right) + e^{\lambda x}\overrightarrow{v} = e^{\lambda x}\left(\lambda x\overrightarrow{v} + A \cdot \overrightarrow{p}\right)$$

- Simplify equation

$$\lambda\overrightarrow{p} + \overrightarrow{v} = A \cdot \overrightarrow{p}$$

- Make use of the identity matrix I

$$(\lambda \cdot I) \cdot \overrightarrow{p} + \overrightarrow{v} = A \cdot \overrightarrow{p}$$

- Condition $\overrightarrow{p}$ must meet for $\overrightarrow{y}_2(x)$ to be a solution to the homogeneous system

$$(A - \lambda \cdot I) \cdot \overrightarrow{p} = \overrightarrow{v}$$

- Choose $\overrightarrow{p}$ to use in the second solution to the homogeneous system from eigenvalue 1

$$\left(\begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & 2 & -2 & 2 \end{bmatrix} - 1 \cdot \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}\right) \cdot \overrightarrow{p} = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$$

- Choice of $\overrightarrow{p}$

$$\overrightarrow{p} = \begin{bmatrix} -1 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

- Second solution from eigenvalue 1

$$\vec{y}_2(x) = e^x \cdot \left(x \cdot \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} + \begin{bmatrix} -1 \\ 0 \\ 0 \\ 0 \end{bmatrix}\right)$$

- Consider complex eigenpair, complex conjugate eigenvalue can be ignored

$$\left[-I, \begin{bmatrix} -I \\ -1 \\ I \\ 1 \end{bmatrix}\right]$$

- Solution from eigenpair

$$e^{-Ix} \cdot \begin{bmatrix} -I \\ -1 \\ I \\ 1 \end{bmatrix}$$

- Use Euler identity to write solution in terms of $\sin$ and $\cos$

$$(\cos(x) - I\sin(x)) \cdot \begin{bmatrix} -I \\ -1 \\ I \\ 1 \end{bmatrix}$$

- Simplify expression

$$\begin{bmatrix} -I(\cos(x) - I\sin(x)) \\ -\cos(x) + I\sin(x) \\ I(\cos(x) - I\sin(x)) \\ \cos(x) - I\sin(x) \end{bmatrix}$$

- Both real and imaginary parts are solutions to the homogeneous system

$$\left[\vec{y}_3(x) = \begin{bmatrix} -\sin(x) \\ -\cos(x) \\ \sin(x) \\ \cos(x) \end{bmatrix}, \vec{y}_4(x) = \begin{bmatrix} -\cos(x) \\ \sin(x) \\ \cos(x) \\ -\sin(x) \end{bmatrix}\right]$$

- General solution to the system of ODEs

$$\vec{y} = c_1\vec{y}_1(x) + c_2\vec{y}_2(x) + c_3\vec{y}_3(x) + c_4\vec{y}_4(x)$$

- Substitute solutions into the general solution

$$\vec{y} = c_1 e^x \cdot \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} + c_2 e^x \cdot \left(x \cdot \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} + \begin{bmatrix} -1 \\ 0 \\ 0 \\ 0 \end{bmatrix}\right) + \begin{bmatrix} -c_3\sin(x) - \cos(x) c_4 \\ -c_3\cos(x) + \sin(x) c_4 \\ c_3\sin(x) + \cos(x) c_4 \\ c_3\cos(x) - \sin(x) c_4 \end{bmatrix}$$

- First component of the vector is the solution to the ODE

$$y = ((x-1) c_2 + c_1) e^x - c_3 \sin(x) - \cos(x) c_4$$

Maple trace

```
`Methods for high order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
<- constant coefficients successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 21

```
dsolve(diff(y(x),x$4)-2*diff(y(x),x$3)+2*diff(y(x),x$2)-2*diff(y(x),x)+y(x)=0,y(x), singsol=
```

$$y(x) = e^x (c_2 x + c_1) + c_3 \sin(x) + c_4 \cos(x)$$

✓ Solution by Mathematica

Time used: 0.003 (sec). Leaf size: 27

```
DSolve[y''''[x]-2*y'''[x]+2*y''[x]-2*y'[x]+y[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to e^x(c_4x + c_3) + c_1 \cos(x) + c_2 \sin(x)$$

2.10 problem 10

Internal problem ID [3251]
Internal file name [`OUTPUT/2743_Sunday_June_05_2022_08_39_56_AM_70210603/index.tex`]

Book: Differential equations for engineers by Wei-Chau XIE, Cambridge Press 2010
Section: Chapter 4. Linear Differential Equations. Page 183
Problem number: 10.
ODE order: 4.
ODE degree: 1.

The type(s) of ODE detected by this program : **"higher_order_linear_constant_coefficients_ODE"**

Maple gives the following as the ode type

`[[_high_order, _missing_x]]`

$$\boxed{y'''' - 5y''' + 5y'' + 5y' - 6y = 0}$$

The characteristic equation is

$$\lambda^4 - 5\lambda^3 + 5\lambda^2 + 5\lambda - 6 = 0$$

The roots of the above equation are

$$\lambda_1 = 1$$
$$\lambda_2 = 2$$
$$\lambda_3 = 3$$
$$\lambda_4 = -1$$

Therefore the homogeneous solution is

$$y_h(x) = c_1 e^{-x} + c_2 e^x + c_3 e^{2x} + e^{3x} c_4$$

The fundamental set of solutions for the homogeneous solution are the following

$$y_1 = e^{-x}$$
$$y_2 = e^x$$
$$y_3 = e^{2x}$$
$$y_4 = e^{3x}$$

Summary
The solution(s) found are the following

$$y = c_1 \mathrm{e}^{-x} + c_2 \mathrm{e}^{x} + c_3 \mathrm{e}^{2x} + \mathrm{e}^{3x} c_4 \tag{1}$$

Verification of solutions

$$y = c_1 \mathrm{e}^{-x} + c_2 \mathrm{e}^{x} + c_3 \mathrm{e}^{2x} + \mathrm{e}^{3x} c_4$$

Verified OK.

2.10.1 Maple step by step solution

Let's solve

$y'''' - 5y''' + 5y'' + 5y' - 6y = 0$

- Highest derivative means the order of the ODE is 4

  y''''

- □ Convert linear ODE into a system of first order ODEs
  - Define new variable $y_1(x)$

    $y_1(x) = y$
  - Define new variable $y_2(x)$

    $y_2(x) = y'$
  - Define new variable $y_3(x)$

    $y_3(x) = y''$
  - Define new variable $y_4(x)$

    $y_4(x) = y'''$
  - Isolate for $y_4'(x)$ using original ODE

    $y_4'(x) = 5y_4(x) - 5y_3(x) - 5y_2(x) + 6y_1(x)$

  Convert linear ODE into a system of first order ODEs

  $[y_2(x) = y_1'(x), y_3(x) = y_2'(x), y_4(x) = y_3'(x), y_4'(x) = 5y_4(x) - 5y_3(x) - 5y_2(x) + 6y_1(x)]$

- Define vector

$$\vec{y}(x)=\begin{bmatrix} y_1(x) \\ y_2(x) \\ y_3(x) \\ y_4(x) \end{bmatrix}$$

- System to solve

$$\vec{y}'(x)=\begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 6 & -5 & -5 & 5 \end{bmatrix}\cdot \vec{y}(x)$$

- Define the coefficient matrix

$$A=\begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 6 & -5 & -5 & 5 \end{bmatrix}$$

- Rewrite the system as

$$\vec{y}'(x)=A\cdot \vec{y}(x)$$

- To solve the system, find the eigenvalues and eigenvectors of A
- Eigenpairs of A

$$\left[\left[-1, \begin{bmatrix} -1 \\ 1 \\ -1 \\ 1 \end{bmatrix}\right], \left[1, \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}\right], \left[2, \begin{bmatrix} \frac{1}{8} \\ \frac{1}{4} \\ \frac{1}{2} \\ 1 \end{bmatrix}\right], \left[3, \begin{bmatrix} \frac{1}{27} \\ \frac{1}{9} \\ \frac{1}{3} \\ 1 \end{bmatrix}\right]\right]$$

- Consider eigenpair

$$\left[-1, \begin{bmatrix} -1 \\ 1 \\ -1 \\ 1 \end{bmatrix}\right]$$

- Solution to homogeneous system from eigenpair

$$\vec{y}_1 = \mathrm{e}^{-x} \cdot \begin{bmatrix} -1 \\ 1 \\ -1 \\ 1 \end{bmatrix}$$

- Consider eigenpair

$$\left[1, \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}\right]$$

- Solution to homogeneous system from eigenpair

$$\vec{y}_2 = \mathrm{e}^{x} \cdot \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$$

- Consider eigenpair

$$\left[2, \begin{bmatrix} \frac{1}{8} \\ \frac{1}{4} \\ \frac{1}{2} \\ 1 \end{bmatrix}\right]$$

- Solution to homogeneous system from eigenpair

$$\vec{y}_3 = \mathrm{e}^{2x} \cdot \begin{bmatrix} \frac{1}{8} \\ \frac{1}{4} \\ \frac{1}{2} \\ 1 \end{bmatrix}$$

- Consider eigenpair

$$\left[3, \begin{bmatrix} \frac{1}{27} \\ \frac{1}{9} \\ \frac{1}{3} \\ 1 \end{bmatrix}\right]$$

- Solution to homogeneous system from eigenpair

$$\vec{y}_4 = e^{3x} \cdot \begin{bmatrix} \frac{1}{27} \\ \frac{1}{9} \\ \frac{1}{3} \\ 1 \end{bmatrix}$$

- General solution to the system of ODEs

$$\vec{y} = c_1 \vec{y}_1 + c_2 \vec{y}_2 + c_3 \vec{y}_3 + c_4 \vec{y}_4$$

- Substitute solutions into the general solution

$$\vec{y} = c_1 e^{-x} \cdot \begin{bmatrix} -1 \\ 1 \\ -1 \\ 1 \end{bmatrix} + c_2 e^{x} \cdot \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} + c_3 e^{2x} \cdot \begin{bmatrix} \frac{1}{8} \\ \frac{1}{4} \\ \frac{1}{2} \\ 1 \end{bmatrix} + e^{3x} c_4 \cdot \begin{bmatrix} \frac{1}{27} \\ \frac{1}{9} \\ \frac{1}{3} \\ 1 \end{bmatrix}$$

- First component of the vector is the solution to the ODE

$$y = -c_1 e^{-x} + c_2 e^{x} + \frac{c_3 e^{2x}}{8} + \frac{e^{3x} c_4}{27}$$

Maple trace

```
`Methods for high order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
<- constant coefficients successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 27

```
dsolve(diff(y(x),x$4)-5*diff(y(x),x$3)+5*diff(y(x),x$2)+5*diff(y(x),x)-6*y(x)=0,y(x), singso
```

$$y(x) = c_1 e^{3x} + c_2 e^{2x} + c_3 e^{-x} + c_4 e^{x}$$

✓ Solution by Mathematica

Time used: 0.003 (sec). Leaf size: 36

```
DSolve[y''''[x]-5*y'''[x]+5*y''[x]+5*y'[x]-6*y[x]==0,y[x],x,IncludeSingularSolutions -> True
```

$$y(x) \to c_1 e^{-x} + c_2 e^{x} + c_3 e^{2x} + c_4 e^{3x}$$

2.11 problem 11

Internal problem ID [3252]
Internal file name [`OUTPUT/2744_Sunday_June_05_2022_08_39_56_AM_32753829/index.tex`]

Book: Differential equations for engineers by Wei-Chau XIE, Cambridge Press 2010
Section: Chapter 4. Linear Differential Equations. Page 183
Problem number: 11.
ODE order: 5.
ODE degree: 1.

The type(s) of ODE detected by this program : **"higher_order_linear_constant_coefficients_ODE"**

Maple gives the following as the ode type

```
[[_high_order, _missing_x]]
```

$$\boxed{y^{(5)} - 6y'''' + 9y''' = 0}$$

The characteristic equation is

$$\lambda^5 - 6\lambda^4 + 9\lambda^3 = 0$$

The roots of the above equation are

$$\lambda_1 = 0$$
$$\lambda_2 = 0$$
$$\lambda_3 = 0$$
$$\lambda_4 = 3$$
$$\lambda_5 = 3$$

Therefore the homogeneous solution is

$$y_h(x) = c_3x^2 + c_2x + c_1 + \mathrm{e}^{3x}c_4 + x\,\mathrm{e}^{3x}c_5$$

The fundamental set of solutions for the homogeneous solution are the following

$$
\begin{aligned}
y_1 &= 1 \\
y_2 &= x \\
y_3 &= x^2 \\
y_4 &= \mathrm{e}^{3x} \\
y_5 &= x\,\mathrm{e}^{3x}
\end{aligned}
$$

<u>Summary</u>
The solution(s) found are the following

$$y = c_3x^2 + c_2x + c_1 + \mathrm{e}^{3x}c_4 + x\,\mathrm{e}^{3x}c_5 \tag{1}$$

<u>Verification of solutions</u>

$$y = c_3x^2 + c_2x + c_1 + \mathrm{e}^{3x}c_4 + x\,\mathrm{e}^{3x}c_5$$

Verified OK.

2.11.1 Maple step by step solution

Let's solve
$y^{(5)} - 6y'''' + 9y''' = 0$

- Highest derivative means the order of the ODE is 5
  $y^{(5)}$

□ Convert linear ODE into a system of first order ODEs

  ◦ Define new variable $y_1(x)$
    $y_1(x) = y$
  ◦ Define new variable $y_2(x)$
    $y_2(x) = y'$
  ◦ Define new variable $y_3(x)$
    $y_3(x) = y''$
  ◦ Define new variable $y_4(x)$
    $y_4(x) = y'''$
  ◦ Define new variable $y_5(x)$
    $y_5(x) = y''''$

- Isolate for $y_5'(x)$ using original ODE

  $y_5'(x) = 6y_5(x) - 9y_4(x)$

  Convert linear ODE into a system of first order ODEs

  $[y_2(x) = y_1'(x), y_3(x) = y_2'(x), y_4(x) = y_3'(x), y_5(x) = y_4'(x), y_5'(x) = 6y_5(x) - 9y_4(x)]$

- Define vector

$$\vec{y}(x) = \begin{bmatrix} y_1(x) \\ y_2(x) \\ y_3(x) \\ y_4(x) \\ y_5(x) \end{bmatrix}$$

- System to solve

$$\vec{y}'(x) = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & -9 & 6 \end{bmatrix} \cdot \vec{y}(x)$$

- Define the coefficient matrix

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & -9 & 6 \end{bmatrix}$$

- Rewrite the system as

  $\vec{y}'(x) = A \cdot \vec{y}(x)$

- To solve the system, find the eigenvalues and eigenvectors of A
- Eigenpairs of A

$$\left[\left[0, \begin{bmatrix}1\\0\\0\\0\\0\end{bmatrix}\right], \left[0, \begin{bmatrix}0\\0\\0\\0\\0\end{bmatrix}\right], \left[0, \begin{bmatrix}0\\0\\0\\0\\0\end{bmatrix}\right], \left[3, \begin{bmatrix}\frac{1}{81}\\\frac{1}{27}\\\frac{1}{9}\\\frac{1}{3}\\1\end{bmatrix}\right], \left[3, \begin{bmatrix}0\\0\\0\\0\\0\end{bmatrix}\right]\right]$$

- Consider eigenpair

$$\left[0, \begin{bmatrix}1\\0\\0\\0\\0\end{bmatrix}\right]$$

- Solution to homogeneous system from eigenpair

$$\vec{y}_1 = \begin{bmatrix}1\\0\\0\\0\\0\end{bmatrix}$$

- Consider eigenpair

$$\left[0, \begin{bmatrix}0\\0\\0\\0\\0\end{bmatrix}\right]$$

- Solution to homogeneous system from eigenpair

$$\vec{y}_2 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

- Consider eigenpair

$$\left[0, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}\right]$$

- Solution to homogeneous system from eigenpair

$$\vec{y}_3 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

- Consider eigenpair, with eigenvalue of algebraic multiplicity 2

$$\left[3, \begin{bmatrix} \frac{1}{81} \\ \frac{1}{27} \\ \frac{1}{9} \\ \frac{1}{3} \\ 1 \end{bmatrix}\right]$$

- First solution from eigenvalue 3

$$\overrightarrow{y}_{4}(x) = \mathrm{e}^{3x} \cdot \begin{bmatrix} \frac{1}{81} \\ \frac{1}{27} \\ \frac{1}{9} \\ \frac{1}{3} \\ 1 \end{bmatrix}$$

- Form of the 2nd homogeneous solution where $\overrightarrow{p}$ is to be solved for, $\lambda = 3$ is the eigenvalue, an

  $\overrightarrow{y}_{5}(x) = \mathrm{e}^{\lambda x}\left(x\overrightarrow{v} + \overrightarrow{p}\right)$

- Note that the x multiplying $\overrightarrow{v}$ makes this solution linearly independent to the 1st solution obt
- Substitute $\overrightarrow{y}_{5}(x)$ into the homogeneous system

  $\lambda \mathrm{e}^{\lambda x}\left(x\overrightarrow{v} + \overrightarrow{p}\right) + \mathrm{e}^{\lambda x}\overrightarrow{v} = \left(\mathrm{e}^{\lambda x} A\right) \cdot \left(x\overrightarrow{v} + \overrightarrow{p}\right)$

- Use the fact that $\overrightarrow{v}$ is an eigenvector of A

  $\lambda \mathrm{e}^{\lambda x}\left(x\overrightarrow{v} + \overrightarrow{p}\right) + \mathrm{e}^{\lambda x}\overrightarrow{v} = \mathrm{e}^{\lambda x}\left(\lambda x\overrightarrow{v} + A \cdot \overrightarrow{p}\right)$

- Simplify equation

  $\lambda\overrightarrow{p} + \overrightarrow{v} = A \cdot \overrightarrow{p}$

- Make use of the identity matrix I

  $(\lambda \cdot I) \cdot \overrightarrow{p} + \overrightarrow{v} = A \cdot \overrightarrow{p}$

- Condition $\overrightarrow{p}$ must meet for $\overrightarrow{y}_{5}(x)$ to be a solution to the homogeneous system

  $(A - \lambda \cdot I) \cdot \overrightarrow{p} = \overrightarrow{v}$

- Choose $\overrightarrow{p}$ to use in the second solution to the homogeneous system from eigenvalue 3

  $$\left(\begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & -9 & 6 \end{bmatrix} - 3 \cdot \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}\right) \cdot \overrightarrow{p} = \begin{bmatrix} \frac{1}{81} \\ \frac{1}{27} \\ \frac{1}{9} \\ \frac{1}{3} \\ 1 \end{bmatrix}$$

- Choice of $\overrightarrow{p}$

$$\vec{p} = \begin{bmatrix} -\frac{1}{243} \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

- Second solution from eigenvalue 3

$$\vec{y}_5(x) = e^{3x} \cdot \left(x \cdot \begin{bmatrix} \frac{1}{81} \\ \frac{1}{27} \\ \frac{1}{9} \\ \frac{1}{3} \\ 1 \end{bmatrix} + \begin{bmatrix} -\frac{1}{243} \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \right)$$

- General solution to the system of ODEs

$$\vec{y} = c_1 \vec{y}_1 + c_2 \vec{y}_2 + c_3 \vec{y}_3 + c_4 \vec{y}_4(x) + c_5 \vec{y}_5(x)$$

- Substitute solutions into the general solution

$$\vec{y} = e^{3x} c_4 \cdot \begin{bmatrix} \frac{1}{81} \\ \frac{1}{27} \\ \frac{1}{9} \\ \frac{1}{3} \\ 1 \end{bmatrix} + c_5 e^{3x} \cdot \left(x \cdot \begin{bmatrix} \frac{1}{81} \\ \frac{1}{27} \\ \frac{1}{9} \\ \frac{1}{3} \\ 1 \end{bmatrix} + \begin{bmatrix} -\frac{1}{243} \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \right) + \begin{bmatrix} c_1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

- First component of the vector is the solution to the ODE

$$y = \frac{((3x-1)c_5 + 3c_4) e^{3x}}{243} + c_1$$

Maple trace

```
`Methods for high order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
<- constant coefficients successful`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 24

```
dsolve(diff(y(x),x$5)-6*diff(y(x),x$4)+9*diff(y(x),x$3)=0,y(x), singsol=all)
```

$$y(x) = (c_5x + c_4)\,\mathrm{e}^{3x} + c_3x^2 + c_2x + c_1$$

✓ Solution by Mathematica

Time used: 0.143 (sec). Leaf size: 35

```
DSolve[y'''''[x]-6*y''''[x]+9*y'''[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \frac{1}{27}e^{3x}(c_2(x-1)+c_1) + x(c_5x + c_4) + c_3$$

2.12 problem 12

Internal problem ID [3253]
Internal file name [`OUTPUT/2745_Sunday_June_05_2022_08_39_57_AM_29914027/index.tex`]

Book: Differential equations for engineers by Wei-Chau XIE, Cambridge Press 2010
Section: Chapter 4. Linear Differential Equations. Page 183
Problem number: 12.
ODE order: 6.
ODE degree: 1.

The type(s) of ODE detected by this program : **"higher_order_linear_constant_coefficients_ODE"**

Maple gives the following as the ode type

```
[[_high_order, _missing_x]]
```

$$\boxed{y^{(6)} - 64y = 0}$$

The characteristic equation is

$$\lambda^6 - 64 = 0$$

The roots of the above equation are

$$\begin{aligned}
\lambda_1 &= 2 \\
\lambda_2 &= -2 \\
\lambda_3 &= \sqrt{-2 + 2i\sqrt{3}} \\
\lambda_4 &= -\sqrt{-2 + 2i\sqrt{3}} \\
\lambda_5 &= \sqrt{-2i\sqrt{3} - 2} \\
\lambda_6 &= -\sqrt{-2i\sqrt{3} - 2}
\end{aligned}$$

Therefore the homogeneous solution is

$$y_h(x) = c_1 \mathrm{e}^{-2x} + c_2 \mathrm{e}^{2x} + \mathrm{e}^{\sqrt{-2i\sqrt{3}-2}\,x} c_3 + \mathrm{e}^{-\sqrt{-2+2i\sqrt{3}}\,x} c_4 + \mathrm{e}^{\sqrt{-2+2i\sqrt{3}}\,x} c_5 + \mathrm{e}^{-\sqrt{-2i\sqrt{3}-2}\,x} c_6$$

The fundamental set of solutions for the homogeneous solution are the following

$$
\begin{aligned}
y_1 &= e^{-2x} \\
y_2 &= e^{2x} \\
y_3 &= e^{\sqrt{-2i\sqrt{3}-2}\,x} \\
y_4 &= e^{-\sqrt{-2+2i\sqrt{3}}\,x} \\
y_5 &= e^{\sqrt{-2+2i\sqrt{3}}\,x} \\
y_6 &= e^{-\sqrt{-2i\sqrt{3}-2}\,x}
\end{aligned}
$$

Summary

The solution(s) found are the following

$$
y = c_1 e^{-2x} + c_2 e^{2x} + e^{\sqrt{-2i\sqrt{3}-2}\,x} c_3 + e^{-\sqrt{-2+2i\sqrt{3}}\,x} c_4 + e^{\sqrt{-2+2i\sqrt{3}}\,x} c_5 + e^{-\sqrt{-2i\sqrt{3}-2}\,x} c_6 \tag{1}
$$

Verification of solutions

$$
y = c_1 e^{-2x} + c_2 e^{2x} + e^{\sqrt{-2i\sqrt{3}-2}\,x} c_3 + e^{-\sqrt{-2+2i\sqrt{3}}\,x} c_4 + e^{\sqrt{-2+2i\sqrt{3}}\,x} c_5 + e^{-\sqrt{-2i\sqrt{3}-2}\,x} c_6
$$

Verified OK.

2.12.1 Maple step by step solution

Let's solve

$y^{(6)} - 64y = 0$

- Highest derivative means the order of the ODE is 6

  $y^{(6)}$

- □ Convert linear ODE into a system of first order ODEs
  - Define new variable $y_1(x)$

    $y_1(x) = y$
  - Define new variable $y_2(x)$

    $y_2(x) = y'$
  - Define new variable $y_3(x)$

    $y_3(x) = y''$
  - Define new variable $y_4(x)$

    $y_4(x) = y'''$

- Define new variable $y_5(x)$

  $y_5(x) = y''''$
- Define new variable $y_6(x)$

  $y_6(x) = y^{(5)}$
- Isolate for $y_6'(x)$ using original ODE

  $y_6'(x) = 64y_1(x)$

  Convert linear ODE into a system of first order ODEs

  $[y_2(x) = y_1'(x), y_3(x) = y_2'(x), y_4(x) = y_3'(x), y_5(x) = y_4'(x), y_6(x) = y_5'(x), y_6'(x) = 64y_1(x)]$

- Define vector

$$\vec{y}(x) = \begin{bmatrix} y_1(x) \\ y_2(x) \\ y_3(x) \\ y_4(x) \\ y_5(x) \\ y_6(x) \end{bmatrix}$$

- System to solve

$$\vec{y}'(x) = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 64 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \cdot \vec{y}(x)$$

- Define the coefficient matrix

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 64 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

- Rewrite the system as

  $\overrightarrow{y}'(x) = A \cdot \overrightarrow{y}(x)$

- To solve the system, find the eigenvalues and eigenvectors of A
- Eigenpairs of A

  $$\left[\left[-2, \begin{bmatrix} -\frac{1}{32} \\ \frac{1}{16} \\ -\frac{1}{8} \\ \frac{1}{4} \\ -\frac{1}{2} \\ 1 \end{bmatrix}\right], \left[2, \begin{bmatrix} \frac{1}{32} \\ \frac{1}{16} \\ \frac{1}{8} \\ \frac{1}{4} \\ \frac{1}{2} \\ 1 \end{bmatrix}\right], \left[-1 - \mathrm{I}\sqrt{3}, \begin{bmatrix} \frac{1}{\left(-1-\mathrm{I}\sqrt{3}\right)^5} \\ \frac{1}{\left(-1-\mathrm{I}\sqrt{3}\right)^4} \\ \frac{1}{\left(-1-\mathrm{I}\sqrt{3}\right)^3} \\ \frac{1}{\left(-1-\mathrm{I}\sqrt{3}\right)^2} \\ \frac{1}{-1-\mathrm{I}\sqrt{3}} \\ 1 \end{bmatrix}\right], \left[-1 + \mathrm{I}\sqrt{3}, \begin{bmatrix} \frac{1}{\left(-1+\mathrm{I}\sqrt{3}\right)^5} \\ \frac{1}{\left(-1+\mathrm{I}\sqrt{3}\right)^4} \\ \frac{1}{\left(-1+\mathrm{I}\sqrt{3}\right)^3} \\ \frac{1}{\left(-1+\mathrm{I}\sqrt{3}\right)^2} \\ \frac{1}{-1+\mathrm{I}\sqrt{3}} \\ 1 \end{bmatrix}\right], \right.$$

- Consider eigenpair

  $$\left[-2, \begin{bmatrix} -\frac{1}{32} \\ \frac{1}{16} \\ -\frac{1}{8} \\ \frac{1}{4} \\ -\frac{1}{2} \\ 1 \end{bmatrix}\right]$$

- Solution to homogeneous system from eigenpair

  $$\overrightarrow{y}_1 = \mathrm{e}^{-2x} \cdot \begin{bmatrix} -\frac{1}{32} \\ \frac{1}{16} \\ -\frac{1}{8} \\ \frac{1}{4} \\ -\frac{1}{2} \\ 1 \end{bmatrix}$$

- Consider eigenpair

$$\left[2, \begin{bmatrix} \frac{1}{32} \\ \frac{1}{16} \\ \frac{1}{8} \\ \frac{1}{4} \\ \frac{1}{2} \\ 1 \end{bmatrix}\right]$$

- Solution to homogeneous system from eigenpair

$$\overrightarrow{y}_2 = e^{2x} \cdot \begin{bmatrix} \frac{1}{32} \\ \frac{1}{16} \\ \frac{1}{8} \\ \frac{1}{4} \\ \frac{1}{2} \\ 1 \end{bmatrix}$$

- Consider complex eigenpair, complex conjugate eigenvalue can be ignored

$$\left[-1 - I\sqrt{3}, \begin{bmatrix} \frac{1}{\left(-1-I\sqrt{3}\right)^5} \\ \frac{1}{\left(-1-I\sqrt{3}\right)^4} \\ \frac{1}{\left(-1-I\sqrt{3}\right)^3} \\ \frac{1}{\left(-1-I\sqrt{3}\right)^2} \\ \frac{1}{-1-I\sqrt{3}} \\ 1 \end{bmatrix}\right]$$

- Solution from eigenpair

$$e^{\left(-1-I\sqrt{3}\right)x} \cdot \left[\begin{array}{c} \frac{1}{\left(-1-I\sqrt{3}\right)^{5}} \\ \frac{1}{\left(-1-I\sqrt{3}\right)^{4}} \\ \frac{1}{\left(-1-I\sqrt{3}\right)^{3}} \\ \frac{1}{\left(-1-I\sqrt{3}\right)^{2}} \\ \frac{1}{-1-I\sqrt{3}} \\ 1 \end{array}\right]$$

- Use Euler identity to write solution in terms of sin and cos

$$e^{-x} \cdot \left(\cos\left(\sqrt{3}\,x\right) - I\sin\left(\sqrt{3}\,x\right)\right) \cdot \left[\begin{array}{c} \frac{1}{\left(-1-I\sqrt{3}\right)^{5}} \\ \frac{1}{\left(-1-I\sqrt{3}\right)^{4}} \\ \frac{1}{\left(-1-I\sqrt{3}\right)^{3}} \\ \frac{1}{\left(-1-I\sqrt{3}\right)^{2}} \\ \frac{1}{-1-I\sqrt{3}} \\ 1 \end{array}\right]$$

- Simplify expression

$$e^{-x} \cdot \left[\begin{array}{c} \frac{\cos\left(\sqrt{3}\,x\right)-I\sin\left(\sqrt{3}\,x\right)}{\left(-1-I\sqrt{3}\right)^{5}} \\ \frac{\cos\left(\sqrt{3}\,x\right)-I\sin\left(\sqrt{3}\,x\right)}{\left(-1-I\sqrt{3}\right)^{4}} \\ \frac{\cos\left(\sqrt{3}\,x\right)-I\sin\left(\sqrt{3}\,x\right)}{\left(-1-I\sqrt{3}\right)^{3}} \\ \frac{\cos\left(\sqrt{3}\,x\right)-I\sin\left(\sqrt{3}\,x\right)}{\left(-1-I\sqrt{3}\right)^{2}} \\ \frac{\cos\left(\sqrt{3}\,x\right)-I\sin\left(\sqrt{3}\,x\right)}{-1-I\sqrt{3}} \\ \cos\left(\sqrt{3}\,x\right) - I\sin\left(\sqrt{3}\,x\right) \end{array}\right]$$

- Both real and imaginary parts are solutions to the homogeneous system

$$\left[\vec{y}_3(x) = e^{-x} \cdot \begin{bmatrix} \Re\left(\frac{\cos(\sqrt{3}\,x) - I\sin(\sqrt{3}\,x)}{(-1-I\sqrt{3})^5}\right) \\ \Re\left(\frac{\cos(\sqrt{3}\,x) - I\sin(\sqrt{3}\,x)}{(-1-I\sqrt{3})^4}\right) \\ \frac{\cos(\sqrt{3}\,x)}{8} \\ -\frac{\cos(\sqrt{3}\,x)}{8} - \frac{\sin(\sqrt{3}\,x)\sqrt{3}}{8} \\ -\frac{\cos(\sqrt{3}\,x)}{4} + \frac{\sin(\sqrt{3}\,x)\sqrt{3}}{4} \\ \cos(\sqrt{3}\,x) \end{bmatrix}, \vec{y}_4(x) = e^{-x} \cdot \begin{bmatrix} \Im\left(\frac{\cos(\sqrt{3}\,x) - I\sin(\sqrt{3}\,x)}{(-1-I\sqrt{3})^5}\right) \\ \Im\left(\frac{\cos(\sqrt{3}\,x) - I\sin(\sqrt{3}\,x)}{(-1-I\sqrt{3})^4}\right) \\ -\frac{\sin(\sqrt{3}\,x)}{8} \\ -\frac{\cos(\sqrt{3}\,x)\sqrt{3}}{8} + \frac{\sin(\sqrt{3}\,x)}{8} \\ \frac{\cos(\sqrt{3}\,x)\sqrt{3}}{4} + \frac{\sin(\sqrt{3}\,x)}{4} \\ -\sin(\sqrt{3}\,x) \end{bmatrix}\right]$$

- Consider complex eigenpair, complex conjugate eigenvalue can be ignored

$$\left[1 - I\sqrt{3}, \begin{bmatrix} \frac{1}{(1-I\sqrt{3})^5} \\ \frac{1}{(1-I\sqrt{3})^4} \\ \frac{1}{(1-I\sqrt{3})^3} \\ \frac{1}{(1-I\sqrt{3})^2} \\ \frac{1}{1-I\sqrt{3}} \\ 1 \end{bmatrix}\right]$$

- Solution from eigenpair

$$e^{(1-I\sqrt{3})x} \cdot \begin{bmatrix} \frac{1}{(1-I\sqrt{3})^5} \\ \frac{1}{(1-I\sqrt{3})^4} \\ \frac{1}{(1-I\sqrt{3})^3} \\ \frac{1}{(1-I\sqrt{3})^2} \\ \frac{1}{1-I\sqrt{3}} \\ 1 \end{bmatrix}$$

- Use Euler identity to write solution in terms of $\sin$ and $\cos$

$$e^{x} \cdot \left(\cos\left(\sqrt{3}\,x\right) - I\sin\left(\sqrt{3}\,x\right)\right) \cdot \left[\begin{array}{c} \frac{1}{\left(1-I\sqrt{3}\right)^{5}} \\ \frac{1}{\left(1-I\sqrt{3}\right)^{4}} \\ \frac{1}{\left(1-I\sqrt{3}\right)^{3}} \\ \frac{1}{\left(1-I\sqrt{3}\right)^{2}} \\ \frac{1}{1-I\sqrt{3}} \\ 1 \end{array}\right]$$

- Simplify expression

$$e^{x} \cdot \left[\begin{array}{c} \frac{\cos\left(\sqrt{3}\,x\right)-I\sin\left(\sqrt{3}\,x\right)}{\left(1-I\sqrt{3}\right)^{5}} \\ \frac{\cos\left(\sqrt{3}\,x\right)-I\sin\left(\sqrt{3}\,x\right)}{\left(1-I\sqrt{3}\right)^{4}} \\ \frac{\cos\left(\sqrt{3}\,x\right)-I\sin\left(\sqrt{3}\,x\right)}{\left(1-I\sqrt{3}\right)^{3}} \\ \frac{\cos\left(\sqrt{3}\,x\right)-I\sin\left(\sqrt{3}\,x\right)}{\left(1-I\sqrt{3}\right)^{2}} \\ \frac{\cos\left(\sqrt{3}\,x\right)-I\sin\left(\sqrt{3}\,x\right)}{1-I\sqrt{3}} \\ \cos\left(\sqrt{3}\,x\right) - I\sin\left(\sqrt{3}\,x\right) \end{array}\right]$$

- Both real and imaginary parts are solutions to the homogeneous system

$$\left[\vec{y}_{5}(x) = e^{x} \cdot \left[\begin{array}{c} \Re\left(\frac{\cos\left(\sqrt{3}\,x\right)-I\sin\left(\sqrt{3}\,x\right)}{\left(1-I\sqrt{3}\right)^{5}}\right) \\ \Re\left(\frac{\cos\left(\sqrt{3}\,x\right)-I\sin\left(\sqrt{3}\,x\right)}{\left(1-I\sqrt{3}\right)^{4}}\right) \\ -\frac{\cos\left(\sqrt{3}\,x\right)}{8} \\ -\frac{\cos\left(\sqrt{3}\,x\right)}{8} + \frac{\sin\left(\sqrt{3}\,x\right)\sqrt{3}}{8} \\ \frac{\cos\left(\sqrt{3}\,x\right)}{4} + \frac{\sin\left(\sqrt{3}\,x\right)\sqrt{3}}{4} \\ \cos\left(\sqrt{3}\,x\right) \end{array}\right], \vec{y}_{6}(x) = e^{x} \cdot \left[\begin{array}{c} \Im\left(\frac{\cos\left(\sqrt{3}\,x\right)-I\sin\left(\sqrt{3}\,x\right)}{\left(1-I\sqrt{3}\right)^{5}}\right) \\ \Im\left(\frac{\cos\left(\sqrt{3}\,x\right)-I\sin\left(\sqrt{3}\,x\right)}{\left(1-I\sqrt{3}\right)^{4}}\right) \\ \frac{\sin\left(\sqrt{3}\,x\right)}{8} \\ \frac{\cos\left(\sqrt{3}\,x\right)\sqrt{3}}{8} + \frac{\sin\left(\sqrt{3}\,x\right)}{8} \\ \frac{\cos\left(\sqrt{3}\,x\right)\sqrt{3}}{4} - \frac{\sin\left(\sqrt{3}\,x\right)}{4} \\ -\sin\left(\sqrt{3}\,x\right) \end{array}\right]\right]$$

- General solution to the system of ODEs
  $\overrightarrow{y} = c_1 \overrightarrow{y}_1 + c_2 \overrightarrow{y}_2 + c_3 \overrightarrow{y}_3(x) + c_4 \overrightarrow{y}_4(x) + c_5 \overrightarrow{y}_5(x) + c_6 \overrightarrow{y}_6(x)$
- Substitute solutions into the general solution

$$\overrightarrow{y} = c_1 \mathrm{e}^{-2x} \cdot \begin{bmatrix} -\frac{1}{32} \\ \frac{1}{16} \\ -\frac{1}{8} \\ \frac{1}{4} \\ -\frac{1}{2} \\ 1 \end{bmatrix} + c_2 \mathrm{e}^{2x} \cdot \begin{bmatrix} \frac{1}{32} \\ \frac{1}{16} \\ \frac{1}{8} \\ \frac{1}{4} \\ \frac{1}{2} \\ 1 \end{bmatrix} + c_3 \mathrm{e}^{-x} \cdot \begin{bmatrix} \Re\left(\frac{\cos(\sqrt{3}\,x) - \mathrm{I}\sin(\sqrt{3}\,x)}{(-1-\mathrm{I}\sqrt{3})^5}\right) \\ \Re\left(\frac{\cos(\sqrt{3}\,x) - \mathrm{I}\sin(\sqrt{3}\,x)}{(-1-\mathrm{I}\sqrt{3})^4}\right) \\ \frac{\cos(\sqrt{3}\,x)}{8} \\ -\frac{\cos(\sqrt{3}\,x)}{8} - \frac{\sin(\sqrt{3}\,x)\sqrt{3}}{8} \\ -\frac{\cos(\sqrt{3}\,x)}{4} + \frac{\sin(\sqrt{3}\,x)\sqrt{3}}{4} \\ \cos(\sqrt{3}\,x) \end{bmatrix} + c_4 \mathrm{e}^{-x} \cdot \left[\Im\left(\frac{\cos \ldots}{\ldots}\right) \ldots \right.$$

- First component of the vector is the solution to the ODE

$$y = \left(c_6 \mathrm{e}^{3x} \Im\left(\frac{\sin(\sqrt{3}\,x) + \mathrm{I}\cos(\sqrt{3}\,x)}{(\sqrt{3}+\mathrm{I})^5}\right) + c_5 \mathrm{e}^{3x} \Re\left(\frac{\sin(\sqrt{3}\,x) + \mathrm{I}\cos(\sqrt{3}\,x)}{(\sqrt{3}+\mathrm{I})^5}\right) - c_4 \Im\left(\frac{\sin(\sqrt{3}\,x) + \mathrm{I}\cos(\sqrt{3}\,x)}{(\mathrm{I}-\sqrt{3})^5}\right) \mathrm{e} \ldots \right.$$

Maple trace

```
`Methods for high order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
<- constant coefficients successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 53

```
dsolve(diff(y(x),x$6)-64*y(x)=0,y(x), singsol=all)
```

$$y(x) = \mathrm{e}^{-2x}\left(\left(c_4 \mathrm{e}^{3x} + c_6 \mathrm{e}^{x}\right)\cos\left(\sqrt{3}\,x\right) + \left(c_3 \mathrm{e}^{3x} + c_5 \mathrm{e}^{x}\right)\sin\left(\sqrt{3}\,x\right) + \mathrm{e}^{4x} c_1 + c_2\right)$$

✓ Solution by Mathematica

Time used: 0.004 (sec). Leaf size: 68

```
DSolve[y''''''[x]-64*y[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to e^{-2x}\left(c_1e^{4x} + e^x\left(c_2e^{2x} + c_3\right)\cos\left(\sqrt{3}x\right) + e^x\left(c_6e^{2x} + c_5\right)\sin\left(\sqrt{3}x\right) + c_4\right)$$

2.13 problem 13

Internal problem ID [3254]
Internal file name [`OUTPUT/2746_Sunday_June_05_2022_08_39_57_AM_76916014/index.tex`]

Book: Differential equations for engineers by Wei-Chau XIE, Cambridge Press 2010
Section: Chapter 4. Linear Differential Equations. Page 183
Problem number: 13.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_linear_constant_coeff"**

Maple gives the following as the ode type

```
[[_2nd_order, _linear, _nonhomogeneous]]
```

$$\boxed{y'' + 6y' + 10y = 3x\,\mathrm{e}^{-3x} - 2\cos(x)\,\mathrm{e}^{3x}}$$

2.13.1 Solving as second order linear constant coeff ode

This is second order non-homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = f(x)$$

Where $A = 1, B = 6, C = 10, f(x) = 3x\,\mathrm{e}^{-3x} - 2\cos(x)\,\mathrm{e}^{3x}$. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the non-homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = f(x)$. y_h is the solution to

$$y'' + 6y' + 10y = 0$$

This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = 0$$

Where in the above $A = 1, B = 6, C = 10$. Let the solution be $y = e^{\lambda x}$. Substituting this into the ODE gives

$$\lambda^2 \mathrm{e}^{\lambda x} + 6\lambda\, \mathrm{e}^{\lambda x} + 10\, \mathrm{e}^{\lambda x} = 0 \tag{1}$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda x}$ gives

$$\lambda^2 + 6\lambda + 10 = 0 \tag{2}$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form.Using the quadratic formula

$$\lambda_{1,2} = \frac{-B}{2A} \pm \frac{1}{2A}\sqrt{B^2 - 4AC}$$

Substituting $A = 1, B = 6, C = 10$ into the above gives

$$\begin{aligned}\lambda_{1,2} &= \frac{-6}{(2)(1)} \pm \frac{1}{(2)(1)}\sqrt{6^2 - (4)(1)(10)} \\ &= -3 \pm i\end{aligned}$$

Hence

$$\begin{aligned}\lambda_1 &= -3 + i \\ \lambda_2 &= -3 - i\end{aligned}$$

Which simplifies to

$$\begin{aligned}\lambda_1 &= -3 + i \\ \lambda_2 &= -3 - i\end{aligned}$$

Since roots are complex conjugate of each others, then let the roots be

$$\lambda_{1,2} = \alpha \pm i\beta$$

Where $\alpha = -3$ and $\beta = 1$. Therefore the final solution, when using Euler relation, can be written as

$$y = e^{\alpha x}(c_1 \cos(\beta x) + c_2 \sin(\beta x))$$

Which becomes

$$y = e^{-3x}(c_1 \cos(x) + c_2 \sin(x))$$

Therefore the homogeneous solution y_h is

$$y_h = \mathrm{e}^{-3x}(c_1 \cos(x) + c_2 \sin(x))$$

The particular solution is now found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$3x\,\mathrm{e}^{-3x} - 2\cos(x)\,\mathrm{e}^{3x}$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$[\{x\,\mathrm{e}^{-3x}, \mathrm{e}^{-3x}\}, \{\cos(x)\,\mathrm{e}^{3x}, \mathrm{e}^{3x}\sin(x)\}]$$

While the set of the basis functions for the homogeneous solution found earlier is

$$\{\mathrm{e}^{-3x}\cos(x), \mathrm{e}^{-3x}\sin(x)\}$$

Since there is no duplication between the basis function in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis in the UC_set.

$$y_p = A_1 x\,\mathrm{e}^{-3x} + A_2\mathrm{e}^{-3x} + A_3\cos(x)\,\mathrm{e}^{3x} + A_4\mathrm{e}^{3x}\sin(x)$$

The unknowns $\{A_1, A_2, A_3, A_4\}$ are found by substituting the above trial solution y_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$\begin{aligned}&A_1 x\,\mathrm{e}^{-3x} + A_2\mathrm{e}^{-3x} + 36A_3\cos(x)\,\mathrm{e}^{3x} - 12A_3\sin(x)\,\mathrm{e}^{3x}\\&+ 36A_4\mathrm{e}^{3x}\sin(x) + 12A_4\mathrm{e}^{3x}\cos(x) = 3x\,\mathrm{e}^{-3x} - 2\cos(x)\,\mathrm{e}^{3x}\end{aligned}$$

Solving for the unknowns by comparing coefficients results in

$$\left[A_1 = 3, A_2 = 0, A_3 = -\frac{1}{20}, A_4 = -\frac{1}{60}\right]$$

Substituting the above back in the above trial solution y_p, gives the particular solution

$$y_p = 3x\,\mathrm{e}^{-3x} - \frac{\cos(x)\,\mathrm{e}^{3x}}{20} - \frac{\mathrm{e}^{3x}\sin(x)}{60}$$

Therefore the general solution is

$$\begin{aligned}y &= y_h + y_p\\&= \left(\mathrm{e}^{-3x}(c_1\cos(x) + c_2\sin(x))\right) + \left(3x\,\mathrm{e}^{-3x} - \frac{\cos(x)\,\mathrm{e}^{3x}}{20} - \frac{\mathrm{e}^{3x}\sin(x)}{60}\right)\end{aligned}$$

Summary

The solution(s) found are the following

$$y = e^{-3x}(c_1 \cos(x) + c_2 \sin(x)) + 3x\,e^{-3x} - \frac{\cos(x)\,e^{3x}}{20} - \frac{e^{3x}\sin(x)}{60} \tag{1}$$

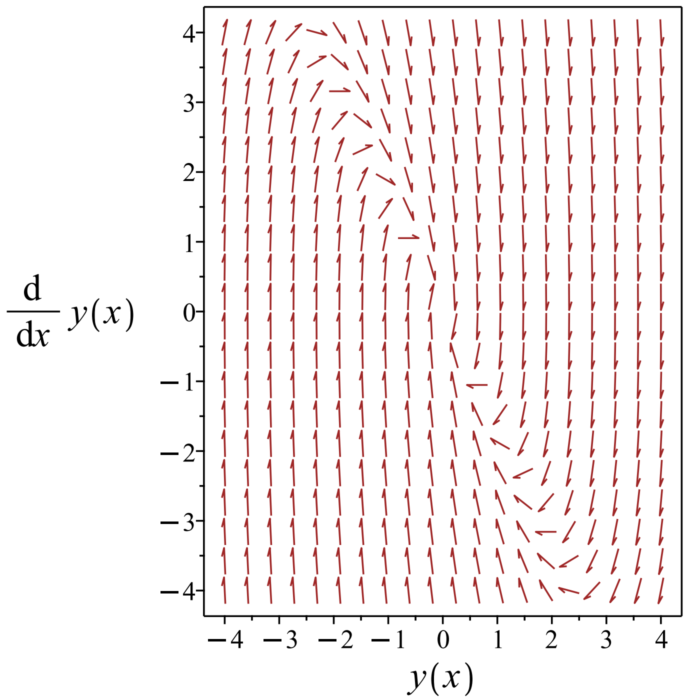


Figure 131: Slope field plot

Verification of solutions

$$y = e^{-3x}(c_1 \cos(x) + c_2 \sin(x)) + 3x\,e^{-3x} - \frac{\cos(x)\,e^{3x}}{20} - \frac{e^{3x}\sin(x)}{60}$$

Verified OK.

2.13.2 Solving using Kovacic algorithm

Writing the ode as

$$y'' + 6y' + 10y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1 \\ B &= 6 \\ C &= 10 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = y e^{\int \frac{B}{2A}\, dx}$$

Then (2) becomes

$$z''(x) = r z(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{-1}{1} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= -1 \\ t &= 1 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = -z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0,1,2,4,6,8,\cdots\}$	$\{\cdots,-6,-4,-2,0,2,3,4,5,6,\cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1,2\}$	$\{2,3,4,5,6,7,\cdots\}$

Table 55: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 0 - 0 \\ &= 0 \end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Since $r = -1$ is not a function of x, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$z_1(x) = \cos(x)$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$\begin{aligned} y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx} \\ &= z_1 e^{-\int \frac{1}{2}\frac{6}{1}\,dx} \\ &= z_1 e^{-3x} \\ &= z_1 \left(\mathrm{e}^{-3x}\right) \end{aligned}$$

Which simplifies to

$$y_1 = e^{-3x} \cos(x)$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A} dx}}{y_1^2} dx$$

Substituting gives

$$\begin{aligned} y_2 &= y_1 \int \frac{e^{\int -\frac{6}{1} dx}}{(y_1)^2} dx \\ &= y_1 \int \frac{e^{-6x}}{(y_1)^2} dx \\ &= y_1(\tan(x)) \end{aligned}$$

Therefore the solution is

$$\begin{aligned} y &= c_1 y_1 + c_2 y_2 \\ &= c_1 \left(e^{-3x} \cos(x)\right) + c_2 \left(e^{-3x} \cos(x) (\tan(x))\right) \end{aligned}$$

This is second order nonhomogeneous ODE. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the nonhomogeneous ODE $Ay''(x) + By'(x) + Cy(x) = f(x)$. y_h is the solution to

$$y'' + 6y' + 10y = 0$$

The homogeneous solution is found using the Kovacic algorithm which results in

$$y_h = e^{-3x} \cos(x) c_1 + e^{-3x} \sin(x) c_2$$

The particular solution is now found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$3x e^{-3x} - 2\cos(x) e^{3x}$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$[\{x\,\mathrm{e}^{-3x}, \mathrm{e}^{-3x}\}, \{\cos(x)\,\mathrm{e}^{3x}, \mathrm{e}^{3x}\sin(x)\}]$$

While the set of the basis functions for the homogeneous solution found earlier is

$$\{\mathrm{e}^{-3x}\cos(x), \mathrm{e}^{-3x}\sin(x)\}$$

Since there is no duplication between the basis function in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis in the UC_set.

$$y_p = A_1 x\,\mathrm{e}^{-3x} + A_2 \mathrm{e}^{-3x} + A_3 \cos(x)\,\mathrm{e}^{3x} + A_4 \mathrm{e}^{3x}\sin(x)$$

The unknowns $\{A_1, A_2, A_3, A_4\}$ are found by substituting the above trial solution y_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$\begin{aligned}&A_1 x\,\mathrm{e}^{-3x} + A_2 \mathrm{e}^{-3x} + 36A_3 \cos(x)\,\mathrm{e}^{3x} - 12A_3 \sin(x)\,\mathrm{e}^{3x}\\&+ 36A_4 \mathrm{e}^{3x}\sin(x) + 12A_4 \mathrm{e}^{3x}\cos(x) = 3x\,\mathrm{e}^{-3x} - 2\cos(x)\,\mathrm{e}^{3x}\end{aligned}$$

Solving for the unknowns by comparing coefficients results in

$$\left[A_1 = 3, A_2 = 0, A_3 = -\frac{1}{20}, A_4 = -\frac{1}{60}\right]$$

Substituting the above back in the above trial solution y_p, gives the particular solution

$$y_p = 3x\,\mathrm{e}^{-3x} - \frac{\cos(x)\,\mathrm{e}^{3x}}{20} - \frac{\mathrm{e}^{3x}\sin(x)}{60}$$

Therefore the general solution is

$$\begin{aligned}y &= y_h + y_p\\&= \left(\mathrm{e}^{-3x}\cos(x)\,c_1 + \mathrm{e}^{-3x}\sin(x)\,c_2\right) + \left(3x\,\mathrm{e}^{-3x} - \frac{\cos(x)\,\mathrm{e}^{3x}}{20} - \frac{\mathrm{e}^{3x}\sin(x)}{60}\right)\end{aligned}$$

Which simplifies to

$$y = \mathrm{e}^{-3x}(c_1\cos(x) + c_2\sin(x)) + 3x\,\mathrm{e}^{-3x} - \frac{\cos(x)\,\mathrm{e}^{3x}}{20} - \frac{\mathrm{e}^{3x}\sin(x)}{60}$$

Summary
The solution(s) found are the following

$$y = e^{-3x}(c_1 \cos(x) + c_2 \sin(x)) + 3x\,e^{-3x} - \frac{\cos(x)\,e^{3x}}{20} - \frac{e^{3x} \sin(x)}{60} \tag{1}$$

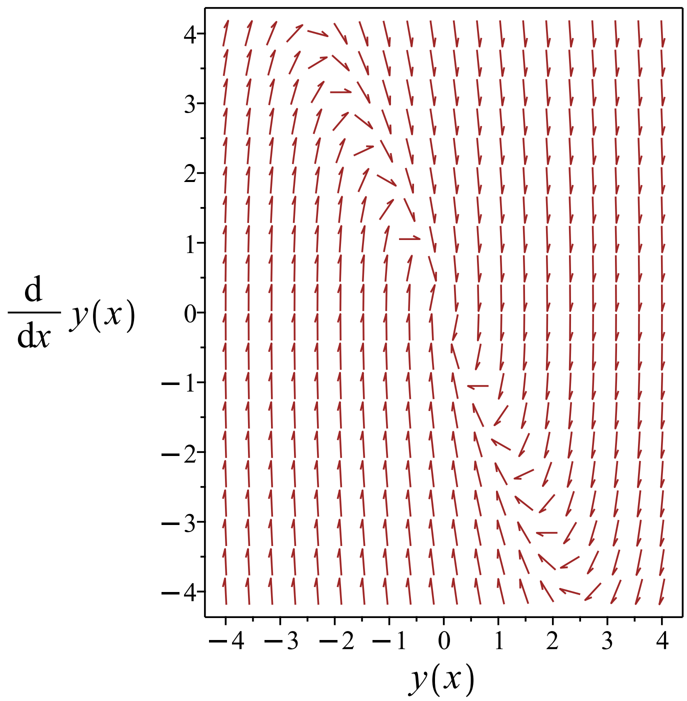


Figure 132: Slope field plot

Verification of solutions

$$y = e^{-3x}(c_1 \cos(x) + c_2 \sin(x)) + 3x\,e^{-3x} - \frac{\cos(x)\,e^{3x}}{20} - \frac{e^{3x} \sin(x)}{60}$$

Verified OK.

2.13.3 Maple step by step solution

Let's solve

$y'' + 6y' + 10y = 3x\,e^{-3x} - 2\cos(x)\,e^{3x}$

- Highest derivative means the order of the ODE is 2

  y''

- Characteristic polynomial of homogeneous ODE

$r^2+6r+10=0$

- Use quadratic formula to solve for r

$r=\frac{(-6)\pm(\sqrt{-4})}{2}$

- Roots of the characteristic polynomial

$r=(-3-\mathrm{I},-3+\mathrm{I})$

- 1st solution of the homogeneous ODE

$y_1(x)=\mathrm{e}^{-3x}\cos(x)$

- 2nd solution of the homogeneous ODE

$y_2(x)=\mathrm{e}^{-3x}\sin(x)$

- General solution of the ODE

$y=c_1y_1(x)+c_2y_2(x)+y_p(x)$

- Substitute in solutions of the homogeneous ODE

$y=\mathrm{e}^{-3x}\cos(x)c_1+\mathrm{e}^{-3x}\sin(x)c_2+y_p(x)$

□ Find a particular solution $y_p(x)$ of the ODE

- Use variation of parameters to find y_p here $f(x)$ is the forcing function

$\left[y_p(x)=-y_1(x)\left(\int\frac{y_2(x)f(x)}{W(y_1(x),y_2(x))}dx\right)+y_2(x)\left(\int\frac{y_1(x)f(x)}{W(y_1(x),y_2(x))}dx\right),f(x)=3x\,\mathrm{e}^{-3x}-2\cos(x)\,\mathrm{e}^{3x}\right]$

- Wronskian of solutions of the homogeneous equation

$W(y_1(x),y_2(x))=\begin{bmatrix}\mathrm{e}^{-3x}\cos(x) & \mathrm{e}^{-3x}\sin(x)\\ -3\,\mathrm{e}^{-3x}\cos(x)-\mathrm{e}^{-3x}\sin(x) & -3\,\mathrm{e}^{-3x}\sin(x)+\mathrm{e}^{-3x}\cos(x)\end{bmatrix}$

- Compute Wronskian

$W(y_1(x),y_2(x))=\mathrm{e}^{-6x}$

- Substitute functions into equation for $y_p(x)$

$y_p(x)=\mathrm{e}^{-3x}\left(\cos(x)\left(\int\sin(x)\left(2\cos(x)\,\mathrm{e}^{6x}-3x\right)dx\right)-\sin(x)\left(\int\left(2\cos(x)^2\,\mathrm{e}^{6x}-3\cos(x)\,x\right)dx\right)\right)$

- Compute integrals

$y_p(x)=\frac{(-3\cos(x)-\sin(x))\mathrm{e}^{3x}}{60}+3x\,\mathrm{e}^{-3x}$

- Substitute particular solution into general solution to ODE

$y=\mathrm{e}^{-3x}\cos(x)c_1+\mathrm{e}^{-3x}\sin(x)c_2+\frac{(-3\cos(x)-\sin(x))\mathrm{e}^{3x}}{60}+3x\,\mathrm{e}^{-3x}$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 2; linear nonhomogeneous with symmetry [0,1]
trying a double symmetry of the form [xi=0, eta=F(x)]
-> Try solving first the homogeneous part of the ODE
   checking if the LODE has constant coefficients
   <- constant coefficients successful
<- solving first the homogeneous part of the ODE successful`
```

✓ Solution by Maple

Time used: 0.015 (sec). Leaf size: 35

```
dsolve(diff(y(x),x$2)+6*diff(y(x),x)+10*y(x)=3*x*exp(-3*x)-2*exp(3*x)*cos(x),y(x), singsol=a
```

$$y(x) = (\cos (x) c_1 + \sin (x) c_2 + 3x) e^{-3x} - \frac{e^{3x}\left(\cos (x) + \frac{\sin(x)}{3}\right)}{20}$$

✓ Solution by Mathematica

Time used: 0.426 (sec). Leaf size: 46

```
DSolve[y''[x]+6*y'[x]+10*y[x]==3*x*Exp[-3*x]-2*Exp[3*x]*Cos[x],y[x],x,IncludeSingularSolutio
```

$$y(x) \to \frac{1}{60} e^{-3x} \left(180x - 3\left(e^{6x} - 20c_2\right) \cos(x) - \left(e^{6x} - 60c_1\right) \sin(x)\right)$$

2.14 problem 14

Internal problem ID [3255]
Internal file name [`OUTPUT/2747_Sunday_June_05_2022_08_39_58_AM_23095541/index.tex`]

Book: Differential equations for engineers by Wei-Chau XIE, Cambridge Press 2010
Section: Chapter 4. Linear Differential Equations. Page 183
Problem number: 14.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_linear_constant_coeff"**

Maple gives the following as the ode type

```
[[_2nd_order, _linear, _nonhomogeneous]]
```

$$\boxed{y'' - 8y' + 17y = e^{4x}\left(x^2 - 3x\sin(x)\right)}$$

2.14.1 Solving as second order linear constant coeff ode

This is second order non-homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = f(x)$$

Where $A = 1, B = -8, C = 17, f(x) = e^{4x}x(-3\sin(x) + x)$. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the non-homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = f(x)$. y_h is the solution to

$$y'' - 8y' + 17y = 0$$

This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = 0$$

Where in the above $A = 1, B = -8, C = 17$. Let the solution be $y = e^{\lambda x}$. Substituting this into the ODE gives

$$\lambda^2 e^{\lambda x} - 8\lambda e^{\lambda x} + 17 e^{\lambda x} = 0 \tag{1}$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda x}$ gives

$$\lambda^2 - 8\lambda + 17 = 0 \tag{2}$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form.Using the quadratic formula

$$\lambda_{1,2} = \frac{-B}{2A} \pm \frac{1}{2A}\sqrt{B^2 - 4AC}$$

Substituting $A = 1, B = -8, C = 17$ into the above gives

$$\begin{aligned}\lambda_{1,2} &= \frac{8}{(2)(1)} \pm \frac{1}{(2)(1)}\sqrt{-8^2 - (4)(1)(17)} \\ &= 4 \pm i\end{aligned}$$

Hence

$$\begin{aligned}\lambda_1 &= 4 + i \\ \lambda_2 &= 4 - i\end{aligned}$$

Which simplifies to

$$\begin{aligned}\lambda_1 &= 4 + i \\ \lambda_2 &= 4 - i\end{aligned}$$

Since roots are complex conjugate of each others, then let the roots be

$$\lambda_{1,2} = \alpha \pm i\beta$$

Where $\alpha = 4$ and $\beta = 1$. Therefore the final solution, when using Euler relation, can be written as

$$y = e^{\alpha x}(c_1 \cos(\beta x) + c_2 \sin(\beta x))$$

Which becomes

$$y = e^{4x}(c_1 \cos(x) + c_2 \sin(x))$$

Therefore the homogeneous solution y_h is

$$y_h = e^{4x}(c_1 \cos(x) + c_2 \sin(x))$$

The particular solution is now found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$e^{4x}x(-3\sin(x)+x)$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$[\{x\,e^{4x}, x^2e^{4x}, e^{4x}\}, \{e^{4x}\cos(x), e^{4x}\sin(x), x\cos(x)\,e^{4x}, e^{4x}\sin(x)\,x\}]$$

While the set of the basis functions for the homogeneous solution found earlier is

$$\{e^{4x}\cos(x), e^{4x}\sin(x)\}$$

Since $e^{4x}\cos(x)$ is duplicated in the UC_set, then this basis is multiplied by extra x. The UC_set becomes

$$[\{x\,e^{4x}, x^2e^{4x}, e^{4x}\}, \{x\cos(x)\,e^{4x}, x^2\cos(x)\,e^{4x}, x^2e^{4x}\sin(x), e^{4x}\sin(x)\,x\}]$$

Since there was duplication between the basis functions in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis function in the above updated UC_set.

$$\begin{aligned} y_p = {} & A_1x\,e^{4x} + A_2x^2e^{4x} + A_3e^{4x} + A_4x\cos(x)\,e^{4x} \\ & + A_5x^2\cos(x)\,e^{4x} + A_6x^2e^{4x}\sin(x) + A_7e^{4x}\sin(x)\,x \end{aligned}$$

The unknowns $\{A_1, A_2, A_3, A_4, A_5, A_6, A_7\}$ are found by substituting the above trial solution y_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$\begin{aligned} & A_3e^{4x} + 2A_6e^{4x}\sin(x) + 4A_6x\,e^{4x}\cos(x) + 2A_7e^{4x}\cos(x) - 4A_5x\sin(x)\,e^{4x} \\ & + 2A_5\cos(x)\,e^{4x} - 2A_4\sin(x)\,e^{4x} + 2A_2e^{4x} + A_2x^2e^{4x} + A_1x\,e^{4x} = e^{4x}x(-3\sin(x)+x) \end{aligned}$$

Solving for the unknowns by comparing coefficients results in

$$\left[A_1 = 0, A_2 = 1, A_3 = -2, A_4 = 0, A_5 = \frac{3}{4}, A_6 = 0, A_7 = -\frac{3}{4}\right]$$

Substituting the above back in the above trial solution y_p, gives the particular solution

$$y_p = x^2e^{4x} - 2\,e^{4x} + \frac{3x^2\cos(x)\,e^{4x}}{4} - \frac{3\,e^{4x}\sin(x)\,x}{4}$$

Therefore the general solution is

$$
\begin{aligned}
y &= y_h + y_p \\
&= \left(\mathrm{e}^{4x}(c_1 \cos(x) + c_2 \sin(x))\right) + \left(x^2 \mathrm{e}^{4x} - 2\,\mathrm{e}^{4x} + \frac{3x^2 \cos(x)\,\mathrm{e}^{4x}}{4} - \frac{3\,\mathrm{e}^{4x} \sin(x)\,x}{4}\right)
\end{aligned}
$$

Summary

The solution(s) found are the following

$$
y = \mathrm{e}^{4x}(c_1 \cos(x) + c_2 \sin(x)) + x^2 \mathrm{e}^{4x} - 2\,\mathrm{e}^{4x} + \frac{3x^2 \cos(x)\,\mathrm{e}^{4x}}{4} - \frac{3\,\mathrm{e}^{4x} \sin(x)\,x}{4} \tag{1}
$$

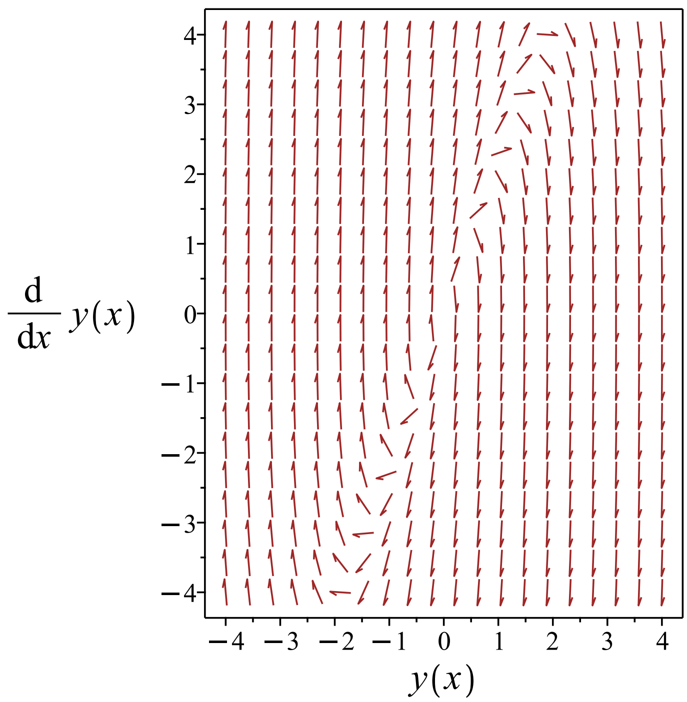


Figure 133: Slope field plot

Verification of solutions

$$
y = \mathrm{e}^{4x}(c_1 \cos(x) + c_2 \sin(x)) + x^2 \mathrm{e}^{4x} - 2\,\mathrm{e}^{4x} + \frac{3x^2 \cos(x)\,\mathrm{e}^{4x}}{4} - \frac{3\,\mathrm{e}^{4x} \sin(x)\,x}{4}
$$

Verified OK.

2.14.2 Solving using Kovacic algorithm

Writing the ode as

$$y'' - 8y' + 17y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1 \\ B &= -8 \\ C &= 17 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\, dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{-1}{1} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= -1 \\ t &= 1 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = -z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 57: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$
\begin{aligned}
O(\infty) &= \deg(t) - \deg(s) \\
&= 0 - 0 \\
&= 0
\end{aligned}
$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Therefore

$$
L = [1]
$$

Since $r = -1$ is not a function of x, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$
z_1(x) = \cos(x)
$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$
\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{-8}{1}\,dx}
\end{aligned}
$$

$$
\begin{aligned}
&= z_1 e^{4x} \\
&= z_1 \left(\mathrm{e}^{4x}\right)
\end{aligned}
$$

Which simplifies to

$$
y_1 = \mathrm{e}^{4x} \cos(x)
$$

The second solution y_2 to the original ode is found using reduction of order

$$
y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\, dx}}{y_1^2}\, dx
$$

Substituting gives

$$
\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{-8}{1}\, dx}}{(y_1)^2}\, dx \\
&= y_1 \int \frac{e^{8x}}{(y_1)^2}\, dx \\
&= y_1(\tan(x))
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1 \left(\mathrm{e}^{4x} \cos(x)\right) + c_2 \left(\mathrm{e}^{4x} \cos(x) (\tan(x))\right)
\end{aligned}
$$

This is second order nonhomogeneous ODE. Let the solution be

$$
y = y_h + y_p
$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the nonhomogeneous ODE $Ay''(x) + By'(x) + Cy(x) = f(x)$. y_h is the solution to

$$
y'' - 8y' + 17y = 0
$$

The homogeneous solution is found using the Kovacic algorithm which results in

$$
y_h = \mathrm{e}^{4x} \cos(x)\, c_1 + \mathrm{e}^{4x} \sin(x)\, c_2
$$

The particular solution is now found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$e^{4x}x(-3\sin(x)+x)$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$[\{x\,e^{4x}, x^2e^{4x}, e^{4x}\}, \{e^{4x}\cos(x), e^{4x}\sin(x), x\cos(x)\,e^{4x}, e^{4x}\sin(x)\,x\}]$$

While the set of the basis functions for the homogeneous solution found earlier is

$$\{e^{4x}\cos(x), e^{4x}\sin(x)\}$$

Since $e^{4x}\cos(x)$ is duplicated in the UC_set, then this basis is multiplied by extra x. The UC_set becomes

$$[\{x\,e^{4x}, x^2e^{4x}, e^{4x}\}, \{x\cos(x)\,e^{4x}, x^2\cos(x)\,e^{4x}, x^2e^{4x}\sin(x), e^{4x}\sin(x)\,x\}]$$

Since there was duplication between the basis functions in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis function in the above updated UC_set.

$$\begin{aligned}y_p = {} & A_1x\,e^{4x} + A_2x^2e^{4x} + A_3e^{4x} + A_4x\cos(x)\,e^{4x} \\ & + A_5x^2\cos(x)\,e^{4x} + A_6x^2e^{4x}\sin(x) + A_7e^{4x}\sin(x)\,x\end{aligned}$$

The unknowns $\{A_1, A_2, A_3, A_4, A_5, A_6, A_7\}$ are found by substituting the above trial solution y_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$\begin{aligned}& 2A_6e^{4x}\sin(x) + 4A_6x\,e^{4x}\cos(x) + 2A_7e^{4x}\cos(x) + 2A_5\cos(x)\,e^{4x} - 4A_5x\sin(x)\,e^{4x} \\ & + A_3e^{4x} + 2A_2e^{4x} - 2A_4\sin(x)\,e^{4x} + A_2x^2e^{4x} + A_1x\,e^{4x} = e^{4x}x(-3\sin(x)+x)\end{aligned}$$

Solving for the unknowns by comparing coefficients results in

$$\left[A_1 = 0, A_2 = 1, A_3 = -2, A_4 = 0, A_5 = \frac{3}{4}, A_6 = 0, A_7 = -\frac{3}{4}\right]$$

Substituting the above back in the above trial solution y_p, gives the particular solution

$$y_p = x^2e^{4x} - 2\,e^{4x} + \frac{3x^2\cos(x)\,e^{4x}}{4} - \frac{3\,e^{4x}\sin(x)\,x}{4}$$

Therefore the general solution is

$$
\begin{aligned}
y &= y_h + y_p \\
&= \left(e^{4x} \cos(x) c_1 + e^{4x} \sin(x) c_2\right) + \left(x^2 e^{4x} - 2 e^{4x} + \frac{3x^2 \cos(x) e^{4x}}{4} - \frac{3 e^{4x} \sin(x) x}{4}\right)
\end{aligned}
$$

Which simplifies to

$$
y = e^{4x}(c_1 \cos(x) + c_2 \sin(x)) + x^2 e^{4x} - 2 e^{4x} + \frac{3x^2 \cos(x) e^{4x}}{4} - \frac{3 e^{4x} \sin(x) x}{4}
$$

Summary

The solution(s) found are the following

$$
y = e^{4x}(c_1 \cos(x) + c_2 \sin(x)) + x^2 e^{4x} - 2 e^{4x} + \frac{3x^2 \cos(x) e^{4x}}{4} - \frac{3 e^{4x} \sin(x) x}{4} \quad (1)
$$

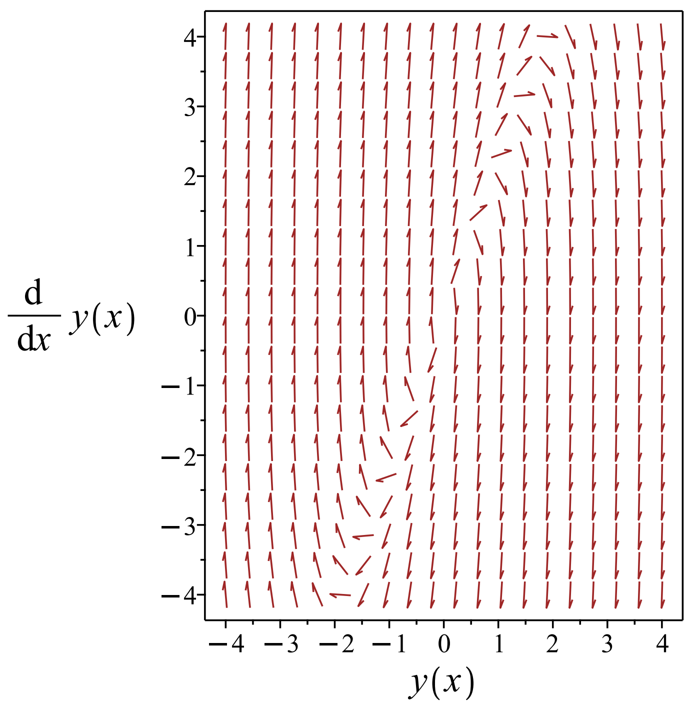


Figure 134: Slope field plot

Verification of solutions

$$
y = e^{4x}(c_1 \cos(x) + c_2 \sin(x)) + x^2 e^{4x} - 2 e^{4x} + \frac{3x^2 \cos(x) e^{4x}}{4} - \frac{3 e^{4x} \sin(x) x}{4}
$$

Verified OK.

2.14.3 Maple step by step solution

Let's solve

$y'' - 8y' + 17y = e^{4x}x(-3\sin(x) + x)$

- Highest derivative means the order of the ODE is 2

  y''
- Isolate 2nd derivative

  $y'' = -3\,e^{4x}\sin(x)\,x + x^2e^{4x} + 8y' - 17y$
- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y'' - 8y' + 17y = -e^{4x}x(3\sin(x) - x)$
- Characteristic polynomial of homogeneous ODE

  $r^2 - 8r + 17 = 0$
- Use quadratic formula to solve for r

  $r = \frac{8\pm(\sqrt{-4})}{2}$
- Roots of the characteristic polynomial

  $r = (4 - \mathrm{I}, 4 + \mathrm{I})$
- 1st solution of the homogeneous ODE

  $y_1(x) = e^{4x}\cos(x)$
- 2nd solution of the homogeneous ODE

  $y_2(x) = e^{4x}\sin(x)$
- General solution of the ODE

  $y = c_1y_1(x) + c_2y_2(x) + y_p(x)$
- Substitute in solutions of the homogeneous ODE

  $y = e^{4x}\cos(x)\,c_1 + e^{4x}\sin(x)\,c_2 + y_p(x)$

☐ Find a particular solution $y_p(x)$ of the ODE

- ○ Use variation of parameters to find y_p here $f(x)$ is the forcing function

  $\left[y_p(x) = -y_1(x)\left(\int \frac{y_2(x)f(x)}{W(y_1(x),y_2(x))}dx\right) + y_2(x)\left(\int \frac{y_1(x)f(x)}{W(y_1(x),y_2(x))}dx\right), f(x) = -e^{4x}x(3\sin(x) - x)\right]$
- ○ Wronskian of solutions of the homogeneous equation

$$W(y_1(x), y_2(x)) = \begin{bmatrix} e^{4x}\cos(x) & e^{4x}\sin(x) \\ 4e^{4x}\cos(x) - e^{4x}\sin(x) & 4e^{4x}\sin(x) + e^{4x}\cos(x) \end{bmatrix}$$

- Compute Wronskian

  $W(y_1(x), y_2(x)) = e^{8x}$
- Substitute functions into equation for $y_p(x)$

  $y_p(x) = e^{4x}\left(\cos(x)\left(\int -x\sin(x)(-3\sin(x)+x)\,dx\right) - \sin(x)\left(\int -x\cos(x)(-3\sin(x)+x)\,dx\right)\right)$
- Compute integrals

  $y_p(x) = -\frac{(-3x^2\cos(x)+3x\sin(x)-4x^2+8)e^{4x}}{4}$

- Substitute particular solution into general solution to ODE

  $y = e^{4x}\cos(x)c_1 + e^{4x}\sin(x)c_2 - \frac{(-3x^2\cos(x)+3x\sin(x)-4x^2+8)e^{4x}}{4}$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 2; linear nonhomogeneous with symmetry [0,1]
trying a double symmetry of the form [xi=0, eta=F(x)]
-> Try solving first the homogeneous part of the ODE
   checking if the LODE has constant coefficients
   <- constant coefficients successful
<- solving first the homogeneous part of the ODE successful`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 39

```
dsolve(diff(y(x),x$2)-8*diff(y(x),x)+17*y(x)=exp(4*x)*(x^2-3*x*sin(x)),y(x), singsol=all)
```

$$y(x) = \frac{((3x^2+4c_1)\cos(x) + (-3x+4c_2)\sin(x) + 4x^2 - 8)e^{4x}}{4}$$

✓ Solution by Mathematica

Time used: 0.263 (sec). Leaf size: 47

```
DSolve[y''[x]-8*y'[x]+17*y[x]==Exp[4*x]*(x^2-3*x*Sin[x]),y[x],x,IncludeSingularSolutions ->
```

$$y(x) \to \frac{1}{8}e^{4x}\left(8\left(x^2-2\right)+\left(6x^2-3+8c_2\right)\cos(x)+\left(-6x+8c_1\right)\sin(x)\right)$$

2.15 problem 15

Internal problem ID [3256]
Internal file name [`OUTPUT/2748_Sunday_June_05_2022_08_39_59_AM_87006663/index.tex`]

Book: Differential equations for engineers by Wei-Chau XIE, Cambridge Press 2010
Section: Chapter 4. Linear Differential Equations. Page 183
Problem number: 15.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_linear_constant_coeff"**

Maple gives the following as the ode type

```
[[_2nd_order, _linear, _nonhomogeneous]]
```

$$\boxed{y'' - 2y' + 2y = (x + e^x) \sin(x)}$$

2.15.1 Solving as second order linear constant coeff ode

This is second order non-homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = f(x)$$

Where $A = 1, B = -2, C = 2, f(x) = (x + e^x) \sin(x)$. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the non-homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = f(x)$. y_h is the solution to

$$y'' - 2y' + 2y = 0$$

This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = 0$$

Where in the above $A = 1, B = -2, C = 2$. Let the solution be $y = e^{\lambda x}$. Substituting this into the ODE gives

$$\lambda^2 \mathrm{e}^{\lambda x} - 2\lambda\,\mathrm{e}^{\lambda x} + 2\,\mathrm{e}^{\lambda x} = 0 \tag{1}$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda x}$ gives

$$\lambda^2 - 2\lambda + 2 = 0 \tag{2}$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form.Using the quadratic formula

$$\lambda_{1,2} = \frac{-B}{2A} \pm \frac{1}{2A}\sqrt{B^2 - 4AC}$$

Substituting $A = 1, B = -2, C = 2$ into the above gives

$$\begin{aligned}\lambda_{1,2} &= \frac{2}{(2)\,(1)} \pm \frac{1}{(2)\,(1)}\sqrt{-2^2 - (4)\,(1)\,(2)}\\ &= 1 \pm i\end{aligned}$$

Hence

$$\begin{aligned}\lambda_1 &= 1 + i\\ \lambda_2 &= 1 - i\end{aligned}$$

Which simplifies to

$$\begin{aligned}\lambda_1 &= 1 + i\\ \lambda_2 &= 1 - i\end{aligned}$$

Since roots are complex conjugate of each others, then let the roots be

$$\lambda_{1,2} = \alpha \pm i\beta$$

Where $\alpha = 1$ and $\beta = 1$. Therefore the final solution, when using Euler relation, can be written as

$$y = e^{\alpha x}(c_1 \cos(\beta x) + c_2 \sin(\beta x))$$

Which becomes

$$y = e^{x}(c_1 \cos(x) + c_2 \sin(x))$$

Therefore the homogeneous solution y_h is

$$y_h = \mathrm{e}^{x}(c_1 \cos(x) + c_2 \sin(x))$$

The particular solution is now found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$(x + e^x) \sin(x)$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$[\{\cos(x) e^x, e^x \sin(x)\}, \{x \sin(x), \cos(x) x, \cos(x), \sin(x)\}]$$

While the set of the basis functions for the homogeneous solution found earlier is

$$\{\cos(x) e^x, e^x \sin(x)\}$$

Since $\cos(x) e^x$ is duplicated in the UC_set, then this basis is multiplied by extra x. The UC_set becomes

$$[\{e^x x \cos(x), e^x x \sin(x)\}, \{x \sin(x), \cos(x) x, \cos(x), \sin(x)\}]$$

Since there was duplication between the basis functions in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis function in the above updated UC_set.

$$y_p = A_1 e^x x \cos(x) + A_2 e^x x \sin(x) + A_3 x \sin(x) + A_4 \cos(x) x + A_5 \cos(x) + A_6 \sin(x)$$

The unknowns $\{A_1, A_2, A_3, A_4, A_5, A_6\}$ are found by substituting the above trial solution y_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$\begin{aligned} &A_6 \sin(x) + A_5 \cos(x) - 2A_4 \sin(x) - 2A_4 \cos(x) + 2A_4 \sin(x) x \\ &\quad - 2A_1 e^x \sin(x) + 2A_2 e^x \cos(x) - 2A_3 \sin(x) - 2A_3 x \cos(x) + A_3 x \sin(x) \\ &\quad + A_4 \cos(x) x - 2A_6 \cos(x) + 2A_3 \cos(x) + 2A_5 \sin(x) = (x + e^x) \sin(x) \end{aligned}$$

Solving for the unknowns by comparing coefficients results in

$$\left[A_1 = -\frac{1}{2}, A_2 = 0, A_3 = \frac{1}{5}, A_4 = \frac{2}{5}, A_5 = \frac{14}{25}, A_6 = \frac{2}{25}\right]$$

Substituting the above back in the above trial solution y_p, gives the particular solution

$$y_p = -\frac{e^x x \cos(x)}{2} + \frac{x \sin(x)}{5} + \frac{2 \cos(x) x}{5} + \frac{14 \cos(x)}{25} + \frac{2 \sin(x)}{25}$$

Therefore the general solution is

$$
\begin{aligned}
y &= y_h + y_p \\
&= (e^x(c_1 \cos(x) + c_2 \sin(x))) \\
&\quad + \left(-\frac{e^x x \cos(x)}{2} + \frac{x \sin(x)}{5} + \frac{2\cos(x)\, x}{5} + \frac{14\cos(x)}{25} + \frac{2\sin(x)}{25}\right)
\end{aligned}
$$

Summary

The solution(s) found are the following

$$
\begin{aligned}
y = e^x(c_1 \cos(x) + c_2 \sin(x)) - \frac{e^x x \cos(x)}{2} \\
+ \frac{x \sin(x)}{5} + \frac{2\cos(x)\, x}{5} + \frac{14\cos(x)}{25} + \frac{2\sin(x)}{25}
\end{aligned} \tag{1}
$$

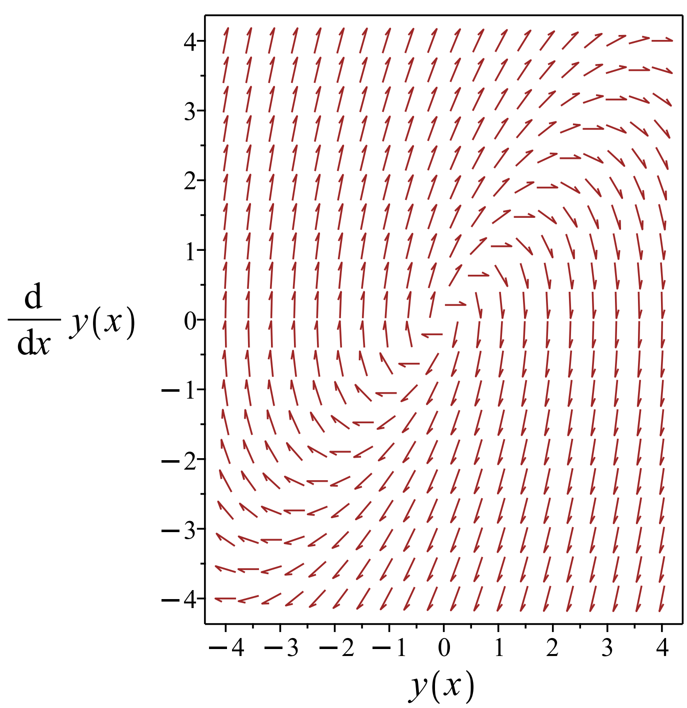


Figure 135: Slope field plot

Verification of solutions

$$
y = e^x(c_1 \cos(x) + c_2 \sin(x)) - \frac{e^x x \cos(x)}{2} + \frac{x \sin(x)}{5} + \frac{2\cos(x)\, x}{5} + \frac{14\cos(x)}{25} + \frac{2\sin(x)}{25}
$$

Verified OK.

2.15.2 Solving using Kovacic algorithm

Writing the ode as

$$y'' - 2y' + 2y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1 \\ B &= -2 \\ C &= 2 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A} \, dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{-1}{1} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= -1 \\ t &= 1 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = -z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A} \, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 59: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$
\begin{aligned}
O(\infty) &= \deg(t) - \deg(s) \\
&= 0 - 0 \\
&= 0
\end{aligned}
$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Therefore

$$
L = [1]
$$

Since $r = -1$ is not a function of x, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$
z_1(x) = \cos(x)
$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$
\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{-2}{1}\,dx}
\end{aligned}
$$

$$
\begin{aligned}
&= z_1 e^x \\
&= z_1(\mathrm{e}^x)
\end{aligned}
$$

Which simplifies to

$$
y_1 = \cos(x)\,\mathrm{e}^x
$$

The second solution y_2 to the original ode is found using reduction of order

$$
y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\, dx}}{y_1^2}\, dx
$$

Substituting gives

$$
\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{-2}{1}\, dx}}{(y_1)^2}\, dx \\
&= y_1 \int \frac{e^{2x}}{(y_1)^2}\, dx \\
&= y_1(\tan(x))
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1(\cos(x)\,\mathrm{e}^x) + c_2(\cos(x)\,\mathrm{e}^x(\tan(x)))
\end{aligned}
$$

This is second order nonhomogeneous ODE. Let the solution be

$$
y = y_h + y_p
$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the nonhomogeneous ODE $Ay''(x) + By'(x) + Cy(x) = f(x)$. y_h is the solution to

$$
y'' - 2y' + 2y = 0
$$

The homogeneous solution is found using the Kovacic algorithm which results in

$$
y_h = \mathrm{e}^x \cos(x)\, c_1 + \mathrm{e}^x \sin(x)\, c_2
$$

The particular solution is now found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$(x+e^x)\sin(x)$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$[\{\cos(x)\,e^x, e^x\sin(x)\}, \{x\sin(x), \cos(x)\,x, \cos(x), \sin(x)\}]$$

While the set of the basis functions for the homogeneous solution found earlier is

$$\{\cos(x)\,e^x, e^x\sin(x)\}$$

Since $\cos(x)\,e^x$ is duplicated in the UC_set, then this basis is multiplied by extra x. The UC_set becomes

$$[\{e^x x\cos(x), e^x x\sin(x)\}, \{x\sin(x), \cos(x)\,x, \cos(x), \sin(x)\}]$$

Since there was duplication between the basis functions in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis function in the above updated UC_set.

$$y_p = A_1 e^x x\cos(x) + A_2 e^x x\sin(x) + A_3 x\sin(x) + A_4\cos(x)\,x + A_5\cos(x) + A_6\sin(x)$$

The unknowns $\{A_1, A_2, A_3, A_4, A_5, A_6\}$ are found by substituting the above trial solution y_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$\begin{aligned}&-2A_3\sin(x) - 2A_3x\cos(x) - 2A_4\cos(x) + 2A_4\sin(x)\,x - 2A_1e^x\sin(x)\\ &+ 2A_2e^x\cos(x) - 2A_4\sin(x) + A_5\cos(x) + A_6\sin(x) + 2A_5\sin(x)\\ &- 2A_6\cos(x) + 2A_3\cos(x) + A_4\cos(x)\,x + A_3x\sin(x) = (x+e^x)\sin(x)\end{aligned}$$

Solving for the unknowns by comparing coefficients results in

$$\left[A_1 = -\frac{1}{2}, A_2 = 0, A_3 = \frac{1}{5}, A_4 = \frac{2}{5}, A_5 = \frac{14}{25}, A_6 = \frac{2}{25}\right]$$

Substituting the above back in the above trial solution y_p, gives the particular solution

$$y_p = -\frac{e^x x\cos(x)}{2} + \frac{x\sin(x)}{5} + \frac{2\cos(x)\,x}{5} + \frac{14\cos(x)}{25} + \frac{2\sin(x)}{25}$$

Therefore the general solution is

$$
\begin{aligned}
y &= y_h + y_p \\
&= \left(e^{x} \cos(x) c_1 + e^{x} \sin(x) c_2\right) \\
&\quad + \left(-\frac{e^{x} x \cos(x)}{2} + \frac{x \sin(x)}{5} + \frac{2 \cos(x) x}{5} + \frac{14 \cos(x)}{25} + \frac{2 \sin(x)}{25}\right)
\end{aligned}
$$

Which simplifies to

$$
y = e^{x}\left(c_1 \cos(x) + c_2 \sin(x)\right) - \frac{e^{x} x \cos(x)}{2} + \frac{x \sin(x)}{5} + \frac{2 \cos(x) x}{5} + \frac{14 \cos(x)}{25} + \frac{2 \sin(x)}{25}
$$

<u>Summary</u>
The solution(s) found are the following

$$
\begin{aligned}
y = e^{x}\left(c_1 \cos(x) + c_2 \sin(x)\right) - \frac{e^{x} x \cos(x)}{2} \\
+ \frac{x \sin(x)}{5} + \frac{2 \cos(x) x}{5} + \frac{14 \cos(x)}{25} + \frac{2 \sin(x)}{25}
\end{aligned} \tag{1}
$$

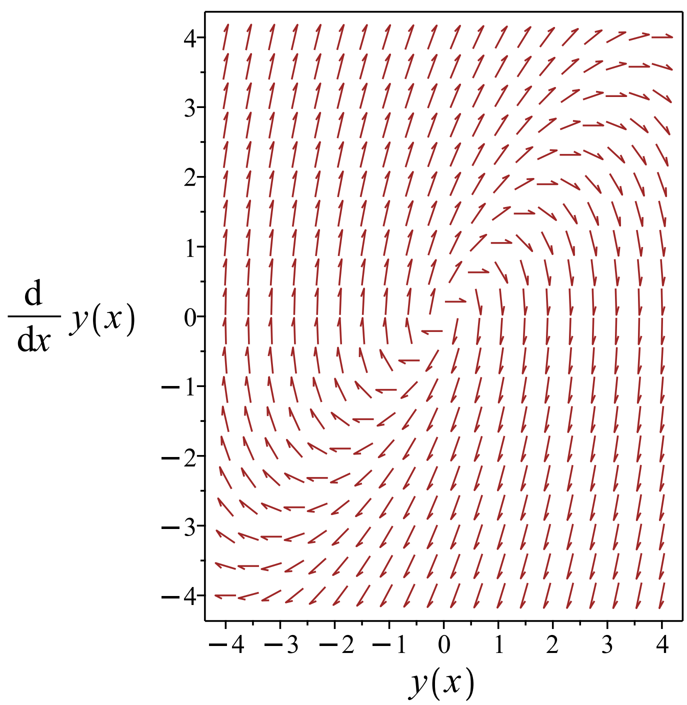


Figure 136: Slope field plot

Verification of solutions

$$y = e^x(c_1 \cos(x) + c_2 \sin(x)) - \frac{e^x x \cos(x)}{2} + \frac{x \sin(x)}{5} + \frac{2\cos(x) x}{5} + \frac{14\cos(x)}{25} + \frac{2\sin(x)}{25}$$

Verified OK.

2.15.3 Maple step by step solution

Let's solve

$y'' - 2y' + 2y = (x + e^x)\sin(x)$

- Highest derivative means the order of the ODE is 2

  y''

- Characteristic polynomial of homogeneous ODE

  $r^2 - 2r + 2 = 0$

- Use quadratic formula to solve for r

  $r = \frac{2 \pm (\sqrt{-4})}{2}$

- Roots of the characteristic polynomial

  $r = (1 - I, 1 + I)$

- 1st solution of the homogeneous ODE

  $y_1(x) = \cos(x) e^x$

- 2nd solution of the homogeneous ODE

  $y_2(x) = e^x \sin(x)$

- General solution of the ODE

  $y = c_1 y_1(x) + c_2 y_2(x) + y_p(x)$

- Substitute in solutions of the homogeneous ODE

  $y = e^x \cos(x) c_1 + e^x \sin(x) c_2 + y_p(x)$

□ Find a particular solution $y_p(x)$ of the ODE

- ◦ Use variation of parameters to find y_p here $f(x)$ is the forcing function

  $\left[y_p(x) = -y_1(x)\left(\int \frac{y_2(x) f(x)}{W(y_1(x), y_2(x))} dx\right) + y_2(x)\left(\int \frac{y_1(x) f(x)}{W(y_1(x), y_2(x))} dx\right), f(x) = (x + e^x)\sin(x)\right]$

- ◦ Wronskian of solutions of the homogeneous equation

$$W(y_1(x), y_2(x)) = \begin{bmatrix} \cos(x)\,\mathrm{e}^x & \mathrm{e}^x \sin(x) \\ -\mathrm{e}^x \sin(x) + \cos(x)\,\mathrm{e}^x & \mathrm{e}^x \sin(x) + \cos(x)\,\mathrm{e}^x \end{bmatrix}$$

- Compute Wronskian

  $$W(y_1(x), y_2(x)) = \mathrm{e}^{2x}$$

- Substitute functions into equation for $y_p(x)$

  $$y_p(x) = \frac{\mathrm{e}^x \left(\sin(x)\left(\int \sin(2x)\left(x\,\mathrm{e}^{-x}+1\right)dx\right) - 2\cos(x)\left(\int \sin(x)^2\left(x\,\mathrm{e}^{-x}+1\right)dx\right)\right)}{2}$$

- Compute integrals

  $$y_p(x) = \frac{(-25x\,\mathrm{e}^x + 20x + 28)\cos(x)}{50} + \frac{\sin(x)(2+5x)}{25}$$

- Substitute particular solution into general solution to ODE

  $$y = \mathrm{e}^x \cos(x)\,c_1 + \mathrm{e}^x \sin(x)\,c_2 + \frac{(-25x\,\mathrm{e}^x + 20x + 28)\cos(x)}{50} + \frac{\sin(x)(2+5x)}{25}$$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 2; linear nonhomogeneous with symmetry [0,1]
trying a double symmetry of the form [xi=0, eta=F(x)]
-> Try solving first the homogeneous part of the ODE
   checking if the LODE has constant coefficients
   <- constant coefficients successful
<- solving first the homogeneous part of the ODE successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 36

```
dsolve(diff(y(x),x$2)-2*diff(y(x),x)+2*y(x)=(x+exp(x))*sin(x),y(x), singsol=all)
```

$$y(x) = \frac{((-25x + 50c_1)\,\mathrm{e}^x + 20x + 28)\cos(x)}{50} + \frac{\left(5c_2\mathrm{e}^x + x + \frac{2}{5}\right)\sin(x)}{5}$$

✓ <u>Solution by Mathematica</u>

Time used: 0.333 (sec). Leaf size: 48

```
DSolve[y''[x]-2*y'[x]+2*y[x]==(x+Exp[x])*Sin[x],y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \frac{1}{50}((-5(5e^x - 4)\,x + 50c_2e^x + 28)\cos(x) + 2(5x + 25c_1e^x + 2)\sin(x))$$

2.16 problem 16

Internal problem ID [3257]
Internal file name [`OUTPUT/2749_Sunday_June_05_2022_08_40_00_AM_6769952/index.tex`]

Book: Differential equations for engineers by Wei-Chau XIE, Cambridge Press 2010
Section: Chapter 4. Linear Differential Equations. Page 183
Problem number: 16.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_linear_constant_coeff"**

Maple gives the following as the ode type

```
[[_2nd_order, _linear, _nonhomogeneous]]
```

$$\boxed{y'' + 4y = \sinh(x)\sin(2x)}$$

2.16.1 Solving as second order linear constant coeff ode

This is second order non-homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = f(x)$$

Where $A = 1, B = 0, C = 4, f(x) = \sinh(x)\sin(2x)$. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the non-homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = f(x)$. y_h is the solution to

$$y'' + 4y = 0$$

This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = 0$$

Where in the above $A = 1, B = 0, C = 4$. Let the solution be $y = e^{\lambda x}$. Substituting this into the ODE gives

$$\lambda^2 e^{\lambda x} + 4 e^{\lambda x} = 0 \tag{1}$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda x}$ gives

$$\lambda^2 + 4 = 0 \tag{2}$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form.Using the quadratic formula

$$\lambda_{1,2} = \frac{-B}{2A} \pm \frac{1}{2A}\sqrt{B^2 - 4AC}$$

Substituting $A = 1, B = 0, C = 4$ into the above gives

$$\begin{aligned}\lambda_{1,2} &= \frac{0}{(2)(1)} \pm \frac{1}{(2)(1)}\sqrt{0^2 - (4)(1)(4)} \\ &= \pm 2i\end{aligned}$$

Hence

$$\begin{aligned}\lambda_1 &= +2i \\ \lambda_2 &= -2i\end{aligned}$$

Which simplifies to

$$\begin{aligned}\lambda_1 &= 2i \\ \lambda_2 &= -2i\end{aligned}$$

Since roots are complex conjugate of each others, then let the roots be

$$\lambda_{1,2} = \alpha \pm i\beta$$

Where $\alpha = 0$ and $\beta = 2$. Therefore the final solution, when using Euler relation, can be written as

$$y = e^{\alpha x}(c_1 \cos(\beta x) + c_2 \sin(\beta x))$$

Which becomes

$$y = e^0(c_1 \cos(2x) + c_2 \sin(2x))$$

Or

$$y = c_1 \cos(2x) + c_2 \sin(2x)$$

Therefore the homogeneous solution y_h is

$$y_h = c_1 \cos(2x) + c_2 \sin(2x)$$

The particular solution y_p can be found using either the method of undetermined coefficients, or the method of variation of parameters. The method of variation of parameters will be used as it is more general and can be used when the coefficients of the ODE depend on x as well. Let

$$y_p(x) = u_1 y_1 + u_2 y_2 \tag{1}$$

Where u_1, u_2 to be determined, and y_1, y_2 are the two basis solutions (the two linearly independent solutions of the homogeneous ODE) found earlier when solving the homogeneous ODE as

$$\begin{aligned} y_1 &= \cos(2x) \\ y_2 &= \sin(2x) \end{aligned}$$

In the Variation of parameters u_1, u_2 are found using

$$u_1 = -\int \frac{y_2 f(x)}{aW(x)} \tag{2}$$

$$u_2 = \int \frac{y_1 f(x)}{aW(x)} \tag{3}$$

Where $W(x)$ is the Wronskian and a is the coefficient in front of y'' in the given ODE. The Wronskian is given by $W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$. Hence

$$W = \begin{vmatrix} \cos(2x) & \sin(2x) \\ \frac{d}{dx}(\cos(2x)) & \frac{d}{dx}(\sin(2x)) \end{vmatrix}$$

Which gives

$$W = \begin{vmatrix} \cos(2x) & \sin(2x) \\ -2\sin(2x) & 2\cos(2x) \end{vmatrix}$$

Therefore

$$W = (\cos(2x))(2\cos(2x)) - (\sin(2x))(-2\sin(2x))$$

Which simplifies to

$$W = 2\cos(2x)^2 + 2\sin(2x)^2$$

Which simplifies to

$$W = 2$$

Therefore Eq. (2) becomes

$$u_1 = -\int \frac{\sin(2x)^2 \sinh(x)}{2} \, dx$$

Which simplifies to

$$u_1 = -\int \frac{\sin(2x)^2 \sinh(x)}{2} dx$$

Hence

$$u_1 = -\frac{e^x}{8} + \frac{e^x \cos(4x)}{136} + \frac{\sin(4x)\, e^x}{34} - \frac{e^{-x}}{8} + \frac{e^{-x}\cos(4x)}{136} - \frac{e^{-x}\sin(4x)}{34}$$

And Eq. (3) becomes

$$u_2 = \int \frac{\sin(2x)\sinh(x)\cos(2x)}{2} \, dx$$

Which simplifies to

$$u_2 = \int \frac{\sinh(x)\sin(4x)}{4} dx$$

Hence

$$\begin{aligned} u_2 = &-\frac{e^x \cos(4x)}{34} + \frac{\sin(4x)\, e^x}{136} - \frac{e^x \cos(2x)}{10} + \frac{e^x \sin(2x)}{20} \\ &- \frac{e^x(-2\cos(2x) + \sin(2x))}{20} + \frac{e^{-x}\cos(4x)}{34} + \frac{e^{-x}\sin(4x)}{136} \\ &+ \frac{e^{-x}\cos(2x)}{10} + \frac{e^{-x}\sin(2x)}{20} + \frac{e^{-x}(-\sin(2x) - 2\cos(2x))}{20} \end{aligned}$$

Which simplifies to

$$
\begin{aligned}
u_1 &= \frac{(-17+\cos(4x)-4\sin(4x))\,\mathrm{e}^{-x}}{136}+\frac{\mathrm{e}^{x}(-17+\cos(4x)+4\sin(4x))}{136} \\
u_2 &= \frac{(-\mathrm{e}^{x}+\mathrm{e}^{-x})\cos(4x)}{34}+\frac{\sin(4x)\,(\mathrm{e}^{x}+\mathrm{e}^{-x})}{136}
\end{aligned}
$$

Therefore the particular solution, from equation (1) is

$$
\begin{aligned}
y_p(x) = {} & \left(\frac{(-17+\cos(4x)-4\sin(4x))\,\mathrm{e}^{-x}}{136}+\frac{\mathrm{e}^{x}(-17+\cos(4x)+4\sin(4x))}{136}\right)\cos(2x) \\
& + \left(\frac{(-\mathrm{e}^{x}+\mathrm{e}^{-x})\cos(4x)}{34}+\frac{\sin(4x)\,(\mathrm{e}^{x}+\mathrm{e}^{-x})}{136}\right)\sin(2x)
\end{aligned}
$$

Which simplifies to

$$
y_p(x) = \frac{(-4\,\mathrm{e}^{x}-4\,\mathrm{e}^{-x})\cos(2x)}{34}+\frac{\sin(2x)\,(\mathrm{e}^{x}-\mathrm{e}^{-x})}{34}
$$

Therefore the general solution is

$$
\begin{aligned}
y &= y_h + y_p \\
&= (c_1\cos(2x)+c_2\sin(2x)) + \left(\frac{(-4\,\mathrm{e}^{x}-4\,\mathrm{e}^{-x})\cos(2x)}{34}+\frac{\sin(2x)\,(\mathrm{e}^{x}-\mathrm{e}^{-x})}{34}\right)
\end{aligned}
$$

<u>Summary</u>

The solution(s) found are the following

$$
y = c_1\cos(2x)+c_2\sin(2x)+\frac{(-4\,\mathrm{e}^{x}-4\,\mathrm{e}^{-x})\cos(2x)}{34}+\frac{\sin(2x)\,(\mathrm{e}^{x}-\mathrm{e}^{-x})}{34} \tag{1}
$$

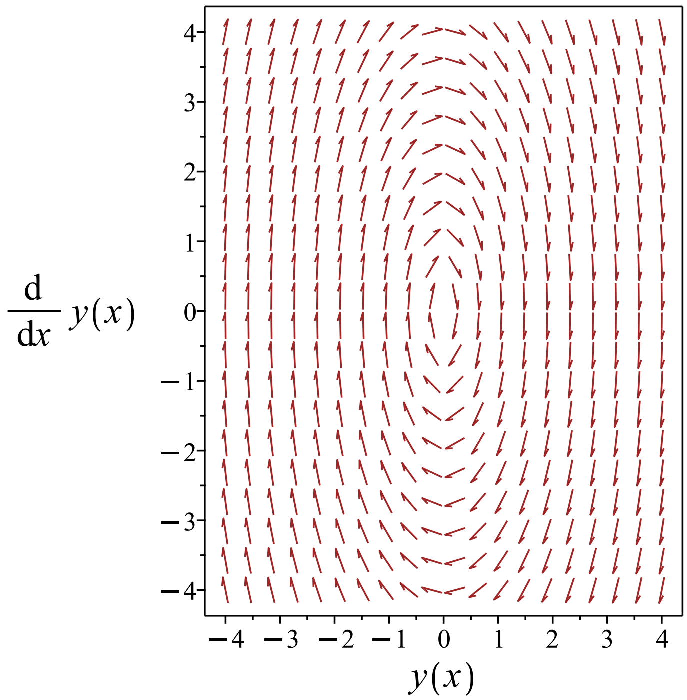


Figure 137: Slope field plot

Verification of solutions

$$y = c_1 \cos(2x) + c_2 \sin(2x) + \frac{(-4\,e^x - 4\,e^{-x})\cos(2x)}{34} + \frac{\sin(2x)\,(e^x - e^{-x})}{34}$$

Verified OK.

2.16.2 Solving using Kovacic algorithm

Writing the ode as

$$y'' + 4y = 0 \tag{1}$$

$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1 \\ B &= 0 \\ C &= 4 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = y e^{\int \frac{B}{2A}\, dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{-4}{1} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= -4 \\ t &= 1 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = -4z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 61: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 0 - 0 \\ &= 0 \end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Since $r = -4$ is not a function of x, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$z_1(x) = \cos(2x)$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$y_1 = z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx}$$

Since $B = 0$ then the above reduces to

$$\begin{aligned} y_1 &= z_1 \\ &= \cos(2x) \end{aligned}$$

Which simplifies to

$$y_1 = \cos(2x)$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx$$

Since $B = 0$ then the above becomes

$$\begin{aligned} y_2 &= y_1 \int \frac{1}{y_1^2}\,dx \\ &= \cos(2x) \int \frac{1}{\cos(2x)^2}\,dx \\ &= \cos(2x) \left(\frac{\tan(2x)}{2}\right) \end{aligned}$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1(\cos(2x)) + c_2\left(\cos(2x)\left(\frac{\tan(2x)}{2}\right)\right)
\end{aligned}
$$

This is second order nonhomogeneous ODE. Let the solution be

$$
y = y_h + y_p
$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x)=0$, and y_p is a particular solution to the nonhomogeneous ODE $Ay''(x)+By'(x)+Cy(x)=f(x)$. y_h is the solution to

$$
y'' + 4y = 0
$$

The homogeneous solution is found using the Kovacic algorithm which results in

$$
y_h = c_1 \cos(2x) + \frac{c_2 \sin(2x)}{2}
$$

The particular solution y_p can be found using either the method of undetermined coefficients, or the method of variation of parameters. The method of variation of parameters will be used as it is more general and can be used when the coefficients of the ODE depend on x as well. Let

$$
y_p(x) = u_1 y_1 + u_2 y_2 \tag{1}
$$

Where u_1, u_2 to be determined, and y_1, y_2 are the two basis solutions (the two linearly independent solutions of the homogeneous ODE) found earlier when solving the homogeneous ODE as

$$
\begin{aligned}
y_1 &= \cos(2x) \\
y_2 &= \frac{\sin(2x)}{2}
\end{aligned}
$$

In the Variation of parameters u_1, u_2 are found using

$$
u_1 = -\int \frac{y_2 f(x)}{aW(x)} \tag{2}
$$

$$
u_2 = \int \frac{y_1 f(x)}{aW(x)} \tag{3}
$$

Where $W(x)$ is the Wronskian and a is the coefficient in front of y'' in the given ODE. The Wronskian is given by $W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$. Hence

$$W = \begin{vmatrix} \cos(2x) & \frac{\sin(2x)}{2} \\ \frac{d}{dx}(\cos(2x)) & \frac{d}{dx}\left(\frac{\sin(2x)}{2}\right) \end{vmatrix}$$

Which gives

$$W = \begin{vmatrix} \cos(2x) & \frac{\sin(2x)}{2} \\ -2\sin(2x) & \cos(2x) \end{vmatrix}$$

Therefore

$$W = (\cos(2x))(\cos(2x)) - \left(\frac{\sin(2x)}{2}\right)(-2\sin(2x))$$

Which simplifies to

$$W = \cos(2x)^2 + \sin(2x)^2$$

Which simplifies to

$$W = 1$$

Therefore Eq. (2) becomes

$$u_1 = -\int \frac{\frac{\sin(2x)^2 \sinh(x)}{2}}{1} dx$$

Which simplifies to

$$u_1 = -\int \frac{\sin(2x)^2 \sinh(x)}{2} dx$$

Hence

$$u_1 = -\frac{e^x}{8} + \frac{e^x \cos(4x)}{136} + \frac{\sin(4x)\, e^x}{34} - \frac{e^{-x}}{8} + \frac{e^{-x}\cos(4x)}{136} - \frac{e^{-x}\sin(4x)}{34}$$

And Eq. (3) becomes

$$u_2 = \int \frac{\sin(2x)\sinh(x)\cos(2x)}{1} dx$$

Which simplifies to

$$u_2 = \int \frac{\sinh(x)\sin(4x)}{2}dx$$

Hence

$$\begin{aligned} u_2 = &-\frac{e^x \cos(4x)}{17} + \frac{\sin(4x)e^x}{68} - \frac{e^x \cos(2x)}{5} + \frac{e^x \sin(2x)}{10} \\ &- \frac{e^x(-2\cos(2x) + \sin(2x))}{10} + \frac{e^{-x}\cos(4x)}{17} + \frac{e^{-x}\sin(4x)}{68} \\ &+ \frac{e^{-x}\cos(2x)}{5} + \frac{e^{-x}\sin(2x)}{10} + \frac{e^{-x}(-\sin(2x) - 2\cos(2x))}{10} \end{aligned}$$

Which simplifies to

$$u_1 = \frac{(-17 + \cos(4x) - 4\sin(4x))e^{-x}}{136} + \frac{e^x(-17 + \cos(4x) + 4\sin(4x))}{136}$$

$$u_2 = \frac{(-4e^x + 4e^{-x})\cos(4x)}{68} + \frac{\sin(4x)(e^x + e^{-x})}{68}$$

Therefore the particular solution, from equation (1) is

$$\begin{aligned} y_p(x) = &\left(\frac{(-17 + \cos(4x) - 4\sin(4x))e^{-x}}{136} + \frac{e^x(-17 + \cos(4x) + 4\sin(4x))}{136}\right)\cos(2x) \\ &+ \frac{\left(\frac{(-4e^x+4e^{-x})\cos(4x)}{68} + \frac{\sin(4x)(e^x+e^{-x})}{68}\right)\sin(2x)}{2} \end{aligned}$$

Which simplifies to

$$y_p(x) = \frac{(-4e^x - 4e^{-x})\cos(2x)}{34} + \frac{\sin(2x)(e^x - e^{-x})}{34}$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \left(c_1 \cos(2x) + \frac{c_2 \sin(2x)}{2}\right) + \left(\frac{(-4e^x - 4e^{-x})\cos(2x)}{34} + \frac{\sin(2x)(e^x - e^{-x})}{34}\right) \end{aligned}$$

Summary

The solution(s) found are the following

$$y = c_1 \cos(2x) + \frac{c_2 \sin(2x)}{2} + \frac{(-4e^x - 4e^{-x})\cos(2x)}{34} + \frac{\sin(2x)(e^x - e^{-x})}{34} \quad (1)$$

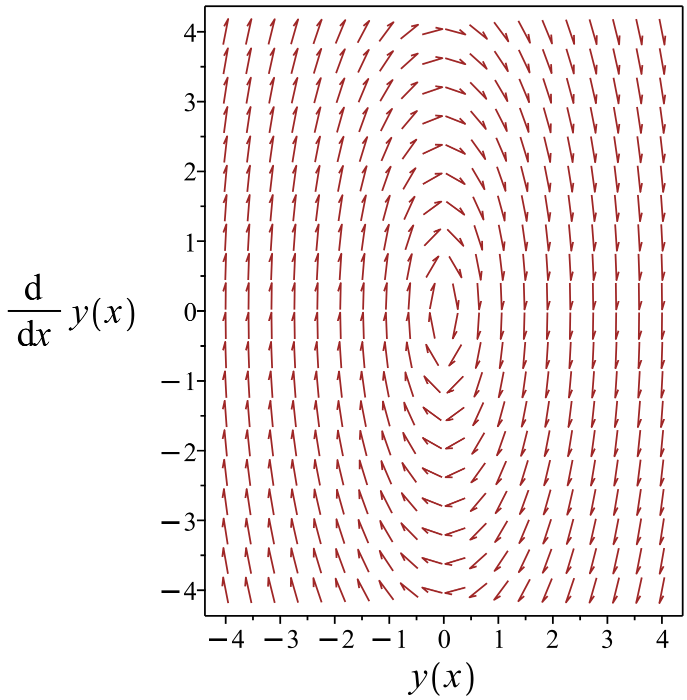


Figure 138: Slope field plot

<u>Verification of solutions</u>

$$y = c_1 \cos(2x) + \frac{c_2 \sin(2x)}{2} + \frac{(-4\,e^{x} - 4\,e^{-x})\cos(2x)}{34} + \frac{\sin(2x)\,(e^{x} - e^{-x})}{34}$$

Verified OK.

2.16.3 Maple step by step solution

Let's solve

$y'' + 4y = \sinh(x)\sin(2x)$

- Highest derivative means the order of the ODE is 2

  y''

- Characteristic polynomial of homogeneous ODE

  $r^2 + 4 = 0$

- Use quadratic formula to solve for r

  $r = \frac{0 \pm (\sqrt{-16})}{2}$

- Roots of the characteristic polynomial

$r = (-2\,\mathrm{I}, 2\,\mathrm{I})$

- 1st solution of the homogeneous ODE

  $y_1(x) = \cos(2x)$

- 2nd solution of the homogeneous ODE

  $y_2(x) = \sin(2x)$

- General solution of the ODE

  $y = c_1 y_1(x) + c_2 y_2(x) + y_p(x)$

- Substitute in solutions of the homogeneous ODE

  $y = c_1 \cos(2x) + c_2 \sin(2x) + y_p(x)$

☐ Find a particular solution $y_p(x)$ of the ODE

- Use variation of parameters to find y_p here $f(x)$ is the forcing function

  $\left[y_p(x) = -y_1(x)\left(\int \frac{y_2(x) f(x)}{W(y_1(x), y_2(x))} dx\right) + y_2(x)\left(\int \frac{y_1(x) f(x)}{W(y_1(x), y_2(x))} dx\right), f(x) = \sinh(x)\sin(2x)\right]$

- Wronskian of solutions of the homogeneous equation

  $W(y_1(x), y_2(x)) = \begin{bmatrix} \cos(2x) & \sin(2x) \\ -2\sin(2x) & 2\cos(2x) \end{bmatrix}$

- Compute Wronskian

  $W(y_1(x), y_2(x)) = 2$

- Substitute functions into equation for $y_p(x)$

  $y_p(x) = -\frac{\cos(2x)\left(\int \sin(2x)^2 \sinh(x) dx\right)}{2} + \frac{\sin(2x)\left(\int \sinh(x)\sin(4x) dx\right)}{4}$

- Compute integrals

  $y_p(x) = \frac{(-4\,\mathrm{e}^{x} - 4\,\mathrm{e}^{-x})\cos(2x)}{34} + \frac{\sin(2x)(\mathrm{e}^{x} - \mathrm{e}^{-x})}{34}$

- Substitute particular solution into general solution to ODE

  $y = c_1 \cos(2x) + c_2 \sin(2x) + \frac{(-4\,\mathrm{e}^{x} - 4\,\mathrm{e}^{-x})\cos(2x)}{34} + \frac{\sin(2x)(\mathrm{e}^{x} - \mathrm{e}^{-x})}{34}$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 2; linear nonhomogeneous with symmetry [0,1]
trying a double symmetry of the form [xi=0, eta=F(x)]
-> Try solving first the homogeneous part of the ODE
   checking if the LODE has constant coefficients
   <- constant coefficients successful
<- solving first the homogeneous part of the ODE successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 42

```
dsolve(diff(y(x),x$2)+4*y(x)=sinh(x)*sin(2*x),y(x), singsol=all)
```

$$y(x) = \frac{(34c_1 - 4\,\mathrm{e}^{x} - 4\,\mathrm{e}^{-x})\cos(2x)}{34} + \left(c_2 + \frac{\mathrm{e}^{x}}{34} - \frac{\mathrm{e}^{-x}}{34}\right)\sin(2x)$$

✓ Solution by Mathematica

Time used: 0.119 (sec). Leaf size: 46

```
DSolve[y''[x]+4*y[x]==Sinh[x]*Sin[2*x],y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \frac{1}{34}(-(4-i)\cos((2+i)x) - (4+i)\cosh((1+2i)x) + 34c_1\cos(2x) + 34c_2\sin(2x))$$

2.17 problem 17

Internal problem ID [3258]
Internal file name [`OUTPUT/2750_Sunday_June_05_2022_08_40_00_AM_3854497/index.tex`]

Book: Differential equations for engineers by Wei-Chau XIE, Cambridge Press 2010
Section: Chapter 4. Linear Differential Equations. Page 183
Problem number: 17.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_linear_constant_coeff"**

Maple gives the following as the ode type

```
[[_2nd_order, _linear, _nonhomogeneous]]
```

$$\boxed{y'' + 2y' + 2y = \cosh(x)\sin(x)}$$

2.17.1 Solving as second order linear constant coeff ode

This is second order non-homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = f(x)$$

Where $A = 1, B = 2, C = 2, f(x) = \cosh(x)\sin(x)$. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the non-homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = f(x)$. y_h is the solution to

$$y'' + 2y' + 2y = 0$$

This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = 0$$

Where in the above $A = 1, B = 2, C = 2$. Let the solution be $y = e^{\lambda x}$. Substituting this into the ODE gives

$$\lambda^2 e^{\lambda x} + 2\lambda e^{\lambda x} + 2 e^{\lambda x} = 0 \tag{1}$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda x}$ gives

$$\lambda^2 + 2\lambda + 2 = 0 \tag{2}$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form.Using the quadratic formula

$$\lambda_{1,2} = \frac{-B}{2A} \pm \frac{1}{2A}\sqrt{B^2 - 4AC}$$

Substituting $A = 1, B = 2, C = 2$ into the above gives

$$\begin{aligned}\lambda_{1,2} &= \frac{-2}{(2)(1)} \pm \frac{1}{(2)(1)}\sqrt{2^2 - (4)(1)(2)}\\ &= -1 \pm i\end{aligned}$$

Hence

$$\begin{aligned}\lambda_1 &= -1 + i\\ \lambda_2 &= -1 - i\end{aligned}$$

Which simplifies to

$$\begin{aligned}\lambda_1 &= -1 + i\\ \lambda_2 &= -1 - i\end{aligned}$$

Since roots are complex conjugate of each others, then let the roots be

$$\lambda_{1,2} = \alpha \pm i\beta$$

Where $\alpha = -1$ and $\beta = 1$. Therefore the final solution, when using Euler relation, can be written as

$$y = e^{\alpha x}(c_1 \cos(\beta x) + c_2 \sin(\beta x))$$

Which becomes

$$y = e^{-x}(c_1 \cos(x) + c_2 \sin(x))$$

Therefore the homogeneous solution y_h is

$$y_h = e^{-x}(c_1 \cos(x) + c_2 \sin(x))$$

The particular solution y_p can be found using either the method of undetermined coefficients, or the method of variation of parameters. The method of variation of parameters will be used as it is more general and can be used when the coefficients of the ODE depend on x as well. Let

$$y_p(x) = u_1 y_1 + u_2 y_2 \tag{1}$$

Where u_1, u_2 to be determined, and y_1, y_2 are the two basis solutions (the two linearly independent solutions of the homogeneous ODE) found earlier when solving the homogeneous ODE as

$$y_1 = e^{-x} \cos(x)$$
$$y_2 = e^{-x} \sin(x)$$

In the Variation of parameters u_1, u_2 are found using

$$u_1 = -\int \frac{y_2 f(x)}{a W(x)} \tag{2}$$

$$u_2 = \int \frac{y_1 f(x)}{a W(x)} \tag{3}$$

Where $W(x)$ is the Wronskian and a is the coefficient in front of y'' in the given ODE. The Wronskian is given by $W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$. Hence

$$W = \begin{vmatrix} e^{-x} \cos(x) & e^{-x} \sin(x) \\ \frac{d}{dx}(e^{-x} \cos(x)) & \frac{d}{dx}(e^{-x} \sin(x)) \end{vmatrix}$$

Which gives

$$W = \begin{vmatrix} e^{-x} \cos(x) & e^{-x} \sin(x) \\ -e^{-x} \cos(x) - e^{-x} \sin(x) & -e^{-x} \sin(x) + e^{-x} \cos(x) \end{vmatrix}$$

Therefore

$$W = (e^{-x} \cos(x)) (-e^{-x} \sin(x) + e^{-x} \cos(x)) - (e^{-x} \sin(x)) (-e^{-x} \cos(x) - e^{-x} \sin(x))$$

Which simplifies to

$$W = e^{-2x} \sin(x)^2 + e^{-2x} \cos(x)^2$$

Which simplifies to

$$W = \mathrm{e}^{-2x}$$

Therefore Eq. (2) becomes

$$u_1 = -\int \frac{\mathrm{e}^{-x} \sin(x)^2 \cosh(x)}{\mathrm{e}^{-2x}}\, dx$$

Which simplifies to

$$u_1 = -\int \cosh(x) \sin(x)^2 \mathrm{e}^{x} dx$$

Hence

$$u_1 = -\frac{(2\sin(x) - 2\cos(x))\, \mathrm{e}^{2x} \sin(x)}{16} - \frac{\mathrm{e}^{2x}}{16} + \frac{\sin(x)\cos(x)}{4} - \frac{x}{4}$$

And Eq. (3) becomes

$$u_2 = \int \frac{\mathrm{e}^{-x} \cos(x) \cosh(x) \sin(x)}{\mathrm{e}^{-2x}}\, dx$$

Which simplifies to

$$u_2 = \int \frac{\cosh(x) \sin(2x)\, \mathrm{e}^{x}}{2} dx$$

Hence

$$u_2 = \frac{\mathrm{e}^{2x}(2\sin(2x) - 2\cos(2x))}{32} - \frac{\cos(x)^2}{4}$$

Which simplifies to

$$u_1 = \frac{(-2 + \cos(2x) + \sin(2x))\, \mathrm{e}^{2x}}{16} - \frac{x}{4} + \frac{\sin(2x)}{8}$$

$$u_2 = \frac{\left(-2\cos(x)^2 + 2\sin(x)\cos(x) + 1\right) \mathrm{e}^{2x}}{16} - \frac{\cos(x)^2}{4}$$

Therefore the particular solution, from equation (1) is

$$
\begin{aligned}
y_p(x) &= \left(\frac{(-2+\cos(2x)+\sin(2x))\,\mathrm{e}^{2x}}{16}-\frac{x}{4}+\frac{\sin(2x)}{8}\right)\mathrm{e}^{-x}\cos(x) \\
&\quad+\left(\frac{\left(-2\cos(x)^2+2\sin(x)\cos(x)+1\right)\mathrm{e}^{2x}}{16}-\frac{\cos(x)^2}{4}\right)\mathrm{e}^{-x}\sin(x)
\end{aligned}
$$

Which simplifies to

$$
y_p(x) = -\frac{\mathrm{e}^{-x}\cos(x)\,x}{4}-\frac{\mathrm{e}^{x}(\cos(x)-\sin(x))}{16}
$$

Therefore the general solution is

$$
\begin{aligned}
y &= y_h + y_p \\
&= \left(\mathrm{e}^{-x}(c_1\cos(x)+c_2\sin(x))\right)+\left(-\frac{\mathrm{e}^{-x}\cos(x)\,x}{4}-\frac{\mathrm{e}^{x}(\cos(x)-\sin(x))}{16}\right)
\end{aligned}
$$

Summary
The solution(s) found are the following

$$
y = \mathrm{e}^{-x}(c_1\cos(x)+c_2\sin(x))-\frac{\mathrm{e}^{-x}\cos(x)\,x}{4}-\frac{\mathrm{e}^{x}(\cos(x)-\sin(x))}{16} \tag{1}
$$

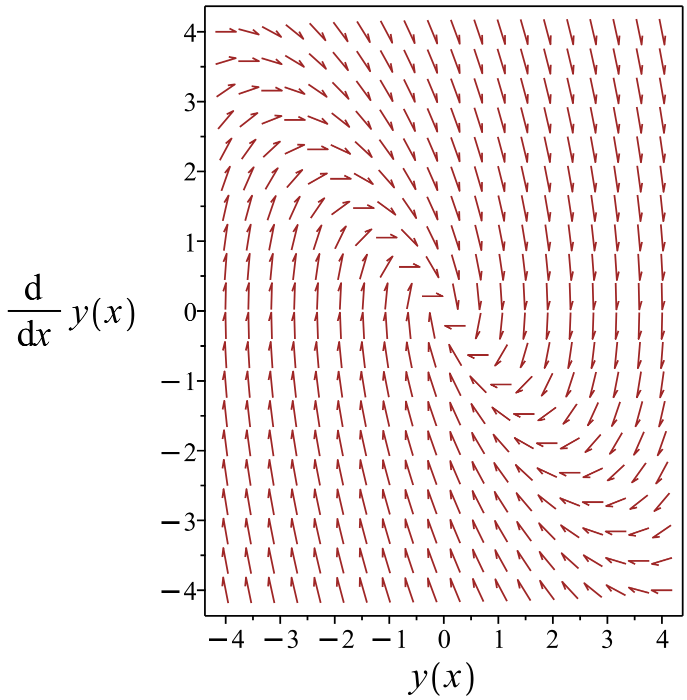


Figure 139: Slope field plot

Verification of solutions

$$y = e^{-x}(c_1 \cos(x) + c_2 \sin(x)) - \frac{e^{-x} \cos(x) x}{4} - \frac{e^{x}(\cos(x) - \sin(x))}{16}$$

Verified OK.

2.17.2 Solving using Kovacic algorithm

Writing the ode as

$$y'' + 2y' + 2y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1 \\ B &= 2 \\ C &= 2 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\, dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{-1}{1} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= -1 \\ t &= 1 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = -z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 63: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$
\begin{aligned}
O(\infty) &= \deg(t) - \deg(s) \\
&= 0 - 0 \\
&= 0
\end{aligned}
$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Therefore

$$
L = [1]
$$

Since $r = -1$ is not a function of x, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$
z_1(x) = \cos(x)
$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$
\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{2}{1}\,dx} \\
&= z_1 e^{-x} \\
&= z_1\left(\mathrm{e}^{-x}\right)
\end{aligned}
$$

Which simplifies to

$$
y_1 = \mathrm{e}^{-x}\cos(x)
$$

The second solution y_2 to the original ode is found using reduction of order

$$
y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx
$$

Substituting gives

$$
\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{2}{1}\,dx}}{(y_1)^2}\,dx \\
&= y_1 \int \frac{e^{-2x}}{(y_1)^2}\,dx \\
&= y_1(\tan(x))
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1 \left(\mathrm{e}^{-x} \cos(x)\right) + c_2 \left(\mathrm{e}^{-x} \cos(x) \left(\tan(x)\right)\right)
\end{aligned}
$$

This is second order nonhomogeneous ODE. Let the solution be

$$
y = y_h + y_p
$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the nonhomogeneous ODE $Ay''(x) + By'(x) + Cy(x) = f(x)$. y_h is the solution to

$$
y'' + 2y' + 2y = 0
$$

The homogeneous solution is found using the Kovacic algorithm which results in

$$
y_h = c_1 \mathrm{e}^{-x} \cos(x) + c_2 \mathrm{e}^{-x} \sin(x)
$$

The particular solution y_p can be found using either the method of undetermined coefficients, or the method of variation of parameters. The method of variation of parameters will be used as it is more general and can be used when the coefficients of the ODE depend on x as well. Let

$$
y_p(x) = u_1 y_1 + u_2 y_2 \tag{1}
$$

Where u_1, u_2 to be determined, and y_1, y_2 are the two basis solutions (the two linearly independent solutions of the homogeneous ODE) found earlier when solving the homogeneous ODE as

$$
\begin{aligned}
y_1 &= \mathrm{e}^{-x} \cos(x) \\
y_2 &= \mathrm{e}^{-x} \sin(x)
\end{aligned}
$$

In the Variation of parameters u_1, u_2 are found using

$$
u_1 = -\int \frac{y_2 f(x)}{a W(x)} \tag{2}
$$

$$
u_2 = \int \frac{y_1 f(x)}{a W(x)} \tag{3}
$$

Where $W(x)$ is the Wronskian and a is the coefficient in front of y'' in the given ODE.

The Wronskian is given by $W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$. Hence

$$W = \begin{vmatrix} e^{-x}\cos(x) & e^{-x}\sin(x) \\ \frac{d}{dx}\left(e^{-x}\cos(x)\right) & \frac{d}{dx}\left(e^{-x}\sin(x)\right) \end{vmatrix}$$

Which gives

$$W = \begin{vmatrix} e^{-x}\cos(x) & e^{-x}\sin(x) \\ -e^{-x}\cos(x) - e^{-x}\sin(x) & -e^{-x}\sin(x) + e^{-x}\cos(x) \end{vmatrix}$$

Therefore

$$W = \left(e^{-x}\cos(x)\right)\left(-e^{-x}\sin(x)+e^{-x}\cos(x)\right) - \left(e^{-x}\sin(x)\right)\left(-e^{-x}\cos(x)-e^{-x}\sin(x)\right)$$

Which simplifies to

$$W = e^{-2x}\sin(x)^2 + e^{-2x}\cos(x)^2$$

Which simplifies to

$$W = e^{-2x}$$

Therefore Eq. (2) becomes

$$u_1 = -\int \frac{e^{-x}\sin(x)^2\cosh(x)}{e^{-2x}}\,dx$$

Which simplifies to

$$u_1 = -\int \cosh(x)\sin(x)^2 e^{x}\,dx$$

Hence

$$u_1 = -\frac{\left(2\sin(x) - 2\cos(x)\right)e^{2x}\sin(x)}{16} - \frac{e^{2x}}{16} + \frac{\sin(x)\cos(x)}{4} - \frac{x}{4}$$

And Eq. (3) becomes

$$u_2 = \int \frac{e^{-x}\cos(x)\cosh(x)\sin(x)}{e^{-2x}}\,dx$$

Which simplifies to

$$u_2 = \int \frac{\cosh(x)\sin(2x)\,\mathrm{e}^{x}}{2}dx$$

Hence

$$u_2 = \frac{\mathrm{e}^{2x}(2\sin(2x) - 2\cos(2x))}{32} - \frac{\cos(x)^2}{4}$$

Which simplifies to

$$u_1 = \frac{(-2+\cos(2x)+\sin(2x))\,\mathrm{e}^{2x}}{16} - \frac{x}{4} + \frac{\sin(2x)}{8}$$
$$u_2 = \frac{\left(-2\cos(x)^2 + 2\sin(x)\cos(x) + 1\right)\mathrm{e}^{2x}}{16} - \frac{\cos(x)^2}{4}$$

Therefore the particular solution, from equation (1) is

$$\begin{aligned} y_p(x) = &\left(\frac{(-2+\cos(2x)+\sin(2x))\,\mathrm{e}^{2x}}{16} - \frac{x}{4} + \frac{\sin(2x)}{8}\right)\mathrm{e}^{-x}\cos(x) \\ &+ \left(\frac{\left(-2\cos(x)^2 + 2\sin(x)\cos(x) + 1\right)\mathrm{e}^{2x}}{16} - \frac{\cos(x)^2}{4}\right)\mathrm{e}^{-x}\sin(x) \end{aligned}$$

Which simplifies to

$$y_p(x) = -\frac{\mathrm{e}^{-x}\cos(x)\,x}{4} - \frac{\mathrm{e}^{x}(\cos(x) - \sin(x))}{16}$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \left(c_1\mathrm{e}^{-x}\cos(x) + c_2\mathrm{e}^{-x}\sin(x)\right) + \left(-\frac{\mathrm{e}^{-x}\cos(x)\,x}{4} - \frac{\mathrm{e}^{x}(\cos(x) - \sin(x))}{16}\right) \end{aligned}$$

Which simplifies to

$$y = \mathrm{e}^{-x}(c_1\cos(x) + c_2\sin(x)) - \frac{\mathrm{e}^{-x}\cos(x)\,x}{4} - \frac{\mathrm{e}^{x}(\cos(x) - \sin(x))}{16}$$

Summary

The solution(s) found are the following

$$y = e^{-x}(c_1 \cos(x) + c_2 \sin(x)) - \frac{e^{-x}\cos(x)\, x}{4} - \frac{e^{x}(\cos(x) - \sin(x))}{16} \tag{1}$$

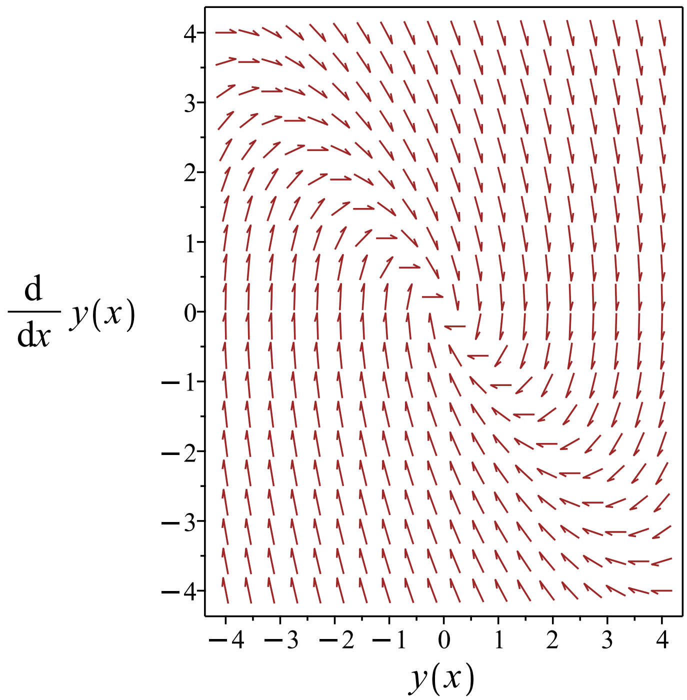


Figure 140: Slope field plot

Verification of solutions

$$y = e^{-x}(c_1 \cos(x) + c_2 \sin(x)) - \frac{e^{-x}\cos(x)\, x}{4} - \frac{e^{x}(\cos(x) - \sin(x))}{16}$$

Verified OK.

2.17.3 Maple step by step solution

Let's solve

$y'' + 2y' + 2y = \cosh(x)\sin(x)$

- Highest derivative means the order of the ODE is 2

  y''

- Characteristic polynomial of homogeneous ODE

$r^2+2r+2=0$

- Use quadratic formula to solve for r

$r=\frac{(-2)\pm(\sqrt{-4})}{2}$

- Roots of the characteristic polynomial

$r=(-1-\mathrm{I},-1+\mathrm{I})$

- 1st solution of the homogeneous ODE

$y_1(x)=\mathrm{e}^{-x}\cos(x)$

- 2nd solution of the homogeneous ODE

$y_2(x)=\mathrm{e}^{-x}\sin(x)$

- General solution of the ODE

$y=c_1y_1(x)+c_2y_2(x)+y_p(x)$

- Substitute in solutions of the homogeneous ODE

$y=c_1\mathrm{e}^{-x}\cos(x)+c_2\mathrm{e}^{-x}\sin(x)+y_p(x)$

☐ Find a particular solution $y_p(x)$ of the ODE

  - Use variation of parameters to find y_p here $f(x)$ is the forcing function

$\left[y_p(x)=-y_1(x)\left(\int\frac{y_2(x)f(x)}{W(y_1(x),y_2(x))}dx\right)+y_2(x)\left(\int\frac{y_1(x)f(x)}{W(y_1(x),y_2(x))}dx\right),f(x)=\cosh(x)\sin(x)\right]$

  - Wronskian of solutions of the homogeneous equation

$W(y_1(x),y_2(x))=\left[\begin{array}{cc}\mathrm{e}^{-x}\cos(x) & \mathrm{e}^{-x}\sin(x)\\ -\mathrm{e}^{-x}\cos(x)-\mathrm{e}^{-x}\sin(x) & -\mathrm{e}^{-x}\sin(x)+\mathrm{e}^{-x}\cos(x)\end{array}\right]$

  - Compute Wronskian

$W(y_1(x),y_2(x))=\mathrm{e}^{-2x}$

  - Substitute functions into equation for $y_p(x)$

$y_p(x)=\frac{\mathrm{e}^{-x}\left(\sin(x)\left(\int\cosh(x)\sin(2x)\mathrm{e}^{x}dx\right)-2\cos(x)\left(\int\cosh(x)\sin(x)^2\mathrm{e}^{x}dx\right)\right)}{2}$

  - Compute integrals

$y_p(x)=-\frac{\mathrm{e}^{-x}\cos(x)x}{4}-\frac{\mathrm{e}^{x}(\cos(x)-\sin(x))}{16}$

- Substitute particular solution into general solution to ODE

$y=c_1\mathrm{e}^{-x}\cos(x)+c_2\mathrm{e}^{-x}\sin(x)-\frac{\mathrm{e}^{-x}\cos(x)x}{4}-\frac{\mathrm{e}^{x}(\cos(x)-\sin(x))}{16}$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 2; linear nonhomogeneous with symmetry [0,1]
trying a double symmetry of the form [xi=0, eta=F(x)]
-> Try solving first the homogeneous part of the ODE
   checking if the LODE has constant coefficients
   <- constant coefficients successful
<- solving first the homogeneous part of the ODE successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 38

```
dsolve(diff(y(x),x$2)+2*diff(y(x),x)+2*y(x)=cosh(x)*sin(x),y(x), singsol=all)
```

$$y(x) = \frac{((-x+4c_1)\cos(x)+4\sin(x)\,c_2)\,\mathrm{e}^{-x}}{4} - \frac{\mathrm{e}^{x}(-\sin(x)+\cos(x))}{16}$$

✓ Solution by Mathematica

Time used: 0.199 (sec). Leaf size: 47

```
DSolve[y''[x]+2*y'[x]+2*y[x]==Cosh[x]*Sin[x],y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \frac{1}{16}e^{-x}\left(\left(e^{2x}+2+16c_1\right)\sin(x) - \left(e^{2x}+4(x-4c_2)\right)\cos(x)\right)$$

2.18 problem 18

Internal problem ID [3259]
Internal file name [OUTPUT/2751_Sunday_June_05_2022_08_40_01_AM_96669158/index.tex]

Book: Differential equations for engineers by Wei-Chau XIE, Cambridge Press 2010
Section: Chapter 4. Linear Differential Equations. Page 183
Problem number: 18.
ODE order: 3.
ODE degree: 1.

The type(s) of ODE detected by this program : **"higher_order_linear_constant_coefficients_ODE"**

Maple gives the following as the ode type

```
[[_3rd_order, _missing_y]]
```

$$\boxed{y''' + y' = \sin(x) + \cos(x) x}$$

This is higher order nonhomogeneous ODE. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE And y_p is a particular solution to the nonhomogeneous ODE. y_h is the solution to

$$y''' + y' = 0$$

The characteristic equation is

$$\lambda^3 + \lambda = 0$$

The roots of the above equation are

$$\lambda_1 = 0$$
$$\lambda_2 = i$$
$$\lambda_3 = -i$$

Therefore the homogeneous solution is

$$y_h(x) = c_1 + e^{ix} c_2 + e^{-ix} c_3$$

The fundamental set of solutions for the homogeneous solution are the following

$$
\begin{aligned}
y_1 &= 1 \\
y_2 &= \mathrm{e}^{ix} \\
y_3 &= \mathrm{e}^{-ix}
\end{aligned}
$$

Now the particular solution to the given ODE is found

$$
y''' + y' = \sin(x) + \cos(x)\, x
$$

Let the particular solution be

$$
y_p = U_1 y_1 + U_2 y_2 + U_3 y_3
$$

Where y_i are the basis solutions found above for the homogeneous solution y_h and $U_i(x)$ are functions to be determined as follows

$$
U_i = (-1)^{n-i} \int \frac{F(x) W_i(x)}{a W(x)}\, dx
$$

Where $W(x)$ is the Wronskian and $W_i(x)$ is the Wronskian that results after deleting the last row and the i-th column of the determinant and n is the order of the ODE or equivalently, the number of basis solutions, and a is the coefficient of the leading derivative in the ODE, and $F(x)$ is the RHS of the ODE. Therefore, the first step is to find the Wronskian $W(x)$. This is given by

$$
W(x) = \begin{vmatrix} y_1 & y_2 & y_3 \\ y_1' & y_2' & y_3' \\ y_1'' & y_2'' & y_3'' \end{vmatrix}
$$

Substituting the fundamental set of solutions y_i found above in the Wronskian gives

$$
\begin{aligned}
W &= \begin{bmatrix} 1 & \mathrm{e}^{ix} & \mathrm{e}^{-ix} \\ 0 & i\mathrm{e}^{ix} & -i\mathrm{e}^{-ix} \\ 0 & -\mathrm{e}^{ix} & -\mathrm{e}^{-ix} \end{bmatrix} \\
|W| &= -2i\mathrm{e}^{ix}\mathrm{e}^{-ix}
\end{aligned}
$$

The determinant simplifies to

$$
|W| = -2i
$$

Now we determine W_i for each U_i.

$$\begin{aligned}W_1(x) &= \det \begin{bmatrix} e^{ix} & e^{-ix} \\ ie^{ix} & -ie^{-ix} \end{bmatrix} \\ &= -2i\end{aligned}$$

$$\begin{aligned}W_2(x) &= \det \begin{bmatrix} 1 & e^{-ix} \\ 0 & -ie^{-ix} \end{bmatrix} \\ &= -ie^{-ix}\end{aligned}$$

$$\begin{aligned}W_3(x) &= \det \begin{bmatrix} 1 & e^{ix} \\ 0 & ie^{ix} \end{bmatrix} \\ &= ie^{ix}\end{aligned}$$

Now we are ready to evaluate each $U_i(x)$.

$$\begin{aligned}U_1 &= (-1)^{3-1} \int \frac{F(x)W_1(x)}{aW(x)}\,dx \\ &= (-1)^2 \int \frac{(\sin(x) + \cos(x)\,x)(-2i)}{(1)(-2i)}\,dx \\ &= \int \frac{-2i(\sin(x) + \cos(x)\,x)}{-2i}\,dx \\ &= \int (\sin(x) + \cos(x)\,x)\;dx \\ &= x\sin(x)\end{aligned}$$

$$\begin{aligned}U_2 &= (-1)^{3-2} \int \frac{F(x)W_2(x)}{aW(x)}\,dx \\ &= (-1)^1 \int \frac{(\sin(x) + \cos(x)\,x)(-ie^{-ix})}{(1)(-2i)}\,dx \\ &= -\int \frac{-i(\sin(x) + \cos(x)\,x)\,e^{-ix}}{-2i}\,dx \\ &= -\int \left(\frac{(\sin(x) + \cos(x)\,x)\,e^{-ix}}{2}\right)dx \\ &= -\left(\int \frac{(\sin(x) + \cos(x)\,x)\,e^{-ix}}{2}dx\right)\end{aligned}$$

$$
\begin{aligned}
U_3 &= (-1)^{3-3} \int \frac{F(x) W_3(x)}{a W(x)}\, dx \\
&= (-1)^{0} \int \frac{(\sin(x) + \cos(x)\, x)\,(i \mathrm{e}^{ix})}{(1)\,(-2i)}\, dx \\
&= \int \frac{i(\sin(x) + \cos(x)\, x)\, \mathrm{e}^{ix}}{-2i}\, dx \\
&= \int \left(-\frac{(\sin(x) + \cos(x)\, x)\, \mathrm{e}^{ix}}{2}\right) dx \\
&= -\frac{x^2}{8} - \frac{ix}{4} + \frac{i(-i + 2x)\, \mathrm{e}^{2ix}}{16}
\end{aligned}
$$

Now that all the U_i functions have been determined, the particular solution is found from

$$
y_p = U_1 y_1 + U_2 y_2 + U_3 y_3
$$

Hence

$$
\begin{aligned}
y_p &= (x \sin(x)) \\
&+ \left(-\left(\int \frac{(\sin(x) + \cos(x)\, x)\, \mathrm{e}^{-ix}}{2} dx\right)\right) (\mathrm{e}^{ix}) \\
&+ \left(-\frac{x^2}{8} - \frac{ix}{4} + \frac{i(-i + 2x)\, \mathrm{e}^{2ix}}{16}\right) (\mathrm{e}^{-ix})
\end{aligned}
$$

Therefore the particular solution is

$$
y_p = \frac{(-4x^2 - 1)\cos(x)}{16} - \frac{3\sin(x)\left(i - \frac{4x}{3}\right)}{16}
$$

Which simplifies to

$$
y_p = \frac{(-4x^2 - 1)\cos(x)}{16} - \frac{3\sin(x)\left(i - \frac{4x}{3}\right)}{16}
$$

Therefore the general solution is

$$
\begin{aligned}
y &= y_h + y_p \\
&= \left(c_1 + \mathrm{e}^{ix} c_2 + \mathrm{e}^{-ix} c_3\right) + \left(\frac{(-4x^2 - 1)\cos(x)}{16} - \frac{3\sin(x)\left(i - \frac{4x}{3}\right)}{16}\right)
\end{aligned}
$$

Summary

The solution(s) found are the following

$$
y = c_1 + \mathrm{e}^{ix} c_2 + \mathrm{e}^{-ix} c_3 + \frac{(-4x^2 - 1)\cos(x)}{16} - \frac{3\sin(x)\left(i - \frac{4x}{3}\right)}{16} \tag{1}
$$

Verification of solutions

$$y = c_1 + e^{ix}c_2 + e^{-ix}c_3 + \frac{(-4x^2 - 1)\cos(x)}{16} - \frac{3\sin(x)\left(i - \frac{4x}{3}\right)}{16}$$

Verified OK.

2.18.1 Maple step by step solution

Let's solve

$y''' + y' = \sin(x) + \cos(x)\,x$

- Highest derivative means the order of the ODE is 3

  y'''

- □ Convert linear ODE into a system of first order ODEs

  - ○ Define new variable $y_1(x)$

    $y_1(x) = y$

  - ○ Define new variable $y_2(x)$

    $y_2(x) = y'$

  - ○ Define new variable $y_3(x)$

    $y_3(x) = y''$

  - ○ Isolate for $y_3'(x)$ using original ODE

    $y_3'(x) = \cos(x)\,x + \sin(x) - y_2(x)$

  Convert linear ODE into a system of first order ODEs

  $[y_2(x) = y_1'(x), y_3(x) = y_2'(x), y_3'(x) = \cos(x)\,x + \sin(x) - y_2(x)]$

- Define vector

  $$\vec{y}(x) = \begin{bmatrix} y_1(x) \\ y_2(x) \\ y_3(x) \end{bmatrix}$$

- System to solve

  $$\vec{y}'(x) = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{bmatrix} \cdot \vec{y}(x) + \begin{bmatrix} 0 \\ 0 \\ \sin(x) + \cos(x)\,x \end{bmatrix}$$

- Define the forcing function

$$\vec{f}(x) = \begin{bmatrix} 0 \\ 0 \\ \sin(x) + \cos(x)\, x \end{bmatrix}$$

- Define the coefficient matrix

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{bmatrix}$$

- Rewrite the system as

$$\vec{y}'(x) = A \cdot \vec{y}(x) + \vec{f}$$

- To solve the system, find the eigenvalues and eigenvectors of A
- Eigenpairs of A

$$\left[\left[0, \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}\right], \left[-\mathrm{I}, \begin{bmatrix} -1 \\ \mathrm{I} \\ 1 \end{bmatrix}\right], \left[\mathrm{I}, \begin{bmatrix} -1 \\ -\mathrm{I} \\ 1 \end{bmatrix}\right]\right]$$

- Consider eigenpair

$$\left[0, \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}\right]$$

- Solution to homogeneous system from eigenpair

$$\vec{y}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

- Consider complex eigenpair, complex conjugate eigenvalue can be ignored

$$\left[-\mathrm{I}, \begin{bmatrix} -1 \\ \mathrm{I} \\ 1 \end{bmatrix}\right]$$

- Solution from eigenpair

$$e^{-Ix} \cdot \begin{bmatrix} -1 \\ I \\ 1 \end{bmatrix}$$

- Use Euler identity to write solution in terms of sin and cos

$$(\cos(x) - I\sin(x)) \cdot \begin{bmatrix} -1 \\ I \\ 1 \end{bmatrix}$$

- Simplify expression

$$\begin{bmatrix} -\cos(x) + I\sin(x) \\ I(\cos(x) - I\sin(x)) \\ \cos(x) - I\sin(x) \end{bmatrix}$$

- Both real and imaginary parts are solutions to the homogeneous system

$$\left[\vec{y}_2(x) = \begin{bmatrix} -\cos(x) \\ \sin(x) \\ \cos(x) \end{bmatrix}, \vec{y}_3(x) = \begin{bmatrix} \sin(x) \\ \cos(x) \\ -\sin(x) \end{bmatrix}\right]$$

- General solution of the system of ODEs can be written in terms of the particular solution $\vec{y}_p($

$$\vec{y}(x) = c_1\vec{y}_1 + c_2\vec{y}_2(x) + c_3\vec{y}_3(x) + \vec{y}_p(x)$$

□ Fundamental matrix

- Let $\phi(x)$ be the matrix whose columns are the independent solutions of the homogeneous syst

$$\phi(x) = \begin{bmatrix} 1 & -\cos(x) & \sin(x) \\ 0 & \sin(x) & \cos(x) \\ 0 & \cos(x) & -\sin(x) \end{bmatrix}$$

- The fundamental matrix, $\Phi(x)$ is a normalized version of $\phi(x)$ satisfying $\Phi(0) = I$ where I is t

$$\Phi(x) = \phi(x) \cdot \frac{1}{\phi(0)}$$

- Substitute the value of $\phi(x)$ and $\phi(0)$

$$\Phi(x) = \begin{bmatrix} 1 & -\cos(x) & \sin(x) \\ 0 & \sin(x) & \cos(x) \\ 0 & \cos(x) & -\sin(x) \end{bmatrix} \cdot \frac{1}{\begin{bmatrix} 1 & -1 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}}$$

- Evaluate and simplify to get the fundamental matrix

$$\Phi(x) = \begin{bmatrix} 1 & \sin(x) & -\cos(x)+1 \\ 0 & \cos(x) & \sin(x) \\ 0 & -\sin(x) & \cos(x) \end{bmatrix}$$

☐ Find a particular solution of the system of ODEs using variation of parameters

- Let the particular solution be the fundamental matrix multiplied by $\overrightarrow{v}(x)$ and solve for $\overrightarrow{v}(x)$

  $\overrightarrow{y}_p(x) = \Phi(x) \cdot \overrightarrow{v}(x)$

- Take the derivative of the particular solution

  $\overrightarrow{y}'_p(x) = \Phi'(x) \cdot \overrightarrow{v}(x) + \Phi(x) \cdot \overrightarrow{v}'(x)$

- Substitute particular solution and its derivative into the system of ODEs

  $\Phi'(x) \cdot \overrightarrow{v}(x) + \Phi(x) \cdot \overrightarrow{v}'(x) = A \cdot \Phi(x) \cdot \overrightarrow{v}(x) + \overrightarrow{f}(x)$

- The fundamental matrix has columns that are solutions to the homogeneous system so its der

  $A \cdot \Phi(x) \cdot \overrightarrow{v}(x) + \Phi(x) \cdot \overrightarrow{v}'(x) = A \cdot \Phi(x) \cdot \overrightarrow{v}(x) + \overrightarrow{f}(x)$

- Cancel like terms

  $\Phi(x) \cdot \overrightarrow{v}'(x) = \overrightarrow{f}(x)$

- Multiply by the inverse of the fundamental matrix

  $\overrightarrow{v}'(x) = \frac{1}{\Phi(x)} \cdot \overrightarrow{f}(x)$

- Integrate to solve for $\overrightarrow{v}(x)$

  $\overrightarrow{v}(x) = \int_0^x \frac{1}{\Phi(s)} \cdot \overrightarrow{f}(s)\, ds$

- Plug $\overrightarrow{v}(x)$ into the equation for the particular solution

  $\overrightarrow{y}_p(x) = \Phi(x) \cdot \left(\int_0^x \frac{1}{\Phi(s)} \cdot \overrightarrow{f}(s)\, ds\right)$

- Plug in the fundamental matrix and the forcing function and compute

$$\vec{y}_p(x) = \begin{bmatrix} \frac{x(\sin(x)-\cos(x)x)}{4} \\ \frac{x^2\sin(x)}{4} + \frac{\sin(x)}{4} - \frac{\cos(x)x}{4} \\ \frac{x(\cos(x)x+3\sin(x))}{4} \end{bmatrix}$$

- Plug particular solution back into general solution

$$\vec{y}(x) = c_1\vec{y}_1 + c_2\vec{y}_2(x) + c_3\vec{y}_3(x) + \begin{bmatrix} \frac{x(\sin(x)-\cos(x)x)}{4} \\ \frac{x^2\sin(x)}{4} + \frac{\sin(x)}{4} - \frac{\cos(x)x}{4} \\ \frac{x(\cos(x)x+3\sin(x))}{4} \end{bmatrix}$$

- First component of the vector is the solution to the ODE

$$y = \frac{(-x^2-4c_2)\cos(x)}{4} + \frac{(4c_3+x)\sin(x)}{4} + c_1$$

Maple trace

```
`Methods for third order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 3; linear nonhomogeneous with symmetry [0,1]
-> Calling odsolve with the ODE`, diff(diff(_b(_a), _a), _a) = cos(_a)*_a+sin(_a)-_b(_a), _b
   Methods for second order ODEs:
   --- Trying classification methods ---
   trying a quadrature
   trying high order exact linear fully integrable
   trying differential order: 2; linear nonhomogeneous with symmetry [0,1]
   trying a double symmetry of the form [xi=0, eta=F(x)]
   -> Try solving first the homogeneous part of the ODE
      checking if the LODE has constant coefficients
      <- constant coefficients successful
   <- solving first the homogeneous part of the ODE successful
<- differential order: 3; linear nonhomogeneous with symmetry [0,1] successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 29

```
dsolve(diff(y(x),x$3)+diff(y(x),x)=sin(x)+x*cos(x),y(x), singsol=all)
```

$$y(x) = \frac{(-x^2 - 4c_2 + 2)\cos(x)}{4} + \frac{(x + 4c_1)\sin(x)}{4} + c_3$$

✓ Solution by Mathematica

Time used: 0.237 (sec). Leaf size: 36

```
DSolve[y'''[x]+y'[x]==Sin[x]+x*Cos[x],y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to -\frac{1}{8}(2x^2 - 3 + 8c_2)\cos(x) + \left(\frac{x}{4} + c_1\right)\sin(x) + c_3$$

2.19 problem 19

Internal problem ID [3260]
Internal file name [`OUTPUT/2752_Sunday_June_05_2022_08_40_02_AM_40663654/index.tex`]

Book: Differential equations for engineers by Wei-Chau XIE, Cambridge Press 2010
Section: Chapter 4. Linear Differential Equations. Page 183
Problem number: 19.
ODE order: 3.
ODE degree: 1.

The type(s) of ODE detected by this program : **"higher_order_linear_constant_coefficients_ODE"**

Maple gives the following as the ode type

```
[[_3rd_order, _linear, _nonhomogeneous]]
```

$$\boxed{y''' - 2y'' + 4y' - 8y = e^{2x} \sin(2x) + 2x^2}$$

This is higher order nonhomogeneous ODE. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE And y_p is a particular solution to the nonhomogeneous ODE. y_h is the solution to

$$y''' - 2y'' + 4y' - 8y = 0$$

The characteristic equation is

$$\lambda^3 - 2\lambda^2 + 4\lambda - 8 = 0$$

The roots of the above equation are

$$\lambda_1 = 2$$
$$\lambda_2 = 2i$$
$$\lambda_3 = -2i$$

Therefore the homogeneous solution is

$$y_h(x) = e^{2x} c_1 + e^{2ix} c_2 + e^{-2ix} c_3$$

The fundamental set of solutions for the homogeneous solution are the following

$$
\begin{aligned}
y_1 &= \mathrm{e}^{2x} \\
y_2 &= \mathrm{e}^{2ix} \\
y_3 &= \mathrm{e}^{-2ix}
\end{aligned}
$$

Now the particular solution to the given ODE is found

$$
y''' - 2y'' + 4y' - 8y = \mathrm{e}^{2x} \sin(2x) + 2x^2
$$

The particular solution is found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$
\mathrm{e}^{2x} \sin(2x) + 2x^2
$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$
[\{\mathrm{e}^{2x} \cos(2x), \mathrm{e}^{2x} \sin(2x)\}, \{1, x, x^2\}]
$$

While the set of the basis functions for the homogeneous solution found earlier is

$$
\{\mathrm{e}^{2x}, \mathrm{e}^{-2ix}, \mathrm{e}^{2ix}\}
$$

Since there is no duplication between the basis function in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis in the UC_set.

$$
y_p = A_1 \mathrm{e}^{2x} \cos(2x) + A_2 \mathrm{e}^{2x} \sin(2x) + A_3 + A_4 x + A_5 x^2
$$

The unknowns $\{A_1, A_2, A_3, A_4, A_5\}$ are found by substituting the above trial solution y_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$
\begin{aligned}
&-8A_1 \mathrm{e}^{2x} \sin(2x) - 16A_1 \mathrm{e}^{2x} \cos(2x) + 8A_2 \mathrm{e}^{2x} \cos(2x) - 16A_2 \mathrm{e}^{2x} \sin(2x) \\
&\quad - 4A_5 + 4A_4 + 8A_5 x - 8A_3 - 8A_4 x - 8A_5 x^2 = \mathrm{e}^{2x} \sin(2x) + 2x^2
\end{aligned}
$$

Solving for the unknowns by comparing coefficients results in

$$
\left[A_1 = -\frac{1}{40}, A_2 = -\frac{1}{20}, A_3 = 0, A_4 = -\frac{1}{4}, A_5 = -\frac{1}{4}\right]
$$

Substituting the above back in the above trial solution y_p, gives the particular solution

$$y_p = -\frac{e^{2x}\cos(2x)}{40} - \frac{e^{2x}\sin(2x)}{20} - \frac{x}{4} - \frac{x^2}{4}$$

Therefore the general solution is

$$\begin{aligned}y &= y_h + y_p \\ &= \left(e^{2x}c_1 + e^{2ix}c_2 + e^{-2ix}c_3\right) + \left(-\frac{e^{2x}\cos(2x)}{40} - \frac{e^{2x}\sin(2x)}{20} - \frac{x}{4} - \frac{x^2}{4}\right)\end{aligned}$$

Summary
The solution(s) found are the following

$$y = e^{2x}c_1 + e^{2ix}c_2 + e^{-2ix}c_3 - \frac{e^{2x}\cos(2x)}{40} - \frac{e^{2x}\sin(2x)}{20} - \frac{x}{4} - \frac{x^2}{4} \tag{1}$$

Verification of solutions

$$y = e^{2x}c_1 + e^{2ix}c_2 + e^{-2ix}c_3 - \frac{e^{2x}\cos(2x)}{40} - \frac{e^{2x}\sin(2x)}{20} - \frac{x}{4} - \frac{x^2}{4}$$

Verified OK.

2.19.1 Maple step by step solution

Let's solve

$y''' - 2y'' + 4y' - 8y = e^{2x}\sin(2x) + 2x^2$

- Highest derivative means the order of the ODE is 3

  y'''

- □ Convert linear ODE into a system of first order ODEs
  - Define new variable $y_1(x)$

    $y_1(x) = y$
  - Define new variable $y_2(x)$

    $y_2(x) = y'$
  - Define new variable $y_3(x)$

    $y_3(x) = y''$
  - Isolate for $y_3'(x)$ using original ODE

$y_3'(x) = e^{2x}\sin(2x) + 2x^2 + 2y_3(x) - 4y_2(x) + 8y_1(x)$

Convert linear ODE into a system of first order ODEs

$[y_2(x) = y_1'(x), y_3(x) = y_2'(x), y_3'(x) = e^{2x}\sin(2x) + 2x^2 + 2y_3(x) - 4y_2(x) + 8y_1(x)]$

- Define vector

$$\vec{y}(x) = \begin{bmatrix} y_1(x) \\ y_2(x) \\ y_3(x) \end{bmatrix}$$

- System to solve

$$\vec{y}'(x) = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 8 & -4 & 2 \end{bmatrix} \cdot \vec{y}(x) + \begin{bmatrix} 0 \\ 0 \\ e^{2x}\sin(2x) + 2x^2 \end{bmatrix}$$

- Define the forcing function

$$\vec{f}(x) = \begin{bmatrix} 0 \\ 0 \\ e^{2x}\sin(2x) + 2x^2 \end{bmatrix}$$

- Define the coefficient matrix

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 8 & -4 & 2 \end{bmatrix}$$

- Rewrite the system as

$$\vec{y}'(x) = A \cdot \vec{y}(x) + \vec{f}$$

- To solve the system, find the eigenvalues and eigenvectors of A
- Eigenpairs of A

$$\left[\left[2, \begin{bmatrix} \frac{1}{4} \\ \frac{1}{2} \\ 1 \end{bmatrix}\right], \left[-2\,\mathrm{I}, \begin{bmatrix} -\frac{1}{4} \\ \frac{\mathrm{I}}{2} \\ 1 \end{bmatrix}\right], \left[2\,\mathrm{I}, \begin{bmatrix} -\frac{1}{4} \\ -\frac{\mathrm{I}}{2} \\ 1 \end{bmatrix}\right]\right]$$

- Consider eigenpair

$$\left[2, \left[\begin{array}{c} \frac{1}{4} \\ \frac{1}{2} \\ 1 \end{array}\right]\right]$$

- Solution to homogeneous system from eigenpair

$$\overrightarrow{y}_1 = \mathrm{e}^{2x} \cdot \left[\begin{array}{c} \frac{1}{4} \\ \frac{1}{2} \\ 1 \end{array}\right]$$

- Consider complex eigenpair, complex conjugate eigenvalue can be ignored

$$\left[-2\,\mathrm{I}, \left[\begin{array}{c} -\frac{1}{4} \\ \frac{\mathrm{I}}{2} \\ 1 \end{array}\right]\right]$$

- Solution from eigenpair

$$\mathrm{e}^{-2\,\mathrm{I}x} \cdot \left[\begin{array}{c} -\frac{1}{4} \\ \frac{\mathrm{I}}{2} \\ 1 \end{array}\right]$$

- Use Euler identity to write solution in terms of $\sin$ and $\cos$

$$\left(\cos \left(2x\right) - \mathrm{I} \sin \left(2x\right)\right) \cdot \left[\begin{array}{c} -\frac{1}{4} \\ \frac{\mathrm{I}}{2} \\ 1 \end{array}\right]$$

- Simplify expression

$$\left[\begin{array}{c} -\frac{\cos(2x)}{4} + \frac{\mathrm{I} \sin(2x)}{4} \\ \frac{\mathrm{I}}{2}\left(\cos \left(2x\right) - \mathrm{I} \sin \left(2x\right)\right) \\ \cos \left(2x\right) - \mathrm{I} \sin \left(2x\right) \end{array}\right]$$

- Both real and imaginary parts are solutions to the homogeneous system

$$\left[\overrightarrow{y}_2(x) = \left[\begin{array}{c} -\frac{\cos(2x)}{4} \\ \frac{\sin(2x)}{2} \\ \cos \left(2x\right) \end{array}\right], \overrightarrow{y}_3(x) = \left[\begin{array}{c} \frac{\sin(2x)}{4} \\ \frac{\cos(2x)}{2} \\ -\sin \left(2x\right) \end{array}\right]\right]$$

- General solution of the system of ODEs can be written in terms of the particular solution $\vec{y}_p($

  $\vec{y}(x) = c_1\vec{y}_1 + c_2\vec{y}_2(x) + c_3\vec{y}_3(x) + \vec{y}_p(x)$

☐ Fundamental matrix

- Let $\phi(x)$ be the matrix whose columns are the independent solutions of the homogeneous syst

$$\phi(x) = \begin{bmatrix} \frac{e^{2x}}{4} & -\frac{\cos(2x)}{4} & \frac{\sin(2x)}{4} \\ \frac{e^{2x}}{2} & \frac{\sin(2x)}{2} & \frac{\cos(2x)}{2} \\ e^{2x} & \cos(2x) & -\sin(2x) \end{bmatrix}$$

- The fundamental matrix, $\Phi(x)$ is a normalized version of $\phi(x)$ satisfying $\Phi(0) = I$ where I is t

  $\Phi(x) = \phi(x) \cdot \frac{1}{\phi(0)}$

- Substitute the value of $\phi(x)$ and $\phi(0)$

$$\Phi(x) = \begin{bmatrix} \frac{e^{2x}}{4} & -\frac{\cos(2x)}{4} & \frac{\sin(2x)}{4} \\ \frac{e^{2x}}{2} & \frac{\sin(2x)}{2} & \frac{\cos(2x)}{2} \\ e^{2x} & \cos(2x) & -\sin(2x) \end{bmatrix} \cdot \frac{1}{\begin{bmatrix} \frac{1}{4} & -\frac{1}{4} & 0 \\ \frac{1}{2} & 0 & \frac{1}{2} \\ 1 & 1 & 0 \end{bmatrix}}$$

- Evaluate and simplify to get the fundamental matrix

$$\Phi(x) = \begin{bmatrix} \frac{e^{2x}}{2} + \frac{\cos(2x)}{2} - \frac{\sin(2x)}{2} & \frac{\sin(2x)}{2} & \frac{e^{2x}}{8} - \frac{\cos(2x)}{8} - \frac{\sin(2x)}{8} \\ e^{2x} - \sin(2x) - \cos(2x) & \cos(2x) & \frac{e^{2x}}{4} + \frac{\sin(2x)}{4} - \frac{\cos(2x)}{4} \\ 2e^{2x} - 2\cos(2x) + 2\sin(2x) & -2\sin(2x) & \frac{e^{2x}}{2} + \frac{\cos(2x)}{2} + \frac{\sin(2x)}{2} \end{bmatrix}$$

☐ Find a particular solution of the system of ODEs using variation of parameters

- Let the particular solution be the fundamental matrix multiplied by $\vec{v}(x)$ and solve for $\vec{v}(x)$

  $\vec{y}_p(x) = \Phi(x) \cdot \vec{v}(x)$

- Take the derivative of the particular solution

  $\vec{y}_p'(x) = \Phi'(x) \cdot \vec{v}(x) + \Phi(x) \cdot \vec{v}'(x)$

- Substitute particular solution and its derivative into the system of ODEs

  $\Phi'(x) \cdot \vec{v}(x) + \Phi(x) \cdot \vec{v}'(x) = A \cdot \Phi(x) \cdot \vec{v}(x) + \vec{f}(x)$

- The fundamental matrix has columns that are solutions to the homogeneous system so its der

  $A \cdot \Phi(x) \cdot \vec{v}(x) + \Phi(x) \cdot \vec{v}'(x) = A \cdot \Phi(x) \cdot \vec{v}(x) + \vec{f}(x)$

- Cancel like terms

  $\Phi(x)\cdot \vec{v}'(x)=\vec{f}(x)$

- Multiply by the inverse of the fundamental matrix

  $\vec{v}'(x)=\frac{1}{\Phi(x)}\cdot \vec{f}(x)$

- Integrate to solve for $\vec{v}(x)$

  $\vec{v}(x)=\int_0^x \frac{1}{\Phi(s)}\cdot \vec{f}(s)\,ds$

- Plug $\vec{v}(x)$ into the equation for the particular solution

  $\vec{y}_p(x)=\Phi(x)\cdot\left(\int_0^x \frac{1}{\Phi(s)}\cdot \vec{f}(s)\,ds\right)$

- Plug in the fundamental matrix and the forcing function and compute

$$\vec{y}_p(x)=\left[\begin{array}{c}\frac{(5-\cos(2x)-2\sin(2x))\mathrm{e}^{2x}}{40}-\frac{x^2}{4}-\frac{x}{4}-\frac{\cos(2x)}{10}+\frac{3\sin(2x)}{40}\\ \frac{(5-3\cos(2x)-\sin(2x))\mathrm{e}^{2x}}{20}-\frac{x}{2}+\frac{3\cos(2x)}{20}+\frac{\sin(2x)}{5}-\frac{1}{4}\\ \frac{(5-4\cos(2x)+2\sin(2x))\mathrm{e}^{2x}}{10}+\frac{2\cos(2x)}{5}-\frac{3\sin(2x)}{10}-\frac{1}{2}\end{array}\right]$$

- Plug particular solution back into general solution

$$\vec{y}(x)=c_1\vec{y}_1+c_2\vec{y}_2(x)+c_3\vec{y}_3(x)+\left[\begin{array}{c}\frac{(5-\cos(2x)-2\sin(2x))\mathrm{e}^{2x}}{40}-\frac{x^2}{4}-\frac{x}{4}-\frac{\cos(2x)}{10}+\frac{3\sin(2x)}{40}\\ \frac{(5-3\cos(2x)-\sin(2x))\mathrm{e}^{2x}}{20}-\frac{x}{2}+\frac{3\cos(2x)}{20}+\frac{\sin(2x)}{5}-\frac{1}{4}\\ \frac{(5-4\cos(2x)+2\sin(2x))\mathrm{e}^{2x}}{10}+\frac{2\cos(2x)}{5}-\frac{3\sin(2x)}{10}-\frac{1}{2}\end{array}\right]$$

- First component of the vector is the solution to the ODE

$$y=\frac{(10c_1-\cos(2x)-2\sin(2x)+5)\mathrm{e}^{2x}}{40}+\frac{(-2-5c_2)\cos(2x)}{20}+\frac{(10c_3+3)\sin(2x)}{40}-\frac{x^2}{4}-\frac{x}{4}$$

Maple trace

```
`Methods for third order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 3; linear nonhomogeneous with symmetry [0,1]
trying high order linear exact nonhomogeneous
trying differential order: 3; missing the dependent variable
checking if the LODE has constant coefficients
<- constant coefficients successful`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 58

```
dsolve(diff(y(x),x$3)-2*diff(y(x),x$2)+4*diff(y(x),x)-8*y(x)=exp(2*x)*sin(2*x)+2*x^2,y(x), s
```

$$y(x) = \frac{(80c_2 - 2\cos(2x) - 4\sin(2x) - 5)\,e^{2x}}{80} + \frac{(80c_1 - 5)\cos(2x)}{80} + \frac{(80c_3 + 5)\sin(2x)}{80} - \frac{x^2}{4} - \frac{x}{4}$$

✓ Solution by Mathematica

Time used: 0.241 (sec). Leaf size: 61

```
DSolve[y'''[x]-2*y''[x]+4*y'[x]-8*y[x]==Exp[2*x]*Sin[2*x]+2*x^2,y[x],x,IncludeSingularSoluti
```

$$y(x) \to \frac{1}{80}\left(-20x(x+1) + 5(-1+16c_3)e^{2x} - 2\left(e^{2x} - 40c_1\right)\cos(2x) - 4\left(e^{2x} - 20c_2\right)\sin(2x)\right)$$

2.20 problem 20

Internal problem ID [3261]
Internal file name [OUTPUT/2753_Sunday_June_05_2022_08_40_04_AM_76247354/index.tex]

Book: Differential equations for engineers by Wei-Chau XIE, Cambridge Press 2010
Section: Chapter 4. Linear Differential Equations. Page 183
Problem number: 20.
ODE order: 3.
ODE degree: 1.

The type(s) of ODE detected by this program : **"higher_order_linear_constant_coefficients_ODE"**

Maple gives the following as the ode type

```
[[_3rd_order, _missing_y]]
```

$$\boxed{y''' - 4y'' + 3y' = x^2 + x\,e^{2x}}$$

This is higher order nonhomogeneous ODE. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE And y_p is a particular solution to the nonhomogeneous ODE. y_h is the solution to

$$y''' - 4y'' + 3y' = 0$$

The characteristic equation is

$$\lambda^3 - 4\lambda^2 + 3\lambda = 0$$

The roots of the above equation are

$$\lambda_1 = 0$$
$$\lambda_2 = 3$$
$$\lambda_3 = 1$$

Therefore the homogeneous solution is

$$y_h(x) = c_1 + c_2 e^x + e^{3x} c_3$$

The fundamental set of solutions for the homogeneous solution are the following

$$y_1 = 1$$
$$y_2 = e^x$$
$$y_3 = e^{3x}$$

Now the particular solution to the given ODE is found

$$y''' - 4y'' + 3y' = x^2 + x\,e^{2x}$$

The particular solution is found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$x^2 + x\,e^{2x}$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$[\{x\,e^{2x}, e^{2x}\}, \{1, x, x^2\}]$$

While the set of the basis functions for the homogeneous solution found earlier is

$$\{1, e^x, e^{3x}\}$$

Since 1 is duplicated in the UC_set, then this basis is multiplied by extra x. The UC_set becomes

$$[\{x\,e^{2x}, e^{2x}\}, \{x, x^2, x^3\}]$$

Since there was duplication between the basis functions in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis function in the above updated UC_set.

$$y_p = A_1 x\,e^{2x} + A_2 e^{2x} + A_3 x + A_4 x^2 + A_5 x^3$$

The unknowns $\{A_1, A_2, A_3, A_4, A_5\}$ are found by substituting the above trial solution y_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$-A_1 e^{2x} - 2A_1 x\,e^{2x} - 2A_2 e^{2x} + 6A_5 - 8A_4 - 24A_5 x + 3A_3 + 6A_4 x + 9A_5 x^2 = x^2 + x\,e^{2x}$$

Solving for the unknowns by comparing coefficients results in

$$\left[A_1 = -\frac{1}{2}, A_2 = \frac{1}{4}, A_3 = \frac{26}{27}, A_4 = \frac{4}{9}, A_5 = \frac{1}{9}\right]$$

Substituting the above back in the above trial solution y_p, gives the particular solution

$$y_p = -\frac{x\,e^{2x}}{2} + \frac{e^{2x}}{4} + \frac{26x}{27} + \frac{4x^2}{9} + \frac{x^3}{9}$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \left(c_1 + c_2 e^{x} + e^{3x} c_3\right) + \left(-\frac{x\,e^{2x}}{2} + \frac{e^{2x}}{4} + \frac{26x}{27} + \frac{4x^2}{9} + \frac{x^3}{9}\right) \end{aligned}$$

Summary
The solution(s) found are the following

$$y = c_1 + c_2 e^{x} + e^{3x} c_3 - \frac{x\,e^{2x}}{2} + \frac{e^{2x}}{4} + \frac{26x}{27} + \frac{4x^2}{9} + \frac{x^3}{9} \tag{1}$$

Verification of solutions

$$y = c_1 + c_2 e^{x} + e^{3x} c_3 - \frac{x\,e^{2x}}{2} + \frac{e^{2x}}{4} + \frac{26x}{27} + \frac{4x^2}{9} + \frac{x^3}{9}$$

Verified OK.

2.20.1 Maple step by step solution

Let's solve

$y''' - 4y'' + 3y' = x^2 + x\,e^{2x}$

- Highest derivative means the order of the ODE is 3

  y'''

□ Convert linear ODE into a system of first order ODEs

  ◦ Define new variable $y_1(x)$

    $y_1(x) = y$

  ◦ Define new variable $y_2(x)$

    $y_2(x) = y'$

  ◦ Define new variable $y_3(x)$

    $y_3(x) = y''$

  ◦ Isolate for $y_3'(x)$ using original ODE

$y_3'(x) = x\,\mathrm{e}^{2x} + x^2 + 4y_3(x) - 3y_2(x)$

Convert linear ODE into a system of first order ODEs

$[y_2(x) = y_1'(x), y_3(x) = y_2'(x), y_3'(x) = x\,\mathrm{e}^{2x} + x^2 + 4y_3(x) - 3y_2(x)]$

- Define vector

$$\vec{y}(x) = \begin{bmatrix} y_1(x) \\ y_2(x) \\ y_3(x) \end{bmatrix}$$

- System to solve

$$\vec{y}'(x) = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & -3 & 4 \end{bmatrix} \cdot \vec{y}(x) + \begin{bmatrix} 0 \\ 0 \\ x^2 + x\,\mathrm{e}^{2x} \end{bmatrix}$$

- Define the forcing function

$$\vec{f}(x) = \begin{bmatrix} 0 \\ 0 \\ x^2 + x\,\mathrm{e}^{2x} \end{bmatrix}$$

- Define the coefficient matrix

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & -3 & 4 \end{bmatrix}$$

- Rewrite the system as

$\vec{y}'(x) = A \cdot \vec{y}(x) + \vec{f}$

- To solve the system, find the eigenvalues and eigenvectors of A
- Eigenpairs of A

$$\left[\left[0, \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}\right], \left[1, \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}\right], \left[3, \begin{bmatrix} \frac{1}{9} \\ \frac{1}{3} \\ 1 \end{bmatrix}\right]\right]$$

- Consider eigenpair

$$\left[0, \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}\right]$$

- Solution to homogeneous system from eigenpair

$$\overrightarrow{y}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

- Consider eigenpair

$$\left[1, \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}\right]$$

- Solution to homogeneous system from eigenpair

$$\overrightarrow{y}_2 = \mathrm{e}^{x} \cdot \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

- Consider eigenpair

$$\left[3, \begin{bmatrix} \frac{1}{9} \\ \frac{1}{3} \\ 1 \end{bmatrix}\right]$$

- Solution to homogeneous system from eigenpair

$$\overrightarrow{y}_3 = \mathrm{e}^{3x} \cdot \begin{bmatrix} \frac{1}{9} \\ \frac{1}{3} \\ 1 \end{bmatrix}$$

- General solution of the system of ODEs can be written in terms of the particular solution $\overrightarrow{y}_p($

$$\overrightarrow{y}(x) = c_1\overrightarrow{y}_1 + c_2\overrightarrow{y}_2 + c_3\overrightarrow{y}_3 + \overrightarrow{y}_p(x)$$

□ Fundamental matrix

- Let $\phi(x)$ be the matrix whose columns are the independent solutions of the homogeneous syst

$$\phi(x) = \begin{bmatrix} 1 & e^{x} & \frac{e^{3x}}{9} \\ 0 & e^{x} & \frac{e^{3x}}{3} \\ 0 & e^{x} & e^{3x} \end{bmatrix}$$

- The fundamental matrix, $\Phi(x)$ is a normalized version of $\phi(x)$ satisfying $\Phi(0) = I$ where I is t

  $\Phi(x) = \phi(x) \cdot \frac{1}{\phi(0)}$
- Substitute the value of $\phi(x)$ and $\phi(0)$

$$\Phi(x) = \begin{bmatrix} 1 & e^{x} & \frac{e^{3x}}{9} \\ 0 & e^{x} & \frac{e^{3x}}{3} \\ 0 & e^{x} & e^{3x} \end{bmatrix} \cdot \frac{1}{\begin{bmatrix} 1 & 1 & \frac{1}{9} \\ 0 & 1 & \frac{1}{3} \\ 0 & 1 & 1 \end{bmatrix}}$$

- Evaluate and simplify to get the fundamental matrix

$$\Phi(x) = \begin{bmatrix} 1 & -\frac{4}{3} + \frac{3\,e^{x}}{2} - \frac{e^{3x}}{6} & \frac{1}{3} - \frac{e^{x}}{2} + \frac{e^{3x}}{6} \\ 0 & \frac{3\,e^{x}}{2} - \frac{e^{3x}}{2} & -\frac{e^{x}}{2} + \frac{e^{3x}}{2} \\ 0 & \frac{3\,e^{x}}{2} - \frac{3\,e^{3x}}{2} & -\frac{e^{x}}{2} + \frac{3\,e^{3x}}{2} \end{bmatrix}$$

□ Find a particular solution of the system of ODEs using variation of parameters

- Let the particular solution be the fundamental matrix multiplied by $\vec{v}(x)$ and solve for $\vec{v}(x)$

  $\vec{y}_p(x) = \Phi(x) \cdot \vec{v}(x)$
- Take the derivative of the particular solution

  $\vec{y}'_p(x) = \Phi'(x) \cdot \vec{v}(x) + \Phi(x) \cdot \vec{v}'(x)$
- Substitute particular solution and its derivative into the system of ODEs

  $\Phi'(x) \cdot \vec{v}(x) + \Phi(x) \cdot \vec{v}'(x) = A \cdot \Phi(x) \cdot \vec{v}(x) + \vec{f}(x)$
- The fundamental matrix has columns that are solutions to the homogeneous system so its der

  $A \cdot \Phi(x) \cdot \vec{v}(x) + \Phi(x) \cdot \vec{v}'(x) = A \cdot \Phi(x) \cdot \vec{v}(x) + \vec{f}(x)$
- Cancel like terms

  $\Phi(x) \cdot \vec{v}'(x) = \vec{f}(x)$
- Multiply by the inverse of the fundamental matrix

  $\vec{v}'(x) = \frac{1}{\Phi(x)} \cdot \vec{f}(x)$

- Integrate to solve for $\vec{v}(x)$

  $\vec{v}(x) = \int_0^x \frac{1}{\Phi(s)} \cdot \vec{f}(s)\, ds$

- Plug $\vec{v}(x)$ into the equation for the particular solution

  $\vec{y}_p(x) = \Phi(x) \cdot \left(\int_0^x \frac{1}{\Phi(s)} \cdot \vec{f}(s)\, ds\right)$

- Plug in the fundamental matrix and the forcing function and compute

$$\vec{y}_p(x) = \begin{bmatrix} \frac{347}{324} + \frac{(1-2x)e^{2x}}{4} + \frac{x^3}{9} + \frac{4x^2}{9} + \frac{26x}{27} - \frac{3\,e^x}{2} + \frac{29\,e^{3x}}{162} \\ -x\,e^{2x} + \frac{x^2}{3} + \frac{8x}{9} + \frac{29\,e^{3x}}{54} + \frac{26}{27} - \frac{3\,e^x}{2} \\ \frac{8}{9} + (-1-2x)\,e^{2x} + \frac{2x}{3} - \frac{3\,e^x}{2} + \frac{29\,e^{3x}}{18} \end{bmatrix}$$

- Plug particular solution back into general solution

$$\vec{y}(x) = c_1\vec{y}_1 + c_2\vec{y}_2 + c_3\vec{y}_3 + \begin{bmatrix} \frac{347}{324} + \frac{(1-2x)e^{2x}}{4} + \frac{x^3}{9} + \frac{4x^2}{9} + \frac{26x}{27} - \frac{3\,e^x}{2} + \frac{29\,e^{3x}}{162} \\ -x\,e^{2x} + \frac{x^2}{3} + \frac{8x}{9} + \frac{29\,e^{3x}}{54} + \frac{26}{27} - \frac{3\,e^x}{2} \\ \frac{8}{9} + (-1-2x)\,e^{2x} + \frac{2x}{3} - \frac{3\,e^x}{2} + \frac{29\,e^{3x}}{18} \end{bmatrix}$$

- First component of the vector is the solution to the ODE

  $y = \frac{347}{324} + \frac{(1-2x)e^{2x}}{4} + \frac{(29+18c_3)e^{3x}}{162} + \frac{(-3+2c_2)e^x}{2} + \frac{x^3}{9} + \frac{4x^2}{9} + \frac{26x}{27} + c_1$

Maple trace

```
`Methods for third order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 3; linear nonhomogeneous with symmetry [0,1]
-> Calling odsolve with the ODE`, diff(diff(_b(_a), _a), _a) = exp(2*_a)*_a+_a^2-3*_b(_a)+4*
   Methods for second order ODEs:
   --- Trying classification methods ---
   trying a quadrature
   trying high order exact linear fully integrable
   trying differential order: 2; linear nonhomogeneous with symmetry [0,1]
   trying a double symmetry of the form [xi=0, eta=F(x)]
   -> Try solving first the homogeneous part of the ODE
      checking if the LODE has constant coefficients
      <- constant coefficients successful
   <- solving first the homogeneous part of the ODE successful
<- differential order: 3; linear nonhomogeneous with symmetry [0,1] successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 41

```
dsolve(diff(y(x),x$3)-4*diff(y(x),x$2)+3*diff(y(x),x)=x^2+x*exp(2*x),y(x), singsol=all)
```

$$y(x) = \frac{(1-2x)\,\mathrm{e}^{2x}}{4} + \frac{x^3}{9} + \frac{4x^2}{9} + \frac{c_1\mathrm{e}^{3x}}{3} + c_2\mathrm{e}^{x} + \frac{26x}{27} + c_3$$

✓ Solution by Mathematica

Time used: 0.239 (sec). Leaf size: 58

```
DSolve[y'''[x]-4*y''[x]+3*y'[x]==x^2+x*Exp[2*x],y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \frac{x^3}{9} + \frac{4x^2}{9} + \frac{26x}{27} + \frac{1}{4}e^{2x}(1-2x) + c_1e^x + \frac{1}{3}c_2e^{3x} + c_3$$

2.21 problem 21

Internal problem ID [3262]
Internal file name [`OUTPUT/2754_Sunday_June_05_2022_08_40_05_AM_24257591/index.tex`]

Book: Differential equations for engineers by Wei-Chau XIE, Cambridge Press 2010
Section: Chapter 4. Linear Differential Equations. Page 183
Problem number: 21.
ODE order: 4.
ODE degree: 1.

The type(s) of ODE detected by this program : **"higher_order_linear_constant_coefficients_ODE"**

Maple gives the following as the ode type

```
[[_high_order, _missing_y]]
```

$$\boxed{y'''' + 2y'' = 7x - 3\cos(x)}$$

This is higher order nonhomogeneous ODE. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE And y_p is a particular solution to the nonhomogeneous ODE. y_h is the solution to

$$y'''' + 2y'' = 0$$

The characteristic equation is

$$\lambda^4 + 2\lambda^2 = 0$$

The roots of the above equation are

$$\begin{aligned} \lambda_1 &= 0 \\ \lambda_2 &= 0 \\ \lambda_3 &= i\sqrt{2} \\ \lambda_4 &= -i\sqrt{2} \end{aligned}$$

Therefore the homogeneous solution is

$$y_h(x) = c_2 x + c_1 + e^{-i\sqrt{2}\,x} c_3 + e^{i\sqrt{2}\,x} c_4$$

The fundamental set of solutions for the homogeneous solution are the following

$$
\begin{aligned}
y_1 &= 1 \\
y_2 &= x \\
y_3 &= \mathrm{e}^{-i\sqrt{2}\,x} \\
y_4 &= \mathrm{e}^{i\sqrt{2}\,x}
\end{aligned}
$$

Now the particular solution to the given ODE is found

$$y'''' + 2y'' = 7x - 3\cos(x)$$

The particular solution is found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$7x - 3\cos(x)$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$[\{1, x\}, \{\cos(x), \sin(x)\}]$$

While the set of the basis functions for the homogeneous solution found earlier is

$$\left\{1, x, \mathrm{e}^{i\sqrt{2}\,x}, \mathrm{e}^{-i\sqrt{2}\,x}\right\}$$

Since 1 is duplicated in the UC_set, then this basis is multiplied by extra x. The UC_set becomes

$$[\{x, x^2\}, \{\cos(x), \sin(x)\}]$$

Since x is duplicated in the UC_set, then this basis is multiplied by extra x. The UC_set becomes

$$[\{x^2, x^3\}, \{\cos(x), \sin(x)\}]$$

Since there was duplication between the basis functions in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis function in the above updated UC_set.

$$y_p = A_2x^3 + A_1x^2 + A_3\cos(x) + A_4\sin(x)$$

The unknowns $\{A_1, A_2, A_3, A_4\}$ are found by substituting the above trial solution y_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$-A_3\cos(x) - A_4\sin(x) + 12A_2x + 4A_1 = 7x - 3\cos(x)$$

Solving for the unknowns by comparing coefficients results in

$$\left[A_1 = 0, A_2 = \frac{7}{12}, A_3 = 3, A_4 = 0\right]$$

Substituting the above back in the above trial solution y_p, gives the particular solution

$$y_p = \frac{7x^3}{12} + 3\cos(x)$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \left(c_2 x + c_1 + e^{-i\sqrt{2}\,x} c_3 + e^{i\sqrt{2}\,x} c_4\right) + \left(\frac{7x^3}{12} + 3\cos(x)\right) \end{aligned}$$

Summary

The solution(s) found are the following

$$y = c_2 x + c_1 + e^{-i\sqrt{2}\,x} c_3 + e^{i\sqrt{2}\,x} c_4 + \frac{7x^3}{12} + 3\cos(x) \tag{1}$$

Verification of solutions

$$y = c_2 x + c_1 + e^{-i\sqrt{2}\,x} c_3 + e^{i\sqrt{2}\,x} c_4 + \frac{7x^3}{12} + 3\cos(x)$$

Verified OK.

Maple trace

```
`Methods for high order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 4; linear nonhomogeneous with symmetry [0,1]
-> Calling odsolve with the ODE`, diff(diff(_b(_a), _a), _a) = -2*_b(_a)+7*_a-3*cos(_a), _b(
   Methods for second order ODEs:
   --- Trying classification methods ---
   trying a quadrature
   trying high order exact linear fully integrable
   trying differential order: 2; linear nonhomogeneous with symmetry [0,1]
   trying a double symmetry of the form [xi=0, eta=F(x)]
   -> Try solving first the homogeneous part of the ODE
      checking if the LODE has constant coefficients
      <- constant coefficients successful
   <- solving first the homogeneous part of the ODE successful
<- differential order: 4; linear nonhomogeneous with symmetry [0,1] successful`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 36

```
dsolve(diff(y(x),x$4)+2*diff(y(x),x$2)=7*x-3*cos(x),y(x), singsol=all)
```

$$y(x)=\frac{7x^3}{12}-\frac{\cos\left(\sqrt{2}\,x\right)c_1}{2}-\frac{c_2\sin\left(\sqrt{2}\,x\right)}{2}+3\cos\left(x\right)+c_3x+c_4$$

✓ Solution by Mathematica

Time used: 0.603 (sec). Leaf size: 51

```
DSolve[y''''[x]+2*y''[x]==7*x-3*Cos[x],y[x],x,IncludeSingularSolutions -> True]
```

$$y(x)\to\frac{7x^3}{12}+3\cos(x)+c_4x-\frac{1}{2}c_1\cos\left(\sqrt{2}x\right)-\frac{1}{2}c_2\sin\left(\sqrt{2}x\right)+c_3$$

2.22 problem 22

Internal problem ID [3263]
Internal file name [`OUTPUT/2755_Sunday_June_05_2022_08_40_05_AM_87500085/index.tex`]

Book: Differential equations for engineers by Wei-Chau XIE, Cambridge Press 2010
Section: Chapter 4. Linear Differential Equations. Page 183
Problem number: 22.
ODE order: 4.
ODE degree: 1.

The type(s) of ODE detected by this program : **"higher_order_linear_constant_coefficients_ODE"**

Maple gives the following as the ode type

```
[[_high_order, _linear, _nonhomogeneous]]
```

$$\boxed{y'''' + 5y'' + 4y = \sin(x)\cos(2x)}$$

This is higher order nonhomogeneous ODE. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE And y_p is a particular solution to the nonhomogeneous ODE. y_h is the solution to

$$y'''' + 5y'' + 4y = 0$$

The characteristic equation is

$$\lambda^4 + 5\lambda^2 + 4 = 0$$

The roots of the above equation are

$$\lambda_1 = 2i$$
$$\lambda_2 = -2i$$
$$\lambda_3 = i$$
$$\lambda_4 = -i$$

Therefore the homogeneous solution is

$$y_h(x) = e^{2ix}c_1 + e^{ix}c_2 + e^{-2ix}c_3 + e^{-ix}c_4$$

The fundamental set of solutions for the homogeneous solution are the following

$$y_1 = e^{2ix}$$
$$y_2 = e^{ix}$$
$$y_3 = e^{-2ix}$$
$$y_4 = e^{-ix}$$

Now the particular solution to the given ODE is found

$$y'''' + 5y'' + 4y = \sin(x)\cos(2x)$$

Let the particular solution be

$$y_p = U_1y_1 + U_2y_2 + U_3y_3 + U_4y_4$$

Where y_i are the basis solutions found above for the homogeneous solution y_h and $U_i(x)$ are functions to be determined as follows

$$U_i = (-1)^{n-i}\int \frac{F(x)W_i(x)}{aW(x)}\,dx$$

Where $W(x)$ is the Wronskian and $W_i(x)$ is the Wronskian that results after deleting the last row and the i-th column of the determinant and n is the order of the ODE or equivalently, the number of basis solutions, and a is the coefficient of the leading derivative in the ODE, and $F(x)$ is the RHS of the ODE. Therefore, the first step is to find the Wronskian $W(x)$. This is given by

$$W(x) = \begin{vmatrix} y_1 & y_2 & y_3 & y_4 \\ y_1' & y_2' & y_3' & y_4' \\ y_1'' & y_2'' & y_3'' & y_4'' \\ y_1''' & y_2''' & y_3''' & y_4''' \end{vmatrix}$$

Substituting the fundamental set of solutions y_i found above in the Wronskian gives

$$W = \begin{bmatrix} e^{2ix} & e^{ix} & e^{-2ix} & e^{-ix} \\ 2ie^{2ix} & ie^{ix} & -2ie^{-2ix} & -ie^{-ix} \\ -4\,e^{2ix} & -e^{ix} & -4\,e^{-2ix} & -e^{-ix} \\ -8ie^{2ix} & -ie^{ix} & 8ie^{-2ix} & ie^{-ix} \end{bmatrix}$$

$$|W| = 72\,e^{2ix}e^{ix}e^{-2ix}e^{-ix}$$

The determinant simplifies to

$$|W| = 72$$

Now we determine W_i for each U_i.

$$W_1(x) = \det \begin{bmatrix} e^{ix} & e^{-2ix} & e^{-ix} \\ ie^{ix} & -2ie^{-2ix} & -ie^{-ix} \\ -e^{ix} & -4\,e^{-2ix} & -e^{-ix} \end{bmatrix}$$
$$= -6ie^{-2ix}$$

$$W_2(x) = \det \begin{bmatrix} e^{2ix} & e^{-2ix} & e^{-ix} \\ 2ie^{2ix} & -2ie^{-2ix} & -ie^{-ix} \\ -4\,e^{2ix} & -4\,e^{-2ix} & -e^{-ix} \end{bmatrix}$$
$$= -12ie^{-ix}$$

$$W_3(x) = \det \begin{bmatrix} e^{2ix} & e^{ix} & e^{-ix} \\ 2ie^{2ix} & ie^{ix} & -ie^{-ix} \\ -4\,e^{2ix} & -e^{ix} & -e^{-ix} \end{bmatrix}$$
$$= 6ie^{2ix}$$

$$W_4(x) = \det \begin{bmatrix} e^{2ix} & e^{ix} & e^{-2ix} \\ 2ie^{2ix} & ie^{ix} & -2ie^{-2ix} \\ -4\,e^{2ix} & -e^{ix} & -4\,e^{-2ix} \end{bmatrix}$$
$$= 12ie^{ix}$$

Now we are ready to evaluate each $U_i(x)$.

$$\begin{aligned} U_1 &= (-1)^{4-1} \int \frac{F(x)W_1(x)}{aW(x)}\, dx \\ &= (-1)^3 \int \frac{(\sin(x)\cos(2x))\,(-6ie^{-2ix})}{(1)\,(72)}\, dx \\ &= -\int \frac{-6i\sin(x)\cos(2x)\,e^{-2ix}}{72}\, dx \\ &= -\int \left(-\frac{i\sin(x)\cos(2x)\,e^{-2ix}}{12}\right) dx \\ &= \frac{i\left(-\frac{e^{-2ix}\cos(x)}{6} - \frac{ie^{-2ix}\sin(x)}{3} - \frac{3\,e^{-2ix}\cos(3x)}{10} - \frac{ie^{-2ix}\sin(3x)}{5}\right)}{12} \end{aligned}$$

$$
\begin{aligned}
U_2 &= (-1)^{4-2} \int \frac{F(x)W_2(x)}{aW(x)}\,dx \\
&= (-1)^2 \int \frac{(\sin(x)\cos(2x))\,(-12i\mathrm{e}^{-ix})}{(1)\,(72)}\,dx \\
&= \int \frac{-12i\sin(x)\cos(2x)\,\mathrm{e}^{-ix}}{72}\,dx \\
&= \int \left(-\frac{i\sin(x)\cos(2x)\,\mathrm{e}^{-ix}}{6}\right)dx \\
&= \int -\frac{i\sin(x)\cos(2x)\,\mathrm{e}^{-ix}}{6}dx
\end{aligned}
$$

$$
\begin{aligned}
U_3 &= (-1)^{4-3} \int \frac{F(x)W_3(x)}{aW(x)}\,dx \\
&= (-1)^1 \int \frac{(\sin(x)\cos(2x))\,(6i\mathrm{e}^{2ix})}{(1)\,(72)}\,dx \\
&= -\int \frac{6i\sin(x)\cos(2x)\,\mathrm{e}^{2ix}}{72}\,dx \\
&= -\int \left(\frac{i\sin(x)\cos(2x)\,\mathrm{e}^{2ix}}{12}\right)dx \\
&= -\frac{i\left(-\frac{\mathrm{e}^{2ix}\cos(x)}{6}+\frac{i\mathrm{e}^{2ix}\sin(x)}{3}-\frac{3\,\mathrm{e}^{2ix}\cos(3x)}{10}+\frac{i\mathrm{e}^{2ix}\sin(3x)}{5}\right)}{12}
\end{aligned}
$$

$$
\begin{aligned}
U_4 &= (-1)^{4-4} \int \frac{F(x)W_4(x)}{aW(x)}\,dx \\
&= (-1)^0 \int \frac{(\sin(x)\cos(2x))\,(12i\mathrm{e}^{ix})}{(1)\,(72)}\,dx \\
&= \int \frac{12i\sin(x)\cos(2x)\,\mathrm{e}^{ix}}{72}\,dx \\
&= \int \left(\frac{i\sin(x)\cos(2x)\,\mathrm{e}^{ix}}{6}\right)dx \\
&= \int \frac{i\sin(x)\cos(2x)\,\mathrm{e}^{ix}}{6}dx
\end{aligned}
$$

Now that all the U_i functions have been determined, the particular solution is found from

$$
y_p = U_1y_1 + U_2y_2 + U_3y_3 + U_4y_4
$$

Hence

$$y_p = \left(\frac{i\left(-\frac{e^{-2ix}\cos(x)}{6} - \frac{ie^{-2ix}\sin(x)}{3} - \frac{3\,e^{-2ix}\cos(3x)}{10} - \frac{ie^{-2ix}\sin(3x)}{5}\right)}{12}\right)\left(e^{2ix}\right) + \left(\int -\frac{i\sin(x)\cos(2x)\,e^{-ix}}{6}dx\right)\left(e^{ix}\right) + \left(-\frac{i\left(-\frac{e^{2ix}\cos(x)}{6} + \frac{ie^{2ix}\sin(x)}{3} - \frac{3\,e^{2ix}\cos(3x)}{10} + \frac{ie^{2ix}\sin(3x)}{5}\right)}{12}\right)\left(e^{-2ix}\right) + \left(\int \frac{i\sin(x)\cos(2x)\,e^{ix}}{6}dx\right)\left(e^{-ix}\right)$$

Therefore the particular solution is

$$y_p = \frac{2\cos(x)^2\sin(x)}{15} + \frac{\sin(x)}{45} - \frac{i\left(\int \sin(x)\cos(2x)\,e^{-ix}dx\right)e^{ix}}{6} + \frac{i\left(\int \sin(x)\cos(2x)\,e^{ix}dx\right)e^{-ix}}{6}$$

Which simplifies to

$$y_p = -\frac{\left(\int \sin(x)^2\cos(2x)\,dx\right)\cos(x)}{3} + \frac{2\left(\cos(x)^2 + \frac{5\left(\int \sin(4x)dx\right)}{8} + \frac{1}{6}\right)\sin(x)}{15}$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \left(e^{2ix}c_1 + e^{ix}c_2 + e^{-2ix}c_3 + e^{-ix}c_4\right) \\ &\quad + \left(-\frac{\left(\int \sin(x)^2\cos(2x)\,dx\right)\cos(x)}{3} + \frac{2\left(\cos(x)^2 + \frac{5\left(\int \sin(4x)dx\right)}{8} + \frac{1}{6}\right)\sin(x)}{15}\right)\end{aligned}$$

Summary

The solution(s) found are the following

$$y = e^{2ix}c_1 + e^{ix}c_2 + e^{-2ix}c_3 + e^{-ix}c_4 - \frac{\left(\int \sin(x)^2\cos(2x)\,dx\right)\cos(x)}{3} + \frac{2\left(\cos(x)^2 + \frac{5\left(\int \sin(4x)dx\right)}{8} + \frac{1}{6}\right)\sin(x)}{15} \tag{1}$$

Verification of solutions

$$y = e^{2ix}c_1 + e^{ix}c_2 + e^{-2ix}c_3 + e^{-ix}c_4 - \frac{\left(\int \sin(x)^2 \cos(2x) \, dx\right) \cos(x)}{3} + \frac{2\left(\cos(x)^2 + \frac{5\left(\int \sin(4x) dx\right)}{8} + \frac{1}{6}\right) \sin(x)}{15}$$

Verified OK.

2.22.1 Maple step by step solution

Let's solve

$y'''' + 5y'' + 4y = \sin(x)\cos(2x)$

- Highest derivative means the order of the ODE is 4

  y''''

□ Convert linear ODE into a system of first order ODEs

  ◦ Define new variable $y_1(x)$

    $y_1(x) = y$

  ◦ Define new variable $y_2(x)$

    $y_2(x) = y'$

  ◦ Define new variable $y_3(x)$

    $y_3(x) = y''$

  ◦ Define new variable $y_4(x)$

    $y_4(x) = y'''$

  ◦ Isolate for $y_4'(x)$ using original ODE

    $y_4'(x) = \sin(x)\cos(2x) - 5y_3(x) - 4y_1(x)$

  Convert linear ODE into a system of first order ODEs

  $[y_2(x) = y_1'(x), y_3(x) = y_2'(x), y_4(x) = y_3'(x), y_4'(x) = \sin(x)\cos(2x) - 5y_3(x) - 4y_1(x)]$

- Define vector

$$\vec{y}(x) = \begin{bmatrix} y_1(x) \\ y_2(x) \\ y_3(x) \\ y_4(x) \end{bmatrix}$$

- System to solve

$$\overrightarrow{y}'(x) = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -4 & 0 & -5 & 0 \end{bmatrix} \cdot \overrightarrow{y}(x) + \begin{bmatrix} 0 \\ 0 \\ 0 \\ \sin(x)\cos(2x) \end{bmatrix}$$

- Define the forcing function

$$\overrightarrow{f}(x) = \begin{bmatrix} 0 \\ 0 \\ 0 \\ \sin(x)\cos(2x) \end{bmatrix}$$

- Define the coefficient matrix

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -4 & 0 & -5 & 0 \end{bmatrix}$$

- Rewrite the system as

$$\overrightarrow{y}'(x) = A \cdot \overrightarrow{y}(x) + \overrightarrow{f}$$

- To solve the system, find the eigenvalues and eigenvectors of A
- Eigenpairs of A

$$\left[\left[-2\,\mathrm{I}, \begin{bmatrix} -\frac{\mathrm{I}}{8} \\ -\frac{1}{4} \\ \frac{\mathrm{I}}{2} \\ 1 \end{bmatrix}\right], \left[-\mathrm{I}, \begin{bmatrix} -\mathrm{I} \\ -1 \\ \mathrm{I} \\ 1 \end{bmatrix}\right], \left[\mathrm{I}, \begin{bmatrix} \mathrm{I} \\ -1 \\ -\mathrm{I} \\ 1 \end{bmatrix}\right], \left[2\,\mathrm{I}, \begin{bmatrix} \frac{\mathrm{I}}{8} \\ -\frac{1}{4} \\ -\frac{\mathrm{I}}{2} \\ 1 \end{bmatrix}\right]\right]$$

- Consider complex eigenpair, complex conjugate eigenvalue can be ignored

$$\left[-2\,\mathrm{I}, \begin{bmatrix} -\frac{\mathrm{I}}{8} \\ -\frac{1}{4} \\ \frac{\mathrm{I}}{2} \\ 1 \end{bmatrix}\right]$$

- Solution from eigenpair

$$e^{-2 I x} \cdot \begin{bmatrix} -\frac{I}{8} \\ -\frac{1}{4} \\ \frac{I}{2} \\ 1 \end{bmatrix}$$

- Use Euler identity to write solution in terms of sin and cos

$$(\cos(2x) - I\sin(2x)) \cdot \begin{bmatrix} -\frac{I}{8} \\ -\frac{1}{4} \\ \frac{I}{2} \\ 1 \end{bmatrix}$$

- Simplify expression

$$\begin{bmatrix} -\frac{I}{8}(\cos(2x) - I\sin(2x)) \\ -\frac{\cos(2x)}{4} + \frac{I\sin(2x)}{4} \\ \frac{I}{2}(\cos(2x) - I\sin(2x)) \\ \cos(2x) - I\sin(2x) \end{bmatrix}$$

- Both real and imaginary parts are solutions to the homogeneous system

$$\left[\vec{y}_1(x) = \begin{bmatrix} -\frac{\sin(2x)}{8} \\ -\frac{\cos(2x)}{4} \\ \frac{\sin(2x)}{2} \\ \cos(2x) \end{bmatrix}, \vec{y}_2(x) = \begin{bmatrix} -\frac{\cos(2x)}{8} \\ \frac{\sin(2x)}{4} \\ \frac{\cos(2x)}{2} \\ -\sin(2x) \end{bmatrix}\right]$$

- Consider complex eigenpair, complex conjugate eigenvalue can be ignored

$$\left[-I, \begin{bmatrix} -I \\ -1 \\ I \\ 1 \end{bmatrix}\right]$$

- Solution from eigenpair

$$e^{-Ix} \cdot \begin{bmatrix} -I \\ -1 \\ I \\ 1 \end{bmatrix}$$

- Use Euler identity to write solution in terms of sin and cos

$$(\cos(x) - I\sin(x)) \cdot \begin{bmatrix} -I \\ -1 \\ I \\ 1 \end{bmatrix}$$

- Simplify expression

$$\begin{bmatrix} -I(\cos(x) - I\sin(x)) \\ -\cos(x) + I\sin(x) \\ I(\cos(x) - I\sin(x)) \\ \cos(x) - I\sin(x) \end{bmatrix}$$

- Both real and imaginary parts are solutions to the homogeneous system

$$\left[\vec{y}_3(x) = \begin{bmatrix} -\sin(x) \\ -\cos(x) \\ \sin(x) \\ \cos(x) \end{bmatrix}, \vec{y}_4(x) = \begin{bmatrix} -\cos(x) \\ \sin(x) \\ \cos(x) \\ -\sin(x) \end{bmatrix}\right]$$

- General solution of the system of ODEs can be written in terms of the particular solution $\vec{y}_p($

$$\vec{y}(x) = c_1\vec{y}_1(x) + c_2\vec{y}_2(x) + c_3\vec{y}_3(x) + c_4\vec{y}_4(x) + \vec{y}_p(x)$$

☐ Fundamental matrix

- Let $\phi(x)$ be the matrix whose columns are the independent solutions of the homogeneous syst

$$\phi(x) = \begin{bmatrix} -\frac{\sin(2x)}{8} & -\frac{\cos(2x)}{8} & -\sin(x) & -\cos(x) \\ -\frac{\cos(2x)}{4} & \frac{\sin(2x)}{4} & -\cos(x) & \sin(x) \\ \frac{\sin(2x)}{2} & \frac{\cos(2x)}{2} & \sin(x) & \cos(x) \\ \cos(2x) & -\sin(2x) & \cos(x) & -\sin(x) \end{bmatrix}$$

- The fundamental matrix, $\Phi(x)$ is a normalized version of $\phi(x)$ satisfying $\Phi(0) = I$ where I is t

$\Phi(x) = \phi(x) \cdot \frac{1}{\phi(0)}$

- Substitute the value of $\phi(x)$ and $\phi(0)$

$$\Phi(x) = \begin{bmatrix} -\frac{\sin(2x)}{8} & -\frac{\cos(2x)}{8} & -\sin(x) & -\cos(x) \\ -\frac{\cos(2x)}{4} & \frac{\sin(2x)}{4} & -\cos(x) & \sin(x) \\ \frac{\sin(2x)}{2} & \frac{\cos(2x)}{2} & \sin(x) & \cos(x) \\ \cos(2x) & -\sin(2x) & \cos(x) & -\sin(x) \end{bmatrix} \cdot \frac{1}{\begin{bmatrix} 0 & -\frac{1}{8} & 0 & -1 \\ -\frac{1}{4} & 0 & -1 & 0 \\ 0 & \frac{1}{2} & 0 & 1 \\ 1 & 0 & 1 & 0 \end{bmatrix}}$$

- Evaluate and simplify to get the fundamental matrix

$$\Phi(x) = \begin{bmatrix} -\frac{2\cos(x)^2}{3} + \frac{1}{3} + \frac{4\cos(x)}{3} & -\frac{\sin(2x)}{6} + \frac{4\sin(x)}{3} & -\frac{2\cos(x)^2}{3} + \frac{1}{3} + \frac{\cos(x)}{3} & -\frac{\sin(2x)}{6} + \mathrm{si} \\ \frac{2\sin(2x)}{3} - \frac{4\sin(x)}{3} & -\frac{2\cos(x)^2}{3} + \frac{1}{3} + \frac{4\cos(x)}{3} & \frac{2\sin(2x)}{3} - \frac{\sin(x)}{3} & -\frac{2\cos(x)^2}{3} + \frac{1}{3} \\ \frac{8\cos(x)^2}{3} - \frac{4}{3} - \frac{4\cos(x)}{3} & \frac{2\sin(2x)}{3} - \frac{4\sin(x)}{3} & \frac{8\cos(x)^2}{3} - \frac{4}{3} - \frac{\cos(x)}{3} & \frac{2\sin(2x)}{3} - \mathrm{si} \\ -\frac{8\sin(2x)}{3} + \frac{4\sin(x)}{3} & \frac{8\cos(x)^2}{3} - \frac{4}{3} - \frac{4\cos(x)}{3} & -\frac{8\sin(2x)}{3} + \frac{\sin(x)}{3} & \frac{8\cos(x)^2}{3} - \frac{4}{3} - \end{bmatrix}$$

☐ Find a particular solution of the system of ODEs using variation of parameters

- Let the particular solution be the fundamental matrix multiplied by $\vec{v}(x)$ and solve for $\vec{v}(x)$

  $\vec{y}_p(x) = \Phi(x) \cdot \vec{v}(x)$

- Take the derivative of the particular solution

  $\vec{y}_p'(x) = \Phi'(x) \cdot \vec{v}(x) + \Phi(x) \cdot \vec{v}'(x)$

- Substitute particular solution and its derivative into the system of ODEs

  $\Phi'(x) \cdot \vec{v}(x) + \Phi(x) \cdot \vec{v}'(x) = A \cdot \Phi(x) \cdot \vec{v}(x) + \vec{f}(x)$

- The fundamental matrix has columns that are solutions to the homogeneous system so its der

  $A \cdot \Phi(x) \cdot \vec{v}(x) + \Phi(x) \cdot \vec{v}'(x) = A \cdot \Phi(x) \cdot \vec{v}(x) + \vec{f}(x)$

- Cancel like terms

  $\Phi(x) \cdot \vec{v}'(x) = \vec{f}(x)$

- Multiply by the inverse of the fundamental matrix

  $\vec{v}'(x) = \frac{1}{\Phi(x)} \cdot \vec{f}(x)$

- Integrate to solve for $\vec{v}(x)$

$$\vec{v}(x) = \int_0^x \frac{1}{\Phi(s)} \cdot \vec{f}(s)\, ds$$

- Plug $\vec{v}(x)$ into the equation for the particular solution

$$\vec{y}_p(x) = \Phi(x) \cdot \left(\int_0^x \frac{1}{\Phi(s)} \cdot \vec{f}(s)\, ds\right)$$

- Plug in the fundamental matrix and the forcing function and compute

$$\vec{y}_p(x) = \begin{bmatrix} \frac{\cos(x)^2 \sin(x)}{20} + \frac{(15x-28\sin(x))\cos(x)}{180} + \frac{\sin(x)}{45} \\ \frac{3\cos(x)^3}{20} - \frac{14\cos(x)^2}{45} - \frac{x\sin(x)}{12} + \frac{\cos(x)}{180} + \frac{7}{45} \\ -\frac{9\cos(x)^2\sin(x)}{20} + \frac{(-15x+112\sin(x))\cos(x)}{180} - \frac{4\sin(x)}{45} \\ -\frac{27\cos(x)^3}{20} + \frac{56\cos(x)^2}{45} + \frac{x\sin(x)}{12} + \frac{131\cos(x)}{180} - \frac{28}{45} \end{bmatrix}$$

- Plug particular solution back into general solution

$$\vec{y}(x) = c_1\vec{y}_1(x) + c_2\vec{y}_2(x) + c_3\vec{y}_3(x) + c_4\vec{y}_4(x) + \begin{bmatrix} \frac{\cos(x)^2 \sin(x)}{20} + \frac{(15x-28\sin(x))\cos(x)}{180} + \frac{\sin(x)}{45} \\ \frac{3\cos(x)^3}{20} - \frac{14\cos(x)^2}{45} - \frac{x\sin(x)}{12} + \frac{\cos(x)}{180} + \frac{7}{45} \\ -\frac{9\cos(x)^2\sin(x)}{20} + \frac{(-15x+112\sin(x))\cos(x)}{180} - \frac{4\sin(}{45} \\ -\frac{27\cos(x)^3}{20} + \frac{56\cos(x)^2}{45} + \frac{x\sin(x)}{12} + \frac{131\cos(x)}{180} - \end{bmatrix}$$

- First component of the vector is the solution to the ODE

$$y = \frac{(-5c_2+\sin(x))\cos(x)^2}{20} + \frac{((-28-45c_1)\sin(x)+15x-180c_4)\cos(x)}{180} + \frac{(1-45c_3)\sin(x)}{45} + \frac{c_2}{8}$$

Maple trace

```
`Methods for high order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 4; linear nonhomogeneous with symmetry [0,1]
trying high order linear exact nonhomogeneous
trying differential order: 4; missing the dependent variable
checking if the LODE has constant coefficients
<- constant coefficients successful`
```

✓ Solution by Maple

Time used: 0.015 (sec). Leaf size: 43

```
dsolve(diff(y(x),x$4)+5*diff(y(x),x$2)+4*y(x)=sin(x)*cos(2*x),y(x), singsol=all)
```

$$y(x) = \frac{(40c_3 + \sin(x))\cos(x)^2}{20} + \frac{(24c_4 \sin(x) + x + 12c_1)\cos(x)}{12} + \frac{(360c_2 - 7)\sin(x)}{360} - c_3$$

✓ Solution by Mathematica

Time used: 0.04 (sec). Leaf size: 50

```
DSolve[y''''[x]+5*y''[x]+4*y[x]==Sin[x]*Cos[2*x],y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \frac{\sin(x)}{72} + \frac{1}{80}\sin(3x) + \left(\frac{x}{12} + c_3\right)\cos(x) + c_1\cos(2x) + c_4\sin(x) + c_2\sin(2x)$$